Simplest Possible Self-Organized Critical System

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In order to pinpoint the nature of *self-organized criticality*, a simplest possible system exhibiting the phenomenon is introduced and analyzed. Its phase space is fully parametrized by two integer variables, one describing the state of a medium (*sandpile*), the other describing the state of a disturbance (*avalanche*) propagating in the medium, modifying it in the process. For asymptotically large systems, a scaling limit is obtained in which the system's state and dynamics is given by two real numbers and a simple partial differential equation. These results provide a full and transparent description of the dynamics that drives this system critical and keeps it in that state.

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The idea that some extended dissipative systems may be self-organized critical (SOC) was proposed some years ago, and demonstrated theoretically with the so-called *sandpile models* [1]. Since then, much theoretical and experimental work has been done on the subject. A number of SOC systems have been discovered, but have been received with varying degrees of consensus regarding the justifiability of the claims to SOC. Such differences of opinion can be traced to the lack of a general agreement on what constitutes a SOC system.

It is generally agreed that *some* systems are SOC: In the simplest sandpile model, the so-called *height model* [1], the subject has its conceptual equivalent to the Ising model of equilibrium statistical mechanics. Like the latter, the height model has been solved analytically for a number of properties [2]. However, despite its simple definition, its complex critical behavior is not easy to understand—even with exact analytical results available. In particular, the *mechanism* of self-organization to criticality is not all that transparent. This mechanism is, of course, of key interest, since it must embody what constitutes a SOC system. More interesting than definitions, an understanding of the key mechanism could be helpful in discovering new SOC systems, and in modeling and analyzing new and old SOC systems.

For these reasons we introduce and analyze the simplest possible SOC system here—a system that may serve in the field of SOC as the conceptual equivalent to the harmonic oscillator [3]. It will be useful for its discussion to have a general definition of what qualifies a system as being SOC. We use the following definition.

A self-organizing critical system is a driven, dissipative system consisting of

(1) a *medium* which has
(2) *disturbances* propagating through it, causing
(3) a *modification* of the medium, such that eventually
(4) the medium is in a *critical state*, and
(5) the medium is *modified no more*.

With this definition one can, for example, clearly see why the simple random walk is *not* SOC—a question that is sometimes raised. We can think of the space in which the walk is done as the "medium," and of the walker as the "disturbance" propagating through it. But the medium has no degrees of freedom, and is obviously *never modified* by the walker. It is a matter of taste whether one chooses to think of the simple random walk as being "critical" because it has no inherent time or length scale. But this absence of scales is not acquired by a modification of the medium. It is built in *ab initio.* The third requirement in our definition above is not met by the simple random walk.

The random neighbor model:—The random neighbor model, which we introduce now, is just a random neighbor version of the height model. It should not be confused with mean field theory, however [7]. Introduction of random neighbor relations is often used as a first step in the derivation of a mean field description. The next step consists in neglecting fluctuations. We do *not* take that second step, since self-organization to criticality is a fluctuation phenomenon, as we shall see.

While the height model in [1] is defined on an L^d lattice, the random neighbor model just has N "dynamical" sites. As in the height model, we associate a nonnegative integer variable z_i with each of these sites, and refer to it as the *height* of the *column of sand* on site i. The values of these variables define the state of the "*medium,*" the "*sandpile,*" at any instant of time.

Time is discrete, and the system is driven by choosing a random site i and incrementing the value of z_i by 1. This is called *to drop one grain of sand on site i*. If z_i thereby reaches a threshold value $z^{(\max)}$ the column on that site is said to *topple*, and an *avalanche* has been started: z_i's value is decremented by $z^{(\max)}$, and the $z^{(\max)}$ grains of sand thus removed from site i now make up an avalanche. We define an avalanche to be a number of grains which are not part of any columns, but, so to speak, are in the air, yet to be dropped on columns. This definition is chosen for its ensuing simplicity. An avalanche evolves by dropping its grains one at a time, i.e., *sequentially*, on sites chosen at random among a total of $N + M$ sites. N

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of these sites are the dynamical sites already introduced, while other M *absorbing* sites are introduced with the sole purpose of absorbing sand falling on them. This absorption is analogous to sand "falling off the edge of the lattice" in the height model. The randomness introduced here is *"annealed"*: Every time a site topples, the $z^{(\max)}$ sites that its sand is dropped on are chosen at random anew [8].

When an avalanche thus deposits a grain of sand on a dynamical site i, the height z_i of the column on i is incremented by 1. If, as a consequence, it reaches the threshold height, it also topples and adds its contents to the ongoing avalanche. An avalanche may continue for some time this way. It terminates when all columns are below the threshold height after the last grain of sand in the avalanche has been dropped.

Since we wish to describe the simplest possible model, we assume $z^{(\max)} = 2$ from now on. Little is lost in generality by this choice, and the random neighbor model for generic $z^{(\max)}$ is easily obtained, if desired. With this value for $z^{(\max)}$, a site can at most support one grain of sand without toppling, and the state of the medium at time τ is fully described by one integer $N_1(\tau)$, the number of dynamical sites i containing one grain of sand at that time. For convenience, we also introduce the notation N_0 for the number of empty dynamical sites. Obviously, $N_0 + N_1 = N$.

The instantaneous state of an avalanche is also fully described by one integer $n(\tau)$, the amount of sand in the avalanche at that time. This simple description of the state of an avalanche results from our choosing to deposit the sand in an avalanche sequentially. As a consequence, the duration of an avalanche is twice the total number of topplings it involves, the latter quantity usually referred to as its *size*. The elementary time step of an avalanche, the dropping of one grain of sand from the avalanche on a random site, has one of three possible outcomes.

(1) The grain falls on an empty site. This occurs with probability $N_0/(N + M)$. The result of this event is that n and N_0 are decremented by 1, and N_1 is incremented by 1.

(2) The grain falls on a site where there is already a grain. This occurs with probability $N_1/(N + M)$. Consequently, the column on this site topples, and N_1 is decremented by 1, while N_0 and n are incremented by 1.

(3) The grain falls on an absorbing site and is lost. This occurs with probability $M/(N + M)$, and decrements n by 1 unit, while leaving N_0 and N_1 unchanged.

Since $N_0 + N_1 = N$, the system has only two independent degrees of freedom, parametrized, e.g., by N_1 and n. No SOC system can be simpler than this system [3], because, according to the definition given above, both medium and avalanches must be dynamical parts of a SOC system, and therefore each require at least one degree of freedom for their description. We proceed to demonstrate that this random neighbor model is indeed SOC.

Mean field theory and criticality [10–13]:—In the limit $N \to \infty$, $M \to \infty$, $M/N \to 0$, the random neighbor model will gradually be driven to a state which is critical, as signaled by the absence of a characteristic scale in the distribution of avalanche sizes. This is seen by convincing oneself that $N_1 = N/2$ is an attractive fixed point for the dynamics of the medium up to errors and fluctuations of order $\sqrt{N}$. If one approximates the state of the medium N_1 with the constant value $N/2$, one has the mean field approximation, and in the limit above, avalanches can be identified with unbiased random walks on the positive integers according to outcomes (1) and (2) above. The size of an avalanche is just the time it takes a walker to return to zero after leaving zero in the first step, and is power law distributed with exponent $-3/2$ [10–13]. However, when one excludes fluctuations from the state of the medium, one also loses the self-organization from the description. Thus we see that mean field theory can describe the criticality of avalanches, once it is established for the medium, but cannot describe how the medium interacts with the avalanches, and is kept in its critical state by this interaction. This is a fluctuation phenomenon, and consequently missed by mean field theory.

Master equation:—The time evolution of avalanches and medium in the random neighbor model can be described with a master equation. We have already seen that the state of the system is fully specified by the pair of integers (N_1, n). An avalanche starts from size $n = 0$ and a number $N_1(0)$ of sites containing grains. The avalanche starts when a grain dropped on a randomly chosen site happens to fall on a site already containing a grain. Thus, at time $\tau = 1$ during an avalanche, $(N_1, n) = (N_1(0) - 1, 2)$. We use this state as the initial condition.

During an avalanche, the pair of values (N_1, n) perform a biased random walk on phase space, the set $\{0, 1, \ldots, N\} \times \{0, 1, \ldots, N + 1\}$. The nonvanishing transition probabilities for this walk have already been given above. Let $P(N_1, n; \tau)$ denote the probability that the system is in the state (N_1, n) at time τ. Then the time evolution of the system during an avalanche is described by the master equation

$$\begin{aligned} P(N_1, n; \tau + 1) = {} & \frac{N_1 + 1}{N + M} P(N_1 + 1, n - 1; \tau) \\ & + \frac{N - N_1 + 1}{N + M} P(N_1 - 1, n + 1; \tau) \\ & + \frac{M}{N + M} P(N_1, n + 1; \tau) . \end{aligned} \tag{1}$$

This equation is hardly soluble by analytical means, since its simpler scaling version discussed below is found not to be soluble. For a given initial state, Eq. (1) is easily solved numerically for not too large values of N, and can be simulated for yet larger values.

Once arrived at $n = 0$, the avalanche is over. We note that the number of grains in the system, $N_1 + n$, either remain constant during a time step or, with probability $M/(N + M)$, is decreased by 1 unit.

Scaling limit $N \to \infty$:—To bring out the key role of fluctuations in SOC in the present model, we retain only fluctuations of leading order while taking the limit $N \to \infty$. To this end we introduce the *scaling variables* x, y, and t, and the *scaling function* $f(x, y; t)$ with the definitions

$$x = (N_1 - N/2)/\sqrt{N}\,, \tag{2}$$

$$y = n/\sqrt{N}\,, \tag{3}$$

$$t = \tau/N\,, \tag{4}$$

$$\mu = M/\sqrt{N}\,, \tag{5}$$

$$f(x, y; t) = NP(N_1, n; \tau)\,. \tag{6}$$

Inserting these definitions into Eq. (1), and ignoring all subdominant powers of N, we obtain the following master equation for the probability density $f(x, y; t)$ that the medium is in state x, and an avalanche has size y at time t:

$$\partial_t f = \left[\tfrac{1}{2}(\partial_x - \partial_y)^2 + 2(\partial_x - \partial_y)x + \mu\partial_y\right] f\,. \tag{7}$$

On the right-hand side of this equation, the first term is diffusive, and reflects the stochastic nature of the process being described: An avalanche repeatedly loses grains to subthreshold columns and gains grains from toppling columns in a random process that conserves the total number of grains, $x + y$. Were it not for the other two terms, Eq. (7) would just describe a random walk on a straight line $x + y = \text{const}$ in the (x, y) half plane having $y > 0$. As already discussed, one may choose to think of this random walk as being critical, but it is not SOC. The self-organization to criticality requires the next two terms as well.

The second term describes the self-organization to criticality: It is a convective term describing transport with velocity $-2\sqrt{2}x$ along lines in the (x, y) plane characterized by $x + y = \text{const}$. Since the velocity has the opposite sign of x, this term transports the probability in f towards the point $x = 0$ on the line $x + y = \text{const}$. If $x + y > 0$, points in phase space with $x = 0$ are accessible to the biased random walk. At such points the second term on the right-hand side in Eq. (7) vanishes, and the random walker would be unbiased there, were it not for the third term. If $x + y < 0$, the second term biases the random walker towards $y = 0$, i.e., towards a termination of the avalanche.

The third term is convective in form and dispersive in nature. It describes the transport of probability towards $y = 0$ with constant velocity $-\mu$. Without this term the value of $x + y$, the total amount of sand in the system, would be conserved during an avalanche. With this term the sand added to the system between avalanches can be lost during avalanches. As sand is lost at a fixed rate μ, an avalanche of duration t causes a loss of sand from the system equal to μt. On the average, avalanches starting with $x_0 = x(t = 0) < 0$ lose less sand from the system than must be added to get them started, because these avalanches are biased towards early termination. Avalanches starting with $x_0 > 0$ are biased oppositely, hence they lose more sand. This is how the system remains critical through its dynamics.

If for a moment we *neglect* the first term on the right-hand side of Eq. (7) (the *noise* term), we are left with a first-order partial differential equation (PDE) describing deterministic translation of probability along characteristic lines. These lines are given by $x + y - \mu/2 \log(x) = \text{const}$. Some of them are shown in Fig. 1 for the case of $\mu = 1$. The noise term superposes this translation with an unbiased random walk in the direction parallel to $(1, -1)$.

The dynamics of the system in the scaling limit is completely specified by adding that *between* avalanches the driving mechanism described above for the discrete system corresponds to x being increased when $y = 0$. Using the variable transformations given in Eqs. (2)–(6), one can show that an increase of x by the infinitesimal amount dx initiates an avalanche in $y = 2/\sqrt{N}$ with probability $\sqrt{N}/2\,dx$. Taking the absorbing boundary condition at $y = 0$ into account, this driving mechanism is represented by the boundary condition $f(x, y, 0) = -2\delta'(y)$. Here, $-2\delta'(y)$ is a source term for f per unit increment of x. $\delta(y)$ is Dirac's delta function, and the prime on it indicates the derivative with respect to y. The form of this source term is just that for random walkers initiated at an absorbing boundary. The combination of Eq. (7) and this prescription for how to drive the system, describes SOC "in a nutshell."

In the SOC state, the state x of the medium *between* avalanches changes value with each avalanche. Its prob-

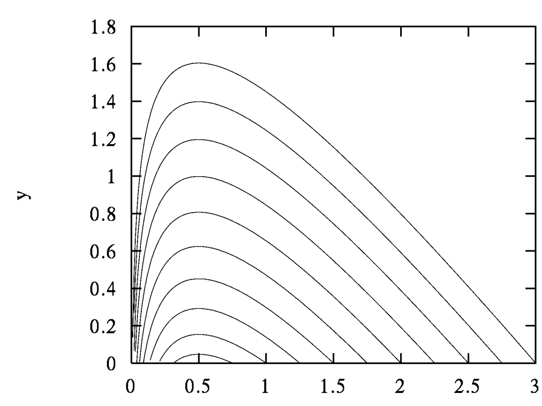

FIG. 1. Some characteristic lines for Eq. (7) with noise term neglected. Case of $\mu = 1$. Avalanches can get started only for $x > \mu/2$, and terminate having $0 < x < \mu/2$.

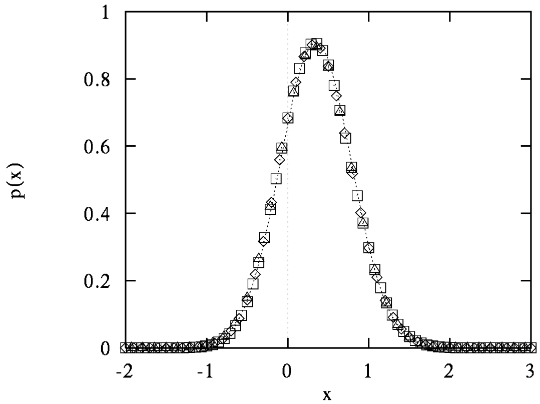

FIG. 2. Probability distribution $p(x)$ for the state of the medium x between avalanches for $\mu = 1$ obtained by integration of Eq. (1). Triangles: $(N, M) = (49, 7)$. Diamonds: $(N, M) = (100, 10)$. Boxes: $(N, M) = (196, 14)$. Dotted curve: Normal distribution with the same mean and variance as boxes.

ability distribution $p(x)$ is shown in Fig. 2 for $\mu = 1$. It seems to be a normal distribution, as one would expect from the central limit theorem, since the change in the state of the medium caused by an avalanche is a Markov process. But the mean of this Gaussian is not zero, showing that the mean field theory result for the average state of the medium, $N_1 = N/2$, has a positive correction of order $\sqrt{N}$.

Partial analytical solution:—Using the method of characteristic lines, we partially solve Eq. (7) by writing f as

$$f(x, y; \tau) = \delta(x + y - x_0 + \mu\tau) g(y, \tau). \quad (8)$$

The resulting equation for g is

$$\partial_t g(y, t) = \Big[\tfrac{1}{2}\partial_y^2 + 2(y - x_0 + \mu t + \mu/2)\partial_y + 2\Big] g(y, t), \quad (9)$$

with boundary condition $g(0, t) = 0$ and initial condition $g(y, 0) = -2\delta'(y)$. It may be rewritten in terms of

$$\psi(y, t) = \exp[(y - x_0 + \mu t + \mu/2)^2] g(y, t), \quad (10)$$

which then must satisfy

$$\partial_t \psi(y, t) = \Big[\tfrac{1}{2}\partial_y^2 - 2(y - x_0 + \mu t + \mu/2)^2 + 1\Big] \psi(y, t). \quad (11)$$

This equation has the form of the Schrödinger equation in imaginary time for a particle restricted to the positive part of the y axis, and there bound in a harmonic oscillator potential with center at $y = x_0 - \mu t - \mu/2$, i.e., with a center moving with constant velocity $-\mu$. This equation is not soluble by analytical means, not even in the adiabatic approximation obtained for small values of μ by treating μt as a constant in Eq. (11). The reason for the nonsolubility is the boundary condition, the condition that the solution must vanish at $y = 0$ for $t > 0$. This absence of a complete analytical solution is not crucial to what we can learn about SOC from this system, as we hope to have demonstrated above.

In conclusion, we have established by example that a dynamical system with only two independent degrees of freedom can be SOC. We have seen that no simpler system can be SOC. We have described the example's dynamics by a master equation, a simple partial differential equation which can be partially solved analytically, and displays the mechanism of self-organization to criticality.

The fact that such a simple description is possible, is encouraging for the investigation of SOC systems in general. It indicates that it may be possible to model and analyze SOC systems with many degrees of freedom in terms of just a few, relevant degrees of freedom, and yet capture their nature.

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