3
Martingales in Discrete Time

3.1 Sequences of Random Variables

When the index set $T = \mathbb{N}$, the stochastic process $\xi_1, \xi_2, \ldots$ is also called a *stochastic sequence*, or a *sequence of random variables*.

Definition 3.1. Let $\xi_1, \xi_2, \ldots$ be a stochastic sequence defined on probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The sequence of numbers $\xi_1(\omega), \xi_2(\omega), \ldots$ for any fixed $\omega \in \Omega$ is called a sample path of stochastic sequence $\xi_1, \xi_2, \ldots$.

A sample path for a sequence of coin tosses (w) is presented in Figure 3.1 (+1 stands for heads and −1 for tails). Figure 3.2 shows the sample path of the FTSE All-Share Index up to 1997. Strictly speaking the pictures should con-

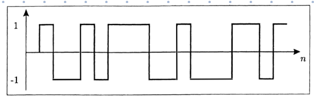


Figure 3.1. Sample path for a sequence of coin tosses (ω)

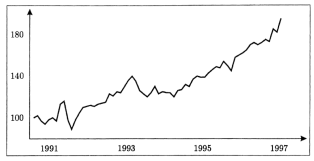


Figure 3.2. Sample path representing the FTSE All-Share Index up to 1997 (ω)

sist of dots, representing the values $\xi_1(\omega), \xi_2(\omega), \ldots$, but it is customary to connect them by a broken line for illustration purposes.

3.2 Filtrations

Definition 3.2. A sequence of sub-σ-fields $\mathcal{F}_1, \mathcal{F}_2, \ldots$ (or, $\{\mathcal{F}_n\}$) on Ω such that

$$\mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots$$

is called a *filtration*.

We recall (Definition 1.1) that a collection $\mathcal{F}$ of subsets of a non-empty set Ω is called a *σ-field* (or, *σ-algebra*) if it satisfies:

- the empty set $\emptyset \in \mathcal{F}$;
- If $A \in \mathcal{F}$, then its complement $A^c = \Omega \setminus A \in \mathcal{F}$, where $A^c = \Omega \setminus A$;
- If $A_n \in \mathcal{F}$ for every $n = 1, 2, \ldots$, then

$$\bigcup_{n=1}^{\infty} A_n = A_1 \cup A_2 \cup \cdots \in \mathcal{F}.$$

⇒ 若在discrete time point n
知$A_1 \cdots A_n$分别是否发生,
则知$A_1 \cdots A_n$中至少1个是否发生

Example 3.1. For a sequence $\xi_1, \xi_2, \ldots$ of coin tosses we take $\mathcal{F}_n$ to be the a σ-field generated by $\xi_1, \ldots, \xi_n$

$$\mathcal{F}_n = \sigma\{\xi_1, \ldots, \xi_n\}.$$

Let

$$A = \{\text{the first 5 tosses produce at least 2 heads}\}.$$

At discrete time $n = 5$, i.e. once the coin has been tossed five times, it will be possible to decide whether A has occurred or not. This means that $A \in \mathcal{F}_5$. However, at $n = 4$ it is not always possible to tell if A has occurred or not. If the outcomes of the first four tosses are, say,

tails, tails, heads, tails,

then the event A remains undecided. We will have to toss the coin once more to see what happens. Therefore $A \notin \mathcal{F}_4$.

This example illustrates another relevant issue. Suppose that the outcomes of the first four coin tosses are

tails, heads, tails, heads.

In this case it is possible to tell that A has occurred already at $n = 4$, whatever the outcome of the fifth toss will be. It does not mean, however, that A belongs to $\mathcal{F}_4$. The point is that for A to belong to $\mathcal{F}_4$ it must be possible to tell whether A has occurred or not after the first four tosses, *no matter what the first four outcomes are*. This is clearly not so in the example in hand.

No matter what actually happened in the first 5 times.

If $A \in \mathcal{F}_n$ then at the time point n, no matter what happened in the last $n-1$ time points, we are able to know whether A has occured or not.

Exercise 3.1

Let $\xi_1, \xi_2, \ldots$ be a sequence of coin tosses and let $\mathcal{F}_n$ be the σ-field generated by $\xi_1, \ldots, \xi_n$. For each of the following events find the smallest n such that the event belongs to $\mathcal{F}_n$:

$(\underbrace{TT\cdots T}_{10}H) \in A$ ≤

$A = \{$the first occurrence of heads is preceeded by no more than 10 tails$\}$,
$B = \{$there is at least 1 head in the sequence $\xi_1, \xi_2, \ldots\}$,
$C = \{$the first 100 tosses produce the same outcome$\}$,
$D = \{$there are no more than 2 heads and 2 tails among the first 5 tosses$\}$.
≤ 2H ≤ 2T

Hint To find the smallest element in a set of numbers you need to make sure that the set is non-empty in the first place.

"Preceeded"

Soln: ① $n = \min\{\cancel{10}, 11, 12, \cdots\} = \cancel{10}\ 11$ $A \in \mathcal{F}_{11}$ but $A \notin \mathcal{F}_{10}$.

② $\cancel{n = \min\{1, 2, 3, \cdots\} = 1}$

B doesn't belong to $\mathcal{F}_n$ for any $n \in \mathbb{N}$. as $\mathbb{N}$ is an infinit sequence. So there's no smallest n s.t. $B \in \mathcal{F}_n$.

③ $n = \min\{100, 101, \cdots\} = 100$ $C \in \mathcal{F}_{100}$. but $C \notin \mathcal{F}_{99}$.

④ $D = \{\emptyset\}$. As $\emptyset \in \mathcal{F}_n$ $\forall n \in \mathbb{N}$ so $n_{min} = 1$.

The condition in the definition below means that $\mathcal{F}_n$ contains everything that can be learned from the values of $\xi_1, \ldots, \xi_n$. In general, it may and often does contain more information.

Definition 3.3. We say that a stochastic sequence $\xi_1, \xi_2, \ldots$ is *adapted* to a filtration $\mathcal{F}_1, \mathcal{F}_2, \ldots$ if ξ_n is $\mathcal{F}_n$-measurable for each $n = 1, 2, \ldots$.

Example 3.2. If $\mathcal{F}_n = \sigma\{\xi_1, \ldots, \xi_n\}$ is the σ-field generated by $\xi_1, \ldots, \xi_n$ for each n, then the stochastic sequence $\xi_1, \xi_2 \ldots$ is adapted to $\mathcal{F}_1, \mathcal{F}_2, \ldots$.

3.3 Martingales

Definition 3.4. A stochastic sequence $\xi_1, \xi_2, \ldots$ is called a *martingale* with respect to a filtration $\mathcal{F}_1, \mathcal{F}_2, \ldots$ if

1) ξ_n is integrable for each $n = 1, 2, \ldots$; i.e. $E(|\xi_n|) < \infty$.
2) $\xi_1, \xi_2, \ldots$ is adapted to $\mathcal{F}_1, \mathcal{F}_2, \ldots$;
3) $\mathbb{E}[\xi_{n+1} \mid \mathcal{F}_n] = \xi_n$ a.s. for each $n = 1, 2, \ldots$.

i.e. $\forall s \leq t$, $E[Z_t | F_s] = Z_s$ for all $t \geq 0$.

i.e. Wherever we are, if we conditioning on whatever we have at the time n (now), we don't expect to be any better or any worse off, we expect to be exactly the place we are now, in any place of the future.

i.e. fair game $\Leftrightarrow$ shouldn't expect to win or loose

Initial Observations:

Let $F_n := \sigma(\xi_1, \xi_2, \ldots, \xi_n)$ and $\{\xi_n\}$ are F_n-MG

then $\xi_n = E[\xi_{n+1} | F_n] = E[\xi_{n+1} | \xi_1, \xi_2, \ldots, \xi_n]$ (*)

by (*) for any $k \geq 1 \Rightarrow k-1 \geq 0 \Rightarrow n+k-1 \geq n$:

$$\begin{aligned} E[\xi_{n+k} | F_n] &= E[E[\xi_{n+k} | F_{n+k-1}] | F_n] \quad \text{by } F_n \subseteq F_{n+k-1} \\ &= E[\xi_{n+k-1} | F_n] \quad \text{and the tower property.} \\ &= E[E(\xi_{n+k-1} | F_{n+k-2}) | F_n] \\ &= E[\xi_{n+k-2} | F_n] \\ &\ \vdots \\ &= E[\xi_n | F_n] \\ &= \xi_n \end{aligned}$$

Example 3.3

Let $\eta_1, \eta_2, \ldots$ be a sequence of independent integrable random variables such that $E(\eta_n) = 0$ for all $n = 1, 2, \ldots$. We put

$$\begin{aligned} \xi_n &= \eta_1 + \cdots + \eta_n, \\ \mathcal{F}_n &= \sigma(\eta_1, \ldots, \eta_n) = \sigma(\xi_1, \xi_2, \ldots, \xi_n) \end{aligned}$$

η_{n+1} is independent of $F_n \Leftarrow$; $\Longrightarrow$ ξ_n is F_n-integrable

Then ξ_n is adapted to the filtration $\mathcal{F}_n$, and it is integrable because

$$\begin{aligned} E(|\xi_n|) &= E(|\eta_1 + \cdots + \eta_n|) \\ &\leq E(|\eta_1|) + \cdots + E(|\eta_n|) \\ &< \infty. \end{aligned}$$

Moreover,

$$\begin{aligned} E(\xi_{n+1}|\mathcal{F}_n) &= E(\eta_{n+1}|\mathcal{F}_n) + E(\xi_n|\mathcal{F}_n) \\ &= E(\eta_{n+1}) + \xi_n \\ &= \xi_n, \end{aligned}$$

Independence 不去 E.

since η_{n+1} is independent of $\mathcal{F}_n$ ('and independent condition drop ξ_n is $\mathcal{F}_n$-measurable ('taking out what is known'). This means t martingale with respect to $\mathcal{F}_n$.

since ξ_n is $\mathcal{F}_n$-measurable ('taking out what is known'), η_{n+1} is independent of $\mathcal{F}_n$ ('and independent condition drops out') and $\mathbb{E}[\eta_{n+1}] = 0$. This means that $\xi_1, \xi_2, \ldots$ is a martingale with respect to $\mathcal{F}_1, \mathcal{F}_2 \ldots$. □

Example 3.4. Let ξ be an integrable random variable and let $\mathcal{F}_1, \mathcal{F}_2, \ldots$ be a filtration. We put

$$\xi_n = \mathbb{E}[\xi \mid \mathcal{F}_n], \quad n = 1, 2, \ldots.$$

Then ξ_n is $\mathcal{F}_n$-measurable according to Definition 3.4, $\Rightarrow$ ② ξ_n is adapted to F_n.

$$|\xi_n| = |\mathbb{E}[\xi \mid \mathcal{F}_n]| \leq \mathbb{E}[|\xi| \mid \mathcal{F}_n]$$

① by Jensen's Inequality, which implies that

$E|\xi_n| = E[|E[\xi|F_n]|] \leq E[E[|\xi||F_n]]$

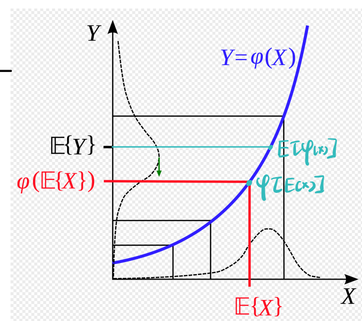
$$\mathbb{E}[|\xi_n|] \leq \mathbb{E}\Big[\mathbb{E}[|\xi| \mid \mathcal{F}_n]\Big] = \mathbb{E}[|\xi|] < \infty.$$

③ Since $\mathcal{F}_n \subset \mathcal{F}_{n+1}$, we have

$$\mathbb{E}[\xi_{n+1} \mid \mathcal{F}_n] = \mathbb{E}\Big[\mathbb{E}[\xi \mid \mathcal{F}_{n+1}] \mid \mathcal{F}_n\Big] = \mathbb{E}[\xi \mid \mathcal{F}_n] = \xi_n.$$

Therefore, $\xi_1, \xi_2, \ldots$ is a martingale with respect to $\mathcal{F}_1, \mathcal{F}_2, \ldots$. □

$$\varphi(\mathrm{E}[X]) \leq \mathrm{E}[\varphi(X)].$$



Exercise 3.3

Show that if ξ_n is a martingale with respect to $\mathcal{F}_n$, then

$$E(\xi_1) = E(\xi_2) = \cdots.$$

Hint What is the expectation of $E(\xi_{n+1}|\mathcal{F}_n)$?

Solution 3.3

Taking the expectation on both sides of the equality

$$\xi_n = E(\xi_{n+1}|\mathcal{F}_n),$$

we obtain

$$E(\xi_n) = E(E(\xi_{n+1}|\mathcal{F}_n)) = E(\xi_{n+1})$$

for each n. This proves the claim.

Exercise 3.4

Suppose that ξ_n is a martingale with respect to a filtration $\mathcal{F}_n$. Show that ξ_n is a martingale with respect to the filtration

$$\mathcal{G}_n = \sigma(\xi_1, \ldots, \xi_n).$$

Hint Observe that $\mathcal{G}_n \subset \mathcal{F}_n$ and use the tower property of conditional expectation.

Solution 3.4

The random variables 1. ξ_n are integrable because ξ_n is a martingale with respect to $\mathcal{F}_n$. Since $\mathcal{G}_n$ is the σ-field generated by $\xi_1, \ldots, \xi_n$, it follows that 2. ξ_n is adapted to $\mathcal{G}_n$. Finally, since $\mathcal{G}_n \subset \mathcal{F}_n$,

$$\begin{aligned} \xi_n &= E(\xi_n|\mathcal{G}_n) \\ &= E(E(\xi_{n+1}|\mathcal{F}_n)|\mathcal{G}_n) \\ 3.\ &= E(\xi_{n+1}|\mathcal{G}_n) \end{aligned}$$

by the tower property of conditional expectation (Proposition 2.4). This proves that ξ_n is a martingale with respect to $\mathcal{G}_n$.

Exercise 3.5

Let ξ_n be a symmetric random walk, that is,

$$\xi_n = \eta_1 + \cdots + \eta_n,$$

i.i.d

where $\eta_1, \eta_2, \ldots$ is a sequence of independent identically distributed random variables such that

$$P\{\eta_n = 1\} = P\{\eta_n = -1\} = \frac{1}{2}$$

(a sequence of coin tosses, for example). Show that $\xi_n^2 - n$ is a martingale with respect to the filtration

$$\mathcal{F}_n = \sigma(\eta_1, \ldots, \eta_n).$$

Solution 3.5

Because

$$\xi_n^2 - n = (\eta_1 + \cdots + \eta_n)^2 - n$$

is a function of $\eta_1, \ldots, \eta_n$, it is measurable with respect to the σ-field $\mathcal{F}_n$ generated by $\eta_1, \ldots, \eta_n$, i.e. $\xi_n^2 - n$ is adapted to $\mathcal{F}_n$. Since

$$|\xi_n| = |\eta_1 + \cdots + \eta_n| \le |\eta_1| + \cdots |\eta_n| = n,$$

it follows that

$$E(|\xi_n^2 - n|) \le E(\xi_n^2) + n \le n^2 + n < \infty,$$

so $\xi_n^2 - n$ is integrable for each n. Because

$$\xi_{n+1}^2 = \eta_{n+1}^2 + 2\eta_{n+1}\xi_n + \xi_n^2,$$

where ξ_n and ξ_n^2 are $\mathcal{F}_n$-measurable and η_{n+1} is independent of $\mathcal{F}_n$, we can use Proposition 2.4 ('taking out what is known' and 'independent condition drops out') to obtain

$$\begin{aligned} E(\xi_{n+1}^2|\mathcal{F}_n) &= E(\eta_{n+1}^2|\mathcal{F}_n) + 2E(\eta_{n+1}\xi_n|\mathcal{F}_n) + E(\xi_n^2|\mathcal{F}_n) \\ &= E(\eta_{n+1}^2) + \underbrace{2\xi_n E(\eta_{n+1})}_{=0} + \xi_n^2 \\ &= 1 + \xi_n^2. \end{aligned}$$

This implies that

$$E(\xi_{n+1}^2 - n - 1|\mathcal{F}_n) = \xi_n^2 - n,$$

so $\xi_n^2 - n$ is a martingale.

Exercise 3.6

Let ξ_n be a symmetric random walk and $\mathcal{F}_n$ the filtration defined in Exercise 3.5. Show that

$$\zeta_n = (-1)^n \cos(\pi \xi_n)$$

is a martingale with respect to $\mathcal{F}_n$.

Hint You want to transform $E((-1)^{n+1}\cos(\pi\xi_{n+1})|\mathcal{F}_n)$ to obtain $(-1)^n \cos(\pi\xi_n)$. Use a similar argument as in Exercise 3.5 to achieve this. But, first of all, make sure that ζ_n is integrable and adapted to $\mathcal{F}_n$.

#

4) $E(\xi|\mathcal{G}) = E(\xi)$ if ξ is independent of $\mathcal{G}$ (*an independent condition drops out*);

r.v. $\Leftrightarrow$ function $\Leftrightarrow$ $\cos(\pi\xi_n)$, $\sin(\pi\eta_{n+1})$

Exercise 3.6

As ξ_n is $\mathcal{F}_n$-measurable,

so $E[\cos(\pi\xi_n)|\mathcal{F}_n] = \cos(\pi\xi_n)$

As η_{n+1} is independent of $\mathcal{F}_n$.

so $E[\sin(\pi\eta_{n+1})|\mathcal{F}_n] = E[\sin(\pi\eta_{n+1})]$.

$\Rightarrow$ $E[f(\xi_n)|\mathcal{F}_n] = f(\xi_n)$.

$E[g(\eta_{n+1})|\mathcal{F}_n] = E[g(\eta_{n+1})]$.

*需清晰认识ξ是

Solution 3.6

Being a function of ξ_n, the random variable ζ_n is $\mathcal{F}_n$-measurable for each n, since ξ_n is. Because $|\zeta_n| \le 1$, it is clear that ζ_n is integrable. Because η_{n+1} is independent of $\mathcal{F}_n$ and ξ_n is $\mathcal{F}_n$-measurable, it follows that

$$\begin{aligned} E(\zeta_{n+1}|\mathcal{F}_n) &= E\left((-1)^{n+1}\cos[\pi(\xi_n + \eta_{n+1})]|\mathcal{F}_n\right) \\ &= (-1)^{n+1} E\left(\cos(\pi\xi_n)\cos(\pi\eta_{n+1})|\mathcal{F}_n\right) \\ &\quad -(-1)^{n+1} E\left(\sin(\pi\xi_n)\sin(\pi\eta_{n+1})|\mathcal{F}_n\right) \\ &= (-1)^{n+1}\cos(\pi\xi_n)\, E\left(\cos(\pi\eta_{n+1})\right) \\ &\quad -(-1)^{n+1}\sin(\pi\xi_n)\, E\left(\sin(\pi\eta_{n+1})\right) \\ &= (-1)^n \cos(\pi\xi_n) \\ &= \zeta_n, \end{aligned}$$

using the formula

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta.$$

To compute $E(\cos(\pi\eta_{n+1}))$ and $E(\sin(\pi\eta_{n+1}))$ observe that $\eta_{n+1} = 1$ or -1 and

$$\begin{aligned} \cos\pi &= \cos(-\pi) = -1, \\ \sin\pi &= \sin(-\pi) = 0. \end{aligned}$$

It follows that ζ_n is a martingale with respect to the filtration $\mathcal{F}_n$.

Definition 3.5

We say that $\xi_1, \xi_2, \ldots$ is a *supermartingale* (*submartingale*) with respect to a filtration $\mathcal{F}_1, \mathcal{F}_2, \ldots$ if

1) ξ_n is integrable for each $n = 1, 2, \ldots$;

2) $\xi_1, \xi_2, \ldots$ is adapted to $\mathcal{F}_1, \mathcal{F}_2, \ldots$;

3) $E(\xi_{n+1}|\mathcal{F}_n) \le \xi_n$ (respectively, $E(\xi_{n+1}|\mathcal{F}_n) \ge \xi_n$) a.s. for each $n = 1, 2, \ldots$.

Exercise 3.7

Let ξ_n be a sequence of square integrable random variables. Show that if ξ_n is a martingale with respect to a filtration $\mathcal{F}_n$, then ξ_n^2 is a submartingale with respect to the same filtration.

Hint Use Jensen's inequality with convex function $\varphi(x) = x^2$.

看到证不等关系
想几个 inequality

$2ab \le a^2 + b^2$

$|a+b| \le |a| + |b|$

For convex φ:

$\varphi[E(x)] \le E[\varphi(x)]$

$\Rightarrow$ Remark: 若ξ_n为MG, ξ_n^2不一定为MG.

Solution 3.7

If ξ_n is adapted to $\mathcal{F}_n$, then so is ξ_n^2. Since $\xi_n = E(\xi_{n+1}|\mathcal{F}_n)$ for each n and $\varphi(x) = x^2$ is a convex function, we can apply Jensen's inequality (Theorem 2.2) to obtain

$$\xi_n^2 = \left[E(\xi_{n+1}|\mathcal{F}_n)\right]^2 \le E\left(\xi_{n+1}^2|\mathcal{F}_n\right)$$

for each n. This means that ξ_n^2 is a submartingale with respect to $\mathcal{F}_n$.

If $\mathcal{F}_n = \sigma(\eta_1, \dots, \eta_n)$, ξ_n is a function of $\eta_1, \dots, \eta_n$

for example $\xi_n = \sum_{i=1}^{n} \eta_i +$ 常数 or $\xi_n = \prod_{i=1}^{n} \eta_i +$ 常数.

then ξ_n and Any function of ξ_n: $f(\xi_n)$ e.g. ξ_n^2

is $\mathcal{F}_n$-measurable. Because they are all functions of $\eta_1, \dots, \eta_n$.

- (A symmetric random walk) Let $\eta_1, \eta_2, \dots$ be a sequence of independent identically distributed random variables with $\mathbb{P}(\eta_1 = 1) = \frac{1}{2}$ and $\mathbb{P}(\eta_1 = -1) = \frac{1}{2}$. Define

$$\xi_n = \sum_{k=1}^{n} \eta_k = \eta_1 + \cdots + \eta_n, \quad n = 1, 2, \dots,$$

and denote $\mathcal{F}_n = \sigma\{\eta_1, \dots, \eta_n\}$ for each $n = 1, 2, \dots$. Then, $\xi_1, \xi_2, \dots$ is called a *symmetric random walk*. Example 3.3 implies that the symmetric random walk $\xi_1, \xi_2, \dots$ is a martingale with respect to the filtration $\mathcal{F}_1, \mathcal{F}_2, \dots$, and Exercise 3.5 shows that $\xi_1^2 - 1$, $\xi_2^2 - 2, \dots$ is also a martingale with respect to the filtration $\mathcal{F}_1, \mathcal{F}_2, \dots$,

3.4 Games of Chance

α_n: Stake in game n.

Suppose that you can vary the stake to be α_n, in game n. Here, α_n, may be zero if you refrain from playing the nth game; it may even be negative if you own the casino and can accept other people's bets.

When the time comes to decide your stake α_n, you will know the outcomes of the first $n-1$ games. Therefore it is reasonable to assume that α_n is $\mathcal{F}_{n-1}$-measurable, where $\mathcal{F}_{n-1}$ represents your knowledge accumulated up to and including game $n-1$. In particular, since nothing is known before the first game, we take $\mathcal{F}_0 = \{\emptyset, \Omega\}$.

玩家 — 玩 — 下注 $\alpha_n > 0$

玩家 — 不玩 — 不下注 $\alpha_n = 0$

庄家 — 收赌注 $\alpha_n < 0$.

Definition 3.6

A *gambling strategy* $\alpha_1, \alpha_2, \dots$ (with respect to a filtration $\mathcal{F}_1, \mathcal{F}_2, \dots$) is a sequence of random variables such that α_n is $\mathcal{F}_{n-1}$-measurable for each $n = 1, 2, \dots$, where $\mathcal{F}_0 = \{\emptyset, \Omega\}$. (Outside the context of gambling such a sequence of random variables α_n is called *previsible*.)

第 n-1 轮结束后已经可决定第 n 轮要下注多少了.

If you follow a strategy $\alpha_1, \alpha_2, \ldots$, then your *total winnings* after n games will be

$$\begin{cases} \zeta_n = \alpha_1 \eta_1 + \cdots + \alpha_n \eta_n \\ \quad = \alpha_1 (\xi_1 - \xi_0) + \cdots + \alpha_n (\xi_n - \xi_{n-1}). \\ \zeta_0 = 0 \end{cases}$$

ξ_0 = 初始本金

We also put $\zeta_0 = 0$ for convenience.

赌注

α_n is a bounded sequence.

The following proposition has important consequences for gamblers. It means that a fair game will always turn into a fair one, no matter which gambling strategy is used. If one is not in a position to wager negative sums of money (e.g. to run a casino), it will be impossible to turn an unfavourable game into a favourable one or vice versa. You cannot beat the system! The boundedness of the sequence α_n means that your available capital is bounded and so is your credit limit.

#在本金、时间都有限的前提下，玩家永不可能赢过庄家.

Proposition 3.1

$\boxed{\text{Let } \alpha_1, \alpha_2, \ldots \text{ be a gambling strategy.}}$

1) If $\alpha_1, \alpha_2, \ldots$ is a bounded sequence and $\xi_0, \xi_1, \xi_2, \ldots$ is a martingale, then $\zeta_0, \zeta_1, \zeta_2, \ldots$ is a martingale (a fair game turns into a fair one no matter what you do);
2) If $\alpha_1, \alpha_2, \ldots$ is a non-negative bounded sequence and $\xi_0, \xi_1, \xi_2, \ldots$ is a supermartingale, then $\zeta_0, \zeta_1, \zeta_2, \ldots$ is a supermartingale (an unfavourable game turns into an unfavourable one).
3) If $\alpha_1, \alpha_2, \ldots$ is a non-negative bounded sequence and $\xi_0, \xi_1, \xi_2, \ldots$ is a submartingale, then $\zeta_0, \zeta_1, \zeta_2, \ldots$ is a submartingale (a favourable game turns into a favourable one).

$\alpha_n - \mathcal{F}_{n-1}$ Predictable

$\xi_n - \mathcal{F}_n - MG$

$\Downarrow$

$\zeta_n = \sum_{i=1}^{n} \alpha_i (\xi_i - \xi_{i-1})$

$|$

$\mathcal{F}_n - MG.$

Proof

Because α_n and ζ_{n-1} are $\mathcal{F}_{n-1}$-measurable, we can take them out of the expectation conditioned on $\mathcal{F}_{n-1}$ ('taking out what is known', Proposition 2.4). Thus, we obtain

$$\begin{aligned} E(\zeta_n | \mathcal{F}_{n-1}) &= E(\zeta_{n-1} + \alpha_n (\xi_n - \xi_{n-1}) | \mathcal{F}_{n-1}) \\ &= \zeta_{n-1} + \alpha_n (E(\xi_n | \mathcal{F}_{n-1}) - \xi_{n-1}). \end{aligned}$$

If ξ_n is a martingale, then

$$\alpha_n (E(\xi_n | \mathcal{F}_{n-1}) - \xi_{n-1}) = 0,$$

which proves assertion 1). If ξ_n is a supermartingale and $\alpha_n \geq 0$, then

$$\alpha_n (E(\xi_n | \mathcal{F}_{n-1}) - \xi_{n-1}) \leq 0,$$

proving assertion 2). Finally, assertion 3) follows because

$$\alpha_n (E(\xi_n | \mathcal{F}_{n-1}) - \xi_{n-1}) \geq 0$$

if ξ_n is a submartingale and $\alpha_n \geq 0$. □

3.5 Stopping Times

to stop playing after 10 rounds, no matter what happens. But in general the decision whether to quit or not will be made after each round depending on the knowledge accumulated so far. Therefore τ is assumed to be a random variable with values in the set $\{1, 2, \ldots\} \cup \{\infty\}$. Infinity is included to cover the theoretical possibility (and a dream scenario of some casinos) that the game never stops. At each step n one should be able to decide whether to stop playing or not, i.e. whether or not $\tau = n$. Therefore the event that $\tau = n$ should be in the σ-field $\mathcal{F}_n$ representing our knowledge at time n. This gives rise to the following definition.

第n次已经完了,知道结果了 ⇒ 可判断是否收手.

Definition 3.7

is a function of ω: τ(ω)

而非{(-∞, ∞)}

A random variable τ with values in the set $\{1, 2, \ldots\} \cup \{\infty\}$ is called a *stopping time* (with respect to a filtration $\mathcal{F}_n$) if for each $n = 1, 2, \ldots$

The event of stopping after n^{th} round $= \{\tau = n\} \in \mathcal{F}_n$.

每一步

At each step n one is able to decide whether to stop playing or not.

Exercise 3.8

Show that the following conditions are equivalent:

1) $\{\tau \le n\} \in \mathcal{F}_n$ for each $n = 1, 2, \ldots$;
2) $\{\tau = n\} \in \mathcal{F}_n$ for each $n = 1, 2, \ldots$.

} Axioms

Hint Can you express $\{\tau \le n\}$ in terms of the events $\{\tau = k\}$, where $k = 1, \ldots, n$? Can you express $\{\tau = n\}$ in terms of the events $\{\tau \le k\}$, where $k = 1, \ldots, n$?

Solution 3.8

1)⇒2). If τ has property 1), then

$$\{\tau \le n\} \in \mathcal{F}_n$$

and

$$\{\tau \le n-1\} \in \mathcal{F}_{n-1} \subset \mathcal{F}_n,$$

so

$$\{\tau = n\} = \{\tau \le n\} \setminus \{\tau \le n-1\} \in \mathcal{F}_n.$$

2)⇒1). If τ has property 2), then

$$\{\tau = k\} \in \mathcal{F}_k \subset \mathcal{F}_n$$

for each $k = 1, \ldots, n$. Therefore

$$\{\tau \le n\} = \{\tau = 1\} \cup \cdots \cup \{\tau = n\} \in \mathcal{F}_n.$$

Example 3.5 (First hitting time)

Suppose that a coin is tossed repeatedly and you win or lose £1, depending on which way it lands. Suppose that you start the game with, say, £5 in your pocket and decide to play until you have £10 or you lose everything. If ξ_n is the amount you have at step n, then the time when you stop the game is

$$\tau = \min\{n : \xi_n = 10 \text{ or } 0\},$$

and is called the *first hitting time* (of 10 or 0 by the random sequence ξ_n). It is a stopping time in the sense of Definition 3.7 with respect to the filtration

$\mathcal{F}_n = \sigma(\xi_1, \ldots, \xi_n)$. This is because

$$\{\tau = n\} = \{0 < \xi_1 < 10\} \cap \cdots \cap \{0 < \xi_{n-1} < 10\} \cap \{\xi_n = 10 \text{ or } 0\}.$$

Now each of the sets on the right-hand side belongs to $\mathcal{F}_n$, so their intersection does too. This proves that

$$\{\tau = n\} \in \mathcal{F}_n$$

for each n, so τ is a stopping time.

Let ξ_n be a sequence of random variables adapted to a filtration $\mathcal{F}_n$ and let τ be a stopping time (with respect to the same filtration). Suppose that ξ_n represents your winnings (or losses) after n rounds of a game. If you decide to quit after τ rounds, then your total winnings will be ξ_τ. In this case your winnings after n rounds will in fact be $\xi_{\tau \wedge n}$. Here $a \wedge b$ denotes the smaller of two numbers a and b,

$$a \wedge b = \min(a, b).$$

$\tau(\omega)$ 为 random variable

(⇒ 可有τ会有自己的 field, 会 be measurable by certain fields)

Exercise 3.9

Let ξ_n be a sequence of random variables adapted to a filtration $\mathcal{F}_n$ and let $B \subset \mathbb{R}$ be a Borel set. Show that the *time of first entry* of ξ_n into B,

$$\tau = \min\{n : \xi_n \in B\}$$

is a stopping time.

Hint Example 3.5 covers the case when $B = (-\infty, 0] \cup [10, \infty)$. Extend the argument to an arbitrary Borel set B.

Solution 3.9

If

$$\tau = \min\{n : \xi_n \in B\},$$

then for any n

$$\{\tau = n\} = \{\xi_1 \notin B\} \cap \cdots \cap \{\xi_{n-1} \notin B\} \cap \{\xi_n \in B\}.$$

Because B is a Borel set, each of the sets on the right-hand side belongs to the σ-field $\mathcal{F}_n = \sigma(\xi_1, \ldots, \xi_n)$, and their intersection does too. This proves that $\{\tau = n\} \in \mathcal{F}_n$ for each n, so τ is a stopping time.

Definition 3.8. We call $\xi_{\tau\wedge 1}, \xi_{\tau\wedge 2}, \ldots$ the *sequence stopped* at τ. It is often denoted by $\xi_1^\tau, \xi_2^\tau, \ldots$. That is, for each $\omega \in \Omega$

$$\xi_n^\tau(\omega) = \xi_{\tau(\omega)\wedge n}(\omega) = \begin{cases} \xi_n(\omega), & \text{if } n < \tau(\omega). \\ \xi_{\tau(\omega)}(\omega), & \text{if } n \geq \tau(\omega). \end{cases}$$

Exercise 3.10

Show that if ξ_n is a sequence of random variables adapted to a filtration $\mathcal{F}_n$, then so is the sequence $\xi_{\tau\wedge n}$.

Hint For any Borel set B express $\{\xi_{\tau\wedge n} \in B\}$ in terms of the events $\{\xi_k \in B\}$ and $\{\tau = k\}$, where $k = 1, \ldots, n$.

Proof: Let $B \subset R$ be any Borel set,

as $\{\xi_{\tau\wedge n} \in B\} = (\{\xi_n \in B\} \cap \{n < \tau\}) \cup (\{\xi_\tau \in B\} \cap \{\tau \leq n\})$ #

$= (\{\xi_n \in B\} \cap \{n < \tau\}) \cup (\{\xi_\tau \in B\} \cap (\{\tau = n\} \cup \{\tau = n-1\} \cup \cdots \cup \{\tau = 1\}))$

$= (\{\xi_n \in B\} \cap \{n < \tau\}) \cup \left[\{\xi_\tau \in B\} \cap \left(\bigcup_{k=1}^{n} \{\tau = k\}\right)\right]$

$= (\{\xi_n \in B\} \cap \{n < \tau\}) \cup \bigcup_{k=1}^{n} \left[\{\xi_\tau \in B\} \cap \{\tau = k\}\right]$

and $\{\xi_n \in B\} \in \mathcal{F}_n$, $\{\tau \cancel{<} n\} \in \mathcal{F}_n$. $\{\tau = k\} \in \mathcal{F}_k \subseteq \mathcal{F}_n$ for $k = 1, 2, \ldots n$.

Thus, $\{\xi_{\tau\wedge n} \in B\} \in \mathcal{F}_n$. $\xi_{\tau\wedge n}$ is adapted to $\mathcal{F}_n$.

□.

Solution 3.10

Let $B \subset \mathbb{R}$ be a Borel set. We can write

$$\{\xi_{\tau\wedge n} \in B\} = \{\xi_n \in B, \tau > n\} \cup \bigcup_{k=1}^{n} \{\xi_k \in B, \tau = k\},$$

已知τ 故{τ>n}∈$\mathcal{F}_n$.

where

$$\{\xi_n \in B, \tau > n\} = \{\xi_n \in B\} \cap \{\tau > n\} \in \mathcal{F}_n$$

and for each $k = 1, \ldots, n$

$$\{\xi_k \in B, \tau = k\} = \{\xi_k \in B\} \cap \{\tau = k\} \in \mathcal{F}_k \subset \mathcal{F}_n.$$

It follows that for each n

$$\{\xi_{\tau\wedge n} \in B\} \in \mathcal{F}_n,$$

as required.

⇒ Remark: If ξ_n is $\mathcal{F}_n$-measurable then so is $\xi_{\tau(\omega)\wedge n}$

⇓ Extends from Measurability to Martingale.

Proposition 3.2

Let τ be a stopping time.

Stopping Time (essentially a Gambling Strategy)
不可致命.

1) If ξ_n is a martingale, then so is $\xi_{\tau\wedge n}$.
2) If ξ_n is a supermartingale, then so is $\xi_{\tau\wedge n}$.
3) If ξ_n is a submartingale, then so is $\xi_{\tau\wedge n}$.

Proof

重点: DON'T FORGET

Let $\alpha_n := \begin{cases} 1, & \text{if } n \leq \tau \\ 0, & \text{if } n > \tau \end{cases}$

证 α_n is $\mathcal{F}_{n-1}$-measurable.

then $\zeta_n = \alpha_1(\xi_1-\xi_0)+\cdots+\alpha_n(\xi_n-\xi_{n-1})$

τ已知

This is in fact a consequence of Proposition 3.1. Given a stopping time τ, we put

$$\alpha_n = \begin{cases} 1 & \text{if } \tau \geq n, \text{若} \leq \text{stopping time, 下注1} \\ 0 & \text{if } \tau < n. \text{ stopping time 之后不再下注} \end{cases}$$

We claim that α_n is a gambling strategy (that is, α_n is $\mathcal{F}_{n-1}$-measurable). This is because the inverse image $\{\alpha_n \in B\}$ of any Borel set $B \subset \mathbb{R}$ is equal to

$$\emptyset \in \mathcal{F}_{n-1}$$

if $0, 1 \notin B$, or to

$$\Omega \in \mathcal{F}_{n-1}$$

if $0, 1 \in B$, or to

$$\{\alpha_n = 1\} = \{\tau \geq n\} = \{\tau > n-1\} \in \mathcal{F}_{n-1}$$ as $\{\tau \leq n-1\} \in \mathcal{F}_{n-1}$

if $1 \in B$ and $0 \notin B$, or to

$$\{\alpha_n = 0\} = \{\tau < n\} = \{\tau \leq n-1\} \in \mathcal{F}_{n-1}$$

if $1 \notin B$ and $0 \in B$. For this gambling strategy

$$\boxed{\xi_{\tau\wedge n} = \alpha_1(\xi_1 - \xi_0) + \cdots + \alpha_n(\xi_n - \xi_{n-1}).} = \text{the winnings after the game ends}$$

因为使 $n>\tau$ 时的 $\alpha_n:=0$，故 $\xi_{\tau\wedge n}$ 可放心为 $(\tau\wedge n)=\min\{\tau,n\}$ 轮后正确的 winnings.

Therefore Proposition 3.1 implies assertions 1), 2) and 3) above. □

As ① α_n is a bounded Gambling Strategy.
② ξ_n is $\mathcal{F}_n$-MG
③ $\zeta_n = \alpha_1(\xi_1-\xi_0)+\cdots+\alpha_n(\xi_n-\xi_{n-1})$.

$\boxed{\text{If time \& capital are unlimited}: \xi_{\tau\wedge n} = \xi_\tau.}$

As $P(\tau<\infty)=1$ i.e. 一定会在有限次内投中 head.

而 $2^n > 2^n-1 = \frac{1-2^n}{1-2} = 1+2^1+2^2+\cdots+2^{n-1}$

故一定能赢(+1)

Example 3.6

$n \to \infty$

$E[\zeta_\tau] = E[1] = 1$ ✗

$E[\zeta_1] = 1 \times \frac{1}{2} - 1 \times \frac{1}{2} = 0$

(You could try to beat the system if you had unlimited capital and unlimited time.) The following gambling strategy is called 'the martingale'. (Do not confuse this with the general definition of a martingale earlier in this section.) Suppose a coin is flipped repeatedly. Let us denote the outcomes by $\eta_1, \eta_2, \ldots$, which can take values $+1$ (heads) or -1 (tails). You wager £1 on heads. If you win, you quit. If you lose, you double the stake and play again. If you win this time round, you quit. Otherwise you double the stake once more, and so on. Thus, your gambling strategy is

$$\alpha_n = \begin{cases} 2^{n-1} & \text{if } \eta_1 = \cdots = \eta_{n-1} = \text{tails}, \\ 0 & \text{otherwise}. \end{cases}$$

Let us put

$$\zeta_n = \eta_1 + 2\eta_2 + \cdots + 2^{n-1}\eta_n$$

and consider the stopping time

$$\tau = \min\{n : \eta_n = \text{heads}\}. \quad (*)$$

Remark ① ②.

Then $\zeta_{\tau \wedge n}$ will be your winnings after n rounds. It is a martingale (check it!). ① It can be shown that $\boxed{P\{\tau < \infty\} = 1}$ (heads will eventually appear in the sequence $\eta_1, \eta_2, \ldots$ with probability one). Therefore it makes sense to consider ζ_τ. This would be your total winnings if you were able to continue to play the game no matter how long it takes for the first heads to appear. It would require unlimited time and capital. If you could afford these, you would be bound to win eventually because ② $\boxed{\zeta_\tau = 1}$ identically, since

$$-1 - 2 - \cdots - 2^{n-1} + 2^n = 1$$

for any n.

赢 $2^n > 1 + 2 + \cdots + 2^{n-1}$ 输.

Exercise 3.11

Show that if a gambler plays 'the martingale', his expected loss just before the ultimate win is infinite, that is,

$n = 1, 2, \cdots, \infty$ + $\tau = n$

$$E(\zeta_{\tau-1}) = -\infty.$$

Hint What is the probability that the game will terminate at step n, i.e. that $\tau = n$? If $\tau = n$, what is $\zeta_{\tau-1}$ equal to? This will give you all possible values of $\zeta_{\tau-1}$ and their probabilities. Now compute the expectation of $\zeta_{\tau-1}$.

Proof: $P(\{\tau = n\}) = P(\{\eta_1 = \text{tail}\} \cap \cdots \cap \{\eta_{n-1} = \text{tail}\} \cap \{\eta_n = \text{head}\})$

$$= \left(\frac{1}{2}\right)^{(n-1)} \cdot \frac{1}{2}$$

$$= \frac{1}{2^n}$$

When $\tau = n$, $\zeta_{\tau-1} = -2^0 - 2^1 - 2^2 - \cdots - \cancel{2^{n-2}} = -\frac{1-2^{n-1}}{1-2} = 1 - 2^{n-1}$

$E(\zeta_{\tau-1}) = \zeta_{\tau-1} \cdot \cancel{P}\{\tau = \cancel{n-1}\} = (1 - 2^{n-1}) \cdot \frac{1}{2^{n-1}}$

若求 $P\{\tau = n-1\}$ 则在第 $n-1$ 次停,违背题意.

When $\tau = n$, n ranges from 1 to infinity,

$$E(\zeta_{\tau-1}) = \sum_{n=1}^{\infty} \zeta_{n-1} P\{\tau = n\}$$

$\tau = n \to \zeta_{n-1}$ 事件即事件 $\tau = n$.

$$= \sum_{n=1}^{\infty} (1 - 2^{n-1}) \cdot \frac{1}{2^n}$$

$$= \sum_{n=1}^{\infty} \left(\frac{1}{2^n} - \frac{1}{2}\right) = 0 - \infty \cdot \frac{1}{2} = 0 - \infty$$

$$= -\infty.$$

Solution 3.11

The probability that 'the martingale' terminates at step n is

$$P\{\tau = n\} = \frac{1}{2^n}$$

($n - 1$ tails followed by heads at step n). Therefore

$$\begin{aligned} E(\zeta_{\tau-1}) &= \sum_{n=1}^{\infty} \zeta_{n-1} P\{\tau = n\} \\ &= \sum_{n=1}^{\infty} (-1 - 2 - \cdots - \underset{(n-1)\text{th time}}{2^{n-2}}) \frac{1}{2^n} \\ &= -\sum_{n=1}^{\infty} \frac{2^{n-1} - 1}{2^n} = -\infty. \end{aligned}$$

3.6 Optional Stopping Theorem

If ξ_n is a martingale, then, in particular,

$$E(\xi_n) = E(\xi_1)$$

for each n. Example 3.6 shows that $E(\xi_\tau)$ is not necessarily equal to $E(\xi_1)$ for a stopping time τ. However, if the equality

$$E(\xi_\tau) = E(\xi_1)$$

does hold, it can be very useful. The Optional Stopping Theorem provides sufficient conditions for this to happen.

In Example 3.6, we have known that the stochastic sequence $\zeta_{\tau \wedge 1}, \zeta_{\tau \wedge 2}, \ldots$ is a martingale with respect to a filtration $\mathcal{F}_1, \mathcal{F}_2, \ldots$. Hence, we have

$$\boxed{\mathbb{E}[zeta_{\tau \wedge n}] = \mathbb{E}[\zeta_{\tau \wedge 1}]} = \mathbb{E}[\zeta_1] = \mathbb{E}[\eta_1] = 0$$

for each n. But, from the discussion in Example 3.6 we have known that $\zeta_\tau \equiv 1$, and so that $\mathbb{E}[\zeta_\tau] = 1$. Hence, we have

$$1 = \mathbb{E}[\zeta_\tau] \neq \lim_{n \to \infty} \mathbb{E}[\zeta_{\tau \wedge n}] = \mathbb{E}[\zeta_1] = \mathbb{E}[\eta_1] = 0.$$

However, if the equality $\mathbb{E}[\xi_\tau] = \mathbb{E}[\xi_1]$ does hold, it can be very useful. The following theorem provides a sufficient condition for this to happen.

Theorem 3.1 (Optional Stopping Theorem)

Let ξ_n be a martingale and τ a stopping time with respect to a filtration $\mathcal{F}_n$ such that the following conditions hold:

1) $\tau < \infty$ a.s., 只能玩有限次

2) ξ_τ is integrable, 预期累积资本有限

3) $E(\xi_n 1_{\{\tau>n\}}) \to 0$ as $n \to \infty$. 当$n\to\infty$, $\tau\le n$.

Then

$$E(\xi_\tau) = E(\xi_1).$$

应用Stopping Time策略的预期最终财富与预期第一轮结束财富一样

Proof

Because

$$\xi_\tau = \xi_{\tau\wedge n} + (\xi_\tau - \xi_n)\, 1_{\{\tau>n\}}, = \begin{cases} \xi_n + \xi_\tau - \xi_n = \xi_\tau, & \tau > n \\ \xi_\tau, & \tau \le n \end{cases}$$

it follows that

$$E(\xi_\tau) = E(\xi_{\tau\wedge n}) + E(\xi_\tau 1_{\{\tau>n\}}) - E(\xi_n 1_{\{\tau>n\}}).$$

Since $\xi_{\tau\wedge n}$ is a martingale by Proposition 3.2, the first term on the right-hand side is equal to

$$E(\xi_{\tau\wedge n}) = E(\xi_1).$$

The last term tends to zero by assumption 3). The middle term

$$\boxed{E(\xi_\tau 1_{\{\tau>n\}}) = \sum_{k=n+1}^{\infty} E(\xi_k 1_{\{\tau=k\}})}$$

tends to zero as $n \to \infty$ because the series

$$E(\xi_\tau) = \sum_{k=1}^{\infty} E(\xi_k 1_{\{\tau=k\}})$$

is convergent by 2). It follows that $E(\xi_\tau) = E(\xi_1)$, as required. □

Proof: Because $\xi_\tau = \xi_{\tau\wedge n} + (\xi_\tau - \xi_n)\cdot(1_{\{\tau>n\}})$

so $E(\xi_\tau) = E(\xi_{\tau\wedge n}) + E(\xi_\tau \cdot 1_{\{\tau>n\}}) - E[\xi_n \cdot 1_{\{\tau>n\}}]$

As $\xi_{\tau\wedge n}$ is a martingale so $\xi_{\tau\wedge n} = \xi_{\tau\wedge 1} = \xi_1$ as $1 \le \tau$.

By (3), $E[\xi_n 1_{\{\tau>n\}}] \to 0$ when $n \to \infty$;

From ξ_τ is integrable, i.e. $E[|\xi_\tau|] < \infty$.

$\Rightarrow E[\xi_\tau] = \sum_{k=1}^{\infty} E[\xi_k 1_{\{\tau=k\}}] < \infty$

infers when $n \to \infty$, $\sum_{k=n+1}^{\infty} E[\xi_k 1_{\{\tau=k\}}] \to 0$

as $E[\xi_\tau 1_{\{\tau>n\}}] = \sum_{k=n+1}^{\infty} E[\xi_k 1_{\{\tau=k\}}]$

so $E[\xi_\tau 1_{\{\tau>n\}}] \to 0$ when $n \to \infty$.

Thus, $E(\xi_\tau) = E(\xi_1)$ □.

Example 3.7 (Expectation of the first hitting time for a random walk)

Let ξ_n be a symmetric random walk as in Exercise 3.5 and let K be a positive integer. We define the first hitting time (of $\pm K$ by ξ_n) to be

$$\boxed{\tau = \min\{n : |\xi_n| = K\}.}$$

By Exercise 3.9 τ is a stopping time. By Exercise 3.5 we know that $\xi_n^2 - n$ is a martingale. If the Optional Stopping Theorem can be applied, then

$$E(\xi_\tau^2 - \tau) = E(\xi_1^2 - 1) = 0.$$

This allows us to find the expectation

$$E(\tau) = E(\xi_\tau^2) = K^2,$$

since $|\xi_\tau| = K$.

Let us verify conditions 1)–3) of the Optional Stopping Theorem.

1) We shall show that $P\{\tau = \infty\} = 0$. To this end we shall estimate $P\{\tau > 2Kn\}$. We can think of $2Kn$ tosses of a coin as n sequences of $2K$ tosses. A necessary condition for $\tau > 2Kn$ is that no one of these n sequences contains heads only. Therefore

$$P\{\tau > 2Kn\} \le \left(1 - \frac{1}{2^{2K}}\right)^n \to 0$$

as $n \to \infty$. Because $\{\tau > 2Kn\}$ for $n = 1, 2, \ldots$ is a contracting sequence of sets (i.e. $\{\tau > 2Kn\} \supset \{\tau > 2K(n+1)\}$), it follows that

$$\begin{aligned} P\{\tau = \infty\} &= P\left(\bigcap_{n=1}^{\infty} \{\tau > 2Kn\}\right) \\ &= \lim_{n\to\infty} P\{\tau > 2Kn\} = 0, \end{aligned}$$

completing the argument.

2) We need to show that

$$E(|\xi_\tau^2 - \tau|) < \infty.$$

Indeed,

$$\begin{aligned} E(\tau) &= \sum_{n=1}^{\infty} nP\{\tau = n\} \\ &= \sum_{n=0}^{\infty}\sum_{k=1}^{2K} (2Kn + k)\, P\{\tau = 2Kn + k\} \\ &\le \sum_{n=0}^{\infty}\sum_{k=1}^{2K} 2K(n+1)\, P\{\tau > 2Kn\} \\ &\le 4K^2 \sum_{n=0}^{\infty} (n+1)\left(1 - \frac{1}{2^{2K}}\right)^n \\ &< \infty, \end{aligned}$$

since the series $\sum_{n=1}^{\infty} (n+1)\, q^n$ is convergent for any $q \in (-1, 1)$. Here we have recycled the estimate for $P\{\tau > 2Kn\}$ used in 2). Moreover, $\xi_\tau^2 = K^2$, so

$$\begin{aligned} E(|\xi_\tau^2 - \tau|) &\le E(\xi_\tau^2) + E(\tau) \\ &= K^2 + E(\tau) \\ &< \infty. \end{aligned}$$

3) Since $\xi_n^2 \le K^2$ on $\{\tau > n\}$,

$$E(\xi_n^2 1_{\{\tau>n\}}) \le K^2 P\{\tau > n\} \to 0$$

as $n \to \infty$. Moreover,

$$E(n 1_{\{\tau>n\}}) \le E(\tau 1_{\{\tau>n\}}) \to 0$$

as $n \to \infty$. Convergence to 0 holds because $E(\tau) < \infty$ by 2) and $\{\tau > n\}$ is a contracting sequence of sets with intersection $\{\tau = \infty\}$ of measure zero. It follows that

$$E((\xi_n^2 - n)\, 1_{\{\tau>n\}}) \to 0,$$

as required.

τ的criteria已知，则ξ_τ的取值，$n<\tau$时ξ_n的取范可知.

1) WTS: $\tau < \infty$ i.e. $P(\tau = \infty) = 0$

Proof: We shall know $P(\tau=\infty)=0$, to this end we shall estimate $P\{\tau > 2kn\}$. We can think of $2kn$ tosses of a coin as n sequences of $2k$ tosses.

$$n\text{个}\left\{\begin{matrix} 00\cdots\cdots 00 \\ 00\cdots\cdots 00 \\ \vdots \\ \underbrace{00\cdots\cdots 00}_{2k\text{个}} \end{matrix}\right. \qquad \begin{aligned} &P(\tau > 2k\times 1) = 1 - P(\text{heads only}) = 1-\left(\tfrac{1}{2}\right)^{2k} \\ &P(\tau > 2k\times 2) = 1 - P(\text{heads only}) = 1-\left(\tfrac{1}{2}\right)^{2k} \\ &\quad\vdots \\ &P(\tau > 2kn) \le \left(1-\frac{1}{2^{2k}}\right)^n \to 0 \text{ as } n\to\infty. \end{aligned}$$

Because $\{\tau > 2kn\}$ for $n=1,2,\cdots$ is a contracting sequence of sets (i.e. $\{\tau>2kn\} \supset \{\tau > 2k(n+1)\}$), so

$$P(\{\tau=\infty\}) = P\left(\bigcap_{n=1}^{\infty}\{\tau > 2kn\}\right)$$

$$= \lim_{n\to\infty} P\{\tau > 2kn\} = \lim_{n\to\infty}\left(1-\frac{1}{2^{2k}}\right)^n = 0$$

so $P(\{\tau<\infty\}) = 1$ i.e. $\tau < \infty$. □

2) WTS: $\varepsilon_\tau^2 - \tau$ is integrable. i.e. $E[|\varepsilon_\tau^2 - \tau|] < \infty$

Since
$$E[\tau] = \sum_{n=1}^{\infty} nP(\tau=n) = \sum_{n=1}^{\infty}\sum_{k=1}^{2k}(2kn+k)P(\tau=2kn+k)$$
$$\le \sum_{n=1}^{\infty}\sum_{k=1}^{2k}(2kn+2k)P(\tau=2kn+k)$$
$$\le 4k^2\sum_{n=1}^{\infty}(n+1)P(\tau>2kn)$$
$$= 4k^2\sum_{n=1}^{\infty}(n+1)\left[1-\frac{1}{2^{2k}}\right]^n < \infty$$

as $1-\frac{1}{2^{2k}} \in (-1,1)$, and $\sum_{n=1}^{\infty}(n+1)q^n$ is convergent if $q\in(-1,1)$.

As stop when $|\varepsilon_\tau| = k$ then $\varepsilon_\tau^2 = k^2$.

so $E[|\varepsilon_\tau^2 - \tau|] \le E[\varepsilon_\tau^2] + E[\tau] = k^2 + E(\tau) < \infty$. □

3) WTS: $E[(\varepsilon_n^2 - n)1_{\{\tau>n\}}] \to 0$ as $n\to\infty$.

Proof: Since when $\tau > n$, i.e. within n times, the criteria $|\varepsilon_n| = k$ hasn't been met, so $\varepsilon_n^2 \le k^2$ on $\{\tau > n\}$.

$\Rightarrow E[\varepsilon_n^2 1_{\{\tau>n\}}] \le k^2 E[1_{\{\tau>n\}}] = k^2 P(\tau>n) \to 0$ as $n\to\infty$.

and $E[n1_{\{\tau>n\}}] \le E[\tau 1_{\{\tau>n\}}] \to 0$ when $n\to\infty$ as $E[\tau]<\infty$.

and $\{\tau>n\}$ is a contracting sequence of sets with intersection $\{\tau=\infty\}$ of measure zero.

It follows that $E[(\varepsilon_n^2 - n)1_{\{\tau>n\}}] \to 0$ as $n\to\infty$. □

□ ε_n, ε_τ, $\varepsilon_{\tau\wedge n}$ 是三个不同的 Sequence

- 已知 If ε_n is F_n-MG then $E[\varepsilon_n] = E[\varepsilon_1]$.
  - $\varepsilon_{\tau\wedge n}$ 被证为 F_n-MG，故 $E[\varepsilon_{\tau\wedge n}] = E[\varepsilon_{\tau\wedge 1}] = E[\varepsilon_1]$
- 由 Optional Stopping Theorem.

  $\varepsilon_\tau = \varepsilon_{\tau\wedge n} + (\varepsilon_\tau - \varepsilon_n)1_{\{\tau>n\}}$ 在 ① $\tau<\infty$. ② ε_τ integrable

③ $E[\varepsilon_n 1_{\{\tau>n\}}] \to 0$ when $n\to\infty$ 的情况下可充分推出

$E[\varepsilon_\tau] = E[\varepsilon_1]$.