DSK814: Algorithms and Data Structures

SDU - Kolding

Analysis of algorithms for swap puzzles

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Try the swap puzzle on the website.

https://cdn.tutsplus.com/active/uploads/legacy/tuts/403_html5TileSwapPuzzle/demo/puzzle.html

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- Can you say something about the best and worst running time of your algorithm for puzzles with **n** pieces?
- Is the running time related to the number of pieces that are correctly aligned to start with?

Analysis of algorithms for swap puzzles

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- More generally,can we accurately describe all best possible algorithms?

Model of puzzle

We model the pieces of a puzzle as the numbers 1,2,3, . . . ,n, numbered according to the layout when the puzzle is solved:

5	10	14	3
1	11	9	15
8	7	2	12
4	13	6	16

$\rightarrow$

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Note: this can also be seen as an array/list:

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
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A list of numbers 1,2,3, . . . ,n in an array of length n also called a **permutation**

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For a puzzle with n pieces, of which t already in place, what is the minimum number of swaps?
For each swap: at most two pieces will be in place.
Therefore at least $(n - t)/2$ swaps.

Cycles

Can we do more? even better(i.e. even more precise) analysis than

“between $(n - t)/2$ and $n - t$ swaps”?

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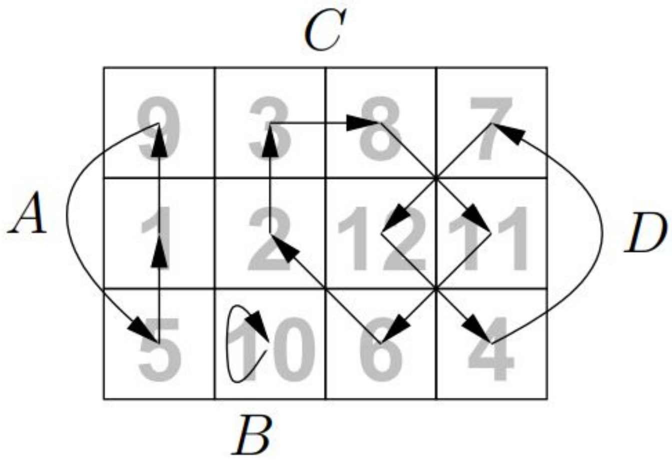
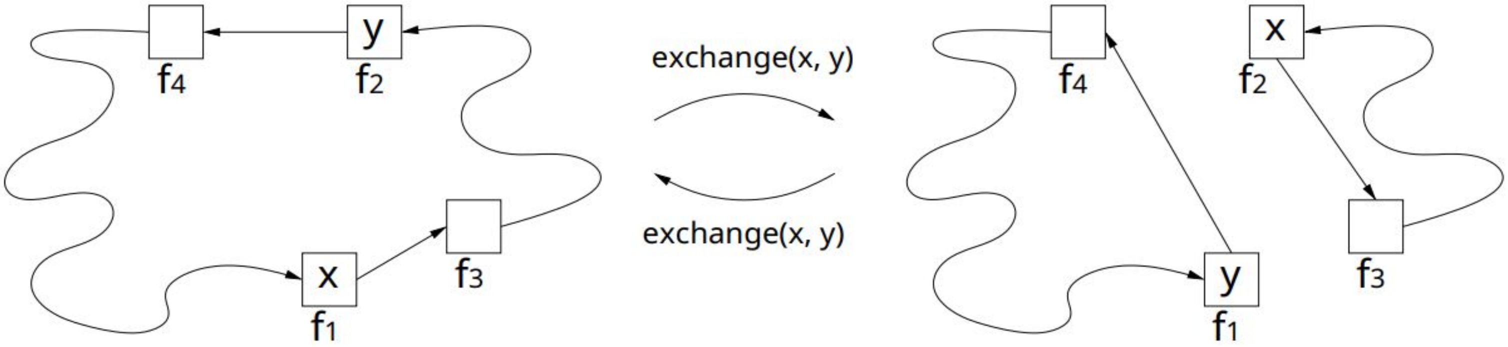
Observation: a permutation naturally gives rise to a collection of cycles:
A number (a piece)t points to the place where it should be, namely the place with number t.



Cycles and swaps

Observation:
Swapping two pieces in the same cycles increases the number of cycles by exactly one. Swapping two pieces in different cycles decreases the number of cycles by exactly one:



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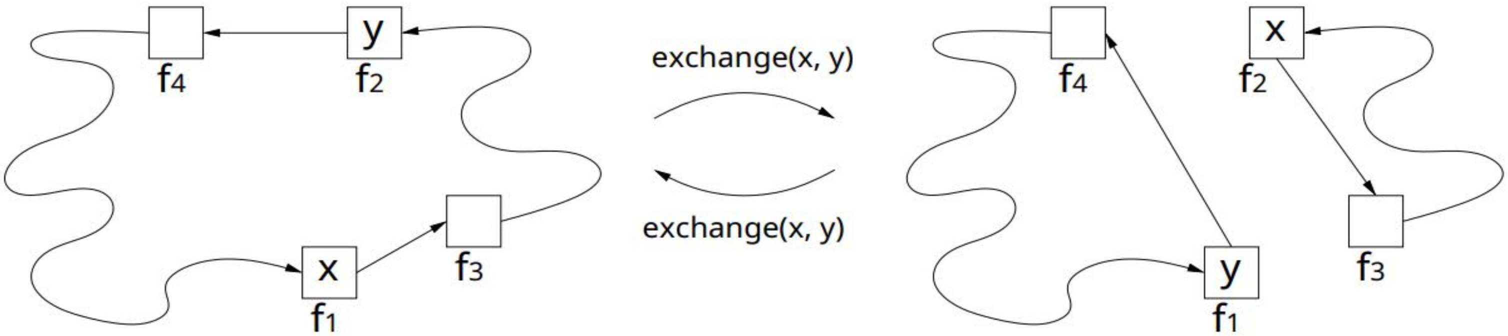

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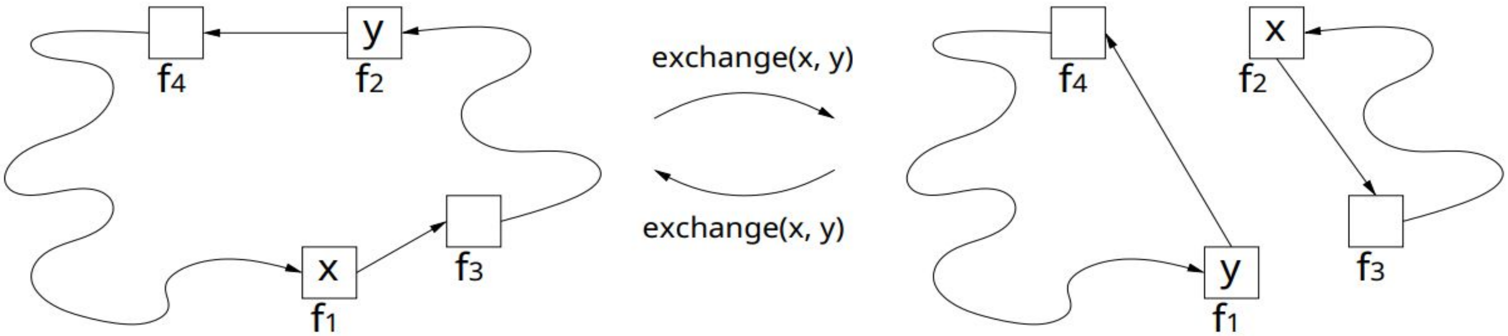

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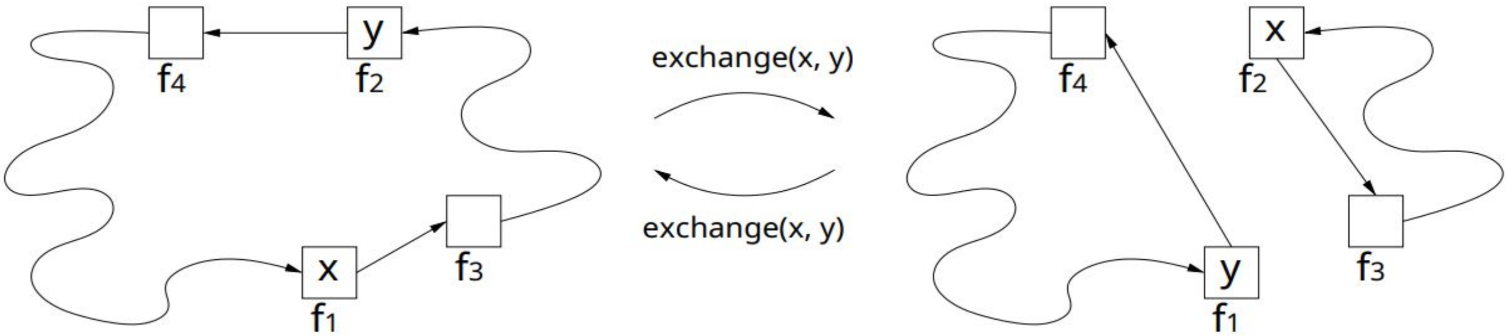

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Conclusion: A puzzle with n pieces and k cycles in the initial setup requires at least n − k swaps, and it can always be done with n − k swaps (for example, using the greedy algorithm, which at each step splits a circle of length t into two cycles of length t − 1 and 1).

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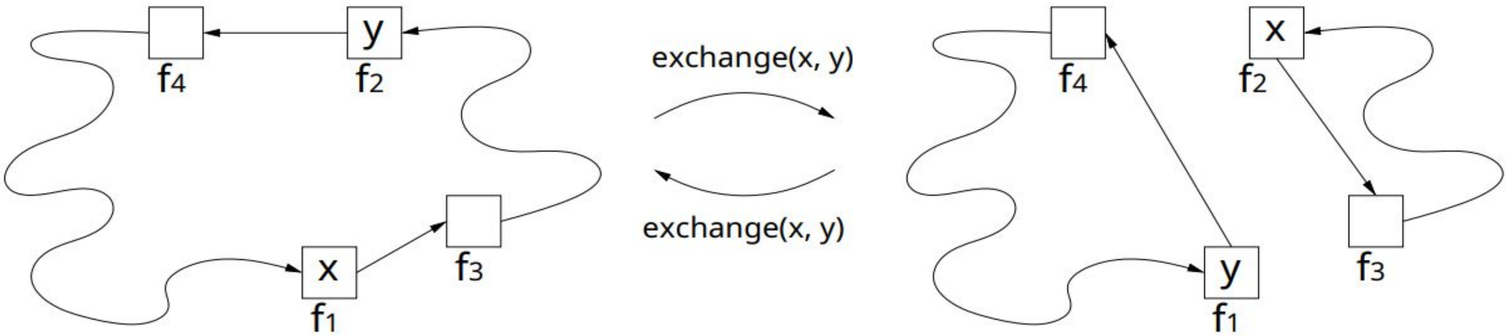

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Conclusion: An algorithm uses the optimal number of swaps (n − k) if and only if each swap is with two pieces that are in the same cycles.

Expected number of cycles

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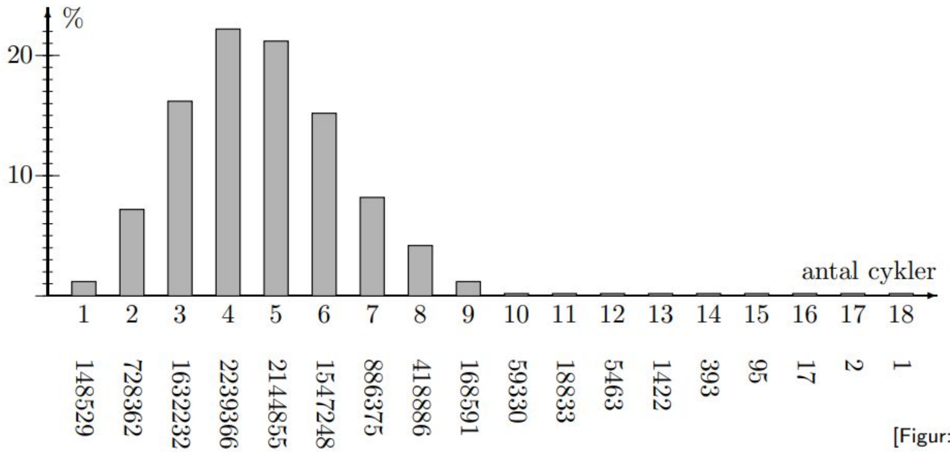
With simulation (testing, 10,000,000 random permutations) for n = 64, the following distribution of the number of permutations is observed:



[Figur: Gerth Brodal]