Exercises 5

$$Q = \alpha P^{\beta}$$

$$\log Q = \log(\alpha P^{\beta})$$

$$= \log \alpha + \beta \log P$$

$$\log Q_{ij} = \beta_{0i} + \beta_{1i} \log P_{ij}$$

$$+ \delta_{ij} \beta_{2i} + \beta_{3i} \delta_{ij} \log P_{ij}$$

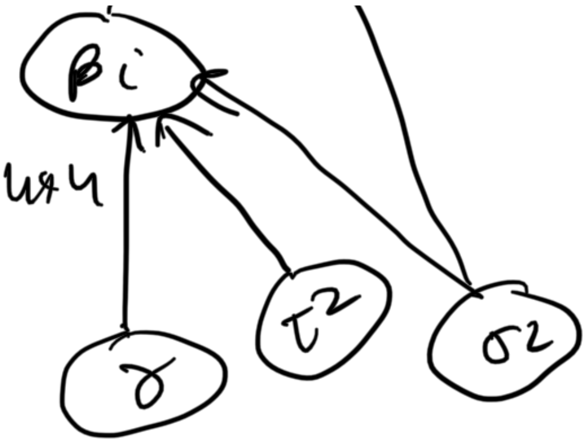
We can define $\beta_i = [\beta_{0i}, \beta_{1i}, \beta_{2i}, \beta_{3i}]^T$

and $X_i^T = \begin{bmatrix} 1 & \log P_{ij} & \delta_{ij} & \delta_{ij} \log P_{ij} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & \log P_{in_i} & \delta_{in_i} & \delta_{in_i} \log P_{in_i} \end{bmatrix}$

Where j is the sample number and n_i is the number of samples in each store i.

and $\log Q_i = \underbrace{X_i^T}_{(j \times 4)} \underbrace{\beta_i}_{(4 \times 1)} + \underbrace{e_i}_{(j \times 1)} \leftarrow MVN(0, \sigma_i^2 I)$

$$y_i = \underbrace{X_i'}_{} \beta_i + \epsilon_i$$

$y_i \sim N(X\beta_i, \sigma^2 I)$ 

$\beta_i \sim N(\underbrace{\gamma}_{4}, \sigma^2 \tau) \to 4\times 4$

$\gamma \propto c$

$4\times 4 \to \tau_{ii} \propto IG\left(\frac{1}{2}, \frac{1}{2}\right)$

$\sigma^2 \propto \frac{1}{\sigma^2}$

$\tau^2 \propto IG\left(\frac{1}{2}, \frac{1}{2}\right)$

$p(\beta_i|-) \propto p(y_i|\beta_i)\, p(\beta_i)$

$\propto \exp\left(-\frac{1}{2}(y_i - X_i\beta_i)^T(\sigma^2 I)^{-1}(y_i - X_i\beta_i)\right)$

$\times \exp\left(-\frac{1}{2}(\beta_i - \gamma)^T(\sigma^2\tau^2 I)^{-1}(\beta_i - \gamma)\right)$

$\propto \exp\left(-\frac{1}{2}\left(\cancel{y_i^T(\sigma^2)^{-1} I y_i} - 2(X_i\beta_i)^T(\sigma^2 I)^{-1} y_i + (X_i\beta_i)^T(\sigma^2)^{-1} I (X_i\beta_i)\right.\right.$

$+ \beta_i^T(\sigma^2\tau^2)^{-1} I \beta_i - 2\beta_i^T(\sigma^2\tau^2)^{-1} I \gamma$

$\left.\left. + \cancel{\gamma^T(\sigma^2\tau^2)^{-1} I \gamma}\right)\right)$

$\propto \exp\left(-\frac{1}{2}\left(-2\beta_i^T\left(X_i^T(\sigma^2)^{-1} I y_i\right.\right.\right.$

$\left. + (\sigma^2\tau^2)\gamma\right) + \beta_i^T\left(X_i^T(\sigma^2)^{-1} I X_i\right.$

$\left.\left.\left. + (\sigma^2\tau^2)^{-1}\right)\beta_i\right)\right)$

$\propto \exp\left(-\frac{1}{2}(\beta_i - m^*)^T K^* (\beta_i - m^*)^T\right)$

$$K^* = X_i^T (\sigma^2)^{-1} I X_i + (\sigma^2 \tau^2)^{-1} I$$

$$= \frac{1}{\sigma^2} \left(\underbrace{X_i^T X_i} + (\tau^2)^{-1} \right)$$

$$\boxed{\begin{aligned} K^* m^* &= X_i^T (\sigma^2)^{-1} I y_i + (\sigma^2 \tau^2) \gamma \\ &= \frac{1}{\sigma^2} \left(X_i^T y_i + (\tau^2)^{-1} \gamma \right) \\ m^* &= K^{*-1} \times \frac{1}{\sigma^2} \left(\underbrace{X_i^T y_i}_{4 \times 1} + (\tau^2)^{-1} \gamma \right) \end{aligned}}$$

$$P(\sigma^2 | -) \propto P(\sigma^2, \beta_i, y_i)$$

$$\propto P(y_i | \beta_i, \sigma^2) P(\beta_i | \sigma^2) P(\sigma^2) \propto \prod^{P}$$

$$(\sigma^2)^{-\frac{N_i}{2}} \exp\left(-\frac{1}{2} (y_i - X_i \beta_i)^T (\sigma^2)^{-1} I (y_i - X_i \beta_i) \right)$$

$$(\sigma^2)^{-\frac{P}{2}} \exp\left(-\frac{1}{2} (\beta_i - \gamma)^T (\sigma^2 \tau^2)^{-1} (\beta_i - \gamma) \right)$$

$$\times \quad \frac{1}{\sigma^2}$$

$$\propto \exp\left((\sigma^2)^{-1} \left[(y_i - X_i \beta_i)^T I (y_i - X_i \beta_i) \right.\right.$$

$$\left.\left. + (\beta_i - \gamma)^T (\tau^2)^{-1} I (\beta_i - \gamma) \right] \right) \times (\sigma^2)^{-1-P-N}$$

$$\left\{ \propto IG\left(\frac{N+P}{2}, \sum_{1}^{P=88} \frac{1}{2} (y_i - X_i \beta_i)^T (y_i - X_i \beta_i) + (\beta_i - \gamma)^T (\tau^2)^{-1} (\beta_1 - \gamma) \right\}$$

$$\propto \exp\left(-\frac{1}{2}\sum^{p}(\beta_i-\gamma)^T(\sigma^2\tau^2)^{-1}I(\beta_i-\gamma)\right)\times C$$

$$\propto \exp\left(-\frac{1}{2\sigma^2}\sum^{p}\left(\cancel{\beta_i^T(\tau^2)^{-1}\beta_i}-2\gamma^T(\tau^2)^{-1}\beta_i\right.\right.$$
$$\left.\left.+\gamma^T(\tau^2)^{-1}\gamma\right)\right)$$

$$\propto \exp\left(-\frac{1}{2\sigma^2}\left(p\gamma^T(\tau^2)^{-1}\gamma-2\gamma^T(\tau^2)^{-1}\sum^{p}\beta_i\right)\right)$$

$$\left\{\begin{array}{l}\propto MVN(m^*, K^*)\\ K^* = \dfrac{pI(\tau^2)^{-1}}{\sigma^2} \quad m^* = K^{*-1}\dfrac{(\tau^2)^{-1}\sum^{p}\beta_i}{\sigma^2}\end{array}\right\}$$

$$P(\tau^2\mid -) \propto P(\beta\mid\tau^2)\,P(\tau^2)$$

$$\prod^{p}(\tau^2\sigma^2)^{-\frac{4}{2}}\times\exp\left(-\frac{1}{2}(\beta_i-\gamma)^T(\tau^2\sigma^2)^{-1}(\beta_i-\gamma)\right)$$
$$\times(\tau^2)^{-\frac{1}{2}-1}\times\exp\left(-\frac{1}{2}(\tau^2)^{-1}\right)$$

$$\propto(\tau^2)^{-\frac{4p}{2}-\frac{1}{2}-1}\exp\left(-\frac{1}{2}(\tau^2)^{-1}\left[\sum(\beta_i-\gamma)^T(\sigma^2)^{-1}(\beta_i-\gamma)\right.\right.$$
$$\left.\left.+1\right]\right)$$

$$IG\left(\frac{4p+1}{2}, \frac{\sum^{p}(\beta_i-\gamma)^T(\sigma^2)^{-1}(\beta_i-\gamma)+1}{2}\right)$$

$$P(\tau_i^2\mid -)\propto P(\beta_i\mid\tau_i^2)\,P(\tau_i^2)$$

$$\propto\prod^{p}\left[\left(\frac{1}{\tau_i^2}\right)^{1/2}\exp\left(-\frac{1}{2\tau_i^2}\frac{(\beta_i-\gamma_i)^2}{\sigma^2}\right)\right]$$

$$\times \left(\tau_i^2\right)^{-\frac{1}{2}-1} \exp\left(-\frac{1}{2}(\tau_i^2)\quad\right)$$

$$\propto \left(\tau_i^2\right)^{-\frac{p}{2}} \exp\left(-\frac{1}{2\tau_i^2} \sum^{p} \frac{(\beta_i-\gamma_i)^2}{\sigma^2}\right)$$

$$\times \left(\tau_i^2\right)^{-\frac{1}{2}-1} \exp\left(-\frac{1}{2}(\tau_i^2)^{-1}\right)$$

$$\propto \left(\tau_i^2\right)^{\underbrace{-\frac{p}{2}-\frac{1}{2}}_{-\alpha}-1} \exp\left(-\frac{1}{2\tau_i^2}\left(\sum^{p} \frac{(\beta_i-\gamma_i)^2}{\sigma^2}+1\right)\right)$$

$$IG\left(\frac{1+p}{2},\ \frac{1}{2}\left(\sum^{p}\frac{(\beta_i-\gamma_i)^2}{\sigma^2}+1\right)\right)$$