$x \neq c$

6.1.1 Definition Let $I \subseteq \mathbb{R}$ be an interval, let $f : I \to \mathbb{R}$, and let $c \in I$. We say that a real number L is the **derivative of f at c** if given any $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that if $x \in I$ satisfies $0 < |x - c| < \delta(\varepsilon)$, then

#

(1) $$\left| \frac{f(x) - f(c)}{x - c} - L \right| < \varepsilon.$$

In this case we say that f is **differentiable** at c, and we write $f'(c)$ for L.

In other words, the derivative of f at c is given by the limit

(2) $$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

provided this limit exists. (We allow the possibility that c may be the endpoint of the interval.)

6.1.2 Theorem *If $f : I \to \mathbb{R}$ has a derivative at $c \in I$, then f is continuous at c.*

妙

Proof. For all $x \in I$, $x \neq c$, we have

$$f(x) - f(c) = \left(\frac{f(x) - f(c)}{x - c} \right) (x - c).$$

Since $f'(c)$ exists, we may apply Theorem 4.2.4 concerning the limit of a product to conclude that

$$\begin{aligned} \lim_{x \to c} (f(x) - f(c)) &= \lim_{x \to c} \left(\frac{f(x) - f(c)}{x - c} \right) \left(\lim_{x \to c} (x - c) \right) \\ &= f'(c) \cdot 0 = 0. \end{aligned}$$

Therefore, $\lim_{x \to c} f(x) = f(c)$ so that f is continuous at c. Q.E.D.

Differentiable => Continuous

6.1.3 Theorem *Let $I \subseteq \mathbb{R}$ be an interval, let $c \in I$, and let $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ be functions that are differentiable at c. Then:*

"组合"后的Fun diff. at c

(a) *If $\alpha \in \mathbb{R}$, then the function αf is differentiable at c, and*

(3) $$(\alpha f)'(c) = \alpha f'(c).$$

(b) *The function $f + g$ is differentiable at c, and*

(4) $$(f + g)'(c) = f'(c) + g'(c).$$

(c) *(Product Rule) The function fg is differentiable at c, and*

(5) $$(fg)'(c) = f'(c)g(c) + f(c)g'(c).$$

(d) *(Quotient Rule) If $g(c) \neq 0$, then the function f/g is differentiable at c, and*

(6) $$\left(\frac{f}{g} \right)'(c) = \frac{f'(c)g(c) - f(c)g'(c)}{(g(c))^2}.$$

(c)

$$\lim_{x \to c} \frac{(fg)(x) - (fg)(c)}{x - c} = \lim_{x \to c} \frac{f(x)g(x) - f(c)g(c)}{x - c}$$

$$= \lim_{x \to c} \frac{f(x) - f(c)}{x - c} g(x) + \lim_{x \to c} \frac{g(x) - g(c)}{x - c} f(c)$$

As f, g is conts. at $c \in I$ so

$\lim_{x \to c} g(x) = g(c)$, $\lim_{x \to c} f(x) = f(c)$.

$\Rightarrow$ L.H.S $= f'(c) \cdot g(c) + f(c) \cdot g'(c)$

CHAPTER 6

DIFFERENTIATION

6.1.4 Corollary *If $f_1, f_2, \dots, f_n$ are functions on an interval I to $\mathbb{R}$ that are differentiable at $c \in I$, then:*

(a) *The function $f_1 + f_2 + \cdots + f_n$ is differentiable at c and*

$$(f_1 + f_2 + \cdots + f_n)'(c) = f_1'(c) + f_2'(c) + \cdots + f_n'(c). \tag{7}$$

(b) *The function $f_1 f_2 \cdots f_n$ is differentiable at c, and*

$$(f_1 f_2 \cdots f_n)'(c) = f_1'(c) f_2(c) \cdots f_n(c) + f_1(c) f_2'(c) \cdots f_n(c) + \cdots + f_1(c) f_2(c) \cdots f_n'(c). \tag{8}$$

An important special case of the extended product rule (8) occurs if the functions are equal, that is, $f_1 = f_2 = \cdots = f_n = f$. Then (8) becomes

$$(f^n)'(c) = n(f(c))^{n-1} f'(c). \tag{9}$$

$\Downarrow$

$(x^n)' = n x^{n-1}$

For any c on I

多用于证明diff.

6.1.5 Carathéodory's Theorem *Let f be defined on an interval I containing the point c. Then f is differentiable at c if and only if there exists a function φ on I that is continuous at c and satisfies*

$$f(x) - f(c) = \varphi(x)(x - c) \quad \text{for} \quad x \in I. \tag{10}$$

satisfy

In this case, we have $\varphi(c) = f'(c)$.

Proof: $\boxed{\Leftarrow}$ As φ is conts. so $\lim_{x \to c} \varphi(x) = \varphi(c)$

When $x \neq c$, $\dfrac{f(x) - f(c)}{x - c} = \varphi(x)$

so $\varphi(c) = \lim_{x \to c} \varphi(x) = \lim_{x \to c} \dfrac{f(x) - f(c)}{x - c}$

$\Rightarrow$ the limit exists $\Rightarrow f'(c) = \lim_{x \to c} \dfrac{f(x) - f(c)}{x - c}$

$\Rightarrow$ $f'(c) = \varphi(c)$, and f is diff. at c.

The Chain Rule

$\boxed{\Rightarrow}$ Define $\varphi(x) = \begin{cases} \dfrac{f(x)-f(c)}{x-c}, & x \neq c \\ f'(c), & x = c \end{cases}$

then $\lim\limits_{x\to c} \dfrac{f(x)-f(c)}{x-c} = f'(c) = \varphi(c)$

So $\varphi(x)$ is differentiable at c and $f'(c) = \varphi(c)$.

Proof. $(\Rightarrow)$ If $f'(c)$ exists, we can define φ by

$$\varphi(x) := \begin{cases} \dfrac{f(x)-f(c)}{x-c} & \text{for} \quad x \neq c,\ x \in I, \\ f'(c) & \text{for} \quad x = c. \end{cases}$$

The continuity of φ follows from the fact that $\lim\limits_{x\to c} \varphi(x) = f'(c)$. If $x = c$, then both sides of (10) equal 0, while if $x \neq c$, then multiplication of $\varphi(x)$ by $x - c$ gives (10) for all other $x \in I$.

$(\Leftarrow)$ Now assume that a function φ that is continuous at c and satisfying (10) exists. If we divide (10) by $x - c \neq 0$, then the continuity of φ implies that

$$\varphi(c) = \lim_{x\to c} \varphi(x) = \lim_{x\to c} \frac{f(x)-f(c)}{x-c}$$

exists. Therefore f is differentiable at c and $f'(c) = \varphi(c)$. Q.E.D.

6.1.6 Chain Rule *Let I, J be intervals in $\mathbb{R}$, let $g : I \to \mathbb{R}$ and $f : J \to \mathbb{R}$ be functions such that $f(J) \subseteq I$, and let $c \in J$. If f is differentiable at c and if g is differentiable at $f(c)$, then the composite function $g \circ f$ is differentiable at c and*

$$(g \circ f)'(c) = g'(f(c)) \cdot f'(c). \tag{11}$$

Proof: As f is differentiable at $x=c$

~~then~~ by 6.1.5,

$\exists\, \varphi_1(x)$ that is continuous at c and $\varphi_1'(c)=f'(c)$

$f(x)-f(c) = \varphi_1(x)(x-c)$

As g is differentiable at $y=f(c)$

$\exists\, \varphi_2(y)$ that is continuous at $f(c)$ and $\varphi_2'[f(c)]=g'(f(c))$

$$g(f(x))-g(f(c)) = \varphi_2(y)(y-f(c))$$

$$= \varphi_2[f(x)]\,[f(x)-f(c)]$$

$$= \varphi_2[f(x)]\,\varphi_1(x)(x-c)$$

$$\Rightarrow \lim_{x\to c}\frac{g\circ f(x)-g\circ f(c)}{x-c} = \lim_{x\to c}\left[\varphi_2(f(x))\,\varphi_1(x)\right]$$

As φ_1, φ_2 are continuous,

so L.H.S. $= \varphi_2(f(c))\,\varphi_1(c)$

Apply Caratheo- Thm. again,

$\varphi_1(c)=f'(c)$ and $\varphi_2(f(c))=g'(f(c))$

Thus, L.H.S. $= g'(f(c))\cdot f'(c)$

Thus, $\displaystyle\lim_{x\to c}\frac{g\circ f(x)-g\circ f(c)}{x-c}$ exists,

i.e. $\displaystyle g\circ f'(c) = \lim_{x\to c}\frac{g\circ f(x)-g\circ f(c)}{x-c}$

$\Rightarrow$ $g\circ f$ is differentiable at c.

Q.E.D.

Proof. Since $f'(c)$ exists, Carathéodory's Theorem 6.1.5 implies that there exists a function φ on J such that φ is continuous at c and $f(x)-f(c)=\varphi(x)(x-c)$ for $x\in J$, and where $\varphi(c)=f'(c)$. Also, since $g'(f(c))$ exists, there is a function ψ defined on I such that ψ is continuous at $d := f(c)$ and $g(y)-g(d)=\psi(y)(y-d)$ for $y\in I$, where $\psi(d)=g'(d)$. Substitution of $y=f(x)$ and $d=f(c)$ then produces

$$g(f(x))-g(f(c)) = \psi(f(x))(f(x)-f(c)) = [(\psi\circ f(x))\cdot\varphi(x)](x-c)$$

for all $x\in J$ such that $f(x)\in I$. Since the function $(\psi\circ f)\cdot\varphi$ is continuous at c and its value at c is $g'(f(c))\cdot f'(c)$, Carathéodory's Theorem gives (11). Q.E.D.

6.1.7 Examples **(a)** If $f : I \to \mathbb{R}$ is differentiable on I and $g(y) := y^n$ for $y \in \mathbb{R}$ and $n \in \mathbb{N}$, then since $g'(y) = ny^{n-1}$, it follows from the Chain Rule 6.1.6 that

$$(g \circ f)'(x) = g'(f(x)) \cdot f'(x) \quad \text{for} \quad x \in I.$$

Therefore we have $(f^n)'(x) = n(f(x))^{n-1} f'(x)$ for all $x \in I$ as was seen in (9).

(b) Suppose that $f : I \to \mathbb{R}$ is differentiable on I and that $f(x) \neq 0$ and $f'(x) \neq 0$ for $x \in I$. If $h(y) := 1/y$ for $y \neq 0$, then it is an exercise to show that $h'(y) = -1/y^2$ for $y \in \mathbb{R}$, $y \neq 0$. Therefore we have

$$\left(\frac{1}{f}\right)'(x) = (h \circ f)'(x) = h'(f(x))f'(x) = -\frac{f'(x)}{(f(x))^2} \quad \text{for} \quad x \in I.$$

若 f、g 同为严格单调且连续的函数，

g 的定义域为 f 的 Range 且 g 为 f 的逆；若 c 为 f 的 Domain I 中任意一点，f 在 c 点可导且 $f'(c) \neq 0$，则 g 在 $f(c) = d$ 点，亦可导，

其导数 $g'(d)$ 为 $f'(c)$ 的倒数，即 $g'(d) = \frac{1}{f'(c)}$

故 $f'(c) \cdot g'(d) = 1$

Inverse Functions

6.1.8 Theorem *Let I be an interval in $\mathbb{R}$ and let $f : I \to \mathbb{R}$ be strictly monotone and continuous on I. Let $J := f(I)$ and let $g : J \to \mathbb{R}$ be the strictly monotone and continuous function inverse to f. If f is differentiable at $c \in I$ and $\boxed{f'(c) \neq 0}$, then g is differentiable at $d := f(c)$ and*

$$g'(d) = \frac{1}{f'(c)} = \frac{1}{f'(g(d))}. \tag{12}$$

Proof. Given $c \in \mathbb{R}$, we obtain from Carathéodory's Theorem 6.1.5 a function φ on I with properties that φ is continuous at c, $f(x) - f(c) = \varphi(x)(x - c)$ for $x \in I$, and $\boxed{\varphi(c) = f'(c)}$. Since $\varphi(c) \neq 0$ by hypothesis, there exists a neighborhood $V := (c - \delta,\ c + \delta)$ such that $\varphi(x) \neq 0$ for all $x \in V \cap I$. (See Theorem 4.2.9.) If $U := f(V \cap I)$, then the inverse function g satisfies $f(g(y)) = y$ for all $y \in U$, so that

$$y - d = f(g(y)) - f(c) = \varphi(g(y)) \cdot (g(y) - g(d)).$$

所需的

Since $\varphi(g(y)) \neq 0$ for $y \in U$, we can divide to get

$$g(y) - g(d) = \frac{1}{\varphi(g(y))} \cdot (y - d).$$

Since the function $1/(\varphi \circ g)$ is continuous at d, we apply Theorem 6.1.5 to conclude that $g'(d)$ exists and $g'(d) = 1/\varphi(g(d)) = 1/\varphi(c) = 1/f'(c)$. Q.E.D.

6.1.9 Theorem *Let I be an interval and let $f : I \to \mathbb{R}$ be strictly monotone on I. Let $J := f(I)$ and let $g : J \to \mathbb{R}$ be the function inverse to f. If f is differentiable on I and $f'(x) \neq 0$ for $x \in I$, then g is differentiable on J and*

$$g' = \frac{1}{f' \circ g}. \tag{13}$$

6.2.1 Interior Extremum Theorem *Let c be an interior point of the interval I at which $f : I \to \mathbb{R}$ has a relative extremum. If the derivative of f at c exists, then $f'(c) = 0$.*

法 1

反证法

Better

Proof. We will prove the result only for the case that f has a relative maximum at c; the proof for the case of a relative minimum is similar.

If $f'(c) > 0$, then by Theorem 4.2.9 there exists a neighborhood $V \subseteq I$ of c such that

$$\frac{f(x) - f(c)}{x - c} > 0 \quad \text{for} \quad x \in V,\ x \neq c.$$

If $x \in V$ and $x > c$, then we have

$$f(x) - f(c) = (x - c) \cdot \frac{f(x) - f(c)}{x - c} > 0.$$

But this contradicts the hypothesis that f has a relative maximum at c. Thus we cannot have $f'(c) > 0$. Similarly (how?), we cannot have $f'(c) < 0$. Therefore we must have $f'(c) = 0$. Q.E.D.

法 2:

两边夹

Proof: Suppose f has a relative max at $c \in (a, b)$.

Then $\exists \delta > 0$ s.t. $\exists U_\delta(c)$ s.t.

1) $(c - \delta, c + \delta) \subseteq (a, b)$ $\forall x \in U_\delta(c)$, $f(x) \le f(c)$

2) $\forall x \in U_\delta(c)$, $f(x) \le f(c)$

let $x_n = c - \frac{\delta}{2n} \in (c - \delta, c)$ 取 $x < c$

$c - \delta < c - \frac{\delta}{2n} < c$

凑 $\geq$

Then $\forall n \in \mathbb{N}$, $x_n \in U_\delta(c)$,

$$f'(c) = \lim_{n \to \infty} \frac{f(x_n) - f(c)}{x_n - c} \geqslant 0$$

let $x_n = c + \frac{\delta}{2n} \in (c, c + \delta)$ 取 $x_n > c$

$c < c - \frac{\delta}{2n} < c + \delta$

凑 $\leq$

Then $\forall n \in \mathbb{N}$, $x_n \in U_\delta(c)$,

$$f'(c) = \lim_{n \to \infty} \frac{f(x_n) - f(c)}{x_n - c} \leqslant 0$$

Thus, $0 \le f'(c) \le 0$

(By Trichotomy)

$\Rightarrow$ $f'(c) = 0$

Q.E.D.

Section 6.2 The Mean Value Theorem

6.2.2 Corollary *Let $f : I \to \mathbb{R}$ be continuous on an interval I and suppose that f has a relative extremum at an interior point c of I. Then either the derivative of f at c does not exist, or it is equal to zero.*

We note that if $f(x) := |x|$ on $I := [-1, 1]$, then f has an interior minimum at $x = 0$; however, the derivative of f fails to exist at $x = 0$.

闭区间上连续，开区间上可导

6.2.3 Rolle's Theorem *Suppose that f is continuous on a closed interval $I := [a, b]$, that the derivative f' exists at every point of the open interval (a, b), and that $f(a) = f(b) = 0$. Then there exists at least one point c in (a, b) such that $f'(c) = 0$.*

* Differentiable at every point in (a, b)

Proof: Since f is conts on $[a,b]$, let f achieves a relative max at $c_1 \in [a,b]$ and relative min $c_2 \in [a,b]$

If $f(c_1) > 0$, then c_1 cannot be endpoints

so $c_1 \in (a,b)$, c is a interior point.

By Interior Extreme Theorem,

$f'(c_1) = 0 \Rightarrow$ Take $c = c_1$, c exists.

If $f(c_2) < 0$ then $c_2 \in (a,b) \Rightarrow f'(c_2) = 0$

Take $c = c_2$ then c exists.

$$\left.\begin{array}{ll} \text{If} & f(c_1) \le 0 \le f(c_2) \\ \text{as} & f(c_2) \le f(c_1) \end{array}\right\} \Rightarrow f(c_2) = f(c_1)$$

As $\forall x \in [a,b]$, $f(c_2) \le f(x) \le f(c_1)$

so f is a constant function

As $f(a) = f(b) = 0$ so $f(x) = 0$, $x \in [a,b]$

$\Rightarrow$ c exists.

Q.E.D.

核心定理，
其后证明常需

意义：将方程值的大小关系、元的大小关系和导数的正负联系起来

6.2.4 Mean Value Theorem *Suppose that f is continuous on a closed interval $I := [a, b]$, and that f has a derivative in the open interval (a, b). Then there exists at least one point c in (a, b) such that*

$$f(b) - f(a) = f'(c)(b - a).$$

Proof: Define $g: [a,b] \to \mathbb{R}$

$\#$ $g(x) = f(x) - f(b) + \dfrac{f(b) - f(a)}{b - a}(b - x)$

$\Rightarrow$ when $x = a$ or b, $g(x) = 0$

Goal for constructing g: i.e. $g(a) = g(b) = 0$

Also, g is conts. on $[a,b]$ and differentiable on (a,b)

By Rolle's Theorem,

$\exists c \in (a,b)$ s.t. $g'(c) = 0$

$g'(x) = f'(x) + \dfrac{f(b) - f(a)}{b - a}(-1)$

$\Rightarrow g'(c) = f'(c) - \dfrac{f(b) - f(a)}{b - a} = 0$

$\Rightarrow f(b) - f(a) = f'(c)(b - a)$

Q.E.D.

Proof. Consider the function φ defined on I by

$$\varphi(x) := f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a).$$

核心

[The function φ is simply the difference of f and the function whose graph is the line segment joining the points $(a, f(a))$ and $(b, f(b))$; see Figure 6.2.2.] The hypotheses of

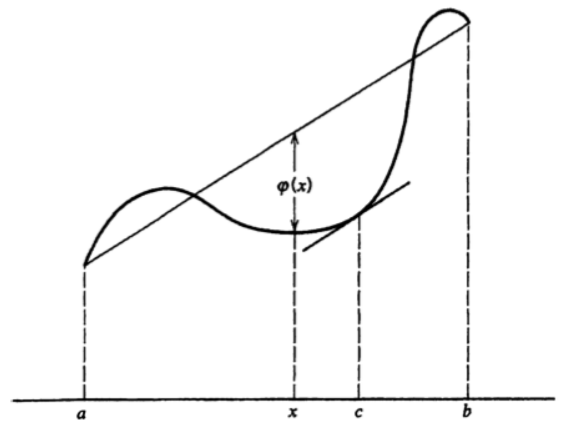

Figure 6.2.2 The Mean Value Theorem

Rolle's Theorem are satisfied by φ since φ is continuous on $[a, b]$, differentiable on (a, b), and $\varphi(a) = \varphi(b) = 0$. Therefore, there exists a point c in (a, b) such that

$$0 = \varphi'(c) = f'(c) - \frac{f(b) - f(a)}{b - a}.$$

Hence, $f(b) - f(a) = f'(c)(b - a)$. Q.E.D.

6.2.5 Theorem *Suppose that f is continuous on the closed interval $I := [a, b]$, that f is differentiable on the open interval (a, b), and that $f'(x) = 0$ for $x \in (a, b)$. Then f is constant on I.*

Proof. We will show that $f(x) = f(a)$ for all $x \in I$. Indeed, if $x \in I$, $x > a$, is given, we apply the Mean Value Theorem to f on the closed interval $[a, x]$. We obtain a point c (depending on x) between a and x such that $f(x) - f(a) = f'(c)(x - a)$. Since $f'(c) = 0$ (by hypothesis), we deduce that $f(x) - f(a) = 0$. Hence, $f(x) = f(a)$ for any $x \in I$. Q.E.D.

6.2.6 Corollary *Suppose that f and g are continuous on $I := [a, b]$, that they are differentiable on (a, b), and that $f'(x) = g'(x)$ for all $x \in (a, b)$. Then there exists a constant C such that $f = g + C$ on I.* No Proof

$\Rightarrow (f-g)'(x) = 0$ for $\forall x \in (a,b) \Rightarrow (f-g)(x) = C$

6.2.7 Theorem *Let $f : I \to \mathbb{R}$ be differentiable on the interval I. Then:*

(a) *f is increasing on I if and only if $f'(x) \geq 0$ for all $x \in I$.*

(b) *f is decreasing on I if and only if $f'(x) \leq 0$ for all $x \in I$.*

Proof: $\boxed{\Leftarrow}$ Suppose $f'(x) \geq 0$, $\forall x \in I$,

let $a, b \in I$ with $a < b$ then f is conts on $[a,b] \subset I$ and differentiable on (a,b)

By M.V.T, $\exists c \in (a,b)$ s.t.

$f(b) - f(a) = f'(c)(b-a) \geq 0$.

$\Rightarrow f(a) \leq f(b)$

Proof. (a) Suppose that $f'(x) \geq 0$ for all $x \in I$. If x_1, x_2 in I satisfy $x_1 < x_2$, then we apply the Mean Value Theorem to f on the closed interval $J := [x_1, x_2]$ to obtain a point c in (x_1, x_2) such that

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1).$$

Since $f'(c) \geq 0$ and $x_2 - x_1 > 0$, it follows that $f(x_2) - f(x_1) \geq 0$. (Why?) Hence, $f(x_1) \leq f(x_2)$ and, since $x_1 < x_2$ are arbitrary points in I, we conclude that f is increasing on I.

For the converse assertion, we suppose that f is differentiable and increasing on I. Thus, for any point $x \neq c$ in I, we have $(f(x) - f(c))/(x - c) \geq 0$. (Why?) Hence, by Theorem 4.2.6 we conclude that

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \geq 0.$$

(b) The proof of part (b) is similar and will be omitted. Q.E.D.

f(x)在c处是否可导不重要

6.2.8 First Derivative Test for Extrema *Let f be continuous on the interval $I := [a, b]$ and let c be an interior point of I. Assume that f is differentiable on (a, c) and (c, b). Then:*

(a) *If there is a neighborhood $(c-\delta, c+\delta) \subseteq I$ such that $f'(x) \geq 0$ for $c-\delta < x < c$ and $f'(x) \leq 0$ for $c < x < c+\delta$, then f has a relative maximum at c.*

(b) *If there is a neighborhood $(c-\delta, c+\delta) \subseteq I$ such that $f'(x) \leq 0$ for $c-\delta < x < c$ and $f'(x) \geq 0$ for $c < x < c+\delta$, then f has a relative minimum at c.*

Proof. (a) If $x \in (c-\delta, c)$, then it follows from the Mean Value Theorem that there exists a point $c_x \in (x, c)$ such that $f(c) - f(x) = (c-x)f'(c_x)$. Since $f'(c_x) \geq 0$ we infer that $f(x) \leq f(c)$ for $x \in (c-\delta, c)$. Similarly, it follows (how?) that $f(x) \leq f(c)$ for $x \in (c, c+\delta)$. Therefore $f(x) \leq f(c)$ for all $x \in (c-\delta, c+\delta)$ so that f has a relative maximum at c.

(b) The proof is similar. Q.E.D.

6.2.9

Example

(b) We can apply the Mean Value Theorem for approximate calculations and to obtain error estimates. For example, suppose it is desired to evaluate $\sqrt{105}$. We employ the Mean Value Theorem with $f(x) := \sqrt{x}$, $a = 100$, $b = 105$, to obtain

$f'(x) = \frac{1}{2\sqrt{x}}$

$$\sqrt{105} - \sqrt{100} = \frac{5}{2\sqrt{c}} = f'(c)(b-a)$$

for some number c with $100 < c < 105$. Since $10 < \sqrt{c} < \sqrt{105} < \sqrt{121} = 11$, we can assert that

$$\frac{5}{2(11)} < \sqrt{105} - 10 < \frac{5}{2(10)},$$

whence it follows that $10.2272 < \sqrt{105} < 10.2500$. This estimate may not be as sharp as desired. It is clear that the estimate $\sqrt{c} < \sqrt{105} < \sqrt{121}$ was wasteful and can be improved by making use of our conclusion that $\sqrt{105} < 10.2500$. Thus, $\sqrt{c} < 10.2500$ and we easily determine that

$$0.2439 < \frac{5}{2(10.2500)} < \sqrt{105} - 10.$$

Our improved estimate is $10.2439 < \sqrt{105} < 10.2500$. □

6.2.11 Lemma *Let $I \subseteq \mathbb{R}$ be an interval, let $f : I \to \mathbb{R}$, let $c \in I$, and assume that f has a derivative at c. Then:*

(a) *If $f'(c) > 0$, then there is a number $\delta > 0$ such that $f(x) > f(c)$ for $x \in I$ such that $c < x < c + \delta$.*

(b) *If $f'(c) < 0$, then there is a number $\delta > 0$ such that $f(x) > f(c)$ for $x \in I$ such that $c - \delta < x < c$.*

Proof. (a) Since

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = f'(c) > 0,$$

it follows from Theorem 4.2.9 that there is a number $\delta > 0$ such that if $x \in I$ and $0 < |x - c| < \delta$, then

$$\frac{f(x) - f(c)}{x - c} > 0.$$

If $x \in I$ also satisfies $x > c$, then we have

$$f(x) - f(c) = (x - c) \cdot \frac{f(x) - f(c)}{x - c} > 0.$$

Hence, if $x \in I$ and $c < x < c + \delta$, then $f(x) > f(c)$.

The proof of (b) is similar. Q.E.D.

6.2.12 Darboux's Theorem *If f is differentiable on $I = [a, b]$ and if k is a number between $f'(a)$ and $f'(b)$, then there is at least one point c in (a, b) such that $f'(c) = k$.*

Proof. Suppose that $f'(a) < k < f'(b)$. We define g on I by $g(x) := kx - f(x)$ for $x \in I$. Since g is continuous, it attains a maximum value on I. Since $g'(a) = k - f'(a) > 0$, it follows from Lemma 6.2.11(a) that the maximum of g does not occur at $x = a$. Similarly, since $g'(b) = k - f'(b) < 0$, it follows from Lemma 6.2.11(b) that the maximum does not occur at $x = b$. Therefore, g attains its maximum at some c in (a, b). Then from Theorem 6.2.1 we have $0 = g'(c) = k - f'(c)$. Hence, $f'(c) = k$. Q.E.D.

\#   * Important

A Preliminary Result

6.3.1 Theorem *Let f and g be defined on $[a, b]$, let $f(a) = g(a) = 0$, and let $g(x) \neq 0$ for $a < x < b$. If f and g are differentiable at a and if $g'(a) \neq 0$, then the limit of f/g at a exists and is equal to $f'(a)/g'(a)$. Thus*

$$\lim_{x \to a+} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}.$$

Proof. Since $f(a) = g(a) = 0$, we can write the quotient $f(x)/g(x)$ for $a < x < b$ as follows:

$$\frac{f(x)}{g(x)} = \frac{f(x) - f(a)}{g(x) - g(a)} = \frac{\dfrac{f(x) - f(a)}{x - a}}{\dfrac{g(x) - g(a)}{x - a}}.$$

Applying Theorem 4.2.4(b), we obtain

$$\lim_{x \to a+} \frac{f(x)}{g(x)} = \frac{\lim\limits_{x \to a+} \dfrac{f(x) - f(a)}{x - a}}{\lim\limits_{x \to a+} \dfrac{g(x) - g(a)}{x - a}} = \frac{f'(a)}{g'(a)}.$$

Q.E.D.

Warning The hypothesis that $f(a) = g(a) = 0$ is essential here. For example, if $f(x) := x + 17$ and $g(x) := 2x + 3$ for $x \in \mathbb{R}$, then

$$\lim_{x \to 0} \frac{f(x)}{g(x)} = \frac{17}{3}, \quad \text{while} \quad \frac{f'(0)}{g'(0)} = \frac{1}{2}.$$

The preceding result enables us to deal with limits such as

$$\lim_{x \to 0} \frac{x^2 + x}{\sin 2x} = \frac{2 \cdot 0 + 1}{2 \cos 0} = \frac{1}{2}.$$

用于替换掉不想要的 $f'(c) / g'(c) / \frac{f'(c)}{g'(c)}$

6.3.2 Cauchy Mean Value Theorem *Let f and g be continuous on $[a, b]$ and differentiable on (a, b), and assume that $g'(x) \neq 0$ for all x in (a, b). Then there exists c in (a, b) such that*

Slope of the parametric Curve $\frac{\Delta u}{\Delta v} =$

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$

$$\begin{cases} u = f(t) \\ v = g(t) \end{cases}$$

Proof. As in the proof of the Mean Value Theorem, we introduce a function to which Rolle's Theorem will apply. First we note that since $g'(x) \neq 0$ for all x in (a, b), it follows from Rolle's Theorem that $g(a) \neq g(b)$. For x in $[a, b]$, we now define

$$h(x) := \frac{f(b) - f(a)}{g(b) - g(a)} (g(x) - g(a)) - (f(x) - f(a)).$$

Then h is continuous on $[a, b]$, differentiable on (a, b), and $h(a) = h(b) = 0$. Therefore, it follows from Rolle's Theorem 6.2.3 that there exists a point c in (a, b) such that

$$0 = h'(c) = \frac{f(b) - f(a)}{g(b) - g(a)} g'(c) - f'(c).$$

Since $g'(c) \neq 0$, we obtain the desired result by dividing by $g'(c)$. Q.E.D.