Exercises 4

i : school
j : student
θ : school level means

$y_{ij} \sim \mathcal{N}(\theta_i, \sigma^2)$

(A)

$$\mathcal{L}(\theta_i) = \prod_i^n f(y_{ij} \mid \theta_i, \sigma^2)$$

$$= \prod_1^n \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2\sigma^2}(y_{ij} - \theta_i)^2\right)$$

$$= \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n \prod_{j=1} \exp\left(-\frac{1}{2\sigma^2}(y_{ij} - \theta_i)^2\right)$$

$$= \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n \exp\left(-\frac{1}{2\sigma^2}\sum_{j=1}^n (y_{ij} - \theta_i)^2\right)$$

$$= \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n \exp\left[-\frac{1}{2\sigma^2}\left(\bar{y}^2 n - 2\theta_i \bar{y} n + n\theta_i^2\right)\right]$$

$$\frac{\partial}{\partial\theta_i} \log\mathcal{L}(\theta_i) = 0$$

$$\frac{\partial}{\partial\theta_i}\left[-\frac{1}{2\sigma^2}\left(\bar{y}^2 n - 2\theta_i \bar{y} n + n\theta_i^2\right)\right] = 0$$

$$-\cancel{2}\bar{y}n + \cancel{2}\theta_i n = 0$$

$$\theta_i = \bar{y}$$

For all schools, we have $\boxed{\theta_{MLE} = (\bar{y}_1, \bar{y}_2 \dots \bar{y}_P)}$

(B) This is because fewer samples will result in a poorer estimate of the true mean of the school

(C)

Fit model using Gibbs sampling:

$y_{ij} \sim \mathcal{N}(\theta_i, \sigma^2) \quad | \Rightarrow P(y_{ij} \mid -)$

$\theta_i \sim N(\mu, \tau^2\sigma^2)$

$\mu \sim$ flat (μ)

$\sigma^2 \sim$ Jeffrey's prior

$\tau^2 \sim IG(1/2, 1/2)$

$P(\theta_i|-)$

$P(\mu|-)$

$P(\sigma^2|-)$

$P(\tau^2|-)$

Note $\bar{y}_i \sim N(\theta_i, \frac{\sigma^2}{n})$

*$P(\theta_i|\bar{y}, \mu, \sigma^2, \tau^2)$

$\propto P(\bar{y}_i|-)P(-)$

$\propto P(\bar{y}_i|-)\, P(\theta_i|\tau^2,\sigma^2,\mu)\, \underbrace{P(\tau^2)\cancel{P(\sigma^2)}P(\mu)}_{\tau^2 \perp\!\!\!\perp \sigma^2 \perp\!\!\!\perp \mu}$

$\propto P(\bar{y}_i|-)\, P(\theta_i|\tau^2,\sigma^2,\mu)$

$\propto \exp\left(\frac{-1}{2\sigma^2/n}(\bar{y}_i - \theta_i)\right)^2 \times \exp\left(\frac{-1}{2\sigma^2\tau^2}(\theta_i-\mu)^2\right)$

$\propto \exp\left(\frac{-1}{2\sigma^2/n}(\cancel{\bar{y}_i^2} - 2\bar{y}_i\theta_i + \theta_i^2) - \frac{1}{2\sigma^2\tau^2}(\theta_i^2 - 2\theta_i\mu + \cancel{\mu^2})\right)$

$\propto \exp\left(\frac{-1}{2\sigma^2\tau^2/n}\left(-2\bar{y}_i\theta_i\tau^2 + \theta_i^2\tau^2 + \theta_i^2 n^{-1} - 2\theta_i\mu n^{-1}\right)\right)$

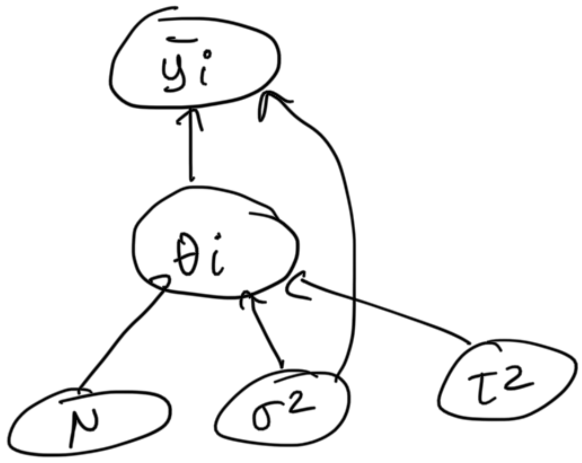
$\propto \exp\left(-\frac{n}{2\sigma^2\tau^2}\left(-2\theta_i(\bar{y}_i\tau^2 + \mu n^{-1}) + \theta_i^2(\tau^2+n^{-1})\right)\right)$

$\propto \exp\left(-\frac{n}{2\sigma^2\tau^2}\left(\theta_i^2 - 2\theta_i\left(\frac{\bar{y}_i\tau^2+\mu n^{-1}}{\tau^2+n^{-1}}\right)\right)(\tau^2+n^{-1})\right)$

$\propto \exp\left(\frac{-n(\tau^2+n^{-1})}{2\sigma^2\tau^2}\left(\theta_i - \left(\frac{\bar{y}_i\tau^2+\mu n^{-1}}{\tau^2+n^{-1}}\right)\right)^2\right)$

This is a $\boxed{N\left(\frac{\bar{y}_i\tau^2+\mu n_i^{-1}}{\tau^2+n^{-1}}, \frac{\sigma^2\tau^2}{(\tau^2 n_i+1)}\right)}$

$$\hookrightarrow \frac{\bar{y}\tau^2 n_i + N}{\tau^2 n_i + 1}$$



* $P(N|-)$

$$\propto P(N|\theta_i)$$

$$\propto P(\theta_i|N,\sigma^2,\tau^2)\,P(N)$$

$$\propto \prod_{1}^{p} \exp\left(-\frac{1}{2\tau^2\sigma^2}(\theta_i-N)^2\right)$$

$$\propto \exp\left(-\frac{1}{2\tau^2\sigma^2}\sum_{p}(\theta_i-N)^2\right)$$

$$\propto \exp\left(-\frac{1}{2\tau^2\sigma^2}\sum_{1}(\theta_i^2-2\theta_i N+N^2)\right)$$

$$\propto \exp\left(\frac{-1}{2\tau^2\sigma^2}(\cancel{p}\bar{\theta}^2-2\bar{\theta}Np+pN^2)\right)$$

$$\propto \exp\left(\frac{-p}{2\tau^2\sigma^2}(-2\bar{\theta}N+N^2)\right)$$

$$\propto \exp\left(\frac{-p}{2\tau\sigma^2}(\bar{\theta}-N)^2\right)$$

This is a normal $\boxed{N\left(\bar{\theta}, \frac{\sigma^2\tau^2}{p}\right)}$

* $P(\tau^2|-) \propto P(\tau^2|\theta) \propto P(\theta|\tau^2)P(\tau^2)$

$$\propto (\tau\cancel{\sigma})^{-p}\exp\left(-\frac{1}{2}(\theta-N)^T(\tau^2\sigma^2 I)^{-1}(\theta-N)\right)(\tau^2)^{-\frac{1}{2}-1}\exp\left(-\frac{1}{2}\tau^{-2}\right)$$

$$\propto (\tau^2)^{-\frac{p}{2}-\frac{1}{2}-1}\exp\left(-\frac{1}{2}(\theta-N)^T(\tau^2\sigma^2 I)^{-1}(\theta-N)+\tau^{-2}\right)$$

$$\propto (\tau^2)^{-\frac{p+1}{2}-1}\exp\left(-\frac{\tau^{-2}}{2}\left[(\theta-N)^T(\sigma^2 I)^{-1}(\theta-N)+1\right]\right)$$

This is an $\boxed{IG\left(\frac{p+1}{2}, \frac{(\theta-N)^T(\sigma^2 I)^{-1}(\theta-N)+1}{2}\right)}$

* $P(\sigma^2|-) \propto P(\sigma^2|\theta_i, y_i)$

$$\propto P(\sigma^2|\theta,y) \propto P(y|\theta)P(\theta)P(\sigma^2)$$

First find $P(y|\theta_i \ldots)$

First Find $P(y|\theta_i, \dots)$

$$\propto \prod_{i=1}^{p}\prod_{j=1}^{n_i} \sigma^{-1}\exp\left(-\frac{1}{2\sigma^2}(\theta_i - y_{ij})\right)^2$$

$$\propto \sigma^{-N}\exp\left(-\frac{1}{2\sigma^2}\sum_{1}^{p}\sum_{1}^{n_i}(\theta_i - y_{ij})^2\right)$$

so
$$P(\sigma^2|-) \propto (\sigma^2)^{-\frac{N}{2}}\exp\left(-\frac{1}{2\sigma^2}\sum_{1}^{p}\sum_{1}^{n_i}(\theta_i - y_{ij})^2\right)$$
$$(\sigma^2\tau^2)^{-\frac{p}{2}}\exp\left(-\frac{1}{2}(\theta-\mu)^T(\sigma^2\tau^2 I)^{-1}(\theta-\mu)\right)\times\frac{1}{\sigma^2}$$

$$\propto (\sigma^2)^{-\frac{N+p}{2}-1}\exp\left(-\frac{1}{2\sigma^2}\left((\theta-\mu)^T(\tau^2 I)^{-1}(\theta-\mu) + \left(\sum_{1}^{p}\sum_{1}^{n_i}(\theta_i - y_{ij})\right)^2\right)\right)$$

This is an IG $\left(\frac{N+p}{2}, \frac{(\theta-\mu)^T(\tau^2 I)^{-1}(\theta-\mu)}{2} + \frac{\sum^{p}\sum^{n_i}(\theta_i - y_{ij})^2}{2}\right.$

R

D) $E(\theta_i \mid -) = \dfrac{\bar{y}_i \tau^2 + \mu n_i^{-1}}{\tau^2 + n_i^{-1}}$

$= \dfrac{\bar{y}_i \tau^2 n_i + \mu}{\tau^2 n_i + 1}$

$= \underbrace{\dfrac{1}{\tau^2 n_i + 1}}_{k_i} \mu + \underbrace{\dfrac{\tau^2 n_i}{\tau^2 n_i + 1}}_{1-k_i} \bar{y}_i$

$k_i = \dfrac{1}{\tau^2 n_i + 1}$

(E) $y_{ij} = \mu + \delta_i + e_{ij}$

$\delta_i \sim N(0, \tau^2 \sigma^2)$

$e_{ij} \sim N(0, \sigma^2)$

$y_{ij} \sim N(\theta_i, \sigma^2)$

$\theta_i \sim N(\mu, \tau^2, \sigma^2)$

$\theta_i = \mu + \delta_i$ $\qquad$ $\delta_i \sim N(0, \tau^2 \sigma^2)$

$y_{ij} = \theta_i + \epsilon_{ij}$ $\qquad$ $\epsilon_{ij} \sim N(0, \sigma^2)$

$y_{ij} = \mu + \delta_i + \epsilon_{ij}$

$\operatorname{cor}(y_{ij}, y_{ik})$ $\quad j \neq k$

$$E\left((\mu + \delta_i + \epsilon_{ij} - E(y_{ij}))\right.$$
$$\left.(\mu + \delta_i + \epsilon_{ik} - E(y_{ik}))\right)$$
$$= E\left((\cancel{\mu} + \delta_i + \epsilon_{ij} - \cancel{\mu})(\cancel{\mu} + \delta_i + \epsilon_{ik} - \cancel{\mu})\right)$$
$$= E\left([\delta_i + \epsilon_{ij}][\delta_i + \epsilon_{ik}]\right)$$
$$= E\left(\delta_i 2 + \delta_i \epsilon_{ik} + \epsilon_{ij}\delta_i + \epsilon_{ij}\epsilon_{ik}\right)$$
$$= var(\delta_i) + \cancel{cov(\delta_i, \epsilon_{ik})} + \cancel{cov(\delta_i, \epsilon_{ij})}$$
$$+ \cancel{cov(\epsilon_{ij}, \epsilon_{ik})}$$
$$= \boxed{\tau 2 \sigma 2}$$

$$cov(y_{ij}, y_{i'k})$$

$$= E\left((\cancel{\mu} + \delta_i + \epsilon_{ij} \cancel{\mu})(\cancel{\mu} + \delta_{i'} + \epsilon_{i'k} \cancel{\mu})\right)$$
$$= E(\delta_i \delta_{i'} + \delta_i \epsilon_{i'k} + \epsilon_{ij}\delta_{i'} + \epsilon_{ij}\epsilon_{i'k})$$
$$= \boxed{\emptyset}$$

(F) common variance for all schools?

$$\hat{S}_i = \sqrt{\frac{\Sigma(y_{ij} - \bar{y}_i)^2}{N_i - 1}}$$ estimate of the variance

Blood pressure

A) t-test

$H_0: N_2 - N_1 = 0$ similar variances

$H_1: N_2 - N_1 \neq 0$

if H_0 is true $N_2 - N_1 \sim t$ with

$loc = 0$, $s = \sqrt{\dfrac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}}$

$dof = n_1 + n_2 - 2$

Reject H_0 with $\alpha = 0.05$

Not appropriate because there are several observations per person & people sampled different number of times.

$$SE = \sqrt{\frac{1}{n_1}\sigma_1^2 + \frac{1}{n_2}\sigma_2^2}$$

B) Still reject H_0. Standard error is bigger because we have less samples. It is still problematic that all subjects don't have the same number of tests.

(c) $y_{ij} \sim N(\theta_i, \sigma^2)$

$\theta_i \sim N(\mu + \beta x_i, \tau^2 \sigma^2)$

j = measurement number

i = person

Flat prior on $\beta \rightarrow$ doesn't change anything, replace μ with $\mu + \beta x_i$ in conditionals

$$P(\beta \mid -) \propto P(\beta \mid \theta_i)$$

$$\propto P(\theta_i \mid \beta \ldots) P(\beta)$$

$$\propto \prod_i^p \exp\left(\frac{-1}{2\sigma^2\tau^2} (\theta_i - (\mu + \beta x_i))^2 \right)$$

$$\propto \exp\left(-\frac{1}{2\sigma^2\tau^2} \sum_i^p (\theta_i - (\mu + \beta x_i))^2 \right)$$

$$\propto \exp\left(\frac{-1}{2\sigma^2\tau^2} \sum_i^p (\theta_i^2 - 2\theta_i(\mu + \beta x_i) + (\mu + \beta x_i)^2 \right)$$

$$\propto \exp\left(-\frac{1}{2\sigma^2\tau^2} \left(\cancel{\mu \theta_i^2} - 2 \sum^p \theta_i \beta x_i + \sum \cancel{\mu^2} + 2\mu\beta \bar{x}_i p + \beta^2 \overline{x_i^2} p \right) \right)$$

$$\propto \exp\left(\frac{-1}{2\sigma^2\tau^2} \left(-2\, \overline{\theta_i x_i}\, p\beta + 2\mu\beta p \bar{x}_i + \beta^2 \overline{x_i^2} p \right) \right)$$

$$\propto \exp\left(\frac{-1}{2\sigma^2\tau^2}\left(-2\beta\left(\overline{\theta_i x_i}\, p - N p \bar{x}_i\right) + \beta^2 \overline{x_i^2}\, p\right)\right)$$

$$\propto \exp\left(-\frac{\overline{x_i^2}\, p}{2\sigma^2\tau^2}\left(-2\beta\frac{\left(\overline{\theta_i x_i}\, p - N p \bar{x}_i\right)}{\overline{x_i^2}\, p} + \beta^2\right)\right)$$

$$\propto \exp\left(-\frac{\overline{x_i^2}\, p}{2\sigma^2\tau^2}\left(\beta - \left(\frac{\overline{\theta_i x_i}\,\cancel{p} - N \cancel{p}\bar{x}_i}{\cancel{p}\,\overline{x_i^2}}\right)\right)^2\right)$$

This is a $\boxed{N\left(\frac{\overline{\theta x} - N\bar{x}}{\overline{x^2}}, \frac{\sigma^2\tau^2}{p\overline{x^2}}\right)}$

$$P(N \mid -) \propto P(N \mid \theta)$$

$$\propto P(\theta \mid N)\, P(N)$$

$$\propto \prod^{p} \exp\left(-\frac{1}{2\tau^2\sigma^2}\left(\theta_i - (N + \beta x_i)\right)^2\right)$$

$$\propto \exp\left(-\frac{1}{2\tau^2\sigma^2}\sum\left(\theta_i - (N + \beta x_i)\right)^2\right)$$

$$\propto \exp\left(\frac{-1}{2\tau^2\sigma^2}\sum\left(\theta_i^2 - 2(N + \beta x_i)\theta_i + (N + \beta x_i)^2\right)\right)$$

$$\propto \exp\left(-\frac{1}{2\tau^2\sigma^2}\sum\left(\cancel{\theta_i^2} - 2(N + \cancel{\beta x_i})\theta_i + N^2 + 2\beta x_i N + \beta^2 x_i^2\right)\right)$$

$2\mu N + \beta x_i^2)$

$\propto \exp\left(\frac{-1}{2\tau^2\sigma^2}\sum\left(-2(N\theta_i - \beta x_i N) + N^2\right)\right)$

$\propto \exp\left(-\frac{1}{2\tau^2\sigma^2}\left(-2(Np\bar{\theta}_i - \beta\bar{x}_i p N) + N^2 p\right)\right)$

$\propto \exp\left(\frac{-1}{2\tau^2\sigma^2}\left(-2N(p\bar{\theta} - \beta\bar{x}_i p) + N^2 p\right)\right)$

$\propto \exp\left(-\frac{p}{2\tau^2\sigma^2}\left(N - (\bar{\theta} - \beta\bar{x}_i)\right)^2\right)$

$$\boxed{N\left(\bar{\theta} - \beta\bar{x}, \frac{\tau^2\sigma^2}{p}\right)}$$

$P(\theta_i \mid -) \propto P(\theta_i \mid \bar{y}_i)$

$\propto P(\bar{y}_i \mid \theta_i)\, P(\theta_i)$

$\propto \exp\left(-\frac{n}{2\sigma^2}(\bar{y}_i - \theta_i)^2\right)$

$\times \exp\left(-\frac{1}{2\sigma^2\tau^2}(\theta_i - (N + \beta x_i))^2\right)$

$\propto \exp\left(-\frac{n}{2\sigma^2}\left(\cancel{\bar{y}_i^2} - 2\bar{y}_i\theta_i + \theta_i^2\right)\right)$

$(\theta_i^2 - 2\theta_i(N + \beta x_i) + (N + \cancel{\beta x_i})^2)$

$$-\frac{}{2\sigma^2\tau^2}\left(\theta_i - 2\theta_i(\mu+\beta x_i) + (\mu+\beta x_i)^2\right)$$

$$\propto \exp\left(-\frac{n}{2\sigma^2\tau^2}\left(-2\bar{y}_i\theta_i\tau^2 + \theta_i^2\tau^2 + \theta_i^2 n_i^{-1} - 2\theta_i n_i^{-1}(\mu+\beta x_i)\right)\right)$$

$$\propto \exp\left(\frac{-n}{2\sigma^2\tau^2}\left(-2\theta_i\left(\bar{y}_i\tau^2 + n_i^{-1}(\mu+\beta x_i)\right) + \theta_i^2(\tau^2+n_i^{-1})\right)\right)$$

$$\propto \exp\left(-\frac{n_i(\tau^2+n_i^{-1})}{2\sigma^2\tau^2}\left(\theta_i - \frac{(\bar{y}_i\tau^2 + n_i^{-1}(\mu+\beta x_i))}{\tau^2+n_i^{-1}}\right)\right)$$

$$\boxed{N\left(\frac{\bar{y}_i\tau^2 n_i + (\mu+\beta x_i)}{n_i\tau^2+1}, \frac{\sigma^2\tau^2}{\tau^2 n_i+1}\right)}$$

$X\gamma = \mu + \beta x_i$ * other possible parametrization

$$P(\gamma|-) \propto \prod^{p} P(\theta_i|\gamma,\sigma^2,\tau^2)\cancel{P(\gamma)}$$

$$\propto \exp\left(-\frac{1}{2}(\theta - X\ )^T(\tau^2\sigma^2 I)^{-1}(\theta - \ )\right)$$

$$\boxed{\begin{aligned} &\gamma|- \sim N(m,v) \\ &m = (X^TX)^{-1}X^T\theta \\ &v = \tau^2\sigma^2(X^TX)^{-1} \end{aligned}}$$

$SE = \pm\ 0.067$

D) $(y_{ij} \perp y_{ik} | \theta_i)$

Says that the u_{ij}'s are independent for a given i. This seems ok to me.