Nonlinear stochastic differential equation generating $1/f$ noise

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1 Nonlinear stochastic differential equations with multiplicative noise

In Refs. [1, 2] nonlinear stochastic differential equations (SDEs) of the form

$$dx_t = \sigma^2 \left(\eta - \frac{\lambda}{2}\right) x_t^{2\eta-1} dt + \sigma x_t^{\eta} dW_t \tag{1}$$

have been proposed. Here W_t is a standard Wiener process (the Brownian motion), η is the power-law exponent of multiplicative noise and σ is the amplitude of the noise. In order to avoid the divergence of the steady-state probability density function (PDF), Eq. (1) should be considered together with appropriate restriction of the diffusion of the stochastic variable x. Here we investigate the SDE with the exponential restriction of diffusion at $x = x_{\min}$

$$dx_t = \sigma^2 \left(\eta - \frac{\lambda}{2} + \frac{m}{2}\left(\frac{x_{\min}}{x}\right)^m\right) x_t^{2\eta-1} dt + \sigma x_t^{\eta} dW_t\,. \tag{2}$$

Equation (2) has stationary probability distribution function (PDF) of the form

$$P_0(x) \sim x^{-\lambda} \exp\left(-\left(\frac{x_{\min}}{x}\right)^m\right)\,. \tag{3}$$

Based on scaling consideration [3] it is predicted that the power spectral density (PSD) of the signal x_t has $f^{-\beta}$ behaviour in a wide range of frequencies, with

$$\beta = 1 + \frac{\lambda - 3}{2(\eta - 1)}\,. \tag{4}$$

Using the parameters $\lambda = 2\eta$ and $m = 2\eta - 2$, equation (2) takes the form of the *Constant Elasticity of Variance* (CEV) process:

$$dx_t = \sigma^2(\eta - 1) x_{\min}^{2(\eta-1)} x_t dt + \sigma x_t^{\eta} dW_t\,. \tag{5}$$

If $\lambda = 3$ then SDE (2) gives $1/f$ spectrum. Thus the CEV process has $1/f$ spectrum when $\eta = \frac{3}{2}$:

$$dx_t = \mu x_t dt + \sigma x_t^{\frac{3}{2}} dW_t\,. \tag{6}$$

Here

$$\mu = \frac{1}{2}\sigma^2 x_{\min}\,. \tag{7}$$

1.1 Derivation of the analytical expression for the power spectral density

The analytical expression for the transition probability $P_x(x', t|x, 0)$ (the conditional probability that at time t the signal has value x with the condition that at time $t = 0$ the signal had the value x_0) of the CEV process is [4]

$$P_x(x, t|x_0, 0) = \frac{x_{\min}}{(1 - e^{-\mu t})}\sqrt{\frac{x_0}{x^5}} \exp\left(\frac{1}{2}\mu t - \frac{x_{\min}}{(1 - e^{-\mu t})}\left(\frac{1}{x} + \frac{1}{x_0}e^{-\mu t}\right)\right) I_1\left(\frac{x_{\min}}{\sinh\left(\frac{1}{2}\mu t\right)}\frac{1}{\sqrt{x_0 x}}\right)\,. \tag{8}$$

Here I_z is the modified Bessel function with index z. The steady-state PDF has the form

$$P_0(x) = \frac{x_{\min}^2}{x^3} \exp\left(-\frac{x_{\min}}{x}\right)\,. \tag{9}$$

The average of the signal is

$$\bar{x} = \int_0^{\infty} x P_0(x) dx = x_{\min}\,. \tag{10}$$

The autocorrelation function can be calculated using the expression

$$C(t) = \int dx \int dx' \, (x - \bar{x})(x' - \bar{x}) P_0(x) P_x(x', t|x, 0) \,. \tag{11}$$

Using Eq. (8) and performing the integration we obtain the autocorrelation function

$$C(t) = x_{\mathrm{min}}^2 \left[-e^{\mu t} \ln\left(1 - e^{-\mu t}\right) - 1 \right] \,. \tag{12}$$

When $\mu t \ll 1$ we get

$$C(t) \approx -x_{\mathrm{min}}^2 - x_{\mathrm{min}}^2 \ln(\mu t) \,. \tag{13}$$

Similar expansion has been obtained for the autocorrelation function in the case of $1/f$ spectrum.

According to Wiener-Khintchine relations, the power spectral density is connected with the autocorrelation function via the transformation

$$S(f) = 2 \int_{-\infty}^{\infty} C(t) e^{i\omega t} dt = 4 \int_{0}^{\infty} C(t) \cos(\omega t) dt \,, \tag{14}$$

where $\omega = 2\pi f$. Using Eq. (12) for the autocorrelation function we get the following expression for the power spectral density:

$$S(f) = 2x_{\mathrm{min}}^2 \left[\frac{-\gamma - \psi\left(-i\frac{\omega}{\mu}\right)}{\mu + i\omega} + \frac{-\gamma - \psi\left(i\frac{\omega}{\mu}\right)}{\mu - i\omega} \right] \,, \tag{15}$$

where $\gamma \approx 0.577216$ is the Euler's constant and $\psi(z) = \Gamma'(z)/\Gamma(z)$ is the digamma function. When $\omega \gg \mu$ then the power spectral density is

$$S(f) \approx \frac{2\pi x_{\mathrm{min}}^2}{\omega} \,. \tag{16}$$

2 Method of numerical solution

Method of numerical solution with a variable time step is described in Ref. [5]. For the numerical solution, we use Euler-Marujama approximation, transforming differential equations to difference equations. If the time step is $\Delta t = h$ then the difference equations, corresponding to Eq. (6) are

$$x_{k+1} = x_k + \mu x_k h + \sigma x_k^{\frac{3}{2}} \sqrt{h} \varepsilon_k \tag{17}$$

$$t_{k+1} = t_k + h \tag{18}$$

Here ε_k are normally distributed uncorrelated random variables with a zero expectation and unit variance. Variable time step of integration

$$h_k = \frac{\kappa^2}{\sigma^2 x_k}$$

leads to the equations

$$x_{k+1} = x_k + \frac{1}{2}\kappa^2 x_{\mathrm{min}} + \kappa x_k \varepsilon_k \tag{19}$$

$$t_{k+1} = t_k + \frac{\kappa^2}{\sigma^2 x_k} \tag{20}$$

Here $\kappa \ll 1$ is a small parameter.

References

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