LP Homework 1

吴嘉骜 21307130203

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Let matrix $A = [a_{ij}]$, a saddle point (i^*, j^*) of A satisfies
$A_{i^*j} \geq A_{i^*j^*} \geq A_{ij^*}, \forall (i, j)$.
Prove: (a) $\max\limits_i \min\limits_j A_{ij} = \min\limits_j \max\limits_i A_{ij}$ has a solution (i^*, j^*) if and only if A has a saddle point (i^*, j^*).
(b) If $(i^*, j^*) \neq (s^*, t^*)$ are both saddle points of A, then (i^*, t^*) is also a saddle point, with $A_{i^*j^*} = A_{i^*t^*}$.

Proof.
(a)If A has a saddle point (i^*, j^*), then by definition,
$\min\limits_j \max\limits_i A_{ij} \leq \max\limits_i A_{ij^*} = A_{i^*j^*} = \min\limits_j A_{i^*j} \leq \max\limits_i \min\limits_j A_{ij}$
From Weak Duality we have $\max\limits_i \min\limits_j A_{ij} \leq \min\limits_j \max\limits_i A_{ij}$
Thus, $\max\limits_i \min\limits_j A_{ij} = \min\limits_j \max\limits_i A_{ij} = A_{i^*j^*}$.
If there exists (i^*, j^*) such that $\max\limits_i \min\limits_j A_{ij} = \min\limits_j \max\limits_i A_{ij} = A_{i^*j^*}$,
from $\min\limits_j \max\limits_i A_{ij} = A_{i^*j^*}$ we know that $\max\limits_i A_{ij^*} = A_{i^*j^*}$;
from $\max\limits_i \min\limits_j A_{ij} = A_{i^*j^*}$ we know that $\min\limits_j A_{i^*j} = A_{i^*j^*}$.
Since $A_{ij^*} \leq \max\limits_i A_{ij^*} = A_{i^*j^*} = \min\limits_j A_{i^*j} \leq A_{i^*j}, \forall (i, j)$,
(i^*, j^*) is a saddle point.
(b)By definition,we have:
$A_{i^*j^*} \leq A_{i^*t^*} \leq A_{s^*t^*}$, $A_{i^*j^*} \geq A_{s^*j^*} \geq A_{s^*t^*}$,
So $A_{i^*j^*} = A_{s^*t^*} = A_{i^*t^*} = A_{s^*j^*}$.
We could have $A_{it^*} \leq A_{s^*t^*} = A_{i^*t^*} = A_{i^*j^*} \leq A_{i^*t}, \forall (i, t)$, which means
(i^*, t^*) is also a saddle point.

2

Prove that the LP General Form

$$\min_{s.t. Ax \leq b} c^T x \tag{1}$$

is equivalent to the LP Standard Form

$$\min_{s.t. \substack{Ax=b \\ x \geq 0}} c^T x \tag{2}$$

Proof.
(a) If we have (2), then we could rewrite the requirements as the equivalent form: $Ax \leqslant b, -Ax \leqslant -b, -x \leq 0$, which is the general form.
(b) If we have (1), we prove that it is equivalent to the problem:

$$\min_{s.t. \substack{A(x_1 - x_2) + s = b \\ x_1, x_2, s \geq 0}} c^T (x_1 - x_2) \tag{3}$$

Let $\mathcal{F}_1, \mathcal{F}_3$ be the feasible sets of (1) and (3) respectively. Assume that both of them are not $\varnothing$.
(b.1) Let x^* be optimal to (1), then (x_1^*, x_2^*, s^*) is optimal to (3), where $x^* = x_1^* - x_2^*, s = b^* - Ax^*$.
Otherwise, consider $(x_1', x_2', s') \in \mathcal{F}_3$, with $c^T(x_1' - x_2') < c^T(x_1^* - x_2^*) = c^T x^*$. From $s' \geq 0$, we have $A(x_1' - x_2') \leq b$. We then choose $x' = x_1' - x_2'$, $x' \in \mathcal{F}_1$. Therfore, $c^T x' < c^T x^*$, which contradicts x^* being optimal to (1).
(b.2) Let (x_1^*, x_2^*, s^*) be optimal to (3), then $x^* = x_1^* - x_2^*$ is optimal to (1).(It's obvious that $x^* \in \mathcal{F}_1$)
Otherwise, consider $x' \in \mathcal{F}_1$, and let $x_1' = \frac{|x'| + x'}{2}, x_2' = \frac{|x'| - x'}{2}, s' = b - Ax'$, then $(x_1', x_2', s') \in \mathcal{F}_3$.
We have $c^T x' < c^T x^*$, i.e. $c^T(x_1' - x_2') < c^T(x_1^* - x_2^*)$, which contradicts (x_1^*, x_2^*, s^*) being optimal to (3).
By (b.1) and (b.2), we show that (1) and (3) are equivalent, and have the same optimal value.
Since (3) is in the form of (2), we can also conclude that from (1) we can get (2).

3

Prove that the 1-norm objective optimization problem

$$\min_{s.t. Ax=b} \parallel x \parallel_1 \tag{4}$$

is equivalent to

$$\min_{\substack{s.t. A(x^1-x^2)=b \\ x^1,x^2 \geq 0}} \sum_{i=1}^{n} (x_i^1 + x_i^2) \tag{5}$$

Proof.
Assume that the two feasible sets $\mathcal{F}_4, \mathcal{F}_5 \neq \varnothing$.
(a) We prove that if x^* is optimal to (4), then (x^{*1}, x^{*2}) is optimal to (5), where $x_i^{*1} = \max\{x_i^*, 0\}$ and $x_i^{*2} = \max\{-x_i^*, 0\}$.
Assume that $\exists (\hat{x^1}, \hat{x^2}) \in \mathcal{F}_5$,

$$\sum_{i=1}^{n} (\hat{x_i^1} + \hat{x_i^2}) < \sum_{i=1}^{n} (x_i^{*1} + x_i^{*2}) \tag{6}$$

Let $\hat{x} = \hat{x^1} - \hat{x^2} \in \mathcal{F}_4$, and we assert that at least one of $\hat{x_i^1}$ and $\hat{x_i^2}$ is zero for $i \in \{1, 2, \cdots, n\}$.
Or, for $\hat{x_i^1} \geq \hat{x_i^2} \geq 0$, assign $\hat{x_i^1} - \hat{x_i^2}$ to $\hat{x_i^1}$, and set $\hat{x_i^2}$ as 0. $\hat{x^1}, \hat{x^2}$ are still feasible, but $\sum\limits_{i=1}^{n} (\hat{x_i^1} + \hat{x_i^2})$ gets smaller.
Thus, we have $|\hat{x}_i| = |\hat{x_i^1} - \hat{x_i^2}| = \hat{x_i^1} + \hat{x_i^2}$, for all i.
From (6) we get $\sum\limits_{i=1}^{n} \hat{x}_i < \sum\limits_{i=1}^{n} x_i^*$, which contradicts x^* being optimal to (4).
(b) If (x^{1*}, x^{2*}) is optimal to (5), then by (a) we know that there is at least one zero in x_i^{1*} and x_i^{2*}, for all i.
We assert that $x^* = x^{1*} - x^{2*}$ is optimal to (4), and we know that $x_i^{1*} = \max\{x_i^*, 0\}$ and $x_i^{*2} = \max\{-x_i^*, 0\}$, $|x_i^*| = x_i^{1*} + x_i^{2*}$.
Otherwise, $\exists x' \in \mathcal{F}_4$,

$$\parallel x' \parallel_1 < \parallel x^* \parallel_1 \tag{7}$$

Let $x_i^{'1} = \max\{x_i', 0\}$, $x_i^{'2} = \max\{-x_i', 0\}$. Then $(x^{'1}, x^{'2}) \in \mathcal{F}_5$.
We get from (7) that $\sum\limits_{i=1}^{n} (x_i^{'1} + x_i^{'2}) < \sum\limits_{i=1}^{n} (x_i^{1*} + x_i^{2*})$, which contradicts (x^{1*}, x^{2*}) being optimal to (5).