Milestone 4

AST5220

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1 Introduction

In this projet we wish to follow in the footsteps of Petter Callin[1] who numerically reproduces the power spectrum obtained by the CMB data. This will be done in several steps, where each step simulates the different physical processes that make up the power spectrum.

The previous milestones have consisted of calculating the background cosmology, the recombination history of the universe and the evolution of the different perturbations that make up the power spectrum. This final milestone consists of actually tusing all these previously calculated qunatities to create the CMB power spectrum.

As with previous milestones, all numerical solutions will be obtained by utilising the C++ code base provided by our lecturer, Hans Winther.

2 Theoretical background

2.1 Spherical harmonics

To understand what the power spectrum represents, we have to start with understanding spherical harmonics. The temperature field that makes up the CMB can be represented using spherical harmonics, which reads as

$$T(\hat{n}) = \sum_{\ell m} a_{\ell m} Y_{\ell m}(\hat{n}). \tag{1}$$

Here, $T(\hat{n})$ represents the temperature in direction $\hat{n}$, a_{lm} are the spherical harmonic coefficients, and $Y_{\ell m}$ are the spherical harmonic functions themselves. Spherical harmonics are wave functions on the sphere, they are completely analogous to fourier transformations in flat space.

The ℓs refer to scale, with smaller ℓs being bigger scales. For each ℓ we have $m = 2\ell + 1$. Now, the CMB power spectrum shows us the expectation value for each $a_{\ell m}$, or

$$C_\ell \equiv \left\langle |a_{\ell m}|^2 \right\rangle = \langle a_{\ell m} a^*_{\ell m} \rangle . \tag{2}$$

where we, for each ℓ, take the average over all m. This is due to the universe being isotropic.

2.2 The source function and line of sight integration

In Milestone 3, we calculated the photon temperature fluctuations Θ_ℓ for $\ell \in [0, 6]$. But we are really interested in the interval $\ell \in [0, 1200]$, at least! Luckily, we dont have to do the calculations from the last milestone 1200 times. Thanks to Zaldarriaga and Seljak, we can instead do something called line of sight-integration (los-integration)! This integration takes on the form

$$\Theta_\ell(k, x = 0) = \int_{-\infty}^{0} \tilde{S}(k, x) j_\ell \left[k \left(\eta_0 - \eta\right)\right] dx, \tag{3}$$

where $\tilde{S}(k, x)$ is the source function, and $j_\ell \left[k \left(\eta_0 - \eta\right)\right]$ are Bessel functions. The source function looks like

$$\tilde{S}(k,x) = \tilde{g}\left[\Theta_0 + \Psi + \frac{1}{4}\Pi\right] + e^{-\tau}\left[\Psi' - \Phi'\right] - \frac{1}{ck}\frac{d}{dx}\left(\mathcal{H}\tilde{g}v_b\right) + \frac{3}{4c^2k^2}\frac{d}{dx}\left[\mathcal{H}\frac{d}{dx}(\mathcal{H}\tilde{g}\Pi)\right]. \quad (4)$$

In essense, the source function explains the different physics that affect a photon on its journey from last scattering until we measure it as a CMB photon. The first term, which we can see is relevant at last scattering due to it being weighted by the visibility function, explains how a photon is affected at last scattering, when it climbs out of the gravitational wells created by the baryons before free-streaming towards us. This is called the Sachs-Wolfe term. The next term is the integrated Sachs-Wolfe term, which explains how a photon is affected when traveling throug changing gravitational potentials. The third term is a Doppler term, and the fourth term is the term-who-must-not-be-named, apparently!

With los-integration, we dont have to solve all the coupled differential equations for each ell, we only need to solve eq. (3) instead, which greatly reduces computational time!

2.3 The temperature and matter power spectrums

For this milestone we are interested in both the temperature power spectrum and the matter power spectrum.

The temperature power spectrum takes on the form

$$C_\ell = \frac{2}{\pi}\int k^2 P_{\text{primordial}}(k)\Theta_\ell^2(k)dk \quad (5)$$

where $P_{\text{primordial}}(k)$, the primordial power spectrum, looks like

$$\frac{k^3}{2\pi^2}P_{\text{primordial}}(k) = A_s\left(\frac{k}{k_{\text{pivot}}}\right)^{n_s-1}. \quad (6)$$

Here, n_s is the spectral index for scalar perturbation, which takes on the value $n_s \approx 0.96$. k_{pivot} is some scale where the amplitude is A_s For our universe, we have $A_s \approx 2\times 10^{-9}$ and $k_{pivot} \approx 0.05/\text{Mpc}$.

The primordial power spectrum sets up the anisotropies from inflation?? idk

Adding this back eq. (5), we get

$$C_\ell = 4\pi\int_0^\infty A_s\left(\frac{k}{k_{\text{pivot}}}\right)^{n_s-1}\Theta_\ell^2(k)\frac{dk}{k}. \quad (7)$$

This is the integral we wish to solve.

For the matter power spectrum, we simply have

$$P(k,x) = \left|\Delta_M(k,x)\right|^2 P_{\text{primordial}}(k) \quad (8)$$

where

$$\Delta_M(k,x) \equiv \frac{c^2k^2\Phi(k,x)}{\frac{3}{2}\Omega_{M0}a^{-1}H_0^2}. \tag{9}$$

3 Method

3.1 Code structure and parameters

All main coding was done in the `PowerSpectrum.cpp` file. The differntial equations were solved using the ODESolver found in the GSL library.All visualisation was done in `Milestone4_plot.py`.

All solutions where found in the interval $x \in [-12, 0]$. Ideally, this intervall should have been bigger, but due to limitations from milestone 2, the optical depth was only solved for $x > -12$. We solved the equations for 1000 k-values logarithmically spaced, in the interval $k \in [0.00005/\text{Mpc}, 0.3/\text{Mpc}]$. We solved for a set of ℓ's in the range $\ell \in [2, 2000]$.

3.2 Bessel splines

First of all, we needed to spline the bessel functions for each ℓ. We did this logarithmically spaced with values ragning from $log(1e-8)$ to $log(4e4)$.

3.3 LOS-integration to LOS-ODE

To solve the line of sight integration, we wrote eq. (3) as an ordinary differential equation on the form

$$\frac{d\Theta_\ell(k,x)}{dx} = \tilde{S}(k,x)j_\ell\left[k\left(\eta_0 - \eta\right)\right], \quad \Theta_\ell(k,-\infty) = 0, \tag{10}$$

with initial conditions $\Theta_\ell(k, x = -\infty) \approx \Theta_\ell(k, x = -12) = 0$. This was solved using GSLs ODESolver.

While we mainly wanted to find $Theta_\ell$ to compute the temperature power spectrum, we also plotted the transfer function $\Theta_\ell(k)$ and the spectrum integrand $\Theta_\ell^2(k)/k$ for a diverse set of ℓ-values.

3.4 Temperature power spectrum

We also solved eq. (7) using the ODESolver. To do this we rewrote the equation as

$$\frac{dC_\ell}{d\text{log}k} = 4\pi A_s\left(\frac{k}{k_{\text{pivot}}}\right)^{n_s-1}\Theta_\ell^2(k) \tag{11}$$

where we have used the fact that $\frac{dC_\ell}{d\text{log}k} = k\frac{dC_\ell}{dk}$, with initial conditions $C_\ell(k_{\min}) = 0$.

The result from this integration was normalized by a factor $\frac{\ell(\ell+1)}{2\pi}(10^6 \times T_{\text{CMB}})^2$.

The first result of the temperature power spectrum was too large for large values of ℓ. To investigate the source of this error, we plotted the contribution from each term of the source function separately. In doing so we found that the last term was several orders of magnitudes larger than it should be. This is probably due to some error in the derivatives of Θ_2, which are the only variable unique to this last term. Unable to find this error within reasonable time, this term was dropped in the final results. Originally this term is the smallest of all the terms in the source function, most of the effect on the power spectrum stems from the Sachs-Wolfe term, so not including this last term leads to less errors than including it the way it looks now.

3.5 Matter power spectrum

Calculating the matter power spectrum was pretty straight forward. We only needed to calculate eq. (8). We also wanted to find $k_{\text{peak}} = \mathcal{H}(a_{eq})$. This was done by blablablaba

4 Results

4.1 Thetas

In fig. 1 we see a plot of the transfer function $\Theta_\ell(k)$.

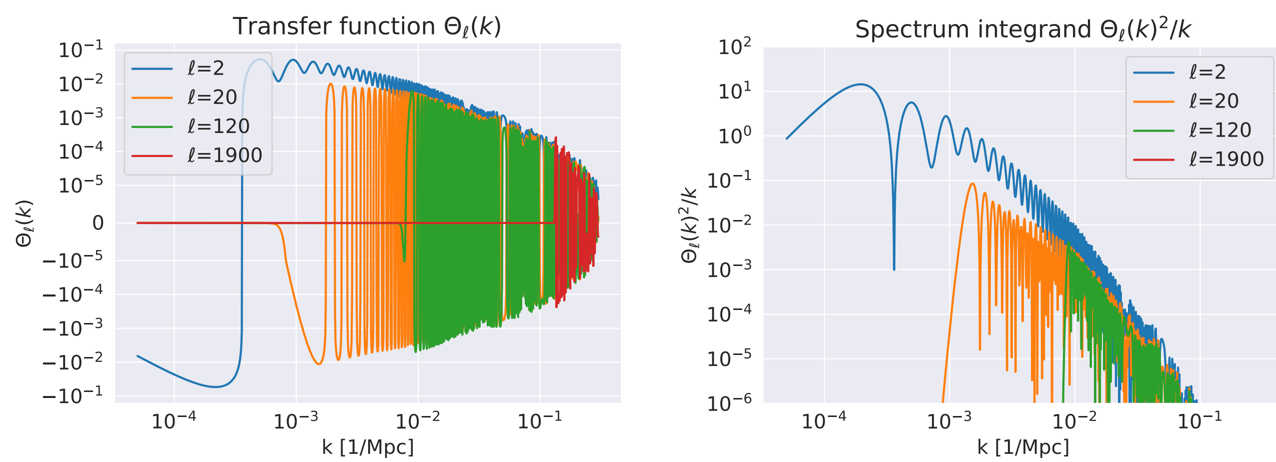

Figure 1: Plot showing the transfer function for a few selected values of ℓ. The Θ's shown here is calculated

4.2 Temperature power spectrum

Now for the star of the show. In fig. 2 we see how the power spectrum looks when including all terms in the source function. As mentioned earlier, there is a pretty severe bug somewhere in in the code calculating the perturbations. The result of including this term is a way too high amplitude for low ℓ's, the well known Sachs-Wolfe plateau is nowhere to be seen.

To investigate this error, we plotted all the different contributions from the source function, as seen in the left figure in fig. 3. Here we see, as expected, that the biggest contributor is the Sachs-Wolfe term. We also observe the late integrated Sachs-Wolfe effect wich causes an upturn in the power spectrum for the very largest scales. However, the most important thing this figure shows is that the contribution from the last source function term. This is off by several orders of magnitude for the largest scales.

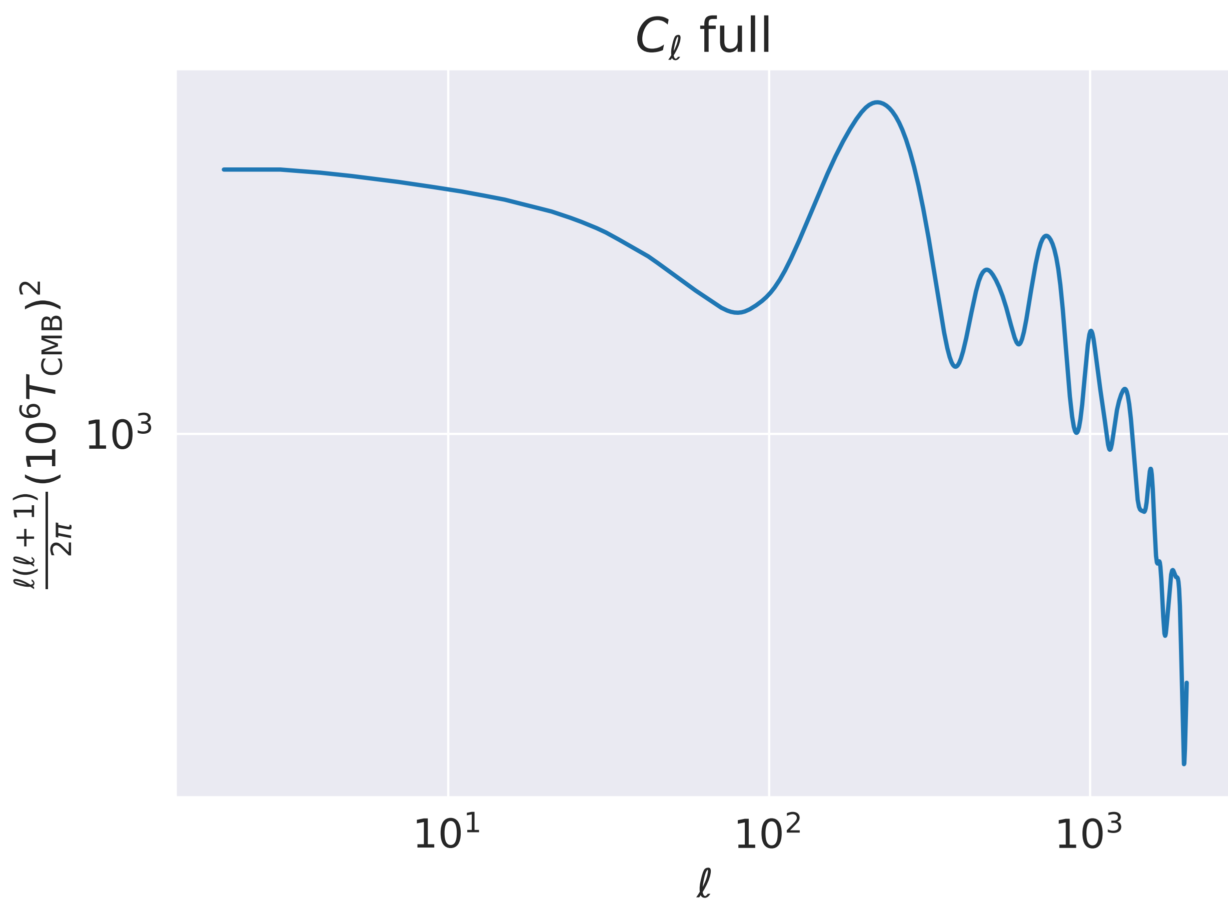

Figure 2: Plot showing the temperature power spectrum when including all four terms in the source function. Here we see that the Sachs-Wolfe plateu is nonexsistent, as the power spectrum is increasing at large ℓ's. This means something is wrong.

The right figure in fig. 3 shows a comparison of the power spectrum with and without this last term. Here we see that they are more or less equal for small scales, after the first peak, but they diverge completely on large scales. This shows that, in our case, excluding the quadrupole term is the right decision.

So, in fig. 4 we see a plot of only the temperature power spectrum without the errors from fig. 2. When analyzing the temperature power spectrum, we can start by dividing the scales that were outside or inside the horizon during recombination.

The scales outside the horizon at recombination was never affected by the coupling between baryons or photons, or any casual physics for that matter. These perturbations were set up by inflation and remained more or less the same untill entering the horizon at later times. This is representet by the Sachs-Wolfe plateau in the power spectrum. We see an upturn in this plateau for they very large scales, this upturn stems from the late integrated Sachs-Wolfe effect. These scales entered the horizon after the universe became dark energy-dominated, so they are affected by the decaying gravitational potential due to the accelerated expansion of the universe.

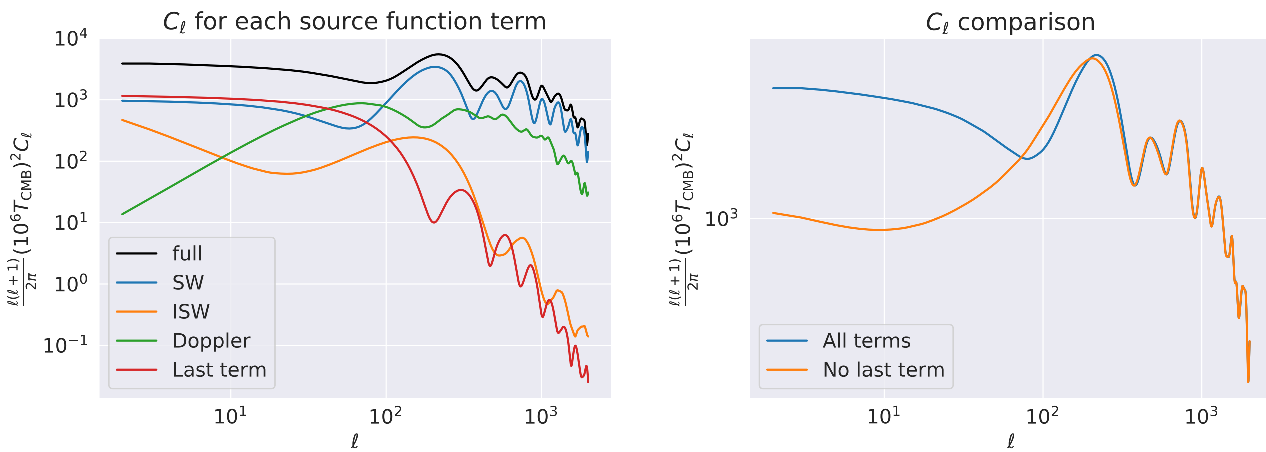

Figure 3: Plots showing how the different terms in the source function affect the power spectrum. The left plot shows the temperature perturbations would look if only one of the terms contributed to the power spectrum. The right plot shows the difference between including and excluding the last source function term. Here we see that the source of the error in the full solution is definietly this last term, as it raises the amplitude of the power spectrum for large scales.

For the samller scales that entered the horizon before recombination however, we see there's more physics at play. We see that these scales oscillate

4.3 Matter power spectrum

References

[1] Petter Callin. How to calculate the cmb spectrum, 2006.

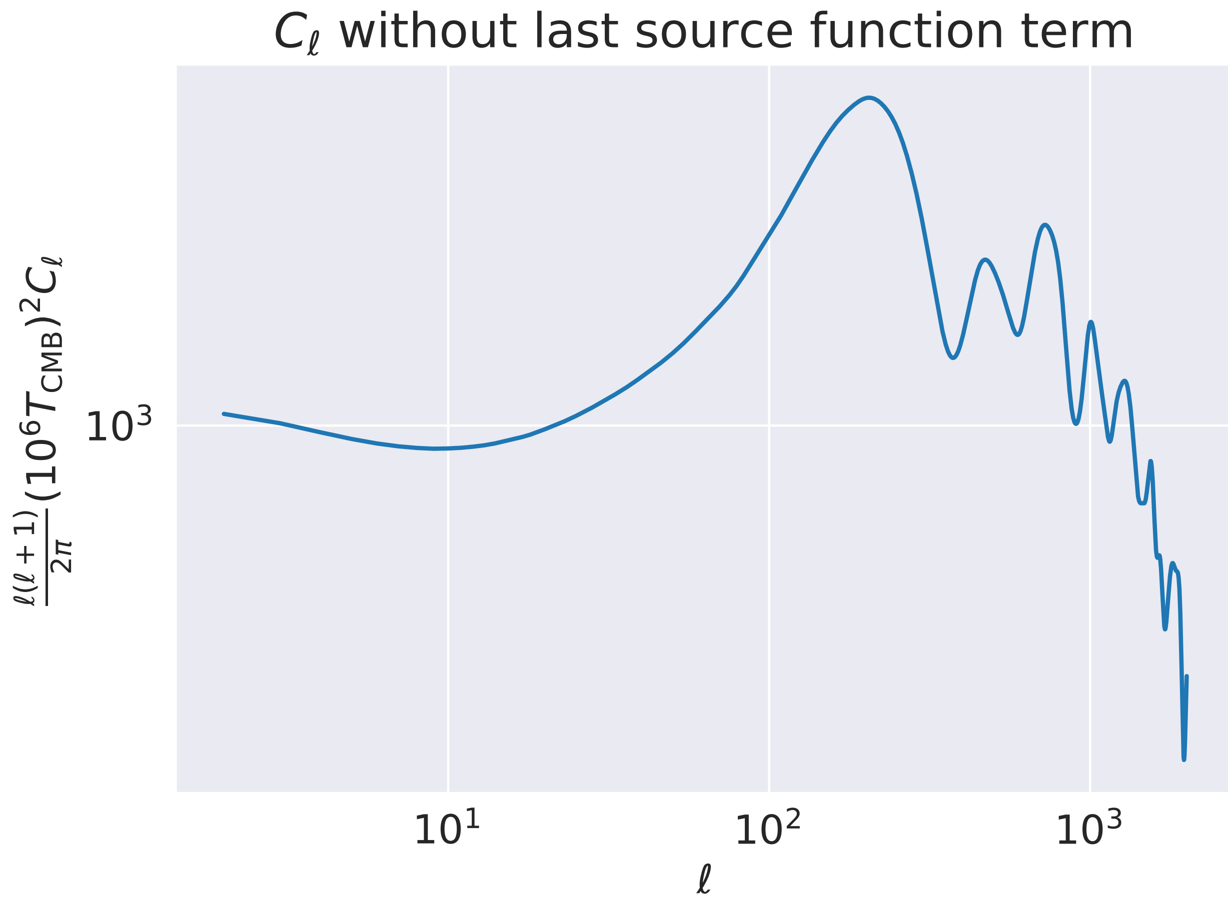

Figure 4: Plot showing the temperature power spectrum when excluding the fourth term in the source function.