Intro to AI Project 3: Car

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Problem 1: Bayesian network basics

(a)

In (a), we want to calculate $P\left(C_2=1 \mid D_2=0\right)$. To delay normalization until the very end, we can calculate $P\left(C_2, D_2=0\right)$ and then normalize it. By Bayes' rule, we have the following reasoning process:

$$
\begin{aligned}
P(C_2 \mid D_2=0) &= \alpha P(C_2, D_2=0) \\
&= \alpha \sum_{c_1} \sum_{d_1} \sum_{c_3} \sum_{d_3} P(c_1, d_1, C_2, D_2=0, c_3, d_3) \\
&= \alpha \sum_{c_1} \sum_{d_1} \sum_{c_3} \sum_{d_3} P(c_1) P(d_1 \mid c_1) P(C_2 \mid c_1) P(D_2=0 \mid C_2) P(c_3 \mid C_2) P(d_3 \mid c_3) \\
&= \alpha P(D_2=0 \mid C_2) \sum_{c_1} P(c_1) P(C_2 \mid c_1) \sum_{d_1} P(d_1 \mid c_1) \sum_{c_3} P(c_3 \mid C_2) \sum_{d_3} P(d_3 \mid c_3) \\
&= \alpha P(D_2=0 \mid C_2) \sum_{c_1} P(c_1) P(C_2 \mid c_1)
\end{aligned}
$$

By the setup of the problem, we know that $\sum_{c_1} P(c_1) P(C_2 \mid c_1) = 0.5 \sum_{c_1} P(C_2 \mid c_1) = (0.5, 0.5)$. Thus, we can derive that:

$$
P\left(C_2=1, D_2=0\right)=\frac{\eta}{2} \quad , P\left(C_2=0, D_2=0\right)=\frac{1-\eta}{2}
$$

Then we conduct normalization and get the target value:

$$
P\left(C_2=1 \mid D_2=0\right)=\frac{P\left(C_2=1, D_2=0\right)}{P\left(C_2=1, D_2=0\right)+P\left(C_2=0, D_2=0\right)}=\eta
$$

(b)

In (b), we want to calculate $P\left(C_2=1 \mid D_2=0, D_3=1\right)$. We perform the similar reasoning process as in (a):

$$
\begin{aligned}
P(C_2 \mid D_2=0, D_3=1) &= \alpha P(C_2, D_2=0, D_3=1) \\
&= \alpha \sum_{c_1} \sum_{d_1} \sum_{c_3} P(c_1, d_1, C_2, D_2=0, c_3, d_3) \\
&= \alpha \sum_{c_1} \sum_{d_1} \sum_{c_3} P\left(c_1\right) P\left(d_1 \mid c_1\right) P\left(C_2 \mid c_1\right) P\left(D_2=0 \mid C_2\right) P\left(c_3 \mid C_2\right) P\left(D_3=1 \mid c_3\right) \\
&= \alpha P\left(D_2=0 \mid C_2\right) \sum_{c_1} P\left(c_1\right) P\left(C_2 \mid c_1\right) \sum_{c_3} P\left(c_3 \mid C_2\right) P\left(D_3=1 \mid c_3\right) \sum_{d_1} P\left(d_1 \mid c_1\right) \\
&= \alpha P\left(D_2=0 \mid C_2\right) \sum_{c_1} P\left(c_1\right) P\left(C_2 \mid c_1\right) \sum_{c_3} P\left(c_3 \mid C_2\right) P\left(D_3=1 \mid c_3\right)
\end{aligned}
$$

We have:

$$
\begin{aligned}
P\left(C_2=0, D_2=0, D_3=1\right) &= (1-\eta) \cdot 0.5 \cdot [(1-\epsilon) \cdot \eta+\epsilon \cdot(1-\eta)] = \frac{2 \epsilon \eta^2-\eta^2-3 \epsilon \eta+\epsilon+\eta}{2} \\
P\left(C_2=1, D_2=0, D_3=1\right) &= \eta \cdot 0.5 \cdot [\epsilon \cdot \eta+(1-\epsilon) \cdot(1-\eta)] = \frac{2 \epsilon \eta^2-\eta^2-\epsilon \eta+\eta}{2}
\end{aligned}
$$

Normalize the above two equations, we get:

$$
\begin{aligned}
P\left(C_2=1 \mid D_2=0, D_3=1\right) &= \frac{P\left(C_2=1 \mid D_2=0, D_3=1\right)}{P\left(C_2=1 \mid D_2=0, D_3=1\right)+P\left(C_2=0 \mid D_2=0, D_3=1\right)} \\
&= \frac{2 \epsilon \eta^2-\eta^2-\epsilon \eta+\eta}{4 \epsilon \eta^2-2 \eta^2-4 \epsilon \eta+\epsilon+2 \eta}
\end{aligned}
$$

(c)

By assumption, $\epsilon=0.1$ and $\eta=0.2$.

(c1).

From (a), we know:

$$
P\left(C_2=1 \mid D_2=0\right)=\eta=0.2
$$

From (b), we know:

$$
P\left(C_2=1 \mid D_2=0, D_3=1\right)=\frac{2 \epsilon \eta^2-\eta^2-\epsilon \eta+\eta}{4 \epsilon \eta^2-2 \eta^2-4 \epsilon \eta+\epsilon+2 \eta} \approx 0.4157
$$

(c2).

With the addition of the new information that $D_3=1$, it can be observed that $P(C_2=1|D_2=0, D_3=1)$ surpasses $P(C_2=1|D_2=0)$. This suggests that knowing $D_3=1$ significantly increases the likelihood of $C_2=1$. With $\epsilon=0.1$ indicating a high chance of the car staying still, which is equivalent to $P(C_2=C_3)$ being high, and $\eta=0.2$ suggesting an acceptable likelihood of accurate sensor reports, i.e. $P(D_3=C_3)$. Therefore, it's more likely that $C_2=D_3$. Hence, if $\mathrm{D}_3=1$ is reported, it's rational to believe $C_2=1$, reflected in the probability increase from $P(C_2=1|D_2=0)$ to $P(C_2=1|D_2=0, D_3=1)$.

(c3).

Let $P\left(C_2=1 \mid D_2=0\right)=P\left(C_2=1 \mid D_2=0, D_3=1\right)$, which is equivalent to:

$$
\begin{aligned}
\eta= & \frac{2 \epsilon \eta^2-\eta^2-\epsilon \eta+\eta}{4 \epsilon \eta^2-2 \eta^2-4 \epsilon \eta+\epsilon+2 \eta} \\
\Longleftrightarrow & 4 \epsilon \eta^2-2 \eta^2-4 \epsilon \eta+\epsilon+2 \eta=2 \epsilon \eta-\eta-\epsilon+1 \\
\Longleftrightarrow & 4 \epsilon \eta^2-2 \eta^2-6 \epsilon \eta+2 \epsilon+3 \eta-1=0 \\
\Longleftrightarrow & \epsilon=0.5
\end{aligned}
$$

From above, we should set ϵ be 0.5 while keeping $\eta=0.2$.

Assigning a value of 0.5 to ϵ implies that the car has an equal likelihood of either moving or staying stationary, essentially indicating no correlation between C_2 and C_3. Consequently, observing $\mathrm{D}_3=1$ provides insight only into the state of C_3, being 1, but offers no valuable information regarding C_2. Therefore, in terms of intuitive understanding, $P(C_2=1|D_2=0)$ remains the same as $P(C_2=1|D_2=0, D_3=1)$, as the observation of $\mathrm{D}_3=1$ does not influence the assessment of C_2.