4
Martingale Inequalities and Convergence

4.1 Doob's Martingale Inequalities

Recall

We recall that, if $\{\xi_n\}$ is a submartingale with respect to a filtraction $\{\mathcal{F}_n\}$, then

$$\mathbb{E}[\xi_n \mid \mathcal{F}_k] \geq \xi_k \Longrightarrow \mathbb{E}[\xi_n] \geq \mathbb{E}[\xi_k], \quad n \geq k,$$

and $\tau \wedge n = \min\{\tau, n\} \leq n$.

Theorem 3.1 (Optional Stopping Theorem)

Let ξ_n be a martingale and τ a stopping time with respect to a filtration $\mathcal{F}_n$ such that the following conditions hold:

1) $\tau < \infty$ a.s., 只能玩有限次
2) ξ_τ is integrable, 预期累积资本有限
3) $E(\xi_n 1_{\{\tau > n\}}) \to 0$ as $n \to \infty$. 当$n\to\infty$, $\tau \le n$.

①

Then

$$E(\xi_\tau) = E(\xi_1).$$

应用Stopping Time策略的预期最终财富与预期第一轮结束财富一样

Proposition A4.1 (Option stopping theorem for submartingales)

Suppose that $\{\xi_n\}$ is a submartingale with respect to a filtration $\{\mathcal{F}_n\}$, and τ is a stopping time with respect to $\{\mathcal{F}_n\}$. Then, for each time n,

$$\mathbb{E}[\xi_n] \geq \mathbb{E}[\xi_{\tau \wedge n}].$$

Proof. For any time n we have

$$\xi_n = \xi_{\tau\wedge n} + (\xi_n - \xi_\tau)1_{\{\tau<n\}}.$$

Taking expectation, we get

$$\mathbb{E}[\xi_n] = \mathbb{E}[\xi_{\tau\wedge n}] + \mathbb{E}[(\xi_n - \xi_\tau)1_{\{\tau<n\}}].$$

Since $\{\xi_n\}$ is a submartingale, we have that, for any $1 \leq k \leq n$,

$$\mathbb{E}[\xi_n \mid \mathcal{F}_k] \geq \xi_k.$$

Thus, since τ is a stopping time, $\{\tau = k\} \in \mathcal{F}_k$. Therefore, for the last term, we get

$$\begin{aligned}
\mathbb{E}[(\xi_n - \xi_\tau)1_{\{\tau<n\}}] &= \mathbb{E}\Big[\sum_{k=1}^{n-1}(\xi_n - \xi_k)1_{\{\tau=k\}}\Big] \\
&= \sum_{k=1}^{n-1}\mathbb{E}\Big[\mathbb{E}\big[(\xi_n - \xi_k)1_{\{\tau=k\}} \mid \mathcal{F}_k\big]\Big] \\
&= \sum_{k=1}^{n-1}\mathbb{E}\Big[\Big(\mathbb{E}[\xi_n \mid \mathcal{F}_k] - \xi_k\Big)1_{\{\tau=k\}}\Big] \geq 0
\end{aligned}$$

Therefore, we prove this proposition.□

Proof: As for each time n, we have

$$\xi_n = \xi_{\tau\wedge n} + (\xi_n - \xi_\tau) I_{\{\tau<n\}}.$$

Taking expectation, then

$$E[\xi_n] = E[\xi_{\tau\wedge n}] + E[(\xi_n - \xi_\tau) I_{\{\tau<n\}}].$$

Since $\{\xi_n\}$ is a submartingale, then $\forall n$ s.t. $1 \le k \le n$,

$$E[\xi_n \mid F_k] \geq \xi_k$$

Thus, since τ is a stopping time, $\{\tau=k\} \in F_k$.

Therefore, for $E[(\xi_n - \xi_\tau) I_{\{\tau<n\}}]$, we get

$$\begin{aligned}
E[(\xi_n - \xi_\tau) I_{\{\tau<n\}}] &= E\Big[\sum_{k=1}^{n-1}(\xi_n - \xi_\tau) I_{\{\tau=k\}}\Big] \\
&= \sum_{k=1}^{n-1} E[(\xi_n - \xi_\tau) I_{\{\tau=k\}}] \\
&= \sum_{k=1}^{n-1} E\big[E[(\xi_n - \xi_k) I_{\{\tau=k\}} \mid F_k]\big] \quad \text{as } \{\tau=k\} \in F_k. \\
&= \sum_{k=1}^{n-1} E[(E[\xi_n \mid F_k] - \xi_k) I_{\{\tau=k\}}] \\
&\geq 0 \quad \text{as } E[\xi_n \mid F_k] \geq \xi_k.
\end{aligned}$$

Thus, $E[\xi_n] \geq E[\xi_{\tau\wedge n}]$ □.

Proposition 4.1 (Doob's maximal inequality)

Suppose that $\{\xi_n\}$ is a non-negative submartingale with respect to a filtration $\{\mathcal{F}_n\}$. Then, for any $\lambda > 0$,

$$\lambda \mathbb{P}(\xi_n^* \geq \lambda) \leq \mathbb{E}[\xi_n 1_{\{\xi_n^* \geq \lambda\}}],$$

where $\xi_n^* = \max_{1 \leq k \leq n} \xi_k$ for each n.

i.e. ξ_n^* 是n个imperfect observation of ξ中最大的

而 $n = 1, 2, 3, \cdots$ 故 $\xi_1^* \le \xi_2^* \le \xi_3^* \le \cdots$

$$\xi_n I_{\{\xi_n^* \ge \lambda\}} = \begin{cases} \xi_n & \text{if } \xi_n^* \ge \lambda \\ 0 & \text{else } \xi_1^* = \xi_1 < \lambda \end{cases} \quad \text{for } n = 1, 2, 3, \cdots$$

i.e. $P(\xi_n^* \ge \lambda) \le \frac{1}{\lambda} E[\xi_n I_{\{\xi_n^* \ge \lambda\}}]$

"penalty"

Proof

We put $\xi_n^* = \max_{k \le n} \xi_k$ for brevity. For $\lambda > 0$ let us define

$$\tau = \min\{k \le n : \xi_k \ge \lambda\},$$ 第1个超过(等于)λ的 ξ_k

若在n前达到(超过)λ,则在该点停,
若没有,则在n停.

if there is a $k \le n$ such that $\xi_k \ge \lambda$, and $\tau = n$ otherwise. Then τ is a stopping time such that $\tau \le n$ a.s. Since ξ_n is a submartingale,

$$E(\xi_n) \ge E(\xi_\tau).$$ By $E[\xi_n] \geqslant E[\xi_{\tau\wedge n}]$ and $\tau := \min\{k \le n : \xi_k \geqslant \lambda\}$

But

$$E(\xi_\tau) = E\left(\xi_\tau 1_{\{\xi_n^* \ge \lambda\}}\right) + E\left(\xi_\tau 1_{\{\xi_n^* < \lambda\}}\right).$$

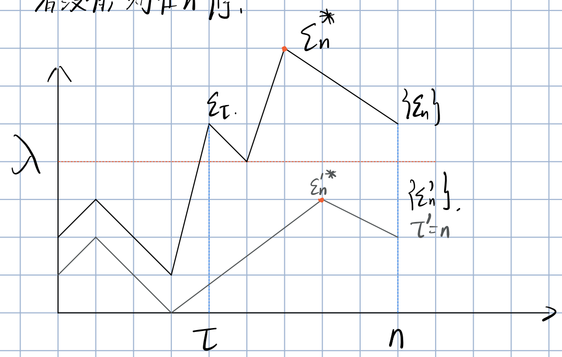
Observe that if $\xi_n^* \ge \lambda$, then $\xi_\tau \ge \lambda$. Moreover, if $\xi_n^* < \lambda$, then $\tau = n$, and so $\xi_\tau = \xi_n$. Therefore

$$E(\xi_n) \ge E(\xi_\tau) \ge \lambda P(\xi_n^* \ge \lambda) + E\left(\xi_n 1_{\{\xi_n^* < \lambda\}}\right),$$

It follows that

$$\lambda P(\xi_n^* \ge \lambda) \le E(\xi_n) - E\left(\xi_n 1_{\{\xi_n^* < \lambda\}}\right) = E\left(\xi_n 1_{\{\xi_n^* \ge \lambda\}}\right),$$

completing the proof. □



Proof:

$\forall \lambda > 0$, define $\tau = \begin{cases} \min\{k : k \le n,\ \xi_k \geqslant \lambda\}, & \text{if exists such } k. \\ n, & \text{otherwise.} \end{cases}$

as $\{\tau = k\} \in \mathcal{F}_k$ by the def. of stopping time.

$\Rightarrow$ τ is a stopping time s.t. $\tau \le n$.

Since $\{\xi_k\}$ is a submartingale, and $\tau \le n$,

so $E[\xi_n] \geqslant E[\xi_{\tau\wedge n}] = E[\xi_\tau]$ (1)

Note that $\boxed{\text{if } \xi_n^* \geqslant \lambda}$ ①, then $\max\{\xi_k,\ 1 \le k \le n\} \geqslant \lambda$

then there exists k s.t. $k \le n$ and $\xi_k \geqslant \lambda$

so τ is equal to such k. i.e. $\xi_\tau \geqslant \lambda$. (2)

$\boxed{\text{if } \xi_n^* < \lambda}$ ② then there doesn't exist such k,

so $\tau = n$ i.e. $\xi_\tau = \xi_n$. (3)

Also, $E[\xi_\tau] = E[\xi_\tau 1_{\{\xi_n^* \geqslant \lambda\}}] + E[\xi_\tau 1_{\{\xi_n^* < \lambda\}}]$ (4)

Therefore, $E[\xi_n] \geqslant E[\xi_\tau]$

$= E[\xi_\tau 1_{\{\xi_n^* \geqslant \lambda\}}] + E[\xi_\tau 1_{\{\xi_n^* < \lambda\}}]$

$\geqslant E[\lambda 1_{\{\xi_n^* \geqslant \lambda\}}] + E[\xi_n 1_{\{\xi_n^* \geqslant \lambda\}}]$

$= \lambda P(\xi_n^* \geqslant \lambda) + E[\xi_n 1_{\{\xi_n^* < \lambda\}}]$.

as $E[1_A] = P(A)$

Thus, $\lambda P(\xi_n^* \geqslant \lambda) \le E[\xi_n] - E[\xi_n 1_{\{\xi_n^* < \lambda\}}]$

$= E[\xi_n 1_{\{\xi_n^* \geqslant \lambda\}}]$

as $E[\xi_n] = E[\xi_n 1_{\{\xi_n^* \geqslant \lambda\}}] + E[\xi_n 1_{\{\xi_n^* < \lambda\}}]$. □.

Theorem 4.1 (Doob's maximal L^2 inequality)

If $\{\xi_n\}$ is a non-negative square integrable submartingale with respect to a filtration $\{\mathcal{F}_n\}$. Then,

$$\mathbb{E}\left[\left|\xi_n^*\right|^2\right] \le 4\mathbb{E}\left[\left|\xi_n\right|^2\right],$$

where $\xi_n^* = \max_{1 \le k \le n} \xi_k$ for each n.

Proof. Since $\{\xi_n\}$ is non-negative, by Exercise 1.9, Proposition 4.1, and the Fubini theorem, we have

$$\begin{aligned}
\mathbb{E}\left[\left|\xi_n^*\right|^2\right] &\overset{1.9}{=} 2\int_0^\infty t\mathbb{P}(\xi_n^* > t)\mathrm{d}t \le 2\int_0^\infty t\mathbb{P}(\xi_n^* \ge t)\mathrm{d}t \\
&\overset{4.1}{\le} 2\int_0^\infty \mathbb{E}\left[\xi_n 1_{\{\xi_n^* \ge t\}}\right]\mathrm{d}t = 2\int_0^\infty \Big(\int_{\{\xi_n^* \ge t\}} \xi_n \mathbb{P}(\mathrm{d}\omega)\Big)\mathrm{d}t \\
&= 2\int_\Omega \xi_n \Big(\int_0^{\xi_n^*} \mathrm{d}t\Big)\mathbb{P}(\mathrm{d}\omega) = 2\int_\Omega \xi_n \xi_n^* \mathbb{P}(\mathrm{d}\omega) \\
&= 2\mathbb{E}\left[\xi_n \xi_n^*\right].
\end{aligned}$$

Applying the Cauchy-Schwarz inequality, we get

$$\mathbb{E}\left[\left|\xi_n^*\right|^2\right] \le 2\mathbb{E}\left[\xi_n \xi_n^*\right] \le 2\left(\mathbb{E}\left[\left|\xi_n\right|^2\right]\right)^{\frac{1}{2}}\left(\mathbb{E}\left[\left|\xi_n^*\right|^2\right]\right)^{\frac{1}{2}}.$$

Dividing $\left(\mathbb{E}\left[\left|\xi_n^*\right|^2\right]\right)^{\frac{1}{2}}$, we have

$$\left(\mathbb{E}\left[\left|\xi_n^*\right|^2\right]\right)^{\frac{1}{2}} \le 2\left(\mathbb{E}\left[\left|\xi_n\right|^2\right]\right)^{\frac{1}{2}}.$$

That implies the Doob's maximal L^2 inequality, and the theorem is proved. □

Exercise 1.9

Show that if $\eta : \Omega \to [0, \infty)$ is a non-negative square integrable random variable, then

$$E(\eta^2) = 2\int_0^\infty tP(\eta > t)\,dt.$$

Hint Express $E(\eta^2)$ in terms of the distribution function $F_\eta(t)$ of η and then integrate by parts.

In mathematical analysis, **Fubini's theorem** is a result that gives conditions under which it is possible to compute a double integral by using an iterated integral, introduced by Guido Fubini in 1907. One may switch the order of integration if the double integral yields a finite answer when the integrand is replaced by its absolute value.

$$\iint_{X\times Y} f(x,y)\,\mathrm{d}(x,y) = \int_X\left(\int_Y f(x,y)\,\mathrm{d}y\right)\mathrm{d}x = \int_Y\left(\int_X f(x,y)\,\mathrm{d}x\right)\mathrm{d}y \quad \text{if} \quad \iint_{X\times Y} |f(x,y)|\,\mathrm{d}(x,y) < +\infty.$$

Proof: Since $\{\xi_n\}$ is non-negative,

$$E[|\xi_n^*|^2] = 2\int_0^\infty t P(\xi_n^* > t)\,dt \quad \text{by Ex.1.9}$$
$$\le 2\int_0^\infty t P(\xi_n^* \ge t)\,dt$$
$$\le 2\int_0^\infty E[\xi_n 1_{\{\xi_n^* \ge t\}}]\,dt \quad \text{by Proposition 4.1}$$
$$= 2\int_0^\infty \left(\int_{\{\xi_n^* \ge t\}} \xi_n\,dP\right) dt$$
$$= 2\int_\Omega \xi_n \left(\int_0^{\xi_n^*} dt\right) dP \quad \text{by Fubini Theorem}$$
$$= 2\int_\Omega \xi_n \xi_n^*\,dP$$
$$= 2E[\xi_n \xi_n^*]$$

By Cauchy-Schwartz Inequality, $\varphi(E(X)) \le E(\varphi(X))$

$$E[|\xi_n^*|^2] \le 2E[\xi_n\xi_n^*] \le 2\left(E[|\xi_n|^2]\right)^{\frac{1}{2}}\left(E[|\xi_n^*|^2]\right)^{\frac{1}{2}}$$

$\varphi := x^2$

Dividing $\left(E[|\xi_n^*|^2]\right)^{\frac{1}{2}}$, get

$$\left(E[|\xi_n^*|^2]\right)^{\frac{1}{2}} \le 2\left(E[|\xi_n|^2]\right)^{\frac{1}{2}}$$
$$E[|\xi_n^*|^2] \le 4E[|\xi_n|^2]. \quad \square$$

由4.1，知：若$\{\xi_n\}$为submartingale：

$E[\xi_n] \ge E[\xi_{\tau\wedge n}]$. $\forall n \in \mathbb{N}$.

i.e. 在任意time point n的自然收获 ≥ 在stopping time strategy干预下的收获.

$\lambda P(\xi_n^* \ge \lambda) \le E[\xi_n 1_{\{\xi_n^* \ge \lambda\}}]$. $\forall \lambda > 0$

i.e. 截止至n时，曾有过的最高收获 $\xi_n^* \ge$ 任意 $\lambda > 0$ 的概率有上界，$P(\xi_n^* \ge \lambda) \le \frac{1}{\lambda} E[\xi_n 1_{\{\xi_n^* \ge \lambda\}}]$.

⇒ λ越大概率越小

$E[|\xi_n^*|^2] \le 4E[|\xi_n|^2]$

i.e. ξ_n^* 在L^2内可积

4.2 Doob's Martingale Convergence Theorem

用了upcrossing，不考证明.

Theorem 4.2 (Doob's martingale convergence theorem)

Suppose that $\xi_1, \xi_2, \ldots$ is a supermartingale with respect to a filtration $\mathcal{F}_1, \mathcal{F}_2, \ldots$ such that

expectation 的上确界 $\sup_n \mathbb{E}[|\xi_n|] < \infty.$ 有限 ⇒ Expect 的完整买卖次数 $E[U_n[a,b]] < \infty$

条件

Then, there is an integrable random variable ξ such that

$\lim_{n\to\infty} \xi_n = \xi \quad$ a.s. 则submartingale收敛

结论

Remark 4.1. In particular, the theorem is valid for martingales because every martingale is a supermartingale. It is also valid for submartingales, since $\xi_1, \xi_2, \ldots$ is a submartingale if and only if $-\xi_1, -\xi_2, \ldots$ is a supermartingale. $\sup E[|-\xi_n|] = \sup E[|\xi_n|] < \infty$

Remark 4.2. Observe that even though all the ξ_n as well as the limit ξ are integrable random variables, it is claimed only that ξ_n tends to ξ a.s. Note that no convergence in L^1 is asserted.

Proof (of Doob's Martingale Convergence Theorem)

By the Upcrossings Inequality

$$E\left(U_n[a,b]\right) \le \frac{E\left((\xi_n - a)^-\right)}{b-a} \le \frac{M+|a|}{b-a} < \infty,$$

where

$$M = \sup_n E\left(|\xi_n|\right) < \infty.$$

Since $U_n[a,b]$ is a non-decreasing sequence, it follows that

$$E\left(\lim_{n\to\infty} U_n[a,b]\right) = \lim_{n\to\infty} E\left(U_n[a,b]\right) \le \frac{M+|a|}{b-a} < \infty.$$

This implies that

$$P\left\{\lim_{n\to\infty} U_n[a,b] < \infty\right\} = 1.$$

for any $a < b$. Since the set of all pairs of rational numbers $a < b$ is countable, the event

$$A = \bigcap_{a<b \text{ rational}} \left\{\lim_{n\to\infty} U_n[a,b] < \infty\right\} \tag{4.3}$$

has probability 1. (The intersection of countably many events has probability 1 if each of these events has probability 1.)

We claim that the sequence ξ_n converges a.s. to a limit ξ. Consider the set

$$B = \{\liminf_n \xi_n < \limsup_n \xi_n\} \subset \Omega$$

on which the sequence ξ_n fails to converge. Then for any $\omega \in B$ there are rational numbers a, b such that

$$\liminf_n \xi_n(\omega) < a < b < \limsup_n \xi_n(\omega),$$

implying that $\lim_{n\to\infty} U_n[a,b](\omega) = \infty$. This means that B and the event A in (4.3) are disjoint, so $P(B) = 0$, since $P(A) = 1$, which proves the claim.

It remains to show that the limit ξ is an integrable random variable. By Fatou's lemma

$$\begin{aligned} E\left(|\xi|\right) &= E\left(\liminf_n |\xi_n|\right) \\ &\le \liminf_n E\left(|\xi_n|\right) \\ &< \sup_n E\left(|\xi_n|\right) < \infty. \end{aligned}$$

This completes the proof. □

Proof: By the Upcrossing Inequality,

$(b-a)\, E[U_n[a,b]] \le E[(\xi_n-a)^-]$

Let $M := \sup_n E[|\xi_n|] < \infty$ then

$$E[U_n[a,b]] \le \frac{1}{b-a} E[(\xi_n-a)^-] = \begin{cases} 0 & \text{if } \xi_n \ge a \\ \frac{1}{b-a} E[a-\xi_n] & \text{if } \xi_n < a \end{cases}$$

$$\le \frac{1}{b-a}\left(E[|\xi_n - a|]\right)$$

$$\le \frac{1}{b-a}\left(E[|\xi_n|] + E[|a|]\right)$$

$$\le \frac{1}{b-a}(M+|a|)$$

$$= \frac{M+|a|}{b-a} < \infty.$$

$E[U_n[a,b]] < \infty$.

完整实实次数有限.

Since $U_n[a,b]$ is non-decreasing, so

$$E[\lim_{n\to\infty} U_n[a,b]] = \lim_{n\to\infty} E[U_n[a,b]] \le \frac{M+|a|}{b-a} < \infty.$$

This implies that for any $a<b$,

$$P(\lim_{n\to\infty} U_n[a,b] < \infty) = 1$$

Denote the set of all rational numbers by $\mathbb{Q}$, since all pairs of rational numbers $a<b$ is countable,

the event $A = \bigcap_{\substack{a<b \\ a,b\in\mathbb{Q}}} \{\lim_{n\to\infty} U_n[a,b] < \infty\}$, $P(A) = 1$

as the intersection of countably many events has probability 1 if each of these events has probability 1.

Denote $A_{a,b} = \{\lim_{n\to\infty} U_n[a,b] < \infty\}$.

then for any $a<b$, $P(A_{a,b}) = 1 \Rightarrow P(A_{a,b}^c) = 0$.

So $P(A^c) = P[\bigcup_{\substack{a<b \\ a,b\in\mathbb{Q}}} A_{a,b}^c] \le \sum_{\substack{a<b \\ a,b\in\mathbb{Q}}} P(A_{a,b}^c) = 0$.

$\Rightarrow P(A) = 1$

Another approach to prove $P(A)=1$.

① Now, claim the stochastic sequence ξ_n converges a.s. to a limit ξ.

Consider the set $B = \{\liminf_n \xi_n < \limsup_n \xi_n\} \subset \Omega$

on which $\{\xi_n\}$ fail to converge

Then, $\forall \omega \in B$, there are $a, b \in \mathbb{Q}$ s.t.

$$\liminf_n \xi_n(\omega) < a < b < \limsup_n \xi_n(\omega)$$

implying that $\lim_{n\to\infty} U_n[a,b] = \infty$ for any $\omega \in B$.

Therefore, $B \subset A^c$, so $P(B) = 0$. as $P(A^c) = 1 - P(A) = 0$.

implies that $\{\xi_n\}$ must converge.

i.e. there's a r.v. ξ s.t. $\lim_{n\to\infty} \xi_n = \xi$.

② Now, prove the limit ξ is an integrable r.v.

By Fatou's Lemma,

Theorem — Fatou's lemma. Given a measure space $(\Omega, \mathcal{F}, \mu)$ and a set $X \in \mathcal{F}$, let $\{f_n\}$ be a sequence of $(\mathcal{F}, \mathcal{B}_{\mathbb{R}_{\ge 0}})$-measurable non-negative functions $f_n : X \to [0, +\infty]$. Define the function $f : X \to [0, +\infty]$ by setting $f(x) = \liminf_{n\to\infty} f_n(x)$, for every $x \in X$

Then f is $(\mathcal{F}, \mathcal{B}_{\mathbb{R}_{\ge 0}})$-measurable, and also $\int_X f\,d\mu \le \liminf_{n\to\infty} \int_X f_n\,d\mu$, where the integrals may be infinite.

$$E[|\xi|] = E[\liminf_n |\xi_n|] \le \liminf_n E[|\xi_n|] < \sup_n E[|\xi_n|] < \infty.$$

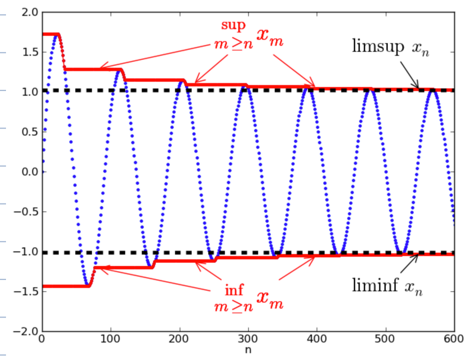
So ξ is integrable. □.

Notices:

i.e, 直到 $n \to \infty$ ξ_n都在上下振荡.

Consider the set $B = \{\liminf_n \xi_n < \limsup_n \xi_n\} \subset \Omega$

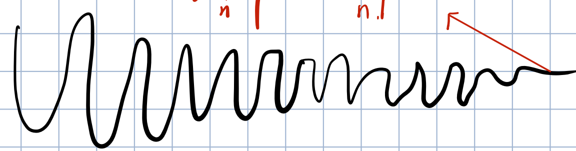
on which $\{\xi_n\}$ fail* to converge



if $\xi_n \to \xi$ when $n \to \infty$,

the it could be like:

$\liminf_n \xi_n = \limsup_n \xi_n$



Exercise 4.2

Show that if ξ_n is a non-negative supermartingale, then it converges a.s. to an integrable random variable.

Hint To apply Doob's Theorem all you need to verify is that the sequence ξ_n is bounded in L^1, i.e. the supremum of $E(|\xi_n|)$ is less than ∞.

Solution 4.2

For a non-negative supermartingale $\Rightarrow$ $\{E(\xi_n)\}$ is upper bounded by $E[\xi_1]$

$$\sup_n E(|\xi_n|) = \sup_n E(\xi_n) \leq E(\xi_1) = E(|\xi_1|) < \infty,$$

since

$$E(\xi_n) \leq E(\xi_1)$$

for each $n = 1, 2, \ldots$. Thus Doob's Martingale Convergence Theorem implies that ξ_n converges a.s. to an integrable limit.

通篇意在说明二者互为充要

4.3 Uniform Integrability and L^1 Convergence of Martingales

对$\xi_1, \xi_2 \cdots$ $\Longleftrightarrow$

We recall that a random variable ξ is said to be *integrable* if

$$\mathbb{E}[|\xi|] = \int_\Omega |\xi(\omega)| \mathbb{P}(d\omega) < \infty.$$

And the set of all integrable random variable is denoted by L^1, or, $L^1(\Omega, \mathcal{F}, \mathbb{P})$.

Exercise 4.3. A random variable ξ is integrable if and only if for every $\varepsilon > 0$ there exists an $M > 0$ such that

$$\int_{\{|\xi|>M\}} |\xi(\omega)| \mathbb{P}(d\omega) < \varepsilon.$$

Solution 4.3

Necessity. Suppose that ξ is integrable. It follows that

$$P\{|\xi| < \infty\} = 1.$$

The sequence of random variables $|\xi| 1_{\{|\xi|>M\}}$ indexed by $M = 1, 2, \ldots$ is monotone and

$$|\xi| 1_{\{|\xi|>M\}} \searrow 0 \quad \text{as } M \to \infty$$

on the set $\{|\xi| < \infty\}$, i.e. a.s. By the monotone convergence theorem for integrals

$$\int_{\{|\xi|>M\}} |\xi| \, dP \searrow 0 \quad \text{as } M \to \infty.$$

It follows that for every $\varepsilon > 0$ there exists an $M > 0$ such that

$$\int_{\{|\xi|>M\}} |\xi| \, dP < \varepsilon.$$

Sufficiency. Take $\varepsilon = 1$. There exists an $M > 0$ such that

$$\int_{\{|\xi|>M\}} |\xi| \, dP < 1.$$

Then

$$\begin{aligned} E(|\xi|) &= \int_\Omega |\xi| \, dP \\ &= \int_{\{|\xi|>M\}} |\xi| \, dP + \int_{\{|\xi|\leq M\}} |\xi| \, dP \\ &< 1 + MP\{|\xi| \leq M\} \\ &\leq 1 + M < \infty. \end{aligned}$$

$$\int_{\{|\xi|\leq M\}} |\xi| \, dP = \int_\Omega |\xi| I_{\{|\xi|\leq M\}} dP \leq M P(\{|\xi| \leq M\})$$

Solution. Necessity. Since ξ is integrable, we have that $\mathbb{P}(\{|\xi| < \infty\}) = 1$. In fact, if there exists a $\delta > 0$ such that $\mathbb{P}(\{|\xi| = \infty\}) = \delta$, then, by the Exercise 1.1, we have

$$\lim_{n\to\infty} \mathbb{P}(\{|\xi| \geq n\}) = \mathbb{P}\Big(\bigcap_{n=1}^{\infty}\{|\xi| \geq n\}\Big) = \mathbb{P}(\{|\xi| = \infty\}) = \delta.$$

Thus, there exists a number n_δ such that $\mathbb{P}(\{|\xi| \geq n\}) \geq \frac{1}{2}\delta > 0$ for all $n \geq n_\delta$, and hence we have

$$\int_\Omega |\xi(\omega)|\mathbb{P}(\mathrm{d}\omega) \geq \int_{\{|\xi|\geq n\}} |\xi(\omega)|\mathbb{P}(\mathrm{d}\omega) \geq n\mathbb{P}(\{|\xi| \geq n\}) \geq n\cdot\frac{\delta}{2} \to \infty,$$

as $n \to \infty$. This contradicts the given condition that ξ is integrable. Now, we set $\eta_m = |\xi|1_{\{|\xi|>m\}}$ for each $m = 1, 2, \ldots$. We have

$$\eta_m = |\xi|1_{\{|\xi|>m\}} \geq \eta_{m+1} = |\xi|1_{\{|\xi|>m+1\}} \to 0, \quad \text{as} \quad m \to \infty,$$

on the set $\{|\xi| < \infty\}$. That is, $\eta_m \downarrow 0$ a.s. on Ω. By the monotone convergence theorem for integral, we have

$$\lim_{m\to\infty} \int_{\{|\xi|>m\}} |\xi(\omega)|\mathbb{P}(\mathrm{d}\omega) = \lim_{m\to\infty} \int_\Omega \eta_m(\omega)\mathbb{P}(\mathrm{d}\omega) = \int_\Omega \lim_{m\to\infty} \eta_m(\omega)\mathbb{P}(\mathrm{d}\omega) = 0.$$

Thus, for every $\varepsilon > 0$ there exists an $M > 0$ such that the inequality $(*)$ holds.

Sufficiency. Assume that the inequality $(*)$ holds for every $\varepsilon > 0$. Taking $\varepsilon = 1$, there exists an $M > 0$ such that

$$\int_{\{|\xi|>M\}} |\xi(\omega)|\mathbb{P}(\mathrm{d}\omega) < 1.$$

Then, we have

$$\begin{aligned}\mathbb{E}[|\xi|] &= \int_\Omega |\xi(\omega)|\mathbb{P}(\mathrm{d}\omega) = \int_{\{|\xi|>M\}} |\xi(\omega)|\mathbb{P}(\mathrm{d}\omega) + \int_{\{|\xi|\leq M\}} |\xi(\omega)|\mathbb{P}(\mathrm{d}\omega) \\ &< 1 + M\cdot\mathbb{P}(\{|\xi| \leq M\}) \leq 1 + M < \infty.\end{aligned}$$

The sufficiency is proved. □

Generalize from a single Variable to a sequence.

Comes the Def. of uniformly integrable.

Thus, for any sequence ξ_n of integrable random variables and any $\varepsilon > 0$ there is a sequence of numbers $M_n > 0$ such that

$$\int_{\{|\xi_n|>M_n\}} |\xi_n| \, dP < \varepsilon.$$

If the M_n are independent of n, then we say that the sequence ξ_n is uniformly integrable.

$\forall n\in\mathbb{N}$, $M_n = M$. 一统所有 ξ_n

Definition 4.2. A sequence $\xi_1, \xi_2, \ldots$ of random variables is called *uniformly integrable* if for every $\varepsilon > 0$ there exists an $M > 0$ such that

$$\int_{\{|\xi_n|>M\}} |\xi_n(\omega)|\mathbb{P}(\mathrm{d}\omega) < \varepsilon$$

for all $n = 1, 2, \ldots$.

Exercise 4.4

Let $\Omega = [0, 1]$ with the σ-field of Borel sets and Lebesgue measure. Take

$$\xi_n = n1_{(0,\frac{1}{n})}.$$

Show that the sequence $\xi_1, \xi_2, \ldots$ is not uniformly integrable.

Hint What is the integral of ξ_n over $\{\xi_n > M\}$ if $n > M$?

* 因为 $n>M$，所以若 $n1_{(0,\frac{1}{n})}$ 要大于 M 则 $1_{(0,\frac{1}{n})}$ 需 $=1$.

Solution. Let $I_n = (0, \frac{1}{n})$. For any $M > 0$ and any $n > M$, we have that

$$\{\xi_n > M\} = \{n1_{(0,\frac{1}{n})} > M\} \underset{*}{=} I_n,$$

so

$$\int_{\{\xi_n>M\}} \xi_n \mathrm{d}\mathbb{P} = \int_{(0,\frac{1}{n})} n1_{(0,\frac{1}{n})}\mathrm{d}\mathbb{P} = \int_{(0,\frac{1}{n})} n\mathrm{d}\mathbb{P} = n\cdot\mathbb{P}(I_n) = 1.$$

This means that there does not exists an $M > 0$ such that for all n

$$\int_{\{\xi_n>M\}} \xi_n \mathrm{d}\mathbb{P} < \frac{1}{2}.$$

According to the definition, the sequence $\xi_1, \xi_2, \ldots$ is not uniformly integrable. □

Lemma 4.2 条件

If a random variable ξ is integrable, then, for every $\varepsilon > 0$, there is a $\delta > 0$ such that, for any $A \in \mathcal{F}$,

$$\text{if} \quad \mathbb{P}(A) < \delta \Longrightarrow \int_A |\xi(\omega)| \mathbb{P}(d\omega) < \varepsilon.$$

Proof. Let $\varepsilon > 0$. Since ξ is integrable, by Exercise 4.3 there exists an $M > 0$ such that

$$\int_{\{|\xi|>M\}} |\xi(\omega)| \mathbb{P}(d\omega) < \frac{\varepsilon}{2}.$$

Now, for any event A, we have

$$\begin{aligned} &\int_A |\xi(\omega)| \mathbb{P}(d\omega) \\ = &\int_{A\cap\{|\xi|\le M\}} |\xi(\omega)| \mathbb{P}(d\omega) + \int_{A\cap\{|\xi|>M\}} |\xi(\omega)| \mathbb{P}(d\omega) \\ \le &\int_A M\mathbb{P}(d\omega) + \int_{\{|\xi|>M\}} |\xi(\omega)| \mathbb{P}(d\omega) \; < \; M\mathbb{P}(A) + \frac{\varepsilon}{2}. \end{aligned}$$

Let $\delta = \varepsilon/(2M)$. Then,

$$\mathbb{P}(A) < \delta \Longrightarrow \int_A |\xi(\omega)| \mathbb{P}(d\omega) < \varepsilon.$$

The lemma is proved.□

Proof. Let $\epsilon > 0$, since ξ is integrable, by Ex. 4.3 there exists $M > 0$ s.t. $\int_{\{|\xi|>M\}} |\xi| dP < \frac{\epsilon}{2}$

For any event $A \in \mathcal{F}$,

$$\int_A |\xi| dP = \int_{A\cap\{|\xi|\le M\}} |\xi| dP + \int_{A\cap\{|\xi|>M\}} |\xi| dP$$

直接换

$$\le \int_A M dP + \int_{\{|\xi|>M\}} |\xi| dP$$

$$< MP(A) + \frac{\epsilon}{2}$$ 取M使其 $< \epsilon$

Let $\delta = \frac{\epsilon}{2M}$ then if $P(A) < \delta$ then

$$\int_A |\xi| dp < M \cdot \frac{\epsilon}{2M} + \frac{\epsilon}{2} = \frac{\epsilon}{2}$$

□.

Exercise 4.5

Let ξ be an integrable random variable and $\mathcal{F}_1, \mathcal{F}_2, \ldots$ a filtration. Show that $E(\xi|\mathcal{F}_n)$ is a uniformly integrable martingale.

Hint Use Lemma 4.2.

Solution 4.5

In Example 3.4 it was verified that $\xi_n = E(\xi|\mathcal{F}_n)$ is a martingale. Let $\varepsilon > 0$. By Lemma 4.2 there is a $\delta > 0$ such that

$$P(A) < \delta \Longrightarrow \int_A |\xi| \, dP < \varepsilon.$$

By Jensen's inequality $|\xi_n| \le E(|\xi| \,|\mathcal{F}_n)$ a.s., so

$$E(|\xi|) \ge E(|\xi_n|) \ge \int_{\{|\xi_n|\ge M\}} |\xi_n| \, dP \ge MP\{|\xi_n| > M\}.$$

If we take $M > E(|\xi|)/\delta$, then

$$P\{|\xi_n| > M\} < \delta.$$

Since $\{|\xi_n| > M\} \in \mathcal{F}_n$, it follows that

$$\int_{\{|\xi_n|>M\}} |\xi_n| \, dP \le \int_{\{|\xi_n|>M\}} E(|\xi| \,|\mathcal{F}_n) \, dP = \int_{\{|\xi_n|>M\}} |\xi| \, dP < \varepsilon,$$

proving that $\xi_n = E(\xi|\mathcal{F}_n)$ is a uniformly integrable sequence.

Proof:

From Example 3.4, $\xi_n = E[\xi|F_n]$ is a Martingale.

By Lemma 4.2, as ξ is integrable so

$\forall \epsilon>0, \exists \delta>0$ s.t. if A is any event in F and

$P(A)<\delta$ then $\int_A |\xi|\, dP < \epsilon$

By Jasen's inequality,

$$|\xi_n| = \big|E[\xi|F_n]\big| \le E\big[|\xi|\,\big|\, F_n\big]$$

$$\Rightarrow E[|\xi_n|] \le E[E[|\xi||F_n]] = E[|\xi|]$$

As $E[|\xi_n|] = \int_\Omega |\xi_n|\, dP = \int_{\{|\xi_n|\ge M\}} |\xi_n|\, dP + \int_{\{|\xi_n|\le M\}} |\xi_n|\, dP$

$$\ge \int_{\{|\xi_n|>M\}} |\xi_n|\, dP$$

$$\ge M \int 1_{\{|\xi_n|\ge M\}}\, dP$$

$$= M P(\{|\xi_n|\ge M\})$$

so $P(\{|\xi_n|\ge M\}) \le \frac{E[|\xi_n|]}{M} < \frac{E[|\xi|]}{M}$ let $M > \frac{E[|\xi|]}{\delta}$

then $P(\{|\xi_n|>M\}) < \delta$ 使$\{|\xi_n|>M\}$满足A的条件.

Since $\{|\xi_n|>M\} \in F_n$ so $\int_{\{|\xi_n|>M\}} |\xi|\, dP < \epsilon$ $(\{|\xi_n|>M\}=A)$

$$\int_{\{|\xi_n|>M\}} |\xi_n|\, dP \le \int_{\{|\xi_n|>M\}} E[|\xi||F_n] = \int_{\{|\xi_n|>M\}} |\xi|\, dP < \epsilon$$

Thus, $\xi_n = E[\xi|F_n]$ is uniformly convergent. □

a stronger condition called ***uniform integrability***. Proposition 4.2 shows that it is a necessary condition for L^1 convergence. In Theorem 4.2 we prove that uni-

Proposition 4.2

Let $\xi_1, \xi_2, \dots$ be a sequence of integrable random variables. If ξ_n converges to ξ in L^1, i.e.

$$\mathbb{E}[|\xi_n - \xi|] = \int_\Omega |\xi_n(\omega) - \xi(\omega)| \mathbb{P}(d\omega) \longrightarrow 0,$$

as $n \to \infty$, then $\xi_1, \xi_2, \dots$ is uniformly integrable.

if ξ_n converges to ξ in L^1
then
$\xi_1, \xi_2, \cdots$ is uniformly integrable.

Remark. Proposition 4.2 means that the unform integrability is a necessary condition for the convergence in L^1.

思路: ① → ② → ③

Proof: Since $\xi_n \to \xi$ in L^1 so $\forall \epsilon>0$, there's an integer

N s.t. $E[|\xi_n - \xi|] < \frac{\epsilon}{2}$, $\forall n \ge N$.

By Lemma 4.2, $\exists \delta>0$ s.t. $\forall A \in F$, as ξ in integrable.

then if $P(A)<\delta \Rightarrow \int_A |\xi|\, dP < \frac{\epsilon}{2}$, $\int_A |\xi_n|\, dP < \frac{\epsilon}{2}$ ②

for each $n \ge N$.

(Now claim there's an $M>0$ s.t. $P(|\xi_n|>M) < \delta$, $n=1,2,\dots$)

Since $\xi_1, \xi_2, \dots$ converges in L^1, so it is bounded in L^1,

i.e. $\sup_n E[|\xi_n|] < \infty$.

Let $M = \frac{\sup_n E[|\xi_n|]}{\delta}$ then from the inequality

$E[|\xi_n|] \ge \int_{\{|\xi_n|>M\}} |\xi_n|\, dP \ge M P(\{|\xi_n|>M\})$ obtain

$$P(\{|\xi_n|>M\}) \le \frac{E[|\xi_n|]}{M} \le \frac{\sup_n E[|\xi_n|]}{M} = \delta \quad ③$$

Thus, $\forall n>N$,

$$\int_{\{|\xi_n|>M\}} |\xi_n|\, dP \le \int_{\{|\xi_n|>M\}} |\xi|\, dP + \int_{\{|\xi_n|>M\}} |\xi_n - \xi|\, dP$$

$$\le \int_{\{|\xi_n|>M\}} |\xi|\, dP + \int_\Omega |\xi_n - \xi|\, dP$$

$$= \int_{\{|\xi_n|>M\}} |\xi|\, dP + E[|\xi_n - \xi|] \quad ①$$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

i.e. $\int_{\{|\xi_n|>M\}} |\xi_n|\, dP < \epsilon$ $\forall n = 1, 2, \cdots$

So ξ_n is uniformly integrable for each $n = 1, 2, 3, \cdots$

□

Exercise 4.6

Show that a uniformly integrable sequence of random variables is bounded in L^1, i.e.

$$\sup_n E(|\xi_n|) < \infty.$$

Hint Write $E(|\xi_n|)$ as the sum of the integrals of $|\xi_n|$ over $\{|\xi_n| > M\}$ and $\{|\xi_n| \le M\}$.

$\Rightarrow$ Any uniformly integrable martingale is bounded in L^1.

$\Rightarrow \sup_n E[|E[\xi|F_n]|] < \infty.$

Solution 4.6

Because ξ_n is a uniformly integrable sequence, there is an $M > 0$ such that for all n

$$\int_{\{|\xi_n|>M\}} |\xi_n| \, dP < 1.$$

It follows that

$$\begin{aligned} E(|\xi_n|) &= \int_{\{|\xi_n|>M\}} |\xi_n| \, dP + \int_{\{|\xi_n|\le M\}} |\xi_n| \, dP \\ &< 1 + MP\{|\xi_n| \le M\} \\ &< 1 + M < \infty \end{aligned}$$

for all n, proving that ξ_n is a bounded sequence in L^1.

Proof: As ξ_n is uniformly integrable then by Exercise 4.3,

$\forall \epsilon > 0$, $\exists M > 0$ s.t. $\int_{\{|\xi_n|>M\}} |\xi_n| \, dP < \epsilon$. $\forall n \in N$.

As ϵ is an arbitrary number in R^+, so let $\epsilon := 1$

then $\int_{\{|\xi_n|>M\}} |\xi_n| \, dP < 1.$

Also, $\int_{\{|\xi_n|\le M\}} |\xi_n| dP = \int_\Omega |\xi_n| I_{\{|\xi_n|\le M\}} dP$

$\le \int_\Omega M \, dP$

$= M$

Thus, $E[|\xi_n|] = \int_\Omega |\xi_n| dP = \int_{\{|\xi_n|\le M\}} |\xi_n| dP + \int_{\{|\xi_n|>M\}} |\xi_n| dP$

$\le M + 1$

so $\{E[|\xi_n|]\}$ is bounded $\Rightarrow \sup_n E[|\xi_n|]$ exists,

i.e. $\sup_n E[|\xi_n|] < \infty$. □

If $\xi_1, \xi_2, \dots$ is uniformly integrable, (if super/sub martingale $E[\xi|F_n]$)

then

$\xi_1, \xi_2, \dots$ converges in L^1

Theorem 4.3

Every uniformly integrable supermartingale (submartingale) (Martingale) $\xi_1, \xi_2, \dots$ converges in L^1. i.e. $E[|\xi_n|] \to$ a limit. or $E[|\xi_n - \xi|] \to 0$

Proof: By Ex. 4.6, $\{\xi_n\}$ is bounded in L^1, then by Thm. 4.2

Theorem 4.2 (Doob's Martingale Convergence Theorem)

Suppose that $\xi_1, \xi_2, \dots$ is a supermartingale (with respect to a filtration $\mathcal{F}_1, \mathcal{F}_2, \dots$) (Sub MG) such that

$$\sup_n E(|\xi_n|) < \infty.$$

Then there is an integrable random variable ξ such that

$$\lim_{n\to\infty} \xi_n = \xi \quad \text{a.s.}$$

pointwise, not real convergence

there's an integrable r.v. ξ s.t. $\lim_{n\to\infty} \xi_n = \xi$. a.s.

Since $\xi_n - \xi$ can be taken in place of ξ_n, so without loss of generality, $\xi = 0$.

$\Rightarrow P(\lim_{n\to\infty} \xi_n = 0) = 1 \Rightarrow \xi_n \to 0$ when $n \to \infty$.

i.e. $\forall \epsilon > 0$, $P(|\xi_n| > \epsilon) \to 0$ as $n \to \infty$.

This is because by Fatou's Lemma,

$0 \le \liminf_n P(|\xi_n| > \epsilon) \le \limsup_n P(|\xi_n| > \epsilon)$

$= \limsup_n \int_\Omega I_{\{|\xi_n|>\epsilon\}} dP$

$\le \int_\Omega \limsup_n I_{\{|\xi_n|>\epsilon\}} dP$

$= \int_\Omega I_{\{\limsup_n |\xi_n| > \epsilon\}} dP$

$= P(\{\limsup_n |\xi_n| > \epsilon\})$

$\le P(\Omega \setminus \{\lim_n \xi_n = 0\})$

$= 0$ as $P(\lim_{n\to\infty} \xi_n = 0) = 1$

Let $\epsilon > 0$ be fixed, as ξ_n is uniformly integrable for each $n = 1, 2, 3, \dots$

so $\exists M > 0$ s.t. $\int_{\{|\xi_n| > M\}} |\xi_n| \, dP \le \frac{\epsilon}{3}$ ①, $\forall n \in \mathbb{N}$.

Since $\xi_n \to 0$ when $n \to \infty$ so $\exists k \in \mathbb{N}$ s.t. $\forall n \ge k$,

$P(|\xi_n| > \frac{\epsilon}{3}) < \frac{\epsilon}{3M}$ ② where M is assumed to be larger than $\frac{\epsilon}{3}$ i.e. $\frac{\epsilon}{3M} < 1$

Thus,

$$E[|\xi_n|] = \int_{\{|\xi_n| > M\}} |\xi_n| \, dP + \int_{\{\frac{\epsilon}{3} < |\xi_n| \le M\}} |\xi_n| \, dP + \int_{\{|\xi_n| \le \frac{\epsilon}{3}\}} |\xi_n| \, dP$$

$$\le \frac{\epsilon}{3} + \int_{\{|\xi_n| \le M\}} |\xi_n| I_{\{|\xi_n| > \frac{\epsilon}{3}\}} \, dP + \frac{\epsilon}{3} \underbrace{P(\{|\xi_n| \le \frac{\epsilon}{3}\})}_{\le 1}$$

$$\le \frac{\epsilon}{3} + M P(\{|\xi_n| > \frac{\epsilon}{3}\}) + \frac{\epsilon}{3}$$

$$< \frac{\epsilon}{3} + M \cdot \frac{\epsilon}{3M} + \frac{\epsilon}{3}$$

$$= \epsilon$$

$\Rightarrow E[|\xi_n|] \to 0$ as $n \to \infty$, i.e. ξ_n convergents to 0 in L^1. $\square$.

WTS: $\xi := 0$ $\forall \epsilon > 0$

$E[|\xi_n - 0|] < \epsilon$

i.e. $\int_\Omega |\xi_n| \, dP < \epsilon$

拆 Ω.

Theorem 4.4

Let $\xi_1, \xi_2, \dots$ be a uniformly integrable martingale. Then,

$$\boxed{\xi_n = \mathbb{E}[\xi \mid \mathcal{F}_n], \quad n = 1, 2, \dots} \text{ 结论.}$$

where $\xi = \lim_{n\to\infty} \xi_n$ is the limit of ξ_n in L^1, i.e.

$$\lim_{n\to\infty} \mathbb{E}[|\xi_n - \xi|] = 0,$$

and $\mathcal{F}_n = \sigma\{\xi_1, \dots, \xi_n\}$ is the filtration generated by $\xi_1, \xi_2, \dots$.

Def. of Conditional Prob.:

$\forall A \in \mathcal{F}_n$, $E[\xi | \mathcal{F}_n] := \int_A \xi \, dP$

WTS: $\forall A \in \mathcal{F}_n$, $\int_A \xi_n \, dP = \int_A \xi \, dP$.

Proof: Since $\{\xi_n\}$ is a martingale so $\forall m > n$,

$E[\xi_m | \mathcal{F}_n] = \xi_n$

as $\forall$ event $A \in \mathcal{F}_n$, $\int_A E[\xi_m | \mathcal{F}_n] \, dP = \int_A \xi_m \, dP$

so $\int_A \xi_m \, dP = \int_A \xi_n \, dP$

Since n is an arbitrary natural number, m is any natural number larger than n, and A is an arbitrary event in $\mathcal{F}_n$, let $\xi = \lim_{n\to\infty} \xi_n$ then

$0 \le \left| \int_A (\xi_n - \xi) \, dP \right| = \left| \int_A (\xi_m - \xi) \, dP \right|$

$\le \int_A |\xi_m - \xi| \, dP$

$\le \int_\Omega |\xi_m - \xi| \, dP$

$= E[|\xi_m - \xi|] \to 0$ when $m \to \infty$

therefore $\left| \int_A (\xi_n - \xi) \, dP \right| = 0$ when $n \to \infty$

$\Rightarrow \int_A \xi_n \, dP = \int_A \xi \, dP$

for any $A \in \mathcal{F}_n$ when $n \to \infty$.

Recall that $\int_A \xi \, dP = \int_A E[\xi | \mathcal{F}_n] \, dP$

for any $A \in \mathcal{F}_n$.

so $\xi_n = E[\xi | \mathcal{F}_n]$ $\square$.

Proof. Since $\xi_1, \xi_2, \dots$ is a martingale, for any $m > n$,

$$\mathbb{E}[\xi_m \mid \mathcal{F}_n] = \xi_n,$$

i.e. for any $A \in \mathcal{F}_n$,

$$\int_A \xi_m(\omega)\mathbb{P}(d\omega) = \int_A \xi_n(\omega)\mathbb{P}(d\omega).$$

Let n be an arbitrary integer and $A \in \mathcal{F}_n$. For any $m > n$, we have

$$\left|\int_A (\xi_n(\omega) - \xi(\omega))\mathbb{P}(d\omega)\right| = \left|\int_A (\xi_m(\omega) - \xi(\omega))\mathbb{P}(d\omega)\right|$$
$$\leq \int_A |\xi_m(\omega) - \xi(\omega)|\mathbb{P}(d\omega) \leq \mathbb{E}[|\xi_m - \xi|] \to 0$$

as $m \to \infty$.

It follows that

$$\int_A \xi_n(\omega)\mathbb{P}(d\omega) = \int_A \xi(\omega)\mathbb{P}(d\omega)$$

holds for any $A \in \mathcal{F}_n$. According to the definition of conditional expectation, we conclude that $\xi_n = \mathbb{E}[\xi \mid \mathcal{F}_n]$. The theorem is proved.□

ξ_n is MG + ξ_n converges in L^1

$\Downarrow$

ξ_n is uniformly integrable $\forall n \in \mathbb{N}$

$\Downarrow$

$\xi_n = E[\xi | F_n]$ where ξ is the limit of ξ_n in L^1.

Corollary of Theorem 4.4 (Exercise 4.7)

If $\xi_1, \xi_2, \dots$ is a martingale and $\xi_n \to a$ in L^1 for some $a \in \mathbb{R}$, then $\xi_n = a$ a.s. for each n.

Proof: Since $\xi_n \to a$ in L^1 for some $a \in \mathbb{R}$, so by Proposition 4.2, ξ_n is uniformly integrable for $\forall n \in \mathbb{N}$.

As proved before, $\xi_n = E[\xi | F_n]$* is a martingale $\forall n \in \mathbb{N}$.

so by Theorem 4.4, $\xi_n = E[a | F_n] = a$ a.s. $\forall n \in \mathbb{N}$.

as $a \in \mathbb{R}$ □

* where ξ is the limit of ξ_n in L^1

Proof. Since $\xi_n \to a$ in L^1, according to Proposition 4.2, we have that $\xi_1, \xi_2, \dots$ is uniformly integrable. Hence, ξ_n is a uniformly integrable martingale. By Theorem 4.4, we have

$$\xi_n = \mathbb{E}[a \mid \mathcal{F}_n] = a, \quad \text{a.s.}$$

for each n. □

Remark: 遇见证 $\xi_1, \xi_2 \dots$ uniformly convergent 需要证 $\exists \delta > 0$ s.t. $P(\{|\xi_n| > m\}) < \delta$ 时:

常用垫脚石:

$$\int_\Omega |\xi_n| \, dP = \int_{\{|\xi_n| > m\}} |\xi_n| \, dP + \int_{\{|\xi_n| < m\}} |\xi_n| \, dP$$

$$\Rightarrow \int_\Omega |\xi_n| \, dP \geq \int_{\{|\xi_n| > m\}} |\xi_n| \, dP$$

As $\int_{\{|\xi_n| > M\}} |\xi_n| \, dP = \int_\Omega |\xi_n| \cdot \mathbb{1}_{\{|\xi_n| > M\}} \, dP > M \, P(\{|\xi_n| > M\})$

and $\quad E[|\xi_n|] = \int_\Omega |\xi_n| \, dP$

so $\quad E[|\xi_n|] \geq \int_{\{|\xi_n| > M\}} |\xi_n| \, dP \geq M \, P(\{|\xi_n| > M\})$

$$\Rightarrow P(\{|\xi_n| > M\}) \leq \frac{E[|\xi_n|]}{M} \quad (1)$$

if ξ_n converges in L^1 so $E[|\xi_n|]$ is bounded above, therefore, $\sup E[|\xi_n|] < \infty$. exists.

$\Rightarrow E[|\xi_n|] \leq \sup_n E[|\xi_n|] < \infty$.

Then, from (1),

$$P(\{|\xi_n| \geq M\}) \leq \frac{\sup E[|\xi_n|]}{M}$$

4.4 Kolmogorov's 0-1 Law

都不考

Theorem 4.5 (Kolmogorov's 0-1 law)

Let $\eta_1, \eta_2, \dots$ be a sequence of independent random variables. Define the tail σ-field:

$$\mathcal{T} = \mathcal{T}_1 \cap \mathcal{T}_2 \cap \cdots,$$

where $\mathcal{T}_n = \sigma\{\eta_n, \eta_{n+1}, \dots\}$. Then, for any $A \in \mathcal{T}$,

$$\mathbb{P}(A) = 0 \quad \text{or} \quad \mathbb{P}(A) = 1.$$

$\mathcal{T}_n$ is a contracting sequence

Eg. $A = \{\lim_{n \to \infty} X_n \text{ exists}\}$ is a tail event as proved above.

则 $\lim_{n \to \infty} X_n$ 只有存在/不存在2种状态.

即 $P(A) = 1$ or $P(A) = 0$.

Exercise 4.8. Let $\eta_1, \eta_2, \dots$ be a sequence of independent random variables. Assume that $A_n \in \sigma(\eta_n)$ for each n, and define

$$\operatorname{limsup}_n A_n = \bigcap_{j=1}^{\infty} \bigcup_{i=j}^{\infty} A_i \quad \text{and} \quad \operatorname{liminf}_n A_n = \bigcup_{j=1}^{\infty} \bigcap_{i=j}^{\infty} A_i.$$

Show that $\operatorname{limsup}_n A_n$ and $\operatorname{liminf}_n A_a$ belong to the tail σ-field $\mathcal{T}$.

Remark. Note that

$$\operatorname{limsup}_n A_n = \{\text{There are infinitely many } A_n \text{ occur.}\}.$$

Exercise 4.9. Show that in a sequence of coin tosses there are a.s. infinitely many heads.