LP Homework 2

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2023 年 3 月 31 日

2.1

For each one of the following sets, determine whether it is a polyhedron.
(a) The set of all $(x, y) \in \mathbb{R}^2$ satisfying the constraints

$$\begin{aligned} xcos\theta + ysin\theta \leq 1, \forall\theta \in [0, \pi/2] \\ x \geq 0, \\ y \geq 0. \end{aligned}$$

(b) The set of all $x \in \mathbb{R}$ satisfying the constraint $x^2 - 8x + 15 \leq 0$.
(c) The empty set.
Solution.
(a) It is not a polyhedron, as we will prove that it is actually the set
$B = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 \leq 1, x \geq 0, y \geq 0\}$, which is not a polyhedron.
Let A be the set defined by the primal constraints. For any $(x, y) \in A$, if it is $(0, 0)$, then $(x, y) \in B$. Assume $(x, y) > 0$, then there exists a $\theta^* \in [0, \pi/2]$ such that $cos\theta^* = \frac{x}{\sqrt{x^2+y^2}}, sin\theta^* = \frac{y}{\sqrt{x^2+y^2}}$.
Then from $xcos\theta^* + ysin\theta^* \leq 1$ we have $x^2 + y^2 \leq 1$, so $(x, y) \in B$.
On the other hand, For any $(x, y) \in B$ and $\theta \in [0, \pi/2]$, by Cauchy's inequality, we have $(xcos\theta + ysin\theta)^2 \leq (x^2 + y^2)(cos^2\theta + sin^2\theta) \leq 1$,
So $xcos\theta + ysin\theta \leq 1$, namely $(x, y) \in A$.
Then we get $A = B$.
(b) It is a polyhedron, since it can be written equivalently as
$\{x \in \mathbb{R} | x \leq 5, x \geq 3\}$.

(c) Empty set is a polyhedron, since it can be written as
$\{x \in \mathbb{R} | x \leq 0, x \geq 1\}$, which fits the definition.

2.2

Let $f : \mathbb{R}^n \mapsto \mathbb{R}$ be a convex function and let c be some constant.
Show that the set $S = \{x \in \mathbb{R}^n | f(x) \leq c\}$ is convex.
Proof.
Suppose $x, y \in S$ and $\alpha \in [0, 1]$. By the convexity of f, we have
$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha)f(y) \leq \alpha c + (1-\alpha)c = c$,
which indicates $\alpha x + (1-\alpha)y \in S$, i.e. S is convex.

2.3

(Basic feasible solutions in standard form polyhedra with upper bounds) Consider a polyhedron defined by the constraints $\mathbf{Ax} = \mathbf{b}$ and $\mathbf{0} \leq \mathbf{x} \leq \mathbf{u}$, and assume that the matrix $\mathbf{A}$ has linearly independent rows. Provide a procedure analogous to the one in Section 2.3 for constructing basic solutions, and prove an analog of Theorem 2.4.
Solution.
An analog of Theorem 2.4 is:
Consider the constraints $\mathbf{Ax} = \mathbf{b}$ and $\mathbf{0} \leq \mathbf{x} \leq \mathbf{u}$ and assume that the $m \times n$ matrix $\mathbf{A}$ has linearly independent rows. A vector $x \in \mathbb{R}^n$ is a basic solution if and only if we have $\mathbf{Ax} = \mathbf{b}$, and there exist indices $B(1), \ldots, B(m)$ such that:
(a) The columns $\mathbf{A}_{B(1)}, \ldots, \mathbf{A}_{B(m)}$ are linearly independent;
(b) If $i \neq B(1), \ldots, B(m)$, then $x_i = 0$ or $x_i = u_i$.
Proof.
(1) Consider some $\mathbf{x} : \mathbf{Ax} = \mathbf{b}$ and there exist indices $B(1), \ldots, B(m)$ satisfying (a) and (b). Let $\mathcal{N} = \{i \notin B(1), \ldots, B(m) | x_i = 0\}$ and $\mathcal{U} = \{i \notin B(1), \ldots, B(m) | x_i = u_i\}$. The active constraints and $\mathbf{Ax} = \mathbf{b}$ imply that
$\sum_{i=1}^{m} \mathbf{A}_{B(i)} x_{B(i)} = \sum_{i=1}^{n} \mathbf{A}_i x_i - \sum_{i \in \mathcal{N}} \mathbf{A}_i 0 - \sum_{i \in \mathcal{U}} \mathbf{A}_i u_i = \mathbf{b} - \sum_{i \in \mathcal{U}} \mathbf{A}_i u_i$.
Since the columns $\mathbf{A}_{B(i)}, i = 1, \ldots, m$ are linearly independent, $x_{B(1)}, \ldots, x_{B(m)}$

are uniquely determined. Thus, the system of equations formed by the active constraints has a unique solution. This is equivalent to saying that there are n linearly independent active constraints, and this implies that $\mathbf{x}$ is a basic solution.
(2) For the converse, we assume that $\mathbf{x}$ is a basic solution of $\{x|\mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{0} \leq \mathbf{x} \leq \mathbf{u}\}$ and we will show that conditions (a) and (b) in the statement of the theorem are satisfied. Define $\mathcal{N} = \{1 \leq i \leq n | x_i = 0\}$ and $\mathcal{U} = \{1 \leq i \leq n | x_i = u_i\}$, and denote the elements that are not included by either of the two sets by $x_{B(1)}, \ldots, x_{B(k)}$ for some k. Since $\mathbf{x}$ is a basic solution, the system of equations formed by the active constraints $\sum_{i=1}^{n} \mathbf{A}_i x_i = \mathbf{b}$, $x_i = 0, i \in \mathcal{N}$ and $x_i = u_i, i \in \mathcal{U}$, have a unique solution; equivalently, the equation
$\sum_{i=1}^{k} \mathbf{A}_{B(i)} x_{B(i)} = \sum_{i=1}^{n} \mathbf{A}_i x_i - \sum_{i \in \mathcal{N}} \mathbf{A}_i 0 - \sum_{i \in \mathcal{U}} \mathbf{A}_i u_i = \mathbf{b} - \sum_{i \in \mathcal{U}} \mathbf{A}_i u_i.$
has a unique solution. It follows that the columns $\mathbf{A}_{B(1)}, \ldots, \mathbf{A}_{B(k)}$ are linearly independent, which implies $k \leq m$. Since A has m linearly independent rows, it also has m linearly independent columns. It follows that we can find $m - k$ additional columns $\mathbf{A}_{B(k+1)}, \ldots, \mathbf{A}_{B(m)}$ so that the columns $\mathbf{A}_{B(i)}, i = 1, \ldots, m$ are linearly independent. In addition, if $i \neq B(1), \ldots, B(m)$, then $i \neq B(1), \ldots, B(k)$ (because $k \leq m$) , and $x_i = 0$ or $x_i = u_i$. Therefore, both conditions (a) and (b) in the statement of the theorem are satisfied.
By the above analogous theorem, all basic solutions to a bounded form polynomial can be constructed according to the following procedure.
1. Choose m linearly independent columns $\mathbf{A}_{B(1)}, \ldots, \mathbf{A}_{B(m)}$.
2. Let $x_i = 0$ or $x_i = u_i$ for all $i \neq B(1), \ldots, B(m)$.
3. Solve the system of m equations $\mathbf{A}\mathbf{x} = \mathbf{b}$ for the unknowns $x_{B(1)}, \ldots, x_{B(m)}$.
If a basic solution constructed according to this procedure satisfies $\mathbf{0} \leq \mathbf{x} \leq \mathbf{u}$, then it is feasible, and it is a basic feasible solution. Conversely, since every basic feasible solution is a basic solution, it can be obtained from this procedure.

2.7

Suppose that $\{x \in \mathbb{R}^n | \mathbf{a_i'x} \geq b_i, i = 1, \ldots, m\}$ and $\{x \in \mathbb{R}^n | \mathbf{g_i'x} \geq h_i, i = 1, \ldots, k\}$ are two representations of the same nonempty polyhedron. Suppose that the vectors $\mathbf{a_1}, \ldots, \mathbf{a_m}$ span $\mathbb{R}^n$. Show that the same must be true for the vectors $\mathbf{g_1}, \ldots, \mathbf{g_k}$.

Proof.

Let P_1 denote the polyhedron $\{x \in \mathbb{R}^n | \mathbf{a_i'x} \geq b_i, i = 1, \ldots, m\}$, and $P_2 = \{x \in \mathbb{R}^n | \mathbf{g_i'x} \geq h_i, i = 1, \ldots, k\}$.

Since $\mathbf{a_1}, \ldots, \mathbf{a_m}$ span $\mathbb{R}^n$, there exist n linearly independent vectors in $\mathbf{a_1}, \ldots, \mathbf{a_m}$. Then P_1 has at least one extreme point.

From $P_1 = P_2$, we know that P_2 also has at least one extreme point. Then there exist n linearly independent vectors in $\mathbf{g_1}, \ldots, \mathbf{g_k}$, which shows that $\mathbf{g_1}, \ldots, \mathbf{g_k}$ can span $\mathbb{R}^n$ as well.

2.8

Consider the standard form polyhedron $\{x | \mathbf{Ax} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$, and assume that the rows of the matrix $\mathbf{A}$ are linearly independent. Let $\mathbf{x}$ be a basic solution, and let $J = \{i | x_i \neq 0\}$. Show that a basis is associated with the basic solution $\mathbf{x}$ if and only if every column $\mathbf{A}_i, i \in J$, is in the basis.

Proof.

If a basis $\mathbf{B} = [\mathbf{A}_{B(1)}, \ldots, \mathbf{A}_{B(m)}]$ is associated with the basic solution $\mathbf{x}$, then any index $i \in \{1, \ldots, n\} \setminus \{B(1), \ldots, B(m)\}$ is associated with $x_i = 0$. Therefore, $J = \{i | x_i \neq 0\}$ must be a subset of $\{B(1), \ldots, B(m)\}$, which indicates that every column $\mathbf{A}_i, i \in J$, is in the basis.

If every column $\mathbf{A}_i, i \in J$, is in the basis $\mathbf{B}$, then for all the columns $\mathbf{A}_j$ outside of $\mathbf{B}$ we must have $j \notin J$, i.e. $x_j = 0$. Then $\mathbf{B}$ is associated with $\mathbf{x}$.

2.10

Consider the standard form polyhedron $P = \{x | \mathbf{Ax} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$. Suppose that the matrix $\mathbf{A}$ has dimensions $m \times n$ and that its rows are linearly independent. For each one of the following statements, state whether it is true

or false. If true, provide a proof, else, provide a counterexample.
(a) If $n = m + 1$, then P has at most two basic feasible solutions.
(b) The set of all optimal solutions is bounded.
(e) At every optimal solution, no more than m variables can be positive.
(d) If there is more than one optimal solution, then there are uncountably many optimal solutions.
(e) If there are several optimal solutions, then there exist at least two basic feasible solutions that are optimal.
(f) Consider the problem of minimizing $\max\{\mathbf{c^T x}, \mathbf{d^T x}\}$ over the set P. If this problem has an optimal solution, it must have an optimal solution which is an extreme point of P.

Solution.
(a) It is true. Consider solving the linear equation $\mathbf{Ax} = \mathbf{b}$. By $rank(\mathbf{A}) = m = n - 1$, we know that the solution space can be written as $x_0 + W$, where x_0 is a special solution, and W denotes the null space of $\mathbf{A}$, $dimW = n - (n-1) = 1$. Therefore, P is the subset of a subspace of dimension 1, namely a line. Hence P has at most two extreme points, equivalently at most two basic feasible solutions.
(b) It is false. Consider $P = \{[x_1, x_2]^T | x_2 = 2x_1 + 1, x_1, x_2 \geq 0\}$. Here $\mathbf{A} = [-2, 1]$, and suppose the problem is to minimize $\mathbf{c^T x}$ with $\mathbf{c} = [-2, 1]^T$. Then every point of P is an optimal solution. Since P is unbounded, the set of optimal solutions is unbounded.
(c) It is false. Consider the same conditions in (b), namely $m = 1, n = 2$, and $[1, 3]^T$ with 2 positive variables is an optimal solution.
(d) It is true. If there are two optimal solutions, every convex combination of them is an optimal solution too.
Suppose x, y are two optimal solutions, meaning $c^T x = c^T y = p^*$, the optimal value. Let $\alpha \in (0, 1)$, and then
$c^T(\alpha x + (1-\alpha)y) = \alpha c^T x + (1-\alpha)c^T y = p^*$. As P is a convex set, $z = \alpha x + (1-\alpha)y \in P$, so z is also an optimal solution. It is easy to see that there are uncountably many optimal solutions.
(e) It is false. Consider the same counterexample in (b), where there are

uncountably many optimal solutions. In fact, there is only one basic feasible solution $[0,1]^T$.

(f) It is false. Consider $P = \{[x_1, x_2]^T | x_1 + x_2 = 1, x_1, x_2 \geq 0\}$. Let $\mathbf{c} = [1, -1]^T, \mathbf{d} = [-1, 1]^T$. Then the problem is that minimizing $\max\{\mathbf{x_1} - \mathbf{x_2}, \mathbf{x_2} - \mathbf{x_1}\}$ over the set P. This is equivalently to minimize $|x_1 - x_2|$ over P, and the optimal value is 0 with the solitary optimal solution $x^* = [\frac{1}{2}, \frac{1}{2}]^T$, which is not an extreme point of P.