LP Homework 7

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4.4

Let A be a symmetric square matrix. Consider the linear programming problem

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & Ax \geq c \\ & x \geq 0 \end{aligned}$$

Prove that if x^* satisfies $Ax^* = c$ and $x^* \geq 0$, then x^* is an optimal solution.
Proof.
The dual problem is

$$\begin{aligned} \text{maximize} \quad & c^T p \\ \text{subject to} \quad & A^T p \leq c \\ & p \geq 0 \end{aligned}$$

If x^* satisfies $Ax^* = c$ and $x^* \geq 0$, it is feasible for both primal and dual problems. Note that the cost $c^T x^*$ is the same for two problems, and this indicates by Strong Duality that x^* is an optimal solution.

4.11

Consider a linear programming problem in standard form which is infeasible, but which becomes feasible and has finite optimal cost when the last equality

constraint is omitted. Show that the dual of the original (infeasible) problem is feasible and the optimal cost is infinite.
Proof.
Consider the following linear programming problem in standard form:

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & Ax = b \\ & x \geq 0 \end{aligned} \tag{1}$$

If the last equality constraint is omitted, the problem becomes

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & \tilde{A}x \geq \tilde{b} \\ & x \geq 0 \end{aligned} \tag{2}$$

The dual problem of (1) is

$$\begin{aligned} \text{maximize} \quad & b^T p \\ \text{subject to} \quad & A^T p \leq c \end{aligned} \tag{3}$$

The dual problem of (2) becomes

$$\begin{aligned} \text{maximize} \quad & \tilde{b}^T \tilde{p} \\ \text{subject to} \quad & \tilde{A}^T \tilde{p} \leq c \end{aligned} \tag{4}$$

Since (2) is feasible with finite optimal cost, so is the dual problem (4).
Let $\tilde{p} \in \mathbb{R}^{m-1}$ be a feasible solution of (4), then we have $p = [\tilde{p}; 0] \in \mathbb{R}^m$ is a feasible solution of (3), since $p^T A = [\tilde{p}^T, 0][\tilde{A}; a_m^T] = \tilde{p}^T \tilde{A} \leq c^T$.
Due to the infeasibliity of (1) and Strong Duality, the dual problem (3) cannot have a finite optimal solution.
However, it is feasible, which means the optimal cost of (3) is infinite.

4.25

This exercise shows that if we bring the dual problem into standard form and then apply the primal simplex method, the resulting algorithm is not identical to the dual simplex method. Consider the following standard form problem and its dual.

$$\begin{aligned} \text{minimize} \quad & x_1 + x_2 \\ \text{subject to} \quad & x_1 = 1 \\ & x_2 = 1 \\ & x_1, x_2 \geq 0 \end{aligned}$$

$$\begin{aligned} \text{maximize} \quad & p_1 + p_2 \\ \text{subject to} \quad & p_1 \leq 1 \\ & p_2 \leq 1 \end{aligned}$$

Here, there is only one possible basis and the dual simplex method must terminate immediately. Show that if the dual problem is converted into standard form and the primal simplex method is applied to it, one or more changes of basis may be required.
Proof.
Let $p_1 = s_1 - s_2$, $p_2 = s_3 - s_4$, the dual problem in standard form is:

$$\begin{aligned} \text{minimize} \quad & -s_1 + s_2 - s_3 + s_4 \\ \text{subject to} \quad & s_1 - s_2 + s_5 = 1 \\ & s_3 - s_4 + s_6 = 1 \\ & s_1, s_2, s_3, s_4, s_5, s_6 \geq 0 \end{aligned}$$

Then A is

$$\begin{bmatrix} 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{bmatrix}$$

If we choose the initial basis as $\{s_5, s_6\}$, the initial tableau is:

	s_1	s_2	s_3	s_4	s_5	s_6
0	-1	1	-1	1	0	0
$s_5 = 1$	1	-1	0	0	1	0
$s_6 = 1$	0	0	1	-1	0	1

Apparently, we can change the basis to $\{s_1, s_6\}$, and the new tableau is:

	s_5	s_2	s_3	s_4	s_1	s_6
1	0	0	-1	1	1	0
$s_1 = 1$	1	-1	0	0	1	0
$s_6 = 1$	0	0	1	-1	0	1

Then we can change the basis to $\{s_1, s_3\}$, and the new tableau is:

	s_5	s_6	s_3	s_4	s_1	s_2
2	0	0	0	0	1	1
$s_1 = 1$	1	-1	0	0	1	0
$s_3 = 1$	0	0	1	-1	0	1

The optimal solution is $s_1 = 1, s_3 = 1$, and the optimal cost is -2, which means that the optimal solution of the dual problem is $p_1 = 1, p_2 = 1$, and the optimal cost is 2.

From this example, we can see that the dual simplex method and the primal simplex method are different.