LP Homework 3

吴嘉骜 21307130203

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2.9

Consider the standard form polyhedron $\{\mathbf{x}|\mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$, and assume that the rows of the matrix $\mathbf{A}$ are linearly independent.
(a) Suppose that two different bases lead to the same basic solution. Show that the basic solution is degenerate.
(b) Consider a degenerate basic solution. Is it true that it corresponds to two or more distinct bases? Prove or give a counterexample.
(c) Suppose that a basic solution is degenerate. Is it true that there exists an adjacent basic solution which is degenerate? Prove or give a counterexample.
Solution.
(a) Assume that the basic solution is nondegenerate. Suppose its entries are zero at $\{1, \dots, n\} \backslash \{B(1), \dots, B(m)\}$. Then it has exactly m nonzero components at $\{B(1), \dots, B(m)\}$. Therefore, it is associated with the unique base $[A_{B(1)}, \dots, A_{B(m)}]$, which constraints that there exist two different bases with the same basic solution.
(b) No, it is false. Consider a polyhedron defined by the following constraints:

$$\left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 \middle| \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \geq 0 \right\}$$

A_2 and A_3 are linearly dependent. Thus, we have only two distinct bases $[A_1, A_2]$ with $[0, 1, 0]^T$ as its basic solution and $[A_1, A_3]$ with $[0, 0, \frac{1}{2}]^T$ as its

basic solution. Both of the two basic solutions are degenerate, yet correspond to distinct base. Thus, a degenerate solution can correspond to a unique base under certain cases.
(c) No, it is false.Consider a polyhedron defined by the following constraints:

$$\left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 \middle| \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \geq 0 \right\}$$

Any two of A_1, A_2 and A_3 are linearly independent. Thus, we have three distinct bases $[A_1, A_2]$ with $\mathbf{u} = [0, 1, 0]^T$ as its basic solution, $[A_2, A_3]$ with $\mathbf{v} = [0, 1, 0]^T$, and $[A_1, A_3]$ with $\mathbf{w} = [1, 0, 1]^T$. Since $\mathbf{u}$ and $\mathbf{v}$ are the same basic solution, they are not adjacent. It is easy to see that $\mathbf{u}$ and $\mathbf{w}$ are adjacent, because there are two common equality constraints which are linearly independent. However, $\mathbf{w}$ is nondegenerate, so for $\mathbf{u}$ we cannot find an adjacent degenerate basic solution.

2.12

Consider a nonempty polyhedron P and suppose that for each variable x_i we have either the constraint $x_i \geq 0$ or the constraint $x_i \leq 0$. Is it true that P has at least one basic feasible solution?
Solution.
Yes, P has at least one basic feasible solution. Let $P = \{\mathbf{x} \in \mathbb{R}^n | \mathbf{A}\mathbf{x} \leq \mathbf{b}\}$. We will see that there are n linearly independent rows in A, which is equivalent to the former assertation. In fact,

$$\begin{bmatrix} \pm 1 & 0 & \cdots & 0 \\ 0 & \pm 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \pm 1 \end{bmatrix}$$

must be a $n \times n$ submatrix of A, and its n rows are linearly independent, so there are n linearly independent rows in A.

2.13

Consider the standard form polyhedron $P = \{\mathbf{x}|\mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$. Suppose that the matrix $\mathbf{A}$, of dimensions $m \times n$, has linearly independent rows, and that all basic feasible solutions are nondegenerate. Let $\mathbf{x}$ be an element of P that has exactly m positive components.

(a) Show that $\mathbf{x}$ is a basic feasible solution.

(b) Show that the result of part (a) is false if the nondegeneracy assumption is removed.

Proof.

(a) Let $B(1), \ldots, B(m)$ denote the indices of the m positive components of $\mathbf{x}$. Since $\mathbf{x} \in P$, it is feasible. To show that $\mathbf{x}$ is a basic feasible solution, it suffices to argue that the columns $A_{B(1)}, \ldots, A_{B(m)}$ are linearly independent, together with $x_i = 0$ for $i \neq B(1), \ldots, B(m)$. The latter is true by assumption.

Suppose that $A_{B(1)}, \ldots, A_{B(m)}$ are linearly dependent. Without loss of generality, we may assume that $A_{B(m)} = k_1 A_{B(1)} + \ldots + k_{m-1} A_{B(m-1)}$,where $k_1, \ldots, k_{m-1} \in \mathbb{R}$. Then from $\sum\limits_{i=1}^{m} x_{B(i)} A_{B(i)} = b$ we have $\sum\limits_{i=1}^{m-1} x_{B(i)} A_{B(i)} + x_{B(m)}(k_1 A_{B(1)} + \ldots + k_{m-1} A_{B(m-1)}) = b$, i.e.

$$\begin{bmatrix} A_{B(1)} & \cdots & A_{B(m-1)} & 0 \end{bmatrix} \begin{bmatrix} x_{B(1)} + k_1 x_{B(m)} \\ \vdots \\ x_{B(m-1)} + k_{m-1} x_{B(m)} \\ 0 \end{bmatrix} = b$$

If $A_{B(1)}, \ldots, A_{B(m-1)}$ are linearly independent, then by $rank(\mathbf{A}) = m$, we can find another column $A_j, j \neq B(1), \ldots, B(m)$, such that $A_{B(1)}, \ldots, A_{B(m-1)}, A_j$ are linearly independent. Thus, $\begin{bmatrix} x_{B(1)} + k_1 x_{B(m)} \\ \vdots \\ x_{B(m-1)} + k_{m-1} x_{B(m)} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ is a degenerate basic solution associated with the base $\begin{bmatrix} A_{B(1)} & \cdots & A_{B(m-1)} & A_j \end{bmatrix}$, which contradicts to the fact that all basic feasible solutions in P are non-degenerate.

If $A_{B(1)}, \ldots, A_{B(m-1)}$ are still linearly dependent, repeat the procedure above until a linearly independent set of columns is achieved. Then add the remaining columns from $\{1, \ldots, n\} \setminus \{B(1), \ldots, B(m)\}$ to get a linearly independent basis and the corresponding degenerate basic solution, where contradiction lies.

(b) For a counterexample, consider a polyhedron defined by the following constraints:

$$\left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 \middle| \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \geq 0 \right\}$$

Then the vector $[1, 0, 1]^T$ is an element of P with exactly 2 positive components, but it is not a basic solution because A_1 and A_3 are linearly dependent. In addition, there exist a vector $[3, 0, 0]^T$ that is a basic feasible degenerate solution.

2.17

Consider the polyhedron $\{\mathbf{x} \in \mathbb{R}^n | \mathbf{A}\mathbf{x} \leq \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$ and a nondegenerate basic feasible solution $\mathbf{x}^*$. We introduce slack variables $\mathbf{z}$ and construct a corresponding polyhedron $\{(\mathbf{x}, \mathbf{z}) | \mathbf{A}\mathbf{x} + \mathbf{z} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}, \mathbf{z} \geq \mathbf{0}\}$ in standard form. Show that $(\mathbf{x}^*, \mathbf{b} - \mathbf{A}\mathbf{x}^*)$ is a nondegenerate basic feasible solution for the new polyhedron.

Proof.

It is clear that $p = (\mathbf{x}^*, \mathbf{b} - \mathbf{A}\mathbf{x}^*)$ is in the new polyhedron, so we have feasibility.

Without loss of generality, we may assume that after reordering the inequalities of $\mathbf{A}\mathbf{x} \leq \mathbf{b}$, $a_i^T x = b_i$ for $i = 1, \ldots, k$ are the active constraints. Let $N(1), \ldots, N(l)$ are the indices of zero components in $\mathbf{x}^*$. Let $\mathbf{z}^*$ denote $\mathbf{b} - \mathbf{A}\mathbf{x}^*$. Then, we have $z_1^*, \ldots, z_k^* = 0$.

For any $(\mathbf{x}, \mathbf{z})$ satisfying $\mathbf{A}\mathbf{x} + \mathbf{z} = \mathbf{b}$, $z_i = 0$ for $i = 1, \ldots, k$ and $x_j = 0$ for $j = N(1), \ldots, N(l)$, we must have $\mathbf{x}$ satisfying $a_i^T x = b_i$ for $i = 1, \ldots, k$

since $z_j = 0$ for $j = 1, \dots, k$. Therefore, $\mathbf{x}$ is a solution to $a_i^T x = b_i$ for $i = 1, \dots, k$ and $x_j = 0$ for $j = N(1), \dots, N(l)$. This implies that $\mathbf{x}$ must be $\mathbf{x}^*$ since $\mathbf{x}^*$ is a basic feasible solution, leading to the uniqueness. It follows that $\mathbf{z} = \mathbf{b} - \mathbf{A}\mathbf{x}^*$. Hence, the original system has a unqiue solution, implying that p is a basic solution.

Since $\mathbf{x}^*$ is nondegenerate and baisc, the number of active constraints at it is exactly n, namely $k + l = n$. Notice that the number of active constraints at p is exactly $m + k + l = m + n$, which implies that $(\mathbf{x}^*, \mathbf{b} - \mathbf{A}\mathbf{x}^*)$ is nondegenerate.

In sum, we demonstrate that $(\mathbf{x}^*, \mathbf{b} - \mathbf{A}\mathbf{x}^*)$ is a nondegenerate basic feasible solution for the new polyhedron.