Chapter 3 Sequence

3.1.1 Definition A **sequence of real numbers** (or a **sequence in** $\mathbb{R}$) is a function defined on the set $\mathbb{N} = \{1, 2, \ldots\}$ of natural numbers whose range is contained in the set $\mathbb{R}$ of real numbers.

$\boxed{N \longrightarrow X \subseteq R}$

The Limit of a Sequence

3.1.3 Definition A sequence $X = (x_n)$ in $\mathbb{R}$ is said to **converge** to $x \in \mathbb{R}$, or x is said to be a **limit** of (x_n), if for every $\varepsilon > 0$ there exists a natural number $K(\varepsilon)$ such that for all $n \geq K(\varepsilon)$, the terms x_n satisfy $|x_n - x| < \varepsilon$.

$$\lim X = x \quad \text{or} \quad \lim(x_n) = x.$$

We will sometimes use the symbolism $x_n \to x$, which indicates the intuitive idea that the values x_n "approach" the number x as $n \to \infty$.

If a sequence has a limit, we say that the sequence is **convergent**; if it has no limit, we say that the sequence is **divergent**.

3.1.4 Uniqueness of Limits *A sequence in $\mathbb{R}$ can have at most one limit.*

3.1.5 Theorem *Let $X = (x_n)$ be a sequence of real numbers, and let $x \in \mathbb{R}$. The following statements are equivalent.*

(a) *X converges to x.*
(b) *For every $\varepsilon > 0$, there exists a natural number K such that for all $n \geq K$, the terms x_n satisfy $|x_n - x| < \varepsilon$.*
(c) *For every $\varepsilon > 0$, there exists a natural number K such that for all $n \geq K$, the terms x_n satisfy $x - \varepsilon < x_n < x + \varepsilon$.*
(d) *For every ε-neighborhood $V_\varepsilon(x)$ of x, there exists a natural number K such that for all $n \geq K$, the terms x_n belong to $V_\varepsilon(x)$.*

} Definitions

Proof of (b) $\Leftrightarrow$ (c) $\Leftrightarrow$ (d):

$$|u - x| < \varepsilon \iff -\varepsilon < u - x < \varepsilon \iff x - \varepsilon < u < x + \varepsilon \iff u \in V_\varepsilon(x).$$

Q.E.D.

3.1.6 Examples

证明当 $n \to \infty$ 时，

$\lim(x_n) = 0$

$\Updownarrow$

当 $n > k \in N$ 时，

$x_n \to x$

Choose k

$\Downarrow$

证的是符合要求的自然数 K

(a) $\lim(1/n) = 0.$

If $\varepsilon > 0$ is given, then $1/\varepsilon > 0$. By the Archimedean Property 2.4.3, there is a natural number $K = K(\varepsilon)$ such that $1/K < \varepsilon$. Then, if $n \geq K$, we have $1/n \leq 1/K < \varepsilon$. Conse-

Choose

(b) $\lim(1/(n^2 + 1)) = 0.$

Now choose K such that $1/K < \varepsilon$, as in (a) above. Then $n \geq K$ implies that $1/n < \varepsilon$,

(c) $\lim\left(\dfrac{3n+2}{n+1}\right) = 3.$

Choose

Now if the inequality $1/n < \varepsilon$ is satisfied, then the inequality (1) holds. Thus if $1/K < \varepsilon$, then for any $n \geq K$, we also have $1/n < \varepsilon$ and hence (1) holds. Therefore the limit of the sequence is 3.

(d) $\lim(\sqrt{n+1} - \sqrt{n}) = 0.$

(d) $\lim(\sqrt{n+1} - \sqrt{n}) = 0.$
We multiply and divide by $\sqrt{n+1} + \sqrt{n}$ to get

$$\frac{(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{\sqrt{n+1} + \sqrt{n}} = \frac{n+1-n}{\sqrt{n+1} + \sqrt{n}}$$
$$= \frac{1}{\sqrt{n+1} + \sqrt{n}} \leq \frac{1}{\sqrt{n}}$$

For a given $\varepsilon > 0$, we obtain $1/\sqrt{n} < \varepsilon$ if and only if $1/n < \varepsilon^2$ or $n > 1/\varepsilon^2$. Thus if we take / Choose $K > 1/\varepsilon^2$, then $\sqrt{n+1} - \sqrt{n} < \varepsilon$ for all $n > K$. (For example, if we are given $\varepsilon = 1/10$, then $K > 100$ is required.)

Start: If $b^n < \varepsilon$ for all $\varepsilon > 0$.

的存在性

(e) If $0 < b < 1$, then $\lim(b^n) = 0$.

$\ln b < 0$ *

Method ①

We will use elementary properties of the natural logarithm function. If $\varepsilon > 0$ is given, we see that

$$b^n < \varepsilon \iff n \ln b < \ln \varepsilon \iff n > \ln \varepsilon / \ln b.$$

(The last inequality is reversed because $\ln b < 0$.) Thus if we choose K to be a number such that $K > \ln \varepsilon / \ln b$, then we will have $0 < b^n < \varepsilon$ for all $n \geq K$. Thus we have $\lim(b^n) = 0$.

Method ②

Proof: As $0 < b < 1$

let $b = \frac{1}{1+a_n}$ $(a_n > 0)$

so $b^n = \frac{1}{(1+a_n)^n} \leq \frac{1}{1+na_n} < \frac{1}{n} \cdot \frac{1}{a_n}$

Choose $K \in \mathbb{N}$, s.t. $\forall n \geq K$,

$\frac{1}{n} < a_n \cdot \varepsilon$ $(\varepsilon > 0)$

$\Rightarrow 0 < b^n < \frac{1}{n} \cdot \frac{1}{a_n} < a_n \cdot \varepsilon \cdot \frac{1}{a_n} = \varepsilon$

$\Rightarrow \lim(b^n) = 0$ □.

Tails of Sequences

3.1.8 Definition If $X = (x_1, x_2, \ldots, x_n, \ldots)$ is a sequence of real numbers and if m is a given natural number, then the m-**tail** of X is the sequence

$$X_m := (x_{m+n} : n \in \mathbb{N}) = (x_{m+1}, x_{m+2}, \ldots)$$

3.1.9 Theorem *Let $X = (x_n : n \in \mathbb{N})$ be a sequence of real numbers and let $m \in \mathbb{N}$. Then the m-tail $X_m = (x_{m+n} : n \in \mathbb{N})$ of X converges if and only if X converges. In this case,* $\lim X_m = \lim X$.

#稍分类讨论 $k(\varepsilon)$ 与 m 的大小关系

Proof. We note that for any $p \in \mathbb{N}$, the pth term of X_m is the $(p+m)$th term of X. Similarly, if $q > m$, then the qth term of X is the $(q-m)$th term of X_m.

Assume X converges to x. Then given any $\varepsilon > 0$, if the terms of X for $n \geq K(\varepsilon)$ satisfy $|x_n - x| < \varepsilon$, then the terms of X_m for $k \geq K(\varepsilon) - m$ satisfy $|x_k - x| < \varepsilon$. Thus we can take $K_m(\varepsilon) = K(\varepsilon) - m$, so that X_m also converges to x.

Conversely, if the terms of X_m for $k \geq K_m(\varepsilon)$ satisfy $|x_k - x| < \varepsilon$, then the terms of X for $n \geq K(\varepsilon) + m$ satisfy $|x_n - x| < \varepsilon$. Thus we can take $K(\varepsilon) = K_m(\varepsilon) + m$.

Therefore, X converges to x if and only if X_m converges to x. Q.E.D.

因为若 $k(\varepsilon) \leq m$ 则 $k(\varepsilon) - m \leq 0$

$\Rightarrow \forall n \geq k(\varepsilon) - m \geq 0$，合理.

3.1.10 Theorem *Let (x_n) be a sequence of real numbers and let $x \in \mathbb{R}$. If (a_n) is a sequence of positive real numbers with $\lim(a_n) = 0$ and if for some constant $C > 0$ and some $m \in \mathbb{N}$ we have*

$$|x_n - x| \leq C a_n \quad \textit{for all} \quad n \geq m,$$

then it follows that $\lim(x_n) = x$.

Proof. If $\varepsilon > 0$ is given, then since $\lim(a_n) = 0$, we know there exists $K = K(\varepsilon/C)$ such that $n \geq K$ implies

$$a_n = |a_n - 0| < \varepsilon/C.$$

Therefore it follows that if both $n \geq K$ and $n \geq m$, then

$$|x_n - x| \leq C a_n < C(\varepsilon/C) = \varepsilon.$$

Prove directly

Since $\varepsilon > 0$ is arbitrary, we conclude that $x = \lim(x_n)$. Q.E.D.

#很重要

3.1.11 Examples

(a) If $a > 0$, then $\lim\left(\frac{1}{1+na}\right) = 0$.

(b) If $0 < b < 1$, then $\lim(b^n) = 0$.

(c) If $c > 0$, then $\lim(c^{1/n}) = 1$.

The case $c = 1$ is trivial, since then $(c^{1/n})$ is the constant sequence $(1, 1, \ldots)$, which evidently converges to 1.

\# If $c > 1$, then $c^{1/n} = 1 + d_n$ for some $d_n > 0$. Hence by Bernoulli's Inequality 2.1.13(c),

$d_n > 0$

$$c = (1 + d_n)^n \geq 1 + nd_n \quad \text{for} \quad n \in \mathbb{N}.$$

Therefore we have $c - 1 \geq nd_n$, so that $d_n \leq (c-1)/n$. Consequently we have

$$|c^{1/n} - 1| = d_n \leq (c-1)\frac{1}{n} \quad \text{for} \quad n \in \mathbb{N}.$$

We now invoke Theorem 3.1.10 to infer that $\lim(c^{1/n}) = 1$ when $c > 1$.

不熟

Now suppose that $0 < c < 1$; then $c^{1/n} = 1/(1 + h_n)$ for some $h_n > 0$. Hence Bernoulli's Inequality implies that

$$c = \frac{1}{(1+h_n)^n} \leq \frac{1}{1 + nh_n} < \frac{1}{nh_n},$$

from which it follows that $0 < h_n < 1/nc$ for $n \in \mathbb{N}$. Therefore we have

$$0 < 1 - c^{1/n} = \boxed{\frac{h_n}{1 + h_n} < h_n} < \frac{1}{nc}$$

so that

$$\boxed{|c^{1/n} - 1|} < \left(\frac{1}{c}\right)\frac{1}{n} \quad \text{for} \quad n \in \mathbb{N}.$$

We now apply Theorem 3.1.10 to infer that $\lim(c^{1/n}) = 1$ when $0 < c < 1$.

\# $n>1$ 很重要

(d) $\lim(n^{1/n}) = 1$

Since $n^{1/n} > 1$ for $n > 1$, we can write $n^{1/n} = 1 + k_n$ for some $k_n > 0$ when $n > 1$. Hence $n = (1 + k_n)^n$ for $n > 1$. By the Binomial Theorem, if $n > 1$ we have

$$n = 1 + nk_n + \tfrac{1}{2}n(n-1)k_n^2 + \cdots \geq 1 + \tfrac{1}{2}n(n-1)k_n^2,$$

whence it follows that

$$n - 1 \geq \tfrac{1}{2}n(n-1)k_n^2.$$

Hence $k_n^2 \leq 2/n$ for $n > 1$. If $\varepsilon > 0$ is given, it follows from the Archimedean Property that there exists a natural number N_ε such that $2/N_\varepsilon < \varepsilon^2$. It follows that if $n \geq \sup\{2, N_\varepsilon\}$ then $2/n < \varepsilon^2$, whence

$$0 < n^{1/n} - 1 = k_n \leq (2/n)^{1/2} < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we deduce that $\lim(n^{1/n}) = 1$. □

通过 证 $k_n < \varepsilon$

证 $\lim(n^{\frac{1}{n}} - 1) = 0$

作业题: $\lim(nb^n) = 0$, $0 < b < 1$

设 $b = \frac{1}{1+a}$, some $a \in P$

$\frac{1}{b^n} = (1+a)^n \geq 1 + na + \frac{1}{2}n(n-1)a^2 + \cdots \geq \frac{1}{2}n(n-1)a^2$

$\Rightarrow \quad nb^n \leq \frac{2}{(n-1)a^2} < \frac{1}{(n-1)a^2} = \left(\frac{1}{a}\right)^2 \cdot \boxed{\frac{1}{n-1}} < \varepsilon$

Section 3.2 Limit Theorems

3.2.1 Definition A sequence $X = (x_n)$ of real numbers is said to be **bounded** if there exists a real number $M > 0$ such that $|x_n| \le M$ for all $n \in \mathbb{N}$.

Thus, the sequence (x_n) is bounded if and only if the set $\{x_n : n \in \mathbb{N}\}$ of its values is a bounded subset of $\mathbb{R}$.

证明 bounded 时也往 $|x_n| \le M$ 上靠；
别拆绝对值

3.2.2 Theorem *A convergent sequence of real numbers is bounded.*

Proof. Suppose that $\lim(x_n) = x$ and let $\varepsilon := 1$. Then there exists a natural number $K = K(1)$ such that $|x_n - x| < 1$ for all $n \ge K$. If we apply the Triangle Inequality with $n \ge K$ we obtain

$$|x_n| = |x_n - x + x| \le |x_n - x| + |x| < 1 + |x|.$$

依旧属于已知数

$$\big||a|-|b|\big| \le |a-b| \le \underbrace{|a|+|b|}_{\varepsilon_0}$$

If we set

$$M := \sup\{|x_1|, |x_2|, \ldots, |x_{K-1}|, 1 + |x|\},$$

直接代替 $|x_n|$

then it follows that $|x_n| \le M$ for all $n \in \mathbb{N}$. Q.E.D.

3.2.3 Theorem **(a)** *Let $X = (x_n)$ and $Y = (y_n)$ be sequences of real numbers that converge to x and y, respectively, and let $c \in \mathbb{R}$. Then the sequences $X + Y$, $X - Y$, $X \cdot Y$, and cX converge to $x + y$, $x - y$, xy, and cx, respectively.*

To show that $X \cdot Y = (x_n y_n)$ converges to xy, we make the estimate

$$\begin{aligned} |x_n y_n - xy| &= |(x_n y_n - x_n y) + (x_n y - xy)| \\ &\le |x_n(y_n - y)| + |(x_n - x)y| \\ &= |x_n||y_n - y| + |x_n - x||y|. \end{aligned}$$

According to Theorem 3.2.2 there exists a real number $M_1 > 0$ such that $|x_n| \le M_1$ for all $n \in \mathbb{N}$ and we set $M := \sup\{M_1, |y|\}$. Hence we have

$$|x_n y_n - xy| \le M|y_n - y| + M|x_n - x|.$$

#

From the convergence of X and Y we conclude that if $\varepsilon > 0$ is given, then there exist natural numbers K_1 and K_2 such that if $n \ge K_1$ then $|x_n - x| < \varepsilon/2M$, and if $n \ge K_2$ then $|y_n - y| < \varepsilon/2M$. Now let $K(\varepsilon) = \sup\{K_1, K_2\}$; then, if $n \ge K(\varepsilon)$ we infer that

$$\begin{aligned} |x_n y_n - xy| &\le M|y_n - y| + M|x_n - x| \\ &< M(\varepsilon/2M) + M(\varepsilon/2M) = \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, this proves that the sequence $X \cdot Y = (x_n y_n)$ converges to xy.

$\big||z|-|z_n|\big|$ 和 $\big||z_n|-|z|\big|$ 都可

取 $\varepsilon = \frac{1}{2}|z|$, $\big||z|-|z_n|\big| \le |z_n - z| < \varepsilon = \frac{1}{2}|z|$

$-\frac{1}{2}|z| = -\varepsilon < -|z_n - z| \le \big||z_n|-|z|\big| \Rightarrow |z_n| \ge \frac{1}{2}|z|$

不熟、笔记 P148

(b) *If $X = (x_n)$ converges to x and $Z = (z_n)$ is a sequence of nonzero real numbers that converges to z and if $z \ne 0$, then the quotient sequence X/Z converges to x/z.*

Extention

Some of the results of Theorem 3.2.3 can be extended, by Mathematical Induction, to a finite number of convergent sequences. For example, if $A = (a_n)$, $B = (b_n)$, $\ldots$, $Z = (z_n)$ are convergent sequences of real numbers, then their sum $A + B + \cdots + Z = (a_n + b_n + \cdots + z_n)$ is a convergent sequence and

$$\text{(1)} \qquad \lim(a_n + b_n + \cdots + z_n) = \lim(a_n) + \lim(b_n) + \cdots + \lim(z_n).$$

Also their product $A \cdot B \cdots Z := (a_n b_n \cdots z_n)$ is a convergent sequence and

$$\text{(2)} \qquad \lim(a_n b_n \cdots z_n) = (\lim(a_n))\,(\lim(b_n)) \cdots (\lim(z_n)).$$

Hence, if $k \in \mathbb{N}$ and if $A = (a_n)$ is a convergent sequence, then

$$\text{(3)} \qquad \lim(a_n^k) = (\lim(a_n))^k.$$

$\lim(\sqrt{a_n}) = \sqrt{\lim(a_n)}$

3.2.4 Theorem *If $X = (x_n)$ is a convergent sequence of real numbers and if $x_n \ge 0$ for all $n \in \mathbb{N}$, then $x = \lim(x_n) \ge 0$.*

Proof: Assume that
$x < 0$ so $-x > 0$
let $\varepsilon_0 := -x$
$\Rightarrow |x - x_n| < \varepsilon_0 = -x$
$\Rightarrow x_n < 0$. Contradiction.

Generalize

3.2.5 Theorem *If $X = (x_n)$ and $Y = (y_n)$ are convergent sequences of real numbers and if $x_n \le y_n$ for all $n \in \mathbb{N}$, then $\lim(x_n) \le \lim(y_n)$.*

小于等于

Remark:
If $x_n < y_n \not\Rightarrow \lim x_n < \lim y_n$

3.2.6 Theorem *If $X = (x_n)$ is a convergent sequence and if $a \le x_n \le b$ for all $n \in \mathbb{N}$, then $a \le \lim(x_n) \le b$.*

Generalize

3.2.7 Squeeze Theorem *Suppose that $X = (x_n)$, $Y = (y_n)$, and $Z = (z_n)$ are sequences of real numbers such that*

$$x_n \le y_n \le z_n \quad \text{for all} \quad n \in \mathbb{N},$$

and that $\lim(x_n) = \lim(z_n)$. Then $Y = (y_n)$ is convergent and

$$\lim(x_n) = \lim(y_n) = \lim(z_n).$$

Q：书上的证明方法：
(从收敛定义入手)

$-\varepsilon < x_n - L \le y_n - L \le z_n - L < \varepsilon$

$\Rightarrow |y_n - L| < \varepsilon$

3.2.9 Theorem *Let the sequence $X = (x_n)$ converge to x. Then the sequence $(|x_n|)$ of absolute values converges to $|x|$. That is, if $x = \lim(x_n)$, then $|x| = \lim(|x_n|)$.*

3.2.10 Theorem *Let $X = (x_n)$ be a sequence of real numbers that converges to x and suppose that $x_n \ge 0$. Then the sequence $(\sqrt{x_n})$ of positive square roots converges and $\lim(\sqrt{x_n}) = \sqrt{x}$.*

Proof. It follows from Theorem 3.2.4 that $x = \lim(x_n) \ge 0$ so the assertion makes sense. We now consider the two cases: (i) $x = 0$ and (ii) $x > 0$.

Case (i) If $x = 0$, let $\varepsilon > 0$ be given. Since $x_n \to 0$ there exists a natural number K such that if $n \ge K$ then

$$0 \le x_n = x_n - 0 < \varepsilon^2.$$

Therefore [see Example 2.1.13(a)], $0 \le \sqrt{x_n} < \varepsilon$ for $n \ge K$. Since $\varepsilon > 0$ is arbitrary, this implies that $\sqrt{x_n} \to 0$.

Case (ii) If $x > 0$, then $\sqrt{x} > 0$ and we note that

$$\sqrt{x_n} - \sqrt{x} = \frac{(\sqrt{x_n} - \sqrt{x})(\sqrt{x_n} + \sqrt{x})}{\sqrt{x_n} + \sqrt{x}} = \frac{x_n - x}{\sqrt{x_n} + \sqrt{x}}$$

Since $\sqrt{x_n} + \sqrt{x} \ge \sqrt{x} > 0$, it follows that

$$\left|\sqrt{x_n} - \sqrt{x}\right| \le \left(\frac{1}{\sqrt{x}}\right)|x_n - x|.$$

The convergence of $\sqrt{x_n} \to \sqrt{x}$ follows from the fact that $x_n \to x$. Q.E.D.

3.2.11 Theorem *Let (x_n) be a sequence of positive real numbers such that $L := \lim(x_{n+1}/x_n)$ exists. If $L < 1$, then (x_n) converges and $\lim(x_n) = 0$.*

Proof. By 3.2.4 it follows that $L \ge 0$. Let r be a number such that $L < r < 1$, and let $\varepsilon := r - L > 0$. There exists a number $K \in \mathbb{N}$ such that if $n \ge K$ then

$$\left|\frac{x_{n+1}}{x_n} - L\right| < \varepsilon.$$

It follows from this (why?) that if $n \ge K$, then

$$\frac{x_{n+1}}{x_n} < L + \varepsilon = L + (r - L) = r.$$

Therefore, if $n \ge K$, we obtain

$$0 < x_{n+1} < x_n r < x_{n-1} r^2 < \cdots < x_K r^{n-K+1}.$$

If we set $C := x_K / r^K$, we see that $0 < x_{n+1} < C r^{n+1}$ for all $n \ge K$. Since $0 < r < 1$, it follows from 3.1.11(b) that $\lim(r^n) = 0$ and therefore from Theorem 3.1.10 that $\lim(x_n) = 0$. Q.E.D.

As an illustration of the utility of the preceding theorem, consider the sequence (x_n) given by $x_n := n/2^n$. We have

$$\frac{x_{n+1}}{x_n} = \frac{n+1}{2^{n+1}} \cdot \frac{2^n}{n} = \frac{1}{2}\left(1 + \frac{1}{n}\right),$$

so that $\lim(x_{n+1}/x_n) = \frac{1}{2}$. Since $\frac{1}{2} < 1$, it follows from Theorem 3.2.11 that $\lim(n/2^n) = 0$.

(i) We can use Definition 3.1.3 or Theorem 3.1.5 directly. This is often (but not always) difficult to do.

(ii) We can dominate $|x_n - x|$ by a multiple of the terms in a sequence (a_n) known to converge to 0, and employ Theorem 3.1.10.

(iii) We can identify X as a sequence obtained from other sequences that are known to be convergent by taking tails, algebraic combinations, absolute values, or square roots, and employ Theorems 3.1.9, 3.2.3, 3.2.9, or 3.2.10.

(iv) We can "squeeze" X between two sequences that converge to the same limit and use Theorem 3.2.7.

(v) We can use the "ratio test" of Theorem 3.2.11.

All of these methods require us to know the value of limit in advance.

Section 3.3 Monotone Sequences

3.3.1 Definition Let $X = (x_n)$ be a sequence of real numbers. We say that X is **increasing** if it satisfies the inequalities

$$x_1 \le x_2 \le \cdots \le x_n \le x_{n+1} \le \cdots.$$

We say that X is **decreasing** if it satisfies the inequalities

$$x_1 \ge x_2 \ge \cdots \ge x_n \ge x_{n+1} \ge \cdots.$$

We say that X is **monotone** if it is either increasing or decreasing.

3.3.2 Monotone Convergence Theorem *A monotone sequence of real numbers is convergent if and only if it is bounded. Further:*

∀ Sequence — Convergent → Bounded

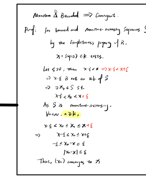
Monotone Sequence — Convergent ⇌ Bounded

(a) *If $X = (x_n)$ is a bounded increasing sequence, then*

$$\lim(x_n) = \sup\{x_n : n \in \mathbb{N}\}.$$

(b) *If $Y = (y_n)$ is a bounded decreasing sequence, then*

$$\lim(y_n) = \inf\{y_n : n \in \mathbb{N}\}.$$



All bounded & monotone Sequences have a Limit

3.3.3 Examples

(a) $\lim(1/\sqrt{n}) = 0.$

On the other hand, once we know that $X := (1/\sqrt{n})$ is bounded and decreasing, we know that it converges to some real number x. Since $X = (1/\sqrt{n})$ converges to x, it follows from Theorem 3.2.3 that $X \cdot X = (1/n)$ converges to x^2. Therefore $x^2 = 0$, whence $x = 0$.

存在一子序列发散，则母序列发散

(b) Let $h_n := 1 + 1/2 + 1/3 + \cdots + 1/n$ for $n \in \mathbb{N}$.

有通式的Seq. 尝试用Subseq.解决问题

$$\begin{aligned} h_{2^n} &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \cdots + \left(\frac{1}{2^{n-1}+1} + \cdots + \frac{1}{2^n}\right) \\ &> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \cdots + \left(\frac{1}{2^n} + \cdots + \frac{1}{2^n}\right) \\ &= 1 + \frac{1}{2} + \frac{1}{2} + \cdots + \frac{1}{2} \\ &= 1 + \frac{n}{2}. \end{aligned}$$

Since (h_n) is unbounded, Theorem 3.2.2 implies that it is divergent. (This proves that the infinite series known as the *harmonic series* diverges. See Example 3.7.6(b) in

Sequences that are defined inductively must be treated differently. If such a sequence is known to converge, then the value of the limit can sometimes be determined by using the inductive relation.

For example, suppose that convergence has been established for the sequence (x_n) defined by

$$x_1 = 2, \qquad x_{n+1} = 2 + \frac{1}{x_n}, \quad n \in \mathbb{N}.$$

从x_2开始单增

If we let $x = \lim(x_n)$, then we also have # $x = \lim(x_{n+1})$ since the 1-tail (x_{n+1}) converges to the same limit. Further, we see that $x_n \geq 2$, so that $x \neq 0$ and $x_n \neq 0$ for all $n \in \mathbb{N}$. Therefore, we may apply the limit theorems for sequences to obtain

$$x = \boxed{\lim(x_{n+1}) = 2 + \frac{1}{\lim(x_n)}} = 2 + \frac{1}{x}.$$

妙啊

Thus, the limit x is a solution of the quadratic equation $x^2 - 2x - 1 = 0$, and since x must be positive, we find that the limit of the sequence is $x = 1 + \sqrt{2}$.

Of course, the issue of convergence must not be ignored or casually assumed. For example, if we assumed the sequence (y_n) defined by $y_1 := 1$, $y_{n+1} := 2y_n + 1$ is convergent with limit y, then we would obtain $y = 2y + 1$, so that $y = -1$. Of course, this is absurd.

3.3.4

(b) Let $Z = (z_n)$ be the sequence of real numbers defined by $z_1 := 1$, $z_{n+1} := \sqrt{2z_n}$ for $n \in \mathbb{N}$. We will show that $\lim(z_n) = 2$.

Note that $z_1 = 1$ and $z_2 = \sqrt{2}$; hence $1 \leq z_1 < z_2 < 2$. We claim that the sequence Z is increasing and bounded above by 2. To show this we will show, by Induction, that $1 \leq z_n < z_{n+1} < 2$ for all $n \in \mathbb{N}$. This fact has been verified for $n = 1$. Suppose that it is true for $n = k$; then $2 \leq 2z_k < 2z_{k+1} < 4$, whence it follows (why?) that

$$1 < \sqrt{2} \leq z_{k+1} = \sqrt{2z_k} < z_{k+2} = \sqrt{2z_{k+1}} < \sqrt{4} = 2.$$

[In this last step we have used Example 2.1.13(a).] Hence the validity of the inequality $1 \leq z_k < z_{k+1} < 2$ implies the validity of $1 \leq z_{k+1} < z_{k+2} < 2$. Therefore $1 \leq z_n < z_{n+1} < 2$ for all $n \in \mathbb{N}$.

Since $Z = (z_n)$ is a bounded increasing sequence, it follows from the Monotone Convergence Theorem that it converges to a number $z := \sup\{z_n\}$. It may be shown directly that $\sup\{z_n\} = 2$, so that $z = 2$. Alternatively we may use the method employed in part (a). The relation $z_{n+1} = \sqrt{2z_n}$ gives a relation between the nth term of the 1-tail Z_1 of Z and the nth term of Z. By Theorem 3.1.9, we have $\lim Z_1 = z = \lim Z$. Moreover, by Theorems 3.2.3 and 3.2.10, it follows that the limit z must satisfy the relation

$$z = \sqrt{2z}.$$

Hence z must satisfy the equation $z^2 = 2z$, which has the roots $z = 0, 2$. Since the terms of $z = (z_n)$ all satisfy $1 \leq z_n \leq 2$, it follows from Theorem 3.2.6 that we must have $1 \leq z \leq 2$. Therefore $z = 2$. □

The Calculation of Square Roots

3.3.5 Example Let $a > 0$; we will construct a sequence (s_n) of real numbers that converges to $\sqrt{a}$.

Claim: Monotone & Bounded.

Let $s_1 > 0$ be arbitrary and define $s_{n+1} := \frac{1}{2}(s_n + a/s_n)$ for $n \in \mathbb{N}$.

We first show that $s_n^2 \geq a$ for $n \geq 2$. Since s_n satisfies the quadratic equation $s_n^2 - 2s_{n+1}s_n + a = 0$, this equation has a real root. Hence the discriminant $4s_{n+1}^2 - 4a$ must be nonnegative; that is, $s_{n+1}^2 \geq a$ for $n \geq 1$.

To see that (s_n) is ultimately decreasing, we note that for $n \geq 2$ we have

$$s_n - s_{n+1} = s_n - \frac{1}{2}\left(s_n + \frac{a}{s_n}\right) = \frac{1}{2} \cdot \frac{(s_n^2 - a)}{s_n} \geq 0.$$

Hence, $s_{n+1} \leq s_n$ for all $n \geq 2$. The Monotone Convergence Theorem implies that $s := \lim(s_n)$ exists. Moreover, from Theorem 3.2.3, the limit s must satisfy the relation #

$$s = \frac{1}{2}\left(s + \frac{a}{s}\right),$$

whence it follows (why?) that $s = a/s$ or $s^2 = a$. Thus $s = \sqrt{a}$.

For the purposes of calculation, it is often important to have an estimate of *how rapidly* the sequence (s_n) converges to $\sqrt{a}$. As above, we have $\sqrt{a} \leq s_n$ for all $n \geq 2$, whence it follows that $a/s_n \leq \sqrt{a} \leq s_n$. Thus we have

$$0 \leq s_n - \sqrt{a} \leq s_n - a/s_n = (s_n^2 - a)/s_n \quad \text{for} \quad n \geq 2.$$

Using this inequality we can calculate $\sqrt{a}$ to any desired degree of accuracy. □

Subsequences and the Bolzano-Weierstrass Theorem

Section 3.4

3.4.1 Definition Let $X = (x_n)$ be a sequence of real numbers and let $n_1 < n_2 < \cdots < n_k < \cdots$ be a strictly increasing sequence of natural numbers. Then the sequence $X' = (x_{n_k})$ given by

$$(x_{n_1}, x_{n_2}, \ldots, x_{n_k}, \ldots)$$

is called a **subsequence** of X.

(无限性)

Subsequence 也具有所有 Sequence 应拥有的合法性质

3.4.2 Theorem *If a sequence $X = (x_n)$ of real numbers converges to a real number x, then any subsequence $X' = (x_{n_k})$ of X also converges to x.*

Proof. Let $\varepsilon > 0$ be given and let $K(\varepsilon)$ be such that if $n \geq K(\varepsilon)$, then $|x_n - x| < \varepsilon$. Since $n_1 < n_2 < \cdots < n_k < \cdots$ is an increasing sequence of natural numbers, it is easily proved (by Induction) that $n_k \geq k$. Hence, if $k \geq K(\varepsilon)$, we also have $n_k \geq k \geq K(\varepsilon)$ so that $|x_{n_k} - x| < \varepsilon$. Therefore the subsequence (x_{n_k}) also converges to x. Q.E.D.

单向 ⇒

收敛的序列没有不收敛的子序列.

Find the Limit

3.4.3 Examples

(a) $\lim(b^n) = 0$ if $0 < b < 1$.

$0 < b < 1$, then $x_{n+1} = b^{n+1} < b^n = x_n$ so that the sequence (x_n) is decreasing. It is also clear that $0 \leq x_n \leq 1$, so it follows from the Monotone Convergence Theorem 3.3.2 that the sequence is convergent. Let $x := \lim x_n$. Since (x_{2n}) is a subsequence of (x_n) it follows from Theorem 3.4.2 that $x = \lim(x_{2n})$. Moreover, it follows from the relation $x_{2n} = b^{2n} = (b^n)^2 = x_n^2$ and Theorem 3.2.3 that

$$x = \lim(x_{2n}) = \left(\lim(x_n)\right)^2 = x^2.$$

Therefore we must have either $x = 0$ or $x = 1$. Since the sequence (x_n) is decreasing and bounded above by $b < 1$, we deduce that $x = 0$.

证收敛

bernoulli Inequality

ratio test

Mo-Con Theorem

求极限

(b) $\lim(c^{1/n}) = 1$ for $c > 1$.

This limit has been obtained in Example 3.1.11(c) for $c > 0$, using a rather ingenious argument. We give here an alternative approach for the case $c > 1$. Note that if $z_n := c^{1/n}$, then $z_n > 1$ and $z_{n+1} < z_n$ for all $n \in \mathbb{N}$. (Why?) Thus by the Monotone Convergence Theorem, the limit $z := \lim(z_n)$ exists. By Theorem 3.4.2, it follows that $z = \lim(z_{2n})$. In addition, it follows from the relation

$$z_{2n} = c^{1/2n} = (c^{1/n})^{1/2} = z_n^{1/2}$$

and Theorem 3.2.10 that

$$z = \lim(z_{2n}) = \left(\lim(z_n)\right)^{1/2} = z^{1/2}.$$

Therefore we have $z^2 = z$ whence it follows that either $z = 0$ or $z = 1$. Since $z_n > 1$ for all $n \in \mathbb{N}$, we deduce that $z = 1$.

We leave it as an exercise to the reader to consider the case $0 < c < 1$. □

Negation of Convergent

3.4.4 Theorem *Let $X = (x_n)$ be a sequence of real numbers, Then the following are equivalent:*

(i) *The sequence $X = (x_n)$ does not converge to $x \in \mathbb{R}$.*

(ii) *There exists an $\varepsilon_0 > 0$ such that for any $k \in \mathbb{N}$, there exists $n_k \in \mathbb{N}$ such that $n_k \geq k$ and $|x_{n_k} - x| \geq \varepsilon_0$.* $(x_n) \cap V_{\varepsilon_0}(x) = \emptyset$

(iii) *There exists an $\varepsilon_0 > 0$ and a subsequence $X' = (x_{n_k})$ of X such that $|x_{n_k} - x| \geq \varepsilon_0$ for all $k \in \mathbb{N}$.*

* 母序列发散 ⇅ 存在一子序列发散

Proof. (i) ⇒ (ii) If (x_n) does not converge to x, then for some $\varepsilon_0 > 0$ it is impossible to find a natural number k such that for all $n \geq k$ the terms x_n satisfy $|x_n - x| < \varepsilon_0$. That is, for each $k \in \mathbb{N}$ it is *not-true* that for *all* $n \geq k$ the inequality $|x_n - x| < \varepsilon_0$ holds. In other words, for each $k \in \mathbb{N}$ there exists a natural number $n_k \geq k$ such that $|x_{n_k} - x| \geq \varepsilon_0$.

$\boxed{\text{(ii)} \Rightarrow \text{(iii)}}$ Let ε_0 be as in (ii) and let $n_1 \in \mathbb{N}$ be such that $n_1 \geq 1$ and $|x_{n_1} - x| \geq \varepsilon_0$. Now let $n_2 \in \mathbb{N}$ be such that $n_2 > n_1$ and $|x_{n_2} - x| \geq \varepsilon_0$; let $n_3 \in \mathbb{N}$ be such that $n_3 > n_2$ and $|x_{n_3} - x| \geq \varepsilon_0$. Continue in this way to obtain a subsequence $X' = (x_{n_k})$ of X such that $|x_{n_k} - x| \geq \varepsilon_0$ for all $k \in \mathbb{N}$.

(iii) ⇒ (i) Suppose $X = (x_n)$ has a subsequence $X' = (x_{n_k})$ satisfying the condition in (iii). Then X cannot converge to x; for if it did, then, by Theorem 3.4.2, the subsequence X' would also converge to x. But this is impossible, since none of the terms of X' belongs to the ε_0-neighborhood of x. Q.E.D.

3.4.5 Divergence Criteria *If a sequence $X = (x_n)$ of real numbers has either of the following properties, then X is divergent.*

(i) *X has two convergent subsequences $X' = (x_{n_k})$ and $X'' = (x_{r_k})$ whose limits are not equal.*

(ii) *X is unbounded.*

无证明

3.4.6 Examples

(a) The sequence $X := ((-1)^n)$ is divergent.

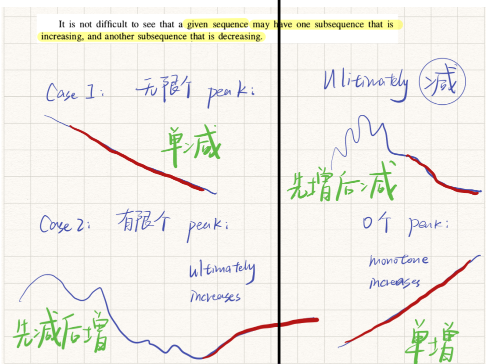
(b) The sequence $\left(1, \frac{1}{2}, 3, \frac{1}{4}, \ldots\right)$ is divergent.

(c) The sequence $S := (\sin n)$ is divergent.

“无条件” Mono tone

3.4.7 Monotone Subsequence Theorem *If $X = (x_n)$ is a sequence of real numbers, then there is a subsequence of X that is monotone.*

Proof. For the purpose of this proof, we will say that the mth term x_m is a "peak" if $x_m \ge x_n$ for all n such that $n \ge m$. (That is, x_m is never exceeded by any term that follows it



in the sequence.) Note that, in a decreasing sequence, every term is a peak, while in an increasing sequence, no term is a peak.

We will consider two cases, depending on whether X has infinitely many, or finitely many, peaks.

Case 1: X has infinitely many peaks. In this case, we list the peaks by increasing subscripts: $x_{m_1}, x_{m_2}, \ldots, x_{m_k}, \ldots$. Since each term is a peak, we have

$$x_{m_1} \ge x_{m_2} \ge \cdots \ge x_{m_k} \ge \cdots.$$

Therefore, the subsequence (x_{m_k}) of peaks is a decreasing subsequence of X.

Case 2: X has a finite number (possibly zero) of peaks. Let these peaks be listed by increasing subscripts: $x_{m_1}, x_{m_2}, \ldots, x_{m_r}$. Let $s_1 := m_r + 1$ be the first index beyond the last peak. Since x_{s_1} is not a peak, there exists $s_2 > s_1$ such that $x_{s_1} < x_{s_2}$. Since x_{s_2} is not a peak, there exists $s_3 > s_2$ such that $x_{s_2} < x_{s_3}$. Continuing in this way, we obtain an increasing subsequence (x_{s_k}) of X. Q.E.D.

Directly infered from Monotone Subsequence Thm.

3.4.8 The Bolzano-Weierstrass Theorem *A bounded sequence of real numbers has a convergent subsequence.*

First Proof. It follows from the Monotone Subsequence Theorem that if $X = (x_n)$ is a bounded sequence, then it has a subsequence $X' = (x_{n_k})$ that is monotone. Since this subsequence is also bounded, it follows from the Monotone Convergence Theorem 3.3.2 that the subsequence is convergent. Q.E.D.

Nested Interval Property

~~*Second Proof.* Since the set of values $\{x_n : n \in \mathbb{N}\}$ is bounded, this set is contained in an interval $I_1 := [a, b]$. We take $n_1 := 1$.~~

~~We now bisect I_1 into two equal subintervals I_1' and I_1'', and divide the set of indices $\{n \in \mathbb{N} : n > 1\}$ into two parts:~~

$$A_1 := \{n \in \mathbb{N} : n > n_1, x_n \in I_1'\}, \quad B_1 = \{n \in \mathbb{N} : n > n_1, x_n \in I_1''\}.$$

~~If A_1 is infinite, we take $I_2 := I_1'$ and let n_2 be the smallest natural number in A_1. If A_1 is a finite set, then B_1 must be infinite, and we take $I_2 := I_1''$ and let n_2 be the smallest natural number in B_1.~~

~~We now bisect I_2 into two equal subintervals I_2' and I_2'', and divide the set $\{n \in \mathbb{N} : n > n_2\}$ into two parts:~~

$$A_2 = \{n \in \mathbb{N} : n > n_2, x_n \in I_2'\}, \quad B_2 := \{n \in \mathbb{N} : n > n_2, x_n \in I_2''\}$$

~~If A_2 is infinite, we take $I_3 := I_2'$ and let n_3 be the smallest natural number in A_2. If A_2 is a finite set, then B_2 must be infinite, and we take $I_3 := I_2''$ and let n_3 be the smallest natural number in B_2.~~

~~We continue in this way to obtain a sequence of nested intervals $I_1 \supseteq I_2 \supseteq \cdots \supseteq I_k \supseteq \cdots$ and a subsequence (x_{n_k}) of X such that $x_{n_k} \in I_k$ for $k \in \mathbb{N}$. Since the length of I_k is equal to $(b-a)/2^{k-1}$, it follows from Theorem 2.5.3 that there is a (unique) common point $\xi \in I_k$ for all $k \in \mathbb{N}$. Moreover, since x_{n_k} and ξ both belong to I_k, we have~~

$$|x_{n_k} - \xi| \le (b-a)/2^{k-1},$$

~~whence it follows that the subsequence (x_{n_k}) of X converges to ξ. Q.E.D.~~

3.4.9 Theorem *Let $X = (x_n)$ be a bounded sequence of real numbers and let $x \in \mathbb{R}$ have the property that every convergent subsequence of X converges to x. Then the sequence X converges to x.*

Proof. Suppose $M > 0$ is a bound for the sequence X so that $|x_n| \le M$ for all $n \in \mathbb{N}$. If X does not converge to x, then Theorem 3.4.4 implies that there exist $\varepsilon_0 > 0$ and a subsequence $X' = (x_{n_k})$ of X such that

(1) $\boxed{|x_{n_k} - x| \ge \varepsilon_0 \quad \text{for all} \quad k \in \mathbb{N}.}$

不能说明矛盾 —— (x_{n_k}) divergent

可以 —— $\lim (x_{n_k}) \neq x$ but exists.

Since X' is a subsequence of X, the number M is also a bound for X'. Hence the Bolzano-Weierstrass Theorem implies that X' has a convergent subsequence X''. Since X'' is also a subsequence of X, it converges to x by hypothesis. Thus, its terms ultimately belong to the ε_0-neighborhood of x, contradicting (1). Q.E.D.

not X' itself

Remark:

每个 sequence 都可能同时拥有 convergent 和 divergent 的 subsequence.

故即使 Thm 中说 "every convergent subsequence of X convergent to x",
存在1个 divergent 的 X' 并不能构成矛盾。

且需致成的是 *every convergent subsequence of X converges to x.*
Divergent 也不符合条件.

$|x_n - x| \geq \varepsilon_0 \Rightarrow$ 构建全部落于 $V_{\varepsilon_0}(x)$ 的 (x_{n_k})

矛盾

mid-term 不考

Limit Superior and Limit Inferior

Section 3.5 The Cauchy Criterion

3.5.1 Definition A sequence $X = (x_n)$ of real numbers is said to be a **Cauchy sequence** if for every $\varepsilon > 0$ there exists a natural number $H(\varepsilon)$ such that for all natural numbers $n, m \geq H(\varepsilon)$, the terms x_n, x_m satisfy $|x_n - x_m| < \varepsilon$.

3.5.2 Examples

(a) The sequence $(1/n)$ is a Cauchy sequence.

(b) The sequence $(1 + (-1)^n)$ is *not* a Cauchy sequence.

Remark We emphasize that to prove a sequence (x_n) is a Cauchy sequence, we may not assume a relationship between m and n, since the required inequality $|x_n - x_m| < \varepsilon$ must hold for *all* $n, m \geq H(\varepsilon)$. But to prove a sequence is *not* a Cauchy sequence, we may specify a relation between n and m as long as arbitrarily large values of n and m can be chosen so that $|x_n - x_m| \geq \varepsilon_0$.

3.5.3 Lemma *If $X = (x_n)$ is a convergent sequence of real numbers, then X is a Cauchy sequence.*

Proof. If $x := \lim X$, then given $\varepsilon > 0$ there is a natural number $K(\varepsilon/2)$ such that if $n \geq K(\varepsilon/2)$ then $|x_n - x| < \varepsilon/2$. Thus, if $H(\varepsilon) := K(\varepsilon/2)$ and if $n, m \geq H(\varepsilon)$, then we have

$$\begin{aligned} |x_n - x_m| &= |(x_n - x) + (x - x_m)| \\ &\leq |x_n - x| + |x_m - x| < \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, it follows that (x_n) is a Cauchy sequence. Q.E.D.

将未知范围

缩小到前

3.5.4 Lemma *A Cauchy sequence of real numbers is bounded.*

finite n个

Proof. Let $X := (x_n)$ be a Cauchy sequence and let $\varepsilon := 1$. If $H := H(1)$ and $n \geq H$, then $|x_n - x_H| < 1$. Hence, by the Triangle Inequality, we have $|x_n| \leq |x_H| + 1$ for all $n \geq H$. If we set

$$M := \sup\{|x_1|, |x_2|, \ldots, |x_{H-1}|, |x_H| + 1\},$$

仍属有限数

then it follows that $|x_n| \leq M$ for all $n \in \mathbb{N}$. Q.E.D.

$|x_H|, |x_{H+1}| \cdots |x_n|$ 都小于等于 $|x_H|+$

3.5.5 Cauchy Convergence Criterion *A sequence of real numbers is convergent if and only if it is a Cauchy sequence.*

Proof. We have seen, in Lemma 3.5.3, that a convergent sequence is a Cauchy sequence.

Conversely, let $X = (x_n)$ be a Cauchy sequence; we will show that X is convergent to some real number. First we observe from Lemma 3.5.4 that the sequence X is bounded. Therefore, by the Bolzano-Weierstrass Theorem 3.4.8, there is a subsequence $X' = (x_{n_k})$ of X that converges to some real number x^*. We shall complete the proof by showing that X converges to x^*.

Proof:

Since $X = (x_n)$ is a Cauchy sequence, given $\varepsilon > 0$ there is ∃ a natural number $H(\varepsilon/2)$ such that if $n, m \geq H(\varepsilon/2)$ then

(1) $$|x_n - x_m| < \varepsilon/2.$$

Since the subsequence $X' = (x_{n_k})$ converges to x^*, there is a natural number $K \geq H(\varepsilon/2)$ belonging to the set $\{n_1, n_2, \ldots\}$ such that

$$|x_K - x^*| < \varepsilon/2.$$

Since $K \geq H(\varepsilon/2)$, it follows from (1) with $m = K$ that

$$|x_n - x_K| < \varepsilon/2 \quad \text{for} \quad n \geq H(\varepsilon/2).$$

Therefore, if $n \geq H(\varepsilon/2)$, we have

$$\begin{aligned} |x_n - x^*| &= |(x_n - x_K) + (x_K - x^*)| \\ &\leq |x_n - x_K| + |x_K - x^*| \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

Sincee $\varepsilon > 0$ is arbitrary, we infer that $\lim(x_n) = x^*$. Therefore the sequence X is convergent. Q.E.D.

取 Convergent 的 M

$M < H(\varepsilon)$

$\forall n \geq M$, $|x_n - x^*| < \varepsilon$

由 Sequence 的无限性，$\exists k$ s.t. $k \geq H(\varepsilon) > M$

$|x_m - x_k| < \varepsilon$

$M \geq H(\varepsilon)$

Choose $k = M$

$|x_m - x_k| < \varepsilon$

3.5.6 Examples **(a)** Let $X = (x_n)$ be defined by

$$x_1 := 1, \quad x_2 := 2, \quad \text{and} \quad x_n := \frac{1}{2}(x_{n-2} + x_{n-1}) \quad \text{for} \quad n > 2.$$

It can be shown by Induction that $1 \leq x_n \leq 2$ for all $n \in \mathbb{N}$. (Do so.) Some calculation shows that the sequence X is not monotone. However, since the terms are formed by averaging, it is readily seen that

$$|x_n - x_{n+1}| = \frac{1}{2^{n-1}} \quad \text{for} \quad n \in \mathbb{N}.$$

(Prove this by Induction.) Thus, if $m > n$, we may employ the Triangle Inequality to obtain

$$\begin{aligned} |x_n - x_m| &\leq |x_n - x_{n+1}| + |x_{n+1} - x_{n+2}| + \cdots + |x_{m-1} - x_m| \\ &= \frac{1}{2^{n-1}} + \frac{1}{2^n} + \cdots + \frac{1}{2^{m-2}} \\ &= \frac{1}{2^{n-1}}\left(1 + \frac{1}{2} + \cdots + \frac{1}{2^{m-n-1}}\right) < \frac{1}{2^{n-2}}. \end{aligned}$$

Therefore, given $\varepsilon > 0$, if n is chosen so large that $1/2^n < \varepsilon/4$ and if $m \geq n$, then it follows that $|x_n - x_m| < \varepsilon$. Therefore, X is a Cauchy sequence in $\mathbb{R}$. By the Cauchy Criterion 3.5.5 we infer that the sequence X converges to a number x.

To evaluate the limit x, we might first "pass to the limit" in the rule of definition $x_n = \frac{1}{2}(x_{n-1} + x_{n-2})$ to conclude that x must satisfy the relation $x = \frac{1}{2}(x + x)$, which is true, but not informative. Hence we must try something else.

\# Since X converges to x, so does the subsequence X' with odd indices. By Induction, the reader can establish that [see 1.2.4(f)]

$$\begin{aligned} x_{2n+1} &= 1 + \frac{1}{2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{2n-1}} \\ &= 1 + \frac{2}{3}\left(1 - \frac{1}{4^n}\right). \end{aligned}$$

$n=1$ 时 $2n+1=3$

It follows from this (how?) that $x = \lim X = \lim X' = 1 + \frac{2}{3} = \frac{5}{3}$.

递归证 Convergent: I. 先看 monotone-convergence Thm.

若不单调且有界 ⇒ II. 证 Contractive therefore cauchy

3.5.7 Definition We say that a sequence $X = (x_n)$ of real numbers is **contractive** if there exists a constant C, $0 < C < 1$, such that

$$|x_{n+2} - x_{n+1}| \le C|x_{n+1} - x_n|$$

for all $n \in \mathbb{N}$. The number C is called the **constant** of the contractive sequence.

3.5.8 Theorem *Every contractive sequence is a Cauchy sequence, and therefore is convergent.*

Proof. If we successively apply the defining condition for a contractive sequence, we can work our way back to the beginning of the sequence as follows:

$$\begin{aligned} |x_{n+2} - x_{n+1}| \le C|x_{n+1} - x_n| &\le C^2|x_n - x_{n-1}| \\ &\le C^3|x_{n-1} - x_{n-2}| \le \cdots \le C^n|x_2 - x_1|. \end{aligned}$$

For $m > n$, we estimate $|x_m - x_n|$ by first applying the Triangle Inequality and then using the formula for the sum of a geometric progression (see 1.2.4(f)). This gives

$$\begin{aligned} |x_m - x_n| &\le |x_m - x_{m-1}| + |x_{m-1} - x_{m-2}| + \cdots + |x_{n+1} - x_n| \\ &\le (C^{m-2} + C^{m-3} + \cdots + C^{n-1})|x_2 - x_1| \\ &= C^{n-1}\left(\frac{1 - C^{m-n}}{1 - C}\right)|x_2 - x_1| \\ &\le C^{n-1}\boxed{\left(\frac{1}{1 - C}\right)|x_2 - x_1|.} \end{aligned}$$

$\varepsilon_0 > 0$

Since $0 < C < 1$, we know $\lim(C^n) = 0$ [see 3.1.11(b)]. Therefore, we infer that (x_n) is a Cauchy sequence. It now follows from the Cauchy Convergence Criterion 3.5.5 that (x_n) is a convergent sequence. Q.E.D.

Proof: $|x_m - x_n| = |x_m - x_{m-1} + x_{m-1} - x_{m-2} + \cdots - x_n|$

$\le |x_m - x_{m-1}| + |x_{m-1} - x_{m-2}| + \cdots + |x_{n+1} - x_n|$

$\because |x_{n+2} - x_{n+1}| \le C|x_{n+1} - x_n|$

$\le C^2|x_n - x_{n-1}|$

$\le \cdots \le C^n|x_2 - x_1|$

so $|x_m - x_n| \le (C^{m-2} + C^{m-3} + \cdots + C^{n-1})|x_2 - x_1|$

$[m-2-(n-1)+1] = m-n$

$= \dfrac{C^{n-1}(1 - C^{m-n})}{1 - C}|x_2 - x_1|$

$(C<0)$ $\le \underbrace{C^{n-1}}_{\downarrow 0}\left(\frac{1}{1-C}|x_2 - x_1|\right)$

证其收敛，求其极限

3.5.9 Example We consider the sequence of Fibonacci fractions $x_n := f_n/f_{n+1}$ where $f_1 = f_2 = 1$ and $f_{n+1} = f_n + f_{n-1}$. (See Example 3.1.2(d).) The first few terms are $x_1 = 1$, $x_2 = 1/2$, $x_3 = 2/3$, $x_4 = 3/5$, $x_5 = 5/8$, and so on. It is shown that the sequence (x_n) is given inductively by the equation $x_{n+1} = 1/(1 + x_n)$ as follows:

$$x_{n+1} = \frac{f_{n+1}}{f_{n+2}} = \frac{f_{n+1}}{f_{n+1} + f_n} = \frac{1}{1 + \dfrac{f_n}{f_{n+1}}} = \frac{1}{1 + x_n}.$$

3.5.10 Corollary *If $X := (x_n)$ is a contractive sequence with constant C, $0 < C < 1$, and if $x^* := \lim X$, then*

(i) $|x^* - x_n| \le \dfrac{C^{n-1}}{1 - C}|x_2 - x_1|$,

(ii) $|x^* - x_n| \le \dfrac{C}{1 - C}|x_n - x_{n-1}|$.

Proof. From the preceding proof, if $m > n$, then $|x_m - x_n| \le \left(C^{n-1}/(1 - C)\right)|x_2 - x_1|$. If we let $m \to \infty$ in this inequality, we obtain (i).

Proof: $|x^* - x_n| = \lim\limits_{m\to\infty} |x_m - x_n|$

$= |x_m - x_{m-1} + x_{m-1} - \cdots + x_{n+1} - x_n|$

$\le |x_m - x_{m-1}| + |x_{m-1} - x_{m-2}| + \cdots + |x_{n+1} - x_n|$

$\le (\underbrace{C^{\boxed{m-2}} + C^{m-3} + \cdots + C^{n-1}}_{m-n}) |x_2 - x_1|$

$= C^{n-1} \cdot \dfrac{1 + C^{m-n}}{1 - C} |x_2 - x_1|$

$\le \dfrac{C^{n-1}}{1 - C} |x_2 - x_1|$

Proof: $|x^* - x_n| = \lim\limits_{m\to\infty} |x_m - x_n|$

$= |x_m - x_{m-1} + x_{m-1} - \cdots + x_{n+1} - x_n|$

$\le |x_m - x_{m-1}| + |x_{m-1} - x_{m-2}| + \cdots + |x_{n+1} - x_n|$

$\le \left[C^{\boxed{m-n}} + C^{m-n-1} + \cdots + C^2 + C\right] |x_n - x_{n-1}|$

$= C \cdot \dfrac{1 - C^n}{1 - C} |x_n - x_{n-1}|$

$\le \dfrac{C}{1 - C} |x_n - x_{n-1}|$

3.5.11 Example We are told that the cubic equation $x^3 - 7x + 2 = 0$ has a solution between 0 and 1 and we wish to approximate this solution. This can be accomplished by means of an iteration procedure as follows. We first rewrite the equation as $x = (x^3 + 2)/7$ and use this to define a sequence. We assign to x_1 an arbitrary value between 0 and 1, and then define

Construct: $x_{n+1} := \frac{1}{7}(x_n^3 + 2)$ for $n \in \mathbb{N}$.

By M.I.

Because $0 < x_1 < 1$, it follows that $0 < x_n < 1$ for all $n \in \mathbb{N}$. (Why?) Moreover, we have

$$\begin{aligned} \left|x_{n+2} - x_{n+1}\right| &= \left|\tfrac{1}{7}(x_{n+1}^3 + 2) - \tfrac{1}{7}(x_n^3 + 2)\right| = \tfrac{1}{7}\left|x_{n+1}^3 - x_n^3\right| \\ &= \tfrac{1}{7}\left|x_{n+1}^2 + x_{n+1}x_n + x_n^2\right|\left|x_{n+1} - x_n\right| \le \tfrac{3}{7}\left|x_{n+1} - x_n\right|. \end{aligned}$$

Therefore, (x_n) is a contractive sequence and hence there exists r such that $\lim(x_n) = r$. If we pass to the limit on both sides of the equality $x_{n+1} = (x_n^3 + 2)/7$, we obtain $r = (r^3 + 2)/7$ and hence $r^3 - 7r + 2 = 0$. Thus r is a solution of the equation.

3.6.1 Definition Let (x_n) be a sequence of real numbers.

(i) We say that (x_n) **tends to** $\pm\infty$, and write $\lim(x_n) = +\infty$, if for every $\alpha \in \mathbb{R}$ there exists a natural number $K(\alpha)$ such that if $n \ge K(\alpha)$, then $x_n > \alpha$.

(ii) We say that (x_n) **tends to** $-\infty$, and write $\lim(x_n) = -\infty$, if for every $\beta \in \mathbb{R}$ there exists a natural number $K(\beta)$ such that if $n \ge K(\beta)$, then $x_n < \beta$.

We say that (x_n) is **properly divergent** in case we have either $\lim(x_n) = +\infty$ or $\lim(x_n) = -\infty$.

证 unbounded 就取 $\forall a \in P\ (P^-)$, 证 $\exists n \in N$, $x_n > a\ (x_n < a)$

3.6.3 Theorem *A monotone sequence of real numbers is properly divergent if and only if it is unbounded.*

(a) *If (x_n) is an unbounded increasing sequence, then $\lim(x_n) = +\infty$.*

(b) *If (x_n) is an unbounded decreasing sequence, then $\lim(x_n) = -\infty$.*

Proof:

$\Leftarrow$ When (x_n) is monotone increasing,
if (x_n) is unbounded,
then $\forall a \in \mathbb{R}$, $\exists n_t \in \mathbb{N}$ $x_{n_t} > a$

As (x_n) is increasing
so $\forall n \geq n_t$, $x_n \geq x_{n_t} > a$
$\Rightarrow$ $\lim(x_n) \to +\infty$
As a is an arbitrary number in $\mathbb{R}$
so (x_n) is divergent.

$\Rightarrow$ If (x_n) is a divergent monotone increasing sequence, then unbounded

3.6.4 Theorem *Let (x_n) and (y_n) be two sequences of real numbers and suppose that*

(1) $$x_n \leq y_n \quad \textit{for all} \quad n \in \mathbb{N}.$$

(a) *If $\lim(x_n) = +\infty$, then $\lim(y_n) = +\infty$.*
(b) *If $\lim(y_n) = -\infty$, then $\lim(x_n) = -\infty$.*

Proof. (a) If $\lim(x_n) = +\infty$, and if $\alpha \in \mathbb{R}$ is given, then there exists a natural number $K(\alpha)$ such that if $n \geq K(\alpha)$, then $\alpha < x_n$. In view of (1), it follows that $\alpha < y_n$ for all $n \geq K(\alpha)$. Since α is arbitrary, it follows that $\lim(y_n) = +\infty$.

The proof of (b) is similar. Q.E.D.

3.6.5 Theorem *Let (x_n) and (y_n) be two sequences of positive real numbers and suppose that for some $L \in \mathbb{R}, L > 0$, we have*

(2) $$\lim(x_n/y_n) = L.$$

Then $\lim(x_n) = +\infty$ if and only if $\lim(y_n) = +\infty$.

Proof. If (2) holds, there exists $K \in \mathbb{N}$ such that

$$\tfrac{2}{3} \quad \cancel{\tfrac{1}{2}}L < x_n/y_n < \cancel{\tfrac{3}{2}}L \quad \tfrac{4}{3} \quad \text{for all} \quad n \geq K.$$

Hence we have $(\frac{1}{2}L)y_n < x_n < (\frac{3}{2}L)y_n$ for all $n \geq K$. The conclusion now follows from a slight modification of Theorem 3.6.4. We leave the details to the reader. Q.E.D.