CHAPTER 4

LIMITS

Section 4.1 Limits of Functions

4.1.1 Definition Let $A \subseteq \mathbb{R}$. A point $c \in \mathbb{R}$ is a **cluster point** of A if for every $\delta > 0$ there exists at least one point $x \in A$, $x \neq c$ such that $|x - c| < \delta$.

4.1.2 Theorem *A number $c \in \mathbb{R}$ is a cluster point of a subset A of $\mathbb{R}$ if and only if there exists a sequence (a_n) in A such that $\lim(a_n) = c$ and $a_n \neq c$ for all $n \in \mathbb{N}$.*

之后证明的基石：
不必证 (x_n) 的存在性

Proof: $\boxed{\Rightarrow}$ If c is a cluster point of A

then $\forall n \in \mathbb{N}$, $\exists a_n \in A \backslash \{c\}$ s.t.

$a_n \in V_{\frac{1}{n}}(c)$

i.e. $c - \frac{1}{n} < a_n < c + \frac{1}{n}$

$\Rightarrow \quad 0 < |a_n - c| < \frac{1}{n}$

When $n \to \infty$,

$0 < |a_n - c| < \lim(\frac{1}{n}) = 0$

so $\lim(a_n) = c$ and $a_n \neq c$

$\boxed{\Leftarrow}$ Define a seq. (a_n) in A

s.t. $\lim(a_n) = c$ and $a_n \neq c$ for all $n \in \mathbb{N}$

then $\forall \delta > 0$ $\exists K \in \mathbb{N}$ s.t. $\forall n \geq K$,

$0 < |a_n - c| < \delta$

Recall that $\forall n \in \mathbb{N}$, $a_n \in A$

so $a_n \in V_\delta(c) \cap A \backslash \{c\}$

Thus, $\forall \delta > 0$, $\exists x \in A \backslash \{c\}$ s.t.

$|x - c| < \delta$

$\Rightarrow$ c is a cluster point of A □.

Proof. If c is a cluster point of A, then for any $n \in \mathbb{N}$ the $\boxed{(1/n)\text{-neighborhood } V_{1/n}(c)}$ contains at least one point a_n in A distinct from c. Then $a_n \in A$, $a_n \neq c$, and $|a_n - c| < 1/n$ implies $\lim(a_n) = c$.

δ的任意性的衍生物，
多次用到.

Conversely, if there exists a sequence (a_n) in $A \backslash \{c\}$ with $\lim(a_n) = c$, then for any $\delta > 0$ there exists K such that if $n \geq K$, then $a_n \in V_\delta(c)$. Therefore the δ-neighborhood $V_\delta(c)$ of c contains the points a_n, for $n \geq K$, which belong to A and are distinct from c. Q.E.D.

4.1.3 Examples **(a)** For the open interval $A_1 := (0, 1)$, every point of the closed interval $[0,1]$ is a cluster point of A_1. Note that the points 0, 1 are cluster points of A_1, but do not belong to A_1. All the points of A_1 are cluster points of A_1.

(b) A finite set has no cluster points.

(c) The infinite set $\mathbb{N}$ has no cluster points.

$\Rightarrow$ Countable sets have no cluster point

(d) The set $A_4 := \{1/n : n \in \mathbb{N}\}$ has only the point 0 as a cluster point. None of the points in A_4 is a cluster point of A_4.

(e) If $I := [0, 1]$, then the set $A_5 := I \cap \mathbb{Q}$ consists of all the rational numbers in I. It follows from the Density Theorem 2.4.8 that every point in I is a cluster point of A_5. □

4.1.4 Definition Let $A \subseteq \mathbb{R}$, and let c be a cluster point of A. For a function $f : A \to \mathbb{R}$, a real number L is said to be a **limit of f at c** if, given any $\varepsilon > 0$, there exists a $\delta > 0$ such that if $x \in A$ and $0 < |x - c| < \delta$, then $|f(x) - L| < \varepsilon$.

The Definition of the Limit

4.1.5 Theorem *If $f : A \to \mathbb{R}$ and if c is a cluster point of A, then f can have only one limit at c.*

Proof. Suppose that numbers L and L' satisfy Definition 4.1.4. For any $\varepsilon > 0$, there exists $\delta(\varepsilon/2) > 0$ such that if $x \in A$ and $0 < |x - c| < \delta(\varepsilon/2)$, then $|f(x) - L| < \varepsilon/2$. Also there exists $\delta'(\varepsilon/2)$ such that if $x \in A$ and $0 < |x - c| < \delta'(\varepsilon/2)$, then $|f(x) - L'| < \varepsilon/2$. Now let $\delta := \inf\{\delta(\varepsilon/2), \delta'(\varepsilon/2)\}$. Then if $x \in A$ and $0 < |x - c| < \delta$, the Triangle Inequality implies that

$$|L - L'| \le |L - f(x)| + |f(x) - L'| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we conclude that $L - L' = 0$, so that $L = L'$. Q.E.D.

4.1.6 Theorem *Let $f : A \to \mathbb{R}$ and let c be a cluster point of A. Then the following statements are equivalent.*

(i) $\lim\limits_{x \to c} f(x) = L$.

(ii) *Given any ε-neighborhood $V_\varepsilon(L)$ of L, there exists a δ-neighborhood $V_\delta(c)$ of c such that if $x \neq c$ is any point in $V_\delta(c) \cap A$, then $f(x)$ belongs to $V_\varepsilon(L)$.*

Sequential Criterion for Limits

4.1.8 Theorem *(Sequential Criterion) Let $f : A \to \mathbb{R}$ and let c be a cluster point of A. Then the following are equivalent.*

(i) $\lim\limits_{x \to c} f = L$.

(ii) *For every sequence (x_n) in A that converges to c such that $x_n \neq c$ for all $n \in \mathbb{N}$, the sequence $(f(x_n))$ converges to L.*

Proof. (i) $\Rightarrow$ (ii). Assume f has limit L at c, and suppose (x_n) is a sequence in A with $\lim(x_n) = c$ and $x_n \neq c$ for all n. We must prove that the sequence $(f(x_n))$ converges to L. Let $\varepsilon > 0$ be given. Then by Definition 4.1.4, there exists $\delta > 0$ such that if $x \in A$ satisfies $0 < |x - c| < \delta$, then $f(x)$ satisfies $|f(x) - L| < \varepsilon$. We now apply the definition of convergent sequence for the given δ to obtain a natural number $K(\delta)$ such that if $n > K(\delta)$ then $|x_n - c| < \delta$. But for each such x_n we have $|f(x_n) - L| < \varepsilon$. Thus if $n > K(\delta)$, then $|f(x_n) - L| < \varepsilon$. Therefore, the sequence $(f(x_n))$ converges to L.

(ii) $\Rightarrow$ (i). [The proof is a contrapositive argument.] If (i) is not true, then there exists an ε_0-neighborhood $V_{\varepsilon_0}(L)$ such that no matter what δ-neighborhood of c we pick, there will be at least one number x_δ in $A \cap V_\delta(c)$ with $x_\delta \neq c$ such that $f(x_\delta) \notin V_{\varepsilon_0}(L)$. Hence for every $n \in \mathbb{N}$, the $(1/n)$-neighborhood of c contains a number x_n such that

$$0 < |x_n - c| < 1/n \qquad \text{and} \qquad x_n \in A,$$

but such that

$$|f(x_n) - L| \ge \varepsilon_0 \qquad \text{for all} \qquad n \in \mathbb{N}.$$

We conclude that the sequence (x_n) in $A \backslash \{c\}$ converges to c, but the sequence $(f(x_n))$ does not converge to L. Therefore we have shown that if (i) is not true, then (ii) is not true. We conclude that (ii) implies (i). Q.E.D.

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(ii) $\Rightarrow$ (i) Proof: Assume that $\lim\limits_{x\to c} f \neq L$

Then $\forall \delta > 0$, $|x - c| < \delta$

$\exists\, \varepsilon_0 > 0$ s.t. $|f(x) - L| \geq \varepsilon_0$ $*$

As c is a cluster point of A

then $\forall n \in \mathbb{N}$, $\exists\, x_n \in A \backslash \{c\}$ s.t.

$c - \frac{1}{n} < x_n < c + \frac{1}{n}$

$|x_n - c| < \frac{1}{n} \to 0$

so $\lim(x_n) \to c$

by (ii), then $|f(x_n) - L| < \varepsilon$ $(\varepsilon > 0)$

Contradicts to $*$

Thus, $\lim\limits_{x\to c} f = L$

$\square$.

4.1.9 Divergence Criteria *Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$ and let $c \in \mathbb{R}$ be a cluster point of A.*

(a) *If $L \in \mathbb{R}$, then f does not have limit L at c if and only if there exists a sequence (x_n) in A with $x_n \neq c$ for all $n \in \mathbb{N}$ such that the sequence (x_n) converges to c but the sequence $(f(x_n))$ does **not** converge to L.*

(b) *The function f does **not** have a limit at c if and only if there exists a sequence (x_n) in A with $x_n \neq c$ for all $n \in \mathbb{N}$ such that the sequence (x_n) converges to c but the sequence $(f(x_n))$ does **not** converge in $\mathbb{R}$.*

4.2.1 Definition Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$, and let $c \in \mathbb{R}$ be a cluster point of A. We say that f is **bounded on a neighborhood of** c if there exists a δ-neighborhood $V_\delta(c)$ of c and a constant $M > 0$ such that we have $|f(x)| \le M$ for all $x \in A \cap V_\delta(c)$.

f(c) 即 L

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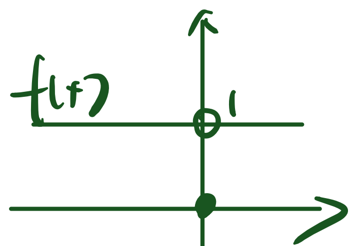
4.2.2 Theorem *If $A \subseteq \mathbb{R}$ and $f : A \to \mathbb{R}$ $\boxed{\text{has a limit at } c}$ $\in \mathbb{R}$, then f is bounded on some neighborhood of c.*



lim f(x) at x=0 is 1 but not f(c)=0

Proof. If $L := \lim_{x \to c} f$, then for $\varepsilon = 1$, there exists $\delta > 0$ such that if $0 < |x - c| < \delta$, then $|f(x) - L| < 1$; hence (by Corollary 2.2.4(a)),

$$|f(x)| - |L| \le |f(x) - L| < 1.$$

Therefore, if $x \in A \cap V_\delta(c)$, $x \ne c$, then $|f(x)| \le |L| + 1$. If $c \notin A$, we take $M = |L| + 1$, while if $c \in A$ we take $M := \sup\{|f(c)|, |L| + 1\}$. It follows that if $x \in A \cap V_\delta(c)$, then $|f(x)| \le M$. This shows that f is bounded on the neighborhood $V_\delta(c)$ of c. Q.E.D.

证 bounded 类：

别用 $|f(x)-L|$ 去拆得 $|f(x)|$

Triangle Inequality 已给 $|f(x)|$

Proof: As $\lim_{x\to c} f(x) = L$

then $\forall \varepsilon > 0$, for some $\delta > 0$,

when $|x-c| < \delta$,

$|f(x) - L| < \varepsilon$

Choose $\varepsilon = 1$

then $|f(x)| - |L| < |f(x) - L| < 1$

$\Rightarrow |f(x)| < 1 + |L|$

Case 1: if $c \notin A$ then

let $M := |1+|L||$ is fine.

$|f(x)| \le M$. bounded.

Case 2: if $c \in A$ then

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let $M := \sup\{1+|L|, |f(c)|\}$

$|f(x)| \le M$ bounded □.

Section 4.2 Limit Theorems

4.2.3 Definition Let $A \subseteq \mathbb{R}$ and let f and g be functions defined on A to $\mathbb{R}$. We define the **sum** $f + g$, the **difference** $f - g$, and the **product** fg on A to $\mathbb{R}$ to be the functions given by

$$(f+g)(x) := f(x) + g(x), \qquad (f-g)(x) := f(x) - g(x),$$
$$(fg)(x) := f(x)g(x)$$

for all $x \in A$. Further, if $b \in \mathbb{R}$, we define the **multiple** bf to be the function given by

$$(bf)(x) := bf(x) \quad \text{for all} \quad x \in A\,.$$

Finally, if $h(x) \ne 0$ for $x \in A$, we define the **quotient** f/h to be the function given by

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$$\boxed{\left(\frac{f}{h}\right)(x) := \frac{f(x)}{h(x)}} \quad \text{for all} \quad x \in A\,.$$

4.2.4 Theorem *Let $A \subseteq \mathbb{R}$, let f and g be functions on A to $\mathbb{R}$, and let $c \in \mathbb{R}$ be a cluster point of A. Further, let $b \in \mathbb{R}$.*

(a) *If $\lim_{x\to c} f = L$ and $\lim_{x\to c} g = M$, then:*

$$\lim_{x\to c}(f+g) = L+M, \qquad \lim_{x\to c}(f-g) = L-M,$$
$$\lim_{x\to c}(fg) = LM, \qquad \lim_{x\to c}(bf) = bL.$$

(b) *If $h : A \to \mathbb{R}$, if $h(x) \neq 0$ for all $x \in A$, and if $\lim_{x\to c} h = H \neq 0$, then*

$$\lim_{x\to c}\left(\frac{f}{h}\right) = \frac{L}{H}.$$

Proof: $\lim\left[(f\pm g)(x_n)\right] = \lim\left[f(x_n) \pm g(x_n)\right]$
$= \lim\left[f(x_n)\right] \pm \lim\left[g(x_n)\right]$
$= L \pm M$

As $(x_n) \to c$ and $x_n \neq c$, $x_n \in A$.

so $\lim_{x\to c}(f\pm g)(x) = \lim_{x\to c}\left[(f\pm g)(x_n)\right] = L \pm M$.

Proof. One proof of this theorem is exactly similar to that of Theorem 3.2.3. Alternatively, it can be proved by making use of Theorems 3.2.3 and 4.1.8. For example, let (x_n) be any sequence in A such that $x_n \neq c$ for $n \in \mathbb{N}$, and $c = \lim(x_n)$. It follows from Theorem 4.1.8 that

$$\lim(f(x_n)) = L, \qquad \lim(g(x_n)) = M.$$

On the other hand, Definition 4.2.3 implies that

$$(fg)(x_n) = f(x_n)g(x_n) \qquad \text{for} \quad n \in \mathbb{N}.$$

Therefore an application of Theorem 3.2.3 yields

$$\begin{aligned}\lim((fg)(x_n)) &= \lim(f(x_n)g(x_n)) \\ &= [\lim(f(x_n))]\,[\lim(g(x_n))] = LM.\end{aligned}$$

Consequently, it follows from Theorem 4.1.8 that

$$\lim_{x\to c}(fg) = \lim((fg)(x_n)) = LM.$$

The other parts of this theorem are proved in a similar manner. We leave the details to the reader. Q.E.D.

(b) Proof: Let (x_n) be any seq. in A s.t.
$x_n \to c$ and $x_n \neq c$
then $\lim(h(x_n)) = H \neq 0$

$$0 \leq \left|\frac{1}{h(x_n)} - \frac{1}{H}\right| = \left|\frac{H - h(x_n)}{h(x_n)H}\right|$$
$$= \left|H - h(x_n)\right| \cdot \left|\frac{1}{h(x_n)H}\right|$$
$$< \frac{1}{\frac{1}{2}|H|^2} \cdot \left|H - h(x_n)\right|$$

As $\left|H - h(x_n)\right| \to 0$

So by S.T. $\lim\left(\frac{1}{h(x_n)}\right) = \frac{1}{H}$

Let $\varepsilon = \frac{1}{2}|H|$ then

$$|h(x_n)| - |H| < |h(x_n) - H| < \frac{1}{2}|H|$$

So $|H| = |H - h(x_n) + h(x_n)|$
$\leq |H - h(x_n)| + |h(x_n)|$
$< \frac{1}{2}|H| + |h(x_n)|$

$\Rightarrow |h(x_n)| > \frac{1}{2}|H|$

Remark Let $A \subseteq \mathbb{R}$, and let $f_1, f_2, \ldots, f_n$ be functions on A to $\mathbb{R}$, and let c be a cluster point of A. If $L_k := \lim_{x \to c} f_k$ for $k = 1, \ldots, n$, then it follows from Theorem 4.2.4 by an Induction argument that

$$L_1 + L_2 + \cdots + L_n = \lim_{x \to c} (f_1 + f_2 + \cdots + f_n),$$

and

$$L_1 \cdot L_2 \cdots L_n = \lim (f_1 \cdot f_2 \cdots f_n).$$

In particular, we deduce that if $L = \lim_{x \to c} f$ and $n \in \mathbb{N}$, then

$$L^n = \lim_{x \to c} (f(x))^n.$$

(f) If p is a polynomial function, then $\lim_{x \to c} p(x) = p(c)$.

Let p be a polynomial function on $\mathbb{R}$ so that $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$ for all $x \in \mathbb{R}$. It follows from Theorem 4.2.4 and the fact that $\lim_{x \to c} x^k = c^k$ that

$$\begin{aligned}
\lim_{x \to c} p(x) &= \lim_{x \to c} \left[a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0\right] \\
&= \lim_{x \to c} (a_n x^n) + \lim_{x \to c} (a_{n-1} x^{n-1}) + \cdots + \lim_{x \to c} (a_1 x) + \lim_{x \to c} a_0 \\
&= a_n c^n + a_{n-1} c^{n-1} + \cdots + a_1 c + a_0 \\
&= p(c).
\end{aligned}$$

Hence $\lim_{x \to c} p(x) = p(c)$ for any polynomial function p.

(g) If p and q are polynomial functions on $\mathbb{R}$ and if $q(c) \neq 0$, then

$$\lim_{x \to c} \frac{p(x)}{q(x)} = \frac{p(c)}{q(c)}.$$

Since $q(x)$ is a polynomial function, it follows from a theorem in algebra that there are at most a finite number of real numbers $\alpha_1, \ldots, \alpha_m$ [the real zeroes of $q(x)$] such that $q(\alpha_j) = 0$ and such that if $x \notin \{\alpha_1, \ldots, \alpha_m\}$, then $q(x) \neq 0$. Hence, if $x \notin \{\alpha_1, \ldots, \alpha_m\}$, we can define

$$r(x) := \frac{p(x)}{q(x)}.$$

If c is not a zero of $q(x)$, then $q(c) \neq 0$, and it follows from part (f) that $\lim_{x \to c} q(x) = q(c) \neq 0$. Therefore we can apply Theorem 4.2.4(b) to conclude that

$$\lim_{x \to c} \frac{p(x)}{q(x)} = \frac{\lim_{x \to c} p(x)}{\lim_{x \to c} q(x)} = \frac{p(c)}{q(c)}. \qquad \square$$

\# Thm 4.1.8 could apply.

4.2.6 Theorem *Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$, and let $c \in \mathbb{R}$ be a cluster point of A. If*

$$a \leq f(x) \leq b \quad \text{for all} \quad x \in A, x \neq c,$$

\#

and if $\lim_{x \to c} f$ exists, then $a \leq \lim_{x \to c} f \leq b$.

Proof. Indeed, if $L = \lim_{x \to c} f$, then it follows from Theorem 4.1.8 that if (x_n) is any sequence of real numbers such that $c \neq x_n \in A$ for all $n \in \mathbb{N}$ and if the sequence (x_n) converges to c, then the sequence $(f(x_n))$ converges to L. Since $a \leq f(x_n) \leq b$ for all $n \in \mathbb{N}$, it follows from Theorem 3.2.6 that $a \leq L \leq b$. Q.E.D.

宗旨：所有function中解决不了的
通过thm 4.1.8 放到 sequence中解决

then apply 4.1.8 $ii \Rightarrow i$, $a \leq \lim_{x \to c} f \leq b$

4.2.7 Squeeze Theorem *Let $A \subseteq \mathbb{R}$, let $f, g, h: A \to \mathbb{R}$, and let $c \in \mathbb{R}$ be a cluster point of A. If*

$$f(x) \le g(x) \le h(x) \quad \textit{for all} \quad x \in A,\ x \ne c,$$

and if $\lim_{x\to c} f = L = \lim_{x\to c} h$, then $\lim_{x\to c} g = L$.

(e) $\lim_{x\to 0} \left(\dfrac{\sin x}{x}\right) = 1$.

Again we cannot use Theorem 4.2.4(b) to evaluate this limit. However, it will be proved later (see Theorem 8.4.8) that

$$x - \tfrac{1}{6}x^3 \le \sin x \le x \quad \text{for} \quad x \ge 0$$

and that

$$x \le \sin x \le x - \tfrac{1}{6}x^3 \quad \text{for} \quad x \le 0.$$

Therefore it follows (why?) that

$$1 - \tfrac{1}{6}x^2 \le (\sin x)/x \le 1 \quad \text{for all} \quad x \ne 0.$$

But since $\lim_{x\to 0}\left(1 - \frac{1}{6}x^2\right) = 1 - \frac{1}{6}\cdot \lim_{x\to 0} x^2 = 1$, we infer from the Squeeze Theorem that $\lim_{x\to 0} (\sin x)/x = 1$.

(f) $\lim_{x\to 0} (x \sin(1/x)) = 0$.

Let $f(x) = x\sin(1/x)$ for $x \ne 0$. Since $-1 \le \sin z \le 1$ for all $z \in \mathbb{R}$, we have the inequality

$$-|x| \le f(x) = x\sin(1/x) \le |x|$$

for all $x \in \mathbb{R}$, $x \ne 0$. Since $\lim_{x\to 0} |x| = 0$, it follows from the Squeeze Theorem that $\lim_{x\to 0} f = 0$.

4.2.9 Theorem *Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$ and let $c \in \mathbb{R}$ be a cluster point of A. If*

$$\lim_{x\to c} f > 0 \quad \left[\textit{respectively, } \lim_{x\to c} f < 0\right],$$

then there exists a neighborhood $V_\delta(c)$ of c such that $f(x) > 0$ [respectively, $f(x) < 0$] for all $x \in A \cap V_\delta(c)$, $x \ne c$.

Proof. Let $L := \lim_{x\to c} f$ and suppose that $L > 0$. We take $\varepsilon = \frac{1}{2}L > 0$ in Definition 4.1.4, and obtain a number $\delta > 0$ such that if $0 < |x - c| < \delta$ and $x \in A$, then $|f(x) - L| < \frac{1}{2}L$. Therefore (why?) it follows that if $x \in A \cap V_\delta(c)$, $x \ne c$, then $f(x) > \frac{1}{2}L > 0$.

If $L < 0$, a similar argument applies. Q.E.D.

Proof: As $\lim_{x\to c} f = L > 0$

so $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t.

$\exists x \in V_\delta(c)$, s.t. $|f(x) - L| < \varepsilon$

Let $\varepsilon_0 = \frac{1}{2}L > 0$ — obtain $\delta_0 > 0$ s.t. when $0 < |x-c| < \delta_0$

then $|f(x) - L| < \frac{1}{2}L$

$-\frac{1}{2}L < f(x) - L < \frac{1}{2}L$

$0 < \frac{1}{2}L < f(x) < \frac{3}{2}L$ $\square$.