EE364a Homework 1 solutions

2.7 *Voronoi description of halfspace.* Let a and b be distinct points in $\mathbf{R}^n$. Show that the set of all points that are closer (in Euclidean norm) to a than b, *i.e.*, $\{x \mid \|x-a\|_2 \leq \|x-b\|_2\}$, is a halfspace. Describe it explicitly as an inequality of the form $c^T x \leq d$. Draw a picture.

Solution. Since a norm is always nonnegative, we have $\|x-a\|_2 \leq \|x-b\|_2$ if and only if $\|x-a\|_2^2 \leq \|x-b\|_2^2$, so

$$\begin{aligned} \|x-a\|_2^2 \leq \|x-b\|_2^2 &\iff (x-a)^T(x-a) \leq (x-b)^T(x-b) \\ &\iff x^Tx - 2a^Tx + a^Ta \leq x^Tx - 2b^Tx + b^Tb \\ &\iff 2(b-a)^Tx \leq b^Tb - a^Ta. \end{aligned}$$

Therefore, the set is indeed a halfspace. We can take $c = 2(b-a)$ and $d = b^Tb - a^Ta$. This makes good geometric sense: the points that are equidistant to a and b are given by a hyperplane whose normal is in the direction $b-a$.

2.10 *Solution set of a quadratic inequality.* Let $C \subseteq \mathbf{R}^n$ be the solution set of a quadratic inequality,

$$C = \{x \in \mathbf{R}^n \mid x^TAx + b^Tx + c \leq 0\},$$

with $A \in \mathbf{S}^n$, $b \in \mathbf{R}^n$, and $c \in \mathbf{R}$.

(a) Show that C is convex if $A \succeq 0$.

(b) Show that the intersection of C and the hyperplane defined by $g^Tx + h = 0$ (where $g \neq 0$) is convex if $A + \lambda gg^T \succeq 0$ for some $\lambda \in \mathbf{R}$.

Are the converses of these statements true?

Solution. A set is convex if and only if its intersection with an arbitrary line $\{\hat{x} + tv \mid t \in \mathbf{R}\}$ is convex.

(a) We have

$$(\hat{x} + tv)^TA(\hat{x} + tv) + b^T(\hat{x} + tv) + c = \alpha t^2 + \beta t + \gamma$$

where

$$\alpha = v^TAv, \qquad \beta = b^Tv + 2\hat{x}^TAv, \qquad \gamma = c + b^T\hat{x} + \hat{x}^TA\hat{x}.$$

The intersection of C with the line defined by $\hat{x}$ and v is the set

$$\{\hat{x} + tv \mid \alpha t^2 + \beta t + \gamma \leq 0\},$$

which is convex if $\alpha \geq 0$. This is true for any v, if $v^TAv \geq 0$ for all v, *i.e.*, $A \succeq 0$. The converse does not hold; for example, take $A = -1$, $b = 0$, $c = -1$. Then $A \not\succeq 0$, but $C = \mathbf{R}$ is convex.

(b) Let $H = \{x \mid g^T x + h = 0\}$. We define α, β, and γ as in the solution of part (a), and, in addition,

$$\delta = g^T v, \qquad \epsilon = g^T \hat{x} + h.$$

Without loss of generality we can assume that $\hat{x} \in H$, *i.e.*, $\epsilon = 0$. The intersection of $C \cap H$ with the line defined by $\hat{x}$ and v is

$$\{\hat{x} + tv \mid \alpha t^2 + \beta t + \gamma \leq 0,\ \delta t = 0\}.$$

If $\delta = g^T v \neq 0$, the intersection is the singleton $\{\hat{x}\}$, if $\gamma \leq 0$, or it is empty. In either case it is a convex set. If $\delta = g^T v = 0$, the set reduces to

$$\{\hat{x} + tv \mid \alpha t^2 + \beta t + \gamma \leq 0\},$$

which is convex if $\alpha \geq 0$. Therefore $C \cap H$ is convex if

$$g^T v = 0 \Longrightarrow v^T A v \geq 0.$$

This is true if there exists λ such that $A + \lambda g g^T \succeq 0$; then the above condition holds, because then

$$v^T A v = v^T (A + \lambda g g^T) v \geq 0$$

for all v satisfying $g^T v = 0$.

Again, the converse is not true.

2.12 Which of the following sets are convex?

(a) A *slab*, *i.e.*, a set of the form $\{x \in \mathbf{R}^n \mid \alpha \leq a^T x \leq \beta\}$.

(b) A *rectangle*, *i.e.*, a set of the form $\{x \in \mathbf{R}^n \mid \alpha_i \leq x_i \leq \beta_i,\ i = 1, \ldots, n\}$. A rectangle is sometimes called a *hyperrectangle* when $n > 2$.

(c) A *wedge*, *i.e.*, $\{x \in \mathbf{R}^n \mid a_1^T x \leq b_1,\ a_2^T x \leq b_2\}$.

(d) The set of points closer to a given point than a given set, *i.e.*,

$$\{x \mid \|x - x_0\|_2 \leq \|x - y\|_2 \text{ for all } y \in S\}$$

where $S \subseteq \mathbf{R}^n$.

(e) The set of points closer to one set than another, *i.e.*,

$$\{x \mid \mathbf{dist}(x, S) \leq \mathbf{dist}(x, T)\},$$

where $S, T \subseteq \mathbf{R}^n$, and

$$\mathbf{dist}(x, S) = \inf\{\|x - z\|_2 \mid z \in S\}.$$

(f) The set $\{x \mid x + S_2 \subseteq S_1\}$, where $S_1, S_2 \subseteq \mathbf{R}^n$ with S_1 convex.

(g) The set of points whose distance to a does not exceed a fixed fraction θ of the distance to b, *i.e.*, the set $\{x \mid \|x-a\|_2 \le \theta\|x-b\|_2\}$. You can assume $a \ne b$ and $0 \le \theta \le 1$.

Solution.

(a) A slab is an intersection of two halfspaces, hence it is a convex set (and a polyhedron).

(b) As in part (a), a rectangle is a convex set and a polyhedron because it is a finite intersection of halfspaces.

(c) A wedge is an intersection of two halfspaces, so it is convex set. It is also a polyhedron. It is a cone if $b_1 = 0$ and $b_2 = 0$.

(d) This set is convex because it can be expressed as

$$\bigcap_{y \in S} \{x \mid \|x - x_0\|_2 \le \|x - y\|_2\},$$

i.e., an intersection of halfspaces. (For fixed y, the set

$$\{x \mid \|x - x_0\|_2 \le \|x - y\|_2\}$$

is a halfspace; see exercise 2.9).

(e) In general this set is not convex, as the following example in $\mathbf{R}$ shows. With $S = \{-1, 1\}$ and $T = \{0\}$, we have

$$\{x \mid \mathbf{dist}(x, S) \le \mathbf{dist}(x, T)\} = \{x \in \mathbf{R} \mid x \le -1/2 \text{ or } x \ge 1/2\}$$

which clearly is not convex.

(f) This set is convex. $x + S_2 \subseteq S_1$ if $x + y \in S_1$ for all $y \in S_2$. Therefore

$$\{x \mid x + S_2 \subseteq S_1\} = \bigcap_{y \in S_2} \{x \mid x + y \in S_1\} = \bigcap_{y \in S_2} (S_1 - y),$$

the intersection of convex sets $S_1 - y$.

(g) The set is convex, in fact a ball.

$$\begin{aligned} &\{x \mid \|x-a\|_2 \le \theta\|x-b\|_2\} \\ &\quad = \{x \mid \|x-a\|_2^2 \le \theta^2\|x-b\|_2^2\} \\ &\quad = \{x \mid (1-\theta^2)x^Tx - 2(a-\theta^2 b)^T x + (a^Ta - \theta^2 b^T b) \le 0\} \end{aligned}$$

If $\theta = 1$, this is a halfspace. If $\theta < 1$, it is a ball

$$\{x \mid (x - x_0)^T(x - x_0) \le R^2\},$$

with center x_0 and radius R given by

$$x_0 = \frac{a - \theta^2 b}{1 - \theta^2}, \qquad R = \left(\frac{\theta^2\|b\|_2^2 - \|a\|_2^2}{1-\theta^2} + \|x_0\|_2^2\right)^{1/2}.$$

2.15 *Some sets of probability distributions.* Let x be a real-valued random variable with $\mathbf{prob}(x = a_i) = p_i$, $i = 1, \ldots, n$, where $a_1 < a_2 < \cdots < a_n$. Of course $p \in \mathbf{R}^n$ lies in the standard probability simplex $P = \{p \mid \mathbf{1}^T p = 1,\ p \succeq 0\}$. Which of the following conditions are convex in p? (That is, for which of the following conditions is the set of $p \in P$ that satisfy the condition convex?)

(a) $\alpha \leq \mathbf{E} f(x) \leq \beta$, where $\mathbf{E} f(x)$ is the expected value of $f(x)$, *i.e.*, $\mathbf{E} f(x) = \sum_{i=1}^n p_i f(a_i)$. (The function $f : \mathbf{R} \to \mathbf{R}$ is given.)

(b) $\mathbf{prob}(x > \alpha) \leq \beta$.

(c) $\mathbf{E}\,|x^3| \leq \alpha\, \mathbf{E}\,|x|$.

(d) $\mathbf{E}\, x^2 \leq \alpha$.

(e) $\mathbf{E}\, x^2 \geq \alpha$.

(f) $\mathbf{var}(x) \leq \alpha$, where $\mathbf{var}(x) = \mathbf{E}(x - \mathbf{E}\, x)^2$ is the variance of x.

(g) $\mathbf{var}(x) \geq \alpha$.

(h) $\mathbf{quartile}(x) \geq \alpha$, where $\mathbf{quartile}(x) = \inf\{\beta \mid \mathbf{prob}(x \leq \beta) \geq 0.25\}$.

(i) $\mathbf{quartile}(x) \leq \alpha$.

Solution. We first note that the constraints $p_i \geq 0$, $i = 1, \ldots, n$, define halfspaces, and $\sum_{i=1}^n p_i = 1$ defines a hyperplane, so P is a polyhedron.

The first five constraints are, in fact, linear inequalities in the probabilities p_i.

(a) $\mathbf{E} f(x) = \sum_{i=1}^n p_i f(a_i)$, so the constraint is equivalent to two linear inequalities

$$\alpha \leq \sum_{i=1}^n p_i f(a_i) \leq \beta.$$

(b) $\mathbf{prob}(x \geq \alpha) = \sum_{i:\, a_i \geq \alpha} p_i$, so the constraint is equivalent to a linear inequality

$$\sum_{i:\, a_i \geq \alpha} p_i \leq \beta.$$

(c) The constraint is equivalent to a linear inequality

$$\sum_{i=1}^n p_i (|a_i^3| - \alpha |a_i|) \leq 0.$$

(d) The constraint is equivalent to a linear inequality

$$\sum_{i=1}^n p_i a_i^2 \leq \alpha.$$

(e) The constraint is equivalent to a linear inequality

$$\sum_{i=1}^{n} p_i a_i^2 \geq \alpha.$$

The first five constraints therefore define convex sets.

(f) The constraint

$$\mathbf{var}(x) = \mathbf{E}\, x^2 - (\mathbf{E}\, x)^2 = \sum_{i=1}^{n} p_i a_i^2 - (\sum_{i=1}^{n} p_i a_i)^2 \leq \alpha$$

is not convex in general. As a counterexample, we can take $n = 2$, $a_1 = 0$, $a_2 = 1$, and $\alpha = 1/5$. $p = (1, 0)$ and $p = (0, 1)$ are two points that satisfy $\mathbf{var}(x) \leq \alpha$, but the convex combination $p = (1/2, 1/2)$ does not.

(g) This constraint is equivalent to

$$\sum_{i=1}^{n} p_i a_i^2 - (\sum_{i=1}^{n} p_i a_i)^2 = b^T p - p^T A p \geq \alpha,$$

where $b_i = a_i^2$ and $A = aa^T$. We write this as

$$p^T A p - b^T p + \alpha \leq 0.$$

This defines a convex set, since the matrix aa^T is positive semidefinite.
To show this set is convex for $A \in \mathbf{S}_+$, we first make the observation that

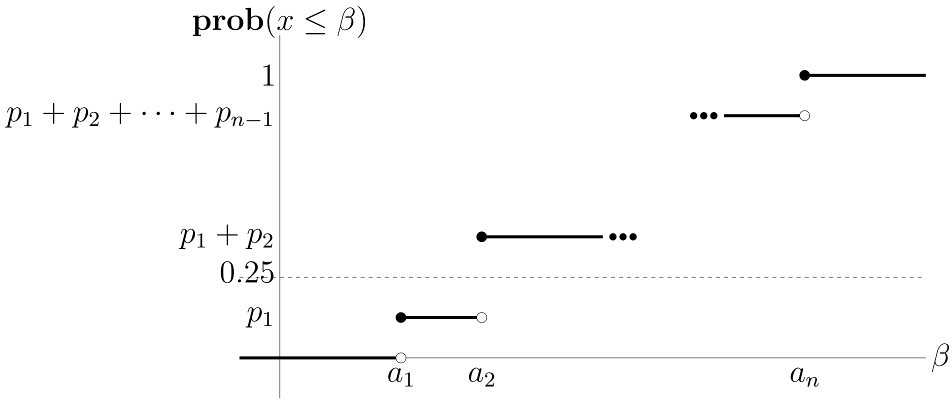
$$x^T A y \leq \frac{x^T A x + y^T A y}{2}$$

which follows from $(x - y)^T A (x - y) \geq 0$. Now choose $x, y \in \mathcal{C}$ and show $z = \theta x + (1 - \theta) y \in \mathcal{C}$:

$$\begin{aligned}
&z^T A z + b^T z + \alpha \\
&= \theta^2 x^T A x + (1-\theta)^2 y^T A y + 2\theta(1-\theta) x^T A y + b^T(\theta x + (1-\theta) y) + \alpha \\
&\leq \theta^2 x^T A x + (1-\theta)^2 y^T A y + \theta(1-\theta)(x^T A x + y^T A y) + b^T(\theta x + (1-\theta) y) + \alpha \\
&= \theta x^T A x + \theta b^T x + (1-\theta) y^T A y + (1-\theta) b^T y + \theta\alpha + (1-\theta)\alpha \\
&= \theta(x^T A x + b^T x + \alpha) + (1-\theta)(y^T A y + b^T y + \alpha) \\
&\leq 0
\end{aligned}$$

Therefore $z \in \mathcal{C}$ and the set is convex.

Let us denote $\mathbf{quartile}(x) = f(p)$ to emphasize it is a function of p. The figure illustrates the definition. It shows the cumulative distribution for a distribution p with $f(p) = a_2$.



(h) The constraint $f(p) \geq \alpha$ is equivalent to

$$\mathbf{prob}(x \leq \beta) < 0.25 \text{ for all } \beta < \alpha.$$

If $\alpha \leq a_1$, this is always true. Otherwise, define $k = \max\{i \mid a_i < \alpha\}$. This is a fixed integer, independent of p. The constraint $f(p) \geq \alpha$ holds if and only if

$$\mathbf{prob}(x \leq a_k) = \sum_{i=1}^{k} p_i < 0.25.$$

This is a strict linear inequality in p, which defines an open halfspace.

(i) The constraint $f(p) \leq \alpha$ is equivalent to

$$\mathbf{prob}(x \leq \beta) \geq 0.25 \text{ for all } \beta \geq \alpha.$$

Here, let us define $k = \max\{i \mid a_i \leq \alpha\}$. Again, this is a fixed integer, independent of p. The constraint $f(p) \leq \alpha$ holds if and only if

$$\mathbf{prob}(x \leq a_k) = \sum_{i=1}^{k} p_i \geq 0.25.$$

If $\alpha < a_1$, then no p satisfies $f(p) \leq \alpha$, which means that the set is empty. Thus, the constraint $f(p) \leq \alpha$ is a linear inequality on p.