Tarea 04

Matemáticas para las Ciencias Aplicadas II
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1.

Verifique que la función $z = \ln\left[e^x + e^y\right]$ es una solución de las ecuaciones diferenciales:

- $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 1$

  Primero calculamos las derivadas parciales de primer orden necesarias:

  Para $\frac{\partial z}{\partial x}$

$$\begin{aligned}\frac{\partial z}{\partial x} &= \frac{1}{e^x + e^y} \cdot \frac{\partial}{\partial x}(e^x + e^y) \\ &= \frac{1}{e^x + e^y} \cdot e^x \\ &= \frac{e^x}{e^x + e^y}\end{aligned}$$

  Para $\frac{\partial z}{\partial y}$

$$\begin{aligned}\frac{\partial z}{\partial x} &= \frac{1}{e^x + e^y} \cdot \frac{\partial}{\partial y}(e^x + e^y) \\ &= \frac{1}{e^x + e^y} \cdot e^y \\ &= \frac{e^y}{e^x + e^y}\end{aligned}$$

Así, $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}$

$$\begin{aligned}\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} &= \frac{e^x}{e^x + e^y} + \frac{e^y}{e^x + e^y} \\ &= \frac{e^x + e^y}{e^x + e^y} \\ &= 1\end{aligned}$$

$\therefore z = \ln\left[e^x + e^y\right]$ satisface la ecuación $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 1$.

- $\frac{\partial^2 z}{\partial x^2}\frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 = 0$

  Primero, a partir de $\frac{\partial z}{\partial x} = \frac{e^x}{e^x+e^y}$ y $\frac{\partial z}{\partial y} = \frac{e^y}{e^x+e^y}$, calculamos las derivadas parciales de segundo orden necesarias:

  Para $\frac{\partial^2 z}{\partial^2 x}$

$$\begin{aligned}\frac{\partial^2 z}{\partial^2 x} &= \frac{\partial}{\partial x}\left(\frac{e^x}{e^x + e^y}\right) \\ &= \frac{\left[(e^x + e^y)\cdot e^x\right] - \left[e^x \cdot e^x\right]}{(e^x + e^y)^2} \\ &= \frac{e^{2x} + e^{x+y} - e^{2x}}{(e^x + e^y)^2} \\ &= \frac{e^{x+y}}{(e^x + e^y)^2}\end{aligned}$$

  Para $\frac{\partial^2 z}{\partial^2 y}$

$$\begin{aligned}\frac{\partial^2 z}{\partial^2 y} &= \frac{\partial}{\partial y}\left(\frac{e^y}{e^x + e^y}\right) \\ &= \frac{\left[(e^x + e^y)\cdot e^y\right] - \left[e^y \cdot e^y\right]}{(e^x + e^y)^2} \\ &= \frac{e^{x+y} + e^{2y} - e^{2y}}{(e^x + e^y)^2} \\ &= \frac{e^{x+y}}{(e^x + e^y)^2}\end{aligned}$$

Para $\frac{\partial^2 z}{\partial x \partial y}$

$$\begin{aligned}
\frac{\partial^2 z}{\partial x \partial y} &= \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{e^y}{e^x + e^y}\right) \\
&= \frac{[(e^x + e^y)\cdot 0] - (e^y \cdot e^x)}{(e^x + e^y)^2} \\
&= -\frac{e^{x+y}}{(e^x + e^y)^2}
\end{aligned}$$

Así, $\frac{\partial^2 z}{\partial x^2}\frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2$

$$\begin{aligned}
\frac{\partial^2 z}{\partial x^2}\frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 &= \left[\frac{e^{x+y}}{(e^x + e^y)^2} \cdot \frac{e^{x+y}}{(e^x + e^y)^2}\right] - \left[-\frac{e^{x+y}}{(e^x + e^y)^2}\right]^2 \\
&= \frac{(e^{x+y})^2}{(e^{x+y})^4} - \frac{(e^{x+y})^2}{(e^{x+y})^4} \\
&= 0
\end{aligned}$$

$\therefore z = \ln\left[e^x + e^y\right]$ satisface la ecuación $\frac{\partial^2 z}{\partial x^2}\frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 = 0$.

2.

La *energía cinética* de un cuerpo de masa m y velocidad v es $K = \frac{1}{2}mv^2$. Demuestre que $K = \frac{\partial K}{\partial m}\frac{\partial^2 K}{\partial v^2}$.

Primero calculamos las derivadas parciales de primer orden necesarias:
Para $\frac{\partial K}{\partial m}$

$$\begin{aligned}
\frac{\partial K}{\partial m} &= \frac{1}{2}\left(\frac{\partial}{\partial m} mv^2\right) \\
&= \frac{1}{2}v^2
\end{aligned}$$

Para $\frac{\partial K}{\partial v}$

$$\begin{aligned}\frac{\partial K}{\partial v} &= \frac{1}{2}\left(\frac{\partial}{\partial v} m v^2\right) \\ &= \frac{1}{2} \cdot 2vm \\ &= vm\end{aligned}$$

Ahora, calculamos la derivada parcial de segundo orden necesaria:
Para $\frac{\partial K}{\partial v^2}$

$$\begin{aligned}\frac{\partial K}{\partial v^2} &= \frac{\partial K}{\partial v}(vm) \\ &= m\end{aligned}$$

Así, $\frac{\partial K}{\partial m}\frac{\partial^2 K}{\partial v^2}$

$$\begin{aligned}\frac{\partial K}{\partial m}\frac{\partial^2 K}{\partial v^2} &= \frac{1}{2} v^2 \cdot m \\ &= \frac{1}{2} m v^2 \\ &= K\end{aligned}$$

$\therefore K = \frac{1}{2} m v^2$ satisface la ecuación $K = \frac{\partial K}{\partial m}\frac{\partial^2 K}{\partial v^2}$.

$\blacksquare$

3.

Determine una ecuación del plano tangente a la función $z = xe^{xy}$ en el punto $(x_0, y_0) = (5, 0)$.

Sea $z = xe^{xy}$. Entonces

$$
\begin{aligned}
f_x(x,y) &= x\frac{\partial}{\partial x}e^{xy} + e^{xy}\frac{\partial}{\partial x}x \\
&= xye^{xy} + e^{xy} \\
&= (xy+1)e^{xy} \\
f_x(5,0) &= (5\cdot 0+1)e^{5\cdot 0} \\
&= 1\cdot e^0 \\
&= 1
\end{aligned}
$$

$$
\begin{aligned}
f_y(x,y) &= x\frac{\partial}{\partial y}e^{xy} + e^{xy}\frac{\partial}{\partial y}x \\
&= x^2e^{xy} + 0 \\
&= x^2e^{xy} \\
f_y(5,0) &= 5^2\cdot e^{5\cdot 0} \\
&= 5^2\cdot e^0 \\
&= 25\cdot 1 \\
&= 25
\end{aligned}
$$

Dado que $x = 5$ y $y = 0$, se tiene que $z = 5\cdot e^{5\cdot 0} = 5\cdot e^0 = 5$ Entonces, da la ecuación del plano tangente en $(5,0,5)$ como

$$z - 5 = 1(x-5) + 25(y-0)$$

o bien,

$$z = x + 25y$$

4.

Compruebe que la **aproximación lineal** en (0, 0).

$$\frac{2x+3}{4y+1} \approx 2x - 12y + 3$$

Sea $f(x,y)=\frac{2x+3}{4y+1}$. Tenemos que las derivadas parciales son

$$\begin{aligned}
f_x(x,y) &= \frac{\left[(4y+1)\cdot\frac{\partial}{\partial x}(2x+3)\right]-\left[(2x+3)\frac{\partial}{\partial x}(4y+1)\right]}{(4y+1)^2} \\
&= \frac{2(4y+1)-0}{(4y+1)^2} \\
&= \frac{2(4y+1)}{(4y+1)(4y+1)} \\
&= \frac{2}{4y+1} \\
f_x(0,0) &= 2
\end{aligned}$$

$$\begin{aligned}
f_y(x,y) &= \frac{\left[(4y+1)\cdot\frac{\partial}{\partial y}(2x+3)\right]-\left[(2x+3)\frac{\partial}{\partial y}(4y+1)\right]}{(4y+1)^2} \\
&= \frac{0-4(2x+3)}{(4y+1)^2} \\
&= \frac{-8x-12}{(4y+1)^2} \\
f_y(0,0) &= -12
\end{aligned}$$

Tanto f_x como f_y son continuas y existen cerca de $(0,0)$, de modo que f es diferenciable en $(0,0)$. La linealización es

$$\begin{aligned}
L(x,y) &= f(0,0)+2(x-0)+(-12)(y-0) \\
&= 2x-12y+3
\end{aligned}$$

5.

Utilice la **regla de la cadena** para calcular $\frac{\partial z}{\partial s}$ y $\frac{\partial z}{\partial t}$. Dado que

$$z=\sin\theta\cos\phi,\ \theta=st^2,\ \phi=s^2t$$

Primero calculamos las derivadas parciales de primer orden necesarias:

$$
\begin{aligned}
\frac{\partial z}{\partial \theta} &= \frac{\partial}{\partial \theta}(\sin\theta\cos\phi) \\
&= \left(\sin\theta \cdot \frac{\partial}{\partial \theta}\cos\phi\right) + \left(\cos\phi \cdot \frac{\partial}{\partial \theta}\sin\theta\right) \\
&= 0 + \cos\phi\cos\theta \\
&= \cos\theta\cos\phi \\
\frac{\partial z}{\partial \phi} &= \frac{\partial}{\partial \phi}(\sin\theta\cos\phi) \\
&= \left(\sin\theta \cdot \frac{\partial}{\partial \phi}\cos\phi\right) + \left(\cos\phi \cdot \frac{\partial}{\partial \phi}\sin\theta\right) \\
&= (\sin\theta \cdot -\sin\phi) + 0 \\
&= -\sin\theta\sin\phi
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \theta}{\partial s} &= \frac{\partial}{\partial s}(st^2) \\
&= \left(s \cdot \frac{\partial}{\partial s}t^2\right) + \left(t^2 \cdot \frac{\partial}{\partial s}s\right) \\
&= 0 + t^2 \\
&= t^2 \\
\frac{\partial \phi}{\partial s} &= \frac{\partial}{\partial s}(s^2 t) \\
&= \left(s^2 \cdot \frac{\partial}{\partial s}t\right) + \left(t \cdot \frac{\partial}{\partial s}s^2\right) \\
&= 0 + 2st \\
&= 2st
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial\theta}{\partial t} &= \frac{\partial}{\partial t}(st^2) \\
&= \left(s\cdot\frac{\partial}{\partial t}t^2\right) + \left(t^2\cdot\frac{\partial}{\partial t}s\right) \\
&= 2ts + 0 \\
&= 2st \\
\frac{\partial\phi}{\partial t} &= \frac{\partial}{\partial t}(s^2t) \\
&= \left(s^2\cdot\frac{\partial}{\partial t}t\right) + \left(t\cdot\frac{\partial}{\partial t}s^2\right) \\
&= s^2 + 0 \\
&= s^2
\end{aligned}
$$

Al aplicar el caso 2 de la regla de la cadena, obtenemos

$$
\begin{aligned}
\frac{\partial z}{\partial s} &= \frac{\partial z}{\partial\theta}\frac{\partial\theta}{\partial s} + \frac{\partial z}{\partial\phi}\frac{\partial\phi}{\partial s} \\
&= (\cos\theta\cos\phi\cdot t^2) + (-\sin\theta\sin\phi\cdot 2st) \\
&= t^2\cos\theta\cos\phi - 2st\sin\theta\sin\phi \\
\therefore \frac{\partial z}{\partial s} &= t^2\cos\theta\cos\phi - 2st\sin\theta\sin\phi
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial z}{\partial t} &= \frac{\partial z}{\partial\theta}\frac{\partial\theta}{\partial t} + \frac{\partial z}{\partial\phi}\frac{\partial\phi}{\partial t} \\
&= (\cos\theta\cos\phi\cdot 2st) + (-\sin\theta\sin\phi\cdot s^2) \\
&= 2st\cos\theta\cos\phi - s^2\sin\theta\sin\phi \\
\therefore \frac{\partial z}{\partial t} &= 2st\cos\theta\cos\phi - s^2\sin\theta\sin\phi
\end{aligned}
$$

6.

Sea $z = x^4 + x^2y$, con $x = s + 2t - u$, $y = stu^2$, utilice la **regla de la cadena** para calcular: $\frac{\partial z}{\partial s}$ $\frac{\partial z}{\partial t}$ $\frac{\partial z}{\partial u}$, donde $s = 4$ $t = 2$, $u = 1$.

Primero calculamos las derivadas parciales de primer orden necesarias:

$$
\begin{aligned}
\frac{\partial z}{\partial x} &= \frac{\partial}{\partial x}(x^4 + x^2 y) \\
&= 4x^3 + 2xy
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial z}{\partial y} &= \frac{\partial}{\partial y}(x^4 + x^2 y) \\
&= x^2
\end{aligned}
$$

Así, al aplicar el caso 2 de la regla de la cadena, obtenemos

$$
\begin{aligned}
\frac{\partial z}{\partial s} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \\
&= \left[(4x^3 + 2xy) \cdot \frac{\partial}{\partial s}(s + 2t - u)\right] + \left[x^2 \cdot \frac{\partial}{\partial s}(stu^2)\right] \\
&= 4x^3 + 2xy + x^2 tu^2
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial z}{\partial t} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} \\
&= \left[(4x^3 + 2xy) \cdot \frac{\partial}{\partial t}(s + 2t - u)\right] + \left[x^2 \cdot \frac{\partial}{\partial t}(stu^2)\right] \\
&= 2(4x^3 + 2xy) + x^2 su^2 \\
&= 8x^3 + 4xy + x^2 su^2
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} \\
&= \left[(4x^3 + 2xy) \cdot \frac{\partial}{\partial u}(s + 2t - u)\right] + \left[x^2 \cdot \frac{\partial}{\partial u}(stu^2)\right] \\
&= -(4x^3 + 2xy) + 2x^2 stu \\
&= -4x^3 - 2xy + 2x^2 stu
\end{aligned}
$$

Cuando $s = 4$, $t = 2$, y $u = 1$, tenemos $x = 4 + 2(2) - 1 = 7$ y $y =$

$(4)(2)(1^2) = 8$, de modo que

$$\begin{aligned}\frac{\partial z}{\partial s} &= 4x^3 + 2xy + x^2tu^2 \\ &= 4(7^3) + 2(7)(8) + (7^2)(2)(1^2) \\ &= 1372 + 112 + 98 \\ &= 1582\end{aligned}$$

$$\begin{aligned}\frac{\partial z}{\partial t} &= 8x^3 + 4xy + x^2su^2 \\ &= 8(7^3) + 4(7)(8) + (7^2)(4)(1^2) \\ &= 2744 + 224 + 196 \\ &= 3164\end{aligned}$$

$$\begin{aligned}\frac{\partial z}{\partial u} &= -4x^3 - 2xy + 2x^2stu \\ &= -4(7^3) - 2(7)(8) + 2(7^2)(4)(2)(1) \\ &= -1372 - 112 + 784 \\ &= -700\end{aligned}$$

$\therefore \frac{\partial z}{\partial s} = 1582$, $\frac{\partial z}{\partial t} = 3164$ y $\frac{\partial z}{\partial u} = -700$

7.

Sea $f(x,y,z) = x^2yz - xyz^3$, $P(2,-1,1)$, $\hat{u} = \left(0, \frac{4}{5}, \frac{-3}{5}\right)$:

- Determine el **gradiente** de la función escalar $f(x,y,z)$.
- Evalúe el **gradiente** en el punto P.
- Encuentre la *razón de cambio* de $f(x,y,z)$ en el punto P en la dirección del vector $\hat{u}$.

8.

Determine la máxima **razón de cambio** de $f(x,y) = 4y\sqrt{x}$ en el punto $P(4,1)$ y la dirección en la cuál se presenta.

9.

Sea $f(x,y) = x^2 + xy + y^2 + y$. Calcule los valores **máximo** y **mínimo** *locales*, y *punto(s)* silla de la función.

10.

Sea $f(x,y) = x^2 + y^2 2x$, donde D es la región triángular cerrada con vértices $A(2,O)$, $B(O,2)$, $C(0,-2)$.

Determine los **valores máximos absolutos**, **valores mínimos absolutos** de $f(x,y)$ sobre el conjunto D.

11.

Encuentre tres números positivos cuya suma es 100 y cuyo producto es un máximo.

12.

Utilizando **multiplicadores de Lagrange**, encuentre los valores **máximo** y **mínimo** de la función sujeta a la **restricción(es)** dadas.

- $f(x,y) = x^2 + y^2$, sujeto a la restricción $xy = 1$.
- $f(x,y) = xyz$, sujeto a la restricción $x^2 + 2y^2 + 3z^2 = 6$.