Nevertheless, the dual simplex method also has a geometric interpretation. It keeps moving from one dual basic feasible solution to an adjacent one and, in this respect, it is similar to the primal simplex method applied to the dual problem.

All of duality theory can be developed by exploiting the termination conditions of the simplex method, and this was our initial approach to the subject. We also pursued an alternative line of development that proceeded from first principles and used geometric arguments. This is a more direct and more general approach, but requires more abstract reasoning.

Duality theory provided us with some powerful tools based on which we were able to enhance our geometric understanding of polyhedra. We derived a few theorems of the alternative (like Farkas' lemma), which are surprisingly powerful and have applications in a wide variety of contexts. In fact, Farkas' lemma can be viewed as the core of linear programming duality theory. Another major result that we derived is the resolution theorem, which allows us to express any element of a nonempty polyhedron with at least one extreme point as a convex combination of its extreme points plus a nonnegative linear combination of its extreme rays; in other words, every polyhedron is "finitely generated." The converse is also true, and every finitely generated set is a polyhedron (can be represented in terms of linear inequality constraints). Results of this type play a key role in confirming our intuitive geometric understanding of polyhedra and linear programming. They allow us to develop alternative views of certain situations and lead to deeper understanding. Many such results have an "obvious" geometric content and are often taken for granted. Nevertheless, as we have seen, rigorous proofs can be quite elaborate.

4.12 Exercises

Exercise 4.1 Consider the linear programming problem:

$$\begin{array}{rl}
\text{minimize} & x_1 - x_2 \\
\text{subject to} & 2x_1 + 3x_2 - x_3 + x_4 \le 0 \\
& 3x_1 + x_2 + 4x_3 - 2x_4 \ge 3 \\
& -x_1 - x_2 + 2x_3 + x_4 = 6 \\
& x_1 \le 0 \\
& x_2, x_3 \ge 0.
\end{array}$$

Write down the corresponding dual problem.

Exercise 4.2 Consider the primal problem

$$\begin{array}{rl}
\text{minimize} & \mathbf{c}'\mathbf{x} \\
\text{subject to} & \mathbf{A}\mathbf{x} \ge \mathbf{b} \\
& \mathbf{x} \ge \mathbf{0}.
\end{array}$$

Form the dual problem and convert it into an equivalent minimization problem. Derive a set of conditions on the matrix $\mathbf{A}$ and the vectors $\mathbf{b}$, $\mathbf{c}$, under which the dual is identical to the primal, and construct an example in which these conditions are satisfied.

Exercise 4.3 The purpose of this exercise is to show that solving linear programming problems is no harder than solving systems of linear inequalities.

Suppose that we are given a subroutine which, given a system of linear inequality constraints, either produces a solution or decides that no solution exists. Construct a simple algorithm that uses a single call to this subroutine and which finds an optimal solution to any linear programming problem that has an optimal solution.

Exercise 4.4 Let $\mathbf{A}$ be a symmetric square matrix. Consider the linear programming problem

$$\begin{array}{ll} \text{minimize} & \mathbf{c}'\mathbf{x} \\ \text{subject to} & \mathbf{A}\mathbf{x} \geq \mathbf{c} \\ & \mathbf{x} \geq \mathbf{0}. \end{array}$$

Prove that if $\mathbf{x}^*$ satisfies $\mathbf{A}\mathbf{x}^* = \mathbf{c}$ and $\mathbf{x}^* \geq \mathbf{0}$, then $\mathbf{x}^*$ is an optimal solution.

Exercise 4.5 Consider a linear programming problem in standard form and assume that the rows of $\mathbf{A}$ are linearly independent. For each one of the following statements, provide either a proof or a counterexample.

(a) Let $\mathbf{x}^*$ be a basic feasible solution. Suppose that for every basis corresponding to $\mathbf{x}^*$, the associated basic solution to the dual is infeasible. Then, the optimal cost must be strictly less that $\mathbf{c}'\mathbf{x}^*$.

(b) The dual of the auxiliary primal problem considered in Phase I of the simplex method is always feasible.

(c) Let p_i be the dual variable associated with the ith equality constraint in the primal. Eliminating the ith primal equality constraint is equivalent to introducing the additional constraint $p_i = 0$ in the dual problem.

(d) If the unboundedness criterion in the primal simplex algorithm is satisfied, then the dual problem is infeasible.

Exercise 4.6* (Duality in Chebychev approximation) Let $\mathbf{A}$ be an $m \times n$ matrix and let $\mathbf{b}$ be a vector in $\Re^m$. We consider the problem of minimizing $\|\mathbf{A}\mathbf{x} - \mathbf{b}\|_\infty$ over all $\mathbf{x} \in \Re^n$. Here $\|\cdot\|_\infty$ is the vector norm defined by $\|\mathbf{y}\|_\infty = \max_i |y_i|$. Let v be the value of the optimal cost.

(a) Let $\mathbf{p}$ be any vector in $\Re^m$ that satisfies $\sum_{i=1}^m |p_i| \leq 1$ and $\mathbf{p}'\mathbf{A} = \mathbf{0}'$. Show that $\mathbf{p}'\mathbf{b} \leq v$.

(b) In order to obtain the best possible lower bound of the form considered in part (a), we form the linear programming problem

$$\begin{array}{ll} \text{maximize} & \mathbf{p}'\mathbf{b} \\ \text{subject to} & \mathbf{p}'\mathbf{A} = \mathbf{0}' \\ & \displaystyle\sum_{i=1}^m |p_i| \leq 1. \end{array}$$

Show that the optimal cost in this problem is equal to v.

Exercise 4.7 (Duality in piecewise linear convex optimization) Consider the problem of minimizing $\max_{i=1,\ldots,m}(\mathbf{a}_i'\mathbf{x} - b_i)$ over all $\mathbf{x} \in \Re^n$. Let v be the value of the optimal cost, assumed finite. Let $\mathbf{A}$ be the matrix with rows $\mathbf{a}_1, \ldots, \mathbf{a}_m$, and let $\mathbf{b}$ be the vector with components $b_1, \ldots, b_m$.

(a) Consider any vector $\mathbf{p} \in \Re^m$ that satisfies $\mathbf{p}'\mathbf{A} = \mathbf{0}'$, $\mathbf{p} \geq \mathbf{0}$, and $\sum_{i=1}^m p_i = 1$. Show that $-\mathbf{p}'\mathbf{b} \leq v$.

(b) In order to obtain the best possible lower bound of the form considered in part (a), we form the linear programming problem

$$\begin{array}{ll} \text{maximize} & -\mathbf{p}'\mathbf{b} \\ \text{subject to} & \mathbf{p}'\mathbf{A} = \mathbf{0}' \\ & \mathbf{p}'\mathbf{e} = 1 \\ & \mathbf{p} \geq \mathbf{0}, \end{array}$$

where $\mathbf{e}$ is the vector with all components equal to 1. Show that the optimal cost in this problem is equal to v.

Exercise 4.8 Consider the linear programming problem of minimizing $\mathbf{c}'\mathbf{x}$ subject to $\mathbf{A}\mathbf{x} = \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$. Let $\mathbf{x}^*$ be an optimal solution, assumed to exist, and let $\mathbf{p}^*$ be an optimal solution to the dual.

(a) Let $\tilde{\mathbf{x}}$ be an optimal solution to the primal, when $\mathbf{c}$ is replaced by some $\tilde{\mathbf{c}}$. Show that $(\tilde{\mathbf{c}} - \mathbf{c})'(\tilde{\mathbf{x}} - \mathbf{x}^*) \leq 0$.

(b) Let the cost vector be fixed at $\mathbf{c}$, but suppose that we now change $\mathbf{b}$ to $\tilde{\mathbf{b}}$, and let $\tilde{\mathbf{x}}$ be a corresponding optimal solution to the primal. Prove that $(\mathbf{p}^*)'(\tilde{\mathbf{b}} - \mathbf{b}) \leq \mathbf{c}'(\tilde{\mathbf{x}} - \mathbf{x}^*)$.

Exercise 4.9 (Back-propagation of dual variables in a multiperiod problem) A company makes a product that can be either sold or stored to meet future demand. Let $t = 1, \ldots, T$ denote the periods of the planning horizon. Let b_t be the production volume during period t, which is assumed to be known in advance. During each period t, a quantity x_t of the product is sold, at a unit price of d_t. Furthermore, a quantity y_t can be sent to long-term storage, at a unit transportation cost of c. Alternatively, a quantity w_t can be retrieved from storage, at zero cost. We assume that when the product is prepared for long-term storage, it is partly damaged, and only a fraction f of the total survives. Demand is assumed to be unlimited. The main question is whether it is profitable to store some of the production, in anticipation of higher prices in the future. This leads us to the following problem, where z_t stands for the amount kept in long-term storage, at the end of period t:

$$\begin{array}{lll} \text{maximize} & \displaystyle\sum_{t=1}^{T} \alpha^{t-1}(d_t x_t - c y_t) + \alpha^T d_{T+1} z_T & \\ \text{subject to} & x_t + y_t - w_t = b_t, & t = 1, \ldots, T, \\ & z_t + w_t - z_{t-1} - f y_t = 0, & t = 1, \ldots, T, \\ & z_0 = 0, & \\ & x_t, y_t, w_t, z_t \geq 0. & \end{array}$$

Here, d_{T+1} is the salvage prive for whatever inventory is left at the end of period T. Furthermore, α is a discount factor, with $0 < \alpha < 1$, reflecting the fact that future revenues are valued less than current ones.

(a) Let p_t and q_t be dual variables associated with the first and second equality constraint, respectively. Write down the dual problem.

(b) Assume that $0 < f < 1$, $b_t \geq 0$, and $c \geq 0$. Show that the following formulae provide an optimal solution to the dual problem:

$$\begin{aligned} q_T &= \alpha^T d_{T+1}, \\ p_T &= \max\left\{\alpha^{T-1} d_T,\ f q_T - \alpha^{T-1} c\right\}, \\ q_t &= \max\left\{q_{t+1}, \alpha^{t-1} d_t\right\}, \qquad & t = 1, \ldots, T-1, \\ p_t &= \max\left\{\alpha^{t-1} d_t,\ f q_t - \alpha^{t-1} c\right\}, \qquad & t = 1, \ldots, T-1. \end{aligned}$$

(c) Explain how the result in part (b) can be used to compute an optimal solution to the original problem. Primal and dual nondegeneracy can be assumed.

Exercise 4.10 (Saddle points of the Lagrangean) Consider the standard form problem of minimizing $\mathbf{c}'\mathbf{x}$ subject to $\mathbf{A}\mathbf{x} = \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$. We define the *Lagrangean* by

$$L(\mathbf{x}, \mathbf{p}) = \mathbf{c}'\mathbf{x} + \mathbf{p}'(\mathbf{b} - \mathbf{A}\mathbf{x}).$$

Consider the following "game": player 1 chooses some $\mathbf{x} \geq \mathbf{0}$, and player 2 chooses some $\mathbf{p}$; then, player 1 pays to player 2 the amount $L(\mathbf{x}, \mathbf{p})$. Player 1 would like to minimize $L(\mathbf{x}, \mathbf{p})$, while player 2 would like to maximize it.

A pair $(\mathbf{x}^*, \mathbf{p}^*)$, with $\mathbf{x}^* \geq \mathbf{0}$, is called an *equilibrium* point (or a *saddle point*, or a *Nash equilibrium*) if

$$L(\mathbf{x}^*, \mathbf{p}) \leq L(\mathbf{x}^*, \mathbf{p}^*) \leq L(\mathbf{x}, \mathbf{p}^*), \qquad \forall\, \mathbf{x} \geq \mathbf{0},\ \forall\, \mathbf{p}.$$

(Thus, we have an equilibrium if no player is able to improve her performance by unilaterally modifying her choice.)

Show that a pair $(\mathbf{x}^*, \mathbf{p}^*)$ is an equilibrium if and only if $\mathbf{x}^*$ and $\mathbf{p}^*$ are optimal solutions to the standard form problem under consideration and its dual, respectively.

Exercise 4.11 Consider a linear programming problem in standard form which is infeasible, but which becomes feasible and has finite optimal cost when the last equality constraint is omitted. Show that the dual of the original (infeasible) problem is feasible and the optimal cost is infinite.

Exercise 4.12* (Degeneracy and uniqueness – I) Consider a general linear programming problem and suppose that we have a nondegenerate basic feasible solution to the primal. Show that the complementary slackness conditions lead to a system of equations for the dual vector that has a unique solution.

Exercise 4.13* (Degeneracy and uniqueness – II) Consider the following pair of problems that are duals of each other:

$$\begin{array}{llll} \text{minimize} & \mathbf{c}'\mathbf{x} & \qquad \text{maximize} & \mathbf{p}'\mathbf{b} \\ \text{subject to} & \mathbf{A}\mathbf{x} = \mathbf{b} & \qquad \text{subject to} & \mathbf{p}'\mathbf{A} \leq \mathbf{c}'. \\ & \mathbf{x} \geq \mathbf{0}, & & \end{array}$$

(a) Prove that if one problem has a nondegenerate and unique optimal solution, so does the other.

(b) Suppose that we have a nondegenerate optimal basis for the primal and that the reduced cost for one of the nonbasic variables is zero. What does the result of part (a) imply? Is it true that there must exist another optimal basis?

Exercise 4.14 (Degeneracy and uniqueness – III) Give an example in which the primal problem has a degenerate optimal basic feasible solution, but the dual has a unique optimal solution. (The example need not be in standard form.)

Exercise 4.15 (Degeneracy and uniqueness – IV) Consider the problem

$$\begin{array}{rl} \text{minimize} & x_2 \\ \text{subject to} & x_2 = 1 \\ & x_1 \geq 0 \\ & x_2 \geq 0. \end{array}$$

Write down its dual. For both the primal and the dual problem determine whether they have unique optimal solutions and whether they have nondegenerate optimal solutions. Is this example in agreement with the statement that nondegeneracy of an optimal basic feasible solution in one problem implies uniqueness of optimal solutions for the other? Explain.

Exercise 4.16 Give an example of a pair (primal and dual) of linear programming problems, both of which have multiple optimal solutions.

Exercise 4.17 This exercise is meant to demonstrate that knowledge of a primal optimal solution does not necessarily contain information that can be exploited to determine a dual optimal solution. In particular, determining an optimal solution to the dual is as hard as solving a system of linear inequalities, even if an optimal solution to the primal is available.

Consider the problem of minimizing $\mathbf{c}'\mathbf{x}$ subject to $\mathbf{A}\mathbf{x} \geq \mathbf{0}$, and suppose that we are told that the zero vector is optimal. Let the dimensions of $\mathbf{A}$ be $m \times n$, and suppose that we have an algorithm that determines a dual optimal solution and whose running time $O\big((m+n)^k\big)$, for some constant k. (Note that if $\mathbf{x} = \mathbf{0}$ is not an optimal primal solution, the dual has no feasible solution, and we assume that in this case our algorithm exits with an error message.) Assuming the availability of the above algorithm, construct a new algorithm that takes as input a system of m linear inequalities in n variables, runs for $O\big((m+n)^k\big)$ time, and either finds a feasible solution or determines that no feasible solution exists.

Exercise 4.18 Consider a problem in standard form. Suppose that the matrix $\mathbf{A}$ has dimensions $m \times n$ and its rows are linearly independent. Suppose that all basic solutions to the primal and to the dual are nondegenerate. Let $\mathbf{x}$ be a feasible solution to the primal and let $\mathbf{p}$ be a dual vector (not necessarily feasible), such that the pair $(\mathbf{x}, \mathbf{p})$ satisfies complementary slackness.

(a) Show that there exist m columns of $\mathbf{A}$ that are linearly independent and such that the corresponding components of $\mathbf{x}$ are all positive.

(b) Show that $\mathbf{x}$ and $\mathbf{p}$ are basic solutions to the primal and the dual, respectively.

(c) Show that the result of part (a) is false if the nondegeneracy assumption is removed.

Exercise 4.19 Let $P = \{\mathbf{x} \in \Re^n \mid \mathbf{Ax} = \mathbf{b},\ \mathbf{x} \geq \mathbf{0}\}$ be a nonempty polyhedron, and let m be the dimension of the vector $\mathbf{b}$. We call x_j a *null variable* if $x_j = 0$ whenever $\mathbf{x} \in P$.

(a) Suppose that there exists some $\mathbf{p} \in \Re^m$ for which $\mathbf{p}'\mathbf{A} \geq \mathbf{0}'$, $\mathbf{p}'\mathbf{b} = 0$, and such that the jth component of $\mathbf{p}'\mathbf{A}$ is positive. Prove that x_j is a null variable.

(b) Prove the converse of (a): if x_j is a null variable, then there exists some $\mathbf{p} \in \Re^m$ with the properties stated in part (a).

(c) If x_j is not a null variable, then by definition, there exists some $\mathbf{y} \in P$ for which $y_j > 0$. Use the results in parts (a) and (b) to prove that there exist $\mathbf{x} \in P$ and $\mathbf{p} \in \Re^m$ such that:

$$\mathbf{p}'\mathbf{A} \geq \mathbf{0}', \qquad \mathbf{p}'\mathbf{b} = 0, \qquad \mathbf{x} + \mathbf{A}'\mathbf{p} > \mathbf{0}.$$

Exercise 4.20 * (Strict complementary slackness)

(a) Consider the following linear programming problem and its dual

$$\begin{array}{llll} \text{minimize} & \mathbf{c}'\mathbf{x} & \qquad \text{maximize} & \mathbf{p}'\mathbf{b} \\ \text{subject to} & \mathbf{Ax} = \mathbf{b} & \qquad \text{subject to} & \mathbf{p}'\mathbf{A} \leq \mathbf{c}', \\ & \mathbf{x} \geq \mathbf{0}, & & \end{array}$$

and assume that both problems have an optimal solution. Fix some j. Suppose that every optimal solution to the primal satisfies $x_j = 0$. Show that there exists an optimal solution $\mathbf{p}$ to the dual such that $\mathbf{p}'\mathbf{A}_j < c_j$. (Here, $\mathbf{A}_j$ is the jth column of $\mathbf{A}$.) *Hint:* Let d be the optimal cost. Consider the problem of minimizing $-x_j$ subject to $\mathbf{Ax} = \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$, and $-\mathbf{c}'\mathbf{x} \geq -d$, and form its dual.

(b) Show that there exist optimal solutions $\mathbf{x}$ and $\mathbf{p}$ to the primal and to the dual, respectively, such that for every j we have either $x_j > 0$ or $\mathbf{p}'\mathbf{A}_j < c_j$. *Hint:* Use part (a) for each j, and then take the average of the vectors obtained.

(c) Consider now the following linear programming problem and its dual:

$$\begin{array}{llll} \text{minimize} & \mathbf{c}'\mathbf{x} & \qquad \text{maximize} & \mathbf{p}'\mathbf{b} \\ \text{subject to} & \mathbf{Ax} \geq \mathbf{b} & \qquad \text{subject to} & \mathbf{p}'\mathbf{A} \leq \mathbf{c}' \\ & \mathbf{x} \geq \mathbf{0}, & & \mathbf{p} \geq \mathbf{0}. \end{array}$$

Assume that both problems have an optimal solution. Show that there exist optimal solutions to the primal and to the dual, respectively, that satisfy *strict complementary slackness*, that is:

(i) For every j we have either $x_j > 0$ or $\mathbf{p}'\mathbf{A}_j < c_j$.

(ii) For every i, we have either $\mathbf{a}_i'\mathbf{x} > b_i$ or $p_i > 0$. (Here, $\mathbf{a}_i'$ is the ith row of $\mathbf{A}$.) *Hint:* Convert the primal to standard form and apply part (b).

(d) Consider the linear programming problem

$$\begin{array}{rrcrl} \text{minimize} & 5x_1 & + & 5x_2 & \\ \text{subject to} & x_1 & + & x_2 & \geq 2 \\ & 2x_1 & - & x_2 & \geq 0 \\ & & x_1, x_2 \geq 0. & & \end{array}$$

Does the optimal primal solution $(2/3, 4/3)$, together with the corresponding dual optimal solution, satisfy strict complementary slackness? Determine all primal and dual optimal solutions and identify the set of *all* strictly complementary pairs.

Exercise 4.21* (Clark's theorem) Consider the following pair of linear programming problems:

$$\begin{array}{rl} \text{minimize} & \mathbf{c}'\mathbf{x} \\ \text{subject to} & \mathbf{A}\mathbf{x} \geq \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0}, \end{array} \qquad \begin{array}{rl} \text{maximize} & \mathbf{p}'\mathbf{b} \\ \text{subject to} & \mathbf{p}'\mathbf{A} \leq \mathbf{c}' \\ & \mathbf{p} \geq \mathbf{0}. \end{array}$$

Suppose that at least one of these two problems has a feasible solution. Prove that the set of feasible solutions to at least one of the two problems is unbounded. *Hint:* Interpret boundedness of a set in terms of the finiteness of the optimal cost of some linear programming problem.

Exercise 4.22 Consider the dual simplex method applied to a standard form problem with linearly independent rows. Suppose that we have a basis which is primal infeasible, but dual feasible, and let i be such that $x_{B(i)} < 0$. Suppose that all entries in the ith row in the tableau (other than $x_{B(i)}$) are nonnegative. Show that the optimal dual cost is $+\infty$.

Exercise 4.23 Describe in detail the mechanics of a revised dual simplex method that works in terms of the inverse basis matrix $\mathbf{B}^{-1}$ instead of the full simplex tableau.

Exercise 4.24 Consider the lexicographic pivoting rule for the dual simplex method and suppose that the algorithm is initialized with each column of the tableau being lexicographically positive. Prove that the dual simplex method does not cycle.

Exercise 4.25 This exercise shows that if we bring the dual problem into standard form and then apply the primal simplex method, the resulting algorithm is not identical to the dual simplex method.

Consider the following standard form problem and its dual.

$$\begin{array}{rl} \text{minimize} & x_1 + x_2 \\ \text{subject to} & x_1 = 1 \\ & x_2 = 1 \\ & x_1, x_2 \geq 0 \end{array} \qquad \begin{array}{rl} \text{maximize} & p_1 + p_2 \\ \text{subject to} & p_1 \leq 1 \\ & p_2 \leq 1. \end{array}$$

Here, there is only one possible basis and the dual simplex method must terminate immediately. Show that if the dual problem is converted into standard form and the primal simplex method is applied to it, one or more changes of basis may be required.

Exercise 4.26 Let $\mathbf{A}$ be a given matrix. Show that exactly one of the following alternatives must hold.

(a) There exists some $\mathbf{x} \neq \mathbf{0}$ such that $\mathbf{Ax} = \mathbf{0}$, $\mathbf{x} \geq \mathbf{0}$.

(b) There exists some $\mathbf{p}$ such that $\mathbf{p}'\mathbf{A} > \mathbf{0}'$.

Exercise 4.27 Let $\mathbf{A}$ be a given matrix. Show that the following two statements are equivalent.

(a) Every vector such that $\mathbf{Ax} \geq \mathbf{0}$ and $\mathbf{x} \geq \mathbf{0}$ must satisfy $x_1 = 0$.

(b) There exists some $\mathbf{p}$ such that $\mathbf{p}'\mathbf{A} \leq \mathbf{0}$, $\mathbf{p} \geq \mathbf{0}$, and $\mathbf{p}'\mathbf{A}_1 < 0$, where $\mathbf{A}_1$ is the first column of $\mathbf{A}$.

Exercise 4.28 Let $\mathbf{a}$ and $\mathbf{a}_1, \ldots, \mathbf{a}_m$ be given vectors in $\Re^n$. Prove that the following two statements are equivalent:

(a) For all $\mathbf{x} \geq \mathbf{0}$, we have $\mathbf{a}'\mathbf{x} \leq \max_i \mathbf{a}_i'\mathbf{x}$.

(b) There exist nonnegative coefficients λ_i that sum to 1 and such that $\mathbf{a} \leq \sum_{i=1}^{m} \lambda_i \mathbf{a}_i$.

Exercise 4.29 (Inconsistent systems of linear inequalities) Let $\mathbf{a}_1, \ldots, \mathbf{a}_m$ be some vectors in $\Re^n$, with $m > n + 1$. Suppose that the system of inequalities $\mathbf{a}_i'\mathbf{x} \geq b_i$, $i = 1, \ldots, m$, does not have any solutions. Show that we can choose $n + 1$ of these inequalities, so that the resulting system of inequalities has no solutions.

Exercise 4.30 (Helly's theorem)

(a) Let $\mathcal{F}$ be a finite family of polyhedra in $\Re^n$ such that every $n+1$ polyhedra in $\mathcal{F}$ have a point in common. Prove that all polyhedra in $\mathcal{F}$ have a point in common. *Hint:* Use the result in Exercise 4.29.

(b) For $n = 2$, part (a) asserts that the polyhedra $P_1, P_2, \ldots, P_K$ $(K \geq 3)$ in the plane have a point in common if and only if every three of them have a point in common. Is the result still true with "three" replaced by "two"?

Exercise 4.31 (Unit eigenvectors of stochastic matrices) We say that an $n \times n$ matrix $\mathbf{P}$, with entries p_{ij}, is *stochastic* if all of its entries are nonnegative and

$$\sum_{j=1}^{n} p_{ij} = 1, \qquad \forall\, i,$$

that is, the sum of the entries of each row is equal to 1.

Use duality to show that if $\mathbf{P}$ is a stochastic matrix, then the system of equations

$$\mathbf{p}'\mathbf{P} = \mathbf{p}', \qquad \mathbf{p} \geq \mathbf{0},$$

has a nonzero solution. (Note that the vector $\mathbf{p}$ can be normalized so that its components sum to one. Then, the result in this exercise establishes that every finite state Markov chain has an invariant probability distribution.)

Exercise 4.32 * (Leontief systems and Samuelson's substitution theorem) A *Leontief matrix* is an $m \times n$ matrix $\mathbf{A}$ in which every column has at most one positive element. For an interpretation, each column $\mathbf{A}_j$ corresponds to a production process. If a_{ij} is negative, $|a_{ij}|$ represents the amount of goods of type i consumed by the process. If a_{ij} is positive, it represents the amount of goods of type i produced by the process. If x_j is the intensity with which process j is used, then $\mathbf{Ax}$ represents the net output of the different goods. The matrix $\mathbf{A}$ is called *productive* if there exists some $\mathbf{x} \geq \mathbf{0}$ such that $\mathbf{Ax} > \mathbf{0}$.

(a) Let $\mathbf{A}$ be a square productive Leontief matrix ($m = n$). Show that every vector $\mathbf{z}$ that satisfies $\mathbf{Az} \geq \mathbf{0}$ must be nonnegative. *Hint:* If $\mathbf{z}$ satisfies $\mathbf{Az} \geq \mathbf{0}$ but has a negative component, consider the smallest nonnegative θ such that some component of $\mathbf{x} + \theta\mathbf{z}$ becomes zero, and derive a contradiction.

(b) Show that every square productive Leontief matrix is invertible and that all entries of the inverse matrix are nonnegative. *Hint:* Use the result in part (a).

(c) We now consider the general case where $n \geq m$, and we introduce a constraint of the form $\mathbf{e}'\mathbf{x} \leq 1$, where $\mathbf{e} = (1, \ldots, 1)$. (Such a constraint could capture, for example, a bottleneck due to the finiteness of the labor force.) An "output" vector $\mathbf{y} \in \Re^m$ is said to be *achievable* if $\mathbf{y} \geq \mathbf{0}$ and there exists some $\mathbf{x} \geq \mathbf{0}$ such that $\mathbf{Ax} = \mathbf{y}$ and $\mathbf{e}'\mathbf{y} \leq 1$. An achievable vector $\mathbf{y}$ is said to be *efficient* if there exists no achievable vector $\mathbf{z}$ such that $\mathbf{z} \geq \mathbf{y}$ and $\mathbf{z} \neq \mathbf{y}$. (Intuitively, an output vector $\mathbf{y}$ which is not efficient can be improved upon and is therefore uninteresting.) Suppose that $\mathbf{A}$ is productive. Show that there exists a positive efficient vector $\mathbf{y}$. *Hint:* Given a positive achievable vector $\mathbf{y}^*$, consider maximizing $\sum_i y_i$ over all achievable vectors $\mathbf{y}$ that are larger than $\mathbf{y}^*$.

(d) Suppose that $\mathbf{A}$ is productive. Show that there exists a set of m production processes that are capable of generating all possible efficient output vectors $\mathbf{y}$. That is, there exist indices $B(1), \ldots, B(m)$, such that every efficient output vector $\mathbf{y}$ can be expressed in the form $\mathbf{y} = \sum_{i=1}^{m} \mathbf{A}_{B(i)} x_{B(i)}$, for some nonnegative coefficients $x_{B(i)}$ whose sum is bounded by 1. *Hint:* Consider the problem of minimizing $\mathbf{e}'\mathbf{x}$ subject to $\mathbf{Ax} = \mathbf{y}$, $\mathbf{x} \geq \mathbf{0}$, and show that we can use the same optimal basis for all efficient vectors $\mathbf{y}$.

Exercise 4.33 (Options pricing) Consider a market that operates for a single period, and which involves three assets: a stock, a bond, and an option. Let S be the price of the stock, in the beginning of the period. Its price $\overline{S}$ at the end of the period is random and is assumed to be equal to either Su, with probability β, or Sd, with probability $1 - \beta$. Here u and d are scalars that satisfy $d < 1 < u$. Bonds are assumed riskless. Investing one dollar in a bond results in a payoff of r, at the end of the period. (Here, r is a scalar greater than 1.) Finally, the option gives us the right to purchase, at the end of the period, one stock at a fixed price of K. If the realized price $\overline{S}$ of the stock is greater than K, we exercise the option and then immediately sell the stock in the stock market, for a payoff of $\overline{S} - K$. If on the other hand we have $\overline{S} < K$, there is no advantage in exercising the option, and we receive zero payoff. Thus, the value of the option at the end of the period is equal to $\max\{0, \overline{S} - K\}$. Since the option is itself an asset, it should have a value in the beginning of the time period. Show that under the absence of arbitrage condition, the value of the option must be equal to

$$\gamma \max\{0, Su - K\} + \delta \max\{0, Sd - K\},$$

where γ and δ are a solution to the following system of linear equations:

$$\begin{aligned} u\gamma + d\delta &= 1 \\ \gamma + \delta &= \frac{1}{r}. \end{aligned}$$

Hint: Write down the payoff matrix $\mathbf{R}$ and use Theorem 4.8.

Exercise 4.34 (Finding separating hyperplanes) Consider a polyhedron P that has at least one extreme point.

(a) Suppose that we are given the extreme points $\mathbf{x}^i$ and a complete set of extreme rays $\mathbf{w}^j$ of P. Create a linear programming problem whose solution provides us with a separating hyperplane that separates P from the origin, or allows us to conclude that none exists.

(b) Suppose now that P is given to us in the form $P = \{\mathbf{x} \mid \mathbf{a}_i'\mathbf{x} \geq b_i,\ i = 1, \ldots, m\}$. Suppose that $\mathbf{0} \notin P$. Explain how a separating hyperplane can be found.

Exercise 4.35 (Separation of disjoint polyhedra) Consider two nonempty polyhedra $P = \{\mathbf{x} \in \Re^n \mid \mathbf{A}\mathbf{x} \leq \mathbf{b}\}$ and $Q = \{\mathbf{x} \in \Re^n \mid \mathbf{D}\mathbf{x} \leq \mathbf{d}\}$. We are interested in finding out whether the two polyhedra have a point in common.

(a) Devise a linear programming problem such that: if $P \cap Q$ is nonempty, it returns a point in $P \cap Q$; if $P \cap Q$ is empty, the linear programming problem is infeasible.

(b) Suppose that $P \cap Q$ is empty. Use the dual of the problem you have constructed in part (a) to show that there exists a vector $\mathbf{c}$ such that $\mathbf{c}'\mathbf{x} < \mathbf{c}'\mathbf{y}$ for all $\mathbf{x} \in P$ and $\mathbf{y} \in Q$.

Exercise 4.36 (Containment of polyhedra)

(a) Let P and Q be two polyhedra in $\Re^n$ described in terms of linear inequality constraints. Devise an algorithm that decides whether P is a subset of Q.

(b) Repeat part (a) if the polyhedra are described in terms of their extreme points and extreme rays.

Exercise 4.37 (Closedness of finitely generated cones) Let $\mathbf{A}_1, \ldots, \mathbf{A}_n$ be given vectors in $\Re^m$. Consider the cone $C = \left\{\sum_{i=1}^n \mathbf{A}_i x_i \mid x_i \geq 0\right\}$ and let $\mathbf{y}^k$, $k = 1, 2, \ldots$, be a sequence of elements of C that converges to some $\mathbf{y}$. Show that $\mathbf{y} \in C$ (and hence C is closed), using the following argument. With $\mathbf{y}$ fixed as above, consider the problem of minimizing $\|\mathbf{y} - \sum_{i=1}^n \mathbf{A}_i x_i\|_\infty$, subject to the constraints $x_1, \ldots, x_n \geq 0$. Here $\|\cdot\|_\infty$ stands for the maximum norm, defined by $\|\mathbf{x}\|_\infty = \max_i |x_i|$. Explain why the above minimization problem has an optimal solution, find the value of the optimal cost, and prove that $\mathbf{y} \in C$.

Exercise 4.38 (From Farkas' lemma to duality) Use Farkas' lemma to prove the duality theorem for a linear programming problem involving constraints of the form $\mathbf{a}_i'\mathbf{x} = b_i$, $\mathbf{a}_i'\mathbf{x} \geq b_i$, and nonnegativity constraints for some of the variables x_j. *Hint:* Start by deriving the form of the set of feasible directions at an optimal solution.

Exercise 4.39 (Extreme rays of cones) Let us define a nonzero element $\mathbf{d}$ of a pointed polyhedral cone C to be an *extreme ray* if it has the following property: if there exist vectors $\mathbf{f} \in C$ and $\mathbf{g} \in C$ satisfying $\mathbf{d} = \mathbf{f} + \mathbf{g}$, then both $\mathbf{f}$ and $\mathbf{g}$ are scalar multiples of $\mathbf{d}$. Prove that this definition of extreme rays is equivalent to Definition 4.2.

Exercise 4.40 (Extreme rays of a cone are extreme points of its sections) Consider the cone $C = \{\mathbf{x} \in \Re^n \mid \mathbf{a}_i'\mathbf{x} \geq 0,\ i = 1, \ldots, m\}$ and assume that the first n constraint vectors $\mathbf{a}_1, \ldots, \mathbf{a}_n$ are linearly independent. For any nonnegative scalar r, we define the polyhedron P_r by

$$P_r = \left\{ \mathbf{x} \in C \;\middle|\; \sum_{i=1}^{n} \mathbf{a}_i'\mathbf{x} = r \right\}.$$

(a) Show that the polyhedron P_r is bounded for every $r \geq 0$.

(b) Let $r > 0$. Show that a vector $\mathbf{x} \in P_r$ is an extreme point of P_r if and only if $\mathbf{x}$ is an extreme ray of the cone C.

Exercise 4.41 (Carathéodory's theorem) Show that every element $\mathbf{x}$ of a bounded polyhedron $P \subset \Re^n$ can be expressed as a convex combination of at most $n+1$ extreme points of P. *Hint:* Consider an extreme point of the set of all possible representations of $\mathbf{x}$.

Exercise 4.42 (Problems with side constraints) Consider the linear programming problem of minimizing $\mathbf{c}'\mathbf{x}$ over a bounded polyhedron $P \subset \Re^n$ and subject to additional constraints $\mathbf{a}_i'\mathbf{x} = b_i$, $i = 1, \ldots, L$. Assume that the problem has a feasible solution. Show that there exists an optimal solution which is a convex combination of $L+1$ extreme points of P. *Hint:* Use the resolution theorem to represent P.

Exercise 4.43

(a) Consider the minimization of $c_1x_1 + c_2x_2$ subject to the constraints

$$x_2 - 3 \leq x_1 \leq 2x_2 + 2, \qquad x_1, x_2 \geq 0.$$

Find necessary and sufficient conditions on (c_1, c_2) for the optimal cost to be finite.

(b) For a general feasible linear programming problem, consider the set of all cost vectors for which the optimal cost is finite. Is it a polyhedron? Prove your answer.

Exercise 4.44

(a) Let $P = \{(x_1, x_2) \mid x_1 - x_2 = 0,\ x_1 + x_2 = 0\}$. What are the extreme points and the extreme rays of P?

(b) Let $P = \{(x_1, x_2) \mid 4x_1 + 2x_2 \geq 8,\ 2x_1 + x_2 \leq 8\}$. What are the extreme points and the extreme rays of P?

(c) For the polyhedron of part (b), is it possible to express each one of its elements as a convex combination of its extreme points plus a nonnegative linear combination of its extreme rays? Is this compatible with the resolution theorem?

Exercise 4.45 Let P be a polyhedron with at least one extreme point. Is it possible to express an arbitrary element of P as a convex combination of its extreme points plus a nonnegative multiple of a single extreme ray?

Exercise 4.46 (Resolution theorem for polyhedral cones) Let C be a nonempty polyhedral cone.

(a) Show that C can be expressed as the union of a finite number $C_1, \ldots, C_k$ of pointed polyhedral cones. *Hint:* Intersect with orthants.

(b) Show that an extreme ray of C must be an extreme ray of one of the cones $C_1, \ldots, C_k$.

(c) Show that there exists a finite number of elements $\mathbf{w}^1, \ldots, \mathbf{w}^r$ of C such that

$$C = \left\{ \sum_{i=1}^{r} \theta_i \mathbf{w}^i \;\middle|\; \theta_1, \ldots, \theta_r \geq 0 \right\}.$$

Exercise 4.47 (Resolution theorem for general polyhedra) Let P be a polyhedron. Show that there exist vectors $\mathbf{x}^1, \ldots, \mathbf{x}^k$ and $\mathbf{w}^1, \ldots, \mathbf{w}^r$ such that

$$P = \left\{ \sum_{i=1}^{k} \lambda_i \mathbf{x}^i + \sum_{j=1}^{r} \theta_j \mathbf{w}^j \;\middle|\; \lambda_i \geq 0,\ \theta_j \geq 0,\ \sum_{i=1}^{k} \lambda_i = 1 \right\}.$$

Hint: Generalize the steps in the preceding exercise.

Exercise 4.48 * (Polar, finitely generated, and polyhedral cones) For any cone C, we define its *polar* $C^\perp$ by

$$C^\perp = \{\mathbf{p} \mid \mathbf{p}'\mathbf{x} \leq 0, \text{ for all } \mathbf{x} \in C\}.$$

(a) Let F be a finitely generated cone, of the form

$$F = \left\{ \sum_{i=1}^{r} \theta_i \mathbf{w}^i \;\middle|\; \theta_1, \ldots, \theta_r \geq 0 \right\}.$$

Show that $F^\perp = \{\mathbf{p} \mid \mathbf{p}'\mathbf{w}^i \leq 0,\ i = 1, \ldots, r\}$, which is a polyhedral cone.

(b) Show that the polar of $F^\perp$ is F and conclude that the polar of a polyhedral cone is finitely generated. *Hint:* Use Farkas' lemma.

(c) Show that a finitely generated pointed cone F is a polyhedron. *Hint:* Consider the polar of the polar.

(d) **(Polar cone theorem)** Let C be a closed, nonempty, and convex cone. Show that $(C^{\perp})^{\perp} = C$. *Hint:* Mimic the derivation of Farkas' lemma using the separating hyperplane theorem (Section 4.7).

(e) Is the polar cone theorem true when C is the empty set?

Exercise 4.49 Consider a polyhedron, and let $\mathbf{x}$, $\mathbf{y}$ be two basic feasible solutions. If we are only allowed to make moves from any basic feasible solution to an adjacent one, show that we can go from $\mathbf{x}$ to $\mathbf{y}$ in a finite number of steps. *Hint:* Generalize the simplex method to nonstandard form problems: starting from a nonoptimal basic feasible solution, move along an extreme ray of the cone of feasible directions.

Exercise 4.50 We are interested in the problem of deciding whether a polyhedron

$$Q = \left\{\mathbf{x} \in \Re^n \mid \mathbf{A}\mathbf{x} \leq \mathbf{b},\ \mathbf{D}\mathbf{x} \geq \mathbf{d},\ \mathbf{x} \geq \mathbf{0}\right\}$$

is nonempty. We assume that the polyhedron $P = \{\mathbf{x} \in \Re^n \mid \mathbf{A}\mathbf{x} \leq \mathbf{b},\ \mathbf{x} \geq \mathbf{0}\}$ is nonempty and bounded. For any vector $\mathbf{p}$, of the same dimension as $\mathbf{d}$, we define

$$g(\mathbf{p}) = -\mathbf{p}'\mathbf{d} + \max_{\mathbf{x} \in P} \mathbf{p}'\mathbf{D}\mathbf{x}.$$

(a) Show that if Q is nonempty, then $g(\mathbf{p}) \geq 0$ for all $\mathbf{p} \geq \mathbf{0}$.

(b) Show that if Q is empty, then there exists some $\mathbf{p} \geq \mathbf{0}$, such that $g(\mathbf{p}) < 0$.

(c) If Q is empty, what is the minimum of $g(\mathbf{p})$ over all $\mathbf{p} \geq \mathbf{0}$?

4.13 Notes and sources

4.3. The duality theorem is due to von Neumann (1947), and Gale, Kuhn, and Tucker (1951).

4.6. Farkas' lemma is due to Farkas (1894) and Minkowski (1896). See Schrijver (1986) for a comprehensive presentation of related results. The connection between duality theory and arbitrage was developed by Ross (1976, 1978).

4.7. Weierstrass' Theorem and its proof can be found in most texts on real analysis; see, for example, Rudin (1976). While the simplex method is only relevant to linear programming problems with a finite number of variables, the approach based on the separating hyperplane theorem leads to a generalization of duality theory that covers more general convex optimization problems, as well as infinite-dimensional linear programming problems, that is, linear programming problems with infinitely many variables and constraints; see, e.g., Luenberger (1969) and Rockafellar (1970).

4.9. The resolution theorem and its converse are usually attributed to Farkas, Minkowski, and Weyl.