Exercises 5 – continued

* How to check hierarchical model – example with cheese problem. We want to check that the model can produce data similar to the actual data.

For cheese:

$$y_i \sim N(X_i \beta_i, \sigma^2 I)$$

$$\sigma^2 \propto \frac{1}{\sigma^2}$$

$$\beta_i \sim N(\gamma, \sigma^2 \Lambda)$$

$$\gamma \propto C$$

$$\Lambda_{ii} \sim IG\left(\frac{1}{2}, \frac{1}{2}\right)$$

From each trace:

→ Take β_i

→ Take σ^2

→ Draw y_i for X_i

→ Plot y_i for each store & compare to real data.

A hierarchical probit model

$$P(y_{ij} = 1) = \Phi(z_{ij})$$

$\Phi \Rightarrow$ CDF of $N(0,1)$

$$z_{ij} = \mu_i + x_{ij}^T \beta_i$$

$j \equiv$ respondent

$i \equiv$ state

$X_i \Rightarrow$ demographic predictors

$$P(y_{ij} = 1) = P(z_{ij} > 0) = \Phi(X_{ij} \beta_i)$$

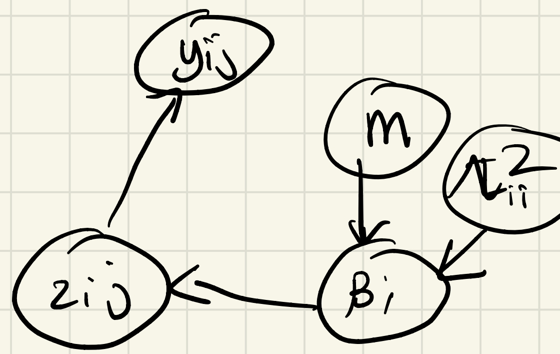
$$y_{ij} = \mathbb{1}(z_{ij} \geq 0) \qquad\qquad y_{ij} = \Phi(z_{ij})$$

$$z_{ij} \sim N(\underbrace{x_{ij}\beta_i}_{\text{includes intercept}}, 1)$$

$$\beta_i \sim N(m, \tau^2 I)$$

$$m \propto c$$

$$\tau \sim \propto \mathrm{IG}\left(\tfrac{1}{2}, \tfrac{1}{2}\right)$$



* conditionals

$$P(z_{ij} \mid -) \propto P(y_{ij} \mid z_{ij}) P(z_{ij})$$

$$= \begin{cases} \mathbb{1}(y_{ij} = 1)\, N(x_{ij}\beta_i, 1) \\ \mathbb{1}(y_{ij} = 0)\, N(x_{ij}\beta_i, 1) \end{cases} \leftarrow \text{truncated normal}$$

$$P(\beta_i \mid -) \propto P(z_i \mid \beta_i) P(\beta_i)$$

$$\propto P(\beta_i) \prod_{j}^{n_i} P(z_{ij} \mid \beta_i) P(\beta_i)$$

$$\propto \exp\left(-\frac{1}{2} (\beta_i - m)^T (\tau^2 I)^{-1} (\beta_i - m)\right)$$

$$\times \exp\left(-\frac{1}{2} (z_i - \beta_i)^T (z_i - \beta_i)\right)$$

$$\propto \exp\Big(-\frac{1}{2}\Big[(\beta_i - m)^T (\tau^2 I)^{-1} (\beta_i - m)$$

$$+ (z_i - X_i \beta_i)^T (z_i - X_i \beta_i)\Big)$$

$$\propto \exp\Big(-\frac{1}{2}\Big(\beta_i^T (\tau^2 I)^{-1} \beta_i - 2\beta_i^T (\tau^2 I)^{-1} m + \cancel{m^T} (\tau^2 I)^{-1}$$

$$m + z_i^T \cancel{z_i} - 2(\beta_i^T X_i^T)(z_i) + (\beta_i^T X_i^T)$$

$$(X_i \beta_i)\Big)\Big)$$

$$\propto \exp\Big(-\frac{1}{2}\Big(\beta_i^T \big((\tau^2 I)^{-1} + X_i^T X_i\big)\beta_i$$

$$- 2\beta_i^T \big((\tau^2 I)^{-1} m + X_i^T z_i\big)\Big)\Big)$$

$$\propto \exp\Big(-\frac{1}{2}\big(\beta_i - m^*\big)^T K^* \big(\beta_i - m^*\big)\Big)$$

$$\boxed{\begin{aligned} K^* &= (\tau^2 I)^{-1} + X_i^T X_i \\ m^* &= (K^*)^{-1}\Big[(\tau^2 I)^{-1} m + \underbrace{X_i^T z_i}_{(12\times n)\,(n\times 1)}\Big] \\ \beta_i &\sim \underset{\text{predictors}}{N_p}\big(m^*, (K^*)^{-1}\big) \end{aligned}}$$

$$P(m \mid -) \propto P(m) \prod^{state} P(\beta_i \mid m)$$

$$\propto \prod^{state} \exp\left(-\frac{1}{2} (\beta_i - m)^T (\tau^2 I)^{-1} (\beta_i - m) \right)$$

$$\propto \exp\left(-\frac{1}{2} \sum^{state} (\beta_i - m)^T (\tau^2 I)^{-1} (\beta_i - m) \right)$$

$$\propto \exp\left(-\frac{1}{2} \sum^{state} \left(\beta_i^T (\tau^2 I) \beta_i - 2 m^T (\tau^2 I) m \right.\right.$$

$$+ m^T (\tau^2 I) m$$

$$\boxed{\begin{aligned} &\sim N_p(m^*, K^{*-1}) \\ &K^* = S(\tau^2 I)^{-1} \\ &m^* = (K^*)^{-1} \sum^{state} (\tau^2 I)^{-1} \beta_i \end{aligned}}$$

$$P(\tau^2|-) \propto P(\tau^2) \prod^{state} P(\beta_i|\tau^2)$$

$$\propto (\tau^2)^{-1-\frac{1}{2}} \exp\left(-\frac{1}{2}(\tau^2)^{-1}\right) \prod^{state} \frac{1}{(\tau^2)^{+\frac{1}{2}}} \exp\left(-\frac{1}{2}(\beta_i - m)(\tau^2 I)^{-1}(\beta_i - m)\right)$$

$$\propto (\tau^2)^{-1-\frac{1}{2}} \exp\left(-\frac{1}{2}(\tau^2)^{-1}\right) \times (\tau^2)^{-\frac{1}{2}\times S} \times \exp\left(-\frac{1}{2}\sum^{state}(\beta_i - m)(\tau^2 I)^{-1}(\beta_i - m)\right)$$

$$\propto (\tau^2)^{-1-\frac{1}{2}} \exp\left(-\frac{1}{2}(\tau^2)^{-1}\right) \exp\left(-\frac{1}{2\tau^2}\sum^{state}(\beta_i - m)^T(\beta_i - m)\right)$$

$$\propto (\tau^2)^{-1-\frac{1}{2}-\frac{S}{2}} \exp\left(-\frac{1}{2\tau^2}\sum^{S}(\beta_i - m)^T(\beta_i - m) - \frac{1}{2}(\tau^2)^{-1}\right)$$

$$\boxed{\propto IG\left(\frac{S+1}{2}, \frac{\sum^{S}(\beta_i - m)^T(\beta_i - m) + 1}{2}\right)}$$

$$\sum^{50}(1\times 12)(12\times 1)$$