Car modelling

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January 11, 2021

1 System description

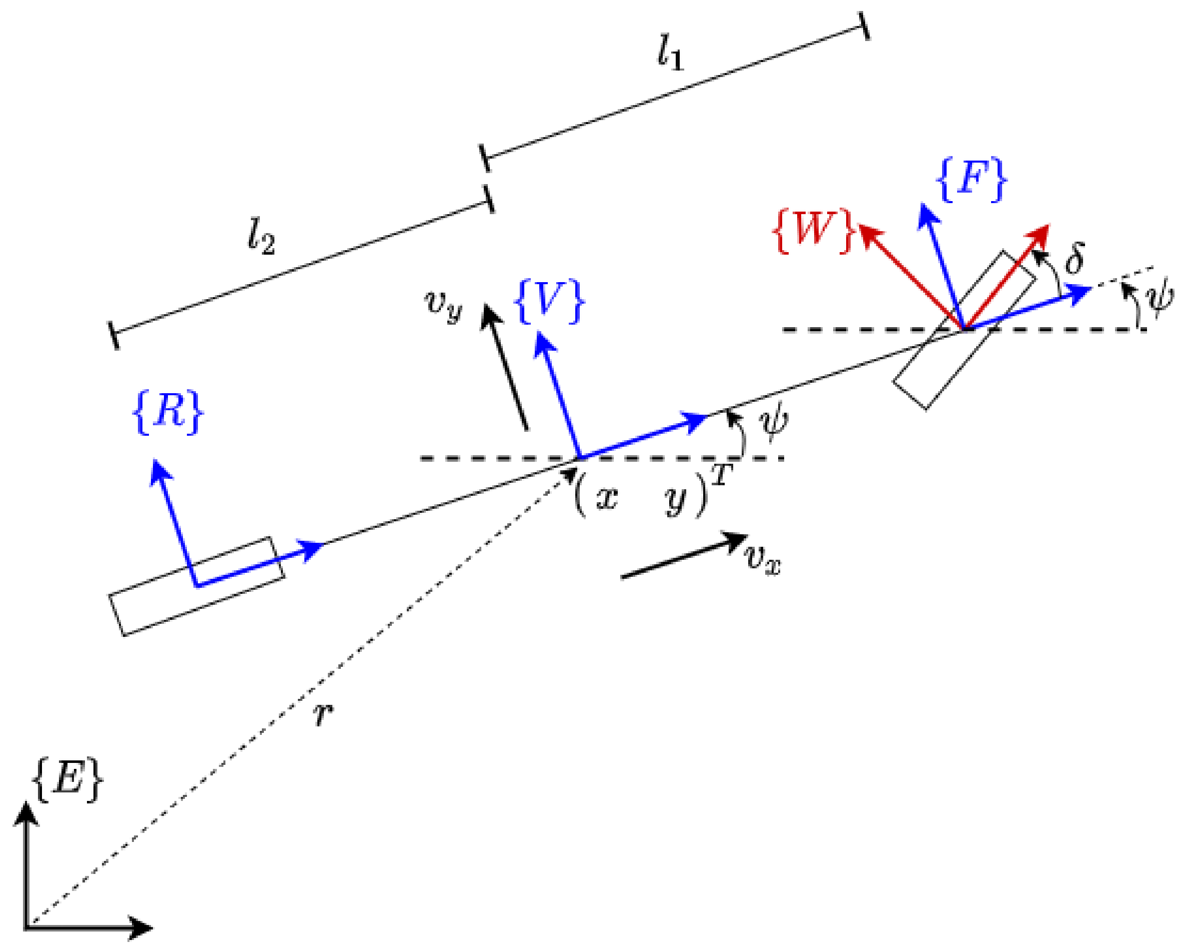

Figure 1: Car model: Single axle bicycle model

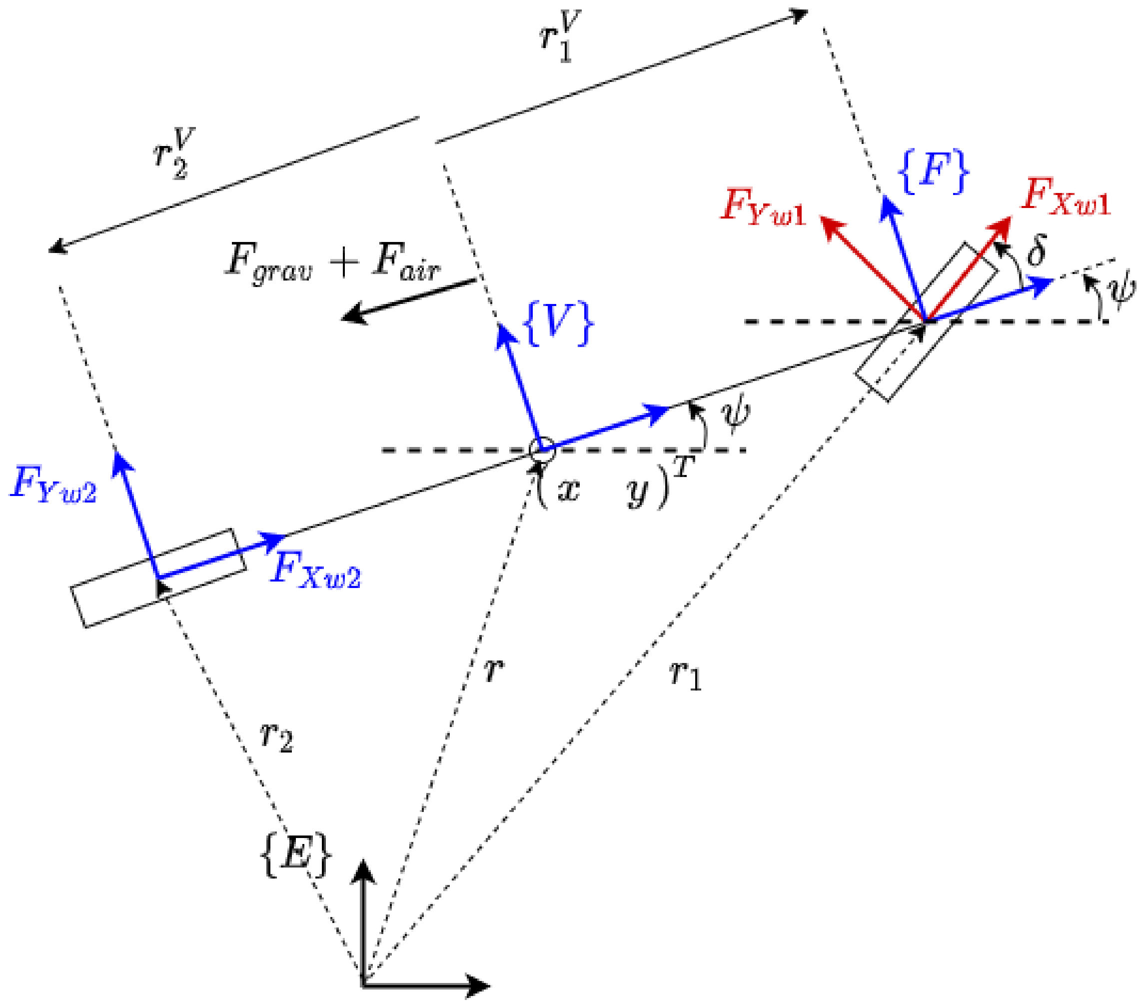

Figure 2: Car model: Position and forces vectors.

Frame $\{E\}$	: Axis system fixed in the earth frame or reference frame.
Frame $\{V\}$	: Axis system fixed in the center of gravity (CoG) of the car.
Frame $\{W\}$	: Axis system fixed in the front wheel.
Frame $\{F\}$	: Axis system fixed in the front axle (axle 1).
Frame $\{R\}$	: Axis system fixed in the rear axle (axle 2).
R_V^E	: Rotation matrix for Frame $\{V\}$ (axis system fixed in the Cog of car) w.r.t. Frame $\{E\}$.
R_W^E	: Rotation matrix for front wheel w.r.t. Frame $\{E\}$.
$r = \begin{pmatrix} x & y \end{pmatrix}^T$	: Position vector of CoG of car w.r.t Frame $\{E\}$. Unit: m
$v = \dot{r} = \begin{pmatrix} \dot{x} & \dot{y} \end{pmatrix}^T$	: Velocity vector of CoG of car w.r.t Frame $\{E\}$. Unit: $\frac{m}{s}$
$v^V = \begin{pmatrix} v_x & v_y \end{pmatrix}^T$	: Velocity vector of CoG of car w.r.t Frame $\{V\}$. Unit: $\frac{m}{s}$
$a = \ddot{r} = \begin{pmatrix} \ddot{x} & \ddot{y} \end{pmatrix}^T$	: Velocity vector of CoG of car w.r.t Frame $\{E\}$. Unit: $\frac{m}{s^2}$
$r_1 = \begin{pmatrix} x_1 & y_1 \end{pmatrix}^T$	: Position vector of axle 1 w.r.t Frame $\{E\}$. Unit: m
$v_1 = \dot{r}_1 = \begin{pmatrix} \dot{x}_1 & \dot{y}_1 \end{pmatrix}^T$	: Velocity vector of axle 1 w.r.t Frame $\{E\}$. Unit: $\frac{m}{s}$
$v_1^W = \dot{r}_1 = \begin{pmatrix} v_{Xw1} & v_{Yw1} \end{pmatrix}^T$	: Velocity vector of axle 1 w.r.t Frame $\{W\}$. Unit: $\frac{m}{s}$
$r_2 = \begin{pmatrix} x_2 & y_2 \end{pmatrix}^T$	: Position vector of axle 2 w.r.t Frame $\{E\}$. Unit: [m]
$v_2 = \dot{r} = \begin{pmatrix} \dot{x}_2 & \dot{y}_2 \end{pmatrix}^T$	: Velocity vector of axle 2 w.r.t Frame $\{E\}$. Unit: $\frac{m}{s}$
$v_2^V = \begin{pmatrix} v_{Xw2} & v_{Yw2} \end{pmatrix}^T$	: Velocity vector of axle 2 w.r.t Frame $\{E\}$. Unit: $\frac{m}{s}$
ψ	: Inclination of Unit 1 w.r.t x-axis of Frame $\{E\}$. Unit: rad
δ	: Wheel steer angle of front wheel w.r.t x-axis of Frame $\{E\}$. Unit: rad

l_1	: Distance between axle 1 (front axle) and centre of gravity. Unit: m
l_2	: Distance between axle 2 (rear axle) and centre of gravity. Unit: m
C_{y1}	: Cornering stiffness axle 1. Unit: $\frac{N}{rad}$
C_{y2}	: Cornering stiffness axle 2. Unit: $\frac{N}{rad}$
m	: Car mass. Unit: Kg
J	: Inertia around z-axis of vehicle Unit. Unit: $kg.m^2$

2 Rotation matrices

The following rotation matrices of Frame $\{V\}$ and $\{W\}$ with respect to Frame $\{E\}$ are calculated by,

$$R_V^E = R_F^E = R_R^E = R_z(\psi) = \begin{pmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{pmatrix} \tag{1}$$

$$R_W^E = R_z(\psi+\delta) = \begin{pmatrix} \cos(\psi+\delta) & -\sin(\psi+\delta) \\ \sin(\psi+\delta) & \cos(\psi+\delta) \end{pmatrix} \tag{2}$$

Additionally, we can calculate the rotation matrix of Frame $\{W\}$ with respect to Frame $\{E\}$ by,

$$R_W^V = R_z(\delta) = \begin{pmatrix} \cos\delta & -\sin\delta \\ \sin\delta & \cos\delta \end{pmatrix} \tag{3}$$

Some properties for rotation matrices,

$$R^{-1} = R^T \tag{4}$$

$$R_p^n = R_m^n R_p^m \tag{5}$$

$$r^n = R_p^n r^p \tag{6}$$

where r^i is the vector r represented in the Frame i and R_j^i is the rotation matrix of Frame $\{j\}$ with respect to Frame $\{i\}$.

3 Car dynamics using Euler-Lagrange equation

Some manipulations were done using the Euler-Lagrange equation in order to obtain the state-space equation. This steps are a bit different than other steps published in some articles and books. These steps help to compute the differential equation use for the state-space expression.

The Euler-Lagrange equation is defined by,

$$\frac{d}{dt}\Big(\frac{\partial \mathcal{L}(q,\dot{q})}{\partial \dot{q}}\Big) - \frac{\partial \mathcal{L}(q,\dot{q})}{\partial q} = Q \tag{7}$$

where q is the generalized coordinates, $\mathcal{L}(q,\dot{q})$ is the Lagrangian and Q are the generalized forces of the system.

The Lagrangian is defined as,

$$\mathcal{L}(q,\dot{q}) = T(q,\dot{q}) - V(q) \tag{8}$$

where T is the kinetic energy and V is the potential energy. For car modeling, the potential energy can be neglected, therefore the Euler-Lagrange equation can be expressed as,

$$\frac{d}{dt}\Big(\frac{\partial T(q,\dot{q})}{\partial \dot{q}}\Big) - \frac{\partial T(q,\dot{q})}{\partial q} = Q \tag{9}$$

Then by using chain-rule, the following equivalent expressions can be obtained,

$$\frac{d}{dt}\Big(\frac{\partial T(q,\dot{q})}{\partial \dot{q}}\Big) = \frac{\partial}{\partial q}\Big(\frac{\partial T(q,\dot{q})}{\partial \dot{q}}\Big)\dot{q} + \frac{\partial}{\partial \dot{q}}\Big(\frac{\partial T(q,\dot{q})}{\partial \dot{q}}\Big)\ddot{q} \tag{10}$$

using (10) in (9) and solving for $\ddot{q}$,

$$\ddot{q} = \left(\frac{\partial}{\partial \dot{q}}\Big(\frac{\partial T(q,\dot{q})}{\partial \dot{q}}\Big)\right)^{-1}\left(\frac{\partial T(q,\dot{q})}{\partial q} - \frac{\partial}{\partial q}\Big(\frac{\partial T(q,\dot{q})}{\partial \dot{q}}\Big)\dot{q} + Q\right) \tag{11}$$

3.1 Generalized coordinates

$$q = \begin{pmatrix} x \\ y \\ \psi \end{pmatrix} \quad \longrightarrow \quad \dot{q} = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\psi} \end{pmatrix} \quad \longrightarrow \quad \ddot{q} = \begin{pmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{\psi} \end{pmatrix} \tag{12}$$

3.2 Kinetic energy

The total kinetic energy for the semi-trailer combination is given by,

$$T(q, \dot{q}) = \frac{1}{2} m \dot{r}^T \dot{r} + \frac{1}{2} J \dot{\psi}^2 \tag{13}$$

using chain rule, $\dot{r}$ can be calculated by,

$$\dot{r} = \frac{\partial r}{\partial q} \dot{q} \tag{14}$$

3.3 Generalized forces

The generalized forces can be calculated by,

$$Q = \left(\frac{\partial r}{\partial q}\right)^T F + \left(\frac{\partial r_1}{\partial q}\right)^T F_1 + \left(\frac{\partial r_2}{\partial q}\right)^T F_2 \tag{15}$$

where F is the force applied on the center of gravity of the car and F_j is the force applied on axle j. All of these forces are represented in Frame $\{E\}$.

In this case, we will neglected the forces by gravity (roads with slope) or aerodynamics (air) on the car. Therefore,

$$F = -F_{grav} - F_{air} = 0 \tag{16}$$

In order to calculate r_j, were j is the axle number, we can define the position vector of each axle in Frame $\{V\}$,

$$r_1^V = \begin{pmatrix} l_1 \\ 0 \end{pmatrix} \tag{17}$$

$$r_2^V = \begin{pmatrix} -l_2 \\ 0 \end{pmatrix} \tag{18}$$

Then, the position vector of each axle in Frame E can be calculated by,

$$r_1 = r_1^E = r + R_V^E r_1^V \tag{19}$$

$$r_2 = r_2^E = r + R_V^E r_2^V \tag{20}$$

The forces applied on each tyre (or wheel) with respect to the corresponding tyre frame can be calculated by,

$$F_1^W = \begin{pmatrix} F_{Xw1} \\ F_{Yw1} \end{pmatrix} = \begin{pmatrix} F_{prop,1} + F_{brake,1} \\ -C_{Y1}\frac{v_{Yw1}}{v_{Xw1}} \end{pmatrix} \tag{21}$$

$$F_2^R = \begin{pmatrix} F_{Xw2} \\ F_{Yw2} \end{pmatrix} = \begin{pmatrix} F_{prop,2} + F_{brake,2} \\ -C_{Y2}\frac{v_{Yw2}}{v_{Xw2}} \end{pmatrix} \tag{22}$$

where $F_{prop,j}$ is the propulsion force on tyre j, $F_{brake,j}$ the braking force on tyre j, C_{Yj} is the cornering stiffness of tyre j and $\begin{pmatrix} v_{Xwj} & v_{Ywj} \end{pmatrix}^T$ is the velocity vector of tyre j with respect to the frame fixed to the tyre.

Those values are obtained using,

$$\begin{pmatrix} v_{Xw1} \\ v_{Yw1} \end{pmatrix} = R_E^W \dot{r}_1 \tag{23}$$

$$\begin{pmatrix} v_{Xw2} \\ v_{Yw2} \end{pmatrix} = R_E^R \dot{r}_2 \tag{24}$$

where,

$$\dot{r}_1 = \frac{\partial r_1}{\partial q}\dot{q} \tag{25}$$

$$\dot{r}_2 = \frac{\partial r_2}{\partial q}\dot{q} \tag{26}$$

Additionally, it is assumed that propulsion and braking is only applied in tyre 2. Then,

$$F_{prop,1} = F_{brake,1} = 0 \tag{27}$$

Forces applied on each tyre can be obtained by,

$$F_1 = R_W^E F_1^W \tag{28}$$

$$F_2 = R_R^E F_2^R \tag{29}$$

After all the operations, $\ddot{q}$ is estimated as a function of ψ, $\dot{x}$, $\dot{y}$ and $\dot{\psi}$,

$$\ddot{q} = \begin{pmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{\psi} \end{pmatrix} = \mathcal{F}(\psi, \dot{x}, \dot{y}, \dot{\psi}) \tag{30}$$

Since the velocity sensors on the car measure the longitudinal and lateral velocity of the car with respect of Frame $\{V\}$. So we want something like,

$$\ddot{q}_v = \begin{pmatrix} \dot{v}_x \\ \dot{v}_y \\ \ddot{\psi} \end{pmatrix} = \mathcal{F}(\psi, v_x, v_y, \dot{\psi}) \tag{31}$$

then we should transform,

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = R_V^E \begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} v_x \cos\psi - v_y \sin\psi \\ v_x \sin\psi + v_y \cos\psi \end{pmatrix} \tag{32}$$

Moreover, calculating the derivative with respect to time of (32),

$$\begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix} = \dot{R}_V^E \begin{pmatrix} v_x \\ v_y \end{pmatrix} + R_V^E \begin{pmatrix} \dot{v}_x \\ \dot{v}_y \end{pmatrix} \tag{33}$$

Rewriting (33),

$$\begin{pmatrix} \dot{v}_x \\ \dot{v}_y \end{pmatrix} = R_E^V \begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix} - R_E^V \dot{R}_V^E \begin{pmatrix} v_x \\ v_y \end{pmatrix} \tag{34}$$

4 State-space model

4.1 Nonlinear state-space model: Lateral and longitudinal dynamics

The following state and input vectors are selected for the semi-trailer nonlinear model,

$$\mathcal{X}_{NL} = \begin{pmatrix} v_x \\ v_y \\ \dot{\psi} \end{pmatrix} \quad , \quad U_{NL} = \begin{pmatrix} \delta \\ F_{Xw2} \end{pmatrix} \tag{35}$$

The nonlinear state-space model is defined by,

$$\dot{\mathcal{X}}_{NL} = \begin{pmatrix} \dot{v}_{x1} \\ \dot{v}_{y1} \\ \ddot{\psi}_1 \end{pmatrix} = f_{NL}(\mathcal{X}_{NL}, U_{NL}) \tag{36}$$

4.2 Nonlinear state-space model: Only lateral dynamics

In order to consider only lateral dynamics, it is possible to assume constant longitudinal velocity,

$$v_x = \text{Constant} \tag{37}$$

For constant longitudinal velocity, it is assumed that the sum of the projections to x-axis of Frame $\{V\}$ of all the forces apply to the car is zero, i.e.,

$$F_{Xw1} \cos\delta - F_{Yw1} \sin\delta + F_{Xw2} = 0 \tag{38}$$

The following state and input vectors are selected for the semi-trailer nonlinear model,

$$\mathcal{X}_{nl} = \begin{pmatrix} v_y \\ \dot{\psi} \end{pmatrix} \quad , \quad U_{nl} = \delta \tag{39}$$

The nonlinear state-space model is defined by,

$$\dot{\mathcal{X}}_{nl} = \begin{pmatrix} \dot{v}_{y1} \\ \ddot{\psi}_1 \end{pmatrix} = f_{nl}(\mathcal{X}_{nl}, U_{nl}) \tag{40}$$

4.3 Linear state-space model: Lateral and longitudinal dynamics

In order to linearize f_{NL} (showed in (36)), the following equilibrium $\begin{pmatrix} \mathcal{X}_e & \mathcal{Y}_e \end{pmatrix}^T$ point is selected,

$$\mathcal{X}_e = \begin{pmatrix} v_{xe} \\ v_{ye} \\ \dot{\psi}_e \end{pmatrix} = \begin{pmatrix} v_{xd} \\ 0 \\ 0 \end{pmatrix} \quad , \quad U_e = \begin{pmatrix} \delta_e \\ F_{Xw2e} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{41}$$

where v_{xd} is the desired longitudinal velocity of the car. Additionally, around the equilibrium point no acceleration or braking is needed in order to keep constant longitudinal velocity. Therefore, F_{Xw2e} can be approximated to zero,

$$F_{Xw2e} = 0 \tag{42}$$

Then, f_{NL} can be linearized around the equilibrium point ($\mathcal{X}_e$,U_e),

$$\dot{\mathcal{X}} = f_{LINEAR} = A(v_{xd})\mathcal{X} + B(v_{xd})U \tag{43}$$

4.4 Linear state-space model: Lateral dynamics

In order to linearize f_{nl} (showed in (40)), the following equilibrium $\begin{pmatrix} \mathcal{X}_e & \mathcal{Y}_e \end{pmatrix}^T$ point is selected,

$$\mathcal{X}_e = \begin{pmatrix} v_{ye} \\ \dot{\psi}_e \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad , \quad U_e = \delta_e = 0 \tag{44}$$

Then, f_{nl} can be linearized around the equilibrium point ($\mathcal{X}_e$,U_e),

$$\dot{\mathcal{X}} = f_{linear} = A(v_{xd})\mathcal{X} + B(v_{xd})U \tag{45}$$