课上提了

CHAPTER 7

THE RIEMANN INTEGRAL

Section 7.1 Riemann Integral

7.1.1 Definition A function $f : [a, b] \to \mathbb{R}$ is said to be **Riemann integrable** on $[a, b]$ if there exists a number $L \in \mathbb{R}$ such that for every $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that if $\dot{\mathcal{P}}$ is any tagged partition of $[a, b]$ with $\|\dot{\mathcal{P}}\| < \delta_\varepsilon$, then

$$\left|S(f; \dot{\mathcal{P}}) - L\right| < \varepsilon.$$

The set of all Riemann integrable functions on $[a, b]$ will be denoted by $\mathcal{R}[a, b]$.

证 limit时 回顾定义: $\exists K \in \mathbb{N}$ s.t. $\forall n \geq k$ ...

7.1.2 Theorem *If $f \in \mathcal{R}[a, b]$, then the value of the integral is uniquely determined.*

Proof. Assume that L' and L'' both satisfy the definition and let $\varepsilon > 0$. Then there exists $\delta'_{\varepsilon/2} > 0$ such that if $\dot{\mathcal{P}}_1$ is any tagged partition with $\|\dot{\mathcal{P}}_1\| < \delta'_{\varepsilon/2}$, then

$$\left|S(f; \dot{\mathcal{P}}_1) - L'\right| < \varepsilon/2.$$

Also there exists $\delta''_{\varepsilon/2} > 0$ such that if $\dot{\mathcal{P}}_2$ is any tagged partition with $\|\dot{\mathcal{P}}_2\| < \delta''_{\varepsilon/2}$, then

$$\left|S(f; \dot{\mathcal{P}}_2) - L''\right| < \varepsilon/2.$$

别忘

Now let $\boxed{\delta_\varepsilon := \min\{\delta'_{\varepsilon/2}, \delta''_{\varepsilon/2}\}} > 0$ and let $\dot{\mathcal{P}}$ be a tagged partition with $\|\dot{\mathcal{P}}\| < \delta_\varepsilon$. Since both $\|\dot{\mathcal{P}}\| < \delta'_{\varepsilon/2}$ and $\|\dot{\mathcal{P}}\| < \delta''_{\varepsilon/2}$, then

$$\left|S(f; \dot{\mathcal{P}}) - L'\right| < \varepsilon/2 \quad \text{and} \quad \left|S(f; \dot{\mathcal{P}}) - L''\right| < \varepsilon/2,$$

whence it follows from the Triangle Inequality that

$$\begin{aligned} |L' - L''| &= \left|L' - S(f; \dot{\mathcal{P}}) + S(f; \dot{\mathcal{P}}) - L''\right| \\ &\leq \left|L' - S(f; \dot{\mathcal{P}})\right| + \left|S(f; \dot{\mathcal{P}}) - L''\right| \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, it follows that $L' = L''$. Q.E.D.

7.1.3 Theorem If g is Riemann integrable on $[a, b]$ and if $f(x) = g(x)$ except for a finite number of points in $[a, b]$, then f is Riemann integrable and $\int_a^b f = \int_a^b g$.

Proof. If we prove the assertion for the case of one exceptional point, then the extension to a finite number of points is done by a standard induction argument, which we leave to the reader.

#

Let c be a point in the interval and let $L = \int_a^b g$. Assume that $f(x) = g(x)$ for all $x \neq c$. For any tagged partition $\dot{\mathcal{P}}$, the terms in the two sums $S(f; \dot{\mathcal{P}})$ and $S(g; \dot{\mathcal{P}})$ are identical with the exception of at most two terms (in the case that $c = x_i = x_{i-1}$ is an endpoint). Therefore, we have

$$\left|S(f; \dot{\mathcal{P}}) - S(g; \dot{\mathcal{P}})\right| = |\Sigma(f(x_i) - g(x_i))(x_i - x_{i-1})| \leq 2\left(|g(c)| + |f(c)|\right) \|\dot{\mathcal{P}}\|.$$

Now, given $\varepsilon > 0$, we let $\delta_1 > 0$ satisfy $\delta_1 < \varepsilon/(4(|f(c)| + |g(c)|)$, and let $\delta_2 > 0$ be such that $\|\dot{\mathcal{P}}\| < \delta_2$ implies $\left|S(g; \dot{\mathcal{P}}) - L\right| < \varepsilon/2$. We now let $\delta := \min\{\delta_1, \delta_2\}$. Then, if $\|\dot{\mathcal{P}}\| < \delta$, we obtain

$$\left|S(f; \dot{\mathcal{P}}) - L\right| \leq \left|S(f; \dot{\mathcal{P}}) - S(g; \dot{\mathcal{P}})\right| + \left|S(g; \dot{\mathcal{P}}) - L\right| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Hence, the function f is integrable with integral L. Q.E.D.

7.1.7 Example We consider Thomae's function $h : [0, 1] \to \mathbb{R}$ defined, as in Example 5.1.6(h), by $h(x) := 0$ if $x \in [0, 1]$ is irrational, $h(0) := 1$ and by $h(x) := 1/n$ if $x \in [0, 1]$ is the rational number $x = m/n$ where $m, n \in \mathbb{N}$ have no common integer factors except 1. It was seen in 5.1.6(h) that h is continuous at every irrational number and discontinuous at every rational number in [0, 1]. See Figure 5.1.2. We will now show that $h \in \mathcal{R}[0, 1]$.

For $\varepsilon > 0$, the set $\boxed{E := \{x \in [0,1] : h(x) \geq \varepsilon/2\}}$ is a finite set. (For example, if $\varepsilon/2 = 1/5$, then there are eleven values of x such that $h(x) \geq 1/5$, namely, $E = \{0, 1, 1/2, 1/3, 2/3, 1/4, 3/4, 1/5, 2/5, 3/5, 4/5\}$. (Sketch a graph.) We let n be the number of elements in E and take $\delta := \varepsilon/(4n)$. If $\dot{\mathcal{P}}$ is a given tagged partition such that $||\dot{\mathcal{P}}|| < \delta$, then we separate $\dot{\mathcal{P}}$ into two subsets. We let $\dot{\mathcal{P}}_1$ be the collection of tagged intervals in $\dot{\mathcal{P}}$ that have their tags in E, and we let $\dot{\mathcal{P}}_2$ be the subset of tagged intervals in $\dot{\mathcal{P}}$ that have their tags elsewhere in [0, 1]. Allowing for the possibility that a tag of $\dot{\mathcal{P}}_1$ may be an endpoint of adjacent intervals, we see that $\dot{\mathcal{P}}_1$ has at most $2n$ intervals and the total length of these intervals can be at most $2n\delta = \varepsilon/2$. Also, we have $0 < h(t_i) \leq 1$ for each tag t_i in $\dot{\mathcal{P}}_1$. Consequently, we have $S(h; \dot{\mathcal{P}}_1) \leq 1 \cdot 2n\delta \leq \varepsilon/2$. For tags t_i in $\dot{\mathcal{P}}_2$, we have $h(t_i) < \varepsilon/2$ and the total length of the subintervals in $\dot{\mathcal{P}}_2$ is clearly less than 1, so that $S(h; \dot{\mathcal{P}}_2) < (\varepsilon/2) \cdot 1 = \varepsilon/2$. Therefore, combining these results, we get

$$0 \leq S(h; \dot{\mathcal{P}}) = S(h; \dot{\mathcal{P}}_1) + S(h; \dot{\mathcal{P}}_2) < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we infer that $h \in \mathcal{R}[0, 1]$ with integral 0. □

$$h(x) = \begin{cases} 0, & x \in [0,1] \cap \mathbb{Q}^c \\ 1, & x = 0 \\ \frac{1}{n}, & x \in [0,1] \cap \mathbb{Q} \ (x = \frac{m}{n}) \end{cases} \qquad h \in R[0,1].$$

Prove by Definition:

$\forall \varepsilon > 0$, take $\varepsilon := \frac{1}{5}$ then there're finit of numbers x_k that $h(x_k) \geq \frac{1}{5} = \frac{\varepsilon}{2}$. Gather these x_k into set E.

e.g. in this case, $E = \{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}\}$.

Let n be the number of elements in E

Define $\dot{P}$ on $[0,1]$, $\delta := \frac{\varepsilon}{4n}$ and seperate it into 2:

① $\dot{P}_1$: collection of tagged ptt. in $\dot{P}$ that have their tags in E.

② $\dot{P}_2$: collection of tagged ptt. in $\dot{P}$ that have their tags out side E $\Rightarrow h(t_i) < \frac{\varepsilon}{2}$

$\Rightarrow S(h;\dot{P}) = S(h;\dot{P}_1) + S(h;\dot{P}_2)$

$$S(h;\dot{P}_1) \leq \sum_{k=1}^{n} 2 \times |h(x_k)||x_k - x_{k-1}|$$
$$\leq \sum_{k=1}^{n} 2||\dot{P}|| \, |h(x_k)| \quad \text{for } x_k \in E, \ h(x_k) \leq 1$$
$$\leq \sum_{k=1}^{n} 2\delta$$
$$= 2n\delta = 2n \cdot \frac{\varepsilon}{4n} = \frac{\varepsilon}{2}$$

$S(h;\dot{P}_2) < \frac{\varepsilon}{2} \cdot 1 = \frac{\varepsilon}{2}$ as $h(t_i) < 1$

$\Rightarrow 0 \leq S(h;\dot{P}_1) + S(h;\dot{P}_2) = S(h;\dot{P}) < \varepsilon$

$\Rightarrow h \in R[0,1]$, $\int_0^1 h = 0$ □.

Case 1: when c is not a tagged point.

then $S(f;\dot{P}) = S(g;\dot{P})$

$\Rightarrow \int_a^b f = \int_a^b g$

Case 2: when c is a tagged point.

The worst case is c is then endpoint of two partitions: $x_{i-1} \quad x_i \ (c) \quad x_{i+1}$

in this case, $g(c)(x_i - x_{i-1}) \neq f(c)(x_i - x_{i-1})$

$g(c)(x_{i+1} - x_i) \neq f(c)(x_{i+1} - x_i)$

$\Rightarrow |S(f;\dot{P}) - S(g;\dot{P})| = |(f(c) - g(c))(x_i - x_{i-1}) + (f(c) - g(c))(x_{i+1} - x_i)|$

$||a+b|| \leq |a| + |b|$

$\leq |f(c) - g(c)| \, \big| |x_i - x_{i-1}| + |x_{i+1} - x_i| \big|$

$\leq |f(c) - g(c)| \, |\, ||\dot{P}|| + ||\dot{P}|| \,|$

$\leq 2||\dot{P}|| (|f(c)| + |g(c)|)$ (1)

[Clue: $||\dot{P}|| < \frac{\varepsilon}{2[|f(c)| + |g(c)|]}$ but not yet

Goal: $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. if $||\dot{P}|| < \delta$ then $\exists\, L \in R$ s.t. $|S(f;\dot{P}) - L| < \varepsilon$]

$\forall \varepsilon > 0$, choose $\delta_1 > 0$ s.t. $\delta_1 < \frac{\varepsilon}{4[|f(c)| + |g(c)|]}$

choose $\delta_2 > 0$ s.t. if $||\dot{P}|| < \delta_2$, $|S(g;\dot{P}) - L| < \frac{\varepsilon}{2}$.

Then when $||\dot{P}|| < \delta = \min\{\delta_1, \delta_2\}$,

(1): $\leq 2||\dot{P}|| \, [|f(c)| + |g(c)|]$

$< \frac{\varepsilon}{2}$

As $|S(f;\dot{P}) - L| = |S(f;\dot{P}) - S(g;\dot{P}) + S(g;\dot{P}) - L|$

$\leq |S(f;\dot{P}) - S(g;\dot{P})| + |S(g;\dot{P}) - L|$

$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$

$= \varepsilon$

Thus, $f \in R[a,b]$ and $\int_a^b f = \int_a^b g$.

7.1.4 Examples

(a) Every constant function on $[a, b]$ is in $\mathcal{R}[a, b]$.

期中考了

(b) Let $g : [0, 3] \to \mathbb{R}$ be defined by $g(x) := 2$ for $0 \leq x \leq 1$, and $g(x) := 3$ for $1 < x \leq 3$. A preliminary investigation, based on the graph of g (see Figure 7.1.4), suggests that we might expect that $\int_0^3 g = 8$.

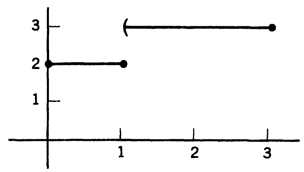


Figure 7.1.4 Graph of g

7.1.5 Theorem *Suppose that f and g are in $\mathcal{R}[a, b]$. Then:*

(a) *If $k \in \mathbb{R}$, the function kf is in $\mathcal{R}[a, b]$ and*

$$\int_a^b kf = k \int_a^b f.$$

(b) *The function $f + g$ is in $\mathcal{R}[a, b]$ and*

$$\int_a^b (f + g) = \int_a^b f + \int_a^b g.$$

(c) *If $f(x) \leq g(x)$ for all $x \in [a, b]$, then*

$$\int_a^b f \leq \int_a^b g.$$

Proof. If $\dot{\mathcal{P}} = \{([x_{i-1}, x_i], t_i)\}_{i=1}^n$ is a tagged partition of $[a, b]$, then it is an easy exercise to show that

$$S(kf; \dot{\mathcal{P}}) = kS(f; \dot{\mathcal{P}}), \qquad S(f + g; \dot{\mathcal{P}}) = S(f; \dot{\mathcal{P}}) + S(g; \dot{\mathcal{P}}),$$
$$S(f; \dot{\mathcal{P}}) \leq S(g; \dot{\mathcal{P}}).$$

We leave it to the reader to show that the assertion (a) follows from the first equality. As an example, we will complete the proofs of (b) and (c).

Given $\varepsilon > 0$, we can use the argument in the proof of the Uniqueness Theorem 7.1.2 to construct a number $\delta_\varepsilon > 0$ such that if $\dot{\mathcal{P}}$ is any tagged partition with $||\dot{\mathcal{P}}|| < \delta_\varepsilon$, then both

$$(5) \qquad \left|S(f; \dot{\mathcal{P}}) - \int_a^b f\right| < \varepsilon/2 \quad \text{and} \quad \left|S(g; \dot{\mathcal{P}}) - \int_a^b g\right| < \varepsilon/2.$$

To prove (b), we note that

$$\left|S(f + g; \dot{\mathcal{P}}) - \left(\int_a^b f + \int_a^b g\right)\right| = \left|S(f; \dot{\mathcal{P}}) + S(g; \dot{\mathcal{P}}) - \int_a^b f - \int_a^b g\right|$$
$$\leq \left|S(f; \dot{\mathcal{P}}) - \int_a^b f\right| + \left|S(g; \dot{\mathcal{P}}) - \int_a^b g\right|$$
$$< \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we conclude that $f + g \in \mathcal{R}[a, b]$ and that its integral is the sum of the integrals of f and g.

To prove (c), we note that the Triangle Inequality applied to (5) implies

$$\int_a^b f - \varepsilon/2 < S(f; \dot{\mathcal{P}}) \quad \text{and} \quad S(g; \dot{\mathcal{P}}) < \int_a^b g + \varepsilon/2.$$

If we use the fact that $S(f; \dot{\mathcal{P}}) \leq S(g; \dot{\mathcal{P}})$, we have

$$\int_a^b f \leq \int_a^b g + \varepsilon.$$

But, since $\varepsilon > 0$ is arbitrary, we conclude that $\int_a^b f \leq \int_a^b g$. Q.E.D.

7.1.6 Theorem *If $f \in \mathcal{R}[a, b]$, then f is bounded on $[a, b]$.*

Proof: Assume that f is not bounded.

As $f \in R[a,b]$ so let $L := \int_a^b f$, L exists.

Define $\dot{P}$ to be any tagged partition of $[a,b]$

then $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. if $\|\dot{P}\| < \delta$ then

$$|S(f;\dot{P}) - L| < \varepsilon$$

Choose $\varepsilon = 1$ then $|S(f;\dot{P}) - L| < 1$

$$\Rightarrow |S(f;\dot{P})| < |L| + 1 \quad ①$$

Let $Q = \{[x_{i-1}, x_i]\}_{i=1}^{n}$ be a partition for $[a,b]$ with $\|Q\| < \delta$.

因为要新选 t_k，故需定义新 Q

As f is not bounded so there exists at least one interval on $[a,b]$ s.t. $|f|$ is not bounded name it $[x_{k-1}, x_k]$.

Tag Q by $t_i = x_i$ for $i \neq k$, pick $t_k \in [x_{k-1}, x_k]$ s.t. (t_k可任选，与k^*相同)

$$|f(t_k)(x_k - x_{k-1})| > |L| + 1 + \left|\sum_{i \neq k} f(t_i)(x_i - x_{i-1})\right|$$

As $S(f;\dot{P}) = \left|\sum_{i \neq k} f(t_i)(x_i - x_{i-1}) + f(t_k)(x_k - x_{k-1})\right|$

corollary of Triangle Inequality: $|a| - |b| \leq |a+b| \leq |a| + |b|$

$$\geq |f(t_k)(x_k - x_{k-1})| - \left|\sum_{i \neq k} f(t_i)(x_i - x_{i-1})\right|$$

$$= |L| + 1$$

Contradicts to ①.

Thus, f is bounded.

Q.E.D.

7.2.1 Cauchy Criterion *A function: $[a, b] \to \mathbb{R}$ belongs to $\mathcal{R}[a,b]$ if and only if for every $\varepsilon > 0$ there exists $\eta_\varepsilon > 0$ such that if $\dot{\mathcal{P}}$ and $\dot{\mathcal{Q}}$ are any tagged partitions of $[a, b]$ with $\|\dot{\mathcal{P}}\| < \eta_\varepsilon$ and $\|\dot{\mathcal{Q}}\| < \eta_\varepsilon$, then*

$$\left|S(f; \dot{\mathcal{P}}) - S(f; \dot{\mathcal{Q}})\right| < \varepsilon.$$

□

(b) The Cauchy Criterion can be used to show that a function $f : [a, b] \to \mathbb{R}$ is *not* Riemann integrable. To do this we need to show that: *There exists $\varepsilon_0 > 0$ such that for any $\eta > 0$ there exists tagged partitions $\dot{\mathcal{P}}$ and $\dot{\mathcal{Q}}$ with $\|\dot{\mathcal{P}}\| < \eta$ and $\|\dot{\mathcal{Q}}\| < \eta$ such that* $\left|S(f; \dot{\mathcal{P}}) - S(f; \dot{\mathcal{Q}})\right| \geq \varepsilon_0$.

We will apply these remarks to the Dirichlet function, considered in 5.1.6(g), defined by $f(x) := 1$ if $x \in [0, 1]$ is rational and $f(x) := 0$ if $x \in [0, 1]$ is irrational.

Here we take $\varepsilon_0 := \frac{1}{2}$. If $\dot{\mathcal{P}}$ is any partition all of whose tags are rational numbers then $S(f; \dot{\mathcal{P}}) = 1$, while if $\dot{\mathcal{Q}}$ is any tagged partition all of whose tags are irrational numbers then $S(f; \dot{\mathcal{Q}}) = 0$. Since we are able to take such tagged partitions with arbitrarily small norms, we conclude that the Dirichlet function is *not* Riemann integrable. □

$[\Rightarrow]$ Proof: As $f \in R[a,b]$ so let $L := \int_a^b f$

$\forall \varepsilon > 0$, $\exists \eta_\varepsilon > 0$ s.t. for any tagged partitions

$\dot{P}, \dot{Q}$ of $[a,b]$, if $\|\dot{P}\| < \eta_\varepsilon$, $\|\dot{Q}\| < \eta_\varepsilon$

then $|S(f;\dot{P}) - L| < \frac{\varepsilon}{2}$ and $|S(f;\dot{Q}) - L| < \frac{\varepsilon}{2}$

$$\Rightarrow |S(f;\dot{P}) - S(f;\dot{Q})| = |S(f;\dot{P}) - L + L - S(f;\dot{Q})|$$
$$\leq |S(f;\dot{P}) - L| + |L - S(f;\dot{Q})|$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$
$$= \varepsilon$$

Q.E.D.

$[\Leftarrow]$ WTS: Limit of $S(f;\dot{P})$ exists (1) & unique (2).

Cauchy 普遍性检验

Suppose $\forall \varepsilon > 0$, $\exists \eta_\varepsilon > 0$ s.t. if $\|\dot{P}\| < \eta_\varepsilon$, $\|\dot{Q}\| < \eta_\varepsilon$, then $|S(f;\dot{P}) - S(f;\dot{Q})| < \varepsilon$

As ε is an arbitrary number in $\mathbb{P}$ $(k > 0)$

δ that depends on n

So $\forall n \in \mathbb{N}$, $\exists \delta_n > 0$ s.t. if $\|\dot{P}\| < \delta_n$, $\|\dot{Q}\| < \delta_n$

then $|S(f_n;\dot{P}) - S(f_n;\dot{Q})| < \frac{1}{n}$ (*)

$\varepsilon = 1$ $(n=1)$, $\exists \eta_1$ s.t. if $\dot{P}_1, \dot{Q}_1$ with $\|\dot{P}_1\| < \eta_1$, $\|\dot{Q}_1\| < \eta_1$,

then $|S(f;\dot{P}_1) - S(f;\dot{Q}_1)| < 1$

$\varepsilon = \frac{1}{2}$ $(n=2)$, $\exists \eta_2$ s.t. if ...

$\varepsilon = \frac{1}{n}$ $(n=n)$, $\exists \eta_n$ s.t. if $\dot{P}_n, \dot{Q}_n$ with $\|\dot{P}_n\| < \eta_n$, $\|\dot{Q}_n\| < \eta_n$ then $\boxed{|S(f;\dot{P}_n) - S(f;\dot{Q}_n)| < \frac{1}{n}}$ (※)

(1) Proof:

(此时越fine越精准并没有被证明)

Assume that $\delta_{n+1} \leq \delta_n$ for $n \in \mathbb{N}$,

Otherwise, replace δ_n by $\min\{\delta_1, \cdots \delta_n\}$

若 $\eta_{n+1} > \eta_n$ 都可满足(※)则缩小 η_{n+1} 也可满足

$\forall n \in \mathbb{N}$, let $\|P_n\|$ be a tagged partition with $\|\dot{P}_n\| \leq \delta_n$, then $\forall m > n$, $m \in \mathbb{N}$,

$\|\dot{P}_m\| \leq \delta_m \leq \delta_n$

$|S(f;\dot{P}_n) - S(f;\dot{P}_m)| < \frac{1}{n}$ for $m > n$ (※)

$\Rightarrow \{S(f;\dot{P}_m)\}_{m=1}^{\infty}$ is a cauchy sequence

therefore convergent. Let $A := \lim_{m\to\infty}\{S(f;\dot{P}_m)\}$

When $m \to \infty$,

$\Rightarrow \boxed{|S(f;\dot{P}_n) - A| < \frac{1}{n} \text{ for } \forall n \in \mathbb{N}}$ (有极限)

(2) Proof:

Define $\dot{Q}$ be any tagged partition of $[a,b]$ s.t. $\|\dot{Q}\| \leq \delta_K$ then by (※), $K > \frac{2}{\varepsilon}$

$$|S(f;\dot{Q}) - S(f;\dot{P}_K)| < \frac{1}{K}$$

Thus, $|S(f;\dot{Q}) - A| = |S(f;\dot{Q}) - S(f;\dot{P}_K) + S(f;\dot{P}_K) - A|$
$$\leq |S(f;\dot{Q}) - S(f;\dot{P}_K)| + |S(f;\dot{P}_K) - A|$$
$$< \frac{1}{K} + \frac{1}{K}$$
$$< \varepsilon$$

(且有唯一极限)

The limit is unique.

Q.E.D.

Proof. $(\Rightarrow)$ If $f \in \mathcal{R}[a, b]$ with integral L, let $\eta_\varepsilon := \delta_{\varepsilon/2} > 0$ be such that if $\dot{\mathcal{P}}$, $\dot{\mathcal{Q}}$ are tagged partitions such that $||\dot{\mathcal{P}}|| < \eta_\varepsilon$ and $||\dot{\mathcal{Q}}|| < \eta_\varepsilon$, then

$$\left|S(f;\dot{\mathcal{P}}) - L\right| < \varepsilon/2 \quad \text{and} \quad \left|S(f;\dot{\mathcal{Q}}) - L\right| < \varepsilon/2.$$

Therefore we have

$$\begin{aligned}\left|S(f;\dot{\mathcal{P}}) - S(f;\dot{\mathcal{Q}})\right| &\leq \left|S(f;\dot{\mathcal{P}}) - L + L - S(f;\dot{\mathcal{Q}})\right| \\ &\leq \left|S(f;\dot{\mathcal{P}}) - L\right| + \left|L - S(f;\dot{\mathcal{Q}})\right| \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon.\end{aligned}$$

$(\Leftarrow)$ For each $n \in \mathbb{N}$, let $\delta_n > 0$ be such that if $\dot{\mathcal{P}}$ and $\dot{\mathcal{Q}}$ are tagged partitions with norms $< \delta_n$, then

$$\left|S(f;\dot{\mathcal{P}}) - f(f;\dot{\mathcal{Q}})\right| < 1/n.$$

Evidently we may assume that $\delta_n \geq \delta_{n+1}$ for $n \in \mathbb{N}$; otherwise, we replace δ_n by $\delta_n' := \min\{\delta_1, \ldots, \delta_n\}$.

For each $n \in \mathbb{N}$, let $\dot{\mathcal{P}}_n$ be a tagged partition with $||\dot{\mathcal{P}}_n|| < \delta_n$. Clearly, if $m > n$ then both $\dot{\mathcal{P}}_m$ and $\dot{\mathcal{P}}_n$ have norms $< \delta_n$, so that

(1) $$\left|S(f;\dot{\mathcal{P}}_n) - S(f;\dot{\mathcal{P}}_m)\right| < 1/n \qquad \text{for} \qquad m > n.$$

Consequently, the sequence $\left(S(f;\dot{\mathcal{P}}_m)\right)_{m=1}^{\infty}$ is a Cauchy sequence in $\mathbb{R}$. Therefore (by Theorem 3.5.5) this sequence converges in $\mathbb{R}$ and we let $A := \lim_m S(f;\dot{\mathcal{P}}_m)$.

Passing to the limit in (1) as $m \to \infty$, we have

$$\left|S(f;\dot{\mathcal{P}}_n) - A\right| \leq 1/n \qquad \text{for all} \qquad n \in \mathbb{N}.$$

To see that A is the Riemann integral of f, given $\varepsilon > 0$, let $K \in \mathbb{N}$ satisfy $K > 2/\varepsilon$. If $\dot{\mathcal{Q}}$ is any tagged partition with $||\dot{\mathcal{Q}}|| < \delta_K$, then

$$\begin{aligned}\left|S(f;\dot{\mathcal{Q}}) - A\right| &\leq \left|S(f;\dot{\mathcal{Q}}) - S(f;\dot{\mathcal{P}}_K)\right| + \left|S(f;\dot{\mathcal{P}}_K) - A\right| \\ &\leq 1/K + 1/K < \varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, then $f \in \mathcal{R}[a, b]$ with integral A. Q.E.D.

7.2.3 Squeeze Theorem *Let $f : [a,b] \to \mathbb{R}$. Then $f \in \mathcal{R}[a,b]$ if and only if for every $\varepsilon > 0$ there exist functions α_ε and ω_ε in $\mathcal{R}[a,b]$ with*

$$\alpha_\varepsilon(x) \le f(x) \le \omega_\varepsilon(x) \qquad \text{for all} \qquad x \in [a,b], \tag{2}$$

and such that

$$\int_a^b (\omega_\varepsilon - \alpha_\varepsilon) < \varepsilon. \tag{3}$$

Proof. ($\Rightarrow$) Take $\alpha_\varepsilon = \omega_\varepsilon = f$ for all $\varepsilon > 0$.

($\Leftarrow$) Let $\varepsilon > 0$. Since α_ε and ω_ε belong to $\mathcal{R}[a,b]$, there exists $\delta_\varepsilon > 0$ such that if $\dot{\mathcal{P}}$ is any tagged partition with $||\dot{\mathcal{P}}|| < \delta_\varepsilon$ then

$$\left|S(\alpha_\varepsilon;\dot{\mathcal{P}}) - \int_a^b \alpha_\varepsilon\right| < \varepsilon \qquad \text{and} \qquad \left|S(\omega_\varepsilon;\dot{\mathcal{P}}) - \int_a^b \omega_\varepsilon\right| < \varepsilon.$$

It follows from these inequalities that

$$\int_a^b \alpha_\varepsilon - \varepsilon < S(\alpha_\varepsilon;\dot{\mathcal{P}}) \qquad \text{and} \qquad S(\omega_\varepsilon;\dot{\mathcal{P}}) < \int_a^b \omega_\varepsilon + \varepsilon.$$

直接去绝对值

In view of inequality (2), we have $S(\alpha_\varepsilon;\dot{\mathcal{P}}) \le S(f;\dot{\mathcal{P}}) \le S(\omega_\varepsilon;\dot{\mathcal{P}})$, whence

$$\int_a^b \alpha_\varepsilon - \varepsilon < S(f;\dot{\mathcal{P}}) < \int_a^b \omega_\varepsilon + \varepsilon.$$

If $\dot{\mathcal{Q}}$ is another tagged partition with $||\dot{\mathcal{Q}}|| < \delta_\varepsilon$, then we also have

$$\int_a^b \alpha_\varepsilon - \varepsilon < S(f;\dot{\mathcal{Q}}) < \int_a^b \omega_\varepsilon + \varepsilon.$$

If we subtract these two inequalities and use (3), we conclude that

$$\begin{aligned} |S(f;\dot{\mathcal{P}}) - S(f;\dot{\mathcal{Q}})| &< \int_a^b \omega_\varepsilon - \int_a^b \alpha_\varepsilon + 2\varepsilon \\ &= \int_a^b (\omega_\varepsilon - \alpha_\varepsilon) + 2\varepsilon < 3\varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, the Cauchy Criterion implies that $f \in \mathcal{R}[a,b]$. Q.E.D.

7.2.4 Lemma *If J is a subinterval of $[a,b]$ having endpoints $c < d$ and if $\varphi_J(x) := 1$ for $x \in J$ and $\varphi_J(x) := 0$ elsewhere in $[a,b]$, then $\varphi_J \in \mathcal{R}[a,b]$ and $\int_a^b \varphi_J = d - c$.*

Proof. If $J = [c,d]$ with $c \le d$, this is Exercise 7.1.13 and we can choose $\delta_\varepsilon := \varepsilon/4$.

There are three other subintervals J having the same endpoints c and d, namely, $[c,d)$, $(c,d]$, and (c,d). Since, by Theorem 7.1.3, we can change the value of a function at finitely many points without changing the integral, we have the same result for these other three subintervals. Therefore, we conclude that all four functions φ_J are integrable with integral equal to $d - c$. Q.E.D.

Classes of R.I.

7.2.5 Theorem *If $\varphi : [a, b] \to \mathbb{R}$ is a step function, then $\varphi \in \mathcal{R}[a, b]$.*

Proof. Step functions of the type appearing in 7.2.4 are called "elementary step functions." In Exercise 5 it is shown that an arbitrary step function φ can be expressed

as a linear combination of such elementary step functions:

$$\varphi = \sum_{j=1}^{m} k_j \varphi_{J_j}, \tag{4}$$

where J_j has endpoints $c_j < d_j$. The lemma and Theorem 7.1.5(a, b) imply that $\varphi \in \mathcal{R}[a, b]$ and that

$$\int_a^b \varphi = \sum_{j=1}^{m} k_j (d_j - c_j). \tag{5}$$

Q.E.D.

7.2.6 Examples **(a)** The function g in Example 7.1.4(b) is defined by $g(x) = 2$ for $0 \le x \le 1$ and $g(x) = 3$ for $1 < x \le 3$. We now see that g is a step function and therefore we calculate its integral to be $\int_0^3 g = 2\cdot(1 - 0) + 3\cdot(3 - 1) = 2 + 6 = 8$.

(b) Let $h(x) := x$ on [0,1] and let $P_n := (0, 1/n, 2/n, \ldots, (n-1)/n, n/n = 1)$. We define the step functions α_n and ω_n on the disjoint subintervals $[0, 1/n), [1/n, 2/n), \ldots, [(n-2)/n), (n-1)/n), [(n-1)/n, 1]$ as follows: $\alpha_n(x) := h((k-1)/n) = (k-1)/n$ for x in $[(k-1)/n, k/n)$ for $k = 1, 2, \ldots, n-1$, and $\alpha_n(x) := h((n-1)/n) = (n-1)/n$ for x in $[(n-1)/n, 1]$. That is, α_n has the minimum value of h on each subinterval. Similarly, we define ω_n to be the maximum value of h on each subinterval, that is, $\omega_n(x) := k/n$ for x in $[(k-1)/n, k/n)$ for $k = 1, 2, \ldots, n-1$, and $\omega_n(x) := 1$ for x in $[(n-1)/n, 1]$. (The reader should draw a sketch for the case $n = 4$.)

Then we get

$$\begin{aligned}
\int_0^1 \alpha_n &= \frac{1}{n}(0 + 1/n + 2/n + \cdots + (n-1)/n) \\
&= \frac{1}{n^2}(1 + 2 + \cdots + (n-1)) \\
&= \frac{1}{n^2}\frac{(n-1)n}{2} = \frac{1}{2}(1 - 1/n).
\end{aligned}$$

In a similar manner, we also get $\int_0^1 \omega_n = \frac{1}{2}(1 + 1/n)$. Thus we have

$$\alpha_n(x) \le h(x) \le \omega_n(x)$$

for $x \in [0, 1]$ and

$$\int_0^1 (\omega_n - \alpha_n) = \frac{1}{n}.$$

Since for a given $\varepsilon > 0$, we can choose n so that $\frac{1}{n} < \varepsilon$, it follows from the Squeeze Theorem that h is integrable. We also see that the value of the integral of h lies between the integrals of α_n and ω_n for all n and therefore has value $\frac{1}{2}$. □

Uniform Continuity Max-Min

7.2.7 Theorem *If $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$, then $f \in \mathcal{R}[a, b]$.*

Proof. It follows from Theorem 5.4.3 that f is uniformly continuous on $[a, b]$. Therefore, given $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that if $u, v \in [a, b]$ and $|u - v| < \delta_\varepsilon$, then we have $|f(u) - f(v)| < \varepsilon/(b - a)$.

Let $\mathcal{P} = \{I_i\}_{i=1}^n$ be a partition such that $||\mathcal{P}|| < \delta_\varepsilon$. Applying Theorem 5.3.4 we let $u_i \in I_i$ be a point where f attains its minimum value on I_i, and let $v_i \in I_i$ be a point where f attains its maximum value on I_i.

Let α_ε be the step function defined by $\alpha_\varepsilon(x) := f(u_i)$ for $x \in [x_{i-1}, x_i)$ $(i = 1, \ldots, n - 1)$ and $\alpha_\varepsilon(x) := f(u_n)$ for $x \in [x_{n-1}, x_n]$. Let ω_ε be defined similarly using the points v_i instead of the u_i. Then one has

$$\alpha_\varepsilon(x) \le f(x) \le \omega_\varepsilon(x) \qquad \text{for all} \qquad x \in [a, b].$$

Moreover, it is clear that

$$0 \le \int_a^b (\omega_\varepsilon - \alpha_\varepsilon) = \sum_{i=1}^n (f(v_i) - f(u_i))(x_i - x_{i-1})$$
$$< \sum_{i=1}^n \left(\frac{\varepsilon}{b - a}\right)(x_i - x_{i-1}) = \varepsilon.$$

Therefore it follows from the Squeeze Theorem that $f \in \mathcal{R}[a, b]$. Q.E.D.

Proof: As f is continuous on a closed & bounded interval $[a,b]$, so f is uniformly ~~convergent~~ continuous

$\Rightarrow \forall \varepsilon > 0, \exists \delta > 0$ s.t. $\forall u, v \in [a,b]$, if $|u-v| < \delta$,

then $|f(u) - f(v)| < \frac{\varepsilon}{b-a}$.

let $P = \{I_i\}_{i=1}^n$ be any partition of $[a,b]$ with $\|P\| < \delta$. then I_i is a closed and bounded interval for $i = 1, \cdots, n$.

therefore, by Max-Min Thm, for each I_i,

$\exists u_i, v_i \in I_i$ s.t. $\forall x \in I_i$,

$f(u_i) \le f(x) \le f(v_i)$. 初步目的.

Then, define functions

$\alpha(x) = f(u_i)$ for $x \in [x_{i-1}, x_i)$, $i = 1, 2, \cdots, n-1$

and

$\alpha(x) = f(u_n)$ $x \in [x_{n-1}, x_n]$.

$\omega(x) = f(v_i)$ for $x \in [x_{i-1}, x_i)$, $i = 1, 2, \cdots, n-1$

$\omega(x) = f(v_n)$ $x \in [x_{n-1}, x_n]$

$\Rightarrow$ $\alpha(x) \le f(x) \le \omega(x)$ $\forall x \in [a,b]$

and $\alpha(x)$, $\omega(x)$ are step functions

$\Rightarrow$ $\alpha(x), \omega(x) \in R[a,b]$

Also, as

$$0 \le \int_a^b (\omega(x) - \alpha(x)) = \sum_{i=1}^n [f(v_i) - f(u_i)][x_i - x_{i-1}]$$
$$< \frac{\varepsilon}{b-a} \sum_{i=1}^n [x_i - x_{i-1}]$$
$$= \varepsilon$$

By Squeeze Thm, $f \in R[a,b]$

Q.E.D.

7.2.8 Theorem *If $f : [a, b] \to \mathbb{R}$ is monotone on $[a, b]$, then $f \in \mathcal{R}[a, b]$.*

Proof. Assume that f is increasing on $I = [a, b]$. Partitioning the interval into n equal subintervals $I_k = [x_{k-1}, x_k]$ gives us $x_k - x_{k-1} = (b-a)/n$, $k = 1, 2, \ldots, n$. Since f is increasing on I_k, its minimum value is attained at the left endpoint x_{k-1} and its maximum value is attained at the right endpoint x_k. Therefore, we define the step functions $\alpha(x) := f(x_{k-1})$ and $\omega(x) := f(x_k)$ for $x \in [x_{k-1}, x_k)$, $k = 1, 2, \ldots, n-1$, and $\alpha(x) := f(x_{n-1})$ and $\omega(x) := f(x_n)$ for $x \in [x_{n-1}, x_n]$. Then we have $\alpha(x) \le f(x) \le \omega(x)$ for all $x \in I$, and

$$\int_a^b \alpha = \frac{b-a}{n}(f(x_0) + f(x_1) + \cdots + f(x_{n-1}))$$

$$\int_a^b \omega = \frac{b-a}{n}(f(x_1) + \cdots + f(x_{n-1}) + f(x_n)).$$

Subtracting, and noting the many cancellations, we obtain

$$\int_a^b (\omega - \alpha) = \frac{b-a}{n}(f(x_n) - f(x_0)) = \frac{b-a}{n}(f(b) - f(a)).$$

Thus for a given $\varepsilon > 0$, we choose n such that $n > (b-a)(f(b)-f(a))/\varepsilon$. Then we have $\int_a^b (\omega - \alpha) < \varepsilon$ and the Squeeze Theorem implies that f is integrable on I. Q.E.D.

Proof: As f is monotone on $[a,b]$

assume that f is monotone increasing

Define P to be a partition,

$P = \{I_k\}_{k=1}^{n}$ $\quad I = [x_{k-1}, x_k]$.

then $x_k - x_{k-1} = \frac{b-a}{n}$, $k = 1, 2, \cdots n$.

As f is monotone increasing so $\forall x \in I_k$,

$f(x_{k-1}) \le f(x) \le f(x_k)$

⇒ 规定其为 step func.

⇓

w, α are R.I.

Define $\alpha(x) = f(x_{k-1})$, $w(x) = f(x_k)$

for $x \in [x_{k-1}, x_k)$, $k = 1, 2, \cdots n-1$ 最后: $[x_{n-1}, x_n]$ 终.

$\alpha(x) = f(x_{n-1})$ $\quad w(x) = f(x_n)$ for $x \in [x_{n-1}, x_n]$

Thus, $\alpha(x) \le f(x) \le w(x)$ for $\forall x \in [a,b]$

and $\int_a^b \alpha = \frac{b-a}{n}[f(x_0) + f(x_1) + \cdots + f(x_{n-1})]$ ①

$\int_a^b w = \frac{b-a}{n}[f(x_1) + f(x_2) + \cdots + f(x_n)]$ ②

① - ②: $\int_a^b (w - \alpha) = \frac{b-a}{n}[f(x_n) - f(x_0)]$

$= \frac{b-a}{n}[f(b) - f(a)]$

$\boxed{\text{Thus, } \forall \varepsilon > 0, \text{ Choose } n \in \mathbb{N} \text{ s.t. } n > \frac{(b-a)(f(b)-f(a))}{\varepsilon}}$

then $\int_a^b (w - \alpha) < \frac{(b-a)\varepsilon}{(b-a)(f(b)-f(a))}[f(b) - f(a)]$

$= \varepsilon$

Thus, by Squeeze Thm, $f \in R[a,b]$.

Q.E.D.

7.2.9 Additivity Theorem *Let $f := [a, b] \to \mathbb{R}$ and let $c \in (a, b)$. Then $f \in \mathcal{R}[a, b]$ if and only if its restrictions to $[a, c]$ and $[c, b]$ are both Riemann integrable. In this case*

$$\int_a^b f = \int_a^c f + \int_c^b f. \tag{6}$$

结论有2:

1) $f \in$ R.I.

2) R 数值上等.

设 Step Func.

构造不等关系

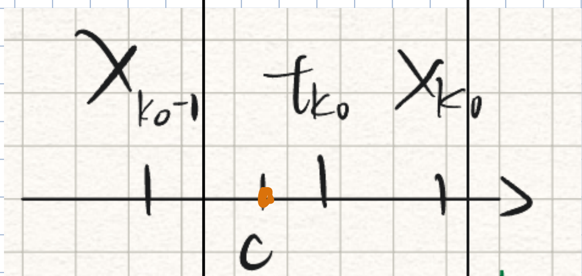
⇓

Squeeze 和等式

⇓

Additivity

得 R.I.



$\Leftarrow$ Let $f = \begin{cases} f_1, & x \in [a, c] \\ f_2, & x \in [c, b] \end{cases}$

So $f_1 \in R[a, c]$, $f_2 \in R[c, b]$

Define $L_1 := \int_a^c f_1$, $L_2 := \int_c^b f_2$

then Define $\dot{P}_1$ to be any tagged partition on $[a, c]$

$\forall \varepsilon > 0$, $\exists \delta_1 > 0$ s.t. if $\|\dot{P}_1\| < \delta_1$ then

$$|S(f_1; \dot{P}_1) - L_1| < \frac{\varepsilon}{3}$$

Define $\dot{P}_2$ to be any tagged partition on $[c, b]$

$\forall \varepsilon > 0$, $\exists \delta_2 > 0$ s.t. if $\|\dot{P}_2\| < \delta_2$ then

$$|S(f_2; \dot{P}_2) - L_2| < \frac{\varepsilon}{3}$$

As f is defined on $[a, b]$ so f is bounded

i.e. $\exists M > 0$ s.t. $|f(x)| \le M$, $\forall x \in [a, b]$

Define tagged partition $\dot{Q}$ on $[a, b]$

with $\|\dot{Q}\| < \delta = \min\{\delta_1, \delta_2, \frac{\varepsilon}{6M}\}$

Case I: If c is a tag of $[a, b]$

Split $\dot{Q}$ into $\dot{Q}_1$ on $[a, c]$ and $\dot{Q}_2$ on $[c, b]$

前提

then $S(f; \dot{Q}) = S(f; \dot{Q}_1) + S(f; \dot{Q}_2)$

$\Rightarrow \left|S(f; \dot{Q}) - (L_1 + L_2)\right| \le \frac{\varepsilon}{3} + \frac{\varepsilon}{3} < \varepsilon$

$\Rightarrow f \in R[a, b]$. $\int_a^b f = L_1 + L_2$.

Case 2: If c is not a tag of $[a, b]$

as $\dot{Q} = \{I_k, t_k\}_{k=1}^m$ then $\exists k_0 \in \mathbb{N}$

$k_0 \le m$ s.t. $c \in I_{k_0}$

Define $\dot{Q}_1 := \{(I_1, t_1), \dots, (I_{k_0-1}, t_{k_0-1}), ([x_{k_0-1}, c], c)\}$

$\dot{Q}_2 := \{([c, x_{k_0}], c), (I_{k_0+1}, t_{k_0+1}), \cdots (I_m, t_m)\}$

$\Rightarrow \boxed{S(f; \dot{Q}) - S(f; \dot{Q}_1) - S(f; \dot{Q}_2)}$ $(c - x_{k_0-1}) + (x_{k_0} - c)$

$= f(t_{k_0})(x_{k_0} - x_{k_0-1}) - f(c)(x_{k_0} - x_{k_0-1})$

$= [f(t_{k_0}) - f(c)](x_{k_0} - x_{k_0-1})$

$\le (|f(t_{k_0})| + |f(c)|)(x_{k_0} - x_{k_0-1})$

$\le 2M(x_{k_0} - x_{k_0-1})$

$\le 2M \cdot \delta$

$< 2M \cdot \frac{\varepsilon}{6M}$

$= \frac{\varepsilon}{3}$

Goal

$$\Rightarrow \boxed{\begin{aligned} |S(f; \dot{Q}) - (L_1 + L_2)| = |&S(f; \dot{Q}) - S(f; \dot{Q}_1) - S(f; \dot{Q}_2) \\ &+ S(f; \dot{Q}_1) - L_1 + S(f; \dot{Q}_2) - L_2| \end{aligned}}$$

$$\le \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3}$$

$$= \varepsilon$$

Thus, $\int_a^b f = L_1 + L_2 = \int_a^c f + \int_c^b f$

Q.E.D.

$\Rightarrow$. Suppose that $f \in R[a,b]$

Define any tagged partitions $\dot{K}$ and $\dot{R}$ on $[a,b]$. $\forall \varepsilon > 0$, $\exists \delta_\varepsilon > 0$ s.t. if $\|\dot{K}\| < \delta_\varepsilon$, $\|\dot{R}\| < \delta_\varepsilon$

then $|S(f;\dot{K}) - S(f;\dot{R})| < \varepsilon$

Cauchy

Let $\dot{P}_1, \dot{Q}_1$ be any tagged ptts of $[a,c]$

with $\|\dot{P}_1\| < \delta_\varepsilon$ $\|\dot{Q}_1\| < \delta_\varepsilon$

令$[a,c]$不同,
$[c,b]$相同.

Extend $\dot{P}_1$ and $\dot{Q}_1$ to $\dot{P}$ and $\dot{Q}$ by adding same additional points and tags from $[c,b]$ that satisfy $\|\dot{P}\| < \delta_\varepsilon$, $\|\dot{Q}\| < \delta_\varepsilon$

then $S(f_1;\dot{P}_1) - S(f_1;\dot{Q}_1) = S(f;\dot{P}) - S(f;\dot{Q})$

$$\boxed{\begin{aligned}&\Rightarrow |S(f_1;\dot{P}_1) - S(f_1;\dot{Q}_1)| = |S(f;\dot{P}) - S(f;\dot{Q})| < \varepsilon \\ &\Rightarrow f_1 \in R[a,c]\end{aligned}}$$

Goal.

Similarly, $f_2 \in R[c,b]$

$$\Rightarrow \int_a^b f = \int_a^c f + \int_c^b f$$

Q.E.D.

Summary: 关于黎曼可积的判定:

1. 定义 → 7. Step Function → 6. 可加性 $f_1 \in R[a,c], f_2 \in R[c,b] \Leftrightarrow f \in R[a,b]$

5. 闭区间上单调函数 $\in R[a,b]$ (Monotone)

2. Cauchy

3. Squeeze. → 4. 闭区间上连续函数 $\in R[a,b]$. (Continuous)

Generalization of Additivity Theorem:

7.2.10 Corollary *If $f \in \mathcal{R}[a,b]$, and if $[c,d] \subseteq [a,b]$, then the restriction of f to $[c,d]$ is in $\mathcal{R}[c,d]$.*

Proof. Since $f \in \mathcal{R}[a,b]$ and $c \in [a,b]$, it follows from the theorem that its restriction to $[c,b]$ is in $\mathcal{R}[c,b]$. But if $d \in [c,b]$, then another application of the theorem shows that the restriction of f to $[c,d]$ is in $\mathcal{R}[c,d]$. Q.E.D.

7.2.11 Corollary *If $f \in \mathcal{R}[a,b]$ and if $a = c_0 < c_1 < \cdots < c_m = b$, then the restrictions of f to each of the subintervals $[c_{i-1}, c_i]$ are Riemann integrable and*

$$\int_a^b f = \sum_{i=1}^{m} \int_{c_{i-1}}^{c_i} f.$$

Generalization of the interval:

7.2.12 Definition If $f \in \mathcal{R}[a,b]$ and if $\alpha, \beta \in [a,b]$ with $\alpha < \beta$, we define

$$\int_\beta^\alpha f := -\int_\alpha^\beta f \qquad \text{and} \qquad \int_\alpha^\alpha f := 0.$$

7.2.13 Theorem *If $f \in \mathcal{R}[a,b]$ and if α, β, γ are any numbers in $[a,b]$, then*

$$\int_\alpha^\beta f = \int_\alpha^\gamma f + \int_\gamma^\beta f, \tag{8}$$

用 $\int_\alpha^\alpha f = 0$

in the sense that the existence of any two of these integrals implies the existence of the third integral and the equality (8).

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(求)

Proof: Case I: Any 2 of α, β, γ are equal

then as $\int_a^a f = 0$ so $\int_\alpha^\beta f = \int_\alpha^\gamma f + \int_\gamma^\beta f$.

Case 2: $\alpha \neq \beta \neq \gamma$. distinct values.

There are $A_3^3 = 6$ kinds of permutations:

Choose $\alpha < \gamma < \beta$ then

By Additive law

$$L(\alpha, \beta, \gamma) = \int_\alpha^\beta f + \int_\beta^\gamma f + \int_\gamma^\alpha f = 0$$

Similarly, $L = 0$ for all configurations

Thus, $\int_\alpha^\beta f = \int_\alpha^\gamma f + \int_\gamma^\beta f$

Q.E.D.

7.3.1 Fundamental Theorem of Calculus (First Form) *Suppose there is a **finite** set E in $[a, b]$ and functions $f, F := [a, b] \to \mathbb{R}$ such that:*

(a) *F is continuous on $[a, b]$,*
(b) *$F'(x) = f(x)$ for all $x \in [a, b] \backslash E$,*
(c) *f belongs to $\mathcal{R}[a, b]$.*

Then we have

$$\int_a^b f = F(b) - F(a). \tag{1}$$

1. Extend the proof of the Fundamental Theorem 7.3.1 to the case of an arbitrary finite set E.

Proof. Assume that the set $E = \{a = c_0 < c_1 \cdots < c_m = b\}$ contains the points in $[a, b]$ where the derivative $F'(x)$ doesn't exist or $F'(x) \neq f(x)$.

Then for every subset

$E_{(i)} = \{c_{i-1}, c_i\}$, $i = 0, 1, 2, 3 \cdots$

they by 7.2.11,

$$\int_a^b f = \sum_{i=1}^{m} \int_{c_{i-1}}^{c_i} f$$

As by 7.3.1, $\int_{c_{i-1}}^{c_i} f = F(c_i) - F(c_{i-1})$

Thus, $\int_a^b f = \sum_{i=1}^{m} [F(c_i) - F(c_{i-1})]$

$= F(c_m) - F(c_0)$

$= F(b) - F(a)$ Q.E.D.

Proof: In the case $E := \{a, b\}$

(a) F is continuous on $[a, b]$,

$\forall \varepsilon > 0$, Since $f \in \mathcal{R}[a, b]$ then $\exists \delta_\varepsilon > 0$ s.t.

if $\dot{\mathcal{P}}$ is any partition with $\|\dot{\mathcal{P}}\| < \delta_\varepsilon$ then

$$\left| S(f; \dot{\mathcal{P}}) - \int_a^b f \right| < \varepsilon$$

If the subintervals of $\dot{\mathcal{P}}$ are $[x_{i-1}, x_i]$ then

Mean Value Theorem $\Rightarrow$ $\forall i \in \mathbb{N}$, $\exists u_i \in (x_{i-1}, x_i)$

s.t. $F(x_i) - F(x_{i-1}) = F'(u_i)[x_i - x_{i-1}]$

$\Rightarrow \sum [F(x_i) - F(x_{i-1})] = \sum F'(u_i)[x_i - x_{i-1}]$

$\Rightarrow F(b) - F(a) = \sum_{i=1}^{n} f(u_i)(x_i - x_{i-1})$

(b) $F'(x) = f(x)$ for all $x \in [a, b] \backslash E$,

Let $\dot{\mathcal{P}}_u := \{[x_{i-1}, x_i], u_i\}_{i=1}^{n}$ so

$$\boxed{\begin{aligned} &F(b) - F(a) = S(f; \dot{\mathcal{P}}_u) \\ \Rightarrow\ &\left| F(b) - F(a) - \int_a^b f \right| < \varepsilon \end{aligned}}$$

Goal

Q.E.D.

The general case can be obtained by breaking $[a, b]$ into the union of finite number of intervals.

Proof. We will prove the theorem in the case where $E := \{a, b\}$. The general case can be obtained by breaking the interval into the union of a finite number of intervals (see Exercise 1).

Let $\varepsilon > 0$ be given. Since $f \in \mathcal{R}[a, b]$ by assumption (c), there exists $\delta_\varepsilon > 0$ such that if $\dot{\mathcal{P}}$ is *any* tagged partition with $\|\dot{\mathcal{P}}\| < \delta_\varepsilon$, then

$$\left| S(f; \dot{\mathcal{P}}) - \int_a^b f \right| < \varepsilon. \tag{2}$$

If the subintervals in $\dot{\mathcal{P}}$ are $[x_{i-1}, x_i]$, then the Mean Value Theorem 6.2.4 applied to F on $[x_{i-1}, x_i]$ implies that there exists $u_i \in (x_{i-1}, x_i)$ such that

$$F(x_i) - F(x_{i-1}) = F'(u_i) \cdot (x_i - x_{i-1}) \qquad \text{for} \qquad i = 1, \ldots, n.$$

If we add these terms, note the telescoping of the sum, and use the fact that $F'(u_i) = f(u_i)$, we obtain

$$F(b) - F(a) = \sum_{i=1}^{n} (F(x_i) - F(x_{i-1})) = \sum_{i=1}^{n} f(u_i)(x_i - x_{i-1}).$$

Now let $\dot{\mathcal{P}}_u := \{([x_{i-1}, x_i], u_i)\}_{i=1}^{n}$, so the sum on the right equals $S(f; \dot{\mathcal{P}}_u)$. If we substitute $F(b) - F(a) = S(f; \dot{\mathcal{P}}_u)$ into (2), we conclude that

$$\left| F(b) - F(a) - \int_a^b f \right| < \varepsilon.$$

But, since $\varepsilon > 0$ is arbitrary, we infer that equation (1) holds. Q.E.D.

7.3.3 Definition If $f \in \mathcal{R}[a, b]$, then the function defined by

$$F(z) := \int_a^z f \qquad \text{for} \qquad z \in [a, b], \tag{3}$$

is called the **indefinite integral** of f with **basepoint** a. (Sometimes a point other than a is used as a basepoint; see Exercise 6.)

若$f \in R[a,b]$则其不定积分连续且满足Lipstchiz.

7.3.4 Theorem *The indefinite integral F defined by (3) is continuous on $[a, b]$. In fact, if $|f(x)| \le M$ for all $x \in [a, b]$, then $|F(z) - F(w)| \le M|z - w|$ for all $z, w \in [a, b]$.*

Proof. The Additivity Theorem 7.2.9 implies that if $z, w \in [a, b]$ and $w \le z$, then

$$\boxed{F(z) = \int_a^z f = \int_a^w f + \int_w^z f = F(w) + \int_w^z f,}$$

whence we have

$$F(z) - F(w) = \int_w^z f.$$

Now if $-M \le f(x) \le M$ for all $x \in [a, b]$, then Theorem 7.1.5(c) implies that

$$-M(z - w) \le \int_w^z f \le M(z - w),$$

whence it follows that

$$|F(z) - F(w)| \le \left| \int_w^z f \right| \le M|z - w|,$$

as asserted. Q.E.D.

Proof: $\forall x, y \in [a, b]$, $F(x) = \int_a^x f$, $F(y) = \int_a^y f$

$\Rightarrow |F(x) - F(y)| = \left| \int_a^x f - \int_a^y f \right|$

$= \left| \int_a^x f + \int_y^a f \right|$

$= \left| \int_y^x f \right|$

As $f \in R[a, b]$ so f is bounded

$\Rightarrow \exists M > 0$ s.t. $\forall x \in [a, b]$, $|f(x)| \le M$

$\Rightarrow -M \le f(x) \le M$

$-M(x-y) \le \int_y^x f(x) \le M(x-y)$

$\left| \int_y^x f(x) \right| \le M(x-y)$

$\Rightarrow |F(x) - F(y)| \le M(x-y)$

Q.E.D.

1点

7.3.5 Fundamental Theorem of Calculus (Second Form) *Let $f \in \mathcal{R}[a, b]$ and let f be continuous at a point $c \in [a, b]$. Then the indefinite integral, defined by (3), is differentiable at c and $F'(c) = f(c)$.*

$F(c) = \int_a^c f.$ $\quad c \in [a,b]$

\#

$F(x) = \int_a^x f$

$F'(x) = f(x)$

条件: $f \in R[a,b]$

f continuous at c

(隐含: $F(x) = \int_a^x f$ continuous on $[a,b]$)

结论: F is differentiable at c

$F'(c) = f(c)$

Proof. We will suppose that $c \in [a, b)$ and consider the right-hand derivative of F at c. Since f is continuous at c, given $\varepsilon > 0$ there exists $\eta_\varepsilon > 0$ such that if $c \le x < c + \eta_\varepsilon$, then

$$f(c) - \varepsilon < f(x) < f(c) + \varepsilon. \tag{4}$$

Let h satisfy $0 < h < \eta_\varepsilon$. The Additivity Theorem 7.2.9 implies that f is integrable on the intervals $[a, c]$, $[a, c + h]$ and $[c, c + h]$ and that

$$F(c + h) - F(c) = \int_c^{c+h} f.$$

Now on the interval $[c, c + h]$ the function f satisfies inequality (4), so that we have

$$(f(c) - \varepsilon) \cdot h \le F(c + h) - F(c) = \int_c^{c+h} f \le (f(c) + \varepsilon) \cdot h.$$

If we divide by $h > 0$ and subtract $f(c)$, we obtain

$$\left| \frac{F(c + h) - F(c)}{h} - f(c) \right| \le \varepsilon.$$

But, since $\varepsilon > 0$ is arbitrary, we conclude that the right-hand limit is given by

$$\lim_{x \to 0+} \frac{F(c + h) - F(c)}{h} = f(c).$$

It is proved in the same way that the left-hand limit of this difference quotient also equals $f(c)$ when $c \in (a, b]$, whence the assertion follows. Q.E.D.

Proof: As f is continuous at $c \in [a,b]$ so $\forall \varepsilon > 0$, $\exists \delta > 0$

s.t. if $|x-c| < \delta$ then $|f(x) - f(c)| < \varepsilon$

$\Rightarrow \quad f(c) - \varepsilon < f(x) < f(c) + \varepsilon$

In the region of $|x-c| < \delta$: $c - \delta < x < c + \delta$

Consider $[c, c+\delta)$, define $0 < h < \delta$

then from $f \in R[a,b]$, by Additive Laws

$f \in R\,[a,c],\ [a,c+h],\ [c,c+h]$

and $\int_a^{c+h} f - \int_a^c f = \int_c^{c+h} f$

$F(c+h) - F(c) = \int_c^{c+h} f$

On $[c, c+h]$, $f(c) - \varepsilon < f(x) < f(c) + \varepsilon$

$\Rightarrow \int_c^{c+h} (f(c) - \varepsilon) < \int_c^{c+h} f(x) \le \int_c^{c+h} (f(c) + \varepsilon)$

$h(f(c) - \varepsilon) < F(c+h) - F(c) \le h(f(c) + \varepsilon)$

$f(c) - \varepsilon < \frac{F(c+h) - F(c)}{h} \le f(c) + \varepsilon$

$\Rightarrow \quad \left| \frac{F(c+h) - F(c)}{h} - f(c) \right| < \varepsilon$ Goal

$\Rightarrow \quad f(c) = \lim_{h \to 0^+} \frac{F(c+h) - F(c)}{h}$

Similarly, $f(c) = \lim_{h \to 0^-} \frac{F(c+h) - F(c)}{h}$

$\Rightarrow \quad f(c) = F'(c)$

Q.E.D.

7.3.6 Theorem *If f is continuous on $[a, b]$, then the indefinite integral F, defined by (3), is differentiable on $[a, b]$ and $F'(x) = f(x)$* for all *$x \in [a, b]$.*

7.3.7 Examples **(a)** If $f(x) := \operatorname{sgn} x$ on $[-1, 1]$, then $f \in \mathcal{R}[-1, 1]$ and has the indefinite integral $F(x) := |x| - 1$ with the basepoint -1. However, since $F'(0)$ does not exist, F is not an antiderivative of f on $[-1, 1]$.

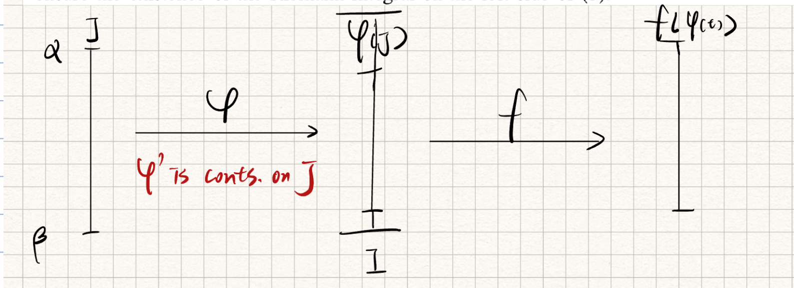
(b) If h denotes Thomae's function, considered in 7.1.7, then its indefinite integral $H(x) := \int_0^x h$ is identically 0 on $[0, 1]$. Here, the derivative of this indefinite integral exists at every point and $H'(x) = 0$. But $H'(x) \ne h(x)$ whenever $x \in \mathbb{Q} \cap [0, 1]$, so that H is not an antiderivative of h on $[0, 1]$. □

由 7.3.4. 不满足F(x)在 [a,b]上可导就不是 antiderivative. ⇒ 不能用 FTC.

7.3.8 Substitution Theorem *Let $J := [\alpha, \beta]$ and let $\varphi : J \to \mathbb{R}$ have a continuous derivative on J. If $f : I \to \mathbb{R}$ is continuous on an interval I containing $\varphi(J)$, then*

$$(5) \qquad \int_{\alpha}^{\beta} f(\varphi(t)) \cdot \varphi'(t)\,dt = \int_{\varphi(\alpha)}^{\varphi(\beta)} f(x)\,dx. \qquad x = \varphi$$

The proof of this theorem is based on the Chain Rule 6.1.6, and will be outlined in Exercise 17. The hypotheses that f and φ' are continuous are restrictive, but are used to ensure the existence of the Riemann integral on the left side of (5).



$\left\{ \begin{array}{l} \varphi \text{ is continuously differentiable on } I. \\ f \text{ is continuous. on Range}(\varphi(x)) \end{array} \right.$

内函数 Continuously Differentiable

7.3.9 Examples **(a)** Consider the integral $\displaystyle\int_1^4 \frac{\sin\sqrt{t}}{\sqrt{t}}\,dt$.

Here we substitute $\varphi(t) := \sqrt{t}$ for $t \in [1, 4]$ so that $\varphi'(t) = 1/(2\sqrt{t})$ is continuous on $[1, 4]$. If we let $f(x) := 2\sin x$, then the integrand has the form $(f \circ \varphi) \cdot \varphi'$ and the Substitution Theorem 7.3.8 implies that the integral equals $\int_1^2 2\sin x\,dx = -2\cos x|_1^2 = 2(\cos 1 - \cos 2)$.

(b) Consider the integral $\displaystyle\int_0^4 \frac{\sin\sqrt{t}}{\sqrt{t}}\,dt$.

Since $\varphi(t) := \sqrt{t}$ does not have a continuous derivative on $[0, 4]$, the Substitution Theorem 7.3.8 is not applicable, at least with this substitution. (In fact, it is not obvious that

19. Explain why Theorem 7.3.8 and/or Exercise 7.3.17 cannot be applied to evaluate the following integrals, using the indicated substitution.

(a) $\displaystyle\int_0^4 \frac{\sqrt{t}\,dt}{1+\sqrt{t}} \quad \varphi(t) = \sqrt{t},$ (b) $\displaystyle\int_0^4 \frac{\cos\sqrt{t}\,dt}{\sqrt{t}} \quad \varphi(t) = \sqrt{t},$

(c) $\displaystyle\int_{-1}^1 \sqrt{1+2|t|}\,dt \quad \varphi(t) = |t|,$ (d) $\displaystyle\int_0^1 \frac{dt}{\sqrt{1-t^2}} \quad \varphi(t) = \text{Arcsin}\, t.$

7.3.10 Definition **(a)** A set $Z \subset \mathbb{R}$ is said to be a **null set** if for every $\varepsilon > 0$ there exists a countable collection $\{(a_k, b_k)\}_{k=1}^{\infty}$ of open intervals such that

$$(6) \qquad Z \subseteq \bigcup_{k=1}^{\infty} (a_k, b_k) \quad \text{and} \quad \sum_{k=1}^{\infty} (b_k - a_k) \le \varepsilon.$$

(b) If $Q(x)$ is a statement about the point $x \in I$, we say that $Q(x)$ holds **almost everywhere** on I (or for **almost every** $x \in I$), if there exists a null set $Z \subset I$ such that $Q(x)$ holds for all $x \in I \backslash Z$. In this case we may write

$$Q(x) \quad \text{for a.e.} \quad x \in I.$$

It is trivial that any subset of a null set is also a null set, and it is easy to see that the union of two null sets is a null set. We will now give an example that may be very surprising.

We now state Lebesgue's Integrability Criterion. It asserts that a bounded function on an interval is Riemann integrable if and only if its points of discontinuity form a null set.

7.3.12 Lebesgue's Integrability Criterion *A bounded function $f : [a, b] \to \mathbb{R}$ is Riemann integrable if and only if it is continuous almost everywhere on $[a, b]$.*

bounded f

$f \in R[a,b] \rightleftharpoons f$ continuous almost everywhere.

7.3.13 Examples **(a)** The step function g in Example 7.1.4(b) is continuous at every point except the point $x = 1$. Therefore it follows from the Lebesgue Integrability Criterion that g is Riemann integrable.

In fact, since every step function has at most a finite set of points of discontinuity, then: *Every step function on $[a, b]$ is Riemann integrable.*

(b) Since it was seen in Theorem 5.6.4 that the set of points of discontinuity of a monotone function is countable, we conclude that: *Every monotone function on $[a, b]$ is Riemann integrable.*

Properties of Rieman Integrals.

7.3.14 Composition Theorem *Let $f \in \mathcal{R}[a, b]$ with $f([a, b]) \subseteq [c, d]$ and let $\varphi : [c, d] \to \mathbb{R}$ be continuous. Then the composition $\varphi \circ f$ belongs to $\mathcal{R}[a, b]$.*

Proof. If f is continuous at a point $u \in [a, b]$, then $\varphi \circ f$ is also continuous at u. Since the set D of points of discontinuity of f is a null set, it follows that the set $\boxed{D_1 \subseteq D}$ of points of discontinuity of $\varphi \circ f$ is also a null set. Therefore the composition $\varphi \circ f$ also belongs to $\mathcal{R}[a, b]$. Q.E.D.

7.3.15 Corollary *Suppose that $f \in \mathcal{R}[a, b]$. Then its absolute value $|f|$ is in $\mathcal{R}[a, b]$, and*

$$\left| \int_a^b f \right| \le \int_a^b |f| \le M(b - a),$$

where $|f(x)| \le M$ for all $x \in [a, b]$.

P: ① $f \in R[a,b] \Rightarrow \exists M$. s.t. $|f| \le M$, $\forall x \in [a,b]$.

② Let $\varphi(t) = |t|$, then $\varphi \in R[a,b] \Rightarrow \varphi \circ f = |f| \in R[a,b]$.

③ $-|f| \le f \le |f| \Rightarrow -\int_a^b |f| \le \int_a^b f \le \int_a^b |f|$

$\Rightarrow \left|\int_a^b f\right| \le \int_a^b |f|$

$|f| \le M \Rightarrow \int_a^b |f| \le \int_a^b M = M(b-a)$

$\Big\} \Rightarrow \left|\int_a^b f\right| \le \int_a^b |f| \le M(b-a)$ #.

7.3.16 The Product Theorem *If f and g belong to $\mathcal{R}[a,b]$, then the product fg belongs to $\mathcal{R}[a,b]$.*

Proof. If $\varphi(t) := t^2$ for $t \in [-M, M]$, it follows from the Composition Theorem that $f^2 = \varphi \circ f$ belongs to $\mathcal{R}[a,b]$. Similarly, $(f+g)^2$ and g^2 belong to $\mathcal{R}[a,b]$. But since we can write the product as

$$fg = \frac{1}{2}\left[(f+g)^2 - f^2 - g^2\right],$$

it follows that $fg \in \mathcal{R}[a,b]$. Q.E.D.

7.3.17 Integration by Parts *Let F, G be differentiable on $[a, b]$ and let $f := F'$ and $g := G'$ belong to $\mathcal{R}[a,b]$. Then*

$$(7) \qquad \int_a^b fG = FG\Big|_a^b - \int_a^b Fg.$$

Proof. By Theorem 6.1.3(c), the derivative $(FG)'$ exists on $[a, b]$ and

$$(FG)' = F'G + FG' = fG + Fg.$$

Since F, G are continuous and f, g belong to $\mathcal{R}[a,b]$, the Product Theorem 7.3.16 implies that fG and Fg are integrable. Therefore the Fundamental Theorem 7.3.1 implies that

$$FG\Big|_a^b = \int_a^b (FG)' = \int_a^b fG + \int_a^b Fg,$$

from which (7) follows. Q.E.D.

Proof: Since $(F \cdot G)' = f \cdot G + F \cdot g$

$\Rightarrow \int_a^b (f \cdot G + F \cdot g) = \int_a^b (F \cdot G)' \overset{\text{F.T.C.}}{=} F(b)G(b) - F(a)G(a)$

$\Rightarrow \int_a^b fG = FG\Big|_a^b - \int_a^b Fg$

Q.E.D.

7.3.18 Taylor's Theorem with the Remainder *Suppose that $f', \dots, f^{(n)}, f^{(n+1)}$ exist on $[a, b]$ and that $f^{(n+1)} \in \mathcal{R}[a, b]$. Then we have*

不同于 R^*，
需要规定 $f^{(n+1)} \in \mathcal{R}^*[a,b]$
因为在 F.T.C.(I) 中为
条件而非结论.

$$f(b) = f(a) + \frac{f'(a)}{1!}(b-a) + \cdots + \frac{f^{(n)}(a)}{n!}(b-a)^n + R_n, \tag{8}$$

where the remainder is given by

$$R_n = \frac{1}{n!}\int_a^b \underbrace{f^{(n+1)}(t)}_{F'} \cdot \overbrace{(b-t)^n}^{G} dt. \tag{9}$$

Proof. Apply Integration by Parts to equation (9), with $F(t) := f^{(n)}(t)$ and $G(t) := (b-t)^n/n!$, so that $g(t) = -(b-t)^{n-1}/(n-1)!$, to get

$$R_n = \frac{1}{n!}f^{(n)}(t) \cdot (b-t)^n \Big|_{t=a}^{t=b} + \frac{1}{(n-1)!}\int_a^b f^{(n)}(t) \cdot (b-a)^{n-1} dt$$

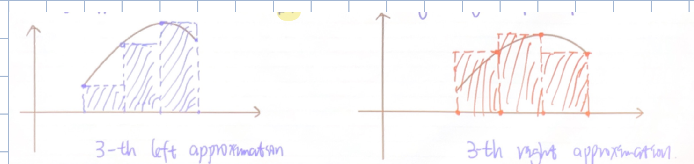
$$= -\frac{f^{(n)}(a)}{n!} \cdot (b-a)^n + \frac{1}{(n-1)!}\int_a^b f^{(n)}(t) \cdot (b-t)^{n-1} dt.$$

If we continue to integrate by parts in this way, we obtain (8). Q.E.D.

Section 7.5 Approximate Integration

(关注作业)

Equal Partitions



7.5.1 Theorem *If $f : [a, b] \to \mathbb{R}$ is monotone and if $T_n(f)$ is given by (1), then*

$$\left|\int_a^b f - T_n(f)\right| \le |f(b) - f(a)| \cdot \frac{(b-a)}{2n}. \tag{3}$$

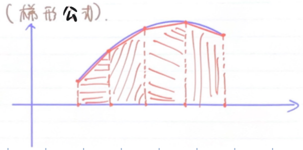
$= \text{max. error} = \dfrac{|f(b)-f(a)| \cdot h_n}{2}$

7.5.2 Example If $f(x) := e^{-x^2}$ on $[0, 1]$, then f is decreasing. It follows from (3) that if $n = 8$, then $|\int_0^1 e^{-x^2}dx - T_8(f)| \le (1 - e^{-1})/16 < 0.04$, and if $n = 16$, then $|\int_0^1 e^{-x^2}dx - T_{16}(f)| \le (1 - e^{-1})/32 < 0.02$. Actually, the approximation is considerably better, as we will see in Example 7.5.5. □

The Trapezoidal Rule

The Trapezoidal Rule: (梯形公式).

Geometric meaning:



Since the area of a trapezoid with horizontal base h and vertical sides l_1 and l_2 is known to be $\frac{1}{2}h(l_1 + l_2)$, we have

$$\int_{a+kh_n}^{a+(k+1)h_n} g_n = \frac{1}{2}h_n \cdot [f(a + kh_n) + f(a + (k+1)h_n)],$$

for $k = 0, 1, \ldots, n-1$. Summing these terms and noting that each partition point in $\mathcal{P}_n$ except a and b belongs to two adjacent subintervals, we obtain

最方便计算

$$* \quad \int_a^b g_n = h_n\left(\frac{1}{2}f(a) + f(a + h_n) + \cdots + f(a + (k-1)h_n) + \frac{1}{2}f(b)\right).$$

But the term on the right is precisely $T_n(f)$, found in (1) as the mean of $L_n(f)$ and $R_n(f)$. We call $T_n(f)$ the **nth Trapezoidal Approximation** of f.

7.5.3 Theorem *Let f, f' and f'' be continuous on $[a, b]$ and let $T_n(f)$ be the nth Trapezoidal Approximation (1). Then there exists $c \in [a, b]$ such that*

$$(4) \qquad \boxed{T_n(f) - \int_a^b f} = \frac{(b-a)h_n^2}{12} \cdot f''(c). \Rightarrow \begin{cases} f''(c) \geq 0,\ T_n \geq \int_a^b f & \text{Convex } \cup \\ f''(c) \leq 0,\ T_n \leq \int_a^b f & \cap \text{ Concave} \end{cases}$$

7.5.4 Corollary *Let f, f', and f'' be continuous, and let $|f''(x)| \leq B_2$ for all $x \in [a, b]$. Then*

$$(5) \qquad \left|T_n(f) - \int_a^b f\right| \leq \frac{(b-a)h_n^2}{12} \cdot B_2 = \frac{(b-a)^3}{12n^2} \cdot B_2.$$

则变动的只剩n.

When an upper bound B_2 can be found, (5) can be used to determine how large n must be chosen in order to be certain of a desired accuracy.

The Midpoint Rule

计算式

$$(6) \qquad M_n(f) := h_n\left(f\left(a+\frac{1}{2}h_n\right)+f\left(a+\frac{3}{2}h_n\right)+\cdots+f\left(a\left(n-\frac{1}{2}\right)h_n\right)\right)$$
$$= h_n\sum_{k=1}^{n} f\left(a+\left(k-\frac{1}{2}\right)h_n\right).$$

7.5.6 Theorem *Let f, f', and f'' be continuous on $[a, b]$ and let $M_n(f)$ be the nth Midpoint Approximation (6). Then there exists $\gamma \in [a, b]$ such that*

$$(7) \qquad \boxed{\int_a^b f - M_n(f)} = \frac{(b-a)h_n^2}{24}\cdot f''(\gamma). \Rightarrow \begin{cases} f''(\gamma)\geq 0, \int_a^b f \geq M_n & \cup \\ f''(\gamma)\leq 0, \int_a^b f \leq M_n & \cap \end{cases}$$

The proof of this result is in Appendix D.

$f''>0$ convex:

Proof. Use the fact that $f''(x) \geq 0$ in (4) and (7). Geometrically the inequality is reasonable since if the function is **convex**, then the chord of the trapezoid lies above the tangent to the graph. If $f''(x) \leq 0$, then the graph is **concave** and the inequality is revised. The key point is to understand and describe the concept of **convex** and **concave**. □

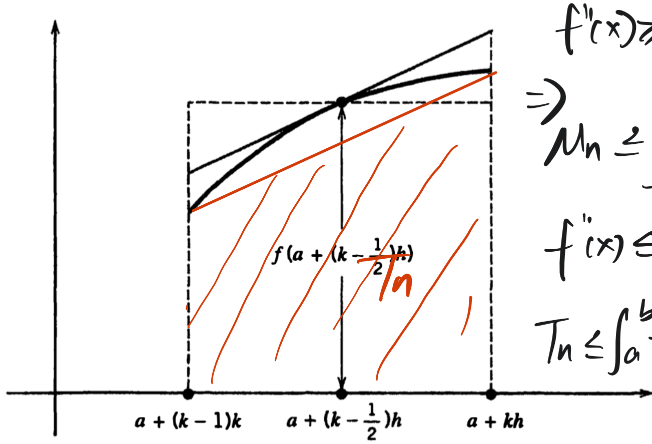


Figure 7.5.1 The tangent trapezoid

$f''(x)\geq 0$ (convex): $\cup$

$\Rightarrow M_n \leq \int_a^b f \leq T_n$

$f''(x)\leq 0$, concave $\cap$

$T_n \leq \int_a^b f \leq M_n$

7.5.7 Corollary *Let f, f', and f'' be continuous, and let $|f''(x)| \leq B_2$ for all $x \in [a, b]$. Then*

$$(8) \qquad \left|M_n(f) - \int_a^b f\right| \leq \frac{(b-a)h_n^2}{24}\cdot B_2 = \frac{(b-a)^3}{24n^2}\cdot B_2.$$

$h_n = \frac{b-a}{n}$

Simpson's Rule

Whereas the Trapezoidal and Midpoint Rules were based on the approximation of f by piecewise linear functions, Simpson's Rule approximates the graph of f by parabolic arcs. To help motivate the formula, the reader may show that if three points

$$(-h,\ y_0),\ (0,\ y_1),\ \text{and}\ (h, y_2)$$

are given, then the quadratic function $q(x) := Ax^2 + Bx + C$ that passes through these points has the property that

*推导过程

$$\int_{-h}^{h} q = \frac{1}{3}h(y_0 + 4y_1 + y_2).$$

Now let f be a continuous function on $[a,\ b]$ and let $n \in \mathbb{N}$ be *even*, and let $h_n := (b-a)/n$. On each "double subinterval"

$$[a,\ a+2h_n],\ [a+2h_n,\ a+4h_n], \ldots, [b-2h_n, b],$$

we approximate f by $n/2$ quadratic functions that agree with f at the points

$$y_0 := f(a),\quad y_1 := f(a+h_n),\quad y_2 := f(a+2h_n), \ldots,\quad y_n := f(b).$$

These considerations lead to the **nth Simpson Approximation**, defined by

计算式

$$\begin{aligned} S_n(f) := \frac{1}{3}h_n\big(&f(a) + 4f(a+h_n) + 2f(a+2h_n) + 4f(a+3h_n) \\ &+ 2f(a+4h_n) + \cdots + 2f(b-2h_n) + 4f(b-h_n) + f(b)\big). \end{aligned} \tag{9}$$

Note that the coefficients of the values of f at the $n+1$ partition points follow the pattern $1,\ 4,\ 2,\ 4,\ 2, \ldots, 4,\ 2,\ 4,\ 1$.

We now state a theorem that gives an estimate about the accuracy of the Simpson Approximation; it involves the *fourth derivative* of f.

h_n^4 四阶导!

7.5.8 Theorem *Let $f, f', f'', f^{(3)}$, and $f^{(4)}$ be continuous on $[a,\ b]$ and let $n \in \mathbb{N}$ be even. If $S_n(f)$ is the nth Simpson Approximation (9), then there exists $c \in [a,\ b]$ such that*

$$S_n(f) - \int_a^b f = \frac{(b-a)h_n^4}{180} \cdot f^{(4)}(c). \tag{10}$$

A proof of this result is given in Appendix D.

7.5.9 Corollary *Let $f, f', f'', f^{(3)}$, and $f^{(4)}$ be continuous on $[a,\ b]$ and let $\left|f^{(4)}(x)\right| \le B_4$ for all $x \in [a,\ b]$. Then*

$$\left|S_n(f) - \int_a^b f\right| \le \frac{(b-a)h_n^4}{180} \cdot B_4 = \frac{(b-a)^5}{180n^4} \cdot B_4. \tag{11}$$

Successful use of the estimate (11) depends on being able to find an upper bound for the fourth derivative.

Remark The nth Midpoint Approximation $M_n(f)$ can be used to "step up" to the $(2n)$th Trapezoidal and Simpson Approximations by using the formulas

$$T_{2n}(f) = \frac{1}{2}M_n(f) + \frac{1}{2}T_n(f) \qquad \text{and} \qquad \boxed{S_{2n}(f) = \frac{2}{3}M_n(f) + \frac{1}{3}T_n(f),}$$

that are given in the exercises. Thus once the initial Trapezoidal Approximation $T_1 = T_1(f)$ has been calculated, only the Midpoint Approximations $M_n = M_n(f)$ need be found. That is, we employ the following sequence of calculations:

$$T_1 = \frac{1}{2}(b-a)(f(a)+f(b));$$

$$\begin{array}{rlll} M_1 = (b-a)f\left(\frac{1}{2}(a+b)\right), & T_2 = \frac{1}{2}M_1 + \frac{1}{2}T_1, & S_2 = \frac{2}{3}M_1 + \frac{1}{3}T_1; \\ M_2, & T_4 = \frac{1}{2}M_2 + \frac{1}{2}T_2, & S_4 = \frac{2}{3}M_2 + \frac{1}{3}T_2; \\ M_4, & T_8 = \frac{1}{2}M_4 + \frac{1}{2}T_4, & S_8 = \frac{2}{3}M_4 + \frac{1}{3}T_4; \\ & \cdots & \cdots \end{array}$$