EE364a Homework 2 solutions

2.24 *Supporting hyperplanes.*

(a) Express the closed convex set $\{x \in \mathbf{R}^2_+ \mid x_1 x_2 \geq 1\}$ as an intersection of halfspaces.

Solution. The set is the intersection of all supporting halfspaces at points in its boundary, which is given by $\{x \in \mathbf{R}^2_+ \mid x_1 x_2 = 1\}$. The supporting hyperplane at $x = (t, 1/t)$ is given by

$$x_1/t^2 + x_2 = 2/t,$$

so we can express the set as

$$\bigcap_{t>0} \{x \in \mathbf{R}^2 \mid x_1/t^2 + x_2 \geq 2/t\}.$$

(b) Let $C = \{x \in \mathbf{R}^n \mid \|x\|_\infty \leq 1\}$, the ℓ_∞-norm unit ball in $\mathbf{R}^n$, and let $\hat{x}$ be a point in the boundary of C. Identify the supporting hyperplanes of C at $\hat{x}$ explicitly.

Solution. $s^T x \geq s^T \hat{x}$ for all $x \in C$ if and only if

$$\begin{array}{ll} s_i < 0 & \hat{x}_i = 1 \\ s_i > 0 & \hat{x}_i = -1 \\ s_i = 0 & -1 < \hat{x}_i < 1. \end{array}$$

2.27 *Converse supporting hyperplane theorem.* Suppose the set C is closed, has nonempty interior, and has a supporting hyperplane at every point in its boundary. Show that C is convex.

Solution. Let H be the set of all halfspaces that contain C. H is a closed convex set, and contains C by definition.

The support function S_C of a set C is defined as $S_C(y) = \sup_{x \in C} y^T x$. The set H and its interior can be defined in terms of the support function as

$$H = \bigcap_{y \neq 0} \{x \mid y^T x \leq S_C(y)\}, \qquad \mathbf{int}\, H = \bigcap_{y \neq 0} \{x \mid y^T x < S_C(y)\},$$

and the boundary of H is the set of all points in H with $y^T x = S_C(y)$ for at least one $y \neq 0$.

By definition $\mathbf{int}\, C \subseteq \mathbf{int}\, H$. We also have $\mathbf{bd}\, C \subseteq \mathbf{bd}\, H$: if $\bar{x} \in \mathbf{bd}\, C$, then there exists a supporting hyperplane at $\bar{x}$, *i.e.*, a vector $a \neq 0$ such that $a^T \bar{x} = S_C(a)$, *i.e.*, $\bar{x} \in \mathbf{bd}\, H$.

We now show that these properties imply that C is convex. Consider an arbitrary line intersecting $\mathbf{int}\, C$. The intersection is a union of disjoint open intervals I_k, with endpoints in $\mathbf{bd}\, C$ (hence also in $\mathbf{bd}\, H$), and interior points in $\mathbf{int}\, C$ (hence also in $\mathbf{int}\, H$). Now $\mathbf{int}\, H$ is a convex set, so the interior points of two different intervals I_1 and I_2 can not be separated by boundary points (since boundary points are in $\mathbf{bd}\, H$, not in $\mathbf{int}\, H$). Therefore there can be at most one interval, *i.e.*, $\mathbf{int}\, C$ is convex.

2.29 *Cones in* $\mathbf{R}^2$. Suppose $K \subseteq \mathbf{R}^2$ is a closed convex cone.

(a) Give a simple description of K in terms of the polar coordinates of its elements ($x = r(\cos\phi, \sin\phi)$ with $r \geq 0$).

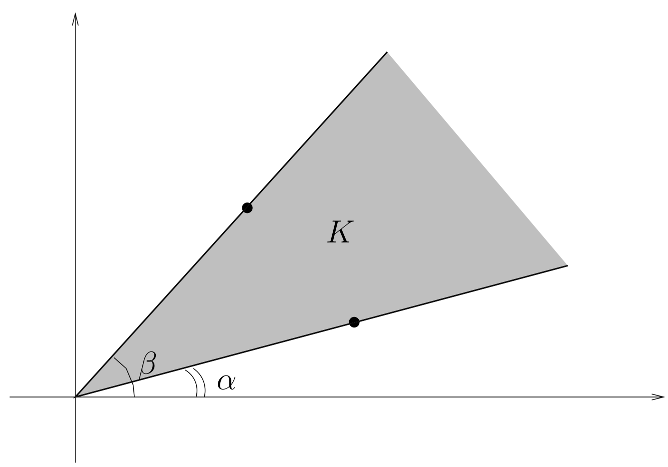
(b) Give a simple description of K^*, and draw a plot illustrating the relation between K and K^*.

(c) When is K pointed?

(d) When is K proper (hence, defines a generalized inequality)? Draw a plot illustrating what $x \preceq_K y$ means when K is proper.

Solution.

(a) In $\mathbf{R}^2$ a cone K is a "pie slice" (see figure).



In terms of polar coordinates, a pointed closed convex cone K can be expressed

$$K = \{(r\cos\phi, r\sin\phi) \mid r \geq 0, \alpha \leq \phi \leq \beta\}$$

where $0 \leq \beta - \alpha < 180°$. When $\beta - \alpha = 180°$, this gives a non-pointed cone (a halfspace). Other possible non-pointed cones are the entire plane

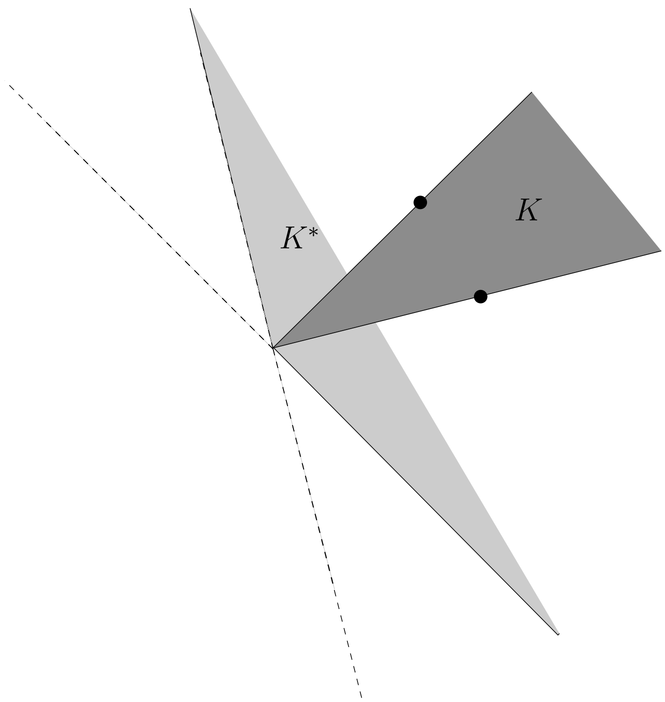
$$K = \{(r\cos\phi, r\sin\phi) \mid r \geq 0, 0 \leq \phi \leq 2\pi\} = \mathbf{R}^2,$$

and lines through the origin

$$K = \{(r\cos\alpha, r\sin\alpha) \mid r \in \mathbf{R}\}.$$

(b) By definition, K^* is the intersection of all halfspaces $x^T y \geq 0$ where $x \in K$. However, as can be seen from the figure, if K is pointed, the two halfspaces defined by the extreme rays are sufficient to define K^*, *i.e.*,

$$K^* = \{y \mid y_1 \cos\alpha + y_2 \sin\alpha \geq 0, y_1 \cos\beta + y_2 \sin\beta \geq 0\}.$$



If K is a halfspace, $K = \{x \mid v^T x \geq 0\}$, the dual cone is the ray

$$K^* = \{tv \mid t \geq 0\}.$$

If $K = \mathbf{R}^2$, the dual cone is $K^* = \{0\}$. If K is a line $\{tv \mid t \in \mathbf{R}\}$ through the origin, the dual cone is the line perpendicular to v

$$K^* = \{y \mid v^T y = 0\}.$$

(c) See part (a).

(d) K must be closed convex and pointed, and have nonempty interior. From part (a), this means K can be expressed as

$$K = \{(r\cos\phi, r\sin\phi) \mid r \geq 0, \alpha \leq \phi \leq \beta\}$$

where $0 < \beta - \alpha < 180°$.

$x \preceq_K y$ means $y \in x + K$.

2.36 *Euclidean distance matrices.* Let $x_1, \ldots, x_n \in \mathbf{R}^k$. The matrix $D \in \mathbf{S}^n$ defined by $D_{ij} = \|x_i - x_j\|_2^2$ is called a *Euclidean distance matrix*. It satisfies some obvious properties such as $D_{ij} = D_{ji}$, $D_{ii} = 0$, $D_{ij} \geq 0$, and (from the triangle inequality) $D_{ik}^{1/2} \leq D_{ij}^{1/2} + D_{jk}^{1/2}$. We now pose the question: When is a matrix $D \in \mathbf{S}^n$ a Euclidean distance matrix (for some points in $\mathbf{R}^k$, for some k)? A famous result answers this question: $D \in \mathbf{S}^n$ is a Euclidean distance matrix if and only if $D_{ii} = 0$ and $x^T D x \leq 0$ for all x with $\mathbf{1}^T x = 0$. (See §8.3.3.)

Show that the set of Euclidean distance matrices is a convex cone.

Solution. The set of Euclidean distance matrices in $\mathbf{S}^n$ is a closed convex cone because it is the intersection of (infinitely many) halfspaces defined by the following homogeneous inequalities:

$$e_i^T D e_i \leq 0, \qquad e_i^T D e_i \geq 0, \qquad x^T D x = \sum_{j,k} x_j x_k D_{jk} \leq 0,$$

for all $i = 1, \ldots, n$, and all x with $\mathbf{1}^T x = 0$.

It follows that dual cone is given by

$$K^* = \mathbf{cone}\left(\{-xx^T \mid \mathbf{1}^T x = 0\} \bigcup \{e_1 e_1^T, -e_1 e_1^T, \ldots, e_n e_n^T, -e_n e_n^T\}\right),$$

where **cone** means conic hull. This can be made more explicit as follows. Define $V \in \mathbf{R}^{n \times (n-1)}$ as

$$V_{ij} = \begin{cases} 1 - 1/n & i = j \\ -1/n & i \neq j. \end{cases}$$

The columns of V form a basis for the set of vectors orthogonal to $\mathbf{1}$, *i.e.*, a vector x satisfies $\mathbf{1}^T x = 0$ if and only if $x = Vy$ for some y. The dual cone is

$$K^* = \{VWV^T + \mathbf{diag}(u) \mid W \preceq 0, u \in \mathbf{R}^n\}.$$

A1.1 Is the set $\{a \in \mathbf{R}^k \mid p(0) = 1,\ |p(t)| \leq 1 \text{ for } \alpha \leq t \leq \beta\}$, where

$$p(t) = a_1 + a_2 t + \cdots + a_k t^{k-1},$$

convex?

Solution. Yes, it is convex; it is the intersection of an infinite number of slabs,

$$\{a \mid -1 \leq a_1 + a_2 t + \cdots + a_k t^{k-1} \leq 1\},$$

parametrized by $t \in [\alpha, \beta]$, and the hyperplane

$$\{a \mid a_0 = 1\}.$$

3.10 *An extension of Jensen's inequality.* One interpretation of Jensen's inequality is that randomization or dithering hurts, *i.e.*, raises the average value of a convex function: For f convex and v a zero mean random variable, we have $\mathbf{E}\, f(x_0 + v) \geq f(x_0)$. This leads to the following conjecture. If f is convex, then the larger the variance of v, the larger $\mathbf{E}\, f(x_0 + v)$.

(a) Give a counterexample that shows that this conjecture is false. Find zero mean random variables v and w, with $\mathbf{var}(v) > \mathbf{var}(w)$, a convex function f, and a point x_0, such that $\mathbf{E}\, f(x_0 + v) < \mathbf{E}\, f(x_0 + w)$.

(b) The conjecture is true when v and w are scaled versions of each other. Show that $\mathbf{E}\, f(x_0 + tv)$ is monotone increasing in $t \geq 0$, when f is convex and v is zero mean.

Solution.

(a) Define $f : \mathbf{R} \to \mathbf{R}$ as

$$f(x) = \begin{cases} 0, & x \leq 0 \\ x, & x > 0, \end{cases}$$

$x_0 = 0$, and scalar random variables

$$w = \begin{cases} 1 & \text{with probability } 1/2 \\ -1 & \text{with probability } 1/2 \end{cases} \qquad v = \begin{cases} 4 & \text{with probability } 1/10 \\ -4/9 & \text{with probability } 9/10. \end{cases}$$

w and v are zero-mean and

$$\mathbf{var}(v) = 16/9 > 1 = \mathbf{var}(w).$$

However,

$$\mathbf{E}\, f(v) = 2/5 < 1/2 = \mathbf{E}\, f(w).$$

(b) $f(x_0 + tv)$ is convex in t for fixed v, hence if v is a random variable, $g(t) = \mathbf{E}\, f(x_0 + tv)$ is a convex function of t. From Jensen's inequality,

$$g(t) = \mathbf{E}\, f(x_0 + tv) \geq f(x_0) = g(0).$$

Now consider two points a, b, with $0 < a < b$. If $g(b) < g(a)$, then

$$\frac{b-a}{b} g(0) + \frac{a}{b} g(b) < \frac{b-a}{b} g(a) + \frac{a}{b} g(a) = g(a)$$

which contradicts Jensen's inequality. Therefore we must have $g(b) \geq g(a)$.