Virtual focus of a ray data source

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Abstract

Ray data sources are a common way to model light sources, like LEDs, in optical design software. Often, it is convenient to know the coordinates of just a single point which is representative of the light source location, similar to the center of mass for a distribution of matter. In some communities, this point is known as the *virtual focus*, and some LED vendors publish the virtual focus coordinates with their ray data sources. However, a clear definition of what the virtual focus is, and how it can be computed in an efficient way, is hard to find. In this paper, we explain the definition and provide an efficient algorithm.

1 Virtual focus definition

A ray data source consists, at minimum, of a number of rays r_i, each of which is given by three coordinates $\mathbf{x}_i = (x_i^1, x_i^2, x_i^3)$ for a starting point, three more coordinates $\mathbf{k}_i = (k_i^1, k_i^2, k_i^3)$ for direction, and a weight w_i, which generally denotes the radiant or luminous flux associated with this ray. In the following, we assume that $|\mathbf{k}_i| = 1$; if this is not the case, $\mathbf{k}_i$ shall be normalized to length 1 before we proceed. Accordingly, any point $\mathbf{p}$ along the ray r_i is given by

$$\mathbf{p} = \mathbf{x}_i + \mu \mathbf{k}_i, \quad \mu \in \mathbb{R} \tag{1}$$

In general, the rays do not pass through a single (focal) point. If they did, the source would be a true point source. Instead, they come from an extended source. In the simple case of a Lambertian source (which is by definition a planar source with constant luminance), a "center of mass" of the source would be just the actual center of mass of the light emitting surface. However, when the source is not Lambertian, i.e. luminance varies both over angle and location, and especially when the source has a more complex geometry, like an HID bulb, or an LED with a silicone dome, the concept of center of mass can not be applied in a straightforward way. What we would look for is the point to which all rays come as close as possible: the *virtual focus*.

Let $\mathbf{F}$ be such a candidate virtual focus point. The vector $\mathbf{d}_i$, connecting $\mathbf{F}$ with the closest point $\mathbf{p}$ on the ray r_i given by

$$\mathbf{d}_i = (\mathbf{F} - \mathbf{x}_i) - [(\mathbf{F} - \mathbf{x}_i) \cdot \mathbf{k}_i]\mathbf{k}_i \tag{2}$$

i.e. $\mathbf{d}_i$ connects $\mathbf{F}$ with its projection onto the ray r_i.

But which candidate point is the best one? For motivation, let us look to the definition of *center of mass* in mechanics. Here, the center of mass $\mathbf{c}$ of a number of p oint masses with mass m_i located at $\mathbf{r}_i$ is given by

$$\mathbf{c} = \frac{\sum_i m_i \mathbf{r}_i}{\sum_i m_i} \tag{3}$$

i.e. the usual weighted sum of the locations of the point masses. It is now easy to see that of all possible points, $\mathbf{c}$ is just that unique point which minimizes

$$f(\mathbf{c}) = \sum_i m_i(\mathbf{c} - \mathbf{r}_i) \cdot (\mathbf{c} - \mathbf{r}_i) \tag{4}$$

Proof: When $\mathbf{c}$ is a minimum of f, then $\nabla_{\mathbf{c}} f = 0$. Now,

$$\nabla_{\mathbf{c}} f = 2 \sum_i m_i(\mathbf{c} - \mathbf{r}_i) = 0 \tag{5}$$

from which eq. 3 follows with elementary algebra.

In other words, the center of mass is the point which minimizes the *weighted sum of squared distances*.

We now apply this concept to the virtual focus of a ray set. Here, it is the minimum distance between a candidate virtual focus and each ray which enters the definition:

$$\mathbf{F} = \min_{\mathbf{G}} \left[\sum_i w_i \left| (\mathbf{G} - \mathbf{x}_i) - ((\mathbf{G} - \mathbf{x}_i) \cdot \mathbf{k}_i)\mathbf{k}_i \right|^2 \right] \tag{6}$$

In matrix notation, where $\mathbf{1}$ is the 3×3 identity matrix and $\otimes$ is the outer product, we can also write

$$\mathbf{F} = \min_{\mathbf{G}} (g(\mathbf{G})) \tag{7}$$

with

$$g(\mathbf{G}) = \sum_i w_i (\mathbf{G} - \mathbf{x}_i)^{\mathrm{T}} (\mathbf{1} - \mathbf{k}_i \otimes \mathbf{k}_i)(\mathbf{G} - \mathbf{x}_i) \tag{8}$$

2 Computing the virtual focus

There is no need for numerical, iterative minimization of g. Instead, the minimum can be computed in a straightforward way as follows.

First we define the matrix $\mathbf{A}$ and the vector $\mathbf{b}$ as

$$\mathbf{A} = \sum_i w_i (\mathbf{1} - \mathbf{k}_i \otimes \mathbf{k}_i), \qquad \mathbf{b} = \sum_i w_i (\mathbf{1} - \mathbf{k}_i \otimes \mathbf{k}_i)\mathbf{x}_i \tag{9}$$

and obtain (since $\mathbf{A}$ is symmetric)

$$g(\mathbf{G}) = \mathbf{G}^{\mathrm{T}} \mathbf{A} \mathbf{G} - 2\mathbf{G}^{\mathrm{T}} \mathbf{b} + \mathrm{const} \tag{10}$$

Since g is quadratic in $\mathbf{G}$, there is a single global minimum $\mathbf{F}$, which is located where the gradient of g vanishes, as long as $\mathbf{A}$ is not singular. Thus, the virtual focus is well defined for nonsingular $\mathbf{A}$; we will see later that $\mathbf{A}$ is singular only if all rays have the same direction (are fully collimated).

Setting the gradient of g to zero gives

$$\nabla g = 2 \sum_i w_i (\mathbf{1} - \mathbf{k}_i \otimes \mathbf{k}_i)(\mathbf{F} - \mathbf{x}_i) = 0 \tag{11}$$

or

$$\underbrace{\left[\sum_i w_i (\mathbf{1} - \mathbf{k}_i \otimes \mathbf{k}_i) \right]}_{\mathbf{A}} \mathbf{F} = \underbrace{\sum_i w_i (\mathbf{1} - \mathbf{k}_i \otimes \mathbf{k}_i)\mathbf{x}_i}_{\mathbf{b}} \tag{12}$$

or, finally,

$$\mathbf{AF} = \mathbf{b} \tag{13}$$

with the definitions of the 3×3 matrix $\mathbf{A}$ and the 3×1 vector $\mathbf{b}$ from eq. **??**.

Thus, we can compute the virtual focus $\mathbf{F}$ of a ray set r_i by simply solving the set of three linear equations, eq. 13.

We note that this definition of the virtual focus has some desirable properties:

- The result is invariant against translation of any $\mathbf{x}_i$ along $\mathbf{k}_i$ (as it should be – such a translation does not change the ray). Proof: Since $|\mathbf{k}_i| = 1$, the matrix $\mathbf{k}_i \otimes \mathbf{k}_i$ is the projection operator which maps any vector to its component along $\mathbf{k}_i$. Therefore, $(\mathbf{k}_i \otimes \mathbf{k}_i)\mathbf{k}_i = \mathbf{k}_i$, and $(\mathbf{1} - \mathbf{k}_i \otimes \mathbf{k}_i)\mathbf{k}_i = 0$. Accordingly, adding any $\mu \mathbf{k}_i, \mu \in \mathbb{R}$ to $\mathbf{x}_i$ does not change the right hand side of eq. 12.

- The virtual focus of a ray set with a true focus $\mathbf{F}'$, i.e. a ray set where all rays pass through $\mathbf{F}'$, is just that true focus $\mathbf{F}'$(as it should be). This can be seen by moving each $\mathbf{x}_i$ along its ray to $\mathbf{F}'$ (this is allowed, see previous item). Then both sides of eq. 12 are identical if $\mathbf{F} = \mathbf{F}'$.

- There is no unique virtual focus point when the rays are all parallel (a fully collimated bundle). Instead, any virtual focus point can be translated along the collimated direction without changing the geometry at all. This behavior is reproduced by our definition. Proof: All $\mathbf{k}_i$ are equal to the overall direction $\mathbf{k}$, and $\mathbf{A} = \text{const} \times (\mathbf{1} - \mathbf{k} \otimes \mathbf{k})$. Then, $\mathbf{A}$ is a linear operator which removes from any vector its component along $\mathbf{k}$. The image of $\mathbf{A}$ is then the plane through the origin perpendicular to $\mathbf{k}$, $\mathbf{A}$ is singular and has rank 2, and its null space is just $\mu\mathbf{k}$. In general, there is no solution for an arbitrary nonzero right hand side if the matrix is singular. However, in this special case, eq. 12 does have a solution, namely:

$$F = \frac{\sum_i w_i \mathbf{x}_i}{\sum_i w_i}, \tag{14}$$

  i.e. the center of mass of all ray starting points. All solutions of the now underdetermined set of equations are virtual foci, which differ by $\mu\mathbf{k}$. This is as it should be: For a fully collimated ray bundle, the virtual focus is in fact a line passing through the "center of mass" of all ray starting points, with direction $\mathbf{k}$. However, the actual center of mass according to eq. 14 is probably the best candidate for a given ray set, since it is guaranteed to lie within the convex hull of all ray starting points.

3 Conclusion

We have presented a well formed definition of the virtual focus point of a ray set, which is motivated by the analogous, common computation of the center of mass. This definition has all the properties we would like it to have: the virtual focus is invariant against moving ray starting points along its rays, it is equal to a true focal point, when the rays all pass through that focal point, and for a fully collimated bundle, the virtual focus is in fact a line, as it should be. We have also shown how the virtual focus can be determined via solving a simple 3×3 set of linear equations, where the matrix and the right hand side are computed via straightforward vector operations on the ray starting points, ray directions and ray powers.

We plan to provide an open source implementation of software which reads a ray data set from a TM25 file, and computes the virtual focus. TM25 is the excellent ray file definition standard provided in IES-TM25-13, see `https://www.ies.org/product/ray-file-format-for-the-description-of-the-emission-property-of-light-sources/`), which is increasingly being provided by LED manufacturers.