*El Inge de Control*

# Taller
# Control Robusto y Estocástico

## Tema:
## Control $\mathcal{H}_\infty$ de información completa

Dr. Julio A. García-Rodríguez

# Introduction

- The main goal of a control system is to meet **performance specifications** in addition to ensuring **internal stability**. One way to express performance is by measuring the size of certain system signals, such as the tracking error.
- To do this, the Hardy spaces $\mathcal{H}_2$ and $\mathcal{H}_\infty$ are introduced, which provide different norms for measuring signals depending on the context.

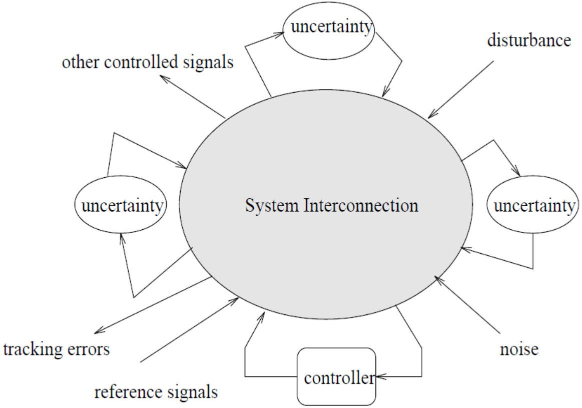
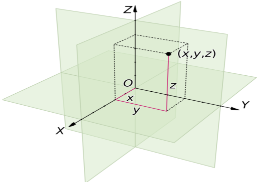


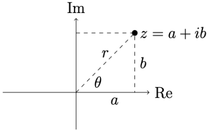

Figure 1: General system interconnection

- Recall the inner product of vectors defined on a Euclidean space $\mathbb{C}^n$:

$$\langle x, y\rangle := x^* y = \sum_{i=1}^{n} \bar{x}_i y_i \quad \forall x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \quad y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \in \mathbb{C}^n.$$




(a) Euclidean space overview.




(b) Complex vector example.

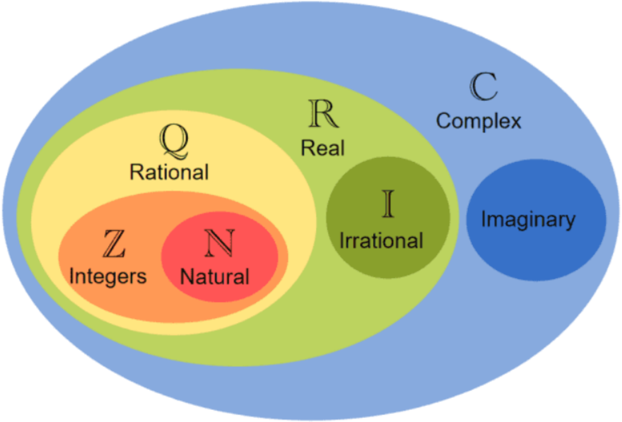



Figure 3: Complex Plane

***Example 1***: Consider the complex vectors $x$ and $y$ and compute the inner product $\langle x, y\rangle$

$$x = \begin{bmatrix} 1+i \\ 2 \end{bmatrix}, \quad y = \begin{bmatrix} 3 \\ i \end{bmatrix}$$

**Step 1:** Take the conjugate transpose of $x$

$$x^* = \begin{bmatrix} \overline{1+i} & \overline{2} \end{bmatrix} = \begin{bmatrix} 1-i & 2 \end{bmatrix}$$

**Step 2:** Compute the inner product $\langle x, y\rangle = x^* y$

$$\langle x, y\rangle = (1-i)(3) + (2)(i) = 3 - 3i + 2i = 3 - i$$

**Answer:**

$$\boxed{\langle x, y\rangle = 3 - i}$$

$\blacksquare$

- Many important metric notions and geometrical properties, such as **length**, **distance**, **angle**, and the **energy of physical systems**, can be deduced from this inner product. For instance, the length of a vector $x \in \mathbb{C}^n$ is defined as

$$\|x\| := \sqrt{\langle x, x\rangle}$$

- The angle between two vectors $x, y \in \mathbb{C}^n$

$$\cos(\angle(x, y)) = \frac{\langle x, y\rangle}{\|x\|\|y\|}, \quad \angle(x, y) \in [0, \pi].$$

- Two vectors are said to be *orthogonal* if $\angle(x, y) = \frac{\pi}{2}$.

**Definition 1.** Let $V$ be a vector space over $\mathbb{C}$. An *inner product* on $V$ is a complex-valued function,

$$\langle \cdot, \cdot \rangle : V \times V \longrightarrow \mathbb{C}$$

such that for any $x, y, z \in V$ and $\alpha, \beta \in \mathbb{C}$, the following properties hold:

1. $\langle x, \alpha y + \beta z \rangle = \alpha \langle x, y \rangle + \beta \langle x, z \rangle$
2. $\langle x, y \rangle = \overline{\langle y, x \rangle}$
3. $\langle x, x \rangle > 0$ if $x \neq 0$

A vector space $V$ with an inner product is called an *inner product space*.

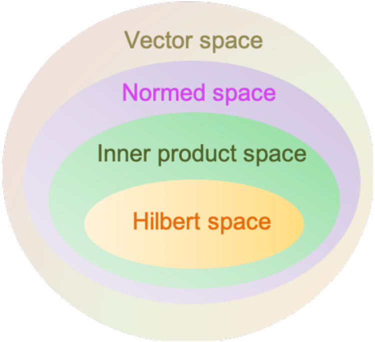



Figure 4: Space vectors diagram

- It is clear that the inner product induces a norm
$$\|x\| := \sqrt{\langle x, x\rangle},$$
- Also, the distance between vectors $x$ and $y$ is
$$d(x, y) = \|x - y\|.$$
- Two vectors $x$ and $y$ in an inner product space $V$ are said to be *orthogonal* if $\langle x, y\rangle = 0$, denoted $x \perp y$.

- More generally, a vector $x$ is said to be orthogonal to a set $S \subset V$, denoted by $x \perp S$, if $x \perp y$ for all $y \in S$.

The **inner product** and the **inner product induced norm** have the following familiar properties.

**Theorem 1.** Let $V$ be an inner product space and let $x, y \in V$. Then:

1. $|\langle x, y\rangle| \leq \|x\| \, \|y\|$ (Cauchy-Schwarz inequality).
2. $\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$ (Parallelogram law).
3. $\|x+y\|^2 = \|x\|^2 + \|y\|^2$ if $x \perp y$.

A **Hilbert space** is a *complete inner product space* with the norm induced by its inner product. For example, $\mathbb{C}^n$ with the usual inner product is a finite dimensional Hilbert space (Recall Example 1).

It is possible to verify that $\mathbb{C}^{n\times m}$ with the inner product defined as

$$\langle A, B\rangle := \text{trace}(A^*B) = \sum_{i=1}^{n}\sum_{j=1}^{m}\overline{a}_{ij}b_{ij} \quad \forall A, B \in \mathbb{C}^{n\times m}$$

is also a finite-dimensional Hilbert Space.