Exercises 4

i: school
j: student
θ: school level means

$$y_{ij} \sim N(\theta_i, \sigma^2)$$

(A)
$$\mathcal{L}(\theta_i) = \prod_i^n f(y_{ij} \mid \theta_i, \sigma^2)$$
$$= \prod_1^n \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2\sigma^2}(y_{ij} - \theta_i)^2\right)$$
$$= \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n \prod_{j=1} \exp\left(-\frac{1}{2\sigma^2}(y_{ij} - \theta_i)^2\right)$$
$$= \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n \exp\left(-\frac{1}{2\sigma^2}\sum_{j=1}^n (y_{ij} - \theta_i)^2\right)$$
$$= \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n \exp\left[-\frac{1}{2\sigma^2}\left(\bar{y}^2 n - 2\theta_i \bar{y} n + n\theta_i^2\right)\right]$$

$$\frac{\partial}{\partial\theta_i} \log \mathcal{L}(\theta_i) = 0$$
$$\frac{\partial}{\partial\theta_i}\left[-\frac{1}{2\sigma^2}\left(\bar{y}^2 n - 2\theta_i \bar{y} n + n\theta_i^2\right)\right] = 0$$
$$-\cancel{2}\bar{y} n + \cancel{2}\theta_i n = 0$$
$$\theta_i = \bar{y}$$

For all schools, we have $\boxed{\theta_{MLE} = (\bar{y}_1, \bar{y}_2 \dots \bar{y}_P)}$

(B) This is because fewer samples will result in a poorer estimate of the true mean of the school

(C)

Fit model using Gibbs sampling:

$$y_{ij} \sim N(\theta_i, \sigma^2) \qquad \Big| \Rightarrow P(y_{ij} \mid -)$$

$\theta_i \sim N(\mu, \tau^2\sigma^2)$

$\mu \sim \text{flat}(\mu)$

$\sigma^2 \sim$ Jeffrey's prior

$\tau^2 \sim IG(1/2, 1/2)$

$p(\theta_i | -)$

$p(\mu | -)$

$p(\sigma^2 | -)$

$p(\tau^2 | -)$

Note $\bar{y}_i \sim N(\theta_i, \frac{\sigma^2}{n})$

* $P(\theta_i | \bar{y}, \mu, \sigma^2, \tau^2)$

$\propto P(\bar{y}_i | -) P(-)$

$\propto P(\bar{y}_i | -) P(\theta_i | \tau^2, \sigma^2, \mu) \underbrace{P(\tau^2) \cancel{P(\sigma^2) P(\mu)}}_{\tau^2 \perp\!\!\!\perp \sigma^2 \perp\!\!\!\perp \mu}$

$\propto P(\bar{y}_i | -) P(\theta_i | \tau^2, \sigma^2, \mu)$

$\propto \exp\left(\frac{-1}{2\sigma^2/n}(\bar{y}_i - \theta_i)\right)^2 \times \exp\left(\frac{-1}{2\sigma^2\tau^2}(\theta_i - \mu)^2\right)$

$\propto \exp\left(\frac{-1}{2\sigma^2/n}(\cancel{\bar{y}_i^2} - 2\bar{y}_i\theta_i + \theta_i^2) \frac{-1}{2\sigma^2\tau^2}(\theta_i^2 - 2\theta_i\mu + \cancel{\mu^2})\right)$

$\propto \exp\left(\frac{-1}{2\sigma^2\tau^2/n}\left(-2\bar{y}_i\theta_i\tau^2 + \theta_i^2\tau^2 + \theta_i^2 n^{-1} - 2\theta_i\mu n^{-1}\right)\right)$

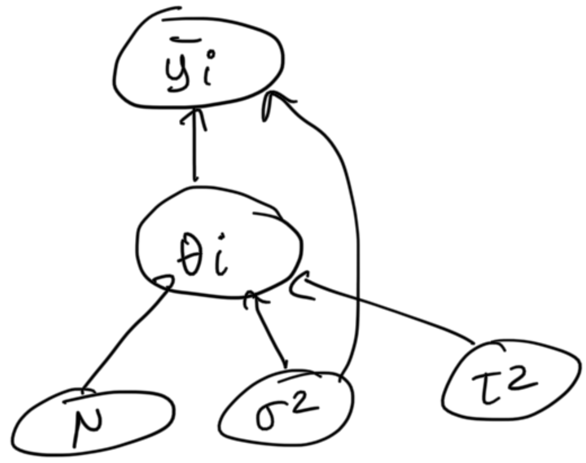
$\propto \exp\left(-\frac{n}{2\sigma^2\tau^2}\left(-2\theta_i(\bar{y}_i\tau^2 + \mu n^{-1}) + \theta_i^2(\tau^2 + n^{-1})\right)\right)$

$\propto \exp\left(-\frac{n}{2\sigma^2\tau^2}\left(\theta_i^2 - 2\theta_i\left(\frac{\bar{y}_i\tau^2 + \mu n^{-1}}{\tau^2 + n^{-1}}\right)\right)(\tau^2 + n^{-1})\right)$

$\propto \exp\left(\frac{-n(\tau^2 + n^{-1})}{2\sigma^2\tau^2}\left(\theta_i - \left(\frac{\bar{y}_i\tau^2 + \mu n^{-1}}{\tau^2 + n^{-1}}\right)\right)^2\right)$

This is a $N\left(\frac{\bar{y}_i\tau^2 + \mu n_i^{-1}}{\tau^2 + n^{-1}}, \frac{\sigma^2\tau^2}{(\tau^2 n_i + 1)}\right)$

$$\hookrightarrow \frac{\bar{y}\tau^2 n_i + N}{\tau^2 n_i + 1}$$



* $P(N|-)$

$\propto P(N|\theta_i)$

$\propto P(\theta_i|N,\sigma^2,\tau^2)\,P(N)$

$\propto \prod_{1}^{p} \exp\left(-\frac{1}{2\tau^2\sigma^2}(\theta_i - N)^2\right)$

$\propto \exp\left(-\frac{1}{2\tau^2\sigma^2}\sum_{p}(\theta_i - N)^2\right)$

$\propto \exp\left(-\frac{1}{2\tau^2\sigma^2}\sum_{1}\left((\theta_i^2 - 2\theta_i N + N^2)\right)\right)$

$\propto \exp\left(\frac{-1}{2\tau^2\sigma^2}(\cancel{p}\bar{\theta}^2 - 2\bar{\theta}Np + pN^2)\right)$

$\propto \exp\left(\frac{-p}{2\tau^2\sigma^2}(-2\bar{\theta}N + N^2)\right)$

$\propto \exp\left(\frac{-p}{2\tau\sigma^2}(\bar{\theta} - N)^2\right)$

This is a normal $\boxed{N\left(\bar{\theta}, \frac{\sigma^2\tau^2}{p}\right)}$

* $P(\tau^2|-) \propto P(\tau^2|\theta) \propto P(\theta|\tau^2)\,P(\tau^2)$

$\propto (\tau\cancel{\sigma})^{-p}\exp\left(-\frac{1}{2}(\theta - N)^T(\tau^2\sigma^2 I)^{-1}(\theta - N)\right)(\tau^2)^{-\frac{1}{2}-1}\exp\left(-\frac{1}{2}\tau^{-2}\right)$

$\propto (\tau^2)^{-\frac{p}{2}-\frac{1}{2}-1}\exp\left(-\frac{1}{2}(\theta - N)^T(\tau^2\sigma^2 I)^{-1}(\theta - N) + \tau^{-2}\right)$

$\propto (\tau^2)^{-\frac{p+1}{2}-1}\exp\left(-\frac{\tau^{-2}}{2}\left[(\theta - N)^T(\sigma^2 I)^{-1}(\theta - N) + 1\right]\right)$

This is an $\boxed{IG\left(\frac{p+1}{2}, \frac{(\theta - N)^T(\sigma^2 I)^{-1}(\theta - N) + 1}{2}\right)}$

* $P(\sigma^2|-) \propto P(\sigma^2|\theta_i, y_i)$

$\propto P(\sigma^2|\theta, y) \propto P(y|\theta)P(\theta)P(\sigma^2)$

First find $P(y|\theta_i\ldots)$

First Find $P(y|\theta_i, \ldots)$

$$\propto \prod_{i=1}^{p} \prod_{j=1}^{n_i} \sigma^{-1} \exp\left(-\frac{1}{2\sigma^2}(\theta_i - y_{ij})\right)^2$$

$$\propto \sigma^{-N} \exp\left(-\frac{1}{2\sigma^2} \sum_{1}^{p} \sum_{1}^{n_i} (\theta_i - y_{ij})^2\right)$$

so $p(\sigma^2 | -) \propto (\sigma^2)^{-\frac{N}{2}} \exp\left(-\frac{1}{2\sigma^2} \sum_{1}^{p} \sum_{1}^{n_i} (\theta_i - y_{ij})^2\right)$

$$(\sigma^2 \tau^2)^{-\frac{p}{2}} \exp\left(-\frac{1}{2}(\theta - \mu)^T (\sigma^2 \tau^2 I)^{-1} (\theta - \mu)\right) \times \frac{1}{\sigma^2}$$

$$\propto (\sigma^2)^{-\frac{N+p}{2} - 1} \exp\left(-\frac{1}{2\sigma^2}\left((\theta - \mu)^T (\tau^2 I)^{-1} (\theta - \mu) + \left(\sum_{1}^{p} \sum_{1}^{n_i} (\theta_i - y_{ij})^2\right)\right)\right)$$

This is an IG $\left(\frac{N+p}{2}, \frac{(\theta-\mu)^T (\tau^2 I)^{-1} (\theta - \mu)}{2} + \frac{\sum^{p} \sum^{n_i} (\theta_i - y_{ij})^2}{2}\right)$

D) $E(\theta_i \mid -) = \dfrac{\bar{y}_i \tau^2 + \mu n_i^{-1}}{\tau^2 + n_i^{-1}}$

$$= \frac{\bar{y}_i \tau^2 n_i + \mu}{\tau^2 n_i + 1}$$

$$= \underbrace{\frac{1}{\tau^2 n_i + 1}}_{k_i} \mu + \underbrace{\frac{\tau^2 n_i}{\tau^2 n_i + 1}}_{1-k_i} \bar{y}_i$$

$$k_i = \frac{1}{\tau^2 n_i + 1}$$

(E) $y_{ij} = \mu + \delta_i + e_{ij}$

$\delta_i \sim N(0, \tau^2 \sigma^2)$

$e_{ij} \sim N(0, \sigma^2)$

$y_{ij} \sim N(\theta_i, \sigma^2)$

$\theta_i \sim N(\mu, \tau^2, \sigma^2)$

$\theta_i = \mu + \delta_i$ $\qquad$ $\delta_i \sim N(0, \tau^2 \sigma^2)$

$\varepsilon_{ij} \sim N(0, \sigma^2)$

$y_{ij} = \theta_i + \varepsilon_{ij}$

$y_{ij} = \delta_i + \epsilon_{ij}$

$y_{ij} = \mu + \delta_i + \epsilon_{ij}$

$i \equiv$ school
$j \equiv$ student

$\text{cov}(y_{ij}, y_{ik}) \quad j \neq k$

between students

$E\big((\mu + \delta_i + \epsilon_{ij} - E(y_{ij}))$
$(\mu + \delta_i + \epsilon_{ik} - E(y_{ik}))\big)$

$= E\big((\cancel{\mu} + \delta_i + \epsilon_{ij} - \cancel{\mu})(\cancel{\mu} + \delta_i + \epsilon_{ik} - \cancel{\mu})\big)$

$= E\big([\delta_i + \epsilon_{ij}][\delta_i + \epsilon_{ik}]\big)$

$= E(\delta_i^2 + \delta_i\epsilon_{ik} + \epsilon_{ij}\delta_i + \epsilon_{ij}\epsilon_{ik})$

$= \text{var}(\delta_i) + \cancel{\text{cov}(\delta_i, \epsilon_{ik})} + \cancel{\text{cov}(\delta_i, \epsilon_{ij})}$
$+ \cancel{\text{cov}(\epsilon_{ij}, \epsilon_{ik})}$

$= \boxed{\tau^2 \sigma^2}$

$\text{cov}(y_{ij}, y_{i'k})$

different schools

$= E\big((\cancel{\mu} + \delta_i + \epsilon_{ij} \cancel{\mu})(\cancel{\mu} + \delta_{i'} + \epsilon_{i'k} \cancel{\mu})\big)$

$= E(\delta_i\delta_{i'} + \delta_i\epsilon_{i'k} + \epsilon_{ij}\delta_{i'} + \epsilon_{ij}\epsilon_{i'k})$

$= \boxed{\emptyset}$

(F) No?

Blood pressure

A) t-test

$H_0: N_2 - N_1 = 0$

$H_1: N_2 - N_1 \neq 0$

similar variances

if H_0 is true $N_2 - N_1 \sim t$ with

loc $= 0$, $s = \sqrt{\dfrac{(n_1-1)s_1^2 + (n_2-1)s_2^2}{n_1+n_2-2}}$

dof $= n_1 + n_2 - 2$

Reject H_0 with $\alpha = 0.05$

Not appropriate because there are several observations per person & people sampled different number of times.

B) Still reject H_0. Standard error is bigger because we have less samples. it is still problematic that all subjects don't have the same number of tests.

(c) $y_{ij} \sim N(\theta_i, \sigma^2)$ * wrong, probably

$\theta_i \sim N(\mu + \beta x_i, \tau^2 \sigma^2)$

j = measurement number

i = person

Flat prior on $\beta \to$ doesn't change anything, replace μ with $\mu + \beta x_i$ in conditionals

$P(\beta | -) \propto P(\beta | \theta_i)$

$\propto P(\theta_i | \beta \ldots) P(\beta)$

$\propto \prod_i^p \exp\left(\frac{-1}{2\sigma^2 \tau^2} (\theta_i - (\mu + \beta x_i))^2 \right)$

$\propto \exp\left(-\frac{1}{2\sigma^2\tau^2} \sum_i^p (\theta_i - (\mu + \beta x_i))^2 \right)$

$\propto \exp\left(\frac{-1}{2\sigma^2\tau^2} \sum_i^p (\theta_i^2 - 2\theta_i(\mu + \beta x_i) + (\mu + \beta x_i)^2) \right)$

$\propto \exp\left(-\frac{1}{2\sigma^2\tau^2} \left(\cancel{p\theta_i^2} - 2 \sum^p \theta_i \beta x_i + \cancel{\sum \mu^2} + 2\mu\beta \bar{x_i} p + \beta^2 \bar{x_i^2} p \right) \right)$

$$\propto \exp\left(\frac{-1}{2\sigma^2\tau^2}\left(-2\,\overline{\theta_i x_i p}\,\beta + 2N\beta p\,\overline{x_i} + \beta^2\,\overline{x_i^2}\,p\right)\right)$$

$$\propto \exp\left(\frac{-1}{2\sigma^2\tau^2}\left(-2\beta\left(\overline{\theta_i x_i}\,p - Np\overline{x_i}\right) + \beta^2\,\overline{x_i^2}\,p\right)\right)$$

$$\propto \exp\left(-\frac{\overline{x_i^2}\,p}{2\sigma^2\tau^2}\left(-2\beta\,\frac{\left(\overline{\theta_i x_i}\,p - Np\overline{x_i}\right)}{\overline{x_i^2}\,p} + \beta^2\right)\right)$$

$$\propto \exp\left(-\frac{\overline{x_i^2}\,p}{2\sigma^2\tau^2}\left(\beta - \left(\frac{\overline{\theta_i x_i}\,\cancel{p} - N\cancel{p}\,\overline{x_i}}{\cancel{p}\,\overline{x_i^2}}\right)\right)^2\right)$$

This is a $\boxed{N\left(\frac{\overline{\theta x} - N\overline{x}}{\overline{x^2}},\ \frac{\sigma^2\tau^2}{p\overline{x^2}}\right)}$

$$\frac{0.\cancel{5}(\theta_2 - N)}{\cancel{0}.5} \qquad \frac{\sigma^2\tau^2}{1}$$

$$P(N \mid -) \propto P(N \mid \theta)$$

$$\propto P(\theta_i \mid N)\,P(N)$$

$$\propto \prod^{p} \exp\left(-\frac{1}{2\tau^2\sigma^2}\left(\theta_i - (N + \beta x_i)\right)^2\right)$$

$$\propto \exp\left(-\frac{1}{2\tau^2\sigma^2}\sum\left(\theta_i - (N + \beta x_i)\right)^2\right)$$

$$\propto \exp\left(\frac{-1}{2\tau^2\sigma^2}\sum\left(\theta_i^2 - 2(N + \beta x_i)\theta_i + (N + \beta x_i)^2\right)\right)$$

$$\propto \exp\left(-\frac{1}{2\tau^2\sigma^2}\sum\left(\cancel{\theta_i^2} - 2(\mu + \beta x_i)\theta_i\right.\right.$$

$$\left.\left. + \mu^2 + 2\beta x_i \mu + \cancel{\beta^2 x_i^2}\right)\right)$$

$$\propto \exp\left(\frac{-1}{2\tau^2\sigma^2}\sum\left(-2(\mu\theta_i - \beta x_i \mu)\right.\right.$$

$$\left.\left. + \mu^2\right)\right)$$

$$\propto \exp\left(-\frac{1}{2\tau^2\sigma^2}\left(-2(\mu p\bar{\theta}_i - \beta\bar{x}_i p\mu)\right.\right.$$

$$\left.\left. + \mu^2 p\right)\right)$$

$$\propto \exp\left(\frac{-1}{2\tau^2\sigma^2}\left(-2\mu(p\bar{\theta} - \beta\bar{x}_i p)\right.\right.$$

$$\left.\left. + \mu^2 p\right)\right)$$

$$\propto \exp\left(-\frac{p}{2\tau^2\sigma^2}\left(\mu - (\bar{\theta} - \beta\bar{x}_i)\right)^2\right.$$

$$\boxed{\mathcal{N}\left(\bar{\theta} - \beta\bar{x}, \frac{\tau^2\sigma^2}{p}\right)}$$

$$P(\theta_i \mid -) \propto P(\theta_i \mid \bar{y}_i)$$

$$\propto P(\bar{y}_i \mid \theta_i)\, P(\theta_i)$$

$$\propto \exp\left(-\frac{n}{2\sigma^2}(\bar{y}_i - \theta_i)^2\right)$$

$$\times \exp\left(-\frac{1}{2\sigma^2\tau^2}\left(\theta_i - (\mu + \beta x_i)\right)^2\right.$$

$$\propto \exp\left(-\frac{n}{2\sigma^2}\left(\cancel{\bar{y}_i^2} - 2\bar{y}_i\theta_i + \theta_i^2\right)\right.$$

$$\left. - \frac{1}{2\sigma^2\tau^2}\left(\theta_i^2 - 2\theta_i(\mu + \beta x_i) + \cancel{(\mu + \beta x_i)^2}\right)\right)$$

$$\propto \exp\left(-\frac{n}{2\sigma^2\tau^2}\left(-2\bar{y}_i\theta_i\tau^2 + \theta_i^2\tau^2\right.\right.$$

$$\left.\left. + \theta_i^2 n_i^{-1} - 2\theta_i n_i^{-1}(\mu + \beta x_i)\right)\right)$$

$$\propto \exp\left(\frac{-n}{2\sigma^2\tau^2}\left(-2\theta_i\left(\bar{y}_i\tau^2 + n_i^{-1}\right.\right.\right.$$

$$\left.\left.\left.(\mu + \beta x_i)\right) + \theta_i^2(\tau^2 + n_i^{-1})\right)\right)$$

$$\propto \exp\left(-\frac{n_i(\tau^2 + n_i^{-1})}{2\sigma^2\tau^2}\left(\theta_i - \frac{(\bar{y}_i\tau^2}{\tau^2 + n_i^{-1}}\right.\right.$$

$$\left.\left. + n_i^{-1}(\mu + \beta x_i))\right)\right)$$

$$\boxed{N\left(\frac{\bar{y}_i\tau^2 n_i + (\mu + \beta x_i)}{n_i\tau^2 + 1}, \frac{\sigma^2\tau^2}{\tau^2 n_i + 1}\right)}$$

$\boxed{Q.4}$

$$\mu_1 = \mu_2 + \beta$$

$$\beta = \Delta_\mu$$