5
Markov Chains

5.1.1 An Example of Markov Chain

Example 5.1. In some homes the use of the telephone can become quite a sensitive issue. Suppose:

- If the phone is free during some period of time, say the nth minute, then with probability p, where $0 < p < 1$, it will be busy during the next minute.
- If the phone has been busy during the nth minute, it will become free during the next minute with probability q, where $0 < q < 1$.

The situation described in Example 5.1 is quite typical. Often the probability of a certain event at time $n+1$ depends only on what happens at time n, but not further into the past. Example 5.1 provides us with a simple case of a Markov chain.

Assume that the phone is free in the 0th minute. We would like to answer the following two questions:

1) What is the probability x_n, that the telephone will be free in the nth minute? $\Rightarrow$ $P(\text{busy}) = P(B_n) = 1 - X_n$

   $P(\text{free}) = P(A_n) = X_n$
2) What is $\lim_{n\to\infty} x_n$, if it exists?

Denote by A_n the event that the phone is free during the nth minute and let $B_n = \Omega \setminus A_n$ be its complement, i.e. the event that the phone is busy during the nth minute, The conditions of the example give us

A_n: 在n时 free $\Rightarrow P(A_n) = X_n$

B_n: 在n时 busy $\Rightarrow P(B_n) = 1 - X_n$

$$\mathbb{P}(B_{n+1} \mid A_n) = p, \tag{1}$$
$$\mathbb{P}(A_{n+1} \mid B_n) = q. \tag{2}$$

We also assume that $\mathbb{P}(A_0) = 1$, i.e. $x_0 = 1$. Using this notation, we have $x_n = \mathbb{P}(A_n)$.

Then the total probability formula, see Exercise 1.10, together with (1) - (2) imply that

total = conditional · marginal

核心:
$$\begin{aligned} x_{n+1} &= \mathbb{P}(A_{n+1}) \\ &= \mathbb{P}(A_{n+1} \mid A_n)\mathbb{P}(A_n) + \mathbb{P}(A_{n+1} \mid B_n)\mathbb{P}(B_n) \\ &= (1-p)x_n + q(1-x_n) = q + (1-p-q)x_n. \end{aligned} \tag{3}$$

$\Rightarrow$ 只与上一步的状态有关

It's a bit tricky to find an explicit formula for x_n. To do so we suppose first that the sequence $\{x_n\}$ is convergent, i.e.

$$\lim_{n\to\infty} x_n = x. \quad \text{to be proved later} \tag{4}$$

The element properties of limits and equation (3), i.e.

$$x_{n+1} = q + (1-p-q)x_n,$$

yield

$$x = q + (1-p-q)x. \quad \text{when } n\to\infty \tag{5}$$

The unique solution to the last equation is

$$x = \frac{q}{q+p}. \tag{6}$$

In particular, from (5) and (6) we have

$$\frac{q}{q+p} = q + (1-q-p)\frac{q}{q+p}. \tag{7}$$

Subtracting (7) from (3), we infer that

$$(3)-(7) = \quad x_{n+1} - \frac{q}{q+p} = (1-q-p)\Big(x_n - \frac{q}{q+p}\Big). \tag{8}$$

Thus, $\{x_n - \frac{q}{q+p}\}$ is a geometric sequence and therefore, for all $n \in \mathbb{N}$,

$$x_n - \frac{q}{q+p} = (1-q-p)^n\Big(x_0 - \frac{q}{q+p}\Big).$$

Hence, by taking into account the initial condition $x_0 = 1$, we have

$$x_n = \frac{q}{q+p} + (1-q-p)^n \frac{p}{q+p}. \tag{9}$$

Let us point out that although we have used the assumption (4) to derive (8), the proof of the latter is now complete.

Indeed, having proven (9), we can show that the assumption (4) is indeed satisfied. This is because the conditions $0 < p, q < 1$ imply that $|1-p-q| < 1$, and so

$\Rightarrow 0 < p+q < 2$

$\Rightarrow -1 < 1-(p+q) < 1$

$$(1-p-q)^n \to 0 \quad \text{as} \quad n \to \infty. \Longrightarrow \{Z_n\} \text{ converges as } Z_n = \alpha Z_{n-1}, |\alpha| < 1.$$

Thus, (4) holds. This provides an answer to the second part of the example, i.e.

$$\lim_{n\to\infty} x_n = \frac{q}{p+q}. \quad \square$$

与之前假设{x_n}收敛时
算出的limit值相同

Exercise 5.1

In the framework of Example 5.1, let y_n denote the probability that the telephone is busy in the nth minute. Supposing that $y_0 = 1$, find an explicit formula for y_n and, if it exists, $\lim_{n\to\infty} y_n$.

Hint This exercise can be solved directly by repeating the above argument, or indirectly by using some of the results in Example 5.1.

Remark 5.1. The formulae (3) and (10) can be written collectively in a compact form by using vector and matrix notation. First of all, since $x_n + y_n = 1$, we get

$$\begin{aligned} \text{free} \qquad x_{n+1} &= (1-p)x_n + qy_n, \\ \text{busy} \qquad y_{n+1} &= px_n + (1-q)y_n. \end{aligned}$$

Hence, the matrix version takes the form:

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} 1-p & q \\ p & 1-q \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix}.$$

5.1.2 Definitions of Markov Chain

Definition 5.1. Suppose that $\mathcal{S}$ is a finite or a countable set. Suppose also that a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is given. An $\mathcal{S}$-valued sequence of random variables $\xi_0, \xi_1, \xi_2, \ldots$ is called an $\mathcal{S}$-valued *Markov chain* or a *Markov chain* on $\mathcal{S}$ if for all $n = 0, 1, 2, \ldots$ and all $s \in \mathcal{S}$

$$\mathbb{P}(\xi_{n+1} = s \mid \xi_0, \xi_1, \ldots, \xi_n) = \mathbb{P}(\xi_{n+1} = s \mid \xi_n). \qquad (11)$$

Property (11) will usually be referred to as the *Markov property* of the Markov chain $\xi_0, \xi_1 \xi_2, \ldots$. The set $\mathcal{S}$ is called the *state space* and the elements of $\mathcal{S}$ are called *states*.

Proposition 5.1

The model in Example 5.1 and Exercise 5.1 is a Markov chain.

Proof: Let $S = \{0, 1\}$. where 0 and 1 represent the state of telephone being "free" or "busy".

Construct a probability space:

Let Ω be the set of all sequences $w = \{w_0, w_1 \cdots\}$ with values in S. Let μ_0 be any probability measure on S. For examples $\mu_0 = \delta_0$ corresponds to the case when telephone is free at time 0.

Define P by induction:

For any S-valued sequence $s_1, s_2 \cdots$ we put

Define $\xi_n(w) = w_n$, $w = \{w_0, w_1 \cdots\} \in \Omega$.

Show that the transition probabilities of ξ_n are what they should be:

$$P(\xi_{n+1} = 1 \mid \xi_n = 0) = p \quad (1);$$
$$P(\xi_{n+1} = 0 \mid \xi_n = 1) = q \quad (2)$$

By the def. of conditional probability:

$$P(\xi_{n+1} = 1 \mid \xi_n = 0) = \frac{P[\xi_{n+1} = 1, \xi_n = 0]}{P(\xi_n = 0)}$$

$P(\{w \in \Omega: w_0 = s_0\}) = \mu_0(\{s_0\})$ and

$P(\{w \in \Omega: w_i = s_i, i = 0, \dots, n+1\})$ $\boxed{\text{def. of } P}$

$= p(s_{n+1} \mid s_n) P(\{w \in \Omega: w_i = s_i, i = 0, \dots, n\})$ $(*)$

where $p(s \mid r)$ are entries of the 2×2 matrix.

$$\begin{bmatrix} p(0|0) & p(0|1) \\ p(1|0) & p(1|1) \end{bmatrix} = \begin{bmatrix} 1-p & q \\ p & 1-q \end{bmatrix}.$$

$\Rightarrow$ P is a probability measure respect to the trival σ-field of all subsets of Ω.

By $(*)$ def. of P:

$$P(\xi_{n+1} = 1, \xi_n = 0) = P(\{w: w_n = 0, w_{n+1} = 1\})$$

$s_0, \dots, s_{n-1}$ 每个都可能取每个值. 将每一种"可能路径"加在一起.

$$= \sum_{s_0, \dots, s_{n-1} \in S} P(\{w: w_0 = s_0, \dots, w_{n-1} = s_{n-1}, w_n = 0, w_{n+1} = 1\})$$

随便什么 $s_0, \dots, s_{n-1} \in S$

$$= \sum_{s_0, \dots, s_{n-1} \in S} p \cdot P(\{w: w_0 = s_0, \dots, w_{n-1} = s_{n-1}, w_n = 0\})$$

$$= p \cdot P(\xi_n = 0)$$

$\Rightarrow$ $P(\xi_{n+1} = 1 \mid \xi_n = 0) = p$. (1) is proved.

Similarly, $P(\xi_{n+1} = 0 \mid \xi_n = 1) = q$.

To verify that for any $n = 1, 2, \cdots$ and any $s_0, s_1, \cdots, s_{n+1} \in S$,

$$P(\xi_{n+1} = s_{n+1} \mid \xi_0 = s_0, \dots, \xi_n = s_n) = P(\xi_{n+1} = s_{n+1} \mid \xi_n = s_n)$$

i.e. Markov Property:

Firstly, by using def. of P $(*)$:

$P(\xi_0 = s_0, \dots, \xi_n = s_n, \xi_{n+1} = s_{n+1})$

$= P(\{w: w_0 = s_0, \dots, w_n = s_n, w_{n+1} = s_{n+1}\})$

$= p(s_{n+1} \mid s_n) P(\{w: w_0 = s_0, \dots, w_n = s_n\})$

$= p(s_{n+1} \mid s_n) P(\xi_0 = s_0, \dots, \xi_n = s_n)$ as $\xi_n(w) = w_n$

$$\Rightarrow \frac{P(\xi_0 = s_0, \dots, \xi_n = s_n, \xi_{n+1} = s_{n+1})}{P(\xi_0 = s_0, \dots, \xi_n = s_n)} = p(s_{n+1} \mid s_n)$$

as $P[\xi_{n+1} = s_{n+1} \mid \xi_0 = s_0, \dots, \xi_n = s_n] = \dfrac{P(\xi_0 = s_0, \dots, \xi_n = s_n, \xi_{n+1} = s_{n+1})}{P(\xi_0 = s_0, \dots, \xi_n = s_n)}$

so $P[\xi_{n+1} = s_{n+1} \mid \xi_0 = s_0, \dots, \xi_n = s_n] = p(s_{n+1} \mid s_n)$

On the other hand, by the generalization of

$P(\xi_{n+1} = 1 \mid \xi_n = 0) = p$ and $P(\xi_{n+1} = 0 \mid \xi_n = 1) = q$,

$P(\xi_{n+1} = s_{n+1} \mid \xi_n = s_n) = p(s_{n+1} \mid s_n)$

Thus, $P(\xi_{n+1} = s \mid \xi_0, \xi_1, \dots, \xi_n) = P(\xi_{n+1} = s \mid \xi_n)$

so S-valued sequence of r.v. $\{\xi_n\}$ is a Markov Chain on S.

□.

"time homogeneous" Markov Chain:

Definition 5.2. An $\mathcal{S}$-valued Markov chain $\xi_0, \xi_1, \xi_2, \dots$ is called *time-homogeneous* or *homogeneous* if for all $n = 1, 2, \dots$ and all $i, j \in \mathcal{S}$

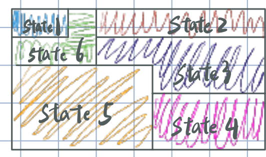
$$\mathbb{P}(\xi_{n+1} = j \mid \xi_n = i) = \mathbb{P}(\xi_1 = j \mid \xi_0 = i). \qquad (17)$$

The number $\mathbb{P}(\xi_1 = j \mid \xi_0 = i)$ is denoted by $p(j \mid i)$ and called the *transition probability* from state i to state j. The matrix $P = [p(j \mid i)]_{i,j \in \mathcal{S}}$ is called the *transition matrix* of the chain ξ_n.

5.1.3 Transition Matrices

$\forall s,t \in S$, $\{\xi_n = s\} \amalg \{\xi_n = t\}$ 互斥且ξ_n一定落于某个state.

Ω



ξ_n不可能既$=s$又$=t$. $(s \neq t)$

故 $P(\bigcup_n \{\xi_n = j\}) = \sum_n P(\{\xi_n = j\})$

or: η_i's are i.i.d r.v.

Exercise 5.2

In the discussion so far we have seen an example of a transition matrix, $P = \begin{bmatrix} 1-p & q \\ p & 1-q \end{bmatrix}$. Obviously the sum of the entries in each column of P is equal to 1. Prove that this is true in general.

Hint Remember that $P(\Omega|A) = 1$ for any event A.

Definition 5.3. A matrix $A = [a_{ji}]_{i,j \in \mathcal{S}}$ is called a *stochastic matrix* if

1) $a_{ji} \geq 0$ for all $i, j \in \mathcal{S}$;
2) the sum of the entries in each column is 1, i.e. $\sum_{j \in \mathcal{S}} a_{ji} = 1$ for any $i \in \mathcal{S}$.

A matrix $A = [a_{ji}]_{i,j \in \mathcal{S}}$ is called a *double stochastic matrix* if both A and its transpose A^t are stochastic matrices.

In the context of Markov chains, a **transition matrix** is a specific type of stochastic matrix that describes the transitions of a Markov chain [1] [2]. It is important to note that every stochastic matrix corresponds to a Markov chain for which it is the one-step transition matrix [3]. However, not every stochastic matrix is the two-step transition matrix of a Markov Chain [3].

In summary, a transition matrix is a type of stochastic matrix, and they are often used interchangeably in the context of Markov chains. However, not all stochastic matrices can serve as transition matrices for all types of Markov chains [3].

Proposition 5.2 of a doubly stochastic matrix

Show that a stochastic matrix is doubly stochastic if and only if the sum of the entries in each row is 1, i.e. $\sum_{i \in \mathcal{S}} a_{ji} = 1$ for any $j \in \mathcal{S}$.

Proof. Put $A^t = [b_{ij}]$. Then, by the definition of the transposed matrix, $b_{ij} = a_{ji}$. Therefore, A^t is a stochastic matrix if and only if

$$\sum_{i \in \mathcal{S}} a_{ji} = \sum_{i \in \mathcal{S}} b_{ij} = 1.$$

completing the proof. □

Remarks. (1) For any non-negative double sequence $\{a_{ij}\}_{i,j=1}^{\infty}$,

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij}.$$

(2) If $A = [a_{ji}]_{j,i \in \mathcal{S}}$ and $B = [b_{ji}]_{j,i \in \mathcal{S}}$ are two stochastic matrices. Then $C = BA$ is also a stochastic matrix, where $C = [c_{ji}]_{j,i \in \mathcal{S}}$ with $c_{ji} = \sum_k b_{jk} a_{ki}$. $= 1 \times \sum_k a_{ki} = \sum_k a_{ki} =$

Exercise 5.3

Show that if $P = [p_{ji}]_{j,i \in \mathcal{S}}$ is a stochastic matrix, then any natural power P^n of P is a stochastic matrix. Is the corresponding result true for a double stochastic matrix? Yes

Hint Show that if A and B are two stochastic matrices, then so is BA. For the second problem, recall that $(BA)^t = A^t B^t$.

If P is a (doubly) stochastic matrix then so is P^n. ⇒ Somehow related to the memoryless property?

$P(\xi_n = j \mid \xi_{n-1} = i, \xi_{n-2} \cdots \xi_0) = P(\xi_n = j \mid \xi_{n-1} = i)$

Exercise 5.4

Let $P = \begin{bmatrix} 1-p & q \\ p & 1-q \end{bmatrix}$. Show that

$$\mathbf{P \cdot P =} P^2 = \begin{bmatrix} 1+p^2-2p+pq & 2q-pq-q^2 \\ 2p-pq-p^2 & 1+q^2-2q+pq \end{bmatrix}.$$

Hint This is just simple matrix multiplication.

We see that there is a problem with finding higher powers of the matrix P. When multiplying P^2 by P, P^2, and so on, we obtain more and more complicated expressions.

Exercise 5.5

Find an exact formula for P_n for the matrix P from Exercise 5.4.

Hint Put $x_n = P(\xi_n = 0|\xi_0 = 0)$ and $y_n = P(\xi_n = 1|\xi_0 = 1)$. Is it correct to suppose that $p_n(0|0) = x_n$ and $p_n(1|1) = y_n$? If yes, you may be able use Example 5.1 and Exercise 5.1.

$$P_n = \begin{bmatrix} p_n(0|0) & p_n(0|1) \\ p_n(1|0) & p_n(1|1) \end{bmatrix}$$

而 当 $X_0=1, y_0=1$, 代入 $\frac{x_n}{y_n}$ 表达示得

$$x_n = \frac{q}{q+p} + (1-q-p)^n \frac{p}{q+p}.$$

$$y_n = \frac{p}{q+p} + (1-q-p)^n \frac{q}{p+q}.$$

Soln: Put $x_n = P(\xi_n=0|\xi_0=0)$ and $y_n = P(\xi_n=1|\xi_0=1)$

By example 5.1,

$$P(\xi_n=1|\xi_0=0) = 1-x_n = \frac{p}{p+q} - \frac{p}{p+q}(1-p-q)^n$$

$$P(\xi_n=0|\xi_0=1) = 1-y_n = \frac{q}{p+q} - \frac{q}{p+q}(1-p-q)^n$$

$$\Rightarrow \quad x_n = \frac{q}{p+q} + \frac{p}{p+q}(1-p-q)^n$$

$$y_n = \frac{p}{p+q} + \frac{q}{p+q}(1-p-q)^n$$

then derive the n-step transition matrix:

Spps. $p_n(j|i) = P(\xi_n=j|\xi_0=i)$

$$P_n = \begin{bmatrix} P(\xi_n=0|\xi_0=0) & P(\xi_n=0|\xi_0=1) \\ P(\xi_n=1|\xi_0=0) & P(\xi_n=1|\xi_0=1) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{q}{p+q} + \frac{p}{p+q}(1-p-q)^n & \frac{q}{p+q} - \frac{q}{p+q}(1-p-q)^n \\ \frac{p}{p+q} - \frac{p}{p+q}(1-p-q)^n & \frac{p}{p+q} + \frac{q}{p+q}(1-p-q)^n \end{bmatrix}$$

$\square$.

Exercise 5.6 Based on Ex. 5.5

You may suspect that P_n equals P^n, the nth power of the matrix P. This holds for $n = 1$. Check if it is true for $n = 2$. If this is the case, try to prove that $P_n = P^n$ for all $n \in \mathbb{N}$.

Hint Once again, this is an exercise in matrix multiplication.

Generalize

Definition 5.4. The n-step transition matrix of a Markov chain ξ_n, $n \in \mathbb{N}$, with n-step transition probabilities $p_n(j \mid i)$, $j, i \in \mathcal{S}$ is the matrix P_n with entries

$$p_n(j \mid i) = \mathbb{P}(\xi_n = j \mid \xi_0 = i) \tag{18}$$

Remark. In the following Chapman-Kolmogorov equation[2], we should show that, if P is the transition matrix of a Markov chain ξ_n, $n \in \mathbb{N}$, then for each $n \in \mathbb{N}$, the n-step transition matrix of ξ_n is given by

$$P_n = P^n.$$

Proposition 5.3 (Chapman-Kolmogorov equation)

Suppose that ξ_n, $n \in \mathbb{N}$, is an $\mathcal{S}$-valued Markov chain with n-step transition probabilities $p_n(j \mid i)$. Then for all $k, n \in \mathbb{N}$

$$\underset{(k+n)}{p_{n+k}}(j \mid i) = \sum_{s \in \mathcal{S}} p_n(j \mid s) p_k(s \mid i), \quad i, j \in \mathcal{S}. \tag{19}$$

Exercise 5.7

Prove Proposition 5.3.

Hint $p_{n+k}(j|i)$ are the entries of the matrix $P_{n+k} = P^{n+k}$.

Proof: Let P be the transition matrix,

P_n is the n-step transition matrix.

Since $p(j|i)$ are entries of P_n, we only need to show that $P_n = P^n$ for all $n \in N$. by induction.

The assertion is clearly true for $n=1$. i.e. $P_1 = P = P^1$.

Suppose that $P_n = P^n$, then $\forall i, j \in S$, by total probability formula and Markov Property,

$$P_{n+1}(j|i) = P(\xi_{n+1} = j \mid \xi_0 = i)$$

#

$$= \sum_{\forall s \in S} P(\xi_{n+1} = j \mid \xi_n = s, \xi_0 = i) P(\xi_n = s \mid \xi_0 = i)$$

$$= \sum_{s \in S} P(\xi_{n+1} = j \mid \xi_n = s) P(\xi_n = s \mid \xi_0 = i)$$

$$= \sum_{s \in S} p(j|s) P_n(s|i)$$

so $P_{n+1} = P \cdot P_n$, Similarly, $P_{n+k} = P_k \cdot P_n$ $\forall k, n \in N$.

□.

$$\begin{array}{c} k+n \\ \begin{bmatrix} a_{k+n} \\ \vdots \\ z_{k+n} \end{bmatrix} \end{array} = \underbrace{\overset{P}{[\ \ ]}\overset{P}{[\ \ ]}\cdots\overset{P}{[\ \ ]}}_{n\text{个}} \begin{array}{c} k \\ \begin{bmatrix} a_k \\ \vdots \\ z_k \end{bmatrix} \end{array}$$

而 $$\begin{bmatrix} a_k \\ \vdots \\ z_k \end{bmatrix} = \underbrace{\overset{P}{[\ \ ]}\cdots\overset{P}{[\ \ ]}}_{k\text{个}} \begin{bmatrix} a_0 \\ \vdots \\ z_0 \end{bmatrix}$$

故 $P_{n+k} = P^n \cdot P^k \cdot \begin{bmatrix} a_0 \\ \vdots \\ z_0 \end{bmatrix}$

$\Rightarrow$ $\forall s \in S$, fix s, then $i \to s \to j$ 路径

通过 $P_n(j|s) P_k(s|i)$

$\Rightarrow$ $\bigcup_s$ 所有路径:

$$p_{n+k}(j|i) = \sum_{s \in S} P_n(j|s) P_k(s|i)$$

5.1.4 Random Walks and Branching Processes

Example:

$S=\{0,\pm1,\pm2,\cdots\}$

Exercise 5.8. (Random Walk) Suppose that $\mathcal{S}=\mathbb{Z}$. Let $\eta_1,\eta_2,\dots$ be a sequence of independent identically distributed random variables with $\mathbb{P}(\eta_1=1)=p$ and $\mathbb{P}(\eta_1=-1)=q=1-p$, and let ξ_0 be $\mathbb{Z}$-valued random variable such that ξ_0 and $\eta_1,\eta_2,\dots$ are independent. Define

$$\xi_n=\xi_0+\sum_{k=1}^{n}\eta_k,\quad n=1,2,\dots.$$

Then, $\xi_0,\xi_1,\xi_2,\dots$ a Markov chain with transition probabilities

$$p(j\mid i)=\begin{cases}p, & \text{if } j=i+1,\\ q, & \text{if } j=i-1,\\ 0, & \text{otherwise.}\end{cases}$$

If $x_0=i$ is given, $\xi_1,\xi_2\dots$ is called a *random walk* starting at the state i. □

$\Rightarrow$ $\xi_{n+1}=\xi_n+\eta_{k+1}$

As η_{k+1} is random and $\eta_{k+1}\perp\!\!\!\perp\xi_n$ $\Rightarrow$ Markov Property.

So ξ_{n+1} only depends on ξ_n.

Def.

Replacing $\xi_0=0$ with $\xi_0=i$ in the random walk $\xi_1,\xi_2,\dots$ defined in Exercise 5.8, we get a random walk starting at $\xi_0=i$.

$$\xi_n=\xi_0+\sum_{k=1}^{n}\eta_k,\quad n=1,2,\dots.$$

η_k are i.i.d r.v. $\forall k\in\mathbb{N}$.

一维 Random Walk 的通式:

Exercise 5.9. Let $\xi_0,\xi_1,\xi_2,\dots$ be a random walk starting at $\xi_0=i$. Then

$$\boxed{\mathbb{P}(\xi_n=j\mid\xi_0=i)=\binom{n}{\frac{n+j-i}{2}}p^{\frac{n+j-i}{2}}q^{\frac{n-j+i}{2}},}\qquad(20)$$

for $n+j-i$ is an even non-negative integer, and

$$\mathbb{P}(\xi_n=j\mid\xi_0=i)=0,\quad\text{otherwise.}$$

Prove by induction.

一维 Random Walk 的重要特征:

$\xi_{n+1}=\begin{cases}\xi_n+1\\ \xi_n-1\end{cases}$ i.e. for $p(j|i)$, $\begin{cases}j=i+1\\ j=i-1\end{cases}$

Hint Use induction. Note that $\binom{n}{\frac{n+j-i}{2}}p^{\frac{n+j-i}{2}}$ equals 0 if $|j-i|\geq n+1$. i.e. $\frac{n+j-i}{2}>n$

$$p_n(j|i)=\begin{cases}P_{2k}(j|i)=\binom{2k}{k}(p)^{\frac{n+j-i}{2}}q^{\frac{n-j+i}{2}}\\ P_{2k+1}(j|i)=0 \quad \text{as } 2k+1 \text{ is odd.}\end{cases}$$

$\Downarrow$

Proposition 5.4

Let $\xi_0,\xi_1,\xi_2,\dots$ be a random walk starting at $\xi_0=i$. For all $p\in(0,1)$,

$$\mathbb{P}(\xi_n=i\mid\xi_0=i)\to0,\quad\text{as}\quad n\to\infty.\qquad(21)$$

symmetric: $p=q=\frac{1}{2}$ —— 超纲

assymmetric: $p\neq\frac{1}{2},q\neq\frac{1}{2}$. $p+q=1$

When $n=2k$

when $k \geq k+1$

i.e. 时间越久/transit 的次数越多，越难回到原点。

Proof:

As $0<p,q<1$ and by Ex. 5.9.

$$P(\xi_n=i \mid \xi_0=i) = \binom{n}{\frac{n+i-i}{2}} p^{\frac{n+i-i}{2}} q^{\frac{n-i+i}{2}}$$

$$= \binom{n}{\frac{n}{2}} p^{\frac{n}{2}} q^{\frac{n}{2}}$$

$$= \frac{n!}{(n-\frac{n}{2})!\,(\frac{n}{2})!} p^{\frac{n}{2}} q^{\frac{n}{2}} \quad (1)$$

$$= \frac{n\cdot(n-1)\cdots(\frac{n}{2}+1)}{(\frac{n}{2})!} (pq)^{\frac{n}{2}}$$

$$(1) \Rightarrow P(\xi_n=i \mid \xi_0=i) = \begin{cases} \frac{(2k)!}{(k!)^2}(pq)^k & \text{if } n=2k \\ 0, & \text{if } n \text{ is odd.} \end{cases}$$

因为是 $\infty \cdot 0$ 型，故具体是否 tends to 0 需证明。

Denoting $a_k = \frac{(2k)!}{(k!)^2}(pq)^k$ then

$$\frac{a_{k+1}}{a_k} = \frac{(2k+1)(2k+2)}{(k+1)^2} pq = \frac{4k^2+6k+2}{k^2+2k+1} pq$$

$$= \frac{4(k^2+2k+1)-2k-2}{k^2+2k+1} pq$$

$$= \left(4-\frac{2(k+1)}{(k+1)^2}\right) pq$$

$$= \left(4-\frac{2}{k+1}\right) pq \to 4pq \text{ as } k\to\infty.$$

Assume $p \neq \frac{1}{2}$ and as $p+q = p+(1-p) = 1$,

we have $4pq < 4\left(\frac{p+q}{2}\right)^2 = 4\times\frac{1}{4} = 1$

then $\frac{a_{k+1}}{a_k} < 1$ when $k\to\infty$. （若不设 $p\neq\frac{1}{2}$ 则 $\frac{a_{k+1}}{a_k}\leq 1$，$\frac{a_{k+1}}{a_k}=1$ 时不可判断敛散性）.

So $a_k \to 0$ as $k\to\infty$.

Thus, $P(\xi_{2k}=i \mid \xi_0=i) \to 0$ as $k\to\infty$.

the result follows since $P(\xi_{2k+1}=i \mid \xi_0=i) = 0 \to 0$. as $k\to\infty$.
$n=2k+1$ is odd.

(As $\frac{a_{k+1}}{a_k}<1$ so $\frac{a_{2k+2}}{a_{2k+1}}<1$ when $k\to\infty$. so $0<a_{2k+2}<a_{2k+1}=0$).

Proposition 5.5

Let $\xi_0, \xi_1, \xi_2, \ldots$ be a random walk starting at $\xi_0 = i$. The probability that the random walk $\xi_0, \xi_1, \xi_2, \ldots$ ever returns to the starting point is $1-|p-q|$, i.e.

$$\mathbb{P}(\exists n \text{ s.t. } \xi_n = i \mid \xi_0 = i) = 1-|p-q|.$$

$\Rightarrow$ symmetric random walk 的起始 state 一定是 recurrent 的。

证明很长，应该不考。

背

Exercise 5.10

Prove formula (5.24).
$$\sum_{k=0}^{\infty}\binom{2k}{k}x^k = (1-4x)^{-1/2}, \quad |x|<\frac{1}{4}. \qquad (24)$$

Hint Use the Taylor formula to expand the right-hand side of (5.24) into a power series.

Solution 5.10

Denote the right-hand side of (5.24) by $h(x)$. It follows by induction that

$$h^{(k)}(x) = \frac{(2k)!}{k!}(1-4x)^{-1/2-k}, \quad |x|<\frac{1}{4}.$$

On the other hand, h is analytic and

$$h(x) = \sum_{k=0}^{\infty}\frac{1}{k!}h^{(k)}(0)x^k, \quad |x|<\frac{1}{4}.$$

The most basic kind of branching processes, the Galton–Watson process[3] can be described as follows:

Exercise 5.11. On an island there lives an almost extinct species of plant. The males of this species can produce zero, one, two, ..., male offspring with probability $p_0, p_1, p_2, \ldots$ respectively, where $p_i \geq 0$ and

$$\sum_{i=0}^{\infty} p_i = 1.$$

虾睐活直至成年

A challenging problem would be to find chances of survival for this species assuming that each individual lives exactly one year. Rewrite the problem in the language of Markov chains. → 无需证明 Markov Property.

Solution. Denote by S the set of all non-negative integers $\{0, 1, 2, \ldots\}$. Let ξ_n denote the number of males in the nth year (or generation), where the present year is called year 0. If $\xi_n = i$, i.e. there are exactly i males in year n, then the probability that there will be j males in the next year is given by

$$\mathbb{P}(\xi_{n+1} = j \mid \xi_n = i) = \mathbb{P}(X_1 + \cdots + X_i = j), \qquad (26)$$

$X_k = m$: 编号为k的 male 可生 m 个 male.

where $\{X_k\}_{k=1}^{\infty}$ is a sequence of independent identically distributed random variances with common distribution:

$$\mathbb{P}(X_1 = m) = p_m, \quad m \in \mathbb{N}.$$

Hence $\xi_0, \xi_1, \xi_2, \ldots$ is a Markov chain on S with transition probabilities

$$p(j \mid i) = \mathbb{P}(\underbrace{X_1 + \cdots + X_i = j}_{i\text{个 male 共生 }j\text{个}}). \qquad (27)$$

Notice that $p(0 \mid 0) = 1$, i.e. if $\xi_n = 0$, then $\xi_m = 0$ for all $m \geq n$. ⇒ 若最开始一只没有, 也没人生.
Dying out means that eventually $\xi_n = 0$, starting from some $n \in \mathbb{N}$, Once this happens, ξ_n will stay at 0 forever. □

生了N个, 活了m个

$\sum_{k=1}^{i} X_k \sim binomial(Ni, j, p)$ 活的规律 follow binomial

活的规律 follow poisson. $\sum_{k=1}^{i} X_k \sim Possion(i\lambda)$

Exercise 5.12. Consider two special cases in Exercise 5.11.

(1) Let $p \in (0, 1)$ and $N \in \mathbb{N}$, and define

$$P(X_i = m) = p_m = \binom{N}{m} p^m (1-p)^{N-m}, \quad m = 0, 1, \ldots, N,$$

and $p_m = 0$ for $m > N$. Then

$$p(j \mid i) = \binom{Ni}{j} p^j (1-p)^{Ni-j}, \quad j = 0, 1, \ldots, Ni, \qquad (28)$$

and $p(j \mid i) = 0$ for all $j > Ni$.

$$p(j|i) = P(\xi_{n+1} = j \mid \xi_n = i)$$
$$= P(X_1 + X_2 + \cdots + X_i = j)$$
$$= C_j^{iN} p^j (1-p)^{iN-j}$$

(2) Let $\lambda > 0$, and define $X_k \sim poisson(\lambda)$

$$p_m = \frac{\lambda^m}{m!} e^{-\lambda}, \quad m = 0, 1, 2, \ldots.$$

In other wards, assume that each X_j has the Poisson distribution with mean λ. Then

$$p(j \mid i) = \frac{(\lambda i)^j}{j!} e^{\lambda i}, \quad i, j = 0, 1, 2 \ldots. \qquad (29)$$

$(X_1 + \cdots + X_i) \sim poisson(\mu = \lambda i)$

$$\Rightarrow P(j|i) = P(X_1 + X_2 + \cdots + X_i = j)$$
$$= \frac{(\lambda i)^j}{j!} e^{-\lambda i}$$

Proposition 5.6

The probability of survival in Exercise 5.12 (2), equals 0 if $\lambda \le 1$, and $1 - \hat{r}^k$ if $\lambda > 1$, where k is the initial population, i.e. $\xi_0 = k$, and $\hat{r} \in (0, 1)$ is a solution to the equation:

$$r = r^{(r-1)\lambda}. \tag{30}$$

That is, for any k, if $\lambda \le 1$ then the probability of extinct:

$$\mathbb{P}(\exists n \text{ s.t. } \xi_n = 0 \mid \xi_0 = k) = 1;$$

and if $\lambda > 1$ then the probability of extinct:

$$\mathbb{P}(\exists n \text{ s.t. } \xi_n = 0 \mid \xi_0 = k) = \hat{r}^k.$$

平均每个male
成活不到1个崽儿

Exercise 5.14. (Queuing model). A car wash machine can serve at most one customer at a time. With probability p, $0 < p < 1$, the machine can finish serving a customer in a unit time. If this happens, the next waiting car can be served at the beginning of the next unit of time.

During the time interval between the nth and $(n+1)$st unit of time the number of cars arriving has the Poisson distribution with parameter $\lambda > 0$.

$\Rightarrow$ $\Delta\xi_n \sim \text{poisson}(\lambda)$

$$P(\Delta\xi_n \text{ arrive}) = \frac{\lambda^{\Delta\xi_n}}{(\Delta\xi_n)!} e^{-\lambda}$$

Let ξ_n denote the number of cars being served or waiting to be served at the beginning of unit n. Then, ξ_n, $n = 0, 1, 2, \ldots$, is a Markov chain. Find its transition probabilities

Soln: Let Z_n be number of arrivals in time unit n.

then $\begin{cases} \xi_{n+1} = \xi_n + Z_n & \text{if } n\text{时在洗的没洗完.} \\ \xi_{n+1} = \xi_n + Z_n - 1 & \text{if } n\text{时在洗的洗完走了.} \end{cases}$

$\Leftrightarrow$ $j - i = \Delta\xi_n = \xi_{n+1} - \xi_n = \begin{cases} Z_n & \text{if } n\text{时在洗的没洗完.} \\ Z_n - 1 & \text{if } n\text{时在洗的洗完走了.} \end{cases}$

$\Leftrightarrow$ $Z_n = \begin{cases} j - i & \text{if } n\text{时在洗的没洗完.} \\ j - i + 1 & \text{if } n\text{时在洗的洗完走了.} \end{cases}$

Also, $p = P[\Delta\xi_n = Z_n - 1] = P[Z_n = j - i + 1 \mid \xi_n = i, \xi_{n+1} = j]$

$1 - p = P[\Delta\xi_n = Z_n] = P[Z_n = j - i \mid \xi_n = i, \xi_{n+1} = j]$.

① When n时排+洗共 i人, $i \ge 1$; $n+1$时一共 j人, $j \ge i - 1 \ge 0$.

则 $p(j|i) = P([\xi_{n+1}=j \mid \xi_n=i])$

$$= P([\Delta\xi_n = \overset{j-i}{\cancel{Z_n}} \mid \xi_n=i]) + \cancel{P([\Delta\xi_n = Z_n-1 \mid \xi_n=i])}$$

$$= P([Z_n=j-i \mid \Delta\xi_n=Z_n,\ \xi_n=i]) + P([Z_n=j-i+1 \mid \Delta\xi_n=Z_n-1,\ \xi_n=i])$$

$$= P([Z_n=j-i \mid \Delta\xi_n=Z_n,\ \xi_n=i)\cdot P(\Delta\xi_n=Z_n \mid \xi_n=i)$$

$$+ P([Z_n=j-i+1 \mid \Delta\xi_n=Z_n-1,\ \xi_n=i)\ P(\Delta\xi_n=Z_n-1 \mid \xi_n=i)$$

$$= P(Z_n=j-i)\cdot P(\Delta\xi_n=Z_n) + P(Z_n=j-i+1)\cdot P(\Delta\xi_n=Z_n-1)$$

$$= \frac{\lambda^{(j-i)}}{(j-i)!}e^{-\lambda}\cdot P(\{n\text{时在洗的没洗完.}\})$$

$$+ \frac{\lambda^{j-i+1}}{(j-i+1)!}e^{-\lambda}\,P(\{n\text{时在洗的洗完走了.}\})$$

$$= p\frac{\lambda^{(j-i)}}{(j-i)!}e^{-\lambda} + (1-p)\frac{\lambda^{j-i+1}}{(j-i+1)!}e^{-\lambda}$$

② When $i=0$, $j\geq 0$ 则不用考虑是否有没洗完的车.

$\Delta\xi_n = Z_n$

$$p(j|i) = P[\xi_{n+1}=j \mid \xi_n=i]$$

$$= P[\Delta\xi_n=j-i \mid \xi_n=i]$$

$$= P[Z_n=j-i \mid \xi_n=i] = P[Z_n=j-i]$$

$$= \frac{\lambda^{j-i}}{(j-i)!}e^{-\lambda}$$

$$= \frac{\lambda^{j}}{j!}e^{-\lambda} \quad \text{as } i=0.$$

5.2 Classification of States

作业要考.

5.2.1 Recurrence and Transience

Definition 5.5. A state i is called *recurrent* if the process $\xi_0, \xi_1, \xi_2, \ldots$ will eventually return to i given that it starts at $\xi_0 = i$, i.e.

$$\mathbb{P}(\xi_n = i \text{ for some } n \geq 1 \mid \xi_0 = i) = 1. \tag{31}$$

If the condition (31) is not satisfied, then the state i is called *transient*. n. 瞬变

\# recurrent
transient 形容 state 而非 Chain

Theorem 5.1

Let $\xi_0, \xi_1, \xi_2, \ldots$ be a random walk on $\mathbb{Z}$ with parameter $p \in (0, 1)$. Then any state i in $\mathbb{Z}$ is recurrent if and only if $p = \frac{1}{2}$.

Proof. We know from Proposition 5.5 that

$$\mathbb{P}(\xi_n = i \text{ for some } n \geq 1 \mid \xi_0 = i) = 1 - |p - q|$$

for any $i \in \mathbb{Z}$. $\square$ $1-|p-q| = 1 \Rightarrow |2p-1|=0 \Rightarrow p = \frac{1}{2}$ as $q = 1-p$.

is it because we can start at the any state on Z?

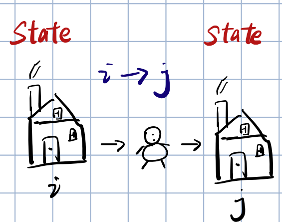
正在为你生成答案...

Yes, exactly! The key idea is that a symmetric random walk on Z is **time and space homogeneous**. This means the behavior of the walk is the same no matter where or when we start. So, if we can prove that the walk returns to its starting state with probability 1 when it starts at a specific state i, then due to the homogeneity, we can generalize this result to any state in Z. Therefore, we can say that any state in Z is recurrent. This is a fundamental property of symmetric random walks on Z.

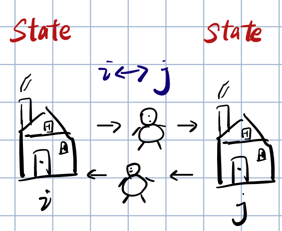
Definition 5.6. We say that a state i *communicates* with a state j if with positive probability the chain will visit the state j having started at i, i.e. 有可能.

$$\mathbb{P}(\xi_n = j \text{ for some } n \geq 0 \mid \xi_0 = i) > 0. \tag{32}$$

∃



If i communicates with j, then we shall write $i \to j$. we say that the state i *intercommunicates* with a state j, and write $i \leftrightarrow j$, if $i \to j$ and $j \to i$.



常用

Exercise 5.15

Show that $i \to j$ if and only if $p_k(j|i) > 0$ for some $k \geq 1$. $\exists$

Hint Recall that $p_k(j|i) = P(\xi_k = j | \xi_0 = i)$.

$\Leftarrow$

Proof. If $p_k(j \mid i) > 0$, then we have

$$\mathbb{P}(\xi_n = j \text{ for some } n \geq 0 \mid \xi_0 = i) = \mathbb{P}\Big(\bigcup_{n=0}^{\infty}\{\xi_n = j\} \mid \xi_0 = i\Big)$$

$$\geq \mathbb{P}(\xi_k = j \mid \xi_0 = i) = p_k(j \mid i) > 0.$$

On the other hand, if $p_n(j \mid i) = 0$ for all $n \geq 0$, then we have

$\Rightarrow$ 反证

$$\mathbb{P}(\xi_n = j \text{ for some } n \geq 0 \mid \xi_0 = i) = \mathbb{P}\Big(\bigcup_{n=0}^{\infty}\{\xi_n = j\} \mid \xi_0 = i\Big)$$

$${}^{*}\boxed{\leq} \sum_{n=0}^{\infty}\mathbb{P}(\xi_n = j \mid \xi_0 = i) = \sum_{n=0}^{\infty} p_n(j \mid i) = 0. \quad \square$$

* as $\{\xi_{n+1} = j\}$ has correlation with $\{\xi_n = j\}$ or $\{\xi_n \neq j\}$.

Exercise 5.16. The relation $\leftrightarrow$ is an equivalence relation on $\mathcal{S}$, i.e. it satisfies the following properties:

1) $i \leftrightarrow i$ (reflexive); = intercommunicates with itself
2) if $i \leftrightarrow j$ then $j \leftrightarrow i$ (symmetric);
3) if $i \leftrightarrow j$ and $j \leftrightarrow k$ then $i \leftrightarrow k$ (transitive).

i — flexible

In other words, show that $\leftrightarrow$ is an equivalence relation on S.

Hint 1) and 2) are obvious. For 3) use the Chapman–Kolmogorov equations.

3) Suppose $i \leftrightarrow j$, $j \leftrightarrow k$ then by Ex. 5.15,

there exists $n, m \geq 1$ s.t. $p_n(j|i) > 0$, $p_m(k|j) > 0$.

then by Chapman-Kolmogorov equations,

$$p_{m+n}(k|i) = \sum_{s \in S} p_m(k|s)\, p_n(s|i) \geq p_m(k|j)\, p_n(j|i) > 0$$

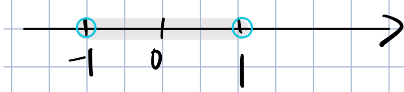
also by Ex 5.15, $k \leftrightarrow i$. $\square$.

DON'T FORGET 证明不要求

Exercise 5.17

For $|x| < 1$ and $j, i \in S$ define

在(-1,1)上 define $P_{ji}(x)$, $F_{ji}(x)$



$$P_{ji}(x) = \sum_{n=0}^{\infty} p_n(j|i)x^n, \to \sum_{n=0}^{\infty} p_n(j|i) \qquad (5.38)$$

n是第一个回到j的

$$F_{ji}(x) = \sum_{n=1}^{\infty} f_n(j|i)x^n, \to \sum_{n=1}^{\infty} f_n(j|i) \qquad (5.39)$$

where $f_n(j|i) = P(\xi_n = j, \xi_k \neq j, k = 1, \cdots, n-1 | \xi_0 = i)$. Show that the power series in (5.38)–(5.39) are absolutely convergent for $|x| < 1$ and that

$$P_{ji}(x) = F_{ji}(x)P_{jj}(x), \text{ if } j \neq i, \qquad (5.40)$$

$$P_{ii}(x) = 1 + F_{ii}(x)P_{ii}(x). \text{ if } j = i \qquad (5.41)$$

Hint Note that $|p_n(j|i)| \leq 1$, so the radius of convergence of the power series (5.38) is ≥ 1.

$$p_n(j|i) = P(\xi_n = j | \xi_0 = i)$$

$$= \sum_{k=1}^{n} P(\xi_n = j, \xi_k = j, \xi_\ell \neq j, 1 \leq \ell \leq k-1 | \xi_0 = i)$$

#依需分配.

$$\boxed{= \sum_{k=1}^{n} P(\xi_k = j, \xi_\ell \neq j, 1 \leq \ell \leq k-1 | \xi_0 = i) \cdot P(\xi_n = j | \xi_k = j, \xi_\ell \neq j, 1 \leq \ell \leq k-1)}$$

$$= \sum_{k=1}^{n} f_k(j|i) P(\xi_n = j | \xi_k = j) \quad \text{by Markov Property}$$

$$= \sum_{k=1}^{n} f_k(j|i) \cdot p_{n-k}(j|j)$$

Abel's Theorem. Let $G(x) = \sum_{k=0}^{\infty} a_k x^k$ be a power series with real coefficients with radius of convergence $R \geq 1$. (1) If the series $\sum_{k=0}^{\infty} a_k$ converges, then $G(x)$ is continuous from the left at $x = 1$, i.e.

$$\lim_{x \uparrow 1} G(x) = \sum_{k=0}^{\infty} a_k.$$

(2) If the series $\sum_{k=0}^{\infty} a_k$ diverge to infinity, i.e. $\sum_{k=0}^{\infty} a_k = \infty$, then

$$\lim_{x \uparrow 1} G(x) = \infty.$$

The radius of convergence of a power series. Given the power series $\sum_{k=0}^{\infty} a_k x^k$, put

$$\alpha = \limsup_{k \to \infty} \sqrt[k]{|a_k|}, \quad \text{and} \quad R = \frac{1}{\alpha}.$$

(If $\alpha = 0$. $R = \infty$, if $\alpha = \infty$, $R = 0$.) Then the power series $\sum_{k=0}^{\infty} a_k x^k$ converges if $|x| < R$, and diverges if $|x| > R$. The number R is called the radius of convergence of the power series $\sum_{k=0}^{\infty} a_k x^k$.

Exercise 5.18

Show that $\lim_{x \nearrow 1^-} P_{jj}(x) = \sum_{n=0}^{\infty} p_n(j|j)$ and $\lim_{x \nearrow 1^-} F_{jj}(x) = \sum_{n=0}^{\infty} f_n(j|j)$.

Hint Apply Abel's lemma[2]: If $a_k \geq 0$ for all $k \geq 0$ and $\limsup_{k \to \infty} \sqrt[k]{|a_k|} \leq 1$, then $\lim_{x \nearrow 1} \sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} a_k$, no matter whether this sum is finite or infinite. → 不用证 $\sum_{k=0}^{\infty} p_k(j|j) < \infty$

Proof. Since $0 \leq p_n(j \mid i) \leq 1$ and $0 \leq f_n(j \mid i) \leq 1$, the result follows redly from Abel's theorem. $\square$

$$\Downarrow$$

$$\alpha = \limsup_{k \to \infty} \sqrt[k]{|p_n(j|i)|} \leq 1$$

$$\Downarrow$$

$$R = \frac{1}{\alpha} \geqslant 1$$

DON'T FORGET 证明不要求.

Exercise 5.19

Show that a state j is recurrent if and only if $\sum_n p_n(j|j) = \infty$. Deduce that the state j is transient if and only if

$$\sum_n p_n(j|j) < \infty. \tag{5.42}$$

$p_n(j|j) = P(\xi_n = j \mid \xi_0 = j)$

i.e. j是recurrent的state iff 从j出发回到j的路径有无穷多条.

i.e. As $j \leftrightarrow j$ iff $p_k(j|j) > 0$ for some $k \geq 1$ then

$\sum_n p_n(j|j) = \infty$ implies even when $k \to \infty$, there's $p_k(j|j) > 0 \Leftrightarrow j \leftrightarrow j$.

i.e. j communicates with itself (reflexible) iff $\sum_n p_n(j|j) = \infty$.

Show that if j is transient, then for each $i \in S$

$$\sum_n p_n(j|i) < \infty. \tag{5.43}$$

Hint If j is recurrent, then $F_{jj}(x) \to \sum_n f_n(j|j) = 1$ as $x \nearrow 1$. Use (5.41) in conjunction with Abel's Lemma.

i.e. 若j是个转瞬即逝的state, 则在无穷的process中,

从任意S中的state出发只有有限的可能造访j.

i.e. 能将无穷的process从任意初始点i引向state j的路径有限.

⇒ j由自身出发造访自己的路径亦有限. i.e. $\sum_n p_n(j|j) < \infty$ 与上文应和.

Exercise 5.19 结论的应用:

Exercise 5.20

For a Markov chain ξ_n with transition matrix $P = \begin{bmatrix} 1-p & q \\ p & 1-q \end{bmatrix}$ show that both states are recurrent.

Hint Use Exercise 5.19 and 5.5.

Proof: Two states: 0 and 1

By Example 5.1, $\boxed{p_n(0|0) = \frac{q}{p+q} + (x_0 - \frac{q}{p+q})(1-p-q)^n}$

$= \frac{q}{p+q} + p(1-p-q)^n$ as $X_0 = 1$

so when $n \to \infty$, $p_n(0|0) \longrightarrow \frac{q}{p+q} > 0$ as $|1-p-q| < 1$

$\Rightarrow \sum_{n=0}^{\infty} p_n(0|0) = \infty$.

By Ex 5.19, 0 is a recurrent state.

Similarly, $\sum_{n=0}^{\infty} p_n(1|1) = \infty$ so 1 is a recurrent state.

$\Rightarrow$ 初始繁忙，终会再次繁忙
(不忙) (不忙).

□

Exercise 5.21

Show that if ξ_n is a Markov chain with finite state space S, then there exists at least one recurrent state $i \in S$.

Hint Argue by contradiction and use (5.43).

Show that if j is transient, then for each $i \in S$

$$\sum_n p_n(j|i) < \infty. \tag{5.43}$$

Proof: Assume $\forall j \in S$, j is transient. then by (5.43),

for each $i \in S$, $\sum_{n=0}^{\infty} p_n(j|i) < \infty$.

$\Rightarrow$ as S is a finit set, so $\sum_{j \in S} \sum_{n=1}^{\infty} p_n(j|i) < \infty$.

Yet, as ξ_n is a Markov Chain on S, so

$\sum_{j \in S} p_n(j|i) = 1$ for each $i \in S$.

$\Rightarrow \sum_{j \in S} \sum_{n=1}^{\infty} p_n(j|i) = \sum_{n=1}^{\infty} \sum_{j \in S} p_n(j|i) = \sum_{n=1}^{\infty} 1 = \infty$.

Thus, contradiction. □

$\forall j \in S$,

$\sum_{j \in S} \sum_{n=1}^{\infty} p_n(j|i) =$ "总线路"

矛盾

Theorem 5.2

The following result is quoted here for reference. The proof is surprisingly difficult and falls beyond the scope of this book.

A state $j \in S$ is recurrent if and only if

$$\mathbb{P}\big(\xi_n = j \text{ for infinitely many } n \mid \xi_0 = j\big) = 1,$$

and it is transient if and only if

$$\mathbb{P}\big(\xi_n = j \text{ for infinitely many } n \mid \xi_0 = j\big) = 0.$$

Split "recurrent" state further:

Definition 5.7. Let $\xi_0, \xi_1, \xi_2, \ldots$ be an S-valued Markov chain, and let the state $i \in S$ be recurrent. Then, its mean recurrence time m_i defined by

m_i: expected recurrent time

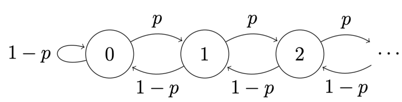
$$m_i := \sum_{n=1}^{\infty} n\, f_n(i \mid i) = \text{n×在第n步第一次回到i的概率} \tag{38}$$

where $f_n(i \mid i) = \mathbb{P}\big(\xi_n = i,\ \xi_k \neq i,\ k = 1, \ldots, n-1 \mid \xi_0 = i\big)$.

(1) The state i is called *null-recurrent* if its mean recurrence time m_i is infinite, i.e. $m_i = \infty$.

(2) The state i is called *positive-recurrent* if its mean recurrence time is finite, i.e., $m_i < \infty$.

Example 5. Consider the reflected random walk where the probability of moving forwards is $p \in (0,1)$.



The chain is irreducible, and it can be classified as:

- positive recurrent, when $p < 1/2$;
- null recurrent, when $p = 1/2$;
- transient, when $p > 1/2$.

Proposition 5.4

Let $\xi_0, \xi_1, \xi_2, \ldots$ be a random walk starting at $\xi_0 = i$. For all $p \in (0,1)$,

$$p_n(i|i) = \mathbb{P}(\xi_n = i \mid \xi_0 = i) \to 0, \quad \text{as} \quad n \to \infty. \tag{21}$$

Remark 5.4

One can show that a recurrent state i is null-recurrent if and only if $p_n(i|i) \to 0$.

We already know that for a random walk on $\mathbb{Z}$ the state 0 is recurrent if and only if $p = 1/2$, i.e. if and only if the random walk is symmetric. In the following problem we shall try to answer if 0 is a null-recurrent or positive-recurrent state (when $p = 1/2$).

Exercise 5.22

Consider a symmetric random walk on $\mathbb{Z}$. Show that 0 is a null-recurrent state. Can you deduce whether other states are positive-recurrent or null-recurrent?

Hint State 0 is null-recurrent if and only if $\sum_n nf_n(0|0) = \infty$. As ~~in Exercise 5.18,~~ $\sum_n nf_n(0|0) = \lim_{x\nearrow 1} F'_{00}(x)$, where F_{00} is defined by (5.39).

Solution. From Exercise 5.9, we have

$$p_{2k}(0 \mid 0) = \mathbb{P}(\xi_{2k} = 0 \mid \xi_0 = 0) = \binom{2k}{k}\left(\frac{1}{2}\right)^k\left(\frac{1}{2}\right)^k, \quad k = 0, 1, 2, \ldots,$$

and

$$p_{2k+1}(0 \mid 0) = \mathbb{P}(\xi_{2k+1} = 0 \mid \xi_0 = 0) = 0, \quad k = 0, 1, 2, \ldots.$$

Thus, for any $x \in (-1, 1)$, from Exercise 5.17 we have

$$P_{00}(x) = \sum_{n=0}^{\infty} p_n(0 \mid 0)x^n = \sum_{k=0}^{\infty}\binom{2k}{k}\left(\frac{x^2}{4}\right)^k = (1-x^2)^{-1/2}.$$

Since from the equation (5.41):

$$P_{00}(x) = 1 + F_{00}(x)P_{00}(x), \quad x \in (-1, 1),$$

we get

$$F_{00}(x) = 1 - \frac{1}{P_{00}(x)} = 1 - (1-x^2)^{1/2}, \quad x \in (-1, 1).$$

Thus, we have

$$F'_{00}(x) = x(1-x^2)^{-1/2} \quad x \in (-1, 1), \qquad (1)$$

and

$$F'_{00}(x) = \sum_{n=1}^{\infty} nf_n(0 \mid 0)x^{n-1}, \quad x \in (-1, 1). \qquad (2)$$

Now, from (1) we have

$$F'_{00}(x) \nearrow \infty \quad \text{as} \quad x \nearrow 1,$$

and by using Abel's lemma, from (2) we get

$$m_0 = \sum_{n=1}^{\infty} nf_n(0 \mid 0) = \infty.$$

This implies that the state 0 is null-recurrent.□

Proof: From exercise 5.9,

$$p_n(0|0) = \begin{cases} p_{2k}(0|0) = \binom{2k}{k}(p)^{\frac12}(q)^{\frac12} \\ p_{2k+1}(0|0) = 0 \end{cases} \quad \text{for } k = 0, 1, 2, \ldots$$

Thus, if any $x \in (-1,1)$, by Exercise 5.17,

$$P_{00}(x) = \sum_{n=0}^{\infty} p_n(0|0)x^n = \sum_{k=0}^{\infty}\binom{2k}{k}\left(\frac12\right)^k\left(\frac12\right)^k x^{2k}$$

$$= \sum_{k=0}^{\infty}\binom{2k}{k}\left(\frac{x}{2}\right)^{2k}$$

$$\sum_{k=0}^{\infty}\binom{2k}{k}x^k = (1-4x)^{-1/2}, \quad |x| < \frac{1}{4}.$$

$$= \sum_{k=0}^{\infty}\binom{2k}{k}\left(\frac{x^2}{4}\right)^k = \frac{1}{\sqrt{1-x^2}}$$

Since $P_{00}(x) = 1 + F_{00}(x)P_{00}(x)$

so $F_{00}(x) = 1 - \frac{1}{P_{00}(x)} = 1 - \sqrt{1-x^2}, \quad x \in (-1,1)$

Thus, $F'_{00}(x) = \frac{1}{2} \times \frac{2x}{\sqrt{1-x^2}} = \frac{x}{\sqrt{1-x^2}}, \quad x \in (-1,1)$ ①

and by the def. of $F_{00}(x) = \sum_{n=1}^{\infty} f_n(0|0)x^n$

$F'_{00}(x) = \sum_{n=1}^{\infty} nf_n(0|0)x^{n-1}, \quad x \in (-1,1)$ ②

As from ①, $\lim_{x \to 1^-} F'_{00}(x) = \infty$ and by Abel's lemma,

② $\Rightarrow \lim_{x \to 1^-} \sum_{n=1}^{\infty} nf_n(0|0)x^{n-1} = \sum_{n=1}^{\infty} nf_n(0|0)$ on $(-1,1)$.

so $m_0 = \sum_{n=1}^{\infty} nf_n(0|0) = \infty$, i.e. 0 is null-recurrent.

□.

Proof: Supplement: The random walk is 'space homogeneous'

i.e. $p(j|i) = p(j-i|0)$

$\Rightarrow p(j|j) = p(j-j|0) = p(0|0)$

As $p_n(j|i) = \sum_{k=1}^{n} f_k(j|i) \cdot p_{n-k}(j|j)$

so $p_n(j-i|0) = \sum_{k=1}^{n} f_k(j|i)\, p_{n-k}(0|0)$

also, $p_n(j-i|0) = \sum_{k=1}^{n} f_k(j-i|0)\, p_{n-k}(j-i|j-i)$

$= \sum_{k=1}^{n} f_k(j-i|0)\, p_{n-k}(0|0)$

therefore, $f_n(j|i) = f_n(j-i|0) \quad \forall n \in \mathbb{N}$.

$\Rightarrow \forall i \in \mathbb{Z}, \quad m_i = \sum_{n=1}^{\infty} nf_n(i|i) = \sum_{n=1}^{\infty} nf_n(i-i|0) = \sum_{n=1}^{\infty} nf_n(0|0) = m_0$

So if 0 is a null-recurrent state or a positive-recurrent state, all states are null-recurrent or positive recurrent accordingly.

Exercise 5.23

For the Markov chain ξ_n from Exercise 5.20 show that not only are all states recurrent, but they are positive-recurrent. i.e, $m_i < \infty$

Hint Calculate $f_n(0|0)$ directly.

For a Markov chain ξ_n with transition matrix $P = \begin{bmatrix} 1-p & q \\ p & 1-q \end{bmatrix}$ show that both states are recurrent.

Proof: $S = \{0, 1\}$. $m_i := \sum_{n=1}^{\infty} n f_n(i|i)$

As $f_n(0|0) = P[\xi_n = 0, \xi_k \neq 0, k = 1, 2, \cdots n-1 \mid \xi_0 = 0]$

$= p(0|1)[p(1|1)]^{n-2} p(1|0)$

$= p(1-q)^{n-2} \cdot q$

Let $a_n := n f_n(0|0)$ then as $\frac{a_{n+1}}{a_n} = \frac{n+1}{n}(1-q) < 1$ when $n \to \infty$.

So $\sum_{n=1}^{\infty} n f_n(0|0)$ is convergent. i.e, $m_0 < \infty$.

Approach 2:

Remark 5.4

One can show that a recurrent state i is null-recurrent if and only if $p_n(i|i) \to 0$.

+

Ex 20. 的结果:

0, 1 are both recurrent.