LP Homework 4

吴嘉骜 21307130203

2023 年 4 月 27 日

3.1

(Local minima of convex functions) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function and let $S \subset \mathbb{R}^n$ be a convex set. Let $\mathbf{x}^*$ be an element of S. Suppose that $\mathbf{x}^*$ is a local optimum for the problem of minimizing $f(\mathbf{x})$ over S; that is, there exists some $\varepsilon > 0$ such that $f(\mathbf{x}^*) \leq f(\mathbf{x})$ for all $\mathbf{x} \in S$ for which $||\mathbf{x} - \mathbf{x}^*|| \leq \epsilon$. Prove that $\mathbf{x}^*$ is globally optimal; that is, $f(\mathbf{x}^*) \leq f(\mathbf{x})$ for all $\mathbf{x} \in S$.

Proof.

Assume that $\mathbf{x}^*$ is not globally optimal. Then there exists $\mathbf{x} \in S$ such that $f(\mathbf{x}^*) > f(\mathbf{x})$.

Let $\mathbf{d} = \mathbf{x} - \mathbf{x}^*$. By convexity of f, we have $f(\mathbf{x}^* + \theta\mathbf{d}) = f(\theta\mathbf{x} + (1-\theta)\mathbf{x}^*) \leq \theta f(\mathbf{x}) + (1-\theta)f(\mathbf{x}^*) < \theta f(\mathbf{x}^*) + (1-\theta)f(\mathbf{x}^*) = f(\mathbf{x}^*)$, for all $\theta \in (0, 1)$.

For θ sufficiently small, we have $||\mathbf{x}^* + \theta\mathbf{d} - \mathbf{x}^*|| = ||\theta\mathbf{d}|| \leq \varepsilon$, which contradicts that $\mathbf{x}^*$ is a local optimum.

3.2

(Optimality conditions) Consider the problem of minimizing $\mathbf{c}^\mathbf{T}\mathbf{x}$ over a polyhedron P. Prove the following:

(a) A feasible solution $\mathbf{x}$ is optimal if and only if $\mathbf{c}^\mathbf{T}\mathbf{d} \geq 0$ for every feasible direction $\mathbf{d}$ at $\mathbf{x}$.

(b) A feasible solution $\mathbf{x}$ is the unique optimal solution if and only if $\mathbf{c^T d} > 0$ for every nonzero feasible direction $\mathbf{d}$ at $\mathbf{x}$.
Proof.
(a)
$\Rightarrow$: Suppose that $\mathbf{x}$ is optimal. For every feasible direction $\mathbf{d}$ at $\mathbf{x}$, we have $\mathbf{y} = \mathbf{x} + \theta\mathbf{d} \in P$ for some positive scalar θ. By the optimality condition, we have $\mathbf{c^T y} = \mathbf{c^T x} + \theta\mathbf{c^T d} \geq \mathbf{c^T x}$. Thus, $\mathbf{c^T d} \geq 0$ for every feasible direction $\mathbf{d}$ at $\mathbf{x}$.
$\Leftarrow$: Suppose that $\mathbf{c^T d} \geq 0$ for every feasible direction $\mathbf{d}$ at $\mathbf{x}$. For any $\mathbf{y} \in P$, let $\mathbf{d} = \mathbf{y} - \mathbf{x}$, $\theta = 1$ and we get a feasible direction $\mathbf{d}$. Then $\mathbf{c^T d} = \mathbf{c^T y} - \mathbf{c^T x} \geq 0$. Thus, $\mathbf{x}$ is optimal.
(b)
$\Rightarrow$: Suppose that $\mathbf{x}$ is uniquely optimal. For every nonzero feasible direction $\mathbf{d}$ at $\mathbf{x}$, we have $\mathbf{y} = \mathbf{x} + \theta\mathbf{d} \in P$ for some positive scalar θ. By the uniqueness of the optimal solution and $\mathbf{y} \neq \mathbf{x}$, we have $\mathbf{c^T y} = \mathbf{c^T x} + \theta\mathbf{c^T d} > \mathbf{c^T x}$. Thus, $\mathbf{c^T d} > 0$ for every feasible nonzero direction $\mathbf{d}$ at $\mathbf{x}$.
$\Leftarrow$: Suppose that $\mathbf{c^T d} > 0$ for every nonzero feasible direction $\mathbf{d}$ at $\mathbf{x}$. For any $\mathbf{y} \in P, \mathbf{y} \neq \mathbf{x}$, let $\mathbf{d} = \mathbf{y} - \mathbf{x} \neq \mathbf{0}$ and $\theta = 1$. Then $\mathbf{d}$ is a nonzero feasible direction, and $\mathbf{c^T d} = \mathbf{c^T y} - \mathbf{c^T x} > 0$. Thus, $\mathbf{x}$ is the unique optimal solution.

3.3

Let $\mathbf{x}$ be an element of the standard form polyhedron $P = \{\mathbf{x} \in \mathbb{R}^n | \mathbf{Ax} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$. Prove that a vector $\mathbf{d} \in \mathbb{R}^n$ is a feasible direction at $\mathbf{x}$ if and only if $\mathbf{Ad} = \mathbf{0}$ and $d_i \geq 0$ for every i such that $x_i = 0$.
Proof.
$\Rightarrow$: Suppose $\mathbf{d}$ is a feasible direction at $\mathbf{x}$, and we have $\mathbf{x} + \theta\mathbf{d} \in P$ for some positive scalar θ. Then $\mathbf{A}(\mathbf{x} + \theta\mathbf{d}) = \mathbf{b}$ and $\mathbf{x} + \theta\mathbf{d} \geq \mathbf{0}$. Note that we have $\mathbf{Ax} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}$, thus $\theta\mathbf{Ad} = \mathbf{0}$ and $\theta d_i \geq 0$ whenever $x_i = 0$. Divided by θ, the result is that $\mathbf{Ad} = \mathbf{0}$ and $d_i \geq 0$ for every i such that $x_i = 0$.
$\Leftarrow$: Suppose $\mathbf{Ad} = \mathbf{0}$ and $d_i \geq 0$ for every i such that $x_i = 0$. Let $\mathbf{y} = \mathbf{x} + \theta\mathbf{d}$, $\theta > 0$. Then $\mathbf{Ay} = \mathbf{A}(\mathbf{x} + \theta\mathbf{d}) = \mathbf{Ax} = \mathbf{b}$. For every i, we have $y_i = x_i + \theta d_i$.

If $x_i = 0$, then $y_i = \theta d_i \geq 0$. If $x_i > 0$, then $y_i = x_i + \theta d_i \geq 0$ for sufficiently small θ. Thus, there exists $\theta > 0$ such that $\mathbf{y} \in P$. So $\mathbf{d}$ is a feasible direction at $\mathbf{x}$.

3.4

Consider the problem of minimizing $\mathbf{c^T x}$ over the set $P = \{\mathbf{x} \in \mathbb{R}^n | \mathbf{Ax} = \mathbf{b}, \mathbf{Dx} \leq \mathbf{f}, \mathbf{Ex} \leq \mathbf{g}\}$. Let $\mathbf{x}^*$ be an element of P that satisfies $\mathbf{Dx}^* = \mathbf{f}, \mathbf{Ex}^* < \mathbf{g}$. Show that the set of feasible directions at the point $\mathbf{x}^*$ is the set $\{\mathbf{d} \in \mathbb{R}^n | \mathbf{Ad} = \mathbf{0}, \mathbf{Dd} \leq \mathbf{0}\}$.
Proof.
Let $\mathbf{d}$ be a feasible direction at $\mathbf{x}^*$. Then we need $\mathbf{x}^* + \theta\mathbf{d} \in P$ for some positive scalar θ. Note that we have $\mathbf{Ax}^* = \mathbf{b}, \mathbf{Dx}^* = \mathbf{f}, \mathbf{Ex}^* < \mathbf{g}$, thus the requirements $\mathbf{A}(\mathbf{x}^* + \theta\mathbf{d}) = \mathbf{b}$, $\mathbf{D}(\mathbf{x}^* + \theta\mathbf{d}) \leq \mathbf{f}$ and $\mathbf{E}(\mathbf{x}^* + \theta\mathbf{d}) \leq \mathbf{g}$ are equivalent to $\mathbf{Ad} = \mathbf{0}, \mathbf{Dd} \leq \mathbf{0}$ and $\mathbf{Ex}^* + \theta\mathbf{Ed} \leq \mathbf{g}$. Since $\mathbf{Ex}^* < \mathbf{g}$, the last condition $\mathbf{Ex}^* + \theta\mathbf{Ed} \leq \mathbf{g}$ can be always achieved by taking sufficiently small $\theta > 0$ and is thus redundant. To sum up, the set of feasible directions at the point $\mathbf{x}^*$ is the set $\{\mathbf{d} \in \mathbb{R}^n | \mathbf{Ad} = \mathbf{0}, \mathbf{Dd} \leq \mathbf{0}\}$.

3.7

(Optimality conditions) Consider a feasible solution $\mathbf{x}$ to a standard form problem, and let $Z = \{i | x_i = 0\}$. Show that $\mathbf{x}$ is an optimal solution if and only if the linear programming problem

$$\begin{aligned} &minimize \quad && \mathbf{c^T d} \\ &subject\,to && \mathbf{Ad} = \mathbf{0} \\ & && d_i \geq 0, \quad i \in Z, \end{aligned}$$

has an optimal cost of zero. (In this sense, deciding optimality is equivalent to solving a new linear programming problem.)
Proof.
Let P denote the standard form polyhedron,$\mathcal{F} = \{\mathbf{d} \,|\, \mathbf{Ad} = \mathbf{0}; d_i \geq 0, i \in Z\}$.

$\Rightarrow$: Suppose $\mathbf{x}$ is an optimal solution. Then for any $\mathbf{d} \in \mathcal{F}$, there exists sufficiently small $\theta > 0$, $\mathbf{A}(\mathbf{x} + \theta\mathbf{d}) = \mathbf{0}, \mathbf{x} + \theta\mathbf{d} \geq \mathbf{0}$, so we have $\mathbf{x} + \theta\mathbf{d} \in P$. Thus, $\mathbf{c}^\mathbf{T}(\mathbf{x} + \theta\mathbf{d}) \geq \mathbf{c}^\mathbf{T}\mathbf{x}$, namely $\mathbf{c}^\mathbf{T}\mathbf{d} \geq 0$ for every $\mathbf{d} \in \mathcal{F}$. It follows that the linear programming problem has an optimal cost of zero since $\mathbf{d} = \mathbf{0}$ is an optimal solution.

$\Leftarrow$: Suppose the linear programming problem has an optimal cost of zero. Then $\mathbf{c}^\mathbf{T}\mathbf{d} \geq \mathbf{0}$ for every $\mathbf{d} \in \mathcal{F}$. For any $\mathbf{y} \in P$, $\mathbf{y} - \mathbf{x} \in \mathcal{F}$ since $\mathbf{A}(\mathbf{y} - \mathbf{x}) = \mathbf{0}$ and $y_i - x_i = y_i \geq 0, i \in Z$. so we have $\mathbf{c}^\mathbf{T}(\mathbf{y} - \mathbf{x}) \geq \mathbf{0}$. Thus, $\mathbf{c}^\mathbf{T}\mathbf{y} \geq \mathbf{c}^\mathbf{T}\mathbf{x}$ and $\mathbf{x}$ is an optimal solution.

3.12

Consider the problem

$$
\begin{aligned}
&\textit{minimize} && -2x_1 - x_2 \\
&\textit{subject to} && x_1 - x_2 \leq 2 \\
& && x_1 + x_2 \leq 6 \\
& && x_1, x_2 \geq 0.
\end{aligned}
$$

(a) Convert the problem into standard form and construct a basic feasible solution at which $(x_1, x_2) = (0, 0)$.

(b) Carry out the full tableau implementation of the simplex method, starting with the basic feasible solution of part (a).

(c) Draw a graphical representation of the problem in terms of the original variables x_1, x_2, and indicate the path taken by the simplex algorithm.

Solution.

(a) Let x_1, x_2 be the original variables and x_3, x_4 be the slack variables. Then the standard form problem becomes:

$$
\begin{aligned}
&\textit{minimize} && -2x_1 - x_2 + 0x_3 + 0x_4 \\
&\textit{subject to} && x_1 - x_2 + x_3 + 0x_4 = 2 \\
& && x_1 + x_2 + 0x_3 + x_4 = 6 \\
& && x_1, x_2, x_3, x_4 \geq 0.
\end{aligned}
$$

Here $A = \begin{bmatrix} 1 & -1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$, and $c^T = [-2, -1, 0, 0]$.

(b) Starting with the basic feasible solution $\mathbf{x} = (0, 0, 2, 6)^T$, we have the following tableau:

$$\begin{array}{c|cccc} 0 & -2 & -1 & 0 & 0 \\ \hline x_3 = 2 & 1 & -1 & 1 & 0 \\ x_4 = 6 & 1 & 1 & 0 & 1 \end{array}$$

Choose x_1 as the entering variable and x_3 as the leaving variable. Then we have the following tableau:

$$\begin{array}{c|cccc} 4 & 0 & -3 & 2 & 0 \\ \hline x_1 = 2 & 1 & -1 & 1 & 0 \\ x_4 = 4 & 0 & 2 & -1 & 1 \end{array}$$

Choose x_2 as the entering variable and x_4 as the leaving variable. Then we have the following tableau:

$$\begin{array}{c|cccc} 10 & 0 & 0 & 1/2 & 3/2 \\ \hline x_1 = 4 & 1 & 0 & 1/2 & 1/2 \\ x_2 = 2 & 0 & 1 & -1/2 & 1/2 \end{array}$$

Since all the reduced costs are nonnegative, the optimal solution is $\mathbf{x} = (2, 4, 0, 0)^T$ and the optimal value is -10.

(c) In terms of the original variables (x_1, x_2), we have moved from the point $(0, 0)$ to the point $(2, 0)$ and finally to the point $(4, 2)$.

The graphical representation of the problem is shown below. The path taken by the simplex algorithm is shown in red.

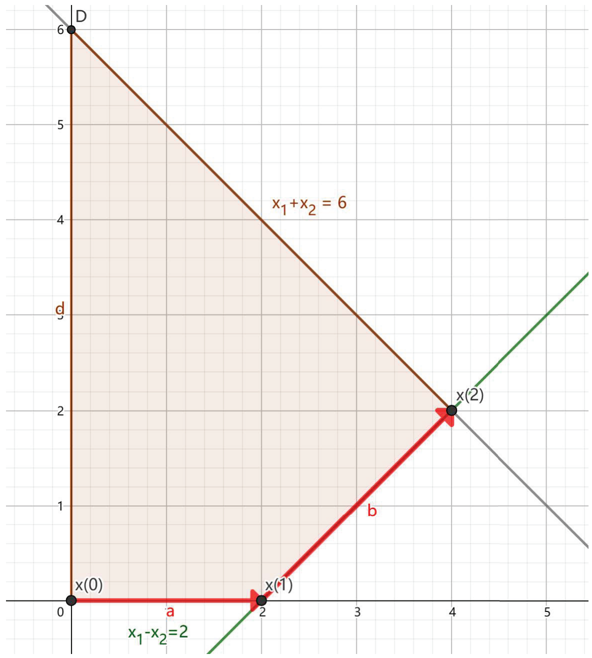

Figure 1. graphical representation of problem 3.12

3.13

This exercise shows that our efficient procedures for updating a tableau can be derived from a useful fact in numerical linear algebra.
(a) (Matrix inversion lemma) Let $\mathbf{C}$ be an $m \times m$ invertible matrix and let $\mathbf{w}, \mathbf{v}$ be vectors in $\mathbb{R}^n$. Show that

$$(\mathbf{C} + \mathbf{w}\mathbf{v}^\mathbf{T})^{-1} = \mathbf{C^{-1}} - \frac{\mathbf{C^{-1}}\mathbf{w}\mathbf{v}^\mathbf{T}\mathbf{C^{-1}}}{1 + \mathbf{v}^\mathbf{T}\mathbf{C^{-1}}\mathbf{w}}.$$

(Note that $\mathbf{w}\mathbf{v}^\mathbf{T}$ is an $m \times m$ matrix). Hint: Multiply both sides by $(\mathbf{C} + \mathbf{w}\mathbf{v}^\mathbf{T})$.
(b) Assuming that $\mathbf{C}^{-1}$ is available, explain how to obtain $(\mathbf{C} + \mathbf{w}\mathbf{v}^\mathbf{T})^{-1}$ using only $O(m^2)$ arithmetic operations.
(c) Let $\mathbf{B}$ and $\overline{\mathbf{B}}$ be basis matrices before and after an iteration of the simplex method. Let $\mathbf{A}_{B(\ell)}, \mathbf{A}_{\overline{B}(\ell)}$ be the exiting and entering column, respectively. Show that

$$\overline{\mathbf{B}} - \mathbf{B} = (\mathbf{A}_{\overline{B}(\ell)} - \mathbf{A}_{B(\ell)})\mathbf{e}_\ell^T.$$

where $\mathbf{e}_\ell$ is the ℓth unit vector.
(d) Note that $\mathbf{e}_i^T\mathbf{B}^{-1}$ is the ith row of $\mathbf{B}^{-1}$ and $\mathbf{e}_\ell^T\mathbf{B}^{-1}$ is the pivot row. Show that

$$\mathbf{e}_i^T\overline{\mathbf{B}}^{-1} = \mathbf{e}_i^T\mathbf{B}^{-1} - g_i\mathbf{e}_\ell^T\mathbf{B}^{-1}, \quad i = 1, \dots, m,$$

for suitable scalars g_i. Provide a formula for g_i. Interpret the above equation in terms of the mechanics for pivoting in the revised simplex method.
(e) Multiply both sides of the equation in part (d) by $[\mathbf{b} \,|\, \mathbf{A}]$ and obtain an interpretation of the mechanics for pivoting in the full tableau implementation.
Solution:
(a) Multiply both sides by $(\mathbf{C}+\mathbf{w}\mathbf{v}^\mathbf{T})$, we have

$$\begin{aligned}
&(\mathbf{C}+\mathbf{w}\mathbf{v}^\mathbf{T})^{-1} = \mathbf{C}^{-1} - \frac{\mathbf{C}^{-1}\mathbf{w}\mathbf{v}^\mathbf{T}\mathbf{C}^{-1}}{1+\mathbf{v}^\mathbf{T}\mathbf{C}^{-1}\mathbf{w}} \\
&\Leftrightarrow \mathbf{I} = \mathbf{C}^{-1}(\mathbf{C}+\mathbf{w}\mathbf{v}^\mathbf{T}) - \frac{\mathbf{C}^{-1}\mathbf{w}\mathbf{v}^\mathbf{T}\mathbf{C}^{-1}}{1+\mathbf{v}^\mathbf{T}\mathbf{C}^{-1}\mathbf{w}}(\mathbf{C}+\mathbf{w}\mathbf{v}^\mathbf{T}) \\
&\Leftrightarrow \mathbf{I} = \mathbf{I} + \mathbf{C}^{-1}\mathbf{w}\mathbf{v}^\mathbf{T} - \frac{\mathbf{C}^{-1}\mathbf{w}\mathbf{v}^\mathbf{T}(\mathbf{I}+\mathbf{C}^{-1}\mathbf{w}\mathbf{v}^\mathbf{T})}{1+\mathbf{v}^\mathbf{T}\mathbf{C}^{-1}\mathbf{w}} \\
&\Leftrightarrow \mathbf{C}^{-1}\mathbf{w}\mathbf{v}^\mathbf{T}(1+\mathbf{v}^\mathbf{T}\mathbf{C}^{-1}\mathbf{w}) = \mathbf{C}^{-1}\mathbf{w}\mathbf{v}^\mathbf{T}(\mathbf{I}+\mathbf{C}^{-1}\mathbf{w}\mathbf{v}^\mathbf{T}) \\
&\Leftrightarrow \mathbf{C}^{-1}\mathbf{w}\mathbf{v}^\mathbf{T}\mathbf{v}^\mathbf{T}\mathbf{C}^{-1}\mathbf{w} = \mathbf{C}^{-1}\mathbf{w}\mathbf{v}^\mathbf{T}\mathbf{C}^{-1}\mathbf{w}\mathbf{v}^\mathbf{T}
\end{aligned}$$

Note that $\mathbf{v}^\mathbf{T}\mathbf{C}^{-1}\mathbf{w}$ is a scalar, so we have

$$\begin{aligned}
&\mathbf{C}^{-1}\mathbf{w}\mathbf{v}^\mathbf{T}\mathbf{v}^\mathbf{T}\mathbf{C}^{-1}\mathbf{w} = \mathbf{C}^{-1}\mathbf{w}\mathbf{v}^\mathbf{T}\mathbf{C}^{-1}\mathbf{w}\mathbf{v}^\mathbf{T} \\
&\Leftrightarrow \mathbf{v}^\mathbf{T}\mathbf{C}^{-1}\mathbf{w}(\mathbf{C}^{-1}\mathbf{w}\mathbf{v}^\mathbf{T} - \mathbf{C}^{-1}\mathbf{w}\mathbf{v}^\mathbf{T}) = 0
\end{aligned}$$

which is obvious.
(b) From part (a) and the assumption that $\mathbf{C}^{-1}$ is available, we have: computing $\mathbf{C}^{-1}\mathbf{w}$ as well as $\mathbf{v}^\mathbf{T}\mathbf{C}^{-1}$ need $O(m^2)$ flops; computing $\mathbf{v}^\mathbf{T}(\mathbf{C}^{-1}\mathbf{w})$ need $O(m)$ flops; computing $(\mathbf{C}^{-1}\mathbf{w})(\mathbf{v}^\mathbf{T}\mathbf{C}^{-1})$ need $O(m)$ flops; computing the last matrix subtraction need $O(m)$ flops; others need $O(1)$ flops. So the total flops is $O(m^2)$.
(c) From the definition of $\mathbf{B}$ and $\overline{\mathbf{B}}$, we assume that
$\mathbf{B} = [\mathbf{A}_{B(1)}, \dots, \mathbf{A}_{B(\ell-1)}, \mathbf{A}_{B(\ell)}, \mathbf{A}_{B(\ell+1)}, \dots, \mathbf{A}_{B(m)}]$
$\overline{\mathbf{B}} = [\mathbf{A}_{B(1)}, \dots, \mathbf{A}_{B(\ell-1)}, \mathbf{A}_{\overline{B}(\ell)}, \mathbf{A}_{B(\ell+1)}, \dots, \mathbf{A}_{B(m)}]$.
then $\overline{\mathbf{B}}-\mathbf{B}$ is a $m \times m$ zero matrix, except that its ℓ th column is $\mathbf{A}_{\overline{B}(\ell)} - \mathbf{A}_{B(\ell)}$.

This indicates that $\overline{\mathbf{B}} - \mathbf{B} = (\mathbf{A}_{\overline{B}(\ell)} - \mathbf{A}_{B(\ell)})\mathbf{e}_\ell^T$.

(d) From (c) and (a) we have

$$\overline{\mathbf{B}}^{-1} = (\mathbf{B} + (\mathbf{A}_{\overline{B}(\ell)} - \mathbf{A}_{B(\ell)})\mathbf{e}_\ell^T)^{-1} = \mathbf{B}^{-1} - \frac{\mathbf{B}^{-1}(\mathbf{A}_{\overline{B}(\ell)} - \mathbf{A}_{B(\ell)})\mathbf{e}_\ell^T\mathbf{B}^{-1}}{1 + \mathbf{e}_\ell^T\mathbf{B}^{-1}(\mathbf{A}_{\overline{B}(\ell)} - \mathbf{A}_{B(\ell)})}$$

Therefore, we have

$$\mathbf{e}_i^T\overline{\mathbf{B}}^{-1} = \mathbf{e}_i^T\mathbf{B}^{-1} - \frac{\mathbf{e}_i^T\mathbf{B}^{-1}(\mathbf{A}_{\overline{B}(\ell)} - \mathbf{A}_{B(\ell)})}{1 + \mathbf{e}_\ell^T\mathbf{B}^{-1}(\mathbf{A}_{\overline{B}(\ell)} - \mathbf{A}_{B(\ell)})}\mathbf{e}_\ell^T\mathbf{B}^{-1}$$

Then $g_i = \frac{\mathbf{e}_i^T\mathbf{B}^{-1}(\mathbf{A}_{\overline{B}(\ell)} - \mathbf{A}_{B(\ell)})}{1 + \mathbf{e}_\ell^T\mathbf{B}^{-1}(\mathbf{A}_{\overline{B}(\ell)} - \mathbf{A}_{B(\ell)})}$.

Note that $1 + \mathbf{e}_\ell^T\mathbf{B}^{-1}(\mathbf{A}_{\overline{B}(\ell)} - \mathbf{A}_{B(\ell)}) = 1 + \mathbf{e}_\ell^T(\mathbf{B}^{-1}\mathbf{A}_{\overline{B}(\ell)} - \mathbf{e}_\ell) = \mathbf{e}_\ell^T\mathbf{B}^{-1}\mathbf{A}_{\overline{B}(\ell)}$.

For $i \neq \ell$, we have $\mathbf{e}_i^T\mathbf{B}^{-1}\mathbf{A}_{B(\ell)} = \mathbf{e}_i^T\mathbf{e}_\ell = 0$,

then $g_i = \frac{\mathbf{e}_i^T\mathbf{B}^{-1}\mathbf{A}_{\overline{B}(\ell)}}{\mathbf{e}_\ell^T\mathbf{B}^{-1}\mathbf{A}_{\overline{B}(\ell)}}$;

$\mathbf{e}_i^T\overline{\mathbf{B}}^{-1} = \mathbf{e}_i^T\mathbf{B}^{-1} - \frac{\mathbf{e}_i^T\mathbf{B}^{-1}\mathbf{A}_{\overline{B}(\ell)}}{\mathbf{e}_\ell^T\mathbf{B}^{-1}\mathbf{A}_{\overline{B}(\ell)}}\mathbf{e}_\ell^T\mathbf{B}^{-1}$.

For $i = \ell$, we have $\mathbf{e}_\ell^T\mathbf{B}^{-1}\mathbf{A}_{B(\ell)} = \mathbf{e}_\ell^T\mathbf{e}_\ell = 1$,

then $g_\ell = \frac{\mathbf{e}_\ell^T\mathbf{B}^{-1}\mathbf{A}_{\overline{B}(\ell)} - 1}{\mathbf{e}_\ell^T\mathbf{B}^{-1}\mathbf{A}_{\overline{B}(\ell)}}$;

$\mathbf{e}_\ell^T\overline{\mathbf{B}}^{-1} = \frac{1}{\mathbf{e}_\ell^T\mathbf{B}^{-1}\mathbf{A}_{\overline{B}(\ell)}}\mathbf{e}_\ell^T\mathbf{B}^{-1}$.

In the revised simplex method, this operation is that we update $[\mathbf{B}^{-1} \mid \mathbf{B}^{-1}\mathbf{A}_{\overline{B}(\ell)}]$ by adding to each one of its rows a multiple g_i of the ℓth row (pivot row) to make the last column equal to the unit vector $\mathbf{e}_\ell$. The first m columns of the result is the matrix $\overline{\mathbf{B}}^{-1}$.

(e) Multiply both sides of the equation in part (d) by $[\mathbf{b} \mid \mathbf{A}]$, we have

$$\mathbf{e}_i^T[\overline{\mathbf{B}}^{-1}\mathbf{b} \mid \overline{\mathbf{B}}^{-1}A] = \mathbf{e}_i^T[\mathbf{B}^{-1}\mathbf{b} \mid \mathbf{B}^{-1}A] - g_i\mathbf{e}_\ell^T[\mathbf{B}^{-1}\mathbf{b} \mid \mathbf{B}^{-1}A]$$

In the full tableau implementation, this operation is that we update the ith row of the tableau $[\mathbf{B}^{-1}\mathbf{b} \mid \mathbf{B}^{-1}A]$ by adding to it a multiple g_i of the ℓth row (pivot row) to set all entries of the pivot column to zero, with the exception of the pivot element which is set to one.

3.17

Solve completely (i.e., both Phase I and Phase II) via the simplex method the following problem:

$$\begin{aligned} &\textit{minimize} \quad 2x_1 + 3x_2 + 3x_3 + x_4 - 2x_5 \\ &\textit{subject to} \quad x_1 + 3x_2 + 4x_4 + x_5 = 2 \\ &\qquad x_1 + 2x_2 - 3x_4 + x_5 = 2 \\ &\qquad - x_1 - 4x_2 + 3x_3 = 1 \\ &\qquad x_1, \dots, x_5 \geq 0. \end{aligned}$$

Solution.
Phase I.
In order to find a feasible solution, we form the auxiliary problem

$$\begin{aligned} &\textit{minimize} \quad x_6 + x_7 + x_8 \\ &\textit{subject to} \quad x_1 + 3x_2 + 4x_4 + x_5 + x_6 = 2 \\ &\qquad x_1 + 2x_2 - 3x_4 + x_5 + x_7 = 2 \\ &\qquad - x_1 - 4x_2 + 3x_3 + x_8 = 1 \\ &\qquad x_1, \dots, x_5, x_6, x_7, x_8 \geq 0. \end{aligned}$$

Starting with the basic feasible solution $\mathbf{x} = (0, 0, 0, 0, 0, 2, 2, 1)^T$, we have the following tableau:

-5	-1	-1	-3	-1	-2	0	0	0
$x_6 = 2$	1	3	0	4	1	1	0	0
$x_7 = 2$	1	2	0	-3	1	0	1	0
$x_8 = 1$	-1	-4	3	0	0	0	0	1

Choose x_3 as the entering variable and x_8 as the leaving variable. Then we have the following tableau:

-4	-2	-5	0	-1	-2	0	0	1
$x_6 = 2$	1	3	0	4	1	1	0	0
$x_7 = 2$	1	2	0	-3	1	0	1	0
$x_3 = 1/3$	$-1/3$	$-4/3$	1	0	0	0	0	$1/3$

Choose x_1 as the entering variable and x_6 as the leaving variable. Then we have the following tableau:

$$
\begin{array}{c|ccccc|ccc}
0 & 0 & 1 & 0 & 7 & 0 & 2 & 0 & 1 \\
\hline
x_1 = 2 & 1 & 3 & 0 & 4 & 1 & 1 & 0 & 0 \\
x_7 = 0 & 0 & -1 & 0 & -7 & 0 & -1 & 1 & 0 \\
x_3 = 1 & 0 & -1/3 & 1 & 4/3 & 1/3 & 1/3 & 0 & 1/3
\end{array}
$$

Choose x_2 as the entering variable and x_7 as the leaving variable. Then we have the following tableau:

$$
\begin{array}{c|ccccc|ccc}
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
\hline
x_1 = 2 & 1 & 0 & 0 & -17 & 1 & -2 & 3 & 0 \\
x_2 = 0 & 0 & 1 & 0 & 7 & 0 & 1 & -1 & 0 \\
x_3 = 1 & 0 & 0 & 1 & 11/3 & 1/3 & 2/3 & -1/3 & 1/3
\end{array}
$$

Since the cost of the auxiliary problem has dropped to zero, which indicates that the original problem is feasible, and all of the artificial variables have been removed, we can move on to the second phase.

Phase II.

Starting with the basic feasible solution $\mathbf{x} = (2, 0, 1, 0, 0)^T$, we have the following tableau:

$$
\begin{array}{c|ccccc}
-7 & 0 & 0 & 0 & 3 & -5 \\
\hline
x_1 = 2 & 1 & 0 & 0 & -17 & 1 \\
x_2 = 0 & 0 & 1 & 0 & 7 & 0 \\
x_3 = 1 & 0 & 0 & 1 & 11/3 & 1/3
\end{array}
$$

Choose x_5 as the entering variable and x_1 as the leaving variable. Then we have the following tableau:

$$
\begin{array}{c|ccccc}
3 & 5 & 0 & 0 & -82 & 0 \\
\hline
x_5 = 2 & 1 & 0 & 0 & -17 & 1 \\
x_2 = 0 & 0 & 1 & 0 & 7 & 0 \\
x_3 = 1/3 & -1/3 & 0 & 1 & 28/3 & 0
\end{array}
$$

Choose x_4 as the entering variable and x_2 as the leaving variable. Then we have the following tableau:

$$
\begin{array}{c|cccccc}
3 & 5 & 82/7 & 0 & 0 & 0 \\
\hline
x_5 = 2 & 1 & 17/7 & 0 & 0 & 17 \\
x_4 = 0 & 0 & 1/7 & 0 & 1 & 0 \\
x_3 = 1/3 & -1/3 & -4/3 & 1 & 0 & 0
\end{array}
$$

Since all the reduced costs are nonnegative, the optimal solution is $\mathbf{x} = (0, 0, 1/3, 0, 2)^T$ and the optimal value is -3.