CHAPTER 5

CONTINUOUS FUNCTIONS

Section 5.1 Continuous Functions

5.1.1 Definition Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$, and let $c \in A$. We say that f is **continuous at** c if, given any number $\varepsilon > 0$, there exists $\delta > 0$ such that if x is any point of A satisfying $|x - c| < \delta$, then $|f(x) - f(c)| < \varepsilon$.

If f fails to be continuous at c, then we say that f is **discontinuous at** c.

5.1.5 Definition Let $A \subseteq \mathbb{R}$ and let $f : A \to \mathbb{R}$. If B is a subset of A, we say that f is **continuous on the set** B if f is continuous at every point of B.

5.1.2 Theorem *A function $f : A \to \mathbb{R}$ is continuous at a point $c \in A$ if and only if given any ε-neighborhood $V_\varepsilon(f(c))$ of $f(c)$ there exists a δ-neighborhood $V_\delta(c)$ of c such that if x is any point of $A \cap V_\delta(c)$, then $f(x)$ belongs to $V_\varepsilon(f(c))$, that is,*

$$f(A \cap V_\delta(c)) \subseteq V_\varepsilon(f(c)).$$

Remarks **(1)** If $c \in A$ is a cluster point of A, then a comparison of Definitions 4.1.4 and 5.1.1 show that f is continuous at c if and only if

$$f(c) = \lim_{x \to c} f(x). \tag{1}$$

Thus, if c is a cluster point of A, then three conditions must hold for f to be continuous at c:

(i) f must be defined at c (so that $f(c)$ makes sense),

(ii) the limit of f at c must exist in $\mathbb{R}$ (so that $\lim_{x \to c} f(x)$ makes sense), and

(iii) these two values must be equal.

5.1.3 Sequential Criterion for Continuity *A function $f : A \to \mathbb{R}$ is continuous at the point $c \in A$ if and only if for every sequence (x_n) in A that converges to c, the sequence $(f(x_n))$ converges to $f(c)$.*

$\boxed{\text{Prove by Contradiction:}}$ 直接借助 VS 构建(ii)中 $(x_n) \to c$

$\boxed{\Leftarrow}$ Proof: Suppose f is $\boxed{\text{not}}$ conts. at c,

then $\exists\, \varepsilon_0 > 0$ s.t. $\forall \delta > 0$, $\exists\, x \in S$

$|x - c| < \delta$ and $|f(x) - f(c)| \geq \varepsilon_0$

Choose $\delta = 1$.

Then $\exists\, x_1 \in S$ s.t. $|x_1 - c| < 1$ and $|f(x_1) - f(c)| \geq \varepsilon_0$.

Choose $\delta = \frac{1}{2}$

Then $\exists\, x_2 \in S$ s.t. $|x_2 - c| < \frac{1}{2}$ and $|f(x_2) - f(c)| \geq \varepsilon_0$

Then $\forall n \in \mathbb{N}$, $\exists\, x_n \in S$ s.t. $|x_n - c| < \frac{1}{n}$

and $|f(x_n) - f(c)| \geq \varepsilon_0$

Thus, $0 \leq |x_n - c| < \frac{1}{n} \to 0 \Rightarrow x_n \to c$ when $n \to \infty$.

Then by the statement, $0 = \lim_{n \to \infty} |f(x_n) - f(c)|$

Yet, $|f(x_n) - f(c)| \geq \varepsilon_0 > 0$

Contradiction.

Whence, f conts. at c □.

与 Thm 4.1.8 (i) ⇒ (ii) 相同

$\boxed{\Rightarrow}$

Function: Let $\varepsilon > 0$, Since f is conts at c

$\exists\, \delta > 0$ s.t. $|x - c| < \delta \Rightarrow |f(x) - f(c)| < \varepsilon$

Sequence: Since (x_n) converges to c then

$\exists\, k \in \mathbb{N}$ s.t. $\forall n \geq k$, $|x_n - c| < \delta$

Choose $M = k$, then $\forall n \geq M$,

$|x_n - c| < \delta \Rightarrow |f(x_n) - f(c)| < \varepsilon$

Thus, $(f(x_n))$ converges to $f(c)$

if x converge to c

□.

5.1.4 Discontinuity Criterion *Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$, and let $c \in A$. Then f is discontinuous at c if and only if there exists a sequence (x_n) in A such that (x_n) converges to c, but the sequence $(f(x_n))$ does not converge to $f(c)$.* To Proof

5.1.6 Examples

(a) The constant function $f(x) := b$ is continuous on $\mathbb{R}$.

(b) $g(x) := x$ is continuous on $\mathbb{R}$.

(c) $h(x) := x^2$ is continuous on $\mathbb{R}$.

Prove by Def.:

Let $h(x) := x^2$ for all $x \in \mathbb{R}$. We want to make the difference

$$|h(x) - c^2| = |x^2 - c^2|$$

less than a preassigned $\varepsilon > 0$ by taking x sufficiently close to c. To do so, we note that $x^2 - c^2 = (x+c)(x-c)$. Moreover, if $|x - c| < 1$, then

$$|x| < |c| + 1 \qquad \text{so that} \qquad |x + c| \le |x| + |c| < 2|c| + 1.$$

Therefore, if $|x - c| < 1$, we have

(1) $$|x^2 - c^2| = |x + c||x - c| < (2|c| + 1)|x - c|.$$

Moreover this last term will be less than ε provided we take $|x - c| < \varepsilon/(2|c| + 1)$. Consequently, if we choose

$$\delta(\varepsilon) := \inf\left\{1, \frac{\varepsilon}{2|c| + 1}\right\},$$

then if $0 < |x - c| < \delta(\varepsilon)$, it will follow first that $|x - c| < 1$ so that (1) is valid, and therefore, since $|x - c| < \varepsilon/(2|c| + 1)$ that

$$|x^2 - c^2| < (2|c| + 1)|x - c| < \varepsilon.$$

Since we have a way of choosing $\delta(\varepsilon) > 0$ for an arbitrary choice of $\varepsilon > 0$, we infer that $\lim\limits_{x \to c} h(x) = \lim\limits_{x \to c} x^2 = c^2$.

(d) $\varphi(x) := 1/x$ is continuous on $A := \{x \in \mathbb{R} : x > 0\}$.

Let $\varphi(x) := 1/x$ for $x > 0$ and let $c > 0$. To show that $\lim\limits_{x \to c} \varphi = 1/c$ we wish to make the difference

$$\left|\varphi(x) - \frac{1}{c}\right| = \left|\frac{1}{x} - \frac{1}{c}\right|$$

less than a preassigned $\varepsilon > 0$ by taking x sufficiently close to $c > 0$. We first note that

$$\left|\frac{1}{x} - \frac{1}{c}\right| = \left|\frac{1}{cx}(c - x)\right| = \frac{1}{cx}|x - c|$$

for $x > 0$. It is useful to get an upper bound for the term $1/(cx)$ that holds in some neighborhood of c. In particular, if $|x - c| < \frac{1}{2}c$, then $\frac{1}{2}c < x < \frac{3}{2}c$ (why?), so that

$$0 < \frac{1}{cx} < \frac{2}{c^2} \qquad \text{for} \qquad |x - c| < \frac{1}{2}c.$$

Therefore, for these values of x we have

(2) $$\left|\varphi(x) - \frac{1}{c}\right| \le \frac{2}{c^2}|x - c|.$$

In order to make this last term less than ε it suffices to take $|x - c| < \frac{1}{2}c^2\varepsilon$. Consequently, if we choose

$$\delta(\varepsilon) := \inf\left\{\frac{1}{2}c, \frac{1}{2}c^2\varepsilon\right\},$$

then if $0 < |x - c| < \delta(\varepsilon)$, it will follow first that $|x - c| < \frac{1}{2}c$ so that (2) is valid, and therefore, since $|x - c| < \left(\frac{1}{2}c^2\right)\varepsilon$, that

$$\left|\varphi(x) - \frac{1}{c}\right| = \left|\frac{1}{x} - \frac{1}{c}\right| < \varepsilon.$$

Since we have a way of choosing $\delta(\varepsilon) > 0$ for an arbitrary choice of $\varepsilon > 0$, we infer that $\lim\limits_{x \to c} \varphi = 1/c$.

(e) $\varphi(x) := 1/x$ is not continuous at $x = 0$.

Indeed, if $\varphi(x) = 1/x$ for $x > 0$, then φ is not defined for $x = 0$, so it cannot be continuous there. Alternatively, it was seen in Example 4.1.10(a) that $\lim_{x \to 0} \varphi$ does not exist in $\mathbb{R}$, so φ cannot be continuous at $x = 0$.

(g) Let $A := \mathbb{R}$ and let f be Dirichlet's "discontinuous function" defined by

$$f(x) := \begin{cases} 1 & \text{if} \quad x \text{ is rational,} \\ 0 & \text{if} \quad x \text{ is irrational.} \end{cases}$$

We claim that *f is not continuous at any point of* $\mathbb{R}$. (This function was introduced in 1829

Indeed, if c is a rational number, let (x_n) be a sequence of irrational numbers that converges to c. (Corollary 2.4.9 to the Density Theorem 2.4.8 assures us that such a sequence does exist.) Since $f(x_n) = 0$ for all $n \in \mathbb{N}$, we have $\lim(f(x_n)) = 0$, while $f(c) = 1$. Therefore f is not continuous at the rational number c.

On the other hand, if b is an irrational number, let (y_n) be a sequence of rational numbers that converge to b. (The Density Theorem 2.4.8 assures us that such a sequence does exist.) Since $f(y_n) = 1$ for all $n \in \mathbb{N}$, we have $\lim(f(y_n)) = 1$, while $f(b) = 0$. Therefore f is not continuous at the irrational number b.

Since every real number is either rational or irrational, we deduce that f is not continuous at any point in $\mathbb{R}$.

(h) Let $A := \{x \in \mathbb{R} : x > 0\}$. For any irrational number $x > 0$ we define $h(x) := 0$. For a rational number in A of the form m/n, with natural numbers m, n having no common factors except 1, we define $h(m/n) := 1/n$. (We also define $h(0) := 1$.)

We claim that *h is continuous at every irrational number in A, and is discontinuous at every rational number in A.* (This function was introduced in 1875 by K. J. Thomae.)

Indeed, if $a > 0$ is rational, let (x_n) be a sequence of irrational numbers in A that converges to a. Then $\lim(h(x_n)) = 0$, while $h(a) > 0$. Hence h is discontinuous at a.

On the other hand, if b is an irrational number and $\varepsilon > 0$, then (by the Archimedean Property) there is a natural number n_0 such that $1/n_0 < \varepsilon$. There are only a finite number of rationals with denominator less than n_0 in the interval $(b - 1, b + 1)$. (Why?) Hence $\delta > 0$ can be chosen so small that the neighborhood $(b - \delta, b + \delta)$ contains no rational numbers with denominator less than n_0. It then follows that for $|x - b| < \delta, x \in A$, we have $|h(x) - h(b)| = |h(x)| \le 1/n_0 < \varepsilon$. Thus h is continuous at the irrational number b.

Consequently, we deduce that Thomae's function h is continuous precisely at the irrational points in A. (See Figure 5.1.2.) □

5.1.7 Remarks

5.1.7 Remarks **(a)** Sometimes a function $f : A \to \mathbb{R}$ is not continuous at a point c because it is not defined at this point. However, if the function f has a limit L at the point c and if we define F on $A \cup \{c\} \to \mathbb{R}$ by

$$F(x) := \begin{cases} L & \text{for} \quad x = c, \\ f(x) & \text{for} \quad x \in A, \end{cases}$$

then F is continuous at c. To see this, one needs to check that $\lim_{x\to c} F = L$, but this follows (why?), since $\lim_{x\to c} f = L$.

(b) If a function $g : A \to \mathbb{R}$ does not have a limit at c, then there is no way that we can obtain a function $G : A \cup \{c\} \to \mathbb{R}$ that is continuous at c by defining

$$G(x) := \begin{cases} C & \text{for} \quad x = c, \\ g(x) & \text{for} \quad x \in A. \end{cases}$$

To see this, observe that if $\lim_{x\to c} G$ exists and equals C, then $\lim_{x\to c} g$ must also exist and equal C.

5.1.8 Examples **(a)** The function $g(x) := \sin(1/x)$ for $x \neq 0$ (see Figure 4.1.3) does not have a limit at $x = 0$ (see Example 4.1.10(c)). Thus there is no value that we can assign at $x = 0$ to obtain a continuous extension of g at $x = 0$.

(b) Let $f(x) := x \sin(1/x)$ for $x \neq 0$. (See Figure 5.1.3.) It was seen in Example 4.2.8(f) that $\lim_{x\to 0}(x\sin(1/x)) = 0$. Therefore it follows from Remark 5.1.7(a) that if we define $F : \mathbb{R} \to \mathbb{R}$ by

$$F(x) := \begin{cases} 0 & \text{for} \quad x = 0, \\ x\sin(1/x) & \text{for} \quad x \neq 0, \end{cases}$$

then F is continuous at $x = 0$. □

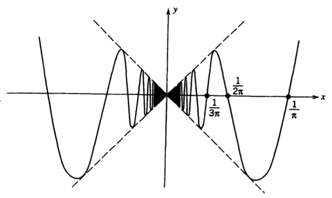

Figure 5.1.3 Graph of $f(x) = x\sin(1/x)$ $(x \neq 0)$

Section 5.2 Combinations of Continuous Functions

5.2.1 Theorem *Let $A \subseteq \mathbb{R}$, let f and g be functions on A to $\mathbb{R}$, and let $b \in \mathbb{R}$. Suppose that $c \in A$ and that f and g are continuous at c.*

(a) *Then $f + g$, $f - g$, fg, and bf are continuous at c.*

(b) *If $h : A \to \mathbb{R}$ is continuous at $c \in A$ and if $h(x) \neq 0$ for all $x \in A$, then the quotient f/h is continuous at c.*

5.2.2 Theorem *Let $A \subseteq \mathbb{R}$, let f and g be continuous on A to $\mathbb{R}$, and let $b \in \mathbb{R}$.*

(a) *The functions $f + g$, $f - g$, fg, and bf are continuous on A.*

(b) *If $h : A \to \mathbb{R}$ is continuous on A and $h(x) \neq 0$ for $x \in A$, then the quotient f/h is continuous on A.*

Proof. If $c \in A$ is not a cluster point of A, then the conclusion is automatic. Hence we assume that c is a cluster point of A.

(a) Since f and g are continuous at c, then

$$f(c) = \lim_{x\to c} f \quad \text{and} \quad g(c) = \lim_{x\to c} g.$$

Hence it follows from Theorem 4.2.4(a) that

$$(f + g)(c) = f(c) + g(c) = \lim_{x\to c}(f + g).$$

Therefore $f + g$ is continuous at c. The remaining assertions in part (a) are proved in a similar fashion.

(b) Since $c \in A$, then $h(c) \neq 0$. But since $h(c) = \lim_{x\to c} h$, it follows from Theorem 4.2.4(b) that

$$\frac{f}{h}(c) = \frac{f(c)}{h(c)} = \frac{\lim_{x\to c} f}{\lim_{x\to c} h} = \lim_{x\to c}\left(\frac{f}{h}\right).$$

Therefore f/h is continuous at c. Q.E.D.

5.2.3 Examples

(a) Polynomial Functions always continuous on R

$\Downarrow$

(b) *a rational function is continuous at every real number for which it is defined.*

$r(x) = \dfrac{P(x)}{Q(x)}$ $(Q(x) \neq 0)$ P, Q are polynomials.

(c) We shall show that the sine function sin is continuous on $\mathbb{R}$.

To do so we make use of the following properties of the sine and cosine functions. (See Section 8.4.) For all $x, y, z \in \mathbb{R}$ we have:

$$|\sin z| \leq |z|, \quad |\cos z| \leq 1,$$
$$\sin x - \sin y = 2 \sin\left[\tfrac{1}{2}(x-y)\right] \cos\left[\tfrac{1}{2}(x+y)\right].$$

Hence if $c \in \mathbb{R}$, then we have

$$|\sin x - \sin c| \leq 2 \cdot \tfrac{1}{2}|x - c| \cdot 1 = |x - c|.$$

Therefore sin is continuous at c. Since $c \in \mathbb{R}$ is arbitrary, it follows that sin is continuous on $\mathbb{R}$.

(d) The cosine function is continuous on $\mathbb{R}$.

We make use of the following properties of the sine and cosine functions. For all $x, y, z \in \mathbb{R}$ we have:

$$|\sin z| \leq |z|, \quad |\sin z| \leq 1,$$
$$\cos x - \cos y = -2 \sin\left[\tfrac{1}{2}(x+y)\right] \sin\left[\tfrac{1}{2}(x-y)\right].$$

Hence if $c \in \mathbb{R}$, then we have

$$|\cos x - \cos c| \leq 2 \cdot 1 \cdot \tfrac{1}{2}|c - x| = |x - c|.$$

5.2.4 Theorem *Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$, and let $|f|$ be defined by $|f|(x) := |f(x)|$ for $x \in A$.*

(a) *If f is continuous at a point $c \in A$, then $|f|$ is continuous at c.*

(b) *If f is continuous on A, then $|f|$ is continuous on A.*

Proof: ① When c is not a cluster point of A

then $\lim_{x\to c}|f|(c) = \lim_{x\to c}|f(c)| = |f(c)|$

is automatically continuous.

② When c is a cluster point of A.

$\exists$ Sequence (x_n) defined on A s.t. $x_n \to c$

As f is continuous at c

$\lim_{x\to c} f(x) = \lim_{n\to\infty}(f(x_n))$

so $\left|\lim_{x\to c} f(x)\right| = \left|\lim_{n\to\infty}(f(x_n))\right| = |f(c)|$

As $\lim_{x\to c}|f(x)| = |\lim_{x\to c} f(x)|$ then

and $|f|(x) = |f(x)|$ $) = f(c)$

then $\lim_{x\to c}|f|(x) = \lim_{x\to c}|f(x)| = \left|\lim_{x\to c} f(x)\right| = |f(c)| = |f|(c)$

Thus, $|f|(x)$ continuous at c

Q.E.D.

2

(b) Let $g(x) = |x|$ defined on A

then $g(x)$ continuous on A

so $|f|(x) = |f(x)| = g(f(x))$ continuous on A

as f continuous on A

5.2.5 Theorem *Let $A \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$, and let $f(x) \geq 0$ for all $x \in A$. We let $\sqrt{f}$ be defined for $x \in A$ by $(\sqrt{f})(x) := \sqrt{f(x)}$.*

(a) *If f is continuous at a point $c \in A$, then $\sqrt{f}$ is continuous at c.*

(b) *If f is continuous on A, then $\sqrt{f}$ is continuous on A.*

Proof: $f(x) = \sqrt{x}$ and $g(x) = x$ continue on R

$\Rightarrow$ $h = f \circ g\,(x)$ continuous on R

Composition of Continuous Functions

5.2.6 Theorem *Let $A, B \subseteq \mathbb{R}$ and let $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$ be functions such that $f(A) \subseteq B$. If f is continuous at a point $c \in A$ and g is continuous at $b = f(c) \in B$, then the composition $g \circ f : A \to \mathbb{R}$ is continuous at c.*

法 ①

Proof. Let $\{x_n\}$ be a sequence in A s.t. $x_n \to c$

Since $x_n \to c$ and f is conts. at c

then $\lim_{x_n \to c} f(x_n) = f(c) \Rightarrow (f(x_n))$ converges to $f(c)$

As $b = f(c) \in B$ and g conts. at $b = f(c)$

so $g(f(x_n)) \to g(f(c))$

i.e. $\lim_{x \to c} g \circ f(x) = g \circ f(c)$ □.

法 ②

Q: Will $|f|$, $\sqrt{f}$, be continuous, when f is continuous?

A: Yes.

☆ Composition function.

Thm: Let $A, B \subseteq \mathbb{R}$, $f: A \to \mathbb{R}$, $g: B \to \mathbb{R}$. be two functions with $f(A) \subseteq B$.

If f is continuous at c, and g is continuous at $f(c)$,

then $g \circ f$ is continuous at c.

P: ① g is continuous at $f(c)$: for a given $V_\varepsilon(g(f(c)))$, $\exists V_\delta(f(c))$, s.t. if $y \in B \cap V_\delta(f(c))$

then $g(y) \in V_\varepsilon(g(f(c)))$.

② For above $V_\delta(f(c))$, since f is continuous at c,

then $\exists V_r(c)$ of c, s.t. if $x \in A \cap V_r(c)$,

then $f(x) \in V_\delta(f(c))$

③ $f(A) \subseteq B \Rightarrow$ If $x \in A \cap V_r(c)$, then $f(x) \in B \cap V_\delta(f(c))$, so $g(f(x)) \in V_\varepsilon(g(f(c)))$.

5.2.7 Theorem *Let $A, B \subseteq \mathbb{R}$, let $f : A \to \mathbb{R}$ be continuous on A, and let $g : B \to \mathbb{R}$ be continuous on B. If $f(A) \subseteq B$, then the composite function $g \circ f : A \to \mathbb{R}$ is continuous on A.*

Proof. The theorem follows immediately from the preceding result, if f and g are continuous at every point of A and B, respectively. Q.E.D.

5.2.8 Examples **(a)** Let $g_1(x) := |x|$ for $x \in \mathbb{R}$. It follows from the Triangle Inequality that

$$|g_1(x) - g_1(c)| \le |x - c|$$

for all $x, c \in \mathbb{R}$. Hence g_1 is continuous at $c \in \mathbb{R}$. If $f : A \to \mathbb{R}$ is any function that is continuous on A, then Theorem 5.2.7 implies that $g_1 \circ f = |f|$ is continuous on A. This gives another proof of Theorem 5.2.4.

(b) Let $g_2(x) := \sqrt{x}$ for $x \ge 0$. It follows from Theorems 3.2.10 and 5.1.3 that g_2 is continuous at any number $c \ge 0$. If $f : A \to \mathbb{R}$ is continuous on A and if $f(x) \ge 0$ for all $x \in A$, then it follows from Theorem 5.2.7 that $g_2 \circ f = \sqrt{f}$ is continuous on A. This gives another proof of Theorem 5.2.5.

Section 5.3 Continuous Functions on Intervals

5.3.1 Definition A function $f : A \to \mathbb{R}$ is said to be **bounded on** A if there exists a constant $M > 0$ such that $|f(x)| \le M$ for all $x \in A$.

5.3.2 Boundedness Theorem[†] *Let $I := [a, b]$ be a closed bounded interval and let $f : I \to \mathbb{R}$ be continuous on I. Then f is bounded on I.*

To show that each hypothesis of the Boundedness Theorem is needed, we can construct examples that show the conclusion fails if any one of the hypotheses is relaxed.

(i) The interval must be bounded. The function $f(x) := x$ for x in the unbounded, closed interval $A := [0, \infty)$ is continuous but not bounded on A.

(ii) The interval must be closed. The function $g(x) := 1/x$ for x in the half-open interval $B := (0, 1]$ is continuous but not bounded on B.

(iii) The function must be continuous. The function h defined on the closed interval $C := [0, 1]$ by $h(x) := 1/x$ for $x \in (0, 1]$ and $h(0) := 1$ is discontinuous and unbounded on C.

Proof: Assume $f: [a,b] \to R$ and f is unbounded

Then $\forall n \in N$, $\boxed{\exists x_n \in [a,b] \text{ s.t. } |f(x_n)| \geq n}$ (1)

$\Downarrow$

Then $\{x_n\}_n$ is bounded

So by Balzono–Weistress,

$\exists$ subsequence $\{x_{n_k}\}_k$ and $x \in R$

s.t. $\lim_{n\to\infty} x_{n_k} = x$

As $\forall k \in N$, $a \leq x_{n_k} \leq b$

So $a \leq \lim_{n\to\infty} x_{n_k} = x \leq b$

i.e. $x \in [a,b]$

As f is conts. on I

so $\lim_{x_{n_k}\to x} (f(x_{n_k})) = f(x) \overset{*}{\Rightarrow} (f(x_{n_k}))$ is bounded

$\Rightarrow \{|f(x_{n_k})|\}_k$ is bounded

and since $n_k \leq |f(x_{n_k})|$

$\Rightarrow \{n_k\}$ is bounded. which is impossible

because a sequence is infinite.

Thus, $f(x)$ is bounded on I

Q.E.D.

*Convergent sequences are bounded.

The Maximum-Minimum Theorem

5.3.3 Definition Let $A \subseteq \mathbb{R}$ and let $f : A \to \mathbb{R}$. We say that f **has an absolute maximum** on A if there is a point $x^* \in A$ such that

$$f(x^*) \ge f(x) \quad \text{for all} \quad x \in A.$$

We say that f **has an absolute minimum** on A if there is a point $x_* \in A$ such that

$$f(x_*) \le f(x) \quad \text{for all} \quad x \in A.$$

We say that x^* is an **absolute maximum point** for f on A, and that x_* is an **absolute minimum point** for f on A, if they exist.

5.3.4 Maximum-Minimum Theorem *Let $I := [a,b]$ be a closed bounded interval and let $f : I \to \mathbb{R}$ be continuous on I. Then f has an absolute maximum and an absolute minimum on I.*

Proof. Consider the nonempty set $f(I) := \{f(x) : x \in I\}$ of values of f on I. In Theorem 5.3.2 it was established that $f(I)$ is a bounded subset of $\mathbb{R}$. Let $s^* := \sup f(I)$ and $s_* := \inf f(I)$. We claim that there exist points x^* and x_* in I such that $s^* = f(x^*)$ and $s_* = f(x_*)$. We will establish the existence of the point x^*, leaving the proof of the existence of x_* to the reader.

Since $s^* = \sup f(I)$, if $n \in \mathbb{N}$, then the number $s^* - 1/n$ is not an upper bound of the set $f(I)$. Consequently there exists a number $x_n \in I$ such that

$$(1) \qquad s^* - \frac{1}{n} < f(x_n) \le s^* \quad \text{for all} \quad n \in \mathbb{N}.$$

Since I is bounded, the sequence $X := (x_n)$ is bounded. Therefore, by the Bolzano-Weierstrass Theorem 3.4.8, there is a subsequence $X' = (x_{n_r})$ of X that converges to some number x^*. Since the elements of X' belong to $I = [a,b]$, it follows from Theorem 3.2.6 that $x^* \in I$. Therefore f is continuous at x^* so that $\lim(f(x_{n_r})) = f(x^*)$. Since it follows from (1) that

$$s^* - \frac{1}{n_r} < f(x_{n_r}) \le s^* \quad \text{for all} \quad r \in \mathbb{N},$$

we conclude from the Squeeze Theorem 3.2.7 that $\lim(f(x_{n_r})) = s^*$. Therefore we have

$$f(x^*) = \lim(f(x_{n_r})) = s^* = \sup f(I).$$

We conclude that x^* is an absolute maximum point of f on I. Q.E.D.

Proof: As f is continuous on a close and interval I

then f is bounded. above and below.

Let $E = \{f(x), x \in I\}$ then $\sup(E)$ and $\inf(E)$ exist, let $M := \sup(E)$, $m := \inf(E)$

Then $\forall n \in N$, $M - \frac{1}{n}$ is not an upper bound for E, so $\exists x_n \in I$ s.t. $M - \frac{1}{n} < f(x_n) < M$

Similarly, $\exists y_n \in I$ s.t. $m < f(y_n) < m + \frac{1}{n}$

As (x_n) and (y_n) are all in I

so they are all bounded by I.

$\Rightarrow \exists X' = (x_{n_k}) \subseteq (x_n)$ s.t. X' is convergent

let $x^* := \lim(x_{n_k})$ then $x^* \in I$

$\exists Y' = (y_{n_k}) \subseteq (y_n)$ s.t. Y' is convergent

let $x_* := \lim(y_{n_k})$ then $x_* \in I$

i.e. (x_{n_k}) converges to x^*,

(y_{n_k}) converges to x_*

As f is continuous on I, $x^*, x_* \in I$

then $\lim\limits_{x_{n_k} \to x^*} f(x_{n_k}) = f(x^*)$

$\lim\limits_{y_{n_k} \to x_*} f(y_{n_k}) = f(x_*)$

$M - \frac{1}{n_k} < f(x_{n_k}) < M$ for $\forall k \in N$

$m < f(y_{n_k}) < m + \frac{1}{n_k}$ for $\forall k \in N$

so $f(x^*) = M$, $f(x_*) = m$

Q.E.D.

5.3.5 Location of Roots Theorem *Let $I = [a, b]$ and let $f : I \to \mathbb{R}$ be continuous on I. If $f(a) < 0 < f(b)$, or if $f(a) > 0 > f(b)$, then there exists a number $c \in (a, b)$ such that $f(c) = 0$.*

Proof. We assume that $f(a) < 0 < f(b)$. We will generate a sequence of intervals by successive bisections. Let $I_1 := [a_1, b_1]$, where $a_1 := a$, $b_1 := b$, and let p_1 be the midpoint $p_1 := \frac{1}{2}(a_1 + b_1)$. If $f(p_1) = 0$, we take $c := p_1$ and we are done. If $f(p_1) \neq 0$, then either $f(p_1) > 0$ or $f(p_1) < 0$. If $f(p_1) > 0$, then we set $a_2 := a_1$, $\boxed{b_2 := p_1}$, while if $f(p_1) < 0$, then we set $\boxed{a_2 := p_1}$, $b_2 := b_1$. In either case, we let $I_2 := [a_2, b_2]$; then we have $I_2 \subset I_1$ and $f(a_2) < 0$, $f(b_2) > 0$.

We continue the bisection process. Suppose that the intervals $I_1, I_2, \ldots, I_k$ have been obtained by successive bisection in the same manner. Then we have $f(a_k) < 0$ and $f(b_k) > 0$, and we set $p_k := \frac{1}{2}(a_k + b_k)$. If $f(p_k) = 0$, we take $c := p_k$ and we are done. If $f(p_k) > 0$, we set $a_{k+1} := a_k$, $b_{k+1} := p_k$, while if $f(p_k) < 0$, we set $a_{k+1} := p_k$, $b_{k+1} := b_k$. In either case, we let $I_{k+1} := [a_{k+1}, b_{k+1}]$; then $I_{k+1} \subset I_k$ and $f(a_{k+1}) < 0$, $f(b_{k+1}) > 0$.

If the process terminates by locating a point p_n such that $f(p_n) = 0$, then we are done. If the process does not terminate, then we obtain a nested sequence of closed bounded intervals $I_n := [a_n, b_n]$ such that for every $n \in \mathbb{N}$ we have

$$f(a_n) < 0 \quad \text{and} \quad f(b_n) > 0.$$

Furthermore, since the intervals are obtained by repeated bisection, the length of I_n is equal to $b_n - a_n = (b - a)/2^{n-1}$. It follows from the Nested Intervals Property 2.5.2 that there exists a point c that belongs to I_n for all $n \in \mathbb{N}$. Since $a_n \leq c \leq b_n$ for all $n \in \mathbb{N}$ and $\lim(b_n - a_n) = 0$, it follows that $\lim(a_n) = c = \lim(b_n)$. Since f is continuous at c, we have

$$\lim(f(a_n)) = f(c) = \lim(f(b_n)).$$

The fact that $f(a_n) < 0$ for all $n \in \mathbb{N}$ implies that $f(c) = \lim(f(a_n)) \leq 0$. Also, the fact that $f(b_n) > 0$ for all $n \in \mathbb{N}$ implies that $f(c) = \lim(f(b_n)) \geq 0$. Thus, we conclude that $f(c) = 0$. Consequently, c is a root of f. Q.E.D.

5.3.6 Example The equation $f(x) = xe^x - 2 = 0$ has a root c in the interval $[0, 1]$, because f is continuous on this interval and $f(0) = -2 < 0$ and $f(1) = e - 2 > 0$. Using a calculator we construct the following table, where the sign of $f(p_n)$ determines the interval at the next step. The far right column is an upper bound on the error when p_n is used to approximate the root c, because we have

$$|p_n - c| \leq \tfrac{1}{2}(b_n - a_n) = 1/2^n.$$

We will find an approximation p_n with error less than 10^{-2}.

n	a_n	b_n	p_n	$f(p_n)$	$\boxed{\frac{1}{2}(b_n - a_n)}$
1	0	1	.5	−1.176	.5
2	.5	1	.75	−.412	.25
3	.75	1	.875	+.099	.125
4	.75	.875	.8125	−.169	.0625
5	.8125	.875	.84375	−.0382	.03125
6	.84375	.875	.859375	+.0296	.015625
7	.84375	.859375	.8515625	—	.0078125

Max Error
=
边界-中点的距离.

We have stopped at $n = 7$, obtaining $c \approx p_7 = .8515625$ with error less than .0078125. < 0.01 This is the first step in which the error is less than 10^{-2}. The decimal place values of p_7 past the second place cannot be taken seriously, but we can conclude that $.843 < c < .860$. □

middle 中 (夹在f(a), f(b)中间的值)

Bolzano's Theorem

5.3.7 Bolzano's Intermediate Value Theorem *Let I be an interval and let $f : I \to \mathbb{R}$ be continuous on I. If $a, b \in I$ and if $k \in \mathbb{R}$ satisfies $f(a) < k < f(b)$, then there exists a point $c \in I$ between a and b such that $f(c) = k$.*

也可保持 f(x)-k

Proof. Suppose that $a < b$ and let $g(x) := f(x) - k$; then $g(a) < 0 < g(b)$. By the Location of Roots Theorem 5.3.5 there exists a point c with $a < c < b$ such that $0 = g(c) = f(c) - k$. Therefore $f(c) = k$.

If $b < a$, let $h(x) := k - f(x)$ so that $h(b) < 0 < h(a)$. Therefore there exists a point c with $b < c < a$ such that $0 = h(c) = k - f(c)$, whence $f(c) = k$. Q.E.D.

$[a,b]$

5.3.8 Corollary *Let $I = [a,b]$ be a closed, bounded interval and let $f : I \to \mathbb{R}$ be continuous on I. If $k \in \mathbb{R}$ is any number satisfying*

$$\inf f(I) \leq k \leq \sup f(I),$$

then there exists a number $c \in I$ such that $f(c) = k$.

Proof. It follows from the Maximum-Minimum Theorem 5.3.4 that there are points c_* and c^* in I such that

$$\inf f(I) = f(c_*) \leq k \leq f(c^*) = \sup f(I).$$

The conclusion now follows from Bolzano's Theorem 5.3.7. Q.E.D.

$[a,b]$

5.3.9 Theorem *Let I be a closed bounded $\boxed{\text{interval}}$ and let $f : I \to \mathbb{R}$ be continuous on I. Then the set $f(I) := \{f(x) : x \in I\}$ is a closed bounded $\boxed{\text{interval}}$.*

$[m,M] = f(I)$

Proof: Let $m := \inf f(I)$ $M := \sup f(I)$ $\Rightarrow f(I) \subseteq [m,M]$

(WTS: $[m,M] = f(I)$)

$\forall k \in f(I)$, $k \in [m,M]$.

By B-IVT, $\exists c \in I$ s.t. $f(c) = k$

Thus, $\forall k \in [m,M]$, $k \in f(I)$

So $[m,M] \subseteq f(I)$

Whence, $[m,M] = f(I)$

Q.E.D.

5.3.10 Preservation of Intervals Theorem *Let I be an interval and let $f : I \to \mathbb{R}$ be continuous on I. Then the set $f(I)$ is an interval.* or constant

Proof. Let $\alpha, \beta \in f(I)$ with $\alpha < \beta$; then there exist points $a, b \in I$ such that $\alpha = f(a)$ and $\beta = f(b)$. Further, it follows from Bolzano's Intermediate Value Theorem 5.3.7 that if $k \in (\alpha, \beta)$ then there exists a number $c \in I$ with $k = f(c) \in f(I)$. Therefore $[\alpha, \beta] \subseteq f(I)$, showing that $f(I)$ possesses property (1) of Theorem 2.5.1. Therefore $f(I)$ is an interval. Q.E.D.

* 意在说明 Function 不会将一个连续的区间 map 为间断的点，
(Interval)

1、定义判定法

5.4.1 Definition Let $A \subseteq \mathbb{R}$ and let $f : A \to \mathbb{R}$. We say that f is **uniformly continuous** on A if for each $\varepsilon > 0$ there is a $\delta(\varepsilon) > 0$ such that if $x, u \in A$ are any numbers satisfying $|x - u| < \delta(\varepsilon)$, then $|f(x) - f(u)| < \varepsilon$.

5.4.2 Nonuniform Continuity Criteria *Let $A \subseteq \mathbb{R}$ and let $f : A \to \mathbb{R}$. Then the following statements are equivalent:*

(i) *f is not uniformly continuous on A.*

(ii) *There exists an $\varepsilon_0 > 0$ such that for every $\delta > 0$ there are points x_δ, u_δ in A such that $|x_\delta - u_\delta| < \delta$ and $|f(x_\delta) - f(u_\delta)| \geq \varepsilon_0$.*

(iii) *There exists an $\varepsilon_0 > 0$ and two sequences (x_n) and (u_n) in A such that $\lim(x_n - u_n) = 0$ and $|f(x_n) - f(u_n)| \geq \varepsilon_0$ for all $n \in \mathbb{N}$.*

D

2、区间 I 判定

$[a,b]$

5.4.3 Uniform Continuity Theorem *Let I be a closed bounded interval and let $f : I \to \mathbb{R}$ be continuous on I. Then f is uniformly continuous on I.*

i.e. For close and bounded interval I, $f: I \to R$.

Continuity $\Leftrightarrow$ Uniform Continuity.

Proof: Assume that when I is closed & bounded, $f: I \to R$ is conts. on I But f is not uniformly conts.

Then $\exists \varepsilon_0 > 0$ s.t. $\forall \delta > 0$, $\exists x, c \in [a,b]$

s.t. $|x-c| < \delta$ but $|f(x) - f(c)| \geq \varepsilon_0$

Then $\forall n \in N$, $\exists x_n, c_n \in [a,b]$ s.t.

$|x_n - c_n| < \frac{1}{n}$ ($\Leftrightarrow \lim |x_n - c_n| = 0$)

yet $|f(x_n) - f(c_n)| \geq \varepsilon_0$

As (x_n) is bounded by a and b,

by B-W Thm. $\exists$ subsequence (x_{n_k}) of (x_n)

$\lim\limits_{k \to \infty} (x_{n_k}) = x$ s.t. $x \in [a,b]$

Since (c_n) is also bounded by a and b

So exists subseq. (c_{n_k}) of (c_n) s.t.

$\lim\limits_{k \to \infty} (c_{n_k}) = c$, $c \in [a,b]$

As $|x - c| = |x - x_{n_k} + x_{n_k} - c|$

$\leq |x - x_{n_k}| + |x_{n_k} - c| < \varepsilon$

So $z := x = c \Leftrightarrow \lim\limits_{x_{n_k} \to z} (f(x_{n_k})) = \lim\limits_{c_{n_k} \to z} (f(c_{n_k})) = f(z)$

$\Rightarrow$ when $x \to c$, $|f(x) - f(c)| < \varepsilon$

Contradicts to assumption.

$\therefore$ QED

① Assume f is not u. Conts. $\Rightarrow$ nonuniform ~ (iii)

② I is bounded $\Rightarrow$ (x_n) is bounded $\Rightarrow \exists$ subseq. (x_{n_k}) s.t. $\exists \lim(x_{n_k})$

$\lim(u_{n_k})$ exists.

$\Rightarrow \lim(x_{n_k} - u_{n_k}) = 0 \Rightarrow \lim(x_{n_k}) = \lim(u_{n_k})$

③ I is closed $\overset{\#}{\Rightarrow} \lim(x_{n_k}) = \lim(u_{n_k}) \in \underset{\#}{I}$

By 4.1.8 Seq. Cri. $f(\lim(x_{n_k})) = \lim(f(x_{n_k}))$

$f(\lim(u_{n_k})) = \lim(f(u_{n_k}))$ Contradicts to Assumption.

Standard:

As f is uniformly continuous on $A = (a,b)$

then $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $\forall x, u \in A$,

with $|x-u| < \delta$, then $|f(x) - f(u)| < \varepsilon$

5.4.4 Definition Let $A \subseteq \mathbb{R}$ and let $f : A \to \mathbb{R}$. If there exists a constant $K > 0$ such that

$$(4) \qquad |f(x) - f(u)| \le K|x - u|$$

for all $x, u \in A$, then f is said to be a **Lipschitz function** (or to satisfy a **Lipschitz condition**) on A.

不可用从A上任意选取的C指代K，K必须是一个已知常数(>0).

The condition (4) that a function $f : I \to \mathbb{R}$ on an interval I is a Lipschitz function can be interpreted geometrically as follows. If we write the condition as

$$\left|\frac{f(x) - f(u)}{x - u}\right| \le K, \qquad x, u \in I, x \neq u,$$

then the quantity inside the absolute values is the slope of a line segment joining the points $(x, f(x))$ and $(u, f(u))$. Thus a function f satisfies a Lipschitz condition if and only if the slopes of all line segments joining two points on the graph of $y = f(x)$ over I are bounded by some number K.

3. Lipschitz 判定: 任意 Interval

Lipschitz Functions

Lip → U. Conts.

5.4.5 Theorem *If $f : A \to \mathbb{R}$ is a Lipschitz function, then f is uniformly continuous on A.*

Proof. If condition (4) is satisfied, then given $\varepsilon > 0$, we can take $\delta := \varepsilon/K$. If $x, u \in A$ satisfy $|x - u| < \delta$, then

$$|f(x) - f(u)| < K \cdot \frac{\varepsilon}{K} = \varepsilon.$$

Therefore f is uniformly continuous on A. Q.E.D.

5.4.6 Examples

5.4.6 Examples **(a)** If $f(x) := x^2$ on $A := [0, b]$, where $b > 0$, then

$$|f(x) - f(u)| = |x + u||x - u| \le 2b|x - u|$$

for all x, u in $[0, b]$. Thus f satisfies (4) with $K := 2b$ on A, and therefore f is uniformly continuous on A. Of course, since f is continuous and A is a closed bounded interval, this can also be deduced from the Uniform Continuity Theorem. (Note that f does *not* satisfy a Lipschitz condition on the interval $[0, \infty)$.)

(b) Not every uniformly continuous function is a Lipschitz function.

Let $g(x) := \sqrt{x}$ for x in the closed bounded interval $I := [0, 2]$. Since g is continuous on I, it follows from the Uniform Continuity Theorem 5.4.3 that g is uniformly continuous on I. However, there is no number $K > 0$ such that $|g(x)| \le K|x|$ for all $x \in I$. (Why not?) Therefore, g is not a Lipschitz function on I.

T: ¬Lip → ¬Un.conts.

F: Uni. Conts. → Lip

(c) The Uniform Continuity Theorem and Theorem 5.4.5 can sometimes be combined to establish the uniform continuity of a function on a set.

We consider $g(x) := \sqrt{x}$ on the set $A := [0, \infty)$. The uniform continuity of g on the interval $I := [0, 2]$ follows from the Uniform Continuity Theorem as noted in (b). If $J := [1, \infty)$, then if both x, u are in J, we have

$$|g(x) - g(u)| = |\sqrt{x} - \sqrt{u}| = \frac{|x-u|}{\sqrt{x} + \sqrt{u}} \le \tfrac{1}{2}|x - u|.$$

Thus g is a Lipschitz function on J with constant $K = \frac{1}{2}$, and hence by Theorem 5.4.5, g is uniformly continuous on $[1, \infty)$. Since $A = I \cup J$, it follows [by taking $\delta(\varepsilon) := \inf\{1, \delta_I(\varepsilon), \delta_J(\varepsilon)\}$] that g is uniformly continuous on A. We leave the details to the reader. □

Uniform continuous function 可将一个Cauchy Sequence (x_n) map 为一个同为Cauchy的Seq. $(f(x_n))$.

5.4.7 Theorem *If $f : A \to \mathbb{R}$ is uniformly continuous on a subset A of* $\mathbb{R}$ and *if (x_n) is a Cauchy sequence in A, then $(f(x_n))$ is a Cauchy sequence in* $\mathbb{R}$.

任意

The Continuous Extension Theorem

4. Bounded 区间

Proof: As (x_n) is a sequence in A and $f: A \to R$ is uniformly continuous,

$\forall m, n \in N$, $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t.

if $|x_m - x_n| < \delta$ then $|f(x_m) - f(x_n)| < \varepsilon$ (1)

As (x_n) is a Cauchy sequence,

then $\exists k \in N$ s.t. $\forall m, n \geq k$,

$|x_m - x_n| < \varepsilon$

So $\exists M \in N$ s.t. $\forall m, n \geq M$,

$|x_m - x_n| < \delta$

from (1), then $|f(x_m) - f(x_n)| < \varepsilon$

Thus, $(f(x_n))$ is a cauchy sequence.

Q.E.D.

Proof. Let (x_n) be a Cauchy sequence in A, and let $\varepsilon > 0$ be given. First choose $\delta > 0$ such that if x, u in A satisfy $|x - u| < \delta$, then $|f(x) - f(u)| < \varepsilon$. Since (x_n) is a Cauchy sequence, there exists $H(\delta)$ such that $|x_n - x_m| < \delta$ for all $n, m > H(\delta)$. By the choice of δ, this implies that for $n, m > H(\delta)$, we have $|f(x_n) - f(x_m)| < \varepsilon$. Therefore the sequence $(f(x_n))$ is a Cauchy sequence. Q.E.D.

5.4.8 Continuous Extension Theorem *A function f is uniformly continuous on the interval (a, b) if and only if it can be defined at the endpoints a and b such that the extended function is continuous on $[a, b]$.*

Proof:

$\boxed{\Rightarrow}$ As f is uniformly continuous on $A = (a,b)$

Also, define convergent sequence (x_n) on A

s.t. $\lim_{n\to\infty}(x_n) = a$

then (x_n) is cauchy

therefore, $(f(x_n))$ is cauchy $\Rightarrow$ convergent.

then $\lim(f(x_n))$ exists, name it L.

Define (u_n) be any other sequence on A

that converges to a

then $\lim(u_n) = \lim(x_n)$

$\Rightarrow \forall \varepsilon > 0, \left|\lim(u_n - x_n)\right| < \varepsilon$

As f is uniformly convergent,

then $\left|\lim(f(u_n) - f(x_n))\right| < \varepsilon$

$\Rightarrow 0 < \left|\lim f(u_n) - \lim f(x_n)\right| = \left|\lim f(u_n) - L\right| < \varepsilon$

Thus, $\lim f(u_n) = L$

As (u_n) is arbitrary, then every sequence on A that converges to a, $f(x_n)$ converges to L

By Sequential Criteria,

$\lim_{x\to a} f(x) = \lim_{x_n \to a} f(x_n) = L$

Define $f(a) = L$ then

$\lim_{x\to a} f(x) = f(a)$.

Thus, f continuous at $x = a$

$\Rightarrow f$ continuous on $[a,b)$

Similarly for $x = b$.

Q.E.D.

$\boxed{\Leftarrow}$ Define $F(x) = \begin{cases} f(a). & x = a \\ f(x). & x \in (a,b) \\ f(b) & x = b \end{cases}$

then F continuous on $[a,b]$

As $[a,b]$ is a close & bounded interval

then F uniformly conts. on $[a,b]$

$\Rightarrow f$ uniformly conts. on (a,b)

Q.E.D.