LP Homework 5

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3.19

While solving a standard form problem, we arrive at the following tableau, with x_3, x_4 and x_5 being the basic variables:

$$\begin{array}{c|ccccc} -10 & \delta & -2 & 0 & 0 & 0 \\ \hline 4 & -1 & \eta & 1 & 0 & 0 \\ 1 & \alpha & -4 & 0 & 1 & 0 \\ \beta & \gamma & 3 & 0 & 0 & 1 \end{array}$$

The entries $\alpha, \beta, \gamma, \delta, \eta$ in the tableau are unknown parameters. For each one of the following statements, find some parameter values that will make the statement true.
(a) The current solution is optimal and there are multiple optimal solutions.
(b) The optimal cost is $-\infty$.
(c) The current solution is feasible but not optimal.
Solution.
(a) Since the current solution is optimal, yet the reduced cost $\overline{c}_2$ is negative, the current solution must be degenerate, indicating $\beta = 0$. In the next iteration, we will choose the x_2 to enter the basis and x_5 to exit. Let $\beta + \frac{2}{3}\gamma \geq 0$ and then the next tableau will be optimal. For the optimal solutions to be multiple, we may set $\delta = 0$ and $\alpha > 0$. Then we can make x_4 an exiting variable and x_1 an entering variable without changing the

optimal cost since $\delta = 0$. The following tableau could be a possible choice:

$$\begin{array}{r|ccccc} -10 & 0 & -2 & 0 & 0 & 0 \\ \hline 4 & -1 & 0 & 1 & 0 & 0 \\ 1 & 1 & -4 & 0 & 1 & 0 \\ 0 & 0 & 3 & 0 & 0 & 1 \end{array}$$

The next tableau will be:

$$\begin{array}{r|ccccc} -10 & 0 & 0 & 0 & 0 & 2/3 \\ \hline 4 & -1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 4/3 \\ 0 & 0 & 1 & 0 & 0 & 1/3 \end{array}$$

Adding x_1 to the basis and removing x_4 from the basis, we get the following tableau:

$$\begin{array}{r|ccccc} -10 & 0 & 0 & 0 & 0 & 2/3 \\ \hline 5 & 0 & 0 & 1 & 0 & 4/3 \\ 1 & 1 & 0 & 0 & 1 & 4/3 \\ 0 & 0 & 1 & 0 & 0 & 1/3 \end{array}$$

Both $\mathbf{x} = (0, 0, 4, 1, 0)^T$ and $\mathbf{x} = (1, 0, 5, 0, 0)^T$ are optimal.

(b) For feasibility, we must demand $\beta \geq 0$. Since the problem is unbounded if no components of an exiting column are positive; by setting $\delta < 0$ and $\alpha, \gamma \leq 0$, we can make x_1 an exiting variable and the optimal cost $-\infty$. The following tableau could be a possible choice:

$$\begin{array}{r|ccccc} -10 & -3 & -2 & 0 & 0 & 0 \\ \hline 4 & -1 & -1 & 1 & 0 & 0 \\ 1 & -1 & -4 & 0 & 1 & 0 \\ 1 & -1 & 3 & 0 & 0 & 1 \end{array}$$

(c) For feasibility, we must demand $\beta \geq 0$. Since the current solution is not optimal, we may set $\beta > 0$ and then move in the direction of x_2 to get a lower cost.The following tableau could be a possible choice:

$$\begin{array}{r|ccccc} -10 & 0 & -2 & 0 & 0 & 0 \\ \hline 4 & -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & -4 & 0 & 1 & 0 \\ 3 & 0 & 3 & 0 & 0 & 1 \end{array}$$

In the next iteration, we will choose the x_2 to enter the basis and x_5 to exit. The next tableau will be:

$$
\begin{array}{c|ccccc}
-8 & 0 & 0 & 0 & 0 & 2/3 \\
\hline
4 & -1 & 0 & 1 & 0 & 0 \\
5 & 0 & 0 & 0 & 1 & 4/3 \\
1 & 0 & 1 & 0 & 0 & 1/3
\end{array}
$$

which is the optimal situation.

3.22

Consider the following linear programming problem with a single constraint:

$$
\begin{aligned}
\text{maximize} \quad & \sum_{i=1}^{n} c_i x_i \\
\text{subject to} \quad & \sum_{i=1}^{n} a_i x_i = b \\
& x_i \geq 0, \quad i = 1, \ldots, n.
\end{aligned}
$$

(a) Derive a simple test for checking the feasibility of this problem.
(b) Assuming that the optimal cost is finite, develop a simple method for obtaining an optimal solution directly.
Solution.
(a)
1. If $b = 0$, then the problem is feasible because $\mathbf{x} = \mathbf{0}$ is feasible.
2. If $b > 0$, then we assert that the problem is feasible if and only if there exists $a_j > 0$ for some $j \in [1, n]$.
Proof.
$\Rightarrow$ If the problem is feasible, then there exists $\mathbf{x} \geq 0$ such that $\sum\limits_{i=1}^{n} a_i x_i = b$.
If $a_j \leq 0$ for all $j \in [1, n]$, then $\sum\limits_{i=1}^{n} a_i x_i \leq 0 < b$, which is a contradiction.
$\Leftarrow$ If there exists $a_j > 0$ for some $j \in [1, n]$, then we can set $x_j = b/a_j$ and $x_i = 0$ for all $i \neq j$. Then $\mathbf{x}$ is feasible.
3. If $b < 0$, then we assert that the problem is feasible if and only if there exists $a_j < 0$ for some $j \in [1, n]$.

The proof is similar to the case $b > 0$.

(b)

Since the optimal cost is finite and the constraint set is a standard form polyhedron, there exists an optimal solution that is a basic feasible solution. By $m = 1$, we know that every BFS has at most one nonzero component.

1. If $b = 0$, then we assert that the optimal cost is 0.

Proof.

Let $\mathbf{x} = \mathbf{0}$, we obtain a cost of 0. If the optimal cost is not 0, then there exists a BFS $\mathbf{x}^*$ such that $\sum_{i=1}^{n} c_i x_i^* < 0$. Since $\mathbf{x}^*$ has at most one nonzero component, it then has exactly one nonzero component, say $x_k^* > 0$ for some $k \in [1, n]$. Then we have $a_k = 0$ and $c_k < 0$. Consider $\mathbf{x} = \alpha \mathbf{e}_k$, $\alpha > 0$, where $\mathbf{e}_k$ is the kth unit vector and feasibility satisfies. For α sufficiently large, the optimal cost $\sum_{i=1}^{n} c_i x_i = \alpha c_k$ is unbounded from below, which is a contradiction.

2. If $b > 0$, then every BFS has exactly one nonzero component. Let $k := \arg\min_{i=1,\dots,n}\{\frac{c_i b}{a_i} \mid a_i > 0\}$, then an optimal solution is $\mathbf{x} = (0, \dots, 0, b/a_k, 0, \dots, 0)$.

3. If $b < 0$, then every BFS has exactly one nonzero component. Let $k := \arg\min_{i=1,\dots,n}\{\frac{c_i b}{a_i} \mid a_i < 0\}$, then an optimal solution is $\mathbf{x} = (0, \dots, 0, b/a_k, 0, \dots, 0)$.

3.26

(The big-M method) Consider the variant of the big-M method in which M is treated as an undetermined large parameter. Prove the following.

(a) If the simplex method terminates with a solution $(\mathbf{x}, \mathbf{y})$ for which $\mathbf{y} = \mathbf{0}$, then $\mathbf{x}$ is an optimal solution to the original problem.

(b) If the simplex method terminates with a solution $(\mathbf{x}, \mathbf{y})$ for which $\mathbf{y} \neq \mathbf{0}$, then the original problem is infeasible.

(c) If the simplex method terminates with an indication that the optimal cost in the auxiliary problem is $-\infty$, show that the original problem is either infeasible or its optimal cost is $-\infty$. *Hint*: When the simplex method terminates, it has discovered a feasible direction $\mathbf{d} = (\mathbf{d}_x, \mathbf{d}_y)$ of cost decrease. Show that $\mathbf{d}_y = \mathbf{0}$.

(d) Provide examples to show that both alternatives in part (c) are possible.

Proof.

The original problem can be written as:

$$\begin{aligned} \textit{minimize} \quad & \mathbf{c}^T\mathbf{x} \\ \textit{subject to} \quad & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq 0. \end{aligned}$$

The auxiliary problem can be written as:

$$\begin{aligned} \textit{minimize} \quad & \mathbf{c}^T\mathbf{x} + M\sum_{i=1}^{n} y_i \\ \textit{subject to} \quad & \mathbf{A}\mathbf{x} + \mathbf{y} = \mathbf{b} \\ & \mathbf{x} \geq 0. \\ & \mathbf{y} \geq 0. \end{aligned}$$

(a) Assume that there exists a feasible primal solution $\mathbf{z}$ such that $\mathbf{c}^T\mathbf{z} < \mathbf{c}^T\mathbf{x}$. It is obvious that $(\mathbf{z}, \mathbf{0})$ is also feasible for the auxiliary problem. Then we have $\mathbf{c}^T\mathbf{z} + M \cdot 0 < \mathbf{c}^T\mathbf{x} + M \cdot 0$, which is a contradiction to the fact that $(\mathbf{x}, \mathbf{0})$ is optimal to the auxiliary problem.

(b) Assume that the big-M method terminates with a solution $(\mathbf{x}, \mathbf{y})$ with $\mathbf{y} \neq \mathbf{0}$, whereas the original problem has a feasible point $\mathbf{z}$. It is obvious that $(\mathbf{z}, \mathbf{0})$ is also feasible for the auxiliary problem. Then we have $\mathbf{c}^T\mathbf{z} + M \cdot 0 < \mathbf{c}^T\mathbf{x} + M \cdot \sum_{i=1}^{n} y_i$ for M sufficiently large and $\mathbf{y} \neq \mathbf{0}$, which is a contradiction to the fact that $(\mathbf{x}, \mathbf{y})$ is optimal to the auxiliary problem.

(c) Assume that the original problem has an optimal solution $\mathbf{z} \in \mathbb{R}^n$. When the big-$M$ method terminates with an indication that the optimal cost is $-\infty$, it has discovered a feasible direction $\mathbf{d} = (\mathbf{d}_x, \mathbf{d}_y)$ of cost decrease

$$\mathbf{c}^T\mathbf{d}_x + M\sum_{i=1}^{n} d_{yi} < 0$$

such that $(\mathbf{x}, \mathbf{y}) + \theta(\mathbf{d}_x, \mathbf{d}_y)$ can be feasible for the auxiliary problem for any $\theta > 0$, which indicates that $\mathbf{d} \geq \mathbf{0}$ with at least one nonzero component.

We assert that $\mathbf{d}_y$ must be $\mathbf{0}$, otherwise we could take M large enough to make $\mathbf{c}^T\mathbf{d}_x + M\sum_{i=1}^{n} d_{yi} > 0$. Then we immediately have $\mathbf{c}^T\mathbf{d}_x < 0$.

Notice that $\mathbf{d}_x$ is a feasible direction of cost decrease for the original problem,

because $\mathbf{A}(\mathbf{x}+\mathbf{d}_x)+(\mathbf{y}+\mathbf{d}_y)=\mathbf{b} \Rightarrow \mathbf{A}\mathbf{d}_x=\mathbf{b}-(\mathbf{A}\mathbf{x}+\mathbf{y})=\mathbf{0}$. Then we have $\mathbf{c}^T\mathbf{d}_x<0$, which is a contradiction to the fact that $\mathbf{z}$ is an optimal solution.
(d) First consider an infeasible original problem:

$$\begin{aligned} &\textit{minimize} && x_1-2x_2 \\ &\textit{subject to} && -x_1-x_3=1 \\ & && -x_1+x_3=1. \\ & && \mathbf{x}\geq 0. \end{aligned}$$

Its big-M problem:

$$\begin{aligned} &\textit{minimize} && x_1-2x_2+M(y_1+y_2) \\ &\textit{subject to} && -x_1-x_3+y_1=1 \\ & && -x_1+x_3+y_2=1. \\ & && \mathbf{x}\geq 0. \\ & && \mathbf{y}\geq 0. \end{aligned}$$

The initial tableau of the big-M problem is:

$$\begin{array}{c|ccccc} -2M & 1+2M & -2 & 0 & 0 & 0 \\ \hline 1 & -1 & 0 & -1 & 1 & 0 \\ 1 & -1 & 0 & 1 & 0 & 1 \end{array}$$

The second column is cost-reducing while none of its components is positive; thus, the big-M problem is unbounded.
Then consider an unbounded original problem:

$$\begin{aligned} &\textit{minimize} && -x_1-x_2 \\ &\textit{subject to} && x_1-x_2=1 \\ & && x_1,x_2\geq 0. \end{aligned}$$

Its big-M problem:

$$\begin{aligned} &\textit{minimize} && -x_1-x_2+My_1 \\ &\textit{subject to} && x_1-x_2+y_1=1 \\ & && x_1,x_2,y_1\geq 0. \end{aligned}$$

The initial tableau of the big-M problem is:

$$\begin{array}{c|ccc}
-M & -1-M & -1+M & 0 \\
\hline
1 & 1 & -1 & 1
\end{array}$$

Let y_1 exit the basis and x_1 enter the basis, we have:

$$\begin{array}{c|ccc}
1 & 0 & -2 & 1+M \\
\hline
1 & 1 & -1 & 1
\end{array}$$

The second column is cost-reducing while none of its components is positive; thus, the big-M problem is unbounded.

3.28

Consider a linear programming problem in standard form with a bounded feasible set. Furthermore, suppose that we know the value of a scalar U such that any feasible solution satisfies $x_i \leq U$, for all i. Show that the problem can be transformed into an equivalent one that contains the constraint $\sum_{i=1}^{n} x_i = 1$.
Proof.
The original problem can be written as:

$$\begin{aligned}
&\textit{minimize} && \mathbf{c}^T\mathbf{x} \\
&\textit{subject to} && \mathbf{A}\mathbf{x} = \mathbf{b} \\
& && \mathbf{x} \geq 0
\end{aligned}$$

Notice that from $x_i \leq U$ for all i we have $\frac{1}{nU}\sum_{i=1}^{n} x_i \leq 1$. Introduce the new variable x_{n+1} and the constraint turns into an equation $\sum_{i=1}^{n} \frac{1}{nU}x_i + x_{n+1} = 1$. We now show that the following problem with $n+1$ variables is equivalent to the original problem:

$$\begin{aligned} & \textit{minimize} \quad \tilde{\mathbf{c}}^T \tilde{\mathbf{x}} \\ & \textit{subject to} \quad \tilde{\mathbf{A}}\tilde{\mathbf{x}} = \tilde{\mathbf{b}} \\ & \qquad\qquad\quad \tilde{\mathbf{x}} \geq 0 \\ & \qquad\qquad\quad \sum_{i=1}^{n+1} \tilde{x}_i = 1. \end{aligned}$$

where $\tilde{\mathbf{c}} = [nU \cdot \mathbf{c}, 0]^T$, $\tilde{\mathbf{A}} = [\mathbf{A}, \mathbf{0}]$, $\tilde{\mathbf{b}} = \frac{1}{nU}\mathbf{b}$.

1. If $\mathbf{x}$ is optimal to the original problem, then $\tilde{\mathbf{x}} = [\frac{1}{nU}\mathbf{x}, 1 - \frac{1}{nU}\sum_{i=1}^{n} x_i]^T$ is optimal to the new problem.

It is easy to verify that $\tilde{\mathbf{x}}$ is feasible for the new problem. Suppose that there exists another feasible solution $\tilde{\mathbf{x}} = [\tilde{\mathbf{x}}_1', \tilde{x}_{n+1}]^T, \tilde{\mathbf{x}}_1' \in \mathbb{R}^n$ such that $\tilde{\mathbf{c}}^T\tilde{\mathbf{x}}' < \tilde{\mathbf{c}}^T\tilde{\mathbf{x}}$. Then let $\mathbf{x}' = nU \cdot \tilde{\mathbf{x}}_1'$ which is feasible to the original problem, we have $\mathbf{c}^T\mathbf{x}' = \tilde{\mathbf{c}}^T\tilde{\mathbf{x}}' < \tilde{\mathbf{c}}^T\tilde{\mathbf{x}} = \mathbf{c}^T\mathbf{x}$, which contradicts the optimality of $\mathbf{x}$.

2. If $\tilde{\mathbf{x}} = [\tilde{\mathbf{x}}_1, \tilde{x}_{n+1}]^T, \tilde{\mathbf{x}}_1 \in \mathbb{R}^n$ is optimal to the new problem, then $\mathbf{x} = nU \cdot \tilde{\mathbf{x}}_1$ is optimal to the original problem. Suppose that there exists another feasible solution $\mathbf{x}'$ to the original problem such that $\mathbf{c}^T\mathbf{x}' < \mathbf{c}^T\mathbf{x}$. Then let $\tilde{\mathbf{x}}' = [\frac{1}{nU}\mathbf{x}', 1 - \frac{1}{nU}\sum_{i=1}^{n} x_i']^T$ which is feasible to the new problem, we have $\tilde{\mathbf{c}}^T\tilde{\mathbf{x}}' = \mathbf{c}^T\mathbf{x}' < \mathbf{c}^T\mathbf{x} = \tilde{\mathbf{c}}^T\tilde{\mathbf{x}}$, which contradicts the optimality of $\tilde{\mathbf{x}}$.

Hence, the two problems are equivalent.