

Introduction to Mathematical Modelling

Lecture 1

Ago Merico

27 January 2025

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On Exactitude in Science

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By Jorge Luis Borges

Mathematical Modelling

there are many reasons why scientists should have at least a working knowledge of mathematical modelling:

- ▶ models are useful for extracting information from large, complicated data sets;
- ▶ models that reflect underlying mechanisms can uncover information that could otherwise not be inferred from the data;
- ▶ models can help determine which experiment to do next: choose treatment, or suggest new observations in time and space;
- ▶ thinking in terms of a 'model' sharpens one's understanding of the hypotheses involved (just trying to formulate a model often throws conceptual difficulties in sharp relief).

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a **mathematical model** is a description of a phenomenon by means of mathematics, with the view of capturing the essential aspects of the phenomenon at hand.

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What is a mathematical model ?

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1. mathematical descriptions are at best **approximation** of a natural phenomenon;
2. even though models are *only* approximations of reality, they are still useful as long as they satisfy some **key requirements**, i.e.:
 - a. consistent;
 - b. coherent;
 - c. efficient (e.g. parsimonious, modular, general);
 - d. statistically tractable.

Elements of a mathematical model

<i>parameter</i>	a parameter of a model is a quantity that remains constant over time;
<i>variable</i>	a variable of a model is a quantity that changes over time;
<i>function</i>	a function is a description of a process that is part of the natural phenomenon a model intends to describe, functions are defined with variables and parameters;
<i>dynamics</i>	the dynamics of a system is the pattern of change that occurs over time;
<i>parsimony</i>	according to the principle of parsimony, a simple explanation (or model) should be preferred over a complex explanation if both are equally compatible with the data.

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this would require using hundreds or thousands equations, measuring thousands of parameters, solving the equations to get numerical predictions, and then comparing these predictions against nature;

The problem of complexity

- ▶ there are too many parameters to measure, some are vaguely defined or unknown, many would require a lifetime for their measurement;
- ▶ the equations would be not solvable analytically or they would exceed the capacity of current best computers;
- ▶ even if analytically solvable, the result would be so complicated that it would have no meaning for us.

The problem of complexity

it is desirable to work with manageable models which maximize **generality**, **realism**, and **precision** toward the goals of *understanding* and *predicting* nature.

but this cannot be achieved !

typically, alternative strategies are considered...

Strategies for addressing complexity

1. **sacrifice generality to realism and precision;**

this is typically the approach of many fishery biologists; this strategy involves reducing parameters to those relevant to the short-term behaviour of the organisms, make fairly accurate measurements, solve numerically on the computer, and conclude with precise, testable predictions specific to the particular situations investigated;

Strategies for addressing complexity

2. sacrifice realism to generality and precision;

this is typically the approach of many physicists who enter population biology; this strategy involves setting up quite general equations from which precise results may be obtained; these equations are unrealistic; for example Lotka-Volterra predator-prey systems, which omit time lags, physiological states, the effect of species' population density on its own rate of increase, the environment, etc. it is hoped that natural biological systems behave like frictionless systems or perfect gases; it is expected that small deviations from realism result in small deviations in the conclusions;

Strategies for addressing complexity

3. sacrifice precision to realism and generality;

this is the approach adopted by those who are interested in qualitative rather than quantitative aspects and results (which are only important in testing hypotheses); such models are typically very flexible, often graphical, and generally assume that functions are increasing or decreasing, convex or concave, greater or less than some values, instead of specifying the mathematical form of an equation; predictions are also expressed as inequalities as between tropical and temperate species, insular *versus* continental fauna, patchy *versus* uniform environments, etc.

Main point

models are essential for understanding reality but they
should not be confused with that reality itself!

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dynamical models can also be distinguished between *discrete time* and *continuous time* models, depending on whether time is represented in discrete steps or as continuous.

How to construct a model – recipe

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major steps to follow when constructing a model are:

1. formulate the question;
2. determine the basic ingredients;
3. qualitatively describe the relevant system;
4. quantitatively describe the relevant system;
5. analyse the equations;
6. checks and balances;
7. relate the results back to the question.

How to construct a model

1. formulate the question

coming up with a question can be more difficult than it sounds; in most natural science disciplines, students are told what the relevant questions are; rarely are students asked to formulate scientific questions; this is unfortunate because, in any scientific research endeavour, the process begin with a question.

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for example, consider the following three questions:

- a. how does the number of branches of a tree change over time?
- b. how does a cat change the number of mice in a yard?
- c. how does the number of people with the flu change over the flu season?

these are "toy" questions that we will use to illustrate the steps of modelling.

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consider the constraints that each variable is subject to; choose the time scale over which changes of variables are measured; we will use "day" as the basic unit of time;

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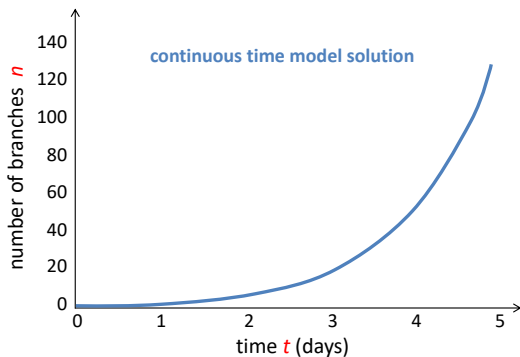
in **continuous-time** models, equations specify the rate of change of the variables over time:

$$\frac{dn(t)}{dt} = \text{"some function of } n(t) \text{"};$$

this equation is called *differential* equation; a differential equation describes the rate at which a variable changes over time.

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tree branching: a hypothetical plot of the number of branches on a tree over time produced using a continuous-time model.

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as with variables, one should figure out the constraints imposed on each parameter, including the range of parameter values that are biologically meaningful.

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we will only use flow diagrams.

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3. qualitatively describe the biological system - flow diagram

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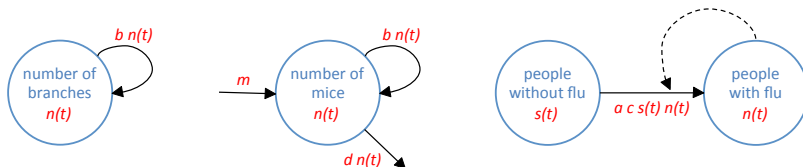
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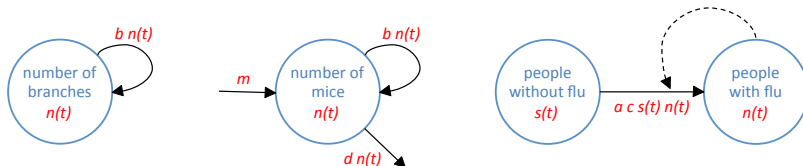


flow diagrams of the three different models considered; a *circle* represents a variable, arrows pointing to a circle indicate processes that drive a variable up over time, while arrows pointing out of a circle indicate processes that drive a variable down over time; returning arrows (those that exit and come back to the same circle) represent variables that can generate more of itself; the arrows represent events (i.e. processes) occurring continuously over time at certain rates; the dashed arrow indicates that a variable influences the flow into another circle but does not represent a decline in the variable from which the arrow begins (e.g. a carrier of the flu does not lose the flu by passing it on).

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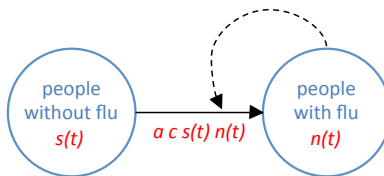
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individuals with the flu interact with susceptible individuals at a rate proportional to $n(t) s(t)$; specifically, an infected individual has a probability c of contacting any given one of the healthy individuals $s(t)$ per day, giving a total number of contacts per day of $c s(t)$ per infected individual; with $n(t)$ infected individuals in the population, we expect a total of $c n(t) s(t)$ contacts per day; the probability that any contact results in the transmission of the virus is a ; therefore, we expect a total of $a c n(t) s(t)$ new cases of flu per day.

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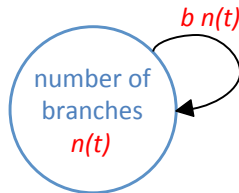
$$\frac{dn(t)}{dt} = \text{rate of increase} - \text{rate of decrease}$$

the general approach is simple: we sum all the factors causing the variable to increase or decrease without worrying about the order of events within a time step, because the time step is infinitesimally small.

How to construct a model

4. quantitatively describe the biological system

for the tree-branching model:



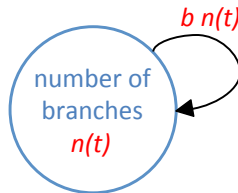
$dn(t) / dt = +$ the flow rates along arrows entering the circle - the flow rates along arrows exiting the circle + the flow rates along arrows leaving and returning to the circle.

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$$\frac{dn(t)}{dt} = b n(t)$$

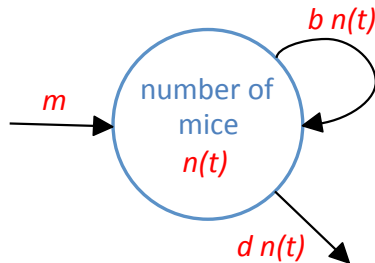


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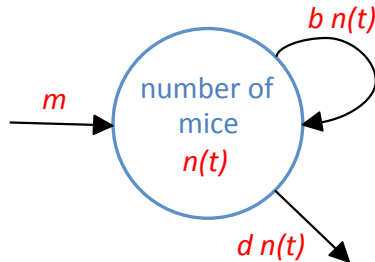
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$$\frac{dn(t)}{dt} = b n(t) - d n(t) + m$$

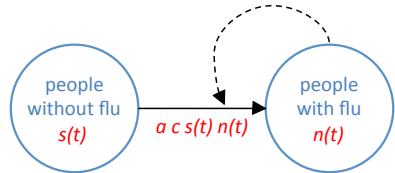


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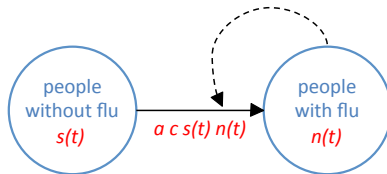
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- ▶ graphical analyses;
- ▶ equilibrium and stability analyses;
- ▶ deriving general solutions;
- ▶ determining long-term or asymptotic behaviours;
- ▶ analysing the model for periodic behaviour.

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at this point, the models are said to be *fully specified*; we know the type of model, the variables, the parameters, and the equations for describing the changes in the variables; the next step is to analyse the model;

there are many different ways of analysing equations, these include:

- ▶ graphical analyses;
- ▶ equilibrium and stability analyses;
- ▶ deriving general solutions;
- ▶ determining long-term or asymptotic behaviours;
- ▶ analysing the model for periodic behaviour.

we will not cover these aspects in this course

How to construct a model

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carry detailed sensitivity analyses; the sensitivity analysis investigates how the uncertainty in the output of a mathematical model can be related to different sources of uncertainty in its inputs (typically, the parameters).

How to construct a model

7. relate the results back to the question

modelling biological processes is worthwhile only if the mathematical results are related back to biological problems;

theory is closely tied to empirical observations and tests; empirical observations suggest a theoretical model, which in turn generates an empirical test, which suggests that the model needs to be refined in particular ways; this interplay is extremely fruitful and could be driven by asking the following questions:

- ▶ how do the results alter the way scientists should think about a problem?
- ▶ what predictions can be made based on the model?
- ▶ what experiments could test these predictions?
- ▶ are there any data that can be explained or better understood in light of the model results?

Final summary

- ▶ a mathematical model is a description of a natural phenomenon by means of mathematics;
- ▶ models are approximation of reality and should not be confused with reality itself;
- ▶ relevant components and aspects of mathematical models are parameters, variables, functions, dynamics, and parsimony;
- ▶ models are constructed by following a number of steps, which include (1) defining a question, (2) gathering the main ingredients, (3) formulate the model schematically, (4) translate the conceptual schemes into equations, (5) analyse the equations, (6) perform checks and balances, and (7) relate the results back to the original question.

Further reading



Hugo van den Berg 2011

Mathematical Models of Biological Systems

Oxford University Press



Sarah P. Otto & Troy Day 2007

A Biologist's Guide to Mathematical Modelling in Ecology and Evolution

Princeton University Press



Richard Levins

The strategy of model building in population biology

American Scientist, 54(4):421–431, 1966