8.1.1 Definition Let (f_n) be a sequence of functions on $A \subseteq \mathbb{R}$ to $\mathbb{R}$, let $A_0 \subseteq A$, and let $f : A_0 \to \mathbb{R}$. We say that the **sequence** (f_n) **converges on** A_0 **to** f if, for each $x \in A_0$, the sequence $(f_n(x))$ converges to $f(x)$ in $\mathbb{R}$. In this case we call f the **limit on** A_0 **of the sequence** (f_n). When such a function f exists, we say that the sequence (f_n) **is convergent on** A_0, or that (f_n) **converges pointwise on** A_0.

In order to symbolize that the sequence (f_n) converges on A_0 to f, we sometimes write

$$f = \lim(f_n) \quad \text{on} \quad A_0, \quad \text{or} \quad f_n \to f \quad \text{on} \quad A_0.$$

Sometimes, when f_n and f are given by formulas, we write

$$f(x) = \lim f_n(x) \quad \text{for} \quad x \in A_0, \quad \text{or} \quad f_n(x) \to f(x) \quad \text{for} \quad x \in A_0.$$

8.1.3 Lemma *A sequence (f_n) of functions on $A \subseteq \mathbb{R}$ to $\mathbb{R}$ converges to a function $f : A_0 \to \mathbb{R}$ on A_0 if and only if for each $\varepsilon > 0$ and each $x \in A_0$ there is a natural number $K(\varepsilon, x)$ such that if $n \geq K(\varepsilon, x)$, then*

$$|f_n(x) - f(x)| < \varepsilon. \tag{3}$$

We leave it to the reader to show that this is equivalent to Definition 8.1.1. We wish to emphasize that the value of $K(\varepsilon, x)$ will depend, in general, on *both* $\varepsilon > 0$ and $x \in A_0$. The reader should confirm the fact that in Examples 8.1.2(a–c), the value of $K(\varepsilon, x)$ required to obtain an inequality such as (3) does depend on both $\varepsilon > 0$ and $x \in A_0$. The intuitive reason for this is that the convergence of the sequence is "significantly faster" at some points than it is at others. However, in Example 8.1.2(d), as we have seen in inequality (2), if we choose n sufficiently large, we can make $|F_n(x) - F(x)| < \varepsilon$ for *all* values of $x \in \mathbb{R}$. It is precisely this rather subtle difference that distinguishes between the notion of the "pointwise convergence" of a sequence of functions (as defined in Definition 8.1.1) and the notion of "uniform convergence."

理解 "converge significantly faster"

8.1.2 Examples **(a)** $\lim(x/n) = 0$ for $x \in \mathbb{R}$.

For $n \in \mathbb{N}$, let $f_n(x) := x/n$ and let $f(x) := 0$ for $x \in \mathbb{R}$. By Example 3.1.6(a), we have $\lim(1/n) = 0$. Hence it follows from Theorem 3.2.3 that

$$\lim(f_n(x)) = \lim(x/n) = x \lim(1/n) = x \cdot 0 = 0$$

for all $x \in \mathbb{R}$. (See Figure 8.1.1.)

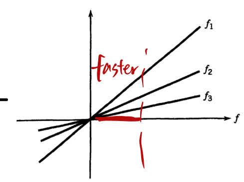


Figure 8.1.1 $f_n(x) = x/n$

(b) $\lim(x^n)$.

Let $g_n(x) := x^n$ for $x \in \mathbb{R}, n \in \mathbb{N}$. (See Figure 8.1.2.) Clearly, if $x = 1$, then the sequence $(g_n(1)) = (1)$ converges to 1. It follows from Example 3.1.11(b) that $\lim(x^n) = 0$ for $0 \leq x < 1$ and it is readily seen that this is also true for $-1 < x < 0$. If $x = -1$, then $g_n(-1) = (-1)^n$, and it was seen in Example 3.2.8(b) that the sequence is divergent. Similarly, if $|x| > 1$, then the sequence (x^n) is not bounded, and so it is not convergent in $\mathbb{R}$. We conclude that if

$$g(x) := \begin{cases} 0 & \text{for} \quad -1 < x < 1, \\ 1 & \text{for} \quad x = 1, \end{cases}$$

then the sequence (g_n) converges to g on the set $(-1, 1]$.

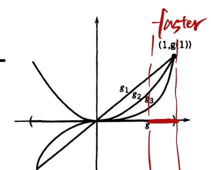


Figure 8.1.2 $g_n(x) = x^n$

K(x,ε)

(c) $\lim((x^2+nx)/n) = x$ for $x \in \mathbb{R}$.

Let $h_n(x) := (x^2+nx)/n$ for $x \in \mathbb{R}$, $n \in \mathbb{N}$, and let $h(x) := x$ for $x \in \mathbb{R}$. (See Figure 8.1.3.) Since we have $h_n(x) = (x^2/n) + x$, it follows from Example 3.1.6(a) and Theorem 3.2.3 that $h_n(x) \to x = h(x)$ for all $x \in \mathbb{R}$.

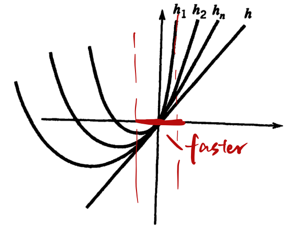


Figure 8.1.3 $h_n(x) = (x^2+nx)/n$

$\leq \frac{1}{n}$ (no x anymore ⇒ K doesn't depend on x)

(d) $\lim((1/n)\sin(nx+n)) = 0$ for $x \in \mathbb{R}$.

Let $F_n(x) := (1/n)\sin(nx+n)$ for $x \in \mathbb{R}$, $n \in \mathbb{N}$, and let $F(x) := 0$ for $x \in \mathbb{R}$. (See Figure 8.1.4.) Since $|\sin y| \leq 1$ for all $y \in \mathbb{R}$ we have

$$(2) \qquad |F_n(x) - F(x)| = \left|\frac{1}{n}\sin(nx+n)\right| \leq \frac{1}{n}$$

for all $x \in \mathbb{R}$. Therefore it follows that $\lim(F_n(x)) = 0 = F(x)$ for all $x \in \mathbb{R}$. The reader should note that, given any $\varepsilon > 0$, if n is sufficiently large, then $|F_n(x) - F(x)| < \varepsilon$ for all values of x simultaneously! □

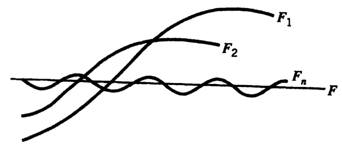


Figure 8.1.4 $F_n(x) = \sin(nx+n)/n$

$K(\varepsilon)$

Uniform Convergence

8.1.4 Definition A sequence (f_n) of functions on $A \subseteq \mathbb{R}$ to $\mathbb{R}$ **converges uniformly on** $A_0 \subseteq A$ to a function $f : A_0 \to \mathbb{R}$ if for each $\varepsilon > 0$ there is a natural number $K(\varepsilon)$ (depending on ε but **not** on $x \in A_0$) such that if $n \geq K(\varepsilon)$, then

$$(4) \qquad |f_n(x) - f(x)| < \varepsilon \quad \text{for all} \quad x \in A_0.$$

Uniform Convergent 判定方法 1

In this case we say that the sequence (f_n) is **uniformly convergent on** A_0. Sometimes we write

≈ uniform continuous 的proof.

$$f_n \rightrightarrows f \quad \text{on} \quad A_0, \qquad \text{or} \qquad f_n(x) \rightrightarrows f(x) \quad \text{for} \quad x \in A_0.$$

Negation:

$f_{n_k}(x_k)$

8.1.5 Lemma *A sequence (f_n) of functions on $A \subseteq \mathbb{R}$ to $\mathbb{R}$ does not converge uniformly on $A_0 \subseteq A$ to a function $f : A_0 \to \mathbb{R}$ if and only if for some $\varepsilon_0 > 0$ there is a subsequence (f_{n_k}) of (f_n) and a sequence (x_k) in A_0 such that*

$$(5) \qquad |f_{n_k}(x_k) - f(x_k)| \geq \varepsilon_0 \quad \text{for all} \quad k \in \mathbb{N}.$$

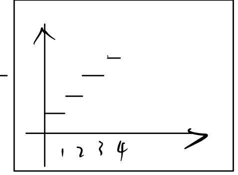
理解为a sequence (x_k). $x_k \in A_0$ [可非连续选择]

It is an immediate consequence of the definitions that if the sequence (f_n) is uniformly convergent on A_0 to f, then this sequence also converges pointwise on A_0 to f in the sense of Definition 8.1.1. That the converse is not always true is seen by a careful examination of Examples 8.1.2(a–c); other examples will be given below.

Uniform ⇒ pointwise
$\cancel{\Leftarrow}$

Proof:

8.1.6 Examples **(a)** Consider Example 8.1.2(a). If we let $n_k := k$ and $x_k := k$, then $f_{n_k}(x_k) = 1$ so that $|f_{n_k}(x_k) - f(x_k)| = |1 - 0| = 1$. Therefore the sequence (f_n) does not converge uniformly on $\mathbb{R}$ to f.



(b) Consider Example 8.1.2(b). If $n_k := k$ and $x_k := \left(\frac{1}{2}\right)^{1/k}$, then

$$|g_{n_k}(x_k) - g(x_k)| = \left|\tfrac{1}{2} - 0\right| = \tfrac{1}{2}.$$

$g_{n_k}(x_k) = \left[\left(\frac{1}{2}\right)^{\frac{1}{k}}\right]^k$

Therefore the sequence (g_n) does not converge uniformly on $(-1, 1]$ to g.

(c) Consider Example 8.1.2(c). If $n_k := k$ and $x_k := -k$, then $h_{n_k}(x_k) = 0$ and $h(x_k) = -k$ so that $|h_{n_k}(x_k) - h(x_k)| = k$. Therefore the sequence (h_n) does not converge uniformly on $\mathbb{R}$ to h. □

The Uniform Norm — 定义对象: A set of Bounded Functions.

8.1.7 Definition If $A \subseteq \mathbb{R}$ and $\varphi : A \to \mathbb{R}$ is a function, we say that φ is **bounded on** A if the set $\varphi(A)$ is a bounded subset of $\mathbb{R}$. If φ is bounded we define the **uniform norm of** φ **on** A by

$$(6) \qquad \|\varphi\|_A := \boxed{\sup}\{|\varphi(x)| : x \in A\}.$$

不一定 $\in \{|\varphi(x)|\}$

Note that it follows that if $\varepsilon > 0$, then

$$(7) \qquad \|\varphi\|_A \le \varepsilon \iff |\varphi(x)| \le \varepsilon \quad \text{for all} \quad x \in A.$$

8.1.8 Lemma *A sequence (f_n) of bounded functions on $A \subseteq \mathbb{R}$ converges uniformly on A to f if and only if $\|f_n - f\|_A \to 0$.*

Uniform Convergent 判定方法 2

Proof. ($\Rightarrow$) If (f_n) converges uniformly on A to f, then by Definition 8.1.4, given any $\varepsilon > 0$ there exists $K(\varepsilon)$ such that if $n \ge K(\varepsilon)$ and $x \in A$ then

$$|f_n(x) - f(x)| \le \varepsilon.$$

From the definition of supremum, it follows that $\|f_n - f\|_A \le \varepsilon$ whenever $n \ge K(\varepsilon)$. Since $\varepsilon > 0$ is arbitrary this implies that $\|f_n - f\|_A \to 0$.

($\Leftarrow$) If $\|f_n - f\|_A \to 0$, then given $\varepsilon > 0$ there is a natural number $H(\varepsilon)$ such that if $n \ge H(\varepsilon)$ then $\|f_n - f\|_A \le \varepsilon$. It follows from (7) that $|f_n(x) - f(x)| \le \varepsilon$ for all $n \ge H(\varepsilon)$ and $x \in A$. Therefore (f_n) converges uniformly on A to f. Q.E.D.

8.1.9 Examples **(a)** We cannot apply Lemma 8.1.8 to the sequence in Example 8.1.2(a) since the function $f_n(x) - f(x) = x/n$ is not bounded on $\mathbb{R}$.

For the sake of illustration, let $A := [0, 1]$. Although the sequence (x/n) did not converge uniformly on $\mathbb{R}$ to the zero function, we shall show that the convergence is uniform on A. To see this, we observe that

$$\|f_n - f\|_A = \sup\{|x/n - 0| : 0 \le x \le 1\} = \frac{1}{n}$$

① 先求 sup||fn-f||

② 再对 sup||fn-f|| 求 n→∞ 极限

so that $\|f_n - f\|_A \to 0$. Therefore (f_n) is uniformly convergent on A to f.

(b) Let $g_n(x) := x^n$ for $x \in A := [0, 1]$ and $n \in \mathbb{N}$, and let $g(x) := 0$ for $0 \le x < 1$ and $g(1) := 1$. The functions $g_n(x) - g(x)$ are bounded on A and

$$||\, g_n - g||_A = \sup\begin{Bmatrix} x^n & \text{for} \quad 0 \le x < 1 \\ 0 & \text{for} \quad x = 1 \end{Bmatrix} = 1$$

for any $n \in \mathbb{N}$. Since $||\, g_n - g||_A$ does *not* converge to 0, we infer that the sequence (g_n) does *not* converge uniformly on A to g.

supreme 不一定属于 $\{||g_n-g||\}$

DON'T FORGET

$n=1$, x很接近1时, $||g_1-g||_A$几乎与1相等.

(e) Let $G(x) := x^n(1-x)$ for $x \in A := [0, 1]$. Then the sequence $(G_n(x))$ converges to $G(x) := 0$ for each $x \in A$. To calculate the uniform norm of $G_n - G = G_n$ on A, we find the derivative and solve

$$G_n'(x) = x^{n-1}(n - (n+1)x) = 0$$

to obtain the point $\boxed{x_n := n/(n+1)}$. This is an interior point of [0, 1], and it is easily verified by using the First Derivative Test 6.2.8 that G_n attains a maximum on [0, 1] at x_n. Therefore, we obtain

$$||\, G_n||_A = G_n(x_n) = \underbrace{\boxed{(1+1/n)^{-n}}}_{e}\, \frac{1}{n+1} \;0,$$

which converges to $(1/e) \cdot 0 = 0$. Thus we see that convergence is uniform on A. □

有Bounded 才能谈 uniform Norm

#

8.1.10 Cauchy Criterion for Uniform Convergence *Let (f_n) be a sequence of bounded functions on $A \subseteq \mathbb{R}$. Then this sequence converges uniformly on A to a bounded function f if and only if for each $\varepsilon > 0$ there is a number $H(\varepsilon)$ in $\mathbb{N}$ such that for all $m, n \ge H(\varepsilon)$, then $||f_m - f_n||_A \le \varepsilon$.* □

Uniform Convergent 判定方法 3

Proof. ($\Rightarrow$) If $f_n \rightrightarrows f$ on A, then given $\varepsilon > 0$ there exists a natural number $K(\frac{1}{2}\varepsilon)$ such that if $n \ge K(\frac{1}{2}\varepsilon)$ then $||f_n - f||_A \le \frac{1}{2}\varepsilon$. Hence, if both $m, n \ge K(\frac{1}{2}\varepsilon)$, then we conclude that

$$|f_m(x) - f_n(x)| \le |f_m(x) - f(x)| + |f_n(x) - f(x)| \le \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon$$

for all $x \in A$. Therefore $||f_m - f_n||_A \le \varepsilon$ for $m, n \ge K(\frac{1}{2}\varepsilon) =: H(\varepsilon)$.

($\Leftarrow$) Conversely, suppose that for $\varepsilon > 0$ there is $H(\varepsilon)$ such that if $m, n \ge H(\varepsilon)$, then $||f_m - f_n||_A \le \varepsilon$. Therefore, for each $x \in A$ we have

$$(8) \qquad |f_m(x) - f_n(x)| \le ||f_m - f_n||_A \le \varepsilon \qquad \text{for} \quad m, n \ge H(\varepsilon).$$

It follows that $(f_n(x))$ is a Cauchy sequence in $\mathbb{R}$; therefore, by Theorem 3.5.5, it is a convergent sequence. We define $f : A \to \mathbb{R}$ by

$$f(x) := \lim(f_n(x)) \qquad \text{for} \quad x \in A.$$

If we let $n \to \infty$ in (8), it follows from Theorem 3.2.6 that for each $x \in A$ we have

$$|f_m(x) - f(x)| \le \varepsilon \qquad \text{for} \quad m \ge H(\varepsilon).$$

Therefore the sequence (f_n) converges uniformly on A to f. Q.E.D.

Proof: $\boxed{\Leftarrow}$ Suppose $\exists H(\varepsilon) \in \mathbb{N}$ s.t. $\forall m, n \geq H(\varepsilon)$,

$$||f_m - f_n||_A \le \varepsilon$$

then $\sup\{|f_m(x) - f_n(x)|\} \le \varepsilon$, $x \in A$.

$\Rightarrow |f_m(x) - f_n(x)| \le \varepsilon$

$\Rightarrow$ $\{f_n(x)\}$ is a cauchy sequence therefore convergent.

Let $f(x) := \lim\limits_{n\to\infty}\{f_n(x)\}$

then $\forall \varepsilon > 0$, $\exists K \in \mathbb{N}$ s.t. $\forall n \geq K$,

$$|f_n(x) - f(x)| < \varepsilon$$

Thus, $\{f_n(x)\}$ is uniformly convergent.

$\boxed{\Rightarrow}$ Suppose $\{f_n(x)\}$ is uniformly convergent on A to a bounded function f

then when $m, n \in \mathbb{N}$, m, n are sufficiently large

i.e. let $n \to \infty$ and $m \geq n$ then $\forall \varepsilon > 0$,

$$|f_n - f| \le ||f_n - f||_A < \frac{\varepsilon}{2}$$

$$|f_m - f| \le ||f_m - f||_A < \frac{\varepsilon}{2}$$

$$\begin{aligned} |f_m - f_n| - ||f_m - f_n|| &= ||f_m - f - f_n + f||_A \\ &\le ||f_n - f||_A + ||f_m - f||_A \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

Thus, $||f_m - f_n||_A \le \varepsilon$

Q.E.D.

Uniform Convergent 的三个判定:

1. 定义: $\forall n \geq K(\varepsilon)$, $|f_n(x) - f| < \varepsilon$

2. Uniform Norm: $||f_n - f||_A \to 0$

3. Cauchy Criterion: $\forall m, n \geq H(\varepsilon)$, $||f_m - f_n||_A < \varepsilon$

Pointwise Convergent (Examples)

8.2.1 Examples **(a)** Let $g_n(x) := x^n$ for $x \in [0,1]$ and $n \in \mathbb{N}$. Then, as we have noted in Example 8.1.2(b), the sequence (g_n) converges pointwise to the function

$$g(x) := \begin{cases} 0 & \text{for} \quad 0 \le x < 1, \\ 1 & \text{for} \quad x = 1. \end{cases}$$

Although all of the functions g_n are continuous at $x = 1$, the limit function g is not continuous at $x = 1$. Recall that it was shown in Example 8.1.6(b) that this sequence does not converge uniformly to g on $[0, 1]$.

Deny

(1) conts.

(b) Each of the functions $g_n(x) = x^n$ in part (a) has a continuous derivative on $[0,1]$. However, the limit function g does not have a derivative at $x = 1$, since it is not continuous at that point.

(2) diff.

$g_n(x) = \frac{x^n}{n}$ on $[0,1]$ — $g_n \to g = 0$

— $g_n' = \frac{n}{n} x^{n-1} = x^{n-1} \to h(x) = \begin{cases} 0, & x \in [0,1) \\ 1, & x = 1 \end{cases}$ $\Rightarrow h \ne g'$ at $x=1$

$\Rightarrow$ The derivative of the limit $\ne$ The limit of derivative

$\lim_{k\to\infty}\left[\lim_{n\to\infty} f_n(x_k)\right]$ pointwise $\lim_{n\to\infty}\left[\lim_{k\to\infty} f_n(x_k)\right]$

(c) Let $f_n : [0,1] \to \mathbb{R}$ be defined for $n \ge 2$ by

Bounded Interval

$$f_n(x) := \begin{cases} n^2 x & \text{for} \quad 0 \le x \le 1/n, \\ -n^2(x - 2/n) & \text{for} \quad 1/n \le x \le 2/n, \\ 0 & \text{for} \quad 2/n \le x \le 1. \end{cases}$$

(3) R.I.

deny

(See Figure 8.2.1.) It is clear that each of the functions f_n is continuous on $[0, 1]$; hence it is Riemann integrable. Either by means of a direct calculation, or by referring to the significance of the integral as an area, we obtain

$$\int_0^1 f_n(x)\,dx = 1 \quad \text{for} \quad n \ge 2. \Rightarrow \lim_{n\to\infty}\int_0^1 f = \lim_{n\to\infty}(1) = 1$$

The reader may show that $f_n(x) \to 0$ for all $x \in [0, 1]$; hence the limit function f vanishes identically and is continuous (and hence integrable), and $\int_0^1 f(x)\,dx = 0$. Therefore we have the uncomfortable situation that:

$$\int_0^1 f(x)\,dx = 0 \ne 1 = \lim \int_0^1 f_n(x)\,dx.$$

Rieman Integral of the limit $\ne$ Limit of Rieman-Integral

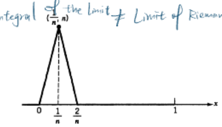


Figure 8.2.1 Example 8.2.1(c)

Interchange of Limit and Continuity

8.2.2 Theorem *Let (f_n) be a sequence of continuous functions on a set $A \subseteq \mathbb{R}$ and suppose that (f_n) converges uniformly on A to a function $f : A \to \mathbb{R}$. Then f is continuous on A.*

1st

2nd

Proof. By hypothesis, given $\varepsilon > 0$ there exists a natural number $H := H(\frac{1}{3}\varepsilon)$ such that if $n \ge H$ then $|f_n(x) - f(x)| < \frac{1}{3}\varepsilon$ for all $x \in A$. Let $c \in A$ be arbitrary; we will show that f is continuous at c. By the Triangle Inequality we have

$$\begin{aligned} |f(x) - f(c)| &\le |f(x) - f_H(x)| + |f_H(x) - f_H(c)| + |f_H(c) - f(c)| \\ &\le \tfrac{1}{3}\varepsilon + |f_H(x) - f_H(c)| + \tfrac{1}{3}\varepsilon. \end{aligned}$$

Since f_H is continuous at c, there exists a number $\boxed{\delta := \delta(\frac{1}{3}\varepsilon, c, f_H) > 0}$ such that if $|x - c| < \delta$ and $x \in A$, then $|f_H(x) - f_H(c)| < \frac{1}{3}\varepsilon$. Therefore, if $|x - c| < \delta$ and $x \in A$, then we have $|f(x) - f(c)| < \varepsilon$. Since $\varepsilon > 0$ is arbitrary, this establishes the continuity of f at the arbitrary point $c \in A$. (See Figure 8.2.2.) Q.E.D.

Uniform Convergent

f_n is continuous $\xrightarrow{\checkmark}$ f is continuous

f_n is continuous $\xleftarrow{\times}$ f is continuous

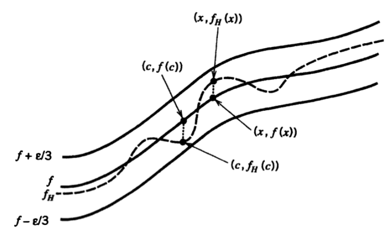


Figure 8.2.2 $|f(x) - f(c)| < \varepsilon$

Interchange of Limit and Derivative

[a,b] or (a,b)

8.2.3 Theorem *Let $J \subseteq \mathbb{R}$ be a bounded interval and let (f_n) be a sequence of functions on J to $\mathbb{R}$. Suppose that there exists $x_0 \in J$ such that $(f_n(x_0))$ converges, and that the sequence (f'_n) of derivatives exists on J and converges uniformly on J to a function g.*

Then the sequence (f_n) converges uniformly on J to a function f that has a derivative at every point of J and $f' = g$.

Mean Value Theorem

闭区间连续，开区间可导.

(f_n) is differentiable at every point of J.

② $f_n'(x)$ exists on $\forall x \in J$.

Thm. Suppose $f_n: [a,b] \to \mathbb{R}$ is ① continuously diff. then $f, g: [a,b] \to \mathbb{R}$. and

此时 f 还未定义.

② $f_n \to \cancel{f}$ pointwisely on $[a,b]$

③ $f_n' \rightrightarrows g$ uniformly on $[a,b]$

Then ① $f_n \rightrightarrows f$

② f is differentiable at every point of J

③ $g = f'$ [i.e. $f_n' \to g = f'$]

Limit of derivatives = Derivative of the Limit

$g = \lim_{n\to\infty} f_n'$ f_n' is uni. conv. f_n is pointwise $[\lim_{n\to\infty} f_n]'$

Proof:

Thm 8.2.3 Change of Limit : Differentiable

条件: ① f_n continuously differentiable on J

② $\exists x_0 \in J$. $f_n(x_0)$ converges, pointwisely.

③ $f_n' \rightrightarrows g$ on J. (此时 f 还未定义)

结论: ① $f_n \rightrightarrows f$ on J

② f is continuously differentiable on J.

③ $f' = g$ on J.

Bounded interval J ⇏ closed (闭)

Proof: As $\forall n \in \mathbb{N}$, f_n is continuously differentiable on the bounded interval J (let $J := (a,b)$ or $[a,b]$)

then $f_m - f_n$ is continuously diff. on J.

⇒ Mean Value Thm. could be applied to closed & bdd intervals:

Define I_x be closed & bdd interval ends with x and x_0. $\forall x \in J$. then by M.V.T,

$\exists y \in I_x$ s.t.

$(f_m - f_n)(x) - (f_m - f_n)(x_0) = (f_m - f_n)'(y)(x - x_0)$

$\Rightarrow (f_m - f_n)(x) = (f_m - f_n)(x_0) + (f_m - f_n)'(y)(x - x_0)$

$\Rightarrow |(f_m - f_n)(x)| \le |(f_m - f_n)(x_0)| + |(f_m' - f_n')(y)(x - x_0)|$

$\le |(f_m - f_n)(x_0)| + |(f_m'(y) - f_n'(y))(b-a)|$

As $f_n \to f$ at $x = x_0$ and $f_n' \rightrightarrows g$ on J

⇓ Cauchy: & $|(f_m - f_n)(x_0)| < \varepsilon/2$ ⇓ Cauchy $\lim_{n\to\infty} |(f_m' - f_n')(y)| < \varepsilon/2(b-a)$

Goal 1

$$\boxed{\Rightarrow |(f_m - f_n)(x)| \le \varepsilon \Rightarrow \|f_m - f_n\|_J \to 0}$$

Thus, (f_n) is uniformly convergent on J

Denote the limit of (f_n) by f.

⇒ f is continuous on J.

(WTS: f' exists on J / $\forall c \in J$.)

For any $c \in J$, Define an interval I_x that ends with c, x $\forall x \in J$. then by M.V.T,

$\exists z \in I_x$ s.t. $(f_m - f_n)(x) - (f_m - f_n)(c) = (f_m' - f_n')(z)(x - c)$

If $x \ne c$:

$$\frac{f_m(x) - f_m(c)}{x - c} - \frac{f_n(x) - f_n(c)}{x - c} = f_m'(z) - f_n'(z)$$

$$\Rightarrow \left|\frac{f_m(x) - f_m(c)}{x - c} - \frac{f_n(x) - f_n(c)}{x - c}\right| \le \|f_m' - f_n'\|_J$$

As (f_n') converges uniformly on J so $\forall \varepsilon > 0$,

$\exists H(\varepsilon) \in \mathbb{N}$ s.t. $\forall m, n \ge H(\varepsilon)$,

$$\left|\frac{f_m(x) - f_m(c)}{x - c} - \frac{f_n(x) - f_n(c)}{x - c}\right| \le \varepsilon$$

$\Rightarrow$ $m \to \infty$, $n \ge H(\varepsilon)$. $\left|\frac{f(x) - f(c)}{x - c} - \frac{f_n(x) - f_n(c)}{x - c}\right| \le \varepsilon/3$

As $f_n'(c) \rightrightarrows g(c)$ so $g(c) = \lim_{n\to\infty}[f_n'(c)]$.

$\Rightarrow \exists k(\varepsilon) \in \mathbb{N}$ s.t. $\forall n \ge k(\varepsilon)$,

$|f_n'(c) - g(c)| \le \varepsilon/3$ (而非 $\frac{f_n(x) - f_n(c)}{x - c}$)

⇒ Let $M := \max\{k(\varepsilon), H(\varepsilon)\}$ then $\forall n \ge M$,

Since $f_m'(c)$ exists. so $\exists \delta_m(\varepsilon) > 0$ s.t. if $0 < |x - c| < \delta_m(\varepsilon)$,

$$\left|\frac{f_m(x) - f_m(c)}{x - c} - f_m'(c)\right| < \frac{\varepsilon}{3}$$

Thus, if $0 < |x - c| < \delta_m(\varepsilon)$, then $\forall n \ge M$,

Goal 2

$$\boxed{\left|\frac{f(x) - f(c)}{x - c} - g(x)\right|} \le \left|\frac{f(x) - f(c)}{x - c} - \frac{f_n(x) - f_n(c)}{x - c}\right| + \left|\frac{f_n(x) - f_n(c)}{x - c} - f_n'(c)\right| + |f_n'(c) - g(c)|$$

$$< \frac{\varepsilon}{3} \times 3 = \varepsilon$$

Thus, $f'(x) = g(x)$ $\forall x \in J$

Q.E.D.

$f_n \to f$, f_n continuously diff. 的应用: Mean Value Theorem (2次)

+

f_n is diff. on $\forall c \in J$

+

$f_n' \rightrightarrows g$ at $\forall c \in J$.

Proof. Let $a < b$ be the endpoints of J and let $x \in J$ be arbitrary. If $m, n \in \mathbb{N}$, we apply the Mean Value Theorem 6.2.4 to the difference $f_m - f_n$ on the interval with endpoints x_0, x. We conclude that there exists a point y (depending on m, n) such that

$$f_m(x) - f_n(x) = f_m(x_0) - f_n(x_0) + (x - x_0)\{f_m'(y) - f_n'(y)\}.$$

Hence we have

$$(1) \qquad \|f_m - f_n\|_J \le |f_m(x_0) - f_n(x_0)| + (b - a)\|f_m' - f_n'\|_J.$$

From Theorem 8.1.10, it follows from (1) and the hypotheses that $(f_n(x_0))$ is convergent and that (f_n') is uniformly convergent on J, that (f_n) is uniformly convergent on J. We denote the limit of the sequence (f_n) by f. Since the f_n are all continuous and the convergence is uniform, it follows from Theorem 8.2.2 that f is continuous on J.

To establish the existence of the derivative of f at a point $c \in J$, we apply the Mean Value Theorem 6.2.4 to $f_m - f_n$ on an interval with end points c, x. We conclude that there exists a point z (depending on m, n) such that

$$\{f_m(x) - f_n(x)\} - \{f_m(c) - f_n(c)\} = (x - c)\{f_m'(z) - f_n'(z)\}.$$

Hence, if $x \ne c$, we have

$$\left|\frac{f_m(x) - f_m(c)}{x - c} - \frac{f_n(x) - f_n(c)}{x - c}\right| \le \|f_m' - f_n'\|_J.$$

Since (f_n') converges uniformly on J, if $\varepsilon > 0$ is given there exists $H(\varepsilon)$ such that if $m, n \ge H(\varepsilon)$ and $x \ne c$, then

$$(2) \qquad \left|\frac{f_m(x) - f_m(c)}{x - c} - \frac{f_n(x) - f_n(c)}{x - c}\right| \le \varepsilon.$$

If we take the limit in (2) with respect to m and use Theorem 3.2.6, we have

$$\left|\frac{f(x) - f(c)}{x - c} - \frac{f_n(x) - f_n(c)}{x - c}\right| \le \varepsilon.$$

provided that $x \ne c$, $n \ge H(\varepsilon)$. Since $g(c) = \lim(f_n'(c))$, there exists $N(\varepsilon)$ such that if $n \ge N(\varepsilon)$, then $|f_n'(c) - g(c)| < \varepsilon$. Now let $K := \sup\{H(\varepsilon), N(\varepsilon)\}$. ~~Since $f_K'(c)$ exists~~, there exists $\delta_K(\varepsilon) > 0$ such that if $0 < |x - c| < \delta_K(\varepsilon)$, then

f_n' 在 J上处处存在

$$\left|\frac{f_K(x) - f_K(c)}{x - c} - f_K'(c)\right| < \varepsilon.$$

Combining these inequalities, we conclude that if $0 < |x - c| < \delta_K(\varepsilon)$, then

Desired :

$$\boxed{\left|\frac{f(x) - f(c)}{x - c} - g(c)\right| < 3\varepsilon.}$$

$\lim_{x\to c} \frac{f(x) - f(c)}{x - c} = g(c)$

Since $\varepsilon > 0$ is arbitrary, this shows that $f'(c)$ exists and equals $g(c)$. Since $c \in J$ is arbitrary, we conclude that $f' = g$ on J. Q.E.D.

Interchange of Limit and Integral

8.2.4 Theorem *Let (f_n) be a sequence of functions in $\mathcal{R}[a,b]$ and suppose that (f_n) converges **uniformly** on $[a, b]$ to f. Then $f \in \mathcal{R}[a,b]$ and* (3) *holds.*

$$\int_a^b f = \lim_{n\to\infty} \int_a^b f_n. = A$$

Goal: $\left|\boxed{S(f;\dot{\mathcal{P}})} - A\right| \le \varepsilon$

$\boxed{\text{Proof:}}$

Rieman Integral 定义法:

$\|\dot{\mathcal{P}}\| \to 0$, $|S(f;\dot{\mathcal{P}}) - L| < \varepsilon$

第一步!

Proof: As $f_n \rightrightarrows f$, so by Cauchy criterion,

$\forall \varepsilon > 0$, $\exists H \in \mathbb{N}$ s.t. $\forall m, n \ge H$, $|f_m - f_n| \le \varepsilon$ for $x \in [a,b]$

By Thm 7.1.5,

$$-\varepsilon(b-a) \le \int_a^b (f_m - f_n) \le \varepsilon(b-a)$$

$$-\varepsilon(b-a) \le \int_a^b f_m - \int_a^b f_n \le \varepsilon(b-a)$$

$$\Rightarrow \quad \left|\int_a^b f_m - \int_a^b f_n\right| \le \varepsilon(b-a)$$

So $\left(\int_a^b f_n\right)$ is a Cauchy therefore convergent sequence.

Let $A := \lim_{n\to\infty} \int_a^b f_n$ i.e. $\exists M \in \mathbb{N}$ s.t. $\forall n \ge M$, $\left|\int_a^b f_n - A\right| < \varepsilon$ (1)

As (f_n) is uniformly convergent, so $\exists K \in \mathbb{N}$ s.t. $\forall n \ge k$,

$|f_n(x) - f(x)| < \varepsilon$ for $\forall x \in [a,b]$

$\Rightarrow$ Define $\dot{\mathcal{P}}$ to be any tagged partition of $[a,b]$

s.t. $\dot{\mathcal{P}} = \{[x_{i-1}, x_i], t_i\}_{i=1}^m$, so $t_i \in [a,b]$ $\forall i \in \mathbb{N}$.

then

$$|S(f_n;\dot{\mathcal{P}}) - S(f;\dot{\mathcal{P}})| = \left|\sum_{i=1}^m (f_n(t_i) - f(t_i))(x_i - x_{i-1})\right|$$

$$\le \sum_{i=1}^m |f_n(t_i) - f(t_i)|\,|x_i - x_{i-1}|$$

$$\le \sum_{i=1}^m \varepsilon(x_i - x_{i-1})$$

$$= \varepsilon \sum_{i=1}^m (x_i - x_{i-1})$$

$$= \varepsilon(b-a) \quad (2)$$

As $f_n \in R[a,b]$ so by the definition of Rieman-Integral,

$\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. if $\|\dot{\mathcal{P}}\| < \delta$ then

$$\left|S(f_n;\dot{\mathcal{P}}) - \int_a^b f_n\right| < \varepsilon \quad (3)$$

Thus, Combine inequalities (1), (2), (3):

$$\underline{\underline{|S(f;\dot{\mathcal{P}}) - A|}} \le |S(f;\dot{\mathcal{P}}) - S(f_n;\dot{\mathcal{P}})| + \left|S(f_n;\dot{\mathcal{P}}) - \int_a^b f_n\right|$$

Goal

$$+ \left|\int_a^b f_n - A\right|$$

$$< \varepsilon(b-a) + \varepsilon + \varepsilon$$

$$= \varepsilon(b-a+2)$$

Thus, $\lim_{\|\dot{\mathcal{P}}\|\to 0} S(f;\dot{\mathcal{P}}) = A$ so $f \in R[a,b]$ and $\int_a^b f = A$

Q.E.D.

Proof. It follows from the Cauchy Criterion 8.1.10 that given $\varepsilon > 0$ there exists $H(\varepsilon)$ such that if $m > n \geq H(\varepsilon)$ then

$$-\varepsilon \leq f_m(x) - f_n(x) \leq \varepsilon \quad \text{for} \quad x \in [a, b].$$

Theorem 7.1.5 implies that

$$-\varepsilon(b-a) \leq \int_a^b f_m - \int_a^b f_n \leq \varepsilon(b-a).$$

Since $\varepsilon > 0$ is arbitrary, the sequence $(\int_a^b f_m)$ is a Cauchy sequence in $\mathbb{R}$ and therefore converges to some number, say $A \in \mathbb{R}$.

We now show $f \in \mathcal{R}[a, b]$ with integral A. If $\varepsilon > 0$ is given, let $K(\varepsilon)$ be such that if $m > K(\varepsilon)$, then $|f_m(x) - f(x)| < \varepsilon$ for all $x \in [a, b]$. If $\dot{\mathcal{P}} := \{([x_{i-1}, x_i], t_i)\}_{i=1}^n$ is any tagged partition of $[a, b]$ and if $m > K(\varepsilon)$, then

$$\begin{aligned}|S(f_m; \dot{\mathcal{P}}) - S(f; \dot{\mathcal{P}})| &= \left|\sum_{i=1}^n \{f_m(t_i) - f(t_i)\}(x_i - x_{i-1})\right| \\ &\leq \sum_{i=1}^n |f_m(t_i) - f(t_i)|(x_i - x_{i-1}) \\ &\leq \sum_{i=1}^n \varepsilon(x_i - x_{i-1}) = \varepsilon(b-a).\end{aligned}$$

We now choose $r \geq K(\varepsilon)$ such that $|\int_a^b f_r - A| < \varepsilon$ and we let $\delta_{r,\varepsilon} > 0$ be such that $|\int_a^b f_r - S(f_r; \dot{\mathcal{P}})| < \varepsilon$ whenever $||\dot{\mathcal{P}}|| < \delta_{r,\varepsilon}$. Then we have

$$\begin{aligned}|S(f; \dot{\mathcal{P}}) - A| &\leq |S(f; \dot{\mathcal{P}}) - S(f_r; \dot{\mathcal{P}})| + \left|S(f_r; \dot{\mathcal{P}}) - \int_a^b f_r\right| + \left|\int_a^b f_r - A\right| \\ &\leq \varepsilon(b-a) + \varepsilon + \varepsilon = \varepsilon(b-a+2).\end{aligned}$$

But since $\varepsilon > 0$ is arbitrary, it follows that $f \in \mathcal{R}[a, b]$ and $\int_a^b f = A$. Q.E.D.

uni.conv.的条件很难达到：

8.2.5 Bounded Convergence Theorem *Let (f_n) be a sequence in $\mathcal{R}[a, b]$ that converges on $[a, b]$ to a function $f \in \mathcal{R}[a, b]$. Suppose also that there exists $B > 0$ such that $|f_n(x)| \leq B$ for all $x \in [a, b]$, $n \in \mathbb{N}$. Then equation (3) holds.*

1. f is bounded
2. $f_n \in \mathcal{R}[a,b]$
3. $f_n \to f$ pointwisely
4. $f \in \mathcal{R}[a,b]$

$\Downarrow$

$\int_a^b f = \lim_{n\to\infty} \int_a^b f_n$

Tel. 3 Uniform Convergent is too strong for normal cases.

8.2.6 Dini's Theorem *Suppose that (f_n) is a monotone sequence of continuous functions on $I := [a, b]$ that converges on I to a continuous function f. Then the convergence of the sequence is uniform.*

Proof. We suppose that the sequence (f_n) is decreasing and let $g_m := f_m - f$. Then (g_m) is a decreasing sequence of continuous functions converging on I to the 0-function. We will show that the convergence is uniform on I.

Given $\varepsilon > 0$, $t \in I$, there exists $m_{\varepsilon,t} \in \mathbb{N}$ such that $0 \le g_{m_{\varepsilon,t}}(t) < \varepsilon/2$. Since $g_{m_{\varepsilon,t}}$ is continuous at t, there exists $\delta_\varepsilon(t) > 0$ such that $0 \le g_{m_{\varepsilon,t}}(x) < \varepsilon$ for all $x \in I$ satisfying $|x - t| \le \delta_\varepsilon(t)$. Thus, δ_ε is a gauge on I, and if $\dot{\mathcal{P}} = \{(I_i, t_i)\}_{i=1}^n$ is a δ_ε-fine partition, we set $M_\varepsilon := \max\{m_{\varepsilon,t_1}, \ldots, m_{\varepsilon,t_n}\}$. If $m \ge M_\varepsilon$ and $x \in I$, then (by Lemma 5.5.3) there exists an index i with $|x - t_i| \le \delta_\varepsilon(t_i)$ and hence

$$0 \le g_m(x) \le g_{m,t_i}(x) < \varepsilon.$$

Therefore, the sequence (g_m) converges uniformly to the 0-function. Q.E.D.

Proof:

As (f_n) is monotone so assume it to be decreasing

As f_n converge to f on I so

Let $g_m(x) := f_m(x) - f(x)$ $\forall x \in I$ then

$(g_m(x))$ is a monotone decreasing sequence of

continuous functions converging on I to $g(x) = 0$. $\forall x \in I$

$\Rightarrow g_m(x) \geq 0$ $\forall m \in \mathbb{N}$

$\Rightarrow$ By density of real numbers,

$\forall \varepsilon > 0$, $\forall t \in I$, $\exists m_{\varepsilon,t} \in \mathbb{N}$ s.t.

$0 \le g_{m_{\varepsilon,t}}(t) < \frac{\varepsilon}{2}$

As $g_{m_{\varepsilon,t}}(x)$ is continuous at t, then

$\exists \delta_\varepsilon(t) > 0$ s.t. $\forall x \in I$ satisfying

$|x - t| \le \delta_\varepsilon(t)$, $0 \le g_{m_{\varepsilon,t}}(x) < \varepsilon$

Define $\dot{P} = \{(I_i, t_i)\}_{i=1}^n$, $\|\dot{P}\| = \delta_\varepsilon$

set $M_\varepsilon := \max\{m_{\varepsilon,t_1}, \ldots m_{\varepsilon,t_n}\}$

If $m \geq M_\varepsilon$ and $x \in I$ then

$\exists i \in \mathbb{N}$ s.t. $|x - t_i| \le \delta_\varepsilon(t_i)$

$\Rightarrow 0 \le g_m(x) \le g_{m_{\varepsilon,t_i}}(x) < \varepsilon$

Thus, (g_m) convergence uniformly to $g(x) = 0$

Q.E.D.

Section 8.3 The Exponential and Logarithmic Functions

E的存在性：

8.3.1 Theorem *There exists a function $E : \mathbb{R} \to \mathbb{R}$ such that:*

(i) $E'(x) = E(x)$ *for all* $x \in \mathbb{R}$. 须可导性的传递

f'

(ii) $E(0) = 1$. 须连续性的传递

Member Function

Limit Function

① 构建 $E_n(x)$

③ Cauchy

$m > n > 2A$, $[-A, A]$

uniform convergent

$|E_m - E_n|$

$E_n \rightrightarrows E$

$E(x)$

② 找关系: $E_{n+1}' = E_n$

$E_n'(x)$

④ E_n 的连续性可传递到 E

$E(0) = 1$

WTS: $\|$

$E'(x)$

④' uniform convergent

⑤ E_n 可导性由 $\begin{cases} E_n \rightrightarrows E \\ E_n' \rightrightarrows g \end{cases}$ 传传递

得 $g = E'$

$\lim_{n\to\infty} E_n' = E'$

$\lim_{n\to\infty}(E_{n-1}) = E'$

$E = E'$

得证

Proof. We inductively define a sequence (E_n) of continuous functions as follows:

$$E_1(x) := 1 + x, \tag{1}$$

$$E_{n+1}(x) := 1 + \int_0^x E_n(t)\,dt, \tag{2}$$

for all $n \in \mathbb{N}$, $x \in \mathbb{R}$. Clearly E_1 is continuous on $\mathbb{R}$ and hence is integrable over any bounded interval. If E_n has been defined and is continuous on $\mathbb{R}$, then it is integrable over any bounded interval, so that E_{n+1} is well-defined by the above formula. Moreover, it follows from the Fundamental Theorem (Second Form) 7.3.5 that E_{n+1} is differentiable at any point $x \in \mathbb{R}$ and that

$$E'_{n+1}(x) = E_n(x) \quad \text{for} \quad n \in \mathbb{N}. \tag{3}$$

An Induction argument (which we leave to the reader) shows that

$$\boxed{E_n(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} \quad \text{for} \quad x \in \mathbb{R}.} = \sum_{k=0}^{n} \frac{x^k}{k!} \tag{4}$$

Let $A > 0$ be given; then if $|x| \le A$ and $m > n > 2A$, we have

$$\begin{aligned} |E_m(x) - E_n(x)| &= \left|\frac{x^{n+1}}{(n+1)!} + \cdots + \frac{x^m}{m!}\right| \\ &\le \frac{A^{n+1}}{(n+1)!}\left[1 + \frac{A}{n} + \cdots + \left(\frac{A}{n}\right)^{m-n-1}\right] \\ &< \frac{A^{n+1}}{(n+1)!}2. \end{aligned} \tag{5}$$

Since $\lim(A^n/n!) = 0$, it follows that the sequence (E_n) converges uniformly on the interval $[-A, A]$ where $A > 0$ is arbitrary. In particular this means that $(E_n(x))$ converges for each $x \in \mathbb{R}$. We define $E : \mathbb{R} \to \mathbb{R}$ by

$$E(x) := \lim E_n(x) \quad \text{for} \quad x \in \mathbb{R}.$$

Since each $x \in \mathbb{R}$ is contained inside some interval $[-A, A]$, it follows from Theorem 8.2.2 that E is continuous at x. Moreover, it is clear from (1) and (2) that $E_n(0) = 1$ for all $n \in \mathbb{N}$. Therefore $E(0) = 1$, which proves (ii).

On any interval $[-A, A]$ we have the uniform convergence of the sequence (E_n). In view of (3), we also have the uniform convergence of the sequence (E_n') of derivatives. It therefore follows from Theorem 8.2.3 that the limit function E is differentiable on $[-A, A]$ and that

$$E'(x) = \lim(E_n'(x)) = \lim(E_{n-1}(x)) = E(x)$$

for all $x \in [-A, A]$. Since $A > 0$ is arbitrary, statement (i) is established. Q.E.D.

Proof: Define sequence of functions Inductively:

$E_1(x) = 1 + x \Rightarrow E_1' = 1$

$E_2(x) = 1 + \int_0^x 1 + t \, dt = 1 + x + \frac{1}{2}x^2 \Rightarrow E_2' = E_1$

$E_3(x) = 1 + \int_0^x 1 + t + \frac{1}{2}t^2 \, dt = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 \Rightarrow E_3' = E_2$

$\vdots$

$$\boxed{E_{n+1}(x) = 1 + \int_0^x E_n(t)\,dt \Rightarrow E_{n+1}' = E_n}$$

As E_1 is continuous on $\mathbb{R}$ so it is integrable over any bounded area.

If E_n is defined and is continuous on $\mathbb{R}$ then it is integrable over any bounded interval.

$\Rightarrow E_{n+1} = 1 + \int_0^x E_n(t)\,dt$ exists on $\mathbb{R}$ (well-defined on $\mathbb{R}$)

By the F.T.C. (Second form), on any $[a,b]$, we showed that $E_{n+1}(x) \in \mathcal{R}[a,b]$. and $E_{n+1}(x)$ continuous on $[a,b]$

Then $E_{n+1}(x)$ is differentiable

and as $E_{n+1}(x) = 1 + \int_0^x E_n(t)\,dt$

So $E_{n+1}'(x) = E_n(x)$ $\forall n \in \mathbb{N}$.

Assume that $E_m(x) = \sum_{m=0}^{n} \frac{x^m}{m!}$

then when $n = 1$, $E_1(x) = \frac{x^0}{0!} + \frac{x^1}{1!} = 1 + x$

assume that when $n = k$,

$$\boxed{E_k(x) = \sum_{m=0}^{k} \frac{x^m}{m!}} = \frac{x^0}{0!} + \frac{x^1}{1!} + \cdots + \frac{x^k}{k!}$$

when $n = k+1$,

$$E_{k+1} = 1 + \int_0^x E_k(t)\,dt = 1 + \int_0^x \sum_{m=0}^{k} \frac{t^m}{m!}\,dt$$

$$= 1 + x + \frac{1}{2}x^2 + \cdots + \frac{x^k}{k!} + \frac{x^{k+1}}{(k+1)!}$$

$$= \sum_{m=0}^{k+1} \frac{x^m}{m!}$$

Thus, $E_n(x) = \sum \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$, $\forall x \in \mathbb{R}$.

$\forall A > 0$, $\forall x \in [-A, A]$, $\forall m, n \in \mathbb{N}$, $m > n > 2A$,

$$|E_m(x) - E_n(x)| = \left| \frac{x^{n+1}}{(n+1)!} + \cdots + \frac{x^m}{m!} \right|$$

$$\leq \sum_{k=n+1}^{k=m} \left| \frac{x^k}{k!} \right|$$

$$\leq \sum_{k=n+1}^{k=m} \left| \frac{A^k}{k!} \right|$$

$$\boxed{= \frac{A^{n+1}}{(n+1)!} \left[1 + \frac{A}{n} + \cdots + \left(\frac{A}{n}\right)^{m-(n+1)} \right]}$$

$\left(= \frac{A^{n+1}}{(n+1)!} \cdot \frac{1 - (\frac{A}{n})^n}{1 - \frac{A}{n}} \right)$ 不好算

$m > n > 2A$

$$\leq \frac{A^{n+1}}{(n+1)!} \left[1 + \frac{A}{2A} + \cdots + \left(\frac{A}{2A}\right)^{m-(n+1)} \right]$$

$$\leq \frac{A^{n+1}}{(n+1)!} \times 2$$

As $\lim_{n \to \infty} \left(\frac{A^n}{n!} \right) = 0$ so $\forall \varepsilon > 0$, $|E_m(x) - E_n(x)| < \varepsilon$

when $m > n > 2A$,

By Cauchy

$\Rightarrow (E_n(x))$ converges uniformly on $[-A, A]$

Recall that A is arbitrary,

so $(E_n(x))$ converges uniformly on $\mathbb{R}$, let $E := \lim(E_n)$

then $E_n \rightrightarrows E$ $\forall x \in \mathbb{R}$

Recall that E_n continuous on $\mathbb{R}$,

with $E_n \rightrightarrows E$, by 8.2.2 we get E is continuous on $\mathbb{R}$

so $\lim_{x \to 0} E(x) = E(0)$

As $\lim_{n \to \infty} E_n(0) = \lim_{n \to \infty} \lim_{x \to 0} E_n(x) = \lim_{x \to 0} \lim_{n \to \infty} E_n(x) = \lim_{x \to 0} E(x)$

and $\lim_{n \to \infty} [E_n(0)] = \lim_{n \to \infty} [1] = 1$

so $E(0) = 1$ proved (ii)

As $E_{n+1}'(x) = E_n(x)$ and $E_n \rightrightarrows E$ on $[-A, A]$

so $(E_n'(x))$ is uniform convergent.

"限定"在闭区间内方便套性质

By 8.2.3, the limit function E is differentiable on $[-A, A]$ and that

$E'(x) = \lim_{n \to \infty} (E_n'(x)) = \lim_{n \to \infty} (E_{n-1}(x)) = E(x)$ $\forall x \in [-A, A]$

Thus, $\forall x \in \mathbb{R}$, $E'(x) = E(x)$. (i) is proved.

Q.E.D.

that satisfies (i) and (ii)

8.3.2 Corollary *The function E has a derivative of every order and $E^{(n)}(x) = E(x)$ for all $n \in \mathbb{N}$, $x \in \mathbb{R}$.*

Proof. If $n = 1$, the statement is merely property (i). It follows for arbitrary $n \in \mathbb{N}$ by Induction. Q.E.D.

8.3.3 Corollary *If $x > 0$, then $1 + x < E(x)$.*

$E_1 < E_n \leq E \Rightarrow E_1 < E$

Proof. It is clear from (4) that if $x > 0$, then the sequence $(E_n(x))$ is strictly increasing. Hence $E_1(x) < E(x)$ for all $x > 0$. Q.E.D.

E的唯一性:

8.3.4 Theorem *The function $E : \mathbb{R} \to \mathbb{R}$ that satisfies* (i) *and* (ii) *of Theorem* 8.3.1 *is unique.*

Proof: Assume there're two functions

$E_1(x): R \to R$ and $E_2(x): R \to R$ satisfy (i) and (ii)

then Define $F(x) := E_1(x) - E_2(x)$

From (ii), $F(0) = E_1(0) - E_2(0) = 1 - 1 = 0$

From (i) $F'(x) = E_1'(x) - E_2'(x) = E_1 - E_2 = F$

By 8.3.2, $E_1^{(n)}$ and $E_2^{(n)}$ are exist on R

and $E_1^{(n)} = E_1$, $E_2^{(n)} = E_2$

$\Rightarrow F^n(x) = F(x) \quad \forall x \in R.$

[n阶可导, 又欲证 $F(x) = 0 \quad \forall x \in R$

联想 Taylor Theorem 重塑 $F(x)$]

用泰勒展式先规定区间

$\forall x \in R$, let I_x be a closed interval ends with 0 and x.

As F continuous on R so F continuous on I_x

$\Rightarrow F$ is bounded on I_x ($[0,x]$ or $[x,0]$)

i.e. $\exists k > 0$ s.t. $|F(t)| \le k \quad \forall t \in I_x$

On I_x: $\forall n \in N$, $\exists c_n \in I_x$ s.t.

$$\boxed{F(x) = F(0) + \frac{F'(0)}{1!}x + \cdots + \frac{F^{(n-1)}(0)}{(n-1)!}x^{n-1} + \frac{F^n(c_n)}{n!}x^n}$$

$$= 0 + 0 + \cdots + 0 + \frac{F^n(c_n)}{n!}x^n$$

$$\Rightarrow |F(x)| = \left|\frac{F^n(c_n)}{n!}x^n\right| \le k\left|\frac{x^n}{n!}\right| \quad \forall n \in N$$

As $\lim_{n\to\infty}\left|\frac{x^n}{n!}\right| = 0$

so $0 \le |F(x)| \to 0 \Rightarrow F(x) = 0$

i.e. $E_1(x) = E_2(x)$ on R.

Q.E.D.

Proof. Let E_1 and E_2 be two functions on $\mathbb{R}$ to $\mathbb{R}$ that satisfy properties (i) and (ii) of Theorem 8.3.1 and let $F := E_1 - E_2$. Then

$$F'(x) = E_1'(x) - E_2'(x) = E_1(x) - E_2(x) = F(x)$$

for all $x \in \mathbb{R}$ and

$$F(0) = E_1(0) - E_2(0) = 1 - 1 = 0.$$

It is clear (by Induction) that F has derivatives of all orders and indeed that $F^{(n)}(x) = F(x)$ for $n \in \mathbb{N}$, $x \in \mathbb{R}$.

Let $x \in \mathbb{R}$ be arbitrary, and let I_x be the closed interval with endpoints 0, x. Since F is continuous on I_x, there exists $K > 0$ such that $|F(t)| \le K$ for all $t \in I_x$. If we apply Taylor's Theorem 6.4.1 to F on the interval I_x and use the fact that $F^{(k)}(0) = F(0) = 0$ for all $k \in \mathbb{N}$, it follows that for each $n \in \mathbb{N}$ there is a point $c_n \in I_x$ such that

$$\begin{aligned} F(x) &= F(0) + \frac{F'(0)}{1!}x + \cdots + \frac{F^{(n-1)}}{(n-1)!}x^{n-1} + \frac{F^{(n)}(c_n)}{n!}x^n \\ &= \frac{F(c_n)}{n!}x^n. \end{aligned}$$

Therefore we have

$$|F(x)| \le \frac{K|x|^n}{n!} \qquad \text{for all} \quad n \in \mathbb{N}.$$

But since $\lim(|x|/n!) = 0$, we conclude that $F(x) = 0$. Since $x \in \mathbb{R}$ is arbitrary, we infer that $E_1(x) - E_2(x) = F(x) = 0$ for all $x \in \mathbb{R}$. Q.E.D.

8.3.5 Definition The unique function $E : \mathbb{R} \to \mathbb{R}$, such that $E'(x) = E(x)$ for all $x \in \mathbb{R}$ and $E(0) = 1$, is called the **exponential function**. The number $e := E(1)$ is called **Euler's number**. We will frequently write

$$\exp(x) := E(x) \qquad \text{or} \qquad e^x := E(x) \qquad \text{for} \quad x \in \mathbb{R}.$$

□ $\boxed{E(0)=1}$ 的必要性：

When $u(x)=0 \Rightarrow u^{(n)}(x)=u(x)$

但 $u(0)\neq 1 \Rightarrow u(x)\neq E(x)=e^x$

8.3.6 Theorem *The exponential function satisfies the following properties:*

(iii) $E(x) \neq 0$ *for all* $x \in \mathbb{R}$;

(iv) $E(x+y) = E(x)E(y)$ *for all* $x, y \in \mathbb{R}$;

(v) $E(r) = e^r$ *for all* $r \in \mathbb{Q}$.

(iii) Proof: Assume that $\exists x_0 \in R$ s.t. $E(x_0)=0$

Let J_{x_0} be a closed interval with end points 0 and x_0

then $E(x)$ bounded on J_{x_0}

i.e. $\exists k>0$ s.t. $\forall t \in J_{x_0}$, $|E(t)| \le k$

As $E^{(n)}(x) = E(x)$ $\forall x \in J_{x_0}$ so

by Taylor's Theorem, $\forall n \in N$, $\exists C_n \in J_{x_0}$ s.t.

$$E(0) = E(x_0) + \frac{E^{(1)}(x_0)}{1!}(0-x_0) + \cdots + \frac{E^{(n-1)}(x_0)}{(n-1)!}(0-x_0)^{n-1} + \frac{E^{(n)}(C_n)}{n!}(0-x_0)^n$$

已知

$$= 0 + \cdots + 0 + \frac{E^{(n)}(C_n)}{n!}(-x_0)^n$$

$$= 1$$

$$\Rightarrow 0 < 1 = \left|\frac{E^{(n)}(C_n)}{n!}(-x_0)^n\right| \le \left|\frac{(-x_0)^n}{n!}\right| k = \frac{k}{n!}|x_0|^n$$

and $\lim \frac{|x_0|^n}{n!} = 0$

So $0 < 1 \le 0$. Contradiction.

Q.E.D.

Proof. (iii) Let $\alpha \in \mathbb{R}$ be such that $E(\alpha) = 0$, and let J_α be the closed interval with endpoints $0, \alpha$. Let $K \ge |E(t)|$ for all $t \in J_\alpha$. Taylor's Theorem 6.4.1 implies that for each $n \in \mathbb{N}$ there exists a point $c_n \in J_\alpha$ such that

$$1 = E(0) = E(\alpha) + \frac{E'(\alpha)}{1!}\overset{(0-\alpha)}{(-\alpha)} + \cdots + \frac{E^{(n-1)}(\alpha)}{(n-1)!}(-\alpha)^{n-1} + \frac{E^{(n)}(\alpha)}{(n)!}(-\alpha)^n = \frac{E(c_n)}{n!}(-\alpha)^n.$$

Thus we have $0 < 1 \le (K/n!)|\alpha|^n$ for $n \in \mathbb{N}$. But since $\lim(|\alpha|^n/n!) = 0$, this is a contradiction.

(iv) $E(x+y) = E(x)E(y)$ *for all* $x, y \in \mathbb{R}$;

Proof. Let y be fixed. by (iii), $E(y) \neq 0$

then define $G: \mathbb{R} \to \mathbb{R}$ by

$$\boxed{G(x) := \frac{E(x+y)}{E(y)} \quad \forall x \in \mathbb{R}}$$

$$\Rightarrow G'(x) = \frac{E'(x+y)}{E(y)} \quad [\,E(y) \text{ is a constant here}\,]$$

$$= \frac{E(x+y)}{E(y)}$$

$$= G(x) \quad \forall x \in \mathbb{R}$$

Also, $G(0) = \dfrac{E(0+y)}{E(y)} = 1$

So by the uniqness of $E(x)$, $G(x) = E(x)$ $\forall x \in \mathbb{R}$.

Thus, $E(x+y) = E(x) \cdot E(y)$ $\forall x \in \mathbb{R}$.

Q.E.D.

(iv) Let y be fixed; by (iii) we have $E(y) \neq 0$. Let $G : \mathbb{R} \to \mathbb{R}$ be defined by

$$G(x) := \frac{E(x+y)}{E(y)} \quad \text{for} \quad x \in \mathbb{R}.$$

Evidently we have $G'(x) = E'(x+y)/E(y) = E(x+y)/E(y) = G(x)$ for all $x \in \mathbb{R}$, and $G(0) = E(0+y)/E(y) = 1$. It follows from the uniqueness of E, proved in Theorem 8.3.4, that $G(x) = E(x)$ for all $x \in \mathbb{R}$. Hence $E(x+y) = E(x)E(y)$ for all $x \in \mathbb{R}$. Since $y \in \mathbb{R}$ is arbitrary, we obtain (iv).

(v) $E(r) = e^r$ *for all* $r \in \mathbb{Q}$.

Proof. $\forall r \in \mathbb{Q}$, $r = \frac{m}{n}$, $n \in \mathbb{Z} \setminus \{0\}$, $m \in \mathbb{N}$.

$$e = E(1) = E(n \cdot \tfrac{1}{n}) = E(\tfrac{1}{n} + \tfrac{1}{n} + \cdots + \tfrac{1}{n}) = \left[E(\tfrac{1}{n})\right]^n$$

$$\Rightarrow E(\tfrac{1}{n}) = e^{\frac{1}{n}}$$

$$E(m) = E(m \times 1) = E(1+1+\cdots+1) = \left[E(1)\right]^m = e^m$$

Thus, $E(\frac{m}{n}) = E(\frac{1}{n} + \frac{1}{n} + \cdots + \frac{1}{n}) = \left[E(\frac{1}{n})\right]^m = e^{\frac{m}{n}}$

Q.E.D.

(v) It follows from (iv) and Induction that if $n \in \mathbb{N}$, $x \in \mathbb{R}$, then

$$E(nx) = E(x)^n.$$

If we let $x = 1/n$, this relation implies that

$$e = E(1) = E\left(n \cdot \frac{1}{n}\right) = \left(E\left(\frac{1}{n}\right)\right)^n,$$

whence it follows that $E(1/n) = e^{1/n}$. Also we have $E(-m) = 1/E(m) = 1/e^m = e^{-m}$ for $m \in \mathbb{N}$. Therefore, if $m \in \mathbb{Z}$, $n \in \mathbb{N}$, we have

$$E(m/n) = (E(1/n))^m = (e^{1/n})^m = e^{m/n}.$$

This establishes (v). Q.E.D.

8.3.7 Theorem *The exponential function E is strictly increasing on $\mathbb{R}$ and has range equal to $\{y \in \mathbb{R} : y > 0\}$. Further, we have*

(vi) $\lim_{x \to -\infty} E(x) = 0$ and $\lim_{x \to \infty} E(x) = \infty$.

Proof. As $E(0) = 1 > 0$ and $E(x) \neq 0$ $\forall x \in R$.

As Proved, E continuous on R

By Bolzano's Intermediate Value Theorem,

$E(x) > 0$ $\forall x \in R$

$\Rightarrow$ $E'(x) = E(x) > 0$ $\forall x \in R$

$E(x)$ monotone increases on R

(vi) As $E(x) > 1 + x$ so $\lim_{x \to \infty} E(x) \geq \lim_{x \to \infty} (1 + x) = \infty$

(vii) $\forall z > 0$, $E(-z) = \frac{1}{E(z)}$

So $\lim_{x \to \infty} E(x) = \frac{1}{\lim_{x \to \infty} E(x)} = 0$

Thus, by intermediate Value Theorem,

$\forall y \in R$ with $y > 0$, $y \in \text{Range}(E(x))$

Q.E.D.

The Logarithm Function

We have seen that the exponential function E is a strictly increasing differentiable function with domain $\mathbb{R}$ and range $\{y \in \mathbb{R} : y > 0\}$. (See Figure 8.3.1.) It follows that $\mathbb{R}$ has an inverse function.

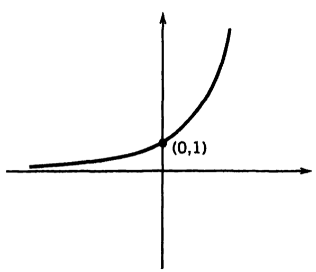


Figure 8.3.1 Graph of E

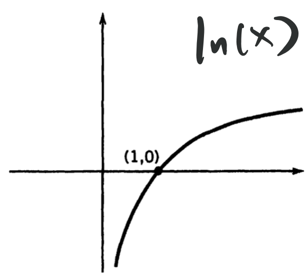


Figure 8.3.2 Graph of L

8.3.8 Definition The function inverse to $E : \mathbb{R} \to \mathbb{R}$ is called the **logarithm** (or the **natural logarithm**). (See Figure 8.3.2.) It will be denoted by L, or by ln.

Since E and L are inverse functions, we have

$$(L \circ E)(x) = x \qquad \text{for all} \quad x \in \mathbb{R}$$

and

$$(E \circ L)(y) = y \qquad \text{for all} \quad y \in \mathbb{R},\ y > 0.$$

These formulas may also be written in the form

$$\ln e^x = x, \qquad e^{\ln y} = y.$$

8.3.9 Theorem *The logarithm is a strictly increasing function L with domain $\{x \in \mathbb{R} : x > 0\}$ and range $\mathbb{R}$. The derivative of L is given by*

(vii) $L'(x) = 1/x \quad for \quad x > 0.$

The logarithm satisfies the functional equation

(viii) $L(xy) = L(x) + L(y)$ *for* $x > 0, y > 0.$

Moreover, we have

(ix) $L(1) = 0$ *and* $L(e) = 1,$

(x) $L(x^r) = rL(x)$ *for* $x > 0, r \in \mathbb{Q}.$

(xi) $\lim\limits_{x \to 0+} L(x) = -\infty$ *and* $\lim\limits_{x \to \infty} L(x) = \infty.$

Proof. That L is strictly increasing with domain $\{x \in \mathbb{R} : x > 0\}$ and range $\mathbb{R}$ follows from the fact that E is strictly increasing with domain $\mathbb{R}$ and range $\{y \in \mathbb{R} : y > 0\}$.

(vii) Since $E'(x) = E(x) > 0$, it follows from Theorem 6.1.9 that L is differentiable on $(0, \infty)$ and that

$$\boxed{L'(x) = \frac{1}{(E' \circ L)(x)} = \frac{1}{(E \circ L)(x)} = \frac{1}{x} \qquad \text{for} \quad x \in (0, \infty).}$$

$$(f^{-1})' = \frac{1}{f'}$$

(viii) If $x > 0, y > 0$, let $u := L(x)$ and $v := L(y)$. Then we have $x = E(u)$ and $y = E(v)$. It follows from property (iv) of Theorem 8.3.6 that

$$xy = E(u)E(v) = E(u + v),$$

so that $L(xy) = (L \circ E)(u + v) = u + v = L(x) + L(y)$. This establishes (viii).

The properties in (ix) follow from the relations $E(0) = 1$ and $E(1) = e$.

(x) This result follows from (viii) and Mathematical Induction for $n \in \mathbb{N}$, and is extended to $r \in \mathbb{Q}$ by arguments similar to those in the proof of 8.3.6(v).

To establish property (xi), we first note that since $2 < e$, then $\lim(e^n) = \infty$ and $\lim(e^{-n}) = 0$. Since $L(e^n) = n$ and $L(e^{-n}) = -n$ it follows from the fact that L is strictly increasing that

$$\lim_{x \to \infty} L(x) = \lim L(e^n) = \infty \quad \text{and} \quad \lim_{x \to 0+} L(x) = \lim L(e^{-n}) = -\infty. \qquad \text{Q.E.D.}$$

Power Functions

arbitrary real powers

8.3.10 Definition If $\alpha \in \mathbb{R}$ and $x > 0$, the number x^α is defined to be

$$x^\alpha := \boxed{e^{\alpha \ln x}} = E(\alpha L(x)).$$

The function $x \mapsto x^\alpha$ for $x > 0$ is called the **power function** with exponent α.

We now state some properties of the power functions. Their proofs are immediate consequences of the properties of the exponential and logarithm functions and will be left to the reader.

8.3.11 Theorem *If $\alpha \in \mathbb{R}$ and x, y belong to $(0, \infty)$, then:*

(a) $1^\alpha = 1$, **(b)** $x^\alpha > 0$,
(c) $(xy)^\alpha = x^\alpha y^\alpha$; **(d)** $(x/y)^\alpha = x^\alpha / y^\alpha$.

套 $x^\alpha = e^{\ln x^\alpha}$

8.3.12 Theorem *If $\alpha, \beta \in \mathbb{R}$ and $x \in (0, \infty)$, then:*

(a) $x^{\alpha+\beta} = x^\alpha x^\beta$ **(b)** $(x^\alpha)^\beta = x^{\alpha\beta} = (x^\beta)^\alpha$;
(c) $x^{-\alpha} = 1/x^\alpha$; **(d)** *if $\alpha < \beta$, then $x^\alpha < x^\beta$ for $x > 1$.*

8.3.13 Theorem *Let $\alpha \in \mathbb{R}$. Then the function $x \mapsto x^\alpha$ on $(0, \infty)$ to $\mathbb{R}$ is continuous and differentiable, and*

$$Dx^\alpha = \alpha x^{\alpha-1} \qquad \text{for} \quad x \in (0, \infty).$$

Proof. By the Chain Rule we have

$$\begin{aligned} Dx^\alpha &= De^{\alpha \ln x} = e^{\alpha \ln x} \cdot D(\alpha \ln x) \\ &= x^\alpha \cdot \frac{\alpha}{x} = \alpha x^{\alpha-1} \qquad \text{for} \quad x \in (0, \infty). \end{aligned}$$

Q.E.D.

The Function $\log_a$

If $a > 0$, $a \neq 1$, it is sometimes useful to define the function $\log_a$.

8.3.14 Definition Let $a > 0$, $a \neq 1$. We define

$$\log_a(x) := \frac{\ln x}{\ln a} \qquad \text{for} \quad x \in (0, \infty).$$

For $x \in (0, \infty)$, the number $\log_a(x)$ is called the **logarithm of x to the base** a. The case $a = e$ yields the logarithm (or natural logarithm) function of Definition 8.3.8. The case $a = 10$ gives the base 10 logarithm (or common logarithm) function $\log_{10}$ often used in computations. Properties of the functions $\log_a$ will be given in the exercises.

Section 8.4 The Trigonometric Functions

8.4.1 Theorem *There exist functions $C : \mathbb{R} \to \mathbb{R}$ and $S : \mathbb{R} \to \mathbb{R}$ such that*

(i) $C''(x) = -C(x)$ *and* $S''(x) = -S(x)$ *for all* $x \in \mathbb{R}$.

(ii) $C(0) = 1, C'(0) = 0,$ *and* $S(0) = 0, S'(0) = 1$.

Proof: Define $C_n(x)$ and $S_n(x)$ inductively:

(ii) $\Rightarrow$ $C_1(x)=1$, $\quad S_1(x)=x$

As C_1 and S_1 are continuous on $\mathbb{R}$ so they are integrable on bounded interval $[0,x]$, $\forall x>0$,

(i) $\Rightarrow$ define

$$\boxed{\begin{aligned} S_n(x) &= \int_0^x C_n(t)\,dt \\ C_{n+1}(x) &= 1-\int_0^x S_n(t)\,dt \qquad \forall n\in\mathbb{N},\ x\in\mathbb{R} \end{aligned}}$$

By Fundamental Theorem of Calculus,

1) S_n, C_{n+1} are differentiable $\forall n\in\mathbb{N}$, $x\in\mathbb{R}$.

2) $S_n'(x)=C_n(x)$, $\quad C_{n+1}'(x)=-S_n(x) \qquad (1)$

By induction:

$$\boxed{\begin{aligned} C_{n+1}(x) &= 1-\frac{1}{2}x^2+\frac{1}{4!}x^4-\frac{1}{6!}x^6+\cdots+(-1)^n\frac{1}{(2n)!}x^{2n} \\ S_{n+1}(x) &= x-\frac{1}{3!}x^3+\frac{1}{5!}x^5-\frac{1}{7!}x^7+\cdots+(-1)^n\frac{1}{(2n+1)!}x^{2n+1} \end{aligned}}$$

$\forall x\in[-A,A]$, $\forall A>0$, let $m,n\in\mathbb{N}$ and $m>n>2A$

then

$$\begin{aligned} |C_m(x)-C_n(x)| &= |x_{n+1}+\cdots+x_m| \\ &= \left|(-1)^n\frac{1}{(2n)!}x^{2n}+\cdots+(-1)^{m-1}\frac{1}{(2m-2)!}x^{2m-2}\right| \\ &\le \left|\frac{1}{(2n)!}x^{2n}\right|+\cdots+\left|\frac{1}{(2(m-1))!}x^{2(m-1)}\right| \\ &= \frac{|x|^{2n}}{(2n)!}\left[1+\frac{1}{2n+1}x+\frac{1}{(2n+2)(2n+1)}x^2+\cdots+\frac{(2n)!}{(2m-2)!}x^{2m-2-2n}\right] \\ &\le \frac{A^{2n}}{(2n)!}\left[1+\frac{A}{2n+1}+\frac{A^2}{(2n+2)(2n+1)}+\cdots+\frac{(2n)!}{(2m-2)!}A^{2(m-n-1)}\right] \\ &< \frac{A^{2n}}{(2n)!}\left[1+\frac{A}{2n}+\frac{A^2}{(2n)^2}+\cdots+\frac{A^{2(m-n-1)}}{(2n)^{2(m-n-1)}}\right] \\ &\le \frac{A^{2n}}{(2n)!}\left[1+\frac{1}{4}+\left(\frac{1}{4}\right)^2+\cdots+\left(\frac{1}{4}\right)^{2(m-n-1)}\right] \\ &< \frac{A^{2n}}{(2n)!}\cdot\cancel{\frac{3}{2}}\ ②\ \frac{16}{15} \end{aligned}$$

$m>n>2A$

$\Rightarrow 4A<2n$

$\frac{A}{2n}<\frac{1}{4}$

As $\lim_{n\to\infty}\frac{A^{2n}}{(2n)!}=0$ so by Cauchy Criterion,

$(C_n(x))$ is a uniform convergent series. on $[-A,A]$

$\Rightarrow$ $(C_n(x))$ uniformly convergent on $\mathbb{R}$.

Define function $C:\mathbb{R}\to\mathbb{R}$ by

$$C(x):=\lim_{n\to\infty}C_n(x), \quad \forall x\in\mathbb{R}$$

then $C_n \rightrightarrows C$

As $C_n(x)$ continuous on $\mathbb{R}$ $\forall n\in\mathbb{N}$

so $C(x)$ continuous on $\mathbb{R}$

$$\Rightarrow C(0)=\lim_{x\to0}C(x)=\lim_{x\to0}\lim_{n\to\infty}C_n(x)=\lim_{n\to\infty}\lim_{x\to0}C_n(x)=\lim_{n\to\infty}(1)=1$$

$$\Rightarrow C(0)=1$$

$\forall x\in[-A,A]$, $\forall A>0$, $m,n\in\mathbb{N}$ and $m\geq n\geq 2A$

$$\begin{aligned} |S_m(x)-S_n(x)| &= \left|\int_0^x C_m(t)\,dt-\int_0^x C_n(t)\,dt\right| \\ &= \left|\int_0^x C_m(t)-C_n(t)\,dt\right| \\ &\le \int_0^x |C_m(t)-C_n(t)|\,dt \\ &< \int_0^x \frac{A^{2n}}{(2n)!}\cdot\cancel{\frac{3}{2}}\,\frac{16}{15}\,dt \\ &\le \frac{A^{2n}}{(2n)!}\cdot\frac{3}{2}\cdot|x| \\ &\le \frac{A^{2n}}{(2n)!}\cdot\cancel{\frac{3}{2}}\,\frac{16}{15}\cdot A \end{aligned}$$

不用再来一遍、

As $\lim_{n\to\infty}\frac{A^{2n+1}}{(2n)!}=0$ so $(S_n(x))$ is uniformly convergent. on $\mathbb{R}$ by $\forall A>0$

Define $S(x)$ on $\mathbb{R}\to\mathbb{R}$:

$$S(x):=\lim_{n\to\infty}S_n(x)$$

As $S_n(x)$ continuous on $\mathbb{R}$, $\forall n\in\mathbb{N}$ so $S(x)$ continuous on $\mathbb{R}$

$$\Rightarrow S(0)=\lim_{x\to0}S(x)=\lim_{x\to0}\lim_{n\to\infty}S_n(x)=\lim_{n\to\infty}\lim_{x\to0}S_n(x)=0$$

$$\Rightarrow S(0)=0$$

From (1), $C_n'(x)=-S_{n-1}(x)$ $\quad \forall n\in\mathbb{N}\setminus\{1\}$

so (C_n') uniformly convergent on $[-A,A]$

$\Rightarrow$ $\lim(C_n)$ is differentiable on $[-A,A]$

and $C'=\lim(C_n')$

$$\Rightarrow C'(x)=\lim(C_n'(x))=\lim(-S_{n-1}(x))=-S(x)$$

$$\Rightarrow C'(x)=-S(x)$$

Similarly, From $S_n'(x)=C_n(x)$, (S_n') is uniformly convergent, then S is differentiable on $\mathbb{R}$

and $\quad S'=\lim(S_n')=\lim(C_n)=C$

$\Rightarrow S'(x)=C(x) \qquad \forall x\in\mathbb{R}$

Thus, $\quad S'(0)=C(0)=1$, $\quad C'(0)=-S(0)=0$

and $\quad S''(x)=C'(x)=-S(x)$

$\quad C''(x)=S'(x)=C(x) \qquad \forall x\in\mathbb{R}$

Q.E.D.

Proof. We define the sequences (C_n) and (S_n) of continuous functions inductively as follows:

$$(1)\qquad C_1(x) := 1, \qquad S_1(x) := x,$$

$$(2)\qquad S_n(x) := \int_0^x C_n(t)\,dt,$$

$$(3)\qquad C_{n+1}(x) := 1 - \int_0^x S_n(t)\,dt,$$

for all $n \in \mathbb{N}$, $x \in \mathbb{R}$.

One sees by Induction that the functions C_n and S_n are continuous on $\mathbb{R}$ and hence they are integrable over any bounded interval; thus these functions are well-defined by the above formulas. Moreover, it follows from the Fundamental Theorem 7.3.5 that S_n and C_{n+1} are differentiable at every point and that

$$(4)\qquad S_n'(x) = C_n(x) \quad \text{and} \quad C_{n+1}'(x) = -S_n(x) \quad \text{for} \quad n \in \mathbb{N}, x \in \mathbb{R}.$$

Induction arguments (which we leave to the reader) show that

$$C_{n+1}(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!},$$

$$S_{n+1}(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!}.$$

Let $A > 0$ be given. Then if $|x| \le A$ and $m > n > 2A$, we have that (since $A/2n < 1/4$):

$$(5)\qquad \begin{aligned} |C_m(x) - C_n(x)| &= \left| \frac{x^{2n}}{(2n)!} - \frac{x^{2n+2}}{(2n+2)!} + \cdots \pm \frac{x^{2m-2}}{(2m-2)!} \right| \\ &\le \frac{A^{2n}}{(2n)!}\left[1 + \left(\frac{A}{2n}\right)^2 + \cdots + \left(\frac{A}{2n}\right)^{2m-2n-2}\right] \\ &< \frac{A^{2n}}{(2n)!}\left(\frac{16}{15}\right). \end{aligned}$$

Since $\lim(A^{2n}/(2n)!) = 0$, the sequence (C_n) converges uniformly on the interval $[-A, A]$, where $A > 0$ is arbitrary. In particular, this means that $(C_n(x))$ converges for each $x \in \mathbb{R}$. We define $C : \mathbb{R} \to \mathbb{R}$ by

$$C(x) := \lim C_n(x) \quad \text{for} \quad x \in \mathbb{R}.$$

It follows from Theorem 8.2.2 that C is continuous on $\mathbb{R}$ and, since $C_n(0) = 1$ for all $n \in \mathbb{N}$, that $C(0) = 1$.

If $|x| \le A$ and $m \ge n > 2A$, it follows from (2) that

$$S_m(x) - S_n(x) = \int_0^x \{C_m(t) - C_n(t)\}\,dt.$$

If we use (5) and Corollary 7.3.15, we conclude that

$$|S_m(x) - S_n(x)| \le \frac{A^{2n}}{(2n)!}\left(\frac{16}{15}A\right),$$

whence the sequence (S_n) converges uniformly on $[-A, A]$. We define $S : \mathbb{R} \to \mathbb{R}$ by

$$S(x) := \lim S_n(x) \quad \text{for} \quad x \in \mathbb{R}.$$

It follows from Theorem 8.2.2 that S is continuous on $\mathbb{R}$ and, since $S_n(0) = 0$ for all $n \in \mathbb{N}$, that $S(0) = 0$.

Since $C_n'(x) = -S_{n-1}(x)$ for $n > 1$, it follows from the above that the sequence (C_n') converges uniformly on $[-A, A]$. Hence by Theorem 8.2.3, the limit function C is differentiable on $[-A, A]$ and

$$C'(x) = \lim C_n'(x) = \lim(-S_{n-1}(x)) = -S(x) \quad \text{for} \quad x \in [-A, A].$$

Since $A > 0$ is arbitrary, we have

$$(6)\qquad C'(x) = -S(x) \quad \text{for} \quad x \in \mathbb{R}.$$

A similar argument, based on the fact that $S_n'(x) = C_n(x)$, shows that S is differentiable on $\mathbb{R}$ and that

$$(7)\qquad S'(x) = C(x) \quad \text{for all} \quad x \in \mathbb{R}.$$

It follows from (6) and (7) that

$$C''(x) = -(S(x))' = -C(x) \quad \text{and} \quad S''(x) = (C(x))' = -S(x)$$

for all $x \in \mathbb{R}$. Moreover, we have

$$C'(0) = -S(0) = 0, \qquad S'(0) = C(0) = 1.$$

Thus statements (i) and (ii) are proved. Q.E.D.

8.4.2 Corollary *If C, S are the functions in Theorem 8.4.1, then*

(iii) $C'(x) = -S(x)$ *and* $S'(x) = C(x)$ *for* $x \in \mathbb{R}$.

Moreover, these functions have derivatives of all orders.

Proof. The formulas (iii) were established in (6) and (7). The existence of the higher order derivatives follows by Induction. Q.E.D.

8.4.3 Corollary *The functions C and S satisfy the Pythagorean Identity:*

(iv) $(C(x))^2 + (S(x))^2 = 1$ for $x \in \mathbb{R}$.

Proof. Let $f(x) := (C(x))^2 + (S(x))^2$ for $x \in \mathbb{R}$, so that

$$f'(x) = 2C(x)(-S(x)) + 2S(x)(C(x)) = 0 \qquad \text{for} \quad x \in \mathbb{R}.$$

Thus it follows that $f(x)$ is a constant for all $x \in \mathbb{R}$. But since $f(0) = 1 + 0 = 1$, we conclude that $f(x) = 1$ for all $x \in \mathbb{R}$. Q.E.D.

We next establish the uniqueness of the functions C and S.

8.4.4 Theorem *The functions C and S satisfying properties* (i) *and* (ii) *of Theorem* 8.4.1 *are unique.*

Proof. Assume that there're C and $\tilde{C}$, S and $\tilde{S}$ that satisfy (i) and (ii)

then define $F_1(x) = C - \tilde{C}$, $F_2(x) = S - \tilde{S}$

then F_1 and F_2 are differentiable on R

$F_1(0) = C(0) - \tilde{C}(0) = 1 - 1 = 0$

$F_2(0) = S(0) - \tilde{S}(0) = 0 - 0 = 0$

$F_1'(x) = C'(x) - \tilde{C}'(x) = -S(x) + \tilde{S}(x) = -F_2$

$F_2'(x) = S'(x) - \tilde{S}'(x) = C(x) - \tilde{C}(x) = F_1$

$\Rightarrow \begin{cases} F_1' = -F_2 \\ F_2' = F_1 \end{cases}$ (1) $\begin{cases} F_1^{(k)}(x) = \pm F_i, \ i = 1,2 \\ F_2^{(k)}(x) = \pm F_i, \ i = 1,2 \end{cases}$

$F_1''(x) = (-F_2)' = -F_1$, $F_2''(x) = F_1' = -F_2$

$\Rightarrow \begin{cases} F_1'' = -F_1 \\ F_2'' = -F_2 \end{cases}$ (2) $\Rightarrow \cdots \Rightarrow \begin{cases} F_1^{(k)}(0) = 0 \\ F_2^{(k)}(0) = 0 \end{cases}$

By Induction, $F_i^{(k)}(x)$ exists on R $\forall k \in N$, $i = 1, 2$.

By Induction, $F_i^{(k)}(x)$ exists on R $\forall k \in N$, $i = 1, 2$.

Thus, $\forall x \in R$, define closed interval I_x with endpoints $0, x$.

As F_1, F_2 continuous on I_x then $\exists M > 0$ s.t.

$|F_1(t)| \le M$, $|F_2(t)| \le M$ $\forall t \in I_x$

By Taylor's Theorem: $\forall n \in N$, $\exists C_n \in I_x$ s.t.

$$F_1(x) = F_1(0) + \frac{F_1'(0)}{1!}x + \frac{F_1''(0)}{2!}x^2 + \cdots + \frac{F_1^{(n-1)}(0)}{(n-1)!}x^{n-1} + \frac{F_1^{(n)}(C_n)}{n!}x^n$$

$$= \frac{F_1^{(n)}(C_n)}{n!}x^n$$

$$\boxed{= \frac{\pm F_1(C_n)}{n!}x^n}$$

$$\le \frac{M}{n!}|x|^n$$

As $\lim_{n \to \infty}\left(\frac{|x|^n}{n!}\right) = 0$ so $F_1(t) = 0$ $\forall t \in I_x$

As x is arbitrary, so $F_1(x) = 0$ $\forall x \in R$,

$\Rightarrow C(x) = \tilde{C}(x)$ $\forall x \in R$

Similarly,

$$F_2(x) = F_2(0) + \frac{F_2'(0)}{1!}x + \frac{F_2''(0)}{2!}x^2 + \cdots + \frac{F_2^{(n-1)}(0)}{(n-1)!}x^{n-1} + \frac{F_2^{(n)}(C_n)}{n!}x^n$$

$$= \frac{F_2^{(n)}(C_n)}{n!}x^n$$

$$= \frac{\pm F_2(C_n)}{n!}x^n$$

$$\le M \cdot \frac{|x|^n}{n!} \to 0$$

Thus, $F_2 = 0 \Rightarrow S(x) = \tilde{S}(x)$, $\forall x \in R$. Q.E.D.

Proof. Let C_1 and C_2 be two functions on $\mathbb{R}$ to $\mathbb{R}$ that satisfy $C_j''(x) = -C_j(x)$ for all $x \in \mathbb{R}$ and $C_j(0) = 1$, $C_j'(0) = 0$ for $j = 1, 2$. If we let $D := C_1 - C_2$, then $D''(x) = -D(x)$ for $x \in \mathbb{R}$ and $D(0) = 0$ and $D^{(k)}(0) = 0$ for all $k \in \mathbb{N}$.

Now let $x \in \mathbb{R}$ be arbitrary, and let I_x be the interval with endpoints $0, x$. Since $D = C_1 - C_2$ and $T := S_1 - S_2 = C_2' - C_1'$ are continuous on I_x, there exists $K > 0$ such that $|D(t)| \le K$ and $|T(t)| \le K$ for all $t \in I_x$. If we apply Taylor's Theorem 6.4.1 to D on I_x and use the fact that $D(0) = 0$, $D^{(k)}(0) = 0$ for $k \in \mathbb{N}$, it follows that for each $n \in \mathbb{N}$ there is a point $c_n \in I_x$ such that

$$D(x) = D(0) + \frac{D'(0)}{1!}x + \cdots + \frac{D^{(n-1)}(0)}{(n-1)!}x^{n-1} + \frac{D^{(n)}(c_n)}{n!}x^n$$
$$= \frac{D^{(n)}(c_n)}{n!}x^n.$$

Now either $D^{(n)}(c_n) = \pm D(c_n)$ or $D^{(n)}(c_n) = \pm T(c_n)$. In either case we have

$$|D(x)| \le \frac{K|x|^n}{n!}.$$

But since $\lim (|x|^n/n!) = 0$, we conclude that $D(x) = 0$. Since $x \in \mathbb{R}$ is arbitrary, we infer that $C_1(x) - C_2(x) = 0$ for all $x \in \mathbb{R}$.

A similar argument shows that if S_1 and S_2 are two functions on $\mathbb{R} \to \mathbb{R}$ such that $S_j''(x) = -S_j(x)$ for all $x \in \mathbb{R}$ and $S_j(0) = 0$, $S_j'(0) = 1$ for $j = 1, 2$, then we have $S_1(x) = S_2(x)$ for all $x \in \mathbb{R}$. Q.E.D.

8.4.6 Theorem *If $f : \mathbb{R} \to \mathbb{R}$ is such that*

$$f''(x) = -f(x) \qquad \textit{for} \quad x \in \mathbb{R},$$

then there exist real numbers α, β such that

存在性证明

$$f(x) = \alpha C(x) + \beta S(x) \quad \textit{for} \quad x \in \mathbb{R}.$$

Prof. Let $\boxed{g(x) := f(0)C(x) + f'(0)S(x) \quad \forall x \in R.}$

$$\Rightarrow g''(x) = (f(0)C'(x) + f'(0)S'(x))'$$
$$= f(0)C''(x) + f'(0)S''(x)$$
$$= f(0)(-C(x)) + f'(0)(-S(x))$$
$$= -g(x) \qquad \forall x \in R$$

and $g(0) = f(0)$, $g'(0) = f'(0)$

Therefore, let $h := f - g$ then

$$h''(x) = f'' - g'' = -f + g = g - f = -h \quad \forall x \in R.$$

and $h(0) = 0$, $h'(0) = 0$

Thus, $h(x) = 0 \quad \forall x \in R$

$\Rightarrow f(x) = g(x). \qquad \forall x \in R.$

$\alpha := f(0) \qquad \beta = f'(0).$

Q.E.D.

Proof. Let $g(x) := f(0)C(x) + f'(0)S(x)$ for $x \in \mathbb{R}$. It is readily seen that $g''(x) = -g(x)$ and that $g(0) = f(0)$, and since

$$g'(x) = -f(0)S(x) + f'(0)C(x),$$

that $g'(0) = f'(0)$. Therefore the function $h := f - g$ is such that $h''(x) = -h(x)$ for all $x \in \mathbb{R}$ and $h(0) = 0$, $h'(0) = 0$. Thus it follows from the proof of the preceding theorem that $h(x) = 0$ for all $x \in \mathbb{R}$. Therefore $f(x) = g(x)$ for all $x \in \mathbb{R}$. Q.E.D.

8.4.7 Theorem *The function C is even and S is odd in the sense that*

(v) $C(-x) = C(x)$ *and* $S(-x) = -S(x)$ *for* $x \in \mathbb{R}$.

If $x, y \in \mathbb{R}$, then we have the "addition formulas"

(vi) $C(x + y) = C(x)C(y) - S(x)S(y), \quad S(x + y) = S(x)C(y) + C(x)S(y).$

(v) Proof: Let $\varphi(x) := C(-x)$ $\forall x \in R$ then

$\varphi''(x) = -C(-x) = -\varphi(x)$ $\forall x \in R$.

Moreover, $\varphi(0) = 1$, $\varphi'(0) = 0$

$\Rightarrow$ $\varphi(x) = C(x) = \cos(x)$

$\Rightarrow$ $C(-x) = C(x)$.

Let $\gamma(x) := S(-x)$ then $\gamma''(x) = -S(-x) = -\gamma(x)$

$\Rightarrow$ $\gamma(0) = S(0) = 0$, $\gamma'(0) = C(0) = 1$

$\Rightarrow$ $\gamma(x) = S(x)$

$\Rightarrow$ $S(-x) = S(x)$ $\forall x \in R$.

Q.E.D.

(vi). As $C''(x) = -C(x)$, $S''(x) = -S(x)$

So $\exists \alpha, \beta \in R$ s.t.

$f(x) = C(x+y) = \alpha C(x) + \beta S(x)$ (1)

and

$f'(x) = -S(x+y) = -\alpha S(x) + \beta C(x)$ $\forall x \in R$

Let $x = 0$,

then $C(y) = \alpha$, $-S(y) = \beta \Rightarrow S(y) = -\beta$

now (1) becomes: $C(x+y) = C(y)C(x) - S(y)S(x)$.

Let $g(x) := S(x+y) = \alpha' S(x) + \beta' C(x)$

then $g'(x) = C(x+y) = \alpha' C(x) - \beta' S(x)$

Let $X = 0$ then

$S(y) = \beta'$, $C(y) = \alpha'$

$\Rightarrow$ $S(x+y) = C(y)S(x) + S(y)C(x)$

Q.E.D.

Proof. (v) If $\varphi(x) := C(-x)$ for $x \in \mathbb{R}$, then a calculation shows that $\varphi''(x) = -\varphi(x)$ for $x \in \mathbb{R}$. Moreover, $\varphi(0) = 1$ and $\varphi'(0) = 0$ so that $\varphi = C$. Hence, $C(-x) = C(x)$ for all $x \in \mathbb{R}$. In a similar way one shows that $S(-x) = -S(x)$ for all $x \in \mathbb{R}$.

(vi) Let $y \in \mathbb{R}$ be given and let $f(x) := C(x + y)$ for $x \in \mathbb{R}$. A calculation shows that $f''(x) = -f(x)$ for $x \in \mathbb{R}$. Hence, by Theorem 8.4.6, there exists real numbers α, β such that

$$f(x) = C(x + y) = \alpha C(x) + \beta S(x) \quad \text{and}$$

$$f'(x) = -S(x + y) = -\alpha S(x) + \beta C(x)$$

for $x \in \mathbb{R}$. If we let $x = 0$, we obtain $C(y) = \alpha$ and $-S(y) = \beta$, whence the first formula in (vi) follows. The second formula is proved similarly.

Q.E.D.

8.4.8 Theorem *If $x \in \mathbb{R}$, $x \geq 0$, then we have*

(vii) $-x \leq S(x) \leq x$;

(viii) $1 - \frac{1}{2}x^2 \leq C(x) \leq 1$;

(ix) $x - \frac{1}{6}x^3 \leq S(x) \leq x$;

(x) $1 - \frac{1}{2}x^2 \leq C(x) \leq 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4$.

Proof. Corollary 8.4.3 implies that $-1 \leq C(t) \leq 1$ for $t \in \mathbb{R}$, so that if $x \geq 0$, then

$$-x \leq \int_0^x C(t)dt \leq x,$$

whence we have (vii). If we integrate (vii), we obtain

$$-\tfrac{1}{2}x^2 \leq \int_0^x S(t)\,dt \leq \tfrac{1}{2}x^2,$$

whence we have

$$-\tfrac{1}{2}x^2 \leq -C(x) + 1 \leq \tfrac{1}{2}x^2.$$

Thus we have $1 - \frac{1}{2}x^2 \leq C(x)$, which implies (viii).

Inequality (ix) follows by integrating (viii), and (x) follows by integrating (ix). Q.E.D.