8.2.6 Dini's Theorem *Suppose that (f_n) is a monotone sequence of continuous functions on $I := [a, b]$ that converges on I to a continuous function f. Then the convergence of the sequence is uniform.*

Proof. We suppose that the sequence (f_n) is decreasing and let $g_m := f_m - f$. Then (g_m) is a decreasing sequence of continuous functions converging on I to the 0-function. We will show that the convergence is uniform on I.

Given $\varepsilon > 0$, $t \in I$, there exists $m_{\varepsilon,t} \in \mathbb{N}$ such that $0 \le g_{m_{\varepsilon,t}}(t) < \varepsilon/2$. Since $g_{m_{\varepsilon,t}}$ is continuous at t, there exists $\delta_\varepsilon(t) > 0$ such that $0 \le g_{m_{\varepsilon,t}}(x) < \varepsilon$ for all $x \in I$ satisfying $|x - t| \le \delta_\varepsilon(t)$. Thus, δ_ε is a gauge on I, and if $\dot{\mathcal{P}} = \{(I_i, t_i)\}_{i=1}^n$ is a δ_ε-fine partition, we set $M_\varepsilon := \max\{m_{\varepsilon,t_1}, \ldots, m_{\varepsilon,t_n}\}$. If $m \ge M_\varepsilon$ and $x \in I$, then (by Lemma 5.5.3) there exists an index i with $|x - t_i| \le \delta_\varepsilon(t_i)$ and hence

$$0 \le g_m(x) \le g_{m,t_i}(x) < \varepsilon.$$

Therefore, the sequence (g_m) converges uniformly to the 0-function. Q.E.D.

Proof:

As (f_n) is monotone so assume it to be decreasing

As f_n converge to f on I so

Let $g_m(x) := f_m(x) - f(x)$ $\forall x \in I$ then

$(g_m(x))$ is a monotone decreasing sequence of

continuous functions converging on I to $g(x) = 0$. $\forall x \in I$

$\Rightarrow g_m(x) \geq 0$ $\forall m \in \mathbb{N}$

$\Rightarrow$ By density of real numbers,

$\forall \varepsilon > 0$, $\forall t \in I$, $\exists m_{\varepsilon,t} \in \mathbb{N}$ s.t.

$$0 \le g_{m_{\varepsilon,t}}(t) < \frac{\varepsilon}{2}$$

As $g_{m_{\varepsilon,t}}(x)$ is continuous at t, then

$\exists \delta_\varepsilon(t) > 0$ s.t. $\forall x \in I$ satisfying

$|x - t| \le \delta_\varepsilon(t)$, $0 \le g_{m_{\varepsilon,t}}(x) < \varepsilon$

Define $\dot{P} = \{(I_i, t_i)\}_{i=1}^n$, $\|\dot{P}\| = \delta_\varepsilon$

set $M_\varepsilon := \max\{m_{\varepsilon,t_1}, \ldots m_{\varepsilon,t_n}\}$

If $m \geq M_\varepsilon$ and $x \in I$ then

$\exists i \in \mathbb{N}$ s.t. $|x - t_i| \le \delta_\varepsilon(t_i)$

$\Rightarrow$ $0 \le g_m(x) \le g_{m_{\varepsilon,t_i}}(x) < \varepsilon$

Thus, (g_m) convergence uniformly to $g(x) = 0$

Q.E.D.

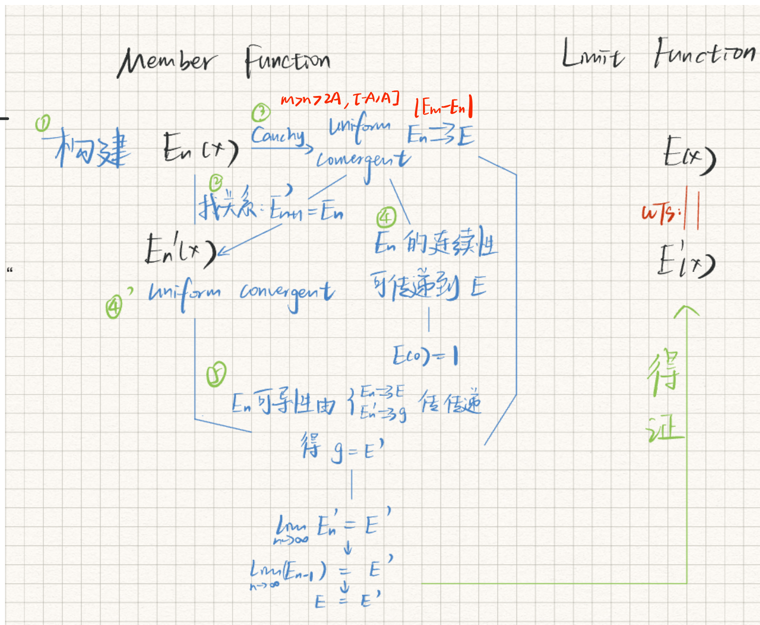
Section 8.3 The Exponential and Logarithmic Functions

E的存在性：

8.3.1 Theorem *There exists a function $E : \mathbb{R} \to \mathbb{R}$ such that:*

(i) $E'(x) = E(x)$ *for all* $x \in \mathbb{R}$. 须可导性的传递

(ii) $E(0) = 1$. 须连续性的传递



Proof. We inductively define a sequence (E_n) of continuous functions as follows:

(1) $$E_1(x) := 1 + x,$$

(2) $$E_{n+1}(x) := 1 + \int_0^x E_n(t)\,dt,$$

for all $n \in \mathbb{N}$, $x \in \mathbb{R}$. Clearly E_1 is continuous on $\mathbb{R}$ and hence is integrable over any bounded interval. If E_n has been defined and is continuous on $\mathbb{R}$, then it is integrable over any bounded interval, so that E_{n+1} is well-defined by the above formula. Moreover, it follows from the Fundamental Theorem (Second Form) 7.3.5 that E_{n+1} is differentiable at any point $x \in \mathbb{R}$ and that

(3) $$E'_{n+1}(x) = E_n(x) \quad \text{for} \quad n \in \mathbb{N}.$$

An Induction argument (which we leave to the reader) shows that

(4) $$\boxed{E_n(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} \quad \text{for} \quad x \in \mathbb{R}.} = \sum_{k=0}^{n} \frac{x^k}{k!}$$

Let $A > 0$ be given; then if $|x| \le A$ and $m > n > 2A$, we have

(5) $$\begin{aligned} |E_m(x) - E_n(x)| &= \left|\frac{x^{n+1}}{(n+1)!} + \cdots + \frac{x^m}{m!}\right| \\ &\le \frac{A^{n+1}}{(n+1)!}\left[1 + \frac{A}{n} + \cdots + \left(\frac{A}{n}\right)^{m-n-1}\right] \\ &< \frac{A^{n+1}}{(n+1)!}2. \end{aligned}$$

Since $\lim(A^n/n!) = 0$, it follows that the sequence (E_n) converges uniformly on the interval $[-A, A]$ where $A > 0$ is arbitrary. In particular this means that $(E_n(x))$ converges for each $x \in \mathbb{R}$. We define $E : \mathbb{R} \to \mathbb{R}$ by

$$E(x) := \lim E_n(x) \quad \text{for} \quad x \in \mathbb{R}.$$

Since each $x \in \mathbb{R}$ is contained inside some interval $[-A, A]$, it follows from Theorem 8.2.2 that E is continuous at x. Moreover, it is clear from (1) and (2) that $E_n(0) = 1$ for all $n \in \mathbb{N}$. Therefore $E(0) = 1$, which proves (ii).

On any interval $[-A, A]$ we have the uniform convergence of the sequence (E_n). In view of (3), we also have the uniform convergence of the sequence (E'_n) of derivatives. It therefore follows from Theorem 8.2.3 that the limit function E is differentiable on $[-A, A]$ and that

$$E'(x) = \lim(E'_n(x)) = \lim(E_{n-1}(x)) = E(x)$$

for all $x \in [-A, A]$. Since $A > 0$ is arbitrary, statement (i) is established. Q.E.D.

Proof: Define sequence of functions Inductively:

$E_1(x) = 1 + x \Rightarrow E_1' = 1$

$E_2(x) = 1 + \int_0^x 1 + t\,dt = 1 + x + \frac{1}{2}x^2 \Rightarrow E_2' = E_1$

$E_3(x) = 1 + \int_0^x 1 + t + \frac{1}{2}t^2\,dt = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 \Rightarrow E_3' = E_2$

$\vdots$

$$\boxed{E_{n+1}(x) = 1 + \int_0^x E_n(t)\,dt \Rightarrow E_{n+1}' = E_n}$$

As E_1 is continuous on R so it is integrable over any bounded area.

If E_n is defined and is continuous on R then it is integrable over any bounded interval.

$\Rightarrow E_{n+1} = 1 + \int_0^x E_n(t)\,dt$ exists on R (well-defined on R)

By the F.T.C. (Second form), on any $[a,b]$, we showed that $E_{n+1}(x) \in R[a,b]$. and $E_{n+1}(x)$ continuous on $[a,b]$

Then $E_{n+1}(x)$ is differentiable

and as $E_{n+1}(x) = 1 + \int_0^x E_n(t)\,dt$

So $E_{n+1}'(x) = E_n(x) \quad \forall n \in N$.

Assume that $E_m(x) = \sum_{m=0}^{n} \frac{x^m}{m!}$

then when $n = 1$, $E_1(x) = \frac{x^0}{0!} + \frac{x^1}{1!} = 1 + x$

assume that when $n = k$,

$$\boxed{E_k(x) = \sum_{m=0}^{k} \frac{x^m}{m!}} = \frac{x^0}{0!} + \frac{x^1}{1!} + \cdots + \frac{x^k}{k!}$$

when $n = k+1$,

$$E_{k+1} = 1 + \int_0^x E_k(t)\,dt = 1 + \int_0^x \sum_{m=0}^{k} \frac{t^m}{m!}\,dt$$
$$= 1 + x + \frac{1}{2}x^2 + \cdots + \frac{x^k}{k!} + \frac{x^{k+1}}{(k+1)!}$$
$$= \sum_{m=0}^{k+1} \frac{x^m}{m!}$$

Thus, $E_n(x) = \sum \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}, \quad \forall x \in R.$

$\forall A > 0$, $\forall x \in [-A, A]$, $\forall m, n \in N$, $m > n > 2A$,

$$|E_m(x) - E_n(x)| = \left| \frac{x^{n+1}}{(n+1)!} + \cdots + \frac{x^m}{m!} \right|$$
$$\le \sum_{k=n+1}^{k=m} \left| \frac{x^k}{k!} \right|$$
$$\le \sum_{k=n+1}^{k=m} \left| \frac{A^k}{k!} \right|$$
$$\boxed{= \frac{A^{n+1}}{(n+1)!} \left[1 + \frac{A}{n} + \cdots + \left(\frac{A}{n}\right)^{m-(n+1)} \right]}$$

$\left(= \frac{A^{n+1}}{(n+1)!} \cdot \frac{1 - \left(\frac{A}{n}\right)^n}{1 - \frac{A}{n}} \right)$ 不好算

$m > n > 2A$
$$\le \frac{A^{n+1}}{(n+1)!} \left[1 + \frac{A}{2A} + \cdots + \left(\frac{A}{2A}\right)^{m-(n+1)} \right]$$
$$\le \frac{A^{n+1}}{(n+1)!} \cdot 2$$

As $\lim_{n\to\infty} \left(\frac{A^n}{n!}\right) = 0$ so $\forall \varepsilon > 0$, $|E_m(x) - E_n(x)| < \varepsilon$

when $m > n > 2A$, By Cauchy

$\Rightarrow (E_n(x))$ converges uniformly on $[-A, A]$

Recall that A is arbitrary,

so $(E_n(x))$ converges uniformly on R, let $E := \lim(E_n)$

then $E_n \rightrightarrows E \quad \forall x \in R$

Recall that E_n continuous on R,

with $E_n \rightrightarrows E$, by 8.2.2 we get E is continuous on R

so $\lim_{x\to 0} E(x) = E(0)$

As $\lim_{n\to\infty} E_n(0) = \lim_{n\to\infty} \lim_{x\to 0} E_n(x) = \lim_{x\to 0} \lim_{n\to\infty} E_n(x) = \lim_{x\to 0} E(x)$

and $\lim_{n\to\infty} [E_n(0)] = \lim_{n\to\infty} [1] = 1$

so $E(0) = 1$ proved (ii)

As $E_{n+1}'(x) = E_n(x)$ and $E_n \rightrightarrows E$ on $[-A, A]$

so $(E_n'(x))$ is uniform convergent. "限定"在闭区间内方便套性质

By 8.2.3, the limit function E is differentiable on $[-A, A]$ and that

$E'(x) = \lim_{n\to\infty} (E_n'(x)) = \lim_{n\to\infty} (E_{n-1}(x)) = E(x) \quad \forall x \in [-A, A]$

Thus, $\forall x \in R$, $E'(x) = E(x)$. (i) is proved.

Q.E.D.

that satisfies (i) and (ii)

8.3.2 Corollary *The function E has a derivative of every order and $E^{(n)}(x) = E(x)$ for all $n \in \mathbb{N}$, $x \in \mathbb{R}$.*

Proof. If $n = 1$, the statement is merely property (i). It follows for arbitrary $n \in \mathbb{N}$ by Induction. Q.E.D.

8.3.3 Corollary *If $x > 0$, then $1 + x < E(x)$.* $\quad E_1 < E_n \le E \Rightarrow E_1 < E$

Proof. It is clear from (4) that if $x > 0$, then the sequence $(E_n(x))$ is strictly increasing. Hence $E_1(x) < E(x)$ for all $x > 0$. Q.E.D.

E的唯一性：

8.3.4 Theorem *The function $E : \mathbb{R} \to \mathbb{R}$ that satisfies* (i) *and* (ii) *of Theorem* 8.3.1 *is unique.*

Proof: Assume there're two functions $E_1(x): R \to R$ and $E_2(x): R \to R$ satisfy (i) and (ii)

then Define $F(x) := E_1(x) - E_2(x)$

From (ii), $F(0) = E_1(0) - E_2(0) = 1 - 1 = 0$

From (i) $F'(x) = E_1'(x) - E_2'(x) = E_1 - E_2 = F$

By 8.3.2, $E_1^{(n)}$ and $E_2^{(n)}$ are exist on R

and $E_1^{(n)} = E_1$, $E_2^{(n)} = E_2$

$\Rightarrow F^{n}(x) = F(x) \quad \forall x \in R.$

[n阶可导，又欲证 $F(x) = 0 \quad \forall x \in R$
联想 Taylor Theorem 重塑 $F(x)$]

用泰勒展式先规定区间

$\forall x \in R$, let I_x be a closed interval ends with 0 and x.

As F continuous on R so F continuous on I_x

$\Rightarrow F$ is bounded on I_x ($[0,x]$ or $[x,0]$)

i.e. $\exists k > 0$ s.t. $|F(t)| \le k \quad \forall t \in I_x$

On I_x: $\forall n \in N$, $\exists c_n \in I_x$ s.t.

$$\boxed{F(x) = F(0) + \frac{F'(0)}{1!}x + \cdots + \frac{F^{(n-1)}(0)}{(n-1)!}x^{n-1} + \frac{F^{n}(c_n)}{n!}x^n}$$

$$= 0 + 0 + \cdots + 0 + \frac{F^{n}(c_n)}{n!}x^n$$

$$\Rightarrow |F(x)| = \left|\frac{F^{n}(c_n)}{n!}x^n\right| \le k\left|\frac{x^n}{n!}\right| \quad \forall n \in N$$

As $\lim_{n\to\infty}\left|\frac{x^n}{n!}\right| = 0$

so $0 \le |F(x)| \to 0 \Rightarrow F(x) = 0$

i.e. $E_1(x) = E_2(x)$ on R.

Q.E.D.

Proof. Let E_1 and E_2 be two functions on $\mathbb{R}$ to $\mathbb{R}$ that satisfy properties (i) and (ii) of Theorem 8.3.1 and let $F := E_1 - E_2$. Then

$$F'(x) = E_1'(x) - E_2'(x) = E_1(x) - E_2(x) = F(x)$$

for all $x \in \mathbb{R}$ and

$$F(0) = E_1(0) - E_2(0) = 1 - 1 = 0.$$

It is clear (by Induction) that F has derivatives of all orders and indeed that $F^{(n)}(x) = F(x)$ for $n \in \mathbb{N}$, $x \in \mathbb{R}$.

Let $x \in \mathbb{R}$ be arbitrary, and let I_x be the closed interval with endpoints 0, x. Since F is continuous on I_x, there exists $K > 0$ such that $|F(t)| \le K$ for all $t \in I_x$. If we apply Taylor's Theorem 6.4.1 to F on the interval I_x and use the fact that $F^{(k)}(0) = F(0) = 0$ for all $k \in \mathbb{N}$, it follows that for each $n \in \mathbb{N}$ there is a point $c_n \in I_x$ such that

$$\begin{aligned} F(x) &= F(0) + \frac{F'(0)}{1!}x + \cdots + \frac{F^{(n-1)}}{(n-1)!}x^{n-1} + \frac{F^{(n)}(c_n)}{n!}x^n \\ &= \frac{F(c_n)}{n!}x^n. \end{aligned}$$

Therefore we have

$$|F(x)| \le \frac{K|x|^n}{n!} \quad \text{for all} \quad n \in \mathbb{N}.$$

But since $\lim(|x|/n!) = 0$, we conclude that $F(x) = 0$. Since $x \in \mathbb{R}$ is arbitrary, we infer that $E_1(x) - E_2(x) = F(x) = 0$ for all $x \in \mathbb{R}$. Q.E.D.

8.3.5 Definition The unique function $E : \mathbb{R} \to \mathbb{R}$, such that $E'(x) = E(x)$ for all $x \in \mathbb{R}$ and $E(0) = 1$, is called the **exponential function**. The number $e := E(1)$ is called **Euler's number**. We will frequently write

$$\exp(x) := E(x) \qquad \text{or} \qquad e^x := E(x) \qquad \text{for} \quad x \in \mathbb{R}.$$

□ $\boxed{E(0)=1}$ 的必要性：

When $u(x)=0 \Rightarrow u^{(n)}(x)=u(x)$

但 $u(0) \neq 1 \Rightarrow u(x) \neq E(x)=e^x$

8.3.6 Theorem *The exponential function satisfies the following properties:*

(iii) $E(x) \neq 0$ *for all* $x \in \mathbb{R}$;

(iv) $E(x+y) = E(x)E(y)$ *for all* $x, y \in \mathbb{R}$;

(v) $E(r) = e^r$ *for all* $r \in \mathbb{Q}$.

(iii) Proof: Assume that $\exists x_0 \in R$ s.t. $E(x_0)=0$

Let J_{x_0} be a closed interval with end points 0 and x_0

then $E(x)$ bounded on J_{x_0}

i.e. $\exists k>0$ s.t. $\forall t \in J_{x_0}, |E(t)| \leq k$

As $E^{(n)}(x) = E(x)$ $\forall x \in J_{x_0}$ so

by Taylor's Theorem, $\forall n \in N$, $\exists C_n \in J_{x_0}$ s.t.

$$E(0) = E(x_0) + \frac{E^{(1)}(x_0)}{1!}(0-x_0) + \cdots + \frac{E^{(n-1)}(x_0)}{(n-1)!}(0-x_0)^{n-1} + \frac{E^{(n)}(C_n)}{n!}(0-x_0)^n$$

已知

$$= 0 + \cdots + 0 + \frac{E^{(n)}(C_n)}{n!}(-x_0)^n$$

$$= 1$$

$$\Rightarrow 0 < 1 = \left| \frac{E^{(n)}(C_n)}{n!}(-x_0)^n \right| \leq \left| \frac{(-x_0^n)}{n!} \right| k = \frac{k}{n!}|x_0|^n$$

and $\lim \frac{|x_0|^n}{n!} = 0$

So $0 < 1 \leq 0$. Contradiction.

Q.E.D.

Proof. (iii) Let $\alpha \in \mathbb{R}$ be such that $E(\alpha) = 0$, and let J_α be the closed interval with endpoints $0, \alpha$. Let $K \geq |E(t)|$ for all $t \in J_\alpha$. Taylor's Theorem 6.4.1 implies that for each $n \in \mathbb{N}$ there exists a point $c_n \in J_\alpha$ such that

$$1 = E(0) = E(\alpha) + \frac{E'(\alpha)}{1!}\overset{(0-\alpha)}{(-\alpha)} + \cdots + \frac{E^{(n-1)}(\alpha)}{(n-1)!}(-\alpha)^{n-1}$$

$$+ \frac{E^{(n)}(\alpha)}{(n)!}(-\alpha)^n = \frac{E(c_n)}{n!}(-\alpha)^n.$$

Thus we have $0 < 1 \leq (K/n!)|\alpha|^n$ for $n \in \mathbb{N}$. But since $\lim(|\alpha|^n/n!) = 0$, this is a contradiction.

(iv) $E(x+y) = E(x)E(y)$ *for all* $x, y \in \mathbb{R}$;

Proof. Let y be fixed. by (iii), $E(y) \neq 0$

then define $G: R \to R$ by

$$\boxed{G(x) := \frac{E(x+y)}{E(y)} \quad \forall x \in R}$$

$$\Rightarrow G'(x) = \frac{E'(x+y)}{E(y)} \quad [E(y) \text{ is a constant here}]$$

$$= \frac{E(x+y)}{E(y)}$$

$$= G(x) \qquad \forall x \in R$$

Also, $G(0) = \frac{E(0+y)}{E(y)} = 1$

So by the uniqness of $E(x)$, $G(x) = E(x)$ $\forall x \in R$.

Thus, $E(x+y) = E(x) \cdot E(y)$ $\forall x \in R$.

Q.E.D.

(iv) Let y be fixed; by (iii) we have $E(y) \neq 0$. Let $G : \mathbb{R} \to \mathbb{R}$ be defined by

$$G(x) := \frac{E(x+y)}{E(y)} \quad \text{for} \quad x \in \mathbb{R}.$$

Evidently we have $G'(x) = E'(x+y)/E(y) = E(x+y)/E(y) = G(x)$ for all $x \in \mathbb{R}$, and $G(0) = E(0+y)/E(y) = 1$. It follows from the uniqueness of E, proved in Theorem 8.3.4, that $G(x) = E(x)$ for all $x \in \mathbb{R}$. Hence $E(x+y) = E(x)E(y)$ for all $x \in \mathbb{R}$. Since $y \in \mathbb{R}$ is arbitrary, we obtain (iv).

(v) $E(r) = e^r$ *for all* $r \in \mathbb{Q}$.

Proof. $\forall r \in \mathbb{Q}$, $r = \frac{m}{n}$, $n \in \mathbb{Z} \setminus \{0\}$, $m \in \mathbb{N}$.

$$e = E(1) = E(n \cdot \tfrac{1}{n}) = E(\tfrac{1}{n} + \tfrac{1}{n} + \cdots + \tfrac{1}{n}) = \left[E(\tfrac{1}{n})\right]^n$$

$$\Rightarrow E(\tfrac{1}{n}) = e^{\frac{1}{n}}$$

$$E(m) = E(m \times 1) = E(1+1+\cdots+1) = [E(1)]^m = e^m$$

Thus, $E(\frac{m}{n}) = E(\frac{1}{n} + \frac{1}{n} + \cdots + \frac{1}{n}) = \left[E(\frac{1}{n})\right]^m = e^{\frac{m}{n}}$

Q.E.D.

(v) It follows from (iv) and Induction that if $n \in \mathbb{N}$, $x \in \mathbb{R}$, then

$$E(nx) = E(x)^n.$$

If we let $x = 1/n$, this relation implies that

$$e = E(1) = E\left(n \cdot \frac{1}{n}\right) = \left(E\left(\frac{1}{n}\right)\right)^n,$$

whence it follows that $E(1/n) = e^{1/n}$. Also we have $E(-m) = 1/E(m) = 1/e^m = e^{-m}$ for $m \in \mathbb{N}$. Therefore, if $m \in \mathbb{Z}$, $n \in \mathbb{N}$, we have

$$E(m/n) = (E(1/n))^m = (e^{1/n})^m = e^{m/n}.$$

This establishes (v). Q.E.D.

8.3.7 Theorem *The exponential function E is strictly increasing on $\mathbb{R}$ and has range equal to $\{y \in \mathbb{R} : y > 0\}$. Further, we have*

(vi) $\lim_{x \to -\infty} E(x) = 0$ and $\lim_{x \to \infty} E(x) = \infty$.

Proof. As $E(0)=1>0$ and $E(x)\neq 0$ $\forall x \in R$.

As proved, E continuous on R

By Bolzano's Intermediate Value Theorem,

$E(x)>0$ $\forall x \in R$

$\Rightarrow$ $E'(x) = E(x) > 0$ $\forall x \in R$

$E(x)$ monotone increases on R

(vi) As $E(x) \geq 1+x$ so $\lim_{x\to\infty} E(x) \geq \lim_{x\to\infty}(1+x) = \infty$

(vii) $\forall z>0$, $E(-z) = \frac{1}{E(z)}$

So $\lim_{x\to\infty} E(x) = \frac{1}{\lim_{x\to\infty} E(x)} = 0$

Thus, by intermediate Value Theorem,

$\forall y \in R$ with $y>0$, $y \in Range(E(x))$

Q.E.D.

The Logarithm Function

We have seen that the exponential function E is a strictly increasing differentiable function with domain $\mathbb{R}$ and range $\{y \in \mathbb{R} : y > 0\}$. (See Figure 8.3.1.) It follows that $\mathbb{R}$ has an inverse function.

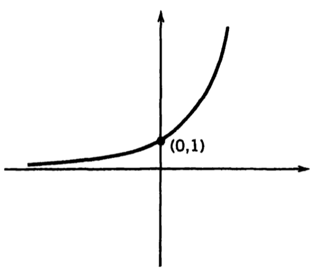


Figure 8.3.1 Graph of E

ln(x)

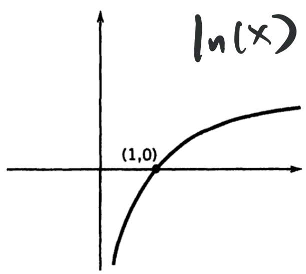


Figure 8.3.2 Graph of L

8.3.8 Definition The function inverse to $E : \mathbb{R} \to \mathbb{R}$ is called the **logarithm** (or the **natural logarithm**). (See Figure 8.3.2.) It will be denoted by L, or by ln.

Since E and L are inverse functions, we have

$$(L \circ E)(x) = x \qquad \text{for all} \quad x \in \mathbb{R}$$

and

$$(E \circ L)(y) = y \qquad \text{for all} \quad y \in \mathbb{R},\ y > 0.$$

These formulas may also be written in the form

$$\ln e^x = x, \qquad e^{\ln y} = y.$$

8.3.9 Theorem *The logarithm is a strictly increasing function L with domain $\{x \in \mathbb{R} : x > 0\}$ and range $\mathbb{R}$. The derivative of L is given by*

(vii) $L'(x) = 1/x$ *for* $x > 0$.

The logarithm satisfies the functional equation

(viii) $L(xy) = L(x) + L(y)$ *for* $x > 0, y > 0$.

Moreover, we have

(ix) $L(1) = 0$ *and* $L(e) = 1$,

(x) $L(x^r) = rL(x)$ *for* $x > 0, r \in \mathbb{Q}$.

(xi) $\lim\limits_{x \to 0+} L(x) = -\infty$ *and* $\lim\limits_{x \to \infty} L(x) = \infty$.

Proof. That L is strictly increasing with domain $\{x \in \mathbb{R} : x > 0\}$ and range $\mathbb{R}$ follows from the fact that E is strictly increasing with domain $\mathbb{R}$ and range $\{y \in \mathbb{R} : y > 0\}$.

(vii) Since $E'(x) = E(x) > 0$, it follows from Theorem 6.1.9 that L is differentiable on $(0, \infty)$ and that

$$\boxed{L'(x) = \frac{1}{(E' \circ L)(x)} = \frac{1}{(E \circ L)(x)} = \frac{1}{x} \qquad \text{for} \quad x \in (0, \infty).}$$

$$(f^{-1})' = \frac{1}{f'}$$

(viii) If $x > 0, y > 0$, let $u := L(x)$ and $v := L(y)$. Then we have $x = E(u)$ and $y = E(v)$. It follows from property (iv) of Theorem 8.3.6 that

$$xy = E(u)E(v) = E(u + v),$$

so that $L(xy) = (L \circ E)(u + v) = u + v = L(x) + L(y)$. This establishes (viii).

The properties in (ix) follow from the relations $E(0) = 1$ and $E(1) = e$.

(x) This result follows from (viii) and Mathematical Induction for $n \in \mathbb{N}$, and is extended to $r \in \mathbb{Q}$ by arguments similar to those in the proof of 8.3.6(v).

To establish property (xi), we first note that since $2 < e$, then $\lim(e^n) = \infty$ and $\lim(e^{-n}) = 0$. Since $L(e^n) = n$ and $L(e^{-n}) = -n$ it follows from the fact that L is strictly increasing that

$$\lim_{x \to \infty} L(x) = \lim L(e^n) = \infty \quad \text{and} \quad \lim_{x \to 0+} L(x) = \lim L(e^{-n}) = -\infty. \qquad \text{Q.E.D.}$$

Power Functions

arbitrary real powers

8.3.10 Definition If $\alpha \in \mathbb{R}$ and $x > 0$, the number x^α is defined to be

$$x^\alpha := \boxed{e^{\alpha \ln x}} = E(\alpha L(x)).$$

The function $x \mapsto x^\alpha$ for $x > 0$ is called the **power function** with exponent α.

We now state some properties of the power functions. Their proofs are immediate consequences of the properties of the exponential and logarithm functions and will be left to the reader.

8.3.11 Theorem *If $\alpha \in \mathbb{R}$ and x, y belong to $(0, \infty)$, then:*

(a) $1^\alpha = 1$, **(b)** $x^\alpha > 0$,
(c) $(xy)^\alpha = x^\alpha y^\alpha$, **(d)** $(x/y)^\alpha = x^\alpha / y^\alpha$.

套 $x^\alpha = e^{\ln x^\alpha}$

8.3.12 Theorem *If $\alpha, \beta \in \mathbb{R}$ and $x \in (0, \infty)$, then:*

(a) $x^{\alpha+\beta} = x^\alpha x^\beta$ **(b)** $(x^\alpha)^\beta = x^{\alpha\beta} = (x^\beta)^\alpha$,
(c) $x^{-\alpha} = 1/x^\alpha$, **(d)** *if $\alpha < \beta$, then $x^\alpha < x^\beta$ for $x > 1$.*

8.3.13 Theorem *Let $\alpha \in \mathbb{R}$. Then the function $x \mapsto x^\alpha$ on $(0, \infty)$ to $\mathbb{R}$ is continuous and differentiable, and*

$$Dx^\alpha = \alpha x^{\alpha-1} \quad \text{for} \quad x \in (0, \infty).$$

Proof. By the Chain Rule we have

$$\begin{aligned} Dx^\alpha &= De^{\alpha \ln x} = e^{\alpha \ln x} \cdot D(\alpha \ln x) \\ &= x^\alpha \cdot \frac{\alpha}{x} = \alpha x^{\alpha-1} \quad \text{for} \quad x \in (0, \infty). \end{aligned}$$

Q.E.D.

The Function $\log_a$

If $a > 0$, $a \neq 1$, it is sometimes useful to define the function $\log_a$.

8.3.14 Definition Let $a > 0$, $a \neq 1$. We define

$$\log_a(x) := \frac{\ln x}{\ln a} \qquad \text{for} \quad x \in (0, \infty).$$

For $x \in (0, \infty)$, the number $\log_a(x)$ is called the **logarithm of x to the base** a. The case $a = e$ yields the logarithm (or natural logarithm) function of Definition 8.3.8. The case $a = 10$ gives the base 10 logarithm (or common logarithm) function $\log_{10}$ often used in computations. Properties of the functions $\log_a$ will be given in the exercises.

Section 8.4 The Trigonometric Functions

8.4.1 Theorem *There exist functions $C : \mathbb{R} \to \mathbb{R}$ and $S : \mathbb{R} \to \mathbb{R}$ such that*

(i) $C''(x) = -C(x)$ *and* $S''(x) = -S(x)$ *for all* $x \in \mathbb{R}$.

(ii) $C(0) = 1, C'(0) = 0,$ *and* $S(0) = 0, S'(0) = 1$.

Proof: Define $C_n(x)$ and $S_n(x)$ inductively:

(ii) $\Rightarrow$ $C_1(x) = 1$, $\quad S_1(x) = x$

As C_1 and S_1 are continuous on $\mathbb{R}$ so they are integrable on bounded interval $[0, x]$, $\forall x > 0$,

(i) $\Rightarrow$ define

$$\boxed{\begin{aligned} S_n(x) &= \int_0^x C_n(t)\,dt \\ C_{n+1}(x) &= 1 - \int_0^x S_n(t)\,dt \qquad \forall n \in \mathbb{N},\ x \in \mathbb{R} \end{aligned}}$$

By Fundamental Theorem of Calculus,

1) S_n, C_{n+1} are differentiable $\forall n \in \mathbb{N}$, $x \in \mathbb{R}$.

2) $S_n'(x) = C_n(x)$, $\quad C_{n+1}'(x) = -S_n(x) \qquad (1)$

By induction:

$$\boxed{\begin{aligned} C_{n+1}(x) &= 1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots + (-1)^n \frac{1}{(2n)!}x^{2n} \\ S_{n+1}(x) &= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots + (-1)^n \frac{1}{(2n+1)!}x^{2n+1} \end{aligned}}$$

$\forall x \in [-A, A]$, $\forall A > 0$, let $m, n \in \mathbb{N}$ and $m > n > 2A$

then
$$\begin{aligned}
|C_m(x) - C_n(x)| &= |x_{n+1} + \cdots + x_m| \\
&= \left| (-1)^n \frac{1}{(2n)!} x^{2n} + \cdots + (-1)^{m-1} \frac{1}{(2m-2)!} x^{2m-2} \right| \\
&\le \left| \frac{1}{(2n)!} x^{2n} \right| + \cdots + \left| \frac{1}{(2(m-1))!} x^{2(m-1)} \right| \\
&= \frac{|x|^{2n}}{(2n)!} \left[1 + \frac{1}{2n+1} x + \frac{1}{(2n+2)(2n+1)} x^2 + \cdots + \frac{(2n)!}{(2m-2)!} x^{2m-2-2n} \right] \\
&\le \frac{A^{2n}}{(2n)!} \left[1 + \frac{A}{2n+1} + \frac{A^2}{(2n+2)(2n+1)} + \cdots + \frac{(2n)!}{(2m-2)!} A^{2(m-n-1)} \right] \\
&< \frac{A^{2n}}{(2n)!} \left[1 + \frac{A}{2n} + \frac{A^2}{(2n)^2} + \cdots + \frac{A^{2(m-n-1)}}{(2n)^{2(m-n-1)}} \right] \\
&\le \frac{A^{2n}}{(2n)!} \left[1 + \frac{1}{4} + \left(\frac{1}{4}\right)^2 + \cdots + \left(\frac{1}{4}\right)^{2(m-n-1)} \right] \\
&< \frac{A^{2n}}{(2n)!} \cdot \cancel{\frac{3}{2}}\ 2\ \frac{16}{15}
\end{aligned}$$

$m > n > 2A$

$\Rightarrow 4A < 2n$

$\frac{A}{2n} < \frac{1}{4}$

As $\lim_{n \to \infty} \frac{A^{2n}}{(2n)!} = 0$ so by Cauchy Criterion,

$(C_n(x))$ is a uniform convergent Series, on $[-A, A]$

$\Rightarrow$ $(C_n(x))$ uniformly convergent on $\mathbb{R}$.

Define function $C: \mathbb{R} \to \mathbb{R}$ by

$$C(x) = \lim_{n \to \infty} C_n(x), \quad \forall x \in \mathbb{R}$$

then $C_n \rightrightarrows C$

As $C_n(x)$ continuous on $\mathbb{R}$ $\forall n \in \mathbb{N}$

So $C(x)$ continuous on $\mathbb{R}$

$$\Rightarrow C(0) = \lim_{x \to 0} C(x) = \lim_{x \to 0} \lim_{n \to \infty} C_n(x) = \lim_{n \to \infty} \lim_{x \to 0} C_n(x) = \lim_{n \to \infty} (1) = 1$$

$\Rightarrow C(0) = 1$

$\forall x \in [-A, A]$, $\forall A > 0$, $m, n \in \mathbb{N}$ and $m \geq n \geq 2A$

$$\begin{aligned}
|S_m(x) - S_n(x)| &= \left| \int_0^x C_m(t)\,dt - \int_0^x C_n(t)\,dt \right| \\
&= \left| \int_0^x C_m(t) - C_n(t)\,dt \right| \\
&\le \int_0^x |C_m(t) - C_n(t)|\,dt \\
&< \int_0^x \frac{A^{2n}}{(2n)!} \cdot \cancel{\frac{3}{2}}\ \frac{16}{15}\,dt \\
&\le \frac{A^{2n}}{(2n)!} \cdot \cancel{\frac{3}{2}}\ \frac{16}{15} \cdot |x| \\
&\le \frac{A^{2n}}{(2n)!} \cdot \cancel{\frac{3}{2}}\ \frac{16}{15} A
\end{aligned}$$

不用再来一遍、

As $\lim_{n \to \infty} \frac{A^{2n+1}}{(2n)!} = 0$ so $(S_n(x))$ is uniformly convergent. on $\mathbb{R}$ by $\forall A > 0$

Define $S(x)$ on $\mathbb{R} \to \mathbb{R}$:

$$S(x) := \lim_{n \to \infty} S_n(x)$$

As $S_n(x)$ continuous on $\mathbb{R}$, $\forall n \in \mathbb{N}$ so $S(x)$ continuous on $\mathbb{R}$

$$\Rightarrow S(0) = \lim_{x \to 0} S(x) = \lim_{x \to 0} \lim_{n \to \infty} S_n(x) = \lim_{n \to \infty} \lim_{x \to 0} S_n(x) = 0$$

$\Rightarrow S(0) = 0$

From (1), $C_n'(x) = -S_{n-1}(x)$ $\quad \forall n \in \mathbb{N} \setminus \{1\}$

so (C_n') uniformly convergent on $[-A, A]$

$\Rightarrow$ $\lim(C_n)$ is differentiable on $[-A, A]$

and $C' = \lim(C_n')$

$\Rightarrow C'(x) = \lim(C_n'(x)) = \lim(-S_{n-1}(x)) = -S(x)$

$\Rightarrow C'(x) = -S(x)$

Similarly, From $S_n'(x) = C_n(x)$, (S_n') is uniformly Convergent, then S is differentiable on $\mathbb{R}$

and $S' = \lim(S_n') = \lim(C_n) = C$

$\Rightarrow S'(x) = C(x) \qquad \forall x \in \mathbb{R}$

Thus, $\quad S'(0) = C(0) = 1, \quad C'(0) = -S(0) = 0$

and $\quad S'(x) = C(x) = -S(x)$

$\quad C''(x) = S'(x) = C(x) \qquad \forall x \in \mathbb{R}$

Q.E.D.

Proof. We define the sequences (C_n) and (S_n) of continuous functions inductively as follows:

$$C_1(x) := 1, \qquad S_1(x) := x, \tag{1}$$

$$S_n(x) := \int_0^x C_n(t)\,dt, \tag{2}$$

$$C_{n+1}(x) := 1 - \int_0^x S_n(t)\,dt, \tag{3}$$

for all $n \in \mathbb{N}$, $x \in \mathbb{R}$.

One sees by Induction that the functions C_n and S_n are continuous on $\mathbb{R}$ and hence they are integrable over any bounded interval; thus these functions are well-defined by the above formulas. Moreover, it follows from the Fundamental Theorem 7.3.5 that S_n and C_{n+1} are differentiable at every point and that

$$S_n'(x) = C_n(x) \quad \text{and} \quad C_{n+1}'(x) = -S_n(x) \quad \text{for} \quad n \in \mathbb{N}, x \in \mathbb{R}. \tag{4}$$

Induction arguments (which we leave to the reader) show that

$$C_{n+1}(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!},$$

$$S_{n+1}(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!}.$$

Let $A > 0$ be given. Then if $|x| \le A$ and $m > n > 2A$, we have that (since $A/2n < 1/4$):

$$\begin{aligned} |C_m(x) - C_n(x)| &= \left| \frac{x^{2n}}{(2n)!} - \frac{x^{2n+2}}{(2n+2)!} + \cdots \pm \frac{x^{2m-2}}{(2m-2)!} \right| \\ &\le \frac{A^{2n}}{(2n)!}\left[1 + \left(\frac{A}{2n}\right)^2 + \cdots + \left(\frac{A}{2n}\right)^{2m-2n-2}\right] \\ &< \frac{A^{2n}}{(2n)!}\left(\frac{16}{15}\right). \end{aligned} \tag{5}$$

Since $\lim(A^{2n}/(2n)!) = 0$, the sequence (C_n) converges uniformly on the interval $[-A, A]$, where $A > 0$ is arbitrary. In particular, this means that $(C_n(x))$ converges for each $x \in \mathbb{R}$. We define $C : \mathbb{R} \to \mathbb{R}$ by

$$C(x) := \lim C_n(x) \quad \text{for} \quad x \in \mathbb{R}.$$

It follows from Theorem 8.2.2 that C is continuous on $\mathbb{R}$ and, since $C_n(0) = 1$ for all $n \in \mathbb{N}$, that $C(0) = 1$.

If $|x| \le A$ and $m \ge n > 2A$, it follows from (2) that

$$S_m(x) - S_n(x) = \int_0^x \{C_m(t) - C_n(t)\}\,dt.$$

If we use (5) and Corollary 7.3.15, we conclude that

$$|S_m(x) - S_n(x)| \le \frac{A^{2n}}{(2n)!}\left(\frac{16}{15}A\right),$$

whence the sequence (S_n) converges uniformly on $[-A, A]$. We define $S : \mathbb{R} \to \mathbb{R}$ by

$$S(x) := \lim S_n(x) \quad \text{for} \quad x \in \mathbb{R}.$$

It follows from Theorem 8.2.2 that S is continuous on $\mathbb{R}$ and, since $S_n(0) = 0$ for all $n \in \mathbb{N}$, that $S(0) = 0$.

Since $C_n'(x) = -S_{n-1}(x)$ for $n > 1$, it follows from the above that the sequence (C_n') converges uniformly on $[-A, A]$. Hence by Theorem 8.2.3, the limit function C is differentiable on $[-A, A]$ and

$$C'(x) = \lim C_n'(x) = \lim(-S_{n-1}(x)) = -S(x) \quad \text{for} \quad x \in [-A, A].$$

Since $A > 0$ is arbitrary, we have

$$C'(x) = -S(x) \quad \text{for} \quad x \in \mathbb{R}. \tag{6}$$

A similar argument, based on the fact that $S_n'(x) = C_n(x)$, shows that S is differentiable on $\mathbb{R}$ and that

$$S'(x) = C(x) \quad \text{for all} \quad x \in \mathbb{R}. \tag{7}$$

It follows from (6) and (7) that

$$C''(x) = -(S(x))' = -C(x) \quad \text{and} \quad S''(x) = (C(x))' = -S(x)$$

for all $x \in \mathbb{R}$. Moreover, we have

$$C'(0) = -S(0) = 0, \qquad S'(0) = C(0) = 1.$$

Thus statements (i) and (ii) are proved. Q.E.D.

8.4.2 Corollary *If C, S are the functions in Theorem 8.4.1, then*

(iii) $C'(x) = -S(x)$ *and* $S'(x) = C(x)$ *for* $x \in \mathbb{R}$.

Moreover, these functions have derivatives of all orders.

Proof. The formulas (iii) were established in (6) and (7). The existence of the higher order derivatives follows by Induction. Q.E.D.

8.4.3 Corollary *The functions C and S satisfy the Pythagorean Identity:*

(iv) $(C(x))^2 + (S(x))^2 = 1$ for $x \in \mathbb{R}$.

Proof. Let $f(x) := (C(x))^2 + (S(x))^2$ for $x \in \mathbb{R}$, so that

$$f'(x) = 2C(x)(-S(x)) + 2S(x)(C(x)) = 0 \qquad \text{for} \quad x \in \mathbb{R}.$$

Thus it follows that $f(x)$ is a constant for all $x \in \mathbb{R}$. But since $f(0) = 1 + 0 = 1$, we conclude that $f(x) = 1$ for all $x \in \mathbb{R}$. Q.E.D.

We next establish the uniqueness of the functions C and S.

8.4.4 Theorem *The functions C and S satisfying properties* (i) *and* (ii) *of Theorem* 8.4.1 *are unique.*

Proof. Assume that there're C and $\tilde{C}$, S and $\tilde{S}$ that satisfy (i) and (ii)

then define $F_1(x) = C - \tilde{C}$, $F_2(x) = S - \tilde{S}$

then F_1 and F_2 are differentiable on R

$F_1(0) = C(0) - \tilde{C}(0) = 1 - 1 = 0$

$F_2(0) = S(0) - \tilde{S}(0) = 0 - 0 = 0$

$F_1'(x) = C'(x) - \tilde{C}'(x) = -S(x) + \tilde{S}(x) = -F_2$

$F_2'(x) = S'(x) - \tilde{S}'(x) = C(x) - \tilde{C}(x) = F_1$

$\Rightarrow \begin{cases} F_1' = -F_2 \\ F_2' = F_1 \end{cases}$ (1) $\quad \begin{cases} F_1^{(k)}(x) = \pm F_i, \ i = 1,2 \\ F_2^{(k)}(x) = \pm F_i, \ i = 1,2 \end{cases}$

$F_1''(x) = (-F_2)' = -F_1$, $\quad F_2''(x) = F_1' = -F_2$

$\Rightarrow \begin{cases} F_1'' = -F_1 \\ F_2'' = -F_2 \end{cases}$ (2) $\Rightarrow \cdots \Rightarrow \begin{cases} F_1^{(k)}(0) = 0 \\ F_2^{(k)}(0) = 0 \end{cases}$

By Induction, $F_i^{(k)}(x)$ exists on R $\forall k \in N$, $i = 1, 2$.

By Induction, $F_i^{(k)}(x)$ exists on R $\forall k \in N$, $i = 1, 2$.

Thus, $\forall x \in R$, define closed interval I_x with endpoints $0, x$.

As F_1, F_2 continuous on I_x then $\exists M > 0$ s.t.

$|F_1(t)| \le M$, $|F_2(t)| \le M$ $\quad \forall t \in I_x$

By Taylor's Theorem: $\forall n \in N$, $\exists C_n \in I_x$ s.t.

$$F_1(x) = F_1(0) + \frac{F_1'(0)}{1!}x + \frac{F_1''(0)}{2!}x^2 + \cdots + \frac{F_1^{(n-1)}(0)}{(n-1)!}x^{n-1} + \frac{F_1^{(n)}(C_n)}{n!}x^n$$
$$= \frac{F_1^{(n)}(C_n)}{n!}x^n$$
$$\boxed{= \frac{\pm F_1(C_n)}{n!}x^n}$$
$$\le \frac{M}{n!}|x|^n$$

As $\lim_{n \to \infty}\left(\frac{|x|^n}{n!}\right) = 0$ so $F_1(t) = 0$ $\forall t \in I_x$

As x is arbitrary, so $F_1(x) = 0$ $\forall x \in R$,

$\Rightarrow C(x) = \tilde{C}(x)$ $\forall x \in R$

Similarly,

$$F_2(x) = F_2(0) + \frac{F_2'(0)}{1!}x + \frac{F_2''(0)}{2!}x^2 + \cdots + \frac{F_2^{(n-1)}(0)}{(n-1)!}x^{n-1} + \frac{F_2^{(n)}(C_n)}{n!}x^n$$
$$= \frac{F_2^{(n)}(C_n)}{n!}x^n$$
$$= \frac{\pm F_2(C_n)}{n!}x^n$$
$$\le M \cdot \frac{|x|^n}{n!} \to 0$$

Thus, $F_2 = 0 \Rightarrow S(x) = \tilde{S}(x)$, $\forall x \in R$. Q.E.D.

Proof. Let C_1 and C_2 be two functions on $\mathbb{R}$ to $\mathbb{R}$ that satisfy $C_j''(x) = -C_j(x)$ for all $x \in \mathbb{R}$ and $C_j(0) = 1$, $C_j'(0) = 0$ for $j = 1, 2$. If we let $D := C_1 - C_2$, then $D''(x) = -D(x)$ for $x \in \mathbb{R}$ and $D(0) = 0$ and $D^{(k)}(0) = 0$ for all $k \in \mathbb{N}$.

Now let $x \in \mathbb{R}$ be arbitrary, and let I_x be the interval with endpoints $0, x$. Since $D = C_1 - C_2$ and $T := S_1 - S_2 = C_2' - C_1'$ are continuous on I_x, there exists $K > 0$ such that $|D(t)| \le K$ and $|T(t)| \le K$ for all $t \in I_x$. If we apply Taylor's Theorem 6.4.1 to D on I_x and use the fact that $D(0) = 0$, $D^{(k)}(0) = 0$ for $k \in \mathbb{N}$, it follows that for each $n \in \mathbb{N}$ there is a point $c_n \in I_x$ such that

$$D(x) = D(0) + \frac{D'(0)}{1!}x + \cdots + \frac{D^{(n-1)}(0)}{(n-1)!}x^{n-1} + \frac{D^{(n)}(c_n)}{n!}x^n$$
$$= \frac{D^{(n)}(c_n)}{n!}x^n.$$

Now either $D^{(n)}(c_n) = \pm D(c_n)$ or $D^{(n)}(c_n) = \pm T(c_n)$. In either case we have

$$|D(x)| \le \frac{K|x|^n}{n!}.$$

But since $\lim(|x|^n/n!) = 0$, we conclude that $D(x) = 0$. Since $x \in \mathbb{R}$ is arbitrary, we infer that $C_1(x) - C_2(x) = 0$ for all $x \in \mathbb{R}$.

A similar argument shows that if S_1 and S_2 are two functions on $\mathbb{R} \to \mathbb{R}$ such that $S_j''(x) = -S_j(x)$ for all $x \in \mathbb{R}$ and $S_j(0) = 0$, $S_j'(0) = 1$ for $j = 1, 2$, then we have $S_1(x) = S_2(x)$ for all $x \in \mathbb{R}$. Q.E.D.

8.4.6 Theorem *If $f : \mathbb{R} \to \mathbb{R}$ is such that*

$$f''(x) = -f(x) \qquad \textit{for} \quad x \in \mathbb{R},$$

then there exist real numbers α, β such that

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$$f(x) = \alpha C(x) + \beta S(x) \quad \textit{for} \quad x \in \mathbb{R}.$$

Prof. Let $\boxed{g(x) := f(0)C(x) + f'(0)S(x) \quad \forall x \in R.}$

$$\Rightarrow g''(x) = (f(0)C'(x) + f'(0)S'(x))'$$
$$= f(0)C''(x) + f'(0)S''(x)$$
$$= f(0)(-C(x)) + f'(0)(-S(x))$$
$$= -g(x) \qquad \forall x \in R$$

and $g(0) = f(0)$, $g'(0) = f'(0)$

Therefore, let $h := f - g$ then

$$h''(x) = f'' - g'' = -f + g = g - f = -h \quad \forall x \in R.$$

and $h(0) = 0$, $h'(0) = 0$

Thus, $h(x) = 0 \quad \forall x \in R$

$$\Rightarrow f(x) = g(x). \qquad \forall x \in R.$$

$\alpha := f(0) \quad \beta = f'(0).$

Q.E.D.

Proof. Let $g(x) := f(0)C(x) + f'(0)S(x)$ for $x \in \mathbb{R}$. It is readily seen that $g''(x) = -g(x)$ and that $g(0) = f(0)$, and since

$$g'(x) = -f(0)S(x) + f'(0)C(x),$$

that $g'(0) = f'(0)$. Therefore the function $h := f - g$ is such that $h''(x) = -h(x)$ for all $x \in \mathbb{R}$ and $h(0) = 0$, $h'(0) = 0$. Thus it follows from the proof of the preceding theorem that $h(x) = 0$ for all $x \in \mathbb{R}$. Therefore $f(x) = g(x)$ for all $x \in \mathbb{R}$. Q.E.D.

8.4.7 Theorem *The function C is even and S is odd in the sense that*

(v) $C(-x) = C(x)$ *and* $S(-x) = -S(x)$ *for* $x \in \mathbb{R}$.

If $x, y \in \mathbb{R}$, then we have the "addition formulas"

(vi) $C(x+y) = C(x)C(y) - S(x)S(y), \quad S(x+y) = S(x)C(y) + C(x)S(y)$.

(v) Proof: Let $\varphi(x) := C(-x)$ $\forall x \in R$ then

$\varphi''(x) = -C(-x) = -\varphi(x)$ $\forall x \in R$.

Moreover, $\varphi(0) = 1$, $\varphi'(0) = 0$

$\Rightarrow$ $\varphi(x) = C(x) = \cos(x)$

$\Rightarrow$ $C(-x) = C(x)$.

Let $\gamma(x) := S(-x)$ then $\gamma''(x) = -S(-x) = -\gamma(x)$

$\Rightarrow$ $\gamma(0) = S(0) = 0$, $\gamma'(0) = C(0) = 1$

$\Rightarrow$ $\gamma(x) = S(x)$

$\Rightarrow$ $S(-x) = S(x)$ $\forall x \in R$.

Q.E.D.

(vi). As $C''(x) = -C(x)$, $S''(x) = -S(x)$

So $\exists \alpha, \beta \in R$ s.t.

$f(x) = C(x+y) = \alpha C(x) + \beta S(x)$ (1)

and

$f'(x) = -S(x+y) = -\alpha S(x) + \beta C(x)$ $\forall x \in R$

Let $x = 0$,

then $C(y) = \alpha$, $-S(y) = \beta \Rightarrow S(y) = -\beta$

now (1) becomes: $C(x+y) = C(y)C(x) - S(y)S(x)$.

Let $g(x) := S(x+y) = \alpha' S(x) + \beta' C(x)$

then $g'(x) = C(x+y) = \alpha' C(x) - \beta' S(x)$

Let $X = 0$ then

$S(y) = \beta'$, $C(y) = \alpha'$

$\Rightarrow$ $S(x+y) = C(y)S(x) + S(y)C(x)$

Q.E.D.

Proof. (v) If $\varphi(x) := C(-x)$ for $x \in \mathbb{R}$, then a calculation shows that $\varphi''(x) = -\varphi(x)$ for $x \in \mathbb{R}$. Moreover, $\varphi(0) = 1$ and $\varphi'(0) = 0$ so that $\varphi = C$. Hence, $C(-x) = C(x)$ for all $x \in \mathbb{R}$. In a similar way one shows that $S(-x) = -S(x)$ for all $x \in \mathbb{R}$.

(vi) Let $y \in \mathbb{R}$ be given and let $f(x) := C(x+y)$ for $x \in \mathbb{R}$. A calculation shows that $f''(x) = -f(x)$ for $x \in \mathbb{R}$. Hence, by Theorem 8.4.6, there exists real numbers α, β such that

$$f(x) = C(x+y) = \alpha C(x) + \beta S(x) \quad \text{and}$$

$$f'(x) = -S(x+y) = -\alpha S(x) + \beta C(x)$$

for $x \in \mathbb{R}$. If we let $x = 0$, we obtain $C(y) = \alpha$ and $-S(y) = \beta$, whence the first formula in (vi) follows. The second formula is proved similarly.

Q.E.D.

8.4.8 Theorem *If $x \in \mathbb{R}$, $x \geq 0$, then we have*

(vii) $-x \leq S(x) \leq x$;

(viii) $1 - \frac{1}{2}x^2 \leq C(x) \leq 1$;

(ix) $x - \frac{1}{6}x^3 \leq S(x) \leq x$;

(x) $1 - \frac{1}{2}x^2 \leq C(x) \leq 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4$.

Proof. Corollary 8.4.3 implies that $-1 \leq C(t) \leq 1$ for $t \in \mathbb{R}$, so that if $x \geq 0$, then

$$-x \leq \int_0^x C(t)dt \leq x,$$

whence we have (vii). If we integrate (vii), we obtain

$$-\tfrac{1}{2}x^2 \leq \int_0^x S(t)\,dt \leq \tfrac{1}{2}x^2,$$

whence we have

$$-\tfrac{1}{2}x^2 \leq -C(x) + 1 \leq \tfrac{1}{2}x^2.$$

Thus we have $1 - \frac{1}{2}x^2 \leq C(x)$, which implies (viii).

Inequality (ix) follows by integrating (viii), and (x) follows by integrating (ix). Q.E.D.