CHAPTER 9

INFINITE SERIES

Convergent $\Rightarrow$ bounded

Positive terms ($S_n := \sum |a_n|$)

欲证 Convergent 先想 Bounded!
(3.7.5)

Section 3.7 Introduction to Infinite Series

3.7.1 Definition If $X := (x_n)$ is a sequence in $\mathbb{R}$, then the **infinite series** (or simply the **series**) **generated by** X is the sequence $S := (s_k)$ defined by

$$\begin{aligned} s_1 &:= x_1 \\ s_2 &:= s_1 + x_2 \quad (= x_1 + x_2) \\ &\cdots \\ s_k &:= s_{k-1} + x_k \quad (= x_1 + x_2 + \cdots + x_k) \\ &\cdots \end{aligned}$$

The numbers x_n are called the **terms** of the series and the numbers s_k are called the **partial sums** of this series. If $\lim S$ exists, we say that this series is **convergent** and call this limit the **sum** or the **value** of this series. If this limit does not exist, we say that the series S is **divergent.**

3.7.2 Examples **(a)** Consider the sequence $X := (r^n)_{n=0}^{\infty}$ where $r \in \mathbb{R}$, which generates the **geometric series**:

$$\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + \cdots + r^n + \cdots. \tag{3}$$

$\Rightarrow S_n = \frac{1\cdot(1-r^{n+1})}{1-r} = \frac{1-r^{n+1}}{1-r}$

We will show that if $|r| < 1$, then this series converges to $1/(1-r)$. (See also Example 1.2.4(f).) Indeed, if $s_n := 1 + r + r^2 + \cdots + r^n$ for $n \geq 0$, and if we multiply s_n by r and subtract the result from s_n, we obtain (after some simplification):

$$s_n(1-r) = 1 - r^{n+1}.$$

Therefore, we have

$$s_n - \frac{1}{1-r} = \frac{r^{n+1}}{1-r},$$

from which it follows that

$$\left| s_n - \frac{1}{1-r} \right| \leq \frac{|r|^{n+1}}{|1-r|}.$$

Since $|r|^{n+1} \to 0$ when $|r| < 1$, it follows that the geometric series converges to $1/(1-r)$ when $|r| < 1$.

(b) Consider the series generated by $((-1)^n)_{n=0}^{\infty}$; that is, the series:

$$\sum_{n=0}^{\infty} (-1)^n = (+1) + (-1) + (+1) + (-1) + \cdots. \tag{4}$$

It is easily seen (by Mathematical Induction) that $s_n = 1$ if $n \geq 0$ is even and $s_n = 0$ if n is odd; therefore, the sequence of partial sums is $(1, 0, 1, 0, \ldots)$. Since this sequence is not convergent, the series (4) is divergent.

(c) Consider the series

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots. \tag{5}$$

By a stroke of insight, we note that

$$\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}.$$

Hence, on adding these terms from $k = 1$ to $k = n$ and noting the telescoping that takes place, we obtain

$$s_n = \frac{1}{1} - \frac{1}{n+1},$$

whence it follows that $s_n \to 1$. Therefore the series (5) converges to 1. □

3.7.3 The *n*th Term Test *If the series $\sum x_n$ converges, then* $\lim(x_n) = 0$.

Proof. By Definition 3.7.1, the convergence of $\sum x_n$ requires that $\lim(s_k)$ exists. Since $x_n = s_n - s_{n-1}$, then $\lim(x_n) = \lim(s_n) - \lim(s_{n-1}) = 0$. Q.E.D.

3.7.4 Cauchy Criterion for Series *The series $\sum x_n$ converges if and only if for every $\varepsilon > 0$ there exists $M(\varepsilon) \in \mathbb{N}$ such that if $m > n \geq M(\varepsilon)$, then*

$$|s_m - s_n| = |x_{n+1} + x_{n+2} + \cdots + x_m| < \varepsilon. \tag{6}$$

The next result, although limited in scope, is of great importance and utility.

3.7.5 Theorem *Let (x_n) be a sequence of nonnegative real numbers. Then the series $\sum x_n$ converges if and only if the sequence $S = (s_k)$ of partial sums is bounded. In this case,*

$$\boxed{\sum_{n=1}^{\infty} x_n = \lim(s_k)} = \sup\{s_k : k \in \mathbb{N}\}.$$

Proof. Since $x_n \geq 0$, the sequence S of partial sums is monotone increasing:

$$s_1 \leq s_2 \leq \cdots \leq s_k \leq \cdots.$$

By the Monotone Convergence Theorem 3.3.2, the sequence $S = (s_k)$ converges if and only if it is bounded, in which case its limit equals $\sup\{s_k\}$. Q.E.D.

3.7.6 Examples **(a)** The geometric series (3) diverges if $|r| \geq 1$.

This follows from the fact that the terms r^n do not approach 0 when $|r| \geq 1$.

(b) The **harmonic series** $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Since the terms $1/n \to 0$, we cannot use the nth Term Test 3.7.3 to establish this divergence. However, it was seen in Examples 3.3.3(b) and 3.5.6(c) that the sequence (s_n) of partial sums is not bounded. Therefore, it follows from Theorem 3.7.5 that the harmonic series is divergent. This series is famous for the very slow growth of its partial sums (see the —法一

discussion in Example 3.3.3(b)) and also for the variety of proofs of its divergence. Here is a proof by contradiction. If we assume the series converges to the number S, then we have —法2:

$$\begin{aligned} S &= \left(1+\frac{1}{2}\right)+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}\right)+\cdots+\left(\frac{1}{2n-1}+\frac{1}{2n}\right)+\cdots \\ &> \left(\frac{1}{2}+\frac{1}{2}\right)+\left(\frac{1}{4}+\frac{1}{4}\right)+\left(\frac{1}{6}+\frac{1}{6}\right)+\cdots+\left(\frac{1}{2n}+\frac{1}{2n}\right)+\cdots \\ &= 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}+\cdots \\ &= S. \end{aligned}$$

The contradiction $S > S$ shows the assumption of convergence must be false and the harmonic series must diverge.

(c) The 2-series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent.

Since the partial sums are monotone, it suffices (why?) to show that some subsequence of (s_k) is bounded. If $k_1 := 2^1 - 1 = 1$, then $s_{k_1} = 1$. If $k_2 := 2^2 - 1 = 3$, then

$$s_{k_2} = \frac{1}{1} + \left(\frac{1}{2^2}+\frac{1}{3^2}\right) < 1 + \frac{2}{2^2} = 1 + \frac{1}{2},$$

and if $k_3 := 2^3 - 1 = 7$, then we have

$$s_{k_3} = s_{k_2} + \left(\frac{1}{4^2}+\frac{1}{5^2}+\frac{1}{6^2}+\frac{1}{7^2}\right) < s_{k_2} + \frac{4}{4^2} < 1 + \frac{1}{2} + \frac{1}{2^2}.$$

By Mathematical Induction, we find that if $k_j := 2^j - 1$, then

$$0 < s_{k_j} < 1 + \tfrac{1}{2} + \left(\tfrac{1}{2}\right)^2 + \cdots + \left(\tfrac{1}{2}\right)^{j-1}.$$

Since the term on the right is a partial sum of a geometric series with $r = \frac{1}{2}$, it is dominated by $1/\left(1-\frac{1}{2}\right) = 2$, and Theorem 3.7.5 implies that the 2-series converges.

(d) The p-**series** $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges when $p > 1$. $= m>n\in\mathbb{N}$ then $m^p > n^p \Rightarrow \frac{1}{m^p} < \frac{1}{n^p}$

Since the argument is very similar to the special case considered in part (c), we will leave some of the details to the reader. As before, if $k_1 := 2^1 - 1 = 1$, then $s_{k_1} = 1$. If $k_2 := 2^2 - 1 = 3$, then since $2^p < 3^p$, we have

$$s_{k_2} = \frac{1}{1^p} + \left(\frac{1}{2^p}+\frac{1}{3^p}\right) < 1 + \frac{2}{2^p} = 1 + \frac{1}{2^{p-1}}.$$

Further, if $k_3 := 2^3 - 1$, then (how?) it is seen that

$$s_{k_3} < s_{k_2} + \frac{4}{4^p} < 1 + \frac{1}{2^{p-1}} + \frac{1}{4^{p-1}}.$$

Finally, we let $\boxed{r := 1/2^{p-1};}$ since $p > 1$, we have $0 < r < 1$. Using Mathematical Induction, we show that if $k_j := 2^j - 1$, then

$$0 < s_{k_j} < 1 + r + r^2 + \cdots + r^{j-1} < \frac{1}{1-r}.$$

Therefore, Theorem 3.7.5 implies that the p-series converges when $p > 1$.

(e) The p-**series** $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges when $0 < p \le 1$. $\Rightarrow n^p \le n \Rightarrow \frac{1}{n^p} \geq \frac{1}{n}$

We will use the elementary inequality $n^p \le n$ when $n \in \mathbb{N}$ and $0 < p \le 1$. It follows that

$$\frac{1}{n} \le \frac{1}{n^p} \quad \text{for} \quad n \in \mathbb{N}.$$

Since the partial sums of the harmonic series are not bounded, this inequality shows that the partial sums of the p-series are not bounded when $0 < p \le 1$. Hence the p-series diverges for these values of p.

充分理解:

对于 Series, 本质是 partial sum, 故 partial sum 的 partial sum (e.g. S_{2n}, S_{n+1} 之于 S_n) 为 (S_n) 的 "subseries"

(f) The **alternating harmonic series**, given by

$$(7) \qquad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \cdots + \frac{(-1)^{n+1}}{n} + \cdots$$

is convergent.

The reader should compare this series with the harmonic series in (b), which is divergent. Thus, the subtraction of some of the terms in (7) is essential if this series is to converge. Since we have

$$s_{2n} = \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{2n-1} - \frac{1}{2n}\right),$$

⇒偶次和 $S_{2n} \geq 0$

S_n的任意"subseries"收敛于同值.

⇒ $0 \leq S_{2n} \leq S_{2n+1} \leq 1$

it is clear that the "even" subsequence (s_{2n}) is increasing. Similarly, the "odd" subsequence (s_{2n+1}) is decreasing since

$$s_{2n+1} = \frac{1}{1} - \left(\frac{1}{2} - \frac{1}{3}\right) - \left(\frac{1}{4} - \frac{1}{5}\right) - \cdots - \left(\frac{1}{2n} - \frac{1}{2n+1}\right).$$

⇒奇次和 $S_{2n+1} \leq 1$

Since $0 < s_{2n} < s_{2n} + 1/(2n+1) = s_{2n+1} \leq 1$, both of these subsequences are bounded below by 0 and above by 1. Therefore they are both convergent and to the same value. Thus the sequence (s_n) of partial sums converges, proving that the alternating harmonic series (7) converges. (It is far from obvious that the limit of this series is equal to $\ln 2$.) □

3.7.7 Comparison Test *Let $X := (x_n)$ and $Y := (y_n)$ be real sequences and suppose that for some $K \in \mathbb{N}$ we have*

$$(8) \qquad 0 \leq x_n \leq y_n \quad \text{for} \quad n \geq K.$$

(a) *Then the convergence of $\sum y_n$ implies the convergence of $\sum x_n$.*

(b) *The divergence of $\sum x_n$ implies the divergence of $\sum y_n$.*

Proof. (a) Suppose that $\sum y_n$ converges and, given $\varepsilon > 0$, let $M(\varepsilon) \in \mathbb{N}$ be such that if $m > n \geq M(\varepsilon)$, then

$$y_{n+1} + \cdots + y_m < \varepsilon.$$

If $m > \sup\{K, M(\varepsilon)\}$, then it follows that

$$0 \leq x_{n+1} + \cdots + x_m \leq y_{n+1} + \cdots + y_m < \varepsilon,$$

from which the convergence of $\sum x_n$ follows.

(b) This statement is the contrapositive of (a). Q.E.D.

3.7.8 Limit Comparison Test *Suppose that $X := (x_n)$ and $Y := (y_n)$ are strictly positive sequences and suppose that the following limit exists in $\mathbb{R}$:*

$$r := \lim\left(\frac{x_n}{y_n}\right). \tag{9}$$

(a) *If $r \neq 0$ then $\sum x_n$ is convergent if and only if $\sum y_n$ is convergent.*
(b) *If $r = 0$ and if $\sum y_n$ is convergent, then $\sum x_n$ is convergent.*

Proof. (a) It follows from (9) and Exercise 3.1.18 that there exists $K \in \mathbb{N}$ such that $\frac{1}{2}r \leq x_n/y_n \leq 2r$ for $n \geq K$, whence

$$\left(\tfrac{1}{2}r\right)y_n \leq x_n \leq (2r)y_n \quad \text{for} \quad n \geq K.$$

If we apply the Comparison Test 3.7.7 twice, we obtain the assertion in (a).
(b) If $r = 0$, then there exists $K \in \mathbb{N}$ such that

$$0 < x_n \leq y_n \quad \text{for} \quad n \geq K,$$

so that Theorem 3.7.7(a) applies. Q.E.D.

$r \neq 0$，敛散性相同
$r = 0$，$\sum y_n$ 的收敛性保证 $\sum x_n$ 的收敛性，

Remark The Comparison Tests 3.7.7 and 3.7.8 depend on having a stock of series that one knows to be convergent (or divergent). The reader will find that the p-series is often useful for this purpose.

3.7.9 Examples **(a)** The series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2+n}$ converges.

It is clear that the inequality

$$0 < \frac{1}{n^2+n} < \frac{1}{n^2} \quad \text{for} \quad n \in \mathbb{N}$$

is valid. Since the series $\sum 1/n^2$ is convergent (by Example 3.7.6(c)), we can apply the Comparison Test 3.7.7 to obtain the convergence of the given series.

(b) The series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2-n+1}$ is convergent.

If the inequality

$$\frac{1}{n^2-n+1} \leq \frac{1}{n^2} \tag{10}$$

False

were true, we could argue as in (a). However, (10) is *false* for all $n \in \mathbb{N}$. The reader can probably show that the inequality

$$0 < \frac{1}{n^2 - n + 1} \le \frac{2}{n^2}$$

式1:

is valid for all $n \in \mathbb{N}$, and this inequality will work just as well. However, it might take some experimentation to think of such an inequality and then establish it.

Instead, if we take $x_n := 1/(n^2 - n + 1)$ and $y_n := 1/n^2$, then we have

$$\frac{x_n}{y_n} = \frac{n^2}{n^2 - n + 1} = \frac{1}{1 - (1/n) + (1/n^2)} \to 1.$$

式2:

Therefore, the convergence of the given series follows from the Limit Comparison Test 3.7.8(a).

(c) The series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}}$ is divergent.

This series closely resembles the series $\sum 1/\sqrt{n}$, which is a p-series with $p = \frac{1}{2}$; by Example 3.7.6(e), it is divergent. If we let $x_n := 1/\sqrt{n+1}$ and $y_n := 1/\sqrt{n}$, then we have

$$\frac{x_n}{y_n} = \frac{\sqrt{n}}{\sqrt{n+1}} = \frac{1}{\sqrt{1 + 1/n}} \to 1.$$

Therefore the Limit Comparison Test 3.7.8(a) applies.

(d) The series $\sum_{n=1}^{\infty} \frac{1}{n!}$ is convergent.

It would be possible to establish this convergence by showing (by Induction) that $n^2 < n!$ for $n \ge 4$, whence it follows that

$$0 < \frac{1}{n!} < \frac{1}{n^2} \quad \text{for} \quad n \ge 4.$$

式1

Alternatively, if we let $x := 1/n!$ and $y_n := 1/n^2$, then (when $n \ge 4$) we have

$$0 \le \frac{x_n}{y_n} = \frac{n^2}{n!} = \frac{n}{1 \cdot 2 \cdots (n-1)} < \frac{1}{n-2} \to 0.$$

式2

Therefore the Limit Comparison Test 3.7.8(b) applies. (Note that this test was a bit troublesome to apply since we do not presently know the convergence of any series for which the limit of x_n/y_n is really easy to determine.) □

Section 9.1 Absolute Convergence

9.1.1 Definition Let $X := (x_n)$ be a sequence in $\mathbb{R}$. We say that the series $\sum x_n$ is **absolutely convergent** if the series $\sum |x_n|$ is convergent in $\mathbb{R}$. A series is said to be **conditionally** (or **nonabsolutely**) **convergent** if it is convergent, but it is not absolutely convergent.

9.1.2 Theorem *If a series in $\mathbb{R}$ is absolutely convergent, then it is convergent.*

Proof. Since $\sum |x_n|$ is convergent, the Cauchy Criterion 3.7.4 implies that, given $\varepsilon > 0$ there exists $M(\varepsilon) \in \mathbb{N}$ such that if $m > n \geq M(\varepsilon)$, then

$$|x_{n+1}| + |x_{n+1}| + \cdots + |x_m| < \varepsilon.$$

However, by the Triangle Inequality, the left side of this expression dominates:

$$|s_m - s_n| = |x_{n+1} + x_{n+2} + \cdots + x_m|.$$

Since $\varepsilon > 0$ is arbitrary, Cauchy's Criterion implies that $\sum x_n$ converges. Q.E.D.

9.1.3 Theorem *If a series $\sum x_n$ is convergent, then any series obtained from it by grouping the terms is also convergent and to the same value.*

Proof. Suppose that we have

$$y_1 := x_1 + \cdots + x_{k_1}, \qquad y_2 := x_{k_1+1} + \cdots + x_{k_2},$$

If s_n denotes the nth partial sum of $\sum x_n$ and t_k denotes the kth partial sum of $\sum y_k$, then we have

$$t_1 = y_1 = s_{k_1}, \qquad t_2 = y_1 + y_2 = s_{k_2},$$

Thus, the sequence (t_k) of partial sums of the grouped series $\sum y_k$ is a subsequence of the sequence (s_n) of partial sums of $\sum x_n$. Since this latter series was assumed to be convergent, so is the grouped series $\sum y_k$. Q.E.D.

9.1.4 Definition A series $\sum y_k$ in $\mathbb{R}$ is a **rearrangement** of a series $\sum x_n$ if there is a bijection f of $\mathbb{N}$ onto $\mathbb{N}$ such that $y_k = x_{f(k)}$ for all $k \in \mathbb{N}$.

9.1.5 Rearrangement Theorem *Let $\sum x_n$ be an absolutely convergent series in $\mathbb{R}$. Then any rearrangement $\sum y_k$ of $\sum x_n$ converges to the same value.*

Proof: Suppose that $\sum x_n$ converges to $x \in \mathbb{R}$

i.e. $\sum x_n \to x$.

$\Rightarrow \forall \varepsilon > 0\ \exists H \in \mathbb{N}$ s.t. $\forall n, q \geq H$, $q > n$

$|x - \sum x_n| = |x - S_n| < \varepsilon$

As $\sum x_n$ absolutely converges so

$$\sum_{i=n}^{q} |x_i| < \varepsilon$$

Now construct the rearrangement of $\sum x_k$: $\sum y_k$

$\forall m \in \mathbb{N}$, $M > H$,

let $y_1 := x_1$, $y_2 := x_2$ ..... $y_H := x_H$

$t_m := y_1 + y_2 + \cdots + y_m$

So all terms $x_1 \dots x_H$ are contained as summands in $t_m := y_1 + \cdots + y_m$

Then $(t_m - S_H)$ should be the sum of a finite terms x_k with index $k > H$

Hence, $\exists q > H$, $q \in \mathbb{N}$ s.t.

$$|t_m - S_H| < \sum_{k=H+1}^{q} |x_k| < \varepsilon$$

Thus, $|t_m - x| \leq |t_m - S_H| + |S_H - x| < 2\varepsilon$

$\sum y_n = t_n \to x$

The rearrangment is still convergent and converges to the same value.

Q.E.D.

Proof. Suppose that $\sum x_n$ converges to $x \in \mathbb{R}$. Thus, if $\varepsilon > 0$, let N be such that if $n, q > N$ and $s_n := x_1 + \cdots + x_n$, then

$$|x - s_n| < \varepsilon \quad \text{and} \quad \sum_{k=N+1}^{q} |x_k| < \varepsilon .$$

Let $M \in \mathbb{N}$ be such that all of the terms $x_1, \ldots, x_N$ are contained as summands in $t_M := y_1 + \cdots + y_M$. It follows that if $m \geq M$, then $t_m - s_n$ is the sum of a finite number of terms x_k with index $k > N$. Hence, for some $q > N$, we have

$$|t_m - s_n| \leq \sum_{k=N+1}^{q} |x_k| < \varepsilon.$$

Therefore, if $m \geq M$, then we have

$$|t_m - x| \leq |t_m - s_n| + |s_n - x| < \varepsilon + \varepsilon = 2\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we conclude that $\sum y_k$ converges to x. Q.E.D.

Section 9.2 Tests for Absolute Convergence

9.2.1 Limit Comparison Test, II *Suppose that $X := (x_n)$ and $Y := (y_n)$ are nonzero real sequences and suppose that the following limit exists in $\mathbb{R}$:*

$$(1) \qquad r := \lim \left| \frac{x_n}{y_n} \right| .$$

(a) *If $r \neq 0$, then $\sum |x_n|$ is ~~absolutely~~ convergent if and only if $\sum |y_n|$ is ~~absolutely~~ convergent.*

(b) *If $r = 0$ and if $\sum |y_n|$ is ~~absolutely~~ convergent, then $\sum |x_n|$ is ~~absolutely~~ convergent.*

本质上与 L.C Test I 的一样,

Proof. This result follows immediately from Theorem 3.7.8. Q.E.D.

3.7.8 Limit Comparison Test *Suppose that $X := (x_n)$ and $Y := (y_n)$ are strictly positive sequences and suppose that the following limit exists in $\mathbb{R}$:*

$$(9) \qquad r := \lim \left(\frac{x_n}{y_n} \right).$$

(a) *If $r \neq 0$ then $\sum x_n$ is convergent if and only if $\sum y_n$ is convergent.*

(b) *If $r = 0$ and if $\sum y_n$ is convergent, then $\sum x_n$ is convergent.*

9.2.2 Root Test *Let $X := (x_n)$ be a sequence in $\mathbb{R}$.*

(a) *If there exist $r \in \mathbb{R}$ with $r < 1$ and $K \in \mathbb{N}$ such that*

$$(2) \qquad |x_n|^{1/n} \leq r \qquad for \quad n \geq K,$$

then the series $\sum x_n$ is absolutely convergent.

(b) *If there exists $K \in \mathbb{N}$ such that*

$$(3) \qquad |x_n|^{1/n} \geq 1 \qquad for \quad n \geq K,$$

then the series $\sum x_n$ is divergent. □

Root Test.

$\lim |x_n|^{\frac{1}{n}}$

注意 n 表示 Series 中第几项

Proof. (a) If (2) holds, then we have $|x_n| \le r^n$ for $n \ge K$. Since the geometric series $\sum r^n$ is convergent for $0 \le r < 1$, the Comparison Test 3.7.7 implies that $\sum |x_n|$ is convergent.

(b) If (3) holds, then $|x_n| \ge 1$ for $n \ge K$, so the terms do not approach 0 and the nth Term Test 3.7.3 applies. Q.E.D.

Root & Ratio Test: 借助几何级数 r^n $(0<r<1)$.

9.2.3 Corollary *Let $X := (x_n)$ be a sequence in $\mathbb{R}$ and suppose that the limit*

$$r := \lim |x_n|^{1/n} \tag{4}$$

exists in $\mathbb{R}$. Then $\sum x_n$ is absolutely convergent when $r < 1$ and is divergent when $r > 1$.

Proof. If the limit in (4) exists and $r < 1$, then there exist r_1 with $r < r_1 < 1$ and $K \in \mathbb{N}$ such that $|x_n|^{1/n} < r_1$ for $n > K$. In this case we can apply 9.2.2(a).

If $r > 1$, then there exists $K \in \mathbb{N}$ such that $|x_n|^{1/n} > 1$ for $n \ge K$ and the nth Term Test applies. Q.E.D.

Note No conclusion is possible in Corollary 9.2.3 when $r = 1$, for either convergence or divergence is possible. See Example 9.2.7(b).

9.2.4 Ratio Test *Let $X := (x_n)$ be a sequence of nonzero real numbers.*

(a) *If there exist $r \in \mathbb{R}$ with $0 < r < 1$ and $K \in \mathbb{N}$ such that*

$$\left|\frac{x_{n+1}}{x_n}\right| \le r \quad \text{for} \quad n \ge K, \tag{5}$$

then the series $\sum x_n$ is absolutely convergent.

(b) *If there exists $K \in \mathbb{N}$ such that*

$$\left|\frac{x_{n+1}}{x_n}\right| \boxed{\ge} 1 \quad \text{for} \quad n \ge K, \tag{6}$$

then the series $\sum x_n$ is divergent. >

Limit Comparison Test vs Ratio Test.

$r = \lim \left|\frac{x_n}{y_n}\right|$, y_n敛散性已知. $r = \lim \left|\frac{x_{n+1}}{x_n}\right|$

Proof. (a) If (5) holds, an Induction argument shows that $|x_{K+m}| \le |x_K| r^m$ for $m \in \mathbb{N}$. Thus, for $n \ge K$ the terms in $\sum |x_n|$ are dominated by a fixed multiple of the terms in the geometric series $\sum r^m$ with $0 < r < 1$. The Comparison Test 3.7.7 then implies that $\sum |x_n|$ is convergent.

(b) If (6) holds, an Induction argument shows that $|x_{K+m}| \ge |x_K|$ for $m \in \mathbb{N}$ and the nth Term Test applies. Q.E.D.

9.2.5 Corollary *Let $X := (x_n)$ be a nonzero sequence in $\mathbb{R}$ and suppose that the limit*

$$r := \lim \left| \frac{x_{n+1}}{x_n} \right| \tag{7}$$

exists in $\mathbb{R}$. Then $\sum x_n$ is absolutely convergent when $r < 1$ and is divergent when $r > 1$.

Proof. If $r < 1$ and if $r < r_1 < 1$, then there exists $K \in \mathbb{R}$ such that $|x_{n+1}/x_n| < r_1$ for $n \geq K$. Thus Theorem 9.2.4(a) applies to give the absolute convergence of $\sum x_n$.

If $r > 1$, then there exists $K \in \mathbb{N}$ such that $|x_{n+1}/x_n| > 1$ for $n \geq K$, whence it follows that $|x_k|$ does not converge to 0 and the nth Term Test applies. Q.E.D.

Note No conclusion is possible in Corollary 9.2.5 when $r = 1$, for either convergence or divergence is possible. See Example 9.2.7(c).

(隐含 absolute convergent)

9.2.6 Integral Test *Let f be a positive, decreasing function on $\{t : t \geq 1\}$. Then the series $\sum_{k=1}^{\infty} f(k)$ converges if and only if the improper integral*

denote it by $\int_1^\infty f(t)dt$

$$\int_1^\infty f(t)dt = \lim_{b\to\infty} \int_1^b f(t)dt$$

exists. In the case of convergence, the partial sum $s_n = \sum_{k=1}^{n} f(k)$ and the sum $s = \sum_{k=1}^{\infty} f(k)$ satisfy the estimate

$$\int_{n+1}^\infty f(t)dt \leq s - s_n \leq \int_n^\infty f(t)dt. \tag{8}$$

Proof. Since f is positive and decreasing on the interval $[k-1, k]$, we have

$$f(k) \leq \int_{k-1}^k f(t)dt \leq f(k-1). \tag{9}$$

By adding this inequality for $k = 2, 3, \ldots, n$, we obtain

$$s_n - f(1) \leq \int_1^n f(t)dt \leq s_{n-1},$$

which shows that either both or neither of the limits

$$\lim_{n\to\infty} s_n \quad \text{and} \quad \lim_{n\to\infty} \int_1^n f(t)dt$$

exist. If they exist, then on adding (9) for $k = n+1, \ldots, m$, we obtain

$$s_m - s_n \leq \int_n^m f(t)dt \leq s_{m-1} - s_{n-1},$$

from which it follows that

$$\int_{n+1}^{m+1} f(t)dt \leq s_m - s_n \leq \int_n^m f(t)dt.$$

If we take the limit in this last inequality as $m \to \infty$, we obtain (8). Q.E.D.

$a_k = f(k)$

$\sum_{k=1}^{\infty} f(k)$ exists $= \lim_{n\to\infty} S_n$ exists.

Proof: Since f is positive and decreasing on $[k-1, k]$, $k \geq 2$, $k \in \mathbb{N}$.

妙啊 so $\int_{k-1}^k f(t)dt \leq \int_{k-1}^k f(k-1)dt = f(k-1)(k-k+1) = f(k-1)$

$\int_{k-1}^k f(t)dt \geq \int_{k-1}^k f(k)dt = f(k)(k-k+1) = f(k)$.

then $f(k) \leq \int_{k-1}^k f(t)dt \leq f(k-1)$ $(*)$

By adding this inequality for $k = 2, 3, 4, \cdots n$,

obtain $\boxed{S_n - f(1) \leq \int_1^n f(t)dt \leq S_{n-1}}$

DON'T FORGET

$\lim S_n$ exists $\Leftrightarrow \lim \int_1^n f(t)dt$ exists.

If they exist, adding $(*)$ for $k = n+1, \cdots, m$,

obtain $S_m - S_n = \int_n^m f(t)dt \leq S_{m-1} - S_{n-1}$

follows that $\int_{n+1}^{m+1} f(t)dt \leq S_m - S_n \leq \int_n^m f(t)dt$.

When $m \to \infty$, $\int_{n+1}^{\infty} f(t)dt \leq S - S_n \leq \int_n^\infty f(t)dt$.

Q.E.D.

□ Apply above Tests to P-series : $\frac{1}{n^p}$

9.2.7 Examples (a) Consider the case $p = 2$; that is, the series $\sum 1/n^2$. We compare it with the convergent series $\sum 1/(n(n+1))$ of Example 3.7.2(c). Since

$$\left|\frac{1}{n^2} \div \frac{1}{n(n+1)}\right| = \frac{n+1}{n} = 1 + \frac{1}{n} \to 1,$$

the Limit Comparison Test 9.2.1 implies that $\sum 1/n^2$ is convergent.

用的时候设 y_n 很重要, y_n 与 x_n 齐次

(b) We demonstrate the failure of the Root Test for the p-series. Note that

$$\left|\frac{1}{n^p}\right|^{1/n} = \frac{1}{(n^p)^{1/n}} = \frac{1}{(n^{1/n})^p}.$$

Since (see Example 3.1.11(d)) we know that $n^{1/n} \to 1$, we have $r = 1$ in Corollary 9.2.3, and the theorem does not give any information.

(c) We apply the Ratio Test to the p-series. Since

$$\left|\frac{1}{(n+1)^p} \div \frac{1}{n^p}\right| = \frac{n^p}{(n+1)^p} = \frac{1}{(1+1/n)^p} \to 1,$$

the Ratio Test, in the form of Corollary 9.2.5, does not give any information.

$t \in \mathbb{N}$ and $t \geq 1$

(d) Finally, we apply the Integral Test to the p-series. Let $f(t) := 1/t^p$ for $t \geq 1$ and recall that

椥数是 n

$$\int_1^n \frac{1}{t}\,dt = \ln n - \ln 1,$$

$$\int_1^n \frac{1}{t^p}\,dt = \frac{1}{1-p}\left(\frac{1}{n^{p-1}} - 1\right) \quad \text{for} \quad p \neq 1.$$

Limit exists if $p > 1$
doesn't exist if $p \leq 1$

From these relations we see that the p-series converges if $p > 1$ and diverges if $p \leq 1$, as we have seen before in 3.7.6(d, e). □

9.2.8 Raabe's Test *Let $X := (x_n)$ be a sequence of nonzero real numbers.*

(a) *If there exist numbers $a > 1$ and $K \in \mathbb{N}$ such that*

$$(10) \qquad \left|\frac{x_{n+1}}{x_n}\right| \leq 1 - \frac{a}{n} \quad \textit{for} \quad n \geq K,$$

then $\sum x_n$ is absolutely convergent.

(b) *If there exist real numbers $a \leq 1$ and $K \in \mathbb{N}$ such that*

$$(11) \qquad \left|\frac{x_{n+1}}{x_n}\right| \geq 1 - \frac{a}{n} \quad \textit{for} \quad n \geq K,$$

\#

then $\sum x_n$ is not absolutely convergent. □

Proof. (a) If the inequality (10) holds, then we have (after replacing n by k and multiplying)

$$k|x_{k+1}| \leq (k-1)|x_k| - (a-1)|x_k| \quad \text{for} \quad k \geq K.$$

On reorganizing the inequality, we have

$$(12) \qquad (k-1)|x_k| - k|x_{k+1}| \geq (a-1)|x_k| > 0 \quad \text{for} \quad k \geq K,$$

from which we deduce that the sequence $(k|x_{k+1}|)$ is decreasing for $k \geq K$. If we add (12) for $k = K, \ldots, n$ and note that the left side telescopes, we get

$$(K-1)|x_K| - n|x_{n+1}| \geq (a-1)(|x_K| + \cdots + |x_n|).$$

This shows (why?) that the partial sums of $\sum |x_n|$ are bounded and establishes the absolute convergence of the series.

(b) If the relation (11) holds for $n \geq K$, then since $a \leq 1$, we have

$$n|x_{n+1}| \geq (n-a)|x_n| \geq (n-1)|x_n| \quad \text{for} \quad n \geq K.$$

Therefore the sequence $(n|x_{n+1}|)$ is increasing for $n \geq K$ and there exists a number $c > 0$ such that $|x_{n+1}| > c/n$ for $n \geq K$. But since the harmonic series $\sum 1/n$ diverges, the series $\sum |x_n|$ also diverges. Q.E.D.

(a) Proof: Suppose $\exists a > 1$ s.t. the inequality

$$(10) \qquad \left|\frac{x_{n+1}}{x_n}\right| \leq 1 - \frac{a}{n} \quad \textit{for} \quad n \geq K,$$

holds then

replace n by k : $\left|\frac{x_{k+1}}{x_k}\right| \leq 1 - \frac{a}{k}$. $\forall k \geq K$.

$\Rightarrow k|x_{k+1}| \leq k|x_k| - |x_k| - a|x_k| + |x_k|$

$\Rightarrow k|x_{k+1}| \leq (k-1)|x_k| - (a-1)|x_k|$

$$\boxed{\Rightarrow (k-1)|x_k| - k|x_{k+1}| \geq (a-1)|x_k| > 0. \ (a>1) \ (\circledast)}$$

$\Rightarrow (k-1)|x_k| > k|x_{k+1}|$

$\Rightarrow$ the sequence $(k|x_{k+1}|)$ is decreasing for $k \geq K$

Adding $(\circledast)$ for $k = K, K+1, \cdots, n-1, n$, obtain

$$\underbrace{(K-1)|x_K|}_{\text{实数}} - \underline{n|x_{n+1}|} \geq (a-1)(|x_K| + \cdots + |x_n|)$$

$(n|x_{n+1}|)\downarrow$ so it is really small when $n \to \infty$

When $n \to \infty$, $0 < (a-1)(|x_K| + \cdots + |x_n|) \leq (K-1)|x_K| = M$

Thus, $\sum |x_n|$ is bounded.

also, as $\sum |x_n|$ is increasing and by $\boxed{\text{Monotone Convergence Thm.}}$

$\sum |x_n|$ is convergent thus $\sum x_n$ is absolute convergent.

Q.E.D.

(b) Proof.

(b) *If there exist real numbers $a \leq 1$ and $K \in \mathbb{N}$ such that*

$$(11) \qquad \left|\frac{x_{n+1}}{x_n}\right| \geq 1 - \frac{a}{n} \quad \textit{for} \quad n \geq K,$$

then

$n|x_{n+1}| \geq (n-a)|x_n|$

also, by $a \leq 1$, $(n-a)|x_n| \geq (n-1)|x_n|$

so $n|x_{n+1}| \geq (n-1)|x_n|$

Thus, the sequence $(n|x_{n+1}|)$ is increasing. $\forall n \geq K$.

$$\boxed{\begin{array}{l} \Rightarrow \exists c > 0 \text{ s.t. } n|x_{n+1}| > c \quad \forall n \geq K, \\ \Rightarrow |x_{n+1}| > \frac{c}{n} \end{array}}$$

Let $y_n := \frac{c}{n}$ then by Comparison Test,

$\sum |x_n|$ is divergent.

Q.E.D.

9.2.9 Corollary *Let $X := (x_n)$ be a nonzero sequence in $\mathbb{R}$ and let*

$$a := \lim\left(n\left(1 - \left|\frac{x_{n+1}}{x_n}\right|\right)\right), \tag{13}$$

whenever this limit exists. Then $\sum x_n$ is absolutely convergent when $a > 1$ and is not absolutely convergent when $a < 1$.

Proof. Suppose the limit in (13) exists and that $a > 1$. If a_1 is any number with $a > a_1 > 1$, then there exists $K \in \mathbb{N}$ such that $a_1 < n(1 - |x_{n+1}/x_n|)$ for $n > K$. Therefore $|x_{n+1}/x_n| < 1 - a_1/n$ for $n \geq K$ and Raabe's Test 9.2.8(a) applies.

The case where $a < 1$ is similar and is left to the reader. Q.E.D.

Note There is no conclusion when $a = 1$; either convergence or divergence is possible, as the reader can show.

9.2.10 Examples **(a)** We reconsider the p-series in the light of Raabe's Test. Applying L'Hospital's Rule when $p \geq 1$, we obtain (why?)

$$\begin{aligned} a &= \lim\left(n\left[1 - \frac{n^p}{(n+1)^p}\right]\right) = \boxed{\lim\left(n\left[\frac{(n+1)^p - n^p}{(n+1)^p}\right]\right)} \\ &= \lim\left(\frac{(1+1/n)^p - 1}{1/n}\right) \cdot \lim\left(\frac{1}{(1+1/n)^p}\right) = \boxed{p \cdot 1 = p.} \end{aligned}$$

We conclude that if $p > 1$ then the p-series is convergent, and if $0 < p < 1$ then the series is divergent (since the terms are positive). However, if $p = 1$ (the harmonic series!), Corollary 9.2.9 yields no information.

(b) We now consider $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+1}$. divergent

An easy calculation shows that $\lim(x_{n+1}/x_n) = 1$, so that Corollary 9.2.5 does not apply. Also, we have $\lim(n(1 - x_{n+1}/x_n)) = 1$, so that Corollary 9.2.9 does not apply either. However, it is an exercise to establish the inequality $x_{n+1}/x_n \geq (n-1)/n$, from which it follows from Raabe's Test 9.2.8(b) that the series is divergent. (Of course, the Integral Test, or the Limit Comparison Test with $(y_n) = (1/n)$, can be applied here.) □

Although the limiting form 9.2.9 of Raabe's Test is much easier to apply, Example 9.2.10(b) shows that the form 9.2.8 is stronger than 9.2.9.

3.7, 9.1 共 7(6)个 Test,
2个 convergent, 5个 abs. conv.

Root & Ratio: abs. conv.
divergent.

Intergral: (an) >0 and ↓. convergent (含abso)

Section 9.3 Tests for Nonabsolute Convergence

9.3.1 Definition A sequence $X := (x_n)$ of nonzero real numbers is said to be **alternating** if the terms $(-1)^{n+1}x_n,\ n \in \mathbb{N}$, are all positive (or all negative) real numbers. If the sequence $X = (x_n)$ is alternating, we say that the series $\sum x_n$ it generates is an **alternating series**.

In the case of an alternating series, it is useful to set $x_n = (-1)^{n+1}z_n$ [or $x_n = (-1)^n z_n$], where $z_n > 0$ for all $n \in \mathbb{N}$.

通过 regroup 可得到许多信息.

9.3.2 Alternating Series Test *Let $Z := (z_n)$ be a decreasing sequence of strictly positive numbers with $\lim(z_n) = 0$. Then the alternating series $\sum(-1)^{n+1}z_n$ is convergent.*

The Alternating Series: $z_1, -z_2, z_3, -z_4 \cdots z_{2n-1}, -z_{2n}$

Proof: $S_{2n} = z_1 - z_2 + z_3 - z_4 + \cdots + z_{2n-1} - z_{2n}$

$= (z_1 - z_2) + (z_3 - z_4) + \cdots + (z_{2n-1} - z_{2n})$

As the series is decreasing so $z_k - z_{k+1} \geq 0$ $\forall k \in \mathbb{N}$.

it follows that the subsequence of S_n (S_{2n}) is increasing.

Since "regroup" S_{2n}:

$S_{2n} = z_1 - (z_2 - z_3) - \cdots - (z_{2n-2} - z_{2n-1}) - z_{2n}$

so $S_{2n} \leq z_1$ $\forall n \in \mathbb{N}$.

$\Rightarrow$ (S_{2n}) is monotone increasing & Bounded.

By Monotone Convergence Theorem,

$\exists S \in \mathbb{R}$ s.t. $S_{2n} \to S$ when $n \to \infty$.

[# (S_{2n}) 只是 Partial Sum S_n 的一个 Subsequence, S_{2n} converges $\neq$ S_n converges.]

$\boxed{\text{* Now show that } S_n \to S}$

As S_{2n} converges so $\forall \varepsilon > 0$, $\exists k \in \mathbb{N}$ s.t. $\forall n \geq k$,

$|S_{2n} - S| < \frac{\varepsilon}{2}$ and $|z_{2n+1}| < \frac{\varepsilon}{2}$ (by ~~n-term test~~ $\lim(z_n) = 0$)

so $\forall n \geq k$,

Transform

$|\boxed{S_{2n+1}} - S| = |\boxed{S_{2n} + z_{2n+1}} - S|$

$\leq |S_{2n} - S| + |z_{2n+1}|$

$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$

$= \varepsilon$

Therefore, every partial sum of an odd number of terms is also converges to S when $n \to \infty$.

Thus, $S_n = \sum(-1)^{n+1}z_n$ converges to S.

Q.E.D.

Proof. Since we have

$$s_{2n} = (z_1 - z_2) + (z_3 - z_4) + \cdots + (z_{2n-1} - z_{2n}),$$

and since $z_k - z_{k+1} \geq 0$, it follows that the subsequence (s_{2n}) of partial sums is increasing. Since

$$s_{2n} = z_1 - (z_2 - z_3) - \cdots - (z_{2n-2} - z_{2n-1}) - z_{2n},$$

it also follows that $s_{2n} \leq z_1$ for all $n \in \mathbb{N}$. It follows from the Monotone Convergence Theorem 3.3.2 that the subsequence (s_{2n}) converges to some number $s \in \mathbb{R}$.

We now show that the entire sequence (s_n) converges to s. Indeed, if $\varepsilon > 0$, let K be such that if $n \geq K$ then $|s_{2n} - s| \leq \frac{1}{2}\varepsilon$ and $|z_{2n+1}| \leq \frac{1}{2}\varepsilon$. It follows that if $n \geq K$ then

$$\begin{aligned} |s_{2n+1} - s| &= |s_{2n} + z_{2n+1} - s| \\ &\leq |s_{2n} - s| + |z_{2n+1}| \leq \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon. \end{aligned}$$

Therefore every partial sum of an odd number of terms is also within ε of s if n is large enough. Since $\varepsilon > 0$ is arbitrary, the convergence of (s_n) and hence of $\sum(-1)^{n+1}z_n$ is established. Q.E.D.

X, Y 无条件, 通式!

9.3.3 Abel's Lemma *Let $X := (x_n)$ and $Y := (y_n)$ be sequences in $\mathbb{R}$ and let the partial sums of $\sum y_n$ be denoted by (s_n) with $s_0 := 0$. If $m > n$, then*

$$\sum_{k=n+1}^{m} x_k y_k = (x_m s_m - x_{n+1} s_n) + \sum_{k=n+1}^{m-1} (x_k - x_{k+1}) s_k. \tag{3}$$

Proof: Since $y_k = s_k - s_{k-1}$ for $k = 1, 2, \dots$

$\Rightarrow \sum_{k=n+1}^{m} x_k y_k = \sum_{k=n+1}^{m} x_k (s_k - s_{k-1})$

$= x_{n+1}(s_{n+1} - s_n) + x_{n+2}(s_{n+2} - s_{n+1}) + \cdots + x_{m-1}(s_{m-1} - s_{m-2}) + x_m(s_m - s_{m-1})$

$= x_{n+1}s_{n+1} - x_{n+1}s_n + x_{n+2}s_{n+2} - x_{n+2}s_{n+1} + \cdots + x_{m-1}s_{m-1} - x_{m-1}s_{m-2} + x_m s_m - x_m s_{m-1}$

$= (x_m s_m - x_{n+1} s_n) + (x_{n+1} - x_{n+2}) s_{n+1} + \cdots + (x_{m-1} - x_m) s_{m-1}$

$= (x_m s_m - x_{n+1} s_n) + \sum_{k=n+1}^{m-1} (x_k - x_{k+1}) s_k$

Proof. Since $y_k = s_k - s_{k-1}$ for $k = 1, 2, \dots$, the left side of (3) is seen to be equal to $\sum_{k=n+1}^{m} x_k (s_k - s_{k-1})$. If we collect the terms multiplying $s_n, s_{n+1}, \dots, s_m$, we obtain the right side of (3). Q.E.D.

$\sum x_n y_n$: ([illegible] test)

1

9.3.4 Dirichlet's Test *If $X := (x_n)$ is a decreasing sequence with $\lim x_n = 0$, and if the partial sums (s_n) of $\sum y_n$ are bounded, then the series $\sum x_n y_n$ is convergent.*

(S_n) is bounded

Proof. Let $|s_n| \le B$ for all $n \in \mathbb{N}$. If $m > n$, it follows from Abel's Lemma 9.3.3 and the fact that $x_k - x_{k+1} \ge 0$ that

$$\begin{aligned}
\left| \sum_{k=n+1}^{m} x_k y_k \right| &\le (x_m + x_{n+1}) B + \sum_{k=n+1}^{m-1} (x_k - x_{k+1}) B \\
&= [(x_m + x_{n+1}) + (x_{n+1} - x_m)] B \\
&= 2 x_{n+1} B.
\end{aligned}$$

Since $\lim(x_k) = 0$, the convergence of $\sum x_k y_k$ follows from the Cauchy Convergence Criterion 3.7.4. Q.E.D.

Proof: As $S_n = \sum y_n$ is bounded so $\exists B > 0$ s.t. $|S_n| \le B$ $\forall n \in \mathbb{N}$.

Let $m > n$. by Abel's Lemma:

$$\sum_{k=n+1}^{m} x_k y_k = (x_m s_m - x_{n+1} s_n) + \sum_{k=n+1}^{m-1} (x_k - x_{k+1}) s_k.$$

Cauchy:

So $\left| \sum_{k=n+1}^{m} x_k y_k \right| = \left| (x_m S_m - x_{n+1} S_n) + \sum_{k=n+1}^{m-1} (x_k - x_{k+1}) S_k \right|$

$\| [T_m - T_{n+1}]$

$\le |x_m S_m - x_{n+1} S_n| + \left| \sum_{k=n+1}^{m-1} (x_k - x_{k+1}) S_k \right|$

$\le |x_m S_m| + |x_{n+1} S_n| + \left| \sum_{k=n+1}^{m-1} (x_k - x_{k+1}) \right| |S_k|$

by decreasing $\le (x_m + x_{n+1}) B + \sum_{k=n+1}^{m-1} (x_k - x_{k+1}) B$

$= (x_m + x_{n+1}) B + B [x_{n+1} - x_{n+2} + x_{n+2} - x_{n+3} + \cdots + x_{m-1} - x_m]$

$= B[(x_m + x_{n+1}) + (x_{n+1} - x_m)]$

$= 2 B x_{n+1}$

As $\lim(x_k) = 0$ so By Cauchy Criterion, $\sum x_n y_n$ converges.

Q.E.D.

9.3.5 Abel's Test *If $X := (x_n)$ is a convergent monotone sequence and the series $\sum y_n$ is convergent, then the series $\sum x_n y_n$ is also convergent.*

Proof. Since x_n is monotone, there are 2 cases:

Case I: If (x_n) is decreasing with limit x,

i.e. $x_n \to x$, let $u_n = x_n - x$ $\forall n \in \mathbb{N}$.

By Convergency, $\lim u_n = \lim(x_n - x) = 0$

and (u_n) monotone decreases.

Then $x_n = x + u_n$ whence $x_n y_n = x y_n + u_n y_n$

As $\sum y_n$ is convergent so $\sum y_n$ is bounded. (+ u_n decreasing + $\lim(u_n) = 0$)

By Dirichlet's Test, $\sum u_n y_n$ is convergent.

Since $\sum y_n$ is convergent, so $\sum x y_n$ is convergent.

Recall that $x_n y_n = x y_n + u_n y_n$,

$\Rightarrow \sum x_n y_n$ is convergent.

Case 2: If (x_n) is increasing and $\lim(x_n) = x$,

let $v_n := x - x_n$ then (v_n) is decreasing

$\lim(v_n) = \lim(x_n - x) = 0$

Write x_n as $x_n = v_n - x$ then

$x_n y_n = (v_n - x) y_n = v_n y_n - x y_n$.

By Dirichlet's Test, $\sum v_n y_n$ is convergent.

$\Rightarrow \sum x_n y_n$ is convergent.

Conclusion: When (x_n) is monotone and convergent, $\sum y_n$ is convergent, $\sum x_n y_n$ is convergent.

Q.E.D.

Proof. If (x_n) is decreasing with limit x, let $u_n := x_n - x$, $n \in \mathbb{N}$, so that (u_n) decreases to 0. Then $x_n = x + u_n$, whence $x_n y_n = x y_n + u_n y_n$. It follows from the Dirichlet Test 9.3.4 that $\sum u_n y_n$ is convergent and, since $\sum x y_n$ converges (because of the assumed convergence of the series $\sum y_n$), we conclude that $\sum x_n y_n$ is convergent.

If (x_n) is increasing with limit x, let $v_n := x - x_n$, $n \in \mathbb{N}$, so that (v_n) decreases to 0. Here $x_n = x - v_n$, whence $x_n y_n = x y_n - v_n y_n$, and the argument proceeds as before. Q.E.D.

9.3.6 Examples (a)

Hence Dirichlet's Test implies that if (a_n) is decreasing with $\lim(a_n) = 0$, then the series $\sum_{n=1}^{\infty} a_n \cos nx$ converges provided $x \neq 2k\pi$.

Since we have

$$2(\sin \tfrac{1}{2}x)(\cos x + \cdots + \cos nx) = \sin(n + \tfrac{1}{2})x - \sin \tfrac{1}{2}x,$$ (此式积和化差)

it follows that if $x \neq 2k\pi$ ($k \in \mathbb{N}$), then

$$|\cos x + \cdots + \cos nx| = \frac{|\sin(n + \frac{1}{2})x - \sin \frac{1}{2}x|}{|2 \sin \frac{1}{2}x|} \leq \frac{1}{|\sin \frac{1}{2}x|}.$$

$\Rightarrow \sum \cos nx$ is bounded.

(b) Since we have

$$2\left(\sin \tfrac{1}{2}x\right)(\sin x + \cdots + \sin nx) = \cos \tfrac{1}{2}x - \cos\left(n + \tfrac{1}{2}\right)x,$$ (此式积和化差)

it follows that if $x \neq 2k\pi$ ($k \in \mathbb{N}$), then

$$|\sin x + \cdots + \sin nx| \leq \frac{1}{|\sin \frac{1}{2}x|}.$$ $\Rightarrow \sum \sin nx$ is bounded.

As before, if (a_n) is decreasing and if $\lim(a_n) = 0$, then the series $\sum_{n=1}^{\infty} a_n \sin nx$ converges for $x \neq 2k\pi$ (and it also converges for these values). □

□ 泰勒展开 $f(x) = \sin x$:

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \quad \forall x \in \mathbb{R}$$

1. Alternating Series 并非 $S_n = \sum x_n$ 在 alternating, 而是 (x_n)

Section 9.4 Series of Functions

9.4.1 Definition If (f_n) is a sequence of functions defined on a subset D of $\mathbb{R}$ with values in $\mathbb{R}$, the sequence of **partial sums** (s_n) of the infinite series $\sum f_n$ is defined for x in D by

$$\begin{aligned}
s_1(x) &:= f_1(x), \\
s_2(x) &:= s_1(x) + f_2(x) \\
&\cdots\cdots\cdots\cdots \\
s_{n+1}(x) &:= s_n(x) + f_{n+1}(x) \\
&\cdots\cdots\cdots\cdots
\end{aligned}$$

In case the sequence (s_n) of functions converges on D to a function f, we say that the infinite series of functions $\sum f_n$ **converges** to f on D. We will often write

$$\sum f_n \quad \text{or} \quad \sum_{n=1}^{\infty} f_n$$

to denote either the series or the limit function, when it exists.

9.4.2 Theorem *If f_n is continuous on $D \subseteq \mathbb{R}$ to $\mathbb{R}$ for each $n \in \mathbb{N}$ and if $\sum f_n$ converges to f uniformly on D, then f is continuous on D.*

8.2.2 Theorem *Let (f_n) be a sequence of continuous functions on a set $A \subseteq \mathbb{R}$ and suppose that (f_n) converges uniformly on A to a function $f : A \to \mathbb{R}$. Then f is continuous on A.*

$\sum f_n \rightrightarrows f.$

9.4.3 Theorem *Suppose that the real-valued functions f_n, $n \in \mathbb{N}$, are Riemann integrable on the interval $J := [a, b]$. If the series $\sum f_n$ converges to f uniformly on J, then f is Riemann integrable and*

$$\int_a^b f = \sum_{n=1}^{\infty} \int_a^b f_n. \tag{1}$$

8.2.4 Theorem *Let (f_n) be a sequence of functions in $\mathcal{R}[a,b]$ and suppose that (f_n) converges **uniformly** on $[a, b]$ to f. Then $f \in \mathcal{R}[a,b]$ and (3) holds.*

$$\int_a^b f = \lim_{n\to\infty} \int_a^b f_n.$$

9.4.4 Theorem *For each $n \in \mathbb{N}$, let f_n be a real-valued junction on $J := [a, b]$ that has a derivative f_n' on J. Suppose that the series $\sum f_n$ converges for at least one point of J and that the series of derivatives $\sum \boxed{f_n'}$ converges uniformly on J.*

Then there exists a real-valued function f on J such that $\sum f_n$ converges uniformly on J to f. In addition, f has a derivative on J and $f' = \sum f_n'$.

Thm:

If
① $\forall n \in \mathbb{N}$, f_n is differentiable on J
② $\sum f_n$ at least pointwise convergent
③ $\sum f_n'$ uniform convergent on J.

Then
① $\exists f$ on J s.t. $f_n \rightrightarrows f$
② $f' = \sum f_n'$

Tests for Uniform Convergence

9.4.5 Cauchy Criterion *Let (f_n) be a sequence of functions on $D \subseteq \mathbb{R}$ to $\mathbb{R}$. The series $\sum f_n$ is uniformly convergent on D if and only if for every $\varepsilon > 0$ there exists an $M(\varepsilon)$ such that if $m > n \geq M(\varepsilon)$, then*

$$|f_{n+1}(x) + \cdots + f_m(x)| < \varepsilon \quad \text{for } \boxed{\text{all} \quad x \in D}.$$

i.e. $|S_m - S_n| = \left| \sum_{k=1}^{m} f_k - \sum_{k=1}^{n} f_k \right| < \varepsilon$

9.4.6 Weierstrass M-Test *Let (M_n) be a sequence of positive real numbers such that $|f_n(x)| \leq M_n$ for $x \in D$, $n \in \mathbb{N}$. If the series $\sum M_n$ is convergent, then $\sum f_n$ is uniformly convergent on D.*

Proof. If $m > n$, we have the relation

$$|f_{n+1}(x) + \cdots + f_m(x)| \leq M_{n+1} + \cdots + M_m \quad \text{for} \quad x \in D.$$

Now apply 3.7.4, 9.4.5, and the convergence of $\sum M_n$. Q.E.D.

Proof: As $\sum M_n$ is convergent then by 3.7.4 Cauchy Criterion,

$\forall \varepsilon > 0$, $\exists K \in \mathbb{N}$ s.t. $\forall m > n > K$, $m, n \in \mathbb{N}$,

$$\left| \sum M_m - \sum M_n \right| = |M_{n+1} + \cdots + M_m| < \varepsilon$$

As $M_n > 0$ $\forall n \in \mathbb{N}$ so $M_{n+1} + \cdots + M_m < \varepsilon$.

As $|f_n(x)| \leq M_n$ $\forall n \in \mathbb{N}$ so

$$|f_{n+1}(x) + \cdots + f_m(x)| \leq |f_{n+1}(x)| + \cdots + |f_m(x)| \leq M_{n+1} + \cdots + M_m < \varepsilon$$

By 9.4.5, $\sum f_n$ is uniformly convergent on D.

Q.E.D.

Power Series

介绍 Power Series 的原因:

Closed & Bounded

5.4.14 Weierstrass Approximation Theorem *Let $I = [a, b]$ and let $f : I \to \mathbb{R}$ be a continuous function. If $\varepsilon > 0$ is given, then there exists a polynomial function p_ε such that $|f(x) - p_\varepsilon(x)| < \varepsilon$ for all $x \in I$.*

与在一点上定义的 Taylor's Theorem 不同.

[联想 .polyfit( )]

=> 为借助 Polynomial 来"逼近" Continuous Function 提供理论依据. (因为一定存在,所以可以用其逼近)

=> "恰"为 Polynomial, 所以来研究 Polynomial 更特殊的 Power Series.

9.4.7 Definition A series of real functions $\sum f_n$ is said to be a **power series around** $x = c$ if the function f_n has the form

$$f_n(x) = a_n(x - c)^n,$$

where a_n and c belong to $\mathbb{R}$ and where $n = 0, 1, 2, \ldots$.

Introduce $x' := x - c$ then $f_n(x') = a_n x'^n$

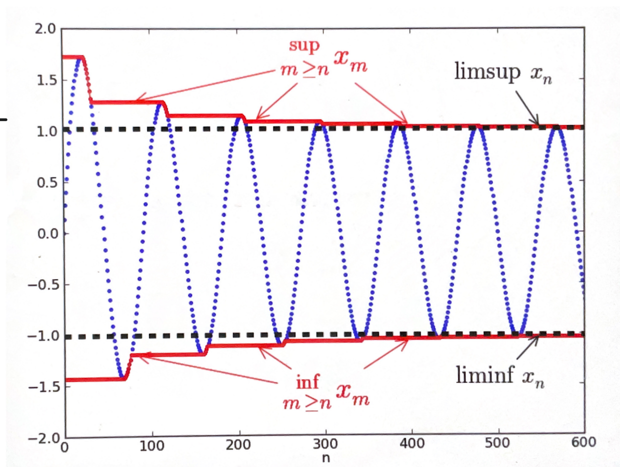
Thus, we could only do research on when $c = 0$, $f_n(x) = a_n x^n$

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + \cdots + a_n x^n + \cdots. \tag{2}$$

9.4.8 Definition Let $\sum a_n x^n$ be a power series. If the sequence $(|a_n|^{1/n})$ is bounded, we set $\rho := \lim\sup(|a_n|^{1/n})$; if this sequence is not bounded we set $\rho = +\infty$. We define the **radius of convergence** of $\sum a_n x^n$ to be given by

$$R := \begin{cases} 0 & \text{if} \quad \rho = +\infty, \\ 1/\rho & \text{if} \quad 0 < \rho < +\infty, \\ +\infty & \text{if} \quad \rho = 0. \end{cases}$$

The **interval of convergence** is the open interval $(-R, R)$.



1. If $w > \limsup(b_n)$ then for all n that is sufficiently large $b_n \le w$

($\exists k \in \mathbb{N}$ s.t. $\forall n \geq k$, $b_n \le w$)

2. If $w < \limsup(b_n)$ then there are infinitely many $n \in \mathbb{N}$ s.t. $b_n \geq w$

9.4.9 Cauchy-Hadamard Theorem *If R is the radius of convergence of the power series $\sum a_n x^n$, then the series is absolutely convergent if $|x| < R$ and is divergent if $|x| > R$.*

Proof. When $0 < R < +\infty$ (R is a finite number and $R \neq 0$)

① If $0 < |x| < R$ then $\exists c > 0$ s.t. $c < 1$ and $|x| < cR$.

$\Rightarrow$ $\boxed{\rho = \frac{1}{R} < \frac{c}{|x|}}$ $\limsup(b_n) < w$

As $\rho = \limsup(|a_n|^{\frac{1}{n}})$ then for sufficiently large n,

$$|a_n|^{\frac{1}{n}} \le \rho < \frac{c}{|x|} \Rightarrow |a_n|^{\frac{1}{n}} \le \frac{c}{|x|}$$

$\Rightarrow$ $|a_n x^n| \le c^n$ for all sufficiently large n

As $c < 1$ so by the property of Geometric Series,

and Comparison Test, $\sum |a_n x^n|$ converges

$\Rightarrow$ $\sum a_n x^n$ converges.

② If $|x| > R = \frac{1}{\rho} = \frac{1}{\limsup(a_n)}$

then $\boxed{\rho = \limsup(a_n) > \frac{1}{|x|}}$ $\limsup(b_n) > w$

so there are infinitely many $n \in \mathbb{N}$ s.t. $|a_n|^{\frac{1}{n}} > \frac{1}{|x|}$

$\Rightarrow$ $|a_n x^n| > 1$

Thus, $\sum a_n x^n$ is divergent.

Q.E.D.

Proof. We shall treat only the case where $0 < R < +\infty$, leaving the cases $R = 0$ and $R = +\infty$ as exercises. If $0 < |x| < R$, then there exists a positive number $c < 1$ such that $|x| < cR$. Therefore $\rho < c/|x|$ and so it follows that if n is sufficiently large, then $|a_n|^{1/n} \le c/|x|$. This is equivalent to the statement that

$$|a_n x^n| \le c^n \tag{3}$$

for all sufficiently large n. Since $c < 1$, the absolute convergence of $\sum a_n x^n$ follows from the Comparison Test 3.7.7.

If $|x| > R = 1/\rho$, then there are infinitely many $n \in \mathbb{N}$ for which $|a_n|^{1/n} > 1/|x|$. Therefore, $|a_n x^n| > 1$ for infinitely many n, so that the sequence $(a_n x^n)$ does not converge to zero. Q.E.D.

It is an exercise to show that the radius of convergence of the series $\sum a_n x^n$ is also given by

$$R = \lim \left| \frac{a_n}{a_{n+1}} \right|, \tag{4}$$

别的级数不能用!

Remark:

The argument used in the proof of the Cauchy-Hadamard Theorem yields the uniform convergence of the power series on any fixed closed and bounded interval in the interval of convergence $(-R, R)$. |x|=R要单独代入计算看是否收敛.

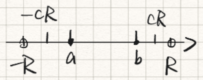
i.e. $\forall a, b$ s.t. $-R < a < b < R$, the uniform convergence is known on $[a, b]$.

9.4.10 Theorem *Let R be the radius of convergence of $\sum a_n x^n$ and let K be a closed and bounded interval contained in the interval of convergence $(-R, R)$. Then the power series converges uniformly on K.*

Proof. The hypothesis on $K \subseteq (-R, R)$ implies that there exists a positive constant $c < 1$ such that $|x| < cR$ for all $x \in K$. (Why?) By the argument in 9.4.9, we infer that for sufficiently large n, the estimate (3) holds for all $x \in K$. Since $c < 1$, the uniform convergence of $\sum a_n x^n$ on K is a direct consequence of the Weierstrass M-test with $M_n := c^n$. Q.E.D.

Proof: As $K \subseteq (-R,R)$ so $\exists c \in (0,1)$ s.t. $\forall x \in K$,

$-cR < x < cR \Rightarrow |x| < cR$



$\Rightarrow \frac{|x|}{c} < R = \frac{1}{\rho} = \frac{1}{\limsup(a_n^{\frac{1}{n}})}$

$\Rightarrow \limsup(|a_n|^{\frac{1}{n}}) < \frac{c}{|x|}$

so for sufficiently large n, $a_n^{\frac{1}{n}} < \frac{c}{|x|}$

$\Rightarrow |a_n x^n| \le c^n$ and $\sum c^n$ converges $(0<c<1)$.

$\Rightarrow \sum a_n x^n$ is uniformly convergent by M-Test.

Q.E.D.

9.4.11 Theorem *The limit of a power series is continuous on the interval of convergence. A power series can be integrated term-by-term over any closed and bounded interval contained in the interval of convergence.*

i.e. Let $f_n(x) = a_n x^n$, $f := \lim f_n$

f is continuous on H (the interval of convergence)

Proof. Let H be a closed and bounded neighborhood of x_0 contained in $(-R,R)$. i.e. $\forall x \in H \subseteq (-R,R)$.

then let x_0 be a element in H,

$\Rightarrow |x_0| < R \Rightarrow \sum f_n = \sum a_n x^n$ is uniformly convergent on x_0.

Since $f_n(x)$ is continuous on x_0 so by 9.4.2,

Limit of f_n: f is continuous on x_0.

Since f_n is polynomial so f_n is Riemann Integrable $\forall n \in \mathbb{N}$, $\forall x \in H$.

by 9.4.3, $\int_a^b f = \sum_{n=1}^{\infty} \int_a^b f_n$ (f_n can be integrated term by term).

Q.E.D.

Proof. If $|x_0| < R$, then the preceding result asserts that $\sum a_n x^n$ converges uniformly on any closed and bounded neighborhood of x_0 contained in $(-R, R)$. The continuity at x_0 then follows from Theorem 9.4.2, and the term-by-term integration is justified by Theorem 9.4.3.

Q.E.D.

Below only for Power Series

We now show that a power series can be differentiated term-by-term. Unlike the situation for general series, we do not need to assume that the differentiated series is uniformly convergent. Hence this result is stronger than Theorem 9.4.4.

9.4.12 Differentiation Theorem *A power series can be differentiated term-by-term within the interval of convergence. In fact, if*

$$f(x) = \sum_{n=0}^{\infty} a_n x^n, \quad \textit{then} \quad f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \qquad \textit{for} \quad |x| < R.$$

Both series have the same radius of convergence.

Proof: Since $\lim(n^{\frac{1}{n}}) = 1$, so the sequence $(|na_n|^{\frac{1}{n}})$ is bounded iff the sequence $(|a_n|^{\frac{1}{n}})$ is bounded.

and $\boxed{\limsup(|na_n|^{\frac{1}{n}}) = \limsup(|a_n|^{\frac{1}{n}}) = \rho}$

As $R = \frac{1}{\rho}$ so the radius of convergence for $f(x)$ and $f'(x)$ is the same.

Define I_x to be any closed and bounded interval s.t. $I_x \subseteq (-R, R)$, (I_x contained in the interval of convergence)

then by $\boxed{9.4.10}$, the Power Series is uniformly convergent on I_x.

i.e. $\left(\sum_{n=0}^{\infty} a_n x^n\right)$ is uniformly convergent on I_x.

$\Rightarrow$ By $\boxed{9.4.4}$, as $\sum_{n=1}^{\infty} a_n x^n$ is differentiable term by term on I_x so

$$D[f(x)] = \left(\sum_{n=0}^{\infty} a_n x^n\right)' = \sum_{n=0}^{\infty} (a_n x^n)' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

Thus, if $f(x) = \sum_{n=0}^{\infty} a_n x^n$ then $f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$ $\forall x \in I_x$

as I_x is an arbitrary closed and bdd interval in $(-R, R)$

So if $f(x) = \sum_{n=0}^{\infty} a_n x^n$ then $f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$ for $|x| < R$

and f, f' have the same radius of convergence.

$[(\sum f_n)' = \sum f_n']$ Q.E.D.

Proof. Since $\lim(n^{1/n}) = 1$, the sequence $(|na_n|^{1/n})$ is bounded if and only if the sequence $(|a_n|^{1/n})$ is bounded. Moreover, it is easily seen that

$$\limsup\left(|na_n|^{1/n}\right) = \limsup\left(|a_n|^{1/n}\right).$$

Therefore, the radius of convergence of the two series is the same, so the formally differentiated series is uniformly convergent on each closed and bounded interval contained in the interval of convergence. We can then apply Theorem 9.4.4 to conclude that the formally differentiated series converges to the derivative of the given series. Q.E.D.

By repeated application of the preceding result, we conclude that if $k \in \mathbb{N}$ then $\sum_{n=0}^{\infty} a_n x^n$ can be differentiated term-by-term k times to obtain

(5)
$$\sum_{n=k}^{\infty} \frac{n!}{(n-k)!} a_n x^{n-k}.$$

Moreover, this series converges absolutely to $f^{(k)}(x)$ for $|x| < R$ and uniformly over any closed and bounded interval in the interval of convergence. If we substitute $x = 0$ in (5), we obtain the important formula

$$f^{(k)}(0) = k! a_k.$$

9.4.13 Uniqueness Theorem *If $\sum a_n x^n$ and $\sum b_n x^n$ converge on some interval $(-r, r)$, $r > 0$, to the same function f, then*

$$a_n = b_n \quad \text{for all} \quad n \in \mathbb{N}.$$

Proof. Our preceding remarks show that $n! a_n = f^{(n)}(0) = n! b_n$ for all $n \in \mathbb{N}$. Q.E.D.