1.1 Derivation of the Γ_{pp}-vertex with Motoharu's frequency convention

The frequency convention used by Motoharu is the following:

$$\mathrm{ph}: \{\nu_1 = \nu, \qquad \nu_2 = \nu + \omega_{\mathrm{ph}}, \qquad \nu_3 = \nu' + \omega_{\mathrm{ph}} \qquad \text{and} \qquad \nu_4 = \nu'\}, \tag{1.1a}$$
$$\mathrm{pp}: \{\nu_1 = \nu', \qquad \nu_2 = \omega_{\mathrm{pp}} + \nu, \qquad \nu_3 = \omega_{\mathrm{pp}} - \nu' \qquad \text{and} \qquad \nu_4 = -\nu\}. \tag{1.1b}$$

It is easy to extend the frequencies by additional momenta, one only needs to write k instead of ν; the shifts and conventions are carried over to the momenta. We have for the full ladder vertex [**GallerThesis2017**, **GeorgRohringer2013**] within DΓA (in ph notation)

$$F^{\mathrm{D\Gamma A};qkk'}_{1234;\uparrow\downarrow} = \frac{1}{2}\left(F^{q\nu\nu'}_{d;1234} - F^{q\nu\nu'}_{m;1234}\right) - F^{(k'-k)(\nu+\omega)\nu}_{m;1432} - \frac{1}{2}\left(F^{\omega\nu\nu'}_{d;1234} - F^{\omega\nu\nu'}_{m;1234}\right). \tag{1.2}$$

The $F^{(k'-k)(\nu+\omega)\nu}_{m;1432}$-term arises from the ph $\rightarrow \overline{\mathrm{ph}}$ transformation to restore crossing symmetry and hence introduces an orbital permutation as well as frequency shifts. This is an equation only written down for the $\uparrow\downarrow$-component of the full vertex. Luckily, one can obtain the $\overline{\uparrow\downarrow}$-component via a crossing symmetry relation of the $\uparrow\downarrow$-component in the following way:

$$F^{qkk'}_{1234;\overline{\uparrow\downarrow}} = -F^{(k'-k)(k+q)k}_{1432;\uparrow\downarrow}. \tag{1.3}$$

One therefore recovers for the $\overline{\uparrow\downarrow}$-component

$$F^{\mathrm{D\Gamma A};qkk'}_{1234;\overline{\uparrow\downarrow}} = -\frac{1}{2}\left(F^{(k'-k)(\nu+\omega)\nu}_{d;1432} - F^{(k'-k)(\nu+\omega)\nu}_{m;1432}\right) + F^{q\nu\nu'}_{m;1234} + \frac{1}{2}\left(F^{(\nu'-\nu)(\nu+\omega)\nu}_{d;1432} - F^{(\nu'-\nu)(\nu+\omega)\nu}_{m;1432}\right). \tag{1.4}$$

This yields for the full vertices in ph-notation in both the singlet ($\uparrow\downarrow - \overline{\uparrow\downarrow}$)

$$\begin{aligned}
F^{\mathrm{D\Gamma A};qkk'}_{s;1234} &= \frac{1}{2}\left(F^{q\nu\nu'}_{d;1234} - F^{q\nu\nu'}_{m;1234}\right) - F^{(k'-k)(\nu+\omega)\nu}_{m;1432} - \frac{1}{2}\left(F^{\omega\nu\nu'}_{d;1234} - F^{\omega\nu\nu'}_{m;1234}\right)\\
&\quad + \frac{1}{2}\left(F^{(k'-k)(\nu+\omega)\nu}_{d;1432} - F^{(k'-k)(\nu+\omega)\nu}_{m;1432}\right) - F^{q\nu\nu'}_{m;1234} - \frac{1}{2}\left(F^{(\nu'-\nu)(\nu+\omega)\nu}_{d;1432} - F^{(\nu'-\nu)(\nu+\omega)\nu}_{m;1432}\right),\\
&= \frac{1}{2}\left(F^{q\nu\nu'}_{d;1234} - 3F^{q\nu\nu'}_{m;1234}\right) - \frac{1}{2}\left(F^{\omega\nu\nu'}_{d;1234} - F^{\omega\nu\nu'}_{m;1234}\right)\\
&\quad + \frac{1}{2}\left(F^{(k'-k)(\nu+\omega)\nu}_{d;1432} - 3F^{(k'-k)(\nu+\omega)\nu}_{m;1432}\right) - \frac{1}{2}\left(F^{(\nu'-\nu)(\nu+\omega)\nu}_{d;1432} - F^{(\nu'-\nu)(\nu+\omega)\nu}_{m;1432}\right)
\end{aligned} \tag{1.5}$$

and triplet ($\uparrow\downarrow + \overline{\uparrow\downarrow}$) channel

$$\begin{aligned}
F^{\mathrm{D\Gamma A};qkk'}_{t;1234} &= \frac{1}{2}\left(F^{q\nu\nu'}_{d;1234} - F^{q\nu\nu'}_{m;1234}\right) - F^{(k'-k)(\nu+\omega)\nu}_{m;1432} - \frac{1}{2}\left(F^{\omega\nu\nu'}_{d;1234} - F^{\omega\nu\nu'}_{m;1234}\right)\\
&\quad - \frac{1}{2}\left(F^{(k'-k)(\nu+\omega)\nu}_{d;1432} - F^{(k'-k)(\nu+\omega)\nu}_{m;1432}\right) + F^{q\nu\nu'}_{m;1234} + \frac{1}{2}\left(F^{(\nu'-\nu)(\nu+\omega)\nu}_{d;1432} - F^{(\nu'-\nu)(\nu+\omega)\nu}_{m;1432}\right),\\
&= \frac{1}{2}\left(F^{q\nu\nu'}_{d;1234} + F^{q\nu\nu'}_{m;1234}\right) - \frac{1}{2}\left(F^{\omega\nu\nu'}_{d;1234} - F^{\omega\nu\nu'}_{m;1234}\right)\\
&\quad - \frac{1}{2}\left(F^{(k'-k)(\nu+\omega)\nu}_{d;1432} + F^{(k'-k)(\nu+\omega)\nu}_{m;1432}\right) + \frac{1}{2}\left(F^{(\nu'-\nu)(\nu+\omega)\nu}_{d;1432} - F^{(\nu'-\nu)(\nu+\omega)\nu}_{m;1432}\right).
\end{aligned} \tag{1.6}$$

We now want to evaluate this expression for $q_{pp} = 0$ by setting $q_{ph} = k_{pp} - k'_{pp}$, $k_{ph} = k'_{pp}$ and $k'_{ph} = -k_{pp}$. Inserting these frequencies and momenta yields for the singlet channel

$$\begin{aligned}
F^{D\Gamma A;(q_{pp}=0)kk'}_{s;1234} &= \frac{1}{2}\left(F^{(k-k')\nu'(-\nu)}_{d;1234} - 3F^{(k-k')\nu'(-\nu)}_{m;1234}\right) - \frac{1}{2}\left(F^{(\nu-\nu')\nu'(-\nu)}_{d;1234} - F^{(\nu-\nu')\nu'(-\nu)}_{m;1234}\right) \\
&\quad + \frac{1}{2}\left(F^{(-k-k')\nu'\nu}_{d;1432} - 3F^{(-k-k')\nu'\nu}_{m;1432}\right) - \frac{1}{2}\left(F^{(-\nu-\nu')\nu'\nu}_{d;1234} - F^{(-\nu-\nu')\nu'\nu}_{m;1234}\right) \\
&= \frac{1}{2}\left(F^{(k-k')\nu'(-\nu)}_{d;1234} - 3F^{(k-k')\nu'(-\nu)}_{m;1234}\right) - \frac{1}{2}\left(F^{(\nu-\nu')\nu'(-\nu)}_{d;1234} - F^{(\nu-\nu')\nu'(-\nu)}_{m;1234}\right) \\
&\quad + (k \to -k \;\&\; 1234 \to 1432) \\
&= \frac{1}{2}\left(F^{(k-k')\nu'(-\nu)}_{d;1234} - 3F^{(k-k')\nu'(-\nu)}_{m;1234}\right) - F^{(\nu-\nu')\nu'(-\nu)}_{1234;\uparrow\downarrow} + (k \to -k \;\&\; 1234 \to 1432)\,.
\end{aligned} \tag{1.7}$$

For the triplet channel we find

$$\begin{aligned}
F^{D\Gamma A;(q_{pp}=0)kk'}_{t;1234} &= \frac{1}{2}\left(F^{(k-k')\nu'(-\nu)}_{d;1234} + F^{(k-k')\nu'(-\nu)}_{m;1234}\right) - \frac{1}{2}\left(F^{(\nu-\nu')\nu'(-\nu)}_{d;1234} - F^{(\nu-\nu')\nu'(-\nu)}_{m;1234}\right) \\
&\quad - \frac{1}{2}\left(F^{(-k-k')\nu'\nu}_{d;1432} + F^{(-k-k')\nu'\nu}_{m;1432}\right) + \frac{1}{2}\left(F^{(-\nu-\nu')\nu'\nu}_{d;1234} - F^{(-\nu-\nu')\nu'\nu}_{m;1234}\right) \\
&= \frac{1}{2}\left(F^{(k-k')\nu'(-\nu)}_{d;1234} + F^{(k-k')\nu'(-\nu)}_{m;1234}\right) - \frac{1}{2}\left(F^{(\nu-\nu')\nu'(-\nu)}_{d;1234} - F^{(\nu-\nu')\nu'(-\nu)}_{m;1234}\right) \\
&\quad - (k \to -k \;\&\; 1234 \to 1432) \\
&= \frac{1}{2}\left(F^{(k-k')\nu'(-\nu)}_{d;1234} + F^{(k-k')\nu'(-\nu)}_{m;1234}\right) - F^{(\nu-\nu')\nu'(-\nu)}_{1234;\uparrow\downarrow} - (k \to -k \;\&\; 1234 \to 1432)\,.
\end{aligned} \tag{1.8}$$

1.2 Derivation of the Γ_{pp}-vertex with Paul's frequency convention

The frequency convention used by Paul is different to the one by Motoharu and reads

$$\mathrm{ph}: \{\tilde{\nu}_1 = \nu, \quad \tilde{\nu}_2 = \nu - \omega_{ph}, \quad \tilde{\nu}_3 = \nu' - \omega_{ph} \quad \text{and} \quad \tilde{\nu}_4 = \nu'\}, \tag{1.9a}$$
$$\mathrm{pp}: \{\tilde{\nu}_1 = \nu, \quad \tilde{\nu}_2 = \omega_{pp} - \nu', \quad \tilde{\nu}_3 = \omega_{pp} - \nu \quad \text{and} \quad \tilde{\nu}_4 = \nu'\}. \tag{1.9b}$$

It is easy to extend the frequencies by additional momenta, one only needs to write k instead of ν; the shifts and conventions are carried over to the momenta. We have for the full vertex in ladder-DΓA [**GallerThesis2017**, **GeorgRohringer2013**]

$$F^{D\Gamma A;qkk'}_{1234;\uparrow\downarrow} = \frac{1}{2}\left(F^{q\nu\nu'}_{d;1234} - F^{q\nu\nu'}_{m;1234}\right) - F^{(k-k')\nu(\nu-\omega)}_{m;1432} - \frac{1}{2}\left(F^{\omega\nu\nu'}_{d;1234} - F^{\omega\nu\nu'}_{m;1234}\right). \tag{1.10}$$

The $F^{(k-k')\nu(\nu-\omega)}_{m;1432}$-term arises from the ph $\rightarrow \overline{\mathrm{ph}}$ transformation to restore crossing symmetry and hence introduces an orbital permutation as well as frequency shifts. This is an equation only written down for the $\uparrow\downarrow$-component of the full vertex. Luckily, one can obtain the $\overline{\uparrow\downarrow}$-component via a crossing symmetry relation of the $\uparrow\downarrow$-component in the following way:

$$F^{qkk'}_{1234;\overline{\uparrow\downarrow}} = -F^{(k-k')k(k-q)}_{1432;\uparrow\downarrow}. \tag{1.11}$$

One therefore recovers for the $\overline{\uparrow\downarrow}$-component

$$F^{D\Gamma A;qkk'}_{1234;\overline{\uparrow\downarrow}} = -\frac{1}{2}\left(F^{(k-k')\nu(\nu-\omega)}_{d;1432} - F^{(k-k')\nu(\nu-\omega)}_{m;1432}\right) + F^{q\nu\nu'}_{m;1234} + \frac{1}{2}\left(F^{(\nu-\nu')\nu(\nu-\omega)}_{d;1432} - F^{(\nu-\nu')\nu(\nu-\omega)}_{m;1432}\right). \tag{1.12}$$

This yields for the full vertices in ph-notation for both the singlet ($\uparrow\downarrow - \overline{\uparrow\downarrow}$)

$$\begin{aligned} F^{D\Gamma A;qkk'}_{s;1234} &= \frac{1}{2}\left(F^{q\nu\nu'}_{d;1234} - 3F^{q\nu\nu'}_{m;1234}\right) - \frac{1}{2}\left(F^{\omega\nu\nu'}_{d;1234} - F^{\omega\nu\nu'}_{m;1234}\right) \\ &\quad + \frac{1}{2}\left(F^{(k-k')\nu(\nu-\omega)}_{d;1432} - 3F^{(k-k')\nu(\nu-\omega)}_{m;1432}\right) - \frac{1}{2}\left(F^{(\nu-\nu')\nu(\nu-\omega)}_{d;1432} - F^{(\nu-\nu')\nu(\nu-\omega)}_{m;1432}\right) \end{aligned} \tag{1.13}$$

and triplet ($\uparrow\downarrow + \overline{\uparrow\downarrow}$) spin combination

$$\begin{aligned} F^{D\Gamma A;qkk'}_{t;1234} &= \frac{1}{2}\left(F^{q\nu\nu'}_{d;1234} + F^{q\nu\nu'}_{m;1234}\right) - \frac{1}{2}\left(F^{\omega\nu\nu'}_{d;1234} - F^{\omega\nu\nu'}_{m;1234}\right) \\ &\quad - \frac{1}{2}\left(F^{(k-k')\nu(\nu-\omega)}_{d;1432} + F^{(k-k')\nu(\nu-\omega)}_{m;1432}\right) + \frac{1}{2}\left(F^{(\nu-\nu')\nu(\nu-\omega)}_{d;1432} - F^{(\nu-\nu')\nu(\nu-\omega)}_{m;1432}\right). \end{aligned} \tag{1.14}$$

We now want to evaluate this expression for $q_{pp} = 0$ by setting $q_{ph} = k_{pp} + k'_{pp}$, $k_{ph} = k_{pp}$ and $k'_{ph} = k'_{pp}$. Inserting these frequencies and momenta yields for the singlet channel

$$\begin{aligned} F^{D\Gamma A;(q_{pp}=0)kk'}_{s;1234} &= \frac{1}{2}\left(F^{(k+k')\nu\nu'}_{d;1234} - 3F^{(k+k')\nu\nu'}_{m;1234}\right) - \frac{1}{2}\left(F^{(\nu+\nu')\nu\nu'}_{d;1234} - F^{(\nu+\nu')\nu\nu'}_{m;1234}\right) \\ &\quad + \frac{1}{2}\left(F^{(k-k')\nu(-\nu')}_{d;1432} - 3F^{(k-k')\nu(-\nu')}_{m;1432}\right) - \frac{1}{2}\left(F^{(\nu-\nu')\nu(-\nu')}_{d;1234} - F^{(\nu-\nu')\nu(-\nu')}_{m;1234}\right) \\ &= \underbrace{\frac{1}{2}\left(F^{(k+k')\nu\nu'}_{d;1234} - 3F^{(k+k')\nu\nu'}_{m;1234}\right) - F^{(\nu+\nu')\nu\nu'}_{1234;\uparrow\downarrow}}_{\tilde{F}^{D\Gamma A;(q=0)kk'}_{s;1234}} + \underbrace{\left(k' \rightarrow -k' \;\&\; 1234 \rightarrow 1432\right)}_{\tilde{F}^{D\Gamma A;(q=0)k(-k')}_{s;1432}}. \end{aligned} \tag{1.15}$$

For the triplet channel we analogously find

$$\begin{aligned} F^{\mathrm{D\Gamma A};(q_{pp}=0)kk'}_{t;1234} &= \frac{1}{2}\left(F^{(k+k')\nu\nu'}_{d;1234} + F^{(k+k')\nu\nu'}_{m;1234}\right) - \frac{1}{2}\left(F^{(\nu+\nu')\nu\nu'}_{d;1234} - F^{(\nu+\nu')\nu\nu'}_{m;1234}\right) \\ &\quad - \frac{1}{2}\left(F^{(k-k')\nu(-\nu')}_{d;1432} + F^{(k-k')\nu(-\nu')}_{m;1432}\right) + \frac{1}{2}\left(F^{(\nu-\nu')\nu(-\nu')}_{d;1234} - F^{(\nu-\nu')\nu(-\nu')}_{m;1234}\right) \\ &= \underbrace{\frac{1}{2}\left(F^{(k+k')\nu\nu'}_{d;1234} + F^{(k+k')\nu\nu'}_{m;1234}\right) - F^{(\nu+\nu')\nu\nu'}_{1234;\uparrow\downarrow}}_{\tilde{F}^{\mathrm{D\Gamma A};(q=0)kk'}_{t;1234}} - \underbrace{\left(k' \to -k' \;\&\; 1234 \to 1432\right)}_{\tilde{F}^{\mathrm{D\Gamma A};(q=0)k(-k')}_{t;1432}}. \end{aligned} \tag{1.16}$$

We can construct the pairing vertex in either singlet or triplet channel by subtracting the purely *local* particle-particle reducible diagrams. We only need to consider the local diagrams here, since neglecting the momentum dependence of particle-particle reducible diagrams is a key approximation in ladder-DΓA. This yields

$$\Gamma^{(q=0)kk'}_{s/t;1234} = F^{\mathrm{D\Gamma A};(q=0)kk'}_{s/t;1234} - \phi^{(\omega=0)\nu\nu'}_{s/t;1234}. \tag{1.17}$$

The local reducible diagrams can be subtracted straightforwardly from $\tilde{F}^{\mathrm{D\Gamma A};(q_{pp}=0)k(\pm k')}_{s/t;1234}$ in the following fashion

$$\begin{aligned} \Gamma^{(q=0)kk'}_{s/t;1234} &= \tilde{F}^{\mathrm{D\Gamma A};(q=0)kk'}_{s/t;1234} \pm \tilde{F}^{\mathrm{D\Gamma A};(q=0)k(-k')}_{s/t;1432} - \phi^{(\omega=0)\nu\nu'}_{s/t;1234} \\ &= \tilde{F}^{\mathrm{D\Gamma A};(q=0)kk'}_{s/t;1234} \pm \tilde{F}^{\mathrm{D\Gamma A};(q=0)k(-k')}_{s/t;1432} - \left(\phi^{(\omega=0)\nu\nu'}_{pp;1234;\uparrow\downarrow} \mp \phi^{(\omega=0)\nu\nu'}_{pp;1234;\overline{\uparrow\downarrow}}\right) \\ &= \tilde{F}^{\mathrm{D\Gamma A};(q=0)kk'}_{s/t;1234} \pm \tilde{F}^{\mathrm{D\Gamma A};(q=0)k(-k')}_{s/t;1432} - \left(\phi^{(\omega=0)\nu\nu'}_{pp;1234;\uparrow\downarrow} \pm \phi^{(\omega=0)\nu(-\nu')}_{pp;1432;\uparrow\downarrow}\right) \\ &= \underbrace{\tilde{F}^{\mathrm{D\Gamma A};(q=0)kk'}_{s/t;1234} - \phi^{(\omega=0)\nu\nu'}_{pp;1234;\uparrow\downarrow}}_{\tilde{\Gamma}^{(q=0)kk'}_{s/t;1234}} \pm \underbrace{\left(\tilde{F}^{\mathrm{D\Gamma A};(q=0)k(-k')}_{s/t;1432} - \phi^{(\omega=0)\nu(-\nu')}_{pp;1432;\uparrow\downarrow}\right)}_{\tilde{\Gamma}^{(q=0)k(-k')}_{s/t;1432}} \\ &= \tilde{\Gamma}^{(q=0)kk'}_{s/t;1234} \pm \left(k' \to -k' \;\&\; 1234 \to 1432\right). \end{aligned} \tag{1.18}$$

1.3 Symmetry of the gap function

We can show that the gap function inherits the correct symmetry upon using the symmetrized vertices of the sections above. Recall that the multiorbital Eliashberg equation is given by

$$\lambda_{s/t}\Delta^{k}_{s/t;12} = \pm\frac{1}{2}\sum_{k';abcd}\Gamma^{(q=0)kk'}_{s/t;1b2a}\chi^{(q=0)k'}_{s/t;0;acbd}\Delta^{k'}_{s/t;dc} \tag{1.19}$$

for $q = 0$ and the singlet and triplet gap functions fulfill $\Delta^{k}_{s;12} = \Delta^{-k}_{s;21}$ and $\Delta^{k}_{t;12} = -\Delta^{-k}_{t;21}$, respectively. We can write Γ in the symmetrized form with the irreducible vertices $\tilde{\Gamma}$ from Eq. (**??**) above, i.e.,

$$\Gamma^{(q=0)kk'}_{s/t;1234} = \tilde{\Gamma}^{(q=0)kk'}_{s/t;1234} \pm \tilde{\Gamma}^{(q=0)k(-k')}_{s/t;1432}, \tag{1.20}$$

where $\tilde{\Gamma}^{(q=0)kk'}_{s/t;1234}$ contains both the non-local and local contribution to the irreducible vertex. To confirm that this combination enforces the correct symmetry of the gap function, we need to show that

$$\Delta^{k}_{s;12} - \Delta^{-k}_{s;21} = 0 \qquad \text{and} \tag{1.21a}$$

$$\Delta^{k}_{t;12} + \Delta^{-k}_{t;21} = 0. \tag{1.21b}$$

This expression for $q = 0$ then reads for both the singlet and triplet channels

$$\Delta^{k}_{s;12} - \Delta^{-k}_{s;21} \propto \sum_{k';abcd}\left[\Gamma^{(q=0)kk'}_{s;1b2a} - \Gamma^{(q=0)(-k)k'}_{s;2b1a}\right]\chi^{(q=0)k'}_{s;0;acbd}\Delta^{k'}_{s;dc} \qquad \text{and} \tag{1.22a}$$

$$\Delta^{k}_{t;12} + \Delta^{-k}_{t;21} \propto \sum_{k';abcd}\left[\Gamma^{(q=0)kk'}_{t;1b2a} + \Gamma^{(q=0)(-k)k'}_{t;2b1a}\right]\chi^{(q=0)k'}_{t;0;acbd}\Delta^{k'}_{t;dc}. \tag{1.22b}$$

The full non-local ladder vertices fulfill the "swapping" symmetry, which is passed along to the pairing vertex $\Gamma^{(q=0)kk'}_{s/t;1234}$ and reads in pp-notation for $q = 0$ [**GallerAbInitio2017**]

$$\Gamma^{(q=0)kk'}_{s/t;1234} = \Gamma^{(q=0)(-k)(-k')}_{s/t;3412} \overset{(??)}{=} \pm\Gamma^{(q=0)(-k)k'}_{s/t;3214}. \tag{1.23}$$

Applying Eq. (**??**) to Eq. (**??**) then yields for the singlet channel

$$\Delta^{k}_{s;12} - \Delta^{-k}_{s;21} \propto \sum_{k';abcd}\left[\Gamma^{(q=0)kk'}_{s;1b2a} - \Gamma^{(q=0)kk'}_{s;1b2a}\right]\chi^{(q=0)k'}_{s;0;acbd}\Delta^{k'}_{s;dc} = 0 \tag{1.24}$$

and analogously for the triplet channel, see Eq. (**??**),

$$\Delta^{k}_{t;12} + \Delta^{-k}_{t;21} \propto \sum_{k';abcd}\left[\Gamma^{(q=0)kk'}_{t;1b2a} - \Gamma^{(q=0)kk'}_{t;1b2a}\right]\chi^{(q=0)k'}_{t;0;acbd}\Delta^{k'}_{t;dc} = 0, \tag{1.25}$$

confirming that the symmetrization in Eq. (**??**) leads to the correct symmetry of the gap function.