Semitrailer modelling

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1 System description

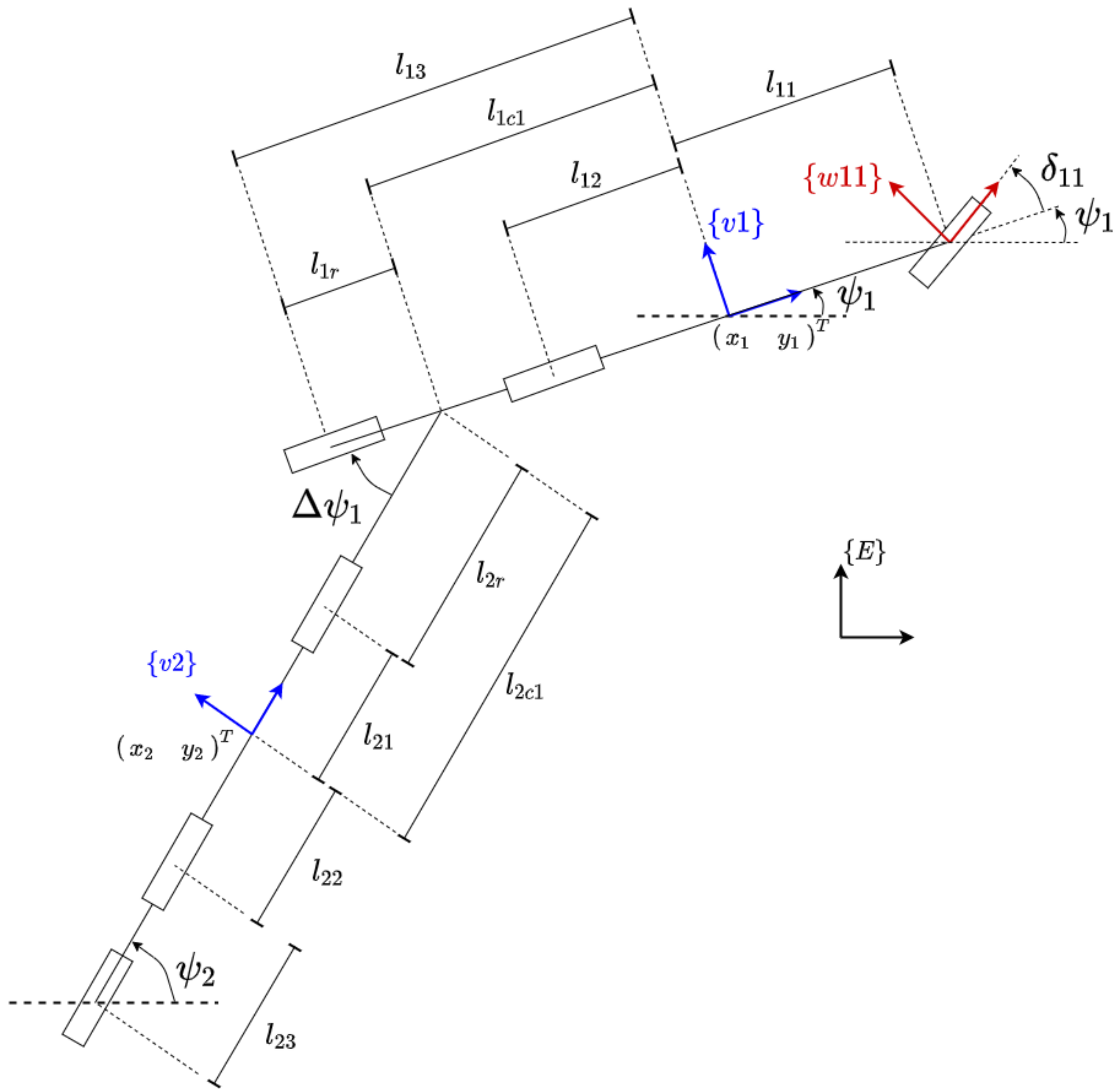

Figure 1: Semitrailer model: Single axle bicycle model

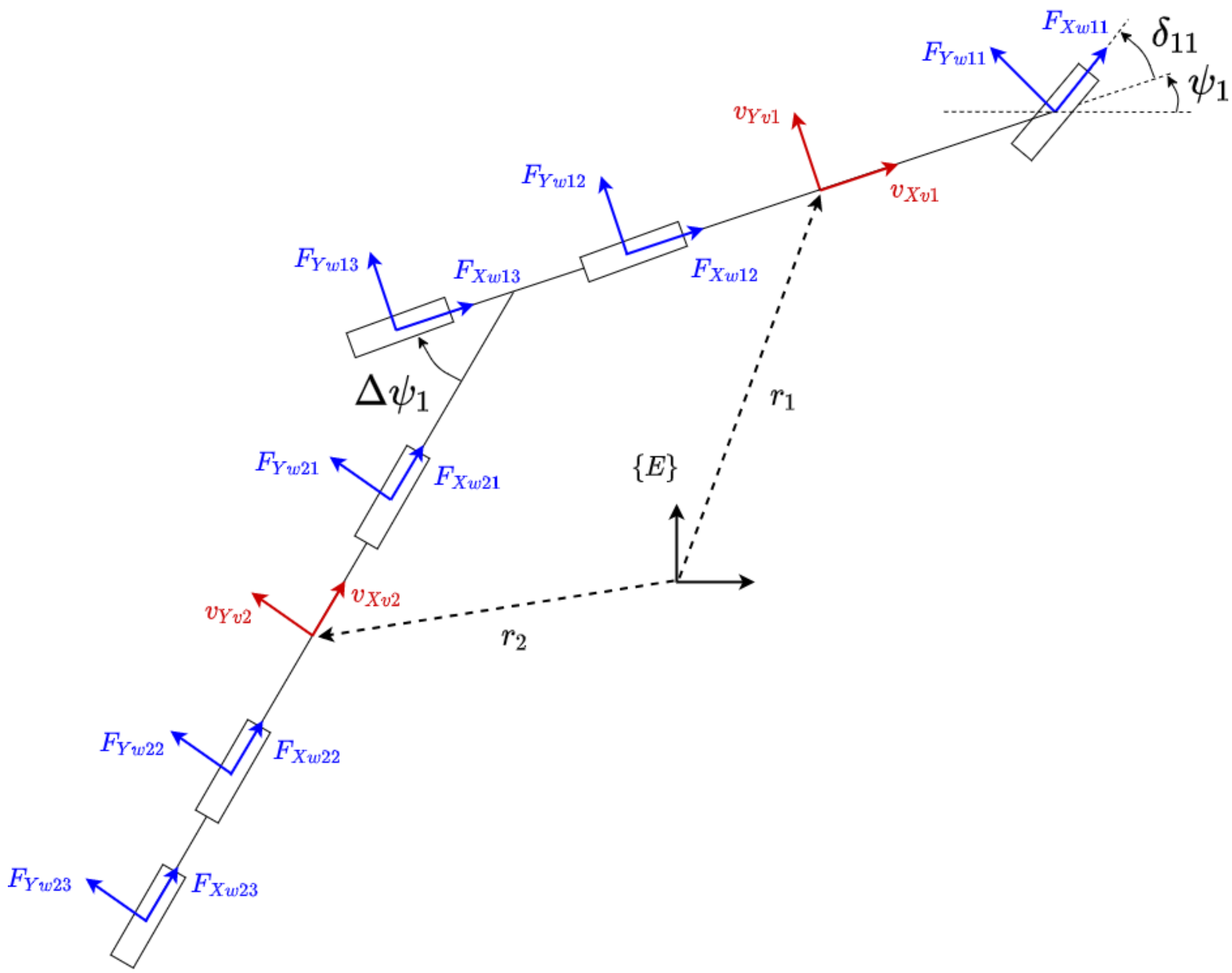

Figure 2: Semitrailer model: Position and forces vectors.

$\mathcal{N}_u$	: Set of numbers of vehicle units.
$\mathcal{N}_a$	: Set of numbers of vehicle axles.
j	: Vehicle unit number variable, where $j \in \mathcal{N}_u$.
k	: Vehicle axle number variable, where $k \in \mathcal{N}_a$.
Frame $\{E\}$	: Axis system fixed in the earth frame or reference frame.
Frame $\{vj\}$	: Axis system fixed in the center of gravity (CoG) in vehicle unit j.
Frame $\{wjk\}$	: Axis system fixed in unit j axle k.
R_z	: Rotation around z-axis.
R_{vj}	: Rotation matrix of Frame $\{vj\}$ with respect to Frame $\{E\}$.
R_{wjk}	: Rotation matrix of Frame $\{wjk\}$ with respect to Frame $\{E\}$.
ψ_j	: Inclination angle of vehicle unit j with respect to x-axis of Frame $\{E\}$. Unit: rad.
$\Delta\psi_j$	: Angle between vehicle unit j and unit $j+1$. Unit: rad.
δ_{jk}	: Inclination angle of Frame $\{wjk\}$ with respect to x-axis of Frame $\{vj\}$. Unit: rad.
$r_j = \begin{pmatrix} x_j & y_j \end{pmatrix}^T$	: Position vector of CoG of vehicle unit j with respect to Frame $\{E\}$. Unit: m
$v_j = \dot{r}_j = \begin{pmatrix} \dot{x}_j & \dot{y}_j \end{pmatrix}^T$	: Velocity vector of CoG of vehicle unit j with respect to Frame $\{E\}$. Unit: $\frac{m}{s}$
$v_j^{vj} = \begin{pmatrix} v_x & v_y \end{pmatrix}^T$	: Velocity vector of CoG of vehicle unit j with respect to Frame $\{vj\}$. Unit: $\frac{m}{s}$
$a_j = \ddot{r}_j = \begin{pmatrix} \ddot{x}_j & \ddot{y}_j \end{pmatrix}^T$	: Velocity vector of CoG of vehicle unit j w.r.t Frame $\{E\}$. Unit: $\frac{m}{s^2}$
r_{jk}	: Position vector of CoG of vehicle axle k unit j with respect to Frame $\{E\}$. Unit: m.

r_{jk}^{vj}	: Position vector of vehicle axle k unit j with respect to Frame $\{vj\}$. Unit: m.
$v_{jk} = \dot{r}_{jk}$	: Velocity vector of vehicle axle k unit j with respect to Frame $\{E\}$. Unit: $\frac{m}{s}$.
$v_{jk}^{vj} = \dot{r}_{jk}^{vj}$	: Velocity vector of vehicle axle k unit j with respect to Frame $\{vj\}$. Unit: $\frac{m}{s}$.
$v_{jk}^{wjk} = \dot{r}_{jk}^{wjk}$	: Velocity vector of vehicle axle k unit j with respect to Frame $\{wjk\}$. Unit: $\frac{m}{s}$.
m_j	: Mass of vehicle unit j. Unit: Kg
J_j	: Inertia around z-axis of vehicle unit j. Unit: $kg.m^2$.
l_{jk}	: Distance from CoG of vehicle unit j to axle k. Unit: m.
C_{yjk}	: Cornering stiffness of vehicle axle k unit j. Unit: $\frac{N}{rad}$.

2 Rotation matrices

$$R_z(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{1}$$

$$R_{v1} = R_z(\psi_1) \tag{2}$$

$$R_{v2} = R_z(\psi_2) \tag{3}$$

$$R_{w11} = R_z(\delta_{11}) \tag{4}$$

Some properties for rotation matrices,

$$R^{-1} = R^T \tag{5}$$

$$R_p^n = R_m^n R_p^m \tag{6}$$

$$r^n = R_p^n r^p \tag{7}$$

where r^n is the vector r represented in the Frame n and R_p^n is the rotation matrix of Frame $\{p\}$ with respect to Frame $\{n\}$.

3 Semitrailer dynamics using Euler-Lagrange equation

Some manipulations were done using the Euler-Lagrange equation in order to obtain the state-space equation. This steps are a bit different than other steps published in some articles and books. These steps help to compute the differential equation use for the state-space expression.

The Euler-Lagrange equation is defined by,

$$\frac{d}{dt}\Big(\frac{\partial \mathcal{L}(q,\dot{q})}{\partial \dot{q}}\Big) - \frac{\partial \mathcal{L}(q,\dot{q})}{\partial q} = Q \tag{8}$$

where q is the generalized coordinates, $\mathcal{L}(q,\dot{q})$ is the Lagrangian and Q are the generalized forces of the system.

The Lagrangian is defined as,

$$\mathcal{L}(q,\dot{q}) = T(q,\dot{q}) - V(q) \tag{9}$$

where T is the kinetic energy and V is the potential energy. For car modeling, the potential energy can be neglected, therefore the Euler-Lagrange equation can be expressed as,

$$\frac{d}{dt}\Big(\frac{\partial T(q,\dot{q})}{\partial \dot{q}}\Big) - \frac{\partial T(q,\dot{q})}{\partial q} = Q \tag{10}$$

Then by using chain-rule, the following equivalent expressions can be obtained,

$$\frac{d}{dt}\Big(\frac{\partial T(q,\dot{q})}{\partial \dot{q}}\Big) = \frac{\partial}{\partial q}\Big(\frac{\partial T(q,\dot{q})}{\partial \dot{q}}\Big)\dot{q} + \frac{\partial}{\partial \dot{q}}\Big(\frac{\partial T(q,\dot{q})}{\partial \dot{q}}\Big)\ddot{q} \tag{11}$$

using (11) in (10) and solving for $\ddot{q}$,

$$\ddot{q} = \left(\frac{\partial}{\partial \dot{q}}\Big(\frac{\partial T(q,\dot{q})}{\partial \dot{q}}\Big)\right)^{-1}\left(\frac{\partial T(q,\dot{q})}{\partial q} - \frac{\partial}{\partial q}\Big(\frac{\partial T(q,\dot{q})}{\partial \dot{q}}\Big)\dot{q} + Q\right) \tag{12}$$

3.1 Generalized coordinates

$$q = \begin{pmatrix} x_1 \\ y_1 \\ \psi_1 \\ \Delta\psi_1 \end{pmatrix} \quad \longrightarrow \quad \dot{q} = \begin{pmatrix} \dot{x}_1 \\ \dot{y}_1 \\ \dot{\psi}_1 \\ \Delta\dot{\psi}_1 \end{pmatrix} \quad \longrightarrow \quad \ddot{q} = \begin{pmatrix} \ddot{x}_1 \\ \ddot{y}_1 \\ \ddot{\psi}_1 \\ \Delta\ddot{\psi}_1 \end{pmatrix} \tag{13}$$

3.2 Kinetic energy

The total kinetic energy for the semi-trailer combination is given by,

$$T(q,\dot{q}) = \sum_{j \in \mathcal{N}_u} \left(\frac{1}{2} m_j v_j^T v_j + \frac{1}{2} J_1 \dot{\psi}_j^2 \right) \tag{14}$$

$$T(q,\dot{q}) = \frac{1}{2} m_1 v_1^T v_1 + \frac{1}{2} m_2 v_2^T v_2 + \frac{1}{2} J_1 \dot{\psi}_1^2 + \frac{1}{2} J_2 \dot{\psi}_2^2 \tag{15}$$

using chain rule, $\dot{r}$ can be calculated by,

$$v_j = \dot{r}_j = \frac{\partial r_j}{\partial q} \dot{q} \tag{16}$$

The following position vectors are defined,

$$r_{1c1}^{vj} = \begin{pmatrix} -l_{1c1} \\ 0 \end{pmatrix} \quad , \quad r_{2c1}^{v2} = \begin{pmatrix} l_{2c1} \\ 0 \end{pmatrix} \tag{17}$$

where r_{jc1}^{v1} is the distance between center of gravity of vehicle unit j and joint 1 represent in Frame $\{vj\}$.

Then, r_2 can be calculated as,

$$r_2 = r_1 + R_{v1} r_{1c1}^{v1} - R_{v2} r_{2c1}^{v2} \tag{18}$$

Therefore, the position vectors of each center of gravity can be expressed as a function of q,

$$r_1(q) = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \tag{19}$$

$$r_2(q) = r_1 + R_{v1} \begin{pmatrix} -l_{1c1} \\ 0 \end{pmatrix} - R_{v2} \begin{pmatrix} l_{2c1} \\ 0 \end{pmatrix} \tag{20}$$

The translational velocities vector for center of gravity of vehicle unit j can be calculated as,

$$v_j(q, \dot{q}) = \dot{r}_j = \frac{\partial r_j}{\partial q}\dot{q} \quad , \quad j \in \mathcal{N}_u \tag{21}$$

Furthermore, you can calculate ψ_2 by,

$$\psi_2(q) = \psi_1 - \Delta\psi_1 \tag{22}$$

3.3 Generalized forces

The generalized forces can be calculated by,

$$Q = \sum_{j \in \mathcal{N}_u} \sum_{k \in \mathcal{N}_a} \left(\frac{\partial r_{jk}}{\partial q}\right)^T F_{jk} + \sum_{j \in \mathcal{N}_u} \left(\frac{\partial r_j}{\partial q}\right)^T F_j \tag{23}$$

$$\begin{aligned} Q =& \left(\frac{\partial r_{11}}{\partial q}\right)^T F_{11} + \left(\frac{\partial r_{12}}{\partial q}\right)^T F_{12} + \left(\frac{\partial r_{13}}{\partial q}\right)^T F_{13} + \\ & \left(\frac{\partial r_{21}}{\partial q}\right)^T F_{21} + \left(\frac{\partial r_{22}}{\partial q}\right)^T F_{22} + \left(\frac{\partial r_{23}}{\partial q}\right)^T F_{23} + \\ & \left(\frac{\partial r_1}{\partial q}\right)^T F_1 + \left(\frac{\partial r_2}{\partial q}\right)^T F_2 \end{aligned} \tag{24}$$

where,

$$\begin{aligned} r_{11}^{v1} &= \begin{pmatrix} l_{11} \\ 0 \end{pmatrix} \quad , \quad r_{21}^{v2} = \begin{pmatrix} l_{21} \\ 0 \end{pmatrix} \\ r_{12}^{v1} &= \begin{pmatrix} -l_{12} \\ 0 \end{pmatrix} \quad , \quad r_{22}^{v2} = \begin{pmatrix} -l_{22} \\ 0 \end{pmatrix} \\ r_{13}^{v1} &= \begin{pmatrix} -l_{13} \\ 0 \end{pmatrix} \quad , \quad r_{23}^{v2} = \begin{pmatrix} -l_{23} \\ 0 \end{pmatrix} \end{aligned} \tag{25}$$

In order to calculate r_{jk}, we can use homogeneous transformation matrices,

$$\begin{pmatrix} r_{jk} \\ 1 \end{pmatrix} = H_j \begin{pmatrix} r_{jk}^{vj} \\ 1 \end{pmatrix} = \begin{pmatrix} R_{vj} & r_j \\ 0_{1\times 3} & 1 \end{pmatrix} \begin{pmatrix} r_{jk}^{vj} \\ 1 \end{pmatrix} \tag{26}$$

where H_j is the homogeneous transformation matrix of Frame $\{vj\}$ with respect to Frame $\{E\}$. Then, the axles position vectors are calculated using,

$$r_{jk} = r_j + R_{vj} r_{jk}^{vj} \quad , \quad (j,k) \in \mathcal{N}_u \times \mathcal{N}_a \tag{27}$$

The translational velocity vector of vehicle unit j axle k with respect to the earth frame (v_{jk}) can be define as,

$$v_{jk} = \frac{\partial r_{jk}}{\partial q} \dot{q} \tag{28}$$

This vector expressed in a wheel fixed frame can be calculated as,

$$\begin{pmatrix} v_{Xwjk} \\ v_{Ywjk} \end{pmatrix} = \begin{cases} R_{wjk} v_{jk} & , \quad (j,k) \in \{(1,1)\} \\ R_{vj} v_{jk} & , \quad (j,k) \in \mathcal{N}_u \times \mathcal{N}_a - \{(1,1)\} \end{cases} \tag{29}$$

Fig. 2 shows the forces acting on the semi-trailer combination. These forces are expressed in the fixed body frame of each unit and wheel. The forces acting on each center of mass are defined by,

$$F_{Xvj} = \begin{cases} F_{air} + F_{grav,j} & j = 1 \\ F_{grav,j} & j = 2 \end{cases} \tag{30}$$

$$F_{Yvj} = 0 \tag{31}$$

Assuming no longitudinal slip, small lateral slip and constant normal load, the lateral tyre forces can be approximated using,

$$F_{Ywjk} = -C_{Yjk} \frac{v_{Ywjk}}{v_{Xwjk}} \tag{32}$$

where C_{Yjk} is the tyre cornering stiffness of unit j axle k.

The longitudinal tyre forces are generated by braking and propulsion and can be calculated as,

$$F_{Xwjk} = \begin{cases} F_{prop,jk} + F_{brake,jk} & , \quad (j,k) \in \{(1,2)\} \\ 0 & , \quad \mathcal{N}_u \times \mathcal{N}_a - \{(1,2)\} \end{cases} \tag{33}$$

In order to calculate F_j and F_{jk}, which are expressed with respect to Frame $\{E\}$, the force vectors (F_{xvj}, F_{yvj}) and (F_{Xwjk}, F_{Ywjk}) should be rotated,

$$F_j = R_{vj} \begin{pmatrix} F_{Xvj} \\ F_{Yvj} \end{pmatrix} \quad , \quad j \in \mathcal{N}_u \tag{34}$$

$$F_{jk} = \begin{cases} R_{wjk} \begin{pmatrix} F_{Xwjk} \\ F_{Ywjk} \end{pmatrix} & , \quad (j,k) \in \{(1,1)\} \\ R_{vj} \begin{pmatrix} F_{Xwjk} \\ F_{Ywjk} \end{pmatrix} & , \quad (j,k) \in \mathcal{N}_u \times \mathcal{N}_a - \{(1,1)\} \end{cases} \tag{35}$$

In this study the aerodynamic drag and gravitational forces are neglected. Therefore,

$$F_{Xvj} = F_{Yvj} = 0 \quad , \quad j \in \mathcal{N}_u \tag{36}$$

After all the operations, $\ddot{q}$ is estimated as a function of $\dot{x}_1$, $\dot{y}_1$, ψ_1, $\dot{\psi}_1$, $\Delta\psi_1$ and $\Delta\dot{\psi}_1$,

$$\ddot{q} = \begin{pmatrix} \ddot{x}_1 \\ \ddot{y}_1 \\ \ddot{\psi}_1 \\ \Delta\ddot{\psi}_1 \end{pmatrix} = \mathcal{F}(\dot{x}_1, \dot{y}_1, \psi_1, , \dot{\psi}_1, \Delta\psi_1, \Delta\dot{\psi}_1) \tag{37}$$

Since the velocity sensors on the car measure the longitudinal and lateral velocity of the car with respect of Frame $\{v1\}$. So we want something like,

$$\ddot{q}_v = \begin{pmatrix} \dot{v}_{Xv1} \\ \dot{v}_{Yv1} \\ \ddot{\psi}_1 \\ \Delta\ddot{\psi}_1 \end{pmatrix} = \mathcal{F}(v_{Xv1}, v_{Yv1}, \psi_1, , \dot{\psi}_1, \Delta\psi_1, \Delta\dot{\psi}_1) \tag{38}$$

In order to expressed the dynamics as a function of v_{Xv1} and v_{Yv1} instead of $\dot{x}_1$ and $\dot{y}_1$, the following rotation is done,

$$\begin{pmatrix} \dot{x}_1 \\ \dot{y}_1 \end{pmatrix} = R_{v1} \begin{pmatrix} v_{Xv1} \\ v_{Yv1} \end{pmatrix} \tag{39}$$

Moreover, calculating the derivative with respect to time,

$$\begin{pmatrix} \ddot{x}_1 \\ \ddot{y}_1 \end{pmatrix} = \dot{R}_{v1} \begin{pmatrix} v_{Xv1} \\ v_{Yv1} \end{pmatrix} + R_{v1} \begin{pmatrix} \dot{v}_{Xv1} \\ \dot{v}_{Yv1} \end{pmatrix} \tag{40}$$

Rewriting (40),

$$\begin{pmatrix} \dot{v}_{Xv1} \\ \dot{v}_{Yv1} \end{pmatrix} = R_{v1}^T \begin{pmatrix} \ddot{x}_1 \\ \ddot{y}_1 \end{pmatrix} - R_{v1}^T \dot{R}_{v1} \begin{pmatrix} v_{Xv1} \\ v_{Yv1} \end{pmatrix} \tag{41}$$

4 State-space model

4.1 Nonlinear state-space model: Lateral and longitudinal dynamics

The following state and input vectors are selected for the semi-trailer nonlinear model,

$$\mathcal{X}_{NL} = \begin{pmatrix} v_{Xv1} \\ v_{Yv1} \\ \dot{\psi}_1 \\ \Delta\dot{\psi}_1 \\ \psi_1 \\ \Delta\psi_1 \end{pmatrix} \quad , \quad U_{NL} = \begin{pmatrix} \delta_{11} \\ F_{Xw12} \end{pmatrix} \tag{42}$$

The nonlinear state-space model is defined by,

$$\dot{\mathcal{X}}_{NL} = \begin{pmatrix} \dot{v}_{Xv1} \\ \dot{v}_{Yv1} \\ \ddot{\psi}_1 \\ \Delta\ddot{\psi}_1 \\ \dot{\psi}_1 \\ \Delta\dot{\psi}_1 \end{pmatrix} = f_{NL}(\mathcal{X}_{NL}, U_{NL}) \tag{43}$$

4.2 Nonlinear state-space model: Only lateral dynamics

In order to consider only lateral dynamics, it is possible to assume constant longitudinal velocity,

$$v_{Xv1} = \text{Constant} \tag{44}$$

For constant longitudinal velocity, it is assumed that the sum of the projections to x-axis of Frame $\{v1\}$ of all the forces apply to the car is zero, i.e.,

$$F_{Xw11}\cos\delta_{11} - F_{Yw11}\sin\delta_{11} + F_{Xw12} + F_{13} = 0 \tag{45}$$

The following state and input vectors are selected for the semi-trailer nonlinear model,

$$\mathcal{X}_{nl} = \begin{pmatrix} v_{Yv1} \\ \dot{\psi}_1 \\ \Delta\dot{\psi}_1 \\ \psi_1 \\ \Delta\psi_1 \end{pmatrix} \quad , \quad U_{nl} = \delta_{11} \tag{46}$$

The nonlinear state-space model is defined by,

$$\dot{\mathcal{X}}_{nl} = \begin{pmatrix} \dot{v}_{Yv1} \\ \ddot{\psi}_1 \\ \Delta\ddot{\psi}_1 \\ \dot{\psi}_1 \\ \Delta\dot{\psi}_1 \end{pmatrix} = f_{nl}(\mathcal{X}_{nl}, U_{nl}) \tag{47}$$

4.3 Linear state-space model: Lateral and longitudinal dynamics

In order to linearize f_{NL} (showed in (43)), the following equilibrium $\begin{pmatrix} \mathcal{X}_e & \mathcal{Y}_e \end{pmatrix}^T$ point is selected,

$$\mathcal{X}_e = \begin{pmatrix} v_{Xv1e} \\ v_{Yv1e} \\ \dot{\psi}_{1e} \\ \Delta\dot{\psi}_{1e} \\ \psi_{1e} \\ \Delta\psi_{1e} \end{pmatrix} = \begin{pmatrix} v_{Xv1d} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad , \quad U_e = \begin{pmatrix} \delta_{11e} \\ F_{Xw12e} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{48}$$

where v_{Xv1d} is the desired longitudinal velocity of the car. Additionally, around the equilibrium point no acceleration or braking is needed in order to keep constant longitudinal velocity. Therefore, F_{Xw12e} can be approximated to zero,

$$F_{Xw12e} = 0 \tag{49}$$

Then, f_{NL} can be linearized around the equilibrium point ($\mathcal{X}_e$,U_e),

$$\dot{\mathcal{X}} = f_{LINEAR} = A(v_{Xv1d})\mathcal{X} + B(v_{Xv1d})U \tag{50}$$

4.4 Linear state-space model: Only lateral dynamics

In order to linearize f_{nl} (showed in (47)), the following equilibrium $\begin{pmatrix} \mathcal{X}_e & \mathcal{Y}_e \end{pmatrix}^T$ point is selected,

$$\mathcal{X}_e = \begin{pmatrix} v_{Yv1e} \\ \dot{\psi}_{1e} \\ \Delta\dot{\psi}_{1e} \\ \psi_{1e} \\ \Delta\psi_{1e} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad , \quad U_e = \delta_e = 0 \tag{51}$$

Then, f_{nl} can be linearized around the equilibrium point ($\mathcal{X}_e$,U_e),

$$\dot{\mathcal{X}} = f_{linear} A(v_{Xv1d})\mathcal{X} + B(v_{Xv1d})U \tag{52}$$