problem. We saw that the changeover from Phase I to Phase II involves some delicate steps whenever some artificial variables are in the final basis constructed by the Phase I algorithm.

The simplex method is a rather efficient algorithm and is incorporated in most of the commercial codes for linear programming. While the number of pivots can be an exponential function of the number of variables and constraints in the worst case, its observed behavior is a lot better, hence the practical usefulness of the method.

3.9 Exercises

Exercise 3.1 (Local minima of convex functions) Let $f : \Re^n \mapsto \Re$ be a convex function and let $S \subset \Re^n$ be a convex set. Let $\mathbf{x}^*$ be an element of S. Suppose that $\mathbf{x}^*$ is a local optimum for the problem of minimizing $f(\mathbf{x})$ over S; that is, there exists some $\epsilon > 0$ such that $f(\mathbf{x}^*) \leq f(\mathbf{x})$ for all $\mathbf{x} \in S$ for which $\|\mathbf{x} - \mathbf{x}^*\| \leq \epsilon$. Prove that $\mathbf{x}^*$ is globally optimal; that is, $f(\mathbf{x}^*) \leq f(\mathbf{x})$ for all $\mathbf{x} \in S$.

Exercise 3.2 (Optimality conditions) Consider the problem of minimizing $\mathbf{c}'\mathbf{x}$ over a polyhedron P. Prove the following:

(a) A feasible solution $\mathbf{x}$ is optimal if and only if $\mathbf{c}'\mathbf{d} \geq 0$ for every feasible direction $\mathbf{d}$ at $\mathbf{x}$.

(b) A feasible solution $\mathbf{x}$ is the unique optimal solution if and only if $\mathbf{c}'\mathbf{d} > 0$ for every nonzero feasible direction $\mathbf{d}$ at $\mathbf{x}$.

Exercise 3.3 Let $\mathbf{x}$ be an element of the standard form polyhedron $P = \{\mathbf{x} \in \Re^n \mid \mathbf{A}\mathbf{x} = \mathbf{b},\ \mathbf{x} \geq \mathbf{0}\}$. Prove that a vector $\mathbf{d} \in \Re^n$ is a feasible direction at $\mathbf{x}$ if and only if $\mathbf{A}\mathbf{d} = \mathbf{0}$ and $d_i \geq 0$ for every i such that $x_i = 0$.

Exercise 3.4 Consider the problem of minimizing $\mathbf{c}'\mathbf{x}$ over the set $P = \big\{\mathbf{x} \in \Re^n \mid \mathbf{A}\mathbf{x} = \mathbf{b},\ \mathbf{D}\mathbf{x} \leq \mathbf{f},\ \mathbf{E}\mathbf{x} \leq \mathbf{g}\big\}$. Let $\mathbf{x}^*$ be an element of P that satisfies $\mathbf{D}\mathbf{x}^* = \mathbf{f}$ and $(\mathbf{E}\mathbf{x}^*)_i < g_i$ for all i. Show that the set of feasible directions at the point $\mathbf{x}^*$ is the set

$$\big\{\mathbf{d} \in \Re^n \mid \mathbf{A}\mathbf{d} = \mathbf{0},\ \mathbf{D}\mathbf{d} \leq \mathbf{0}\big\}.$$

Exercise 3.5 Let $P = \big\{\mathbf{x} \in \Re^3 \mid x_1 + x_2 + x_3 = 1,\ \mathbf{x} \geq \mathbf{0}\big\}$ and consider the vector $\mathbf{x} = (0, 0, 1)$. Find the set of feasible directions at $\mathbf{x}$.

Exercise 3.6 (Conditions for a unique optimum) Let $\mathbf{x}$ be a basic feasible solution associated with some basis matrix $\mathbf{B}$. Prove the following:

(a) If the reduced cost of every nonbasic variable is positive, then $\mathbf{x}$ is the unique optimal solution.

(b) If $\mathbf{x}$ is the unique optimal solution and is nondegenerate, then the reduced cost of every nonbasic variable is positive.

Exercise 3.7 (Optimality conditions) Consider a feasible solution $\mathbf{x}$ to a standard form problem, and let $Z = \{i \mid x_i = 0\}$. Show that $\mathbf{x}$ is an optimal solution if and only if the linear programming problem

$$\begin{aligned} \text{minimize} \quad & \mathbf{c}'\mathbf{d} \\ \text{subject to} \quad & \mathbf{A}\mathbf{d} = \mathbf{0} \\ & d_i \geq 0, \qquad i \in Z, \end{aligned}$$

has an optimal cost of zero. (In this sense, deciding optimality is equivalent to solving a new linear programming problem.)

Exercise 3.8* This exercise deals with the problem of deciding whether a given degenerate basic feasible solution is optimal and shows that this is essentially as hard as solving a general linear programming problem.

Consider the linear programming problem of minimizing $\mathbf{c}'\mathbf{x}$ over all $\mathbf{x} \in P$, where $P = \{\mathbf{x} \in \Re^n \mid \mathbf{A}\mathbf{x} \leq \mathbf{b}\}$ is a given bounded and nonempty polyhedron. Let

$$Q = \left\{(\mathbf{x}, t) \in \Re^{n+1} \;\middle|\; \mathbf{A}\mathbf{x} \leq t\mathbf{b},\ t \in [0, 1]\right\}.$$

(a) Give an example of P and Q, with $n = 2$, for which the zero vector (in $\Re^{n+1}$) is a degenerate basic feasible solution in Q; show the example in a figure.

(b) Show that the zero vector (in $\Re^{n+1}$) minimizes $(\mathbf{c}, 0)'\mathbf{y}$ over all $\mathbf{y} \in Q$ if and only if the optimal cost in the original linear programming problem is greater than or equal to zero.

Exercise 3.9 (Necessary and sufficient conditions for a unique optimum) Consider a linear programming problem in standard form and suppose that $\mathbf{x}^*$ is an optimal basic feasible solution. Consider an optimal basis associated with $\mathbf{x}^*$. Let B and N be the set of basic and nonbasic indices, respectively. Let I be the set of nonbasic indices i for which the corresponding reduced costs $\bar{c}_i$ are zero.

(a) Show that if I is empty, then $\mathbf{x}^*$ is the only optimal solution.

(b) Show that $\mathbf{x}^*$ is the unique optimal solution if and only if the following linear programming problem has an optimal value of zero:

$$\begin{aligned} \text{maximize} \quad & \sum_{i \in I} x_i \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & x_i = 0, \qquad i \in N \setminus I, \\ & x_i \geq 0, \qquad i \in B \cup I. \end{aligned}$$

Exercise 3.10 * Show that if $n - m = 2$, then the simplex method will not cycle, no matter which pivoting rule is used.

Exercise 3.11 * Construct an example with $n - m = 3$ and a pivoting rule under which the simplex method will cycle.

Exercise 3.12 Consider the problem

$$\begin{array}{lrcrcl} \text{minimize} & -2x_1 & - & x_2 & & \\ \text{subject to} & x_1 & - & x_2 & \leq & 2 \\ & x_1 & + & x_2 & \leq & 6 \\ & \multicolumn{5}{l}{x_1, x_2 \geq 0.} \end{array}$$

(a) Convert the problem into standard form and construct a basic feasible solution at which $(x_1, x_2) = (0, 0)$.

(b) Carry out the full tableau implementation of the simplex method, starting with the basic feasible solution of part (a).

(c) Draw a graphical representation of the problem in terms of the original variables x_1, x_2, and indicate the path taken by the simplex algorithm.

Exercise 3.13 This exercise shows that our efficient procedures for updating a tableau can be derived from a useful fact in numerical linear algebra.

(a) **(Matrix inversion lemma)** Let $\mathbf{C}$ be an $m \times m$ invertible matrix and let $\mathbf{u}$, $\mathbf{v}$ be vectors in $\Re^m$. Show that

$$(\mathbf{C} + \mathbf{w}\mathbf{v}')^{-1} = \mathbf{C}^{-1} - \frac{\mathbf{C}^{-1}\mathbf{w}\mathbf{v}'\mathbf{C}^{-1}}{1 + \mathbf{v}'\mathbf{C}^{-1}\mathbf{w}}.$$

(Note that $\mathbf{w}\mathbf{v}'$ is an $m \times m$ matrix). *Hint:* Multiply both sides by $(\mathbf{C} + \mathbf{w}\mathbf{v}')$.

(b) Assuming that $\mathbf{C}^{-1}$ is available, explain how to obtain $(\mathbf{C} + \mathbf{w}\mathbf{v}')^{-1}$ using only $O(m^2)$ arithmetic operations.

(c) Let $\mathbf{B}$ and $\overline{\mathbf{B}}$ be basis matrices before and after an iteration of the simplex method. Let $\mathbf{A}_{B(\ell)}$, $\mathbf{A}_{\overline{B}(\ell)}$ be the exiting and entering column, respectively. Show that

$$\overline{\mathbf{B}} - \mathbf{B} = (\mathbf{A}_{\overline{B}(\ell)} - \mathbf{A}_{B(\ell)})\mathbf{e}'_\ell,$$

where $\mathbf{e}_\ell$ is the ℓth unit vector.

(d) Note that $\mathbf{e}'_i\mathbf{B}^{-1}$ is the ith row of $\mathbf{B}^{-1}$ and $\mathbf{e}'_\ell\mathbf{B}^{-1}$ is the pivot row. Show that

$$\mathbf{e}'_i\overline{\mathbf{B}}^{-1} = \mathbf{e}'_i\mathbf{B}^{-1} - g_i\mathbf{e}'_\ell\mathbf{B}^{-1}, \qquad i = 1, \ldots, m,$$

for suitable scalars g_i. Provide a formula for g_i. Interpret the above equation in terms of the mechanics for pivoting in the revised simplex method.

(e) Multiply both sides of the equation in part (d) by $[\mathbf{b} \mid \mathbf{A}]$ and obtain an interpretation of the mechanics for pivoting in the full tableau implementation.

Exercise 3.14 Suppose that a feasible tableau is available. Show how to obtain a tableau with lexicographically positive rows. *Hint:* Permute the columns.

Exercise 3.15 (Perturbation approach to lexicography) Consider a standard form problem, under the usual assumption that the rows of $\mathbf{A}$ are linearly independent. Let ϵ be a scalar and define

$$\mathbf{b}(\epsilon) = \mathbf{b} + \begin{bmatrix} \epsilon \\ \epsilon^2 \\ \vdots \\ \epsilon^m \end{bmatrix}.$$

For every $\epsilon > 0$, we define the ϵ-perturbed problem to be the linear programming problem obtained by replacing $\mathbf{b}$ with $\mathbf{b}(\epsilon)$.

(a) Given a basis matrix $\mathbf{B}$, show that the corresponding basic solution $\mathbf{x}_B(\epsilon)$ in the ϵ-perturbed problem is equal to

$$\mathbf{B}^{-1}\big[\mathbf{b} \mid \mathbf{I}\big]\begin{bmatrix} 1 \\ \epsilon \\ \vdots \\ \epsilon^m \end{bmatrix}.$$

(b) Show that there exists some $\epsilon^* > 0$ such that all basic solutions to the ϵ-perturbed problem are nondegenerate, for $0 < \epsilon < \epsilon^*$.

(c) Suppose that all rows of $\mathbf{B}^{-1}\big[\mathbf{b} \mid \mathbf{I}\big]$ are lexicographically positive. Show that $\mathbf{x}_B(\epsilon)$ is a basic feasible solution to the ϵ-perturbed problem for ϵ positive and sufficiently small.

(d) Consider a feasible basis for the original problem, and assume that all rows of $\mathbf{B}^{-1}[\mathbf{b} \mid \mathbf{I}]$ are lexicographically positive. Let some nonbasic variable x_j enter the basis, and define $\mathbf{u} = \mathbf{B}^{-1}\mathbf{A}_j$. Let the exiting variable be determined as follows. For every row i such that u_i is positive, divide the ith row of $\mathbf{B}^{-1}[\mathbf{b} \mid \mathbf{I}]$ by u_i, compare the results lexicographically, and choose the exiting variable to be the one corresponding to the lexicographically smallest row. Show that this is the same choice of exiting variable as in the original simplex method applied to the ϵ-perturbed problem, when ϵ is sufficiently small.

(e) Explain why the revised simplex method, with the lexicographic rule described in part (d), is guaranteed to terminate even in the face of degeneracy.

Exercise 3.16 (Lexicography and the revised simplex method) Suppose that we have a basic feasible solution and an associated basis matrix $\mathbf{B}$ such that every row of $\mathbf{B}^{-1}$ is lexicographically positive. Consider a pivoting rule that chooses the entering variable x_j arbitrarily (as long as $\bar{c}_j < 0$) and the exiting variable as follows. Let $\mathbf{u} = \mathbf{B}^{-1}\mathbf{A}_j$. For each i with $u_i > 0$, divide the ith row of $[\mathbf{B}^{-1}\mathbf{b} \mid \mathbf{B}^{-1}]$ by u_i and choose the row which is lexicographically smallest. If row ℓ was lexicographically smallest, then the ℓth basic variable $x_{B(\ell)}$ exits the basis. Prove the following:

(a) The row vector $(-\mathbf{c}_B'\mathbf{B}^{-1}\mathbf{b}, -\mathbf{c}_B'\mathbf{B}^{-1})$ increases lexicographically at each iteration.

(b) Every row of $\mathbf{B}^{-1}$ is lexicographically positive throughout the algorithm.

(c) The revised simplex method terminates after a finite number of steps.

Exercise 3.17 Solve completely (i.e., both Phase I and Phase II) via the simplex method the following problem:

$$\begin{array}{lrcrcrcrcrcl}
\text{minimize} & 2x_1 & + & 3x_2 & + & 3x_3 & + & x_4 & - & 2x_5 & & \\
\text{subject to} & x_1 & + & 3x_2 & & & + & 4x_4 & + & x_5 & = & 2 \\
 & x_1 & + & 2x_2 & & & - & 3x_4 & + & x_5 & = & 2 \\
 & -x_1 & - & 4x_2 & + & 3x_3 & & & & & = & 1 \\
 & \multicolumn{11}{l}{x_1, \ldots, x_5 \geq 0.}
\end{array}$$

Exercise 3.18 Consider the simplex method applied to a standard form problem and assume that the rows of the matrix $\mathbf{A}$ are linearly independent. For each of the statements that follow, give either a proof or a counterexample.

(a) An iteration of the simplex method may move the feasible solution by a positive distance while leaving the cost unchanged.

(b) A variable that has just left the basis cannot reenter in the very next iteration.

(c) A variable that has just entered the basis cannot leave in the very next iteration.

(d) If there is a nondegenerate optimal basis, then there exists a unique optimal basis.

(e) If $\mathbf{x}$ is an optimal solution found by the simplex method, no more than m of its components can be positive, where m is the number of equality constraints.

Exercise 3.19 While solving a standard form problem, we arrive at the following tableau, with x_3, x_4, and x_5 being the basic variables:

-10	δ	-2	0	0	0
4	-1	η	1	0	0
1	α	-4	0	1	0
β	γ	3	0	0	1

The entries α, β, γ, δ, η in the tableau are unknown parameters. For each one of the following statements, find some parameter values that will make the statement true.

(a) The current solution is optimal and there are multiple optimal solutions.

(b) The optimal cost is $-\infty$.

(c) The current solution is feasible but not optimal.

Exercise 3.20 Consider a linear programming problem in standard form, described in terms of the following initial tableau:

0	0	0	0	δ	3	γ	ξ
β	0	1	0	α	1	0	3
2	0	0	1	-2	2	η	-1
3	1	0	0	0	-1	2	1

The entries α, β, γ, δ, η, ξ in the tableau are unknown parameters. Furthermore, let $\mathbf{B}$ be the basis matrix corresponding to having x_2, x_3, and x_1 (in that order) be the basic variables. For each one of the following statements, find the ranges of values of the various parameters that will make the statement to be true.

(a) Phase II of the simplex method can be applied using this as an initial tableau.

(b) The first row in the present tableau (below the row with the reduced costs) indicates that the problem is infeasible.

(c) The corresponding basic solution is feasible, but we do not have an optimal basis.

(d) The corresponding basic solution is feasible and the first simplex iteration indicates that the optimal cost is $-\infty$.

(e) The corresponding basic solution is feasible, x_6 is a candidate for entering the basis, and when x_6 is the entering variable, x_3 leaves the basis.

(f) The corresponding basic solution is feasible, x_7 is a candidate for entering the basis, but if it does, the solution and the objective value remain unchanged.

Exercise 3.21 Consider the oil refinery problem in Exercise 1.16.

(a) Use the simplex method to find an optimal solution.

(b) Suppose that the selling price of heating oil is sure to remain fixed over the next month, but the selling price of gasoline may rise. How high can it go without causing the optimal solution to change?

(c) The refinery manager can buy crude oil B on the spot market at \$40/barrel, in unlimited quantities. How much should be bought?

Exercise 3.22 Consider the following linear programming problem with a single constraint:

$$\begin{array}{ll} \text{minimize} & \displaystyle\sum_{i=1}^{n} c_i x_i \\ \text{subject to} & \displaystyle\sum_{i=1}^{n} a_i x_i = b \\ & x_i \geq 0, \qquad i = 1, \ldots, n. \end{array}$$

(a) Derive a simple test for checking the feasibility of this problem.

(b) Assuming that the optimal cost is finite, develop a simple method for obtaining an optimal solution directly.

Exercise 3.23 While solving a linear programming problem by the simplex method, the following tableau is obtained at some iteration.

	0	$\ldots$	0	$\bar{c}_{m+1}$	$\ldots$	$\bar{c}_n$
x_1	1	$\ldots$	0	$a_{1,m+1}$	$\ldots$	$a_{1,n}$
$\vdots$	$\vdots$	$\ldots$	$\vdots$	$\vdots$	$\ldots$	$\vdots$
x_m	0	$\ldots$	1	$a_{m,m+1}$	$\ldots$	$a_{m,n}$

Assume that in this tableau we have $\bar{c}_j \geq 0$ for $j = m+1, \ldots, n-1$, and $\bar{c}_n < 0$. In particular, x_n is the only candidate for entering the basis.

(a) Suppose that x_n indeed enters the basis and that this is a nondegenerate pivot (that is, $\theta^* \neq 0$). Prove that x_n will remain basic in all subsequent iterations of the algorithm and that x_n is a basic variable in any optimal basis.

(b) Suppose that x_n indeed enters the basis and that this is a degenerate pivot (that is, $\theta^* = 0$). Show that x_n need not be basic in an optimal basic feasible solution.

Exercise 3.24 Show that in Phase I of the simplex method, if an artificial variable becomes nonbasic, it need never again become basic. Thus, when an artificial variable becomes nonbasic, its column can be eliminated from the tableau.

Exercise 3.25 (The simplex method with upper bound constraints) Consider a problem of the form

$$\begin{aligned} \text{minimize} \quad & \mathbf{c}'\mathbf{x} \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{0} \le \mathbf{x} \le \mathbf{u}, \end{aligned}$$

where $\mathbf{A}$ has linearly independent rows and dimensions $m \times n$. Assume that $u_i > 0$ for all i.

(a) Let $\mathbf{A}_{B(1)} \dots, \mathbf{A}_{B(m)}$ be m linearly independent columns of $\mathbf{A}$ (the "basic" columns). We partition the set of all $i \neq B(1), \dots, B(m)$ into two disjoint subsets L and U. We set $x_i = 0$ for all $i \in L$, and $x_i = u_i$ for all $i \in U$. We then solve the equation $\mathbf{A}\mathbf{x} = \mathbf{b}$ for the basic variables $x_{B(1)}, \dots, x_{B(m)}$. Show that the resulting vector $\mathbf{x}$ is a basic solution. Also, show that it is nondegenerate if and only if $x_i \neq 0$ and $x_i \neq u_i$ for every basic variable x_i.

(b) For this part and the next, assume that the basic solution constructed in part (a) is feasible. We form the simplex tableau and compute the reduced costs as usual. Let x_j be some nonbasic variable such that $x_j = 0$ and $\bar{c}_j < 0$. As in Section 3.2, we increase x_j by θ, and adjust the basic variables from $\mathbf{x}_B$ to $\mathbf{x}_B - \theta\mathbf{B}^{-1}\mathbf{A}_j$. Given that we wish to preserve feasibility, what is the largest possible value of θ? How are the new basic columns determined?

(c) Let x_j be some nonbasic variable such that $x_j = u_j$ and $\bar{c}_j > 0$. We decrease x_j by θ, and adjust the basic variables from $\mathbf{x}_B$ to $\mathbf{x}_B + \theta\mathbf{B}^{-1}\mathbf{A}_j$. Given that we wish to preserve feasibility, what is the largest possible value of θ? How are the new basic columns determined?

(d) Assuming that every basic feasible solution is nondegenerate, show that the cost strictly decreases with each iteration and the method terminates.

Exercise 3.26 (The big-M method) Consider the variant of the big-M method in which M is treated as an undetermined large parameter. Prove the following.

(a) If the simplex method terminates with a solution $(\mathbf{x}, \mathbf{y})$ for which $\mathbf{y} = \mathbf{0}$, then $\mathbf{x}$ is an optimal solution to the original problem.

(b) If the simplex method terminates with a solution $(\mathbf{x}, \mathbf{y})$ for which $\mathbf{y} \neq \mathbf{0}$, then the original problem is infeasible.

(c) If the simplex method terminates with an indication that the optimal cost in the auxiliary problem is $-\infty$, show that the original problem is either infeasible or its optimal cost is $-\infty$. *Hint:* When the simplex method terminates, it has discovered a feasible direction $\mathbf{d} = (\mathbf{d}_x, \mathbf{d}_y)$ of cost decrease. Show that $\mathbf{d}_y = \mathbf{0}$.

(d) Provide examples to show that both alternatives in part (c) are possible.

Exercise 3.27 *

(a) Suppose that we wish to find a vector $\mathbf{x} \in \Re^n$ that satisfies $\mathbf{Ax} = \mathbf{0}$ and $\mathbf{x} \geq \mathbf{0}$, and such that the number of positive components of $\mathbf{x}$ is maximized. Show that this can be accomplished by solving the linear programming problem

$$\begin{array}{ll} \text{maximize} & \displaystyle\sum_{i=1}^{n} y_i \\ \text{subject to} & \mathbf{A}(\mathbf{z}+\mathbf{y}) = \mathbf{0} \\ & y_i \leq 1, \qquad \text{for all } i, \\ & \mathbf{z}, \mathbf{y} \geq \mathbf{0}. \end{array}$$

(b) Suppose that we wish to find a vector $\mathbf{x} \in \Re^n$ that satisfies $\mathbf{Ax} = \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$, and such that the number of positive components of $\mathbf{x}$ is maximized. Show how this can be accomplished by solving a single linear programming problem.

Exercise 3.28 Consider a linear programming problem in standard form with a bounded feasible set. Furthermore, suppose that we know the value of a scalar U such that any feasible solution satisfies $x_i \leq U$, for all i. Show that the problem can be transformed into an equivalent one that contains the constraint $\sum_{i=1}^{n} x_i = 1$.

Exercise 3.29 Consider the simplex method, viewed in terms of column geometry. Show that the $m+1$ basic points $(\mathbf{A}_i, c_i)$, as defined in Section 3.6, are affinely independent.

Exercise 3.30 Consider the simplex method, viewed in terms of column geometry. In the terminology of Section 3.6, show that the vertical distance from the dual plane to a point $(\mathbf{A}_j, c_j)$ is equal to the reduced cost of the variable x_j.

Exercise 3.31 Consider the linear programming problem

$$\begin{array}{lrcrcrcrcl} \text{minimize} & x_1 & + & 3x_2 & + & 2x_3 & + & 2x_4 & & \\ \text{subject to} & 2x_1 & + & 3x_2 & + & x_3 & + & x_4 & = & b_1 \\ & x_1 & + & 2x_2 & + & x_3 & + & 3x_4 & = & b_2 \\ & x_1 & + & x_2 & + & x_3 & + & x_4 & = & 1 \\ & \multicolumn{9}{l}{x_1, \ldots, x_4 \geq 0,} \end{array}$$

where b_1, b_2 are free parameters. Let $P(b_1, b_2)$ be the feasible set. Use the column geometry of linear programming to answer the following questions.

(a) Characterize explicitly (preferably with a picture) the set of all (b_1, b_2) for which $P(b_1, b_2)$ is nonempty.

(b) Characterize explicitly (preferably with a picture) the set of all (b_1, b_2) for which some basic feasible solution is degenerate.

(c) There are four bases in this problem; in the ith basis, all variables except for x_i are basic. For every (b_1, b_2) for which there exists a degenerate basic feasible solution, enumerate all bases that correspond to each degenerate basic feasible solution.

(d) For $i = 1, \ldots, 4$, let $S_i = \{(b_1, b_2) \mid \text{the } i\text{th basis is optimal}\}$. Identify, preferably with a picture, the sets $S_1, \ldots, S_4$.

(e) For which values of (b_1, b_2) is the optimal solution degenerate?

(f) Let $b_1 = 9/5$ and $b_2 = 7/5$. Suppose that we start the simplex method with x_2, x_3, x_4 as the basic variables. Which path will the simplex method follow?

Exercise 3.32 * Prove Theorem 3.5.

Exercise 3.33 Consider a polyhedron in standard form, and let $\mathbf{x}$, $\mathbf{y}$ be two different basic feasible solutions. If we are allowed to move from any basic feasible solution to an adjacent one in a single step, show that we can go from $\mathbf{x}$ to $\mathbf{y}$ in a finite number of steps.

3.10 Notes and sources

3.2. The simplex method was pioneered by Dantzig in 1947, who later wrote a comprehensive text on the subject (Dantzig, 1963).

3.3. For more discussion of practical implementations of the simplex method based on products of sparse matrices, instead of $\mathbf{B}^{-1}$, see the books by Gill, Murray, and Wright (1981), Chvátal (1983), Murty (1983), and Luenberger (1984). An excellent introduction to numerical linear algebra is the text by Golub and Van Loan (1983). Example 3.6, which shows the possibility of cycling, is due to Beale (1955).

If we have upper bounds for all or some of the variables, instead of converting the problem to standard form, we can use a suitable adaptation of the simplex method. This is developed in Exercise 3.25 and in the textbooks that we mentioned earlier.

3.4. The lexicographic anticycling rule is due to Dantzig, Orden, and Wolfe (1955). It can be viewed as an outgrowth of a perturbation method developed by Orden and also by Charnes (1952). For an exposition of the perturbation method, see Chvátal (1983) and Murty (1983), as well as Exercise 3.15. The smallest subscript rule is due to Bland (1977). A proof that Bland's rule avoids cycling can also be found in Papadimitriou and Steiglitz (1982), Chvátal (1983), or Murty (1983).

3.6. The column geometry interpretation of the simplex method is due to Dantzig (1963). For further discussion, see Stone and Tovey (1991).