Computing Planck's blackbody spectrum, considering index of refraction

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When blackbody radiation is emitted from the opening of a gas filled cavity (e.g. Argon), and/or detected by a detector in a gas (e.g. air), then the index of refraction of the gas must be considered. See this paper by Gaertner, http://www.bipm.org/cc/CCT/Allowed/25/D11_CCTdraftAAG.pdf which corrects this paper by Hartmann, https://www.osapublishing.org/oe/abstract.cfm?uri=oe-14-18-8121.

We know:

- Basic spectral radiance as function of frequency, $L_{basic,\nu}(\nu) = \dfrac{\mathrm{d}\Phi}{\mathrm{d}A\, n^2\, \cos(\theta)\, d\Omega\, \mathrm{d}\nu}$ is invariant along a ray when this ray undergoes refraction without Fresnel losses (the brightness conservation theorem).
- Integrating blackbody spectral basic radiance, $L_{basic} = \int L_{basic,\nu}(\nu)\,\mathrm{d}\nu$ over frequency yields $L_{basic} = \dfrac{\sigma T^4}{\pi}$, with absolute temperature T and Stefan-Boltzmann constant σ, independent of refractive index.

Note the factor of n^2 in the denominator of $L_{basic,\nu}(\nu)$: this is what makes basic spectral radiance different from standard (geometric) spectral radiance L_ν.

For blackbody radiation, basic spectral radiance as function of frequency is

$$L^{bb}_{basic,\nu}(\nu) = \frac{2h\nu^3}{c^2\left(\exp\left(\frac{h\nu}{kT}\right) - 1\right)}$$

There is no refractive index, correctly, because frequency does not change with index, and because of brightness conservation.

However: In practice, we often need spectra as function of wavelength, and that wavelength is often taken within a refractive medium.

The dispersion free case

Let us now change variables from frequency to wavelength λ in units of "wlu meters", e.g. $\mathrm{wlu} = 10^{-9}$ for nanometers, in medium with constant refractive index n:

$$\nu = \frac{c}{n\,\lambda\,\mathrm{wlu}} \qquad \left|\frac{\mathrm{d}\nu}{\mathrm{d}\lambda}\right| = \frac{c}{n\,\lambda^2\,\mathrm{wlu}}$$

or, with $\lambda_{\mathrm{vac,m}} = n\,\lambda\,\mathrm{wlu}$ (the wavelength in vacuum in meters)

$$\nu = \frac{c}{\lambda_{\mathrm{vac,m}}} \qquad \left|\frac{\mathrm{d}\nu}{\mathrm{d}\lambda}\right| = \frac{c}{\lambda_{\mathrm{vac,m}}\,\lambda}$$

Then, basic spectral radiance as function of wavelength in any units in medium is

$$L_{basic,\lambda}(\lambda) = L_{basic,\nu}(\nu(\lambda))\left|\frac{\mathrm{d}\nu}{\mathrm{d}\lambda}\right| = \frac{c_{1L}}{\lambda\,\lambda_{\mathrm{vac,m}}^4\left[\exp\left(\frac{c_2}{\lambda_{\mathrm{vac,m}}T}\right)-1\right]} = \frac{c_{1L}\,n\,\mathrm{wlu}}{\lambda_{\mathrm{vac,m}}^5\left[\exp\left(\frac{c_2}{\lambda_{\mathrm{vac,m}}T}\right)-1\right]}$$

where $c_{1L} = 2hc^2$ and $c_2 = hc/k$ are CODATA's first and second radiation constants.

This result makes it easy to calculate $L_{basic,\lambda}(\lambda)$ numerically in a constant index medium:

First, compute the standard (non-basic) blackbody spectral radiance for wavelength in meters. Then, multiply with $n\,\mathrm{wlu}$. Finally, check the result by integrating over λ, verifying the result is $L_{basic} = \frac{\sigma T^4}{\pi}$.

The dispersive case

If there is dispersion, $n = n(\lambda)$:

$$\nu = \frac{c}{n(\lambda)\,\lambda\,\mathrm{wlu}} \qquad \left|\frac{\mathrm{d}\nu}{\mathrm{d}\lambda}\right| = \frac{\left(n(\lambda) + \lambda\frac{\mathrm{d}n}{\mathrm{d}\lambda}\right)c}{(n(\lambda)\,\lambda)^2\,\mathrm{wlu}}.$$

Note the additional $\lambda\frac{\mathrm{d}n}{\mathrm{d}\lambda}$ term in the numerator. Then,

$$L_{basic,\lambda}(\lambda) = L_{basic,\nu}(\nu)\left|\frac{\mathrm{d}\nu}{\mathrm{d}\lambda}\right| = \frac{c_{1L}\left(n(\lambda) + \lambda\frac{\mathrm{d}n}{\mathrm{d}\lambda}\right)}{n(\lambda)\,\lambda\,\lambda_{\mathrm{vac,m}}^4\left[\exp\left(\frac{c_2}{\lambda_{\mathrm{vac,m}}T}\right)-1\right]} = \frac{c_{1L}\left(n(\lambda) + \lambda\frac{\mathrm{d}n}{\mathrm{d}\lambda}\right)\mathrm{wlu}}{\lambda_{\mathrm{vac,m}}^5\left[\exp\left(\frac{c_2}{\lambda_{\mathrm{vac,m}}T}\right)-1\right]}$$

The numerical calculation proceeds simlilar to the constant index case: Compute compute the standard (non-basic) blackbody spectral radiance for wavelength in meters. Then, multiply with $\left(n(\lambda) + \lambda\frac{\mathrm{d}n}{\mathrm{d}\lambda}\right)\mathrm{wlu}$. It is the $\frac{\mathrm{d}n}{\mathrm{d}\lambda}$ term which may cause some headaches. The dispersion curve $n(\lambda)$ may be given as a function, in e.g. Sellmeier or Cauchy cofficients. Then, it can be readily differentiated. In case of $n(\lambda)$ given by tabulated values, I would recommend to perform a cubic spline interpolation of the tabulated values and then differentiate the cubic polynomials analytically. In Matlab, for example, `pp = spline(lambda, n)` returns a piecewise polynomial structure that can be evaluated using `ppval`. This structure can be readily differentiated, too, using a function like this:

```
function rv = ipp_deriv(pp)
% Input: pp is piecewise polynomial struct, returned from spline or pchip or makima
% Output: pp struct with derivative coefficients, order - 1
    if strcmp(pp.form,'pp') ~= 1
        error('ipp_deriv: expect piecewise polynomial structure');
    end
    if pp.dim ~= 1
        error('ipp_deriv: expect 1-dim piecewise polynomial structure');
    end
%     breaks: [1 3 4 6 10 11]
%      coefs: [5×4 double]
```

```
%     pieces: 5
%      order: 4
%        dim: 1
    rv = pp;
    rv.order = pp.order - 1;
    rv.coefs = pp.coefs(:,1:end-1) .* (rv.order:-1:1);
end
```