Exercises 4

i : school
j : student
θ : school level means

$y_{ij} \sim \mathcal{N}(\theta_i, \sigma^2)$

(A)
$$\mathcal{L}(\theta_i) = \prod_{i}^{n} f(y_{ij} \mid \theta_i, \sigma^2)$$
$$= \prod_{1}^{n} \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2\sigma^2}(y_{ij} - \theta_i)^2\right)$$
$$= \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n \prod_{j=1} \exp\left(-\frac{1}{2\sigma^2}(y_{ij} - \theta_i)^2\right)$$
$$= \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n \exp\left(-\frac{1}{2\sigma^2}\sum_{j=1}^{n}(y_{ij} - \theta_i)^2\right)$$
$$= \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n \exp\left[-\frac{1}{2\sigma^2}\left(\bar{y}^2 n - 2\theta_i \bar{y} n + n\theta_i^2\right)\right]$$

$$\frac{\partial}{\partial\theta_i} \log \mathcal{L}(\theta_i) = 0$$
$$\frac{\partial}{\partial\theta_i}\left[-\frac{1}{2\sigma^2}\left(\bar{y}^2 n - 2\theta_i \bar{y} n + n\theta_i^2\right)\right] = 0$$
$$-\cancel{2}\bar{y}n + \cancel{2}\theta_i n = 0$$
$$\theta_i = \bar{y}$$

For all schools, we have $\boxed{\theta_{MLE} = (\bar{y}_1, \bar{y}_2 \ldots \bar{y}_P)}$

(B) This is because fewer samples will result in a poorer estimate of the true mean of the school

(C)

Fit model using Gibbs sampling:

$y_{ij} \sim \mathcal{N}(\theta_i, \sigma^2)$ $\quad | \Rightarrow P(y_{ij} \mid -)$

$\theta_i \sim N(\mu, \tau^2\sigma^2)$

$\mu \sim$ flat (μ)

$\sigma^2 \sim$ Jeffrey's prior

$\tau^2 \sim IG(1/2, 1/2)$

$P(\theta_i|-)$

$P(\mu|-)$

$P(\sigma^2|-)$

$P(\tau^2|-)$

Note $\bar{y}_i \sim N(\theta_i, \frac{\sigma^2}{n})$

$* P(\theta_i|\bar{y}, \mu, \sigma^2, \tau^2)$

$\propto P(\bar{y}_i|-)P(-)$

$$\propto P(\bar{y}_i|-)\,P(\theta_i|\tau^2,\sigma^2,\mu)\,\underbrace{P(\tau^2)\cancel{P(\sigma^2)}P(\mu)}_{\tau^2 \perp\!\!\perp \sigma^2 \perp\!\!\perp \mu}$$

$$\propto P(\bar{y}_i|-)\,P(\theta_i|\tau^2,\sigma^2,\mu)$$

$$\propto \exp\left(\frac{-1}{2\sigma^2/n}(\bar{y}_i-\theta_i)\right)^2 \times \exp\left(\frac{-1}{2\sigma^2\tau^2}(\theta_i-\mu)^2\right)$$

$$\propto \exp\left(\frac{-1}{2\sigma^2/n}(\cancel{\bar{y}_i^2}-2\bar{y}_i\theta_i+\theta_i^2)\frac{-1}{2\sigma^2\tau^2}(\theta_i^2 - 2\theta_i\mu + \cancel{\mu^2})\right)$$

$$\propto \exp\left(\frac{-1}{2\sigma^2\tau^2/n}(-2\bar{y}_i\theta_i\tau^2 + \theta_i^2\tau^2 + \theta_i^2 n^{-1} - 2\theta_i\mu n^{-1})\right)$$

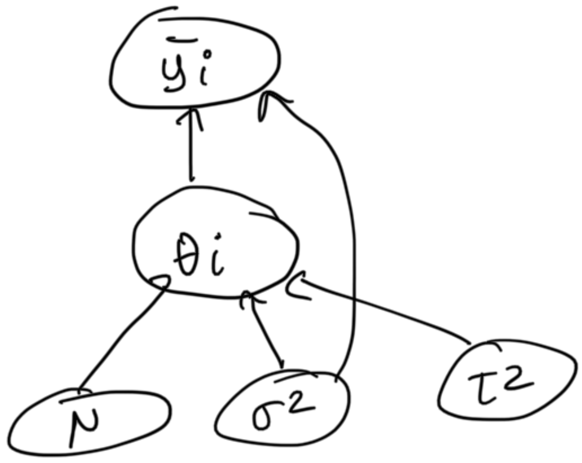
$$\propto \exp\left(-\frac{n}{2\sigma^2\tau^2}\left(-2\theta_i(\bar{y}_i\tau^2 + \mu n^{-1}) + \theta_i^2(\tau^2+n^{-1})\right)\right)$$

$$\propto \exp\left(-\frac{n}{2\sigma^2\tau^2}\left(\theta_i^2 - 2\theta_i\left(\frac{\bar{y}_i\tau^2+\mu n^{-1}}{\tau^2+n^{-1}}\right)\right)(\tau^2+n^{-1})\right)$$

$$\propto \exp\left(\frac{-n(\tau^2+n^{-1})}{2\sigma^2\tau^2}\left(\theta_i - \left(\frac{\bar{y}_i\tau^2+\mu n^{-1}}{\tau^2+n^{-1}}\right)^2\right)\right)$$

This is a $\boxed{N\left(\frac{\bar{y}_i\tau^2+\mu n_i^{-1}}{\tau^2+n^{-1}}, \frac{\sigma^2\tau^2}{(\tau^2 n_i+1)}\right)}$

$$\hookrightarrow \frac{\bar{y}\tau^2 n_i + \mu}{\tau^2 n_i + 1}$$



* $P(\mu \mid -)$

$$\propto P(\mu \mid \theta_i)$$

$$\propto P(\theta_i \mid \mu, \sigma^2, \tau^2)\, P(\mu)$$

$$\propto \prod_{1}^{p} \exp\left(-\frac{1}{2\tau^2\sigma^2}(\theta_i - \mu)^2\right)$$

$$\propto \exp\left(-\frac{1}{2\tau^2\sigma^2}\sum_{p}(\theta_i - \mu)^2\right)$$

$$\propto \exp\left(-\frac{1}{2\tau^2\sigma^2}\sum_{1}\left(\theta_i^2 - 2\theta_i\mu + \mu^2\right)\right)$$

$$\propto \exp\left(\frac{-1}{2\tau^2\sigma^2}\left(\cancel{p}\,\bar{\theta}^2 - 2\bar{\theta}\mu p + p\mu^2\right)\right)$$

$$\propto \exp\left(\frac{-p}{2\tau^2\sigma^2}\left(-2\bar{\theta}\mu + \mu^2\right)\right)$$

$$\propto \exp\left(\frac{-p}{2\tau\sigma^2}\left(\bar{\theta} - \mu\right)^2\right)$$

This is a normal $\boxed{N\left(\bar{\theta}, \frac{\sigma^2\tau^2}{p}\right)}$

* $P(\tau^2 \mid -) \propto P(\tau^2 \mid \theta) \propto P(\theta \mid \tau^2)\, P(\tau^2)$

$$\propto (\tau\cancel{\sigma})^{-p} \exp\left(-\frac{1}{2}(\theta-\mu)^T(\tau^2\sigma^2 I)^{-1}(\theta-\mu)\right)(\tau^2)^{-\frac{1}{2}-1}\exp\left(-\frac{1}{2}\tau^{-2}\right)$$

$$\propto (\tau^2)^{-\frac{p}{2}-\frac{1}{2}-1} \exp\left(-\frac{1}{2}(\theta-\mu)^T(\tau^2\sigma^2 I)^{-1}(\theta-\mu) + \tau^{-2}\right)$$

$$\propto (\tau^2)^{-\frac{p+1}{2}-1} \exp\left(-\frac{\tau^{-2}}{2}\left[(\theta-\mu)^T(\sigma^2 I)^{-1}(\theta-\mu) + 1\right]\right)$$

This is an $\boxed{IG\left(\frac{p+1}{2}, \frac{(\theta-\mu)^T(\sigma^2 I)^{-1}(\theta-\mu)+1}{2}\right)}$

* $P(\sigma^2 \mid -) \propto P(\sigma^2 \mid \theta_i, y_i)$

$$\propto P(\sigma^2 \mid \theta, y) \propto P(y \mid \theta)\, P(\theta)\, P(\sigma^2)$$

First find $P(y \mid \theta_i$ ...

First find $P(y|\theta_i, \ldots)$

$$\propto \prod_{i=1}^{p} \prod_{j=1}^{n_i} \sigma^{-1} \exp\left(-\frac{1}{2\sigma^2} (\theta_i - y_{ij})\right)^2$$

$$\propto \sigma^{-N} \exp\left(-\frac{1}{2\sigma^2} \sum_{1}^{p}\sum_{1}^{n_i} (\theta_i - y_{ij})^2\right)$$

so
$$P(\sigma^2 | -) \propto (\sigma^2)^{-\frac{N}{2}} \exp\left(-\frac{1}{2\sigma^2} \sum_{1}^{p}\sum_{1}^{n_i} (\theta_i - y_{ij})^2\right)$$
$$(\sigma^2\tau^2)^{-\frac{p}{2}} \exp\left(-\frac{1}{2} (\theta - \mu)^T (\sigma^2\tau^2 I)^{-1} (\theta - \mu)\right) \times \frac{1}{\sigma^2}$$

$$\propto (\sigma^2)^{-\frac{N+p}{2} - 1} \exp\left(-\frac{1}{2\sigma^2}\left((\theta - \mu)^T (\tau^2 I)^{-1} (\theta - \mu) + \left(\sum_{1}^{p}\sum_{1}^{n_i} (\theta_i - y_{ij})^2\right)\right)\right)$$

$$\boxed{\text{This is an } IG\left(\frac{N+p}{2}, \frac{(\theta-\mu)^T(\tau^2 I)^{-1}(\theta-\mu)}{2} + \frac{\sum^{p}\sum^{n_i}(\theta_i - y_{ij})^2}{2}\right)}$$

(D) $E(\theta_i | -) = k_i \mu + (1-k_i)\bar{y}_i$

$E(\theta_i | -) = k_i(\mu - \bar{y}_i) + \bar{y}_i$

$$k_i = \frac{E(\theta_i | -) - \bar{y}_i}{\mu - \bar{y}_i}$$

(E) $y_{ij} = \mu + \delta_i + e_{ij}$

$\delta_i \sim N(0, \tau^2\sigma^2)$

$e_{ij} \sim N(0, \sigma^2)$

$y_{ij} \sim N(\theta_i, \sigma^2)$

$\theta_i \sim N(\mu, \tau^2, \sigma^2)$

$\theta_i = \mu + \delta_i$

$y_{ij} = \theta_i + \epsilon_{ij}$

$y_{ij} = \mu + \delta_i + \epsilon_{ij}$

$\delta_i \sim N(0, \tau^2\sigma^2)$

$\epsilon_i \sim N(0, \sigma^2)$

(F) No?

Blood pressure

A) t-test

$H_0: \mu_2 - \mu_1 = 0$

$H_1: \mu_2 - \mu_1 \neq 0$

similar variances

if H_0 is true $\quad N_2 - N_1 \sim t$ with

loc $= 0$, $s = \sqrt{\dfrac{(n_1-1)s_1^2 + (n_2-1)s_2^2}{n_1+n_2-2}}$

dof $= n_1 + n_2 - 2$

Reject H_0 with $\alpha = 0.05$

Not appropriate because there are several observations per person & people sampled different number of times.

B) Still reject H_0. standard error is bigger because we have less samples. It is still problematic that all subjects don't have the same number of tests.

(c) $y_{ij} \sim N(\theta_i, \sigma^2)$ $\quad$ * wrong, probably

$\theta_i \sim N(\mu + \beta x_i, \tau^2\sigma^2)$

j = measurement number

i = person

flat prior on $\beta \rightarrow$ doesn't change anything, replace μ with $\mu + \beta x_i$ in conditionals

$P(\beta | -) \propto P(\beta | \theta_i)$

$\propto P(\theta_i | \beta \ldots) P(\beta)$

P

$$\propto \prod_i \exp\left(\frac{-1}{2\sigma^2\tau^2}\left(\theta_i - (\mu + \beta x_i)\right)^2\right)$$

$$\propto \exp\left(-\frac{1}{2\sigma^2\tau^2}\sum_i^p \left(\theta_i - (\mu + \beta x_i)\right)^2\right)$$

$$\propto \exp\left(\frac{-1}{2\sigma^2\tau^2}\sum_i^p \left(\theta_i^2 - 2\theta_i(\mu + \beta x_i)\right.\right.$$

$$\left.\left. + (\mu + \beta x_i)^2\right)\right)$$

$$\propto \exp\left(-\frac{1}{2\sigma^2\tau^2}\left(\cancel{p\theta_i^2} - 2\sum^p \theta_i \beta x_i\right.\right.$$

$$\left.\left. + \sum \cancel{\mu^2} + 2\mu\beta x_i + \beta^2 x_i^2\right)\right)$$

$$\propto \exp\left(\frac{-1}{2\sigma^2\tau^2}\left(-2\,\overline{\theta_i x_i}\,p\,\beta\right.\right.$$

$$\left.\left. + 2\mu\beta p\,\overline{x_i} + \beta^2\,\overline{x_i^2}\,p\right)\right)$$

$$\propto \exp\left(\frac{-1}{2\sigma^2\tau^2}\left(-2\beta\left(\overline{\theta_i x_i}\,p - \mu p\,\overline{x_i}\right)\right.\right.$$

$$\left.\left. + \beta^2 \right)\right)$$

$$\propto \exp\left(-\frac{\overline{x_i^2}\,p}{2\sigma^2\tau^2}\left(-2\beta\left(\frac{\overline{\theta_i x_i}\,p - \mu p\,\overline{x_i}}{\overline{x_i^2}\,p}\right)\right.\right.$$

$$\left.\left. + \beta^2\right)\right)$$

$$\propto \exp\left(-\frac{\overline{x_i^2}}{2\sigma^2\tau^2}\left(\beta - \left(\frac{\overline{\theta_i x_i}\,p - \mu p\,\overline{x_i}}{\overline{x_i^2}}\right)\right)^2\right)$$

This is a $N\left(\frac{\overline{\theta_i x_i}\,p - \mu p\,\overline{x_i}}{\overline{x_i^2}}, \frac{\sigma^2\tau^2}{\overline{x_i^2}}\right)$