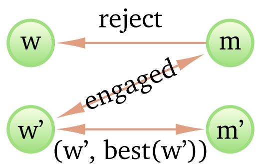
Homework 1

Jiahao Liu March 3, 2024

1. True or False: It is possible to have an instance of the Stable Matching Problem in which two women have the same best valid partner.

False. First prove the Lemma: *Every execution of Gale-Shaply algorithm (when women are proposing) results in the same stable matching regardless of the order in which women propose, denoted as* $S : \{(w, best(w)) : w \in W\}$

Proof. Suppose w is the **first** woman rejected by her first valid partner m in favor of w' in a stable matching S' where (w, m) is a valid pair and m = best(w). The rejection happens because this moment m has formed an engagement with w'. Since m is already a valid partner of w, let's say in S', w' paired with m'.



In the rejection execution, w' had not been rejected by any valid partner when she engaged to m. Since woman proposed in decreading order of preference, w' must prefer m to m'. At the same time, m prefer w' to w for rejection. Since $(w', m) \notin S'$, it follows that (w', m) is an instability of S'. Contradiction. □

To use the above Lemma, let's run Gale-Shapley algorithm on the suppose instance with woman propose, it should end up with a stable matching S where woman is matched with their best valid partner. Say w and w' has the same best valid partner m, then in S, w and w' are both matched with m, which is not a Perfect Matching. Contradiction.

2. In the context of a stable roommates problem involving four students (a, b, c, d), each student ranks the others in a strict order of preference. A matching involves forming two pairs of students, and it is considered stable if no two separated students would prefer each other ove their current roommates. Does a stable matching always exist in this scenario?

No. Consider the following scenario, where d is disliked by everyone and a,b,c are most preferred by someone.

$$RoommatePref = \begin{bmatrix} Pref_a \\ Pref_b \\ Pref_c \\ Pref_d \end{bmatrix} = \begin{bmatrix} b & c & d \\ c & a & d \\ a & b & d \\ b & a & c \end{bmatrix}$$

In matching (d, a) and (b, c), a and c are separated but they prefer each other over their current roommates.
In matching (d, b) and (a, c), b and a are separated but they prefer each other over their current roommates.
In matching (d, c) and (a, b), c and b are separated but they prefer each other over their current roommates.

3. True or False: In every instance of the Stable Matching Problem, there is a stable matching containing a pair (m, w) such that m is ranked first on the preference list of w and w is ranked first on the preference list of m.

False. Consider the following scenario,

$$ManPref = \begin{bmatrix} Pref_{m_1} \\ Pref_{m_2} \end{bmatrix} = \begin{bmatrix} w_1 & w_2 \\ w_2 & w_1 \end{bmatrix} \quad WomanPref = \begin{bmatrix} Pref_{w_1} \\ Pref_{w_2} \end{bmatrix} = \begin{bmatrix} m_2 & m_1 \\ m_1 & m_2 \end{bmatrix}$$

If man propose, the stable matching is $(m_1, w_1), (m_2, w_2)$, there are no pair with man ranked first on the preference list of woman.
If woman propose, the stable matching is $(w_1, m_2), (w_2, m_1)$, there are no pair with woman ranked first on the preference list of man.

4. True or False: Consider an instance of the Stable Matching Problem in which there exists a man m and a woman w such that m is ranked first on the preference list of w and w is ranked first on the preference list of m. Then in every stable matching S for this instance, the pair (m, w) belongs to S.

True. Suppose m rank w first, w rank m first and there is no stable matching containing (m, w). Then there must be pairs (m, w') and (w, m') in the stable matching where w' is ranked higher than w on m's preference list and m' is ranked higher that m on w's preference list. This contradicts with the assumption. Therefore, there must be m' = m, w' = w and a stable matching containing (m, w).

5. True or False: For some $n \geq 2$, there exists a set of preferences for n men and n women such that in the stable matching returned by the G-S algorithm when men are proposing, every woman is matched with their most preferred man, even though that man does not prefer that woman the most.

True. Consider the following scenario,

$$ManPref = \begin{bmatrix} Pref_{m_1} \\ Pref_{m_2} \\ Pref_{m_3} \end{bmatrix} = \begin{bmatrix} w_1 & w_2 & w_3 \\ w_1 & w_2 & w_3 \\ w_2 & w_1 & w_3 \end{bmatrix} \quad WomanPref = \begin{bmatrix} Pref_{w_1} \\ Pref_{w_2} \\ Pref_{w_3} \end{bmatrix} = \begin{bmatrix} m_3 & m_2 & m_1 \\ m_2 & m_1 & m_3 \\ m_1 & m_2 & m_3 \end{bmatrix}$$

we end up with $(m_1, w_3), (m_2, w_2), (m_3, w_1)$, all woman are matched with their most preferred man, while all man are rejected by their highest-ranked woman.

6. True or False: For all $n \geq 2$, there exists a set of preferences for n men and n women such that in the stable matching returned by the G-S algorithm when men are proposing, every woman is matched with their least preferred man.

True. $\forall n \geq 2$, say $m_i (1 \leq i \leq n)$ most prefer $w_j (j = mostPref(m_i) \in [1, n])$. As long as all the w_j are different for n man without conflicts, they will match with their most prefered woman when run G-S algorithm with man propose. At this scenario, the preference list of woman does't matter, so we can make w_j least prefer m_i.

7. Can a man or woman end up better off by lying about his or her preferences? Consider a woman w, who prefers man m to m', but both are low on her list of preferences. Can w end up with a man m" that she truly prefers to both m and m' by falsely claiming that she prefers m' to m?

True. Consider the following scenario,

$$ManPref = \begin{bmatrix} Pref_{m_1} \\ Pref_{m_2} \\ Pref_{m_3} \end{bmatrix} = \begin{bmatrix} w_1 & w_2 & w_3 \\ w_1 & w_2 & w_3 \\ w_2 & w_1 & w_3 \end{bmatrix} \quad WomanPref = \begin{bmatrix} Pref_{w_1} \\ Pref_{w_2} \\ Pref_{w_3} \end{bmatrix} = \begin{bmatrix} m_3 & m_2 & m_1 \\ m_2 & m_3 & m_1 \\ m_1 & m_2 & m_3 \end{bmatrix}$$

we end up with $(w_1, m_2), (w_2, m_3), (w_3, m_1)$, if w_2 lie about her preference and claim that she prefers m_1 to m_3,

$$ManPref = \begin{bmatrix} Pref_{m_1} \\ Pref_{m_2} \\ Pref_{m_3} \end{bmatrix} = \begin{bmatrix} w_1 & w_2 & w_3 \\ w_1 & w_2 & w_3 \\ w_2 & w_1 & w_3 \end{bmatrix} \quad WomanPref = \begin{bmatrix} Pref_{w_1} \\ Pref_{w_2} \\ Pref_{w_3} \end{bmatrix} = \begin{bmatrix} m_3 & m_2 & m_1 \\ m_2 & m_1 & m_3 \\ m_1 & m_2 & m_3 \end{bmatrix}$$

we end up with $(w_1, m_3), (w_2, m_2), (w_3, m_1)$, w_1, w_2 are all better off.

8. There are six students, Harry, Ron, Hermione, Ginny, Draco, and Cho. This class requires them to pair up and work on pair programming. Each has preferences over who they want to be paired with:
Harry: Cho > Ron > Hermione > Ginny > Draco
Ron: Ginny > Harry > Hermione > Cho > Draco
Hermione: Ron > Harry > Ginny > Cho > Draco
Ginny: Harry > Cho > Hermione > Ron > Draco
Draco : Cho > Ron > Ginny > Hermione > Harry
Cho: Hermione > Harry > Ron > Ginny > Draco
Show that there is no stable matching. That means showing that no matter who you put together, there will always be two potential partners who are not matched but prefer each other to the current partner.

Proof. Similar to question 2, Draco is the least preferred partner for the other five students, and any other student is the most preferred partner for at least one of these five students. As long as Draco is paired up with any of the other five students, say X, and Y prefers X most, there must be an instability pair (X, Y) in this matching.
To be more precise, we can divide all the possible matching into five categories, depending on whom Draco is paired with:

1. If (Harry, Draco), then we notice that Ginny ranks Harry first. In this case, no matter whom Ginny is paired up with, (Ginny, Ron) or (Ginny, Hermione) or (Ginny, Cho), there is an instability pair (Ginny, Harry), because Ginny and Harry prefer each other to their current partner.
2. If (Ron, Draco), then we notice that Hermione ranks Ron first. In this case, no matter whom Hermione is paired up with, (Hermione, Harry) or (Hermione, Ginny) or (Hermione, Cho), there is an instability pair (Hermione, Ron), because Hermione and Ron prefer each other to their current partner.
3. If (Hermione, Draco), then we notice that Cho ranks Hermione first. In this case, no matter whom Cho is paired up with, (Cho, Harry) or (Cho, Ron) or (Cho, Ginny), there is an instability pair (Cho, Hermione), because Cho and Hermione prefer each other to their current partner.
4. If (Ginny, Draco), then we notice that Ron ranks Ginny first. In this case, no matter whom Ron is paired up with, (Ron, Harry) or (Ron, Hermione) or (Ron, Cho), there is an instability pair (Ron, Ginny), because Ron and Ginny prefer each other to their current partner.
5. If (Cho, Draco), then we notice that Harry ranks Cho first. In this case, no matter whom Harry is paired up with, (Harry, Ron) or (Harry, Hermione) or (Harry, Ginny), there is an instability pair (Harry, Cho), because Harry and Cho prefer each other to their current partner.

Therefore, we have included all the 15 possible matchings for this problem and there is no stable matching. □