Chapter 2

THE REAL NUMBERS

2.1.1 Algebraic Properties of $\mathbb{R}$ On the set $\mathbb{R}$ of real numbers there are two binary operations, denoted by $+$ and $\cdot$ and called **addition** and **multiplication**, respectively. These operations satisfy the following properties:

(A1) $a + b = b + a$ for all a, b in $\mathbb{R}$ (*commutative property of addition*);

(A2) $(a + b) + c = a + (b + c)$ for all a, b, c in $\mathbb{R}$ (*associative property of addition*);

(A3) there exists an element 0 in $\mathbb{R}$ such that $0 + a = a$ and $a + 0 = a$ for all a in $\mathbb{R}$ (*existence of a zero element*);

(A4) for each a in $\mathbb{R}$ there exists an element $-a$ in $\mathbb{R}$ such that $a + (-a) = 0$ and $(-a) + a = 0$ (*existence of negative elements*);

(M1) $a \cdot b = b \cdot a$ for all a, b in $\mathbb{R}$ (*commutative property of multiplication*);

(M2) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all a, b, c in $\mathbb{R}$ (*associative property of multiplication*);

(M3) there exists an element 1 in $\mathbb{R}$ *distinct from* 0 such that $1 \cdot a = a$ and $a \cdot 1 = a$ for all a in $\mathbb{R}$ (*existence of a unit element*);

(M4) for each $a \neq 0$ in $\mathbb{R}$ there exists an element $1/a$ in $\mathbb{R}$ such that $a \cdot (1/a) = 1$ and $(1/a) \cdot a = 1$ (*existence of reciprocals*);

(D) $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$ and $(b + c) \cdot a = (b \cdot a) + (c \cdot a)$ for all a, b, c in $\mathbb{R}$ (*distributive property of multiplication over addition*). —— 等式里的内容可逆用

2.1.2 Theorem **(a)** *If z and a are elements in $\mathbb{R}$ with $z + a = a$, then $z = 0$.* —— A_3^{-1}

(b) *If u and $b \neq 0$ are elements in $\mathbb{R}$ with $u \cdot b = b$, then $u = 1$.* —— M_3^{-1}

(c) *If $a \in \mathbb{R}$, then $a \cdot 0 = 0$.*

(c) We have (why?)

$$a + a \cdot 0 = a \cdot 1 + a \cdot 0 = a \cdot (1 + 0) = a \cdot 1 = a.$$

2.1.3 Theorem **(a)** *If $a \neq 0$ and b in $\mathbb{R}$ are such that $a \cdot b = 1$, then $b = 1/a$.* —— M_4'

(b) *If $a \cdot b = 0$, then either $a = 0$ or $b = 0$.*

(b) It suffices to assume $a \neq 0$ and prove that $b = 0$. (Why?) We multiply $a \cdot b$ by $1/a$ and apply (M2), (M4), and (M3) to get

$$(1/a) \cdot (a \cdot b) = ((1/a) \cdot a) \cdot b = 1 \cdot b = b.$$

Since $a \cdot b = 0$, by 2.1.2(c) this also equals

$$b = (1/a) \cdot (a \cdot b) = (1/a) \cdot 0 = 0.$$

Thus we have $b = 0$. Q.E.D.

$a \neq 0 \Rightarrow \frac{1}{a}$ 存在

Rational and Irrational Numbers —— **2.1.4 Theorem** *There does not exist a rational number r such that $r^2 = 2$.*

$r = \frac{m}{n}, \Rightarrow r^2 = \frac{m^2}{n^2} = 2 \Rightarrow m^2 = 2n^2$

$\Rightarrow m$ 为偶数 $\rightarrow m$ 有因子为2 $\Rightarrow n$ 不能有因子为2 $\Rightarrow n$ 为奇

$\rightarrow m = 2t \ (t \in N) \Rightarrow \frac{m^2}{n^2} = \frac{4t^2}{n^2} = 2 \Rightarrow n$ 为偶

$n^2 = 2t^2$

矛盾

The Order Properties of $\mathbb{R}$

2.1.5 The Order Properties of $\mathbb{R}$ There is a nonempty subset $\mathbb{P}$ of $\mathbb{R}$, called the set of **positive real numbers**, that satisfies the following properties:

(i) If a, b belong to $\mathbb{P}$, then $a + b$ belongs to $\mathbb{P}$.

(ii) If a, b belong to $\mathbb{P}$, then ab belongs to $\mathbb{P}$.

(iii) If a belongs to $\mathbb{R}$, then $\boxed{\text{exactly one}}$ of the following holds:

$$a \in \mathbb{P}, \qquad a = 0, \qquad -a \in \mathbb{P}.$$

Trichotomy Theorem

} Axioms

Axioms

2.1.6 Definition Let a, b be elements of $\mathbb{R}$.

(a) If $a - b \in \mathbb{P}$, then we write $a > b$ or $b < a$.

(b) If $a - b \in \mathbb{P} \cup \{0\}$, then we write $a \geq b$ or $b \leq a$.

$a > b$ is defined by $a - b \in P$

$a \geqslant b$ is defined by $a - b \in P \cup \{0\}$

2.1.7 Theorem *Let a, b, c be any elements of $\mathbb{R}$.*

(a) *If $a > b$ and $b > c$, then $a > c$.*

(b) *If $a > b$, then $a + c > b + c$.*

(c) *If $a > b$ and $c > 0$, then $ca > cb$.*
If $a > b$ and $c < 0$, then $ca < cb$.

(c) Proof:

If $a > b \Rightarrow a - b \in P$

As $c > 0 \Rightarrow c \in P$

(As $c < 0 \Rightarrow -c \in P$)

$\Rightarrow c(a-b) \in P$

$[(-c)(a-b) \in P]$

As (a) contributes to the proof of (b) and (b) contributes to (c), So we put them together.

2.1.8 Theorem

(a) *If $a \in \mathbb{R}$ and $a \neq 0$, then $a^2 > 0$.* 默认$(-1)^2 = 1$

(b) $1 > 0$. (b) Since $1 = 1^2$, it follows from (a) that $1 > 0$.

(c) *If $n \in \mathbb{N}$, then $n > 0$.*

(c) We use Mathematical Induction. The assertion for $n = 1$ is true by (b). If we suppose the assertion is true for the natural number k, then $k \in \mathbb{P}$, and since $1 \in \mathbb{P}$, we have $k + 1 \in \mathbb{P}$ by 2.1.5(i). Therefore, the assertion is true for all natural numbers. Q.E.D.

2.1.9 Theorem *If $a \in \mathbb{R}$ is such that $0 \leq a < \varepsilon$ for every $\varepsilon > 0$, then $a = 0$.*

2.1.10 Theorem *If $ab > 0$, then either*

(i) $a > 0$ ~~and~~ $\Rightarrow$ $b > 0$, *or*

(ii) $a < 0$ ~~and~~ $\Rightarrow$ $b < 0$.

By 2.1.3 (a)

Proof. First we note that $ab > 0$ implies that $a \neq 0$ and $b \neq 0$. (Why?) From the Trichotomy Property, either $a > 0$ or $a < 0$. If $a > 0$, then $1/a > 0$, and therefore $b = (1/a)\,(ab) > 0$. Similarly, if $a < 0$, then $1/a < 0$, so that $b = (1/a)(ab) < 0$. Q.E.D.

$P \quad P$

2.1.11 Corollary *If $ab < 0$, then either*

(i) *$a < 0$ and $b > 0$, or*

(ii) *$a > 0$ and $b < 0$.*

Inequalities

(a) Let $a \geq 0$ and $b \geq 0$. Then

$$a < b \iff a^2 < b^2 \iff \sqrt{a} < \sqrt{b}$$

(b) If a and b are positive real numbers, then their **arithmetic mean** is $\frac{1}{2}(a+b)$ and their **geometric mean** is $\sqrt{ab}$. The **Arithmetic-Geometric Mean Inequality** for a, b is

(2) $$\boxed{\sqrt{ab} \leq \tfrac{1}{2}(a+b)}$$

#

$(a+b)^2 \geqslant 2ab$

Remark The general Arithmetic-Geometric Mean Inequality for the positive real numbers $a_1, a_2, \ldots, a_n$ is

(3) $$(a_1 a_2 \cdots a_n)^{1/n} \leq \frac{a_1 + a_2 + \cdots + a_n}{n}$$

(c) Bernoulli's Inequality. If $x > -1$, then

(4) $$(1+x)^n \geq 1 + nx \qquad \text{for all} \quad n \in \mathbb{N}$$

Section 2.2 Absolute Value and the Real Line

2.2.1 Definition The **absolute value** of a real number a, denoted by $|a|$, is defined by

$$|a| := \begin{cases} a & \text{if } a > 0, \\ 0 & \text{if } a = 0, \\ -a & \text{if } a < 0. \end{cases}$$

"=" implies "iff"

2.2.2 Theorem **(a)** $|ab| = |a||b|$ *for all* $a, b \in \mathbb{R}$.

(b) $|a|^2 = a^2$ *for all* $a \in \mathbb{R}$.

(c) *If* $c \geq 0$, *then* $|a| \leq c$ *if and only if* $-c \leq a \leq c$.

(d) $-|a| \leq a \leq |a|$ *for all* $a \in \mathbb{R}$.

#可以逆着等式证

(b) Since $a^2 \geq 0$, we have $a^2 = |a^2| = |aa| \overset{(a)}{=} |a||a| = |a|^2$.

(c) If $|a| \leq c$, then we have both $a \leq c$ and $-a \leq c$ (why?), which is equivalent to $-c \leq a \leq c$. Conversely, if $-c \leq a \leq c$, then we have both $a \leq c$ and $-a \leq c$ (why?), so that $|a| \leq c$.

Proof. As $|a| \geqslant 0$

then by (c), $|a| \leq |a|$

iff $-|a| \leq |a| \leq |a|$

① $a > 0$

$\Rightarrow -|a| \leq a \leq |a|$

② $a < 0$

$-|a| \leq -a \leq |a|$

$\Rightarrow -|a| \leq a \leq |a|$

2.2.3 Triangle Inequality *If* $a, b \in \mathbb{R}$, *then* $|a+b| \leq |a| + |b|$.

#这可不能忘,

Proof. From 2.2.2(d), we have $-|a| \leq a \leq |a|$ and $-|b| \leq b \leq |b|$. On adding these inequalities, we obtain

$$\boxed{-(|a|+|b|) \leq a + b \leq |a| + |b|.}$$

Hence, by 2.2.2(c) we have $|a+b| \leq |a| + |b|$. Q.E.D.

2.2.4 Corollary *If* $a, b \in \mathbb{R}$, *then*

(a) $\big||a| - |b|\big| \leq |a - b|$, 两边之差小于第三边

(b) $|a - b| \leq |a| + |b|$. 两边之和大于等于第三边.

2.2.5 Corollary *If* $a_1, a_2, \ldots, a_n$ *are any real numbers, then*

别忘

$$|a_1 + a_2 + \cdots + a_n| \leq |a_1| + |a_2| + \cdots + |a_n|.$$

2.2.7 Definition Let $a \in \mathbb{R}$ and $\varepsilon > 0$. Then the **ε-neighborhood** of a is the set $V_\varepsilon(a) := \{x \in \mathbb{R} : |x - a| < \varepsilon\}$.

For $a \in \mathbb{R}$, the statement that x belongs to $V_\varepsilon(a)$ is equivalent to either of the statements (see Figure 2.2.4)

$$-\varepsilon < x - a < \varepsilon \iff a - \varepsilon < x < a + \varepsilon.$$

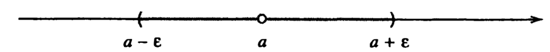


Figure 2.2.4 An ε-neighborhood of a

2.2.8 Theorem *Let $a \in \mathbb{R}$. If x belongs to the neighborhood $V_\varepsilon(a)$ for every $\varepsilon > 0$, then $x = a$.*

i.e. $|x-a| < \varepsilon$
$\Rightarrow x = a$

Section 2.3 The Completeness Property of $\mathbb{R}$

2.3.1 Definition Let S be a nonempty subset of $\mathbb{R}$.

(a) The set S is said to be **bounded above** if there exists a number $u \in \mathbb{R}$ such that $s \le u$ for all $s \in S$. Each such number u is called an **upper bound** of S.

(b) The set S is said to be bounded **below** if there exists a number $w \in \mathbb{R}$ such that $w \le s$ for all $s \in S$. Each such number w is called a **lower bound** of S.

(c) A set is said to be **bounded** if it is both bounded above and bounded below. A set is said to be **unbounded** if it is not bounded.

$(-\infty, a]$, $[b, +\infty)$ are also unbounded.

2.3.2 Definition Let S be a nonempty subset of $\mathbb{R}$.

(a) If S is bounded above, then a number u is said to be a **supremum** (or a **least upper bound**) of S if it satisfies the conditions:

(1) u is an upper bound of S, and

(2) if v is any upper bound of S, then $u \le v$.

(b) If S is bounded below, then a number w is said to be an **infimum** (or a **greatest lower bound**) of S if it satisfies the conditions:

(1′) w is a lower bound of S, and

(2′) if t is any lower bound of S, then $t \le w$.

适合证 sup(S) = upper bound.

[⇔ 证对于任何 u.b. v, v ≥ b].

Direct Proofs of Sup / Inf

sup(S)

The following statements about an upper bound u of a set S are equivalent:

(1) if v is any upper bound of S, then $u \le v$,

(2) if $z < u$, then z is not an upper bound of S,

(3) if $z < u$, then there exists $s_z \in S$ such that $z < s_z$,

(4) if $\varepsilon > 0$, then there exists $s_\varepsilon \in S$ such that $u - \varepsilon < s_\varepsilon$.

(设 z = sup(S))

通过证明 ∀ z < u, ∃ t ∈ S s.t. z < t 来证 z 不是 u.b.
⇒ u = sup(S)

☆ R的非空子集的上界即上确界

2.3.3 Lemma *A number u is the supremum of a nonempty subset S of $\mathbb{R}$ if and only if u satisfies the conditions:*

(1) $s \le u$ *for all* $s \in S$,

(2) *if* $v < u$, *then there exists* $s' \in S$ *such that* $v < s'$.

任取即可 ⇒ 构造

v 具有任意性

[⇒] (2) Prove by Construction

[⇐] Prove by Contradiction

2.3.4 Lemma *An upper bound u of a nonempty set S in $\mathbb{R}$ is the supremum of S if and only if for every $\varepsilon > 0$ there exists an $s_\varepsilon \in S$ such that $u - \varepsilon < s_\varepsilon$.*

Prove by Construction:

$\boxed{\Leftarrow}$ Let $\boxed{\varepsilon = u - v}$, $v = \sup(S) < u \Rightarrow u - v = \varepsilon > 0$

then $u - \varepsilon = u - u + v = v < s_\varepsilon \in S \Rightarrow$ Contradiction

Thus, $u = v \Rightarrow u = \sup(S)$.

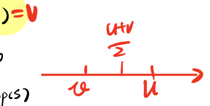
$\boxed{\Rightarrow}$ If $u = \sup(S)$, $\forall \varepsilon > 0$, $u - \varepsilon < u \Rightarrow$ $u - \varepsilon$ is not an upper bound of $S \Rightarrow$ there $\exists s_\varepsilon \in S$ s.t. $s_\varepsilon > u - \varepsilon$

Prove by Contradiction:

$\boxed{\Leftarrow}$ If u is an u.b. and $u - \varepsilon < s_\varepsilon \in S$,

Assume that $u > \sup(S) = v$

let $\varepsilon_0 = \frac{u+v}{2} > 0$



then $u - \varepsilon_0 > v = \sup(S)$

so s_{ε_0} doesn't exist s.t. $u - \varepsilon_0 < s_\varepsilon \in S$.

Contradiction. $\Rightarrow$ $u = \sup(S)$

$\boxed{\Rightarrow}$ If $u = \sup(S)$

Assume that $\forall s_\varepsilon \in S$, $\exists \varepsilon > 0$ s.t. $u - \varepsilon \geq s_\varepsilon \in S$

$\Rightarrow$ $u - \varepsilon$ is an upper bound of S

$\Rightarrow$ $u - \varepsilon \geq \sup(S) = u$

$\Rightarrow$ $\varepsilon \leq 0 \Rightarrow$ Contradicts to $\varepsilon > 0$

Thus, $\forall \varepsilon > 0$, $\exists s_\varepsilon \in S$ s.t. $u - \varepsilon < s_\varepsilon$.

The Completeness Property of $\mathbb{R}$

$\boxed{\text{or}}$ The Supreme Property of $\mathbb{R}$

2.3.6 The Completeness Property of $\mathbb{R}$ *Every nonempty set of real numbers that has an upper bound also has a supremum in $\mathbb{R}$.*

No Proof

i.e. $\phi \neq S \subset \mathbb{R}$, $S \leq B$

$\Downarrow$

exists $\sup(S)$

The Infimum Property of $\mathbb{R}$:

Every nonempty subst that is bounded below has a infimum in $\mathbb{R}$.

所有有上界的R的子集有上确界

(下) (下)

Section 2.4 Applications of the Supremum Property

2.4.1 Examples

(a) compatibility of taking suprema and addition.

Let S be a nonempty subset of $\mathbb{R}$ that is bounded above, and let a be any number in $\mathbb{R}$. Define the set $a+S := \{a+s : s \in S\}$. We will prove that

$$\sup(a+S) = a + \sup S.$$

Proof: ① $a+\sup(S)$ is an u.b. of $(a+S)$
② Assume that $a+\sup(S) > \sup(a+S)$
$\Rightarrow \sup(S) > \sup(a+S) - a$
$\Rightarrow \sup(a+S) - a \in S$
$\Rightarrow \exists S_\varepsilon \in S$ s.t. $\sup(a+S) - a < S_\varepsilon$
$\Rightarrow \sup(a+S) < a + S_\varepsilon$ ①
As $a+S_\varepsilon \in (a+S)$
so $a+S_\varepsilon \le \sup(a+S)$
Contradicts to ①
Thus, $a+\sup(S) = \sup(a+S)$ □.

(b) Suppose that A and B are nonempty subsets of $\mathbb{R}$ that satisfy the property:

$$a \le b \quad \text{for all } a \in A \text{ and all } b \in B.$$

We will prove that

$$\sup A \le \inf B.$$

2.4.3 Archimedean Property *If $x \in \mathbb{R}$, then there exists $n_x \in \mathbb{N}$ such that $x \le n_x$.*

Prove by Contradiction:
Assume that $\exists x \in \mathbb{R}$ s.t. $\forall n_x \in \mathbb{N}$, $x > n_x$
then x is an upper bound of $\mathbb{N}$
by the Completeness property of $\mathbb{R}$
$\boxed{\exists m \in \mathbb{R}}$ s.t. $m = \sup(\mathbb{N})$
$\Rightarrow m-1 < \sup(\mathbb{N})$
then $m-1$ is not an upper bound of $(\mathbb{N})$
therefore $\exists t \in \mathbb{N}$ s.t. $m-1 < t$
$\Rightarrow m = \sup(\mathbb{N}) < t+1 \in \mathbb{N}$
Contradiction
Thus, $\forall x \in \mathbb{R}$, $\exists n_t \in \mathbb{N}$ s.t. $n_t > x$ □.

$\inf(\frac{1}{n}) = 0$ 可直接用

2.4.4 Corollary *If $S := \{1/n : n \in \mathbb{N}\}$, then $\inf S = 0$.*

2.4.5 Corollary *If $t > 0$, there exists $n_t \in \mathbb{N}$ such that $0 < 1/n_t < t$.*

2.4.5 Corollary *If $t > 0$, there exists $n_t \in \mathbb{N}$ such that $0 < 1/n_t < t$.*

Proof. Since $\inf\{1/n : n \in \mathbb{N}\} = 0$ and $t > 0$, then t is not a lower bound for the set $\{1/n : n \in \mathbb{N}\}$. Thus there exists $n_t \in \mathbb{N}$ such that $0 < 1/n_t < t$. Q.E.D.

2.4.6 Corollary *If $y > 0$, there exists $n_y \in \mathbb{N}$ such that $n_y - 1 \le y \le n_y$.*

i.e. y must be between two "neighbor" $\mathbb{N} \cup \{0\}$ $(\mathbb{Z}^+)$

Proof. The Archimedean Property ensures that the subset $E_y := \{m \in \mathbb{N} : y < m\}$ of $\mathbb{N}$ is not empty. By the Well-Ordering Property 1.2.1, E_y has a least element, which we denote by n_y. Then $n_y - 1$ does not belong to E_y, and hence we have $n_y - 1 \le y < n_y$. Q.E.D.

of $\mathbb{N}$

$y \geq n_y - 1$

$n_y \geq y$

y

2.4.7 Theorem *There exists a positive real number x such that $x^2 = 2$.*

A.P. Corollary 的 ② + ③

2.4.8 The Density Theorem *If x and y are any real numbers with $x < y$, then there exists a rational number $r \in \mathbb{Q}$ such that $x < r < y$.*

Proof. It is no loss of generality (why?) to assume that $x > 0$. Since $\boxed{y - x > 0}$, it follows from Corollary 2.4.5 that there exists $n \in \mathbb{N}$ such that $1/n < y - x$. Therefore, we have $nx + 1 < ny$. If we apply Corollary 2.4.6 to $nx > 0$, we obtain $m \in \mathbb{N}$ with $m - 1 \le nx < m$. Therefore, $m \le nx + 1 < ny$, whence $nx < m < ny$. Thus, the rational number $r := m/n$ satisfies $x < r < y$. Q.E.D.

直接消等号.

2.4.9 Corollary *If x and y are real numbers with $x < y$, then there exists an irrational number z such that $x < z < y$.*

Proof. If we apply the Density Theorem 2.4.8 to the real numbers $x/\sqrt{2}$ and $y/\sqrt{2}$, we obtain a rational number $r \neq 0$ (why?) such that

$$\frac{x}{\sqrt{2}} < r < \frac{y}{\sqrt{2}}.$$

Then $z := r\sqrt{2}$ is irrational (why?) and satisfies $x < z < y$. Q.E.D.

∵ exists $x \in S$ s.t. $x < z$
∵ exists $y \in S$ s.t. $z < y$ $\Rightarrow z \in (x, y) \subseteq S$

上/下确界不定在S中

Section 2.5 Intervals

2.5.1 Characterization Theorem *If S is a subset of $\mathbb{R}$ that contains at least two points and has the property*

(1) *if* $x, y \in S$ *and* $x < y$, *then* $[x, y] \subseteq S$,

then S is an interval. i.e. An interval is equal to S.

$\Rightarrow$ [c] 单独一点不能形成 interval

穷举法

Proof. There are four cases to consider: (i) S is bounded, (ii) S is bounded above but not below, (iii) S is bounded below but not above, and (iv) S is neither bounded above nor below.

Case (i): Let $a := \inf S$ and $b := \sup S$. Then $S \subseteq [a, b]$ and we will show that $(a, b) \subseteq S$.

If $a < z < b$, then z is not a lower bound of S, so there exists $x \in S$ with $x < z$. Also, z is not an upper bound of S, so there exists $y \in S$ with $z < y$. Therefore $z \in [x, y]$, so property (1) implies that $z \in S$. Since z is an arbitrary element of (a, b), we conclude that $(a, b) \subseteq S$.

Now if $a \in S$ and $b \in S$, then $S = [a, b]$. (Why?) If $a \notin S$ and $b \notin S$, then $S = (a, b)$. The other possibilities lead to either $S = (a, b]$ or $S = [a, b)$.

Case (ii): S is bounded above but not below:

Let $b := \sup(S)$

then $\forall s \in S$, $s \le b$ so $S \subseteq (-\infty, b]$

$\boxed{\text{Now, WTS: } (-\infty, b) \subseteq S.}$

$\boxed{\forall \alpha \in (-\infty, b)}$, $\alpha < b$

so α is not an u.b. of S

$\Rightarrow \exists y \in S$ s.t. $\alpha < y$

As S is not bounded below,

$\exists x \in S$ s.t. $x < \alpha$

Thus, $\alpha \in [x, y] \Rightarrow \alpha \in S$

As α is an arbitrary element from $(-\infty, b)$

So $(-\infty, b) \subseteq S$

If $\sup(S) = b \in S$ then $(-\infty, b] \subseteq S$

Thus, $S = (-\infty, b)$ or $S = (-\infty, b]$. □.

Nested Intervals

2.5.2 Nested Intervals Property *If $I_n = [a_n, b_n]$, $n \in \mathbb{N}$, is a nested sequence of closed bounded intervals, then there exists a number $\xi \in \mathbb{R}$ such that $\xi \in I_n$ for all $n \in \mathbb{N}$.*

Proof by Construction

#

① 证 Sup(a_n) 的存在 $\Leftarrow$ bounded above by b_1

② 证 b_n 都是 a_n 的 u.b.

Proof. Since the intervals are nested, we have $I_n \subseteq I_1$ for all $n \in \mathbb{N}$, so that $a_n \le b_1$ for all $n \in \mathbb{N}$. Hence, the nonempty set $\{a_n : n \in \mathbb{N}\}$ is bounded above, and we let ξ be its supremum. Clearly $a_n \le \xi$ for all $n \in \mathbb{N}$.

We claim also that $\xi \le b_n$ for all n. This is established by showing that for any particular n, the number b_n is an upper bound for the set $\{a_k : k \in \mathbb{N}\}$. We consider two cases. (i) If $n \le k$, then ~~since~~ $I_n \supseteq I_k$ $\Rightarrow$ we have $a_k \le b_k \le b_n$. (ii) If $k < n$, then since $I_k \supseteq I_n$, we have $a_k \le a_n \le b_n$. (See Figure 2.5.2.) Thus, we conclude that $a_k \le b_n$ for all k, so that b_n is an upper bound of the set $\{a_k : k \in \mathbb{N}\}$. Hence, $\xi \le b_n$ for each $n \in \mathbb{N}$. Since $a_n \le \xi \le b_n$ for all n, we have $\xi \in I_n$ for all $n \in \mathbb{N}$. Q.E.D.

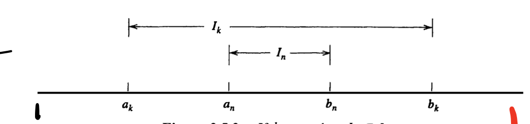
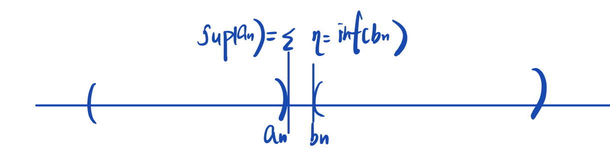


Figure 2.5.2 If $k < n$, then $I_n \subseteq I_k$



$(b_n - a_n)$ converges to 0

$\exists k \in \mathbb{N}$ s.t. $\forall n \ge k$

$|(b_n - a_n) - 0| < \varepsilon$

$(\forall \varepsilon > 0)$

2.5.3 Theorem *If $I_n := [a_n, b_n]$, $n \in \mathbb{N}$, is a nested sequence of closed, bounded intervals such that the lengths $b_n - a_n$ of I_n satisfy*

$$\inf\{b_n - a_n : n \in \mathbb{N}\} = 0,$$

then the number ξ contained in I_n for all $n \in \mathbb{N}$ is unique.

#

Proof. If $\eta := \inf\{b_n : n \in \mathbb{N}\}$, then an argument similar to the proof of 2.5.2 can be used to show that $a_n \le \eta$ for all n, and hence that $\boxed{\xi \le \eta}$. In fact, it is an exercise (see Exercise 10) to show that $x \in I_n$ for all $n \in \mathbb{N}$ if and only if $\xi \le x \le \eta$. If we have $\inf\{b_n - a_n : n \in \mathbb{N}\} = 0$, then for any $\varepsilon > 0$, there exists an $m \in \mathbb{N}$ such that $0 \le \boxed{\eta - \xi \le b_m - a_m} < \varepsilon$. Since this holds for all $\varepsilon > 0$, it follows from Theorem 2.1.9 that $\eta - \xi = 0$. Therefore, we conclude that $\xi = \eta$ is the only point that belongs to I_n for every $n \in \mathbb{N}$. Q.E.D.

key

2.5.4 Theorem *The set $\mathbb{R}$ of real numbers is not countable.*

Proof. We will prove that the unit interval $I := [0, 1]$ is an uncountable set. This implies that the set $\mathbb{R}$ is an uncountable set, for if $\mathbb{R}$ were countable, then the subset I would also be countable. (See Theorem 1.3.9(a).)

The proof is by contradiction. If we assume that I is countable, then we can enumerate the set as $I = \{x_1, x_2, \ldots, x_n, \boxed{\ldots}\}$. We first select a closed subinterval I_1 of I such that $\boxed{x_1 \notin I_1}$, then select a closed subinterval I_2 of I_1 such that $x_2 \notin I_2$, and so on. In this way, we obtain nonempty closed intervals

$$I_1 \supseteq I_2 \supseteq \cdots \supseteq I_n \supseteq \cdots$$

such that $\boxed{I_n \subseteq I \text{ and } x_n \notin I_n}$ for all n. The Nested Intervals Property 2.5.2 implies that there exists a point $\xi \in I$ such that $\xi \in I_n$ for all n. Therefore # $\xi \ne x_n$ for all $n \in \mathbb{N}$, so the enumeration of I is not a complete listing of the elements of I, as claimed. Hence, I is an uncountable set. Q.E.D.

Direct Contradiction to "Countable"

I.

$I_n \subseteq I_{n-1} \subseteq \cdots \subseteq I_2 \subseteq I_1 \subseteq I$

II.

$x_n \notin I_n$