Runge Kutta Method For Stochastical Integration

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Suppose we have the following stochastic process $\mathbf{X}_t$ presented as:

$$d\mathbf{X}_t = \sigma(\mathbf{X}_t, t)d\mathbf{W}_t + \mu(\mathbf{X}_t, t)dt,$$

where $\mathbf{W}_t$ is the Brownian motion.

We aim to perform numerical integration from 0 to T with respect to $f(\mathbf{X}_t)$. Assume f is C^3, and denote $\Pi = \{0 = t_0 < t_1 < \cdots < t_N = T\}$ as a generic partition of the interval $[0, T]$.

To control the local truncation error during the numerical integration, we apply the Runge-Kutta method by comparing the Stratonovich integral and Itô's formula, thereby dynamically updating the time step Δt.

The Stratonovich integral is given as:

$$df(\mathbf{X}_t) = f'(\mathbf{X}_t) \circ d\mathbf{X}_t.$$

Numerically, this can be expressed as:

$$f(\mathbf{X}_{t_i}) - f(\mathbf{X}_{t_{i-1}}) \approx f'\left(\frac{\mathbf{X}_{t_{i-1}} + \mathbf{X}_{t_i}}{2}\right)\left(\mathbf{X}_{t_i} - \mathbf{X}_{t_{i-1}}\right).$$

By applying Taylor's formula with a remainder term, we obtain:

$$\begin{aligned} f'\left(\frac{\mathbf{X}_{t_{i-1}} + \mathbf{X}_{t_i}}{2}\right)\left(\mathbf{X}_{t_i} - \mathbf{X}_{t_{i-1}}\right) = f'(\mathbf{X}_{t_{i-1}})\left(\mathbf{X}_{t_i} - \mathbf{X}_{t_{i-1}}\right) \\ + \frac{1}{2}f''(\mathbf{X}_{ti-1})\left(\mathbf{X}_{t_i} - \mathbf{X}_{t_{i-1}}\right)^2 \\ + \frac{1}{8}f'''(\mathbf{X}_{t_{i-1}})\left(\mathbf{X}_{t_i} - \mathbf{X}_{t_{i-1}}\right)^3 + o\left(\mathbf{X}_{t_i} - \mathbf{X}_{t_{i-1}}\right)^3 \end{aligned}$$

where ξ lies between $\mathbf{X}_{t_{i-1}}$ and $\mathbf{X}_{t_i}$.

Itô's formula provides another expression for the numerical integration as:

$$f(\mathbf{X}_{t_i}) - f(\mathbf{X}_{t_{i-1}}) \approx f'(\mathbf{X}_{t_{i-1}})\left(\mathbf{X}_{t_i} - \mathbf{X}_{t_{i-1}}\right) + \frac{1}{2}f''(\mathbf{X}_{t_{i-1}})\left(\mathbf{X}_{t_i} - \mathbf{X}_{t_{i-1}}\right)^2.$$

Next, we consider the quadratic term $\left(\mathbf{X}_{t_i} - \mathbf{X}_{t_{i-1}}\right)^2$. We first investigate the variation of the process $\langle \mathbf{X}_T, \mathbf{X}_T \rangle$, which is defined as:

$$\langle \mathbf{X}_T, \mathbf{X}_T \rangle = \lim_{N \to \infty} \sum \left(\mathbf{X}_{t_i} - \mathbf{X}_{t_{i-1}} \right)^2 .$$

Since

$$\mathbf{X}_T = \mathbf{X}_0 + \int \sigma(\mathbf{X}_t, t) d\mathbf{W}_t + \int \mu(\mathbf{X}_t, t) dt,$$

we have

$$\begin{aligned} \langle \mathbf{X}_T, \mathbf{X}_T \rangle = {} & \langle \sigma(\mathbf{X}_T, T)\mathbf{W}_T, \sigma(\mathbf{X}_T, T)\mathbf{W}_T \rangle \\ & + 2\langle \sigma(\mathbf{X}_T, T)\mathbf{W}_T, \mu(\mathbf{X}_T, T)T \rangle \\ & + \langle \mu(\mathbf{X}_T, T)T, \mu(\mathbf{X}_T, T)T \rangle. \end{aligned}$$

Due to the relation:

$$\langle W, W \rangle_t \stackrel{p}{=} t,$$

we get

$$(\mathbf{X}_{t_i} - \mathbf{X}_{t_{i-1}})(t_i - t_{i-1}) \stackrel{p}{\approx} (\Delta t)^{\frac{3}{2}},$$

and

$$(t_i - t_{i-1})(t_i - t_{i-1}) \approx (\Delta t)^2.$$

By dropping higher-order terms, we can estimate the local truncation error (LTE) as:

$$\mathbf{LTE} \approx \frac{1}{8} \| \, f'''(\mathbf{X}_{t_{i-1}}) \| \|\sigma(\mathbf{X}_{t_{i-1}}, t_{i-1})\|^3 (\Delta t)^{\frac{3}{2}}.$$

The error **err** is estimated as:

$$\mathbf{err} = \left\| \frac{\text{LTE}}{s} \right\|,$$

where $s = \text{atol} + \text{rtol} \cdot \max(\|f(\mathbf{X}_{i-1})\|, \|f(\mathbf{X}_i)\|)$, with ‘atol‘ and ‘rtol‘ being the prescribed error tolerances.

Thus, we have the following estimated relation:

$$\frac{\mathbf{LTE}_{\text{new}}}{\mathbf{LTE}_{\text{old}}} \sim \frac{(\Delta t_{\text{new}})^{\frac{3}{2}}}{(\Delta t_{\text{old}})^{\frac{3}{2}}}.$$

We choose the Δt_{new} such that

$$\mathbf{err}_{\text{new}} = 1 \Rightarrow \Delta t_{\text{new}} = \frac{\Delta t_{\text{old}}}{(\mathbf{err}_{\text{old}})^{\frac{2}{3}}}.$$

By introducing safety factors α, $\alpha_{\min}$, and $\alpha_{\max}$, along with the above results, we obtain the time step updating formula:

$$\Delta t_{\text{new}} = \Delta t_{\text{old}} \cdot \min \left(\alpha_{\max}, \max \left(\alpha_{\min}, \alpha \cdot (\frac{1}{\mathbf{err}})^{\frac{2}{3}} \right) \right).$$