Section 8.1 Pointwise and Uniform Convergence

8.1.1 Definition Let (f_n) be a sequence of functions on $A \subseteq \mathbb{R}$ to $\mathbb{R}$, let $A_0 \subseteq A$, and let $f : A_0 \to \mathbb{R}$. We say that the **sequence** (f_n) **converges on** A_0 **to** f if, for each $x \in A_0$, the sequence $(f_n(x))$ converges to $f(x)$ in $\mathbb{R}$. In this case we call f the **limit on** A_0 **of the sequence** (f_n). When such a function f exists, we say that the sequence (f_n) **is convergent on** A_0, or that (f_n) **converges pointwise on** A_0.

In order to symbolize that the sequence (f_n) converges on A_0 to f, we sometimes write

$$f = \lim(f_n) \quad \text{on} \quad A_0, \quad \text{or} \quad f_n \to f \quad \text{on} \quad A_0.$$

Sometimes, when f_n and f are given by formulas, we write

$$f(x) = \lim f_n(x) \quad \text{for} \quad x \in A_0, \quad \text{or} \quad f_n(x) \to f(x) \quad \text{for} \quad x \in A_0.$$

8.1.3 Lemma *A sequence (f_n) of functions on $A \subseteq \mathbb{R}$ to $\mathbb{R}$ converges to a function $f : A_0 \to \mathbb{R}$ on A_0 if and only if for each $\varepsilon > 0$ and each $x \in A_0$ there is a natural number $K(\varepsilon, x)$ such that if $n \geq K(\varepsilon, x)$, then*

$$|f_n(x) - f(x)| < \varepsilon. \tag{3}$$

We leave it to the reader to show that this is equivalent to Definition 8.1.1. We wish to emphasize that the value of $K(\varepsilon, x)$ will depend, in general, on *both* $\varepsilon > 0$ and $x \in A_0$. The reader should confirm the fact that in Examples 8.1.2(a–c), the value of $K(\varepsilon, x)$ required to obtain an inequality such as (3) does depend on both $\varepsilon > 0$ and $x \in A_0$. The intuitive reason for this is that the convergence of the sequence is "significantly faster" at some points than it is at others. However, in Example 8.1.2(d), as we have seen in inequality (2), if we choose n sufficiently large, we can make $|F_n(x) - F(x)| < \varepsilon$ for *all* values of $x \in \mathbb{R}$. It is precisely this rather subtle difference that distinguishes between the notion of the "pointwise convergence" of a sequence of functions (as defined in Definition 8.1.1) and the notion of "uniform convergence."

理解 "converge significantly faster"

8.1.2 Examples **(a)** $\lim(x/n) = 0$ for $x \in \mathbb{R}$.

For $n \in \mathbb{N}$, let $f_n(x) := x/n$ and let $f(x) := 0$ for $x \in \mathbb{R}$. By Example 3.1.6(a), we have $\lim(1/n) = 0$. Hence it follows from Theorem 3.2.3 that

$$\lim(f_n(x)) = \lim(x/n) = x \lim(1/n) = x \cdot 0 = 0$$

for all $x \in \mathbb{R}$. (See Figure 8.1.1.)

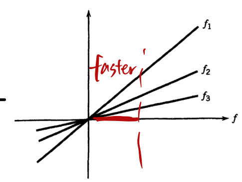


Figure 8.1.1 $f_n(x) = x/n$

(b) $\lim(x^n)$.

Let $g_n(x) := x^n$ for $x \in \mathbb{R}, n \in \mathbb{N}$. (See Figure 8.1.2.) Clearly, if $x = 1$, then the sequence $(g_n(1)) = (1)$ converges to 1. It follows from Example 3.1.11(b) that $\lim(x^n) = 0$ for $0 \leq x < 1$ and it is readily seen that this is also true for $-1 < x < 0$. If $x = -1$, then $g_n(-1) = (-1)^n$, and it was seen in Example 3.2.8(b) that the sequence is divergent. Similarly, if $|x| > 1$, then the sequence (x^n) is not bounded, and so it is not convergent in $\mathbb{R}$. We conclude that if

$$g(x) := \begin{cases} 0 & \text{for} \quad -1 < x < 1, \\ 1 & \text{for} \quad x = 1, \end{cases}$$

then the sequence (g_n) converges to g on the set $(-1, 1]$.

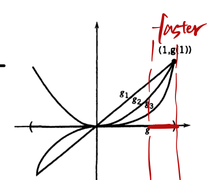


Figure 8.1.2 $g_n(x) = x^n$

$K(x, \varepsilon)$

CHAPTER 8

SEQUENCES OF FUNCTIONS

(c) $\lim((x^2+nx)/n) = x$ for $x \in \mathbb{R}$.

Let $h_n(x) := (x^2+nx)/n$ for $x \in \mathbb{R}$, $n \in \mathbb{N}$, and let $h(x) := x$ for $x \in \mathbb{R}$. (See Figure 8.1.3.) Since we have $h_n(x) = (x^2/n) + x$, it follows from Example 3.1.6(a) and Theorem 3.2.3 that $h_n(x) \to x = h(x)$ for all $x \in \mathbb{R}$.

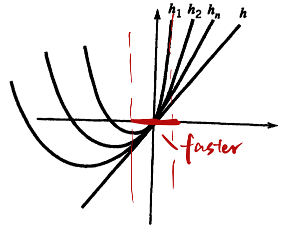


Figure 8.1.3 $h_n(x) = (x^2+nx)/n$

$\leq \frac{1}{n}$ (no x anymore $\Rightarrow$ K doesn't depend on x)

(d) $\lim((1/n)\sin(nx+n)) = 0$ for $x \in \mathbb{R}$.

Let $F_n(x) := (1/n)\sin(nx+n)$ for $x \in \mathbb{R}$, $n \in \mathbb{N}$, and let $F(x) := 0$ for $x \in \mathbb{R}$. (See Figure 8.1.4.) Since $|\sin y| \leq 1$ for all $y \in \mathbb{R}$ we have

$$|F_n(x) - F(x)| = \left|\frac{1}{n}\sin(nx+n)\right| \leq \frac{1}{n} \tag{2}$$

for all $x \in \mathbb{R}$. Therefore it follows that $\lim(F_n(x)) = 0 = F(x)$ for all $x \in \mathbb{R}$. The reader should note that, given any $\varepsilon > 0$, if n is sufficiently large, then $|F_n(x) - F(x)| < \varepsilon$ for all values of x simultaneously! □

$K(\varepsilon)$

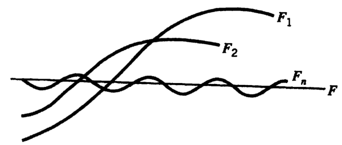


Figure 8.1.4 $F_n(x) = \sin(nx+n)/n$

Uniform Convergence

8.1.4 Definition A sequence (f_n) of functions on $A \subseteq \mathbb{R}$ to $\mathbb{R}$ **converges uniformly on** $A_0 \subseteq A$ to a function $f : A_0 \to \mathbb{R}$ if for each $\varepsilon > 0$ there is a natural number $K(\varepsilon)$ (depending on ε but **not** on $x \in A_0$) such that if $n \geq K(\varepsilon)$, then

$$|f_n(x) - f(x)| < \varepsilon \quad \text{for all} \quad x \in A_0. \tag{4}$$

Uniform Convergent 判定方法 1

In this case we say that the sequence (f_n) is **uniformly convergent on** A_0. Sometimes we write

≈ uniform continuous 的proof.

$$f_n \rightrightarrows f \quad \text{on} \quad A_0, \qquad \text{or} \qquad f_n(x) \rightrightarrows f(x) \quad \text{for} \quad x \in A_0.$$

Negation:

$f_{n_k}(x_k)$

8.1.5 Lemma *A sequence (f_n) of functions on $A \subseteq \mathbb{R}$ to $\mathbb{R}$ does not converge uniformly on $A_0 \subseteq A$ to a function $f : A_0 \to \mathbb{R}$ if and only if for some $\varepsilon_0 > 0$ there is a subsequence (f_{n_k}) of (f_n) and a sequence (x_k) in A_0 such that*

$$|f_{n_k}(x_k) - f(x_k)| \geq \varepsilon_0 \quad \text{for all} \quad k \in \mathbb{N}. \tag{5}$$

理解为a sequence (x_k). $x_k \in A_0$ [可非连续选择]

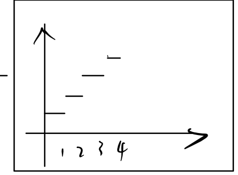
It is an immediate consequence of the definitions that if the sequence (f_n) is uniformly convergent on A_0 to f, then this sequence also converges pointwise on A_0 to f in the sense of Definition 8.1.1. That the converse is not always true is seen by a careful examination of Examples 8.1.2(a–c); other examples will be given below.

Uniform $\Rightarrow$ pointwise

$\not\Leftarrow$

Proof:

8.1.6 Examples **(a)** Consider Example 8.1.2(a). If we let $n_k := k$ and $x_k := k$, then $f_{n_k}(x_k) = 1$ so that $|f_{n_k}(x_k) - f(x_k)| = |1 - 0| = 1$. Therefore the sequence (f_n) does not converge uniformly on $\mathbb{R}$ to f.



(b) Consider Example 8.1.2(b). If $n_k := k$ and $x_k := \left(\frac{1}{2}\right)^{1/k}$, then

$$|g_{n_k}(x_k) - g(x_k)| = \left|\tfrac{1}{2} - 0\right| = \tfrac{1}{2}. \quad g_{n_k}(x_k) = \left[\left(\tfrac{1}{2}\right)^{\frac{1}{k}}\right]^k$$

Therefore the sequence (g_n) does not converge uniformly on $(-1, 1]$ to g.

(c) Consider Example 8.1.2(c). If $n_k := k$ and $x_k := -k$, then $h_{n_k}(x_k) = 0$ and $h(x_k) = -k$ so that $|h_{n_k}(x_k) - h(x_k)| = k$. Therefore the sequence (h_n) does not converge uniformly on $\mathbb{R}$ to h. □

The Uniform Norm — 定义对象: A set of Bounded Functions.

8.1.7 Definition If $A \subseteq \mathbb{R}$ and $\varphi : A \to \mathbb{R}$ is a function, we say that φ is **bounded on** A if the set $\varphi(A)$ is a bounded subset of $\mathbb{R}$. If φ is bounded we define the **uniform norm of** φ **on** A by

$$(6) \qquad ||\varphi||_A := \boxed{\sup}\{|\varphi(x)| : x \in A\}. \longrightarrow \text{不一定} \in \{|\varphi(x)|\}$$

Note that it follows that if $\varepsilon > 0$, then

$$(7) \qquad ||\varphi||_A \le \varepsilon \iff |\varphi(x)| \le \varepsilon \quad \text{for all} \quad x \in A.$$

8.1.8 Lemma *A sequence (f_n) of bounded functions on $A \subseteq \mathbb{R}$ converges uniformly on A to f if and only if $||f_n - f||_A \to 0$.* — Uniform Convergent 判定方法 2

Proof. ($\Rightarrow$) If (f_n) converges uniformly on A to f, then by Definition 8.1.4, given any $\varepsilon > 0$ there exists $K(\varepsilon)$ such that if $n \ge K(\varepsilon)$ and $x \in A$ then

$$|f_n(x) - f(x)| \le \varepsilon.$$

From the definition of supremum, it follows that $||f_n - f||_A \le \varepsilon$ whenever $n \ge K(\varepsilon)$. Since $\varepsilon > 0$ is arbitrary this implies that $||f_n - f||_A \to 0$.

($\Leftarrow$) If $||f_n - f||_A \to 0$, then given $\varepsilon > 0$ there is a natural number $H(\varepsilon)$ such that if $n \ge H(\varepsilon)$ then $||f_n - f||_A \le \varepsilon$. It follows from (7) that $|f_n(x) - f(x)| \le \varepsilon$ for all $n \ge H(\varepsilon)$ and $x \in A$. Therefore (f_n) converges uniformly on A to f. Q.E.D.

8.1.9 Examples **(a)** We cannot apply Lemma 8.1.8 to the sequence in Example 8.1.2(a) since the function $f_n(x) - f(x) = x/n$ is not bounded on $\mathbb{R}$.

For the sake of illustration, let $A := [0, 1]$. Although the sequence (x/n) did not converge uniformly on $\mathbb{R}$ to the zero function, we shall show that the convergence is uniform on A. To see this, we observe that

$$||f_n - f||_A = \sup\{|x/n - 0| : 0 \le x \le 1\} = \frac{1}{n}$$

① 先求 sup||fn-f||

② 再对 sup||fn-f|| 求 n→∞ 极限

so that $||f_n - f||_A \to 0$. Therefore (f_n) is uniformly convergent on A to f.

(b) Let $g_n(x) := x^n$ for $x \in A := [0, 1]$ and $n \in \mathbb{N}$, and let $g(x) := 0$ for $0 \le x < 1$ and $g(1) := 1$. The functions $g_n(x) - g(x)$ are bounded on A and

$$\|g_n - g\|_A = \sup \begin{Bmatrix} x^n & \text{for} \quad 0 \le x < 1 \\ 0 & \text{for} \quad x = 1 \end{Bmatrix} = 1$$

#supreme不一定属于 {|gn-g|}

DON'T FORGET

for any $n \in \mathbb{N}$. Since $\|g_n - g\|_A$ does *not* converge to 0, we infer that the sequence (g_n) does *not* converge uniformly on A to g.

n=1, x很接近1时, $\|g_1 - g\|_A$ 几乎与1相等.

(e) Let $G(x) := x^n(1 - x)$ for $x \in A := [0, 1]$. Then the sequence $(G_n(x))$ converges to $G(x) := 0$ for each $x \in A$. To calculate the uniform norm of $G_n - G = G_n$ on A, we find the derivative and solve

$$G_n'(x) = x^{n-1}(n - (n+1)x) = 0$$

to obtain the point $\boxed{x_n := n/(n+1)}$. This is an interior point of [0, 1], and it is easily verified by using the First Derivative Test 6.2.8 that G_n attains a maximum on [0, 1] at x_n. Therefore, we obtain

$$\|G_n\|_A = G_n(x_n) = \underbrace{\boxed{(1 + 1/n)^{-n}}}_{e} \underbrace{\frac{1}{n+1}}_{0},$$

which converges to $(1/e) \cdot 0 = 0$. Thus we see that convergence is uniform on A. □

有Bounded 才能谈 uniform Norm

#

8.1.10 Cauchy Criterion for Uniform Convergence *Let (f_n) be a sequence of bounded functions on $A \subseteq \mathbb{R}$. Then this sequence converges uniformly on A to a bounded function f if and only if for each $\varepsilon > 0$ there is a number $H(\varepsilon)$ in $\mathbb{N}$ such that for all $m, n \ge H(\varepsilon)$, then $\|f_m - f_n\|_A \le \varepsilon$.* □

Uniform Convergent 判定方法 3

Proof. (⇒) If $f_n \rightrightarrows f$ on A, then given $\varepsilon > 0$ there exists a natural number $K(\frac{1}{2}\varepsilon)$ such that if $n \ge K(\frac{1}{2}\varepsilon)$ then $\|f_n - f\|_A \le \frac{1}{2}\varepsilon$. Hence, if both $m, n \ge K(\frac{1}{2}\varepsilon)$, then we conclude that

$$|f_m(x) - f_n(x)| \le |f_m(x) - f(x)| + |f_n(x) - f(x)| \le \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon$$

for all $x \in A$. Therefore $\|f_m - f_n\|_A \le \varepsilon$ for $m, n \ge K(\frac{1}{2}\varepsilon) =: H(\varepsilon)$.

(⇐) Conversely, suppose that for $\varepsilon > 0$ there is $H(\varepsilon)$ such that if $m, n \ge H(\varepsilon)$, then $\|f_m - f_n\|_A \le \varepsilon$. Therefore, for each $x \in A$ we have

$$(8) \qquad |f_m(x) - f_n(x)| \le \|f_m - f_n\|_A \le \varepsilon \quad \text{for} \quad m, n \ge H(\varepsilon).$$

It follows that $(f_n(x))$ is a Cauchy sequence in $\mathbb{R}$; therefore, by Theorem 3.5.5, it is a convergent sequence. We define $f : A \to \mathbb{R}$ by

$$f(x) := \lim(f_n(x)) \quad \text{for} \quad x \in A.$$

If we let $n \to \infty$ in (8), it follows from Theorem 3.2.6 that for each $x \in A$ we have

$$|f_m(x) - f(x)| \le \varepsilon \quad \text{for} \quad m \ge H(\varepsilon).$$

Therefore the sequence (f_n) converges uniformly on A to f. Q.E.D.

Proof: $\boxed{\Leftarrow}$ Suppose $\exists H(\varepsilon) \in \mathbb{N}$ s.t. $\forall m, n \ge H(\varepsilon)$,

$$\|f_m - f_n\|_A \le \varepsilon$$

then $\sup\{|f_m(x) - f_n(x)|\} \le \varepsilon$, $x \in A$.

$\Rightarrow |f_m(x) - f_n(x)| \le \varepsilon$

$\Rightarrow$ $\{f_n(x)\}$ is a cauchy sequence

therefore convergent.

Let $f(x) := \lim_{n \to \infty} \{f_n(x)\}$

then $\forall \varepsilon > 0$, $\exists K \in \mathbb{N}$ s.t. $\forall n \ge K$,

$$|f_n(x) - f(x)| < \varepsilon$$

Thus, $\{f_n(x)\}$ is uniformly convergent.

$\boxed{\Rightarrow}$ Suppose $\{f_n(x)\}$ is uniformly convergent on A to a bounded function f

then when $m, n \in \mathbb{N}$, m, n are sufficiently large

i.e. let $n \to \infty$ and $m \ge n$ then $\forall \varepsilon > 0$,

$$|f_n - f| \le \|f_n - f\|_A < \frac{\varepsilon}{2}$$

$$|f_m - f| \le \|f_m - f\|_A < \frac{\varepsilon}{2}$$

$$|f_m - f_n| = \|f_m - f_n\| = \|f_m - f - f_n + f\|_A \le \|f_n - f\|_A + \|f_m - f\|_A < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Thus, $\|f_m - f_n\|_A \le \varepsilon$

Q.E.D.

Uniform Convergent 的三个判定:

1. 定义: $\forall n \ge K(\varepsilon)$, $|f_n(x) - f| < \varepsilon$

2. Uniform Norm: $\|f_n - f\|_A \to 0$

3. Cauchy Criterion: $\forall m, n \ge H(\varepsilon)$, $\|f_m - f_n\|_A < \varepsilon$

Pointwise Convergent (Examples)

8.2.1 Examples (a) Let $g_n(x) := x^n$ for $x \in [0,1]$ and $n \in \mathbb{N}$. Then, as we have noted in Example 8.1.2(b), the sequence (g_n) converges pointwise to the function

$$g(x) := \begin{cases} 0 & \text{for } 0 \le x < 1, \\ 1 & \text{for } x = 1. \end{cases}$$

Although all of the functions g_n are continuous at $x = 1$, the limit function g is not continuous at $x = 1$. Recall that it was shown in Example 8.1.6(b) that this sequence does not converge uniformly to g on $[0, 1]$.

Deny

(1) conts.

(b) Each of the functions $g_n(x) = x^n$ in part (a) has a continuous derivative on $[0,1]$. However, the limit function g does not have a derivative at $x = 1$, since it is not continuous at that point.

(2) diff.

$g_n(x) = \frac{x^n}{n}$ on $[0,1]$ $-$ $g_n \to g = 0$

$- g_n' = \frac{n}{n} x^{n-1} = x^{n-1} \to h(x) = \begin{cases} 0, & x \in [0,1) \\ 1, & x = 1 \end{cases}$ $\Rightarrow h \neq g'$ at $x = 1$

$\Rightarrow$ The derivative of the limit $\neq$ The limit of derivative

pointwise

$\lim_{k\to\infty}[\lim_{n\to\infty} f_n(x_k)]$ $\quad$ $\lim_{n\to\infty}[\lim_{k\to\infty} f_n(x_k)]$

(c) Let $f_n : [0,1] \to \mathbb{R}$ be defined for $n \ge 2$ by

Bounded Interval

$$f_n(x) := \begin{cases} n^2 x & \text{for } 0 \le x \le 1/n, \\ -n^2(x - 2/n) & \text{for } 1/n \le x \le 2/n, \\ 0 & \text{for } 2/n \le x \le 1. \end{cases}$$

(3) R.I.

deny

bdd

(See Figure 8.2.1.) It is clear that each of the functions f_n is continuous on $[0, 1]$; hence it is Riemann integrable. Either by means of a direct calculation, or by referring to the significance of the integral as an area, we obtain

$$\int_0^1 f_n(x)\,dx = 1 \quad \text{for} \quad n \ge 2. \quad \Rightarrow \lim_{n\to\infty}\int_0^1 f = \lim_{n\to\infty}(1) = 1$$

The reader may show that $f_n(x) \to 0$ for all $x \in [0,1]$; hence the limit function f vanishes identically and is continuous (and hence integrable), and $\int_0^1 f(x)\,dx = 0$. Therefore we have the uncomfortable situation that:

$$\int_0^1 f(x)\,dx = 0 \neq 1 = \lim \int_0^1 f_n(x)\,dx.$$

Rieman Integral of the limit $\neq$ Limit of Rieman-Integral

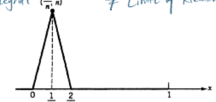


Figure 8.2.1 Example 8.2.1(c)

Interchange of Limit and Continuity

8.2.2 Theorem *Let (f_n) be a sequence of continuous functions on a set $A \subseteq \mathbb{R}$ and suppose that (f_n) converges uniformly on A to a function $f : A \to \mathbb{R}$. Then f is continuous on A.*

Proof. By hypothesis, given $\varepsilon > 0$ there exists a natural number $H := H(\frac{1}{3}\varepsilon)$ such that if $n \ge H$ then $|f_n(x) - f(x)| < \frac{1}{3}\varepsilon$ for all $x \in A$. Let $c \in A$ be arbitrary; we will show that f is continuous at c. By the Triangle Inequality we have

$$\begin{aligned} |f(x) - f(c)| &\le |f(x) - f_H(x)| + |f_H(x) - f_H(c)| + |f_H(c) - f(c)| \\ &\le \tfrac{1}{3}\varepsilon + |f_H(x) - f_H(c)| + \tfrac{1}{3}\varepsilon. \end{aligned}$$

Since f_H is continuous at c, there exists a number $\boxed{\delta := \delta(\frac{1}{3}\varepsilon, c, f_H) > 0}$ such that if $|x - c| < \delta$ and $x \in A$, then $|f_H(x) - f_H(c)| < \frac{1}{3}\varepsilon$. Therefore, if $|x - c| < \delta$ and $x \in A$, then we have $|f(x) - f(c)| < \varepsilon$. Since $\varepsilon > 0$ is arbitrary, this establishes the continuity of f at the arbitrary point $c \in A$. (See Figure 8.2.2.) Q.E.D.

Uniform Convergent

f_n is continuous $\xrightarrow{\checkmark}$ f is continuous

$\xleftarrow{\times}$

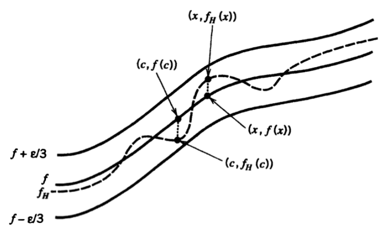


Figure 8.2.2 $|f(x) - f(c)| < \varepsilon$

Interchange of Limit and Derivative

Mean Value Theorem

闭区间连续
开区间可导

[a,b] or (a,b)

8.2.3 Theorem *Let $J \subseteq \mathbb{R}$ be a bounded interval and let (f_n) be a sequence of functions on J to $\mathbb{R}$. Suppose that there exists $x_0 \in J$ such that $(f_n(x_0))$ converges, and that the sequence (f'_n) of derivatives exists on J and converges uniformly on J to a function g.*

Then the sequence (f_n) converges uniformly on J to a function f that has a derivative at every point of J and $f' = g$.

Thm. Suppose $f_n : [a,b] \to R$ is ① continuously diff. ~~when~~

(f_n) is differentiable at every point of J.

② $f_n'(x)$ exists on $\forall x \in J$.

$f, g : [a,b] \to R$. and 此时f还未定义.

② $f_n \to \cancel{f}$ pointwisely on $[a,b]$

③ $f_n' \rightrightarrows g$ uniformly on $[a,b]$

Then ① $f_n \rightrightarrows f$

② f is differentiable at every point of J

③ $g = f'$ [i.e. $f_n' \longrightarrow g = f'$]

Limit of derivatives = Derivative of the Limit

$g = \lim_{n\to\infty} f_n'$ f_n' is uni. conv. f_n is pointwise $[\lim_{n\to\infty} f_n]'$

Proof:

Thm 8.2.3 Change of Limit : Differentiable

条件: ① f_n continuously differentiable on J (Bounded interval J ⇓ closed (闭))
② $\exists x_0 \in J$. $f_n(x_0)$ converges, pointwisely.
③ $f_n' \rightrightarrows g$ on J. (此时 f 还未定义)

结论: ① $f_n \rightrightarrows f$ on J
② f is continuously differentiable on J.
③ $f' = g$ on J.

Proof: As $\forall n \in \mathbb{N}$, f_n is continuously differentiable on the bounded interval J (let $J := (a,b)$ or $[a,b]$) then $f_m - f_n$ is continuously diff. on J.

⇒ Mean Value Thm. could be applied to closed & bdd intervals:

Define I_x be closed & bdd interval ends with x and x_0. $\forall x \in J$. then by M.V.T,

$\exists y \in I_x$ s.t.

$$(f_m - f_n)(x) - (f_m - f_n)(x_0) = (f_m - f_n)'(y)(x - x_0)$$

$$\Rightarrow (f_m - f_n)(x) = (f_m - f_n)(x_0) + (f_m - f_n)'(y)(x - x_0)$$

$$\Rightarrow |(f_m - f_n)(x)| \le |(f_m - f_n)(x_0)| + |(f_m' - f_n')(y)(x - x_0)|$$

$$\le |(f_m - f_n)(x_0)| + |(f_m'(y) - f_n'(y))(b-a)|$$

As $f_n \to f$ at $x = x_0$ and $f_n' \rightrightarrows g$ on J

⇓ Cauchy: & $|(f_m - f_n)(x_0)| < \varepsilon/2$ ⇓ Cauchy $\lim_{n\to\infty} |(f_m' - f_n')(y)| < \varepsilon/2(b-a)$

Goal 1

$$\boxed{\Rightarrow |(f_m - f_n)(x)| \le \varepsilon \Rightarrow \|f_m - f_n\|_J \to 0}$$

Thus, (f_n) is uniformly convergent on J

Denote the limit of (f_n) by f.

⇒ f is continuous on J.

(WTS: f' exists on J / $\forall c \in J$.)

For any $c \in J$, Define an interval I_x that ends with c, x $\forall x \in J$. then by M.V.T,

$\exists z \in I_x$ s.t. $(f_m - f_n)(x) - (f_m - f_n)(c) = (f_m' - f_n')(z)(x - c)$

If $x \ne c$:

$$\frac{f_m(x) - f_m(c)}{x - c} - \frac{f_n(x) - f_n(c)}{x - c} = f_m'(z) - f_n'(z)$$

$$\Rightarrow \left|\frac{f_m(x) - f_m(c)}{x - c} - \frac{f_n(x) - f_n(c)}{x - c}\right| \le \|f_m' - f_n'\|_J$$

As (f_n') converges uniformly on J so $\forall \varepsilon > 0$,

$\exists H(\varepsilon) \in \mathbb{N}$ s.t. $\forall m, n \ge H(\varepsilon)$,

$$\left|\frac{f_m(x) - f_m(c)}{x - c} - \frac{f_n(x) - f_n(c)}{x - c}\right| \le \varepsilon$$

$$\Rightarrow m \to \infty,\ n \ge H(\varepsilon).\ \left|\frac{f(x) - f(c)}{x - c} - \frac{f_n(x) - f_n(c)}{x - c}\right| \le \varepsilon/3$$

As $f_n'(c) \rightrightarrows g(c)$ so $g(c) = \lim_{n\to\infty}[f_n'(c)]$.

⇒ $\exists K(\varepsilon) \in \mathbb{N}$ s.t. $\forall n \ge K(\varepsilon)$,

$$|f_n'(c) - g(c)| \le \varepsilon/3$$

(而非 $\frac{f_n(x) - f_n(c)}{x - c}$)

⇒ Let $M := \max\{K(\varepsilon), H(\varepsilon)\}$ then $\forall n \ge M$,

Since $f_m'(c)$ exists. so $\exists \delta_m(\varepsilon) > 0$ s.t. if $0 < |x - c| < \delta_m(\varepsilon)$,

$$\left|\frac{f_m(x) - f_m(c)}{x - c} - f_m'(c)\right| < \frac{\varepsilon}{3}$$

Thus, if $0 < |x - c| < \delta_m(\varepsilon)$, then $\forall n \ge M$,

Goal 2

$$\boxed{\left|\frac{f(x) - f(c)}{x - c} - g(x)\right|} \le \left|\frac{f(x) - f(c)}{x - c} - \frac{f_n(x) - f_n(c)}{x - c}\right| + \left|\frac{f_n(x) - f(c)}{x - c} - f_n'(c)\right| + |f_n'(c) - g(c)|$$

$$< \frac{\varepsilon}{3} \times 3 = \varepsilon$$

Thus, $f'(x) = g(x)$ $\forall x \in J$

Q.E.D.

$f_n \to f_m - f_n$ continuously diff. 的应用: Mean Value Theorem (2次)

+

f_n is diff. on $\forall c \in J$

+

$f_n' \rightrightarrows g$ at $\forall c \in J$.

Proof. Let $a < b$ be the endpoints of J and let $x \in J$ be arbitrary. If $m, n \in \mathbb{N}$, we apply the Mean Value Theorem 6.2.4 to the difference $f_m - f_n$ on the interval with endpoints x_0, x. We conclude that there exists a point y (depending on m, n) such that

$$f_m(x) - f_n(x) = f_m(x_0) - f_n(x_0) + (x - x_0)\{f_m'(y) - f_n'(y)\}.$$

Hence we have

$$(1) \qquad \|f_m - f_n\|_J \le |f_m(x_0) - f_n(x_0)| + (b - a)\|f_m' - f_n'\|_J.$$

From Theorem 8.1.10, it follows from (1) and the hypotheses that $(f_n(x_0))$ is convergent and that (f_n') is uniformly convergent on J, that (f_n) is uniformly convergent on J. We denote the limit of the sequence (f_n) by f. Since the f_n are all continuous and the convergence is uniform, it follows from Theorem 8.2.2 that f is continuous on J.

To establish the existence of the derivative of f at a point $c \in J$, we apply the Mean Value Theorem 6.2.4 to $f_m - f_n$ on an interval with end points c, x. We conclude that there exists a point z (depending on m, n) such that

$$\{f_m(x) - f_n(x)\} - \{f_m(c) - f_n(c)\} = (x - c)\{f_m'(z) - f_n'(z)\}.$$

Hence, if $x \ne c$, we have

$$\left|\frac{f_m(x) - f_m(c)}{x - c} - \frac{f_n(x) - f_n(c)}{x - c}\right| \le \|f_m' - f_n'\|_J.$$

Since (f_n') converges uniformly on J, if $\varepsilon > 0$ is given there exists $H(\varepsilon)$ such that if $m, n \ge H(\varepsilon)$ and $x \ne c$, then

$$(2) \qquad \left|\frac{f_m(x) - f_m(c)}{x - c} - \frac{f_n(x) - f_n(c)}{x - c}\right| \le \varepsilon.$$

If we take the limit in (2) with respect to m and use Theorem 3.2.6, we have

$$\left|\frac{f(x) - f(c)}{x - c} - \frac{f_n(x) - f_n(c)}{x - c}\right| \le \varepsilon.$$

provided that $x \ne c$, $n \ge H(\varepsilon)$. Since $g(c) = \lim(f_n'(c))$, there exists $N(\varepsilon)$ such that if $n \ge N(\varepsilon)$, then $|f_n'(c) - g(c)| < \varepsilon$. Now let $K := \sup\{H(\varepsilon), N(\varepsilon)\}$. ~~Since $f_K'(c)$ exists~~, there exists $\delta_K(\varepsilon) > 0$ such that if $0 < |x - c| < \delta_K(\varepsilon)$, then

f_n' 在 J 上处处存在

$$\left|\frac{f_K(x) - f_K(c)}{x - c} - f_K'(c)\right| < \varepsilon.$$

Combining these inequalities, we conclude that if $0 < |x - c| < \delta_K(\varepsilon)$, then

Desired :

$$\boxed{\left|\frac{f(x) - f(c)}{x - c} - g(c)\right| < 3\varepsilon.}$$

$$\lim_{x\to c} \frac{f(x) - f(c)}{x - c} = g(c)$$

Since $\varepsilon > 0$ is arbitrary, this shows that $f'(c)$ exists and equals $g(c)$. Since $c \in J$ is arbitrary, we conclude that $f' = g$ on J. Q.E.D.

Interchange of Limit and Integral

8.2.4 Theorem *Let (f_n) be a sequence of functions in $\mathcal{R}[a,b]$ and suppose that (f_n) converges* ***uniformly*** *on $[a, b]$ to f. Then $f \in \mathcal{R}[a,b]$ and (3) holds.*

$$\int_a^b f = \lim_{n\to\infty} \int_a^b f_n. = A$$

Goal: $\left|\boxed{S(f;\dot{\mathcal{P}})} - A\right| \le \varepsilon$

Proof:

Rieman Integral 定义法:

$\|\dot{\mathcal{P}}\| \to 0, \ \left| S(f;\dot{\mathcal{P}}) - L \right| < \varepsilon$

第一步!

Proof: As $f_n \rightrightarrows f$, so by Cauchy criterion,

$\forall \varepsilon > 0, \exists H \in \mathbb{N}$ s.t. $\forall m, n \ge H$, $|f_m - f_n| \le \varepsilon$ for $x \in [a, b]$

By Thm 7.1.5,

$$-\varepsilon(b-a) \le \int_a^b (f_m - f_n) \le \varepsilon(b-a)$$

$$-\varepsilon(b-a) \le \int_a^b f_m - \int_a^b f_n \le \varepsilon(b-a)$$

$$\Rightarrow \quad \left|\int_a^b f_m - \int_a^b f_n\right| \le \varepsilon(b-a)$$

So $\left(\int_a^b f_n\right)$ is a Couchy therefore convergent sequence.

Let $A := \lim_{n\to\infty} \int_a^b f_n$ i.e. $\exists M \in \mathbb{N}$ s.t. $\forall n \ge M$, $\left|\int_a^b f_n - A\right| < \varepsilon$ (1)

As (f_n) is uniformly convergent, so $\exists K \in \mathbb{N}$ s.t. $\forall n \ge k$,

$|f_n(x) - f(x)| < \varepsilon$ for $\forall x \in [a, b]$

$\Rightarrow$ Define $\dot{\mathcal{P}}$ to be any tagged partition of $[a, b]$

s.t. $\dot{\mathcal{P}} = \{[x_{i-1}, x_i], t_i\}_{i=1}^m$, so $t_i \in [a, b] \ \forall i \in \mathbb{N}$.

then

$$\left|S(f_n;\dot{\mathcal{P}}) - S(f;\dot{\mathcal{P}})\right| = \left|\sum_{i=1}^m (f_n(t_i) - f(t_i))(x_i - x_{i-1})\right|$$

$$\le \sum_{i=1}^m |f_n(t_i) - f(t_i)|\,|x_i - x_{i-1}|$$

$$\le \sum_{i=1}^m \varepsilon(x_i - x_{i-1})$$

$$= \varepsilon \sum_{i=1}^m (x_i - x_{i-1})$$

$$= \varepsilon(b-a) \quad (2)$$

As $f_n \in \mathcal{R}[a,b]$ so by the definition of Rieman-Integral,

$\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. if $\|\dot{\mathcal{P}}\| < \delta$ then

$$\left|S(f_n;\dot{\mathcal{P}}) - \int_a^b f_n\right| < \varepsilon \quad (3)$$

Thus, Combine inequalities (1), (2), (3):

$$\underline{\underline{\left|S(f;\dot{\mathcal{P}}) - A\right|}} \le \left|S(f;\dot{\mathcal{P}}) - S(f_n;\dot{\mathcal{P}})\right| + \left|S(f_n;\dot{\mathcal{P}}) - \int_a^b f_n\right| + \left|\int_a^b f_n - A\right|$$

Goal

$$< \varepsilon(b-a) + \varepsilon + \varepsilon$$

$$= \varepsilon(b-a+2)$$

Thus, $\lim_{\|\dot{\mathcal{P}}\|\to 0} S(f;\dot{\mathcal{P}}) = A$ so $f \in \mathcal{R}[a,b]$ and $\int_a^b f = A$

Q.E.D.

Proof. It follows from the Cauchy Criterion 8.1.10 that given $\varepsilon > 0$ there exists $H(\varepsilon)$ such that if $m > n \geq H(\varepsilon)$ then

$$-\varepsilon \leq f_m(x) - f_n(x) \leq \varepsilon \quad \text{for} \quad x \in [a, b].$$

Theorem 7.1.5 implies that

$$-\varepsilon(b-a) \leq \int_a^b f_m - \int_a^b f_n \leq \varepsilon(b-a).$$

Since $\varepsilon > 0$ is arbitrary, the sequence $(\int_a^b f_m)$ is a Cauchy sequence in $\mathbb{R}$ and therefore converges to some number, say $A \in \mathbb{R}$.

We now show $f \in \mathcal{R}[a, b]$ with integral A. If $\varepsilon > 0$ is given, let $K(\varepsilon)$ be such that if $m > K(\varepsilon)$, then $|f_m(x) - f(x)| < \varepsilon$ for all $x \in [a, b]$. If $\dot{\mathcal{P}} := \{([x_{i-1}, x_i], t_i)\}_{i=1}^n$ is any tagged partition of $[a, b]$ and if $m > K(\varepsilon)$, then

$$\begin{aligned} |S(f_m; \dot{\mathcal{P}}) - S(f; \dot{\mathcal{P}})| &= \left| \sum_{i=1}^n \{f_m(t_i) - f(t_i)\}(x_i - x_{i-1}) \right| \\ &\leq \sum_{i=1}^n |f_m(t_i) - f(t_i)|(x_i - x_{i-1}) \\ &\leq \sum_{i=1}^n \varepsilon(x_i - x_{i-1}) = \varepsilon(b-a). \end{aligned}$$

We now choose $r \geq K(\varepsilon)$ such that $|\int_a^b f_r - A| < \varepsilon$ and we let $\delta_{r,\varepsilon} > 0$ be such that $|\int_a^b f_r - S(f_r; \dot{\mathcal{P}})| < \varepsilon$ whenever $||\dot{\mathcal{P}}|| < \delta_{r,\varepsilon}$. Then we have

$$\begin{aligned} |S(f; \dot{\mathcal{P}}) - A| &\leq |S(f; \dot{\mathcal{P}}) - S(f_r; \dot{\mathcal{P}})| + \left| S(f_r; \dot{\mathcal{P}}) - \int_a^b f_r \right| + \left| \int_a^b f_r - A \right| \\ &\leq \varepsilon(b-a) + \varepsilon + \varepsilon = \varepsilon(b-a+2). \end{aligned}$$

But since $\varepsilon > 0$ is arbitrary, it follows that $f \in \mathcal{R}[a, b]$ and $\int_a^b f = A$. Q.E.D.

uni.conv 的条件很难达到：

8.2.5 Bounded Convergence Theorem *Let (f_n) be a sequence in $\mathcal{R}[a, b]$ that converges on $[a, b]$ to a function $f \in \mathcal{R}[a, b]$. Suppose also that there exists $B > 0$ such that $|f_n(x)| \leq B$ for all $x \in [a, b]$, $n \in \mathbb{N}$. Then equation (3) holds.*

i.e., Uniform Convergent is too strong for normal cases.

1. f is bounded
2. $f_n \in \mathcal{R}[a,b]$
3. $f_n \to f$ pointwisely
4. $f \in \mathcal{R}[a,b]$

$\Downarrow$

$\int_a^b f = \lim_{n\to\infty} \int_a^b f_n$

8.2.6 Dini's Theorem *Suppose that (f_n) is a monotone sequence of continuous functions on $I := [a, b]$ that converges on I to a continuous function f. Then the convergence of the sequence is uniform.*

Proof. We suppose that the sequence (f_n) is decreasing and let $g_m := f_m - f$. Then (g_m) is a decreasing sequence of continuous functions converging on I to the 0-function. We will show that the convergence is uniform on I.

Given $\varepsilon > 0$, $t \in I$, there exists $m_{\varepsilon,t} \in \mathbb{N}$ such that $0 \le g_{m_{\varepsilon,t}}(t) < \varepsilon/2$. Since $g_{m_{\varepsilon,t}}$ is continuous at t, there exists $\delta_\varepsilon(t) > 0$ such that $0 \le g_{m_{\varepsilon,t}}(x) < \varepsilon$ for all $x \in I$ satisfying $|x - t| \le \delta_\varepsilon(t)$. Thus, δ_ε is a gauge on I, and if $\dot{\mathcal{P}} = \{(I_i, t_i)\}_{i=1}^n$ is a δ_ε-fine partition, we set $M_\varepsilon := \max\{m_{\varepsilon,t_1}, \ldots, m_{\varepsilon,t_n}\}$. If $m \ge M_\varepsilon$ and $x \in I$, then (by Lemma 5.5.3) there exists an index i with $|x - t_i| \le \delta_\varepsilon(t_i)$ and hence

$$0 \le g_m(x) \le g_{m,t_i}(x) < \varepsilon.$$

Therefore, the sequence (g_m) converges uniformly to the 0-function. Q.E.D.

Proof:

As (f_n) is monotone so assume it to be decreasing

As f_n converge to f on I so

Let $g_m(x) := f_m(x) - f(x)$ $\forall x \in I$ then

$(g_m(x))$ is a monotone decreasing sequence of

continuous functions converging on I to $g(x) = 0$. $\forall x \in I$

$\Rightarrow g_m(x) \geq 0$ $\forall m \in \mathbb{N}$

$\Rightarrow$ By density of real numbers,

$\forall \varepsilon > 0$, $\forall t \in I$, $\exists m_{\varepsilon,t} \in \mathbb{N}$ s.t.

$$0 \le g_{m_{\varepsilon,t}}(t) < \frac{\varepsilon}{2}$$

As $g_{m_{\varepsilon,t}}(x)$ is continuous at t, then

$\exists \delta_\varepsilon(t) > 0$ s.t. $\forall x \in I$ satisfying

$|x - t| \le \delta_\varepsilon(t)$, $0 \le g_{m_{\varepsilon,t}}(x) < \varepsilon$

Define $\dot{P} = \{(I_i, t_i)\}_{i=1}^n$, $\|\dot{P}\| = \delta_\varepsilon$

set $M_\varepsilon := \max\{m_{\varepsilon,t_1}, \ldots m_{\varepsilon,t_n}\}$

If $m \geq M_\varepsilon$ and $x \in I$ then

$\exists i \in \mathbb{N}$ s.t. $|x - t_i| \le \delta_\varepsilon(t_i)$

$\Rightarrow$ $0 \le g_m(x) \le g_{m_{\varepsilon,t_i}}(x) < \varepsilon$

Thus, (g_m) convergence uniformly to $g(x) = 0$

Q.E.D.

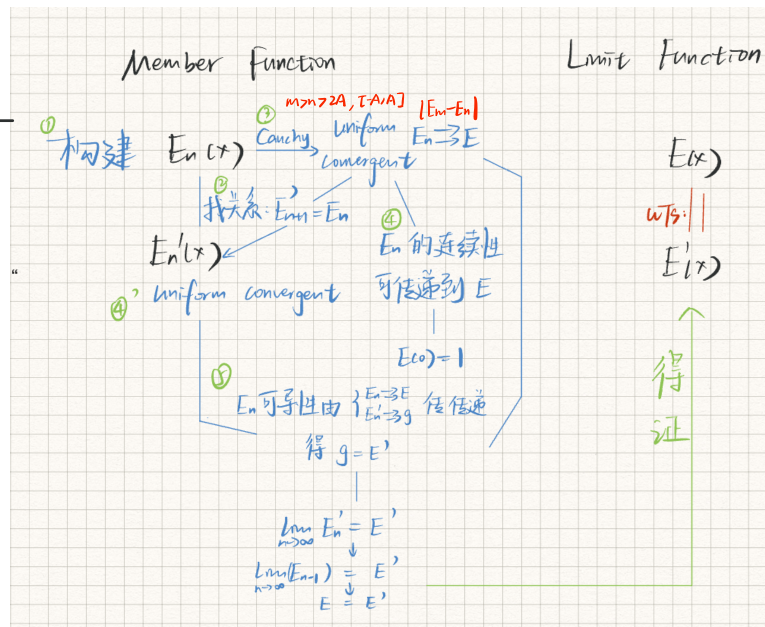
Section 8.3 The Exponential and Logarithmic Functions

E的存在性：

8.3.1 Theorem *There exists a function $E : \mathbb{R} \to \mathbb{R}$ such that:*

(i) $E'(x) = E(x)$ *for all* $x \in \mathbb{R}$. 须可导性的传递

(ii) $E(0) = 1$. 须连续性的传递



Proof. We inductively define a sequence (E_n) of continuous functions as follows:

(1) $$E_1(x) := 1 + x,$$

(2) $$E_{n+1}(x) := 1 + \int_0^x E_n(t)\,dt,$$

for all $n \in \mathbb{N}$, $x \in \mathbb{R}$. Clearly E_1 is continuous on $\mathbb{R}$ and hence is integrable over any bounded interval. If E_n has been defined and is continuous on $\mathbb{R}$, then it is integrable over any bounded interval, so that E_{n+1} is well-defined by the above formula. Moreover, it follows from the Fundamental Theorem (Second Form) 7.3.5 that E_{n+1} is differentiable at any point $x \in \mathbb{R}$ and that

(3) $$E'_{n+1}(x) = E_n(x) \quad \text{for} \quad n \in \mathbb{N}.$$

An Induction argument (which we leave to the reader) shows that

(4) $$\boxed{E_n(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} \quad \text{for} \quad x \in \mathbb{R}.} = \sum_{k=0}^{n} \frac{x^k}{k!}$$

Let $A > 0$ be given; then if $|x| \le A$ and $m > n > 2A$, we have

(5) $$\begin{aligned} |E_m(x) - E_n(x)| &= \left| \frac{x^{n+1}}{(n+1)!} + \cdots + \frac{x^m}{m!} \right| \\ &\le \frac{A^{n+1}}{(n+1)!} \left[1 + \frac{A}{n} + \cdots + \left(\frac{A}{n}\right)^{m-n-1} \right] \\ &< \frac{A^{n+1}}{(n+1)!} 2. \end{aligned}$$

Since $\lim(A^n/n!) = 0$, it follows that the sequence (E_n) converges uniformly on the interval $[-A, A]$ where $A > 0$ is arbitrary. In particular this means that $(E_n(x))$ converges for each $x \in \mathbb{R}$. We define $E : \mathbb{R} \to \mathbb{R}$ by

$$E(x) := \lim E_n(x) \quad \text{for} \quad x \in \mathbb{R}.$$

Since each $x \in \mathbb{R}$ is contained inside some interval $[-A, A]$, it follows from Theorem 8.2.2 that E is continuous at x. Moreover, it is clear from (1) and (2) that $E_n(0) = 1$ for all $n \in \mathbb{N}$. Therefore $E(0) = 1$, which proves (ii).

On any interval $[-A, A]$ we have the uniform convergence of the sequence (E_n). In view of (3), we also have the uniform convergence of the sequence (E_n') of derivatives. It therefore follows from Theorem 8.2.3 that the limit function E is differentiable on $[-A, A]$ and that

$$E'(x) = \lim(E_n'(x)) = \lim(E_{n-1}(x)) = E(x)$$

for all $x \in [-A, A]$. Since $A > 0$ is arbitrary, statement (i) is established. Q.E.D.