02. Probability

Adapted From :

Probability & Statistics for Engineers & Scientists, 9^{th} Ed.

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Applied Statistics and Probablity for Engineers, 6^{rd} Ed.

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Statistics for Managers, 5^{th} Ed

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Uncertainty

- **Life is not deterministic**
- We do not know every detail for sure and have to give decision every day
  - Example: There is 50% of chance of rain today.
  - Should we bring our umbrella?

Methods for Determining Probability

There are three methods for determining the probability of an event:

(1) the classical method

(2) the empirical method

(3) the subjective method

The classical method of computing probabilities requires *equally likely outcomes.*

An experiment is said to have **equally likely outcomes** when each simple event has the same probability of occurring.

Classsical Method

If an experiment has *n* equally likely (simple) events and if the number of ways that an event *E* can occur is *m*, then the probability of *E*, *P*(*E*), is

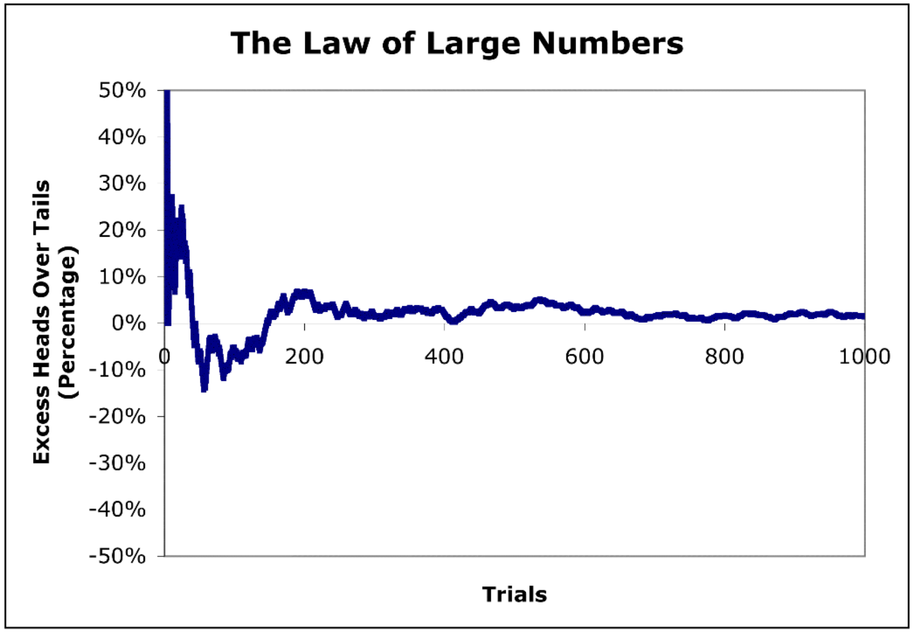
So, if *S* is the sample space of this experiment, then

$$P(E) = \frac{\text{Number of way that E can occur}}{\text{Number of Possible Outcomes}} = \frac{m}{n}$$

$$P(E) = \frac{N(E)}{N(S)}$$

Law of Large Numbers

- As the number of repetitions of a probability experiment increases, the proportion with which a certain outcome is observed gets closer to the probability of the outcome.



Empirical Method

- If outcome of an experiment are not equally likely to occur, **the probabilities must be assigned on the basis of prior knowledge or experimental evidence.**
- For example, if a coin is not balanced, we could **estimate** the probabilities of heads and tails by tossing the coin a large number times and recording the outcomes.
- **According to the relative frequency definition of probability, the true probabilities would be the fractions of heads and tails that occur in the long run**

Empirical Method

The probability of an event *E* is approximately the number of times event *E* is observed divided by the number of repetitions of the experiment.

$$P(E) \approx \text{relative frequency of E}$$

$$= \frac{\text{frequency of E}}{\text{number of trials of experiment}}$$

EXAMPLE *Using Relative Frequencies to Approximate Probabilities*

The following data represent the number of homes with various types of home heating fuels based on a survey of 1,000 homes.

(a) Approximate the probability that a randomly selected home uses electricity as its home heating fuel.

(b) Would it be unusual to select a home that uses coal or coke as its home heating fuel?

HOUSE HEATING FUEL	Frequency
Utility gas	504
Bottled, tank, or LP gas	64
Electricity	307
Fuel oil, kerosene, etc.	94
Coal or coke	2
Wood	17
Solar energy	1
Other fuel	4
No fuel used	7

Subjective Method

Subjective probabilities are probabilities obtained based upon an educated guess.

For example, there is a 40% chance of rain tomorrow.

Probability

- Probability is the numerical measure of the likelihood that an event will occur
- The probability of any event must be between 0 and 1

$$0 \le P(A) \le 1 \quad \text{For any event A}$$

- The sum of the probabilities of all mutually exclusive and collectively exhaustive events is 1

$$P(A) + P(B) + P(C) = 1$$

If A, B, and C are mutually exclusive and collectively exhaustive

Sample Spaces and Events

- An **experiment** is any activity or process whose outcome is subject to uncertainty.
- Although the word *experiment* generally suggests a planned or carefully controlled laboratory testing situation, we use it here in a much wider sense.
- Thus experiments that may be of interest include **tossing a coin once or several times, selecting a card or cards from a deck, weighing a loaf of bread, ascertaining the commuting time from home to work on a particular morning, obtaining blood types from a group of individuals, or measuring the compressive strengths of different steel beams.**

Probability

- The <u>sample space</u> S of an experiment is the set of all possible outcomes.
  - We must understand the sample space in order to determine the probability of each outcome occurring.
  - The sample space is a set, the domain of the probability function.
- Properly enumerating the sample space is key to correctly calculating probabilities.
  - A <u>tree diagram</u> is sometimes useful.

Tree Diagram

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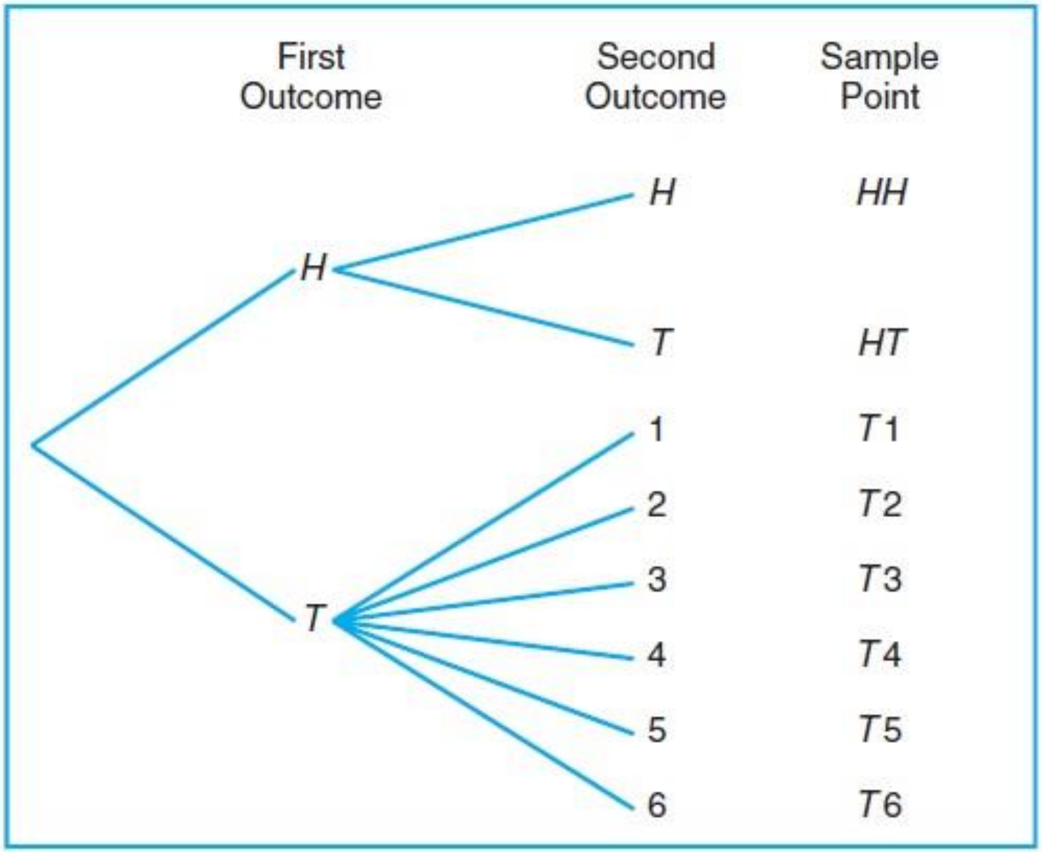

Figure 2.1: Tree diagram for Example 2.2.

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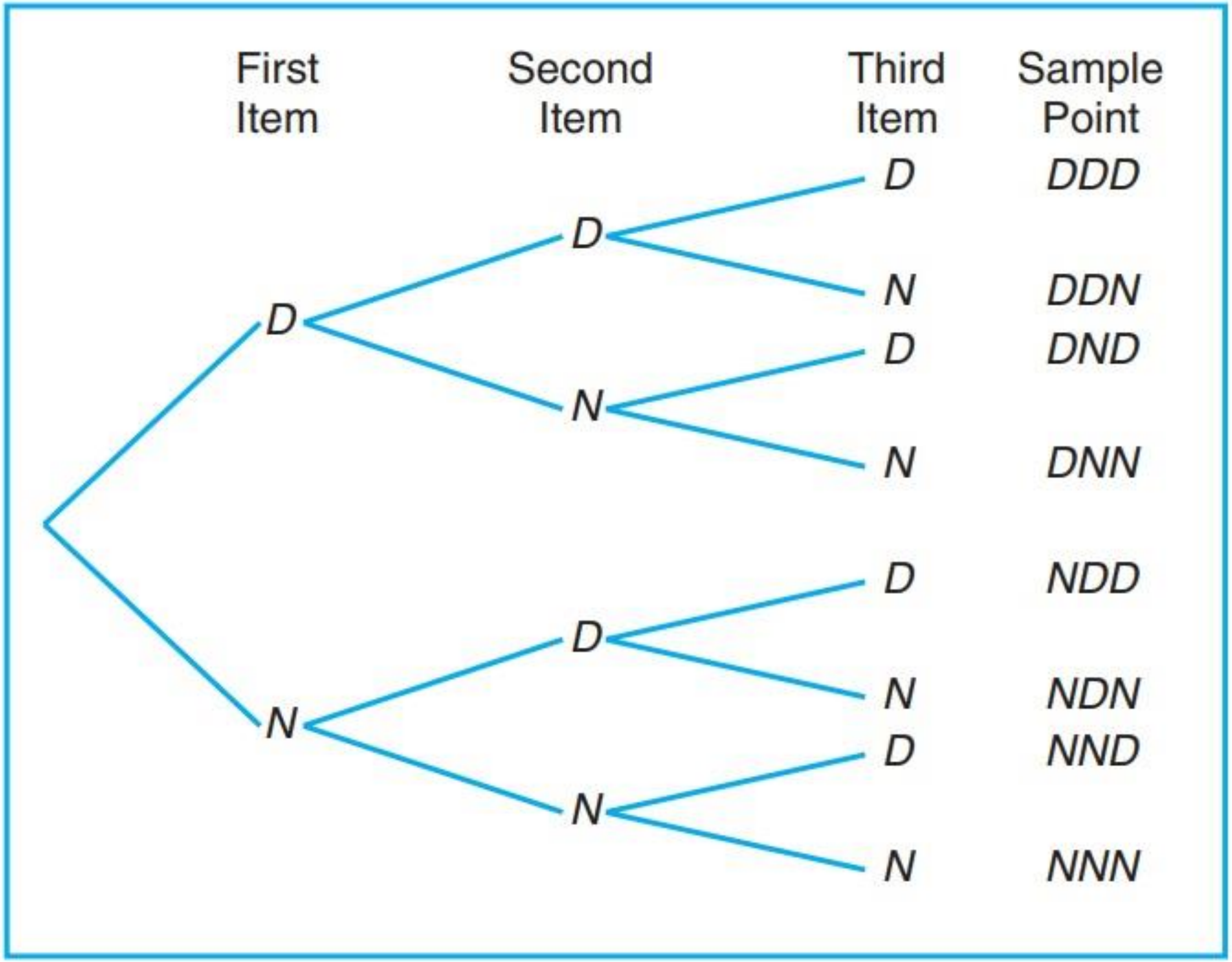

Figure 2.2: Tree diagram for Example 2.3.

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Events

- An <u>event</u> is a subset of a sample space ($E \subseteq S$)
  - Note that both S and $\varnothing$ are events as well.
- Sample spaces can be continuous or discrete.
  - What is a continuous vs. discrete sample space?
- Example: Life in years of a component. S = ?
  - $S = \{t \mid t \geq 0\}$ => "all values of t such that $t \geq 0$"
  - A = component fails before the end of the fifth year.
  - $A = \{t \mid t < 5\}$.
- Example: Flip a coin three times. S = ?
  - S = {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}
- Event A = 1^{st} flip is heads.
  - A = {HHH, HHT, HTH, HTT}

Event/Set Operations

- The complement of an event A?
  - The set of all elements of S not in A. Denoted A’.
  - A = 1^{st} flip is heads. A’ = first flip is not heads.
- The intersection of two events A and B?
  - The set of all elements in both A and B. Denoted $A \cap B$.
  - B = 2^{nd} or 3^{rd} flip, but not both, are heads.
  - B = {HHT, HTH, THT, TTH}. $A \cap B$ = ?
  - $A \cap B$ = {HHT, HTH}
- Two events are mutually exclusive if…?
  - $A \cap B = \varnothing$
- The union of two events, A and B?
  - The set of elements in either A or B. $A \cup B$ = ?
  - $A \cup B$ = {HHH, HHT, HTH, HTT , THT, TTH}.

Diagram Venn

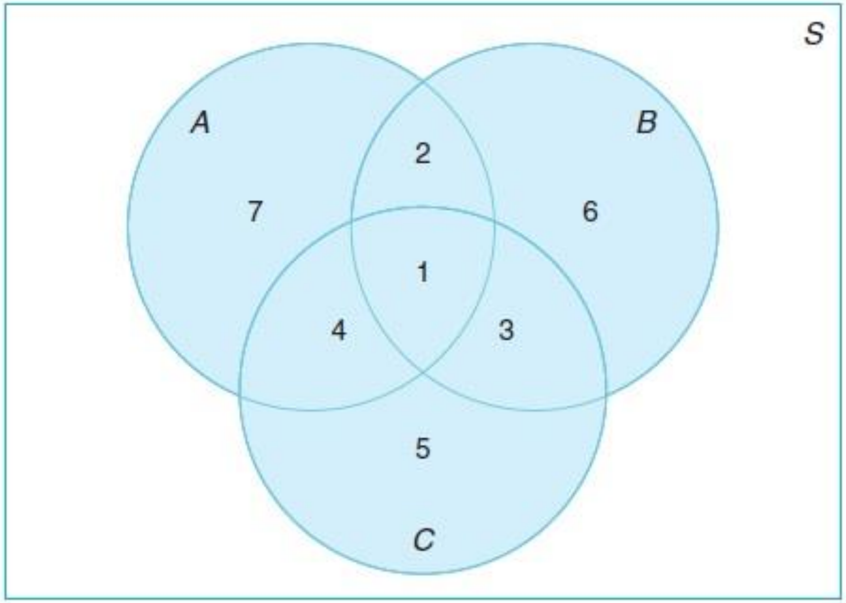


Figure 2.3: Events represented by various regions.

$$
\begin{aligned}
A \cup C &= \text{regions } 1, 2, 3, 4, 5, \text{ and } 7,\\
B' \cap A &= \text{regions } 4 \text{ and } 7,\\
A \cap B \cap C &= \text{region } 1,\\
(A \cup B) \cap C' &= \text{regions } 2, 6, \text{ and } 7,
\end{aligned}
$$

Venn Diagrams

- Venn Diagrams show various events graphically, and are sometimes helpful in understanding set theory problems.
- Standard set theory results hold:
  - $A \cap \varnothing = ?$
  - $A \cup \varnothing = ?$
  - $A \cap A' = ?$
  - $A \cup A' = ?$
  - $S' = ?$
  - $\varnothing' = ?$
  - $(A')' = ?$
  - $(A \cap B)' = ?$
  - $(A \cup B)' = ?$
  - $(A \cap B)'=A' \cup B'$, $(A \cup B)'=A' \cap B'$

Intuitive Sample Point Counting

- If one operation can be performed in n_1 ways, and for each way, a second can be performed in n_2 ways, then the two can be performed a total of $n_1 n_2$ ways.
  - For three operations?
  - $n_1 n_2 n_3$.
  - How many passwords of length 5 need to be checked by a password hacking program if only lower case letters are used?
  - $26^5 = 11,881,376$.
- This is called sampling <u>with replacement</u>.
- A tree diagram can be used to enumerate all of the options.

Multiplication Rule

- Rule 1 : If an operation can be performed in n_1 ways, and if for each of these ways a second operation can be performed in n_2 ways, the the two operations can be performed together in $n_1 n_2$ ways.
  - How many sample points are there in the sample space when a pair of dice is thrown once?
  - A developer of a new subdivision offers prospective home buyers a choice of Tudor, Rustic, Colonial and Traditional exterior styling in ranch, two story, and split level floor plans. In how many different ways can a buyer order one of these homes?

Counting the Sample Points

- A developer of a new subdivision offers prospective home buyers a choice of Tudor, rustic, colonial, and traditional exterior styling in ranch, two-story, and split-level floor plans. In how many different ways can a buyer order one of these homes?
- Since $n1 = 4$ and $n2 = 3$, a buyer must choose from $n1n2 = (4)(3) = 12$ possible homes

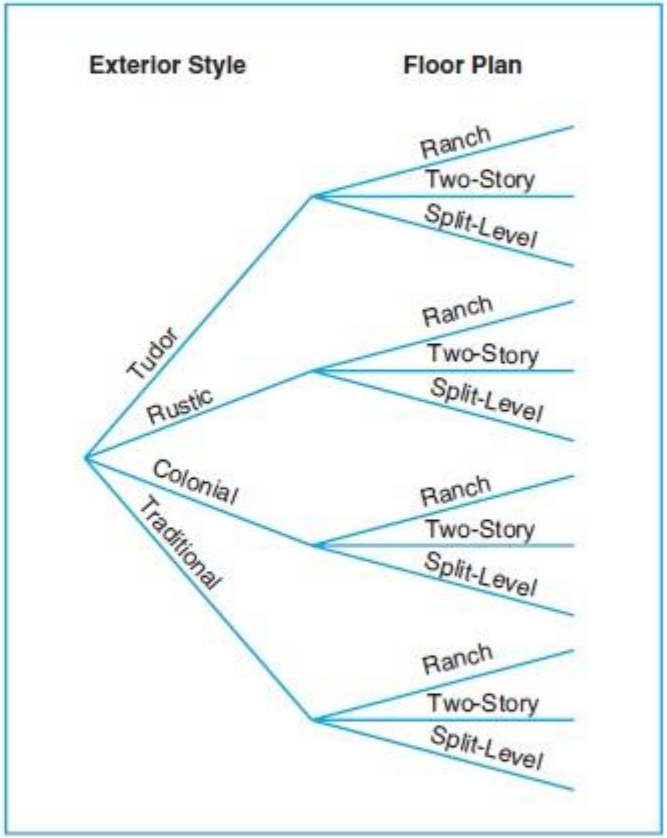


Figure 2.6: Tree diagram for Example 2.14.

Multiplication Rule

Rule 2 : If an operation can be performed in n_1 ways, and if for each of these a second operation can be performed in n_2 ways, and for each of the first two a third operation can be performed in n3 ways, and so forth, then the sequence of k operations can be performed in $n_1 n_2 \ldots n_k$ ways.

- How many **even four-digit numbers** can be formed from the digits 0, 1, 2, 5, 6 and 9, if each digit can be used only once?

Permutation Orderings

- A permutation is an ordering of a set or subset of objects.
- The number of distinct orderings of n items?
  - n items can go in the first position.
  - Once the first item is fixed, n-1 items can go in the 2nd position.
  - Then n-2 items in the third position, etc.
  - Number of orderings is n (n-1) (n-2) … 1, or n!
  - Remember that 1! = 0! = 1.
- This is called sampling without replacement.
  - Once we use a value, it can't be used again.

Permutation

- A Permutation is an arrangement of all or part of a set of objects.
  - E.g. : Three letters a, b, and c. The possible permutations are abc, acb, bac, bca, cab, and cba.
- The number of permutations of n objects is n!
- The number of permutation of n distinct objects taken r at a time is

$$ {}_nP_r = \frac{n!}{(n-r)!} $$

Permutations of Partial Orderings

- Suppose that we will give 3 different awards to three students out of a class of 60 students. How many ways can the awards be given.
  - What if the problem was slightly different and one student could win all three awards?
  - This would be with replacement.
  - The number is $60^3 = 216{,}000$
- Without replacement, 60 students could get the first award, then 59 students are eligible for the 2^{nd} and 58 for the third, or 60*59*58 = 205,320.
- In general, the number of permutations of n things taken r at a time is written ${}_nP_r = n! / (n-r)!$
  - 60! / 57! = 60*59*58.

Circular Permutations

- Circular permutations: n distinct objects arranged in a circle.
  - The position of the first object could be anywhere.
  - If all objects moved one position clockwise, it's still the same permutation.
  - Fix the first object anywhere on the circle, then n-1 objects can go to the left, n-2, next, etc.
  - The number of circular permutations of n objects is (n - 1)!.
  - Another way to look at it: the set of all permutations is n!. For each starting value, there are n orderings that are identical (moving the same ordering around the circle). So the total number of different orderings is n!/n = (n-1)!
  - For example, for the ordering 35142 for n = 5.
  - There are 5 identical orderings (35142, 51423, 14235, 42351, 23514).

Permutations with Identical Objects

- If some objects are identical, with n_1 of type 1, n_2 of type 2, …, n_k of type k, and $n = n_1 + n_2 + \dots + n_k$, the number of distinct permutations is

$$\frac{n!}{n_1!n_2!\dots n_k!}$$

- Example, 3 items of type 1, 1 of type 2 gives 4 permutations.
  - 1112, 1121, 1211, and 2111.
  - For each of the four, there would be 3! orderings if the 1's were distinct (say a, b, and c).
  - For example, 1121 would be ab2c, ac2b, ba2c, bc2a, ca2b, and cb2a. Or, if all items are distinct, there would be 4*3! Or 4! orderings.
  - Three items being identical reduces the number of permutations by a factor of 3!, so we divide by 3!.

Example of Permutation Case

- In a college football training session, the defensive coordinator needs to have 10 players standing in a row. Among these 10 players, there are 1 freshman, 2 sophomores, 4 juniors, and 3 seniors. How many different ways can they be arranged in a row if only their class level will be distinguished?
- Aswer :

$$\frac{10!}{1!2!4!3!} = 12,600$$

Arranging n Objects Into r Cells

- Partitioning n distinct objects into r cells or subsets, each of a given fixed size, where the ordering of objects within a cell doesn't matter.
- Example divide 5 items into two cells, one of size 3 and one of size 2.
  - {(1,2,3), (4,5)}, {(1,2,4), (3,5)}, {(1,2,5), (3,4)}, {(1,3,4), (2,5)}, {(1,3,5), (2,4)}, {(1,4,5), (2,3)}, {(2,3,4), (1,5)}, {(2,3,5), (1,4)}, {(2,4,5), (1,3)}, {(3,4,5), (1,2)}
- There are n! total possible orderings, but n_1! In the 1st cell, and n_2! in the 2nd cell, etc., are identical.
- In general, the number of distinct combinations of n distinct objects into r cells, with n_1 items in the 1^{st} cell, n_2 in the 2^{nd}, …, and n_r in the r^{th} cell is $n! / (n_1!\, n_2! \dots n_r!)$.

Combinations of n Items Taken r at a Time

- To review, there are how many permutations of n items taken r at a time if each ordering is distinct?
  - n (n-1) (n-2) ... (n-r+1), or
  - ${}_nP_r$ = n! / (n – r)! .
  - For any given set of r items, there are r! possible orderings.
- So what if the order of the r items doesn't matter?
  - Divide ${}_nP_r$ by r! to get the number of distinct outcomes.
- The number of <u>combinations</u> of n items taken r at a time, where order doesn't matter, is ${}_nC_r$ = n! / (r!(n-r)!).

Combination

- **Combination** is selecting r objects from n without regard to order.

$$\boxed{\begin{pmatrix} n \\ r, n-r \end{pmatrix} = \begin{pmatrix} n \\ r \end{pmatrix} = \frac{n!}{r!(n-r)!}}$$

- Example: How many distinct poker hands of 5 cards each can be dealt using a deck of 52 cards?
  - ${}_{52}C_5$ = 52! / (5!47!) = 52*51*50*49*48 / 5*4*3*2*1 = 2,598,960.
- Example : A young boy asks his mother to get five Game-Boy cartridges from his collection of 10 arcade and 5 sports games. How many ways are there that his mother will get 3 arcade and 2 sports games, respectively?

Probability of an Event

- For now, we only consider discrete sample spaces.
- Each point in a sample space is assigned a weight or probability value. The higher the probability, the more likely that outcome is to occur.
- The probability of an event A is the sum of the probabilities of the individual points in A. Then,
  - $0 \leq P(A) \leq 1$
  - $P(\varnothing) = ?$
  - $P(S) = ?$
  - If two events are mutually exclusive, (which means?)
  - (that they have no points in common, or $A \cap B = \varnothing$),
  - then $P(A \cup B) = P(A) + P(B)$.

Relative Frequency for Probability

- If an experiment has N different <u>equally likely</u> outcomes, and n outcomes correspond to event A, then P(A) = ?.
    - $P(A) = n / N$.

Additive Probability Rules

- We already know that if A and B are mutually exclusive, $P(A \cup B) = P(A) + P(B)$.
- Also, if <u>more than 2</u> events are mutually exclusive, the probability of the union of all of those events is the sum of all of the individual probabilities.
- What is $P(A \cup B)$ if A and B are not mutually exclusive?
  - Can use a Venn diagram to show this case.
  - The sample points in $P(A \cap B)$ are double counted.
  - So $P(A \cup B) = ?$
  - $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.
- Given that we know P(A), what is P(A')?
  - $P(A') = 1 - P(A)$.

Conditional Probability

- Conditional probability, written $P(B|A)$, is the probability of "B, given A", the probability that B occurs, given that we know that A has occurred.
  - Look at a Venn diagram of A and B.
  - Suppose all sample points are equally likely.
  - $P(B)$ = (# of outcomes in B)/(total # of outcomes in S)
  - $P(B|A) = ?$
  - $P(B|A)$ = (# of outcomes in $A \cap B$)/(total # of outcomes in A)
  - or, $P(B|A) = P(A \cap B) / P(A)$ (as long as $P(A) > 0$).

Conditional Probability Example

- Example. For the following population of 900 people:

	Employed	Unemployed	Total
Male	360	140	500
Female	240	160	400
Total	600	300	900

- If a person is selected at random from this group,
  - $P(E) = ?$
  - $P(M) = ?$
  - $P(E \cap M) = ?$
  - $P(E|M) = ?$
  - $P(M|E) = ?$
  - Answers: $P(E)=600/900=2/3$, $P(M)=500/900=5/9$, $P(E \cap M)=360/900=2/5$, $P(E|M)=360/500=18/25$, $P(M|E)=360/600=3/5$.

Independence

- Conditional probability helps us update the probability of an event given additional information.
- Suppose $P(B|A) = P(B)$. What does this tell us?
  - Whether A occurs or not, the probability of B occurring doesn't change.
- If $P(B|A) = P(B)$, then A and B are <u>independent</u>.
  - Can show that if $P(B|A) = P(B)$ is true, then $P(A|B) = P(A)$ is always also true.
- From the above, and the definition of conditional probability, if A and B are independent,
  - $P(A \cap B) = P(A)\,P(B)$

Multiplicative Rules

- Rearranging the conditional probability formula, if both A and B can occur, then

  $$P(A \cap B) = P(B|A)\,P(A)$$

  Or, the probability of both A and B occurring equals the probability of B given A times the probability of A.

- Note that it is also true that

  $$P(A \cap B) = P(A|B)\,P(B)$$

- If (and only if) events A and B are <u>independent</u>, then from the above formula, we have

  $$P(A \cap B) = P(A)\,P(B)$$

- For more than two independent events, multiply all of the probabilities together.

Theorem of Total Probability

- Suppose the sample space S can be partitioned into events A1, A2, and A3. What does this mean?
  - A1, A2, and A3 are disjoint and between them cover all of S.
- Then the probability of an event B occurring can be calculated using conditional probabilities given that either A1 or A2 or A3 occurred.
- $P(B) = P(B|A1)P(A1) + P(B|A2)P(A2) + P(B|A3)P(A3)$. In words, this means?
  - The probability that B and A1 occur + the probability that B and A2 occur + the probability that B and A3 occur.
- This rule, called the theorem of total probability, or the rule of elimination, holds for any partitioning of S.

Bayes Rule

- Here we can calculate reverse conditional probabilities.
- Using an example from an earlier slide:

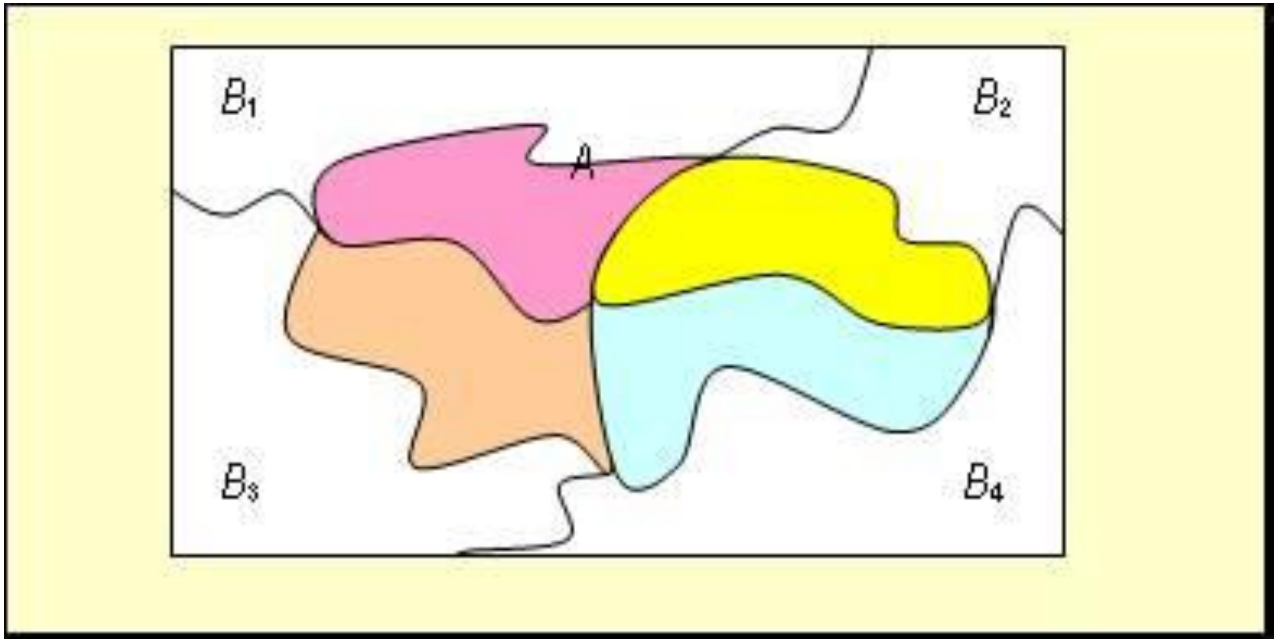
	Employed	Unemployed	Total
Male	360	140	500
Female	240	160	400
Total	600	300	900

- First we show how to use the theorem of total probability to calculate P(M).
  - S can be partitioned into mutually exclusive events E and U.
  - Then $P(M) = P(M|E)P(E) + P(M|U)P(U)$
  - Here $P(M) = (360/600)(600/900) + (140/300)(300/900) = 2/5 + 7/45 = 5/9$ (as expected).
  - Interpret 2/5 and 7/45 above.
  - Works with S partitioned into more than 2 events.

Bayes Theorem

- Bayes Theorem can be used to adjust the probability of a partition subset of S given additional information.
  - In the example suppose we know that the person selected is a male (M). Then what is the probability that the person selected is employed?
  - As before, $P(E|M) = P(E \cap M)/P(M)$, by the definition of conditional probability.
  - By Bayes Theorem,
    $$P(E|M) = \frac{P(M|E)P(E)}{P(M|E)P(E) + P(M|U)P(U)}$$
  - In the example,
    $$P(E|M) = (360/600)(600/900) \,/\, (2/5 + 7/45) = 18/25$$
  - $P(E)$ was 2/3. If we know M occurred, $P(E|M)$ becomes 18/25.
- $P(E)$ is called the prior (or "a priori"), and $P(E|M)$ is called the posterior (or "a posteriori") probabilities.

Bayes Theorem



- where:

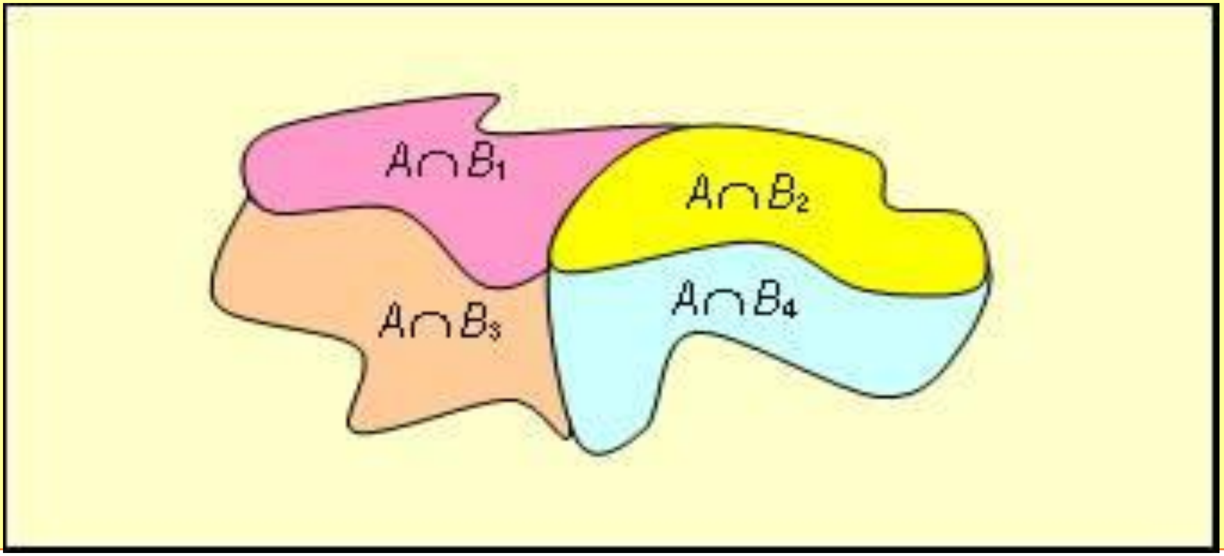
  B_i = i^{th} event of k mutually exclusive and collectively exhaustive events

  A = new event that might impact $P(B_i)$

Formulasi Bayes

- *If A* occurred first
- Probability of B_i is determined using Bayes theorem:

$$P(B_i \mid A) = \frac{P(B_i \cap A)}{P(A)} = \frac{P(B_i)P(A \mid B_i)}{\sum_{i=1}^{n} P(B_i)P(A \mid B_i)} = \frac{P(B_i)P(A \mid B_i)}{\sum_{i=1}^{n} P(B_i \cap A)}$$



Bayes' Theorem

$$P(B_i \mid A) = \frac{P(A \mid B_i)P(B_i)}{P(A \mid B_1)P(B_1) + P(A \mid B_2)P(B_2) + \ldots + P(A \mid B_k)P(B_k)}$$

- where:

  B_i = i^{th} event of k mutually exclusive and collectively exhaustive events

  A = new event that might impact $P(B_i)$

Bayes' Theorem Example

- A drilling company has estimated a 40% chance of striking oil for their new well.
- A detailed test has been scheduled for more information. Historically, 60% of successful wells have had detailed tests, and 20% of unsuccessful wells have had detailed tests.
- Given that this well has been scheduled for a detailed test, what is the probability that the well will be successful?

Bayes' Theorem Example

- Let S = successful well

  U = unsuccessful well
- P(S) = .4 , P(U) = .6 (prior probabilities)
- Define the detailed test event as D
- Conditional probabilities:

  P(D|S) = .6 P(D|U) = .2
- Goal is to find P(S|D)
  - So the revised probability of success, given that this well has been scheduled for a detailed test, is .667

Bayes' Theorem Example *(continued)*

- Given the detailed test, the revised probability of a successful well has risen to .667 from the original estimate of .4

Event	Prior Prob.	Conditional Prob.	Joint Prob.	Revised Prob.
S (successful)	.4	.6	.4*.6 = .24	.24/.36 = .667
U (unsuccessful)	.6	.2	.6*.2 = .12	.12/.36 = .333

Sum = .36

Thank You

"We trust in GOD, all others must bring data"