Master Theorem:
Practice Problems and Solutions

Master Theorem

The Master Theorem applies to recurrences of the following form:

$$T(n) = aT(n/b) + f(n)$$

where $a \geq 1$ and $b > 1$ are constants and $f(n)$ is an asymptotically positive function.

There are 3 cases:

1. If $f(n) = O(n^{\log_b a - \epsilon})$ for some constant $\epsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$.
2. If $f(n) = \Theta(n^{\log_b a} \log^k n)$ with[1] $k \geq 0$, then $T(n) = \Theta(n^{\log_b a} \log^{k+1} n)$.
3. If $f(n) = \Omega(n^{\log_b a + \epsilon})$ with $\epsilon > 0$, and $f(n)$ satisfies the regularity condition, then $T(n) = \Theta(f(n))$.
   Regularity condition: $af(n/b) \leq cf(n)$ for some constant $c < 1$ and all sufficiently large n.

Practice Problems

For each of the following recurrences, give an expression for the runtime $T(n)$ if the recurrence can be solved with the Master Theorem. Otherwise, indicate that the Master Theorem does not apply.

1. $T(n) = 3T(n/2) + n^2$

2. $T(n) = 4T(n/2) + n^2$

3. $T(n) = T(n/2) + 2^n$

4. $T(n) = 2^n T(n/2) + n^n$

5. $T(n) = 16T(n/4) + n$

6. $T(n) = 2T(n/2) + n \log n$

[1] most of the time, $k = 0$

7. $T(n) = 2T(n/2) + n/\log n$

8. $T(n) = 2T(n/4) + n^{0.51}$

9. $T(n) = 0.5T(n/2) + 1/n$

10. $T(n) = 16T(n/4) + n!$

11. $T(n) = \sqrt{2}T(n/2) + \log n$

12. $T(n) = 3T(n/2) + n$

13. $T(n) = 3T(n/3) + \sqrt{n}$

14. $T(n) = 4T(n/2) + cn$

15. $T(n) = 3T(n/4) + n \log n$

16. $T(n) = 3T(n/3) + n/2$

17. $T(n) = 6T(n/3) + n^2 \log n$

18. $T(n) = 4T(n/2) + n/\log n$

19. $T(n) = 64T(n/8) - n^2 \log n$

20. $T(n) = 7T(n/3) + n^2$

21. $T(n) = 4T(n/2) + \log n$

22. $T(n) = T(n/2) + n(2 - \cos n)$

Solutions

1. $T(n) = 3T(n/2) + n^2 \implies T(n) = \Theta(n^2)$ (Case 3)

2. $T(n) = 4T(n/2) + n^2 \implies T(n) = \Theta(n^2 \log n)$ (Case 2)

3. $T(n) = T(n/2) + 2^n \implies \Theta(2^n)$ (Case 3)

4. $T(n) = 2^n T(n/2) + n^n \implies$ Does not apply (a is not constant)

5. $T(n) = 16T(n/4) + n \implies T(n) = \Theta(n^2)$ (Case 1)

6. $T(n) = 2T(n/2) + n \log n \implies T(n) = n \log^2 n$ (Case 2)

7. $T(n) = 2T(n/2) + n/\log n \implies$ Does not apply (non-polynomial difference between $f(n)$ and $n^{\log_b a}$)

8. $T(n) = 2T(n/4) + n^{0.51} \implies T(n) = \Theta(n^{0.51})$ (Case 3)

9. $T(n) = 0.5T(n/2) + 1/n \implies$ Does not apply ($a < 1$)

10. $T(n) = 16T(n/4) + n! \implies T(n) = \Theta(n!)$ (Case 3)

11. $T(n) = \sqrt{2}T(n/2) + \log n \implies T(n) = \Theta(\sqrt{n})$ (Case 1)

12. $T(n) = 3T(n/2) + n \implies T(n) = \Theta(n^{\lg 3})$ (Case 1)

13. $T(n) = 3T(n/3) + \sqrt{n} \implies T(n) = \Theta(n)$ (Case 1)

14. $T(n) = 4T(n/2) + cn \implies T(n) = \Theta(n^2)$ (Case 1)

15. $T(n) = 3T(n/4) + n \log n \implies T(n) = \Theta(n \log n)$ (Case 3)

16. $T(n) = 3T(n/3) + n/2 \implies T(n) = \Theta(n \log n)$ (Case 2)

17. $T(n) = 6T(n/3) + n^2 \log n \implies T(n) = \Theta(n^2 \log n)$ (Case 3)

18. $T(n) = 4T(n/2) + n/\log n \implies T(n) = \Theta(n^2)$ (Case 1)

19. $T(n) = 64T(n/8) - n^2 \log n \implies$ Does not apply ($f(n)$ is not positive)

20. $T(n) = 7T(n/3) + n^2 \implies T(n) = \Theta(n^2)$ (Case 3)

21. $T(n) = 4T(n/2) + \log n \implies T(n) = \Theta(n^2)$ (Case 1)

22. $T(n) = T(n/2) + n(2 - \cos n) \implies$ Does not apply. We are in Case 3, but the regularity condition is violated. (Consider $n = 2\pi k$, where k is odd and arbitrarily large. For any such choice of n, you can show that $c \geq 3/2$, thereby violating the regularity condition.)