Poynting non-linear lemma

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$$\vec{\Pi} = \vec{E} \wedge \vec{H} \tag{1}$$

$$= \frac{\vec{E} + \overrightarrow{E^*}}{2} \wedge \frac{\vec{H} + \overrightarrow{H^*}}{2} \tag{2}$$

Indeed, $\forall z \in \mathbb{C}; \exists (a, b) \in \mathbb{R}^2 | z = a + i\, b$ and thus $\frac{z + \bar{z}}{2} = \frac{a + i\, b + a - i\, b}{2} = a = \Re(z)$

$$\Leftrightarrow 4\, \vec{\Pi} = \vec{E} \wedge \vec{H} + \vec{E} \wedge \overrightarrow{H^*} + \overrightarrow{E^*} \wedge \vec{H} + \overrightarrow{E^*} \wedge \overrightarrow{H^*} \tag{3}$$

Note: $\vec{X} = \vec{E}\, \text{or}\, \vec{H}$ by definition, $\vec{X} = \overrightarrow{X_0}\, e^{-i\omega t}$ such that,

$$4\, \vec{\Pi} = (\overrightarrow{E_0} \wedge \overrightarrow{H_0}) e^{-2\omega t} + \overrightarrow{E_0} \wedge \overrightarrow{H_0^*} + \overrightarrow{E_0^*} \wedge \overrightarrow{H_0} + (\overrightarrow{E_0^*} \wedge \overrightarrow{H_0^*}) e^{2\omega t} \tag{4}$$

And by doing an average over time:

$$\langle \vec{\Pi} \rangle = \frac{1}{4} (\overrightarrow{E_0} \wedge \overrightarrow{H_0^*} + \overrightarrow{E_0^*} \wedge \overrightarrow{H_0}) \tag{5}$$

Note that $z + \bar{z} = a + i\, b + a - i\, b = 2\, a = 2\, \Re(z)$

Thus, we proved that,

$$\langle \vec{\Pi} \rangle = \frac{\Re(\overrightarrow{E_0^*} \wedge \overrightarrow{H_0^*})}{2} \tag{6}$$

And we can define the power $\mathcal{P}$ by:

$$\mathcal{P} = \oiint_S \langle \vec{\Pi} \rangle \cdot \mathrm{d}^2 \vec{S} \tag{7}$$