THE GENERALIZED RIEMANN INTEGRAL

Section 5.5 Continuity and Gauges†

Partition is Finite

f continuous on $[a,b]$ then

$|f|$ continuous on $[a,b]$

$\Rightarrow f, |f| \in R[a,b], R^*[a,b]$

5.5.1 Definition A **partition** of an interval $I := [a, b]$ is a collection $\mathcal{P} = \{I_1, \ldots, I_n\}$ of non-overlapping closed intervals whose union is $[a, b]$. We ordinarily denote the intervals by $I_i := [x_{i-1}, x_i]$, where

$$a = x_0 < \cdots < x_{i-1} < x_i < \cdots < x_n = b.$$

The points x_i $(i = 0, \ldots, n)$ are called the **partition points** of $\mathcal{P}$. If a point t_i has been chosen from each interval I_i, for $i = 1, \ldots, n$, then the points t_i are called the **tags** and the set of ordered pairs

$$\dot{\mathcal{P}} = \{(I_1, t_1), \ldots, (I_n, t_n)\}$$

is called a **tagged partition** of I. (The dot signifies that the partition is tagged.)

5.5.2 Definition A **gauge** on I is a strictly positive function defined on I. If δ is a gauge on I, then a (tagged) partition $\dot{\mathcal{P}}$ is said to be δ-**fine** if

$$(1) \qquad t_i \in I_i \subseteq [t_i - \delta(t_i),\ t_i + \delta(t_i)] \quad \text{for} \quad i = 1, \ldots, n.$$

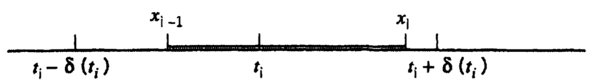


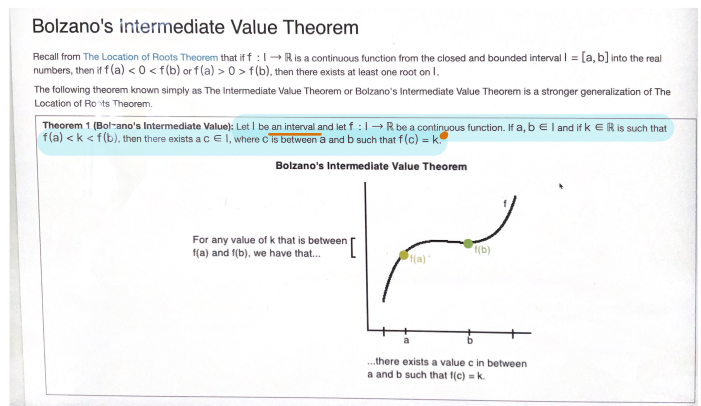
Figure 5.5.1 Inclusion (1)

5.5.3 Lemma *If a partition $\dot{\mathcal{P}}$ of $I := [a, b]$ is δ-fine and $x \in I$, then there exists a tag t_i in $\dot{\mathcal{P}}$ such that $|x - t_i| \le \delta(t_i)$.*

Proof. If $x \in I$, there exists a subinterval $[x_{i-1}, x_i]$ from $\dot{\mathcal{P}}$ that contains x. Since $\dot{\mathcal{P}}$ is δ-fine, then

$$(2) \qquad t_i - \delta(t_i) \le x_{i-1} \le x \le x_i \le t_i + \delta(t_i),$$

whence it follows that $|x - t_i| \le \delta(t_i)$. Q.E.D.



Bolzano's Intermediate Value Theorem

Recall from The Location of Roots Theorem that if $f : I \to \mathbb{R}$ is a continuous function from the closed and bounded interval $I = [a, b]$ into the real numbers, then if $f(a) < 0 < f(b)$ or $f(a) > 0 > f(b)$, then there exists at least one root on I.

The following theorem known simply as The Intermediate Value Theorem or Bolzano's Intermediate Value Theorem is a stronger generalization of The Location of Roots Theorem.

Theorem 1 (Bolzano's Intermediate Value): Let I be an interval and let $f : I \to \mathbb{R}$ be a continuous function. If $a, b \in I$ and if $k \in \mathbb{R}$ is such that $f(a) < k < f(b)$, then there exists a $c \in I$, where c is between a and b such that $f(c) = k$.

5.5.4 Examples **(a)** If δ and γ are gauges on $I := [a, b]$ and if $0 < \delta(x) \le \gamma(x)$ for all $x \in I$, then every partition $\dot{\mathcal{P}}$ that is δ-fine is also γ-fine. This follows immediately from the inequalities

$$t_i - \gamma(t_i) \le t_i - \delta(t_i) \quad \text{and} \quad t_i + \delta(t_i) \le t_i + \gamma(t_i)$$

which imply that

$$t_i \in [t_i - \delta(t_i), t_i + \delta(t_i)] \subseteq [t_i - \gamma(t_i), t_i + \gamma(t_i)] \quad \text{for} \quad i = 1, \ldots, n.$$

(b) If δ_1 and δ_2 are gauges on $I := [a, b]$ and if

$$\delta(x) := \min\{\delta_1(x), \delta_2(x)\} \quad \text{for all} \quad x \in I,$$

then δ is also a gauge on I. Moreover, since $\delta(x) \le \delta_1(x)$, then every δ-fine partition is δ_1-fine. Similarly, every δ-fine partition is also δ_2-fine.

(c) Suppose that δ is defined on $I := [0, 1]$ by

$$\delta(x) := \begin{cases} \frac{1}{10} & \text{if} \quad x = 0, \\ \frac{1}{2}x & \text{if} \quad 0 < x \le 1. \end{cases}$$

Then δ is a gauge on $[0, 1]$. If $0 < t \le 1$, then $[t - \delta(t), t + \delta(t)] = \left[\frac{1}{2}t, \frac{3}{2}t\right]$, which does not contain the point 0. Thus, if $\dot{\mathcal{P}}$ is a δ-fine partition of I, then the only subinterval in $\dot{\mathcal{P}}$ that contains 0 must have the point 0 as its tag.

(d) Let γ be defined on $I := [0, 1]$ by

$$\gamma(x) := \begin{cases} \frac{1}{10} & \text{if} \quad x = 0 \text{ or } x = 1, \\ \frac{1}{2}x & \text{if} \quad 0 < x \le \frac{1}{2}, \\ \frac{1}{2}(1 - x) & \text{if} \quad \frac{1}{2} < x < 1. \end{cases}$$

Then γ is a gauge on I, and it is an exercise to show that the subintervals in any γ-fine partition that contain the points 0 or 1 must have these points as tags. □

Proof. Let E denote the set of all points $x \in [a, b]$ such that there exists a δ-fine partition of the subinterval $[a, x]$. The set E is not empty, since the pair $([a, x], a)$ is a δ-fine partition of the interval $[a, x]$ when $x \in [a, a + \delta(a)]$ and $x \le b$. Since $E \subseteq [a, b]$, the set E is also bounded. Let $u := \sup E$ so that $a < u \le b$. We will show that $u \in E$ and that $u = b$.

We claim that $u \in E$. Since $u - \delta(u) < u = \sup E$, there exists $v \in E$ such that $u - \delta(u) < v < u$. Let $\dot{\mathcal{P}}_1$ be a δ-fine partition of $[a, v]$ and let $\dot{\mathcal{P}}_2 := \dot{\mathcal{P}}_1 \cup ([v, u], u)$. Then $\dot{\mathcal{P}}_2$ is a δ-fine partition of $[a, u]$, so that $u \in E$.

If $u < b$, let $w \in [a, b]$ be such that $u < w < u + \delta(u)$. If $\dot{\mathcal{Q}}_1$ is a δ-fine partition of $[a, u]$, we let $\dot{\mathcal{Q}}_2 := \dot{\mathcal{Q}}_1 \cup ([u, w], u)$. Then $\dot{\mathcal{Q}}_2$ is a δ-fine partition of $[a, w]$, whence $w \in E$. But this contradicts the supposition that u is an upper bound of E. Therefore $u = b$. Q.E.D.

5.5.5 Theorem *If δ is a gauge defined on the interval $[a, b]$, then there exists a δ-fine partition of $[a, b]$.*

不一定是tagged point，而是边界.

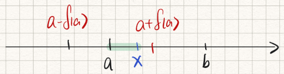
Proof. Let E be the set of all points x in I s.t. $\exists$ δ-fine partition of $[a, x]$. (Construct a δ-fine partition on $[a, x]$)

$E \subseteq I$

[First show: $E \neq \emptyset$].

$\forall x \in [a, a+\delta(a)] \cap [a, b]$, $a < x < a + \delta(a)$ as $\delta > 0$

$\Rightarrow$ $([a, x], a)$ is a tagged partition of $[a, x]$ that is δ-fine. thus, $x \in E$ $(E \neq \emptyset)$



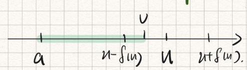
As $E \subseteq [a, b]$, so E is bounded $\Rightarrow$ exists $\sup E$

$\Rightarrow$ let $u := \sup E$ then $a < u \le b$

[Show $u = b$: ① show $u \in E$]

As $u - \delta(u) < u = \sup E$ so $\exists v \in E$ s.t. $v \in (u - \delta(u), u)$

Since $v \in E$, we can find a δ-fine partition of $\dot{\mathcal{P}}_1$ of $[a, v]$



Let $\dot{\mathcal{P}}_2 := \dot{\mathcal{P}}_1 \cup ([v, u], u)$

as $[v, u] \subseteq [u - \delta(u), u + \delta(u)]$ so the $\dot{\mathcal{P}}_2$ is a δ-fine partition of $[a, u]$

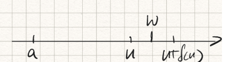
$\Rightarrow$ $u \in E$

[Show $u = b$]

Known: $u \le b$. Assume that $u < b$

then $\exists w \in [a, b]$ s.t. $u < w < u + \delta(u)$.

Then Let $\dot{\mathcal{P}}_3 := \dot{\mathcal{P}}_2 \cup ([u, w], u)$. #而非w



$\Rightarrow$ $[u, w] \subseteq [u, u + \delta(u)]$

then $\dot{\mathcal{P}}_3$ is a tagged partition of $[a, w]$

$\Rightarrow$ $w \in E$

Yet, as $w > u = \sup(E)$ so $w \notin E$. $\Rightarrow$ Contradiction.

Thus, $u = b$. $\Rightarrow$ exists a δ-fine partition on $[a, b]$.

Q.E.D.

Proof:

① $E \neq \emptyset$

② $\sup E \in E$

③ $\sup E = b$

利用[a,b]内partition的有限性，有界性

Alternate Proof of Theorem 5.3.2: Boundedness Theorem. Since f is continuous on I, then for each $t \in I$ there exists $\delta(t) > 0$ such that if $x \in I$ and $|x - t| \le \delta(t)$, then $|f(x) - f(t)| \le 1$. Thus δ is a gauge on I. Let $\{(I_i, t_i)\}_{i=1}^n$ be a δ-fine partition of I and let $K := \max\{|f(t_i)| : i = 1, \ldots, n\}$. By Lemma 5.5.3, given any $x \in I$ there exists i with $|x - t_i| \le \delta(t_i)$, whence

$$|f(x)| \le |f(x) - f(t_i)| + |f(t_i)| \le 1 + K.$$

Since $x \in I$ is arbitrary, then f is bounded by $1 + K$ on I. Q.E.D.

Proof. As f is continuous on I so $\forall t \in I$, f is continuous at t.

i.e. $\forall \varepsilon > 0$, $\exists \delta(t) > 0$ s.t. if $x \in I$ and $|x-t| < \delta(t)$

then $|f(x) - f(t)| < \varepsilon$

so $\delta(t)$ is a guage on $I \overset{5.5.5}{\Rightarrow}$ there's a δ-fine ptt. on I.

call it $\dot{P} := \{[x_{i-1}, x_i], t_i\}_{i=1}^n$

Let $K := \max\{|f(t_i)|; i \in \mathbb{N}\}$ (as n is finite although it is large)

$\boxed{\text{From 5.5.3,}}$ $\forall x \in I$, $\exists i \in \mathbb{N}$ s.t. $|x - t_i| \le \delta(t_i)$

So $|f(x) - f(t_i)| < \varepsilon$, let $\varepsilon = 1$ then $|f(x) - f(t_i)| < 1$

So $|f(x)| < |f(x) - f(t_i)| + |f(t_i)| < 1 + K$

Thus, $f(x)$ is bounded on I. Q.E.D.

5.3.2 Boundedness Theorem[†] *Let $I := [a, b]$ be a closed bounded interval and let $f : I \to \mathbb{R}$ be continuous on I. Then f is bounded on I.*

Contradiction

Alternate Proof of Theorem 5.3.4: Maximum-Minimum Theorem. We will prove the existence of x^*. Let $M := \sup\{f(x) : x \in I\}$ and suppose that $f(x) < M$ for all $x \in I$. Since f is continuous on I, for each $t \in I$ there exists $\delta(t) > 0$ such that if $x \in I$ and $|x - t| \le \delta(t)$, then $f(x) < \frac{1}{2}(M + f(t))$. Thus δ is a gauge on I, and if $\{(I_i, t_i)\}_{i=1}^n$ is a δ-fine partition of I, we let

$$\tilde{M} := \tfrac{1}{2}\max\{M + f(t_1), \ldots, M + f(t_n)\}.$$

By Lemma 5.5.3, given any $x \in I$, there exists i with $|x - t_i| \le \delta(t_i)$, whence

$$f(x) < \tfrac{1}{2}(M + f(t_i)) \le \tilde{M}.$$

Since $x \in I$ is arbitrary, then $\tilde{M}(< M)$ is an upper bound for f on I, contrary to the definition of M as the supremum of f. Q.E.D.

Proof. [Proof the existence of $x^* \in I$]

As f is continuous on a closed and bounded interval I,

So let $M := \sup\{f(x); x \in I\}$ and Assume ~~suppose~~ that $f(x) < M$ $\forall x \in I$ — As. $x^* \notin I$.

$\Downarrow$ there's no $x^* \in I$ s.t. $f(x^*) = M$

$\forall t \in I$, $\exists \delta(t) > 0$ s.t. if $x \in I$ and $|x - t| \le \delta(t)$,

$f(x) < \frac{1}{2}(M + f(t))$

So $\delta(t)$ is a gauge on $I \Rightarrow$ let $\{(I_i, t_i)\}_{i=1}^n$ be a δ-fine partition of I

let $\tilde{M} := \frac{1}{2}\max\{M + f(t_1), \cdots, M + f(t_n)\} < \frac{1}{2}M + \frac{1}{2}M = M$

By 5.5.3, $\forall x \in I$, $\exists i \in \mathbb{N}$ s.t. $|x - t_i| \le \delta(t_i)$

so $\forall x \in I$, $f(x) < \frac{1}{2}(M + f(t_i)) \le \tilde{M}$

$\Rightarrow \tilde{M}$ is an upper bound of $\{f(x), x \in I\}$.

so $\tilde{M} \ge \sup\{f(x)\} = M$

Contradicts to $\tilde{M} < M$

Thus, $M = \sup\{f(x)\} \in \{f(x)\}$

$\Rightarrow \exists x^* \in I$ s.t. $f(x^*) = M$

Q.E.D.

5.3.4 Maximum-Minimum Theorem *Let $I := [a, b]$ be a closed bounded interval and let $f : I \to \mathbb{R}$ be continuous on I. Then f has an absolute maximum and an absolute minimum on I.*

Alternate Proof of Theorem 5.3.5: Location of Roots Theorem. We assume that $f(t) \neq 0$ for all $t \in I$. Since f is continuous at t, Exercise 5.1.7 implies that there exists $\delta(t) > 0$ such that if $x \in I$ and $|x - t| \leq \delta(t)$, then $f(x) < 0$ if $f(t) < 0$, and $f(x) > 0$ if $f(t) > 0$. Then δ is a gauge on I and we let $\{(I_i, t_i)\}_{i=1}^{n}$ be a δ-fine partition. Note that for each i, either $f(x) < 0$ for all $x \in [x_{i-1}, x_i]$ or $f(x) > 0$ for all such x. Since $f(x_0) = f(a) < 0$, this implies that $f(x_1) < 0$, which in turn implies that $f(x_2) < 0$. Continuing in this way, we have $f(b) = f(x_n) < 0$, contrary to the hypothesis that $f(b) > 0$. Q.E.D.

Proof: Assume that $f(t) \neq 0$ $\forall t \in I$

since f is continuous at t so $\exists \delta(t) > 0$ s.t.

if $x \in I$ and $|x-t| \leq \delta(t)$, then $|f(x) - f(t)| < \varepsilon$

$\Rightarrow$ $f(t) - \varepsilon < f(x) < f(t) + \varepsilon$

So if $f(t) > 0$ then $f(x) > 0$

if $f(t) < 0$ then $f(x) < 0$

As δ is a gauge on I so exists δ-fine partition $\{(I_i, t_i)\}_{i=1}^{n}$ on I then $\forall x \in I_i$, if $f(t_i) > 0$ then $f(x) > 0$

if $f(t_i) < 0$ then $f(x) < 0$

Since $f(x_0) = f(a) < 0$ so $f(x_1) < 0$.

$\Rightarrow f(x_2) < 0 \cdots \Rightarrow f(x_n) = f(b) < 0$

Contradicts to $f(b) > 0$.

Thus, $\exists c \in (a,b)$ s.t. $f(c) = 0$.

Q.E.D.

5.3.5 Location of Roots Theorem *Let $I = [a, b]$ and let $f : I \to \mathbb{R}$ be continuous on I. If $f(a) < 0 < f(b)$, or if $f(a) > 0 > f(b)$, then there exists a number $c \in (a, b)$ such that $f(c) = 0$.*

Contradiction

Alternate Proof of Theorem 5.4.3: Uniform Continuity Theorem. Let $\varepsilon > 0$ be given. Since f is continuous at $t \in I$, there exists $\delta(t) > 0$ such that if $x \in I$ and $|x - t| \leq 2\delta(t)$, then $|f(x) - f(t)| \leq \frac{1}{2}\varepsilon$. Thus δ is a gauge on I. If $\{(I_i, t_i)\}_{i=1}^{n}$ is a δ-fine partition of I, let $\delta_\varepsilon := \min\{\delta(t_1), \ldots, \delta(t_n)\}$. Now suppose that $x, u \in I$ and $|x - u| \leq \delta_\varepsilon$, and choose i with $|x - t_i| \leq \delta(t_i)$. Since

$$|u - t_i| \leq |u - x| + |x - t_i| \leq \delta_\varepsilon + \delta(t_i) \leq 2\delta(t_i),$$

then it follows that

$$|f(x) - f(u)| \leq |f(x) - f(t_i)| + |f(t_i) - f(u)| \leq \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon.$$

Therefore, f is uniformly continuous on I. Q.E.D.

Proof: Since f is continuous on I so

f is continuous on $\forall t \in I$.

$\Rightarrow$ $\forall \varepsilon > 0$, $\exists \delta(t) > 0$ s.t. if $x \in I$ and $|x-t| < 2\delta(t)$

then $|f(x) - f(t)| < \frac{\varepsilon}{2}$

so $\delta(t)$ is a gauge on $[a,b]$ as it is

① defined on $[a,b]$ and ② strictly positive.

$\boxed{\text{By J.J.S,}}$ as here's a $\delta(t)$ defined on $[a,b]$ so

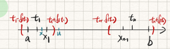
here exists a δ-fine partition on $[a,b]$,

call it $\dot{P} = \{([x_{i-1}, x_i], t_i)\}_{i=1}^{n}$

Let $\delta_\varepsilon := \min\{\delta(t_1), \ldots, \delta(t_n)\}$. (As n is finite)

(Let δ_ε be the minimum of the gauges evaluated at each tagged pt.)

$\forall x, u \in [a,b]$ s.t. $|x-u| < \delta_\varepsilon$



Pick $i \in \mathbb{N}$ s.t. $|x - t_i| \leq \delta(t_i)$

(There must exists an i s.t. $x \in I_i = [x_{i-1}, x_i]$)

so $|f(x) - f(t_i)| < \frac{\varepsilon}{2}$

Then $|u - t_i| = |u - x + x - t_i|$ (How far away is u from t_i?)

$\leq |u-x| + |x - t_i|$

$\leq \delta_\varepsilon + \delta(t_i)$

$\leq \delta(t_i) + \delta(t_i)$

$= 2\delta(t_i)$

$\Rightarrow |f(u) - f(t_i)| < \frac{\varepsilon}{2}$

$\Rightarrow |f(x) - f(u)| = |f(x) - f(t_i) + f(t_i) - f(u)|$

$\leq |f(x) - f(t_i)| + |f(t_i) - f(u)|$

$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$

$= \varepsilon$

Thus, f is uniformly continuous. Q.E.D.

5.4.3 Uniform Continuity Theorem *Let I be a closed bounded interval and let $f : I \to \mathbb{R}$ be continuous on I. Then f is uniformly continuous on I.* $= [a, b]$

Section 10.1 Definition and Main Properties

R.I.	G.R.I.
$\forall \varepsilon>0$, $\exists \delta>0$, $\exists L\in\mathbb{R}$, s.t. for any tagged partition $\dot{\mathcal{P}}$ satisfies $\|\dot{\mathcal{P}}\|<\delta$, $\lvert S(f;\dot{\mathcal{P}})-L\rvert<\varepsilon$	$\forall \varepsilon>0$, $\exists$ a gauge δ_f s.t. for any δ_f-fine partition $\dot{\mathcal{P}}$ on $[a,b]$, $\lvert S(f;\dot{\mathcal{P}})-\int_a^b f\rvert<\varepsilon$

10.1.1 Definition A function $f:[a,b]\to\mathbb{R}$ is said to be ***generalized* Riemann integrable** on $[a,b]$ if there exists a number $L\in\mathbb{R}$ such that for every $\varepsilon>0$ there exists a gauge δ_ε on $[a,b]$ such that if $\dot{\mathcal{P}}$ is any δ_ε-fine partition of $[a,b]$, then

$$|S(f;\dot{\mathcal{P}})-L|<\varepsilon.$$

The collection of all generalized Riemann integrable functions will usually be denoted by $\mathcal{R}^*[a,b]$.

10.1.2 Uniqueness Theorem *If $f\in\mathcal{R}^*[a,b]$, then the value of the integral is uniquely determined.*

Proof. Assume that L' and L'' both satisfy the definition and let $\varepsilon>0$. Thus there exists a gauge $\delta'_{\varepsilon/2}$ such that if $\dot{\mathcal{P}}_1$ is any $\delta'_{\varepsilon/2}$-fine partition, then

$$|S(f;\dot{\mathcal{P}}_1)-L'|<\varepsilon/2.$$

Also there exists a gauge $\delta''_{\varepsilon/2}$ such that if $\dot{\mathcal{P}}_2$ is any $\delta''_{\varepsilon/2}$-fine partition, then

$$|S(f;\dot{\mathcal{P}}_2)-L''|<\varepsilon/2.$$

We define δ_ε by $\delta_\varepsilon(t):=\min\left\{\delta'_{\varepsilon/2}(t),\delta''_{\varepsilon/2}(t)\right\}$ for $t\in[a,b]$, so that δ_ε is a gauge on $[a,b]$. If $\dot{\mathcal{P}}$ is a δ_ε-fine partition, then the partition $\dot{\mathcal{P}}$ is both $\delta'_{\varepsilon/2}$-fine and $\delta''_{\varepsilon/2}$-fine, so that

$$|S(f;\dot{\mathcal{P}})-L'|<\varepsilon/2 \quad\text{and}\quad |S(f;\dot{\mathcal{P}})-L''|<\varepsilon/2,$$

whence it follows that

$$\begin{aligned}|L'-L''| &\le |L'-S(f;\dot{\mathcal{P}})|+|S(f;\dot{\mathcal{P}})-L''|\\ &<\varepsilon/2+\varepsilon/2=\varepsilon.\end{aligned}$$

Since $\varepsilon>0$ is arbitrary, it follows that $L'=L''$. Q.E.D.

Proof: Assume that L' and L'' are both Reneralized Riemann Integral of f over $[a,b]$.

then $\forall\varepsilon>0$, $\exists$ guage $\delta'_{\frac{\varepsilon}{2}}$ s.t. for any $\delta'_{\frac{\varepsilon}{2}}$-fine partition $\dot{\mathcal{P}}_1$ on $[a,b]$, $|S(f;\dot{\mathcal{P}})-L'|<\frac{\varepsilon}{2}$

also exists a guage $\delta''_{\frac{\varepsilon}{2}}$ s.t. for any $\delta''_{\frac{\varepsilon}{2}}$-fine partition $\dot{\mathcal{P}}_2$

$$|S(f;\dot{\mathcal{P}}_2)-L''|<\frac{\varepsilon}{2}$$

Define δ_ε by $\delta_\varepsilon:=\min\{\delta'_{\frac{\varepsilon}{2}}(t),\ \delta''_{\frac{\varepsilon}{2}}(t)\}$ $\forall t\in[a,b]$

So δ_ε is a gauge of $[a,b]$

So there exits a δ_ε-fine partition $\dot{\mathcal{P}}$ on $[a,b]$

which is both $\delta'_{\frac{\varepsilon}{2}}$-fine and $\delta''_{\frac{\varepsilon}{2}}$-fine partition.

$\Rightarrow$ $|S(f;\dot{\mathcal{P}})-L'|<\frac{\varepsilon}{2}$ and $|S(f;\dot{\mathcal{P}})-L''|<\frac{\varepsilon}{2}$

Thus,
$$\begin{aligned}|L'-L''| &\le |S(f;\dot{\mathcal{P}})-L'|+|S(f;\dot{\mathcal{P}})-L''|\\ &<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}\\ &=\varepsilon.\end{aligned}$$

$\Rightarrow$ $L'=L''$ Q.E.D.

10.1.3 Consistency Theorem *If $f \in \mathcal{R}[a, b]$ with integral L, then also $f \in \mathcal{R}^*[a, b]$ with integral L.*

Proof. Given $\varepsilon > 0$, we need to construct an appropriate gauge on $[a, b]$. Since $f \in \mathcal{R}[a, b]$, there exists a number $\delta_\varepsilon > 0$ such that if $\dot{\mathcal{P}}$ is any tagged partition with $\|\dot{\mathcal{P}}\| < \delta_\varepsilon$, then $|S(f; \dot{\mathcal{P}}) - L| < \varepsilon$. We define the function $\delta^*_\varepsilon(t) := \frac{1}{4}\delta_\varepsilon$ for $t \in [a, b]$, so that δ^*_ε is a gauge on $[a, b]$.

If $\dot{\mathcal{P}} = \{(I_i, t_i)\}_{i=1}^n$, where $I_i := [x_{i-1}, x_i]$, is a δ^*_ε-fine partition, then since

$$I_i \subseteq [t_i - \delta^*_\varepsilon(t_i), t_i + \delta^*_\varepsilon(t_i)] = [t_i - \tfrac{1}{4}\delta_\varepsilon, t_i + \tfrac{1}{4}\delta_\varepsilon],$$

it is readily seen that $0 < x_i - x_{i-1} \le \frac{1}{2}\delta_\varepsilon < \delta_\varepsilon$ for all $i = 1, \ldots, n$. Therefore this partition also satisfies $\|\dot{\mathcal{P}}\| < \delta_\varepsilon$ and consequently $|S(f : \dot{\mathcal{P}}) - L| < \varepsilon$.

Thus every δ^*_ε-fine partition $\dot{\mathcal{P}}$ also satisfies $|S(f : \dot{\mathcal{P}}) - L| < \varepsilon$. Since $\varepsilon > 0$ is arbitrary, it follows that f is generalized Riemann integrable to L. Q.E.D.

Proof: Since $f \in R[a,b]$ so $\forall \varepsilon > 0$, $\exists \delta_\varepsilon > 0$ s.t.

if $\dot{P}$ is any tagged partition on $[a,b]$ with $\|\dot{P}\| < \delta_\varepsilon$,

then $|S(f;\dot{P}) - L| < \varepsilon$

使 δ^*_ε-fine 的 $\|\dot{P}\|$ 也小于 δ_ε.

Define $\delta^*_\varepsilon(t) := \frac{1}{4}\delta_\varepsilon$ $\forall t \in [a,b]$. So δ^* is a guage on $[a,b]$ $\Rightarrow$ there exists $\dot{P}_1 := \{(I_i, t_i)\}_{i=1}^n$ on $[a,b]$ s.t. $I_i := [x_{i-1}, x_i]$ is a δ^*_ε-fine partition.

$\Rightarrow I_i \subseteq [t_i - \delta^*_\varepsilon(t_i), t_i + \delta^*_\varepsilon(t_i)] = [t_i - \frac{1}{4}\delta_\varepsilon, t_i + \frac{1}{4}\delta_\varepsilon]$

$\Rightarrow 0 < x_i - x_{i-1} \le \frac{1}{4}\delta_\varepsilon - (-\frac{1}{4}\delta_\varepsilon) = \frac{1}{2}\delta_\varepsilon < \delta_\varepsilon$, $\forall i \in N$

So $\|\dot{P}_1\| < \delta_\varepsilon \Rightarrow |S(f;\dot{P}_1) - L| < \varepsilon$

Thus, any δ^*_ε-fine partition $\dot{P}_1$ also satisfies $|S(f;\dot{P}_1) - L| < \varepsilon$.

$\Rightarrow f \in R^*[a,b]$ and $\int_a^b f = L$

Q.E.D.

From Theorems 7.2.5, 7.2.7, and 7.2.8, we conclude that: *Every step function, every continuous function, and every monotone function belongs to $\mathcal{R}^*[a, b]$*. We will now show that Dirichlet's function, which was shown not to be Riemann integrable in 7.2.2(b) and 7.3.13(d), is *generalized* Riemann integrable.

10.1.4 Examples **(a)** The Dirichlet function f belongs to $\mathcal{R}^*[0, 1]$ and has integral 0.

We enumerate the rational numbers in $[0, 1]$ as $\{r_k\}_{k=1}^\infty$. Given $\varepsilon > 0$ we define $\delta_\varepsilon(r_k) := \varepsilon/2^{k+2}$ and $\delta_\varepsilon(x) := 1$ when x is irrational. Thus δ_ε is a gauge on $[0,1]$ and if the partition $\dot{\mathcal{P}} := \{(I_i, t_i)\}_{i=1}^n$ is δ_ε-fine, then we have $x_i - x_{i-1} \le 2\delta_\varepsilon(t_i)$. Since the only nonzero contributions to $S(f; \dot{\mathcal{P}})$ come from rational tags $t_i = r_k$, where

$$0 < f(r_k)(x_i - x_{i-1}) = 1 \cdot (x_i - x_{i-1}) \le \frac{2\varepsilon}{2^{k+2}} = \frac{\varepsilon}{2^{k+1}},$$

and since each such tag can occur in at most two subintervals, we have

$$0 \le S(f; \dot{\mathcal{P}}) < \sum_{k=1}^\infty \frac{2\varepsilon}{2^{k+1}} = \sum_{k=1}^\infty \frac{\varepsilon}{2^k} = \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, then $f \in \mathcal{R}^*[0, 1]$ and $\int_0^1 f = 0$.

Proof: Construct $\{r_k\}_{k=1}^\infty$ s.t. r_k is all real numbers in $[0,1]$. $\forall \varepsilon > 0$, define $\delta_\varepsilon(r_k) := \frac{\varepsilon}{2^{k+2}}$ and $\delta_\varepsilon(x) := 1$ $\forall x \in [0,1] \cap \mathbb{Q}^c$

Thus, δ_ε is a guage on $[0,1]$, exists a partition that is δ_ε-fine, denote it $\dot{P} := \{(I_i, t_i)\}_{i=1}^n$, $I_i = [x_{i-1}, x_i]$

$\Rightarrow x_i - x_{i-1} \le 2\delta_\varepsilon(t_i)$

Let tags be rational, i.e. let $t_i = r_k$.

then $0 < \boxed{f(t_i)(x_i - x_{i-1})} = f(r_k)(x_i - x_{i-1}) = 1 \times (x_i - x_{i-1})$

$\le 2\delta_\varepsilon(t_i)$

$\le \frac{\varepsilon}{2^{k+1}}$

Since each such tag can ocur in at most 2 intervals. (when they are endpoints)

So $0 \le S(f;\dot{P}) < \sum_{k=1}^\infty 2 \times \frac{\varepsilon}{2^{k+1}} = \sum_{k=1}^\infty \frac{\varepsilon}{2^k} = \varepsilon$

$\Rightarrow f \in R^*[0,1]$ and $\int_0^1 f = 0$.

Q.E.D.

(b) Let $H : [0,1] \to \mathbb{R}$ be defined by $H(1/k) := k$ for $k \in \mathbb{N}$ and $H(x) := 0$ elsewhere on $[0, 1]$.

Since H is not bounded on $[0, 1]$, it follows from the Boundedness Theorem 7.1.6 that it is not Riemann integrable on $[0, 1]$. We will now show that H is *generalized* Riemann integrable to 0.

In fact, given $\varepsilon > 0$, we define $\delta_\varepsilon(1/k) := \varepsilon/(k2^{k+2})$ and set $\delta_\varepsilon(x) := 1$ elsewhere on $[0, 1]$, so δ_ε is a gauge on $[0, 1]$. If $\dot{\mathcal{P}}$ is a δ_ε-fine partition of $[0, 1]$ then $x_i - x_{i-1} \le 2\delta_\varepsilon(t_i)$. Since the only nonzero contributions to $S(H; \dot{\mathcal{P}})$ come from tags $t_i = 1/k$, where

$$0 < H(1/k)(x_i - x_{i-1}) = k \cdot (x_i - x_{i-1}) \le k \cdot \frac{2\varepsilon}{k2^{k+2}} = \frac{\varepsilon}{2^{k+1}},$$

and since each such tag can occur in at most two subintervals, we have

$$0 \le S(H; \dot{\mathcal{P}}) < \sum_{k=1}^{\infty} \frac{\varepsilon}{2^k} = \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, then $H \in \mathcal{R}^*[0, 1]$ and $\int_0^1 H = 0$. □

Proof: $\forall \varepsilon > 0$, define $\delta_\varepsilon(\frac{1}{k}) := \frac{\varepsilon}{k2^{k+2}}$ for $k \in N$

and $\delta_\varepsilon(x) := 1$ elsewhere on $[0,1]$.

$\Rightarrow$ δ_ε is a guage on $[0,1]$.

$\Rightarrow$ there exists a δ-fine partition $\dot{P}$ on $[0,1]$

$\dot{P} := \{(I_i, t_i)\}_{i=1}^{n}$ $I_i = [x_{i-1}, x_i]$,

$\Rightarrow$ $x_i - x_{i-1} \le 2\delta_\varepsilon(t_i)$

Therefore, leg tags be $t_i := \frac{1}{k}$, for $k \in N$.

then $0 < \boxed{H(\frac{1}{k})(x_i - x_{i-1})} = k \cdot (x_i - x_{i-1})$

$\le k \cdot \frac{2\varepsilon}{k2^{k+2}}$

$= \frac{\varepsilon}{2^{k+1}}$

Since each such tag can occur in at most 2 subintervals,

$\Rightarrow$ $0 \le S(H; \dot{P}) \le \sum_{k=1}^{\infty} \frac{2\varepsilon}{2^{k+1}} = \sum_{k=1}^{\infty} \frac{\varepsilon}{2^{k+1}} = \varepsilon$

Thus, $H \in R^*[0,1]$ and $\int_0^1 H = 0$.

Q.E.D.

目标: $S(f;\dot{P})$

过程: Construct δ_ε,

选择 tag t_i.

计算 $S(f;\dot{P}) = \sum f(t_i)(x_i - x_{i-1})$

10.1.5 Theorem *Suppose that f and g are in $\mathcal{R}^*[a, b]$. Then:*

(a) *If $k \in \mathbb{R}$, the function kf is in $\mathcal{R}^*[a, b]$ and*

$$\int_a^b kf = k \int_a^b f.$$

(b) *The function $f + g$ is in $\mathcal{R}^*[a, b]$ and*

$$\int_a^b (f + g) = \int_a^b f + \int_a^b g.$$

(c) *If $f(x) \le g(x)$ for all $x \in [a, b]$, then*

$$\int_a^b f \le \int_a^b g.$$

Proof. (b) Given $\varepsilon > 0$, we can use the argument in the proof of the Uniqueness Theorem 10.1.2 to construct a gauge δ_ε on $[a, b]$ such that if $\dot{\mathcal{P}}$ is any δ_ε-fine partition of $[a, b]$, then

$$\left|S(f;\dot{\mathcal{P}}) - \int_a^b f\right| < \varepsilon/2 \quad \text{and} \quad \left|S(g;\dot{\mathcal{P}}) - \int_a^b g\right| < \varepsilon/2.$$

Since $S(f + g; \dot{\mathcal{P}}) = S(f; \dot{\mathcal{P}}) + S(g; \dot{\mathcal{P}})$, it follows as in the proof of Theorem 7.1.5(b) that

$$\left|S(f+g;\dot{\mathcal{P}}) - \left(\int_a^b f + \int_a^b g\right)\right| \le \left|S(f;\dot{\mathcal{P}}) - \int_a^b f\right| + \left|S(g;\dot{\mathcal{P}}) - \int_a^b g\right| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, then $f + g \in \mathcal{R}^*[a,b]$ and its integral is the sum of the integrals of f and g.

The proofs of (a) and (c) are analogous and are left to the reader. Q.E.D.

(b) Proof: $\forall \varepsilon > 0$, as $f, g \in R^*[a,b]$, use the argument in the proof of Uniqueness Theorem, construct a guage δ_ε on $[a,b]$ s.t. if $\dot{P}$ is any δ_ε-fine partition of $[a,b]$, then $|S(f;\dot{P}) - \int_a^b f| < \frac{\varepsilon}{2}$

$\delta_\varepsilon = \min\{\delta_f, \delta_g\}$.

$|S(g;\dot{P}) - \int_a^b g| < \frac{\varepsilon}{2}$

Since $S(f+g;\dot{P}) = S(f;\dot{P}) + S(g;\dot{P})$ so

$|S(f+g;\dot{P}) - (\int_a^b f + \int_a^b g)| \le |S(f;\dot{P}) - \int_a^b f| + |S(g;\dot{P}) - \int_a^b g|$

$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

Thus, $(f+g) \in R^*[a,b]$ and $\int_a^b (f+g) = \int_a^b f + \int_a^b g$

Q.E.D.

The Cauchy Criterion

10.1.6 Cauchy Criterion *A function $f : [a, b] \to \mathbb{R}$ belongs to $\mathcal{R}^*[a, b]$ if and only if for every $\varepsilon > 0$ there exists a gauge η_ε on $[a, b]$ such that if $\dot{\mathcal{P}}$ and $\dot{\mathcal{Q}}$ are any partitions of $[a, b]$ that are η_ε-fine, then*

$$\left|S(f;\dot{\mathcal{P}}) - S(f;\dot{\mathcal{Q}})\right| < \varepsilon.$$

Proof. ($\Rightarrow$) If $f \in \mathcal{R}^*[a,b]$ with integral L, let $\delta_{\varepsilon/2}$ be a gauge on $[a,b]$ such that if $\dot{\mathcal{P}}$ and $\dot{\mathcal{Q}}$ are $\delta_{\varepsilon/2}$-fine partitions of $[a, b]$, then

$$|S(f;\dot{\mathcal{P}}) - L| < \varepsilon/2 \quad \text{and} \quad |S(f;\dot{\mathcal{Q}}) - L| < \varepsilon/2.$$

We set $\eta_\varepsilon(t) := \delta_{\varepsilon/2}(t)$ for $t \in [a,b]$, so if $\dot{\mathcal{P}}$ and $\dot{\mathcal{Q}}$ are η_ε-fine, then

$$\begin{aligned}|S(f;\dot{\mathcal{P}}) - S(f;\dot{\mathcal{Q}})| &\le |S(f;\dot{\mathcal{P}}) - L| + |L - S(f;\dot{\mathcal{Q}})| \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon.\end{aligned}$$

($\Leftarrow$) For each $n \in \mathbb{N}$, let δ_n be a gauge on $[a, b]$ such that if $\dot{\mathcal{P}}$ and $\dot{\mathcal{Q}}$ are partitions that are δ_n-fine, then

$$\left|S(f;\dot{\mathcal{P}}) - S(f;\dot{\mathcal{Q}})\right| < 1/n.$$

We may assume that $\delta_n(t) \ge \delta_{n+1}(t)$ for all $t \in [a,b]$ and $n \in \mathbb{N}$; otherwise, we replace δ_n by the gauge $\delta_n'(t) = \min\{\delta_1(t), \ldots, \delta_n(t)\}$ for all $t \in [a,b]$.

For each $n \in \mathbb{N}$, let $\dot{\mathcal{P}}_n$ be a partition that is δ_n-fine. Clearly, if $m > n$ then both $\dot{\mathcal{P}}_m$ and $\dot{\mathcal{P}}_n$ are δ_n-fine, so that

$$\text{(2)} \qquad \left|S(f;\dot{\mathcal{P}}_n) - S(f;\dot{\mathcal{P}}_m)\right| < 1/n \qquad \text{for} \qquad m > n.$$

Consequently, the sequence $\left(S(f;\dot{\mathcal{P}}_m)\right)_{m=1}^{\infty}$ is a Cauchy sequence in $\mathbb{R}$, so it converges to some number A. Passing to the limit in (2) as $m \to \infty$, we have

$$\left|S(f;\dot{\mathcal{P}}_n) - A\right| \le 1/n \qquad \text{for all} \qquad n \in \mathbb{N}.$$

To see that A is the generalized Riemann integral of f, given $\varepsilon > 0$, let $K \in \mathbb{N}$ satisfy $K > 2/\varepsilon$. If $\dot{\mathcal{Q}}$ is a δ_k-fine partition, then

$$\begin{aligned}\left|S(f;\dot{\mathcal{Q}}) - A\right| &\le \left|S(f;\dot{\mathcal{Q}}) - S(f;\dot{\mathcal{P}}_K)\right| + \left|S(f;\dot{\mathcal{P}}_K) - A\right| \\ &\le 1/K + 1/K < \varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, then $f \in \mathcal{R}^*[a,b]$ with integral A. Q.E.D.

Proof: $\boxed{\Rightarrow}$ If $f \in R^*[a,b]$ then $\forall \varepsilon > 0$, there exists a gange $\delta_{\frac{\varepsilon}{2}}$ on $[a,b]$ s.t. if $\dot{P}, \dot{Q}$ are $\delta_{\frac{\varepsilon}{2}}$-fine partitions of $[a,b]$,

$$|S(f;\dot{P}) - L| < \frac{\varepsilon}{2}, \quad |S(f;\dot{Q}) - L| < \frac{\varepsilon}{2}$$

Let $\eta_\varepsilon(t) := \delta_{\frac{\varepsilon}{2}}(t)$, for $t \in [a,b]$ then if $\dot{P}, \dot{Q}$ are η_ε-fine,

$$\begin{aligned}|S(f;\dot{P}) - S(f;\dot{Q})| &\le |S(f;\dot{P}) - L| + |S(f;\dot{Q}) - L| \\ &< \varepsilon\end{aligned}$$

$\boxed{\Leftarrow}$ $\forall n \in N$, let δ_n be a gange on $[a,b]$ s.t. if $\dot{P}, \dot{Q}$ are any δ-fine partitions, then

$$|S(f;\dot{P}) - S(f;\dot{Q})| < \frac{1}{n}$$

Assume that $\delta_{n+1}(t) \le \delta(t)$ for $\forall t \in [a,b]$, otherwise, replace δ_n by $\delta_n'(t) := \min\{\delta_1(t), \ldots \delta_n(t)\}$.

$\forall n \in N$, let $\dot{P}_n$ be a δ_n-fine partition, let $m > n$ then $\delta_m(t) \le \delta_n(t)$ so both $\dot{P}_m, \dot{P}_n$ are δ_n-fine ptt.

$$\Rightarrow \quad |S(f;\dot{P}_n) - S(f;\dot{P}_m)| < \frac{1}{n} \quad \text{for } m > n.$$

Consequently, the sequence $(S(f;\dot{P}_m))_{m=1}^{\infty}$ is a Cauchy sequence therefore convergent.

let $A := \lim (S(f;\dot{P}_m))_{m=1}^{\infty}$ so as $m \to \infty$,

$$|S(f;\dot{P}_n) - A| < \frac{1}{n} \quad \forall n \in N.$$

$\Rightarrow$ $\forall \varepsilon > 0$, $\exists K \in N$ s.t. $K > \frac{2}{\varepsilon}$, let $\dot{Q}$ be a δ_K-fine ptt, then

$$\begin{aligned}|S(f;\dot{Q}) - A| &\le |S(f;\dot{Q}) - S(f;\dot{P}_K)| + |S(f;\dot{P}_K) - A| \\ &\le \frac{1}{K} + \frac{1}{K} \\ &< 2 \times \frac{\varepsilon}{2} = \varepsilon\end{aligned}$$

Thus, $f \in R^*[a,b]$ and $\int_a^b f = A$ Q.E.D.

10.1.7 Squeeze Theorem *Let $f : [a, b] \to \mathbb{R}$. Then $f \in \mathcal{R}^*[a, b]$ if and only if for every $\varepsilon > 0$ there exist functions α_ε and ω_ε in $\mathcal{R}^*[a, b]$ with*

$$\alpha_\varepsilon(x) \le f(x) \le \omega_\varepsilon(x) \quad \textit{for all} \quad x \in [a, b],$$

and such that

$$\int_a^b (\omega_\varepsilon - \alpha_\varepsilon) \le \varepsilon.$$

Proof. $\boxed{\Leftarrow}$ let $\alpha_\varepsilon = \omega_\varepsilon = f$ $\forall \varepsilon > 0$.

$\boxed{\Rightarrow}$ $\forall \varepsilon > 0$, as $\alpha_\varepsilon, \omega_\varepsilon \in \mathcal{R}^*[a,b]$ so there exists a gauge δ_ε on $[a,b]$ s.t. if $\dot{P}$ be any δ_ε-fine ptt. on $[a,b]$ then

$$\left| S(\alpha_\varepsilon; \dot{P}) - \int_a^b \alpha_\varepsilon \right| < \varepsilon, \quad \left| S(\omega_\varepsilon; \dot{P}) - \int_a^b \omega_\varepsilon \right| < \varepsilon$$

$$\Rightarrow \quad S(\alpha_\varepsilon; \dot{P}) > \int_a^b \alpha_\varepsilon - \varepsilon, \quad S(\omega_\varepsilon; \dot{P}) < \varepsilon + \int_a^b \omega_\varepsilon$$

As $\alpha_\varepsilon \le f \le \omega_\varepsilon$ so $S(\alpha_\varepsilon; \dot{P}) \le S(f; \dot{P}) \le S(\omega_\varepsilon; \dot{P})$

$$\Rightarrow \quad \int_a^b \alpha_\varepsilon - \varepsilon < S(f; \dot{P}) < \varepsilon + \int_a^b \omega_\varepsilon$$

Let $\dot{Q}$ be another δ_ε-fine ptt. then

$$\int_a^b \alpha_\varepsilon - \varepsilon < S(f; \dot{Q}) < \varepsilon + \int_a^b \omega_\varepsilon$$

$$\Rightarrow \quad \left| S(f; \dot{P}) - S(f; \dot{Q}) \right| < 2\varepsilon + \int_a^b \omega_\varepsilon - \alpha_\varepsilon$$

$$< 3\varepsilon$$

By Cauchy criterion, $f \in \mathcal{R}^*[a,b]$

Q.E.D.

10.1.8 Additivity Theorem *Let $f : [a,b] \to \mathbb{R}$ and let $c \in (a,b)$. Then $f \in \mathcal{R}^*[a,b]$ if and only if its restrictions to $[a, c]$ and $[c, b]$ are both generalized Riemann integrable. In this case*

(3) $$\int_a^b f = \int_a^c f + \int_c^b f.$$ (在两个方向中都是结论).

The Fundamental Theorem (First Form)

10.1.9 The Fundamental Theorem of Calculus (First Form) *Suppose there exists a **countable** set E in $[a, b]$, and functions $f, F : [a,b] \to \mathbb{R}$ such that:*

(a) *F is continuous on $[a, b]$.*

(b) *$F'(x) = f(x)$ for all $x \in [a,b] \setminus E$.*
Then *f belongs to $\mathcal{R}^*[a,b]$ and*

(5) $$\int_a^b f = F(b) - F(a).$$

Proof. We will prove the theorem in the case where $E = \emptyset$, leaving the general case to be handled in the exercises.

Thus, we assume that (b) holds for all $x \in [a, b]$. Since we wish to show that $f \in \mathcal{R}^*[a, b]$, given $\varepsilon > 0$, we need to construct a gauge δ_ε; this will be done by using the differentiability of F on $[a, b]$. If $t \in I$, since the derivative $f(t) = F'(t)$ exists, there exists $\delta_\varepsilon(t) > 0$ such that if $0 < |z - t| \leq \delta_\varepsilon(t)$, $z \in [a, b]$, then

$$\left|\frac{F(z) - F(t)}{z - t} - f(t)\right| < \tfrac{1}{2}\varepsilon.$$

If we multiply this inequality by $|z - t|$, we obtain

$$|F(z) - F(t) - f(t)(z - t)| \leq \tfrac{1}{2}\varepsilon|z - t|$$

whenever $z \in [t - \delta_\varepsilon(t), t + \delta_\varepsilon(t)] \cap [a, b]$. The function δ_ε is our desired gauge.

Now let $u, v \in [a, b]$ with $u < v$ satisfy $t \in [u, v] \subseteq [t - \delta_\varepsilon(t), t + \delta_\varepsilon(t)]$. If we subtract and add the term $F(t) - f(t) \cdot t$ and use the Triangle Inequality and the fact that $v - t \geq 0$ and $t - u \geq 0$, we get

$$\begin{aligned}
&|F(v) - F(u) - f(t)(v - u)| \\
&\leq |F(v) - F(t) - f(t)(v - t)| + |F(t) - F(u) - f(t)(t - u)| \\
&\leq \tfrac{1}{2}\varepsilon(v - t) + \tfrac{1}{2}\varepsilon(t - u) = \tfrac{1}{2}\varepsilon(v - u).
\end{aligned}$$

Therefore, if $t \in [u, v] \subseteq [t - \delta_\varepsilon(t), t + \delta_\varepsilon(t)]$, then we have

(6) $$|F(v) - F(u) - f(t)(v - u)| \leq \tfrac{1}{2}\varepsilon(v - u).$$

We will show that $f \in \mathcal{R}^*[a, b]$ with integral given by the telescoping sum

(7) $$F(b) - F(a) = \sum_{i=1}^{n}\{F(x_i) - F(x_{i-1})\}.$$

For, if the partition $\dot{\mathcal{P}} := \{([x_{i-1}, x_i], t_i)\}_{i=1}^n$ is δ_ε-fine, then

$$t_i \in [x_{i-1}, x_i] \subseteq [t_i - \delta_\varepsilon(t_i), t_i + \delta_\varepsilon(t_i)] \text{ for } i = 1, \ldots, n,$$

and so we can use (7), the Triangle Inequality, and (6) to obtain

$$\begin{aligned}
|F(b) - F(a) - S(f; \dot{\mathcal{P}})| &= \left|\sum_{i=1}^{n}\{F(x_i) - F(x_{i-1}) - f(t_i)(x_i - x_{i-1})\}\right| \\
&\leq \sum_{i=1}^{n}|F(x_i) - F(x_{i-1}) - f(t_i)(x_i - x_{i-1})| \\
&\leq \sum_{i=1}^{n}\tfrac{1}{2}\varepsilon(x_i - x_{i-1}) < \varepsilon(b - a).
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we conclude that $f \in \mathcal{R}^*[a, b]$ and (5) holds. Q.E.D.

Proof: $\boxed{\text{In the Case } E = \emptyset}$

Then F continuous and differentiable on whole $[a,b]$

Show: $f \in R^*[a,b]$

then $\boxed{\forall t \in I, \quad f(t) = F'(t) \text{ exists,}}$ so by the definition of differentiability, $\forall \varepsilon > 0$, $\exists \delta_\varepsilon(t) > 0$ for $t \in [a,b]$ s.t. if $0 < |z - t| \leq \delta_\varepsilon(t)$ $\forall z \in [a,b]$,

then $$\left|\frac{F(z) - F(t)}{z - t} - f(t)\right| < \frac{\varepsilon}{2}$$

$$\Rightarrow |F(z) - F(t) - f(t)(z - t)| < \frac{\varepsilon}{2}|z - t|$$

From the prerequisite: $0 < |z - t| < \delta_\varepsilon(t)$, $z \in [a,b]$

$$\Rightarrow z \in [z - \delta_\varepsilon(t), z + \delta_\varepsilon(t)] \cap [a,b]$$

so the function δ_ε is the desired gauge

(to prove $f \in R^*[a,b]$)

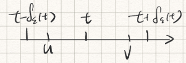
由 Differentiability at $\forall x \in [a,b]$

构建 gauge δ_ε

(to prove $f \in R^*[a,b]$)

$\boxed{\text{Let } u, v \in [a,b] \;\; u < v \text{ satisfy } t \in [u,v] \subseteq [t - \delta_\varepsilon(t), t + \delta_\varepsilon(t)]}$

$\Rightarrow$ $v - t \geq 0$ and $t - u \geq 0$



and $$\begin{aligned}
|F(v) - F(u) - f(t)(v - u)| &= |F(v) - F(t) - f(t)[(v - t) + (t - u)] + F(t) - F(u)| \\
&\leq |F(v) - F(t) - f(t)(v - t)| + |F(t) - F(u) - f(t)(t - u)| \\
&< \frac{\varepsilon}{2}(v - t) + \frac{\varepsilon}{2}(t - u) \\
&= \frac{\varepsilon}{2}(v - u)
\end{aligned}$$

Define the δ_ε-fine partition $\dot{P} := \{[x_{i-1}, x_i], t_i\}_{i=1}^n$

then $\forall i \in N$, $t_i \in [x_{i-1}, x_i] \subseteq [t_i - \delta(t_i), t_i + \delta(t_i)]$

As $F(b) - F(a) = \sum_{i=1}^n \{F(x_i) - F(x_{i-1})\}$

so $$\begin{aligned}
|F(b) - F(a) - S(f; \dot{P})| &= \left|\sum_{i=1}^n [F(x_i) - F(x_{i-1})] - \sum_{i=1}^n f(t_i)(x_i - x_{i-1})\right| \\
&\leq \sum_{i=1}^n |F(x_i) - F(x_{i-1}) - f(t_i)(x_i - x_{i-1})| \\
&\leq \sum_{i=1}^n \frac{\varepsilon}{2}(x_i - x_{i-1}) \\
&= \frac{\varepsilon}{2}(b - a) \\
&< \varepsilon(b - a) \\
&= \varepsilon'
\end{aligned}$$

Thus, $f \in R^*$ and $\int_a^b f = F(b) - F(a)$.

Q.E.D.

The Fundamental Theorem (Second Form)

We now turn to the Second Form of the Fundamental Theorem, in which we wish to differentiate the **indefinite integral** F of f, defined by:

(8) $$F(z) := \int_a^z f(x)\,dx \quad \text{for} \quad z \in [a, b].$$

10.1.11 Fundamental Theorem of Calculus (Second Form) *Let f belong to $\mathcal{R}^*[a, b]$ and let F be the indefinite integral of f. Then we have:*

(a) *F is continuous on $[a, b]$.*

(b) *There exists a null set Z such that if $x \in [a, b] \backslash Z$, then F is differentiable at x and $F'(x) = f(x)$.*

(c) *If f is continuous at $c \in [a, b]$, then $F'(c) = f(c)$.*

(c) Proof: As f is continuous at $c \in [a,b]$,

$\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. if $|x - c| < \delta$ then $|f(x) - f(c)| < \varepsilon$

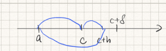
$$\Rightarrow f(c) - \varepsilon < f(x) < f(c) + \varepsilon$$

In the region of $|x - c| < \delta$ (i.e. $x \in (c - \delta, c + \delta)$)

Select the region $[c, c + \delta)$, define $0 < h < \delta$,

then from $f \in R^*[a,b]$, by 10.1.8 Additivity Theorem,

f is generalized Riemann integrable on $[a,c]$, $[a, c+h]$, $[c, c+h]$



and $$\int_a^{c+h} f - \int_a^c f = \int_c^{c+h} f$$

Core: 可加性

By Fundamental Theorem I,

$$\int_c^{c+h} f = F(c+h) - F(c)$$

On $[c, c+h]$, $f(c) - \varepsilon < f(x) < f(c) + \varepsilon$

$\lim_{h \to 0^+}$

$$\Rightarrow \int_c^{c+h} (f(c) - \varepsilon) < \int_c^{c+h} f < \int_c^{c+h} (f(c) + \varepsilon)$$

$$h(f(c) - \varepsilon) < F(c+h) - F(c) < h(f(c) + \varepsilon)$$

$$\Rightarrow \left|\frac{F(c+h) - F(c)}{h} - f(c)\right| < \varepsilon$$

Thus, $f(c) = \lim_{h \to 0^+} \frac{F(c+h) - F(c)}{h}$

Similarly, $f(c) = \lim_{h \to 0^-} \frac{F(c+h) - F(c)}{h}$

$$\Rightarrow f(c) = F'(c)$$

Q.E.D.

10.1.12 Substitution Theorem **(a)** *Let $I := [a, b]$ and $J := [\alpha, \beta]$, and let $F : I \to \mathbb{R}$ and $\varphi : J \to \mathbb{R}$ be continuous functions with $\varphi(J) \subseteq I$.*

(b) *Suppose there exist sets $E_f \subset I$ and $E_\varphi \subset J$ such that $f(x) = F'(x)$ for $x \in I \backslash E_f$, that $\varphi'(t)$ exists for $t \in J \backslash E_\varphi$, and that $E := \varphi^{-1}(E_f) \cup E_\varphi$ is countable.*

(c) *Set $f(x) := 0$ for $x \in E_f$ and $\varphi'(t) := 0$ for $t \in E_\varphi$.*

We conclude that $f \in \mathcal{R}^*(\varphi(J))$, that $(f \circ \varphi) \cdot \varphi' \in \mathcal{R}^*(J)$ and that

(9) 套用F.T.C.I
$$\int_\alpha^\beta (f \circ \varphi) \cdot \varphi' = F \circ \varphi \Big|_\alpha^\beta = \int_{\varphi(\alpha)}^{\varphi(\beta)} f.$$

Proof. Since φ is continuous on J, Theorem 5.3.9 implies that $\varphi(J)$ is a closed interval in I. Also $\varphi^{-1}(E_f)$ is countable, whence $E_f \cap \varphi(J) = \varphi(\varphi^{-1}(E_f))$ is also countable. Since $f(x) = F'(x)$ for all $x \in \varphi(J) \backslash E_f$, the Fundamental Theorem 10.1.9 implies that $f \in \mathcal{R}^*(\varphi(J))$ and that

$$\int_{\varphi(\alpha)}^{\varphi(\beta)} f = F\Big|_{\varphi(\alpha)}^{\varphi(\beta)} = F(\varphi(\beta)) - F(\varphi(\alpha)).$$

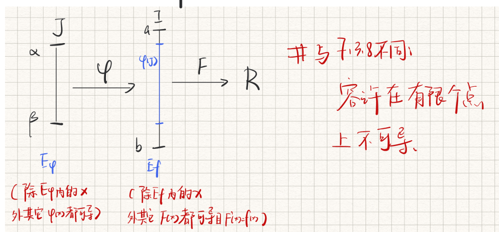
If $t \in J \backslash E$, then $t \in J \backslash E_\varphi$ and $\varphi(t) \in I \backslash E_f$. Hence the Chain Rule 6.1.6 implies that

$$(F \circ \varphi)'(t) = f(\varphi(t)) \cdot \varphi'(t) \quad \text{for} \quad t \in J \backslash E.$$

Since E is countable, the Fundamental Theorem implies that $(f \circ \varphi) \cdot \varphi' \in \mathcal{R}^*(J)$ and that

$$\int_\alpha^\beta (f \circ \varphi) \cdot \varphi' = F \circ \varphi \Big|_\alpha^\beta = F(\alpha(\beta)) - F(\varphi(\alpha)).$$

The conclusion follows by equating these two terms. Q.E.D.



Proof: Since φ is continuous on closed interval J

so $\varphi(J)$ is a closed interval in I.

As E_f is countable so $\varphi^{-1}(E_f)$ is countable,

$\Rightarrow$ $E_f \cap \varphi(J) = \varphi[\varphi^{-1}(E_f)]$ is also countable.

Since $f(x) = F'(x)$ $\forall x \in \varphi(J) \backslash E_f$

by the Fundamental Theorem I,

$f \in R^*(\varphi(J))$ and that

$$\int_{\varphi(\alpha)}^{\varphi(\beta)} f = F\Big|_{\varphi(\alpha)}^{\varphi(\beta)} = F[\varphi(\beta)] - F[\varphi(\alpha)] \quad (1)$$

As $E = \varphi^{-1}(E_f) \cup E_\varphi$

so if $t \in J \backslash E$, then $t \in J \backslash E_\varphi$,

by the chain rule: $(F \circ \varphi)'(t) = F'(\varphi(t)) \cdot \varphi'(t)$

$= f[\varphi(t)] \cdot \varphi(t)$. $\forall t \in J \backslash E$

Since E is countable, so F.T.I. implies that

$(f \circ \varphi) \cdot \varphi' \in R^*(J)$ and $\int_\alpha^\beta (f \circ \varphi) \cdot \varphi' = F \circ \varphi \Big|_\alpha^\beta$

$= F[\varphi(\beta)] - F(\varphi(\alpha))$

from (1) $\Rightarrow$ $= \int_{\varphi(\alpha)}^{\varphi(\beta)} f$

Q.E.D.

10.1.14 Multiplication Theorem *If $f \in \mathcal{R}^*[a, b]$ and if g is a monotone function on $[a, b]$, then the product $f \cdot g$ belongs to $\mathcal{R}^*[a, b]$.*

$f, g \in R^*[a, b]$
$f \cdot g \notin R^*[a, b]$.

10.1.15 Integration by Parts Theorem *Let F and G be differentiable on $[a, b]$. Then $F'G$ belongs to $\mathcal{R}^*[a, b]$ if and only if FG' belongs to $\mathcal{R}^*[a, b]$. In this case we have*

(10)
$$\int_a^b F'G = FG\Big|_a^b - \int_a^b FG'.$$

$$\int_a^b fG = FG\Big|_a^b - \int_a^b Fg$$

10.1.16 Taylor's Theorem *Suppose that $f, f', f'', \ldots, f^{(n)}$ and $f^{(n+1)}$ exist on $[a, b]$. Then we have*

$$(11) \qquad f(b) = f(a) + \frac{f'(a)}{1!}(b-a) + \cdots + \frac{f^{(n)}(a)}{n!}(b-a)^n + R_n,$$

where the remainder is given by

$$(12) \qquad R_n = \frac{1}{n!}\int_a^b f^{(n+1)}(t) \cdot (b-t)^n dt.$$

Proof. Since $f^{(n+1)}$ is a derivative, it belongs to $\mathcal{R}^*[a, b]$. Moreover, since $t \mapsto (b-t)^n$ is monotone on $[a, b]$, the Multiplication Theorem 10.1.14 implies the integral in (12) exists. Integrating by parts repeatedly, we obtain (11). Q.E.D.

R_n

Remark: 当 f 为广义黎曼可积时，$|f|$ 不一定广义黎曼可积，

$\left(f \in \mathcal{R}^*[a,b]$ 且 f is absolute convergent， $\Rightarrow |f| \in \mathcal{R}^*[a,b] \right)$

i.e. 广义黎曼可积空间对取绝对值不封闭，

i.e. Generalized Integral is not always an "absolute integral"

Section 10.2 Improper and Lebesgue Integrals

Hake不证 Lebesgue 而证 R^*

10.2.1 Hake's Theorem *If $f : [a, b] \to \mathbb{R}$, then $f \in \mathcal{R}^*[a, b]$ if and only if for every $\gamma \in (a, b)$ the restriction of f to $[a, \gamma]$ belongs to $\mathcal{R}^*[a, \gamma]$ and*

$$(2) \qquad \lim_{\gamma \to b-} \int_a^\gamma f = A \in \mathbb{R}.$$

In this case $\int_a^b f = A$.

证明 $L[a,b]$：

判定1:

10.2.3 Definition A function $f \in \mathcal{R}^*[a,b]$ such that $|f| \in \mathcal{R}^*[a,b]$ is said to be **Lebesgue integrable** on $[a, b]$. The collection of all Lebesgue integrable functions on $[a, b]$ is denoted by $\mathcal{L}[a,b]$.

判定2:

It is clear that if $f \in \mathcal{R}^*[a,b]$ and if $f(x) \geq 0$ for all $x \in [a,b]$, then we have $|f| = f \in \mathcal{R}^*[a,b]$, so that $f \in \mathcal{L}[a,b]$. That is, a nonnegative function $f \in \mathcal{R}^*[a,b]$ belongs to $\mathcal{L}[a,b]$. The next result gives a more powerful test for a function in $\mathcal{R}^*[a,b]$ to belong to $\mathcal{L}[a,b]$.

判定3:

10.2.4 Comparison Test *If $f, \omega \in \mathcal{R}^*[a,b]$ and $|f(x)| \leq \omega(x)$ for all $x \in [a,b]$, then $f \in \mathcal{L}[a,b]$ and*

(3)
$$\left|\int_a^b f\right| \leq \int_a^b |f| \leq \int_a^b \omega.$$

$\boxed{\text{L 空间对加法和数乘封闭.}}$

10.2.5 Theorem *If $f, g \in \mathcal{L}[a,b]$ and if $c \in \mathbb{R}$, then cf and $f+g$ also belong to $\mathcal{L}[a,b]$. Moreover*

(4)
$$\int_a^b cf = c\int_a^b f \quad \text{and} \quad \int_a^b |f+g| \leq \int_a^b |f| + \int_a^b |g|.$$

Proof. Since $|cf(x)| = |c||f(x)|$ for all $x \in [a,b]$, the hypothesis that $|f|$ belongs to $\mathcal{R}^*[a,b]$ implies that cf and $|cf|$ also belong to $\mathcal{R}^*[a,b]$, whence $cf \in \mathcal{L}[a,b]$.

The Triangle Inequality implies that $|f(x)+g(x)| \le |f(x)|+|g(x)|$ for all $x \in [a,b]$. But since $\omega := |f|+|g|$ belongs to $\mathcal{R}^*[a,b]$, the Comparison Test 10.2.4 implies that $f+g$ belongs to $\mathcal{L}[a,b]$ and that

$$\int_a^b |f+g| \le \int_a^b (|f|+|g|) = \int_a^b |f| + \int_a^b |g|. \qquad \text{Q.E.D.}$$

10.2.6 Theorem *If $f \in \mathcal{R}^*[a,b]$, the following assertions are equivalent:*

(a) $f \in \mathcal{L}[a,b]$.

(b) *There exists $\omega \in \mathcal{L}[a,b]$ such that $f(x) \le \omega(x)$ for all $x \in [a,b]$.*

(c) There *exists $\alpha \in \mathcal{L}[a,b]$ such that $\alpha(x) \le f(x)$ for all $x \in [a,b]$.*

Proof. (a) $\Rightarrow$ (b) Let $\omega := f$.

(b) $\Rightarrow$ (a) Note that $f = \omega - (\omega - f)$. Since $\omega - f \ge 0$ and since $\omega - f$ belongs to $\mathcal{R}^*[a,b]$, it follows that $\omega - f \in \mathcal{L}[a,b]$. Now apply Theorem 10.2.5.

We leave the proof that (a) $\Longleftrightarrow$ (c) to the reader. Q.E.D.

10.2.7 Theorem *If $f, g \in \mathcal{L}[a,b]$, then the functions* $\max\{f,g\}$ *and* $\min\{f,g\}$ *also belong to $\mathcal{L}[a,b]$.*

Proof. It follows from Exercise 2.2.18 that if $x \in [a,b]$, then

$$\boxed{\begin{aligned} \max\{f(x),g(x)\} &= \tfrac{1}{2}(f(x)+g(x)+|f(x)-g(x)|), \\ \min\{f(x),g(x)\} &= \tfrac{1}{2}(f(x)+g(x)-|f(x)-g(x)|). \end{aligned}}$$

The assertions follow from these equations and Theorem 10.2.5. Q.E.D.

10.2.8 Theorem *Suppose that f, g, α, and ω belong to $\mathcal{R}^*[a,b]$. If*

$$f \le \omega, g \le \omega \quad \text{or if} \quad \alpha \le f, \alpha \le g,$$

then $\max\{f,g\}$ and $\min\{f,g\}$ *also belong to* $\mathcal{R}^*[a,b]$.

Proof. Suppose that $f \le \omega$ and $g \le \omega$; then $\max\{f,g\} \le \omega$. It follows from the first equality in the proof of Theorem 10.2.7 that

$$0 \le |f-g| = 2\max\{f,g\} - f - g \le 2\omega - f - g.$$

Since $2\omega - f - g \ge 0$, this function belongs to $\mathcal{L}[a,b]$. The Comparison Test 10.2.4 implies that $2\max\{f,g\} - f - g$ belongs to $\mathcal{L}[a,b]$, and so $\max\{f,g\}$ belongs to $\mathcal{R}^*[a,b]$.

The second part of the assertion is proved similarly. Q.E.D.

The Seminorm in $\mathcal{L}[a,b]$

We will now define the "seminorm" of a function in $\mathcal{L}[a,b]$ and the "distance between" two such functions.

10.2.9 Definition If $f \in \mathcal{L}[a,b]$, we define the **seminorm** of f to be

$$\|f\| := \int_a^b |f|.$$

If $f, g \in \mathcal{L}[a,b]$, we define the **distance between** f and g to be

$$\operatorname{dist}(f,g) := \boxed{\|f-g\| = \int_a^b |f-g|.}$$

10.2.10 Theorem *The seminorm function satisfies:*

(i) $\|f\| \geq 0$ *for all* $f \in \mathcal{L}[a,b]$.

(ii) *If* $f(x) = 0$ *for* $x \in [a,b]$, *then* $\|f\| = 0$.

(iii) *If* $f \in \mathcal{L}[a,b]$ *and* $c \in \mathbb{R}$, *then* $\|cf\| = |c| \cdot \|f\|$.

(iv) *If* $f, g \in \mathcal{L}[a,b]$, *then* $\|f+g\| \leq \|f\| + \|g\|$.

Proof. Parts (i)–(iii) are easily seen. Part (iv) follows from the fact that $|f+g| \leq |f| + |g|$ and Theorem 10.1.5(c). Q.E.D.

10.2.11 Theorem *The distance function satisfies:*

(i) $\operatorname{dist}(f,g) \geq 0$ *for all* $f, g \in \mathcal{L}[a,b]$.

(ii) *If* $f(x) = g(x)$ *for* $x \in [a,b]$, *then* $\operatorname{dist}(f,g) = 0$.

(iii) $\operatorname{dist}(f,g) = \operatorname{dist}(g,f)$ *for all* $f, g \in \mathcal{L}[a,b]$.

(iv) $\operatorname{dist}(f,h) \leq \operatorname{dist}(f,g) + \operatorname{dist}(g,h)$ *for all* $f, g, h \in \mathcal{L}[a,b]$.

10.2.12 Completeness Theorem *A sequence (f_n) of functions in $\mathcal{L}[a,b]$ converges to a function $f \in \mathcal{L}[a,b]$ if and only if it has the property that for every $\varepsilon > 0$ there exists $H(\varepsilon)$ such that if $m, n \geq H(\varepsilon)$ then*

$$\|f_m - f_n\| = \operatorname{dist}(f_m, f_n) < \varepsilon.$$

$\boxed{\text{即}}$ Cauchy