LP Homework 6

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4.7

(Duality in piecewise linear convex optimization) Consider the problem of minimizing $\max_{i=1,\ldots,m} \{a_i^T x - b_i\}$ over all $x \in \mathbb{R}^n$. Let v be the value of the optimal cost, assumed finite. Let A be the matrix with rows $a_1, \ldots, a_m$, and let b be the vector with components $b_1, \ldots, b_m$.
(a) Consider any vector $p \in \mathbb{R}^m$ that satisfies $p^T A = 0^T$, $p \geq 0$, and $\sum_{i=1}^{m} p_i = 1$. Show that $-p^T b \leq v$.
(b) In order to obtain the best possible lower bound of the form considered in part (a), we form the linear programming problem

$$\begin{aligned} \text{maximize} \quad & -p^T b \\ \text{subject to} \quad & p^T A = 0^T \\ & p^T e = 1 \\ & p \geq 0 \end{aligned}$$

where e is the vector with all components equal to 1. Show that the optimal cost in this problem is equal to v.

Proof.
(a) The piecewise linear problem can be writtem as the following equivalent linear programming problem:

$$\begin{aligned} \text{minimize} \quad & t \\ \text{subject to} \quad & a_i^T x - b_i \leq t, \quad i = 1, \ldots, m \end{aligned}$$

We can also transform the problem into the following familiar form:

$$\begin{aligned} \text{minimize} \quad & [0,1][x,t]^T \\ \text{subject to} \quad & [-A \,|\, e][x,t]^T \geq -b \end{aligned}$$

The dual problem is:

$$\begin{aligned} \text{maximize} \quad & -b^T p \\ \text{subject to} \quad & [0,1] = p^T[-A, e] \\ & p \geq 0 \end{aligned}$$

or,

$$\begin{aligned} \text{maximize} \quad & -b^T p \\ \text{subject to} \quad & p^T A = 0^T \\ & p^T e = 1 \\ & p \geq 0 \end{aligned}$$

For any vector $p \in \mathbb{R}^m$ that satisfies $p^T A = 0^T$, $p \geq 0$, and $\sum_{i=1}^{m} p_i = 1$, we have that p is feasible for the dual problem. Then by Weak Duality, we have $-b^T p \leq v$.

(b) We have shown that the problem in (b) is the dual problem of the piecewise linear problem. Then by Strong Duality, we have that the optimal cost in this problem is equal to v.

4.8

Consider the linear programming problem of minimizing $c^T x$ subject to $Ax = b$, $x \geq 0$. Let x^* be an optimal solution, assumed to exist, and let p^* be an optimal solution to the dual.

(a) Let $\tilde{x}$ be an optimal solution to the primal, when c is replaced by some $\tilde{c}$. Show that $(\tilde{c} - c)^T(\tilde{x} - x^*) \leq 0$.

(b) Let the cost vector be fixed at c, but suppose that we now change b to $\tilde{b}$, and let $\tilde{x}$ be a corresponding optimal solution to the primal. Prove that

$(p^*)^T(\tilde{b} - b) \leq c^T(\tilde{x} - x^*)$.

Proof.
(a) The primal problem is:

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

The dual problem is:

$$\begin{aligned} \text{maximize} \quad & b^T p \\ \text{subject to} \quad & p^T A \leq c^T \end{aligned}$$

On one hand, since x^* is an optimal solution to the primal with cost c, we have $c^T x^* \leq c^T \tilde{x}$. On the other hand, since $\tilde{x}$ is an optimal solution to the primal with cost $\tilde{c}$, and x^* is still feasible, we have $\tilde{c}^T \tilde{x} \leq \tilde{c}^T x^*$. Then we have $(\tilde{c} - c)^T(\tilde{x} - x^*) = (\tilde{c}^T \tilde{x} - \tilde{c}^T x^*) - (c^T \tilde{x} - c^T x^*) \leq 0$.
(b) On one hand, by Strong Duality, we have $b^T p^* = c^T x^*$.
Let $\tilde{p}$ be an optimal solution to the dual problem with cost $\tilde{b}$, and p^* is still feasible, we have $\tilde{b}^T \tilde{p} \geq \tilde{b}^T p^*$. On the other hand, by Strong Duality, we have $\tilde{b}^T \tilde{p} = c^T \tilde{x}$. Then we have $c^T \tilde{x} \geq \tilde{b}^T p^*$.
Thus, we obtain $(p^*)^T(\tilde{b} - b) - c^T(\tilde{x} - x^*) = (\tilde{b}^T p^* - c^T \tilde{x}) - (b^T p^* - c^T x^*) \leq 0$.

4.10

(Saddle points of the Lagrangean) Consider the standard form problem of minimizing $c^T x$ subject to $Ax = b$, $x \geq 0$. We define the *Lagrangean* by

$$L(x, p) = c^T x + p^T(b - Ax)$$

Consider the following "game" : player 1 chooses some $x \geq 0$, and player 2 chooses some p; then, player 1 pays to player 2 the amount $L(x, p)$. Player 1 would like to minimize $L(x, p)$, while player 2 would like to maximize it. A pair (x^*, p^*), with $x^* \geq 0$, is called an *equilibrium point* (or a *saddle point*, or a *Nash equilibrium*) if

$$L(x^*, p) \leq L(x^*, p^*) \leq L(x, p^*), \quad \forall x \geq 0, \forall p$$

(Thus, we have an equilibrium if no player is able to improve her performance by unilaterally modifying her choice.)
Show that a pair (x^*, p^*) is an equilibrium if and only if x^* and p^* are optimal solutions to the standard form problem under consideration and its dual, respectively.

Proof.
The primal problem is:

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

The dual problem is:

$$\begin{aligned} \text{maximize} \quad & b^T p \\ \text{subject to} \quad & p^T A \leq c^T \end{aligned}$$

$(\Rightarrow)$ Suppose that (x^*, p^*) is an equilibrium point. Then we have

$$L(x^*, p) \leq L(x^*, p^*) \leq L(x, p^*), \quad \forall x \geq 0, \forall p$$
$$c^T x^* + p^T (b - Ax^*) \leq c^T x^* + p^{*T} (b - Ax^*) \leq (c^T - p^{*T} A)x + p^{*T} b, \quad \forall x \geq 0, \forall p$$

By the first inequality, we must have $b - Ax^* = 0$. Otherwise we can choose p such that $p^T(b - Ax^*)$ sufficiently large. So x^* is feasible for the standard form problem.
By the second inequality, we must have $c^T - p^{*T} A \geq 0$. Otherwise we can choose $x \geq 0$ such that $(c^T - p^{*T} A)x + p^{*T} b$ is sufficiently small. So p^* is feasible for the dual problem.
We know that $c^T x^* = L(x^*, p^*) = \min_{x \geq 0} L(x, p^*) = \max_p L(x^*, p)$. And we also have $\min_{x \geq 0} L(x, p^*) = \min_{x \geq 0} (c^T - p^{*T} A)x + p^{*T} b = p^{*T} b$. Put them together, we have $c^T x^* = p^{*T} b$. Then we obtain Strong Duality, and x^* and p^* are optimal solutions to the standard form problem under consideration and its

dual, respectively.

($\Leftarrow$) Suppose that x^* and p^* are optimal solutions to the standard form problem and its dual, respectively. Since x^* is feasible for the standard form problem, we have $Ax^* = b$. Then we get

$L(x^*, p^*) = c^T x^* + p^{*T}(b - Ax^*) = c^T x^* = L(x^*, p), \quad \forall p.$

Since p^* is feasible for the dual problem, we have $p^{*T}A \leq c^T$. Then we get $\min_{x \geq 0} L(x, p^*) = \min_{x \geq 0}(c^T - p^{*T}A)x + p^{*T}b = p^{*T}b = c^T x^*$, from Strong Duality.

Thus we have $L(x^*, p) \leq \max_p L(x^*, p) = c^T x^* = L(x^*, p^*) = \min_{x \geq 0} L(x, p^*) \leq L(x, p^*), \quad \forall x \geq 0, \forall p.$

Then (x^*, p^*) is an equilibrium point.

4.19

Let $P = \{x \in \mathbb{R}^n | Ax = b, x \geq 0\}$ be a nonempty polyhedron, and let m be the dimension of the vector b. We call x_j a *null variable* if $x_j = 0$ whenever $x \in P$.

(a) Suppose that there exists some $p \in \mathbb{R}^m$ for which $p^T A \geq 0^T$, $p^T b = 0$, and such that the jth component of $p^T A$ is positive. Prove that x_j is a null variable.

(b) Prove the converse of (a) : if x_j is a null variable, then there exists some $p \in \mathbb{R}^m$ with the properties stated in (a).

(c) If x_j is not a null variable, then by definition, there exists some $y \in P$ for which $y_j > 0$. Use the results in parts (a) and (b) to prove that there exist $x \in P$ and $p \in \mathbb{R}^m$ such that:

$$p^T A \geq 0^T, \quad p^T b = 0, \quad x + A^T p > 0.$$

Proof.

(a) Consider the following linear programming problem:

$$\begin{aligned} \text{minimize} \quad & -e_j^T x \\ \text{subject to} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

where e_j is the jth standard basis vector.
The dual problem is:

$$\begin{aligned} \text{maximize} \quad & b^T p \\ \text{subject to} \quad & p^T A \le -e_j^T \end{aligned}$$

Since there exists some $p \in \mathbb{R}^m$ s.t. $p^T A \ge 0^T$, $p^T b = 0$, $\ell := p^T A e_j > 0$, let $q = -\frac{1}{\ell} p$, then we have $q^T A e_j = -1$, $q^T A \le -e_j$.
So q is feasible for the dual problem, and $b^T q = -\frac{1}{\ell} p^T b = 0$.
By Weak Duality, the optimal cost of the primal problem is nonnegative. However, for any feasible x, we have $-e_j^T x \le 0$. So $x_j = 0$ whenever $x \in P$.
(b) Suppose that x_j is a null variable. Consider the same linear programming problem in (a).
Since P is nonempty, there exists some $x \in P$ and the (optimal) cost is always 0. Then by Strong Duality, we have the optimal cost of the dual problem is also 0 at the optimal solution y.
Let $p = -y$, then we have $p^T A \ge e_j^T \ge 0^T$, $p^T b = 0$, which proves (b).
(c) Let $\mathcal{N}$ be the index set of null variables and $\mathcal{N}^c$ be the index set of non-null variables.
If $j \in \mathcal{N}$, then by (b), there exists some $p_{(j)} \in \mathbb{R}^m$ s.t. $p_{(j)}^T A \ge 0^T$, $p_{(j)}^T b = 0$ and $p_{(j)}^T A e_j > 0$.
Let $p = \sum\limits_{j \in \mathcal{N}} p_{(j)}$ ($p = 0$ if $\mathcal{N} = \varnothing$), then we have $p^T A \ge 0^T$, $p^T b = 0$.
If $i \in \mathcal{N}^c$, then, x_i is not a null variable, then there exists some $y_{(i)} \in P$ s.t. $e_i y_{(i)} > 0$.
Let $x = \frac{1}{|\mathcal{N}^c|} \sum\limits_{i \in \mathcal{N}^c} y_{(i)}$ ($x = 0$ if $\mathcal{N}^c = \varnothing$), then we have $x \in P$.
x maintains that $x_i > 0$ for all $i \in \mathcal{N}^c$, and $A^T p$ maintains that $(A^T p)_j > 0$ for all $j \in \mathcal{N}$. Adding them together, we have $x + A^T p > 0$.

4.26

Let A be a given matrix. Show that exactly one of the following alternatives must hold.
(a) There exists some $x \ne 0$ such that $Ax = 0, x \ge 0$.

(b) There exists some p such that $p^T A > 0^T$.

Proof.

Note that (b) is equivalent to $\exists p, p^T A \geq e$, since we can always divide p by $\min_i (p^T A)_i > 0$ to p to make the transformation.

Consider the following linear programming problem:

$$\begin{aligned} \text{minimize} \quad & -e^T x \\ \text{subject to} \quad & Ax = 0 \\ & x \geq 0 \end{aligned}$$

where e is the vector with all components equal to 1.

The dual problem is:

$$\begin{aligned} \text{maximize} \quad & 0^T p \\ \text{subject to} \quad & p^T A \leq -e^T \end{aligned}$$

1. Suppose (b) holds, then there exists some p such that $p^T A \geq e$. Then $-p$ is feasible for the dual problem, and the cost is 0.

By Weak Duality, the cost of the primal problem is nonnegative. However, for any feasible x, we have $-e^T x \leq 0$. So if x is feasible for the primal problem, x must be zero, indicating that (a) does not hold.

2. Suppose (a) holds, then there exists some $x \neq 0$ such that $Ax = 0, x \geq 0$. Then x is feasible for the primal problem, and the cost is negative.

By Weak Duality, the cost of the dual problem is also negative for all feasible p, indicating that the dual problem must be infeasible. Then there exists no p such that $p^T A \geq e$. So (b) does not hold.