Contents

List of Figures

List of Algorithms

List of Open Problems

Preface

This text is under active development, especially due to comments from colleagues, notably the Bath Mathematical Foundations seminar that listened to an explanation of section 4.4, and students on the course CM30070 — Computer Algebra at the University of Bath. David Wilson (now at Facebook) has been a most helpful recorder of errors during this, and he and Matthew England (now at Coventry University) have signalled several problems. David Stoutemyer (Hawaii) has been a very helpful reader. I am grateful to John Abbott (Genoa) for the polynomial f on page 210, and for the encouragement of people at CICM 2010 to write Chapter 8. Figure 3.7 and the subsequent discussion owes much to Chris Brown (US Naval Academy) and Matthew England. I am grateful to Professor Vorobjov for the material in Excursus B.4.3. Professor Sergei Abramov (Russian Academy of Sciences) has been a helpful reader. As always when I write, I am grateful to David Carlisle for his TEXnical wisdom.

It is probably best cited as "J.H. Davenport, *Computer Algebra*. `http://staff.bath.ac.uk/masjhd/JHD-CA.pdf`" with a date.

Mathematical Prerequisites

This book has been used to support teaching at the University of Bath to both Mathematics and Computer Science Students. The main requirement is "mathematical maturity" rather than specific facts: some specific pre-requisites are listed in Table 1. If the students are not familiar with the O notation, it would be as well for the lecturer to explain this, at whatever level of sophistication is appropriate: this book is pretty unsophisticated in its use, and all that is needed is in section 1.4.

Table 1: Specific Prerequisites

Chapter(s)	Subjects
3.2.1	Matrices
4,5	Modular arithmetic
7, 8	Calculus

Changes in Academic Year 2017–18

10.10.2017 Fix typo in Definition 2.

8.11.2017 Fix (3.81) thanks to Zak Tonks.

18.11.2017 Expand (by quoting equation (3.29) — newly numbered) the description of "Interreduce" in the Buchberger algorithms. Expand Algorithm 13 to include solution-counting.

21.11.2017 Fix Theorem 16 thanks to Josh Dutton.

23.11.2017 Improve notation in section A.2.1.

24.11.2017 Clarify that 1 is a monomial on pages pagerefMonomialTerm and 117.

8.12.2017 Fix typos in section 4.6 and in description of VTS.

15.12.2017 Fix error on column labels in Table 4.6.2.

Chapter 1

Introduction

Computer algebra, the use of computers to do algebra rather than simply arithmetic, is almost as old as computing itself, with the first theses [Kah53, Nol53] dating back to 1953. Indeed it was anticipated from the time of Babbage, when Ada Augusta, Countess of Lovelace, wrote

> We may say most aptly that the Analytical Engine weaves algebraical patterns just as the Jacquard loom weaves flowers and leaves. [Ada43]

In fact, it is not even algebra for which we need software packages, computers by themselves can't actually do arithmetic: only a limited subset of it. If we ask Excel[1] to compute $e^{\pi\sqrt{163}} - 262537412640768744$, we will be told that the answer is 256. More mysteriously, if we go back and look at the formula in the cell, we see that it is now $e^{\pi\sqrt{163}} - 262537412640768800$. In fact, 262537412640768744 is too large a whole number (or integer, as mathematicians say) for Excel to handle, and it has converted it into floating-point (what Excel terms "scientific") notation. Excel, or any other software using the IEEE standard [IEE85] representation for floating-point numbers, can only store them to a given accuracy, about[2] 16 decimal places.[3] In fact, it requires twice this precision to show that $e^{\pi\sqrt{163}} \neq 262537412640768744$. Since $e^{\pi\sqrt{163}} = (-1)^{\sqrt{-163}}$, it follows from deep results of transcendental number theory [Bak75], that not only is $e^{\pi\sqrt{163}}$ not an integer, it is not a fraction (or rational number), nor even the solution of a polynomial equation with integer coefficients: essentially it is a 'new' number.

[1]Or any similar software package.

[2]We say 'about' since the internal representation is binary, rather than decimal.

[3]In fact, Excel is more complicated even than this, as the calculations in this table show.

i		1	2	3	4	…	10	11	12	…	15	16
a	10^i	10	100	1000	1…0	…	1…0	10^{11}	10^{12}	…	10^{15}	10^{16}
b	`a-1`	9	99	999	9999	9…9	…	9…9	10^{12}	…	10^{15}	10^{16}
c	`a-b`	1	1	1	1	1	…	1	1	…	1	0

We can see that the printing changes at 12 decimal digits, but that actual accuracy is not lost until we subtract 1 from 10^{16}.

Definition 1 *A number n, or more generally any algebraic object, is said to be* transcendental *over a ring R if there is no non-zero polynomial p with coefficients in R such that $p(n) = 0$.*

With this definition, we can say that $e^{\pi\sqrt{163}}$ is transcendental over the integers. Transcendence is a deep mathematical question with implications for the decidability of computer algebra, see page 299.

We will see throughout this book (for an early example, see page 66) that innocent-seeming problems can give rise to numbers far greater than one would expect. Hence there is a requirement to deal with numbers larger, or to greater precision, than is provided by the hardware manufacturers whose chips underlie our computers. Practically every computer algebra system, therefore, has a package for manipulating arbitrary-sized integers (so-called *bignums*) or real numbers (*bigfloats*). These arbitrary-sized objects, and the fact that mathematical objects in general are unpredictably sized, means that computer algebra systems generally need sophisticated memory management, generally garbage collection (see [JHM11] for a general discussion of this topic).

But the fact that the systems *can* deal with large numbers does not mean that we *should* let numbers increase without doing anything. If we have two numbers with n digits, adding them requires a time proportional to n, or in more formal language (see section 1.4) a time $O(n)$. Multiplying them[4] requires a time $O(n^2)$. Calculating a g.c.d., which is fundamental in the calculation of rational numbers, requires $O(n^3)$, or $O(n^2)$ with a bit of care[5]. This implies that if the numbers become 10 times longer, the time is multiplied by 10, or by 100, or by 1000. So it is always worth reducing the size of these integers. We will see later (an early example is on page 163) that much ingenuity has been well-spent in devising algorithms to compute "obvious" quantities by "non-obvious" ways which avoid, or reduce, the use of large numbers. The phrase *intermediate expression swell* is used to denote the phenomenon where intermediate quantites are much larger than the input to, or outputs from, a calculation.

Notation 1 *We write algorithms in an Algol-like notation, with the Rutishauser symbol* `:=` *to indicate assignment, and* `=` *(as opposed to C's* `==`*) to indicate the equality predicate. We use indentation to indicate grouping*[6]*, rather than clutter the text with* **begin** ... **end**. *Comments are introduced by the # character, running to the end of the line.*

[4]In principle, $O(n \log n \log\log n)$ is enough [AHU74, Chapter 8], but no computer algebra system routinely uses this, for it is more like $20n \log n \log\log n$. However, most systems will use 'Karatsuba arithmetic' (see [KO63, and section B.3]), which takes $O(n^{\log_2 3} \approx n^{1.585})$, once the numbers reach an appropriate length, often 16 words [SGV94].

[5]In principle, $O(n \log^2 n \log\log n)$ [AHU74, Chapter 8], but again no system uses it routinely, but this combined with Karatsuba gives $O(n^{\log_2 3} \log n)$ (section B.3.6), and this is commonly used.

[6]An idea which was tried in Axiom [JS92], but which turns out to be better in books than in real life.

1.1 History and Systems

The first recorded use of computers to do computations of the sort we envisage was in 1951 [MW51], where $180\left(2^{127}-1\right)^2-1$, a 79-digit number, was shown to be prime. In 1952 the great mathematician Emil Artin had the equally great computer scientist John von Neumann perform an extensive calculation relating to elliptic curves on the MANIAC computer [ST92, p. 119]. In 1953, two theses [Kah53, Nol53] kicked off the 'calculus' side of computer algebra with programs to differentiate expressions. In the same year, [Has53] showed that algorithms in group theory could be implemented on computers.

1.1.1 The 'calculus' side

The initial work [Kah53, Nol53] consisted of programs to do one thing, but the focus soon moved on to 'systems', capable of doing a variety of tasks. One early one was Collins' system SAC [Col71], written in Fortran. Its descendants continue in SAC-2 [Col85] and QEPCAD [Bro03][7]. Of course, computer algebra systems *can* be written in any computer language, even Cobol [fHN76].

However, many of the early systems were written in LISP, largely because of its support for garbage collection and large integers. The group at M.I.T., very influential in the 1960s and 70s, developed Macsyma [MF71, PW85]. This system now exists in several versions [Ano07], even on mobile 'phones[8]. At about the same time, Hearn developed Reduce [Hea05], and shortly after a group at IBM Yorktown Heights produced SCRATCHPAD [GJY75]. This group then produced AXIOM [JS92], a system that attempted to match the generality of mathematics with 'generic programming' to allow algorithms to be programmed in the generality in which they are conventionally (as in this book) stated, e.g. "polynomials over a ring" as in Definition 22.

These were, on the whole, very large software systems, and attempts were made to produce smaller ones. muMATH [RS79] and its successor Derive [RS92] were extremely successful systems on the early PC, and paved the way for the computer algebra facilities of many high-end calculators. Much of the work on "compact computer algebra" is described in [Sto11b].

Maple [CGGG83] pioneered a 'kernel+library' design.

> The basic Maple system, the *kernel*, is a relatively small collection of compiled C code. When a Maple session is started, the entire kernel is loaded. It contains the essential facilities required to run Maple and perform basic mathematical operations. These components include the Maple programming language interpreter, arithmetic and simplification routines, print routines, memory management facilities, and a collection of fundamental functions. [MGH⁺03, p. 6]

[7]Also now available on mobile 'phones: `https://sites.google.com/site/maximaonandroid/`.

[8]`https://sites.google.com/site/maximaonandroid/`.

1.1.2 The 'group theory' side

Meanwhile, those interested in computation group theory, and related topics, had not been idle. One major system developed during the 1970s/80s was CAYLEY [BC90]. This team later looked at Axiom, and built the system MAGMA [BCM94], again with a strong emphasis on genericity. Another popular system is GAP [BL98], whose 'kernel+library' design was consciously [Neu95] inspired by Maple.

The reader may well ask 'why two different schools of thought?' The author has often asked himself the same question. The difference seems one of mathematical attitude, if one can call it that. The designer of a calculus system envisages it being used to compute an integral, factor a polynomial, multiply two matrices, or otherwise operate on a mathematical *datum*. The designer of a group theory system, while he will permit the user to multiply, say, permutations or matrices, does not regard this as the object of the system: rather the object is to manipulate whole groups (etc.) of permutations (or matrices, or ...), i.e. a mathematical *structure*.

1.1.3 A synthesis?

While it is too early to say that the division has been erased, it can be seen that MAGMA, for example, while firmly rooted in the group-theory tradition, has many more 'calculus like' features. Conversely, the interest in polynomial ideals, as in Proposition 35, means that systems specialising in this direction, such as SINGULAR [Sch03b] or COCOA [GN90], use polynomial algorithms, but the items of interest are the mathematical structures such as ideals rather than the individual polynomials.

1.2 Expansion and Simplification

These two words, or the corresponding verbs 'expand' and 'simplify' are much used (abused?) in computer algebra, and a preliminary explanation may be in order. Computers of course, do not deal in mathematical objects, but, ultimately, in certain bit patterns which are *representations* of mathematical objects. Just as in ink on paper, a mathematical object may have many representations, just as most of us would say that $x + y$ and $y + x$ are the same mathematical object.

Definition 2 *A correspondence f between a class O of objects (generally we think of the abstract objects of mathematics) and a class R of representations is* a representation *of O by R if each element of O corresponds to one or more elements of R (otherwise it is not represented) and each element of R corresponds to one and only one element of O (otherwise we do not know which element of O is represented).* In other words "is represented by", is the inverse of a surjective function f or "represents" from a subset of R (the "legal representations") to O.

Notation 2 *When discussing the difference between abstract objects and computer representations, we will use mathematical fonts, such as x, for the abstract objects and typewriter font, such as* `x`*, for the representations.*

Hence we could represent any mathematical objects, such as polynomials, by well-formed strings of indeterminates, numbers and the symbols `+,-,*,(,)`. The condition "well-formed" is necessary to prevent nonsense such as `)x+++1y(`, and would typically be defined by some finite grammer [ALSU07, Section 2.2]. With no simplification rules, such a representation would regard `x-x` as just that, rather than as zero.

Definition 3 *A representation of a monoid (i.e. a set with a 0, and an addition operation) is said to be* normal *if the only representation of the object 0 is* `0`.

If we have a normal representation f, then we can tell if two mathematical objects a and b, represented by `a` and `b`, are equal by computing `a-b`: if this is zero (i.e. `0`), then a and b are equal, while if this is not zero, they must be unequal. However, this is an inefficient method, to put it mildly.

Observation 1 *Normal representations are very important in practice, since many algorithms contain tests for zero/non-zero of elements. Sometimes these are explicit, as in Gaussian elimination of a matrix, but often they are implicit, as in Euclid's Algorithm (Algorithm 2, page 63), where we take the remainder after dividing one polynomial by another, which in turn requires that we know the leading* non-zero *coefficient of the divisor.*

Definition 4 *A representation is said to be* canonical *if every object has only one representation, i.e. f is a bijection, or 1–1 mapping.*

With a canonical representation, we can say that two objects "*are* the same if, and only if, they *look* the same". For example, we cannot have *both* `(x+1)^2` and `x^2+2x+1`, since $(x+1)^2 = x^2 + 2x + 1$.

Definition 5 *A representation is said to be* locally canonical *(with respect to a certain context) if every object whose introduction does not change the context has only one representation, i.e. f (restricted to non context-changing expressions) is a bijection, or 1–1 mapping.*

It is possible to build any normal representation into a locally canonical one, using an idea due to [Bro69]. We store every computed (top-level) expression $e_1, e_2, \ldots$, and for a new expression E, we compute the normal form of every $E - e_i$. If this is zero, then $E = e_i$, so we return e_i, otherwise E is a new expression to be stored. This has various objections, both practical (the storage and computation time) and conceptual (the answer printed depends on the context, i.e. the past history of the session), but nevertheless is worth noting.

Another example of locally canonical representations is at page 84. Note that no guarantee is made about comparing expressions in different contexts.

Definition 6 ([Sto11a, Definition 3]) *A* candid *expression is one that is not equivalent to an expression that visibly manifests a simpler expression class.*

In particular, if the "simpler class" is $\{0\}$, "candid" means the same as normal, but the concept is much more general, and useful. In particular, if a candid expressions contains a variable v, then it really depends on v. Untyped systems such as Maple and Macsyma *could* be candid in this sense[9]: typed systems like Axiom have more problems in this area, since subtracting two polynomials in v will produce a polynomial in v, even if the result in a particular case is, say, $v-(v-1)=1$, which doesn't in fact depend on v.

Candidness depends on the class of expressions considered as 'simpler'. Consider

$$F : x \mapsto x - \sqrt{x^2} \tag{1.1}$$

(viewed as a function $\mathbf{R} \to \mathbf{R}$). It might be tempting to consider F to be zero, but in fact $F(-1) = -2$. In fact, F can also be written as $x \mapsto x - |x|$ or $x \mapsto \begin{cases} 0 & x \ge 0 \\ 2x & x \le 0 \end{cases}$. Is (1.1) therefore candid? It depends on the definition of "simpler".

Since practically everyone would agree that $\{0\}$ is a simple class, candid expressions are automatically normal, but need not be canonical (see (1.1)), so this definition provides a useful intermediate ground, which often coïncides with naïve concepts of "simpler". We refer the reader to [Sto11a] for further, very illustrative, examples, to section 2.2.2 for a discussion of candidness for rational functions, and to sections 7.3.2 and 7.3.3 for a discussion of candidness in integration theory.

We should note that 'candidness' is not always achievable. One possible requirement would be "if the final value is real, then no complex numbers should be involved", but this is not generally possible: at (3.10) we see that the three real roots of $x^3 - x$ can only be computed via complex numbers. In this case, they have simple expressions (0, +1, and −1), but that is not true in general: consider $x^3 - x - \frac{1}{1000}$, whose roots are

$$\begin{array}{c}
\frac{1}{60}\sqrt[3]{108+12\,i\sqrt{11999919}} + 20\,\frac{1}{\sqrt[3]{108+12\,i\sqrt{11999919}}}, \\
-\frac{1}{120}\sqrt[3]{108+12\,i\sqrt{11999919}} - 10\,\frac{1}{\sqrt[3]{108+12\,i\sqrt{11999919}}} + \\
\frac{i\sqrt{3}}{20}\left(1/6\sqrt[3]{108+12\,i\sqrt{11999919}} - 200\,\frac{1}{\sqrt[3]{108+12\,i\sqrt{11999919}}}\right), \\
-\frac{1}{120}\sqrt[3]{108+12\,i\sqrt{11999919}} - 10\,\frac{1}{\sqrt[3]{108+12\,i\sqrt{11999919}}} - \\
\frac{i\sqrt{3}}{20}\left(1/6\sqrt[3]{108+12\,i\sqrt{11999919}} - 200\,\frac{1}{\sqrt[3]{108+12\,i\sqrt{11999919}}}\right).
\end{array}$$

1.2.1 A Digression on "Functions"

In colloquial mathematics, the word "function" has many uses and meanings, and one of the distinctive features of computer algebra is that it plays on these

[9]See section 2.2.2 for other aspects of candidness.

various uses.

In principle, (pure) mathematics is clear on the meaning of "function".

> On dit qu'un graphe F est un graphe fonctionnel si, pour tout x, il existe au plus un objet correspondant à x par F (I, p. 40). On dit qu'une correspondance $f = (F, A, B)$ est une fonction si son graphe F est un graphe fonctionnel, et si son ensemble de départ A est égal à son ensemble de définition $\mathrm{pr}_1 F$ [pr_1 is "projection on the first component"].
>
> [Bou70, p. E.II.13]

The present author's loose translation is as follows.

> We say that a graph (i.e. a subset of $A \times B$) F is a functional graph if, for every x, there is at most one pair $(x, y) \in F$ [alternatively $(x, y_1), (x, y_2) \in F$ implies $y_1 = y_2$]. We then say that a correspondance $f = (F, A, B)$ is a function if its graph F is a functional graph and the starting set A is equal to $\mathrm{pr}_1 F = \{x \mid \exists y(x, y) \in F\}$.

So for Bourbaki a function includes the definition of the domain and codomain, and is *total* and *single-valued*.

Notation 3 *We will write $(F, A, B)_{\mathcal{B}}$ for such a function definition. If $C \subset A$, we will write $(F, A, B)_{\mathcal{B}}|_C$, "$(F, A, B)_{\mathcal{B}}$ restricted to C", for $(G, C, B)_{\mathcal{B}}$ where $G = \{(x, y) \in F | x \in C\}$.*

Hence the function sin is, formally, $(\{(x, \sin x) : x \in \mathbf{R}\}, \mathbf{R}, \mathbf{R})_{\mathcal{B}}$, or, equally possibly, $(\{(x, \sin x) : x \in \mathbf{C}\}, \mathbf{C}, \mathbf{C})_{\mathcal{B}}$. But what about $x + 1$? We *might* be thinking of it as a function, e.g. $(\{(x, x + 1) : x \in \mathbf{R}\}, \mathbf{R}, \mathbf{R})_{\mathcal{B}}$, but equally we might just be thinking of it as an abstract polynomial, a member of $\mathbf{Q}[x]$. In fact, the abstract view can be pursued with sin as well, as seen in Chapter 7, especially Observation 18, where sin is viewed as $\frac{1}{2i}(\theta - 1/\theta) \in \mathbf{Q}(i, x, \theta | \theta' = i\theta)$. The relationship between the abstract view and the Bourbakist view is pursued further in Chapter 8.

1.2.2 Expansion

This word is relatively easy to define, as application of the distributive law, as seen in the first bullet point of Maple's description of the `expand` command.

> - The `expand` command distributes products over sums. This is done for all polynomials. For quotients of polynomials, only sums in the numerator are expanded; products and powers [in the denominator[10]] are left alone.
> - The expand command also expands most mathematical functions, including ....

[10]See Section 1.5.2.

In any given system, the precise meaning of expansion depends on the underlying polynomial representation used (recursive/distributed — see page 49), so Maple, which is essentially distributed, would expand $x(y+1)$ into $xy+x$, while Reduce, which is recursive, would not, but would expand $y(x+1)$ into $xy+y$, since its default ordering is 'x before y'.

Expansion can, of course, cause exponential blow-up in the size of the expression: consider $(a+b)(c+d)(e+f)\ldots$, or $\sin(a+b+c+\ldots)$. The second bullet point of Maple's description can lead to even more impressive expansion, as in

```
expand(BesselJ(4,t)^3);
```

(just where did the number 165888 come from?) or

```
expand(WeierstrassP(x+y+z,2,3));
```

1.2.3 Simplification

This word is much used in algebra, particularly at the school level, and has been taken over by computer algebra, which has thereby committed the sin of importing into a precise subject a word without a precise meaning.

> Looking at the standard textbooks on Computer Algebra Systems (CAS) leaves one even more perplexed: it is not even possible to find a proper definition of the problem of simplification [Car04].

Let us first consider a few examples.

1. Does $\frac{x^2-1}{x-1}$ simplify to $x+1$? For most people, the answer would be 'yes', but some would query "what happens when $x=1$", i.e. would ask whether we are dealing with abstract formulae, or representations of functions. This is discussed further for rational functions on pages 59 and 288, and in item 5 below.

2. Assuming the answer to the previous question is 'yes', does $\frac{x^{1000}-1}{x-1}$ simplify to $x^{999}+\cdots+1$? Here the fraction is much shorter than the explicit polynomial, and we have misled the reader by writing $\cdots$ here[11].

3. Does $\sqrt{1-x}\sqrt{1+x}$ simplify to $\sqrt{1-x^2}$? Assuming the mathematical validity, which is not trivial [BD02], then the answer is almost certainly yes, since the second operations involves fewer square roots than the first.

4. Does $\sqrt{x-1}\sqrt{x+1}$ simplify to $\sqrt{x^2-1}$? This might be thought to be equivalent to the previous question, but consider what happens when $x=-2$. The first one simplifies to $\sqrt{-3}\sqrt{-1}=i\sqrt{3}\cdot i=-\sqrt{3}$, while the second simplifies to $\sqrt{3}$. Note that evaluations result in *real* numbers, though they proceed via complex ones. Distinguishing this case from the previous one is a subject of active research [BBDP07, CDJW00].

[11] The construct $\cdots$ is very common in written mathematics, but has had almost (but see [SS06]) no analysis in the computer algebra literature.

5. Assume we are working modulo a prime p, i.e. in the field $\mathbf{F}_p$. Does $x^p - x$ simplify to 0? As polynomials, the answer is no, but as functions $\mathbf{F}_p \to \mathbf{F}_p$, the answer is yes, by Fermat's Little Theorem. Note that, as functions $\mathbf{F}_{p^2} \to \mathbf{F}_{p^2}$, say, the answer is no.

   In terms of Notation 3, we can say that

   $$\left(\{(x, x^p - x) : x \in \mathbf{F}_p\}, \mathbf{F}_p, \mathbf{F}_p\right)_{\mathcal{B}} = \left(\{(x, 0) : x \in \mathbf{F}_p\}, \mathbf{F}_p, \mathbf{F}_p\right)_{\mathcal{B}},$$

   but

   $$\left(\left\{(x, x^p - x) : x \in \mathbf{F}_{p^2}\right\}, \mathbf{F}_{p^2}, \mathbf{F}_{p^2}\right)_{\mathcal{B}} \neq \left(\left\{(x, 0) : x \in \mathbf{F}_{p^2}\right\}, \mathbf{F}_{p^2}, \mathbf{F}_{p^2}\right)_{\mathcal{B}}.$$

Now we give a few illustrations of what simplification means to different audiences.

Teachers At a course on computer algebra systems for teachers of the French 'concours', among the most competitive mathematics examinations in the western world, there was a round table on the meaning of simplification. Everyone agreed that the result should be 'mathematically equivalent', though it was less clear, prompted by example 1, exactly what this meant. The response to example 2 was along the lines of "well, you wouldn't ask such a question". The author wishes he had had examples 3 and 4 to hand at the time!

The general consensus was that 'simplification' meant 'give me the answer I want'. This answer is not *effective*, in the sense that it cannot be converted into a set of rules.

Whether this is an appropriate approach to pedagogy is outside the scope of this book, but we refer the reader to [BDS09, BDS10].

Moses [Mos71] This is a seminal paper, but describes approaches to simplification rather than defining it. Inasmuch as it does define it, it talks about 'utility for further operations', which again is not effective as a definition for top-level simplification, though it's relevant for intermediate operations. However, the principle is important, since a request to factor would find the expression $x^{999} + \cdots + 1$ appropriate[12], whereas $\frac{x^{1000}-1}{x-1}$ is in a field, and factorisation is not a relevant question.

Carette [Car04] He essentially defines simplification in terms of the length of the result, again insisting on mathematical equivalence. This would regard examples 1 (assuming $\mathbf{Q}(x)$, so that the expressions were mathematically equivalent) and 3 as simplifications, but not 2, since the expression becomes longer, or 4, since we don't have mathematical equivalence.

Stoutemyer [Sto11a] sets out 10 goals for simplification, one of which (Goal 6) is that

[12]In fact, knowing the expression *came from* that quotient would be relevant [BD89] to factorisation algorithms, but that's beside the point here.

> Default simplification should produce candid [Definition 6] results for rational expressions and for as many other classes as is practical. Default simplification should try hard even for classes where candidness cannot be guaranteed for all examples.

Candid expressions *tend* be the shorter than others ($\frac{x^{10}-1}{x-1}$ etc. being an obvious family of counter-examples), so this view is relatively close to Carette's, but not identical (see also page 60, item 2).

Numerical Analysts A numerical analyst would be shocked at the idea that $x^2 - y^2$ was 'simpler' than $(x + y)(x - y)$. He would instantly quote an example such as the following [Ham07].

> For simplicity, assume decimal arithmetic, with perfect rounding and four decimal places. Let $x = 543.2$ and $y = 543.1$. Then x^2 evaluates to 295100 and y^2 evaluates to 295000, so $x^2 - y^2$ becomes 100, while $(x + y)(x - y)$ evaluates to 108.6, a perfect rounding of the true answer 108.63.
>
> Furthermore, if we take $x = 913.2$ and $y = 913.1$, $x^2 - y^2$ is still 100, while the true result is 182.63.

This is part of the whole area of *numerical stability*: fascinating, but outside the scope of this text.

One principle that can be extracted from the above is "if the expression is zero, please tell me": this would certainly meet both the teachers' and Carette's views. This can be seen as a call for simplification to return a normal form *where possible* [Joh71, Ric97].

Maple's description of the 'simplify' command is as follows.

- The `simplify` command is used to apply simplification rules to an expression.
- The `simplify/expr` calling sequence searches the expression, `expr`, for function calls, square roots, radicals, and powers. It then invokes the appropriate simplification procedures.
- **symbolic** Specifies that formal symbolic manipulation of expressions is allowed without regard to the analytical issue of branches for multi-valued functions. For example, the expression `sqrt(x^2)` simplifies to `x` under the symbolic option. Without this option, the simplified result must take into account the different possible values of the (complex) sign of `x`.

Maple does its best to return a normal form, but can be fooled: for example

$$RootOf\left(\_Z^4 + b\_Z^2 + d\right) - 1/2\sqrt{-2\,b + 2\sqrt{b^2 - 4\,d}},$$

which is actually zero (applying figure 3.2), does not simplify to zero under Maple 11.

Because simplification may often require expansion, e.g. to take $(x-1)(x+1)$ to x^2-1, the two are often confused, and indeed both Macsyma and Reduce (internally) used `ratsimp` and `*simp` (respectively) to denote what we have called expansion.

1.2.4 An example of simplification

This section is inspired by an example in [Sto13]. Consider

$$\begin{gathered}(\cos(x))^3\sin(x)+\frac{1}{2}(\cos(x))^3\sin(x)+2(\cos(x))^3\cos(2x)\sin(x)+\\ \frac{1}{2}(\cos(x))^3\cos(4x)\sin(x)-\frac{3}{2}\cos(x)(\sin(x))^3-\\ 2\cos(x)\cos(2x)(\sin(x))^3-\frac{1}{2}\cos(x)\cos(4x)(\sin(x))^3.\end{gathered}\tag{1.2}$$

Typing this into Maple simply collects terms, giving

$$\begin{gathered}\frac{3}{2}(\cos(x))^3\sin(x)+2(\cos(x))^3\cos(2x)\sin(x)+\\ \frac{1}{2}(\cos(x))^3\cos(4x)\sin(x)-\frac{3}{2}\cos(x)(\sin(x))^3-\\ 2\cos(x)\cos(2x)(\sin(x))^3-\frac{1}{2}\cos(x)\cos(4x)(\sin(x))^3.\end{gathered}\tag{1.3}$$

`combine(%,trig)`, i.e. using the multiple angle formulae to replace trigonometric powers by sin / cos of multiples of the angles, gives

$$\frac{3}{8}\sin(4x)+\frac{1}{4}\sin(2x)+\frac{1}{4}\sin(6x)+\frac{1}{16}\sin(8x).\tag{1.4}$$

`expand(%,trig)` (i.e. using the multiple angle formulae in the other direction) gives

$$4\sin(x)(\cos(x))^7-4(\cos(x))^5(\sin(x))^3,\tag{1.5}$$

which is also given by Maple's `simplify` applied to (1.3) (`simplify` applied to (1.4) leaves it unchanged). Mathematica's `FullSimplify` gives

$$2(\sin(3x)-\sin(x))(\cos(x))^5.\tag{1.6}$$

However, an even simpler form is found by Eureqa[13]:

$$(\cos(x))^4\sin(4x).\tag{1.7}$$

We note that (1.4) and (1.5) are the results of algorithmic procedures, pushing identities in one direction or the other, applied to (1.3), while (1.6) and (1.7) are half-way positions, which happen to be shorter than the algorithmic results.

[13]`http://creativemachines.cornell.edu/eureqa`. However, we should note that Maple (resp. Mathematica) has *proved* that (1.5) (resp. (1.6)) is equivalent to (1.2), whereas Eureqa merely claims that (1.7) fits the same data points as [a finite sample of] (1.2). Nevertheless, Eureqa's capability to find a short equivalent in essentially Carette's sense [Car04] is impressive, and in any case, knowing *what* to prove, any algebra system can prove equivalence.

1.2.5 Equality

To understand the difficulties that computer algebra systems have with equality, we need to remember the difference between objects and representations (Definition 2. These difficulties are discussed for compter algebra systems in [Dav02] and for computer proof systems in [GKS15],

Notation 4 *We use $=_O$ to stand for equality of (mathematical) objects, and $=_R$ for equality of representations. By slight abuse of notation, we will also regard $=_O$ as a relation on R, meanng "the mathematical objects denoted by these two representations are the same". For a given computer algebra system, we use $=_{\mathrm{CA}}$ to stand for equality in that system, e.g. $=_{\mathrm{Maple}}$.*

Example 1 *Hence, in terms of "ink-and-paper" representations, $(x+1)^2 =_O x^2+2x+1$ but $(x+1)^2 \neq_R x^2+2x+1$.*

In terms of relations on R, i.e. subsets of $R \times R$, $=_R \subseteq =_O$, i.e. if two representations are equal, the corresponding mathematical objects are equal.

Definition 7 (Definition 4 restated) *A representation is said to be* canonical *if $=_O$ is the same as $=_R$.*

What properties might we want $=_{\mathrm{CA}}$ to have?

Reflexive Since $=_R$ is reflexive, this will be achieved if $=_{\mathrm{CA}} \supseteq =_R$, as it should be.

Symmetric This is pretty easy to achieve.

Transitive This is more difficult. If $=_{\mathrm{CA}}$ does more sophisticated processing than $=_R$, it might recognise that $a =_{\mathrm{CA}} b$ and $b =_{\mathrm{CA}} c$, but fail to recognise that $a =_{\mathrm{CA}} c$, especially if b is significantly simpler than a and c.

Congruence If $a =_{\mathrm{CA}} b$, we would like $f(a) =_{\mathrm{CA}} f(b)$. If the system's representation of $f(\mathtt{x})$ is just the uninterpreted `f(x)`, this is relatively easy to achieve. If the implementation of f is more sophisticated, this may be harder to achieve.

Strictly speaking, what we have stated is **unary congruence**, and we really want **binary congruence**. i.e. If $a =_{\mathrm{CA}} b$ and $c =_{\mathrm{CA}} d$, we would like $f(a,c) =_{\mathrm{CA}} f(b,d)$, and in general congruence of all arities. The same remarks apply.

Soundness This is "if $=_{\mathrm{CA}}$ says that two things are equal, then they are": in symbols $=_{\mathrm{CA}} \subseteq =_O$. This is a property we should always have: if there's a counterexample,then two objects, which are mathematically not equal, are declared equal by $=_R$.

Completeness This is "if two things are mathematically equal, then $=_{\mathrm{CA}}$ says so": in symbols $=_{\mathrm{CA}} \supseteq =_O$.

Unfortunately, completeness is impossible in general, essentially as a consequence of the Gödel Incompleteness Theorem.

Neither $=_O$ nor $=_R$ is appropriate for "pedagogical equality" as needed in mathematical education software: see [BDS09, BDS10] for a discussion of this.

1.3 Algebraic Definitions

In this section we give some classic definitions from algebra, which we will return to throughout this book. Other concepts are defined as they occur, but these ones are assumed.

Definition 8 *A set R is said to be a* ring *if it has two binary operations $+$ and $*$, a unary operation $-$ and a distinguished element 0, such that, for all a, b and c in R:*

1. $a + (b + c) = (a + b) + c$ *(associativity of $+$);*
2. $a * (b * c) = (a * b) * c$ *(associativity of $*$);*
3. $a + b = b + a$ *(commutativity of $+$);*
4. $a + (-a) = 0$*;*
5. $a + 0 = a$*;*
6. $a * (b + c) = (a * b) + (a * c)$ *(distributivity of $*$ over $+$);*

6' $(b + c) * a = b * a + c * a$ *(right-distributivity)*

7. $a * b = b * a$ *(commutativity of $*$).*

Not every text includes the last clause, and they would call a 'commutative ring' what we have called simply a 'ring'. In the absence of the last clause, we will refer to a 'non-commutative ring'. 6' is unnecessary if we have commutativity, of course.

Definition 9 *If R is a (possibly non-commutative) ring and $\emptyset \neq I \subseteq R$, then we say that I is a (left-)ideal of R, written $I \triangleleft R$, if the following two conditions are satisfied*[14]*:*

(i) $\forall f, g \in I, f - g \in I,$

(i) $\forall f \in R, g \in I, fg \in I.$

Notation 5 *If $S \subset R$, we write (S) for $\bigcup_{n\in\mathbf{N}} \{\sum_{i=1}^{n} f_i g_i \quad f_i \in R, g_i \in S\}$: the set of all linear combinations (over R) of the elements of S. We tend to abuse notation and write $(a_1, \ldots, a_k)$ instead of $(\{a_1, \ldots, a_k\})$. This is called the* (left-)ideal generated by *S, and clearly is a left-ideal.*

[14]We write $f - g \in I$ since then $0 = f - f \in I$, and then $f + g = f - (0 - g) \in I$.

There are also concepts of *right ideal* and *two-sided ideal*, but all concepts agree in the case of commutative rings. Non-trivial ideals (the trivial ideals are $\{0\}$ and R itself) exist in most rings: for example, the set of multiples of m is an ideal in $\mathbf{Z}$.

Proposition 1 *If I and J are ideals, then $I + J = \{f + g : f \in I, g \in J\}$ and $IJ = \{fg : f \in I, g \in J\}$ are themselves ideals.*

Definition 10 *A ring is said to be* noetherian*, or to satisfy the* ascending chain condition *if every ascending chain $I_1 \subset I_2 \cdots$ of ideals is finite.*

Theorem 1 (Noether) *If R is a commutative noetherian ring, then so is $R[x]$ (Notation 12, page 42).*

Corollary 1 *If R is a commutative noetherian ring, then so is $R[x_1, \ldots, x_n]$.*

Definition 11 *A ring R is said to be an* integral domain *if, in addition to the conditions above, there is a neutral element 1 such that $1 * a = a$ and, whenever $a * b = 0$, at least one of a and b is zero.*

Another way of stating the last is to say that R *has no zero-divisors* (meaning none other than zero itself).

Definition 12 *An element u of a ring R is said to be a* unit *of R if there is an element $u^{-1} \in R$ such that $u * u^{-1} = 1$. u^{-1} is called the* inverse *of u.* Note that the context R matters: 2 is a unit in $\mathbf{Q}$ (with inverse 1/2), but not in $\mathbf{Z}$.

Definition 13 *If $a = u*b$ where u is a unit, we say that a and b are* associates*.*

Proposition 2 *If a and b are associates, and b and c are associates, then a and c are associates. Therefore being associates is an equivalence relation, since it's clearly reflexite and symmetric.*

For the integers, n and $-n$ are associates, whereas for the rational numbers, any two non-zero numbers are associates.

Definition 14 *An ideal I of the form (f), i.e. such that every $h \in I$ is gf for some $g \in R$, is called* principal*. If R is an integral domain such that every ideal I is principal, then R is called a* principal ideal domain*, or P.I.D.*

Classic P.I.D.s are the integers $\mathbf{Z}$, and polynomials in one variable over a field. Inside a principal ideal domain, we have the standard concept of a greatest common divisor (formally defined in Definition 32).

Proposition 3 *Let R be a P.I.D., $a, b \in R$ and $(a, b) = (g)$. Then g is a* greatest common divisor *of a and b, in the sense that any other common divisor divides g, and $g = ca + db$ for $c, d \in R$.*

It is possible to have g.c.d.s without being a P.I.D.: common examples are $\mathbf{Z}[x]$ (where the ideal $(2, x)$ is not principal) and $\mathbf{Q}[x, y]$ (where the ideal (x, y) is not principal).

Definition 15 *If F is a ring in which every non-zero element is a unit, F is said to be a* field*.*

The "language of fields" therefore consists of two constants (0 and 1), four binary operations ($+$, $-$, $*$ and $/$) and two unary operations ($-$ and $^{-1}$, which can be replaced by the binary operations combined with the constants). The rational numbers and real numbers are fields, but the integers are not. For any m, the integers modulo m are a ring, but only if m is prime do they form a field. The only ideals in a field are the trivial ones.

Definition 16 *If R is an integral domain, we can always form a field from it, the so called* field of fractions*, consisting of all formal fractions*[15] $\frac{a}{b} : a, b \in R, b \neq 0$*, where a/b is zero if and only if a is zero. Addition is defined by* $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$*, and multiplication by* $\frac{a}{b} * \frac{c}{d} = \frac{ac}{bd}$*. So* $\frac{a}{b} = \frac{c}{d}$ *if, and only if,* $ad - bc = 0$*.*

In particular, the rational numbers are the field of fractions of the integers.

Definition 17 *If F is a field, the* characteristic *of F, written* $\mathrm{char}(F)$*, is the least positive number n such that* $\underbrace{1 + \cdots + 1}_{n \text{ times}} = 0$*. If there is no such n, we say that the characteristic is zero.*

So the rational numbers have characteristic 0, while the integers modulo p have characteristic p, as do, for example, the rational functions whose coefficients are integers modulo p.

Proposition 4 *The characteristic of a field, if non-zero, is always a prime.*

1.3.1 Algebraic Closures

Some polynomial equations have solutions in a given ring/field, and some do not. For example, $x^2 - 1 = 0$ always has two solutions: $x = -1$ and $x = 1$. The reader may protest that, over a field of characteristic two, there is only one root, since $1 = -1$. However, over a field of characteristic two, $x^2 - 1 = (x-1)^2$, so $x = 1$ is a root with multiplicity two.

Definition 18 *A field F is said to be* algebraically closed *if every polynomial in one variable over F has a root in F.*

Proposition 5 *If F is algebraically closed, then every polynomial of degree n with coefficients in F has, with multplicity, n roots in F.*

Theorem 2 ("Fundamental Theorem of Algebra") [16] $\mathbf{C}$*, the set of complex numbers, is algebraically closed.*

[15] Strictly speaking, it consists of equivalence classes of formal fractions, under the equality we are about to define.

[16] The title is in quotation marks, since $\mathbf{C}$ (and $\mathbf{R}$) are constructs of *analysis*, rather than algebra.

Definition 19 *If F is a field, the* algebraic closure *of F, denoted $\overline{F}$ is the field generated by adjoining to F the roots of all polynomials over F.*

It follows from proposition 84 that the algebraic closure is in fact algebraically closed.

1.4 Some Complexity Theory

As is usual in computing we will use the so-called "Landau notation"[17] to describe the computing time (or space) of operations.

Notation 6 ("Landau") *Let N be some measure of the size of a problem, generally the size of the input to a program, and f some function. If $t(N)$ is the time taken, on a given hardware/software configuration, for a particular program to solve the hardest problem of size N, we say that "t is eventually no bigger than f", in symbols*

$$t(N) = O(f(N)), \tag{1.8}$$

if, from some point onwards as N increases, t is no more than some fixed multiple of f, i.e. $\exists C \in \mathbf{R}, M \in \mathbf{N} : \forall N > M \quad t(N) < Cf(N)$. C (and *M*) are generally referred to as the *implicit constants* of this notation.

Hardware tends to change in speed, but generally linearly, so O-notation is independent of particular hardware choices. If the program implements a particular algorithm, we will say that the algorithm has this O behaviour.

We will also use "soft O" notation.

Notation 7 ("soft *O*") *Let N be some measure of the size of a problem, and f some function. If $t(N)$ is the time taken, on a given hardware/software configuration, for a particular program to solve the hardest problem of size N, we say that*

$$t(N) = \tilde{O}(f(N)) \tag{1.9}$$

if $t(N)$ grows "almost no faster" than $f(N)$, i.e. slower than $f(N)^{1+\epsilon}$ for any positive ϵ: in symbols $\forall \varepsilon > 0 \exists C \in \mathbf{R}, M \in \mathbf{N} : \forall N > M \quad t(N) < Cf(N)^{1+\varepsilon}$.

We should note that "$= O$" and "$= \tilde{O}$" should really be written with "$\in$" rather than "$=$", and this use of "$=$" is not reflexive, symmetric or transitive. Also, many authors use $\tilde{O}$ to mean "up to logarithmic factors", which is included in our, more general, definition. [DL08]. One specific use of $\tilde{O}$ is given in footnote 4, page 165.

The key results of complexity theory for elementary algorithms are that it is possible to multiply two N-bit integers in time $\tilde{O}(N)$, and two degree-N polynomials with coefficients of bit-length at most τ in time $\tilde{O}(N\tau)$. [AHU83] For matrix multiplication, the situation is more complicated: see Notation 45

[17]Though apparently first introduced by[Bac94, p. 401]. See [Knu74].

Definition 20 (Following [BFSS06]) *We say that an algorithm producing output of size N is* optimal *if its running time is $O(N)$, and* almost optimal *if its running time is $\tilde{O}(N)$.*

Addition of numbers and polynomials is therefore optimal, and multiplication almost optimal. Matrix multiplication is not known to be either.

However, the reader should be warned that $\tilde{O}$ expressions are often far from the reality experienced in computer algebra, where data are small enough that the limiting processes in equations (1.8) and (1.9) have not really taken hold (see note [4]), or are quantised (in practice integer lengths are measured in words, not bits, for example). We the therefore often use the phrase "classical arithmetic" to mean $O(n^2)$ integer/polynomial multiplication, and $O(n^3)$ matrix multiplication.

When it comes to measuring the intrinstic difficulty of a problem, rather than the efficiency of a particular algorithm, we need *lower* bounds rather than the upper bounds implied in (1.8) and (1.9).

Notation 8 (Lower bounds) *Consider a problem P, and a given encoding, e.g. "dense polynomials (Definition 26) with coefficients in binary". Let N be the size of a problem instance, and C a particular computing paradigm, and way of counting operations. If we can prove that there is a c such that any algorithm solving this problem must take at least $cf(N)$ operations on at least one problem instance of size N, then we say that this problem has cost* at least of the order of *$f(n)$, written*

$$P_{\mathcal{C}} = \Omega(f(N)) \text{ or loosely } P = \Omega(f(N)). \tag{1.10}$$

Again "=" really ought to be "$\in$".

In some instances (sorting is one of these) we can match upper and lower bounds.

Notation 9 (Θ) *If $P_{\mathcal{C}} = \Omega(f(N))$ and $P_{\mathcal{C}} = O(f(N))$, then we say that $P_{\mathcal{C}}$ is of* order exactly *$f(N)$, and write $P_{\mathcal{C}} = \Theta(f(N))$.*

For example if C is the paradigm in which we only count comparison operations, sorting N objects is $\Theta(N \log N)$.

Notation 10 (Further Abuse) *We will sometimes abuse these notations further, and write, say, $f(N) = 2^{O(N)}$, which can be understood as either of the equivalent forms $\log_2 f(N) = O(N)$ or $\exists C \in \mathbf{R}, M : \forall N > M f(N) < 2^{CN}$.* Note that $2^{O(N)}$ and $O(2^N)$ are very different things. $4^N = 2^{2N}$ is $2^{O(N)}$ but not $O(2^N)$.

Open Problem 1 (Algebra of O) *Manipulating such O-expressions, especially when they depend on several variables, can be very tedious as seen in [Col75, pp. 160–163]. Write a computer algebra package to simplify such expressions, so that, for example,*

```
Osimplify(O(N^3)+O(N^2));
```

would yield $O(N^3)$.

1.4.1 Complexity Hierarchy

If an algorithm has input of size N, it will take time $O(N)$ to read its input, so this is generally the least complexity we consider. We then have various complexities, all of which are (bounded by) polynomials in N:

$$N \prec N\log N \prec N\log N\log\log N \prec N\log^2 N \prec N^{3/2} \prec N^2 \prec N^3 \prec \cdots. \tag{1.11}$$

We have written $f(N) \prec g(N)$ rather than $f(N) < g(N)$ since it depends on the implicit constants whether for a particular N an algorithm whose time is $O(f(N))$ is actually faster than one whose time is $\Omega(g(N))$: all we know is that *eventually*, as N grows, the one with the better complexity will be faster.

All the complexities in class (1.11) are referred to as *polynomial time* complexities, or P.

Definition 21 *An algorithm is* polynomial time*, or* in the class P *if there is a constant c such that the running time of the algorithm is $O(N^c)$.*

Beyond this, we have an exponential class (or EXP):

$$1.01^N = 2^{0.01436N} \prec 2^N \prec 4^N = 2^{2N} \prec 2^{N\log N} \prec 2^{N^2} \prec \cdots \tag{1.12}$$

Again, it depends on the implied constants, but eventually any algorithm whose complexity is polynomial will be faster than one whose complexity is in (1.12).

There are complexities which lie between the two: (B.18) shows $N^{\Theta(1/\log\log N)}$, which goes slower than any c^N.

1.4.2 Probabilistic Algorithms

The traditional definition of an algorithm is "a definite sequence of operations which terminates and produces the desired result" or words to that effect, so the title of this section is, taken literally, an oxymoron. Nevertheless, the concept is very useful, and can be made precise by inserting into that definition "(which may include calls to random number generators)". We can then distinguish various kinds of probabilistic algorithms (where "fast" tends to mean "polynomial time"). In each case, "probably" refers to probability across the possible outputs of the random number generators.

Monte Carlo ("always fast/probably correct") The classic example here is the Miller–Rabin primality testing algorithm [Rab80].

Algorithm 1 (Miller–Rabin primality testing)
Input: *a number N of n bits*
Output: *Either "N is definitely composite" or "N is probably prime".*
The algorithm picks a random $a \in [2, N-1]$ and computes a^{N-1} carefully. The running time, with classical arithmetic, is therefore $O(n^3)$. If N is prime, the algorithm always outputs "N is probably prime". If N is composite, the algorithm outputs "N is probably prime" for at most 1/4 of a-values.

Las Vegas ("always correct/probably fast") If Algorithm A is a Monte Carlo algorithm for problem P, and Algorithm B is a fast verifier for correctness, then the process in Figure 1.1 will give a Las Vegas algorithm for problem P. It is the absence of such a verifier that means that Miller–Rabin primality testing (as opposed to the deterministic, but much more expensive AKS [AKS04] test) is only Monte Carlo.

Figure 1.1: Converting Monte Carlo to Las Vegas

```
do
    ans:=Algorithm A(P)
while Algorithm B(ans,P)=false
return ans
```

Atlantic City ("probably fast/probably correct") Again, the existence of a fast verifier for correctness would let the process in Figure 1.1 convert this to a Las Vegas algorithm. One example of an Atlantic City algorithm is given in Theorem 3: again we lack a fast verifier, so the algorithm is "only" Atlantic City.

1.5 Some Maple

TO BE COMPLETED

In Maple, the synbol `%` refers to the last object *computed*, `%%` to the last but one, and `%%%` to the last but two. Note that this is not necessarily the same as the textually previous item in the worksheet.

1.5.1 Maple polynomials

From Maple 17 onwards, Maple has two possible representations for polynomials.

Original Expressions An expression, whether or not it had been through `expand`, was an n-ary sum of n-ary power products with numerical coefficients. Uniqueness of power products (and hence collection of like terms (8') on page 42) was enforced through hashing, rather than sorting, and hence summands and multiplicands could, and did, appear in any order.

Sparse distributed [MP12, MP14] This is as on page 50, and the order is total degree lexicographic (Section 3.3.3).

There's a Maple worksheet demonstrating this at `http://staff.bath.ac.uk/masjhd/JHD-CA/MaplePoly.html`.

Figure 1.2:

```
> expand((x-1)*(x-2)/((x-3)*(x-4)));
                          2
                         x                3 x               2
                 --------------- - --------------- + ---------------
                 (x - 3) (x - 4)   (x - 3) (x - 4)   (x - 3) (x - 4)
```

Figure 1.3:

```
> normal((x-1)*(x-2)/((x-3)*(x-4)),expanded);
                                         2
                                        x  - 3 x + 2
                                        -------------
                                         2
                                        x  - 7 x + 12
```

1.5.2 Maple rational functions

Maple's `expand` command, as we saw on page 23, only expands numerators of rational functions, so we get results like Figure 1.2. If we actually want the denominators expanded, and the whole placed over a common denominator, the correct tool is `normal(...,expanded)`, as in Figure 1.3. As the name implies, we get a normal representation (Definition 3), and indeed a canonical one, as greatest common divisors are cancelled, and the leading coefficient of the denominator made positive in a consistent way.

1.5.3 The RootOf construct

Maple uses this construct to indicate a solution of an (univariate) equation. It's generally seen in the context of polynomial equations, as in Figure 1.4, where essentially nothing more can be said about these numbers other than they are the roots of the polynomial $z^5 - 5z^3 + 4z - 1$. Note that (over the complex numbers) Maple indexes the result of `RootOf` according to the rules at `http://www.maplesoft.com/support/help/Maple/view.aspx?path=RootOf/indexed`, essentially "closest in argument to the positive real axis first". For the example above, this results in the five roots being

```
[.7418139305, 1.668777593, -.3141413715+.5954413283*I,
 -1.782308780, -.3141413715-.5954413283*I]
```

which actually contradicts the documentation.

The RegularChains package (implementing the ideas of sections 3.4 and 3.5), extends this notation, to allow `index=real[2]` to mean the second *real* root (counting from $-\infty$).

Figure 1.4: An example of Maple's RootOf construct

```
> solve(z^5-3*z^3+1, z);
              /  5      3               \
        RootOf\_Z  - 3 _Z  + 1, index = 1/,

              /  5      3               \
        RootOf\_Z  - 3 _Z  + 1, index = 2/,

              /  5      3               \
        RootOf\_Z  - 3 _Z  + 1, index = 3/,

              /  5      3               \
        RootOf\_Z  - 3 _Z  + 1, index = 4/,

              /  5      3               \
        RootOf\_Z  - 3 _Z  + 1, index = 5/
```

The two notations are clearly not compatible: the first real root may well not be the first complex root. They are also not well-integrated, so for example Maple doesn't see `RootOf(z^5-3*z^3+1,z,index=real[2])` and `RootOf(z^5-3*z^3+1,z,index=real[2])` as being equal, even though they differ by 0 when we ask for numerical evaluation!

1.5.4 Active and Inert Functions

Notation 11 *Maple functions fall into two categories.*

active *is the usual form: a command that tells Maple to perform a computation, as in* `gcd(x^2-1, x^3-1)` *which computes* $x-1$, *or* `cos(Pi)`, *which computes* -1.

inert *is a form which tells Maple just to store the unevaluated concept of the computation, as in* `Gcd(x^2-1, x^3-1)`, *which returns* $Gcd\left(x^2-1, x^3-1\right)$.

Some Maple commands have both active and inert forms — in this case the inert form generally begins with a capital letter. An inert form can always be built by prepending the `%` *symbol to the function name, as in* `%cos(Pi)`, *which returns* `'%cos'` (π).

The `value` *function evaluates the inert forms in its argument, so that*

```
value(Gcd(x^2-1, x^3-1))
```

is equivalent to `gcd(x^2-1, x^3-1)`, *and therefore is* $x-1$.

Inert forms have many uses (see the Maple documentation): one that is particularly relevant to this book is their relationship with the `mod` operator. The Maple documentation says, describing `e mod m`,

> The `mod` operator evaluates the expression e over the integers modulo m.

What may not be apparent here is that the usual Maple rules, that arguments get evaluated before being passed to functions, still apply. Hence

```
gcd(x+2,x-3) mod 5
```

first evaluates `gcd(x+2,x-3)` (getting 1) then passes this in, effectively calling `1 mod 5`, and we get 1. What we probably intended was

```
Gcd(x+2,x-3) mod 5
```

so that the *unevaluated* g.c.d. object is passed to be calculated modulo 5, where the result is $x+2$ (Maple by default uses $0,\ldots,|m|-1$ to store the results modulo m). Similarly

```
factor(x^4+1) mod 2
```

first evaluates `factor(x^4+1)`, getting x^4+1 since this polynomial is irreducible over the integers, effectively calling `x^4+1 mod 2`, which is x^4+1. Had we written

```
Factor(x^4+1) mod 2
```

then Maple would be being asked to factor x^4+1 modulo 2, and the answer would have been $(x+1)^4$.

Note also that `10^100 mod 7` will first calculate the integer 10^{100}. The correct syntax in this case is `10&^100 mod 7` to defer the exponentiation.

1.5.5 The simplify command

This Maple command is been discussed elsewhere, especially in section 1.2.4. It is worth noting that simplification in this sense does not commute with substitution, and it can be argued [Zim07] that it is a 'user-oriented' command that should not be used inside Maple programs, where well-specified commands such as `expand` (Section 1.2.2), or more specific ones such as `combine(%,trig)` are very appropriate.

1.5.6 Equality

There are four operators in Maple which are relevant here, and the first three could all be thought of as Maple's $=_{\text{Maple}}$.

`=` This actually forms a symbolic equation by default, as in

```
> 2=3;
                2=3
```

It is only when this is required to be a Boolean value, either because of the context (`if` or `while` statement), or explicitly by the `evalb` function, that it is converted into true/false. When this happens, it appears to the author that this implements $=_R$ (Notation 4), i.e. equality of data structures.

`testeq` This uses the probabilistic algorithm of [Gon84] and hence *might* be unsound, i.e. say that two things are equal when in fact they are not, but the author has been unable to provoke this. It returns `FAIL` (rather than `false`) if the expressions are of a category that it cannot handle, such as square roots.

- The next two are part of Maple's `assume` facility [WG93], but can be used without any assumptions.

`is` typically has the syntax `is(x1,prop1)` and "returns `true` if all possible values of `x1` satisfy the property `prop1`". It "returns `false` if any possible value of `x1` does not satisfy the property `prop1`", and "returns `FAIL` if it cannot determine whether the property is always satisfied". However, it can also be used as `is(x=y)`, and in this context would seem to do significant amounts of simplification: at least the equivalent of `normal`, and possibly `simplify`. However, it cannot really handle square roots, and returns `false` (rather than `FAIL`, annoyingly) for both examples 3 and 4 (page 24).

`coulditbe` typically has the syntax `coulditbe(x1,prop1)` and "determines whether there is a value of `x1` such that `prop1` is satisfied". However, it cannot really handle square roots, and returns `true` (rather than `FAIL`, annoyingly) for the negations of both examples 3 and 4 (page 24), even though there is no x for which examples 3 is not true.

Chapter 2

Polynomials

Polynomials are fundamental to much of mathematics, even when the objects under discussion are apparently not polynomials, such as differential equations. Equally, polynomials underpin much of computer algebra. But what, precisely, are they?

2.1 What are polynomials?

There are numerous definitions. From our point of view, computer algebra, we will adopt the following definition for *commutative*[1] polynomials, leaving non-commutative polynomials to be discussed in section 2.4.

Definition 22 (Polynomials) *A (commutative)* polynomial *is built up from* coefficients*, which are assumed to form a ring (definition 8), and certain* indeterminates *(often called* variables*), by the* algebraic operations *of addition, subtraction and multiplication. These are subject to the following laws, where a, b, c are polynomials, m, n coefficients, and 0 and 1 certain distinguished coefficients.*

1. $a + b = b + a$;
2. $(a + b) + c = a + (b + c)$;
3. $a + 0 = a$
4. $a + (-a) = 0$;
5. $a * b = b * a$;
6. $a * (b * c) = (a * b) * c$;

[1] "Commutative" meaning $a * b = b * a$. Strictly speaking, we should also worry whether addition is commutative, i.e. whether $a + b = b + a$, but we will always assume that addition is commutative.

7. *$a * 1 = a$;*

8. *$a * (b + c) = (a * b) + (a * c)$;*

9. *$m + n = m \oplus n$;*

10. *$m * n = m \otimes n$;*

where we have used $\oplus$ and $\otimes$ to denote the operations of addition and multiplication on coefficients, which are assumed to be given to us.

The reader can think of the coefficients as being numbers, though they need not be, and may include other indeterminates that are not the "certain indeterminates" of the definition. However, we will use the usual 'shorthand' notation of 2 for $1 \oplus 1$ etc. The associative laws (2 and 6 above) mean that addition and multiplication can be regarded as n-ary operations. A particular consequence of these rules is

8' $m * a + n * a = (m \oplus n) * a$

which we can think of as 'collection of like terms'.

Proposition 6 *Polynomials over a ring form a ring themselves.*

Definition 23 (Free Algebra) *If it is the case that a polynomial is only zero if it can be deduced to be zero by rules 1–10 above, and the properties of $\oplus$ and $\otimes$, then we say that we have a* free *polynomial algebra.*

Free algebras are common, but by no means the only one encountered in computer algebra. For examples, trigonometry is often encoded by regarding $\sin\theta$ and $\cos\theta$ as indeterminates, but subject to $\sin^2\theta + \cos^2\theta = 1$ [Sto77].

Notice what we have *not* mentioned: division and exponentiation.

Definition 24 ((Exact) Division) *If $a = b * c$, then we say that b* divides *a, and we write $b = \frac{a}{c}$.*

Note that, for the moment, division is only defined in this context. We note that, if c is not a zero-divisor, b is unique.

Definition 25 (Exponentiation) *If n is a natural number and a is a polynomial, then we define a^n inductively by:*

- *$a^0 = 1$;*
- *$a^{n+1} = a * a^n$.*

Notation 12 *If K is a set of coefficients, and V a set of variables, we write $K[V]$ for the set of polynomials with coefficients in K and variables in V. We write $K[x]$ instead of $K[\{x\}]$ etc.*

2.1.1 How do we manipulate polynomials?

We have defined the abstract, almost Platonic, concept of polynomials as mathematical objects, and polynomial algebra as rules for these objects. What do we mean by the *representation* of these objects in a computer system?

One option would be for a computer algebra system essentially to do nothing, simply recording the computations requested by the user, so that `a+b` would become simply $a + b$. However, we would not be very happy with a calculator which computed `1+1` as "1+1", as we would rather see "2". In particular, if the answer is 0, we would like to see that shown, i.e. we would like the representation to be normal (definition 3).

2.1.2 Polynomials in one variable

We will first consider polynomials in one variable, say x. If the coefficients come from a domain K, the polynomials in x over K are denoted by $K[x]$ (Notation 12). One obvious route to a canonical representation (definition 4) is to insist that polynomials be expanded, i.e. that multiplication is only applied to coefficients and variables, not to general polynomials. This is achieved by applying distributivity, rule 8 from definition 22, where necessary, ensuring that multiplication is not applied to additions, and rule 8' to collect terms. Once this is done, the polynomial is of the form $\sum_{i=0}^{n} a_i x^i$, where the a_i are coefficients.

Notation 13 *We assume that $a_n \neq 0$, which is easy if the coefficients are represented normally (Definition 3). In this case, n is called the* degree *of the polynomial, denoted* $\deg(f)$*, or* $\deg_x(f)$ *if we wish to make it clear which variable is being considered.* a_n *is called the* leading coefficient *of f, and denoted* $\mathrm{lc}(f)$ *or* $\mathrm{lc}_x(f)$*. If* $\mathrm{lc}(f) = 1$*, we say that f is* monic*.* $f - \mathrm{lc}(f)x^{\deg(f)}$*, i.e. f minus its leading term, is known as the* reductum *of f,* $\mathrm{red}(f)$*. The set* $\{\mathrm{red}(f), \mathrm{red}(\mathrm{red}(f)), \ldots\}$ *is known as the* iterated reducta *of f.*

There is then an important distinction, which does not really occur when doing algebra by hand: does one represent the zero coefficients, or not?

Definition 26 *A representation*[2] *is said to be* dense *if every coefficient, zero or not, is represented, while it is* sparse *if zero coefficients are not stored.*

Hence the polynomial $x^2 + 0x - 1$, normally written as $x^2 - 1$, would be stored as `<1,0,-1>` in a dense representation, but `<<2,1>,<0,-1>>` in a sparse representation. As is implicit in the previous sentence, the normal "human" representation is sparse. Those systems that automatically expand, e.g. Reduce [Hea05], use a sparse representation, since a system would look fairly silly if it was unable to represent $x^{1000000000} + 1$ since it could not store the 999,999,999 zeros. However, dense representations are often used internally in some algorithms.

[2]In the current case, we are dealing with polynomials. But the concept is more general — see section 3.2.2 for sparse matrices, for example.

Definition 27 *For a polynomial $f = \sum_i c_i x^{\alpha_i} \in \mathbf{Z}[x]$, we define the* sparse bit size *of f to be $\sum_i [1 + \log_2(|c_i|) + \log_2(1 + \alpha_i)]$: the number of bits needed to encode f (the* 1+ *term allows for the sign of the coefficient).* This definition ignores the awkward practicalities that bits are whole (we should have $\lceil \log_2 \ldots \rceil$) and come in bytes/words, but has the right O-behaviour. It is also information-theoretically correct, in that the α_i might be 0, hence the 1+, but the c_i can't be 0, so don't need a corresponding 1+, but this is unlikely to be taken advantage of in practice. It also ignores the lengths of length fields, to say which bits mean what, and the lengths of the lengths of the length fields ....

We say that an algorithm has poly-sparse *complexity if the complexity is a polynomial function in the sparse bit size of the inputs and outputs. See also Definition 29.*

Proposition 7 *Both the dense and the sparse expanded representation are canonical (definition 4), provided that:*

- *the coefficients themselves are canonical (otherwise polynomials of degree 0 would not be canonical)*
- *leading (in the dense case) or all (in the sparse case) zeros are suppressed;*
- *(in the sparse case) the individual terms are stored in a specified order, generally[3] sorted.*

Addition is fairly straight-forward in either representation. In Lisp, we can do addition in a sparse representation as follows. We use a representation in which the `CAR` of a polynomial is a term, whilst the `CDR` is another polynomial: the initial polynomial minus the term defined by the `CAR`, i.e. the reductum (Notation 13). A term is a `CONS`, where the `CAR` is the exponent and the `CDR` is the coefficient. Thus the LISP structure of the polynomial $3x^2+1$ is `((2 . 3) (0 . 1))`, and the list `NIL` represents the polynomial 0, which has no non-zero coefficients, and thus nothing to store. In this representation, we must note that the number 1 does not have the same representation as the polynomial 1 (which is `((0 . 1))`), and that the polynomial 0 is represented differently from the other numerical polynomials.

```
(DE ADD-POLY (A B)
  (COND ((NULL A) B)
        ((NULL B) A)
        ((GREATERP (CAAR A) (CAAR B))
            (CONS (CAR A) (ADD-POLY (CDR A) B)))
        ((GREATERP (CAAR B) (CAAR A))
            (CONS (CAR B) (ADD-POLY A (CDR B))))
        ((ZEROP (PLUS (CDAR A) (CDAR B)))
          ; We must not construct a zero term
```

[3]But we should observe that Maple, for example, which uses a hash-based representation, is still canonical, even though it may not seem so to the human eye.

```
              (ADD-POLY (CDR A) (CDR B)))
          (T (CONS (CONS (CAAR A) (PLUS (CDAR A) (CDAR B)))
                   (ADD-POLY (CDR A) (CDR B))))))

(DE MULTIPLY-POLY (A B)
  (COND ((OR (NULL A) (NULL B)) NIL)
;  If a = a0+a1 and b = b0+b1, then ab = a0b0 + a0b1 + a1b
        (T (CONS (CONS (PLUS (CAAR A) (CAAR B))
                       (TIMES (CDAR A) (CDAR B)))
                 (ADD-POLY (MULTIPLY-POLY (LIST (CAR A))
                                          (CDR B))
                           (MULTIPLY-POLY (CDR A) B))))))
```

If A has m terms and B has n terms, the calculating time (i.e. the number of LISP operations) for `ADD-POLY` is bounded by $O(m+n)$, and that for `MULTIPLY-POLY` by $O(m^2n)$ ($(m(m+3)/2-1)n$ to be exact).[4]

There is a technical, but occasionally important, difficulty with this procedure, reported in [ABD88]. We explain this difficulty in order to illustrate the problems which can arise in the translation of mathematical formulae into computer algebra systems. In `MULTIPLY-POLY`, we add a_0b_1 to a_1b. The order in which these two objects are calculated is actually important. Why, since this can affect neither the results nor the time taken? The order *can* dramatically affect the maximum memory space used during the calculations. If a and b are dense of degree n, the order which first calculates a_0b_1 should store all these intermediate results before the recursion finishes. Therefore the memory space needed is $O(n^2)$ words, for there are n results of length between 1 and n. The other order, a_1b calculated before a_0b_1, is clearly more efficient, for the space used at any moment does not exceed $O(n)$. This is not a purely theoretical remark: [ABD88] were able to factorise $x^{1155}-1$ with REDUCE in 2 megabytes of memory[5], but they could not remultiply the factors without running out of memory, which appears absurd.

There are multiplication algorithms which are more efficient than this one: roughly speaking, we ought to sort the terms of the product so that they appear in decreasing order, and the use of `ADD-POLY` corresponds to an insertion sort. We know that the number of coefficient multiplications in such a 'classical' method is mn, so this emphasises that the extra cost is a function of the exponent operations (essentially, comparison) and list manipulation. Of course, the use of a better sorting method (such as "quicksort" or "mergesort") offers a more efficient multiplication algorithm, say $O(mn\log m)$ in terms of time. Na ive construction of all the terms followed by sorting would take $O(mn)$ space to build the unsorted list, and we can do better with "heapsort" [Joh74]. But most systems have tended to use an algorithm similar to the procedure given

[4]It is worth noting the asymmetry in the computing time of what is fundamentally a symmetric operation. Hence we ought to choose A as the one with the fewer terms. If this involves counting the number of terms, many implementors 'cheat' and take A as the one of lower degree, hoping for a similar effect.

[5]A large machine for the time!

above: [MP08] shows the substantial improvements that can be made using a better algorithm.

In general, the product of a sparse polynomial with m terms by one with n terms will have mn terms, so a $O(mn \log m)$ has "nearly optimal" (optimal up to logarithmic factors) complexity in terms of the worst-case output size. However, this worst-case output size is often not achieved in practice. For example, the polynomials may actually be dense in x^{1000}, and then the number of terms is $m+n-1$ rather than mn.

Notation 14 *Define the* support *of a polynomial f,* $\operatorname{supp}(f)$*, to be the set of exponents n such that x^n occurs in f with a non-zero coefficient. The* sparsity *of f is the size of* $\operatorname{supp}(f)$*. Given two polynomials f and g, the* possible exponent set *of $f \cdot g$,* $\operatorname{poss}(f,g)$ *is* $\{e_f + e_g : e_f \in \operatorname{supp}(f), e_g \in \operatorname{supp}(g)\}$.

Hence $\operatorname{supp}(f \cdot g) \subseteq \operatorname{poss}(f,g)$, with strict inequality occurring when cancellation of coefficients gives us an "unexpected" zero.

Theorem 3 ([AR15, Theorem 1.1]) *Given $f, g \in \mathbf{Z}[x]$ with bounds for the degree $D \geq \deg(f) + \deg(g)$ and height $C \geq ||f||_\infty + ||g||_\infty$, and a constant $\mu \in (0,1)$, Algorithm SparseMultZZ of [AR15] correctly computes the product $h = fg$ with probability exceeding $1-\mu$, using expected*[6] $\tilde{O}((\#\operatorname{poss}(f,g)\log D + \#\operatorname{supp}(f \cdot g)\log C)$ *bit operations, where the constants in $\tilde{O}$ depend on μ.*

We can simplify the complexity to $\tilde{O}((\#\operatorname{poss}(f,g)(\log D + \log C))$, which shows that this algorithm, which is based on interpolation, is nearly optimal in the *expected* sparsity of the output. However, it is probabilistic, which limits its use in practice.

Maple's original representation (see section 1.5.1), and methods, are rather different. The terms in a Maple sum might appear to be unordered, so we might ask what prevents $2x^2$ from appearing at one point, and $-2x^2$ at another. Maple uses a hash-based representation [CGGG83], so that insertion in the hash table takes amortised[7] constant time, and the multiplication is $O(mn)$.

In a dense representation, we can use radically different, and more efficient, methods based on Karatsuba's method [KO63, and section B.3], which takes time $O\left(\max(m,n)\min(m,n)^{0.57\cdots}\right)$, or the Fast Fourier Transform [AHU74, chapter 8], where the running time is $O(\max(m,n)\log\min(m,n))$. Since the number of terms in a dense product is $m+n-1 = O(\max(m,n))$, these algorithms are nearly optimal.

Division is fairly straight-forward: to divide f by g, we keep subtracting appropriate (meaning $c_i = (\operatorname{lc}(f)/\operatorname{lc}(g))x^{\deg f - \deg g}$) multiples of g from f until the degree of f is less than the degree of g. If the remaining term (the remainder) is zero, then g divides f, and the quotient can be computed by summing the c_i. This is essentially the process known to schoolchildren as "long division".

[6]This is an Atlantic City ("probably fast/probably correct") algorithm: see section 1.4.2. In fact the authors prove the *existence* of a Monte Carlo algorithm, but cannot describe it as we do not know the number-theoretic constants.

[7]Occasionally the hash table may need to grow, so an individual insertion may be expensive, but the average time is constant.

However, the complexity of sparse polynomial division is not so straightforward [DC10]. We cannot bound the complexity in terms of just the number of terms in the input, because of the example[8] of

$$\frac{x^n-1}{x-1} = x^{n-1} + \cdots + x + 1, \tag{2.1}$$

where two two-term inputs give rise to an n-term output. So imagine that we are dividing f by g, to give a quotient of q and a remainder of r.

Notation 15 *Let a polynomial f have degree d_f, and t_f non-zero terms.*

Since each step in the division algorithm above generates a term of the quotient, there are t_q steps. Each step generates t_g terms, so there are $O(t_g)$ arithmetic operations. But there are also the comparison operations required to organise the subtraction, and it is hard to find a better bound than d_f for the number of these. Hence the total cost is $O(t_q(d_f + t_g))$ operations. Since we don't know t_q until we have done the division, the best estimate is $O(d_f^2 + d_f t_g)$. Again, ideas based on Heapsort can improve this, and the best result is $O(t_f + t_q t_g \log\min(t_q, t_g)) = O(t_f + (d_f - d_g)t_g \log\min(d_f - d_g, t_g))$ commparisons[MP11]. This algorithm therefore has poly-sparse complexity (Definition 27).

Example 2 (Bad Mergesort) *The reason we need to use Heapsort rather than, say, Mergesort is seen is a case like the following*

$$\frac{x^n}{x^{n/2} + x^{n/2-1} + x^{n/4} + 1},$$

where we perform $n/2$ merges of the divisor (4 terms) with polynomials with up to $n/2$ terms, i.e. $O(n^2)$ work. The problem arises because the merges in this case are asymmetrical, and a merge is only efficient when merging things of roughly the same size.

An alternative strategy, known as "geobuckets" for "geometrically increasing buckets", was devised by Yan [Yan98]. Let d be a fixed growth factor (his experiments[9] suggested $d = 4$). The application was Buchberger's Algorithm (9), where we are repeatedly computing $f := f - c_i g_i$, where f typically is a large polynomial and the $c_i g_i$ are polynomials which may be large or small. His intermediate representation for a polynomial f was

$$f := f_1 \oplus f_2 \oplus \cdots \oplus f_l, \tag{2.2}$$

where $\oplus$ signifies a lazy summation that we have yet to perform, and each "bucket" f_i consists of at most d^i terms, sorted normally. Then, when we

[8]See also Excursus B.5.

[9]Geobuckets are also used in CoCoA [Abb15], for polynomial multiplication, division and polynomial reduction, and after experimentation then have also settled on $d = 4$. SINGULAR also uses geobuckets with $d = 4$ [Sch15].

have to subtract $c_i g_i$ from f, we subtract it from f_k, where k is minimal with $d^k \geq t_{g_i}$. It is conceivable that this will cause f_k to 'overflow', i.e. have more than d^k terms, in which case we add f_k to f_{k+1}, and then set f_k to 0. This has the consequence that we (almost) never add (i.e. merge) polynomials of greatly unequal sizes, and also, compared with a heap, the storage requirements are reduced: if $d \geq 2$ the redundancy is bound to be less than 50%, even if every monomial in every other f_i is a duplicate of a monomial in f_l. At the end, we actually perform the summations implicit in (2.2) to get a polynomial in the usual form, taking care to perform them as

$$(\ldots(f_1 \oplus f_2) \oplus \cdots) \oplus f_l$$

so as to preserve the balanced nature of the merges.

2.1.3 A factored representation

Instead of insisting that multiplication not be applied to addition, we could insist that addition not be applied to multiplication. This would mean that a polynomial was represented as a product of polynomials, each the sum of simple terms:

$$f = \prod_i f_i = \prod_i \left(\sum_{j=0}^{n_i} a_{i,j} x^j \right). \tag{2.3}$$

In practice, repeated factors are stored explicitly, as in the following format:

$$f = \prod_i f_i^{d_i} = \prod_i \left(\sum_{j=0}^{n_i} a_{i,j} x^j \right)^{d_i}. \tag{2.4}$$

We have a choice of using sparse or dense representations for the f_i, but usually sparse is chosen. It is common to insist that the f_i are square-free[10] and relatively prime[11] (both of which necessitate only g.c.d. computations[12] — lemma 3), but not necessarily[13] irreducible. Hence this representation is generally known as *partially factored*. In this format, the representation is not canonical, since the polynomial $x^2 - 1$ could be stored either as that (with 2 terms), or as $(x-1)(x+1)$ (with 4 terms): however, it is normal in the sense of definition 3. For equality testing, see excursus B.2 Though an extension of Stoutemyer's

[10]Which almost certainly improves compactness, but see [CD91], where a dense polynomial of degree 12 was produced (13 terms), whose square had only 12 nonzero terms, and the process can be generalised. Twelve is minimal [Abb02].

[11]Which often improves compactness, but consider $(x^p-1)(x^q-1)$ where p and q are distinct primes, which would have to be represented as $(x-1)^2(x^{p-1}+\cdots+1)(x^{q-1}+\cdots+1)$.

[12]These g.c.d. computations, if carried out by modular or p-adic methods (pages 163 and 209), *should* be cheap if the answer is "no simplification", and otherwise should, at least in non-pathological cases, lead to greater efficiency later.

[13]If we were to insist on irreducibility, we would need to store x^p-1 as $(x-1)(x^{p-1}+\cdots+1)$, with $p+2$ terms rather than with 2. Furthermore, irreducibility can be expensive to prove [ASZ00].

original definition of candid (Definition 6), we can say that insisting on square-freeness, so that any repeated part of one factor is explicit in the exponents in (2.4), as in $x^3 + x^2 - x - 1$ having to be written as $(x - 1)(x + 1)^2$, and on relative primeness, so that a repeated factor cannot be hidden in two different factors, as in $(x^2 - 1)(x^2 + x - 2)$ having to be written as $(x - 1)^2(x + 1)(x + 2)$, means that all the repetition is visible in the format of (2.4) and nothing is being hidden.

Multiplication is relatively straight-forward, we check (via g.c.d. computations[14]) for duplicated factors between the two multiplicands, and then combine the multiplicands. Addition can be extremely expensive, and the result of an addition can be exponentially larger than the inputs: consider

$$(x + 1)(x^2 + 1) \cdots (x^{2^k} + 1) + (x + 2)(x^2 + 2) \cdots (x^{2^k} + 2),$$

where the input has $4(k + 1)$ non-zero coefficients, and the output has 2^{k+1} (somewhat larger) ones.

This representation is not much discussed in the general literature, but is used in Redlog [DS97] and Qepcad [CH91], both of which implement cylindrical algebraic decomposition (see section 3.5), where a great deal of use can be made of corollaries 21 and 22.

2.1.4 Polynomials in several variables

Here the first choice is between factored and expanded. The arguments for, and algorithms handling, factored polynomials are much the same[15] as in the case of one variable. The individual factors of a factored form can be stored in any of the ways described below for expanded polynomials, but recursive is more common since it is more suited to g.c.d. computations (chapters 4 and 5), which as we saw above are crucial to manipulating factored representations.

If we choose an expanded form, we have one further choice to make, which we explain in the case of two variables, x and y, and illustrate the choices with $x^2y + x^2 + xy^2 - xy + x - y^2$.

Notation 16 *The* total degree *of a monomial is the sum of the degrees of all the variables. The* total degree *of a polynomial is the greatest total degree of any monomial in it.*

Note that this definition implicitly assumes that we have performed any cancellation, since the total degree of $x^2y^3 - x - x^2y^3$ is 1, not 5. Put another way, we require the representation to be *candid* (Definition 6), at least in this respect.

[14] Again, these should be cheap if there is no factor to detect, and otherwise lead to reductions in size.

[15] In one variable, the space requirements for a typical dense polynomial and its factors are comparable, e.g. 11 terms for a polynomial of degree 10, and 6 each for two factors of degree 5. For multivariates, this is no longer the case. Even for bivariates, we would have 121 terms for a dense polynomial of degree 10, and 36 each for two factors of degree 5. The gain is greater as the number of variables increases.

recursive — $C[x][y]$. We regard the polynomials as polynomials in y, whose coefficients are polynomials in x. Then the sample polynomial would be $(x-1)y^2 + (x^2-x)y + (x^2+x)y^0$. We have made the y^0 term explicit here: in practice detailed representations in different systems differ on this point.

recursive — $C[y][x]$. We regard the polynomials as polynomials in x, whose coefficients are polynomials in y. Then the sample polynomial would be $(y+1)x^2 + (y^2-y+1)x + (-y^2)x^0$.

distributed — $C[x,y]$. We regard the polynomials as polynomials in x and y, whose coefficients are numbers. With 6 terms (as in this example), there are $6! = 720$ possible orders. It is usual to impose two additional constraints on the order on terms, or more accurately on the *monomials*[16], i.e. ignoring the coefficients, which we will denote[17] as $>$.

Definition 28 *An ordering is said to be an* admissible *ordering if it satisfies the following conditions.*

- *Compatibility with multiplication: if $a > b$ then, for all monomials c, $ac > bc$.*
- *Well-foundedness: for all non-trivial monomials a, $a > 1$.*

These requirements greatly reduce the available orders for our sample polynomial. One possibility would be to sort by total degree (i.e. the sum of the degrees in each variable), using degree in x as a tie-breaker. This would give us $x^2y + xy^2 + x^2 - xy - y^2 + x$. There is a fuller discussion of such orderings in Section 3.3.3. However, we should note one important property of admissible orders here.

Theorem 4 (Descending Chain Condition; Dickson's Lemma) *Any decreasing sequence (with respect to an admissible ordering) of monomials in a finite number of variables is finite. [Dic13]*

In general, if there are n variables, there are $n!$ possible recursive representations, but an infinite number of possible distributed representations, though clearly only finitely many different ones for any one given polynomial or finite set of polynomials.

In both cases, we use sparse, rather than dense, representations, since any reasonable multivariate polynomial had better be sparse: degree 6 in each of 6 variables means $7^6 = 117649$ terms.

[16]In this book we use the word *monomial* to mean a product of (possibly repeated) variables, as in xyz or x^2y, without any coefficient. This includes $1 = x^0y^0z^0$ as a monomial. *Term* means a product with a coefficient, as in $3x^2y$. Usage on this point differs.

[17]We are not making any *numerical* evaluation of the monomials, merely saying which order we put the monomials in.

Definition 29 *We say that an algorithm has* poly-sparse *complexity if the complexity is a polynomial function in the sparse bit size (see Definition 27) of the inputs and outputs. We say that an algorithm has* poly-semisparse *complexity if the complexity is a polynomial function in the sparse bit size and the degree of the polynomial. If all that can be said is that the complexity is polynomial in d^n (where d is the total degree and n is the number of indeterminates) we say it has* poly-dense *complexity.*

The same canonicality results as for univariate polynomials apply.

Proposition 8 *For a fixed ordering, both recursive and distributed representations are canonical (definition 3). Partially factored representations are normal, but not canonical.*

It makes no sense to compare polynomials in different representations, or the same representation but different orderings. We have spoken about 'representations', but in fact the division between recursive and distributed goes deeper. While characterisations 3 and 2 of a Gröbner base (theorem 16) can make sense in either view, characterisations 4 and 1 (the only effective one) only make sense in a distributed view. Conversely, while the abstract definitions of factorisation and greatest common divisors (definition 31) make sense whatever the view, the only known algorithms for computing them (Algorithm 2 or the advanced ones in chapters 4 and 5) are inherently recursive[18].

2.1.5 Other representations

Sparse representations take up little space *if* the polynomial is sparse. But shifting the origin from $x = 0$ to $x = 1$, say, will destroy this sparsity, as might many other operations. The following example, adapted from [CGH+03], illustrates this. Let $\Phi(Y, T)$ be

$$\exists X_1 \dots \exists X_n (X_1 = T + 1) \wedge (X_2 = X_1^2) \wedge \cdots \wedge (X_n = X_{n-1}^2) \wedge (Y = X_n^2). \quad (2.5)$$

The technology described in section 3.5.3 will convert this to a polynomial equation

$$\Psi(Y, T) : Y = (1 + T)^{2^n}. \quad (2.6)$$

Dense or sparse representations have problems with this, in the sense that expression (2.5) has length $O(n)$, but expression (2.6) has length $O(2^n)$ or more. A factored representation could handle the right-hand side, *assuming* that we are not representing the equations as polynomial = 0. But changing the last conjunct of Φ to $(Y = (X_n + 1)^2)$ changes Ψ to

$$Y = \left(1 + (1 + T)^{2^{n-1}}\right)^2, \quad (2.7)$$

whose factored representation now has length $O(2^n)$.

[18]At least for commutative polynomials. Factorisation of non-commutative polynomials is best done in a distributed form.

Factored representations display a certain amount of internal structure, but at the cost of an expensive, and possibly data-expanding, process of addition. Are there representations which do not have these 'defects'? Yes, though they may have other 'defects'.

Expression tree This representation "solves" the cost of addition in the factored representation, by storing addition as such, just as the factored representation stored multiplication as such. Hence $\left((x+1)^3-1\right)^2$ would be legal, and represented as such. Equation (2.7) would also be stored compactly *provided* exponentiation is stored as such, e.g. Z^2 requiring one copy of Z, rather than two as in $Z \cdot Z$. This system is not canonical, or even normal: consider $(x+1)(x-1)-(x^2-1)$. This would be described by Moses [Mos71] as a "liberal" system, and generally comes with some kind of `expand` command to convert to a canonical representation. Assuming now that the leaf nodes are constants and variables, and the tree's internal nodes are (binary, i.e. with two arguments) addition, subtraction and multiplication, then a tree with maximal depth p can represent a polynomial with maximum total degree 2^p. It would need to have 2^p-1 internal nodes (all multiplication), and 2^p leaf nodes. The degree is easy to bound, by means of a tree-walk, but harder to compute, especially if cancellation actually occurs. Similarly, the leading coefficient can be computed via a simple tree-walk *if* no cancellation occurs.

Expression DAG also known as **Straight-Line Program (SLP)** [IL80]. This is essentially[19] the representation used by Maple — it looks like the previous representation, but the use of hashing in fact makes it a directed acyclic graph (DAG). Again, a straight-line program of length l (i.e. a DAG of depth $l-1$) can store a polynomial of degree 2^{l-1}. The difference with the expression tree representation above is that we only need l nodes, since the nodes can be reused.

This format is essentially immune to the "change of origin" problem mentioned above, since we need merely replace the x node by a tree to compute $x+1$, thus adding two nodes, and possibly increasing the depth by one, irrespective of the size of the polynomial. The general 'straight-line' formalism has advantages where multi-valued functions such as square root are concerned: see the discussions around figures 3.1 and 3.2.

However, there is one important caveat about straight-line programs: we must be clear what operations are allowed. If the only operations are $+$, $-$ and $\times$, then evidently a straight-line program computes a polynomial. Equally, if division is allowed, the program *might* not compute a polynomial. But might it? If we look at figure 2.1, we see that $p = x^2-1$ and $q = x-1$, so the result is $\frac{p}{q} = \frac{x^2-1}{x-1} = x+1$. Or is it? If we feed in $x=1$, we in fact get $\frac{0}{0}$, rather than 2. This is a singularity of the kind known as

[19]Maple uses n-ary addition and multiplication, rather than binary, as described in section C.3.

a removable singularity, because $\lim_{x\to 1}\frac{p(x)}{q(x)} = 2$. In fact [IL80, Theorem 3], deciding if two straight-line programs are equivalent is undecidable if division is allowed.

Figure 2.1: A polynomial SLP

$$\begin{tikzcd}
 & & & x \arrow[d, Rightarrow] \arrow[dr] & & & \\
 & -1 \arrow[dr] & & * \arrow[dll] & \ \arrow[dr] & & -1 \arrow[dl] \\
p \arrow[r] & + \arrow[dr] & & & & + \arrow[dll] & q \arrow[l] \\
 & & / & & & &
\end{tikzcd}$$

We said earlier that the only known algorithms for computing greatest common divisors were recursive. This is essentially true, and means that the computation of greatest common disivors of straight-line programs is not a straight-forward process [Kal88].

Reconstructing one of the more explicit representations (usually a sparse one!) from a straight-line program representation is not straightforward. The current state of the art over general finite fields is the following[20].

Proposition 9 ([AGR14a, Theorem 1]) *Let $F \in \mathbf{F}_q[z_1, \ldots, z_n]$, and suppose we are given a division-free straight-line program $\mathcal{S}_F$ of length L which evaluates F, an upper bound $D = \max_j \deg_{z_j}(F)$, and an upper bound T on the number of nonzero terms t of F. There exists a probabilistic algorithm which interpolates F with probability at least 3/4. The algorithm requires*

$$\tilde{O}\left(Ln(T\log D + n)(\log D + \log q)\log D + n^{\omega-1}T\log D + n^{\omega}\log D\right)$$

bit operations, where ω is the matrix multiplication exponent.

We note that this is linear in T, which is as good as we could hope for.

If we can choose the finite field, to be $\mathbf{Z}_p$ where p is smooth (has only small prime divisors) then we can do much better: $\tilde{O}(LTn\log D + n^2T\log^2 D)$ [Kal10, for the algorithm], [AGR14a, for the complexity]. Such fields appear to be common [HB78], and this gives us a algorithm for interpolating F over the integers with time $\tilde{O}\left((n^2T\log D + nLT)(n\log D + \log H)\right)$ [Kal10, for the algorithm], [Roc14, for the complexity]. There are further improvements to be had in [AGR14b].

[20]The author is grateful to Dan Roche for explanations here, and for [Roc14].

Figure 2.2: Code fragment A — a graph

```
p:=x+1;
q:=p;
r:=p*q;
```

Figure 2.3: Code fragment B — a tree

```
p:=x+1;
q:=x+1;
r:=p*q;
```

Additive Complexity This [Ris85, Ris88] is similar to a straight-line program, except that we only count the number of (binary) addition/subtraction nodes, i.e. multiplication and exponentiation are 'free'. Hence the degree is unbounded in terms of the additive complexity, but for a given expression (tree/DAG) can be bounded by a tree-walk. A univariate polynomial of additive complexity a has at most C^{a^2} real roots for some absolute constant C: conjecturally this can be reduced to 3^a. These bounds trivially translate to the straight-line program and expression tree cases. See also Open Problem 5.

Specialist There are many possible highly specialist representations of polynomials. One of the most impressive (where it's applicable) is the graph representation, used in [AIR14] to represent a polynomial with 317,881,154 monomials in a half-page graph.

"Additive complexity" is more of a measure of the 'difficulty' of a polynomial than an actual representation. Of the others, the first was used in Macsyma for its "general expression", and the second is used in Maple[21]. In fact, Macsyma would[22] *allow* general DAGs, but would not force them. Consider the two code fragments in figures 2.2 and 2.3. In the case of figure 2.2, both systems would produce the structure in figure 2.4. For figure 2.3, Macsyma would produce the structure[23] in figure 2.5. Maple would still produce the structure of figure 2.4, since the hashing mechanism would recognise that the two $x + 1$ were identical.

[21]Until such time as operations such as `expand` are used!

[22]Not explicitly, but rather as a side-effect of the fact that Macsyma is implemented in Lisp, which cares little for the difference. The basic function `EQUAL` does not distinguish between acyclic and cyclic structures

[23]The x is shown as shared since the Lisp implementation will store symbols unqiuely.

Figure 2.4: DAG representation

$$\begin{tikzcd}
& 1 \arrow[dr] & & x \arrow[dl] & \\
p \arrow[rr] & & + \arrow[d, Rightarrow] & & q \arrow[ll] \\
r \arrow[rr] & & * & &
\end{tikzcd}$$

Figure 2.5: Tree representation

$$\begin{tikzcd}
& 1 \arrow[dr] & & x \arrow[dl] \arrow[dr] & & 1 \arrow[dl] & \\
p \arrow[rr] & & + \arrow[dr] & & + \arrow[dl] & & q \arrow[ll] \\
& & r \arrow[r] & * & & &
\end{tikzcd}$$

2.1.6 The Newton Representation

For simplicity, in this subsection we will only consider the case of charateristic 0: finite characteristic has some serious technical difficulties, and we refer the reader to [BFSS06]. We will also only consider monic polynomials.

Notation 17 *Let* $p = x^n + \sum_{i=0}^{n-1} a_i x^i = \prod_{i=0}^{n}(x - \alpha_i)$ *be a polynomial of degree* n. *Let* $\beta_s = \sum_{i=0}^{n} \alpha_i^s$, *and define the* Newton series *of* p *to be* $\text{Newton}(p) = \sum_{s \geq 0} \beta_s T^s$.

It is well-known that the a_i and α_i are related:

$$\begin{aligned}
a_{n-1} &= -\sum_{i=0}^{n} \alpha_i \\
a_{n-2} &= \sum_{i=0}^{n} \sum_{j=i+1}^{n} \alpha_i \alpha_j \\
\vdots \; &\vdots \quad \vdots \\
a_0 &= (-1)^n \prod_{i=0}^{n} \alpha_i .
\end{aligned}$$

These are then related to the β_i:

$$\beta_1 = -a_{n-1}$$

$$\beta_1^2 = \beta_2 + 2a_{n-2} \vdots \quad \vdots \quad \ddots$$

Hence, in characteristic 0, the β_i ($i \le n$) form an alternative to the a_i, a fact known since 1840 [LV40]. But how do we convert rapidly between these representations?

Proposition 10 ([Sch82], see also [BFSS06, Lemma 1]) Newton$(p) = \frac{\mathrm{rev}(p')}{\mathrm{rev}(p)}$ *as a power series about* $x = 0$*, where* $\mathrm{rev}(p) = \sum_{i=0}^{n} a_{n-i}x^i$.

2.1.7 Representations in Practice

General-purpose (calculus-side) computer algebra systems have to deal with many expressions other than polynomials, but tend to regard polynomials as the basic construct.

2.1.7.1 Representations in Reduce

This is probably one of the most straightforward. The basic Reduce object is a *standard quotient*, i.e. a pair (literally a `CONS cell` in Lisp) of polynomials, or *standard forms*. A standard form is a sparse recursive multivariate polynomial, where the "variables" (known as *kernels* in Reduce) may be variables such as `x`, but equally functions and expressions such as `(cos x)` — a Lisp form representing $\cos(x)$.

If the kernels are genuinely indeterminates, i.e. we have a free algebra (Definition 23), then this is a normal form, and it is canonical subject to the restrictions in Proposition 14. Initially Reduce did not, by default, compute gcds (clause 3), but this changed as more efficient gcd algorithms were implemented.

2.1.7.2 Representations in Macsyma

Macsyma's "general representation" is essentially an expression tree one. There is a special form, known as Canonical Rational Expression[24], which again is a ratio of sparse recursive polynomials.

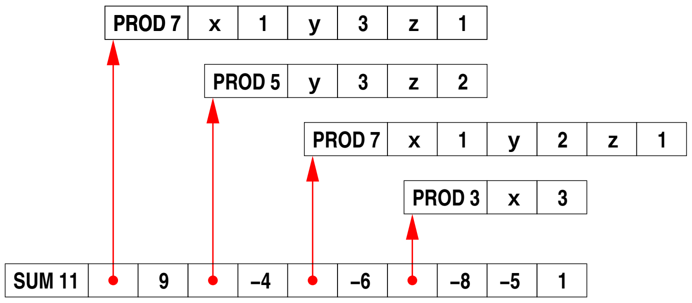
2.1.7.3 Representations in Maple

Maple's original internal representation[25] was an expression tree whose fundamental form was an n-ary sum of n-ary power products, as in the representation of $9xy^3z - 4y^3z^2 - 6xy^2z - 8x^3 - 5$ as in Figure 2.6. The "variables" might in fact be other Maple expressions, so that $x^2 - (x-1)(x+1) - 1$ would be a sum of three terms, one of which was the product of two elements each of which were themselves sums. The elements in `SUM` and `PROD` expressions were stored in hash-code order, which greatly facilitated combining like terms. The advantages and disadvantages of this are described in [MP12, MP13].

[24]See `http://www.ma.utexas.edu/maxima/maxima_11.html`.

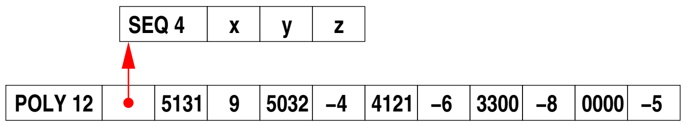
[25]The author is grateful to Michael Monagan of Simon Fraser University for much of this information, and for permisson to reproduce the images. Note that, in the standard format for a `SUM`, all the coefficients are the odd-numbered elements (counting the `SUM` as number 1) *except* for the constant term. This is apparently a historical design decision.

Figure 2.6: Maple's Original Polynomials
`SUM11` means a `sum` expression occupying 11 words (header plus 5 product/coefficient pairs); similarly `PROD7` means a `product` expression occupying 7 words (header plus 3 variable/exponent pairs).



As described there, for genuine polynomials (which would exclude $x^2-(x-1)(x+1)-1$ and expressions involving RootOf etc.), an alternative data structure was introduced at Maple 17: the `POLY` data structure, which is a packed sparse distributed representation: in this case as in Figure 2.7. where the numbers 5131

Figure 2.7: Maple's New-Style Polynomials
`POLY12` means a `poly` data structure of header word, a pointer to the variables, and five exponent/coefficients pairs.



etc. are in fact numbers base 2^{15}, so "5131"$= 5\cdot 2^{45}+2^{30}+3\cdot 2^{15}+1$, meaning 'total degree 5' and then the exponents of the individual variables. This form is only used if the packed exponent field will fit into a 64-bit integer (or a 32-bit integer on 32-bit Maple. Note that a 64-bit Maple integer actually goes up to $2^{62}-1$, as given by `kernelopts(maximmediate)`). In this case, we had four 'variables' (three real ones and the slot for total degree), and $16=\lfloor 62/4\rfloor$ — with four real variables we would have a base 2^{12} expression as $12=\lfloor 62/5\rfloor$, and so on. The fact that total degree is stored, and that the items in a `POLY` data structure are stored in decreasing order of this packed exponent, means that we have a 'graded lexicographic' (see p. 108) ordering.

The difference should be invisible to the casual Maple user, except for performance, but can be seen via the `dismantle` command.

2.1.8 Comparative Sizes

This section is largely of theoretical interest, since practical systems employ a variety of techniques for storing polynomials, and are also constrained by the actual size of machine words etc. Hence an analysis of a practical system will tend to contain $4\lceil\log_{2^{31}}(n)\rceil$ for the size in bytes, rather than $\log_2(n)$ for the size in bits.

Notation 18 *We assume our polynomials have integer coefficients and are in N variables, each of which*[26] *occurs to degree at most D, and the coefficients are at most C in absolute value. Furthermore, suppose there are at most T non-zero terms. Let $n = \log_2 N$, $d = \log_2 D$ and $c = 1 + \log_2 C$ (the "1+" allows for a sign. We ignore the space needed to store N, D etc. themselves.*

Dense At least in principle, dense storage, whether recursive or distributed, just stores the coefficients and no additional structural information. There are $(D+1)^N$ terms, and , hence the storage needed is $s_{\text{dense}} = c(D+1)^N$.

Sparse (Distributed) There are T terms, each needing a coefficient and N exponents, hence $s_{\text{sparse}} = T(c + Nd)$. If T is maximal (i.e. the polynomial is completely dense), $s_{\text{sparse}} = s_{\text{dense}} + (D+1)^N Nd$.

Sparse (Recursive) This is harder to describe, partly because it depends on the variable order, as $x_1^D x_2^D \cdots x_{N-1}^d \left(x_N^D + \cdots + x_N^0\right)$ stores $N + D$ exponents, but if x_N is the main variable, it stores $(D+1)N$ exponents. The coefficient storage is the same as for s_{sparse}, though.

Expression Tree Since this representation is not canonical, we need to discuss the size of a particular representation, not just of an abstract polynomial. Call this s_{tree}. Note that x^N requires $N - 1$ multiplications, whether we use repeated squaring, as in $x^4 = (x * x) * (x * x)$ or iteration as in $x^4 = x * (x * (x * x))$

Expression DAG Again this is not canonical. x^N can now be computed by recursive squaring with reuse, so x^{2^n} only needs n multiplications. Call this s_{DAG}. Since every tree is a DAG, we have $s_{\text{DAG}} \le s_{\text{tree}}$.

Though there are unusual counter-examples, if we assume that $c \gg d \gg n$, then

$$s_{\text{dense}} \gg s_{\text{sparse}} \gg s_{\text{tree}} \gg s_{\text{DAG}}. \qquad (2.8)$$

Proposition 11 *Each of the gaps in (2.8) can be exponentially big.*

[26]Many authors prefer to bound the *total* degree by D. The difference is not great in practice.

2.2 Rational Functions

Of course, we want to manipulate a wider class of expressions, and even $\frac{1}{x}$ is not a polynomial.

Definition 30 *A* rational function *is built up from* coefficients*, which are assumed to form an integral domain (definition 11), and certain* indeterminates*, by the* algebraic operations *of addition, subtraction, multiplication and division (except that division by zero is not permitted). In addition to the laws in definition 22 (but with a, b and c interpreted as rational functions), the following law is obeyed.*

11. $a * (1/a) = 1$.

2.2.1 Canonical Rational Functions

Proposition 12 *Any rational function f can be put over a common denominator, i.e. written as n/d where n and d are polynomials, known as the* numerator *and* denominator *respectively.* We write num(f) and den(f) respectively, noting that in fact these are only defined up to units.

Proposition 13 *In common denominator format, $\frac{a}{b} = \frac{c}{d}$ if, and only if, $ad - bc = 0$.*

We can in fact characterise three simple forms of equality.

common coefficients An example of this would be.

$$\frac{x^2 - 2x + 1}{x^2 - 1} \quad \text{versus} \quad \frac{2x^2 - 4x + 2}{2x^2 - 2}.$$

Here we need to remove the g.c.d. of the contents (definition 35) of the two polynomials.

"up to sign" An example of this would be.

$$\frac{-x^2 + 2x - 1}{x^2 - 1} \quad \text{versus} \quad \frac{x^2 - 2x + 1}{-x^2 + 1}.$$

These are 'clearly equal' but "computers don't do clearly". We need a convention, and the common one[27] is 'leading coefficient positive' in the denominator. However, this does not generalise so easily to other domains of coefficients [DT90].

common factors An example of this would be

$$\frac{x^2 - 2x + 1}{x^2 - 1} \quad \text{versus} \quad \frac{x - 1}{x + 1}.$$

[27]By no means the only possible one: 'leading coefficient negative' would be equally valid, as would 'trailing coefficient positive'.

If we put the difference between the two over a common denominator, we get $\frac{0}{x^2-1}=0$. The reader may complain that $\frac{x^2-2x+1}{x^2-1}$ "is undefined when $x=1$", whereas $\frac{x-1}{x+1}$ "has the value 0". However, we have not defined what we mean by such substitutions, and for the purposes of this chapter, we are concerned with *algebraic equality* in the sense of proposition 13.

Proposition 14 ([DT90]) *A representation n/d where n and d are polynomials is* canonical *if the following conditions are satisfied:*

1. *n and d are polynomials from a free (Definition 23) polynomial algebra;*
2. *these polynomials are themselves represented canonically;*
3. *Any greatest common divisor, whether polynomial or content, is removed from n and d;*
4. *n/d is canonical with respect to units, typically by insisting that d have a canonical-up-to-associates (e.g. positive) leading coefficient.*

The reader might think that condition 1 was unnecessary in view of condition 2. This this is not so is shown by

$$\sqrt{2}-1=\frac{1}{\sqrt{2}+1} \tag{2.9}$$

where each of $\sqrt{2}-1$ and $\sqrt{2}+1$ are represented canonically, but nevertheless we have an equality here, which wouldn't occur for any value of the "indeterminate" except $\sqrt{2}$.

2.2.2 Candidness of rational functions

We have already given (Definition 6) an abstract definition of candidness, which can also be described as "what you see is what you've got" mathematically. What would this mean for rational functions (and therefore for polynomials)? [Sto11a, p.869] gives the following as a sufficient set of conditions[28].

1. there are no compound ratios such as
$$x+\frac{x+1}{1+\frac{1}{x}} \tag{2.10}$$
(note that this is $2x$, and therefore "only" a polynomial, so violates the general definition of candidness),
2. all ratios that occur are reduced (therefore preferring $x^{999}+\cdots+x+1$ to $\frac{x^{1000}-1}{x-1}$, so disagreeing with Carette's definition of 'simplification' on page 25),

[28]He also remarks "There can be other candid forms for rational expressions, including [appropriately reduced, ruling out (2.10)] continued fractions. However, the complexity of implementing a candid simplifier increases with the permissiveness of the allowed result forms."

3. the factors and terms are ordered in an easily discerned traditional way[29], such as lexically by descending degree,
4. all manifestly similar factors and terms are collected,
5. for each variable, the actual degree of every variable in a reduced ratio of an expanded numerator and denominator would be no less than what a user would predict assuming no cancellations. For example, assuming no cancellations, we can predict that at most the degree of x will be 3 in the denominator and 6 in the numerator when
$$x^3 + \frac{1}{x^2-1} + \frac{1}{x+2} \tag{2.11}$$
is reduced over a common denominator. Those *are* the resulting degrees, so (2.11) *is* a candid representation, even though it's probably not one of a class of canonical representations. Conversely (2.10) violates this condition, since we would predict it to be the ratio of a degree 2 and a degree 1 polynomial.

In particular, clause 2 (or 5) implies that any common factors are cancelled, which poses the question, answered in the next section: how do we compute common factors? Given $\frac{f}{g}$, is there an h dividing both f and g that should be cancelled?

2.3 Greatest Common Divisors

The following definition is valid whenever we have a concept of division.

Definition 31 *h is said to be a* greatest common divisor*, or* g.c.d.*, of f and g if, and only if:*

1. *h divides both f and g;*
2. *if h' divides both f and g, then h' divides h.*

This definition clearly extends to any number of arguments. The g.c.d. is normally written $\gcd(f, g)$.

Note that we have defined *a* g.c.d, whereas it is more common to talk of *the* g.c.d. However, 'a' is correct. We normally say that 2 is *the* g.c.d. of 4 and 6, but in fact -2 is equally a g.c.d. of 4 and 6.

Proposition 15 *If h and h' are greatest common divisors of a and b, they are associates (definition 13).*

Example 3 (Greatest common divisors need not exist) *Consider the set of all integers with $\sqrt{-5}$. 2 clearly divides both 6 and and $2+2\sqrt{-5}$. However, so does $1+\sqrt{-5}$ (since $6 = (1+\sqrt{-5})(1-\sqrt{-5})$), yet there is no multiple of both 2 and $1+\sqrt{-5}$ which divides both.*

[29]It is this clause that Maple's sometimes disconcerting hash-based output breaks.

Definition 32 *An integral domain (definition 11) in which any two elements have a greatest common divisor is known*[30] *as a* g.c.d. domain.

If R is a g.c.d. domain, then the elements of the field of fractions (definition 16) can be simplified by cancelling a g.c.d. between numerator and denominator, often called "reducing to lowest terms". While this simplifies fractions, it does not guarantee that they are normal or canonical. One might think that $\frac{0}{1}$ was the unique representation of zero required for normality, but what of $\frac{0}{-1}$? Equally $\frac{-1}{2} = \frac{1}{-2}$, and in general we have to remove the ambiguity caused by units. In the case of rational numbers, we do this automatically by making the denominator positive, but the general case is more difficult [DT90].

Definition 33 *h is said to be a* least common multiple*, or* l.c.m.*, of f and g if, and only if:*

1. *both f and g divide h ;*
2. *if both f and g divide h', then h divides h'.*

This definition clearly extends to any number of arguments. The l.c.m. is normally written $\operatorname{lcm}(f, g)$.

Proposition 16 *If* $\gcd(f, g)$ *exists, then* $fg/\gcd(f, g)$ *is a least common multiple of f and g.*

This result is normally written as $fg = \gcd(f, g)\operatorname{lcm}(f, g)$, but this is only true up to associates. We should also note that this result does *not* extend to any number of arguments: in general $fgh \neq \gcd(f, g, h)\operatorname{lcm}(f, g, h)$.

2.3.1 Polynomials in one variable

For univariate polynomials over a field, we can define a more extended version of division.

Definition 34 *If a and $b \neq 0$ are polynomials in $K[x]$, K a field, and $a = qb+r$ with* $\deg(r) < \deg(b)$*, then we say that b* divides a with quotient q and remainder r*, and q and r are denoted* $\operatorname{quo}(a, b)$ *and* $\operatorname{rem}(a, b)$.

It is clear that q and r exist, and are unique. Division in the previous sense then corresponds to the case $r = 0$.

Theorem 5 (Euclid) *If K is a field, the univariate polynomials $K[x]$ form a g.c.d. domain.*

[30]Normally known as a *unique factorisation domain*, but, while the existence of greatest common divisors is equivalent to the existence of unique factorisation, the ability to *compute* greatest common divisors is not equivalent to the ability to *compute* unique factorisations [FS56, DGT91], and hence we wish to distinguish the two.

Algorithm 2 (Euclid)
Input: $f, g \in K[x]$.
Output: $h \in K[x]$ *a greatest common divisor of* f *and* g

```
$i := 1$;
if $\deg(f) < \deg(g)$
   then $a_0 := g$; $a_1 := f$;
   else $a_0 := f$; $a_1 := g$;
while $a_i \neq 0$ do
        $a_{i+1} = \mathrm{rem}(a_{i-1}, a_i)$;
        $\#q_i :=$the corresponding quotient: $a_{i+1} = a_{i-1} - q_i a_i$
        $i := i + 1$;
return $a_{i-1}$;
```

Proof. We must first show that this is an algorithm, i.e. that the potentially infinite loop actually terminates. But $\deg(a_i)$ is a non-negative integer, strictly decreasing each time round the loop, and therefore the loop must terminate. So $a_i = 0$, but $a_i = \mathrm{rem}(a_{i-2}, a_{i-1})$, so a_{i-1} divides a_{i-2}. In fact, $a_{i-2} = q_{i-1}a_{i-1}$. Now $a_{i-1} = a_{i-3} - q_{i-2}a_{i-2}$, so $a_{i-3} = a_{i-1}(1 + q_{i-2}q_{i-1})$, and so on, until we deduce that a_{i-1} divides a_0 and a_1, i.e. f and g in some order. Hence the result of this algorithm is a common divisor. To prove that it is a greatest common divisor, we must prove that any other common divisor, say d, of f and g divides a_{i-1}. d divides a_0 and a_1. Hence it divides $a_2 = a_0 - q_1a_1$. Hence it divides $a_3 = a_1 - q_2a_2$, and so on until it divides a_{i-1}.

We should note that our algorithm is asymmetric in f and g: if they have the same degree, it is not generally the case that $\gcd(f, g) = \gcd(g, f)$, merely that they are associates.

Lemma 1 *In these circumstances, the result of Euclid's algorithm is a linear combination of* f *and* g*, i.e.* $a_{i-1} = \lambda_{i-1}f + \mu_{i-1}g$*:* $\lambda_{i-1}, \mu_{i-1} \in K[x]$.

Proof. a_0 and a_1 are certainly such combinations: $a_0 = 1 \cdot f + 0 \cdot g$ or $1 \cdot g + 0 \cdot f$ and similarly for a_1. Then $a_2 = a_0 - q_1a_1$ is also such a combination, and so on until a_{i-1}, which is the result.

The above theory, and algorithm, are all very well, but we would like to compute (assuming they exist!) greatest common divisors of polynomials with integer coefficients, polynomials in several variables, etc. So now let R be any g.c.d. domain.

Definition 35 *If* $f = \sum_{i=0}^{n} a_i x^i \in R[x]$*, define the* content *of* f*, written* $\mathrm{cont}(f)$*, or* $\mathrm{cont}_x(f)$ *if we wish to make it clear that* x *is the variable, as* $\gcd(a_0, \ldots, a_n)$*. Technically speaking, we should talk of* a *content, but in the theory we tend to abuse language, and talk of* the *content. Similarly, the* primitive part*, written* $\mathrm{pp}(f)$ *or* $\mathrm{pp}_x(f)$*, is* $f/\mathrm{cont}(f)$*.* f *is said to be* primitive *if* $\mathrm{cont}(f)$ *is a unit.*

Proposition 17 *If* f *divides* g*, then* $\mathrm{cont}(f)$ *divides* $\mathrm{cont}(g)$ *and* $\mathrm{pp}(f)$ *divides* $\mathrm{pp}(g)$*. In particular, any divisor of a primitive polynomial is primitive.*

The following result is in some sense a converse of the previous sentence.

Lemma 2 (Gauss) *The product of two primitive polynomials is primitive.*

Proof. Let $f = \sum_{i=0}^{n} a_i x^i$ and $g = \sum_{j=0}^{m} b_j x^j$ be two primitive polynomials, and $h = \sum_{i=0}^{m+n} c_i x^i$ their product. Suppose, for contradiction, that h is not primitive, and p is a prime[31] dividing $\mathrm{cont}(h)$. Suppose that p divides all the coefficients of f up to, *but not including*, a_k, and similarly for g up to but not including b_l. Now consider

$$c_{k+l} = a_k b_l + \sum_{i=0}^{k-1} a_i b_{k+l-i} + \sum_{i=k+1}^{k+l} a_i b_{k+l-i} \tag{2.12}$$

(where any indices out of range are deemed to correspond to zero coefficients).

Since p divides $\mathrm{cont}(h)$, p divides c_{k+l}. By the definition of k, p divides every a_i in $\sum_{i=0}^{k-1} a_i b_{k+l-i}$, and hence the whole sum. Similarly, by definition of l, p divides every b_{k+l-i} in $\sum_{i=k+1}^{k+l} a_i b_{k+l-i}$, and hence the whole sum. Hence p divides every term in equation (2.12) except $a_k b_l$, and hence has to divide $a_k b_l$. But, by definition of k and l, it does not divide either a_k or b_l, and hence cannot divide the product. Hence the hypothesis, that $\mathrm{cont}(h)$ could be divisible by a prime, is false.

Corollary 2 $\mathrm{cont}(fg) = \mathrm{cont}(f)\mathrm{cont}(g)$.

Theorem 6 ("Gauss' Lemma") *If R is a g.c.d. domain, and $f, g \in R[x]$, then* $\gcd(f, g)$ *exists, and is* $\gcd(\mathrm{cont}(f), \mathrm{cont}(g)) \gcd(\mathrm{pp}(f), \mathrm{pp}(g))$.

Proof. Since R is an integral domain, its field of fractions, say K is a field. Hence, in $K[x]$ where theorem 5 is applicable, $\mathrm{pp}(f)$ and $\mathrm{pp}(g)$ have a greatest common divisor, say h. If c is any non-zero element of R, then ch is also a greatest common divisor of $\mathrm{pp}(f)$ and $\mathrm{pp}(g)$. Hence we can assume that h is in $R[x]$ and, as a polynomial of $R[x]$, is primitive. In $K[x]$, $\mathrm{pp}(f)$ is a multiple of h, say $\mathrm{pp}(f) = hk$ for $k \in K[x]$. We can write $k = dk'$, where $k' \in R[x]$ and is primitive. Then $d^{-1}\mathrm{pp}(f) = hk'$. But h and k' are primitive, so, by the Lemma, their product is primitive, and d is a unit. Hence h is, in $R[x]$, *a* common divisor of $\mathrm{pp}(f)$ and $\mathrm{pp}(g)$.

But, if $\bar{h}$ is a common divisor of $\mathrm{pp}(f)$ and $\mathrm{pp}(g)$ in $R[x]$, it is certainly a common divisor of f and g in $K[x]$, hence divides h in $K[x]$, and so $\mathrm{pp}(\bar{h})$ divides h in R[x]. Hence h is a *greatest* common divisor of $\mathrm{pp}(f)$ and $\mathrm{pp}(g)$ in $R[x]$, and the rest of the theorem is obvious.

This gives us one obvious means of computing g.c.d.s in $R[x]$, which can be described as "compute in $K[x]$ and sort out the contents afterwards". More formally it would be Algorithm 3.

[31]The reader may complain that, in note 30, we said that the ability to compute g.c.d.s was *not* equivalent to the ability to compute unique factors, and hence primes. But we are not asking to factorise $\mathrm{cont}(f)$, merely supposing, *for the sake of contradiction* that it is non-trivial, and therefore has a prime divisor.

Algorithm 3 (General g.c.d.)
Input: $f, g \in R[x]$.
Output: $h \in R[x]$ *a greatest common divisor of* f *and* g

```
1. $f_c := \mathrm{cont}_x(f)$; $f_p := f/f_c$; $g_c := \mathrm{cont}_x(g)$; $g_p := g/g_c$.
2. $h := \gcd(f_p, g_p)$ computed in $K[x]$ by Algorithm 2
3. $h_p := \mathrm{pp}(h \times \text{(enough to remove denominators)})$
4. return $\gcd(f_c, g_c) \times h_p$
# Correct by reasoning above.
```

Certainly this is an algorithm, but is it a good one? Let $k = \max(\deg_x(f), \deg_x(g))$. In terms of the number of coefficient operations, and in the dense representations, we do $O(k^2)$ operations. But how big do these coefficients get?

Consider the computation of the g.c.d. of the following two polynomials (this analysis is mostly taken from [Bro71a, Bro71b], but with one change[32]: -21 instead of $+21$ for the trailing coefficient of B):

$$\begin{aligned} A(x) &= x^8 + x^6 - 3x^4 - 3x^3 + 8x^2 + 2x - 5; \\ B(x) &= 3x^6 + 5x^4 - 4x^2 - 9x - 21. \end{aligned}$$

The first elimination gives $A - (\frac{x^2}{3} - \frac{2}{9})B$, that is

$$\frac{-5}{9}x^4 + \frac{127}{9}x^2 - \frac{29}{3},$$

and the subsequent eliminations give

$$\frac{50157}{25}x^2 - 9x - \frac{35847}{25}$$

$$\frac{93060801700}{1557792607653}x + \frac{23315940650}{173088067517}$$

and, finally,

$$\frac{7610300007338478950486691}{86603128130467228900}.$$

Since this is a number, it follows that no *polynomial* can divide both A and B, i.e. that $\gcd(A, B) = 1$.

It is obvious that these calculations on polynomials with rational coefficients require several g.c.d. calculations on integers, and that the integers in these calculations are not always small.

We can eliminate these g.c.d. calculations by working all the time with polynomials with integer coefficients, and this gives a generalisation of the a_i of algorithm 2, known as *polynomial remainder sequences* or *p.r.s.*, by extending the definition of division.

[32]Originally an error, but it makes the point better.

Definition 36 *Instead of dividing f by g in $K[x]$, we can multiply f by a suitable power of the leading coefficient of g, so that the divisions stay in R. The* pseudo-remainder *of dividing f by g, written* $\mathrm{prem}(f,g)$*, is the remainder when one divides* $\mathrm{lc}(g)^{\deg(f)-\deg(g)+1}f$ *by g, conceptually in $K[x]$, but in fact all the calculations can be performed in R, i.e. all divisions are exact in R. This is denoted*[33] *by* $\mathrm{prem}(f,g)$*.*

In some applications (section 3.1.9) it is necessary to keep track of the signs: we define a signed polynomial remainder sequence *or s.p.r.s. of $f_0 = f$ and $f_1 = g$ to have f_i proportional by a* positive *constant to* $-\mathrm{rem}(f_{i-2}, f_{i-1})$*.*

This gives us a *pseudo-euclidean algorithm*, analogous to algorithm 2 where we replace rem by prem, and fix up the contents afterwards. In the above example, we deduce the following sequence:

$$-15\,x^4 + 381\,x^2 - 261,$$

$$6771195\,x^2 - 30375\,x - 4839345,$$

$$500745295852028212500\,x + 1129134141014747231250$$

and

$$7436622422540486538114177255855890572956445312500.$$

Again, this is a number, so $\gcd(A,B) = 1$. We have eliminated the fractions, but at a cost of even larger numbers. Can we do better?

2.3.2 Subresultant sequences

One option would be to make the a_i primitive at each step, since we are going to fix up the content terms later: giving the so-called *primitive p.r.s.* algorithm, which in this case would give

$$-5x^4 + 127x^2 - 87; 5573x^2 - 25x - 3983; -4196868317x - 1861216034; 1$$

This is a perfectly reasonable algorithm when it is a question of polynomials in one variable, and is essentially equivalent to calculating with rational numbers, but over a common denominator. However, if we come to polynomials in several variables, every step of the g.c.d. for polynomials in n variables would involve the g.c.d. of several polynomials in $n-1$ variables, each step of each of which would involve the g.c.d. of several polynomials in $n-2$ variables, and so on.

The following, slightly mysterious[34], algorithm will do the trick. By the Subresultant Theorem [Loo82], all divisions involved are exact, i.e. we always stay in $R[x]$. Furthermore, the factors β_i that are cancelled are *generically* **as large as possible**, where by "generically" we mean that, if the coefficients of f and g were all independent, nothing more could be cancelled[35]. In the same

[33] This definition agrees with Maple, but not with all software systems, which often use `prem` to denote what Maple calls `sprem`, i.e. only raising $\mathrm{lc}(g)$ to the smallest power necessary.

[34] Some of the mystery is explained by corollary 5 on page 97. In particular the various $-$

Figure 2.8: Subresultant p.r.s. algorithm

Algorithm 4 (Subresultant p.r.s.)
Input: $f, g \in K[x]$.
Output: *$h \in K[x]$ a greatest common divisor of* $\mathrm{pp}(f)$ *and* $\mathrm{pp}(g)$
Comment: *If $f, g \in R[x]$, where R is an integral domain and K is the field of fractions of R, then all computations are exact in $R[x]$. This can therefore fulfil the rôle of steps 2–3 of Algorithm 3.*

```
$i := 1$;
if $\deg(f) < \deg(g)$
  then $a_0 := \mathrm{pp}(g)$; $a_1 := \mathrm{pp}(f)$;
  else $a_0 := \mathrm{pp}(f)$; $a_1 := \mathrm{pp}(g)$;
$\delta_0 := \deg(a_0) - \deg(a_1)$;
$\beta_2 := (-1)^{\delta_0+1}$;
$\psi_2 := -1$;
while $a_i \neq 0$ do
        $a_{i+1} = \mathrm{prem}(a_{i-1}, a_i)/\beta_{i+1}$;
        #$q_i$ :=the corresponding quotient: $a_{i+1} = \mathrm{lc}(a_i)^{\delta_{i-1}} a_{i-1} - q_i a_i$
        $\delta_i := \deg(a_i) - \deg(a_{i+1})$;
        $i := i + 1$;
        $\psi_{i+1} := (-\mathrm{lc}(a_{i-1}))^{\delta_{i-2}} \psi_i^{1-\delta_{i-2}}$;
        $\beta_{i+1} := -\mathrm{lc}(a_{i-1}) \psi_{i+1}^{\delta_{i-1}}$;
return $\mathrm{pp}(a_{i-1})$;
```

The a_i are referred to as a *subresultant* polynomial remainder sequence.

example as before, we get the following:

$$\begin{aligned} a_2 &= 15x^4 - 381x^2 + 261, \\ a_3 &= -27865x^2 + 125x + 19915, \\ a_4 &= -3722432068x - 8393738634, \\ a_5 &= 1954124052188. \end{aligned}$$

Here the numbers are much smaller, and indeed it can be proved that the coefficient growth is only linear in the step number. a_2 has a content of 3, which the primitive p.r.s. would eliminate, but this content in fact disappears later. Similarly a_3 has a content of 5, which again disappears later. Hence we have the following result.

Theorem 7 *Let R be a g.c.d. domain. Then there is an algorithm to calculate the g.c.d. of polynomials in $R[x]$. If the original coefficients have length bounded by B, the length at the i-th step is bounded by iB.* Strictly speaking, this theorem is true for polynomials in several variables, where "length" is replaced by "degree in variables other than x". Over the integers, we need to add $\log(2i!)$ to allow for the fact that the sum of two integers can be larger than either.

This algorithm is the best method known for calculating the g.c.d., of all those based on Euclid's algorithm applied to polynomials with integer coefficients. In chapter 4 we shall see that if we go beyond these limits, it is possible to find better algorithms for this calculation.

2.3.3 The Extended Euclidean Algorithm

We can in fact do more with the Euclidean algorithm. Consider the following variant of Algorithm 2, where we have added a few extra lines, marked (*), manipulating (a, b), (c, d) and (e, e').

Algorithm 5 (Extended Euclidean)
Input: $f, g \in K[x]$.
Output: $h \in K[x]$ *a greatest common divisor of* f *and* g*, and* $c, d \in K[x]$ *such that* $cf + dg = h$.

```
        $i := 1$;
        if $\deg(f) < \deg(g)$
           then $a_0 := g$; $a_1 := f$;
($*_1$)          $a := 1$; $d := 1$; $b := c := 0$
```

signs, which are irrelevant as far as a strict g.c.d. algorithm is concerned, come from corollary 5.

[35]The reader may comment that the example, repeated below with this algorithm, shows a consistent factor of 3 in a_2, and this is true however the *non-zero* coefficients are perturbed. Indeed, if the leading coefficient of a_1 is changed to, say, 4, we get a consistent factor of 4. However, if the coefficient of x^7 in a_0 is made non-zero, then the common factor will generally go away, and that is what we mean by "generically".

```
       else $a_0 := f$; $a_1 := g$;
($*_1$)        $c := 1$; $b := 1$; $a := d := 0$
    while $a_i \neq 0$ do
($*_2$)      #Loop invariant: $a_i = af + bg$; $a_{i-1} = cf + dg$;
             $a_{i+1} := \mathrm{rem}(a_{i-1}, a_i)$;
             $q_i$ :=the corresponding quotient: #$a_{i+1} = a_{i-1} - q_i a_i$
($*_3$)      $e := c - q_i a$; $e' := d - q_i b$;          #$a_{i+1} = ef + e'g$
             $i := i + 1$;
($*_3$)      $(c, d) := (a, b)$;
($*_3$)      $(a, b) := (e, e')$
    return $(a_{i-1}, c, d)$;
```

The comments essentially form the proof of correctness. In particular, if f and g are relatively prime, there exist c and d such that

$$cf + dg = 1 : \tag{2.13}$$

a result often called *Bézout's identity*[36].

For the sake of further developments (section B.3.6), we can express the marked lines as

$$(*_1) \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{2.14}$$

$$(*_2) \qquad \begin{pmatrix} a_i \\ a_{i+1} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix} \tag{2.15}$$

$$(*_3) \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} := \begin{pmatrix} 0 & 1 \\ 1 & -q_i \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{2.16}$$

It is possible to make similar modifications to algorithm 4, and the same theory that shows the division by β_{i+1} is exact shows that we can perform the same division of (e, e'). However, at the end, we return $\mathrm{pp}(a_{i-1})$, and the division by $\mathrm{cont}(a_{i-1})$ is *not* guaranteed to be exact when applied to (a, b).

Algorithm 6 (General extended p.r.s.)
Input: $f, g \in K[x]$.
Output: $h \in K[x]$ *a greatest common divisor of* $\mathrm{pp}(f)$ *and* $\mathrm{pp}(g)$, $h' \in K$ *and* $c, d \in K[x]$ *such that* $cf + dg = h'h$
Comment: *If* $f, g \in R[x]$, *where* R *is an integral domain and* K *is the field of fractions of* R, *then all computations are exact in* $R[x]$, *and* $h' \in R$, $c, d \in R[x]$.

```
    $i := 1$;
    if $\deg(f) < \deg(g)$
       then $a_0 := \mathrm{pp}(g)$; $a_1 := \mathrm{pp}(f)$;
(*)            $a := 1$; $d := 1$; $b := c := 0$
       else $a_0 := \mathrm{pp}(f)$; $a_1 := \mathrm{pp}(g)$;
```

[36]Often spelled Bezout. But the title page of [B́79] does have the acute accent.

```
(*)         $c := 1$; $b := 1$; $a := d := 0$
    $\delta_0 := \deg(a_0) - deg(a_1)$;
    $\beta_2 := (-1)^{\delta_0+1}$;
    $\psi_2 := -1$;
    while $a_i \neq 0$ do
(*)     #Loop invariant: $a_i = af + bg$; $a_{i-1} = cf + dg$;
        $a_{i+1} = \mathrm{prem}(a_{i-1}, a_i)/\beta_{i+1}$;
        #$q_i$ :=the corresponding quotient: $a_{i+1} = \mathrm{lc}(a_i)^{\delta_{i-1}} a_{i-1} - q_i a_i$
        $\delta_i := \deg(a_i) - \deg(a_{i+1})$;
(*)     $e := (c - q_i a)/\beta_{i+1}$;
(*)     $e' := (d - q_i b)/\beta_{i+1}$;        #$a_{i+1} = ef + e'g$
        $i := i + 1$;
(*)     $(c, d) = (a, b)$;
(*)     $(a, b) = (e, e')$
        $\psi_{i+1} := -\mathrm{lc}(a_{i-1})^{\delta_{i-2}} \psi_i^{1-\delta_{i-2}}$;
        $\beta_{i+1} := -\mathrm{lc}(a_{i-1}) \psi_{i+1}^{\delta_{i-1}}$;
    return $(\mathrm{pp}(a_{i-1}), \mathrm{cont}(a_{i-1}), c, d)$;
```

2.3.4 Partial Fractions

Bezout's Identity (2.13) has a useful consequence. Suppose we have a fraction p/q, and $q = fg$ with $\gcd(f, g) = 1$. Then

$$\frac{p}{gf} = \frac{p(cf + dg)}{fg} = \frac{pc}{g} + \frac{pd}{f}. \tag{2.17}$$

Even if p/q is proper, i.e. $\deg p < \deg q$, the same may not be true of pc/g or pd/f. However, if the left-hand side of (2.17) is proper, so must the right-hand side be when collected over a common denominator. Hence the extents to which pc/g and pd/f are improper must cancel. So

$$\frac{p}{q} = \frac{pc \text{ rem } g}{g} + \frac{pd \text{ rem } f}{f}. \tag{2.18}$$

A decomposition of the form of the right-hand side of (2.18) is called a *partial fraction decomposition*. This can clearly be extended to any number of factors of the denominator, i.e.

$$\frac{p}{\prod_{i=1}^n f_i} = \sum_{i=1}^n \frac{p_i}{f_i}. \tag{2.19}$$

There is one important caution: even if $p, q \in \mathbf{Z}[x]$, this need not be the case for the partial fraction decomposition, for example

$$\frac{1}{x^2 - 1} = \frac{1/2}{x - 1} + \frac{-1/2}{x + 1}.$$

2.3.5 Polynomials in several variables

Here it is best to regard the polynomials as recursive, so that $R[x,y]$ is regarded as $R[y][x]$. In this case, we now know how to compute the greatest common divisor of two bivariate polynomials.

Algorithm 7 (Bivariate g.c.d.)
Input: $f,\ g \in R[y][x]$.
Output: $h \in R[y][x]$ *a greatest common divisor of* f *and* g

```
$h_c :=$ the g.c.d. of $\mathrm{cont}_x(f)$ and $\mathrm{cont}_x(g)$
# this is a g.c.d. computation in $R[y]$.
$h_p :=$ algorithm 4 $(\mathrm{pp}_x(f), \mathrm{pp}_x(g))$
# replacing $R$ by $R[y]$, which we know, by theorem 7, is a g.c.d. domain.
return $h_c h_p$
# which by theorem 6 is a g.c.d. of $f$ and $g$.
```

This process generalises.

Theorem 8 *If R is a g.c.d. domain, then $R[x_1,\ldots,x_n]$ is also a g.c.d. domain.*

Proof. Induction on n, with theorem 7 as the building block.

What can we say about the complexity of this process? It is easier to analyse if we split up the division process which computes $\mathrm{rem}(a_{i-1},a_i)$ into a series of repeated subtractions of shifted scaled copies of a_i from a_{i-1}. Each such subtraction reduces $\deg(a_{i-1})$, in general by 1. For simplicity, we shall assume that the reduction is precisely by 1, and that[37] $\deg(a_{i+1}) = \deg(a_i) - 1$. It also turns out that the polynomial manipulation in x is the major cost (this is not the case for the primitive p.r.s, where the recursive costs of the content computations dominates), so we will skip all the other operations (the proof of this is more tedious than enlightening). Let us assume that $\deg_x(f)+\deg_x(g) = k$, and that the coefficients have maximum degree d. Then the first subtraction will reduce k by 1, and replace d by $2d$, and involve k operations on the coefficients. The next step will involve $k-1$ operations on coefficients of size $2d$. The next step combines one of the original polynomials, with coefficients of degree d, with this polynomial with coefficients of degree $2d$, giving a polynomial with coefficients of degree $3d$. Combining this with a polynomial with coefficients of degree $2d$ ought to give us coefficients of degree $5d$, but in fact we divide by the previous leading coefficient (degree d) so the answer is a polynomial with coefficients of degree $4d$, and so on, giving degree id at the i-th step (for a matrix view of this, see Corollary 7 and the discussion after it), and a total cost of $\sum_{i=0}^{k}(k-i)F(id)$, where $F(d)$ is the cost of operating on coefficients of degree d. Let us suppose that there are v variables *in all*: x itself and $v-1$ variables in the coefficients with respect to x.

[37]This assumption is known as assuming that the remainder sequence is *normal*. Note that our example is distinctly non-normal, and that, in the case of a normal p.r.s., $\beta_i = \pm\mathrm{lc}(a_{i-2})^2$. In fact, the sub-resultant algorithm was first developed for normal p.r.s., where it can be seen as a consequence of the Dodgson–Bareiss Theorem (theorem 15).

$v = 2$ Here the coefficients are univariate polynomials. If we assume classic multiplication on dense polynomials, $F(d) = cd^2 + O(d)$. We are then looking at

$$\begin{aligned}
\sum_{i=0}^{k}(k-i)F(id) &\leq c\sum_{i=0}^{k}(k-i)i^2d^2 + \sum_{i=0}^{k} kO(id) \\
&\leq ck\sum_{i=0}^{k} i^2d^2 - c\sum_{i=0}^{k} i^3d^2 + k^3O(d) \\
&= c\left(\frac{1}{3}k^4 + \frac{1}{2}k^3 + \frac{1}{6}k^2\right)d^2 - c\left(\frac{1}{4}k^4 + \frac{1}{2}k^3 + \frac{1}{4}k^2\right)d^2 + k^3O(d) \\
&= c\left(\frac{1}{12}k^4 - \frac{1}{12}k^2\right)d^2 + k^3O(d)
\end{aligned}$$

which we can write as $O(k^4d^2)$. We should note the asymmetry here: this means that we should choose the principal variable (i.e. the x in algorithm 7) to be whichever of x and y minimises

$$\min(\max(\deg_x(f), \deg_x(g)), \max(\deg_y(f), \deg_y(g))).$$

$v = 3$ Here the coefficients are bivariate polynomials. If we assume classic multiplication on dense polynomials, $F(d) = cd^4 + O(d^3)$. We are then looking at

$$\begin{aligned}
\sum_{i=0}^{k}(k-i)F(id) &\leq c\sum_{i=0}^{k}(k-i)i^4d^4 + \sum_{i=0}^{k} kO(i^3d^3) \\
&\leq ck\sum_{i=0}^{k} i^4d^4 - c\sum_{i=0}^{k} i^5d^4 + k^5O(d^3) \\
&= c\left(\frac{1}{5}k^6 + \cdots\right)d^2 - c\left(\frac{1}{6}k^6 + \cdots\right)d^2 + k^5O(d^3) \\
&= c\left(\frac{1}{30}k^6 + \cdots\right)d^4 + k^5O(d^3)
\end{aligned}$$

which we can write as $O(k^6d^4)$. The asymmetry is again obvious.

general v The same analysis produces $O(k^{2v}d^{2v-2})$.

We see that the cost is exponential in v, even though it is polynomial in d and k. This is not a purely theoretical observation: any experiment with several variables will bear this out, even when the inputs (being sparse) are quite small: the reader need merely use his favourite algebra system on

$$a_0 := ax^4 + bx^3 + cx^2 + dx + e; \quad a_1 := fx^4 + gx^3 + hx^2 + ix + j,$$

treating x as the main variable (which of course one would not do in practice), to see the enormous growth of the coefficients involved.

2.3.6 Square-free decomposition

Let us revert to the case of polynomials in one variable, x, over a field K, and let us assume that $\text{char}(K) = 0$ (see definition 17 — the case of characteristic non-zero is more complicated [DT81], and we really ought to talk about 'separable decomposition' [Lec08]).

Definition 37 *The* formal derivative *of* $f(x) = \sum_{i=0}^{n} a_i x^i$ *is written* $f'(x)$ *and computed as* $f'(x) = \sum_{i=1}^{n} i a_i x^{i-1}$.

This is what is usually referred to as the derivative of a polynomial in calculus texts, but we are making no appeal to the theory of differentiation here: merely defining a new polynomial whose coefficients are the old ones (except that a_0 disappears) multiplied by the exponents, and where the exponents are decreased by 1.

Proposition 18 *The formal derivative satisfies the usual laws:*

$$(f+g)' = f' + g' \qquad (fg)' = f'g + fg'.$$

Proof. By algebra from the definition. This is taken up in more generality in Proposition 72.

Let us consider the case $f = g^n h$, where g and h have no common factors. Then $f' = g^n h' + n g^{n-1} g' h$ and is clearly divisible by g^{n-1}. n is not zero in K (by the assumption on $\text{char}(K)$), so g does not divide $f'/g^{n-1} = gh' + ng'h$. Hence $\gcd(f, f')$ is divisible by g^{n-1} but not by g^n. These considerations lead to the following result.

Proposition 19 *Let* $f = \prod_{i=1}^{k} f_i^{n_i}$ *where the* f_i *are relatively prime and have no repeated factors. Then*

$$\gcd(f, f') = \prod_{i=1}^{k} f_i^{n_i - 1}.$$

Definition 38 *The* square-free decomposition *of a polynomial* f *is an expression*

$$f = \prod_{i=1}^{n} f_i^i \tag{2.20}$$

where the f_i *are relatively prime and have no repeated factors.* f *is said to be* square-free *if* $n = 1$.

Note that some of the f_i may be 1.

Lemma 3 *Such a decomposition exists for any non-zero* f*, and can be calculated by means of* gcd *computations and divisions.*

Proof. Let $g = \gcd(f, f') = \prod_{i=1}^n f_i^{i-1}$ by the previous proposition. Then $f/g = \prod_{i=1}^n f_i$ and $\gcd(g, f/g) = \prod_{i=2}^n f_i$. Hence

$$\frac{f/g}{\gcd(g, f/g)} = f_1.$$

Applying the same process to g will compute f_2, and so on.

This is not in fact the most efficient way of computing such a decomposition: a better method was given by Yun [Yun76].

2.3.7 Sparse Complexity

So far we have, implicitly, considered dense polynomials. What if the polynomials are sparse? There are then various embarrassing possibilities that

1. the gcd g might be much denser than the inputs A and B
2. even if the gcd g is not much denser than the inputs A and B, the cofactors A/g and B/g computed as part of the verification might be much denser
3. even if neither of these happens, various intermediate results might be much denser than A and B.

Problem 2 is easy to demonstrate: consider the cofactors in $\gcd(x^p - 1, x^q - 1)$, where p and q are distinct primes. Problem 1 is demonstrated by the following elegant example of [Sch03a] (extended to multivariates in Example 21)

$$\begin{aligned}\gcd(x^{pq} - 1, x^{p+q} - x^p - x^q + 1) &= \tfrac{(x^p-1)(x^q-1)}{x-1} \\ &= \underbrace{x^{p+q-1} + x^{p+q-2} \pm \cdots - 1}_{2\min(p,q)\text{ terms}},\end{aligned} \qquad (2.21)$$

and indeed just knowing *whether* two polynomials have a non-trivial gcd is hard, by the following result.

Theorem 9 ([Pla77]) *It is NP-hard to determine whether two sparse polynomials (in the standard encoding) have a non-trivial common divisor.*

This theorem, like the examples above, relies on the factorisation of $x^p - 1$, and it is an open question [DC10, Challenge 3] whether this is the only obstacle. More precisely, we have the following equivalent of the problem solved for division.

Open Problem 2 (Sparse gcd (strong)) *Find an algorithm for computing $h = \gcd(f, g)$ which is polynomial-time in t_f, t_g and t_h, and independent of the degrees (or possibly polynomial in the logarithms of the degrees). See [DC10, Challenge 5].*

For multivariate polynomials, we seem to have a solution in practice (page 197) in the sense that its running time depends on the number of actual terms in the multivariate g.c.d., rather than the potential number of terms. It is still polynomial in the degree in the main variable, though.

A weaker problem, but still unsolved, is

Open Problem 3 (Sparse gcd (weak)) *Find an algorithm for computing* $h = \gcd(f, g)$ *which is polynomial-time in* t_f, t_g, t_h *and* $t_{f/h}$, $t_{g/h}$.

Observation 2 *It follows from (2.21) that square-free decompositions are also hard in the sense that the number of terms in the output is unbounded in the number of terms in the input:*

$$\begin{aligned}\underbrace{x^{pq+p+q} - x^{pq+p} - x^{pq+q} + x^{pq} - x^{p+q} + x^p + x^q - 1}_{8\text{ terms}} &= \\ (x^{pq} - 1)\left(x^{p+q} - x^p - x^q + 1\right) &= (x-1)^3 \underbrace{\left(x^{p+q-2} + 2x^{p+q-3} \pm \cdots + 1\right)}_{p+q-2\text{ terms}}^2 (\ldots)\end{aligned} \tag{2.22}$$

Furthermore, the largest coefficient in the marked term of multiplicity 2 is $\min(p, q)$*, quashing any hopes that the coefficients of a square-free decomposition might be bounded in terms only of the coefficients of the input.*

2.4 Non-commutative polynomials

2.4.1 Types of non-commutativity

The label "non-commutative polynomials" in fact covers three cases.

1. The indeterminates commute, but the coefficients do not. For definiteness, we will refer to this case as *polynomials with non-commuting coefficients.* In this case, rule 5 of definition 22 has to be replaced by

   5′ $x * y = y * x$;

   where x and y are indeterminates, not general polynomials. This means that some of the traditional laws of algebra cease to operate: for example

   $$(ax + b)(ax - b) = a^2x^2 - b^2$$

   becomes

   $$(ax + b)(ax - b) = a^2x^2 + (-a * b + b * a)x - b^2$$

2. The coefficients commute, but the indeterminates do not. For definiteness, we will refer to this case as *polynomials with non-commuting indeterminates.* In this case, rule 5 of definition 22 has to be replaced by the assumption

   - $m \otimes n = n \otimes m$.

   At this point, many of the traditional laws of algebra cease to operate: even the Binomial Theorem in the form

   $$(x + y)^2 = x^2 + 2xy + y^2$$

has to be replaced by

$$(x+y)^2 = x^2 + (xy + yx) + y^2.$$

A common case of non-commuting indeterminates is in differential algebra, where the variable x and the differentiation operator $\frac{\mathrm{d}}{\mathrm{d}x}$ do not commute, but rather satisfy the equation

$$\frac{\mathrm{d}}{\mathrm{d}x}(xa) = x\frac{\mathrm{d}a}{\mathrm{d}x} + a. \tag{2.23}$$

3. Neither can be assumed to commute, in which case rule 5 of definition 22 is just deleted, with no replacement.

Notation 19 *If the variables do not commute, it is usual to use the notation $R\langle x_1, \ldots, x_n \rangle$ for the ring of polynomials with coefficients in R and the non-commuting variables $x_1, \ldots, x_n$.*

2.4.2 Noncommutativity and Division

It might be tempting to assume that we had an equivalent to the rational functions (Section 2.2). Assume we have an inversion operator, denoted, as usual, by x^{-1}. Then indeed:

1. $\left(x^{-1}\right)^{-1} = x$;
2. $a^{-1}b^{-1} = (ba)^{-1}$ (but note the order).

Definition 39 *Define the* inversion height *of an expression E, denoted $ih_E(E)$ to be the maximum number of nested inversions occurring in that expression, and the* inversion height *of an element e, denoted $ih(e)$, to be the minimum, over all expressions E equal to e, of $ih_E(E)$.*

Example 4 *$ih_E\left(y^{-1} + y^{-1}\left(z^{-1}x^{-1} - y^{-1}\right)^{-1} y^{-1}\right) = 2$ since the z is inside two inversions, and nothing is inside three. But all we can deduce is that $ih\left(y^{-1} + y^{-1}\left(z^{-1}x^{-1} - y^{-1}\right)^{-1} y^{-1}\right) \le 2$, but it might be less.*

We can sometimes clear nested inversions.

Proposition 20 (Hua's Identity) *See [Hua49], [Coh03, (9.1.2)].*

$$\left(a^{-1} + \left(b^{-1} - a\right)^{-1}\right)^{-1} = a - aba.$$

So $ih\left(\left(a^{-1} + \left(b^{-1} - a\right)^{-1}\right)^{-1}\right) = 0$, despite appearances: $ih_E\left(\left(a^{-1} + \left(b^{-1} - a\right)^{-1}\right)^{-1}\right) = 2$.

Corollary 3 *$(a + ab^{-1}a)^{-1} = a^{-1} - (a+b)^{-1}$ (replacing b by $-b^{-1}$ and inverting).*

Proposition 21 ([Reu96, theorem 2.1]) *Each entry of the inverse in the free field of an $n \times n$ generic matrix is of inversion height n.*

In particular, $ih\left(m_{1,1} + m_{1,2}m_{2,2}^{-1}m_{2,1}\right)^{-1} = 2$, but if $m_{1,1} = m_{1,2} = m_{2,1}$, Hua's identity shows that $ih\left(m_{1,1} + m_{1,1}m_{2,2}^{-1}m_{1,1}\right)^{-1} = 1$.

TO BE COMPLETED

Chapter 3

Polynomial Equations

In the first parts of this chapter, we will deal with polynomial equations, either singly or as sets of equations. A preliminary remark is in order. Any polynomial equation

$$A = B, \tag{3.1}$$

where A and B are polynomial equations, can be reduced to one whose right-hand side is zero, i.e.

$$A - B = 0. \tag{3.2}$$

Notation 20 *Henceforth, all polynomial equations will be assumed to be in the form of (3.2).*

3.1 Equations in One Variable

We may as well assume that the unknown variable is x. If the equation is linear in x then, by the notation above, it takes the form

$$ax + b = 0. \tag{3.3}$$

The solution is then obvious: $x = -b/a$.

3.1.1 Quadratic Equations

Again, by the notation above, our equation takes the form

$$ax^2 + bx + c = 0. \tag{3.4}$$

The solutions are well-known to most schoolchildren: there are two of them, of the form

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{3.5}$$

However, if $b^2-4ac=0$, i.e. $c=b^2/4a$ then there is only one solution: $x=\frac{-b}{2a}$. In this case, the equation becomes $ax^2+bx+\frac{b^2}{4a}=0$, which can be re-written as $a\left(x+\frac{b}{2a}\right)^2=0$, making it more obvious that there is a repeated root, and that the polynomial is not square-free (definition 38).

Mathematicians dislike the sort of anomaly in "this equations has two solutions except when $c=b^2/4a$", especially as there are two roots as c tends to the value $b^2/4a$. We therefore say that, in this special case, $x=\frac{-b}{2a}$ is *a double root* of the equation. This can be generalised, and made more formal.

Definition 40 *If, in the equation $f=0$, f has a square-free decomposition $f=\prod_{i=1}^n f_i^i$, and $x=\alpha$ is a root of f_i, we say that $x=\alpha$ is a root of f* of multiplicity *i. When we say we are counting the roots of f* with multiplicity*, we mean that $x=\alpha$ should be counted i times.*

Proposition 22 *The number of roots of a polynomial equation over the complex numbers, counted with multiplicity, is equal to the degree of the polynomial.*

Proof. $\deg(f)=\sum i\deg(f_i)$, and each root of f_i is to be counted i times as a root of f. That f_i has i roots is the so-called Fundamental Theorem of Algebra.

In this case, the two roots are given by the two possible signs of the square root, and $\sqrt{0}$ is assumed to have both positive and negative signs.

3.1.2 Cubic Equations

There is a formula for the solutions of the cubic equation

$$x^3+ax^2+bx+c, \tag{3.6}$$

albeit less well-known to schoolchildren:

$$\frac{1}{6}\sqrt[3]{36\,ba-108\,c-8\,a^3+12\,\sqrt{12\,b^3-3\,b^2a^2-54\,bac+81\,c^2+12\,ca^3}}-\frac{2b-\frac{2}{3}a^2}{\sqrt[3]{36\,ba-108\,c-8\,a^3+12\,\sqrt{12\,b^3-3\,b^2a^2-54\,bac+81\,c^2+12\,ca^3}}}-\frac{1}{3}a.$$

We can simplify this by making a transformation[1] to equation (3.6): replacing x by $x-\frac{a}{3}$. This transforms it into an equation

$$x^3+bx+c \tag{3.7}$$

(where b and c have changed). This has solutions of the form

$$\frac{1}{6}\sqrt[3]{-108\,c+12\,\sqrt{12\,b^3+81\,c^2}}-\frac{2b}{\sqrt[3]{-108\,c+12\,\sqrt{12\,b^3+81\,c^2}}}. \tag{3.8}$$

[1]This is the simplest case of the Tschirnhaus transformation[vT83], which can always eliminate the x^{n-1} term in a polynomial of degree n.

```
        $S$    :=   $\sqrt{12\,b^3 + 81\,c^2}$;
        $T$    :=   $\sqrt[3]{-108\,c + 12\,S}$;
   **return**       $\frac{1}{6}T - \frac{2b}{T}$;
```

Figure 3.1: Program for computing solutions to a cubic

Now a cubic is meant to have three roots, but a naïve look as equation (3.8) shows two cube roots, each with three values, and two square roots, each with two values, apparently giving a total of $3 \times 3 \times 2 \times 2 = 36$ values. Even if we decide that the two occurrences of the square root should have the same sign, and similarly the cube root should have the same value, i.e. we effectively execute the program in figure 3.1, we would still seem to have six possibilities. In fact, however, the choice in the first line is only apparent, since

$$\frac{1}{6}\sqrt[3]{-108\,c - 12\,\sqrt{12\,b^3 + 81\,c^2}} = \frac{2b}{\sqrt[3]{-108\,c + 12\,\sqrt{12\,b^3 + 81\,c^2}}}. \tag{3.9}$$

In the case of the quadratic with real coefficients, there were two real solutions if $b^2 - 4ac > 0$, and complex solutions otherwise. However, the case of the cubic is more challenging. If we consider $x^3 - 1 = 0$, we compute (in figure 3.1)

$$S := 9; \qquad T := 6; \qquad \textbf{return } 1;$$

(or either of the complex cube roots of unity if we choose different values of T). If we consider $x^3 + 1 = 0$, we get

$$S := 9; \qquad T := 0; \qquad \textbf{return } \text{“}\tfrac{0}{0}\text{”};$$

but we can (and must!) take advantage of equation (3.9) and compute

$$S := -9; \qquad T := -6; \qquad \textbf{return } -1;$$

(or either of the complex variants).

For $x^3 + x$, we compute

$$S := \sqrt{12}; \qquad T := \sqrt{12}; \qquad \textbf{return } 0;$$

and the two complex roots come from choosing the complex roots in the computation of T, which is really $\sqrt[3]{12\sqrt{12}}$. $x^3 - x$ is more challenging: we compute

$$S := \sqrt{-12}; \qquad T := \sqrt{-12}; \qquad \textbf{return } \{-1, 0, 1\}; \tag{3.10}$$

i.e. three real roots which can only be computed (at least via this formula) by means of complex numbers. In fact it is clear that any other formula must have the same problem, since the only choices of ambiguity lie in the square and cube roots, and with the cube root, the ambiguity involves complex cube roots of unity.

3.1.3 Quartic Equations

Here the equation would be $x^4 + ax^3 + bx^2 + cx + d$, but after the Tschirnhaus transformation $x \to x - \frac{a}{4}$, analogous to that which took equation (3.6) to (3.7), we can assume that $a = 0$. A truly marvellous solution then looks as follows (but the page is too small to contain it!).

$$\frac{\sqrt{6}}{12}\sqrt{\frac{-4\,b\sqrt[3]{-288\,db + 108\,c^2 + 8\,b^3 + 12\sqrt{-768\,d^3 + 384\,d^2b^2 - 48\,db^4 - 432\,dbc^2 + 81\,c^4 + 12\,c^2b^3}} + \bigl(-}{\sqrt[3]{-288\,db + 108\,c^2 + 8\,b^3 + 12\sqrt{-768\,d^3 +}}}} \tag{3.11}$$

We can adopt the same formulation as in Figure 3.1, as shown in figure 3.2. Here

$$\begin{array}{rcl}
S & := & \sqrt{-768\,d^3 + 384\,d^2b^2 - 48\,db^4 - 432\,dbc^2 + 81\,c^4 + 12\,c^2b^3} \\
T & := & \sqrt[3]{-288\,db + 108\,c^2 + 8\,b^3 + 12\,S} \\
U & := & \sqrt{\dfrac{-4\,bT + T^2 + 48\,d + 4\,b^2}{T}} \\
\textbf{return} & & \dfrac{\sqrt{6}}{12}U + \dfrac{\sqrt{6}}{12}\sqrt{\dfrac{-\left(8\,bTU + UT^2 + 48\,Ud + 4\,Ub^2 + 12\,c\sqrt{6}T\right)}{TU}}
\end{array}$$

Figure 3.2: Program for computing solutions to a quartic

the problem of multiple choices is even more apparent, but in this formulation it turns out that choices cancel, much as in the case of the cubic. We have the same problem as in the case of the cubic, that real solutions can arise from complex intermediates, but also that the answer apparently involves $\sqrt{6}$, even though it clearly need not do so in reality. For example, with $x^4 - 5x^2 + 4$, whose solutions are $\pm 1, \pm 2$, we can evaluate

$$S := 72\sqrt{-3}; \qquad T := 17 + \sqrt{-3}; \qquad U := 3\sqrt{6}; \qquad \textbf{return } 2; \tag{3.12}$$

taking the other square root at the end gives 1, and taking the other square root when computing U gives -1 or -2. We should also note that T was evaluated as $\sqrt[3]{4760 + 864\sqrt{-3}}$: not entirely obvious.

3.1.4 Higher Degree Equations

When it comes to higher degree equations, the situation is very different.

Theorem 10 (Abel, Galois [Gal79]) *The general polynomial equation of degree 5 or more is not soluble in radicals (i.e. in terms of k-th roots).*

In fact,[2] if we choose such a polynomial "at random", the probability of its having a solution that can be expressed in terms of radicals is zero. Of course, any particular quintic, or higher degree equation, may have solutions expressible in radicals, such as $x^5 - 2$, whose solutions are $\sqrt[5]{2}$, but this is the exception rather than the rule.

Hence algebra systems, if they handle such concepts, can only regard the roots of such equations as being defined by the polynomial of which they are a root. A Maple example[3] is given in figure 1.4, where the Maple operator `RootOf` is generated. It is normal to insist that the argument to `RootOf` (or its equivalent) is square-free: then differently-indexed roots are genuinely different. Then α, the first root of $f(x)$, satisfies $f(\alpha) = 0$, the second root β satisfies $f(x)/(x - \alpha) = 0$, and so on. Even if f is irreducible, these later polynomials may not be, but determining the factorisations if they exist is a piece of Galois theory which would take us too far out of our way [FM89]. It is, however, comparatively easy to determine the Monte Carlo question: "such factorisations definitely do not exist"/"they probably do exist" [DS00].

3.1.5 Reducible defining polynomials

It should be noted that handling such constructs when the defining polynomial is not irreducible can give rise to unexpected results. For example, in Maple, if α is `RootOf(x^2-1,x)`, then $\frac{1}{\alpha-1}$ returns that, but attempting to evaluate this numerically gives infinity, which is right if $\alpha = 1$, but wrong if $\alpha = -1$, the other, equally valid, root of $x^2 - 1$. In this case, the mathematical answer to "is $\alpha - 1$ zero?" is neither 'yes' nor 'no', but rather 'it depends which α you mean', and Maple is choosing the 1 value (as we can see from $\frac{1}{\alpha+1}$, which evaluates to 0.5). However, the ability to use polynomials not guaranteed to be irreducible can be useful in some cases — see section 3.3.7. In particular, algorithm 11 asks if certain expressions are invertible, and a 'no' answer here entrains a splitting into cases, just as asking "is $\alpha - 1$ zero?" entrains a splitting of `RootOf(x^2-1,x)`.

In general, suppose we are asking if $g(\alpha)$ is invertible, where $\alpha = \texttt{RootOf}(f(x), x)$, i.e. we are asking for $d(\alpha)$ such that $d(\alpha)g(\alpha) = 1$ after taking account of the fact that $\alpha = \texttt{RootOf}(f(x), x)$. This is tantamount to asking for $d(x)$ such that $d(x)g(x) = 1$ modulo $f(x) = 0$, i.e. $d(x)g(x) + c(x)f(x) = 1$ for some $c(x)$. But applying the Extended Euclidean Algorithm (Algorithm 5) to f and g gives us c and d such that $cf + dg = \gcd(f, g)$. Hence if the gcd is in fact 1, $g(\alpha)$ is invertible, and we have found the inverse.

If in fact the gcd is not 1, say some $h(x) \neq 1$, then we have split f as $f = h\hat{h}$,

[2]The precise statement is as follows. For all $n \geq 5$, the fraction of polynomials of degree n and coefficients at most H which have a root expressible in radicals tends to zero as H tends to infinity.

[3]By default, Maple will also use this formulation for roots of most quartics, and the expression in figure 3.2 is obtained by `convert(%,radical)` and then locating the sub-expressions by hand. This can be seen as an application of Carette's view of simplification (page 25), though historically Carette's paper is a retrospective justification.

where $\hat{h} = f/h$. Now

$$\alpha = \texttt{RootOf}(f(x), x) \Leftrightarrow \alpha = \texttt{RootOf}(h(x), x) \vee \alpha = \texttt{RootOf}(\hat{h}(x), x),$$

and in the first case $g(\alpha)$ is definitely zero, and the second case requires us to consider $\gcd(g, \hat{h})$, and $\hat{h}$ has lower degree than f, so this splitting process terminates.

3.1.6 Multiple Algebraic Numbers

The situation gets more complicated if we have several such algebraic numbers in play. By hand, such situations tend to be bypassed almost instinctively: if there are $\sqrt{2}$ around, we replace $\sqrt{8}$ by $2\sqrt{2}$, and if $\sqrt{2}$ and $\sqrt{3}$ are around, we replace $\sqrt{6}$ by $\sqrt{2}\sqrt{3}$. For positive radicals, such an approach is good enough if correctly formalised.

Proposition 23 *Let the N_i be positive numbers, and the M_j be a square-free basis for the N_i, i.e. the M_j are relatively prime and have no repeated factors, and each $N_i = \prod_j M_j^{n_{i,j}}$. Then the k-th roots of the M_j form a multiplicative basis for the k-th roots of the N_i, and the only relations are $\left(M_j^{1/k}\right)^k = M_j$.*

If we allow negative numbers, we have to deal with the catch that $\sqrt[4]{-4} = 1 + i$, which corresponds to the fact that $x^4 + 4 = (x^2 + 2x + 2)(x^2 - 2x + 2)$, but this is all [Sch00a, Theorem 19].

Hence, given a set S of expressions involving non-nested radicals, we can compute the corresponding square-free basis M_j for the N_i occurring in the radicals, let k be the least common multiple of the denominators of the exponents, and express every number as $N_0 \prod M_j^{\alpha_j/k}$ where $N_0 \in \mathbf{Q}$ and $0 \le \alpha_j < k$. This representation is *locally canonical* (Definition 5): every number has a unique representation *until* a new radical is introduced. However, it is only locally canonical, not canonical: in different contexts we could have $5^{1/2}6^{1/2}$ and $3^{1/2}10^{1/2}$, which are both valid, but equal. If we had them both in the same context, they would both become $2^{1/2}3^{1/2}5^{1/2}$. We note that this representation is not candid (Definition 6), since, if we have some $n^{1/6}$, thereafter we represent $m^{1/2}$ as $m^{3/6}$. This is easily solved by cancelling common factors in α_j/k on printing, while preserving the common denominator internally.

Integer g.c.d. is an efficient process, and can compute a relatively prime basis of the N_i efficiently. However the square-free aspect is more troublesome, and indeed is believed to be as hard as integer factorisation in general. There are various reasons why we want a square-free basis.

1. We certainly want to avoid expressions like $4^{1/2}$ or $27^{1/3}$, and we would probably want to replace $4^{1/4}$ by $2^{1/2}$. This can be achieved by checking the M_j for being perfect powers, and this can be done efficiently: [BS93] shows $O(\log^2 n)$. Failing to do this would mean that our expressions were not even locally canonical, or indeed normal — consider $2 - 4^{1/2}$.

2. Expressions like $z := 72^{1/6}$ are troublesome, as z^2 will be stored internally as $72^{2/6}$ and printed as $72^{1/3}$, whereas a human being would prefer to see $2\cdot 9^{1/3}$, or even better $2\cdot 3^{2/3}$. Hence it could be argued that there is scope for non-candid expressions in this case.

There is further discussion of non-nested radicals in [RS13]: in particular they point out that if we are prepared to do complete factorisation of the N_i into primes we can have truly canonical representations.

In terms of the `RootOf` construct, we see that $\sqrt{2}$ is actually $\alpha = \texttt{RootOf}(x^2-2, x)$ and $\sqrt{8}$ is actually $\beta = \texttt{RootOf}(x^2-8, x)$. Now both x^2-2 and x^2-8 are irreducible polynomials *over the integers*. But, in the presence of α, x^2-8 factors as $(x-2\alpha)(x+2\alpha)$. The "square-free basis" technique of the previous paragraphs spots this factorisation directly for non-nested radicals, but in general we are led to the complications of factorisation in the presence of algebraic numbers (Section 6.3).

3.1.7 Solutions in Real Radicals

We have seen above, both in the case of the cubic, equation (3.10), and the quartic, equation (3.12), that real roots may need to be expressed via complex radicals, even if all the root are real. Indeed, in the case of the cubic, this is necessary. However, the quartic $x^4 + 4\,x^3 + x^2 - 6\,x + 2$, whose roots are

$$\left\{-1+\sqrt{3}, -1-\sqrt{3}, -1+\sqrt{2}, -1-\sqrt{2}\right\}$$

shows that polynomials *can* have real roots expressible in terms of real radicals, and a slightly less obvious example is given by $x^4 + 4\,x^3 - 44\,x^2 - 96\,x + 552$, whose roots are

$$\left\{-1-\sqrt{25+2\sqrt{6}}, -1+\sqrt{25+2\sqrt{6}}, -1-\sqrt{25-2\sqrt{6}}, -1+\sqrt{25-2\sqrt{6}}\right\}.$$

There is a little-known theorem in this area.

Theorem 11 ([Isa85]) *Suppose that all the roots of an irreducible polynomial $f(x)$ over $\mathbf{Q}$ are real. Then if any root of the polynomial is expressible in radicals, the degree of the polynomial must be a power of two.*

3.1.8 Equations of curves

For a fuller description of this topic, see [Ful69]. In particular, we only consider the affine case, whereas the projective case (i.e. allowing for "points at infinity") is in many ways more general.

Definition 41 *An (affine)* algebraic curve *$C(x_1, \ldots, x_n)$ in n dimensions over a field K is the set of solutions $(x_1, \ldots, x_n)$ to $n-1$ independent algebraic equations, i.e. polynomials $g_i(x_1, \ldots, x_n) = 0$.*

If $n = 2$ we say that we have a *plane* algebraic curve.

Of course, the precise curve and equations are often not very interesting: for instance we would like to think that the parabola $x_1^2 - x_2$ was "the same" as $y_1 - y_2^2$, and so on.

Definition 42 *Two curves $C(x_1, \ldots, x_n)$ and $C'(y_1, \ldots, y_m)$ are said to be* birationally equivalent *if there are two families of rational functions*

$$F = (f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n))$$

and

$$G = (g_1(y_1, \ldots, y_m), \ldots, g_n(y_1, \ldots, y_m))$$

such that:

1. *for almost all $(x_1, \ldots, x_n) \in C$, $(f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n))$ is defined and $\in C'$;*

2. *for almost all $(y_1, \ldots, y_m) \in C'$, $g_1(y_1, \ldots, y_m), \ldots, g_n(y_1, \ldots, y_m)$ is defined and $\in C$;*

3. *almost everywhere, F and G are mutually inverse, i.e.*

$$f_i(g_1(y_1, \ldots, y_m), \ldots, g_n(y_1, \ldots, y_m)) = y_i$$

and

$$g_j(f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n)) = x_j.$$

"Almost everywhere" means "on a non-empty Zariski open set" [Ful69,], and can be thought of as "except where we get $\frac{0}{0}$ behaviour".

Theorem 12 *Every algebraic curve is birationally equivalent to a plane curve.*

Proof. If there are more than two variables, there is more than one equation, and we can use resultants to eliminate one variable and one equation.

We then have the concept [Sen08] of a curve being *soluble by radicals*. In this case, the generic curve of degree greater than six is not soluble by radicals [Zar26]. However, many "interesting" curves are soluble by radicals.

Proposition 24 *[Sen08, Corollary 3.2] Every irreducible plane curve of degree at most five is soluble by radicals.*

Proposition 25 *[Sen08, Corollary 3.3] Every irreducible singular plane curve of degree at most six is soluble by radicals.*

Algorithms to compute these expressions are given in [Har11, SS11].

It is also the case[4] that the *offset*, i.e. the curve defined as the set of points a fixed distance d from the original curve, to a curve soluble by radicals is also soluble by radicals.

3.1.9 How many Real Roots?

While a polynomial of degree n has n complex roots, it generally has fewer real ones, though how many, on average, is an interesting question, depending on the definition of "on average". The 'obvious' definitions would be 'normally distributed' or 'uniformly distributed in some range', and for these Kac [Kac43][5] shows that the average number is $\frac{2}{\pi}\log(n+1)$. A definition with better geometric invariance properties gives $\sqrt{\frac{n(n+2)}{3}}$: very different [LL12].

We have seen that it is not obvious how many *real* roots a polynomial has: can we answer that question, or more formally the following?

Problem 1 *Given a square-free polynomial f, determine how many real roots f has, and describe each real root sufficiently precisely to distinguish it from the others.* Many authors have asked the same question about non-square-free polynomials, and have laboured to produce better theoretical complexity bounds, since the square-free part of a polynomial may have larger coefficients than the original polynomial. However, in practice it is always better to compute the square-free part first.

The usual description (but see Section 3.1.10) is to enclose the root in an interval, within which it is the only root.

Definition 36 introduced the concept of a signed polynomial remainder sequence, also called a Sturm–Habicht sequence: f_i is proportional by a *positive* constant to $-\mathrm{rem}(f_{i-2}, f_{i-1})$. The positive constant is normally chosen to keep the coefficients integral and as small as possible.

Definition 43 *If $f(x)$ is a square-free polynomial and $a \in \mathbf{R} \cup \{-\infty, \infty\}$, let $V_f(a)$ denote the number of sign changes in the sequence $f_0(a)$, $f_1(a)$, …, $f_n(a)$, where f_0, …, f_n is the Sturm–Habicht sequence of f and f', also known as the Sturm sequence of f.*

If f is not square-free, we need more careful definitions [BPR06], and to be clear whether we are counting with multiplicity or not.

Theorem 13 (Sturm) *If $a < b$ are not zeros of f, and f is square-free, then $V_f(a) - V_f(b)$ is the number of zeros of f in (a, b).*

[4]Unpublished. Prof. Sendra has supplied this proof for plane curves.

Let (R_1, R_2) be a square parametrization of the curve and $K_0 = \mathbf{C}(t) \subset K_1 \subset \ldots \subset K_s$ be a (radical) field tower such that $R_1, R_2 \in K_s$. Considering the formal derivation with respect to t, one can deduce (for instance by induction on s) that if $R \in K_s$ then its derivative R' is also in K_s.

Now consider $a = (R_1')^2 + (R_2')^2 \in K_s$ and $K_{s+1} = K_s(\sqrt{a})$, then $(O_1, O_2) = (R_1, R_2) \pm d/\sqrt{a}(-(R_2)', (R_1)') \in (K_{s+1})^2$. So (O_1, O_2) is radical with the tower $K_0 \subset \ldots \subset K_s \subset K_{s+1}$.

[5]Kac treats polynomials with n coefficients.

$V_f(\infty)$ (which can be regarded as $\lim_{a\to\infty} V_f(a)$) can be computed as the number of sign changes in the sequence of leading coefficients of the Sturm sequence. Similarly, $V_f(-\infty)$ is the number of sign changes in the sequence of leading coefficients of the Sturm sequence with the signs of the odd-degree terms reversed. Hence $V_f(-\infty)-V_f(\infty)$, the total number of real roots, is easily computed from the Sturm sequence.

Example 5 *Let f be the polynomial $x^5 - 15\,x^4 + 85\,x^3 - 225\,x^2 + 274\,x - 120$. Then a Sturm–Habicht sequence (just taking pseudo-renaimders) is*

$$\begin{array}{rcccl} f_0(x) & = & f & = & x^5 - 15\,x^4 + 85\,x^3 - 225\,x^2 + 274\,x - 120 \\ f_1(x) & = & f(x)' & = & 5\,x^4 - 60\,x^3 + 255\,x^2 - 450\,x + 274 \\ f_2(x) & = & -\mathrm{prem}(f_0, f_1) & = & 50\,x^3 - 450\,x^2 + 1270\,x - 1110 \\ f_3(x) & = & -\mathrm{prem}(f_1, f_2) & = & 17500\,x^2 - 105000\,x + 147500 \\ f_4(x) & = & -\mathrm{prem}(f_2, f_3) & = & 15750000000\,x - 47250000000 \\ f_5(x) & = & -\mathrm{prem}(f_3, f_4) & = & 24806250000000000000000000 \end{array}$$

Since all the leading coefficients are positive, the sign sequence at ∞ is $+,+,+,+,+,+$ and $V_f(\infty) = 0$. Similarly, the sign sequence at $-\infty$ is $-,+,-,+,-,+$, so $V_f(-\infty) = 5$ and there are five real roots. Since $f_i(0)$ is just the trailing coefficient of f_i we see that $V_f(0)$ is the number of variations in $-,+,-,+,-,+$, also 5. Hence there are no roots between $-\infty$ and 0, and five roots between 0 and ∞. A longer version can be found at `http://staff.bath.ac.uk/masjhd/JHD-CA/SHexample.html`.

Theorem 14 *If f is a square-free polynomial of degree d and coefficients less than 2^L, then the number of subdivisions of $(-\infty,\infty)$ required is $O(d(L+\log d))$ [Dav85a]. If $L > \log d$, then the number of subdivisions required can be $\Omega(d(L+\log d))$ [ESY06].* Hence if $L > \log d$, the number of subdivisions required is $\Theta(d(L + \log d))$.

While the obvious way of computing $V_f(a)$ is by the definition, i.e. evaluating $f_0(a)$, …, this turns out not to be the most efficient. Rather, while computing the Sturm sequence f_0 …, we should also store the quotients q_i, so that $f_i(x) = -\left(f_{i-2}(x) - q_i(x)f_{i-1}(x)\right)$. We then compute as follows.

Algorithm 8 (Sturm Sequence evaluation)

	a:	*A number*
Input:	$f_n(x)$*:*	*Last non-zero element of Sturm sequence of f*
	$q_i(x)$*:*	*Quotient sequence from Sturm sequence of f*

Output: *Sequence L of $f_n(a)$, $f_{n-1}(a)$, …, $f_0(a)$.*

```
$L[n] :=$      $f_n(a)$;
$L[n-1] :=$ $q_{n+1}(a)L[n]$;
for            $i = n \ldots 2$
               $L[i-2] := q_i(a)L[i-1] - L[i]$;
return $L$
```

If f has degree n, coefficients of bit-length at most τ, a has numerator and denominator of bit-length σ, this algorithm has asymptotic complexity $\tilde{O}(d^2 \max(\sigma, \tau))$ [LR01].

Since it is possible to say how big the roots of a polynomial can be (propositions 91, 92 and 93), we can determine, as precisely as we wish, the location of the real roots of a univariate polynomial: every time the Sturm sequence says that there are more than one root in an interval, we divide the interval in two, and re-compute $V(a) - V(b)$ for each half.

This is far from the only way of counting and locating real roots, i.e. solving problem 1: other methods are based on Descartes'[6] rule of signs (Theorem 54: the number of roots of f in $(0, \infty)$ is less than or equal to, by an even number, the number of sign changes in the coefficients of f) [CA76], its generalisation the Budan–Fourier theorem [Hur12] (Corollaries 34 and 35: the number of roots of f in[7] $[a, b]$ is less than or equal to, by an even number, the number of sign changes in the derivatives of f evaluated at a (i.e. $f(a)$, $f'(a)$, $f''(a)\ldots$) less the same evaluated at b), on continued fractions [TE07], or on numerical methods [Pan02].

All such algorithms take time polynomial in d, the degree of the polynomial whose real roots are being counted/determined, i.e. they are algorithms for the dense model. The best results are in [Bur13], and are $\tilde{O}(d^4 L^2)$ for coefficients $\leq 2^L$. A good comparison of various methods in practice in given in [HTZ$^+$09].

In the sparse model, we have theorems, but are short on algorithms.

Proposition 26 *A polynomial with t non-zero terms has at most $2t - 1$* real *roots (*not *counted with multiplicity).*

Since the number of sign changes in a sequence of t terms is at most $t-1$, there are at most $2t - 1$ real roots ($t - 1$ positive, $t - 1$ negative and zero: consider $x^3 - x$) for a polynomial with t non-zero terms, irrespective of d.

Open Problem 4 (Roots of Sparse Polynomials) *Find algorithms for counting the number of real roots whose complexity depends polynomially on t alone, or t and $\log d$.* There is recent progress described in [BHPR11]: notably a probabilistic algorithm when $t = 4$. See also [Sag14], which depends polynomially on t and $\log d$, and linearly on the sparse bit size.

In fact, we can say more than Proposition 26.

Proposition 27 ([KPT12, Theorem 9]) *Let g be a polynomial of the form*

$$g(x) = \sum_{i=1}^{k} a_i \prod_{j=1}^{m} f_j^{\alpha_{i,j}}$$

*where $a_i \in \mathbf{R}$, $\alpha_{i,j} \in \mathbf{N}$ and the f_j have at most t non-zero terms. Then the number of real roots of g (*not counted with multiplicity*) is at most $2k^2 t^{k^2 m/2} + 2kmt = 2^{O(k^2 m \log t)}$, and in particular is polynomial in t.*

[6]This rule is always called after Descartes, though the proof actually seems to be due to Gauss [BF93].

[7]We assume neither a nor b are roots.

If all the $\alpha_{i,j} \in \{0,1\}$ the result would be trivial, as g would have at most kt^m non-zero terms. If $k=1$ the result is also trivial. The point is that we are allowing a slightly wider range of polynomials. However, Proposition 27 is probably a long way from the truth. The following example[8] illustrates this.

Open Problem 5 (Roots of $fg+1$) *Suppose f and g each have t terms. Then (Proposition 26) each has at most $2t-1$ real roots (and exactly $2t-1$ only if 0 is included). Hence fg, which might have t^2 terms, has at most $4t-3$ real roots (since we shouldn't count 0 twice). How many real roots does $fg+1$ have? It is not difficult to make it have $4t-2$ roots (since 0 might be a double root of fg, and we can split that into two roots), but nothing more is known.*

This open problem is really about Additive Complexity (p. 54): f and g each have additive complexity at most $t-1$, so $fg+1$ has additive complexity at most $2t-1$, rather than the t^2-1 a general polynomial with t^2 terms might have.

Problem 2 *Having solved Problem 1, we may actually wish to know the roots to a given accuracy, say L bits after the point.* Again, many authors have asked the same question about non-square-free polynomials, and have laboured to produce better theoretical complexity bounds, since the square-free part of a polynomial may have larger coefficients, and potentially more terms [CD91], than the original polynomial. However, in practice the author has always computed the square-free part first.

The best published solution to this problem currently is that in [KS11]. **TO BE COMPLETED**[BK12]

3.1.10 Thom's Lemma

An alternative approach to the description of real roots is provided by Thom's Lemma, and elaborated in [CR88].

Notation 21 *A* sign condition *ϵ_i is any of the symbols "> 0", "< 0", "$= 0$". A* generalised sign condition *is any of these or "≥ 0", "≤ 0". If $\epsilon = (\epsilon_0, \ldots, \epsilon_{n-1})$ is any n-tuple of generalised sign conditions, we denote by $\underline{\epsilon}$ the* relaxation *of ϵ obtained by replacing < 0 by ≤ 0 and > 0 by ≥ 0.*

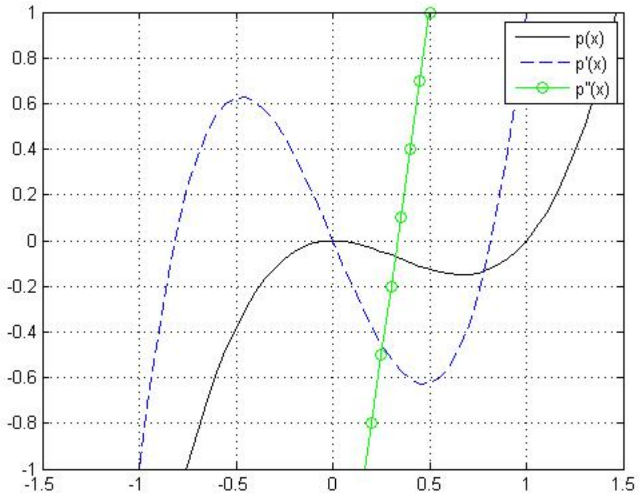
Lemma 4 (Thom's Lemma [CR88, Proposition 1.2]**)** *Let p be a polynomial of degree n, and ϵ an n-tuple of sign conditions. Let*

$$A(\epsilon) = \left\{ x \in \mathbf{R} | \forall i \quad p^{(i)} \epsilon_i \right\},$$

where $p^{(i)}$ denotes the i-th formal derivative (Definition 37) of p. Then:

[8]In [KPT12], but the author is grateful to Professor Koiran for explaining its significance in a June 2015 Dagstuhl seminar.

Figure 3.3: $x^3 - x^2$ illustrating Thom's Lemma



(a) *$A(\epsilon)$ is either empty or connected, i.e. an interval;*

(b) *If $A(\epsilon)$ is an interval, then $A(\underline{\epsilon})$ is the closure of $A(\epsilon)$, obtained by replacing $< r$ by $\leq r$ and $> l$ by $\geq l$.*

The proof is by induction on $\deg(p)$. Item (b) may seem obvious, but is in fact rather subtle. Consider $p = x^3 - x^2$, depicted in Figure 3.3. While $\{x|p(x) > 0\}$ is the interval $(1, \infty)$, $\{x|p(x) \geq 0\}$ is *not* the interval $[1, \infty)$, but rather $\{0\} \cup [1, \infty)$. Thom's Lemma actually talks about $A(\epsilon)$ with all the signs fixed, and the sign of p'' distinguishes $\{0\}$ from $[1, \infty)$.

Corollary 4 *A root x_0 of p, hence a point where $p^{(0)}(x_0) = 0$, is uniquely determined by the signs of all the derivatives of p there.*

It is possible to compute with real algebraic numbers defined this way, and answer questions such as "how many roots does $q(x, y)$ have, when y is the (unique) root of $p(y) = 0$ with the following sign conditions: $p'(y) < 0, p''(y) > 0$ ọts?": again for details see [CR88].

3.2 Linear Equations in Several Variables

We now consider the case of several polynomial equations in several (not necessarily the same number) of variables.

Notation 22 *The variables will be called $x_1, \ldots, x_n$, though in specific examples we may use x, y or x, y, z etc.*

3.2.1 Linear Equations and Matrices

A typical set of 3-by-3 linear equations might look like the following.

$$\begin{aligned} 2x + 3y - 4z &= a; \\ 3x - 2y + 2z &= b; \\ 4x - 3y + 3z &= c. \end{aligned}$$

If we denote by $\mathbf{M}$ the matrix $\begin{pmatrix} 2 & 3 & -4 \\ 3 & -2 & 2 \\ 4 & -3 & 3 \end{pmatrix}$, $\mathbf{x}$ the (column) vector (x, y, z) and $\mathbf{a}$ the (column) vector (a, b, c), then this becomes the single matrix equation

$$\mathbf{M}.\mathbf{x} = \mathbf{a}, \tag{3.13}$$

which has, assuming $\mathbf{M}$ is invertible, the well-known solution

$$\mathbf{x} = \mathbf{M}^{-1}.\mathbf{a}. \tag{3.14}$$

This poses two questions: how do we store matrices, and how do we compute inverses?

3.2.2 Representations of Matrices

The first question that comes to mind here is "dense or sparse?", as in definition 26 (page 43). For a dense representation of matrices, the solution is obvious: we store a two-dimensional array (or one-dimensional array of one-dimensional arrays if our language does not support two-dimensional arrays) containing the values $m_{i,j}$ of the elements of the matrix. The algorithms for adding and multiplying dense matrices are pretty obvious, though in fact it is possible to multiply two 2×2 matrices with seven multiplications of entries rather than the obvious eight [Str69, Win71]: this leads to being able to multiply two $n \times n$ matrices with $O(n^{\log_2 7 \approx 2.807})$ element multiplications rather than $O(n^3)$: see Excursus B.4.

For a sparse representation of matrices, we have more choices.

row-sparse Here M is stored as a one–dimensional array of rows: the i-th row consisting of a list of pairs $(j, m_{i,j})$. This representation is equivalent to that of the sparse polynomial $\sum_j m_{i,j}x^j$, and this technique has been used in practice [CD85] and has a useful analogue in the case of non-linear equations (see section 3.3.5).

column-sparse Here M is stored as a one–dimensional array of columns: the j-th column consisting of a list of pairs $(i, m_{i,j})$.

totally sparse Here M is stored as a list of triples $(i, j, m_{i,j})$.

structured There are a large variety of special structures of matrices familar to numerical analysts, such as 'banded', 'Toeplitz' etc. Each of these can be stored efficiently according to their special form. For example, circulant (a special form of Toeplitz) matrices, of the form

$$\begin{pmatrix} a_1 & a_2 & a_3 & \dots & a_{n-1} & a_n \\ a_n & a_1 & a_2 & a_3 & \dots & a_{n-1} \\ \vdots & \ddots & \dots & \dots & \ddots & \vdots \\ a_2 & a_3 & \dots & a_{n-1} & a_n & a_1 \end{pmatrix},$$

are, strictly speaking, dense, but only have n distinct entries, so are "information-sparse". Recognizing these structures is not just a matter of storage economy: there are also faster algorithms, e.g. [KMS13] can solve an $n \times n$ Toepliz (3.13) in time $O(n \log^2 n)$, rather than the 'obvious' $O(n^3)$ or better $O(n^{2.\cdots})$ (see Section B.4).

Clearly, if the matrix is structured, we should use the corresponding representation. For randomly sparse matrices, the choice depends on what we are doing with the matrix: if it is row operations, then row-sparse is best, and so on. One big issue with sparse matrices is known as *fill-in* — the tendency for operations on sparse matrices to yield less sparse matrices. For example, if we multiply two $n \times n$ matrices, each with e non-zero elements per row, with $n \gg e$, we would expect, assuming the non-zero elements are scattered at random, the resulting matrix to have e^2 non-zero elements per row. This has some apparently paradoxical consequences. Suppose M and N are two such matrices, and we wish to compute MNv for many vectors v. Clearly, we compute MN once and for all, and multiply this by v, and for dense M and N, this is right if there are more than n such vectors v. But, if $n \gg e > 2$, this is not optimal, since, once MN is computed, computing $(MN)v$ requires ne^2 operations, while computing Nv requires ne, as does computing $M(Nv)$, totalling $2ne < ne^2$.

3.2.3 Matrix Inverses: not a good idea!

The first response to the question "how do we compute matrix inverses" ought to be "are you *sure* you want to?" Solving (as opposed to thinking about the solution of) equation (3.13) via equation (3.14) is often not the best way to proceed in computer algebra[9]. Gaussian elimination (possibly using some of the techniques described later in this section) directly on equation (3.13) is generally the best way. This is particularly true if $\mathbf{M}$ is sparse, since $\mathbf{M}^{-1}$ is generally not sparse — an extreme example of fill-in. Indeed, special techniques are generally used for the solution of large sparse systems, particularly those arising in integer factorisation or other cryptographic applications [HD03].

The usual method of solving linear equations, or computing the inverse of a matrix, is via Gaussian elimination, i.e. transforming equation (3.13) into one in which $\mathbf{M}$ is upper triangular, and then back-substituting. This transformation

[9]Or elsewhere: "To most numerical analysts, matrix inversion is a sin" [Hig02, p. 260].

is done by row operations, which amount to adding/subtracting multiples of one row from another, since

$$P = Q \quad \& \quad R = S \quad \text{implies} \quad P + \lambda R = Q + \lambda S. \tag{3.15}$$

If we try this on the above example, we deduce successively that $z = -a + 18b - 13c$, $y = -a + 22b - 16c$ and $x = 3b - 2c$. Emboldened by this we might try a larger matrix:

$$M = \begin{pmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{pmatrix}. \tag{3.16}$$

After clearing out the first column, we get the matrix

$$\begin{pmatrix} a & b & c & d \\ 0 & -\frac{eb}{a} + f & -\frac{ec}{a} + g & -\frac{ed}{a} + h \\ 0 & -\frac{ib}{a} + j & -\frac{ic}{a} + k & -\frac{id}{a} + l \\ 0 & -\frac{mb}{a} + n & -\frac{mc}{a} + o & -\frac{md}{a} + p \end{pmatrix}.$$

Clearing the second column gives us

$$\begin{pmatrix} a & b & c & d \\ 0 & -\frac{eb}{a} + f & -\frac{ec}{a} + g & -\frac{ed}{a} + h \\ 0 & 0 & -\frac{\left(-\frac{ib}{a}+j\right)\left(-\frac{ec}{a}+g\right)}{\left(-\frac{eb}{a}+f\right)} - \frac{ic}{a} + k & \frac{-\left(-\frac{ib}{a}+j\right)\left(-\frac{ed}{a}+h\right)}{\left(-\frac{eb}{a}+f\right)} - \frac{id}{a} + l \\ 0 & 0 & -\frac{\left(-\frac{mb}{a}+n\right)\left(-\frac{ec}{a}+g\right)}{\left(-\frac{eb}{a}+f\right)} - \frac{mc}{a} + o & \frac{\left(\frac{mb}{a}-n\right)\left(-\frac{ed}{a}+h\right)}{\left(-\frac{eb}{a}+f\right)} - \frac{md}{a} + p \end{pmatrix},$$

which we can "simplify" to

$$\begin{pmatrix} a & b & c & d \\ 0 & \frac{-eb+af}{a} & \frac{-ec+ag}{a} & \frac{-ed+ah}{a} \\ 0 & 0 & \frac{afk-agj-ebk+ecj+ibg-icf}{-eb+af} & \frac{afl-ahj-ebl+edj+ibh-idf}{-eb+af} \\ 0 & 0 & \frac{afo-agn-ebo+ecn+mbg-mcf}{-eb+af} & \frac{afp-ahn-ebp+edn+mbh-mdf}{-eb+af} \end{pmatrix}. \tag{3.17}$$

After clearing the third column, the last element of the matrix is

$$-\left(\frac{-\left(-\frac{ib}{a}+j\right)\left(-\frac{ed}{a}+h\right)}{\left(-\frac{eb}{a}+f\right)} - \frac{id}{a} + l\right)\left(\frac{-\left(-\frac{mb}{a}+n\right)\left(-\frac{ec}{a}+g\right)}{\left(-\frac{eb}{a}+f\right)} - \frac{mc}{a} + o\right) \times$$
$$\left(\frac{-\left(-\frac{ib}{a}+j\right)\left(-\frac{ec}{a}+g\right)}{\left(-\frac{eb}{a}+f\right)} - \frac{ic}{a} + k\right)^{-1} - \frac{\left(-\frac{mb}{a}+n\right)\left(-\frac{ed}{a}+h\right)}{\left(-\frac{eb}{a}+f\right)} - \frac{md}{a} + p.$$

This simplifies to

$$-\frac{\begin{array}{c}-afkp+aflo+ajgp-ajho-angl+anhk+ebkp-eblo\\ -ejcp+ejdo+encl-endk-ibgp+ibho+ifcp-ifdo\\ -inch+indg+mbgl-mbhk-mfcl+mfdk+mjch-mjdg\end{array}}{afk-agj-ebk+ecj+ibg-icf}. \tag{3.18}$$

The numerator of this expression is in fact the determinant of the original matrix, $|M|$.

In general, for an $n\times n$ matrix, we would perform $O(n^3)$ computations with rational functions, which would, if we were to simplify, involve g.c.d. computations, often costly.

Can we do better? We could take a leaf out of the calculation on page 66, and not introduce fractions, but rather cross-multiply. If we do this while clearing column one, we get

$$M_2 := \begin{pmatrix} a & b & c & d \\ 0 & -eb+af & -ec+ag & -ed+ah \\ 0 & aj-ib & ak-ic & al-id \\ 0 & -mb+an & ao-mc & ap-md \end{pmatrix}. \tag{3.19}$$

After clearing column two,we get

$$M_3 := \begin{pmatrix} a & b & c & d \\ 0 & -eb+af & -ec+ag & -ed+ah \\ 0 & 0 & \begin{array}{c}(-aj+ib)(-ec+ag)+\\(-eb+af)(ak-ic)\end{array} & \begin{array}{c}(-aj+ib)(-ed+ah)+\\(-eb+af)(al-id)\end{array} \\ 0 & 0 & \begin{array}{c}(-an+mb)(-ec+ag)+\\(-eb+af)(ao-mc)\end{array} & \begin{array}{c}(-an+mb)(-ed+ah)+\\(-eb+af)(ap-md)\end{array} \end{pmatrix}. \tag{3.20}$$

The result of the next step is better contemplated than printed!

However, if we do contemplate the result printed above, we see that rows 3 and 4 contain polynomials of degree four, whereas in the "simplified" form (3.17) we only have polynomials of degree three in the numerators. Indeed, if we were to expand the matrix above, we would observe that rows three and four each had a common factor of a. Similarly, if we were to (or were to get a computer algebra system to) expand and then factor the last step, we would get $a^2(af-eb)|M|$, as in equation (3.18). Such common factors are not a coïncidence (indeed, they cannot be, since M is the most general 4×4 matrix possible).

Theorem 15 (Dodgson–Bareiss [Bar68, Dod66]) [10] *Consider a matrix with*

[10]The Oxford logician Charles Dodgson was better known as Lewis Carroll. Much of this seems to have been known earlier [Chi53].

entries $m_{i,j}$. Let $m_{i,j}^{(k)}$ be the determinant

$$\begin{vmatrix} m_{1,1} & m_{1,2} & \dots & m_{1,k} & m_{1,j} \\ m_{2,1} & m_{2,2} & \dots & m_{2,k} & m_{2,j} \\ \dots & \dots & \dots & \dots & \dots \\ m_{k,1} & m_{k,2} & \dots & m_{k,k} & m_{k,j} \\ m_{i,1} & m_{i,2} & \dots & m_{i,k} & m_{i,j} \end{vmatrix},$$

i.e. that of rows $1 \dots k$ and i, with columns $1 \dots k$ and j. In particular, the determinant of the matrix of size n whose elements are $(m_{i,j})$ is $m_{n,n}^{(n-1)}$ and $m_{i,j} = m_{i,j}^{(0)}$. Then (assuming $m_{0,0}^{(-1)} = 1$):

$$m_{i,j}^{(k)} = \frac{1}{m_{k-1,k-1}^{(k-2)}} \begin{vmatrix} m_{k,k}^{(k-1)} & m_{k,j}^{(k-1)} \\ m_{i,k}^{(k-1)} & m_{i,j}^{(k-1)} \end{vmatrix}.$$

Proof. By fairly tedious induction on k.

How does this relate to what we have just seen? If we do fraction-free elimination on the matrix M of (3.16), we get (3.19), which we can rewrite as

$$M_2 = \begin{pmatrix} a & b & c & d \\ 0 & \begin{vmatrix} a & b \\ e & f \end{vmatrix} & \begin{vmatrix} a & c \\ e & g \end{vmatrix} & \begin{vmatrix} a & d \\ e & h \end{vmatrix} \\ 0 & \begin{vmatrix} a & b \\ i & j \end{vmatrix} & \begin{vmatrix} a & c \\ i & k \end{vmatrix} & \begin{vmatrix} a & d \\ i & l \end{vmatrix} \\ 0 & \begin{vmatrix} a & b \\ m & n \end{vmatrix} & \begin{vmatrix} a & c \\ m & o \end{vmatrix} & \begin{vmatrix} a & d \\ m & p \end{vmatrix} \end{pmatrix}, \tag{3.19$'$}$$

or, in the terminology of Theorem 15,

$$M_2 = \begin{pmatrix} m_{1,1}^{(0)} & m_{1,2}^{(0)} & m_{1,3}^{(0)} & m_{1,4}^{(0)} \\ 0 & m_{2,2}^{(1)} & m_{2,3}^{(1)} & m_{2,4}^{(1)} \\ 0 & m_{3,2}^{(1)} & m_{3,3}^{(1)} & m_{3,4}^{(1)} \\ 0 & m_{4,2}^{(1)} & m_{4,3}^{(1)} & m_{4,4}^{(1)} \end{pmatrix}. \tag{3.19$''$}$$

The next elimination step is

$$M_3 = \begin{pmatrix} m_{1,1}^{(0)} & m_{1,2}^{(0)} & m_{1,3}^{(0)} & m_{1,4}^{(0)} \\ 0 & m_{2,2}^{(1)} & m_{2,3}^{(1)} & m_{2,4}^{(1)} \\ 0 & 0 & \begin{vmatrix} m_{2,2}^{(1)} & m_{2,3}^{(1)} \\ m_{3,2}^{(1)} & m_{3,3}^{(1)} \end{vmatrix} & \begin{vmatrix} m_{2,2}^{(1)} & m_{2,4}^{(1)} \\ m_{3,2}^{(1)} & m_{3,4}^{(1)} \end{vmatrix} \\ 0 & 0 & \begin{vmatrix} m_{2,2}^{(1)} & m_{2,3}^{(1)} \\ m_{4,2}^{(1)} & m_{4,3}^{(1)} \end{vmatrix} & \begin{vmatrix} m_{2,2}^{(1)} & m_{2,4}^{(1)} \\ m_{4,2}^{(1)} & m_{4,4}^{(1)} \end{vmatrix} \end{pmatrix}, \tag{3.20$'$}$$

and Theorem 15 guarantees that $m_{1,1}^{(0)}$ (i.e. a) divides the determinants in rows 3 and 4, so that we have

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1/a & 0 \\ 0 & 0 & 0 & 1/a \end{pmatrix} M_3 = \begin{pmatrix} m_{1,1}^{(0)} & m_{1,2}^{(0)} & m_{1,3}^{(0)} & m_{1,4}^{(0)} \\ 0 & m_{2,2}^{(1)} & m_{2,3}^{(1)} & m_{2,4}^{(1)} \\ 0 & 0 & m_{3,3}^{(2)} & m_{3,4}^{(2)} \\ 0 & 0 & m_{4,3}^{(2)} & m_{4,4}^{(2)} \end{pmatrix}. \tag{3.21}$$

This is a general result.

Corollary 5 (Bareiss' algorithm) *When doing fraction-free Gaussian elimination, after clearing column k, every element of rows $k+1\dots n$ is divisible by $m_{k-1,k-1}^{(k-2)}$.*

This is actually the 'one-step' variant of Bareiss [Bar68]: there are other variants with more advanced look-ahead, but they do not (and can not) cancel any more in general. This result accounts for the factor of a observed in rows 3 and 4 above, and for the factors of a^2 and $af-eb$ in the last step. Cancelling the a in rows 3 and 4 would in fact automatically prevent the a^2 from even being generated — far better than generating it and then cancelling it!

We normally apply Gaussian elimination (fraction-free or not) to solve systems such as (3.13), by transforming them to

$$\mathbf{U}.\mathbf{x} = \mathbf{a}', \tag{3.22}$$

where U' is upper-triangular: performing the same operations on the rows of $\mathbf{a}$ to get $\mathbf{a}'$ as we do on $\mathbf{M}$ to get $\mathbf{U}$. This can be viewed as elimination on the augmented matrix $\overline{\mathbf{M}}$ — $\mathbf{M}$ with $\mathbf{a}$ as an extra column.

Corollary 6 *When doing fraction-free Gaussian elimination in the augmented matrix $\overline{\mathbf{M}}$, after clearing column k, every element of rows $k+1\dots n$, including the "right-hand side", is divisible by $m_{k-1,k-1}^{(k-2)}$.*

The Bareiss–Dodgson calculation pre-supposes that we are indeed doing Gaussian elimination precisely, and building all the sub-determinants specified. It is tempting to assume that we can take "short cuts", and skip zero elements, as after all "they are already zero, and so don't need to be cleared". This is a mistake, as we may miss factors we should cancel. Unfortunately, demonstrating this is not trivial, so there's a Maple worksheet demonstrating this at `http://staff.bath.ac.uk/masjhd/JHD-CA/BDwarning.html`.

Corollary 7 *If the initial entries are integers of length l (resp. polynomials of degree l), then after k steps, the entries will have length (resp. degree) $O(kl)$.*

This is to be contrasted with the $O(2^k l)$ of the naïve fraction-free approach. We mote that a similar approach can be applied to the "*LU* decomposition" of matrices [Jef10].

It is possible to view Euclid's algorithm for polynomials as Gaussian elimination in a matrix (Sylvester's matrix — definition 111) of coefficients, and the factors β_i that are cancelled by the sub-resultant variant for normal polynomial remainder sequences (footnote 37 on page 71) are those predicted by Corollary 5 above.

3.2.4 Complexity

Let the elements of an $n \times n$ matrix have size d (degree for polynomials, or bit-length for integers, ignoring[11] the fact that $a+b$ may have greater bit-length than either a or n), and let the cost of multiplying two elements of size d, or dividing an element of size $2d$ by an element of size $\leq d$, be $M(d)$. We will ignore the cost of addition/substraction. Then the cost of clearing columns is as follows.

column 1 For $n-1$ rows, we do $n-1$ calculations of new elements, which are 2×2 determinants: cost $2M(d)$ each, making a total of $2(n-1)^2M(d)$.

column 2 For $n-2$ rows, we do $n-2$ calculations of new elements, which are 2×2 determinants followed by a cancellation: cost $3M(2d)$ each, making a total of $3(n-2)^2M(2d)$.

column k For $n-k$ rows, we do $n-k$ calculations of new elements, which are 2×2 determinants followed by a cancellation: cost $3M(kd)$ each, making a total of $3(n-k)^2M(kd)$.

If we call the first term $3(n-1)^2M(d)$ rather than $2(n-1)^2M(d)$ (i.e. ignore the fact that there is no cancellation for the first column) we get

$$3\sum_{k=1}^{n-1}(n-k)^2M(kd). \tag{3.23}$$

The value of this depends on M (we'll ignore constant factors, so there is an implied $O(\cdots)$, or $\tilde{O}(\cdots)$, round everything in this analysis).

$M(kd) = (kd)^2$ (classical arithmetic on integers, or univariate polynomials with fixed-length coefficients) we get

$$\frac{1}{30}d^2n^5 - \frac{1}{30}d^2n = O(d^2n^5). \tag{3.24}$$

Allowing for coefficient growth in the integer case[11], we would have

$$O(d^2n^5\log^2 n). \tag{3.25}$$

[11]If we wanted to take it into account, we would use the Hadamard bounds (Propositions 86 and 87) to bound the size of the intermediate objects, which are $k \times k$ determinants by Theorem 15, and get an extra factor of $\log d$ in the length.

$M(kd) = kd$ (the dominant effect in FFT-based arithmetic on integers, or univariate polynomials with fixed-length coefficients) we get $\frac{1}{12}\,dn^4 - \frac{1}{12}\,n^2d = \tilde{O}(dn^4)$.

Suppose we have an $n \times n$ matrix, and the elements are polynomials of degree d in s variables. We have two options for computing the determinant:

1. Fraction-free Gaussian elimination, as described above;
2. Expansion by minors (see Appendix A.6), where we compute all $\frac{n(n-1)}{2}$ $\begin{vmatrix} a_{1,i} & a_{1,j} \\ a_{2,i} & a_{2,j} \end{vmatrix}$, then all $\frac{n(n-1)(n-2)}{6}$ $\begin{vmatrix} a_{1,i} & a_{1,j} & a_{1,k} \\ a_{2,i} & a_{2,j} & a_{2,k} \\ a_{3,i} & a_{3,j} & a_{3,k} \end{vmatrix}$ and so on.

The first involves $O(n^3)$ operations on polynomials, and the second $O(n2^n)$ such operations. The comparison seems clear-cut, we should use the Bareiss-Dodgson Fraction-free method (Corollary 5). Furthermore the intermediate results of Corollary 5 are, apart from the pre-cancellation $\begin{vmatrix} m^{(1)}_{3,2} & m^{(1)}_{3,3} \\ m^{(1)}_{4,2} & m^{(1)}_{4,3} \end{vmatrix}$ etc., sub-minors of the same size as would be computed in the minors method: see (3.21) for an example.

The experimental evidence [Smi76, Smi79] is rather different, and much depends in practice on the sparsity of the matrix. This is analysed in [GJ76], whose conclusions also depend on the sparsity of the polynomials.

Open Problem 6 *The computations in [Smi76, Smi79] should be repeated on modern systems/computers, and the scaling should be re-examined now that larger systems are feasible.*

3.2.5 Over/under-determined Systems

So far we have implicitly assumed that there are as many equations as there are unknowns, and that the equations determine the unknowns precisely (in other words, that the determinant of the corresponding matrix is non-zero). What happens if these assumptions do not hold? There are several cases to be distinguished.

Over-determined and consistent Here the 'extra' equations are consistent with those that determine the solution. A trivial example in one variable would be the pair $2x = 4$, $3x = 6$.

Over-determined and inconsistent Here the 'extra' equations are not consistent with those that determine the solution. A trivial example in one variable would be the pair $2x = 4$, $3x = 9$, where the first implies that $x = 2$, but the second that $x = 3$.

Spuriously over-determined This is a generalisation of "over-determined and consistent" when, after deleting the 'extra' equations that convery no new information, we are left with an under-determined system.

Under-determined and consistent Here there are not enough equations (possibly after deleting spurious ones) to determine all the variables. An example would be $x+y=3$. Here x can be anything, but, once x is chosen, y is fixed as $3-x$. Equally, we could say that y can be anything, but, once y is chosen, x is fixed as $3-y$. The solutions form a k-dimensional hyper-plane, where k is the number of variables minus the number of (non-spurious) equations.

Under-determined yet inconsistent Here the equations (possibly after deleting spurious ones) are still inconsistent. One example would be $x+2y+3z=1$, $2x+4y+6z=3$.

We are then left with three possibilities for the solutions, which can be categorised in terms of the dimension ('dim').

$\dim = -1$ This is the conventional 'dimension' assigned when there are no solutions, i.e. the equations are inconsistent.

$\dim = 0$ Precisely one solution.

$\dim > 0$ An infinite number of solutions, forming a hyperplane of dimension dim.

3.3 Nonlinear Multivariate Equations: Distributed

Most of the section has its origin in the pioneering work of Buchberger [Buc70]. Some good modern texts are [AL94, BW93, CLO06].

If the equations are nonlinear, equation (3.15) is still available to us. So, given the three equations

$$x^2-y=0 \qquad x^2-z=0 \qquad y+z=0,$$

we can subtract the first from the second to get $y-z=0$, hence $y=0$ and $z=0$, and we are left with $x^2=0$, so $x=0$, albeit with multiplicity 2 (definition 40). However, we can do more than this. Given the two equations

$$x^2-1=0 \qquad xy-1=0, \tag{3.26}$$

there might seem to be no row operation available. But in fact we can subtract x times the second equation from y times the first, to get $x-y=0$. Hence the solutions are $x=\pm 1$, $y=x$.

We can generalise equation (3.15) to read as follows: for all polynomials f and g,

$$P=Q \quad \& \quad R=S \quad \text{implies} \quad fP+gR=fQ+gS. \tag{3.27}$$

Lemma 5 *In equation (3.27), it suffices to consider terms (monomials with leading coefficients) for f and g rather than general polynomials.*

Proof. Let f be $\sum a_i m_i$ and g be $\sum b_i m_i$, where the m_i are monomials and the a_i and b_i coefficients (possibly zero, but for a given i, both a_i and b_i should not be zero, since then m_i would be redundant). Then for each i, the monomial version of equation (3.27) gives

$$P = Q \quad \& \quad R = S \quad \text{implies} \quad a_i m_i P + b_i m_i R = a_i m_i Q + b_i m_i S.$$

Then we can use equation (3.15) repeatedly, with $\lambda = 1$, to add these together to get the general form of equation (3.27).

Because of equation (3.2), we can regard equations as synonymous with polynomials. Equation (3.27) then motivates the following definition.

Definition 44 *Let S be a set of polynomials in the variables $x_1, \ldots, x_n$, with coefficients from R. The* ideal generated by *S, denoted (S), is the set of all finite sums $\sum f_i s_i$: $s_i \in S$, $f_i \in R[x_1, \ldots, x_n]$. If S generates I, we say that S is a* basis *for I.*

Observation 3 *In fact we can define ideals in infinitely many variables, but the theory diverges somewhat as Theorem 4 is no longer valid. See [HKL16].*

Proposition 28 *This is indeed an ideal in the sense of definition 9.*

Strictly speaking, what we have defined here is the *left ideal*: there are also concepts of *right ideal* and *two-sided ideal*, but all concepts agree in the case of commutative polynomials, which we will assume until section 3.3.15.

Proposition 29 $((S)) = (S)$.

Definition 45 *Two sets of polynomial equations are* equivalent *if the polynomials defining the left-hand sides generate the same ideal. We will see how to test this in corollary 8.*

Just as an upper triangular matrix is a nice formulation of a set of linear equations, allowing us to "read off", the solutions, so we would like a similarly 'nice' basis for an ideal generated by non-linear equations. In order to do this, we will regard our polynomials in a distributed format, with the terms sorted in some admissible (page 50) ordering $>$. Note that we are *not* requiring that the polynomials are stored this way in an algebra system, though in fact most algebra systems specialising in this area will do so: we are merely discussing the mathematics of such polynomials. Having fixed such an ordering $>$, we can define the following concepts.

Definition 46 *If f is a non-zero polynomial, the* leading term *of f, denoted* $\text{lt}(f)$*, is that term greatest with respect to $>$. The corresponding monomial is called the* leading monomial *of f,* $\text{lm}(f)$*. We will sometimes apply* lm *to sets, where* $\text{lm}(S) = \{\text{lm}(s) | s \in S\}$.

"Monomial algebra" is a particularly simple form of polynomial algebra: in particular

$$\begin{aligned}\gcd\left(\prod_{i=1}^n x_i^{a_i}, \prod_{i=1}^n x_i^{b_i}\right) &= \prod_{i=1}^n x_i^{\min(a_i,b_i)},\\ \operatorname{lcm}\left(\prod_{i=1}^n x_i^{a_i}, \prod_{i=1}^n x_i^{b_i}\right) &= \prod_{i=1}^n x_i^{\max(a_i,b_i)}.\end{aligned}$$

Definition 47 *If* $\mathrm{lm}(g)$ *divides* $\mathrm{lm}(f)$*, then we say that* g *reduces* f *to* $h = \mathrm{lc}(g)f - (\mathrm{lt}(f)/\mathrm{lm}(g))g$*, written* $f \rightarrow^g h$*. Otherwise we say that* f *is* reduced *with respect to* g*.* The Maple user should note that Maple's `Reduce` command actually implements *complete* reduction — see Definition 48.

If R is a field, division is possible, and so it is more usual to reduce f to $f - (\mathrm{lt}(f)/\mathrm{lt}(g))g$. In the construction of h, the leading terms of both $\mathrm{lc}(g)f$ and $(\mathrm{lt}(f)/\mathrm{lm}(g))g$ are $\mathrm{lc}(f)\mathrm{lc}(g)\mathrm{lm}(f)$, and so cancel. Hence $\mathrm{lm}(h) < \mathrm{lm}(f)$. This observation and theorem 4 give us the following result.

Proposition 30 *Any chain* $f_1 \rightarrow^g f_2 \rightarrow^g f_3 \cdots$ *is finite, i.e. terminates in a polynomial* h *reduced with respect to* g*. We write* $f_1 \overset{*}{\rightarrow}^g h$*.*

These concepts and results extend to reduction by a set G of polynomials, where $f \rightarrow^G h$ means $\exists g \in G : f \rightarrow^g h$. We must note that a polynomial can have several reductions with respect to G (one for each element of G whose leading monomial divides the leading monomial of f). For example, let $G = \{g_1 = x-1, g_2 = y-2\}$ and $f = xy$. Then there are two possible reductions of f: $f \rightarrow^{g_1} h_1 = f - yg_1 = y$, and $f \rightarrow^{g_2} h_2 = f - xg_2 = 2x$. In this case $h_1 \rightarrow^{g_2} 2$ and $h_2 \rightarrow^{g_1} 2$, so that $f \overset{*}{\rightarrow}^G 2$ uniquely, but even this need not always be the case. If we let $G = \{g_1 = x-1, g_2 = x^2\}$ and $f = x^2-1$, then $f \rightarrow^{g_2} h_2 = f - g_2 = -1$, whereas $f \rightarrow^{g_1} f - xg_1 = x - 1 \rightarrow^{g_1} 0$: so $f \overset{*}{\rightarrow}^G 0$ or -1.

This definition deals with reduction of the leading monomial of f by g, but it might be that other monomials are reducible. For simplicity we consider the case when R is a field.

Definition 48 *If any term* cm *of* f *is reducible by* g*, i.e. the leading monomial of* g *divides* m*, we say that* g *part-reduces* f*, and write* $f \Rightarrow^g f - (cm/\mathrm{lt}(g))g$*.*

We can continue this process (only finitely often, by repeated application of theorem 4), until no monomial of f is reducible by g, when we write $f \overset{*}{\Rightarrow}^g h$, and say that f is *completely reduced* by g to h. Again, this extends to reduction by a set of polynomials.

Reduction is conceptually fairly easy, but can be expensive if implemented naïvely. Yan [Yan98] observed this, and invented the "geobucket" data structure (see (2.2)), which ensured that we were not repeatedly subtracting polynomials with few terms (as tends to be the case in G) from polynomials with many terms (as f often is). In particular, he observed a factor of over 32 in one large computation: 43 hours instead of 8 weeks!

In section 3.2.1, we performed row operations: subtracting a multiple of one row from another, which is essentially what reduction does, except that the 'multiple' can include a monomial factor. It turns out that we require a more general concept, given in the next definition.

Definition 49 *Let $f, g \in R[x_1, \ldots, x_n]$. The S-polynomial of f and g, written $S(f, g)$ is defined as*

$$S(f,g) = \frac{\mathrm{lt}(g)}{\gcd(\mathrm{lm}(f), \mathrm{lm}(g))} f - \frac{\mathrm{lt}(f)}{\gcd(\mathrm{lm}(f), \mathrm{lm}(g))} g. \tag{3.28}$$

We note that the divisions concerned are exact, and that this generalises reduction in the sense that, if $\mathrm{lm}(g)$ divides $\mathrm{lm}(f)$, then $f \rightarrow^g S(f, g)$. As with reduction, the leading monomials in the two components on the righthand side of equation (3.28) cancel. Another way of thinking of the S-polynomial (when R is a field) is that it is the difference between what you get by reducing $\mathrm{lcm}(\mathrm{lm}(f), \mathrm{lm}(g))$ by f and by g.

Proposition 31 $S(f, g) = -S(g, f)$.

Proposition 32 $S(f, g) \in (\{f, g\})$.

3.3.1 Gröbner Bases

From now until section 3.3.14, we will assume that R is a field. However, we will continue to use R, and not gratuitously make polynomials monic, since this can be expensive.

Theorem 16 *[BW93, Proposition 5.38, Theorem 5.48] The following conditions on a set $G \subset R[x_1, \ldots, x_n]$, with a fixed ordering $>$ on monomials, are equivalent.*

1. *$\forall f, g \in G, S(f, g) \overset{*}{\rightarrow}^G 0$. This is known as the* S-Criterion.
2. *If $f \overset{*}{\rightarrow}^G g_1$ and $f \overset{*}{\rightarrow}^G g_2$, then g_1 and g_2 differ at most by a multiple in R, i.e. $\overset{*}{\rightarrow}^G$ is essentially well-defined.*
3. *$\forall f \in (G), f \overset{*}{\rightarrow}^G 0$.*
4. *$(\mathrm{lm}(G)) = (\mathrm{lm}((G)))$, i.e. the leading monomials of G generate the same ideal as the leading monomials of the whole of (G).*

If G satisfies these conditions, G is called a Gröbner base (or standard basis).

These are very different kinds of conditions, and the strength of Gröbner theory lies in their interplay. Condition 2 underpins the others: $\overset{*}{\to}^G$ is well-defined. Condition 1 looks technical, but has the great advantage that, for finite G, it is finitely checkable: if G has k elements, we take the $k(k-1)/2$ unordered (by proposition 31) pairs from G, compute the S-polynomials, and check that they reduce to zero. This gives us either a proof or an explicit counter-example (which is the key to algorithm 9). Since $f\overset{*}{\to}^G 0$ means that $f \in (G)$, condition 3 means that ideal membership is testable if we have a Gröbner base for the ideal. Condition 4 can be seen as a generalisation of "upper triangular" — see section 3.3.5.

Now let G and H be Gröbner bases, possibly with respect to different orderings.

Proposition 33 *If* $\forall g \in G, g\overset{*}{\to}^H 0$*, then* $(G) \subseteq (H)$.

Proof. Let $f \in (G)$. Then $f\overset{*}{\to}^G 0$, so $f = \sum c_i g_i$. But $g_i\overset{*}{\to}^H 0$, so $g_i = \sum_j d_{ij}h_j$. Therefore $f = \sum_j \left(\sum_i c_i d_{ij}\right) h_j$, and so $f \in (H)$.

Corollary 8 *If* $\forall g \in G, g\overset{*}{\to}^H 0$*, and* $\forall h \in H, h\overset{*}{\to}^G 0$*, then* $(G) = (H)$.

Over a field, a particularly useful Gröbner base is a *completely reduced Gröbner base* (abbreviated *crGb*) G, i.e. one where every element is completely reduced with respect to all the others: in symbols

$$\forall g \in G \qquad g\overset{*}{\Rightarrow}^{G\setminus\{g\}} g. \tag{3.29}$$

For a consistent set of linear polynomials, the crGb would be a set of linear polynomials in one variable each, e.g. $\{x-1, y-2, z-3\}$, effectively the solution. In general, a monic crGb is a canonical (definition 4) form for an ideal: two ideals are equal if, and only if, they have the same crGb (with respect to the same ordering, of course).

Theorem 17 (Buchberger) *Every polynomial ideal has a Gröbner base:* we will show this constructively for finitely-generated[12] ideals over noetherian (definition 10) rings.

Algorithm 9 (Buchberger)
Input: *finite* $G_0 \subset R[x_1, \ldots, x_n]$*; monomial ordering* $>$.
Output: G *a Gröbner base for* (G_0) *with respect to* $>$.

```
$G := G_0$; $n := |G|$;
# we consider $G$ as $\{g_1, \ldots, g_n\}$
$P := \{(i,j) : 1 \le i < j \le n\}$
```

[12]In fact, every polynomial ideal over a noetherian ring *is* finitely generated. However, it is possible to encode undecidability results in infinite descriptions of ideals, hence we say "finitely generated" to avoid this paradox.

```
while $P \neq \emptyset$ do
        Pick $(i,j) \in P$;
        $P := P \setminus \{(i,j)\}$;
        Let $S(g_i, g_j) \overset{*}{\to}^{G} h$
        If $h \neq 0$ then
           # $\mathrm{lm}(h) \notin (\mathrm{lm}(G))$
           $g_{n+1} := h$; # $G := G \cup \{h\}$;
           $P := P \cup \{(i, n+1) : 1 \leq i \leq n\}$;
           $n := n+1$;
Optionally $G :=$Interreduce$(G)$ # (3.29)
```

Proof. The polynomials added to G are reductions of S-polynomials of members of G, and hence are in the same ideal as G, and therefore of G_0. If this process terminates, then the result satisfies condition 1, and so is a Gröbner base for some ideal, and therefore the ideal of G. By proposition 32 and the properties of $\overset{*}{\to}^{G}$, $h \in (G)$, so (G) is constant throughout this process and G has to be a Gröbner base for (G_0). Is it possible for the process of adding new h to G, which implies increasing $(\mathrm{lm}(G))$, to go on for ever? No: corollary 1 says that the increasing chain of $(\mathrm{lm}(G))$ is finite, so at some point we cannot increase $(\mathrm{lm}(G))$ any further, i.e. we cannot add a new h.

Proposition 34 *Every finitely generated polynomial ideal over a field K has a completely reduced Gröbner base with respect to any given ordering, and this is unique up to order of elements and multiplication by elements of K^*.*

Hence, for a fixed ordering, a monic crGb is a "fingerprint" of an ideal, uniquely identifying it. This makes definition 45 algorithmic. It also allows ideal arithmetic.

Proposition 35 *Let G_1 and G_2 be Gröbner bases of the ideals I_1 and I_2 with respect to a fixed ordering. Then:*

1. $I_1 \triangleleft I_2$ *iff* $\forall g \in G_1 \quad g \overset{*}{\to}^{G_2} 0$;
2. $I_1 + I_2 = (G_1 \cup G_2)$;
3. $I_1 I_2 = (\{g_1 g_2 \mid g_1 \in G_1, g_2 \in G_2\})$.

Furthermore, all these processes are algorithmic.

3.3.2 How many Solutions?

Here we will try to give an analysis of the various possibilities for the number of solutions of a set of polynomial equations. We will assume that a crGb for the polynomials has been computed, which therefore cannot be over-determined in the sense of having redundant equations. However, we may still need more equations than variables — see the examples at the start of section 3.3.7.

Unlike section 3.2.5 however, we have to ask ourselves "in which domain are the solutions?" We saw in Theorem 10 that, even for an equation in one variable, the 'solutions' may have no simpler formulation than 'this is a root of $p(x)$'. Fortunately, this is all that we need. We will assume that K is the algebraic closure (definition 19) of (the field of fractions of) R.

Definition 50 *The set of solutions over K of an ideal I is called the* variety *of I, written $V(I)$. If S is a set of polynomials which generates I, so $I = \langle S \rangle$, we will write $V(S)$ as shorthand for $V(\langle S \rangle)$.*

We should note that two different ideals can have the same variety, e.g. (x) and (x^2) both have the variety $x = 0$, but the solution has different multiplicity. However, the two ideals $(x^2 + y^2)$ and (x, y) both have only the solution $x = y = 0$ *over the reals*, but over the complexes the first has the solutions $x = \pm iy$, and hence the varieties are different. See Section 3.5.2.

Proposition 36 *$V(I_1 \cdot I_2) = V(I_1) \cup V(I_2)$.*

Proposition 37 *$V(I_1 \cup I_2) = V(I_1) \cap V(I_2)$.*

Definition 51 *The* radical *of an ideal I, denoted $\sqrt{I}$, is defined as*

$$\sqrt{I} = \{p | \forall \mathbf{x} \in V(I), p(\mathbf{x}) = 0\}.$$

An equivalent definition is

$$\sqrt{I} = \{p | \exists m : p^m \in I\}.$$

If I is generated by a single polynomial p, $\sqrt{I}$ is generated by the square-free part of p.

Example 6 *The ideal $(x^2 + y^2)$ and the ideal (xy) are both radical. However, the ideal $I = (x^2 + y^2, xy)$ is not radical, and in fact $\sqrt{I} = (x, y)$.*

Let us see how this happens. $x^2 + y^2 \in I$ and $xy \in I$, therefore $x^2 + y^2 + 2xy \in I$. But this polynomial is $(x + y)^2$, therefore $x + y \in \sqrt{I}$. Since $xy \in \sqrt{I}$ and $x + y \in \sqrt{I}$, we have $xy - y(x + y) \in \sqrt{I}$. But this is just $-y^2$, so $y^2 \in \sqrt{I}$, and hence $y \in \sqrt{I}$. Similarly $x \in \sqrt{I}$.

In terms of varieties, $V(I)$ is $\{x = 0, y = 0\}$ (which in fact is with multiplicity two, since it is a point of multiplicity 2 in both $(x^2 + y^2)$ and (xy)). A simpler (indeed the simplest) ideal with this variety is (x, y).

Proposition 38 *$\sqrt{I}$ is itself an ideal, and $\sqrt{I} \supset I$.*

Definition 52 *The* dimension *of an ideal I in $S = k[x_1, \ldots, x_n]$ is the maximum number of algebraically independent, over k, elements of the quotient S/I. We can also talk about the dimension of a variety.*

No solutions in K Here the crGb will be $\{1\}$, or more generally $\{c\}$ for some non-zero constant c. The existence of a solution would imply that this constant was zero, so there are no solutions. The dimension is undefined, but normally written as -1.

A finite number of solutions in K There is a neat generalisation of the result that a polynomial of degree n has n roots.

Proposition 39 *If it's finite, the number (counted with multiplicity) of solutions of a system with Gröbner basis G is equal to the number of monomials which are not reducible by G.*

It follows from this that, if (and only if) there are finitely many solutions, every variable x_i must appear alone, to some power, as the leading monomial of some element of G. In this case, the dimension is zero. We return to this case in section 3.3.7.

An infinite number of solutions in K Then some variables do not occur alone, to some power, as the leading monomial of any element of G. In this case, the dimension is greater than zero.

While 'dimension', as defined above, is a convenient generalisation of the linear case, many more things can happen in the non-linear case. If the dimension of the ideal is d, there must be at least d variables which do not occur alone, to some power, as the leading monomial of any element of G. However, if $d > 0$, there may be more. Consider the ideal $(xy - 1) \triangleleft k[x, y]$. $\{xy - 1\}$ is already a Gröbner base, and neither x nor y occur alone, to any power, in a leading term (the only leading term is xy). However, the dimension is 1, not 2, because fixing x determines y, and *vice versa*, so there is only one independent variable. In the case of a triangular set (definition 67), we can do much better, as in Proposition 47.

There are other phenomena that occur with nonlinear equations that cannot occur with linear equations. Consider the ideal

$$\langle (x + 1 - y)(x - 6 + y), (x + 1 - y)(y - 3)\rangle$$

(where the generators we have quoted do in fact form a Gröbner base, at least for `plex(x,y)` and `tdeg(x,y)`, and the leading monomials are x^2 and xy). x occurs alone, but y does not, so in fact this ideal has dimension greater than 0 but at most 1, i.e. dimension 1. But the solutions are $x = y - 1$ (a straight line) *and* the point $(3, 3)$. Such ideals are said to be of *mixed* dimension, and are often quite tedious to work with [Laz09].

3.3.3 Orderings

In section 2.1.4, we defined an admissible ordering on monomials, and the theory so far is valid for all orderings. What sort of orderings are admissible? We first need an ordering on the variables themselves, which we will also denote $>$, and we will assume that $x_1 > \cdots > x_n$ (in examples, $x > y > z$). Suppose the two monomials to be compared are $A = x_1^{a_1} \ldots x_n^{a_n}$ and $B = x_1^{b_1} \ldots x_n^{b_n}$. These monomials have total degree $a = \sum_{i=1}^n a_i$ and $b = \sum_{i=1}^n b_i$.

purely lexicographic — `plex` in Maple We first compare a_1 and b_1. If they differ, this tells us whether $A > B$ ($a_1 > b_1$) or $A < B$ ($a_1 < b_1$). If they are the same, we go on to look at a_2 versus b_2 and so on. The order is similar to looking up words in a dictionary/lexicon — we look at the first letter, and after finding this, look at the second letter, and so on. In this order x^2 is more important than xy^{10}.

total degree, then lexicographic — `grlex` in Maple We first look at the total degrees: if $a > b$, then $A > B$, and $a < b$ means $A < B$. If $a = b$, then we look at lexicographic comparison. In this order xy^{10} is more important than x^2, and x^2y more important than xy^2.

total degree, then reverse lexicographic — `tdeg` in Maple This order is the same as the previous, except that, if the total degrees are equal, we look lexicographically, then *take the opposite*. Many systems, in particular Maple and Mathematica[13], reverse the order of the variables first. The reader may ask "if the order of the variables is reversed, and we then reverse the sense of the answer, what's the difference?". Indeed, for two variables, there is no difference. However, with more variables it does indeed make a difference. For three variables, the monomials of degree three are ordered as

$$x^3 > x^2y > x^2z > xy^2 > xyz > xz^2 > y^3 > y^2z > yz^2 > z^3$$

under `grlex`, but as

$$x^3 > x^2y > xy^2 > y^3 > x^2z > xyz > y^2z > xz^2 > yz^2 > z^3$$

under `tdeg`. One way of seeing the difference is to say that `grlex` with $x > y > z$ discriminates *in favour of* x, whereas `tdeg` with $z > y > x$ discriminates *against* z. This metaphor reinforces the fact that there is no difference with two variables.

It seems that `tdeg` is, in general, the most efficient order, however see Example 7 for a specific counterexample.

k-elimination Here we choose any order $>'$ on $x_1, \ldots, x_k$, and use that. If this cannot decide, we then use a second order $>''$ on $x_{k+1}, \ldots, x_n$. Since $>'$ is admissible, the least monomial is $x_1^0 \ldots x_k^0$, so this order will eliminate $x_1, \ldots, x_k$ as far as possible, in the sense that the polynomials in only $x_{k+1}, \ldots, x_n$ in a Gröbner base computed with such an order are all that can be deduced about these variables. It is common, but by no means required, to use `tdeg` for both $>'$ and $>''$. Note that this is not the

[13] `http://reference.wolfram.com/mathematica/tutorial/PolynomialOrderings.html`

same as simply using `tdeg`, since the exponents of $x_{k+1},\ldots,x_n$ are not considered unless $x_1,\ldots,x_k$ gives a tie.

`plex` is an $(n-1)$-elimination ordering: there's no choice for $>''$ on one variable, and $>'$ is itself `plex`, so is an $(n-2)$-elimination ordering, and so on.

weighted orderings Here we compute the total degree with a weighting factor, e.g. we may weight x twice as much as y, so that the total degree of x^iy^j would be $2i+j$. This can come in lexicographic or reverse lexicographic variants.

matrix orderings These are in fact the most general form of orderings [Rob85]. Let $\mathbf{M}$ be a fixed $n\times n$ matrix of reals, and regard the exponents of A as an n-vector $\mathbf{a}$. Then we compare A and B by computing the two vectors $\mathbf{M}.\mathbf{a}$ and $\mathbf{M}.\mathbf{b}$, and comparing these lexicographically.

lexicographic $\mathbf{M}$ is the identity matrix.

`grlex` $\mathbf{M} = \begin{pmatrix} 1 & 1 & \ldots & 1 & 1 \\ 1 & 0 & \ldots & 0 & 0 \\ 0 & \ddots & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \ldots & 0 & 1 & 0 \end{pmatrix}$.

`tdeg` It would be tempting to say, by analogy with `grlex`, that the matrix is $\begin{pmatrix} 1 & 1 & \ldots & 1 & 1 \\ 0 & 0 & \ldots & 0 & 1 \\ 0 & \ldots & 0 & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & 0 & \ldots & 0 \end{pmatrix}$. However, this is actually `grlex` with the variable order reversed, not genuine reverse lexicographic. To get that, we need the matrix $\begin{pmatrix} 1 & 1 & \ldots & 1 & 1 \\ -1 & 0 & \ldots & 0 & 0 \\ 0 & \ddots & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \ldots & 0 & -1 & 0 \end{pmatrix}$, or, if we are adopting the Maple convention of reversing the variables as well, $\begin{pmatrix} 1 & 1 & \ldots & 1 & 1 \\ 0 & 0 & \ldots & 0 & -1 \\ 0 & \ldots & 0 & -1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & -1 & 0 & \ldots & 0 \end{pmatrix}$.

k-elimination If the matrices are $\mathbf{M}_k$ for $>'$ and $\mathbf{M}_{n-k}$ for $>''$, then $\mathbf{M} = \begin{pmatrix} \mathbf{M}_k & \mathbf{0} \\ \mathbf{0} & \mathbf{M}_{n-k} \end{pmatrix}$.

weighted orderings Here the first row of $\mathbf{M}$ corresponds to the weights, instead of being uniformly 1.

Most "serious" Gröbner systems[14] implement matrix orderings, but have special case implementations for the more common ones listed above, often storing the (weighted) total degree as well as the individual degrees to minimise recomputation.

3.3.4 Complexity of Gröbner Bases

We have proved nothing about the running time of Buchberger's algorithm. Indeed, "algorithm" is almost an over-statement: we have not specified the choice of $(i,j) \in P$ at all, and furthermore $S(g_i, g_j) \overset{*}{\rightarrow}^{G} h$ is not necessarily unique even once (i,j) is chosen. It turns out in practice that the complexity, though *not* the correctness, of the algorithm is strongly dependent on this choice, and on the ordering $>$ being used.

Observation 4 *There have been many improvements and alternative suggestions for computing Gröbner bases. Some improvements are given below. As of writing probably the best algorithm is the F_5 algorithm [Fau02, HA10]. Section 4.6 discusses ways of reducing the coefficient growth in these computations.*

Observation 5 *However many improvements have been, or might be, made, computing Gröbner bases can never be a trivial task. A classic result [MM82] shows that the output degree can be doubly exponential in the number of variables, and this is also the worst case [Dub90].*

Theorem 18 ([MM82], [MR10, Theorem 4.1]) *For any d and k, let $n = 14(k+1)$. There is a set $J_{n,d}$ of polynomials in $k[x_1,\ldots,x_n]$ of degree at most d such that any Gröbner base of $J_{n,d}$ with respect to a total degree ordering contains a polynomial of degree at least $\frac{1}{2}d^{2^k}+4$.*

Therefore this is $O\left(d^{2^{n/14}}\right)$. We can improve this to "nearly" $n/2$, at the cost of being less explicit.

Theorem 19 ([Yap91]) *Fix an admissible monomial ordering. Then there is a family of ideals $I_n \subseteq k[x_1,\ldots,x_n]$ for $n \in \mathbf{N}$, generated by $O(n)$ polynomials of degree $\leq d$, such that any Gröbner base G_n of I_n has degree $\geq d^{2^{(1/2-\epsilon)n}}$ for any epsilon > 0 and sufficiently large d, n.*

Theorem 20 ([Dub90]) *Whatever the ordering, polynomials of total degree $\leq d$ in $k[x_1,\ldots,x_n]$ have a reduced Gröbner base with polynomials of degree $\leq 2\left(\frac{d^2}{2}+d\right)^{2^{n-1}}$.*

Open Problem 7 (Sparse Gröbner Bases) *Even when $k=1$, we are very far away from being able in practice to compute Gröbner bases in 28 variables. Also, of course, these polynomials* could *be very sparse (the example of Theorem 18 involves only binomials), so this result does not* necessarily *prove that*

[14]Such as SINGULAR [Sch03b], CoCoA [Abb04] or Macauley [BS86].

the complexity of computing Gröbner bases is doubly exponential in a sparse encoding. What, then, are the more practical implications of Theorems 18, 19?

We should note that the degree of a Gröbner base depends strongly on the dimension (Definition 52) of the ideal — see section 3.3.8 for the zero-dimensional case, and more generally the following result.

Theorem 21 ([MR11, Theorem 8]) [15] *Let I be an r-dimensional ideal in $k[x_1,\ldots,x_n]$ generated by s polynomials of total degree $d_1 \geq \ldots \geq d_s$. Then, for* any *admissible ordering, the maximum total degree required in a Gröbner base of I is at most $2\left(\frac{1}{2}\left(d_1\cdots d_{n-r}\right)^{2(n-r)}+\frac{1}{2}d_1\right)^{2^r}$.*

The worst case ideals are very far from being radical (Definition 51), and indeed the complexity comes from this property. For radical ideals, we have a much better degree bound.

Theorem 22 TO BE COMPLETED*[Kol88] [Kollar,1988]*

However, this doesn't mean that the Gröbner bases are small, as shown by this example.

Example 7 (van Hoeij [vH15]) *Consider $3n$ variables $\mathcal{V} := \{x_1,\ldots,x_n,y-1,\ldots,y_n,z_1,\ldots,z_n\}$. Let us work modulo 2, and also with the equations $\mathcal{S} := \{c^2-c : \forall c \in \mathcal{V}\}$, so effectively this is the Boolean ring. Let $\mathcal{L} := \{x_iy_i - x_i - y_i - z_i : i \in \{1,\ldots,n\}\}$ and $\mathcal{H} := \mathcal{S}\cup\mathcal{L}\cup\{\prod_{i=1}^n z_i\}$. Intuitively, the last equation in $\mathcal{H}$ says that (at least) one of the z_i is zero, and if $z_i = 0$, the i-th equation in $\mathcal{L}$ says that $x_i = y_i = 0$. If one is familiar with algebraic geometry, it is easy to see that $(\mathcal{H})$ is a radical ideal.*

Let $\mathcal{T} = \{x_iz_i - x_i : i \in \{1,\ldots,n\}\} \cup \{y_iz_i - y_i : i \in \{1,\ldots,n\}\}$ and $\mathcal{P} = \{\prod_{i=1}^n c_i : c_1 \in \{x_1,y_1,z_1\},\ldots,c_n \in \{x_n,y_n,z_n\}\}$. Note that, while all the other sets have $O(n)$ elements, $\mathcal{P}$ has 3^n elements. The equations in $\mathcal{T}$ can be internpreted as "$z_i = 0 \rightarrow x_i = 0$" and "$z_i = 0 \rightarrow y_i = 0$". van Hoeij shows that, with respect to any total degree ordering, the reduced Gröbner base of H is $\mathcal{G} := \mathcal{S}\cup\mathcal{L}\cup\mathcal{T}\cup\mathcal{P}$, i.e. with exponentially more elements.

The author has also observed that, with the lexicographic order and the z_i coming first, the Gröbner base only has those elements of $\mathcal{P}$ containing no z_i, i.e. 2^n of them. So van Hoeij's example is also one where *any* total degree order has exponentially more polynomials than a good lexicographic order.

Having got, as it were, the depressing news out of the way, let's look at some useful results in practice.

Proposition 40 (Buchberger's gcd (or First) Criterion [Buc79]) *If*

$$\gcd(\mathrm{lm}(f),\mathrm{lm}(g)) = 1, \tag{3.30}$$

then $S(f,g)\overset{*}{\rightarrow}^{\{f,g\}}0$.

[15]This was originally published as [MR10, Corollary 3.21], but with an error corrected in [MR11, Theorem 8].

In practice, this is implemented by not even adding to P, either in the initial construction or when augmenting it due to a new h, pairs for which equation (3.30) is satisfied. This proposition also explains why the more general construct of an S-polynomial is not relevant to linear equations: when f and g are linear, if they have the same leading variable, one can reduce the other, and if they do not, then the S-polynomial reduces to zero.

Proposition 41 (Buchberger's lcm (or Third) Criterion [Buc79]) *If $I = (B)$ contains f, g, h and the reductions under $\overset{*}{\rightarrow}^{B}$ of $S(f,g)$ and $S(f,h)$, and if both $\mathrm{lcm}(\mathrm{lm}(f), \mathrm{lm}(g))$ and $\mathrm{lcm}(\mathrm{lm}(f), \mathrm{lm}(h))$ divide $\mathrm{lcm}(\mathrm{lm}(g), \mathrm{lm}(h))$, then $S(g,h)\overset{*}{\rightarrow}^{B} 0$, and hence need not be computed.*

This has been generalised to a chain of polynomials f_i connecting g and h: see [BF91].

Propositions 40 and 41 are therefore *sufficient* to say that we need not compute an S-polynomial: the question of whether they are *necessary* is discussed by [HP07]. Terminology varies in this area, and some refer to Buchberger's *Second* Criterion as well. The more descriptive gcd/lcm terminology is taken from [Per09].

Whereas applying the gcd Criterion (Proposition 40) to $S(f,g)$ depends only on f and g, applying the lcm Criterion (Proposition 41 and its generalisations) to $S(g,h)$ depends on the whole past history of the computation. It might be that the Criterion is not applicable now, but might become applicable in the future. Hence we can ask

> which way of picking elements from P in Algorithm 9 will maximise the effectiveness of the lcm Criterion?

A partial answer was given in [Buc79].

Definition 53 *We say that an implementation of Algorithm 9 follows a* normal selection strategy *if, at each iteration, we pick a pair (i,j) such that $\mathrm{lcm}(\mathrm{lm}(g_i), \mathrm{lm}(g_j))$ is minimal with respect to the ordering in use.*

TO BE COMPLETED(how) This does not quite specify the selection completely: given a tie between (i,j) and (i',j') (with $i < j$, $i' < j'$), we choose the pair (i,j) if $j < j'$, otherwise (i',j') [GMN+91]. Note that here we are actually looking at the numerical values of indices, with larger values meaning "newer" polynomials. Hence we are picking the pair whose newer element is oldest.

For the sake of simplicity, let us assume that we are dealing with a total degree ordering, or a lexicographic ordering. The case of weighted orderings, or so-called multi-graded orderings (e.g. an elimination ordering each of whose components is a total degree ordering) is discussed in [BCR11].

Definition 54 *A polynomial is said to be homogeneous*[16] *if every term has the same total degree. A set of polynomials is said to be homogeneous if each of*

[16]This can be extended to *weighted homogeneous* orderings, and most of the advantages carry through [FSEDT14].

them separately is homogeneous. Note that we are *not* insisting that all terms in the set have the same degree, merely that within each polynomial they have the same total degree.

Definition 55 *If $f = \sum c_i \prod_{j=1}^n x_j^{a_{i,j}} \in K[x_1, \ldots, x_n]$ is not homogeneous, and has total degree d, we can define its* homogenisation *to be $f_0 \in K[x_0, x_1, \ldots, x_n]$ as*

$$f_0 = \sum c_i x_0^{d - \sum_{j=1}^n a_{i,j}} \prod_{j=1}^n x_j^{a_{i,j}}.$$

Proposition 42 *If f and g are homogeneous, so is $S(f, g)$ and h where $f \rightarrow^g h$.*

Corollary 9 *If the input to Algorithm 9 is a set of homogeneous polynomials, then the entire computation is carried out with homogeneous polynomials.*

The normal selection strategy is observed to work well with homogeneous polynomials, but can sometimes be very poor on non-homogeneous polynomials. Hence [GMN$^+$91] introduced the following concept.

Definition 56 *The 'sugar' S_f of a polynomial f in Algorithm 9 is defined inductively as follows:*

1. *For an input $f \in G_0$, S_f is the* total degree *of f (even if we are working in a lexicographic ordering)*
2. *If t is a term, $S_{tf} = \deg(t) + S_f$;*
3. *$S_{f+g} = \max(S_f, S_g)$.*

We define the sugar of a pair of polynomials to be the sugar of their S-polynomial, i.e. (the notation is not optimal here!) $S_{(f,g)} = S_{S(f,g)}$.

The sugar of a polynomial is then the degree it would have had we homogenised all the polynomials before starting Algorithm 9.

Definition 57 *We say that an implementation of Algorithm 9 follows a* sugar selection strategy *if, at each stage, we pick a pair (i, j) such that $S_{(g_i, g_j)}$ is minimal.*

This does not completely specify what to do, and it is usual to break ties with the normal selection strategy (Definition 53), and "sugar then normal" is generally just referred to as "sugar".

3.3.5 A Matrix Formulation

Equation (3.13) showed how a family of linear equations can be represented as a matrix equation. We can do the same with nonlinear equations: (3.26) can be written as

$$\begin{pmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x^2 \\ xy \\ x \\ y \\ 1 \end{pmatrix} = \mathbf{0} \tag{3.31}$$

However, this does not give us an obvious solution. Rather, we need to extend the system, allowing not just the original equations, but also y times the first and x times the second, to give the following.

$$\begin{pmatrix} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x^2y \\ x^2 \\ xy \\ x \\ y \\ 1 \end{pmatrix} = \mathbf{0}. \tag{3.32}$$

Elimination in this gives us

$$\begin{pmatrix} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x^2y \\ x^2 \\ xy \\ x \\ y \\ 1 \end{pmatrix} = \mathbf{0}, \tag{3.33}$$

which produces, as the third row, the equation $y - x$, as we do (up to a change of sign) after (3.26). In pure linear algebra, we can do no further, since we really require y times this equation. This means considering

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x^2y^2 \\ x^2y \\ x^2 \\ xy^2 \\ xy \\ x \\ y^2 \\ y \\ 1 \end{pmatrix} = \mathbf{0}. \tag{3.34}$$

Eliminating here (using row 1 to kill the leading term in row 4, and the same with row 2 against row 5) gives

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x^2y^2 \\ x^2y \\ x^2 \\ xy^2 \\ xy \\ x \\ y^2 \\ y \\ 1 \end{pmatrix} = \mathbf{0}, \tag{3.35}$$

and now row 4 can kill the leading term in row 6, to give

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x^2y^2 \\ x^2y \\ x^2 \\ xy^2 \\ xy \\ x \\ y^2 \\ y \\ 1 \end{pmatrix} = \mathbf{0}. \tag{3.36}$$

The last line of this corresponds to y^2-1. To use this to deduce an equation for x, we would need to consider x times this equation, which would mean adding further rows and columns to the matrix.

In this formulation, the statement "G is a Gröbner base for (F)" corresponds to the existence (*not* the uniqueness, as reduction is not unique) of matrices X, Y and R such that

$$\begin{array}{rcl} X.\mathbf{F} & = & \mathbf{G} \\ Y.\mathbf{G} & = & \mathbf{F} \\ R.\mathbf{G} & = & 0 \end{array} \tag{3.37}$$

where $\mathbf{F}$ and $\mathbf{G}$ are the matrix versions of F and G: the first two lines say that F and G generate the same ideal, and the last that G is a Gröbner basis.

No-one would actually suggest doing this in practice, any more than any-one would compute a g.c.d. in practice by building the Sylvester matrix (which is actually the univariate case of this process), but the fact that it exists can be useful in theory, as we will find that the Sylvester matrix formulation of g.c.d. computation is useful in chapter 4. See section 5.9.3.

3.3.6 Example

Consider the three polynomials below.

$$g_1 = x^3yz - xz^2,$$

$$\begin{aligned} g_2 &= xy^2z - xyz, \\ g_3 &= x^2y^2 - z. \end{aligned}$$

The S-polynomials to be considered are $S(g_1, g_2)$, $S(g_1, g_3)$ and $S(g_2, g_3)$. We use a purely lexicographical ordering with $x > y > z$. The leading terms of $g_2 = xy^2z - xyz$ and $g_3 = x^2y^2 - z$ are xy^2z and x^2y^2, whose l.c.m. is x^2y^2z. Therefore

$$S(g_2, g_3) = xg_2 - zg_3 = (x^2y^2z - x^2yz) - (x^2y^2z - z^2) = -x^2yz + z^2.$$

This polynomial is non-zero and reduced with respect to G, and therefore G is not a Gröbner basis. Therefore we can add this polynomial (or, to make the calculations more readable, its negative) to G — call it g_4. This means that the S-polynomials to be considered are $S(g_1, g_2)$, $S(g_1, g_3)$, $S(g_1, g_4)$, $S(g_2, g_4)$ and $S(g_3, g_4)$.

Fortunately, we can make a simplification, by observing that $g_1 = xg_4$, and therefore the ideal generated by G does not change if we suppress g_1. This simplification leaves us with two S-polynomials to consider: $S(g_2, g_4)$ and $S(g_3, g_4)$.

$$S(g_2, g_4) = xg_2 - yg_4 = -x^2yz + yz^2,$$

and this last polynomial can be reduced (by adding g_4), which gives us $yz^2 - z^2$. As it is not zero, the basis is not Gröbner, and we must enlarge G by adding this new generator, which we call g_5. The S-polynomials to be considered are $S(g_3, g_4)$, $S(g_2, g_5)$, $S(g_3, g_5)$ and $S(g_4, g_5)$.

$$S(g_3, g_4) = zg_3 - yg_4 = -z^2 + yz^2,$$

and this can be reduced to zero (by adding g_5). In fact, this reduction follows from Buchberger's lcm (third) criterion, proposition 41, as we have already computed $S(g_2, g_3)$ and $S(g_2, g_4)$.

$$S(g_2, g_5) = zg_2 - xyg_5 = -xyz^2 + xyz^2 = 0.$$

$$S(g_4, g_5) = zg_4 - x^2g_5 = -z^3 + x^2z^2 = x^2z^2 - z^3,$$

where the last rewriting arranges the monomials in decreasing order (with respect to $<$). This polynomial is already reduced with respect to G, G is therefore not a Gröbner basis, and we must add this new polynomial to G — let us call it g_6. The S-polynomials to be considered are $S(g_3, g_5)$, $S(g_2, g_6)$, $S(g_3, g_6)$, $S(g_4, g_6)$ and $S(g_5, g_6)$. The reader can check that G reduces all these S-polynomials to zero, and that G *is* therefore a Gröbner basis of the ideal, viz.

$$\begin{aligned} g_2 &= xy^2z - xyz, \\ g_3 &= x^2y^2 - z, \\ g_4 &= x^2yz - z^2, \\ g_5 &= yz^2 - z^2, \end{aligned}$$

$$g_6 = x^2z^2 - z^3.$$

No power of x, y or z occurs alone, so we see that the variety is certainly not zero-dimensional, even though we started with three equations in three variables, and z is undetermined. If $z \neq 0$, then g_5 can be divided by z^2 to give $y = 1$ and then g_3 becomes $x^2 - z$, hence this part of the solution variety is a parabola. But if $z = 0$, all equations except g_3 collapse, and we have $x^2y^2 = 0$. Hence this part of the solution variety is two straight lines $x = z = 0$ and $y = z = 0$, each in fact of multiplicity four. Hence the solution is in fact of dimension one, a fact that was not evident when we started.

3.3.7 The Gianni–Kalkbrener Theorem

In this section, we will consider the case of dimension 0, i.e. finitely many solutions over K. We first remark that the situation can be distinctly more challenging than in the case of linear equations, which we illustrate by means of two examples.

1. $G = \{x^2 - 1, y^2 - 1\}$. This is a Gröbner base with respect to any ordering. There are four irreducible monomials $\{1 = x^0y^0, x^1y^0, x^0y^1, x^1y^1\}$, and hence four solutions, $x = \pm 1, y = \pm 1$.

2. $G = \{x^2 - 1, y^2 - 1, (x-1)(y-1)\}$. This is also a Gröbner base with respect to any ordering. There are three irreducible monomials $\{1, x, y\}$, and hence three solutions. There are $x = 1, y = \pm 1$, but when $x = -1$, we only have $y = 1$. The additional polynomial $(x-1)(y-1)$, which rules out the monomial xy, rules out the solution $x = y = -1$. Another way of looking at this is that, when $x = 1$, the polynomial $(x-1)(y-1)$ vanishes, but when $x = -1$, it adds an extra constraint.

Can we generalise this? The answer is 'yes', at least for *purely lexicographical* Gröbner bases of *zero-dimensional* ideals. If the order is $x_n < x_{n-1} < \cdots < x_1$ then such a Gröbner base G must have the form

$$\begin{gathered} p_n(x_n) \\ p_{n-1,1}(x_{n-1}, x_n), \ldots, p_{n-1,k_{n-1}}(x_{n-1}, x_n), \\ p_{n-2,1}(x_{n-2}, x_{n-1}, x_n), \ldots, p_{n-2,k_{n-2}}(x_{n-2}, x_{n-1}, x_n), \\ \cdots \\ p_{1,1}(x_1, \cdots, x_{n-1}, x_n), \ldots, p_{1,k_1}(x_1, \cdots, x_{n-1}, x_n), \end{gathered}$$

where k_i is the number of polynomials involving x_i but not any x_j for $j < i$ and[17]

$$\deg_{x_i}(p_{i,j}) \leq \deg_{x_i}(p_{i,j+1}) \tag{3.38}$$

[17]It is tempting to write $<$ rather than $\leq$ in (3.38). However, this is not always possible, even though, by the time we come to use the $p_{k,j}$ in Algorithm 11, the non-zero $p_{k,j}(\alpha)$ will indeed be in a strict order of $\deg_{x_k}$. We can, of course sort the $p_{k,j}$ by the purely lexicographic order.

Figure 3.4: Gianni–Kalkbrener Algorithm

Algorithm 10 (Gianni–Kalkbrener) *$GK(G,n)$*
Input: *A Gröbner base G for a zero-dimensional ideal I in n variables with respect to lexicographic order.*
Output: *A list of solutions of G.*

```
$S := \{x_n = \mathrm{RootOf}(p_n)\}$
for $k = n-1, \ldots, 1$ do
    $S := \mathrm{GKstep}(G,k,S)$
return $S$
```

Algorithm 11 (Gianni–Kalkbrener Step) $\mathrm{GKstep}(G,k,A)$
Input: *A Gröbner base G for a zero-dimensional ideal I with respect to lexicographic order, an integer k, and A a list of solutions of G_{k+1}.*
Output: *A list of solutions of G_k.*

```
$B := \emptyset$
for each $\alpha \in A$
        $i := 1$
        while $(L := (\mathrm{lc}_{x_k}(p_{k,i}))(\alpha)) = 0$ do $i := i+1$
        if $L$ is invertible with respect to $\alpha$
        # see section 3.1.5
            then $B := B \cup \{(\alpha \cup \{x_k = \mathrm{RootOf}(p_{k,i}(\alpha))\})\}$
            else # $\alpha$ is split as $\alpha_1 \cup \alpha_2$
                $B := B \cup \mathrm{GKstep}(G,k,\{\alpha_1\}) \cup \mathrm{GKstep}(G,k,\{\alpha_2\})$
return $B$
```

and p_{i,k_i} is monic in x_i. Let $G_k = G \cap k[x_k,\ldots,x_n]$, i.e. those polynomials in $x_k,\ldots,x_n$ only.

Theorem 23 (Gianni–Kalkbrener [Gia89, Kal89a]) *Let α be a solution of G_{k+1}. Then if $\mathrm{lc}_{x_k}(p_{k,i})$ vanishes at α, then $(p_{k,i})$ vanishes at α. Furthermore, the lowest degree (in x_k) polynomial of the $p_{k,i}$ not to vanish at α, say p_{k,m_α}, divides all of the other $p_{k,j}$ at α. Hence we can extend α to solutions of G_k by adding $x_k = \mathrm{RootOf}(p_{k,m_\alpha})$.*

This gives us an algorithm (Figure 3.4) to describe the solutions of a zero-dimensional ideal from such a Gröbner base G. This is essentially a generalisation of back-substitution into triangularised linear equations, except that there may be more than one solution, since the equations are non-linear, and possibly more than one equation to substitute into.

In practice, particularly if we are interested in keeping track of multiplicities of solutions, it may be more efficient to perform a square-free decomposition (Definition 38) of p_n, and initialise S to a list of the RootOf of each of its square-free factors, and also associate the multiplicity to each solution.

In case 2. above, the three equations are $G = \{p_2 := x^2-1, p_{1,1} := (x-1)(y-1), p_{1,2} := y^2-1\}$. Taking $x_n = x$, we start off with $S = \{x = \text{RootOf}(x^2-1)\}$, and we call GKstep on this. The initial value of L is $\text{RootOf}(x^2-1)-1$, and we ask whether this is invertible. Adopting a common-sense (i.e. heuristic) approach for the moment, we see that this depends on *which* root we take: for $+1$ it is not invertible, and for -1 it is. Hence GKstep makes two recursive calls to itself, on $x = 1$ and $x = -1$.

GKstep($G, 1, \{x = 1\}$) Here $L := \text{lc}_{x_1}(p_{1,1}(x = 1))$ is 0, so we consider $p_{1,2}$, whose leading coefficient is 1, so $y = \text{RootOf}(y^2-1)$.

GKstep($G, 1, \{x = -1\}$) Here $L := \text{lc}_{x_1}(p_{1,1}(x = -1))$ is -2, and $y = 1$.

There is a larger worked example of this later, at equation (3.48), and a generalisation of 2 above at `http://staff.bath.ac.uk/masjhd/JHD-CA/WorkedGK.html`.

Theorem 23 relies on a special case of a more general result.

Theorem 24 (Elimination Theorem) *Suppose $\{u_1, \ldots, u_k\} \subset \{x_1, \ldots, x_n\}$ and $\prec$ is an elimination ordering (page 108), only considering the u_i if the ordering on the rest of the variables is a tie. If G is a Gröbner basis for an ideal I with respect to $\prec$, then $G \cap k[u_1, \ldots, u_k]$ is a Gröbner basis for $I \cap k[u_1, \ldots, u_k]$ [BW93, Proposition 6.15]. $I \cap k[u_1, \ldots, u_k]$ is called an* elimination ideal *for I with respect to the variables $\{u_1, \ldots, u_k\}$.*

Theorem 23 can be generalised in several ways to non-zero-dimensional ideals, but not completely [FGT01, Examples 3.6, 3.11]. The first is particualrly instructive for us.

Example 8 *Let $I = \langle ax^2 + x + y, bx + y\rangle$ with the order $a \prec b \prec y \prec x$. The Gröbner base is $B_1 \cup B_2$ and there are no polynomials in (a, b) only (so the ideal is at least two-dimensional), in (a, b, y) we have $B_2 := \{ay^2 + b^2y - by\}$, and in all variables $B_1 := \{ax^2 + x + y, axy - by + y, bx + y\}$. We then have the following situation for values of a and b.*

normally *B_2 determines y (generally as the solution of a quadratic), then $x = -y/b$ except when $b = 0$, when $ay^2 = 0$ so $y = 0$, and $ax^2 + x = 0$, so $x = 0$ or $x = -1/a$.*

$a = b = 0$ *B_2 vanishes, so we would be tempted, by analogy with Theorem 23, to deduce that y is undetermined. But in fact $B_1|_{a=b=0} = \{x + y, y, y\}$, so $y = 0$ (and then $x = 0$).*

$a = 0, b = 1$ *Again B_2 vanishes. This time, $B_1|_{a=0,b=1} = \{x + y, 0, x + y\}$, and y is undetermined, with $x = -y$.*

This example is taken up again as Example 12.

3.3.8 The Faugère–Gianni–Lazard–Mora Algorithm

We have seen in the previous section that, for a zero-dimensional ideal, a *purely lexicographical* Gröbner base is a very useful concept. But these are generally the most expensive to compute, with a worst-case complexity of $O(d^{n^3})$ for polynomials of degree d in n variables [CGH88]. A *total degree, reverse lexicographic* Gröbner base, on the other hand, has complexity $O(d^{n^2})$, or $O(d^n)$ if the number of solutions at infinity is also finite [Laz83]. If one prefers practical evidence, we can look at (4.27), which has a simple Gröbner base of (4.28), but 120-digit numbers occur in the intermediate calculations if we compute the Gröbner base in a total degree order, and numbers with tens of thousands of digits when computed in a lexicographic order. The computing time[18] is 1.02 or 2.04 seconds in total degree (depending on the order of variables), but 40126 seconds (over 11 hours) in lexicographic.

Hence the following algorithm [FGLM93] can be very useful, with $>'$ being *total degree, reverse lexicographic* and $>''$ being *purely lexicographical*, though it does have uses in other settings as well.

Algorithm 12 (FGLM)
Input: *A Gröbner base G for a zero-dimensional ideal I with respect to $>'$; an ordering $>''$.*
Output: *A Gröbner base H for I with respect to $>''$.*

```
$H := \emptyset$; $i := j := 0$
Enumerate the monomials irreducible under $H$ in increasing order for $>''$
#When $H$ is a Gröbner base, this is finite by proposition 39
#This enumeration needs to be done lazily — see (*)
**for each** such $m$
        Let $m \overset{*}{\rightarrow}^{G} v$
        **if** $v = \sum_{k=1}^{j} c_k v_k$
          **then** $h_{i+1} := m - \sum_{k=1}^{j} c_k m_k$
                $H := H \cup \{h_{i+1}\}$; $i := i + 1$
                # Changes "irreducible under $H$" (*)
          **else** $j := j + 1$; $m_j := m$; $v_j := v$
**return** $H$
#It is not totally trivial that $H$ is a Gröbner base, but it is [FGLM93].
```

Since this algorithm is basically doing linear algebra in the space spanned by the irreducible monomials under G, whose dimension D is the number of solutions (proposition 39), it is not surprising that the running time seems to be $O(D^3)$, whose worst case is $O(d^{3n})$. Strictly speaking, this analysis refers to the number of arithmetic operations, ignoring any growth in coefficient sizes.

Open Problem 8 (Complexity of the FGLM Algorithm (I)) *The complexity of the FGLM algorithm (Algorithm 12) is $O(D^3)$ where D is the number*

[18] Axiom 3.4 on a 3GHz Intel P4.

of solutions. Can faster matrix algorithms such as Strassen–Winograd [Str69, Win71] (see Notation 45) speed this up? We note that this is not trivial, since the rows are "arriving one at a time" rather than being presented al at once. See [FGHR13, FSEDT14] for recent progress: in particular the latter claims $O(nD^\omega)$.

Open Problem 9 (Complexity of the FGLM Algorithm (II)) *The complexity of the FGLM algorithm (Algorithm 12) is $O(D^3)$ where D is the number of solutions. Can we do any better* in practice *if the linear algebra is sparse? Some progress in this direction has been made in [FM13].*

As an example of the FGLM algorithm, we take the system `Aux` from their paper[19], with three polynomials

$$\begin{gathered} abc + a^2bc + ab^2c + abc^2 + ab + ac + bc \\ a^2bc + a^2b^2c + b^2c^2a + abc + a + c + bc \\ a^2b^2c + a^2b^2c^2 + ab^2c + ac + 1 + c + abc \end{gathered}$$

The total degree Gröbner basis has fifteen polynomials, whose leading monomials are

$$c^4, bc^3, ac^3, b^2c^2, abc^2, a^2c^2, b^3c, ab^2c, a^2bc, a^3c, b^4, ab^3, a^2b^2, a^3b, a^4.$$

This defines a zero-dimensional ideal (c^4, b^4 and a^4 occur in this list), and we can see that the irreducible monomials are

$$1, c, c^2, c^3, b, bc, bc^2, b^2, b^2c, b^3, a, ac, ac^2, ab, abc, ab^2, a^2, a^2c, a^2b, a^3 :$$

twenty in number (as opposed to the 64 we would have if the basis only had the polynomials $a^4 + \cdots, b^4 + \cdots, c^4 + \cdots$). If we wanted a purely lexicographic base to which to apply Gianni-Kalkbrener, we would enumerate the monomials in lexicographic order as

1 (irreducble)

c (irreducble)

c^2 (irreducble)

c^3 (irreducble)

c^4 which reduces to $-\frac{185}{14} - \frac{293}{42}a^3 - \frac{1153}{42}a^2b + \frac{509}{7}ab^2 - \frac{323}{42}b^3 - \frac{2035}{42}a^2c - \frac{821}{21}abc + \frac{173}{6}b^2c - \frac{751}{14}ac^2 + \frac{626}{21}bc^2 + \frac{31}{42}c^3 - \frac{449}{14}a^2 + \frac{1165}{14}ab - \frac{772}{21}b^2 + \frac{550}{21}ac - \frac{429}{7}bc + \frac{184}{21}c^2 - \frac{407}{6}a - \frac{281}{42}b - \frac{4799}{42}c$

$\vdots$

[19]The system is obtained as they describe, except that the substitutions are $x_5 = 1/c$, $x_7 = 1/a$.

c^{20} which reduces to $-\frac{156473200555876438}{7} + \frac{1355257348062243268}{21} bc^2 - \frac{2435043982608847426}{21} a^2c - \frac{455474473888607327}{3} a - \frac{87303768951017165}{21} b - \frac{5210093087753678597}{21} c + \frac{1264966801336921700}{7} ab - \frac{995977348285835822}{7} bc - \frac{2106129034377806827}{21} abc + \frac{136959771343895855}{3} b^2c + \frac{1119856342658748374}{21} ac + \frac{6293517245586787780}{21} c^2 - \frac{774120922299216564}{7} ac^2 - \frac{1416003666295496227}{21} a^2b + \frac{1196637352769448957}{7} ab^2 - \frac{706526575918247673}{7} a^2 - \frac{1536916645521260147}{21} b^2 - \frac{4178712854415094524}{21} a^3 - \frac{3562866593366988974}{21} b^3 + \frac{3738195275477752163}{21} c^3$, which can be expressed in terms of the previous ones as $p = -1 + 6\,c + 41\,c^2 - 71\,c^3 + 41\,c^{18} - 197\,c^{14} - 106\,c^{16} + 6\,c^{19} - 106\,c^4 - 71\,c^{17} - 92\,c^5 - 197\,c^6 - 145\,c^7 - 257\,c^8 - 278\,c^9 - 201\,c^{10} - 278\,c^{11} - 257\,c^{12} - 145\,c^{13} - 92\,c^{15}$. The polynomial $c^{20} - p$ gets added to H: all higher powers of c are therefore expressible, and need not be enumerated.

Open Problem 10 (Coefficient growth in the FGLM Algorithm) *We observe above that the coefficient growth in the expression of c^{20} in terms of the $>'$ monomials is far greater than its expression in terms of the $>''$ monomials (powers of c). Could fraction-free methods as in Theorem 15 do better?*

b which can be expressed in terms of the previous ones as $q = -\frac{9741532}{1645371} - \frac{8270}{343} c + \frac{32325724}{548457} c^2 + \frac{140671876}{1645371} c^3 - \frac{2335702}{548457} c^{18} + \frac{13420192}{182819} c^{14} + \frac{79900378}{1645371} c^{16} + \frac{1184459}{1645371} c^{19} + \frac{3378002}{42189} c^4 - \frac{5460230}{182819} c^{17} + \frac{688291}{4459} c^5 + \frac{1389370}{11193} c^6 + \frac{337505020}{1645371} c^7 + \frac{118784873}{548457} c^8 + \frac{271667666}{1645371} c^9 + \frac{358660781}{1645371} c^{10} + \frac{35978916}{182819} c^{11} + \frac{193381378}{1645371} c^{12} + \frac{553986}{3731} c^{13} + \frac{43953929}{548457} c^{15}$. $b - q$ is added to H: and all multiples of b are therefore expressible, and need not be enumerated.

a which can be expressed in terms of the previous ones as $r = \frac{487915}{705159} c^{18} - \frac{4406102}{705159} c - \frac{16292173}{705159} c^{14} - \frac{17206178}{705159} c^2 - \frac{1276987}{235053} c^{16} - \frac{91729}{705159} c^{19} + \frac{377534}{705159} - \frac{801511}{26117} c^3 - \frac{26686318}{705159} c^4 + \frac{4114333}{705159} c^{17} - \frac{34893715}{705159} c^5 - \frac{37340389}{705159} c^6 - \frac{409930}{6027} c^7 - \frac{6603890}{100737} c^8 - \frac{14279770}{235053} c^9 - \frac{15449995}{235053} c^{10} - \frac{5382578}{100737} c^{11} - \frac{722714}{18081} c^{12} - \frac{26536060}{705159} c^{13} - \frac{13243117}{705159} c^{15}$. $a - r$ is added to H, and there are no more monomials to consider.

These last three give us the Gröbner base in a purely lexicographical order, which looks like $\left\{c^{20} + \cdots, b + \cdots, a + \cdots\right\}$. As there are twenty solutions in reasonably general position (the polynomial in c alone *does* factor, but is square-free), we only need one polynomial per variable, as is often the case. The complete version of this, and another worked example, are given at `http://staff.bath.ac.uk/masjhd/JHD-CA/FGLMexample.html`.

The existence of this algorithm leads to the following process for 'solving' a zero-dimensional set of polynomial equations.

Algorithm 13
Input: *A set S of polynomials*
Output: *A 'description of solutions'*

```
$G$ :=Buchberger($S, >_{\texttt{tdeg}}$)
if $G$ is not zero-dimensional (Proposition 39)
```

```
then return "not zero-dimensional"
else Proposition 39 says how many solutions
     $H$ :=FGLM($G, >_{\texttt{plex}}$)
     Use Gianni–Kalkbrener to solve $H$
     We can check the solution count against Proposition 39
```

In the author's experience, describing the solutions of a set of polynomial equations when the dimension is not zero is still rather an art form, but much aided by the computation of Gröbner bases: see the description at the end of Section 3.3.6.

Other ways of computing Gröbner bases over **Q** will be looked at in section 4.6.

3.3.9 The Gröbner Walk

Though the Gianni–Kalkbrener algorithm only works in dimension 0, we may still want lexicographical-order Gröbner bases in higher dimensions, again balk at the cost of computing them directly, and look for an alternative. The FGLM algorithm instrinsically relies on the dimension being zero. Another method, which does not, is the Gröbner walk, due originally to [CKM97], see also [AGK97, Tra00, FJLT07]. The fundamental observation is in [MR88, Lemma 2.6], that a given ideal only has finitely many possible (across all possible orderings) leading monomial ideals, and hence *for this ideal* there are only finitely many essentially different orderings. The space of orderings is then partitioned into polyhedral sets: a partition known as the *Gröbner Fan*.

We first need a variant of Algorithm 9, given in Figure 3.5, which is to Algorithm 9 as Algorithm 5 is to Algorithm 2.

Algorithm 15 (Gröbner Walk)
Input: *A Gröbner base G for an ideal I with respect to $>'$; an ordering $>''$.*
Output: *A Gröbner base H for I with respect to $>''$.*

The idea is to construct (generally incrementally) a sequence of orders $>_1=>'$, $>_2$, $>_3$, $\ldots >_k=>''$ and corresponding Gröbner bases $G_1 = G$, $\ldots$, $G_k = H$. We use the matrix representation of orderings, so that $>_i$ is given by the matrix M_i — see page 109. M_2 is special — its first row is the first row of M_1, and the remaining rows are the corresponding rows of M_k. All subsequent M_i have the same rows two onwards, and their first rows are a sequence of 'hybrids' between the first row of M_2 and the first row of M_k. Let ω_i be the first row of M_i. The reader who is familar with homotopy methods may care to view this algorithm as a homotopy method between $\omega_1 = \omega_2$ and ω_k. There are then two open questions.

1. Which hybrids, i.e. which intermediate orderings, do we need? The answer, at least for the simple version of the Gröbner walk, is to let $\omega_i = (1 - t_i)\omega_1 + t_i\omega_k$, so $t_1 = t_2 = 0$ and $t_k = 1$. The t_i are then computed by NextCritical.

Figure 3.5: Algorithm 14

Algorithm 14 (Extended Buchberger)
Input: *finite* $G_0 = \{g_1^{(0)}, \ldots, g_m^{(0)}\} \subset R[x_1, \ldots, x_n]$*; monomial ordering* $>$.
Output: $G = \{g_1, \ldots, g_k\}$ *a Gröbner base for* (G_0) *with respect to* $>$.
Matrix $M = (m_{i,j})$ *such that* $g_i = \sum m_{i,j} g_j^{(0)}$

```
$G := G_0$; $n := |G|$; M:=$n \times n$ Identity
# we consider $G$ as $\{g_1, \ldots, g_n\}$
$P := \{(i,j) : 1 \le i < j \le n\}$
while $P \ne \emptyset$ do
    Pick $(i,j) \in P$;
    $P := P \setminus \{(i,j)\}$;
    Let $S(g_i, g_j) \overset{*}{\to}^{G} h$ # tracking $M$
    If $h \ne 0$ then
      # $\mathrm{lm}(h) \notin (\mathrm{lm}(G))$
      $g_{n+1} := h$; $G := G \cup \{h\}$;
      $P := P \cup \{(i, n+1) : 1 \le i \le n\}$;
      $n := n + 1$;
      Add new row to $M$
$G$ :=Interreduce($G$) # (3.29)
```

We note that we have to keep track of the $m_{i,j}$ during the reduction process as well as during the S-polynomial computation. We use the version that does the inter-reduction of the final basis, noting that this has also to update the $m_{i,j}$.

Figure 3.6: Body of Algorithm 15

```
i:=2; $>_1:=>'$; $\omega$ := first row of $M_{>'}$
$>_2$:= ordering whose first row is $\omega$, rest from $>''$
$\tau$ := first row of $M_{>''}$
while ($>_i \neq >''$)
        $G' := \mathrm{init}_\omega(G)$
        $G'', M$ :=Algorithm 14($G', >_i$)
        $H$ :=Transform($G, M, >_i$)
        $t$ :=NextCritical($H, \omega, \tau$)
        $G := H$; $i := i+1$; $\omega := \omega + t(\tau - \omega)$
        $>_i$:=Ordering with matrix as $>''$ but first row $\omega$
```

where

- Algorithm 14 is applied to an ω-homogeneous set of polynomials, many of which will, generically, be monomials. This allows or some efficiency improvements, see [AGK97].
- Transform$(G, M, >)$ computes $h_i = \sum_j m_{i,j} g_j$ in order $>$;
- NextCritical(H, ω, τ) computes the least $t > 0$ such that $\mathrm{init}_\omega(H) \neq \mathrm{init}_{\omega+t(\tau-\omega)}(H)$.

2. How do we compute G_i, given G_{i-1}? Naïvely, we could use Algorithm 9 for $>_i$, ignoring the fact that G_{i-1} is a Gröbner base for $>_{i-1}$, an ordering "fairly similar" to $>_i$. We will see that we can do better.

Definition 58 *Let $\omega = (\omega_1, \ldots, \omega_n)$ be a vector of (non-negative) rational numbers. The ω-degree of a monomial $\prod x_i^{a_i}$ is the sum $\sum \omega_i a_i$. A polynomial is said to be ω-homogeneous if every term has the same ω-degree. A set of polynomials is said to be ω-homogeneous if each of them separately is ω-homogeneous.*[20] *The ω-initial form of a polynomial p, denoted* $\mathrm{init}_\omega(p)$, *is the sum of all terms of maximal ω-degree. If $>$ is a monomial ordering the first row of whose matrix is ω, we write* $\mathrm{init}_>$ *as well as* init_ω.

With Algorithm 14, we can express step 2 in the Gröbner Walk as "when the initials change, run Buchberger's algorithm (14) on the new initials, then use the results to update the whole Gröbner base", as described in Figure 3.6. This depends crucially on the fact that we are only making minimal changes to the order, and the initials: see results in [AGK97].

The theoretical complexity of the Gröbner walk has not been analysed. The experimental results in [AGK97] show that, on zero dimensional ideals, it can be

[20]Note that we are *not* insisting that all terms in the set have the same ω-degree, merely that within each polynomial they have the same ω-degree.

ten or more times faster than their implementation of FGML. Their implementation of the walk contains several improvements over the one we hve outlined. There are two main sources of inefficiency in the algorithm as we have outlined it.

1. If many new terms suddenly appear in $\mathrm{init}_\omega(G)$, the computation of Algorithm 14 can become quite close to a full Gröbner base computation by Algorithm 9. This happens when our walk passes through the intersection of several polyhedral cones in the Gröbner fan, and can be avoided by perturbing the walk [AGK97, §3].
2. In general, the t computed by NextCritical, especially if we perturb to walk to avoid the previous problem, can become quite complicated rational numbers. We "solve" this by clearing denominators, but then have to deal with large integers, as seen in the examples.

The worked versions of the examples of section 3.3.8 for this algorithm are given at `http://staff.bath.ac.uk/masjhd/JHD-CA/GWalkexample.html`. We note that, at least in Maple, for examples of dimension zero the FGLM process seems ten times faster than the Gröbner walk, which rather contradicts the experimental results in [AGK97].

Open Problem 11 *Perform more extensive experiments comparing the FGLM and Gröbner Walk examples.*

3.3.10 Factorization and Gröbner Bases

It may happen, either initially or during the computation of a Gröbner basis, that we observe that a polynomial f_i factors as $f_{i,1}f_{i,2}$ (or more, but for simplicity we consider the case of two factors. Since $f_i(x_1, \ldots, x_n) = 0$ is and only if one of $f_{i,1}(x_1, \ldots, x_n)$ or $f_{i,2}(x_1, \ldots, x_n)$ is zero, we can reduce the problem to two, hopefully simpler, ones. For "random" problems, this is unlikely[21] to happen, but people do not generally ask "random" questions.

This observation[22] can be powerful, but is not as simple to implement as might be expected. One 'clearly' modifies algorithm 9, so that, if $h = h_1h_2$, instead of adding h to G, one forks two copies of the algorithm, one with $G \cup \{h_1\}$ and one with $G \cup \{h_2\}$.

TO BE COMPLETED

3.3.11 The Shape Lemma

Let us look again at example 2 of section 3.3.7. Here we needed three equations to define an ideal in two variables. We note that interchanging the rôles of x

[21] A "random" zero-dimensional system will have a 'shape basis' (Definition 59, and the nonlinear polynomial is a "random" polynomial, and therefore factors with zero probability.

[22] Another case of (near)-simultaneous discovery: [Dav87] (who could solve in 96 seconds problems that could not be solved in two hours), [MMN89, NM92], and [Hie92, Hie93] (solving Quantum Yang–Baxter equations).

and y does not help (in this case, it might in others). However, using other coordinates than x and y definitely does. If we write the equations in terms of $u = x + y, v = x - y$ instead, we get the basis

$$[-4\,v + v^3, v^2 - 4 + 2\,u] : \tag{3.39}$$

three values for v (0, 2 and -2), each with one value of u (2, 0 and 0 respectively), from which the solutions in x, y can be read off. Note that ordering v before u would give the basis

$$[u^2 - 2\,u, uv, v^2 - 4 + 2\,u], \tag{3.40}$$

which is not of this form: it has three polynomials in two variables.

This kind of operation is called, for obvious reasons, a *rotation*. Almost all rotations will place the equations "in general position": and many theoretical approaches to these problems assume a "generic rotation" has been performed. In practice, this is a disaster, since sparsity is lost.

Definition 59 ([BMMT94]) *A basis for a zero-dimensional ideal is a* shape basis *if it is of the form*

$$\{g_1(x_1), x_2 - g_2(x_1), \ldots, x_n - g_n(x_1)\}\,.$$

This is a Gröbner basis for any ordering in which 'degree in x_1' is the first criterion: in the terminology of matrix orderings (page 109), any ordering where the first row of the matrix is $(\lambda, 0 \ldots, 0)$.

For a shape basis, the Gianni–Kalkbrener process is particularly simple: "determine x_1 and the rest follows". Almost all zero-dimensional ideals have shape bases. The precise criterion (•) in the theorem below is somewhat technical, but is satisfied if there are no repeated components.

Theorem 25 (Shape lemma) *[BMMT94, Corollary 3] After a generic rotation, a zero-dimensional ideal has a shape basis if, and only if,*

- *each primary component is simple or of local dimension 1.*

Furthermore [BMMT94, Lemma 2], such a rotation need only be 1-generic, i.e. have matrix $\begin{pmatrix} 1 & \mathbf{v} \\ \mathbf{0} & I \end{pmatrix}$ *for some generic vector* $\mathbf{v}$.

Their paper generalises this to ideals of higher dimension, but the complexity in notation is not worth it for our purposes.

A word of warning is in order here. The Shape Lemma is a powerful theoretical tool, but its application can be costly. Consider example 2 (page 117): $G = \{x^2 - 1, y^2 - 1, (x - 1)(y - 1)\}$. This is certainly not a shape basis, since it has more polynomials than indeterminates. This is inevitable, since the variety is not equiprojectable (see Definition 69 below) onto either x or y. If we write $s = x + y$, $t = x - y$, then the basis becomes $\{-4\,t + t^3, -4 + t^2 + 2\,s\}$ for the ordering[23] $s > t$, which is a shape basis. However, consider the similar basis $G' = \{x^{2n}-1, y^{2n}-1, (x^n-1)(y^n-1)\}$. Similar rotations will work, but t is now the root of a polynomial of degree $3n^2$ with at least $3n+1$ nonzero coefficients, and quite large ones at that, e.g. for $n = 3$

$$\begin{gathered}\bigl\{8469703983104 - 1328571568128\,t^3 + 56109155544\,t^6 - 3387236203\,t^9 \\ +149161506\,t^{12} - 11557977\,t^{15} + 279604\,t^{18} - 1053\,t^{21} - 78\,t^{24} + t^{27}, \\ -586877251095044672\,t + 11229793345003520\,t^4 - 363020550569195\,t^7 \\ +24557528419410\,t^{10} - 3328382464425\,t^{13} + 88786830300\,t^{16} - 417476125\,t^{19} \\ -23303630\,t^{22} + 307217\,t^{25} + 259287804304663680\,s\bigr\}.\end{gathered}$$

3.3.12 The Hilbert function

Let I be any ideal (other than the whole ring) of $k[x_1,\ldots,x_n]$. Let A be the algebra $k[x_1,\ldots,x_n]/I$, i.e. the set of all polynomials under the equivalence relation $f \equiv g$ if there is an $h \in I$ with $f = g + h$.

Proposition 43 *If G is a Gröbner base for I, then A is generated, as a k-vector space, by $M := \{m \text{ monomial } \in k[x_1,\ldots,x_n] | m \overset{*}{\to}^{G} m\}$, i.e. the set of irreducible monomials.*

We have already seen (Proposition 39) that the variety corresponding to I is zero-dimensional if, and only if, M is finite. In this case, $|M|$ is the number of solutions (counted with multiplicity).

The Hilbert function is a way of measuring M when it is infinite.

Definition 60 *Let A_l be the subset of A where there is a representative polynomial of total degree $\le l$. A_l is a finite-dimension vector space over k. Let H_I be the function $\mathbf{N} \to \mathbf{N}$ defined by*

$$H_I(l) = \dim_k A_l. \tag{3.41}$$

Proposition 44 *If G is a Gröbner base for I, then A_l is generated, as a k-vector space, by $M := \{m \text{ monomial } \in k[x_1,\ldots,x_n] | \text{tdeg}(m) \le l \wedge m \overset{*}{\to}^{G} m\}$, i.e. the set of irreducible monomials of total degree at most l.*

Note that, while M itself will depend on G (and therefore on the ordering used), Definition 60 defines an intrinsic property of A, and hence $|M|$ is independent of the order chosen.

Theorem 26 ([BW93, Lemma 9.21]) $\begin{pmatrix} l+d \\ d \end{pmatrix} \le H_I(l) \le \begin{pmatrix} l+n \\ m \end{pmatrix}$.

[23]But not for $t > s$, since s defines the other line, besides the x and y axes, going through two of the points.

Theorem 27 ([BW93, Part of Theorem 9.27]) *Let G be a Gröbner base for $I \leq k[x_1, \ldots, x_n]$ under a total degree order. Let I have dimension d, and let*

$$N = \max\left\{\deg_{x_i}(\text{lm}(g)) \mid g \in G; 1 \leq i \leq n\right\}.$$

Then there is a unique polynomial $h_I \in \mathbf{Q}[X]$ such that $H_I(l) = h_I(l)$ for all $l \geq nN$. h_I is called the Hilbert polynomial *of I.*

3.3.13 Comprehensive Gröbner Bases and Systems

This idea was introduced in [Wei92] (see also [Wei03]).

Example 9 (Comprehensive Gröbner Basis) *Consider[24] first the example of $H_1 := \{x + 1, uy + x\} \subset \mathbf{Q}[u, x, y]$. Under any term order with $x < y$, this forms a (zero-dimensional) Gröbner base in $\mathbf{Q}(u)[x, y]$. However, if we substitute $u = 0$, we get $\{x + 1, x\}$, which is not a Gröbner base at all. If we consider instead $H_2 := \{x + 1, uy - 1\}$, which is equivalent in $\mathbf{Q}(u)[x, y]$, substituting $u = 0$ gives us $\{x + 1, -1\}$, which is a Gröbner basis (admittedly redundant) equivalent to $\{-1\}$ — no solutions. In fact H_2 is what we want — a Gröbner basis which is comprehensive in the informal sense that it is valid, not only for symbolic u, but for all values of u.*

As a formal definition, we have the following.

Definition 61 (Comprehensive Gröbner Basis) *Let K be an integral domain, $R = K[u_1, \ldots, u_m]$ and $T = R[x_1, \ldots, x_n]$, and fix an ordering $\leq$ on the monomials in $x_1, \ldots, x_n$. Let G be a finite subset of T. G is said to be a* Comprehensive Gröbner basis *if, for all fields K' and all ring homomorphisms $\sigma : R \to K'$ (extended to homomorphisms $\sigma : T \to K'[x_1, \ldots, x_n]$), $\sigma(G)$ is a Gröbner basis (under $\leq$) in $K'[x_1, \ldots, x_n]$.*

It is not obvious that these exist, but they do [Wei92, Theorem 2.7]. The construction of these proceeds via the related concept of a Comprehensive Gröbner System, but we need a preliminary definition.

Definition 62 (Algebraic Partition) *Let K be an integral domain, $R = K[u_1, \ldots, u_m]$ and $S \subseteq K^m$. A finite set $\{S_1, \ldots, S_t\}$ of nonempty subsets of S is called an* algebraic partition *of S if it satisfies the following properties*

1. *$\bigcup_{i=1}^t S_i = S$.*
2. *$S_i \cap S_j = \emptyset$ if $i \neq j$.*
3. *For each i, $S_i = V_K(I_i^{(1)}) \setminus V_K(I_i^{(2)})$ for some ideals $I_i^{(1)}$, $I_i^{(2)}$ of R, where $V_K(I)$ is $V(I) \cap K^m$.*

Each S_i is called a segment.

[24] I am grateful to John Abbott for discussions about this example.

Definition 63 (Comprehensive Gröbner System) *Let $\{S_1,\ldots,S_t\}$ be an algebraic partition of $S \subseteq K^m$ as in the previous definition, let $T = R[x_1,\ldots,x_n]$, and fix an ordering $\le$ on the monomials in $x_1,\ldots,x_n$. Let F be a finite subset of T. A finite set $\mathcal{G} := \{(S_1,G_1),\ldots,(S_s,G_s)\}$ satisfying the following properties is called a comprehensive Gröbner system (CGS) of F over S with parameters $u_1,\ldots,u_m$ w.r.t. $\le$:*

1. *Each G_i is a finite subset of (F);*
2. *For each $\overline{c} \in S_i$, $G_i(\overline{c}) := \{g(\overline{c},x_1,\ldots,x_n) | g(u_1,\ldots,u_m,x_1,\ldots,x_n) \in G_i\}$ is a Gröbner basis of the ideal $(F(\overline{c})$ in $C[x_1,\ldots,x_n]$ with respect to $\le$, where $F(\overline{c}) := \{f(\overline{c},x_1,\ldots,x_n) | f(u_1,\ldots,u_m,x_1,\ldots,x_n) \in F\}$*
3. *For each $\overline{c} \in S_i$, $\mathrm{lc}(g)(\overline{c}) \neq 0$ for any element g of G_i.*

In addition, if each $G_i(\overline{c})$ is a minimal (reduced) Gröbner basis, G is said to be minimal (reduced). Being monic is not required. When S is the whole space K^m, the phrase "over S" is usually omitted.

Example 10 (Comprehensive Gröbner System) *In the setting of Example 9, we partition $\mathbf{Q}$ as $\{S_1 := \{0\}, S_2 := \mathbf{Q} \setminus S_1\}$. The Gröbner basis corresponding to S_2 is either H_1 or H_2 (or any other variant), and these are Gröbner bases by the gcd Criterion (Proposition 40)* as long as *the leading term of $uy + x$ is uy. Hence $u = 0$ is a special case, and our polynomials are $\underbrace{uy}_{=0} + x$ and $x + 1$, whose S-polynomial (or indeed reduction) is $\left(\underbrace{uy}_{=0} + x\right) - (x+1) = \underbrace{uy}_{=0} - 1$. So the Gröbner basis corresponding to S_1 is $\{uy - 1\}$.*

Computing a Comprehensive Gröbner System is conceptually straightforward: we start with the trivial partition $\{S\}$, and run Buchberger's Algorithm (9). Every time we have to decide on the zeroness or not of a leading coefficient, either in the $S(g_i,g_j) \overset{*}{\to}^{G} h$ step or in deciding whether $h = 0$ (diretcly of via the Criteria), and that decision depends on the u_i, i.e. whether a polynomial p in the u_i is zero or not, we split our set $S_i = V_K(I_i^{(1)}) \setminus V_K(I_i^{(2)})$ into $S_{i'} = V_K(I_i^{(1)} \cup \{p\}) \setminus V_K(I_i^{(2)})$ and $S_{i''} = V_K(I_i^{(1)}) \setminus V_K(I_i^{(2)} \cup \{p\})$ and continue Algorithm 9 over each set separately, *but keeping* the apparently zero terms. In practice, the same polynomials p keep cropping up, and substantial ingenuity is needed to reduce or eliminate duplication.

Theorem 28 ([Wei92, Proposition 3.4(i)]) *If $\mathcal{G} := \{(S_1,G_1),\ldots,(S_s,G_s)\}$ is a Comprehensive Gröbner System for F over S, then $G' := \bigcup_{i=1}^{s} G_i$ is a Comprehensive Gröbner Basis for F.*

3.3.14 Coefficients other than fields

Most of the theory of this section, notably theorem 16, goes over to the case when R is a P.I.D. (definition 14) rather than a field, as described in [BW93, section 10.1]. However, when it comes to computation, things are not quite so obvious. What is the Gröbner base of $\{2x, 3y\}$ [Pau07]? There are two possible answers to the question.

- $\{2x, 3y\}$ [Tri78].
- $\{2x, 3y, xy\}$ (where xy is computed as $x(3y) - y(2x)$) [Buc84].

We note that $xy = x(3y) - y(2x) \in (2x, 3y)$, so we need xy to reduce to zero. We therefore modify definition 49 as follows

Definition 64 *Let $f, g \in R[x_1, \ldots, x_n]$. Suppose the leading coefficients of f and g are a_f and a_g, and the leading monomials m_f and m_g. Let a be a least common multiple of a_f and a_g, and write $a = a_f b_f = a_g b_g$. Let m be the least common multiple of m_f and m_g. The S-polynomial of f and g, written $S(f, g)$ is defined as*

$$S(f, g) = b_f \frac{m}{m_f} f - b_g \frac{m}{m_g} g. \tag{3.42}$$

Let $c_f a_f + c_g a_g = \gcd(a_f, a_g)$, and define the G-polynomial of f and g, written $G(f, g)$, as

$$G(f, g) = c_f \frac{m}{m_f} f + c_g \frac{m}{m_g} g. \tag{3.43}$$

Note that (3.42) is the same as (3.28) up to a factor of $\gcd(a_f, a_g)$. The S-polynomial is defined up to unit factors, whereas the G-polynomial is much less well-defined, but it turns out not to matter.

For the example quoted above, $S(2x, 3y) = 0$ (which follows from Proposition 40), while $G(2x, 3y) = xy$. Algorithm 9 goes over to this setting, execpt that we have to add G-polynomials as well as S-polynomials, and some care has to be taken to eliminate G-polynomials first — see [BW93, table 10.1].

3.3.15 Non-commutative Ideals

Much of the general mechanism of ideals generalises to the case of non-commutative ideals, provided we are careful to distinguish left, right or two-sided ideals. However, the theory is notably weaker. In particular we have the following opposite of theorem 1 and its corollary.

Proposition 45 *$K\langle x_1, \ldots, x_n\rangle$ is not noetherian for $n \geq 2$.*

Hence Buchberger's algorithm 9 might not terminate, and in general it does not [Mor86].

In fact, not only does this approach not work, but no approach can, as demonstrated by this result.

Proposition 46 ([KRW90]) *Ideal membership is insoluble in $\mathbf{Q}\langle x_1, x_2\rangle$.*

One case of great interest is when R is some field of (expressions representing) functions, and the "indeterminates" are differential or difference operators.

Example 11 *R is $\mathbf{Q}(x,y)$ and the indeterminates are $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$, so that we are working in $R[\frac{\partial}{\partial x},\frac{\partial}{\partial y}]$. Here the "indeterminates" commute with each other, but not with R, since $\frac{\partial}{\partial x}(xf) = f + x\frac{\partial}{\partial x}f$, i.e. $\frac{\partial}{\partial x}x = 1 + x\frac{\partial}{\partial x}$.*

We should note that the result of multiplying a term by an indeterminate is not necessarily a term, e.g. $\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}\right) = \frac{\partial}{\partial x} + x\frac{\partial^2}{\partial x^2}$. This makes characterising a Gröbner base harder, but the following definition is an appropriate generalisation of the last clause of theorem 16 in the setting where the indeterminates commute with each other.

Definition 65 *[Pau07, Definition 4] A finite subset G of $I \setminus \{0\}$, where I is a left-ideal, is a Gröbner basis of I iff, for all monomials m, the R-ideal $(\mathrm{lc}(f)|f \in I \wedge \mathrm{lm}(f) = m)$ is generated by $\{\mathrm{lc}(g)|g \in G \wedge \mathrm{lm}(g)$ divides $m\}$.*

If R is a principal ideal domain, it is possible to define S-polynomials and G-polynomials as in the previous section, but in general we need to consider more complicated (but still finitely many) combinations [Pau07]. This leads to an *effective* test for a Gröbner base in this setting, i.e. we need to check that finitely many combinations reduce to zero. We also get a generalisation of Buchberger's algorithm [Pau07, Proposition 10]: if the combination does not reduce to zero, add it. Termination is non-trivial, however.

3.4 Nonlinear Multivariate Equations: Recursive

Whereas the previous section looked at polynomials as living in $k[x_1,\ldots,x_n]$ thought of (irrespective of implementation) is a distributed way (page 50), with a certain order on the monomials, it is equally possible to think of them in a recursive way (page 49), but now with an order on the variables, rather than the monomials.

3.4.1 Triangular Sets and Regular Chains

An alternative approach to polynomial equation solving is that of characteristic [Rit32, Wu86] or triangular [Laz91] sets, or regular chains.[25] See [ALM99] for a reconciliation of the various theories, and [AM99] for a practical comparison. We can regard regular chains as an approach based on recursive views of polynomials, while Gröbner bases are based on a distributed view. We assume that the variables $x_1,\ldots,x_n$ are ordered as $x_1 < \ldots < x_n$, so that x_n is the most important variable.

Definition 66 *Let p be a polynomial. The* main variable *of p, denoted* $\mathrm{mvar}(p)$*, is the most important variable of p. The* initial *of p, written* $\mathrm{init}(p)$*, is its leading coefficient, when regarded as a univariate polynomial in* $\mathrm{mvar}(p)$.

[25] Terminology in this area has been confused: we are following a recent reconciliation of terminology.

If $S = \{p_1, \ldots, p_k\}$ is a finite set of polynomials, we write

$$\operatorname{init}(S) = \operatorname{lcm}_{1 \le i \le k} \operatorname{init}(p_i).$$

It should be noted that many authors define $\operatorname{init}(S)$ as $\prod_{1 \le i \le k} \operatorname{init}(p_i)$. Since we are normally concerned with the zeros of $\operatorname{init}(S)$, the two definitions have the same consequences, and ours leads to smaller polynomials.

Definition 67 *A set T of polynomials is said to be* triangular *if different polynomials have different main variables.*

Example 2 of section 3.3.7 shows[26] that there may not always be a triangular set generating a particular ideal. *If* we have a triangular set, then the structure of the ideal, and the variety, is relatively obvious.

Definition 68 *Let T be a triangular set generating an ideal I in $k[x_1, \ldots, x_n]$. Then every variable x_i which occurs as a main variable is called* algebraic*, and the set of such variables is denoted* $\operatorname{AlgVar}(T)$.

Proposition 47 *For a triangular set T, the dimension of $I(T)$ is $n - |\operatorname{AlgVar}(T)|$.*

3.4.2 Zero Dimension

Much of the theory applies to positive dimension as well, but we will only consider in this section the case of zero-dimensional ideals/varieties. Let V be a zero-dimensional variety, and V_k be its *projection* onto $x_1, \ldots, x_k$, i.e.

$$V_k = \{(\alpha_1, \ldots, \alpha_k) : \exists (\alpha_1, \ldots, \alpha_n) \in V\}.$$

Definition 69 *A zero-dimensional variety V is* equiprojectable *iff, for all k, the projection $V_k \to V_{k-1}$ is an $n_k : 1$ mapping for some fixed n_k.* Note that this definition depends on the order of the x_i: a variety might be equiprojectable with respect to one order, but not another, as in (3.39) versus (3.40).

Such an equiprojectable variety will have $\prod n_k$ points (i.e. solutions, not counting multiplicity, to the equations).

The variety V of Example 2 of section 3.3.7 is $\{(x = -1, y = 1), (x = 1, y = \pm 1)\}$ and is not equiprojectable. In fact, its equations can be written as $\{(x^2 - 1), (x - 1)(y - 1) + (x + 1)(y^2 - 1)\}$, which is a triangular set with y more important than x (main variables x and y respectively). However, the second polynomial sometimes has degree 1 in y (if $x = -1$), and sometimes degree 2. Hence we need a stronger definition.

Definition 70 *A list, or chain, of polynomials $f_1, \ldots, f_k$ is a* regular chain *if:*

[26] It actually only shows that this set of generators is not triangular. $\{x^2 - 1, (x + 1)(y - 1) + (x - 1)(y^2 - 1)\}$ is triangular [Ron11], but is not a Gröbner basis.

1. *whenever $i < j$,* $\operatorname{mvar}(f_i) \prec \operatorname{mvar}(f_j)$ *(therefore the chain is triangular);*
2. $\operatorname{init}(f_i)$ *is invertible modulo the ideal* $(\operatorname{init}(f_j) : j < i)$.

Proposition 48 *Every equiprojectable variety corresponds to a zero-dimensional regular chain, and vice versa.*

However, V of Example 2 of section 3.3.7 can be written as $V = V_1 \cup V_2$ where $V_1 = \{(x = -1, y = 1)\}$ and $V_2 = \{(x = 1, y = \pm 1)\}$, each of which is equiprojectable. The corresponding regular chains are $T_1 = \{x+1, y-1\}$ and $T_2 = \{x-1, y^2-1\}$.

Theorem 29 (Gianni–Kalkbrener (triangular variant)) *Every zero-dimensional variety can be written as a union of disjoint equiprojectable varieties — an* equiprojectable decomposition.

In fact, each solution description in Algorithm 10 is a description of an equiprojectable variety.

This theorem can be, and was, proved independently, and the decomposition into regular chains (the union of whose varieties is the original variety) can be computed directly. This gives us an alternative to algorithm 13: compute the regular chains corresponding to the equiprojectable decomposition, and solve each one separately [Laz92].

It appears that the triangular decomposition approach is more suitable to modular methods (chapter 4, especially section 4.6) than the Gröbner-base approach, but both aspects are areas of active research.

3.4.3 Positive Dimension

Here we consider the case of solution sets of positive dimension (over the algebraic closure, e.g. over the complexes). As in Theorem 29, the ultimate aim is to express a variety as a union (preferably a disjoint union) of "nicer" varieties, or other sets.

Definition 71 *Quasi-algebraic System If P and Q are two (finite) sets of polynomials, we call the ordered pair (P, Q) a* quasi-algebraic system, *and we write $Z(P, Q)$, the zeros of the quasi-algebraic system, for $V(P) \setminus V(\prod Q)$, with the convention that if Q is empty, $\prod Q = 1$, so $V(Q) = \emptyset$.*

$$Z(P, Q) = \{\mathbf{x} \in K^n | (\forall p \in P \quad p(\mathbf{x}) = 0) \wedge (\forall q \in Q \quad q(\mathbf{x}) \neq 0)\}.$$

We say that (P, Q) is consistent *if $Z(P, Q) \neq \emptyset$.*

In Definition 50, we defined the *variety* of a set of polynomials, but we need some more concepts, all of which depend on having fixed an order of the variables.

Definition 72 *If T is a triangular system, we define the* pseudo-remainder *of p by T to be the pseudo-remainder of dividing p by each $q_i \in T$ in turn (turn defined by decreasing order of* $\operatorname{mvar}(q_i)$*), regarded as univariate polynomials in* $\operatorname{mvar}(q_i)$.

This is a generalization of Definition 36 (page 66).

Definition 73 *Let S be a finite set of polynomials. The set of* regular zeros *of S, written $W(S)$, is $Z(S, \{\text{init}(S)\}) = V(S) \setminus V(\{\text{init}(S)\})$. For $(a_1, \ldots, a_n)$ to be in $W(S)$, where $S = \{p_1, \ldots, p_k\}$, we are insisting that* all *of the p_i vanish at this point, but* none *of the* $\text{init}(p_i)$.

For Example 2 of section 3.3.7, the variety is $\{(x = -1, y = 1), (x = 1, y = \pm 1)\}$. If we take $y > x$, then the inital of the set of polynomials is $\text{lcm}(1, x-1, 1) = x - 1$, so only the zero with $x = -1, y = 1$ is regular. Conversely, if we take $x > y$, the initial is $y - 1$ and only the zero with $y = -1, x = 1$ is regular. This emphasises that W depends on the variable ordering. It is also a property of the precise set S, not just the ideal $\langle S \rangle$.

In this case, $W(S)$ was in fact a variety (as always happens in dimension 0). In general, this is not guaranteed to happen: consider the (trivial) triangular system $S = \{(x-1)y - x + 1\}$ with $y > x$. Since this polynomial is $(x-1)(y-1)$, $V(S)$ is the two lines $x = 1$ and $y = 1$. However, $W(S)$ is the line $y = 1$ *except* for the point $(1, 1)$. In fact this is the only direct description we can give, though we could say that $W(S)$ is "almost" the line $y = 1$. This "almost" is made precise as follows.

Definition 74 *If W is any subset of K^n, the* Zariski closure *of W, written*[27] *$\overline{W}$, is the smallest variety containing it:*

$$\overline{W} = \bigcap \{V(F) \mid W \subseteq V(F)\},$$

which is itself a variety by Proposition 37.

In the example above, $\overline{W(S)} = V(y - 1)$.

3.4.3.1 An example

This example is from [AM99, p. 126][28]. Suppose we have, in two dimensions, a manipulator consisting of an arm of length 1 fixed at the origin, and with another arm, also of length 1, at its other end. We wish the far end of the manipulator to reach the point (a, b) in the plane. Let θ_1 be the angle that the first arm makes with the x axis, and write $c_1 = \cos\theta_1$, $s_1 = \sin\theta_1$. Let θ_2 be the angle that the second arm makes with the first. Then we have the following equations

$$\begin{aligned} c_1 + \cos(\theta_1 + \theta_2) &= a && (3.44)\\ s_1 + \sin(\theta_1 + \theta_2) &= b && (3.45)\\ s_1^2 + c_1^2 &= 1 && (3.46)\\ s_2^2 + c_2^2 &= 1, && (3.47) \end{aligned}$$

[27]Note that we use the same notation for algebraic closure and Zariski closure.

[28]The author is grateful to Russell Bradford for explaining the geometric context.

where the last two equations state that the arms have length 1. We can apply the addition formulae for trigonometric functions to (3.44) and (3.45) to get

$$c_1 + c_1c_2 - s_1s_2 = a \qquad 3.44',$$

$$s_1 + c_1s_2 + c_2s_1 = b \qquad 3.45'.$$

Rewriting these equations as polynomials, assumed to be zero, and using the order

$$c_2 > s_2 > c_1 > s_1 > b > a,$$

we get

$$S = \{c_2c_1 - s_2s_1 + c_1 - a, c_2s_1 + s_2c_1 + s_1 - b, c_1^2 + s_1^2 - 1, c_2^2 + s_2^2 - 1\},$$

which is not triangular since c_2 is the main variable of three different equations.

[AM99] implement the method of [Laz91] to express $V(S)$ as a (disjoint) union

$$W(T_1) \cup W(T_2) \cup W(T_3),$$

where

$$\begin{aligned} T_1 = \{&(b^2 + a^2)(4s_1^2 - 4bs_1 + b^2 + a^2) - 4a^2, 2ac_1 + 2bs_1 - b^2 - a^2, \\ &2as_2 + 2(b^2 + a^2)s_1 - b^2 - a^2b, 2c_2 - b^2 - a^2 + 2\}, \\ T_2 = \{&a, 2s_1 - b, 4c_1^2 + b^2 - 4, s_2 - bc_1, 2c_2 - b^2 + 2\}, \\ T_3 = \{&a, b, c_1^2 + s_1^2 - 1, s_2, c_2 + 1\}. \end{aligned}$$

3.4.3.2 Another Example

This was also considered as Example 8 (page 119).

Example 12 *Let $I = \langle ax^2 + x + y, bx + y\rangle$ with the order $a \prec b \prec y \prec x$. The full triangular decomposition (obtained from Maple's* `RegularChains` *package with* `option=lazard`*) is*

$$\left\{[bx + y, ay + b^2 - b], [x, y], [ax + 1, y, b], [x + y, a, b - 1]\right\}.$$

The first two components correspond to two two-dimensional surfaces (in fact the second one is a plane), whereas the second two correspond to one-dimensional solutions (hyperbola and straight line).

3.4.3.3 Regular Zeros and Saturated Ideals

Definition 75 *If T is a triangular system, define the* saturated ideal *of T to be*

$$\begin{aligned} \mathrm{sat}(T) &= \{p \in K[x_1, \ldots, x_n] | \exists n \in \mathbf{N} \quad \mathrm{init}(T)^n p \in (T)\} \\ &= \{p \in K[x_1, \ldots, x_n] | \mathrm{prem}(p, T) = 0\}. \end{aligned}$$

In other words it is the set of polynomials which can be reduced to zero by T after multiplying by enough of the initials of T so that division works. In terms of the more general concept of saturation $I : S^\infty$ of an ideal ([Bou61, p. 90]), this is $(T) : (\text{init}(T))^\infty$.

Theorem 30 ([ALM99, Theorem 2.1]) *For any non-empty triangular set T,*

$$\overline{W(T)} = V(\text{sat}(T)).$$

In the example motivating Definition 74, $\text{sat}(S)$ is generated by $y-1$, and indeed $\overline{W(S)} = V(\text{sat}(S))$.

3.4.4 Conclusion

Whether we follow the Gianni–Kalkbrener approach directly (algorithm 13) or go via triangular sets, the solutions to a zero-dimensional family of polynomial equations can be expressed as a union of (equiprojectable) sets, each of which can be expressed as a generalised `RootOf` construct. For example, if we take the ideal

$$\{-3\,x-6+x^2-y^2+2\,x^3+x^4, -x^3+x^2y+2\,x-2\,y, -6+2\,x^2-2\,y^2+x^3+y^2x, -6+3\,x^2-xy-2\,y^2+x^3+y^3\},$$

its Gröbner basis (purely lexicographic, $y > x$) is

$$[6-3\,x^2-2\,x^3+x^5, -x^3+x^2y+2\,x-2\,y, 3\,x+6-x^2+y^2-2\,x^3-x^4]. \quad (3.48)$$

There are seven irreducible monomials: 1, x, x^2, x^3, x^4, y and xy. We know that x satisfies a quintic, and y then satisfies $\left(x^2-2\right)y-x^3+2\,x$. When $x^2=2$, this vanishes, so our quintic for x decomposes into $(x^2-2)(x^3-3)$, and the whole solution reduces to

$$\left\langle x^2-2, y^2-x\right\rangle \cup \left\langle x^3-3, y-x\right\rangle. \quad (3.49)$$

Unfortunately, we do not have a convenient syntax to express this other than via the language of ideals. We are also very liable to fall into the 'too many solutions' trap, as in equation (3.8): Maple resolves the first component (in radical form) to

$$\left\{y=\sqrt[4]{2}, x=\sqrt{2}\right\}, \quad (3.50)$$

and the second one to

$$\left\{y=\sqrt[3]{3}, x=\sqrt[3]{3}\right\}, \quad (3.51)$$

both of which lose the connections between x and y ($x=y^2$ in the first case, $x=y$ in the second).

We are also dependent on the choice of order, since with $x > y$ the Gröbner basis is

$$[6-3\,y^4-2\,y^3+y^7, 18-69\,y^2-9\,y^4-46\,y+23\,y^5-2\,y^6+73\,x], \quad (3.52)$$

and no simplification comes to mind, short of factoring the degree seven polynomial in y, which of course is $(y^3-3)(y^4-2)$, and using the choice here to simplify the equation for x into either $x-y$ or $x-y^2$.

Maple's `RegularChains` package, using the technology of section 3.4.1, produces essentially equation (3.49) for the order $y > x$, and for $x > y$ produces

$$[[\left(2\,y+y^3+4\,y^2+2\right)x-8-2\,y^2-2\,y^3-2\,y,y^4-2],$$
$$[\left(5\,y+3+4\,y^2\right)x-12-5\,y^2-3\,y,-3+y^3]],$$

essentially the factored form of 3.52.

3.4.5 Regular Decomposition

TO BE COMPLETED

3.5 Equations and Inequalities

While it is possible to work in more general settings (real closed fields), we will restrict our attention to solving systems over $\mathbf{R}$. Consider the two equations

$$\begin{aligned} x^2+y^2 &= 1 && (3.53)\\ x^2+y^2 &= -1. && (3.54)\end{aligned}$$

Over the complexes, there is little to choose between these two equations, both define a one-dimensional variety. Over $\mathbf{R}$, the situation is very different: (3.53) still defines a one-dimensional variety (a circle), while (3.54) defines the empty set, even though we have only one equation in two variables.

Definition 76 ([ARS⁺13]) *We say that a complex (hyper)-surface* $V := \{(x_1, \ldots, x_n)|p_1(x_1,\ldots,x_n) = \cdots = p_k(x_1,\ldots,x_n) = 0\}$ *is* real *if every complex polynomial vanishing over* $V_{\mathbf{R}} := V \cap \mathbf{R}^n$ *also vanishes over* V. Algebraic Geometers would say that V is the Zariski closure of $V_{\mathbf{R}}$.

With the definition, (3.53) defines a real surface (in fact a curve), but (3.54) does not, since 1 vanishes over $V \cap \mathbf{R}^n$ (which is the empty set), but not over V.

The above example shows that we can essentially introduce the constraint $x \geq 0$ by adding a new variable y and the equation $y^2 - x = 0$. We can also introduce the constraint $x \neq 0$ by adding a new variable z and $xz - 1 = 0$ (essentially insisting that x be invertible). Hence $x > 0$ can be introduced. Having seen that $\geq$ and $>$ can creep in through the back door, we might as well admit them properly, and deal with the language of *real closed fields*, i.e. the language of fields (definition 15) augmented with the binary predicate $>$ and the additional laws:

1. Precisely one of $a = b$, $a > b$ and $b > a$ holds;

2. $a > b$ and $b > c$ imply $a > c$;
3. $a > b$ implies $a + c > b + c$;
4. $a > b$ and $c > 0$ imply $ac > bc$.

This is the domain of *real algebraic geometry*, a lesser-known, but very important, variant of classical algebraic geometry. Suitable texts on the subject are [BPR06, BCR98]. However, we will reserve the word 'algebraic' to mean a set defined by equalities only, and reserve *semi-algebraic* for the case when inequalities (or inequations[29]) are in use. More formally:

Definition 77 *An* algebraic proposition *is one built up from expressions of the form $p_i(x_1, \ldots, x_n) = 0$, where the p_i are polynomials with integer coefficients, by the logical connectives $\neg$ (not), $\wedge$ (and) and $\vee$ (or). A* semi-algebraic proposition *is the same, except that the building blocks are expressions of the form $p_i(x_1, \ldots, x_n)\sigma 0$ where σ is one of $=, \neq, >, \geq, <, \leq$. The language of semi-algebraic propositions is also called the* Tarski language *L.*

This language is in fact redundant, since $\neq, \geq, \leq$ can be replaced with the help of $\neg$, but corresponds more closely to natural usage. The reader will also notice that it is not quite the language of real closed fields described above, since we do not allow division. This is partly for ease of subsequent development, but also allows us to sidestep "division by zero" questions, as raised in problem 1 of section 1.2.3. Hence the proposition $\frac{p}{q} > 0$ has to be translated as

$$(q > 0 \wedge p > 0) \vee (q < 0 \wedge p < 0), \tag{3.55}$$

which is not true when $q = 0$. If this is not what we mean, e.g. when p and q have a common factor, we need to say so.

Open Problem 12 (Better treatment of division) *(3.55) is equivalent to $pq > 0$. However, $\frac{p}{q} \geq 0$ is* not *equivalent to $pq \geq 0$, but rather to $pq \geq 0 \wedge q \neq 0$. In general, the polynomial theorists tend to dismiss the problem as above. The logician tends to worry much more about the problem: consider [AP10, Appendix, lines 1–12] for an example of rational function manipulation.*

3.5.1 Applications

It turns out that many of the problems one wishes to apply computer algebra to can be expressed in terms of real semi-algebraic geometry. This is not totally surprising, since after all, the "real world" is largely real in the sense of **R**. Furthermore, even if problems are posed purely in terms of equations, there may well be implicit inequalities as well. For example, it may be implicit that

[29]Everyone agrees that an *equation* $a = b$ is an *equality*. $a > b$ and its variants are traditionally referred to as *inequalities*. This only leaves the less familiar *inequation* for $a \neq b$. Some treatments ignore inequations, since "$a \neq b$"="$a > b \vee a < b$", but in practice it is useful to regard inequations as first-class objects.

quantities are non-negative, or that concentrations is biochemistry lie in the range $[0,1]$.

Robot motion planning ... **TO BE COMPLETED**

It is also often important to prove *unsatisfiability*, i.e. that a semi-algebraic formula has *no* solutions. [Mon09] gives several examples, ranging from program proving to biological systems. The program proving one is as follows. One wishes to prove that I is an invariant (i.e. if it was true at the start, it is true at the end) of a program which moves from one state to another by a transition relation τ. More formally, one wishes to prove that there do *not* exist two states s, s' such that $s \in I$, $s' \notin I$, but $s \to_\tau s'$. Such a pair (s, s') would be where "the program breaks down", so a proof of unsatisfiability becomes a proof of program correctness. This places stress on the concept of 'proof' — "I can prove that there are no bad cases" is much better than "I couldn't find any bad cases".

3.5.2 Real Radical

. We recall the second part of Definition 51: the *radical* of an ideal I, denoted $\sqrt{I}$, is defined as

$$\sqrt{I} = \{p \mid \exists m : p^m \in I\}.$$

Definition 78 *Let $A = \mathbf{R}[x_1, \ldots, x_k]$ (it is possible to be more general, and talk about* real closed fields*). The* real radical *of an ideal $I \subset A$, denoted $\sqrt[re]{I}$, is defined as*

$$\sqrt[re]{I} = \left\{ p \mid \exists m, k \in \mathbf{N}, r_i \in \mathbf{R}^+, g_i \in A : p^{2m} + \sum_{i=1}^{k} r_i g_i^2 \in I \right\}.$$

3.5.3 Quantifier Elimination

A fundamental result of algebraic geometry is the following, which follows from the existence of resultants (section A.1).

Theorem 31 *A projection of an algebraic set is itself an algebraic set.*

For example, the projection of the set defined by

$$\left\{ (x-1)^2 + (y-1)^2 + (z-1)^2 - 4, x^2 + y^2 + z^2 - 4 \right\} \tag{3.56}$$

on the x, y-plane is the ellipse

$$8\,x^2 + 8\,y^2 - 7 - 12\,x + 8\,xy - 12\,y. \tag{3.57}$$

We can regard equation (3.56) as defining the set

$$\exists z \left((x-1)^2 + (y-1)^2 + (z-1)^2 = 4 \wedge x^2 + y^2 + z^2 = 4 \right) \tag{3.58}$$

and equation (3.57) as the *quantifier-free* equivalent

$$8\,x^2 + 8\,y^2 - 12\,x + 8\,xy - 12\,y = 7. \tag{3.59}$$

Is the same true in real algebraic geometry? If P is a projection operator, and $\Re$ denotes the real part, then clearly

$$P(\Re(U) \cap \Re(V)) \subseteq \Re(P(U \cap V)). \tag{3.60}$$

However, the following example shows that the inclusion can be strict. Consider

$$\left\{(x-3)^2 + (y-1)^2 + z^2 - 1, x^2 + y^2 + z^2 - 1\right\}$$

Its projection is $(10 - 6\,x - 2\,y)^2$, i.e. a straight line (with multiplicity 2). If we substitute in the equation for y in terms of x, we get $z = \sqrt{-10\,x^2 + 30\,x - 24}$, which is never real for real x. In fact $\Re(U) \cap \Re(V) = \emptyset$, as is obvious from the geometric interpretation of two spheres of radius 1 centred at $(0,0,0)$ and $(3,1,0)$. Hence the methods we used for (complex) algebraic geometry will not translate immediately to real algebraic geometry.

The example of $y^2 - x$, whose projection is $x \geq 0$, shows that the projection of an algebraic set need not be an algebraic set, but might be a semi-algebraic set. Is even this guaranteed? What about the projection of a semi-algebraic set? In the language of quantified propositions, we are asking whether, when F is an algebraic or semi-algebraic proposition, the proposition

$$\exists y_1 \ldots \exists y_m F(y_1, \ldots, y_m, x_1, \ldots, x_n) \tag{3.61}$$

has a quantifier-free equivalent $G(x_1, \ldots, x_n)$, where G is a semi-algebraic proposition. We can generalise this.

Problem 3 (Quantifier Elimination) *Given a quantified proposition*[30]

$$Q_1 y_1 \ldots Q_m y_m F(y_1, \ldots, y_m, x_1, \ldots, x_n), \tag{3.62}$$

where F is a semi-algebraic proposition and the Q_i are each either $\exists$ or $\forall$, does there exist a quantifier-free equivalent semi-algebraic proposition $G(x_1, \ldots, x_n)$? If so, can we compute it?

The fact that there is a quantifier-free equivalent is known as the *Tarski–Seidenberg Principle* [Sei54, Tar51]. The first constructive answer to the question was given by Tarski [Tar51], but the complexity of his solution was indescribable[31]. A better (but nevertheless doubly exponential) solution had to await the concept of cylindrical algebraic decomposition (CAD) [Col75] described in the next section.

[30] Any proposition with quantified variables can be converted into one in this form, so-called *prenex normal form* — see any standard logic text.

[31] In the formal sense, that there was no elementary function which could describe it, i.e. no tower of exponentials of fixed height would suffice!

Notation 23 *Since $\exists x \exists y$ is equivalent to $\exists y \exists x$, and similarly for $\forall$, we extend $\exists$ and $\forall$ to operate on blocks of variables, so that, if $\mathbf{x} = (x_1, \ldots, x_n)$, $\exists \mathbf{x}$ is equivalent to $\exists x_1 \ldots \exists x_n$. If we use this notation to rewrite equation 3.62 with the fewest number of quantifiers, the quantifiers then have to alternate, so the formula is (where the $\mathbf{y}_i$ are sets of variables)*

$$\forall \mathbf{y}_1 \exists \mathbf{y}_2 \forall \mathbf{y}_3 \ldots F(\mathbf{y}_1, \mathbf{y}_2, \ldots, x_1, \ldots, x_n), \tag{3.63}$$

or

$$\exists \mathbf{y}_1 \forall \mathbf{y}_2 \exists \mathbf{y}_3 \ldots F(\mathbf{y}_1, \mathbf{y}_2, \ldots, x_1, \ldots, x_n). \tag{3.64}$$

In either form, the number of (block) quantifiers is one more than the number of alternations.

3.5.4 Algebraic Decomposition

Definition 79 *An* algebraic decomposition *of $\mathbf{R}^n$ is an expression of $\mathbf{R}^n$ as the disjoint union of non-empty connected sets, known as* cells*, each defined as*

$$p_1(x_1, \ldots, x_n) \sigma 0 \wedge \cdots \wedge p_m(x_1, \ldots, x_n) \sigma 0, \tag{3.65}$$

where the σ are one of $=, >, <$. Equation (3.65) is known as the defining formula *of the cell C, and denoted* $\mathrm{Def}(C)$.

These should properly be called *semi-algebraic decompositions*, but this terminology has stuck. Note that (3.65) need not define a non-empty connected set — external information is required to show this. We should note that these definitions are a very restricted form of definition 77. Here are some examples.

1. $\mathbf{R}^1$ can be decomposed as $\{x < 0\} \cup \{x = 0\} \cup \{x > 0\}$.
2. $\mathbf{R}^1$ cannot be decomposed as $\{x^2 = 0\} \wedge \{x^2 > 0\}$, as the second set is not connected. Rather, we need the previous decomposition.
3. $\mathbf{R}^1$ cannot be decomposed as

   $$\{(x^2-3)^2 - 2 = 0\} \wedge \{(x^2-3)^2 - 2 > 0\} \wedge \{(x^2-3)^2 - 2 < 0\},$$

   as the sets are not connected. Rather, we need the decomposition (writing $(x^2-3)^2 - 2$ as f)

   $$\begin{gathered} \{f > 0 \wedge x < -2\} \cup \{f = 0 \wedge x < -2\} \cup \{f < 0 \wedge x < 0\} \cup \\ \{f = 0 \wedge x > -2 \wedge x < 0\} \cup \{f > 0 \wedge x > -2 \wedge x < 2\} \cup \\ \{f = 0 \wedge x > 0 \wedge x < 2\} \cup \{f < 0 \wedge x > 0\} \cup \\ \{f = 0 \wedge x > 2\} \cup \{f > 0 \wedge x > 2\}. \end{gathered}$$

4. $\mathbf{R}^2$ can be decomposed as $\{(x^2+y^2) < 0\} \cup \{(x^2+y^2) = 0\} \cup \{(x^2+y^2) > 0\}$.

5. $\mathbf{R}^2$ cannot be decomposed as $\{xy<1\}\cup\{xy=1\}\cup\{xy>1\}$, as the last two sets are not connected. Rather, we need the more complicated
$$\begin{gathered}\{xy<1\}\cup\\ \{xy=1\wedge x>0\}\cup\{xy=1\wedge x<0\}\cup\\ \{xy>1\wedge x<0\}\cup\{xy>1\wedge x>0\}.\end{gathered}$$

6. $\mathbf{R}^2$ cannot be decomposed as $\{f<0\}\cup\{f=0\}\cup\{f>0\}$, where $f=\left(x^2+y^2-1\right)\left((x-3)^2+y^2-1\right)$, as the first two sets are not connected. Rather, we need the more complicated
$$\begin{gathered}\{f<0\wedge x<\tfrac{3}{2}\}\cup\{f<0\wedge x>\tfrac{3}{2}\}\cup\{f=0\wedge x<\tfrac{3}{2}\}\\ \cup\{f=0\wedge x>\tfrac{3}{2}\}\cup\{f>0\}\end{gathered}$$

The reader may complain that example 3 is overly complex: can't we just write
$$\begin{gathered}\{f>0\wedge x<-2\}\cup\{x=-\sqrt{3+\sqrt{2}}\}\cup\{f<0\wedge x<0\}\cup\\ \{x=-\sqrt{3-\sqrt{2}}<0\}\cup\{f>0\wedge x>-2\wedge x<2\}\cup\\ \{x=\sqrt{3-\sqrt{2}}\}\cup\{f<0\wedge x>0\}\cup\{x=\sqrt{3+\sqrt{2}}\}\cup\{f>0\wedge x>2\}?\end{gathered}$$

In this case we could, but in general theorem 10 means that we cannot[32]: we need `RootOf` constructs, and the question then is "which root of ...". In example 3, we chose to use numeric inequalities (and we were lucky that they could be chosen with integer end-points). It is also possible [CR88] to describe the roots in terms of the signs of the derivatives of f, i.e.
$$\begin{gathered}\{f>0\wedge x<-2\}\cup\{f=0\wedge f'<0\wedge f'''<0\}\cup\{f<0\wedge x<0\}\cup\\ \{f=0\wedge f'>0\wedge f'''<0\}\cup\{f>0\wedge x>-2\wedge x<2\}\cup\\ \{f=0\wedge f'<0\wedge f'''>0\}\cup\{f<0\wedge x>0\}\cup\\ \{f=0\wedge f'>0\wedge f'''>0\}\cup\{f>0\wedge x>2\}\end{gathered}$$

(as it happens, the sign of f'' is irrelevant here). This methodology can also be applied to the one-dimensional regions, e.g. the first can also be defined as $\{f>0\wedge f'>0\wedge f''<0\wedge f'''<0\}$.

We may ask how we know that we have a decomposition, and where these extra constraints (such as $x>0$ in example 5 or $x<\frac{3}{2}$ in example 6) come from. This will be addressed in the next section, but the brief answers are:

- we know something is a decomposition because we have constructed it that way;
- $x=0$ came from the leading coefficient (with respect to y) of $xy-1$, whereas $\frac{3}{2}$ in example 6 is a root of $\mathrm{Disc}_y(f)$.

[32]And equation (3.11) demonstrates that we probably wouldn't want to even when we could!

We stated in definition 79 that the cells must be non-empty. How do we know this? For the zero-dimensional cells $\{f = 0 \wedge x > a \wedge x < b\}$, we can rely on the fact that if f changes sign between a and b, there must be at least one zero, and if f' does not[33], there cannot be more than one: such an interval can be called an *isolating interval*. In general, we are interested in the following concept.

Definition 80 *A* sampled algebraic decomposition *of $\mathbf{R}^n$ is an algebraic decomposition together with, for each cell C, an explicit point* Sample(C) *in that cell.*

By 'explicit point' we mean a point each of whose coordinates is either a rational number, or a precise algebraic number: i.e. a defining polynomial[34] together with an indication of which root is meant, an isolating interval, a sufficiently exact[35] numerical approximation or a Thom's Lemma [CR88] list of signs of derivatives.

Definition 81 *A decomposition D of $\mathbf{R}^n$ is said to be sign-invariant for a polynomial $p(x_1, \ldots, x_n)$ if and if only if, for each cell $C \in D$, precisely one of the following is true:*

1. $\forall \mathbf{x} \in C \quad p(\mathbf{x}) > 0$;
2. $\forall \mathbf{x} \in C \quad p(\mathbf{x}) < 0$;
3. $\forall \mathbf{x} \in C \quad p(\mathbf{x}) = 0$;

It is sign-invariant for a set of polynomials if, and only if, for each polynomial, one of the above conditions is true for each cell.

It therefore follows that, for a sampled decomposition, the sign throughout the cell is that at the sample point. A stronger concept is provided by the following definition.

Definition 82 *A decomposition D of $\mathbf{R}^n$ is said to be order-invariant for a polynomial $p(x_1, \ldots, x_n)$ if and if only if, for each cell $C \in D$, precisely one of the following is true:*

1. $\forall \mathbf{x} \in C \quad p(\mathbf{x}) > 0$; *and*
2. $\forall \mathbf{x} \in C \quad p(\mathbf{x}) < 0$;
3. $\exists k \in \mathbf{N}$ *such that:*

(3a) $\forall \mathbf{x} \in C$ *all derivatives of p of order at most k vanish at $\mathbf{x}$;* **and**

[33]Which will involve looking at f'' and so on.

[34]Not necessarily irreducible, though it is normal to insist that it be square-free.

[35]By this, we mean an approximation such that the root cannot be confused with any other, which generally means at least an approximation close enough that Newton's iteration will converge to the indicated root. Maple's `RootOf` supports such a concept.

(3b) $\forall \mathbf{x} \in C$ *there exists a derivative* $\frac{\partial^{k+1} p}{\partial x_{i_1} \ldots \partial x_{i_{k+1}}}$ *which does not vanish at* $\mathbf{x}$

- *note that it may be different order $k+1$-derivatives at different points of C.*

It is order-invariant for a set of polynomials if, and only if, for each polynomial, one of the above conditions is true for each cell. Order-invariance is a strictly stronger concept than sign-invariance.

3.5.5 Cylindrical Algebraic Decomposition

The idea of Cylindrical Algebraic Decomposition is due to [Col75]. The presentation here is more general (see Observation 6), and largely unpublished.

Notation 24 *Let $n > m$ be positive natural numbers, and let $\mathbf{R}^n$ have coordinates $x_1, \ldots, x_n$, with $\mathbf{R}^m$ having coordinates $x_1, \ldots, x_m$.*

Definition 83 *An algebraic decomposition D of $\mathbf{R}^n$ is said to be* cylindrical *over a decomposition D' of $\mathbf{R}^m$ if the projection onto $\mathbf{R}^m$ of every cell of D is a cell of D'. The cells of D which project to $C \in D'$ are said to form the* cylinder *over C, denoted* $\mathrm{Cyl}(C)$. *For a sampled algebraic decomposition, we also insist that the sample point in C be the projection of the sample points of all the cells in the cylinder over C.* This definition is usually stated when $m = n-1$, but the greater generality is theoretically worth having, even though we only currently know how to compute these when $m = n-1$.

Cylindricity is by no means trivial.

Example 13 *Consider the decomposition of $\mathbf{R}^2 = S_1 \cup S_2 \cup S_3$ where*

$$\begin{aligned} S_1 &= \{(x,y) \mid x^2+y^2-1 > 0\}, \\ S_2 &= \{(x,y) \mid x^2+y^2-1 < 0\}, \\ S_3 &= \{(x,y) \mid x^2+y^2-1 = 0\}. \end{aligned}$$

This is an algebraic decomposition, and is sign-invariant for x^2+y^2-1. However, it is not cylindrical over any decomposition of the x-axis $\mathbf{R}^1$. The projection of S_2 is $(-1,1)$, so we need to decompose $\mathbf{R}^1$ as

$$(-\infty,-1) \cup \{-1\} \cup (-1,1) \cup \{1\} \cup (1,\infty). \tag{3.66}$$

S_3 projects onto $[-1,1]$, which is the union of three sets in (3.66). We have to decompose S_3 into four sets:

$$\begin{aligned} S_{3,1} &= \{(-1,0)\}, \quad S_{3,2} = \{(1,0)\}, \\ S_{3,3} &= \{(x,y) \mid x^2+y^2-1 = 0 \wedge y > 0\}, \\ S_{3,4} &= \{(x,y) \mid x^2+y^2-1 = 0 \wedge y < 0\}. \end{aligned}$$

S_1 splits into eight sets, one above each of $(-\infty,-1)$ and $(1,\infty)$ and two above each of the other components of (3.66). It is obvious that this is the minimal

refinement of the original decomposition to possess a cylindric decomposition. Furthermore in this case no linear transformation of the axes can reduce this. If we wanted a sampled decomposition, we could choose x-coordinates of -2, -1, 0, 1 and 2, and y-coordinates to match, from $\{0, \pm 1, \pm 2\}$.

Cylindricity is fundamental to solving problem 3 via the following two propositions.

Proposition 49 *Let*

$$\exists x_n \ldots \exists x_{m+1} P(x_1, \ldots, x_n) \tag{3.67}$$

be an existentially quantified formula, D be a sampled algebraic decomposition of $\mathbf{R}^n$ which is sign-invariant for all the polynomials occurring in P, and D' be a sampled algebraic decomposition of $\mathbf{R}^m$ such that D is cylindrical over D'. Then a quantifier-free form of (3.67) is

$$\bigvee_{C' \in D' : \exists C \in \mathrm{Cyl}(C') P(\mathrm{Sample}(C))} \mathrm{Def}(C'). \tag{3.68}$$

Proposition 50 *Let*

$$\forall x_n \ldots \forall x_{m+1} P(x_1, \ldots, x_n) \tag{3.69}$$

be a universally quantified formula, D be a sampled algebraic decomposition of $\mathbf{R}^n$ which is sign-invariant for all the polynomials occurring in P, and D' be a sampled algebraic decomposition of $\mathbf{R}^m$ such that D is cylindrical over D'. Then a quantifier-free form of (3.69) is

$$\bigvee_{C' \in D' : \forall C \in \mathrm{Cyl}(C') P(\mathrm{Sample}(C))} \mathrm{Def}(C'). \tag{3.70}$$

These two propositions lead to a solution of problem 3.

Theorem 32 ([Col75]) *Let $\mathbf{x}_0, \ldots, \mathbf{x}_k$ be sets of variables, with $\mathbf{x}_i = (x_{i,1}, \ldots, x_{i,n_i})$, and let $N_i = \sum_{j=0}^{i} n_j$. Let $P(\mathbf{x}_0, \ldots, \mathbf{x}_k)$ be a semi-algebraic proposition, D_i be an algebraic decomposition of $\mathbf{R}^{N_i}$ such that each D_i is cylindric over D_{i-1} and D_k is sign-invariant for all the polynomials in P. Then a quantifier-free form of*

$$Q_k \mathbf{x}_k \ldots Q_1 \mathbf{x}_1 P(\mathbf{x}_0, \ldots, \mathbf{x}_k) \tag{3.71}$$

(where the Q_i are $\forall$ or $\exists$) is

$$\bigvee_{C' \in D_0 \forall\exists C \in \mathrm{Cyl}^k(C') P(\mathrm{Sample}(C))} \mathrm{Def}(C'), \tag{3.72}$$

where by $\forall\exists$ we mean that we are quantifying across the coordinates of $\mathrm{Sample}(C)$ *according to the quantifiers in (3.71).*

We can use the (sampled) cylindrical algebraic decomposition in example 13 to answer various questions.

Example 14 $\forall y \quad x^2 + y^2 - 1 > 0$. *For the sampled cells* $\langle(-\infty, -1), (x = -2, y = 0)\rangle$ *and* $\langle(1, \infty), (x = 2, y = 0)\rangle$, *the proposition is true at the sample points, hence true everywhere in the cell. For all the other cells in (3.66), there is a sample point for which it is false (in fact,* $y = 0$ *always works). So the answer is* $(-\infty, -1) \cup (1, \infty)$.

Example 15 $\exists y \quad x^2 + y^2 - 1 > 0$. *For every cell in (3.66), there is a sample point above it for which the proposition is true, hence we deduce that the answer is (3.66), which can be simplified to* **true**.

We should note (and this is both one of the strengths and weaknesses of this approach) that the same cylindrical algebraic decomposition can be used to answer all questions of this form with the same order of (blocks of) quantified variables, irrespective of what the quantifiers actually are.

Example 16 $(\exists y \quad x^2 + y^2 - 1 > 0) \wedge (\exists y \quad x^2 + y^2 - 1 < 0)$. *This formula is not directly amenable to this approach, since it is not in prenex form. In prenex form, it is* $\exists y_1 \exists y_2 \left((x^2 + y_1^2 - 1 > 0) \wedge (x^2 + y_2^2 - 1 < 0)\right)$ *and we need an analogous*[36] *decomposition of* $\mathbf{R}^3$ *cylindric over* $\mathbf{R}^1$. *Fortunately, (3.66) suffices for our decomposition of* $\mathbf{R}^1$, *and the answer is* $(-1 < x < 1)$, *shown by the sample point* $(x = 0, y_1 = 2, y_2 = 0)$, *and by the fact that at other sample points of* $\mathbf{R}^1$, *we do not have* y_1, y_2 *satisfying the conditions.*

We should note that it is *not* legitimate to reduce the formula to

$$\exists y \left((x^2 + y^2 - 1 > 0) \wedge (x^2 + y^2 - 1 < 0)\right),$$

since that is trivially false. It is, however, legitimate to solve two separate problems, $\exists y \quad x^2+y^2-1 > 0$ (which is true, with $y = 2$) and $\exists y \quad x^2+y^2-1 < 0$ (which is $-1 < x < 1$), and combine to get $-1 < x < 1$.

Open Problem 13 (RAG Formulation 1) *It is pretty obviously simpler to solve two two-dimensional problem than one three-dimensional one, but in general the correct translation of an arbitrary statement such as that of Example 16 into the most efficient problem formulation is a hard one.*

However, it could be argued that all we have done is reduce problem 3 to the following one.

Problem 4 *Given a quantified semi-algebraic proposition as in theorem 32, produce a sign-invariant decomposition* D_k *cylindrical over the appropriate* D_i *such that theorem 32 is applicable.* Furthermore, since theorem 32 only talks about “a” quantifier-free form, we would like the simplest possible such D_k (see [Laz88]).

[36]Easier said than done. Above $x = -1$ we have nine cells:$\{y_1 < 0, y_1 = 0, y_1 > 0\} \times \{y_2 < 0, y_2 = 0, y_2 > 0\}$, and the same for $x = 1$, whereas above $(-1, 1)$ we have 25, totalling 45.

Figure 3.7: Cylindrical Decomposition after Collins

$$\begin{tikzcd}
S_n \arrow[d, "\mathcal{P}\mathrm{roj}"'] & \subset & \mathbf{R}[x_1,\ldots,x_n] & \mathbf{R}^n \arrow[d] & & \mathbf{R}^n & \text{decomposed } \mathcal{I} \text{ by} & D_n \\
S_{n-1} \arrow[d, "\mathcal{P}\mathrm{roj}"'] & \subset & \mathbf{R}[x_1,\ldots,x_{n-1}] & \mathbf{R}^{n-1} \arrow[d] & & \mathbf{R}^{n-1} \arrow[u, "\mathcal{L}\mathrm{ift}"] & \text{decomposed } \mathcal{I} \text{ by} & D_{n-1} \arrow[u, phantom, "\mathrm{Cyl}"] \\
\cdots & & \cdots & \cdots \arrow[d] & & \cdots \arrow[u, "\mathcal{L}\mathrm{ift}"] & & \cdots \arrow[u, phantom, "\mathrm{Cyl}"] \\
S_2 \arrow[d, "\mathcal{P}\mathrm{roj}"'] & \subset & \mathbf{R}[x_1,x_2] & \mathbf{R}^2 \arrow[d] & & \mathbf{R}^2 & \text{decomposed } \mathcal{I} \text{ by} & D_2 \\
S_1 & \subset & \mathbf{R}[x_1] & \mathbf{R}^1 \arrow[rr, "\underbrace{\hphantom{xx}}_{\text{Problem 1}}"'] & & \mathbf{R}^1 \arrow[u, "\mathcal{L}\mathrm{ift}"] & \text{decomposed } \mathcal{I} \text{ by} & D_1. \arrow[u, phantom, "\mathrm{Cyl}"]
\end{tikzcd}$$

$\mathcal{P}$roj The projection operator we wish to use.

$\mathcal{L}$ift The corresponding lifting operator.

$\mathcal{I}$ The necessary invariant of the decomposition and polynomials that we wish to preserve.

There is no common term for such a decomposition: we will call it *block-cylindrical*.

Observation 6 *Collins' original presentation of Cylindrical Algebraic Decomposition insisted that every decomposition of $\mathbf{R}^m$, generated by $x_1,\ldots,x_m$ was cylindrical, in the sense of Definition 83, over every $\mathbf{R}^k$ generated by $x_1,\ldots,x_k$.*

Currently, this is the only sort of decomposition we know how to compute, but, since the weaker form of block-cylindrical is all we need for quantifier elimination, we have given the more general definition.

3.5.6 Computing Algebraic Decompositions

Though many improvements have been made to it since, the basic strategy for computing algebraic decompositions is still generally[37] that due to Collins [Col75], and is to compute them cylindrically, as illustrated in the Figure 3.7. From the original proposition, we extract the set of polynomials S_n. We then project this set into S_{n-1} in $n-1$ variables, and so on, until we have a set of univariates S_1. We then isolate, or otherwise describe, the roots of these polynomials, as described in problem 1, to produce a decomposition D_1 of $\mathbf{R}^1$, and then successively lift this to a decomposition D_2 of $\mathbf{R}^2$ and so on, each D_i, and the polynomials at that level, satisfying the invariant $\mathcal{I}$ and cylindrical over D_{i-1}.For Collins, $\mathcal{I}_C$ was the D_k being sign-invariant for S_k and the S_{k+1} being delineable over D_k.

[37]But [CMXY09] have an alternative strategy based on triangular decompositions, as in section 3.4, which may well turn out to have advantages. There are also techniques based on Comprehensive Gröbner Bases (Definition 63) [FIS15].

Definition 84 (Delineable: [Col75, p. 139]) *We say that a set $S_{k+1} \subset \mathbf{R}[x_1,\ldots,x_{k+1}]$ of polynomials (with $A = \prod_{f\in S_{k+1}} f$) is* delineable *over a set $D_k \subset \mathbf{R}^k$ if there are functions $f_1,\ldots,f_m : D_k \to \mathbf{C}$ such that:*

1. *$f_1,\ldots,f_m$ are continuous functions $D_k \to \mathbf{C}$;*
2. *for all $i : 1 \le i \le m$ there is a positive integer e_i such that $f_i(a_1,\ldots,a_k)$ is a root of $A(a_1,\ldots,a_k,x)$ of multiplicity precisely e_i for all $(a_1,\ldots,a_k) \subset D_k$;*
3. *If $(a_1,\ldots,a_k) \in D_k$, $b \in \mathbf{C}$ and $A(a_1,\ldots,a_k,b) = 0$ then for some i, $b = f_i(a_1,\ldots,a_k)$;*
4. *for some $\ell : 0 \le \ell \le m$, $f_1,\ldots f_\ell$ are real-valued with $f_1 < f_2 < \cdots f_\ell$ and $f_{\ell+1},\ldots,f_m$ are all non-real.*

Essentially, $f_1,\ldots,f_\ell$ define the real branches of $A(a_1,\ldots,a_k,b) = 0$ in $D_k \times \mathbf{C}$ and $f_{\ell+1},\ldots,f_m$ define the non-real branches, the real branches have constant multiplicity and do not touch.

Note that the projection from S_{i+1} to S_i must be such that a decomposition D_i satisfying invariant $\mathcal{I}$ can be lifted to a decomposition D_{i+1} satisfying invariant $\mathcal{I}$. Note also that the decomposition thus produced will be block-cylindric for every possible blocking of the variables, since it is block-cylindric for the finest such.

Projection turns out to be a trickier problem than might be expected. One's immediate thought is that one needs the discriminants (with respect to the variable being projected) of all the polynomials in S_{i+1}, since this will give all the critical points where the number of real roots of a given polynomial changes. Then one sees that one needs the resultants of all pairs of such polynomials, since this is where they intersect. Example 5 (page 143) shows that one might need leading coefficients. Then there are issues of what happens when leading coefficients vanish. This led Collins [Col75] to consider the following projection operation $\mathcal{P}\nabla\wr|_C$ for a set $\mathcal{A}$ of polynomials in $x_1,\ldots,x_n$, where x_n is the variable being projected.

Notation 25 *Define the following sets, where we assume $\mathcal{A}$ has k polynomials of degree d.*

$\mathcal{B}$ is the set of non-constant (with respect to x_n) iterated reducta (Notation 13) of all elements of $\mathcal{A}$: $\le kd$ elements.

$\mathcal{L}$ is the set of all leading coefficients of $\mathcal{B}$, which is the same as saying all the coefficients of non-constant terms of $\mathcal{A}$: $\le kd$ elements.

$\mathcal{S}_1$ is the set of all principal subresultant coefficients (Definition 112) $\mathrm{psc}_j(g, \frac{\partial g}{\partial x_n})$ for all $g \in \mathcal{B}$: $\le kd^2$ elements.

$\mathcal{S}_2$ is the set of all principal subresultant coefficients $\mathrm{psc}_j(g_1, g_2)$ for all $g_1, g_2 \in \mathcal{B} : g_1 \ne g_2$: $\le k^2d^3$ elements.

$\mathcal{P}\nabla\wr|_C(\mathcal{A})$ is $\mathcal{L}\cup\mathcal{S}_1\cup\mathcal{S}_2$: $O(k^2d^3)$ elements.

[Col75, essentially Theorem 5] showed that this projection operator is sufficient for Figure 3.7 to be valid: more precisely the following.

Theorem 33 ([Col75]) *If $\mathcal{A}$ is a set of polynomials in $x_1,\ldots,x_n]$ and D is a cylindrical decomposition of $\mathbf{R}^{n-1}$ sign-invariant for $\mathcal{P}\nabla\wr|_C(\mathcal{A})$, then the polynomials of $\mathcal{A}$ are delineable over every cell of D and D can be lifted to a cylindrical decomposition of $\mathbf{R}^n$ sign-invariant for $\mathcal{A}$.*

The problem with this is the size of $\mathcal{P}\nabla\wr|_C(\mathcal{A})$. McCallum [McC84, McC88] saw that one could do better if one used order-invariance (Definition 82) instead.

Notation 26 *Let $\mathcal{B}$ be a square-free basis for the primitive parts of $\mathcal{A}$. Define the following sets, where we assume $\mathcal{A}$ has k polynomials of degree d.*

$\mathcal{C}$ The set of all contents of $\mathcal{A}$.

$\mathcal{B}$ A square-free basis for the primitive parts of $\mathcal{A}$.

$\mathcal{C}_\{$ The set of all coefficients of $\mathcal{B}$.

$\mathcal{D}$ The set of all discriminants of $\mathcal{B}$.

$\mathcal{R}$ The set of all resultants of $\mathcal{B}$, i.e. $\{\mathrm{Res}_{x_n}(B_i,B_j) : 1\le i<j\le|\mathcal{B}|\}$

$\mathcal{P}\nabla\wr|_M(\mathcal{A})$ is $\mathcal{C}\cup\mathcal{C}_\{\cup\mathcal{D}\cup\mathcal{R}$.

It might seem relatively hard to say how big $\mathcal{P}\nabla\wr|_M(\mathcal{A})$ is, since the process of square-free decomposition *might* not decrease d, but *might* increase k.

Definition 85 ([McC84]) *We say that a set of polynomials has the (m,d) property if it can be partitioned into m sets of polynomials such that the product of the elements of any one set has degree at most d (in each variable taken separately.*

Proposition 51 *If $\mathcal{A}$ has the (m,d) property, then so does a square-free basis for it, the set of all discriminants has the $(m,2d^2)$ property, and the set of all resultants ($\mathcal{R}$ above) has the $(\frac{m(m+1)}{2},2d^2)$ property.*

In these terms, there is a good bound for $\mathcal{P}\nabla\wr|_M(\mathcal{A})$.

Lemma 6 ([BDE+14, Lemma 11][38]) *If $\mathcal{A}$ has the (m,d) property, then $\mathcal{P}\nabla\wr|_M(\mathcal{A})$ has the $\left(\left\lfloor\frac{(m+1)^2}{2}\right\rfloor,2d^2\right)$ property.*

Theorem 34 ([McC84, McC88]) *If $\mathcal{A}$ is a set of polynomials in $x_1,\ldots,x_n]$ and D is a cylindrical decomposition of $\mathbf{R}^{n-1}$ order-invariant for $\mathcal{P}\nabla\wr|_M(\mathcal{A})$, then the polynomials of $\mathcal{A}$ are analytic-delineable over every cell of D* on which they do not vanish identically *and D can be lifted to a cylindrical decomposition of $\mathbf{R}^n$ order-invariant for $\mathcal{A}'$: those elements of $\mathcal{A}$ that do not vanish identically on some cell of D.* "Analytic-delineable" is a slightly stronger form of "delineable" (Definition 84: the details do not concern us here.

The condition about elements of $\mathcal{A}$ not vanishing identically is known as stating that $\mathcal{A}$ is *well-oriented*. Hence if $\mathcal{A}$ and all the $\mathcal{P}\nabla\wr|_M(\mathcal{A})$, $\mathcal{P}\nabla\wr|_M(\mathcal{P}\nabla\wr|_M(\mathcal{A}))$, ... are well-oriented, then Theorem 34 means that we can produce an order-invariant decomposition of $\mathbf{R}^n$ for $\mathcal{A}$. Note that, although "order-invariant" is a stronger condition than "sign-invariant", the fact that $\mathcal{P}\nabla\wr|_M$ is much smaller than $\mathcal{P}\nabla\wr|_C$ means that we have a much more efficient algorithm *when it works*, i.e. when it does not detect the failure of well-orientedness.

What do we do when it does fail? In the notation of Figure 3.7, McCallum suggests that, if we detect that a polynomial $f_k \in S_i$ vanishes identically over some cell of D_{i-1}, we should augment S_i with all the partial derivatives $\frac{\partial f_k}{\partial x_j}$ $(1 \le j \le i)$ and project S_i again. To the best of the author's knowledge, this has never been implemented due to the control-flow complexities it introduces, and the usual solution is to give up and use $\mathcal{P}\nabla\wr|_C$ (or a variant due to [Hon90]).

Lazard [Laz94] suggested a further improvement to the projection operator, $\mathcal{P}\nabla\wr|_L$, which is obtained from $\mathcal{P}\nabla\wr|_m$ (Notation 26) by replacing the set $\mathcal{C}_\{$ of all coefficients by just the leading and trailing coefficients, and claimed that this was unconditionally correct, but for an invariant $\mathcal{I}_L$ strictly stronger than $\mathcal{I}_C$ (and incomparable with $\mathcal{I}_M$), and on condition that one had a different lifting procedure $\mathcal{L}\mathrm{ift}_L$.

TO BE COMPLETED

3.5.7 Describing Solutions

If describing the roots of a general polynomial in one variable was tricky (Section 3.5.5), it is more so in two or more dimensions: not that the mathematics doesn't exist, but rather that we are, in general, unused to dealing with it. Consider the example $f := y^3 - 7\,y^2 + 14\,y - x - 8$ in Figure 3.8. Above each value of x, y is given by a univariate polynomial, whose roots can be described is terms of the signs of the derivatives, by Thom's Lemma (Lemma 4). The top and bottom branches of the curve are defined by $f = 0, f' > 0$, but are distinguished by the signs of f''. Similarly, the two inflection points are defined by $f = f' = 0$, but are again distinguished by the signs of f''. However, the middle branch, $f = 0$, $f' < 0$, is unhelpfully split by $f'' = 0$ into two parts. We could (though doing so algorithmically has never, to the author's knowledge, been solved) remove this distinction and just define the branch as $f = 0$, $f' < 0$. Since the Collins projection $\mathcal{P}\nabla\wr|_C$ includes all the derivatives, it is capable of expressing the solutions in the Thom's Lemma manner.

Conversely, and perhaps more naturally, we can try counting the branches, just as we might count the roots of a univariate polynomial (starting from ∞). This is shown in Figure 3.9. Here the top branch is split: it starts off being the third root of f, but as we cross the (x-value of the) inflection point of the *other* branch, it becomes the first root (passing momentarily through the pont at which it is the second root). Unlike the previous description, this split cannot be removed. Note also that "the first root" is not continuous, jumping from the bottom branch to the top branch at this x-value.

Figure 3.8: $y^3 - 7\,y^2 + 14\,y - x - 8$: Thom' Lemma

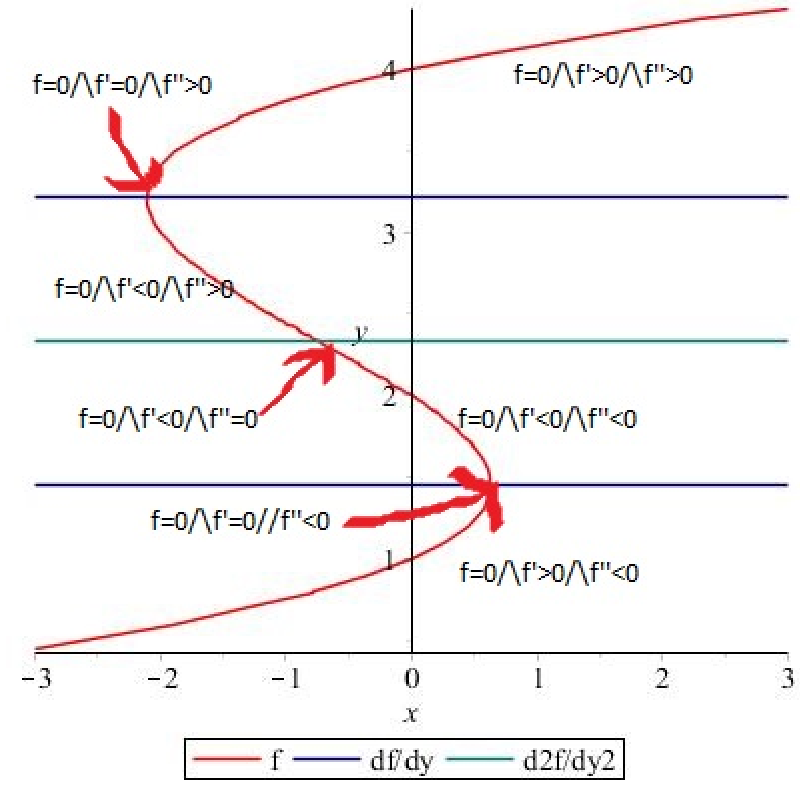
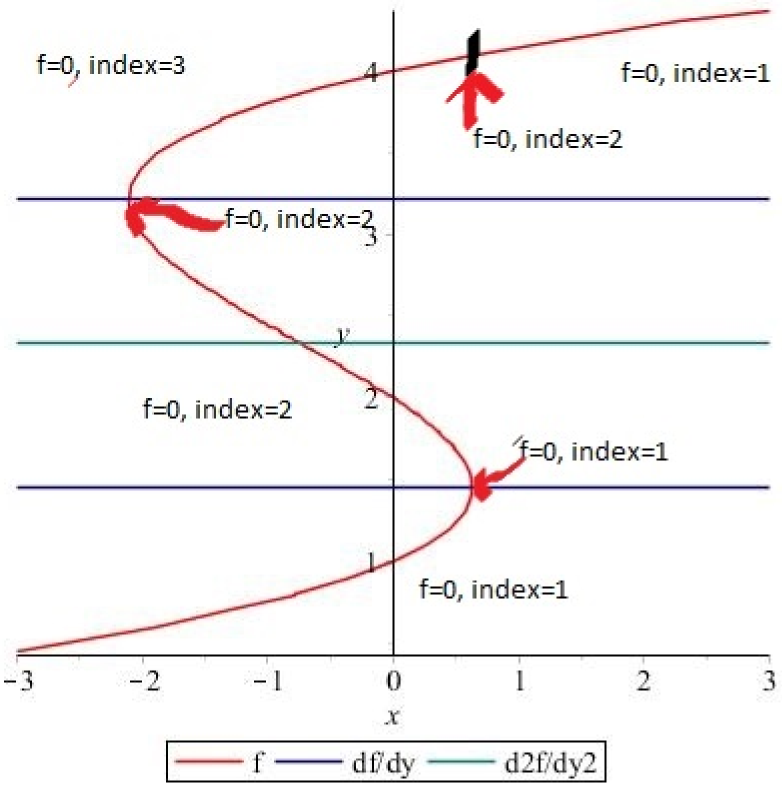


Figure 3.9: $y^3 - 7\,y^2 + 14\,y - x - 8$: indexing



Here is an edited version[39] of the output from Maple on this problem.

$$\left\{\begin{array}{ll}
\left\{\begin{array}{ll}
[RC,[[-4,-4],[-1,-1]]] & y<R\left(f_1,i_1\right)\\
[RC,[[-4,-4],[1/4,1/2]]] & y=R\left(f_1,i_1\right)\\
[RC,[[-4,-4],[2,2]]] & R\left(f_1,i_1\right)<y
\end{array}\right. & x<R\left(f_3,i_1\right)\\
\left\{\begin{array}{ll}
[RC,[[-\frac{271}{128},-\frac{135}{64}],[-1,-1]]] & y<R\left(f_2,i_1\right)\\
[RC,[[-\frac{271}{128},-\frac{135}{64}],[\frac{1}{2},\frac{5}{8}]]] & y=R\left(f_2,i_1\right)\\
[RC,[[-\frac{271}{128},-\frac{135}{64}],[\frac{15}{8},\frac{15}{8}]]] & \wedge\left(\begin{array}{c} R\left(f_2,i_1\right)<y,\\ y<R\left(f_2,i_2\right)\end{array}\right)\\
[RC,[[-\frac{271}{128},-\frac{135}{64}],[\frac{25}{8},\frac{13}{4}]]] & y=R\left(f_2,i_2\right)\\
[RC,[[-\frac{271}{128},-\frac{135}{64}],[5,5]]] & R\left(f_2,i_2\right)<y
\end{array}\right. & x=R\left(f_3,i_1\right)\\
\left\{\begin{array}{ll}
[RC,[[-\frac{95}{128},-\frac{95}{128}],[-1,-1]]] & y<R\left(f_1,i_1\right)\\
[RC,[[-\frac{95}{128},-\frac{95}{128}],[3/4,1]]] & y=R\left(f_1,i_1\right)\\
[RC,[[-\frac{95}{128},-\frac{95}{128}],[\frac{13}{8},\frac{13}{8}]]] & \wedge\left(\begin{array}{c} R\left(f_1,i_1\right)<y,\\ y<R\left(f_1,i_2\right)\end{array}\right)\\
[RC,[[-\frac{95}{128},-\frac{95}{128}],[\frac{9}{4},\frac{5}{2}]]] & y=R\left(f_1,i_2\right)\\
[RC,[[-\frac{95}{128},-\frac{95}{128}],[\frac{25}{8},\frac{25}{8}]]] & \wedge\left(\begin{array}{c} R\left(f_1,i_2\right)<y,\\ y<R\left(f_1,i_3\right)\end{array}\right)\\
[RC,[[-\frac{95}{128},-\frac{95}{128}],[\frac{15}{4},4]]] & y=R\left(f_1,i_3\right)\\
[RC,[[-\frac{95}{128},-\frac{95}{128}],[5,5]]] & R\left(f_1,i_3\right)<y
\end{array}\right. & \wedge\left(\begin{array}{c} R\left(f_3,i_1\right)<x,\\ x<R\left(f_3,i_2\right)\end{array}\right)\\
\left\{\begin{array}{ll}
[RC,[[5/8,\frac{81}{128}],[0,0]]] & y<R\left(f_2,i_1\right)\\
[RC,[[5/8,\frac{81}{128}],[\frac{11}{8},3/2]]] & y=R\left(f_2,i_1\right)\\
[RC,[[5/8,\frac{81}{128}],[\frac{11}{4},\frac{11}{4}]]] & \wedge\left(\begin{array}{c} R\left(f_2,i_1\right)<y,\\ y<R\left(f_2,i_2\right)\end{array}\right)\\
[RC,[[5/8,\frac{81}{128}],[4,\frac{33}{8}]]] & y=R\left(f_2,i_2\right)\\
[RC,[[5/8,\frac{81}{128}],[6,6]]] & R\left(f_2,i_2\right)<y
\end{array}\right. & x=R\left(f_3,i_2\right)\\
\left\{\begin{array}{ll}
[RC,[[2,2],[3,3]]] & y<R\left(f_1,i_1\right)\\
[RC,[[2,2],[\frac{17}{4},9/2]]] & y=R\left(f_1,i_1\right)\\
[RC,[[2,2],[6,6]]] & R\left(f_1,i_1\right)<y
\end{array}\right. & R\left(f_3,i_2\right)<x
\end{array}\right.$$

$$f_1 := \_Z^3 - 7\,\_Z^2 + 14\,\_Z - x - 8;$$
$$f_2 := 14\,\_Z^2 + (-9\,x - 72)\,\_Z + 21\,x + 70$$
$$f_3 := 27\,\_Z^2 + 40\,\_Z - 36$$

This splits $\mathbf{R}^2$ into 23 cells, which is the minimum for any *cylindrical* decomposition with y projected first. In the other direction, we just have three cells: "below the curve", "on the curve" and "above the curve". This sort of variation depending on projection order is not unusual: [BD07] have examples in n variables where the number of cells is constant for some projections, and $2^{2^{O(n))}}$ for others. The variation is not universal though: there are also examples with $2^{2^{O(n))}}$ cells for all projection orders.

The drawback of this representation is that "the ith branch of" is not semi-algebraic in the sense of Definition 77, and we need to convert to the other in order to stay within the semi-algebraic language. One might think that

[39]Common polynomials have been pulled out, and various abbreviations, such as $R(f,i_1)$ for $\mathrm{RootOf}(f, \mathit{index} = \mathit{real}_1)$, have been used.

this involved adding derivatives in order to apply Thom's Lemma (Lemma 4), and that these derivatives wouldneed to enter into the projection phase, thus moving us a long way back to the original Collins projection from the more efficient current ones, but fortunately this is not the case.

3.5.8 Complexity

Let us suppose that there are s polynomials involved in the input formula (3.62), of maximal degree d. Then such a cylindrical algebraic decomposition can be computed in time $O\left((sd)^{2^{O(k))}}\right)$.

There are examples [BD07, DH88], which shows that this behaviour is best-possible, indeed the projection onto $\mathbf{R}^1$ might have a number of components doubly-exponential in k. This is true even for [BD07], where the polynomials are "only" linear.

While this behaviour is intrinsic to cylindrical algebraic decomposition, it is not necessarily intrinsic to quantifier elimination as such. If a is the number of *alternations* of quantifiers (Notation 23) in the problem (so $a < k$), then there are algorithms [Bas99, for example] whose behaviour is singly-exponential in k but doubly-exponential in a; typically $O\left((sd)^{O(k^2)2^{O(a))}}\right)$. The construction of [BD07, DH88] makes extensive use of $\vee$ symbols, and this is essential, as [RESW14] have shown that, with no $\vee$ symbols in (3.62), the linear problem can be solved in *polynomial* time, essentially by a series of reductions to linear programming, which can be solved in polynomial time [Kha79, Kar84]. See also Open Problem 15.

One particular special case is that of *no* alternations. Hence, using the fact that $\exists \mathbf{x}\,(P(\mathbf{x}) \vee Q(\mathbf{x}))$ is equivalent to $(\exists \mathbf{x} P(\mathbf{x})) \vee (\exists \mathbf{x} Q(\mathbf{x}))$, an existential problem is equivalent to a set[40] of problems of the form

$$\exists \mathbf{x} \left(\bigwedge_{f_i \in F} f_i(\mathbf{x}) \geq 0 \right) \wedge \left(\bigwedge_{g_i \in G} g_i(\mathbf{x}) = 0 \right) \wedge \left(\bigwedge_{h_i \in H} h_i(\mathbf{x}) \neq 0 \right). \tag{3.73}$$

This is generally referred to as the *existential theory of the reals*. Since the truth of a universal problem is equivalent to the falsity of an existential problem $(\forall \mathbf{x} P(\mathbf{x}) \Leftrightarrow \neg\exists \mathbf{x} \neg P(\mathbf{x}))$, this is all we need to consider.

Given a problem (3.73), cylindrical algebraic decomposition will yield such an $\mathbf{x}$, if one exists, and failure to yield one is a proof that no such $\mathbf{x}$ exists. However, this is a somewhat unsatisfactory state of affairs in practice, since, computationally, we are relying not just on the correctness of the *theory* of cylindrical algebraic decomposition, but also on the absence of bugs in the implementation.

An alternative is provided by the Positivstellensatz approach [Ste74].

[40]There may be singly-exponential blow-up here as we convert into disjunctive normal form, but this is small compared to the other exponential issues in play!

Theorem 35 ([PQR09, Theorem 3]) *The set of solutions to (3.73) is empty if, and only if, there are:*

$s \in \mathrm{con}(F)$ *where* $\mathrm{con}(F)$*, the* cone *of F, is the smallest set generated by F and the set of squares of all elements of* $\mathbf{R}[\mathbf{x}]$ *wich is closed under multiplication and addition;*

$g \in (G)$ *the ideal generated by G;*

$m \in \mathrm{mon}(H)$ *where* $\mathrm{mon}(H)$*, the (multiplicative) monoid of H is the set of all products (inding $1 =$ the empty product) of elements of H;*

such that $s + g + m^2 = 0$. Furthermore, there is an algorithm to find such s, g and m (if they exist) in $\mathbf{Q}[\mathbf{x}]$ *provided F, G and $H \subset$* $\mathbf{Q}[\mathbf{x}]$.

Partial Proof. If $s + g + m^2 = 0$ but $\mathbf{x}$ is a solution to (3.73), then $s(\mathbf{x}) + g(\mathbf{x}) + m(\mathbf{x})^2$ is of the form "non-negative + zero + strictly positive", so cannot be zero.

We can think of (s, g, m) as a witness to the emptiness of the set of solutions to (3.73). Again, failure to find such an (s, g, m) is a proof of the existence of solutions *provided* we trust the correctness of Theorem 35 *and* the correctness of the implementation.

3.5.9 Further Observations

1. The methodology outlined in figure 3.7, and indeed that of [CMXY09], has the pragmatic drawback that the decomposition computed, as well as solving the given problem (3.62), solves all other problems of the same form with the same polynomials and the variables in the same order. For example, a decomposition which allows us to write down a quantifier-free equivalent of

$$\forall x_4 \exists x_3 p(x_1, x_2, x_3, x_4) > 0 \wedge q(x_1, x_2, x_3, x_4) < 0 \tag{3.74}$$

will also solve

$$\forall x_4 \exists x_3 p(x_1, x_2, x_3, x_4) > 0 \wedge q(x_1, x_2, x_3, x_4) < 0 \tag{3.75}$$

and even

$$\exists x_4 \exists x_3 \forall x_2 p(x_1, x_2, x_3, x_4) < 0 \wedge q(x_1, x_2, x_3, x_4) \geq 0 \tag{3.76}$$

The process of *Partial* Cylindrical Algebraic Decomposition [CH91] can make the lifting process (right hand side $\uparrow$ in Figure 3.7) more efficient, but still doesn't take full account of the structure of the incoming quantified formula.

2. If F is of the form $p(y_1, \ldots, y_m, x_1, \ldots, x_n) = 0 \wedge \hat{\phi}(y_1, \ldots, y_m, x_1, \ldots, x_n)$, or can be transformed into this form, then, as observed in [Col98] we are

only concerned in Theorem 32 with those cells on which p is zero, and the polynomials in $\hat{\phi}$ do not need to be sign-invariant elsewhere. This was formalised in [McC99], where p was described as an *equational constraint*.

This idea has been further generalised in [BDE$^+$13] to cases where every part of F has an equational constraint, but not necessarily the same one, and in [BDE$^+$14] to cases where only parts of F have equational constraints, and/or where there is more than one equational constraint. [EBD15], building on [McC01], shows how to handle more than one equational constraint, using the observation that, in $p_1(y_1, \ldots, y_m, x_1, \ldots, x_n) = 0 \wedge p_2(y_1, \ldots, y_m, x_1, \ldots, x_n) = 0 \wedge \hat{\phi}$, $\mathrm{Res}_{x_n}(p_1, p_2)$ is also an equational constraint after we project out x_n.

3. Yet a further approach to Quantifier Elimination has recently been implemented [FIS15, FIS15], though based on an idea of [Wei98], that a suitable Comprehensive Gröbner System (see Section 3.3.13) can let one write down the quantifier-free equivalent to a formula.

4. It is tempting to think that "the problem is the data structure", and maybe if used Straight-Line Programs (page 52), or some other data structure, we could do better. However, this hope is destroyed by [CGH$^+$03] who state:

   > In this paper we are going to argue that the non-polynomial complexity character of the known symbolic geometric elimination procedures is not a special feature of a particular data structure (like the dense, sparse or arithmetic circuit encoding of polynomials), but rather a consequence of the information encoded by the respective data structure ...

   and their Theorem 4 (which is too complicated to state here in detail), says that any universal elimination procedure for the theory of algebraically closed firleds of charactersitic zero must have non-polynomial complexity.

Open Problem 14 (Not all CADs are outputs of our algorithms) *Consider the H-shaped set*

$$(x = -1 \& y \in (-1, 1)) \cup (-1 < x < 1 \& y = 0) \cup (x = 1 \& y \in (-1, 1)).$$

Then indeed a cylindrical algebraic decomposition is

$x < -1$ *one cell*

$x = -1$ *five cells ($y < -1$, $y = -1$, $-1 < y < 1$, $y = 1$, $y > 1$)*

$-1 < x < 1$ *three cells ($y < 0$, $y = 0$, $y > 0$)*

$x = 1$ *five cells*

$x > 1$ *one cell*

but not one we know how to compute. Indeed, the author is not sure know how to state it formally to QEPCAD or Maple. **TO BE COMPLETED**

3.6 Virtual Term Substitution

The idea of Virtual Term Substitution was introduced by [Wei94][41]: our presentation owes much to [KSD16].

3.6.1 The Weak Case

Problem 5 *Let us consider first a special case of Problem 3, $\exists x \phi(x, \mathbf{u})$, with the following additional constraints:*

1. *ϕ is* positive*, i.e. expressed purely in terms of $\wedge$ and $\vee$, with no $\Rightarrow$ or $\neg$;*

   * *(this can be achieved by standard laws of logic, then converting $\neg(a < b)$ into $a \geq b$ etc.)*

2. *each elementary formula is $p_i(x)\sigma_i 0$ with $\sigma_i \in \{=, \leq, \geq\}$;*

   * *(we will see later how to lift this restriction)*

3. *each polynomial in such a formula is at most quadratic in x;*

   * *(this restriction has been investigated in [Wei94] — see also* **higher** *below, but most practical uses of the Virtual Term Substitution method seem to be with linear/quadratic formulae).*

We wish to rewrite this as a logical formula $\psi(\mathbf{u})$.

We consider the values of $\mathbf{u}$ to be fixed. Then in principle (we will need to worry about degeneracies), each p_i defines one (linear) or two (quadratic) critical points. Furthermore, because of point 2, it is sufficient to consider the truth of ϕ at the critical points, since, if $\phi(x, \mathbf{u})$ is true, $\phi(x_\downarrow, \mathbf{u})$ and $\phi(x_\uparrow, \mathbf{u})$ are also true, where $x_\downarrow$ and $x_\uparrow$ are the critical points immediately $\leq$ and $\geq x$ (we will therefore need to worry later on about the case of x less than, or greater than, any critical point).

We handle the question of degeneracies by considering, not just critical points e, but rather *guarded points*[42] (γ, e) where γ is a logical formula in $\mathbf{u}$, and e is only to be considered if γ is true. We then consider each polynomial p_i in ϕ, depending on its degree in x.

linear Suppose p_i is $f_1 x + f_0$. Then the corresponding guarded point is

$$\left(f_1 \neq 0, \frac{-f_0}{f_1}\right) \tag{3.77}$$

quadratic Suppose p_i is $f_2 x^2 + f_1 x + f_0$, with $\Delta := f_1^2 - 4 f_2 f_0$. Then the corresponding guarded points are

$$\left(f_2 \neq 0 \wedge \Delta \geq 0, \frac{-f_1 - \sqrt{\Delta}}{2 f_2}\right), \tag{3.78}$$

[41] Foreshadowed by [Wei88] for the linear case.

[42] The concept is as old as Virtual Term Substitution, but this terminology is our own.

$$\left(f_2 \neq 0 \wedge \Delta \geq 0, \frac{-f_1 + \sqrt{\Delta}}{2f_2}\right) \tag{3.79}$$

and

$$\left(f_2 = 0 \wedge f_1 \neq 0, \frac{-f_0}{f_1}\right). \tag{3.80}$$

higher One need merely read the discussion in Section 3.1.2 to see that cubic equations would generate many more guarded points. Worse, as seen in (3.10), we may need to involve complex numbers. See [Wei94] for details.

However, the points defined in (3.77)–(3.80) are not defined in the Tarski language L of semi-algebraic propositions (Definition 77), and hence we cannot just substitute them for x in $p_i(x)$. It is for this reason that we speak of *Virtual* Term Substitution. We use the notation $[x//t]$ to denote substituting, in this sense, t for x in the whole proposition $p_i(x)\sigma_i 0$, noting that this takes propositions in L to Boolean combinations of propositions in L.

Example 17 ($\boldsymbol{f(x, \mathbf{u}) = 0[x//\frac{g_1+g_2\sqrt{g_3}}{g_4}]}$) *Since f is a polynomial, it is clear that $f(\frac{g_1+g_2\sqrt{g_3}}{g_4}, \mathbf{u}) = \frac{g_1^*+g_2^*\sqrt{g_3}}{g_4^*}$ for suitable expressions g_1^*, g_2^* and g_4^*.*

$$\begin{aligned}
\frac{g_1^* + g_2^*\sqrt{g_3}}{g_4^*} = 0 \quad &\Rightarrow \quad g_1^* + g_2^*\sqrt{g_3} = 0 \\
&\Rightarrow \quad |g_1^*| = |g_2^*\sqrt{g_3}| \wedge \\
&\qquad\ (\mathrm{sign}(g_1^*) \neq \mathrm{sign}(g_2^*) \vee \mathrm{sign}(g_1^*) = \mathrm{sign}(g_2^*) = 0) \\
&\Rightarrow \quad {g_1^*}^2 - {g_2^*}^2 g_3 = 0 \wedge g_1^* g_2^* \leq 0.
\end{aligned}$$

Substituting the other root gives ${g_1^*}^2 - {g_2^*}^2 g_3 = 0 \wedge g_1^* g_2^* \geq 0$, so the combination reduces to ${g_1^*}^2 - {g_2^*}^2 g_3 = 0$. as we might (with hindsight!) have expected. The cases of $\geq$ and $\leq$ are more complicated. **TO BE COMPLETED**

In general, if the guarded points from the $p_i\sigma_i 0$ in ϕ are $\{(\beta_{i,j}, e_{i,j})\}$, then Virtual Term Substitution (with the restrictions of Problem 5) is

$$\exists x \phi(x, \mathbf{u}) \overset{\text{VTS}}{\Longrightarrow} \bigvee_i \bigvee_j (\beta_{i,j} \wedge (\phi[x//e_{i,j}])) \tag{3.81}$$

Unfortunately the right-hand side of (3.81) does not satisfy the constraints of Problem 5), not least because of the $\neq$ operations in the guards of (3.77)–(3.80), hence we cannot apply two levels of Virtual Term Substitution to $\exists x_1 \exists x_2 \phi(x_1, x_2, \mathbf{u})$ until we can lift restriction 2.

3.6.2 The Strict Case

If we consider a strict inequality ($>$, $<$ or $\neq$), i.e. lifting restriction 2, this again defines a certain number of intervals, but the end-points no longer satisfy the strict inequality: we need interior points.

The initial version of Virtual Term Substitution [Wei88] solved this problem via using the point $\frac{1}{2}(z_i + z_j)$, i.e. the arithmetic mean, for the interval (z_i, z_j). However, this has two serious drawbacks:

1. we do not know the order of the $z_i(\mathbf{u})$, hence we need to add all possible arithmetic means: $\frac{1}{2}n(n-1)$ of them if there are n such z_i;
2. if z_i and z_j are independent quadratic expressions, then the arithmetic mean is
$$\frac{1}{2}\left(\frac{-g_{i,1} \pm \sqrt{g_{i,3}}}{2g_{i,4}} + \frac{-g_{,1} \pm \sqrt{g_{j,3}}}{2g_{j,4}}\right),$$
which is not of the form $\frac{g_1+g_2\sqrt{g_3}}{g_4}$, so the reasoning of Example 17 does not apply.

Hence the alternative approach is to use infinitesimals: for a strict quadratic inequality we would add four points $\frac{-g_{i,1}\pm\sqrt{g_{i,3}}}{2g_{i,4}} \pm \epsilon$. This is in fact overkill as it wil give us two points in most strict intervals, and we only need add $\frac{-g_{i,1}\pm\sqrt{g_{i,3}}}{2g_{i,4}} - \epsilon$, and a $+\infty$ term to allow for upper unbounded intervals where we would otherwise have no point.

However, we need to handle these terms, as is shown in the next two examples.

Example 18 ($p(x) < 0[x//t - \epsilon]$) *(where p is a polynomial and t is a regular (non-infinitesimal) term).*

$$p(x) < 0[x//t - \epsilon] \quad \Rightarrow \quad \Bigg(p(x) < 0 \vee \Big(p(x) = 0 \wedge \Big(p'(x) > 0 \vee \big(p'(x) = 0 \wedge (p''(x) < 0 \vee \cdots)\big)\Big)\Big)\Bigg)[x//t]$$

Example 19 ($ax^2 + bx + c < 0[x//\infty]$)

$$ax^2 + bx + c < 0[x//\infty] \quad \Rightarrow \quad a < 0 \vee \Big(a = 0 \wedge \big(b < 0 \vee (b = 0 \wedge c < 0)\big)\Big)$$

With these extensions, equation (3.81) is still valid, with restriction 2 lifted.

3.6.3 Nested Quantifiers

Consider
$$\exists x_1 \exists x_2 \phi(x_1, x_2, \mathbf{u}). \tag{3.82}$$

We first consider $\exists x_2 \psi(x_2, \mathbf{v})$, where $\mathbf{v}$ is $x_1||\mathbf{u}$ ($||$ signifying concatenation) and ψ is ϕ with the arguments reshuffled. As in (3.81), this is converted, eliminating x_2, into
$$\bigvee_i \bigvee_j \left(\beta_{i,j}(\mathbf{v}) \wedge ((p_i(x_2, \mathbf{v})\sigma_i 0)[x_2//e_{i,j}(\mathbf{v})])\right),$$

which we can write as $\Psi(x_1, \mathbf{u})$ after replacing $\mathbf{v}$ by $x_1||\mathbf{u}$. This satisfies restriction 1, and we have lifted restriction 2. The only problem is restriction 3: there is no guarantee that, even if ϕ is only quadratic in x_1, that Ψ will be. In practice, though, Ψ often does, or can be made to (see [KSD16, §5] for a useful technique), satisfy restriction 3. We then apply the same technique to $\exists x_1 \Psi(x_1, \mathbf{u})$ to obtain

$$\bigvee_{i'} \bigvee_{j'} \left(\beta'_{i',j'}(\mathbf{u}) \wedge \left((p'_{i'}(x_1, \mathbf{u})\sigma'_{i'}0)[x_1 /\!/ e'_{i',j'}(\mathbf{u})]\right)\right),$$

as desired.

There is an important practical[43] point, though. We have suggested applying "the same technique" to

$$\exists x_1 \bigvee_i \bigvee_j \left(\beta_{i,j}(x_1||\mathbf{u}) \wedge ((p_i(x_2, x_1||\mathbf{u})\sigma_i 0)[x_2 /\!/ e_{i,j}(x_1||\mathbf{u})])\right),$$

but we are much better off applying it to the equivalent

$$\bigvee_{i=1}^{N} \bigvee_{j=1}^{N_i} \exists x_1 \left(\beta_{i,j}(x_1||\mathbf{u}) \wedge ((p_i(x_2, x_1||\mathbf{u})\sigma_i 0)[x_2 /\!/ e_{i,j}(x_1||\mathbf{u})])\right), \tag{3.83}$$

so that we are solving $\sum N_i$ problems each the complexity, at least informally speaking, of the original, rather than one problem $\sum N_i$ times as long.

3.6.4 Universal quantifiers

So far we have merely treated existential $\exists$ quantifiers. This is, in one sense, all we need do, since $\forall x \phi(x) \Leftrightarrow \neg \exists x \neg \phi(x)$. The formula $\neg\phi(x)$ breaks constraint 1, but the standard laws of logic, such as

$$\neg(\phi \vee \psi) \Leftrightarrow (\neg\phi) \wedge (\neg\phi) \tag{3.84}$$

will deal with this problem. There are two remarks we can make about the complexity of this process.

1. We may as well work with blocks of quantifiers (as in Notation 23), i.e. transform $\forall \mathbf{x} \phi(\mathbf{x})$ into $\neg \exists \mathbf{x} \neg \phi(\mathbf{x})$.
2. If the innermost block is existential, then after eliminating these quantifiers, we are left with an expression of the form (3.83), i.e. a large disjunction, which we were processing clause-by-clause. When we negate this, we get a large conjunction, which has to be processed in its entirety. Eliminating the $\exists \mathbf{x}$ will give another disjunction, but this again negates to a large conjunction which has to be processed in its entirety. It seems to be this process that makes the complexity heavily dependent on the number of alternations.

[43] And indeed important from the point of complexity theory, as pointed out by [Wei88, p. 25].

3.6.5 Complexity of VTS

The general method we have outlined above was foreshadowed by [Wei88] for the linear case. In particular that paper proved that the complexity of linear quantifier elimination was singly exponential in the number of variables, but doubly exponential in the number of alternations (see Notation 23). A similar claim was made for the quadratic case in [Stu96], but was not proved.

Open Problem 15 *Show, either in the quadratic case, or more generally, that the complexity of quantifier elimination by Virtual Term Substitution is exponential in the number of variables, and only doubly exponential in the number of alternations.*

Sturm wrote to the author as follows. “At the time of writing I had blindly believed that this holds also for the quadratic case. This is still my intuition, but there is no proof.

I had furthermore assumed that the argument would remain correct when generalizing to arbitrary degrees. After [C.W. Brown] told me in Timișoara that there is both a combinatorial and a size-of-polynomials aspect of double exponential complexity, I am not at all certain about this anymore.”

3.7 Conclusions

1. The `RootOf` construct is inevitable (theorem 10), so should be used, as described in footnote 3 (page 83). Such a notation can avoid the “too many solutions” trap — see equations (3.50) and (3.51). We should find a way of extending it to situations such as equation (3.49).

2. While matrix inversion is a valuable concept, it should generally be avoided in practice.

3. Real algebraic geometry is not simply “algebraic geometry writ real”: it has different problems and needs different techniques.

Chapter 4

Modular Methods

In chapter 2, describing the subresultant method of computing greatest common divisors, we said the following.

> This algorithm is the best method known for calculating the g.c.d., of all those based on Euclid's algorithm applied to polynomials with integer coefficients. In chapter 4 we shall see that if we go beyond these limits, it is possible to find better algorithms for this calculation.

Now is the time to fulfil that promise, which we do by describing the historically-first "advanced" algorithm, first with a simple example (section 4.1), and then its greatest success, g.c.d. calculation, first in one variable (section 4.2), then in two (section 4.3) and several (section 4.4) variables. We will then look at other applications (section 4.5), and finally at Gröbner bases (section 4.6).

The basic idea behind these algorithms is shown in Figure 4.1: instead of doing a calculation in some (large) domain R, we do it in several smaller domains R_i, possibly discard some of these, piece the result together to $R'_{1\cdots k}$, regard this

Figure 4.1: Diagrammatic illustration of Modular Algorithms

$$\begin{tikzcd}
R \arrow[rrr, dashed, "\text{calculation}"] \arrow[d, "k\times\text{reduce}"'] & & & R \\
R_1 \arrow[r, "\text{calculation}"] & R_1 & & \\
\vdots & \vdots \arrow[rr, "\text{combine}", "\text{?discard}"'] & & R'_{1\cdots k} \arrow[uu, "\text{interpret}\ \&\ \text{check}"'] \\
R_k \arrow[r, "\text{calculation}"] & R_k & &
\end{tikzcd}$$

$R'_{1\cdots k}$ indicates that some of the R_i may have been rejected by the compatibility checks, so the combination is over a subset of $R_1, \ldots, R_k$.

as being in R and check that it is indeed the right result. The key questions are then the following.

1. Are there domains R_i for which the behaviour of the computation in R_i is sufficiently close to that in R for us to make deductions about the computation in R from that in R_i? Such R_i will generally be called "good".
2. Can we tell, either immediately or with hindsight, whether an R_i is "good"? It will often turn out that we can't, but may be able to say that, given R_i and R_j, one of them is *definitely* "bad", and preferably which.
3. How many reductions should we take? In practice we will only count "good" reductions, so this question is bound up with the previous one.
4. How do we combine the results from the various R_i? The answer to this will often turn out to be a variant of the Chinese Remainder Theorem.
5. How do we check the result? Can we be absolutely certain that this result is in fact the answer to the original question? In category speak, does Figure 4.1 commute?

A common choice for the R_i is given by the following.

Notation 27 *Let n be a positive number. By $\mathbf{Z}_n$ we mean the integers considered modulo n.*

Proposition 52 *$\mathbf{Z}_n$ is a commutative ring with a multiplicative identity (the number 1 itself).*

Example 20 *Note that we can't say that Z_n is an integral domain (Definition 11), since, for example, in $\mathbf{Z}_6$ $2 \times 3 = 0$, even though neither 2 nor 3 are zero in $\mathbf{Z}_6$.*

Proposition 53 *Let p be a prime number. Then $\mathbf{Z}_p$ is a field.*

Proof. Since p is a prime, the problem of Example 20 can't happen, and Z_p is certainly an integral domain. Let $n \in \mathbf{Z}_p$ be nonzero, and apply the Extended Euclidean Algorithm (in $\mathbf{Z}$) to the integers n and p. Since $\gcd(n,p) = 1$, we find a, b such that $an + bp = 1$. Then $an \equiv 1 \pmod p$, so $a = n^{-1}$ and hence inverses exist.

4.1 Matrices: a Simple Example

Suppose we wish to calculate the determinant, or possibly the inverse, D of an $n \times n$ matrix M, and for simplicity we will assume that n is even. If the size of the entries (degree for polynomials, or number of bits/words for integers) is s, then the determinant will have size at most[1] ns, since it is the sum of $n!$ products of n entries from the matrix.

[1]Not quite for integers, since $a + b$ can be bigger than either a or b, but we'll ignore this complication, which does not affect the general point.

4.1.1 Matrices with integer coefficients: Determinants

If we use the fraction-free methods of Theorem 15, then, just before clearing column k, the entries should have size roughly[2] bounded by ks, and at the end, the last entry, which is the determinant, is (roughly) bounded by ns. But consider the half-way state, after we have eliminated $n/2$ columns. The entries are:

row 1 n entries of size s: total ns;

row 2 $n-1$ entries of size $2s$: total $2(n-1)s$;

row 3 $n-2$ entries of size $3s$: total $3(n-3)s$;

. . .

row $n/2$ $n/2$ entries of size $(n/2)s$: total $(n/2)^2s$;

Total $\sum_{i=1}^{n/2} i(n+1-i)s = \left(\frac{n^3}{12}+\frac{n^2}{4}+\frac{n}{6}\right)s.$

next $n/2$ **rows** $n/2$ elements of size $(n/2)s$ in each row, totalling $\frac{n^3}{8}s$

Grand total $\frac{5}{24}n^3s + O(n^2s)$.

Since the final result has size ns, we are storing, and manipulating, $\frac{5}{24}n^2$ times as much data as are needed — a quantifiable case of intermediate expression swell. Can we do better?

Let the p_i be a family of distinct primes[3], chosen so that arithmetic modulo p is fast, typically 31-bit or 63-bit primes[4]. Since D is the sum of products of entries of the matrix, we can evaluate $D \pmod p$ (which we will write as D_p) either by computing D, and then reducing it modulo p, or by computing (modulo p) the determinant of the matrix M_p: in symbols

$$\det(M)_p = \det(M_p). \tag{4.1}$$

The latter takes $O(n^3)$ operations, by classical arithmetic, to which we should add $O(n^2s)$ for the initial reduction modulo p.

The Hadamard bounds (Propositions 86 and 87: see [AM01] for an analysis of the extent to which they are over-estimates — roughly $0.22n$ decimal digits) will give us a number D_0 such that $|D| \le D_0$, $\log_2 D_0 \le \frac{n}{2}s(\log_2 n)$. If we know D_{p_i} for enough primes such that $N := \prod p_i > 2D_0$, we can use the Chinese

[2]Again, we need "roughly" in the integer case, since there is again the question of carries, as in note[1].

[3]For this example, we do not actually need them to be primes, merely relatively prime, but the easiest way to ensure this is for them to be primes.

[4]We therefore take the cost of arithmetic modulo p to be constant — $O(1)$. Strictly speaking, we should allow for the possibility that our problem is so large that there are not enough small primes. But such a problem would not fit in the computer in the first place, and the computer algebra community systematically (and often silently) ignores this possibility. We can replace O by $\tilde{O}$ if we are really worried about it.

Remainder Theorem (Theorem 52 and Algorithm 47) to deduce D_N. If we choose it in the range $\frac{-N}{2} < D_N < \frac{N}{2}$, then $D_N = D$, and we are done.

The number of primes needed is $O(\log D_0) = O(ns \log n)$. Hence the cost of all the determinants is $O((n^3 + n^2 s) ns \log n) = O(n^4 s \log n + n^3 s^2 \log n)$. The cost of the Chinese Remainder Algorithm is, by Proposition 97, $O(\log^2 D_0) = O(n^2 s^2 \log^2 n)$, which is dominated by the second term in the determinant ost. For large n, the first term dominates, but for large s, the second term might dominate. Hence we have proved the following, slightly clumsy, statement.

Proposition 54 *The cost of computing the determinant of an $n \times n$ matrix with integer entries bounded by 2^s is $O\left(\max(n^4 s \log n, n^3 s^2 \log n)\right)$.*

If we attempt to simplify this, we can write $O(n^4 s^2 \log n)$, which doesn't look much different from (3.25)'s $O(n^5 s^2 \log^2 n)$, and this is true in terms of worst-case complexity with respect to each of n and s considered separately. But if, for example, s and n are both k, then (3.25) is $O(k^7 \log^2 k)$ and Proposition 54 is $O(k^5 \log k)$. An even better method is mentioned in Section 5.9.4.

4.1.2 Matrices with polynomial coefficients: Determinants

Assume our matrix entries are polynomials in x, of degree at most d, over a (sufficiently large) ring R which supports the polynomial Chinese Remainder Theorem (Appendix A.4) — in practice $\mathbf{Q}$ or $\mathbf{Q}(y_1, \ldots, y_n)$. One method is the fraction-free computations of Theorem 15, then the same analysis as at the start of the previous section shows that we have intermediate expression swell. Can we do better?

Suppose we know that the degree (in x) of the determinant is less than N, and let $v_1, \ldots, v_N$ be N distinct values in R. Then, using the notation Z_{x-v} to mean "Z, but replacing x by the value v" (which for polynomial Z is the same as the remainder on dividing Z by $x - v$), we have

$$\det(M)_{x-v} = \det(M_{x-v}). \tag{4.2}$$

Proposition 55 *If M is a $n \times n$ matrix whose entries are polynomials in x of degree at most d, then the degree of $\det(M)$ is at most nd.*

If we assume no expression growth in the elements of R (unrealistic, but simplifying), the cost of computing M_{x-v} is $O(n^2 d)$ operations in K, and the cost of evaluating the determinant is $O(n^3)$. The cost of the Chinese Remainder Algorithm is, by Proposition 98, $O(N^2) = O((nd + 1)^2)$.

Proposition 56 *The cost of computing the determinant of an $n \times n$ matrix with polynomial entries of degree bounded by d is $O\left(\max(n^4 d, n^3 d^2)\right)$ coefficient operations.*

In practice, if we had polynomial entries with integer coefficients, we would reduce the polynomial determinant calculation to many integer ones, and do these by the method of section 4.1.1. The analysis becomes rather tedious, but it is still possible to show that the complexity is always better than fraction-free methods.

Open Problem 16 (Matrix Determinant costs) *In Open Problem 6, we saw that the obvious calculation of costs did not seem to have been borne out by (admittedly old) experiments. What is the practical validity of these calculations?*

4.1.3 Conclusion: Determinants

Hence we have simple answer to the questions on page 164.

1. Are there good reductions from R?: yes, every reduction.
2. How can we tell if R_i is good? — always.
3. How many reductions should we take? This depends on bounding the size of $\det(M)$. In the polynomial case, this is easy from the polynomial form of the determinant: Proposition 55. In the integer case, the fact that we are adding $n!$ products makes the estimate more subtle: see Proposition 86.
4. How do we combine the results — one of Algorithms 47 (integer) or 48 (polynomial).
5. How do we check the result: irrelevant.

4.1.4 Linear Equations with integer coefficients

Now suppose that we wish to actually solve a linear system $M.\mathbf{x} = \mathbf{a}$ with coefficients in $\mathbf{Z}$, rather than simply compute the determinant (which we will assume is nonzero). As before, one option is the fraction-free method (Corollary 5) followed by back-substitution, and it can be shown that the cost of the back-substitution is asymptotically the same as that of the fraction-free triangularization.

Let p be a prime not dividing $\det(M)$. We know (3.14) that $\mathbf{x} = M^{-1}.\mathbf{a}$, and hence $\mathbf{x}_p = (M^{-1}.\mathbf{a})_p = (M^{-1})_p.\mathbf{a}_p = {M_p}^{-1}.\mathbf{a}_p$. If we knew the $\mathbf{x}_p$ for enough p, we ought to be able to recover $\mathbf{x}$. However, the entries of $\mathbf{x}$ are rational numbers, not integers. There is a general answer to this problem in Section 4.5.2.3, but in fact we can do better here. We know that the denominators of $\mathbf{x}$ all divide $\det(M)$, so we shall solve the related problem

$$\mathbf{y} = (\det(M))M^{-1}.\mathbf{a}, \tag{4.3}$$

whose solution y is a vector of integers, again bounded by the Hadamard bounds (Propositions 86 and 87).

Algorithm 16 (Modular Linear Equations)
Input: *M an $n \times n$ non-singular matrix, $\mathbf{a}$ an n-vector over $\mathbf{Z}$*
Output: *$\mathbf{x}$ an n-vector over $\mathbf{Q}$ with $M.\mathbf{x} = \mathbf{a}$*

```
1. for  enough ($N$) primes $p_i$
2.      reduce $M$ mod $p_i$ to get $M_i$, and $\mathbf{a}$ to get $\mathbf{a}_i$
3.      triangularize $M_i||\mathbf{a}_i$
4.      if  $M_i$ is non-singular
5.          Backsubstitute to solve $\mathbf{y}_i = (\det(M_i)){M_i}^{-1}.\mathbf{a}_i$.
6. Reconstruct $\det(M)$ from the $\det(M_i)$.
7. Reconstruct $\mathbf{y}$ from the $\mathbf{y}_i$.
8. Return $\mathbf{y}/\det(M)$ (after cancellation).
```

The cost of steps 2 and 3 are $O(n^2 s)$ and $O(n^3)$, as in section 4.1.1. Step 5 costs $O(n^2)$ operations. Step 6 costs, by Proposition 97, $O(\log^2|\det(M)|) = O(n^2s^2\log^2 n)$ operations, and step 7 costs n times as much. Step 8 as the same order of complexity as step 7, as we are computing n gcds of numbers of size bounded by the Hadamard bounds. Hence the total cost is

$$O\left(N(n^2s + n^3) + n(n^2s^2\log^2 n)\right) \tag{4.4}$$

operations. The number of primes we actually need is given by the Hadamard bounds[5], i.e. $ns\log n$, and the number of primes that we have to discard is (at most) the same, as these must divide the determinant. The total cost is therefore

$$O(\underbrace{n^3s^2\log n}_{\text{evaluation}} + \underbrace{n^4s\log n}_{\text{det/solving}} + \underbrace{n^3s^2\log^2 n}_{\text{CRT/simplify}}) = O(n^4s\log n + n^3s^2\log^2 n) \tag{4.5}$$

(where we have annotated the causes of the various summands): almost the same as Proposition 54, and the extra factor of $\log n$ multipliying n^3s^2 is caused by the fact that the Chinese Remainder Theorem is no longer negligeable, as we have to reconstruct n numbers.

4.1.5 Linear Equations with polynomial coefficients

This is sufficiently similar that we will skip the details.

4.1.6 Conclusion: Linear Equations

Hence we have simple answer to the questions on page 164.

1. Are there good reductions from R?: yes, all reductions except those that annihilate the determinant — a finite number.

[5]Though it doesn't change the asymptotics, these logarithms are $\log_B$, where B is the size of our working primes. Hence if $B = 2^{31}$, we have $\log_{2^{31}}$ rather than $\log_e$, saving a factor of 21.

2. How can we tell if R_i is good? — if the determinant is nonzero.

3. How many reductions should we take? This depends on bounding the size of $\det(M)$. In the polynomial case, this is easy from the polynomial form of the determinant: Proposition 55. In the integer case, the fact that we are adding $n!$ products makes the estimate more subtle: see Proposition 86.

4. How do we combine the results — one of Algorithms 47 (integer) or 48 (polynomial).

5. How do we check the result: irrelevant.

4.1.7 Matrix Inverses

The same techniques as linear equations apply: indeed we could (though probably should not) compute the columns of the inverse as the solutions of $M.\mathbf{x} = \mathbf{e}_i$, where $\mathbf{e}_i$ has a 1 in the ith position, and zero elsewhere.

4.2 Gcd in one variable

Let us consider Brown's example (from which page 65 was modified):

$$A(x) = x^8 + x^6 - 3x^4 - 3x^3 + 8x^2 + 2x - 5; \quad (4.6)$$

$$B(x) = 3x^6 + 5x^4 - 4x^2 - 9x + 21. \quad (4.7)$$

Let us suppose that these two polynomials have a common factor, that is a polynomial P (of non-zero degree) which divides A and B. Then there is a polynomial Q such that $A = PQ$. This equation still holds if we take each coefficient as an integer modulo 5. If we write P_5 to signify the polynomial P considered as a polynomial with coefficients modulo 5, this equation implies that P_5 divides A_5. Similarly, P_5 divides B_5, and therefore it is a common factor[6] of A_5 and B_5. But calculating the g.c.d. of A_5 and B_5 is fairly easy:

$$\begin{aligned}
A_5(x) &= x^8 + x^6 + 2x^4 + 2x^3 + 3x^2 + 2x;\\
B_5(x) &= 3x^6 + x^2 + x - 1;\\
C_5(x) &= \text{remainder}(A_5(x), B_5(x)) = A_5(x) + 3(x^2+1)B_5(x) = 4x^2 + 2;\\
D_5(x) &= \text{remainder}(B_5(x), C_5(x)) = B_5(x) + (x^4 + 4x^2 + 3)C_5(x) = x;\\
E_5(x) &= \text{remainder}(C_5(x), D_5(x)) = C_5(x) + xD_5(x) = 3.
\end{aligned}$$

Thus A_5 and B_5 are relatively prime, which implies that $P_5 = 1$. As the leading coefficient of P has to be one, we deduce that $P = 1$.

The concept of modular methods is inspired by this calculation, where there is no possibility of intermediate expression swell, for the integers modulo 5 are bounded (by 4). Obviously, there is no need to use the integers modulo 5: any

[6]Note that we cannot deduce that $P_5 = \gcd(A_5, B_5)$: a counter-example is $A = x - 3$, $B = x + 2$, where $P = 1$, but $A_5 = B_5 = x + 2$, and so $\gcd(A_5, B_5) = x + 2$, whereas $P_5 = 1$.

prime number p will suffice (we chose 5 because the calculation does not work modulo 2, for reasons to be described later, and 3 divides one of the leading coefficients). In this example, the result was that the polynomials are relatively prime. This raises several questions about generalising this calculation to an algorithm capable of calculating the g.c.d. of any pair of polynomials:

1. how do we calculate a non-trivial g.c.d.?
2. what do we do if the modular g.c.d. is not the modular image of the g.c.d. (as in the example in the footnote[6])?
3. how much does this method cost?

4.2.1 Bounds on divisors

Before we can answer these questions, we have to be able to bound the coefficients of the g.c.d. of two polynomials.

Theorem 36 (Landau–Mignotte Inequality [Lan05, Mig74, Mig82]) *Let $Q = \sum_{i=0}^{q} b_i x^i$ be a divisor of the polynomial $P = \sum_{i=0}^{p} a_i x^i$ (where a_i and b_i are integers). Then*

$$\max_{i=0}^{q} |b_i| \le \sum_{i=0}^{q} |b_i| \le 2^q \left|\frac{b_q}{a_p}\right| \sqrt{\sum_{i=0}^{p} a_i^2}.$$

These results are corollaries of statements in Appendix A.2.2.

If we regard P as known and Q as unknown, this formulation does not quite tell us about the unknowns in terms of the knowns, since there is some dependence on Q on the right, but we can use a weaker form:

$$\sum_{i=0}^{q} |b_i| \le 2^p \sqrt{\sum_{i=0}^{p} a_i^2}.$$

When it comes to greatest common divisors, we have the following result.

Corollary 10 *Every coefficient of the g.c.d. of $A = \sum_{i=0}^{\alpha} a_i x^i$ and $B = \sum_{i=0}^{\beta} b_i x^i$ (with a_i and b_i integers) is bounded by*

$$2^{\min(\alpha,\beta)} \gcd(a_\alpha, b_\beta) \min\left(\frac{1}{|a_\alpha|}\sqrt{\sum_{i=0}^{\alpha} a_i^2}, \frac{1}{|b_\beta|}\sqrt{\sum_{i=0}^{\beta} b_i^2}\right).$$

Proof. The g.c.d. is a factor of A and of B, the degree of which is, at most, the minimum of the degrees of the two polynomials. Moreover, the leading coefficient of the g.c.d. has to divide the two leading coefficients of A and B, and therefore has to divide their g.c.d.

A slight variation of this corollary is provided by the following result.

Corollary 11 *Every coefficient of the g.c.d. of* $A = \sum_{i=0}^{\alpha} a_i x^i$ *and* $B = \sum_{i=0}^{\beta} b_i x^i$ *(where* a_i b_i *are integers) is bounded by*

$$2^{\min(\alpha,\beta)} \gcd(a_0, b_0) \min\left(\frac{1}{|a_0|}\sqrt{\sum_{i=0}^{\alpha} a_i^2}, \frac{1}{|b_0|}\sqrt{\sum_{i=0}^{\beta} b_i^2}\right).$$

Proof. If $C = \sum_{i=0}^{\gamma} c_i x^i$ is a divisor of A, then $\hat{C} = \sum_{i=0}^{\gamma} c_{\gamma-i} x^i$ is a divisor of $\hat{A} = \sum_{i=0}^{\alpha} a_{\alpha-i} x^i$, and conversely. Therefore, the last corollary can be applied to $\hat{A}$ and $\hat{B}$, and this yields the bound stated.

It may seem strange that the coefficients of a g.c.d. of two polynomials can be greater than the coefficients of the polynomials themselves. One example which shows this is the following (due to Davenport and Trager):

$$\begin{aligned} A = & \quad x^3 + x^2 - x - 1 & = (x+1)^2(x-1); \\ B = & \quad x^4 + x^3 + x + 1 & = (x+1)^2(x^2 - x + 1); \\ \gcd(A,B) = & \quad x^2 + 2x + 1 & = (x+1)^2. \end{aligned}$$

This example can be generalised, as say

$$\begin{aligned} A = & \quad x^5 + 3x^4 + 2x^3 - 2x^2 - 3x - 1 & = (x+1)^4(x-1); \\ B = & \quad x^6 + 3x^5 + 3x^4 + 2x^3 + 3x^2 + 3x + 1 & = (x+1)^4(x^2 - x + 1); \\ \gcd(A,B) = & \quad x^4 + 4x^3 + 6x^2 + 4x + 1 & = (x+1)^4. \end{aligned}$$

In fact, Mignotte [Mig81] has shown that the number 2 in Theorem 36 is asymptotically the best possible, i.e. it cannot be replaced by any smaller c.

Open Problem 17 *Show*[7] *that the same is true of corollaries 10 and 11.*

4.2.2 The modular – integer relationship

In this sub-section, we answer the question raised above: what do we do if the modular g.c.d. is not the modular image of the g.c.d. calculated over the integers?

Lemma 7 *If* p *does not divide the leading coefficient of* $\gcd(A, B)$*, the degree of* $\gcd(A_p, B_p)$ *is greater than or equal to that of* $\gcd(A, B)$*.*

Proof. Since $\gcd(A, B)$ divides A, then $(\gcd(A, B))_p$ divides A_p. Similarly, it divides B_p, and therefore it divides $\gcd(A_p, B_p)$. This implies that the degree of $\gcd(A_p, B_p)$ is greater than or equal to that of $\gcd(A, B)_p$. But the degree of $\gcd(A, B)_p$ is equal to that of $\gcd(A, B)$, for the leading coefficient of $\gcd(A, B)$ does not cancel when it is reduced modulo p.

This lemma is not very easy to use on its own, for it supposes that we know the g.c.d. (or at least its leading coefficient) before we are able to check whether the modular reduction has the same degree. But this leading coefficient has to divide the two leading coefficients of A and B, and this gives a formulation which is easier to use.

[7] The author is grateful to John Abbott for pointing out that this isn't trivial!

Corollary 12 *If p does not divide the leading coefficients of A and of B (it may divide one, but not both), then the degree of* $\gcd(A_p, B_p)$ *is greater than or equal to that of* $\gcd(A, B)$*.*

As the g.c.d. is the only polynomial (to within an integer multiple) of its degree which divides A and B, we can test the correctness of our calculations of the g.c.d.: if the result has the degree of $\gcd(A_p, B_p)$ (where p satisfies the hypothesis of this corollary) and if it divides A and B, then it is the g.c.d. (to within an integer multiple).

It is quite possible that we could find a $\gcd(A_p, B_p)$ of too high a degree. For example, in the case cited above, $\gcd(A_2, B_2) = x + 1$ (it is obvious that $x+1$ divides the two polynomials modulo 2, because the sum of the coefficients of each polynomial is even). The following lemma shows that this possibility can only arise for a finite number of p.

Lemma 8 *Let $C = \gcd(A, B)$. If p satisfies the condition of the corollary above, and if p does not divide* $\mathrm{Res}_x(A/C, B/C)$*, then* $\gcd(A_p, B_p) = C_p$*.*

Proof. A/C and B/C are relatively prime, for otherwise C would not be the g.c.d. of A and B. By the corollary, C_p does not vanish. Therefore

$$\gcd(A_p, B_p) = C_p \gcd(A_p/C_p, B_p/C_p).$$

For the lemma to be false, the last g.c.d. has to be non-trivial. This implies that the resultant $\mathrm{Res}_x(A_p/C_p, B_p/C_p)$ vanishes, by proposition 78 of the Appendix. This resultant is the determinant of a Sylvester matrix, and $|M_p| = (|M|)_p$, for the determinant is only a sum of products of the coefficients. In the present case, this amounts to saying that $\mathrm{Res}_x(A/C, B/C)_p$ vanishes, that is that p divides $\mathrm{Res}_x(A/C, B/C)$. But the hypotheses of the lemma exclude this possibility.

Definition 86 *If* $\gcd(A_p, B_p) = \gcd(A, B)_p$*, we say that the reduction of this problem modulo p is* good*, or that p is* of good reduction*. If not, we say that p is* of bad reduction*.*

This lemma implies, in particular, that there are only a finite number of values of p such that $\gcd(A_p, B_p)$ does not have the same degree as that of $\gcd(A, B)$, that is the p which divide the g.c.d. of the leading coefficients and the p which divide the resultant of the lemma (the resultant is non-zero, and therefore has only a finite number of divisors). In particular, if A and B are relatively prime, we can always find a p such that A_p and B_p are relatively prime.

Observation 7 *It would be tempting to conclude "the probabiliy that p is bad is the probability that p divides this resultant, i.e. $1/p$. However,* a given *resultant either is, or is not, divisible by p: there is no probability involved. If we consider the space of all (A, B) pairs*[8]*, then we have to allow for p dividing the leading coefficients. This argument would also assume that the distribution of resultants*

[8]Technically, the limit as $H \to \infty$ of all pairs with coefficients at most H.

Figure 4.2: Diagrammatic illustration of Algorithm 17

$$\begin{tikzcd}
\mathbf{Z}[x] \arrow[r, dashed] \arrow[d, "\text{reduce}"'] & \mathbf{Z}[x] \\
\mathbf{Z}_p[x] \arrow[r, "\text{gcd}"] & \mathbf{Z}_p[x] \arrow[u, "\text{interpret}\ \&\ \text{check}"']
\end{tikzcd}$$

is the same as the distribution of integers, and this seems not to be the case: very few resultants are actually prime, for example. Nevertheless, it is an empirical observation that the probability of p being bad does seem to be proportional to $1/p$.

An alternative proof that there are only finitely primes of bad reduction can be deduced from Lemma 11. We can summarize this section in the following.

Theorem 37 (Good Reduction Theorem (Z)) *If p does not divide* $\gcd(a_\alpha, b_\beta)$ *(which can be checked for in advance) or* $\text{Res}_x(A/C, B/C)$*, then p is of good reduction. Furthermore, if p divides* $\text{Res}_x(A/C, B/C)$ *but not* $\gcd(a_\alpha, b_\beta)$*, then the gcd computed modulo p has a* larger *degree than the true result.*

4.2.3 Computing the g.c.d.: one large prime

In this section we answer the question posed earlier: how do we calculate a non-trivial g.c.d.? One obvious method is to use the Landau-Mignotte inequality, which can determine an M such that all the coefficients of the g.c.d. are bounded by M, and to calculate modulo a prime number greater than $2M$. This method translates into Algorithm 17/Figure 4.2 (where `Landau_Mignotte_bound`(A, B) applies corollary 10 and/or corollary 11, and `find_large_prime`$(2M)$ produces a different prime $> 2M$ each tine it is called within a given invocation of the algorithm). We restrict ourselves to monic polynomials, and assume `modular_gcd` gives a monic result, to avoid the problem that a modular g.c.d. is only defined up to a constant multiple.

Algorithm 17 (Modular GCD (Large prime version))
Input: *A, B monic polynomials in* $\mathbf{Z}[x]$.
Output: $\gcd(A, B)$

```
$M$ :=Landau_Mignotte_bound($A, B$);
do  $p$ :=find_large_prime($2M$);
    if  $p$ does not divide $\gcd(\text{lc}(A), \text{lc}(B))$
        then  $C$ :=modular_gcd($A, B, p$);
              if  $C$ divides $A$ and $C$ divides $B$
                  then  return $C$
    forever       #Lemma 8 guarantees termination
```

We can think of the use of `modular_gcd` as a Monte Carlo Algorithm (Section 1.4.2) and Algorithm 17 as an instance of Figure 1.1 converting a Monte Carlo algorithm into a Las Vegas ("always correct/probably fast") one.

If the inputs are not monic, the answer might not be monic. For example, $x + 4$ divides both $2x^2 + x$ and $2x^2 - x - 1$ modulo 7, but the true common divisor over the integers is $2x+1$, which is $2(x+4) \pmod 7$. We do know that the leading coefficient of the g.c.d. divides each leading coefficient $\mathrm{lc}(A)$ and $\mathrm{lc}(B)$, and therefore their g.c.d. $g = \gcd(\mathrm{lc}(A), \mathrm{lc}(B))$. We therefore compute

$$C := \mathrm{pp}(\underbrace{g \times \texttt{modular\_gcd}(A, B)}_{\substack{\text{computed modulo } M,\\ \text{then interpreted in } \mathbf{Z}}}) \tag{4.8}$$

instead, where the pp is needed in case the leading coefficient of the g.c.d. is a proper factor of g.

It is tempting to argue that this algorithm will only handle numbers of the size of twice the Landau–Mignotte bound, but this belief has two flaws.

- While we have proved that there are only finitely many bad primes, we have said nothing about how many there are. The arguments can in fact be made effective, but the results tend to be unduly pessimistic, since it is extremely unlikely that all the bad primes would be clustered just above $2M$.

- In theory, and indeed very often in practice[ABD88], the division tests could yield very large numbers if done as tests of the remainder being zero: for example the remainder on dividing x^{100} by $x - 10$ is 10^{100}. This can be solved by a technique known as "early abort" trial division.

  Proposition 57 *If h, of degree m, is a factor of f of degree n, the coefficient of x^{n-m-i} in the quotient is bounded by $\binom{n-m}{i}\frac{1}{\mathrm{lc}(h)}||f||$.*

  This is basically Corollary 26. Hence, as we are doing the trial division, we can give up as soon as we find a coefficient in the quotient that exceeds this bound, which *is* closely related to M (the difference relates to the leading coefficient terms).

  For example, if we take $p = 7$ in the example at the start of this chapter, we find that the g.c.d. of A_7 and B_7 is $x + 10$ (it could equally well be $x + 3$, but $x + 10$ makes the point better). Does $x + 10$ divide B? We note that $||B|| \approx 23.92$. Successive terms in the quotient are $3x^5$ (and 3 is a permissible coefficient), $-30x^4$ (and $30 < \binom{5}{1} \times 23.92$) and $305x^3$, at which point we observe that $305 > \binom{5}{2} \times 23.92 = 239.2$, so this *cannot* be a divisor of B. Hence 7 was definitely unlucky.

  With this refinement, it is possible to state that the numbers dealt with in this algorithm are "not much larger" than $2M$, though considerable ingenuity is needed to make this statement more precise.

If we apply this algorithm to the polynomials at the start of this section, we deduce that $\sqrt{\sum_{i=0}^{8} a_i^2} = \sqrt{113}$, $\sqrt{\sum_{i=0}^{6} b_i^2} = 2\sqrt{143}$, and hence corollary 10 gives a bound of

$$2^6 \min\left(\sqrt{113}, \frac{2}{3}\sqrt{143}\right) \approx 510.2, \tag{4.9}$$

so our first prime would be 1021, which is indeed of good reduction. In this case, corollary 11 gives a bound of

$$2^6 \min\left(\frac{1}{5}\sqrt{113}, \frac{2}{21}\sqrt{143}\right) \approx 72.8, \tag{4.10}$$

so our first prime would be 149. In general, we cannot tell which gives us the best bound, and it is normal to take the minimum.

Open Problem 18 *[Improving Landau–Mignotte for g.c.d.] A significant factor in the Landau–Mignotte bound here, whether (4.9) or (4.10), was the* $2^{\min(8,6)}$ *contribution from the degree of the putative g.c.d. But in fact the exponent is at most 4, not 6, since the g.c.d. cannot have leading coefficient divisible by 3 (since A does not). Hence the g.c.d. must have at most the degree of the g.c.d. modulo 3, and modulo 3 B has degree 4, so the gc.d. must have degree at most 4.*

Can this be generalised, in particular can we update our estimate "on the fly" as upper bounds on the degree of the g.c.d change, and is it worth it? In view of the 'early success' strategies discussed later, the answer to the last part is probably negative.

4.2.4 Computing the g.c.d.: several small primes

While algorithm 17 does give us some control on the size of the numbers being considered, we are still *often* using numbers larger than those which hindsight would show to be necessary. For example, in (4.6), (4.7) we could deduce coprimeness using the prime 5, rather than 1021 from (4.9) or 149 from (4.10). If instead we consider $(x-1)A$ and $(x-1)B$, the norms change, giving 812.35 in (4.9) (a prime of 1627) and 116.05 in (4.10) (a prime of 239). Yet primes such as 5, 11, 13 etc. will easily show that the result is $x-1$. Before we leap ahead and use such primes, though, we should reflect that, had we taken $(x-10)A$ and $(x-10)B$, 5 would have suggested x as the gcd, 11 would have suggested $x+1$, 13 would have suggested $x+3$ and so on.

The answer to this comes in observing that the smallest polynomial (in terms of coefficient size) which is congruent to x modulo 5 *and* to $x+1$ modulo 11 is $x-10$ (it could be computed by algorithm 48). More generally, we can apply the Chinese Remainder Theorem (Theorem 52) to enough primes of good reduction, as follows. We assume that `find_prime`(g) returns a prime not dividing g, a different one each time. The algorithm is given in Figure 4.3, with a diagram in Figure 4.4.

Figure 4.3: Algorithm 18

```
Algorithm 18 (Modular GCD (Small prime version))
Input: $A, B$ polynomials in $\mathbf{Z}[x]$.
Output: $\gcd(A, B)$

$M$ :=Landau_Mignotte_bound($A, B$);
$g := \gcd(\mathrm{lc}(A), \mathrm{lc}(B))$;
$p :=$ find_prime($g$);
$D := g$modular_gcd($A, B, p$);
if $\deg(D) = 0$ then return 1
$N := p$;       # $N$ is the modulus we will be constructing
while $N < 2M$ repeat               (*)
          $p :=$ find_prime($g$);
          $C := g$modular_gcd($A, B, p$);
          if $\deg(C) = \deg(D)$
             then $D :=$ Algorithm 48($C, D, p, N$);
                      $N := pN$;
             else if $\deg(C) < \deg(D)$
                        # $C$ proves that $D$ is based on primes of bad reduction
                        if $\deg(C) = 0$ then return 1
                        $D := C$;
                        $N := p$;
             else     #$D$ proves that $p$ is of bad reduction, so we ignore it
$D := \mathrm{pp}(D)$;       # In case multiplying by $g$ was overkill
Check that $D$ divides $A$ and $B$, and return it
If not, all primes must have been bad, and we start again
```

The “early abort” of Proposition 57 is needed for these divsibility checks if we are to maintain a “numbers not much larger than $2M$” guarantee.

Figure 4.4: Diagrammatic illustration of Algorithm 18

$$\begin{tikzcd}
\mathbf{Z}[x] \arrow[rrrr, dashed] \arrow[d, "k\times\text{reduce}"'] & & & & \mathbf{Z}[x] \\
\mathbf{Z}_{p_1}[x] \arrow[r, "\text{gcd}"] & \mathbf{Z}_{p_1}[x] & & & \\
\vdots & \vdots & \vdots \Bigg\} \arrow[rr, "\text{C.R.T.}"] & & \mathbf{Z}'_{p_1\cdots p_k}[x] \arrow[uu, "\text{interpret}"', "\text{\& check}"] \\
\mathbf{Z}_{p_k}[x] \arrow[r, "\text{gcd}"] & \mathbf{Z}_{p_k}[x] & & &
\end{tikzcd}$$

$\mathbf{Z}'_{p_1\cdots p_k}[x]$ indicates that some of the p_i may have been rejected by the compatibility checks, so the product is over a subset of $p_1 \cdots p_k$.

Observation 8 *The reader may think that Algorithm 18 is faulty: line (*) in Figure 4.3 iterates until $N \geq 2M$, which would be fine if we were actually computing the g.c.d. But we have forced the leading coefficient to be g, which may be overkill. Hence aren't we in danger of trying to recover g times the true g.c.d., whose coefficients may be* greater *than $2M$?*

In fact there is not a problem. The proof of Corollary 10 relies on estimating the leading coefficient of $\gcd(A,B)$ *by g, and so the bound is in fact a bound for the coefficients* after *this leading coefficient has been imposed.*

Having said that, we can't "mix and match". If we decide that Corollary 11 provides a better lower bound than Corollary 10, then we must go for "imposed trailing coefficients" rather than "imposed leading coefficients", or, and this is the way the author has tended to implement it, compute the g.c.d. of $\hat{A}$ and $\hat{B}$, and reverse that.

We should note the heavy reliance on Corollary 12 to detect bad reduction. We impose g as the leading coefficient throughout, and make the result primitive at the end as in the large prime variant.

4.2.5 Computing the g.c.d.: early success

While Algorithm 18 will detect a g.c.d. of 1 early, it will otherwise compute as far as the Landau–Mignotte bound if the g.c.d. is not 1. While this may be necessary, it would be desirable to terminate earlier if we have already found the g.c.d. This is easily done by replacing the line

then $D :=$ Algorithm $48(C, D, p, N)$;

by the code in Figure 4.5. We should note that we return an E which divides the inputs, and is derived from modular images, and therefore has to be the *greatest* common divisor by Corollary 12.

Figure 4.5: “Early termination” g.c.d. code

```
then $D' := D$
    $D :=$ Algorithm 48$(C, D, p, N)$;
    if $D = D'$     #We *may* have found the answer
      then $E := \mathrm{pp}(D)$;
           if $E$ divides $A$ and $B$
             then return $E$;
           # Otherwise this was a false alert, and we continue as normal.
```

4.2.6 An alternative correctness check

So far we have suggested computing the putative g.c.d. G, then checking that it really divides both, and relying on Corollary 12 to say that G is therefore a *greatest* common divisor. An alternative approach is to compute the co-factors, i.e. A' such that $A = A'G$ and B' such that $B = B'G$ at the same time, and use these as the check. So let us assume that `modular_gcd_cofactors` returns a triple $[G, A', B']$ modulo p. The Algorithm (19) is given in Figure 4.6, and the diagram in Figure 4.4 is still relevant. **TO BE COMPLETED**early termination: here and elsewhere.

Observation 9 *It is tempting to conjecture that we do not need to make both the multiplication checks at the end, but this is false: consider $A = H$, $B = H + 3p_1 \cdots p_k$, when the algorithm will find H as the putative g.c.d., since the Landau–Mignotte bound will ignore the large extra term in B, and only the multiplication check for B will detect this.*

Observation 10 *Early termination can perfectly well be applied to this variant: at any time $gA = DA'$ and $gB = DB'$ over the integers, we can finish.*

4.2.7 Conclusion

Observation 11 *We have presented this material as if there were a choice between one large prime (Algorithm 17) and several small ones (Algorithms 18, 19). In practice, of course, a computer regards all numbers less than 32 bits (and increasingly 64 bits) as ‘small’, so an implementation would generally use the largest ‘small’ primes it could in Algorithms 18, 19, and thus often one prime will suffice, and we have the same effect as Algorithm 17.*

Let us see how we have answered the questions on page 164.

1. Are there “good” reductions from $\mathbf{Z}$ to $\mathbf{Z}_p$? Yes — all primes except those that divide *both* leading coefficients (Lemma 7) and do not divide a certain resultant (Lemma 8). The technique of Lemma 11 (page 202) will show that there are only finitely many bad primes, but does not give a bound.

Figure 4.6: Algorithm 19

Algorithm 19 (Modular GCD (Alternative small prime version))
Input: A, B *polynomials in* $\mathbf{Z}[x]$.
Output: $G := \gcd(A, B), A', B'$ *with* $A = A'G$, $B = B'G$.

```
$M :=$Landau_Mignotte_bound$(A, B)$;
$g := \gcd(\mathrm{lc}(A), \mathrm{lc}(B))$;
$p :=$ find_prime$(g)$;
$[D, A', B'] :=$ modular_gcd_cofactors$(A, B, p)$;
**if** $\deg(D) = 0$ **then** **return** [1,A,B]
$D := gD$     # $g$ is our guess at the leading coefficient of the g.c.d.
$N := p$;     # $N$ is the modulus we will be constructing
**while** $N < 2M$ **repeat**
        $p :=$ find_prime$(g)$;
        $[C, A_1, B_1] :=$ modular_gcd_cofactors$(A, B, p)$;
        **if** $\deg(C) = \deg(D)$
           **then** $D :=$ Algorithm 48$(C, D, p, N)$;
                  $A' :=$ Algorithm 48$(A_1, A', p, N)$;
                  $B' :=$ Algorithm 48$(B_1, B', p, N)$;
                  $N := pN$;
           **else if** $\deg(C) < \deg(D)$
                   # $C$ proves that $D$ is based on primes of bad reduction
                   **if** $\deg(C) = 0$ **then return** [1,A,B]
                   $D := C$; $A' = A_1$; $B' = B_1$;
                   $N := p$;
           **else**    #$D$ proves that $p$ is of bad reduction, so we ignore it
**if** $gA = DA'$ and $gB = DB'$
   **then** $G := \mathrm{pp}(D) \gcd(\mathrm{cont}(A), \mathrm{cont}(B))$;     # Theorem 6
          $A' := A'/\mathrm{lc}(G)$; $B' := B'/\mathrm{lc}(G)$;        # Fix leading coefficients
          **return** $[G, A', B']$
   **else** all primes must have been bad, and we start again
```

2. How can we tell if p is good? We can't, but given two different primes p and q which give different results, we *can* tell which is definitely bad: Corollary 12.
3. How many reductions should we take? We want the apparently good primes to have a product greater than twice the Landau–Mignotte bound (Corollaries 10 and 11). Alternatively we can use early termination as in Figure 4.5.
4. How do we combine? Chinese Remainder Theorem (Algorithm 48).
5. How do we check the result? If it divides both the inputs, then it *is* a common divisor, and hence (Corollary 12) has to be the greatest.

All these algorithms use modular computations as Monte Carlo ("always fast/ probably correct") algorithms (Section 1.4.2), converted into a Las Vegas ("always correct/probably fast") one in the style of Figure 1.1 because we have correctness checks. In fact, because we can bound the number of bad primes (as opposed to just the probability of a prime being bad) we are actually *guaranteed* polynomial running time, so these Las Vegas algorithms are in fact perfectly normal algorithms, though the upper limits on the running times so obtained are *almost* always gross over-estimates.

4.3 Polynomials in two variables

The same techniques can be used to compute the greatest common divisor of polynomials in several variables. This is even more important than in the case of univariate polynomials, since the coeffcient growth observed on page 66 becomes degree growth in the other variables.

4.3.1 Degree Growth in Coefficients

As a trivial example of this, we can consider

$$\begin{aligned} A &= (y^2-y-1)x^2-(y^2-2)x+(2y^2+y+1); \\ B &= (y^2-y+1)x^2-(y^2+2)x+(y^2+y+2). \end{aligned}$$

The first elimination gives

$$C = (2y^2-4y)x+(y^4-y^3+2y^2+3y+3),$$

and the second gives

$$D = -y^{10}+3\,y^9-10\,y^8+11\,y^7-23\,y^6+22\,y^5-37\,y^4+29\,y^3-32\,y^2+15\,y-9.$$

Since this is a polynomial in y only, the greatest common divisor does not depend on x, i.e. $\mathrm{pp}_x(\gcd(A,B))=1$. Since $\mathrm{cont}_x(A)=1$, $\mathrm{cont}_x(\gcd(A,B))=1$, and hence $\gcd(A, B) = 1$, *but* we had to go via a polynomial of degree 10 to show this. Space does not allow us to show bigger examples in full, but it can be shown that, even using the subresultant algorithm (Algorithm 4), computing the g.c.d. of polynomials of degree d in x and y can lead to intermediate[9] polynomials of degree $O(d^2)$.

Suppose A and B both have degree d_x in x and d_y in y, with $A = \sum a_i x^i$ and $B = \sum b_i x^i$). After the first division (which is in fact a scaled subtraction $C := a_c B - b_c A$), the coefficients have, in general, degree $2d_y$ in y. If we assume[10] that each division reduces the degree by 1, then the next result would be $\lambda B - (\mu x + \nu)C$ where $\mu = b_c$ has degree d_y in y and λ and ν have degree $3d_y$. This result has degree $5d_y$ in y, but the subresultant algorithm will divide through by b_c, to leave a result D of degree $d_x - 2$ in x and $4d_y$ in y. Tthe next result would be $\lambda' C - (\mu' x + \nu')D$ where $\mu' = \mathrm{lc}_x(C)$ has degree $2d_y$ in y and λ' and ν' have degree $6d_y$. This result has degree $10d_y$ in y, but the subresultant algorithm will divide by a factor of degree 4, leaving an E of degree $6d_y$ in y. The next time round, we will have degree (after subresultant removal) $8d_y$ in y, and ultimately degree $2d_x d_y$ in y when it has degree 0 in x.

If this is not frigntening enough, consider what happens if, rather than being in x and y, our polynomials were in $n+1$ variables x and $y_1, \ldots, y_n$. Then the coefficients (with respect to x) of the initial polynomials would have degree d_y, and hence (at most) $(1 + d_y)^n$ terms, whereas the result would have $(1 + 2d_x d_y)^n$ terms, roughly $(2d_x)^n$, or exponentially many, times as many terms as the inputs.

Although the author knows of no way of formalising this statement, experience suggests that, even if f and g are sparse, i.e. have many fewer than $(1 + d_y)^n$ terms, the intermediate results are dense, and so the blowup *ratio* is even worse.

If we wish to consider taking greatest common divisors of polynomials in several variables, we clearly have to do better. Fortunately there are indeed better algorithms. The historically first such algorithm is based on Algorithm 18, except that evaluating a variable at a value replaces working modulo a prime.

Open Problem 19 (Alternative Route for Bivariate Polynomial g.c.d.) *Is there any reasonable hope of basing an algorithm on Algorithm 17? Intuition says not, and that, just as Algorithm 18 is preferred to Algorithm 17 for the univariate problem, so should it be here, but to the best of the author's knowledge the question has never been explored.*

The reader will observe that the treatment here is very similar to that of the univariate case, and may well ask "can the treatments be unified?" Indeed they can, and this was done in [Lau82], but the unification requires rather more algebraic machinery than we have at our disposal.

[9]We stress this word. Unlike the integer case, where the coefficients of a g.c.d. can be larger than those of the original polynomials, the degree in y of the final g.c.d. cannot be greater than the (minimum of) the degrees (in y) of the inputs.

[10]This is the "*normal p.r.s.* assumption: see footnote 37 (page 71).

Notation 28 *From now until section 4.4, we will consider the bivariate case,* $\gcd(A, B)$ *with* $A, B \in R[x, y] \equiv R[y][x]$*, and we will be considering evaluation maps replacing* y *by* $v \in R$*, writing* A_{y-v} *for the result of this evaluation.*

This notation is analogous to the previous section, where A_p was the remainder on dividing A by p.

Observation 12 *Clearly* $R[x, y] \equiv R[x][y]$ *also, and the* definition *of g.c.d. is independent of this choice. Algorithmically, though, it seems as if we must make such a choice. Some systems may already have imposed a default choice, but if we have a free hand it is usual to choose as the main variable (*x *in Notation 28) the one which minimises* $\min(\deg(A), \deg(B))$*.*

4.3.2 The evaluation–interpolation relationship

In this sub-section, we answer a question analogous to that in section 4.2.2: what do we do if the g.c.d. of the evaluations is not the image under evaluation of the g.c.d. calculated before evaluation?

Lemma 9 *If* $y-v$ *does not divide the leading coefficient of* $\gcd(A, B)$*, the degree of* $\gcd(A_{y-v}, B_{y-v})$ *is greater than or equal to that of* $\gcd(A, B)$*.*

Proof. Since $\gcd(A, B)$ divides A, then $(\gcd(A, B))_{y-v}$ divides A_{y-v}. Similarly, it divides B_{y-v}, and therefore it divides $\gcd(A_{y-v}, B_{y-v})$. This implies that the degree of $\gcd(A_{y-v}, B_{y-v})$ is greater than or equal to that of $\gcd(A, B)_{y-v}$. But the degree of $\gcd(A, B)_{y-v}$ is equal to that of $\gcd(A, B)$, for the leading coefficient of $\gcd(A, B)$ does not cancel when it is evaluated at v.

This lemma is not very easy to use on its own, for it supposes that we know the g.c.d. (or at least its leading coefficient) before we are able to check whether the modular reduction has the same degree. But this leading coefficient has to divide the two leading coefficients of A and B, and this gives a formulation which is easier to use.

Corollary 13 *If* $y - v$ *does not divide the leading coefficients of* A *and of* B *(it may divide one, but not both), then the degree of* $\gcd(A_{y-v}, B_{y-v})$ *is greater than or equal to that of* $\gcd(A, B)$*.*

As the g.c.d. is the only polynomial (to within a multiple from $R[y]$) of its degree (in x) which divides A and B, we can test the correctness of our calculations of the g.c.d.: if the result has the degree of $\gcd(A_{y-v}, B_{y-v})$ (where v satisfies the hypothesis of this corollary) and if it divides A and B, then it is the g.c.d. (to within a multiple from $R[y]$).

As in section 4.2.2, it is quite possible that we could find a $\gcd(A_{y-v}, B_{y-v})$ of too high a degree: consider $A = x - 1$, $B = x - y$ and the evaluation $y \mapsto 1$. The following lemma shows that this possibility can only arise for a finite number of v.

Lemma 10 *Let $C = \gcd(A, B)$. If v satisfies the condition of the corollary above, and if $y - v$ does not divide* $\mathrm{Res}_x(A/C, B/C)$*, then* $\gcd(A_{y-v}, B_{y-v}) = C_{y-v}$.

Proof. A/C and B/C are relatively prime, for otherwise C would not be the g.c.d. of A and B. By the corollary, C_{y-v} does not vanish. Therefore

$$\gcd(A_{y-v}, B_{y-v}) = C_{y-v} \gcd(A_{y-v}/C_{y-v}, B_{y-v}/C_{y-v}).$$

For the lemma to be false, the last g.c.d. has to be non-trivial. This implies that the resultant $\mathrm{Res}_x(A_{y-v}/C_{y-v}, B_{y-v}/C_{y-v})$ vanishes, by proposition 78 of the Appendix. This resultant is the determinant of a Sylvester matrix[11], and $|M_{y-v}| = (|M|)_{y-v}$, for the determinant is only a sum of products of the coefficients. In the present case, this amounts to saying that $\mathrm{Res}_x(A/C, B/C)_{y-v}$ vanishes, that is that $y - v$ divides $\mathrm{Res}_x(A/C, B/C)$. But the hypotheses of the lemma exclude this possibility.

Definition 87 *If* $\gcd(A_{y-v}, B_{y-v}) = \gcd(A, B)_{y-v}$*, we say that the evaluation of this problem at v is* good*, or that $y - v$ is* of good reduction*. If not, we say that $y - v$ is* of bad reduction.

This lemma implies, in particular, that there are only a finite number of values v such that $\gcd(A_{y-v}, B_{y-v})$ does not have the same degree as that of $\gcd(A, B)$, that is the $y - v$ which divide the g.c.d. of the leading coefficients and the $y - v$ which divide the resultant of the lemma (the resultant is non-zero, and therefore has only a finite number of divisors). In particular, if A and B are relatively prime, we can always find a v such that A_{y-v} and B_{y-v} are relatively prime.

We can summarize this section as follows.

Theorem 38 (Good Reduction Theorem (polynomial)) *If $y-v$ does not divide* $\gcd(a_\alpha, b_\beta)$ *(which can be checked for in advance) or* $\mathrm{Res}_x(A/C, B/C)$*, then v is of good reduction. Furthermore, if $y - v$ divides* $\mathrm{Res}_x(A/C, B/C)$ *but not* $\gcd(a_\alpha, b_\beta)$*, then the gcd computed modulo $y - v$ has a* larger *degree than the true result.*

4.3.3 G.c.d. in $\mathbf{Z}_p[x, y]$

By Gauss' Lemma (Theorem 6),

$$\gcd(A, B) = \gcd(\mathrm{cont}_x(A), \mathrm{cont}_x(B)) \gcd(\mathrm{pp}_x(A), \mathrm{pp}_x(B)),$$

and the real problem is the second factor.

[11]There's a subtle point here. The resultant of polynomials of degrees m and n is the determinant of an $(m + n)^2$ matrix. Hence if $y - v$ divides neither leading coefficient, the Sylvester matrix of A_{y-v} and B_{y-v} is indeed the reduction of the Sylvester matrix of A and B. If $y-v$ divides one leading coefficient, but not the other, the Sylvester matrix of A_{y-v} and B_{y-v} is smaller, and the reader should check that this only makes a difference of a product of that leading coefficient which doesn't vanish when v is substituted for y.

Figure 4.7: Diagrammatic illustration of Algorithm 21

$$\begin{tikzcd}
\mathbf{Z}_p[y][x] \arrow[rrrr, dashed] \arrow[d, "k\times\text{reduce}"'] & & & & \mathbf{Z}_p[y][x] \\
\mathbf{Z}_p[y]_{y-v_1}[x] \arrow[r, "\text{gcd}"] & \mathbf{Z}_p[y]_{y-v_1}[x] & & & \\
\vdots & \vdots & \vdots & \left.\right\} \arrow[r, "\text{C.R.T.}"] & \frac{\mathbf{Z}_p[y][x]}{\prod'(y-v_1)\cdots(y-v_k)} \arrow[uu, "\text{interpret}\ \&\ \text{check}"'] \\
\mathbf{Z}_p[y]_{y-v_k}[x] \arrow[r, "\text{gcd}"] & \mathbf{Z}_p[y]_{y-v_k}[x] & & &
\end{tikzcd}$$

$\prod'$ indicates that some of the v_i may have been rejected by the compatibility checks, so the product is over a subset of $(y-v_1)\cdots(y-v_k)$.

Observation 13 *While the content of $A=\sum_{i=0}^m a_ix^i$ can be computed as*

$$\gcd(a_m, \gcd(a_{m-1}, \gcd(a_{m-2}, \ldots, a_0)\ldots),$$

the following process is more efficient in practice (asymptotically, it's only worth a constant factor on what is asymptotically the cheaper operation, but in practice it is worthwhile), and is valid over any g.c.d. domain R with a concept of 'size' equivalent to degree.

Algorithm 20 (Content)
Input: $A=\sum_{i=0}^m a_ix^i \in R[x]$.
Output: $\text{cont}_x(A)$

```
$S := \{a_i\}$        # Really a set, as duplicates don't matter
$g :=$ minimal degree element of $S$; $S := S \setminus \{g\}$
if $g$ is a unit
  then return 1
$g := \gcd(g, \sum_{h_i\in S} \lambda_i h_i)$        # $\lambda_i$ random
if $g$ is a unit
  then return 1
for $h \in S$
    if $g$ does not divide $h$
        then $g := \gcd(g, h)$
             if $g$ is a unit
                then return 1
return $g$
```

We present an Algorithm (21), analogous to Algorithm 18, for the g.c.d. of primitive polynomials. It would equally be possible to start from Algorithm 19. There is also an 'early termination' version, analogous to the code in Figure 4.5.

We should note the caveat on `find_value` in Figure 4.8: although we know there are only a finite number of bad values v, since $\mathbf{Z}_p$ is finite it is possible for too many, indeed all, values in $\mathbf{Z}_p$ to be bad, and we may need to move to an algebraic extension of $\mathbf{Z}_p$ to have enough good values.

Figure 4.8: Algorithm 21

```
**Algorithm 21 (Bivariate Modular GCD)**
**Input:** $A, B$ primitive polynomials in $\mathbf{Z}_p[y[[x]$.
**Output:** $\gcd(A, B)$

$g := \gcd(\mathrm{lc}_x(A), \mathrm{lc}_x(B))$;
$v :=$ find_value($g$);
$D := g$modular_gcd($A_{y-v}, B_{y-v}, p$);
if $\deg(D) = 0$ then return 1
$N := y - v$;      # $N$ is the modulus we will be constructing
while $\deg_y(N) \le \min(\deg_y(A), \deg_y(B))$ repeat
    $v :=$ find_value($g$);
    $C := g$modular_gcd($A_{y-v}, B_{y-v}, p$);
    if $\deg(C) = \deg(D)$
      then $D :=$ Algorithm 50($C, D, y - v, N$);
           $N := (y - v)N$;
      else if $\deg(C) < \deg(D)$
             # $C$ proves that $D$ is based on values of bad reduction
             if $\deg(C) = 0$ then return 1
             $D := C$;
             $N := y - v$;
      else     #$D$ proves that $v$ is of bad reduction, so we ignore it
$D := \mathrm{pp}(D)$;      # In case multiplying by $g$ was overkill
Check that $D$ divides $A$ and $B$, and return it
If not, all values must have been bad, and we start again
```

`find_value`(g) finds a new value v each time, such that g_{y-v} does not vanish. It is conceivable that we will exhaust $\mathbf{Z}_p$, in which case we have to move to choosing v from an algebraic extension of $\mathbf{Z}_p$. Indeed it is possible that there are no values v in $\mathbf{Z}_p$ such that g_{y-v} does not vanish, e.g. $g = y(y-1)(y-2)$ when $p = 3$.

Figure 4.9: Diagrammatic illustration of g.c.d.s in $\mathbf{Z}[x,y]$ (1)

$$\begin{tikzcd}
\mathbf{Z}[y][x] \arrow[rrr, dashed] \arrow[d, "k\times\text{reduce}"'] & & & \mathbf{Z}[y][x] \\
\mathbf{Z}[y]_{y-v_1}[x] \arrow[r, "\text{gcd}"] & \mathbf{Z}[y]_{y-v_1}[x] & & \\
\vdots & \vdots & \xrightarrow{\text{C.R.T.}} & \dfrac{\mathbf{Z}[y][x]}{\prod'(y-v_1)\cdots(y-v_k)} \arrow[uu, "\text{interpret \& check}"'] \\
\mathbf{Z}[y]_{y-v_k}[x] \arrow[r, "\text{gcd}"] & \mathbf{Z}[y]_{y-v_k}[x] & & \\
& \text{using Algorithm 18/19} \arrow[ul, shorten >=1em] & &
\end{tikzcd}$$

4.3.4 G.c.d. in $\mathbf{Z}[x,y]$

We know how to compute g.c.d.s in $\mathbf{Z}_p[x]$, and wish to compute g.c.d.s in $\mathbf{Z}[x,y]$. We have seen the building blocks in the previous sections, diagrammatically in figures 4.3 and 4.7. We have a choice of ways of combining these constructions, though, depending on what we choose as the intermediate domain.

$\mathbf{Z}[x]$ Here we use an analogue of Figure 4.7 to reduce us to the univariate case. The corresponding diagram then looks like Figure 4.9. There is one complication we should note, though. We are now applying the Chinese Remainder Theorem to polynomials in $\mathbf{Z}[y]$, rather than $\mathbf{Z}_p[y]$, and $\mathbf{Z}$ is not a field. Indeed, the Theorem does not always apply over $\mathbf{Z}$, since, for example, the polynomial satisfying $f(0) = 0$ and $f(2) = 1$ is $f = \frac{1}{2}y$, which is in $\mathbf{Q}[y]$ but not $\mathbf{Z}[y]$.

Should this happen, we actually know that *all* reductions so far are wrong, because they are compatible in degree, but cannot be right. Hence we start again.

$\mathbf{Z}_p[y][x]$ Here we use an analogue of Figure 4.2 to reduce us to the case of Figure 4.7. The overall structure is shown in Figure 4.10.

Open Problem 20 *[Which is the Better Route for Bivariate g.c.d.?] As far as the author can tell, the question of which route to follow has not been systematically explored. The initial literature [Bro71b, Algorithm M] and the more recent survey [Lau82] assume the route in Figure 4.10. [Bro71b] explicitly assumes that p is large enough that we never run out of values, i.e. that the algebraic extension at the end of Figure 4.8 is never needed.*

Implementations the author has seen tend to follow the route in Figure 4.9. This is probably for pragmatic reasons: as one is writing a system one first wishes for univariate g.c.d.s, so implements Algorithm 18 (or 19). Once one has this, Figure 4.9 is less work.

There is a natural tendency to believe that Figure 4.10 is better, as all numbers involved are bounded by p untl the very last Chinese Remainder calculations.

Figure 4.10: Diagrammatic illustration of g.c.d.s in $\mathbf{Z}[x, y]$ (2)

$$\begin{tikzcd}
\mathbf{Z}[y][x] \arrow[rrr, dashed] \arrow[d, "k\times\text{reduce}"'] & & & \mathbf{Z}[y][x] \\
\mathbf{Z}_{p_1}[y][x] \arrow[r, "\text{gcd}"] & \mathbf{Z}_{p_1}[y][x] & & \\
\vdots & \vdots & \text{C.R.T.} \arrow[r] & \dfrac{\mathbf{Z}[y][x]}{\prod' p_1 \cdots p_k} \arrow[uu, "\text{interpret}\ \&\ \text{check}"'] \\
\mathbf{Z}_{p_k}[y][x] \arrow[r, "\text{gcd}"] & \mathbf{Z}_{p_k}[y][x] & & \\
\text{using Algorithm 21} \arrow[u] & & &
\end{tikzcd}$$

On the other hand, if the g.c.d. is actually 1, but the first prime chosen is unlucky, one will reconstruct all the way back to a bivariate before realising the problem.

Let us look at the questions on page 164.

1. Are there "good" reductions from R? Yes, since there are only finitely many bad reductions, but it is possible that we will need to go to an algebraic extension if we are working over a finite field: see the caveat on `find_value` in Figure 4.8.

2. How can we tell if an evaluation v is good? We can't, but given two different evaluations v and w which give different results, we *can* tell which is definitely bad: Corollary 13.

3. How many reductions should we take? If d bounds the degree in y of the g.c.d., then we only need $d + 1$ good reductions. We don't have the same problem as in the univariate case, when our bounds were almost certainly pessimistic, but it is nevertheless common to use early termination as in Figure 4.5.

4. How do we combine the results — Algorithm 50.

5. How do we check the result? If it divides both the inputs, then it *is* a common divisor, and hence (Corollary 13) has to be the greatest.

As on page 180, these algorithms use modular/evaluation computations as Monte Carlo ("always fast/probably correct") algorithms (Section 1.4.2), converted into a Las Vegas ("always correct/probably fast") one in the style of Figure 1.1 because we have correctness checks. In fact, because we can bound the number of bad evaluations and bad primes (as opposed to just the probability of an evaluation/prime being bad) we are actually *guaranteed* polynomial running time. It is even truer here that the upper limits on the running times so obtained are *almost* always gross over-estimates.

4.4 Polynomials in several variables

There is no conceptual difficulty in generalising the work of the previous section to n variables. In principle, there are n possible diagrams corresponding to Figures 4.9-4.10, but in practice the choice is between 'modular last' (Figure 4.9) and 'modular first' (Figure 4.10), and the author has never seen a hybrid algorithm whch reduces, say,

$$\mathbf{Z}[y][z][x] \to \mathbf{Z}[y][z]_{z-v}[x] \to \mathbf{Z}_p[y][z]_{z-v}[x] \to \mathbf{Z}_p[y]_{y-v'}[z]_{z-v}[x].$$

Theorem 39 ([Bro71b, (95)]) *Suppose $f, g \in \mathbf{Z}[x_1, \ldots, x_n]$. Let l bound the lengths of the integer coefficients of f and g, and d bound their degree in each of the x_i. Then under the following assumptions:*

1. *classical $O(N^2)$ multiplication and division of N-word numbers or N-term polynomials;*
2. *no bad values are found, either for primes or for evaluation points;*
3. *that the answer, and corresponding quotients, have coefficient lengths at most l;*
4. *that we can use single-word primes;*

the running time of the 'modular first' (Figure 4.10) algorithm is

$$O\left(l^2(d+1)^n + (d+1)^{n+1}\right).$$

This contrasts with the subresultant algorithm's bound [op. cit. (80)] (assuming the p.r.s. are normal) of

$$O\left(l^2(d+1)^{4n}2^{2n^2}3^n\right),$$

so the dependence on $(d+1)^n$ has essentially gone from quartic to linear. $(d+1)^n$ is the maximal number of terms in the input, so this "modular first" algorithm is "almost optimal" in the sense of Definition 20 for dense inputs.

The real challenge comes with the potential sparsity of the polynomials. The factor $(d+1)^n$ in the running time comes simply from the fact that a dense polynomial of degree d in n variables has that many terms, and we will need $(d+1)^{n-1}$ univariate g.c.d.s (therefore $(d+1)^n$ coefficients) to deduce these potential terms. Furthermore it *is* possible for sparse polynomials to have dense g.c.d.s: the following elegant univariate example is due to [Sch03a] (see also (2.21)):

$$\gcd(\underbrace{x^{pq}-1}_{f(x)}, \underbrace{x^{p+q}-x^p-x^q+1}_{g(x)}) = \frac{(x^p-1)(x^q-1)}{x-1} = \underbrace{x^{p+q-1}-x^{p+q-2}\pm\cdots-1}_{h(x)}. \qquad (4.11)$$

Example 21 (Dense g.c.d.s) *Therefore*

$$\gcd(f(x_1)f(x_2)\cdots f(x_k), g(x_1)g(x_2)\cdots g(x_k)) = h(x_1)h(x_2)\cdots h(x_k), \quad (4.12)$$

and the righthand side has $(2\min(p,q))^k$ *terms, whereas the arguments to* gcd *have* 2^k *and* 4^k *terms.*

In fact, we can be slightly more subtle and take

$$\gcd(f(x_1)g(x_2)f(x_3)g(x_4)\cdots g(x_k), g(x_1)f(x_2)g(x_3)f(x_4)\cdots f(x_k)), \quad (4.13)$$

with the same gcd *but where the arguments have* $8^{k/2} \approx 2.8^k$ *terms.*

Problem 6 (Sparse g.c.d.) *Produce an algorithm for sparse multivariate g.c.d. whose running time is polynomial, ideally linear, in the number of terms in the inputs and output. If we can't find a deterministic algorithm, maybe we can find a Las Vegas (section 1.4.2) one.*

The 'modular first' (Figure 4.10) algorithm does not solve this problem as it will attempt to interpolate $\prod(d_1+1)$ terms if the g.c.d. has degree d_i in x_i. The problem was first addressed, in Las Vegas style, in [Zip79a, Zip79b].

4.4.1 A worked example

Let f be

$$\begin{aligned}
&x^5y^7z^4 + x^4y^8z^4 + x^4y^7z^4 + x^3y^7z^4 + x^2y^8z^4 + x^5y^3z^5 + x^4y^4z^5 +\\
&x^3y^4z^6 + x^2y^7z^4 + x^2y^5z^6 - x^5y^7 + x^5yz^6 - x^4y^8 + x^4y^3z^5 +\\
&x^4y^2z^6 - x^3y^4z^5 - x^2y^5z^5 + x^2y^4z^6 - x^4y^7 + x^4yz^6 - x^2y^4z^5 - x^3y^7 -\\
&x^2y^8 - x^2y^7 - x^5y^3 - x^4y^4 - x^4y^3 - x^5y - x^4y^2 - x^4y + x^3z +\\
&x^2yz - x^3 - x^2y + x^2z - x^2 + xz + yz - x - y + z - 1
\end{aligned} \quad (4.14)$$

and g be

$$\begin{aligned}
&x^6y^7z^4 - x^4y^7z^4 + x^6y^3z^5 + x^4y^7z^3 + x^4y^4z^6 - x^6y^7 + x^6yz^6 + x^4y^7z^2\\
&-x^4y^4z^5 - 2\,x^2y^7z^4 + x^4y^7z - 2\,x^4y^3z^5 + x^2y^7z^3 - 2\,x^2y^4z^6 + 2\,x^4y^7 +\\
&x^4y^3z^4 - 2\,x^4yz^6 + x^2y^7z^2 + 3\,x^2y^4z^5 + x^4y^3z^3 + x^4yz^5 + x^2y^7z -\\
&x^6y^3 + x^4y^3z^2 + x^4yz^4 + 3\,x^2y^7 + x^4y^3z + x^4yz^3 - x^6y + 3\,x^4y^3 +\\
&x^4yz^2 + x^4yz + 3\,x^4y + x^4z - x^4 - x^2z + 2\,x^2 - 2\,z + 3
\end{aligned} \quad (4.15)$$

polynomials with 42 and 39 terms respectively (as against the 378 and 392 they would have if they were dense of the same degree).

Let us regard x as the variable to be preserved throughout. If we compute $\gcd(f|_{z=2}, g|_{z=2})$ (which would require at least[12] 8 y values) we get (assuming that $z=2$ is a good reduction)

$$\gcd(f,g)|_{z=2} = \left(15\,y^7 + 31\,y^3 + 63\,y\right)x^4 + \left(15\,y^7 + 32\,y^4 + 1\right)x^2 + 1. \quad (4.16)$$

[12] In fact, with $y=0$ both f and g drop in x-degree, whereas with $y=-2$ we get a g.c.d. with x-degree 5, as $y=-2$ is a root of the resultant in Lemma 10.

We note that each coefficient (with respect to x) has at most three terms. A dense algorithm would compute five more equivalents of (4.16), at different z values, and then interpolate z polynomials. Each such computation would require 8 y values. Instead, if we believe[13] that (4.16) describes accurately the dependence of the g.c.d. on y, we should be able to deduce these equivalents with only three y values. Consider

$$\left.\begin{array}{rcl} \gcd(f|_{z=3,y=1}, g|_{z=3,y=1}) &=& 525x^4+284x^2+1 \\ \gcd(f|_{z=3,y=-1}, g|_{z=3,y=-1}) &=& -525x^4+204x^2+1 \\ \gcd(f|_{z=3,y=2}, g|_{z=3,y=2}) &=& 6816x^4+9009x^2+1 \end{array}\right\} \qquad (4.17)$$

We should interpolate the coefficients of x^4 to fit the template ay^7+by^3+cy, and those of x^2 to fit the template $a'y^7+b'y^4+c$, while those of x^0 should fit the template $a''y^0$, i.e. be constant. Considering the coefficients of x^4, we have to solve the equations

$$\left.\begin{array}{rcl} a+b+c &=& 525 \\ -a-b-c &=& -525 \\ 2^7a+2^3b+2c &=& 6816 \end{array}\right\}. \qquad (4.18)$$

Unfortunately, these equations are under-determined (the second is minus twice the first), so we have to add another equation to (4.17), e.g.

$$\gcd(f|_{z=3,y=3}, g|_{z=3,y=3}) = 91839x^4+107164x^2+1,$$

which adds $3^7a+3^3b+3c=91839$ to (4.18) and the augmented system is now soluble, as $a=40, b=121, c=364$.

This process gives us an *assumed* (two assumptions are now in play here: that $z=3$ is a good reduction, and that (4.16) describes the sparsity structure $\gcd(f,g)$ accurately) value of

$$\gcd(f,g)|_{z=3} = \left(40\,y^7+121\,y^3+364\,y\right)x^4+\left(40\,y^7+243\,y^4+1\right)x^2+1. \quad (4.19)$$

Similarly we can deduce that

$$\begin{array}{l} \gcd(f,g)|_{z=4} = \left(85\,y^7+341\,y^3+1365\,y\right)x^4+\left(85\,y^7+1024\,y^4+1\right)x^2+1 \\ \gcd(f,g)|_{z=5} = \left(156\,y^7+781\,y^3+3906\,y\right)x^4+\left(156\,y^7+3125\,y^4+1\right)x^2+1 \\ \gcd(f,g)|_{z=6} = \left(259\,y^7+1555\,y^3+9331\,y\right)x^4+\left(259\,y^7+7776\,y^4+1\right)x^2+1 \\ \gcd(f,g)|_{z=7} = \left(400\,y^7+2801\,y^3+19608\,y\right)x^4+\left(400\,y^7+16807\,y^4+1\right)x^2+1 \\ \gcd(f,g)|_{z=8} = \left(585\,y^7+4681\,y^3+37449\,y\right)x^4+\left(585\,y^7+32768\,y^4+1\right)x^2+1 \end{array} \quad (4.20)$$

(each requiring, at least, three y evaluations, but typically no more).

If we now examine the coefficients of x^4y^7, and assume that in $\gcd(f,g)$ there is a corresponding term $x^4y^7p(z)$, we see that $p(2)=15$ (from (4.16)), $p(3)=40$ (from (4.19)), and similarly $p(4),\ldots,p(8)$ from (4.20). Using Algorithm 49, we

[13]We refer to this as the Zippel assumption, after its introduction in [Zip79a, Zip79b], and formalize it in Definition 89.

deduce that $p(z) = z^3 + z^2 + z + 1$. We can do the same for all the other (presumed) $x^i y^j$ terms in the g.c.d., to get the following final result:

$$\begin{aligned} x^4\Big(&y^7\left(z^3+z^2+z+1\right)+y^3\left(z^4+z^3+z^2+z+1\right)+\\ &y\left(z^5+z^4+z^3+z^2+z+1\right)\Big)+x^2\left(y^7\left(z^3+z^2+z+1\right)+y^4z^5+1\right)+1. \end{aligned} \tag{4.21}$$

We still need to check that it *does* divide g and g, but, once we have done so, we are assured by Corollary 13 that it is the *greatest* common divisor.

4.4.2 Converting this to an algorithm

There are, however, several obstacles to converting this to an algorithm: most centring around the linear systems such as (4.18). First, however, we must formalize the assumption made on page 190.

Definition 88 *We define the* skeleton *of a polynomial $f \in R[x_1, \ldots, x_n]$, denoted* Sk$(f)$*, to be the set of monomials in a (notional) distributed representation of f.*

Definition 89 *We say that an evaluation $\phi : R[x_1, \ldots, x_n] \to R[x_i : i \notin S]$, which associates a value in R to every $x_i : i \in S$,* satisfies the Zippel assumption for *f if it preserves the skeleton in the variables not in S, i.e. $\psi(\mathrm{Sk}(f)) = \mathrm{Sk}(\phi(f))$, where ψ deletes the powers of $x_i : i \in S$ from monomials.*

Another way of saying this is to write f (notionally) in $R[x_i : i \in S][x_i : i \notin S]$, where the outer $[\ldots]$ is represented distributedly, and say that ϕ satisfies the Zippel assumption for *f if it maps no non-zero coefficient of ϕ to zero.*

For example, if $f = (y^2 - 1)x^2 - x$, $\mathrm{Sk}(f) = \{x^2y^2, x^2, x\}$. If $\phi : y \mapsto 1$, then $\phi(f) = -x$, so $\mathrm{Sk}(\phi(f)) = \{x\}$, but $\psi(\mathrm{Sk}(f)) = \{x^2, x\}$, so this ϕ does not satisfy the Zippel assumption (it have mapped $y^2 - 1$ to 0). However, if $\phi : y \mapsto 2$, $\phi(f) = 3x^2 - x$ and $\mathrm{Sk}(\phi(f)) = \{x^2, x\}$, so this ϕ satisfies the Zippel assumption.

Hence the assumption on page 190 was that $z \mapsto 2$ satisfies the Zippel assumption. Let us now look at the problems surrounding (4.18). In fact, there are three of them.

1. The equations might be under-determined, as we saw there.
2. We need to solve a $t \times t$ system, where t is the number of terms being interpolated. This will normally take $O(t^3)$ coefficient operations, and, as we know from section 3.2.3, the coefficients may also grow during these operations.
3. The third problem really goes back to (4.17). We stated that

$$\gcd(f|_{z=3,y=-1}, g|_{z=3,y=-1}) = -525x^4 + 204x^2 + 1,$$

Figure 4.11: Diagrammatic illustration of sparse g.c.d.

$$\begin{tikzcd}
R[\mathbf{y}][x] \arrow[rr, dashed] \arrow[d, "k\times\text{reduce}"'] & & R[\mathbf{y}][x] \\
R[\mathbf{y}]_{y_n-v_1}[x] \arrow[r, "\text{gcd}"] & R[\mathbf{y}]_{y_n-v_1}[x] \arrow[dl, "\text{Sk}"] & \\
R[\mathbf{y}]_{y_n-v_2}[x] \arrow[r, "\text{gcd}'"] & R[\mathbf{y}]_{y_n-v_2}[x] \arrow[r, "\text{C.R.T.}"] & \frac{R[\mathbf{y}][x]}{\prod'(y_n-v_1)\cdots(y_n-v_k)} \arrow[uu, "\text{interpret}\ \&\ \text{check}"'] \\
\vdots & \vdots \quad \vdots & \\
R[\mathbf{y}]_{y_n-v_k}[x] \arrow[r, "\text{gcd}'"] & R[\mathbf{y}]_{y_n-v_k}[x] &
\end{tikzcd}$$

$\prod'$ indicates that some of the v_i may have been rejected by the compatibility checks, so the product is over a subset of $(y_n - v_1) \cdots (y_n - v_k)$.
gcd is a recursive call to this algorithm, while gcd$'$ indicates a g.c.d. computation to a prescribed skeleton, indicated by Sk: Algorithm 24.

> but we could equally well have said that it was $525x^4 - 204x^2 - 1$, at which point we would have deduced that the equivalent of (4.18) was inconsistent. In fact, Definition 31 merely defines *a* greatest common divisor, for the reasons outlined there. We therefore, as in Algorithm 18, impose $\gcd(\mathrm{lc}(f), \mathrm{lc}(g))$, as the leading coefficient of the greatest common divisor.

To solve the first two problems, we will use the theory of section A.5, and in particular Corollary 29, which guarantees that a Vandermonde matrix is invertible. In fact, we will need to solve a modified Vandermonde system of form (A.12), and rely on Corollary 30. If, instead of random values for y, we had used powers of a given value, we would get such a system in place of (4.18).

We now give the interlocking set of algorithms, in Figures 4.12–4.14. We present them in the context of the diagram in Figure 4.11.

4.4.3 Worked example continued

We now consider a larger example in four variables. Let f and g be the polynomials in Figures 4.15 and 4.16, with 104 and 83 terms respectively (as against the 3024 and 3136 they would have if they were dense of the same degree).

Let us regard x as the variable to be preserved throughout. $f|_{w=1}$ and $g|_{w=1}$ are, in fact, the polynomials of (4.14) and (4.15), whose g.c.d. we have already computed to be (4.21).

We note that the coefficients (with respect to x) have fifteen, six and one term respectively, and f and g both have degree seven in w. We need to compute seven more equivalents of (4.21), at different w values, and then interpolate w polynomials. Let us consider computing $\gcd(f|_{w=2}, g|_{w=2})$ under the assumptions that 1 and 2 are good values of w, *and* that (4.21) correctly describes the

Figure 4.12: Algorithm 22: Sparse g.c.d.

```
**Algorithm 22 (Sparse g.c.d.)**
**Input:** $A, B$ polynomials in $R[\mathbf{y}][x]$.
**Output:** $\gcd(A, B)$

$A_c := \mathrm{cont}_x(A)$; $A := A/A_c$;        # Using Algorithm 20, which
$B_c := \mathrm{cont}_x(B)$; $B := B/B_c$;        # calls this algorithm recursively
$g :=$Algorithm 22($\mathrm{lc}_x(A), \mathrm{lc}_x(B)$);        # one fewer variable
$G :=$failed
while $G =$failed        # Expect only one iteration
        $G :=$Algorithm 23($g, gA, gB$);
return $\mathrm{pp}_x(G)\times$ Algorithm 22($A_c, B_c$);        # one fewer variable
```

Figure 4.13: Algorithm 23: Inner sparse g.c.d.

```
**Algorithm 23 (Inner sparse g.c.d.)**
**Input:** $g$ leading coefficient; $A, B$ polynomials in $R[\mathbf{y}][x]$.
**Output:** $\gcd(A, B)$, but with leading coefficient $g$
    or failed if we don't have the Zippel assumption

$d_n := \min(\deg_{y_n}(A), \deg_{y_n}(B))$        # (over)estimate for $\deg_{y_n}(\gcd(A, B))$
$v_0 :=$random(lc($A$), lc($B$));        #assumed good and satisfying Zippel for $\gcd(A, B)$;
$P_0 :=$Algorithm 23($g|_{y_n - v_0}, A|_{y_n - v_0}, B|_{y_n - v_0}$)
if $P_0 =$failed
    return failed        # Don't try again
$d_x := \deg_x(P_0)$
$i := 0$
while $i < d$
        $v :=$random(lc($A$), lc($B$));        #assumed good
        $P_{i+1} :=$Algorithm 24(Sk($P_0$), $g|_{y_n - v}, A|_{y_n - v}, B|_{y_n - v}$)
        if $\deg_x(P_{i+1}) > d_x$        # Either $v^j$ in Algorithm 24 or
            continue round the loop        # $|_{y_n - v}$ was not a good evaluation
        if $\deg_x(P_{i+1}) < d_x$
            return failed        # $|_{y_n - v_0}$ was not a good evaluation
        $i := i + 1$; $v_i := v$        # store $v$ and the corresponding $P$
$C :=$Algorithm 50($\{P_i\}, \{v_i\}$)        # Reconstruct
if $C$ divides both $A$ and $B$
    return $C$
    else return failed
```

random(lc(A), lc(B)) chooses a value not annihilating both leading coefficients

Figure 4.14: Algorithm 24: Sparse g.c.d. from skeleton

Algorithm 24 (Sparse g.c.d. from skeleton)
Input: Sk *skeleton, g leading coefficient; A, B polynomials in* $R[\mathbf{y}][x]$.
Output: $\gcd(A,B)$ *assumed to fit* Sk, *but with leading coefficient g*
or *the univariate gcd if this doesn't fit* Sk
Note *that* $\mathbf{y}$ *is* $(y_1,\ldots,y_{n-1})$ *since* y_n *has been evaluated*

```
$d_x := \deg_x(\mathrm{Sk})$
$t := \max_{0\le i\le d_x}$term count$(\mathrm{Sk}, x^i))$
$\mathbf{v} = (v_1,\ldots,v_{n-1})$:= values in $R$
for $j := 1\ldots t$
    $C := \gcd(A_{\mathbf{y}=\mathbf{v}^j}, B_{\mathbf{y}=\mathbf{v}^j})$
    if $\deg_x(C) \ne d_x$
        return $C$          # We return the univariate
    $P_j := \frac{g_{\mathbf{y}=\mathbf{v}^j}}{\mathrm{lc}(C)} C$          # impose leading coefficient
$G := 0$
for $i := 0\ldots d_x$
    $S := [\mathbf{y}^{\mathbf{n}} : x^i\mathbf{y}^{\mathbf{n}} \in \mathrm{Sk}]$
    $R := [m|_{\mathbf{y}=\mathbf{v}} : m \in S]$
    $V := [\mathrm{coeff}(x^i, P_j) : j = 1\ldots\mathrm{length}(S)]$
    $W :=$Algorithm 52$(R, V)$
    for $j = 1\ldots\mathrm{length}(S)$
        $G := G + W_j S_j x^i$
return $G$
```

Figure 4.15: f from section 4.4.3

$$\begin{array}{l}
4\,w^6x^4y^7z+2\,w^5x^5y^7z+2\,w^5x^4y^8z-4\,w^6x^4y^7-2\,w^5x^5y^7-2\,w^5x^4y^8-\\
2\,w^5x^4y^7z+2\,w^5x^4y^7+2\,wx^4y^7z^4+x^5y^7z^4+x^4y^8z^4+6\,w^7x^4y^3z+\\
3\,w^6x^5y^3z+3\,w^6x^4y^4z+4\,w^6x^4yz^4+2\,w^5x^5yz^4+2\,w^5x^4y^2z^4-x^4y^7z^4-\\
6\,w^7x^4y^3-3\,w^6x^5y^3-3\,w^6x^4y^4-3\,w^6x^4y^3z-4\,w^6x^4yz^3-2\,w^5x^5yz^3-\\
2\,w^5x^4y^2z^3-2\,w^5x^4yz^4+2\,wx^2y^7z^4+x^3y^7z^4+x^2y^8z^4+3\,w^6x^4y^3+\\
2\,w^5x^4yz^3-4\,wx^4y^7z+2\,wx^4y^3z^5+2\,wx^2y^4z^6-2\,x^5y^7z+x^5y^3z^5-\\
2\,x^4y^8z+x^4y^4z^5+x^3y^4z^6-x^2y^7z^4+x^2y^5z^6+2\,wx^4y^7+2\,wx^4yz^6-\\
2\,wx^2y^4z^5+x^5y^7+x^5yz^6+x^4y^8+2\,x^4y^7z-x^4y^3z^5+x^4y^2z^6-x^3y^4z^5-\\
x^2y^5z^5-x^2y^4z^6-x^4y^7-x^4yz^6+x^2y^4z^5-4\,wx^4yz^4-2\,wx^2y^7-2\,x^5yz^4-\\
2\,x^4y^2z^4-x^3y^7-x^2y^8-6\,wx^4y^3z+4\,wx^4yz^3-3\,x^5y^3z+2\,x^5yz^3-\\
3\,x^4y^4z+2\,x^4y^2z^3+2\,x^4yz^4+x^2y^7+4\,w^7z+2\,w^6xz+2\,w^6yz+4\,wx^4y^3\\
+2\,x^5y^3+2\,x^4y^4+3\,x^4y^3z-2\,x^4yz^3-4\,w^7-2\,w^6x-2\,w^6y-2\,w^6z\\
-2\,x^4y^3+2\,w^6-2\,wx^4y-x^5y-x^4y^2+x^4y+2\,wx^2z+x^3z+x^2yz-\\
2\,wx^2-x^3-x^2y-x^2z-2\,wz+x^2-xz-yz+2\,w+x+y+z-1
\end{array}$$

Figure 4.16: g from section 4.4.3

$$\begin{array}{l}
2\,w^5x^6y^7z + 2\,w^6x^4y^7z - 2\,w^5x^6y^7 + 2\,w^6x^4y^7 - 2\,w^5x^4y^7z + x^6y^7z^4 + \\
3\,w^6x^6y^3z + 2\,w^5x^6yz^4 + wx^4y^7z^4 + 3\,w^7x^4y^3z - 3\,w^6x^6y^3 + 2\,w^6x^4yz^4 - \\
2\,w^5x^6yz^3 + 2\,wx^4y^7z^3 + 3\,w^7x^4y^3 - 3\,w^6x^4y^3z + 2\,w^6x^4yz^3 - 2\,w^5x^4yz^4 + \\
2\,wx^4y^7z^2 + wx^2y^7z^4 - 2\,x^6y^7z + x^6y^3z^5 - x^4y^7z^3 + x^4y^4z^6 + wx^4y^3z^5 + \\
2\,wx^2y^7z^3 + wx^2y^4z^6 + x^6y^7 + x^6yz^6 - x^4y^7z^2 - x^4y^4z^5 - x^2y^7z^4 - wx^4y^7 + \\
2\,wx^4y^3z^4 + wx^4yz^6 + 2\,wx^2y^7z^2 + wx^2y^4z^5 + x^4y^7z - x^4y^3z^5 - x^2y^7z^3 - \\
x^2y^4z^6 + 2\,wx^4y^3z^3 + 2\,wx^4yz^5 + 2\,wx^2y^7z - 2\,x^6yz^4 - x^4y^7 - x^4y^3z^4 - \\
x^4yz^6 - x^2y^7z^2 + 2\,wx^4y^3z^2 + wx^2y^7 - 3\,x^6y^3z + 2\,x^6yz^3 - x^4y^3z^3 - x^4yz^5 - \\
x^2y^7z + 2\,w^6x^2z - wx^4y^3z + 2\,x^6y^3 - x^4y^3z^2 + x^4yz^4 + 2\,w^7z - 2\,w^6x^2 - \\
2\,wx^4y^3 + 2\,wx^4yz^2 + 2\,x^4y^3z - x^4yz^3 + 2\,w^7 - 2\,w^6z + 2\,wx^4yz - x^6y - \\
x^4yz^2 + wx^4y - x^4yz + x^4z + wx^2z - x^4 + wx^2 - 2\,x^2z - wz + x^2 - w + z
\end{array}$$

Table 4.1: g.c.d.s of univariate images of f and g

y	z	$\gcd(f\vert_{w=2,y,z}, g\vert_{w=2,y,z})$
2	3	$19612\,x^4 + 9009\,x^2 + 127$
4	9	$15381680\,x^4 + 28551425\,x^2 + 127$
8	27	$43407439216\,x^4 + 101638909953\,x^2 + 127$
16	81	$144693004136768\,x^4 + 372950726410241\,x^2 + 127$
32	243	$495206093484836032\,x^4 + 1383508483289120769\,x^2 + 127$
64	729	$1706321451043380811520\,x^4 + 5160513838422975053825\,x^2 + 127$
$\vdots$	$\vdots$	$\vdots$
2^{15}	3^{15}	$198411691765221472092142899333004739661707864775 72096\,x^4 + 821125029267868660742578644189289158324777001819308033\,x^2 +127$

sparsity structure of $\gcd(f,g)$.

We take evaluations with $y = 2^i$ and $z = 3^i$. For simplicity, let us consider the coefficient of x^2, which, we know from (4.21), is assumed to have the shape

$$y^7\left(a_1z^3 + a_2z^2 + a_3z + a_4\right) + a_5y^4z^5 + a_6. \tag{4.22}$$

Hence, taking the coefficients of x^2 from Table 4.1,

$$\begin{array}{lcr}
2^73^3a_1 + 2^73^2a_2 + 2^73a_3 + 2^7a_4 + 2^43^5a_5 + a_6 & = & 9009 \\
\left(2^73^3\right)^2 a_1 + \left(2^73^2\right)^2 a_2 + \left(2^73\right)^2 a_3 + \left(2^7\right)^2 a_4 + \left(2^43^3\right)^2 a_5 + a_6 & = & 28551425 \\
\left(2^73^3\right)^3 a_1 + \left(2^73^2\right)^3 a_2 + \left(2^73\right)^3 a_3 + \left(2^7\right)^3 a_4 + \left(2^43^3\right)^3 a_5 + a_6 & = & 10\ldots \\
\left(2^73^3\right)^4 a_1 + \left(2^73^2\right)^4 a_2 + \left(2^73\right)^4 a_3 + \left(2^7\right)^4 a_4 + \left(2^43^3\right)^4 a_5 + a_6 & = & 37\ldots \\
\left(2^73^3\right)^5 a_1 + \left(2^73^2\right)^5 a_2 + \left(2^73\right)^5 a_3 + \left(2^7\right)^5 a_4 + \left(2^43^3\right)^5 a_5 + a_6 & = & 13\ldots \\
\left(2^73^3\right)^6 a_1 + \left(2^73^2\right)^6 a_2 + \left(2^73\right)^6 a_3 + \left(2^7\right)^6 a_4 + \left(2^43^3\right)^6 a_5 + a_6 & = & 51\ldots
\end{array} \tag{4.23}$$

This is indeed a system of linear equations

4.4.4 Conclusions

Let us look at the questions on page 164.

1. Are there evaluations which are not merely good in the sense of the previous section, but also satisfy the Zippel assumption? The answer is yes. For the evaluation $y_n = v$ to be good, v must not annihilate both leading coefficients (checked for in `random`) and must not be a root of the relevant resultant (Corollary 13). For the Zippel assumption to be violated, a certain coefficient must vanish under the ψ of Definition 89. There are only finitely many such, and each such only has finitely many roots, hence in all there are only finitely many bad values for each x_i.

2. How can we tell if an evaluation should be used? The tests in Algorithm 23 apply.

3. How many reductions should we take? The number of *good* reductions at each stage x_i is $1+\min(\deg_{x_i}(f), \deg_{x_i}(g))$, by the usual argument on the degree of a g.c.d.

   Open Problem 21 (Bad reductions in Zippel's algorithm) *An interesting question is how many* bad *reductions there can be. If the degree in each variable is at most d, then the number of values that annihilate both leading coefficients is at most d, and the number that divide the relevant resultant is at most* $2d^2$*, the degree of the resultant. The challenge is to bound the number that violate the Zippel assumption, which we showed was finite in point 1.*

   It seems to the author, though he has not seen this in the literature, that, as a polynomial with t terms can have at most $2t-1$ *roots (Proposition 26), the number of Zippel-bad values is bounded by twice the number of terms in the g.c.d. being computed. Though we have no good a priori bound on this (Example 21), it means that the number of Zippel-bad reductions is linear in the size of the output.*

   Conjecture 1 *Let* d_f, d_g *be the degrees of* f *in the main variable* x, d *be* $\max_i \min(\deg_{y_i}(f), \deg_{y_i}(g))$, d' *be* $\max_i(\deg_{y_i}(f)+\deg_{y_i}(g))$ *and* t *the number of terms in* $\gcd(f,g)$*. Then the running time of Zippel's algorithm is bounded by*

$$\left(d + \underbrace{(d_f + d_g)d'}_{\text{kills Res}} + 2t\right) \underbrace{\min(d_f, d_g)dt}_{\text{cost}}, \tag{4.24}$$

   where we multiply the number of evaluations (allowing for the maximal number of failed ones) by the cost of a successful calculation. There is also an $O(t^2)$ *cost of the linear algebra, but this is dominated by the cost of the evaluations.*

4. How do we combine? This is the main subtlety of this method, using the theory of section A.5.

5. How do we check the result? We have used the "If C divides both A and B" test, as in Algorithm 17 *et seq*. We could consider using a "reconstruction of cofactors" argument, as in Algorithm 19. To the best of the author's knowledge, this route has not been explored experimentally. However, there is a strong argument against it: it would require the evaluations to satisfy the Zippel assumption for the cofactors as well as for the g.c.d., and intuitively this seems like asking for trouble.

A variant on this algorithm is presented in [HM13], which uses an interpolation technique from [GLL09], and this seems to be the current state of the art, though there is no formal complexity analysis.

In practice these methods seem to solve Open Problem 2, a g.c.d. algorithm whose running time is polynomial in the number of terms in the input *and outputs*, even though (4.24) is $O(t^2)$, rather than the $O(t)$ we would need for the algorithm to be optimal.

4.5 Further Applications

4.5.1 Resultants and Discriminants

The resultant of two polynomials $f = \sum_{i=0}^{n} a_i x^i$ and $g = \sum_{i=0}^{m} b_i x^i$ is defined (Definition 112) as the determinant of a certain $(m+n)\times(m+n)$ matrix $\mathrm{Syl}(f,g)$. Hence if $\mathrm{Syl}(f,g)|_p = \mathrm{Syl}(f|_p, g|_p)$ (or the equivalent for $|_{y-v}$), the theory of Section 4.1 holds. But the only way this can fail to hold is if $\mathrm{Syl}(f|_p, g|_p)$ is no longer an $(m+n)\times(m+n)$ matrix, i.e. if a_n or b_m evaluate to zero. Hence again we have simple answer to the questions on page 164.

1. Are there "good" reductions from R: yes — all evaluations which do not reduce $a_n b_m$ to zero, and this can be tested *a priori*.

2. How can we tell if R_i is good?: in advance!

3–5 As in Section 4.1.

4.5.2 Linear Systems

We consider the matrix equation

$$\mathbf{M}.\mathbf{x} = \mathbf{a}, \tag{3.13}$$

and the case of $\mathbf{M}$ and $\mathbf{a}$ having integer entries, and reduction modulo a prime: the polynomial version is very similar. Conceptually, this has the well-known solution

$$\mathbf{x} = \mathbf{M}^{-1}.\mathbf{a}. \tag{3.14}$$

This formulation shows the fundamental problem: $\mathbf{M}^{-1}$, and $\mathbf{x}$ itself, might not have integer entries. In fact, if p divides $\det(\mathbf{M})$, $(\mathbf{M}|_p)^{-1}$ does not exist. There are two solutions to this problem.

4.5.2.1 Clear Fractions

If we compute $\det(\mathbf{M})$ first, by the method of section 4.1, we can clear denominators in (3.14), and get

$$\widehat{\mathbf{x}} := \det(\mathbf{M})\mathbf{x} = \det(\mathbf{M})\mathbf{M}^{-1}.\mathbf{a}, \tag{4.25}$$

i.e.

$$\mathbf{M}.\widehat{\mathbf{x}} = \det(\mathbf{M})\mathbf{a}. \tag{4.26}$$

If we avoid primes dividing $\det(\mathbf{M})$, we can solve (4.26) for enough primes (suitable bounds are given in Corollary 25), reconstruct integer entries in $\widehat{\mathbf{x}}$, and then divide through by $\det(\mathbf{M})$.

4.5.2.2 Solve with Fractions

If (3.13) is soluble modulo p, its solution $\mathbf{x}_p$ is indeed congruent to $\mathbf{x}$ when evaluated at p, i.e. $\mathbf{x}|_p = \mathbf{x}_p$. If we use many primes p_i (discarding those for which (3.13) is not soluble), and apply Algorithm 47 to the vectors x_{p_i}, we get a vector x_N such that $x_N \equiv x \pmod N$, where $N = \prod p_i$. However, the entries of $\mathbf{x}$ are rationals, with numerator and denominator bounded, say, by B (see Corollary 25), rather than integers. How do we find the entries of $\mathbf{x}$ from $\mathbf{x}_N$? This problem has a long history in number theory, generally under the name Farey fractions, but was first considered in computer algebra in [Wan81]. Since we will have occasion to use this solution elsewhere, we consider it in more generality in the next section.

If we assume this problem solved, we then have the following answer to the questions on page 164.

1. Are there good primes?: yes — all that do not divide $\det(\mathbf{M})$
2. How can we tell if a prime p is bad? Equation (3.13) is not soluble modulo p.
3. How many reductions should we take? Enough such that the product of the good primes is greater than $2B^2$.
4. How do we combine? Algorithm 25.
5. How do we check the result? If we use the bound from Corollary 25), we do not need to check. However, there are "early success" variations, analogous to section 4.2.5, where we do need to check, which can be done by checking that $\mathbf{M}.\mathbf{x} = \mathbf{a}$: an $O(n^2)$ arithmetic operations process rather than the $O(n^3)$ of solving.

4.5.2.3 Farey reconstruction

In this section, we consider the problem of reconstructing an unknown fraction $x = n/d$, with $|n|, |d| < B$, given that we know $x \equiv y \pmod{N}$, i.e. $n \equiv yd \pmod{N}$, where $N > 2B^2$. We first observe that this representation is unique, for if n'/d' (similarly bounded) is also congruent to y, then $nd' \equiv ydd' \equiv n'd$, so $nd' - n'd \equiv 0 \pmod{N}$, and the only solution satisfying the bounds is $nd' - n'd = 0$, i.e. $n/d = n'/d'$.

Actually finding n and d is done with the Extended Euclidean Algorithm (see Algorithm 5).

Algorithm 25 (Farey Reconstruction)
Input: $y, N \in \mathbf{N}$
Output: $n, d \in \mathbf{Z}$ *such that* $|n|, |d| < \sqrt{N/2}$ *and* $n/d \equiv y \pmod{N}$*, or* `failed` *if none such exist.*

```
$i := 1$;
$a_0 := N$; $a_1 := y$; $a := 1$; $d := 1$; $b := c := 0$
#Loop invariant: $a_i = ay + bN$; $a_{i-1} = cy + dN$;
while $a_i > \sqrt{N/2}$ do
    $a_{i+1} = \mathrm{rem}(a_{i-1}, a_i)$;
    $q_i$ :=the corresponding quotient: #$a_{i+1} = a_{i-1} - q_i a_i$
    $e := c - q_i a$; $e' := d - q_i b$;        #$a_{i+1} = ef + e'g$
    $i := i + 1$;
    $(c, d) = (a, b)$;
    $(a, b) = (e, e')$
if $|a| < \sqrt{N/2}$ and $\gcd(b, N) = 1$
  then return $(a_i, a)$
  else return failed
```

Correctness of this algorithm, i.e. the fact that the first $a_i < \sqrt{N/2}$ corresponds to the solution if it exists, is proved in [WGD82], using [HW79, Theorem 184]. The condition $\gcd(b, N) = 1$ was stressed by [CE95], without which we may return meaningless results, such as $(-1, 2)$, when trying to reconstruct 5 (mod 12).

4.6 Gröbner Bases

If coefficient growth is a major problem in g.c.d. computations, it can be even more apparent in Gröbner basis computations. [Arn03] gives the example of

$$\begin{array}{ll}\{8\,x^2y^2 + 5\,xy^3 + 3\,x^3z + x^2yz, & x^5 + 2\,y^3z^2 + 13\,y^2z^3 + 5\,yz^4, \\ \quad 8\,x^3 + 12\,y^3 + xz^2 + 3, & \quad 7\,x^2y^4 + 18\,xy^3z^2 + y^3z^3\}\end{array} \tag{4.27}$$

whose Gröbner base, computed with practically any order, is

$$\left\{x, y^3 + \frac{1}{4}, z^2\right\}. \tag{4.28}$$

Both (4.27) and (4.28) involve very manageable numbers, but computing with total degree orders gives intermediate terms such as

$$\frac{80791641378310759316435307942453674533\ldots 36412673653679966238192054 9}{31979005164817514134306974902709502756\ldots 24739741329561114159211651 6} yx^3$$

(where the ellipses stand for 60 deleted digits)[14], and a lexicographic order gives coefficients with tens of thousands of digits. Can the same ideas allow us to bypass such huge intermediate expressions?

Observation 14 *[Tra88] proposed a very simple efficiency improvement, known as the Gröbner trace idea. We should only compute and reduce an S-polynomial over* $\mathbf{Q}$ *if the equivalent computation modulo p yields a non-zero polynomial. This isn't guaranteed correct, so a final check, either using Theorem 16(1) or on the lines of Theorem 42, is necessary, but can be a great time-saver in practice.*

We should note that there can be various differences between the Gröbner base computed over the rationals and that computed modulo p.

Example 22 *These examples are taken from [Win88].*

1. *(From [Ebe83]). Let $F = \{xy^2 - 2y, x^2y + 3x\}$. Then (under any order) $S(F_1, F_2) = 5xy$. Hence if this is non-zero, it reduces the other elements of F to $\{x, y\}$, which is a Gröbner base. But if this is zero (i.e. if we are working modulo 5), then F is already a Gröbner base (Theorem 16 clause 1).*

2. *Let $F = \{7xy+y+4x, y+2\}$ Then (under any order) $S(F_1, F_2) = -10x+y$.*

   $p = 2$ *This is therefore y, which is F_2. Hence a reduced Gröbner base is $\{y\}$: infinitely many solutions (x unconstrained).*

   $p = 5$ *This is therefore y, which reduces F_2 to 2, hence the Gröbner base is $\{1\}$: no solutions.*

   Otherwise *We get a Gröbner base of $\{y + 2, 5x + 1\}$, with one solution.*

3. *Let $F = \{4\,xy^2 + 16\,x^2 - 4\,z + 1, 2\,y^2z + 4\,x + 1, -2\,x^2z + 2\,y^2 + x\}$ with a purely lexicographic ordering $z > y > x$. Then the Gröbner base over the integers is*

$$\begin{array}{l} 32\,x^7 + 232\,x^6 - 34\,x^4 - 44\,x^3 + x^2 + 30\,x + 8, \\ 2745\,y^2 + 112\,x^6 - 756\,x^5 - 11376\,x^4 - 65\,x^3 + 756\,x^2 + 1772\,x + 2, \\ 10980\,z - 6272\,x^6 - 45504\,x^5 + 216\,x^4 + 3640\,x^3 - 36846\,x^2 - 412\,x - 2857. \end{array} \tag{4.29}$$

   $p = 2$ *Here F is immediately $\{1, x^2 + 1, 1\}$ and there are no solutions, which is a classic "p divides the leading coefficient" issue.*

[14]This computation was performed with Axiom. [Arn03], computing with Macauley, quotes intermediate numbers of 80,000 digits.

$p = 7$ *Though there might not seem to be a problem (none of the leading coefficients above are divisible by 7), in fact* $S(S(F_1, F_3), S(F_1, F_2))$ *reduces* $S(F_1, F_2)$ *to* $14 * x^2 * y^2 + y^2 - 8 * x^4 - 2 * x^3 + 8 * x + 2$, *and the leading monomial here vanishes modulo 7. In fact, though the computation follows a different route, we get out a Gröbner basis of the same shape as (4.29).*

If we reverse the order, though, events take a somewhat different course. The Gröbner base over the integers is

$$\begin{array}{l} 16\,z^7 - 8\,z^6 + z^5 + 52\,z^4 + 75\,z^3 - 342\,z^2 + 266\,z - 60, \\ 1988\,y^2 - 76752\,z^6 + 1272\,z^5 - 4197\,z^4 - 251555\,z^3 - 481837\,z^2 \\ \quad +1407741\,z - 595666, \\ 3976\,x + 37104\,z^6 - 600\,z^5 + 2111\,z^4 + 122062\,z^3 + 232833\,z^2 \\ \quad -680336\,z + 288814. \end{array} \tag{4.30}$$

Here 1988 is *divisible by 7, and we might expect a different result modulo 7, which indeed we get:*

$$\begin{array}{l} z^6 + 4\,z^5 + z^4 + 6\,z^3 + 5\,z^2 + 2\,z + 2, \\ y^2 z + 3\,z^5 + 6\,z^4 + 5\,z^3 + 6\,y^2 + 3\,z + 4, \\ y^4 + 5\,y^2 + 6\,z^5 + 2\,z^4 + 4\,z^3 + 5\,y^2 + 4\,z^2 + 6\,z + 5, \\ x + 2\,z^5 + 4\,z^4 + z^3 + 4\,y^2 + 2\,z. \end{array} \tag{4.31}$$

This example shows that the badness of the prime might depend, not just on the input polynomials, but also on the ordering chosen. Both (4.30) and (4.31) have 14 solutions (as does (4.29), since this does not depend on the order), but the bases have different leading monomial ideals.

The rest of this section is substantially based on [Arn03][15]: a p-adic approach to Gröbner bases is described in section 5.9.3. As in that paper, we assume that our ideal is *homogeneous*, i.e. that it *can*[16] be generated by a set of generators each of which is homogeneous (Definition 54). If we wish to handle non-homogeneous ideals with respect to a total degree ordering[17], we can homogenize, compute the Gröbner basis and dehomogenize, as in [MM84]. This may not be optimal, see [GMN$^+$91, p. 49] and Open Problem 22 later. [IPS11] have shown that the assumption of homogeneity is not strictly necessary.

4.6.1 General Considerations

Given the blow-up in coefficients that can arise when computing Gröbner bases, we might wish to compute them by a modular method.

[15]A somewhat different approach is taken in [IPS11]: they distinguish good/bad primes by majority voting — their algorithm `deleteUnluckyPrimesSB`. This idea is used in the Singular polynomial algebra system. They also use the Gröbner trace idea: subsequent primes do not try an S-polynomial which the first prime reduced to 0.

[16]Note that there may well be sets of generators which are not homogeneous.

[17]And we generally wish to do so, before, if necessary, using the FGLM Algorithm (12) to compute a lexicographical base.

Definition 90 *Given a specific computation $\mathcal{C}$ (by which we mean specifying not just the ordering, but all the choices in Algorithm 9, or any other algorithm for computing a Gröbner base) of a Gröbner base G from an input S over $\mathbf{Q}$, denoted $G := \mathcal{C}(S)$, we say that the prime p is* of good reduction *if $\mathrm{lt}(G) = \mathrm{lt}(\mathcal{C}(S_p))$, i.e. we get the same leading term ideal when we apply $\mathcal{C}$ modulo p as when we apply it over $\mathbf{Q}$. If not, we say that p is* of bad reduction*.*

Lemma 11 *For a given S and $\mathcal{C}$, there are only finitely many primes of bad reduction.*

Proof. If we compare $\mathcal{C}(S)$ and $\mathcal{C}(S_p)$, they can only start differing when a leading coefficient in the computation of $\mathcal{C}(S)$ vanishes when considered modulo p, because the computation of S-polynomials, and reduction, is entirely driven by leading terms. But there are only finitely many such coefficients, and hence only finitely many bad primes: the set of divisors of these coefficients.

We should note that this proof is far less effective than the proof of Lemma 8, but it is sufficient for our purposes. Another proof is given in [Win88, Theorem 1], which in fact shows that the set of bad primes depends only on the input and the ordering, not on the computation.

Note that we cannot follow the plan of section 4.2.3 as we do not know a bound on all the integers appearing (and anyway, the idea is to compute with smaller integers). If we try to follow the plan of section 4.2.4, we hit a snag. Suppose that two computations modulo different primes (say p and q) give us different leading term ideals. Then one is certainly bad, but which? We do not have a simple degree comparison as in the g.c.d. case.

4.6.2 The Hilbert Function and reduction

We recall the Hilbert function from section 3.3.12, which will turn out to be a key tool in comparing Gröbner bases, as earlier we have used degrees for comparing polynomials.

Theorem 40 ([Arn03, Theorem 5.3]) *Let $(f_1, \ldots, f_k)$ generate the ideal I over $\mathbf{Q}$, and the ideal I_p modulo p. Then*

$$\forall n \in \mathbf{N} : H_I(n) \le H_{I_p}(n). \tag{4.32}$$

Definition 91 *p is said to be* Hilbert-good *if, and only if, we have equality in (4.32).*

Observation 15 *Note that we do not have a test for Hilbert-goodness as such, but we do have one for Hilbert-badness: If $H_{I_p}(n) < H_{I_q}(n)$ then q is definitely Hilbert-bad.*

Example 23 (Bigatti [Big15]) *Let S be the set $\{x^2, x\,(x-3\,y)\,, xy^2, xyz, xz\,(y-5\,z)\}$. Over various characteristics its Gröbner bases (always using* `tdeg(x,y,z)`*) and (first few values of) Hilbert functions are:*

0 $\{xy, x^2, xz^2\}$; *(1,3,4,4,5,6,7,8,9,10)*

3 $\{x^2, xz^2, xyz, xy^2\}$; *(1,3,5,4,5,6,7,8,9,10)*

5 $\{xy, x^2\}$; *(1,3,4,5,6,7,8,9,10,11)*

7 $\{xy, x^2, xz^2\}$; *(1,3,4,4,5,6,7,8,9,10)*

$H_{I_3}(2) = 5 > H_{I_5}(2) = 4$, so $p = 3$ is proved to be bad by $p = 5$. But $H_{I_3}(3) = 4 < H_{I_5}(3) = 5$, so $p = 5$ is proved to be bad by $p = 3$. To the best of the author's knowledge, modular Gröbner bases are the only modular calculation in which two primes can prove each other bad.

Proposition 58 *If p is of good reduction for $\mathcal{C}(S)$, then it is Hilbert-good for the ideal generated by S.*

Theorem 41 ([Arn03, Theorem 5.6]) *Let $(g_1, \ldots, g_t)$ be a Gröbner base under $<$ for the ideal generated by $(f_1, \ldots, f_s)$ over $\mathbf{Q}$, and $(g'_1, \ldots, g'_{t'})$ be a Gröbner base under $<$ for the ideal generated by $(f_1, \ldots, f_s)$ modulo p. In both cases these bases are to be ordered by increasing order under $<$. Suppose that p is Hilbert-good for this ideal. Then:*

1. $\mathrm{lt}(g'_1) \le \mathrm{lt}(g_1)$;
2. *If* $\mathrm{lt}(g'_j) = \mathrm{lt}(g_j)$ *for* $1 \le j \le i$, *then* $\mathrm{lt}(g'_{i+1}) \le \mathrm{lt}(g_{i+1})$.

This result means that, *for Hilbert-good primes*, we can compare the sequence of $\mathrm{lt}(g_i)$ for relative luckiness as we compared degrees in the case of g.c.d.s: at the first point i where they differ, the prime with the lesser $\mathrm{lt}(g_i)$ is definitely of bad reduction.

Example 24 (importance of caveat above) *If we consider example 23, and hadn't checked Hilbert functions, we would deduce that $p = 3$ proved that $p = 7$ was bad, since $xy < x^2$. In fact $p = 7$ is good and generates precisely the right ideal.*

Arnold's original example was more complicated.

Example 25 ([Arn03, Example 5.7]) *Let $I = \langle 3\,y^2x - 5\,yx^2 + 2\,x^3, -7\,y^3x + 5\,y^2x^2, 7\,y^6 - 2\,y^3x^3 + yx^5\rangle$. We use the ordering 'total degree then lexicographic, with $y > x$', and consider the primes 5 and 2.*

I_5 has Gröbner base

$$\left\{3\,y^2x + 2\,x^3, 29\,yx^3, x^5, y^6\right\}, \tag{4.33}$$

whereas I_2 has Gröbner base

$$\left\{y^2x + yx^2, y^6 + yx^5\right\}. \tag{4.34}$$

Table 4.2: Hilbert functions for example 25

l	$H_{I_5}(l)$	$H_{I_2}(l)$	M_l
0	1	1	$\{1\}$
1	3	3	$\{1,x,y\}$
2	6	6	$\{1,x,y,x^2,xy,y^2\}$
3	9	9	$\{1,x,y,x^2,xy,y^2,x^3,x^2y,y^3\}$
4		12	$\{1,x,y,x^2,xy,y^2,x^3,x^2y,y^3,x^4,x^3y,y^4\}$
4	11		$\{1,x,y,x^2,xy,y^2,x^3,x^2y,y^3,x^4,y^4\}$

We can tabulate the Hilbert functions as in Table 4.2, though in fact we know they are different, since I_5 has dimension zero and I_2 does not, and hence the Hilbert polynomials have different degree. Either way, Hilbert functions tell us that 2 is definitely not good.

If we just looked at the leading terms, we would note that xy^2, the least leading term, is the same for both I_2 and I_5, but that $x^3y < y^4$ (the next leading terms), leading us, incorrectly, to infer that 5 was of bad reduction.

In fact, 5 is of good reduction, and the true Gröbner base for I itself (as determined by Chinese remainder with 7 and 11) is

$$\left\{3\,y^2x-5\,yx^2+2\,x^3, 29\,yx^3-20\,x^4, x^5, y^6\right\}. \tag{4.35}$$

4.6.3 The Modular Algorithm

We are now most of the way towards a Chinese Remainder Theorem solution to computing Gröbner Bases for an ideal $I\langle f_1,\ldots,f_k\rangle$. We still have to solve three issues.

leading coefficients In the case of g.c.d. computations, the modular answer is only defined up to multiplication by a constant, and the same is true here: each element of the Gröbner base modulo p can be multiplied, independently, by any non-zero constant. For g.c.d.s, we knew a multiple of the leading coefficient of the answer (i.e. the g.c.d. of the leading coefficients of the inputs), which we imposed in (4.8) and thereafter. Here we have no such prior knowledge. We therefore reconstruct a *monic* Gröbner base, with rational number coefficients, using Algorithm 25 above.

When do we stop? In the g.c.d. case we had the Landau–Mignotte bound. For Gröbner bases, while some bounds are known (e.g. [Dah09]), they are not directly relevant. In practice we take a leaf from Figure 4.5, and say that, if the Gröbner base doesn't change when we take a new prime, we are *likely to* have terminated.

How do we know correctness? The first issue is that our reconstructed object G, while a Gröbner base modulo each prime we used, *may* not be a Gröbner base over the rationals. This can be tested by Theorem 16 clause

(1), using, as appropriate, the optimizations from Propositions 40 and 41. But, even if G is a Gröbner base, is it one for I?

Once we know G is a Gröbner base, we can check that $I \subseteq \langle G \rangle$ simply by checking that $f_i \overset{*}{\to}{}^{G} 0$ for $i = 1, \ldots, k$. Inclusion the other way round comes from the following theorem.

Theorem 42 ([Arn03, Theorem 7.1]) *If G is a Gröbner base, $I \subseteq \langle G \rangle$ and $\mathrm{lm}(G) = \mathrm{lm}(G_p)$ for some G_p a Gröbner base obtained from the generators of I modulo p, then $I = \langle G \rangle$.*

It is worth noting that we have produced no complexity bound for this algorithm, just as we have stated none for the original Algorithm 9. In practice it seems to be especially efficient in cases where the final Gröbner basis has small coefficients, despite the internmediate expression swell.

Open Problem 22 (Modular Gröbner Bases for Inhomogeneous Ideals) *This algorithm as described is limited, as is [Arn03], to homogeneous ideals with respect to a total degree ordering. Can it be generalized?* [IPS11, Remark 2.5] claims that it can be, at the cost of either

- being content with a probabilistic algorithm, which may return a G such that $\langle f_1, \ldots, f_k \rangle \subset \langle G \rangle$; or
- homogenizing first — however they say "experiments showed that this is usually not efficient since the standard basis of the homogenized input often has many more elements than the standard basis of the ideal that we started with".

Open Problem 23 (Reconstructed Bases might not be Gröbner) *We stated that it was possible that the reconstruction of modular Gröbner bases might not be a Gröbner base. This certainly seems to be theoretically possible, though we have given no examples of this.*

1. *Give examples where this happens. This is probably trivial, but what might be harder is to give ones where the primes were nevertheless good, and the problem is that we did not take enough of them.*
2. **Or** *show that this cannot in fact happen when the primes are of good reduction.*

4.6.4 Conclusion

Let us see how we have answered the questions on page 164.

1. Are there "good" reductions p? Yes, by Lemma 11 there are only finitely many primes of bad reduction, though we have no bounds on the number.

Figure 4.17: Algorithm 26

Algorithm 26 (Modular Gröbner base)
Input: *$S = \{f_1, \ldots, f_k\}$ homogeneous polynomials in $\mathbf{Z}[x_1, \ldots, x_n]$.*
Output: *G a Gröbner base for $\langle S \rangle$ over $\mathbf{Q}$.*

```
$p :=$ find_prime($S$);
$G :=$ modular_GB($S, p$);
$G_{\mathbf{Q}}$ :=Algorithm 25($G, p$)
$N := p$;      # $N$ is the modulus we will be constructing
while true repeat
    $p :=$ find_prime($S$);
    $G' :=$ modular_GB($S, p$);
    if $H_G, H_{G'}$ are incomparable
      then start again        # All primes are bad (see Example 23)
    else  if $H_G < H_{G'}$
            then       # Do nothing: $p$ is of bad reduction
    else  if $H_G > H_{G'}$
            $G := G'$; $N := p$;       # previous primes were bad
            $G_{\mathbf{Q}}$ :=Algorithm 25($G, p$)
    # Hilbert Functions agree: on to Theorem 41
    else  if $\text{lm}(G) < \text{lm}(G')$      #Comparison in the sense of Theorem 41
            then       # Do nothing: $p$ is of bad reduction
    else  if $\text{lm}(G) > \text{lm}(G')$
            $G := G'$; $N := p$;       previous primes were bad
            $G_{\mathbf{Q}}$ :=Algorithm 25($G, p$)
    else if $G' \equiv G_{\mathbf{Q}} \pmod p$       # Stabilization as in Figure 4.5
            and $G_{\mathbf{Q}}$ is a Gröbner base
            and $\forall i f_i \overset{*}{\to}^{G_{\mathbf{Q}}} 0$
            return $G_{\mathbf{Q}}$       #Correct by Theorem 42
        else $G :=$ Algorithm 48($G', G, p, N$);
             $N := pN$;
             $G_{\mathbf{Q}}$ :=Algorithm 25($G, N$)
```

`find_prime` finds a prime that does not divide any $\text{lc}(f_i)$ (otherwise we're bound to be following a different computation, and almost certain to have bad reduction).
`modular_GB` computes a *monic* modular Gröbner base, either via Algorithm 9 or any other such (Observation 4).

2. How can we tell if p is good? Following [Arn03], we have a two-stage process for comparing p and q when G_p and G_q differ: we first compare Hilbert functions (Theorem 40), then leading monomial ideals (Theorem 41).

3. How many reductions should we take? We have no bound, but rely on stabilization, as in Figure 4.5.

4. How do we combine? Algorithm 47 and Algorithm 25.

5. How do we check the result? Theorem 42. [IPS10] propose doing a further modular check, but this works for them as it is essentially the equivalent of waiting for stabilization in Algorithm 26.

4.7 Conclusions

Let us look generally at our key questions for modular calculations.

1. Are there "good" reductions from R? All the examples we have seen answer this question positively, and indeed, if we are using modular arithmetic to mimic calculations that we could have done without it, then the argument of Lemma 11 shows that there are always only finitely many bad reductions.

2. How can we tell if R_i is good? This tends to have a problem-specific answer, and is always one of the major challenges. Sometimes it may be immediately evident (as when a set of linear equations becomes insoluble (mod p)), sometimes we may have different results in different-looking domains R_1 and R_2 and be able to say that one is definitely wrong (as in the case of gcd), and sometimes we may need to resort to majority voting (see note 15).

3. How many reductions should we take? Sometimes we can compute bounds, but sometimes we have to rely on stabilization as in Figure 4.5 and Algorithm 26. Even if bounds exist, stabilization is often more efficient in practice.

4. How do we combine? Either via a version of Chinese Remainder, or by Farey Reconstruction (Section 4.5.2.3).

5. How do we check the result? This also tends to have a problem-specific answer, and is generally the other major challenge.

How good are these algorithms, and what are the challenges?

Section 4.1 For matrix determinants, these methods are definitely better than the classical methods (see Proposition 54), though there is a further improvement in Section 5.9.4. There is an Open Problem (16) on the difference between theory and practice, though.

gcd The theoretical state of computations in general is hampered by the absence of a solution to Open Problems 2 and 3 (page 74), i.e. the absence of an algorithm whose complexity depends only on the number of terms, not the degrees. The remaining analyses ignore this challenge.

Section 4.2 Univariate polynomial g.c.d. seems to be about as good as it can get algorithmically, though there are useful engineering improvements to the Landau–Mignotte bound in practice (Open Problem 18).

Section 4.3 Bivariate polynomial g.c.d. is also about as good as it can get algorithmically, though we have posed Open Problem 20.

Section 4.4 Multivariate polynomial g.c.d., based on the bivariate work, seems as good as it can get algorithmically for dense polynomials. For sparse polynomials, the work of Zippel explained in this section is good, and empirically the complexity is essentially a function of the number of terms in the output, but there is definitely scope for a proper theoretical analysis, and probably substantial room for practical improvement.

Section 4.5 This depends on the precise application, but the remarks above about Section 4.1 apply.

Section 4.6 The computation of Gröbner bases by modular techniques is probably the area where there is the greatest room for improvement, but it is challenging.

Chapter 5

p-adic Methods

In this chapter, we wish to consider a different problem, that of factoring polynomials. We will see that this cannot reasonably be solved by the methods of the previous chapter, and we need a new technique for solving problems in (large) domains via small ones. The technique is based on the mathematical concept of p-adic numbers, and its fundamental result, Hensel's Lemma. The basic idea behind these algorithms is shown in Figure 5.1: instead of doing a calculation in some (large) domain R, we do it in several smaller domains R_i, pick one of these (say R_l) as the best one, grow the solution to some larger domain $\widehat{R}_l$, regard this as being in R and check that it is indeed the right result.

5.1 Introduction to the factorization problem

For simplicity, we will begin with the case of factoring a univariate polynomial over $\mathbf{Z}$. More precisely, we consider the following.

Problem 7 *Given $f \in \mathbf{Z}[x]$, compute polynomials $f_i \in \mathbf{Z}[x]$ $(1 \le i \le k)$ such that:*

Figure 5.1: Diagrammatic illustration of Hensel Algorithms

$$\begin{tikzcd}
R \arrow[rrrr, dashed, "\text{calculation}"] \arrow[d, "k\times\text{reduce}"'] & & & & R \\
R_1 \arrow[r, "\text{calculation}"] & R_1 & & & \\
\vdots & \vdots & \vdots \left.\right\} \arrow[r, "\text{choose}"] & R_l \arrow[r, "\text{grow}"] & \widehat{R}_l \arrow[uu, "\text{interpret \& check}"'] \\
R_k \arrow[r, "\text{calculation}"] & R_k & & &
\end{tikzcd}$$

1. *$f = \prod_{i=1}^k f_i$;*
2. *each f_i is irreducible in $\mathbf{Z}[x]$, i.e. any polynomial g that divides f_i is either an integer or has the same degree as f_i.*

We might wonder whether we wouldn't be better off considering $f_i \in \mathbf{Q}[x]$, but in fact the answers are the same.

Proposition 59 *Any factorization over $\mathbf{Q}[x]$ of a polynomial $f \in \mathbf{Z}[x]$ is (up to rational multiples) a factorization over $\mathbf{Z}[x]$.*

Proof. Let $f = \prod_{i=1}^k f_i$ with $f_i \in \mathbf{Q}[x]$. By clearing denoninators and removing contents, we can write $f_i = c_i g_i$ with $g_i \in \mathbf{Z}[x]$ and primitive and $c_i \in \mathbf{Q}$. Hence $f = \left(\prod_{i=1}^k c_i\right)\left(\prod_{i=1}^k g_i\right)$, and, since the product of primitive polynomials is primitive (Lemma 2), $\prod_{i=1}^k c_i$ is an integer, and can be absorbed into, say, g_1.

Even knowing that we have only to consider integer coefficients does not seem to help much — we still seem to have an infinite number of possibilities to consider. In fact this is not quite the case.

Notation 29 *Let the polynomial $f = \sum_{i=0}^n a_i x^i$ to be factored have degree n, and coefficients bounded by H. Let us suppose we are looking for factors of degree at most d.*

Corollary 14 (to Theorem 36) *It is sufficient to look for factors of degree $d \le n/2$, whose coefficients are bounded by $2^d H$.*

One might have hoped that it was sufficient to look for factors whose coefficients are bounded by H, but this is not the case. [Abb09] gives the example of

$$\begin{aligned} f &= x^{80} - 2x^{78} + x^{76} + 2x^{74} + 2x^{70} + x^{68} + 2x^{66} + x^{64} + x^{62} + 2x^{60} + 2x^{58} \\ &\quad -2x^{54} + 2x^{52} + 2x^{50} + 2x^{48} - x^{44} - x^{36} + 2x^{32} + 2x^{30} + 2x^{28} - 2x^{26} \\ &\quad +2x^{22} + 2x^{20} + x^{18} + x^{16} + 2x^{14} + x^{12} + 2x^{10} + 2x^6 + x^4 - 2x^2 + 1 \end{aligned}$$

whose factors have coefficients as large as 36, i.e. 18 times as large as the coefficients of f. Non-squarefree polynomials are even worse: [Abb09, p. 18] gives the example of

$$\begin{gathered} -1 - x + x^2 - x^3 + x^4 + x^5 + x^6 + x^8 + x^{10} - x^{11} \\ -x^{12} - x^{13} + x^{14} - x^{15} - x^{17} - x^{18} + x^{20} + x^{21} = \\ \big(1 + 4x + 8x^2 + 14x^3 + 21x^4 + 28x^5 + 34x^6 + 39x^7 + 42x^8 + 43x^9 + 41x^{10} \\ +37x^{11} + 32x^{12} + 27x^{13} + 21x^{14} + 15x^{15} + 9x^{16} + 4x^{17} + x^{18}\big)(x-1)^3 \end{gathered}$$

Although some caution is in order, his table appears to show coefficient growth behaving like 0.7×1.22^d, where d is the degree of a polynomial with coefficients at most ± 1.

Corollary 15 *To detect all irreducible factors of f (except possibly for the last one, which is f divided by all the factors of degree $\le n/2$), it suffices to consider $\left(2^d H\right)^{d+1}$ polynomials.*

We can in fact do better, since the leading coefficient of the factor must divide a_n, and similarly the trailing coefficient must divide a_0, so we get $2^{d(d-1)}H^{d+1}$, and in practice such "brute force" methods[1] can easily factor low-degree polynomials, but the asymptotics are still very poor.

5.2 Modular methods

Hence we might want to use modular methods. Assuming always that p does not divide a_n, we know that, *if* f is irreducible modulo p, it is irreducible over the integers. Since, modulo p, we can assume our polynomials are monic, there are only p^d polynomials of degree d, this gives us a bound that is exponential in d rather than d^2. In fact, we can do better by comparing results of factorizations modulo different primes.

Example 26 *Suppose f is quartic, and factors modulo p into a linear and a cubic, and modulo q into two quadratics (all such factors being irreducible), we can deduce that it is irreducible over the integers, since no factorization over the integers is compatible with both pieces of information.*

Definition 92 *For a square-free polynomial f, define its* factorization shape *to be the multiset*[2] *of the degrees of all the irreducible factors in a given factorization of f into irreducibles.* Roughly speaking, this is the "degrees of all the irreducible factors", but we have to be slightly careful, since if $(x-1)$ is an irreducible factor, so is $(-x+1)$, and we don't want to count both. Note that the factorization shape is independent of which factorization we take.

If f is not known to be square-free, then we have to take a multiset of (degree, multiplicity) pairs. This case is not useful in practice, but because a square-free decomposition can be larger than the starting point (Observation 2, page 75), it can be useful in theory: see section 5.8.1.

In this language, we are saying that the factorization shape modulo p has to be a splitting of the factorization shape over the integers.

5.2.1 The Musser test

Definition 93 *The* sumset *of a factorisation shape is the set of all sums of subsets of the factorisation shape.*

The sumset of a factorisation shape modulo p is therefore the set of all possible degrees of factors over the integers. Example 26 has two factorisation shapes $\{1,3\}$ and $\{2,2\}$, whose sumsets are $\{0,1,3,4\}$ and $\{0,2,4\}$. The intersection of these sumsets is $\{0,4\}$, so no proper factorisation over $\mathbf{Z}$ is possible.

[1] Sometimes known as Newton's algorithm.

[2] That is, a set but allowing repetitions, so $\{2,2,1\}$ and $\{2,1,1,1\}$ would be different multisets, but of course are both the set $\{2,1\}$.

It is an old experimental observation [Mus78] that, *if* a polynomial can be proved irreducible by intersecting sumsets of factorisation shapes, five shapes, i.e. five primes, are nearly always sufficient to prove it.

This test was formally analysed in [PPR15], and very substantial experimental evidence is presented in their Figure 1. The bottom line is that we should probably take seven primes. The rest of this subsection is somewhat more technical, and can be skipped by the reader not familiar with Galois Theory and permutation groups.

Definition 94 *The Galois group of a polynomial $f = a_n \prod_{i=1}^n (x-\alpha_i) \in \mathbf{Z}[x]$ is the set of all automorphisms of $\mathbf{Q}(\alpha_1, \ldots, \alpha_n)$ which leaves $\mathbf{Q}$ fixed. The Galois group is determined by its action on the α_i, and is therefore a subgroup of the symmetric group S_n.*

Proposition 60 *f is irreducible if, and only if, its Galois group is transitive (for all roots α and β, there is a permutation of the Galois group taking α to β).*

Example 27 *Various Galois groups are as follows.*

$x^2 - 2$ Here the roots are $\alpha = \sqrt{2} \approx 1.4142$ and $\beta = -\sqrt{2} \approx -1.4142$. Apart from the identity permutation, the only option is to exchange α and β. Hence the Galois Group is $C_2 = \langle(\alpha, \beta)\rangle$. The cycle types are $(1, 1)$ and (2).

$(x^2 - 2)(x^2 - 3)$ Here the roots are $\alpha = \sqrt{2} \approx 1.4142$, $\beta = -\sqrt{2} \approx -1.4142$, $\gamma = \sqrt{3} \approx 1.732$ and $\delta = -\sqrt{3} \approx -1.732$. Apart from the identity permutation, the only options are to exchange α and β, exchange γ and δ, or possibly both. Sending α to γ, say, is not possible as $\alpha^2 = 2$ but $\gamma^2 = 3$. Hence the Galois Group is $C_2 \times C_2 = \langle(\alpha, \beta), (\gamma, \delta)\rangle$. The cycle types are $(1, 1, 1, 1)$, $(1, 1, 2)$ and $(2, 2)$. Note that this group is not transitive.

$x^4 + 1$ Here the roots are $\alpha = \frac{1+i}{\sqrt{2}}$, $\beta = \frac{1-i}{\sqrt{2}} = \frac{1}{\alpha}$, $\gamma = -\alpha$ and $\delta = -\beta$. One option is to send α to β, which replaces i by $-i$, and is therefore the permutation $(\alpha, \beta)(\gamma, \delta)$, and another is to replace $\sqrt{2}$ by $-\sqrt{2}$, which is the permutation $(\alpha, \gamma)(\beta, \delta)$. Hence the Galois group contains, and is in fact generated by, these two permutations: it is often known as the Klein 4-group, or V. The cycle types are $(1, 1, 1, 1)$ and $(2, 2)$. This group *is* transitive, but every cycle shape in this group is a valid cycle shape in $C_2 \times C_2$ from the previous example, which is not transitive.

$x^8 - 2$ Here the roots are $\alpha_k = \sqrt[8]{2}e^{2\pi ik/8}$, $k = 0..7$. One option is to replace $\sqrt[8]{2}$ by $\sqrt[8]{2}e^{2\pi i/8}$, which cyclically permutes the roots, i.e. $(\alpha_0, \alpha_1, \ldots, \alpha_7)$. Another option is to replace i by $-i$, which corresponds to the permutation $(\alpha_1, \alpha_7), (\alpha_2, \alpha_6), (\alpha_3, \alpha_5)$. In fact the Galois group is generated by these two, and is generally called the dihedral group, with 16 elements.

$x^7 - 2$ Here the roots are $\alpha_k = \sqrt[7]{2}e^{2\pi ik/7}$, $k = 0..6$. One option is to replace $\sqrt[7]{2}$ by $\sqrt[7]{2}e^{2\pi i/7}$, which cyclically permutes the roots, i.e. $\pi_1 :=$

$(\alpha_0, \alpha_1, \ldots, \alpha_6)$. Another option is to replace i by $-i$, which is the permutation $\pi_2 := (\alpha_1, \alpha_6), (\alpha_2, \alpha_5), (\alpha_3, \alpha_4)$. However, $\pi_3 := (\alpha_1, \alpha_3, \alpha_2, \alpha_6\alpha_6, \alpha_5)$ is in fact also a legal permutation, and the group is generated by π_1 and π_3 (since $\pi_2 = \pi_3^3$) and has 42 elements.

Proposition 61 ([vdW34]) *For any n, almost all polynomials of degree n have Galois group S_n, or, more precisely, the probability of a polynomial having Galois group S_n is 1, i.e.*

$$\lim_{H\to\infty} \frac{|\{\textit{polynomials of degree } n \textit{ with coefficients} \le H \textit{ and group } S_n\}|}{|\{\textit{polynomials of degree } n \textit{ with coefficients} \le H\}|} = 1.$$

Corollary 16 *For any n, almost all polynomials of degree n are irreducible, since S_n is transitive.*

Hence, to make a statement about "almost all polynomials", it suffices to consider those with Galois group S_n.

The next question is how the factorisation shapes mod p relate to the Galois group, which is settled by the following result.

Proposition 62 (Frobenius Density Theorem) *The density of prime numbers q for which $f(x) \pmod{q}$ has factorisation shape $\{d_1, \ldots, d_k\}$ is equal to the density in the Galois group $G \le S_n$ of $f(x)$ of elements of S_n with cycle type $(d_1, \ldots, d_k)$.*

Hence, if we assume the Galois group of f is S_n, the factorisation shape of $f(x) \pmod{q}$ is distributed the same way as random permutations from S_n, and the concept of sumset generalises.

Definition 95 (See Definition 93) *The* sumset *$I(\sigma)$ of a permutation σ is the set of all sums of subsets of the cycle shape of the permutation.*

Proposition 63 ([PPR14, Theorem 1.5]; [EFG15, Theorem 1.1]) *There is a constant b_4, independent of n, such that*

$$\Pr_{S_n^4}\left(I(\sigma_1) \cap I(\sigma_2) \cap I(\sigma_3) \cap I(\sigma_4) = \{0, n\}\right) > b_4,$$

where the probability is taken uniformly over quadruples of permutations. This means that, if the Galois group is S_n, four[3] *primes have a nonzero probability of detecting transitivity (i.e. irreducibility) and in fact of detecting that the group is S_n [DS00], [PPR14, Theorem 1.3 and discussion].*

Emprically [PPR14, Figure 1], $b_4 \approx 0.5$. Similarly b_5 seems to be about 0.7 (for small n, the probability is larger, which is presumably why [Mus78] recommends five primes), and $b_7 > 0.9$.

Observation 16 *Since determining that a polynomial is irreducible if one thinks it factors is* much *mure expensive than modulo p factorisation, we should probably take seven primes, rather than the five of [Mus78].*

[3]Four is minimal: cite[EFG15, Theorem 1.2].

5.3 Factoring modulo a prime

Throughout this section, we let p be an odd[4] prime, and f a polynomial of degree n which is square-free modulo p. There are three common methods for factoring f: two due to Berlekamp and one to Cantor & Zassenhaus, and various subsequent improvements, largely in terms of asymptotic complexity. A survey of this field is [vzGP01]. Most implementers of systems use (variants of) the Cantor–Zassenhaus algorithm, for reasons explained at the start of Section 5.7.

Although the author knows of no major implementations, the asymptotically fastest factoring algorithm at the moment of writing seems to be [KU08], which for polynomials of degree n over a field with q elements takes time $\tilde{O}(n^3/2 \log q + n \log^2 q)$.

5.3.1 Berlekamp's small p method

This is due to [Ber67]. We first need to state some facts about arithmetic modulo a prime p.

Proposition 64 *If a and b are two integers modulo p, then $(a+b)^p \equiv a^p + b^p$.*

If we expand $(a+b)^p$ by the Binomial Theorem, all intermediate coefficients are divsible by p.

Proposition 65 *$a^p \equiv a$ modulo p.*

Corollary 17 *Every integer modulo p is a root of $x^p - x$, and therefore*

$$x^p - x = (x-0)(x-1)\dots(x-(p-1)).$$

These facts extend to polynomials.

Proposition 66 *Let $a(x)$ be a polynomial, then $a(x)^p \equiv a(x^p)$ modulo p.*

Let us now suppose that f factorises into r irreducible polynomials:

$$f(x) = f_1(x)f_2(x)\dots f_r(x)$$

(r is unknown for the present). Since f has no multiple factors, the f_i are relatively prime. Let $s_1, \dots, s_r$ be integers modulo p. By the Chinese Remainder Theorem (Algorithm 48)there is a polynomial v such that

$$v \equiv s_i \pmod{p, f_i(x)}, \tag{5.1}$$

and moreover, the degree of v is less than that of the product of the f_i, that is f. Such a polynomial v is useful, for if $s_i \neq s_j$, then $\gcd(f, v - s_i)$ is divisible

[4]It is possible to generalise these methods to the case $p = 2$, but the cost, combined with the unlikelihood of 2 being a good prime, means that computer algebraists (unlike cryptographers) rarely do so.

by f_i, but not by f_j, and therefore leads to a decomposition of f. We have the following relation:

$$v(x)^p \equiv s_j^p \equiv s_j \equiv v(x) \pmod{f_j(x), p},$$

and, by the Chinese remainder theorem,

$$v(x)^p \equiv v(x) \pmod{f(x), p}. \tag{5.2}$$

But, on replacing x by $v(x)$ in Corollary 17

$$v(x)^p - v(x) \equiv (v(x) - 0)(v(x) - 1) \dots (v(x) - (p - 1)) \pmod{p}. \tag{5.3}$$

Thus, if $v(x)$ satisfies (5.2), $f(x)$ divides the left hand side of (5.3), and each of its irreducible factors, the f_i, divides one of the polynomials on the right hand side of (5.3). But this implies that v is equivalent to an integer modulo f_i, that is that v satisfies (5.1). We have proved (by Knuth's method [Knu81, p. 422]) the following result.

Theorem 43 *The solutions v of (5.1) are precisely the solutions of (5.2).*

We have already said that the solutions of (5.1) provide information about the factorisation of f, but we still have the problem of finding them. Berlekamp's basic idea is to note that (5.2) is a *linear* equation for the coefficients of v. This remark may seem strange, but it is a consequence of Proposition 66. In fact, if n is the degree of f, let us consider the matrix

$$Q = \begin{pmatrix} q_{0,0} & q_{0,1} & \cdots & q_{0,n-1} \\ q_{1,0} & q_{1,1} & \cdots & q_{1,n-1} \\ \vdots & \vdots & & \vdots \\ q_{n-1,0} & q_{n-1,1} & \cdots & q_{n-1,n-1} \end{pmatrix},$$

where

$$x^{pk} \equiv q_{k,n-1}x^{n-1} + \cdots + q_{k,1}x + q_{k,0} \pmod{f(x), p}.$$

If we consider a polynomial as a vector of its coefficients, multiplication by Q corresponds to the calculation of the p-th power of the polynomial. The solutions of (5.2) are thus the eigenvectors of the matrix $Q \pmod{p}$ for the eigenvalue 1. Hence the algorithm in Figure 5.2.

Q might seem expensive to compute, but if we precompute $(x^i)^p \pmod{(f(x), p)}$, it takes $O(n^3 + \log(p)n^2)$ operations (with classical linear algebra, and assuming that p is single-word size). Determining r, the number of irreducible factors, similarly takes time $O(n^3)$ The running time of the full algorithm[5] is $O(n^3 + p(r - 1)n^2)$, and the average value of r is $\ln n$. This is very fast if p is small, but may be expensive if p is not small. We note that the algorithm is completely deterministic.

[5]Texts often say $O(n^3 + prn^2)$, but the rth factor needn't be calculated this way, as it follows from knowing all the others.

Figure 5.2: Algorithm 27; Berlekamp for small p

Algorithm 27
Input: *Prime p and f a squarefree (mod p) univariate polynomial*
Output: *The irreducible factors of f*

```
Calculate $Q$
Calculate (a basis for) the eigenvectors of $Q$ for the eigenvalue 1,
          i.e. (a basis for) the solutions $\mathbf{v}$ of $(Q_I)\mathbf{v} = 0$.
# $(1, 0, \dots, 0)^T$ is the trivial eigenvector, as integers solve (5.2) $r$, the size of the basis, is the number o
if $r = 1$
  then return $\{f\}$.
$S = \emptyset$
for $v$ a non-trivial eigenvector
    Let $g$ be the polynomial whose coefficients are $v$
    for $s \in \{0, \dots, p-1\}$
        if $(h := \gcd(g - s, f)) \neq 1$
          then $S := S \cup \{h\}$
                if $|S| = r$
                  then return $S$
```

5.3.2 The Cantor–Zassenhaus method

This method, a combination of Algorithms 28 and 29, is generally attributed to [CZ81][6]. It is based on the following generalization of Fermat's (Little) Theorem/ Corollary 17.

Proposition 67 ([LN97, Theorem 3.20]) *All irreducible polynomials of degree d with coefficients modulo p divide $x^{p^d} - x$, and in fact*

$$x^{p^d} - x = \prod_{e|d} \prod^{*}_{f:\deg(f)=e} f, \tag{5.4}$$

where $\prod^*$ means that we only take irreducible polynomials.

Corollary 18 *All irreducible polynomials of degree d with coefficients modulo p divide $x^{p^d-1} - 1$, except for x itself in the case $d = 1$. Furthermore, no irreducible polynomials of degree more than d divide $x^{p^d-1} - 1$.*

Corollary 19 *Half of the irreducible polynomials of degree d (except for x itself in the case $d = 1$) with coefficients modulo p divide $(x - a)^{(p^d-1)/2} - 1$, for any a (but a different 50%,depending on a).*

Hence we deduce Algorithm 28, which splits a square-free polynomial f as $\prod f_i$, where each f_i is the product of irreducibles of degree i.

[6]Though [Zip93] credits algorithm 28 to [Arw18], and [vzGP01] credits it to Gauß.

Figure 5.3: Algorithm28: Distinct Degree Factorization

Algorithm 28 (Distinct Degree Factorization)
Input: *$f(x)$ a square-free polynomial modulo p, not divisible by x; a prime p*
Output: *A decomposition $f = \prod f_i$, where each f_i is the product of irreducibles of degree i.*

```
$i$:=1
while $2i \le \deg(f)$
        $g := x^{p^i-1} \pmod f$            (*)
        $f_i := \gcd(g-1, f)$
        $f := f/f_i$
        $i := i+1$
if      $f \ne 1$
        then $f_{\deg(f)} := f$
```

Note that the computation in line (*) should be done by the repeated squaring method, reducing modulo f at each stage. We can save time in practice by re-using the previous g. Various improvements (at least in asymptotic complexity) are given in [vzGS92].

If f_i has degree i, then it is clearly irreducible: otherwise we have to split it. This is the purpose of Algorithm 29, which relies on a generalization of Corollary 19.

Proposition 68 ([CZ81, p. 589]) *Let f be a product of $r > 1$ irreducible polynomials of degree d modulo p, and g a random (non-constant) polynomial of degree $< d$. Then the probability that $\gcd(g^{(p^d-1)/2} - 1, f)$ is either 1 or f is at most 2^{1-r}.*

Proposition 69 *In classical arithmetic, the running time of the Cantor–Zassenhaus Algorithm (i.e. Algorithm 28 followed by Algorithm 29) is $O(d^3 \log p)$, where d is the degree of the polynomial being factored.*

We do $O(d \log p)$ operations on polynomials of degree d, where the factor $\log p$ comes from the $x^{p^i} \pmod f$ computations. We note that Algorithm 28 is deterministic, but Algorithm 29 is probabilistic.

5.3.3 Berlekamp's large p method

This method is due to [Ber70]. It has an alternative, matrix-based, form of Algorithm 28, again deterministic, and a probabilistic equivalent of Algorithm 29. We do not go further into it here.

Figure 5.4: Algorithm29: Split a Distinct Degree Factorization

Algorithm 29 (Split a Distinct Degree Factorization)
Input: *A prime p, a degree d and a polynomial $f(x) \pmod p$ known to be the product of irreducibles of degree d (and not divisible by x)*
Output: *The factorization of f (modulo p) into irreducibles of degree d.*

```
if $d = \deg(f)$
  then return $f$              # degree d so is irreducible
$W := \{f\}$
$ans := \emptyset$
while $W \neq \emptyset$            # factors of degree $d$ found
  $h :=$RandomPoly($p$,$d$)
  $h := h^{(p^d-1)/2} \pmod g$            (*)
  $V := \emptyset$            # list of polynomials to be processed
  for $g \in W$ do
    $h_1 := \gcd(h-1, g)$
    if $\deg(h_1) = 0 \vee \deg(h_1) = \deg(g)$
      then $V := V \cup \{g\}$       # we made no progress
      else process($h_1$)
           process($g/h_1$)
  $W := V$
return $ans$
```

Here RandomPoly(p,d) returns a random non-constant polynomial modulo p of degree less than d, and the sub-function process(g_1) takes a polynomial g_1 which is a result of splitting g by h_1, and adds it to *ans* if it has degree d, otherwise adds it to V. Again (*) should be done by repeated squaring, reduing modulo g every time.

5.4 From $\mathbf{Z}_p$ to Z?

Now that we know that factoring over the integers modulo p is possible, the obvious strategy for factoring polynomials over the integers would seem to be to follow one of algorithms 17 or 18. This would depend on having 'good' reduction, which one would naturally define as follows.

Definition 96 (Optimistic) *We say that p is of* good reduction *for the factorization of the polynomial f if the degrees of the irreducible factors of f (modulo p) are the same as those of the factors of f.*

And indeed, if we can find p of good reduction, then we could factorize f. Unfortunately these are rare, and possibly non-existent.

Corollary 20 (of Proposition 62) *If f is a generic (formally speaking, with Galois group S_n) polynomial of degree n, the probability of its remaining irreducible modulo p is $1/n$.*

Nevertheless, we could hope that we can piece together results from several primes to get information. For example, x^5+x+3 factors into irreducibles as

$$\left(x^3+7\,x^2+3\,x+7\right)\left(x^2+4\,x+2\right)\ \bmod 11$$

(and therefore does not have linear factors over $\mathbf{Z}$) and as

$$\left(x^4+5\,x^3+12\,x^2+8\,x+2\right)(x+8)\ \bmod 13$$

(therefore any factorization has a linear factor) and hence must be irreducible over the integers. This test is described in Section 5.2.1. However, there are irreducible polynomials which can *never* be proved so this way.

Example 28 *The polynomial x^4+1, which is irreducible over the integers, factors into two quadratics (and possibly further) modulo* every *prime*[7].

$p=2$ Then $x^4+1=(x+1)^4$.

$p=4k+1$ In this case, -1 is always a square, say $-1=q^2$. This gives us the factorisation $x^4+1=(x^2-q)(x^2+q)$.

$p=8k\pm1$ In this case, 2 is always a square, say $2=q^2$. This gives us the factorisation $x^4+1=(x^2-(2/q)x+1)(x^2+(2/q)x+1)$. In the case $p=8k+1$, we have this factorisation and the factorisation given in the previous case. As these two factorisations are not equal, we can calculate the g.c.d.s of the factors, in order to find a factorisation as the product of four linear factors.

$p=8k+3$ In this case, -2 is always a square , say $-2=q^2$. This is a result of the fact that -1 and 2 are not squares, and so their product must be a square. This property of -2 gives us the factorisation $x^4+1=(x^2-(2/q)x-1)(x^2+(2/q)x-1)$

This polynomial is not an isolated oddity: [SD70] and [KMS83] proved that there are whole families of polynomials with this property of being irreducible, but of factorising compatibly modulo every prime, and indeed compatibly into many quadratics. Several people have said that these polynomials are, nevertheless, "quite rare", which is true if one takes polynomials at random, but [ABD85] showed that they can often occur in the manipulation of algebraic numbers.

Even if we have factorizations modulo several primes, a further problem arises, which we will illustrate with the example of x^4+3. This factors as

$$x^4+3=\left(x^2+2\right)(x+4)(x+3)\ \bmod 7$$

and

$$x^4+3=\left(x^2+x+6\right)\left(x^2+10\,x+6\right)\ \bmod 11. \tag{5.5}$$

[7]The Galois group of this polynomial is discussed on page 212.

In view of the second factorization, the first has too much decomposition, and we need only consider the split

$$x^4 + 3 = \left(x^2 + 2\right)\left(x^2 + 5\right) \bmod 7, \tag{5.6}$$

obtained by combining the two linear factors.

When we come to combine results modulo these two primes by Chinese Remainder Theorem (Theorem 52) to deduce a congruence modulo 77, we have a dilemma: do we pair $\left(x^2 + x + 6\right)$ with $\left(x^2 + 2\right)$ or $\left(x^2 + 5\right)$? Both seem feasible.

In fact, both *are* feasible. The first pairing gives

$$x^4 + 3 = \left(x^2 + 56\,x + 72\right)\left(x^2 - 56\,x - 16\right) \bmod 77, \tag{5.7}$$

and the second gives

$$x^4 + 3 = \left(x^2 + 56\,x + 61\right)\left(x^2 - 56\,x - 5\right) \bmod 77 : \tag{5.8}$$

both of which are correct. The difficulty in this case, as in general, is that, while polynomials over $\mathbf{Z}_7$ have unique factorization, as do those over $\mathbf{Z}_{11}$ (and indeed modulo any prime), polynomials over $\mathbf{Z}_{77}$ (or any product of primes) do not, as (5.7) and (5.8) demonstrate.

5.5 Hensel Lifting

Our attempts to use the Chinese Remainder Theorem seem doomed[8]: we need a different solution, which is provided by what mathematicians call p-adic methods, and computer algebraists call Hensel Lifting. This is the topic of the next four subsections.

5.5.1 Linear Hensel Lifting

This is the simplest implementation of the phase described as 'grow' in Figure 5.1: we grow incrementally $\mathbf{Z}_p \to \mathbf{Z}_{p^2} \to \mathbf{Z}_{p^3} \to \cdots \to \mathbf{Z}_{p^m}$.

For simplicity, we consider first the case of a monic polynomial f, which factorizes modulo p as $f = gh$, where g and h are relatively prime (which implies that f modulo p is square-free, that is, that p does not divide the resultant of f and f'). We use parenthesized superscripts, as in $g^{(1)}$, to indicate the power of p modulo which an object has been calculated. Thus our factorization can be written $f^{(1)} = g^{(1)}h^{(1)} \pmod{p^1}$ and our aim is to calculate a corresponding factorization $f^{(k)} = g^{(k)}h^{(k)} \pmod{p^k}$ such that p^k is sufficiently large.

[8]In fact, they are doomed for two distinct reasons. The first is that we are *not* emulating modulo p a calculation over the integers, so the argument of Lemma 11 does not apply. The second is that we cannot answer question 4 — how do we combine the results, as our results are a set with, possibly, no distinguishing features.

Obviously, $g^{(2)} \equiv g^{(1)} \pmod{p}$, and therefore we can write $g^{(2)} = g^{(1)} + p\hat{g}^{(2)}$ where $\hat{g}^{(2)}$ is a measure of the difference between $g^{(1)}$ and $g^{(2)}$. The same holds for f and h, so that $f^{(2)} = g^{(2)}h^{(2)} \pmod{p^2}$ becomes

$$f^{(1)} + p\hat{f}^{(2)} = (g^{(1)} + p\hat{g}^{(2)})(h^{(1)} + p\hat{h}^{(2)}) \pmod{p^2}.$$

Since $f^{(1)} = g^{(1)}h^{(1)} \pmod{p^1}$, this equation can be rewritten in the form[9]

$$\underbrace{\left[\frac{f^{(1)} - g^{(1)}h^{(1)}}{p}\right]}_{\text{computed mod } p^2} + \hat{f}^{(2)} = \hat{g}^{(2)}h^{(1)} + \hat{h}^{(2)}g^{(1)} \pmod{p}. \tag{5.9}$$

The left hand side of this equation is known, whereas the right hand side depends linearly on the unknowns $\hat{g}^{(2)}$ and $\hat{h}^{(2)}$. Applying the extended Euclidean Algorithm (5)) to $g^{(1)}$ and $h^{(1)}$, which are relatively prime, we can find polynomials $\hat{g}^{(2)}$ and $\hat{h}^{(2)}$ of degree less than $g^{(1)}$ and $h^{(1)}$ respectively, which satisfy this equation modulo p. The restrictions on the degrees of $\hat{g}^{(2)}$ and $\hat{h}^{(2)}$ are valid in the present case, for the leading coefficients of $g^{(k)}$ and $h^{(k)}$ have to be 1. Thus we can determine $g^{(2)}$ and $h^{(2)}$.

Similarly, $g^{(3)} \equiv g^{(2)} \pmod{p^2}$, and we can therefore write $g^{(3)} = g^{(2)} + p^2\hat{g}^{(3)}$ where $\hat{g}^{(3)}$ is a measure of the difference between $g^{(2)}$ and $g^{(3)}$. The same is true for f and h, so that $f^{(3)} = g^{(3)}h^{(3)} \pmod{p^3}$ becomes

$$f^{(2)} + p^2\hat{f}^{(3)} = (g^{(2)} + p^2\hat{g}^{(3)})(h^{(2)} + p^2\hat{h}^{(3)}) \pmod{p^3}.$$

Since $f^{(2)} = g^{(2)}h^{(2)} \pmod{p^2}$, this equation can be rewritten in the form

$$\frac{f^{(2)} - g^{(2)}h^{(2)}}{p^2} + \hat{f}^{(3)} = \hat{g}^{(3)}h^{(2)} + \hat{h}^{(3)}g^{(2)} \pmod{p}. \tag{5.10}$$

Moreover, $g^{(2)} \equiv g^{(1)} \pmod{p}$, so this equation simplifies to

$$\frac{f^{(2)} - g^{(2)}h^{(2)}}{p^2} + \hat{f}^{(3)} = \hat{g}^{(3)}h^{(1)} + \hat{h}^{(3)}g^{(1)} \pmod{p}.$$

The left hand side of this equation is known, whilst the right hand side depends linearly on the unknowns $\hat{g}^{(3)}$ and $\hat{h}^{(3)}$. Applying the extended Euclidean algorithm to $g^{(1)}$ and $h^{(1)}$, which are relatively prime, we can find the polynomials $\hat{g}^{(3)}$ and $\hat{h}^{(3)}$ of degrees less than those of $g^{(1)}$ and $h^{(1)}$ respectively, which satisfy this equation modulo p. Thus we determine $g^{(3)}$ and $h^{(3)}$ starting from $g^{(2)}$ and $h^{(2)}$, and we can continue these deductions in the same way for every power p^k of p until p^k is sufficiently large.

We should note that Euclid's algorithm is always applied to the same polynomials, and therefore it suffices to perform it once. In fact, we can state the algorithm in Figure 5.5.

[9]This comes from dividing the previous congruence by p, and hence the term in $[\ldots]$ must be computed modulo p^2. The same warning applies throughout this chapter. Note also that we have applied the rule that $A \equiv B \pmod{p^2}$ means that $A/p \equiv B/p \pmod{p}$ not $\pmod{p^2}$.

Figure 5.5: Algorithm 30

Algorithm 30 (Univariate Hensel Lifting (Linear Two Factor version))

Input: $f, g^{(1)}, h^{(1)}, p, k$ *with* f *monic and* $\equiv g^{(1)}h^{(1)} \pmod p$
Output: $g^{(k)}, h^{(k)}$ *with* $f \equiv g^{(k)}h^{(k)} \pmod{p^k}$

```
$g := g^{(1)}$
$h := h^{(1)}$
$g^{(r)}, h^{(r)} :=$ Algorithm 5$(g^{(1)}, h^{(1)})$ in $\mathbf{Z}_p[x]$
for $i := 2 \ldots k$
    $\Delta := \frac{f - gh \pmod{p^i}}{p^{i-1}}$
    $g^{(c)} := \Delta * h^{(r)} \pmod{(p, g^{(1)})}$
    $h^{(c)} := \Delta * g^{(r)} \pmod{(p, h^{(1)})}$
    $g := g + p^{i-1}g^{(c)}$
    $h := h + p^{i-1}h^{(c)}$
return $(g, h)$
```

To solve the more general problem of lifting a factorization of a non-monic polynomial, we adopt the same solution as in the g.c.d. case: we impose the leading coefficient in each factor to be the leading coefficient of the orgiinal polynomial (which we may as well assume to be primitive). We also address the problem of lifting n (rather than just 2) factors in Algorithm 31. We have also incorporated an "early termination" test, which is very useful in practice.

Proposition 70 *Assuming classical arithmetic (but ignoring the fact that the length of numbers is quantised into multiples of the wordlength), the cost of the lift from* p^k *to* p^{k+1} *is* $\left(\frac{k+1}{k}\right)^2$ *the cost of the lift from* p^{k-1} *to* p^k*. Hence the cost of a lift to* p^k *is* $O(k^3)$*.*

5.5.2 Quadratic Hensel Lifting

This is an alternative implementation of the phase described as 'grow' in Figure 5.1: we grow incrementally $\mathbf{Z}_p \rightarrow \mathbf{Z}_{p^2} \rightarrow \mathbf{Z}_{p^4} \rightarrow \cdots \rightarrow \mathbf{Z}_{p^{2^m}}$, squaring the modulus each time.

As in the previous section, we first consider the case of a monic polynomial with two factors. The lifting from p to p^2 proceeds exactly as in the previous section, and equation (5.9) is still valid. The lifting from p^2 to p^4 is similar to the lifting from p^2 to p^3 in the previous section, and the analogue of equation (5.10) is the following

$$\frac{f^{(2)} - g^{(2)}h^{(2)}}{p^2} + \hat{f}^{(4)} = \hat{g}^{(4)}h^{(2)} + \hat{h}^{(4)}g^{(2)} \pmod{p^2}. \qquad (5.11)$$

Figure 5.6: Algorithm 31

Algorithm 31 (Univariate Hensel Lifting (Linear version))
Input: $f, g_1^{(1)}, \ldots, g_n^{(1)}, p, k$ *with* $f \in \mathbf{Z}[x]$ *primitive and* $\equiv \prod g_i^{(1)} \pmod p$
Output: $g_1^{(k)}, \ldots, g_n^{(k)}$ *with* $f \equiv \prod g_i^{(k)} \pmod{p^k}$

```
for $j := 1 \ldots n$
    $g_j^{(1)} := \frac{\mathrm{lc}(f)}{\mathrm{lc}(g_j^{(1)})} g_j^{(1)}$
$F := \mathrm{lc}(f)^{n-1} f$        #leading coefficients imposed
for $j := 1 \ldots n$
    $g_j^{(r)}, h_j^{(r)} :=$ Algorithm 5$(g_j^{(1)}, \prod_{i \neq j} g_i^{(1)})$ in $\mathbf{Z}_p[x]$
for $i := 2 \ldots k$
    $\Delta := \frac{F - \prod_j g_j \pmod{p^i}}{p^{i-1}}$
    if $\Delta = 0$
      then break        #True factorization discovered
    for $j := 1 \ldots n$
        $g_j^{(c)} := \Delta * h_j^{(r)} \pmod{(p, g_j^{(1)})}$
        $g_j := g_j + p^{i-1} g_j^{(c)}$
for $j := 1 \ldots n$
    $g_j := \mathrm{pp}(g_j)$        #undo the imposed leading coefficients
return $(g_1, \ldots, g_n)$
```

Figure 5.7: Algorithm 32

Algorithm 32 (Univariate Hensel Lifting (Quadratic Two Factor version))

Input: $f, g^{(1)}, h^{(1)}, p, k$ *with* f *monic and* $\equiv g^{(1)}h^{(1)} \pmod p$
Output: g, h *with* $f \equiv gh \pmod{p^{2^k}}$

```
$g := g^{(1)}$
$h := h^{(1)}$
for $i := 1 \ldots k$                    $\#f \equiv gh \pmod{p^{2^{i-1}}}$
    $g^{(r)}, h^{(r)} :=$ Algorithm 5$(g, h)$ in $\mathbf{Z}_{p^{2^{i-1}}}[x]$
    $\Delta := \frac{f - gh \pmod{p^{2^i}}}{p^{2^{i-1}}}$
    $g^{(c)} := \Delta * h^{(r)} \pmod{(p^{2^{i-1}}, g)}$
    $h^{(c)} := \Delta * g^{(r)} \pmod{(p^{2^{i-1}}, h)}$
    $g := g + p^{2^{i-1}} g^{(c)}$
    $h := h + p^{2^{i-1}} h^{(c)}$          $\#f \equiv gh \pmod{p^{2^i}}$
return $(g, h)$
```

The difference is that this equation is modulo p^2 rather than modulo p, and hence the inverses have to be recomputed, rather than being the same, as (5.10) can re-use the inverses from (5.9). We give the corresponding Algorithm 32 in Figure 5.7, an analogue of Algorithm 30, except that the call to Algorithm 5 is inside the loop, rather than preceding it.

Equally, we can give the general Algorithm 33 in Figure 5.8 as an equivalent of Algorithm 31. Again, the Extended Euclidean Algorithm is in the innermost loop, and pratcical experience confirms that this is indeed the most expensive step. It is also the case that we are repeating work, inasmuch as the calculations are the same as before, except that the inputs are correct to a higher power of p than before, and the resuts are required to a higher power of p than before.

Proposition 71 *Assuming classical arithmetic (but ignoring the fact that the length of numbers is quantised into multiples of the word length), the cost of the lift from* p^{2^k} *to* $p^{2^{k+1}}$ *is 4 times the cost of the lift from* $p^{2^{k-1}}$ *to* p^{2^k}*. Hence the cost of a lift to* p^{2^k} *is dominated by the cost of the last step, which involves arithmetic on numbers of length* $O(2^k)$*, and so is* $O(2^{2k})$*.*

5.5.3 Quadratic Hensel Lifting Improved

The main cost in algorithms 32 and 33 is the repeated calls to Algorithm 5. In fact, for each iteration on k, Algorithm 5 is doing the same calculations, but to a higher power of p. Do we need to do these calculations from scratch each time? The answer is that in fact we can re-use the previous calculations.

Figure 5.8: Algorithm 33

Algorithm 33 (Univariate Hensel Lifting (Quadratic version))
Input: $f, g_1^{(1)}, \ldots, g_n^{(1)}, p, k$ *with* f *primitive and* $\equiv \prod g_i^{(1)} \pmod p$
Output: $g_1, \ldots, g_n$ *with* $f \equiv \prod g_i \pmod{p^{2^k}}$

```
for $j := 1 \ldots n$
    $g_j^{(1)} := \frac{\mathrm{lc}(f)}{\mathrm{lc}(g_j^{(1)})} g_j^{(1)}$
$F := \mathrm{lc}(f)^{n-1} f$        #leading coefficients imposed
for $i := 1 \ldots k$                 #$F = \prod g_j \pmod{p^{2^{i-1}}}$
    $\Delta := \dfrac{F - \prod_j g_j \pmod{p^{2^i}}}{p^{2^{i-1}}}$
    if $\Delta = 0$
      then break        #True factorization discovered
    for $j := 1 \ldots n$
        $g_j^{(r)}, h_j^{(r)} :=$ Algorithm 5$(g_j, \prod_{l \neq j} g_l)$ in $\mathbf{Z}_{p^{2^{i-1}}}[x]$
        $g_j^{(c)} := \Delta * h_j^{(r)} \pmod{(p^{2^{i-1}}, g_j)}$
        $g_j := g_j + p^{2^{i-1}} g_j^{(c)}$
    #$F = \prod g_j \pmod{p^{2^i}}$
for $j := 1 \ldots n$
    $g_j := \mathrm{pp}(g_j)$        #undo the imposed leading coefficients
return $(g_1, \ldots, g_n)$
```

Figure 5.9: Algorithm 34

Algorithm 34 (Univariate Hensel Lifting (Improved Quadratic Two Factor version))

Input: $f, g^{(1)}, h^{(1)}, p, k$ *with* f *monic and* $\equiv g^{(1)}h^{(1)} \pmod p$
Output: g, h *with* $f \equiv gh \pmod{p^{2^k}}$

```
$g := g^{(1)}$
$h := h^{(1)}$
$g^{(r)}, h^{(r)} :=$ Algorithm 5$(g,h)$ in $\mathbf{Z}_p[x]$
for $i := 1 \dots k$                  $\#f \equiv gh \pmod{p^{2^{i-1}}}$
                                      $\#gg^{(r)} \equiv 1 \pmod{h, p^{2^{i-1}}}$
                                      $\#hh^{(r)} \equiv 1 \pmod{g, p^{2^{i-1}}}$
    $\Delta := \frac{f - gh \pmod{p^{2^i}}}{p^{2^{i-1}}}$
    $g^{(c)} := \Delta * h^{(r)} \pmod{(p^{2^{i-1}}, g)}$
    $h^{(c)} := \Delta * g^{(r)} \pmod{(p^{2^{i-1}}, h)}$
    if $i < k$
        $\Delta^{(r)} := -\frac{gg^{(r)}+hh^{(r)}-1}{p^{2^{i-1}}} - g^{(c)}g^{(r)} - h^{(c)}h^{(r)}$
        $g^{(r,c)} := \Delta^{(r)} * g^{(r)} \pmod{(p^{2^{i-1}}, h)}$
        $g^{(r)} := g^{(r)} + p^{2^{i-1}} g^{(r,c)}$
        $h^{(r,c)} := \Delta^{(r)} * h^{(r)} \pmod{(p^{2^{i-1}}, g)}$
        $h^{(r)} := h^{(r)} + p^{2^{i-1}} h^{(r,c)}$
    $g := g + p^{2^{i-1}} g^{(c)}$
    $h := h + p^{2^{i-1}} h^{(c)}$     $\#f \equiv gh \pmod{p^{2^i}}$
return $(g, h)$
```

Consider equation (5.11), where we need the inverses of $g^{(2)}$ and $h^{(2)}$, i.e. $g^{(r,2)}$ and $h^{(r,2)}$ such that $g^{(2)}g^{(r,2)} + h^{(2)}h^{(r,2)} \equiv 1 \pmod{p^2}$. From the previous step we know $g^{(r,1)}$ and $h^{(r,1)}$ such that $g^{(1)}g^{(r,1)} + h^{(1)}h^{(r,1)} \equiv 1 \pmod{p^1}$. Write $g^{(r,2)} = g^{(r,1)} + p\hat{g}^{(r,2)}$ etc., then $\frac{g^{(1)}g^{(r,1)}+h^{(1)}h^{(r,1)}-1}{p} + \hat{g}^{(2)}g^{(r,1)} + g^{(1)}\hat{g}^{(r,2)} + h^{(2)}h^{(r,1)} + h^{(1)}\hat{h}^{(r,2)} \equiv 0 \pmod p$. This is again a linear equation for the unknowns $\hat{g}^{(r,2)}$ and $\hat{h}^{(r,2)}$. If we write it as

$$g^{(1)}\hat{g}^{(r,2)} + h^{(1)}\hat{h}^{(r,2)} = \Delta^{(r)} \pmod p, \tag{5.12}$$

the similarity with (5.10) is obvious, and $\hat{g}^{(r,2)} = \Delta^{(r)}g^{(r,1)} \pmod{h^{(1)}, p}$, $\hat{h}^{(r,2)} = \Delta^{(r)}h^{(r,1)} \pmod{g^{(1)}, p}$.

The analogy of Algorithm 32 is then Algorithm 34, given in Figure 5.9. We do not give an analogy of Algorithm 33, as the notation becomes quite intricate. **TO BE COMPLETED**citation from MooreNorman

5.5.4 Hybrid Hensel Lifting

In theory, quadratic lifting requires many fewer lifting stages than linear lifting, and, though the individual steps are more expensive, quadratic lifting should be faster. This is borne out by simple asymptotic analysis: Proposition 70 says that the cost of linear lifting to p^k is $O(k^3)$, while Proposition 71 says that the cost of quadratic lifting to $p^{2^{k'}}$ is $O((2^{k'})^2)$. Since we expect $k \approx 2^{k'}$, quadratic lifting seems to win.

Reality, though, is more complicated than this asymptotic analysis. Firstly we are lifting to the least power of p, say p^m, greater than $2M$, where M is the appopriate Landau–Mignotte bound. Secondly, number lengths are indeed quantised into units of the length of the machine word, whereas p itself is normally small, often[10] ≤ 19. Thirdly, the cost of quadratic lifting is dominated by that of the last step.

In practice, therefore, we tend to use a hybrid: lifting quadratically to begin with, p, p^2, p^4, $\ldots p^\ell$, and then linearly, $p^{2\ell}$, $p^{4\ell}$ .... There are two main variants of this.

- Let ℓ be such that p^ℓ is the greatest power of p to fit in one machine word. This means Algorithm 5, generally the most expensive step, is only ever applied to polynomials with single-word coefficients. Since ℓ may not be a power of two, the last quadratic lift may be somewhat truncated.

- Let ℓ be the largest power of 2 such that p^m can be reached economically from p^ℓ. The following procedure is suggested. Let m_0 be the greatest power of 2 such that $4m_0 \leq m$, and lift quadratically to p^{m_0}. The next steps depend on where m fits with respect to multiples of m_0. The details are given in [Abb88].

  $4m_0 = m$ Two more quadratic lifts p^{2m_0} and $p^{4m_0} = p^m$.

  $4m_0 < m \leq 5m_0$ Linear lifting p^{2m_0}, p^{3m_0}, p^{4m_0} and $p^{5m_0} \geq p^m$.

  $5m_0 < m \leq 6m_0$ Quadratic lifting to p^{2m_0}, then linear lifting p^{4m_0} and $p^{6m_0} \geq p^m$.

  $6m_0 < m \leq 7m_0$ Linear lifting p^{2m_0}, p^{3m_0}, p^{4m_0}, p^{5m_0}, p^{6m_0} and $p^{7m_0} \geq p^m$.

  $7m_0 < m \leq 8m_0$ Quadratic lifting to p^{2m_0}, then linear lifting p^{4m_0}, p^{6m_0} and $p^{8m_0} \geq p^m$.

  The choice between these, and the fine-tuning of the choices in the second option, are very dependent on details of the implementation in practice.

Further details, and a "balanced factor tree" approach, are given in [vzGG99, §15.5].

TO BE COMPLETED

[10] 19 is the seventh odd prime: see Observation 16.

Figure 5.10: Algorithm 35: Combine Modular Factors

Algorithm 35 (Combine Modular Factors)
Input: *A prime power p^k, a monic polynomial $f(x)$ over the integers, a factorisation $f = \prod_{i=1}^r f_i$ modulo p^k*
Output: *The factorization of f into monic irreducible polynomials over the integers.*

```
$ans := \emptyset$
$M :=$LandauMignotteBound($f$)
$S := \{f_1, \ldots, f_r\}$
**for** subsets $T$ of $S$
    $h := \prod_{g \in T} g \pmod{p^k}$
    **if** $|h| < M$ **and** $h$ divides $f$
        **then** $ans := ans \cup \{h\}$
                $f := f/h$
                $S := S \setminus T$
                $M := \min(M$,LandauMignotteBound($f$))
```

5.6 The recombination problem

However, as pointed out in section 5.4, the fact that we have a factorization of f modulo p^k, where $p^k > 2M$, M being the Landau–Mignotte bound (Theorem 36), or some alternative (see page 309) on the size of any coefficients in any factor of f, does not solve our factorization problem. It is perfectly possibly that f is irreducible, or that f does factor, but less than it does modulo p.

Assuming always that p does not divide lc(f), all that can happen when we reduce f modulo p is that a factor of f that was irreducible over the integers now factors modulo p (and hence modulo p^k). This gives us the algorithm in Figure 5.6, first suggested in [Zas69]. To guarantee irreducibility of the factors found, the loop must try all subsets of T before trying T itself, but this still leaves a lot of choices for the loop: see 1 below.

The running time of this algorithm is, in the worst case, exponential in r since 2^{r-1} subsets of S have to be considered (2^{r-1} since considering T effectively also considers $S \setminus T$). Let n be the degree of f, and H a bound on the coefficients of f, so the Landau–Mignotte bound is at most $2^n(n+1)H$, and $k \le \log_p(2^{n+1}(n+1)H)$.

Many improvements to, or alternatives to, this basic algorithm have been suggested since. In essentially chronological order, the most significant ones are as follows.

1. [Col79] pointed out that there were two obvious ways to code the "**for** subsets T of S" loop: increasing cardinality of T and increasing degree of $\prod_{g \in T} g$. He showed that, subject to two very plausible conjectures, the

average number of products actually formed with the cardinality ordering was $O(n^2)$, thus the *average* running time would be polynomial in n.

2. [LLL82] had a completely different approach to algorithm 35. They asked, for each $d < n$, "given $f_1 \in S$, what is the polynomial g of degree d which divides f over the integers and is divisible by f_1 modulo p^k?". Unfortunately, answering this question needed a k far larger than that implied by the Landau–Mignotte bound, and the complexity, while polynomial in n, was $O(n^{12})$, at least while using classical arithmetic. This paper introduced the 'LLL' lattice reduction algorithm, which has many applications in computer algebra and far beyond.

3. [ABD85] showed that, by a combination of simple divisibility tests and "early abort" trial division (Proposition 57) it was possible to make dramatic reductions, at the time up to four orders of magnitude, in the constant implied in the statement "exponential in r".

4. [ASZ00] much improved this, and the authors were able to eliminate whole swathes of possible T at one go.

5. [vH02] reduces the problem of finding T to a 'knapsack' problem, which, as in method 2, is solved by LLL, but the lattices involved are much smaller — of dimension r rather than n. At the time of writing, this seems to be the best known method. His paper quoted a polynomial of degree $n = 180$, with $r = 36$ factors of degree 5 modulo $p = 19$, but factoring as two polynomials of degree 90 over the integers. This took 152 seconds to factor.

Open Problem 24 (Evaluate [vH02] against [ASZ00]) *How does the factorization algorithm of [vH02] perform on the large factorizations successfully solved by [ASZ00]? Clearly the algorithm of [vH02] is asymptotically faster, but where is the cut-off point? Note also that [vH02]'s example of* $x^{128} - x^{112} + x^{80} - x^{64} + x^{48} - x^{16} + 1$ *is in fact a disguised cyclotomic polynomial (as shown by the methods of [BD89]), being*

$$\frac{\left(x^{240}-1\right)\left(x^{16}-1\right)}{\left(x^{80}-1\right)\left(x^{48}-1\right)} = \prod_{\substack{1 \le k < 15 \\ \gcd(k,15)=1}} \left(x^{16} - e^{2\pi i k/15}\right).$$

5.7 Univariate Factoring Solved

We can put together the components we have seen to deduce an algorithm (Figure 5.11) for factoring square-free polynomials over $\mathbf{Z}[x]$. We have chosen to use the Cantor–Zassenhaus method (Section 5.3.2) for the modular factorisation, as most implementers today do, simply because it gives the information needed for the Musser test.

Figure 5.11: Overview of Factoring Algorithm

Algorithm 36 (Factor over Z)
Input: *A primitive square-free $f(x) \in \mathbf{Z}[x]$*
Output: *The factorization of f into irreducible polynomials over the integers.*

```
$p_1 :=$ find_prime($f$);
$F_1 :=$Corollary 18$(f, p_1)$
if $F_1$ is a singleton
  return $f$
$S :=$AllowableDegrees$(F_1)$
for $i := 2, \ldots, 7$      #7 from Observation 16
    $p_i :=$ find_prime($f$);
    $F_i :=$Corollary 18$(f, p_i)$
    $S := S \cap$AllowableDegrees$(F_i)$
    if $S = \emptyset$
        return $f$
$(p, F) :=$best$(\{(p_i, F_i)\})$;       #'best' in terms of fewest factors
Complete factorization modulo $p$ by Algorithm 29
$F :=$Algorithm 31$(f, F, p, \log_p(2LM(f)))$
return Algorithm 35$(p^{\log_p(2LM(f))}, f, F)$
```

find_prime(f) returns an odd prime p, generally as small as possible, such that f remains square-free modulo p. AllowableDegrees(F) returns the set of allowable *proper* factor degrees from a distinct degree (not needing a complete) modular factorization.
Algorithm 35 can be replaced by any of improvements 1–5 on pages 228–229.

Open Problem 25 (Better Choice of 'Best' Prime) *In algorithm 36, we picked the 'best' prime and the corresponding factorization. In principle, it is possible to do better, as in the following example, but the author has seen no systematic treatment of this.*

Example 29 Suppose f_p factors as polynomials of degree 2, 5 and 5, and f_q factors as polynomials of degree 3, 4 and 5: in terms of number of factors p and q are equivalent. The factorization modulo p implies that the allowable degrees of (proper) factors are 2, 5, 7 and 10. The factorization modulo q implies that the allowable degrees of (proper) factors are 3, 4, 5, 7, 8 and 9. Hence the only possible degrees of proper factors over the integers are 5 and 7. The factorization modulo p can yield this in two ways, but the factorization modulo q can only yield this by combining the factors of degrees 3 and 4. Hence we should do this, and lift this factorization of f_q, rather than the complete factorization.

Open Problem 26 (Low-degree Factorization) *Although no major system to the author's knowledge implements it, it would be possible to use the tools we have described to implement an efficient procedure to find all factors of limited total degree d. This would be efficient for three reasons:*

- *the Cantor–Zassenhaus algorithm (section 5.3.2) need only run up to maximum degree d;*
- *it should be possible to use smaller bounds on the lifting*
- *The potentially-exponential recombination process need only run up to total degree d. In particular, if $d = 1$, no recombination is needed.*

At the moment, the user who only wants low-degree factors has to either program such a search, or rely on the system's `factor`*, which may waste large amounts of time looking for high-degree factors, which the user does not want.*

In fact, [Gre15] described a library implemented in Mathemagix, which finds the factors of degree $\leq d$ of a polynomial f in time which is polynomial in d and the sparse bit size (Definition 27) of f. This is a much more powerful claim than that made above, and relies on key results [Len99a, Len99b] that state, essentially, that if a non-cyclotomic (Definition 117) polynomial g of degree $\leq d$ divides $f_1 + f_2$, and there is a sufficiently large gap between the degrees appearing in f_1 and f_2, then g must divide both f_1 and f_2 separately.

5.8 Multivariate Factoring

Just as in section 4.3, we can use "evaluating y at the value v", i.e. working modulo $(y - v)$, as an analogue of working modulo p. Just as in section 4.3, having worked modulo $(y - v)$, we can lift this to $(y - v)^2$ and so on. There is a fairly obvious generalisation of Algorithm 31, given in Algorithm 37, where we are given $f(x, y, x_1, \ldots, x_m) \in \mathbf{Z}[x_1, \ldots, x_m, y, x] = R[y, x]$, a value $v \in \mathbf{Z}$ and the factorization of $f(x, v, x_1, \ldots, x_m) \in R[x]$, and we wish to lift this to a

Figure 5.12: Algorithm 37

Algorithm 37 (Multivariate Hensel Lifting (Linear version))
Input: $f, g_1^{(1)}, \ldots, g_n^{(1)}, v, k$ *with* $f \in R[y][x]$ *primitive and* $\equiv \prod g_i^{(1)} \pmod{(y-v)}$
Output: $g_1^{(k)}, \ldots, g_n^{(k)}$ *with* $f \equiv \prod g_i^{(k)} \pmod{(y-v)^k}$

```
for $j := 1 \ldots n$
    $g_j^{(1)} := \frac{\mathrm{lc}(f)}{\mathrm{lc}(g_j^{(1)})} g_j^{(1)}$
$F := \mathrm{lc}(f)^{n-1} f$        #leading coefficients imposed
for $j := 1 \ldots n$
    $g_j^{(r)}, h_j^{(r)}$ := Algorithm 5$(g_j^{(1)}, \prod_{i \neq j} g_i^{(1)})$ in $R[x]$
for $i := 2 \ldots k$
    $\Delta := \frac{F - \prod_j g_j \pmod{(y-v)^i}}{(y-v)^{i-1}}$
    if $\Delta = 0$
      then break        #True factorization discovered
    for $j := 1 \ldots n$
        $g_j^{(c)} := \Delta * h_j^{(r)} \pmod{((y-v), g_j^{(1)})}$
        $g_j := g_j + (y-v)^{i-1} g_j^{(c)}$
for $j := 1 \ldots n$
    $g_j := \mathrm{pp}(g_j)$        #undo the imposed leading coefficients
if $f \neq \prod_j g_j$
  then return "bad evaluation"
return $(g_1, \ldots, g_n)$
```

factorization modulo $(y-v)^k$. In practice, $k = 1 + \deg_y(f)$, for then this lifted factorization should be the true factorization in $R[y, x]$.

This is linear lifting: there are also analogies of quadratic lifting (Algorithm 33) and hybrid lifting (section 5.5.4),but in practice these are rarely implemented, as in the multivariate setting most of the cost is in the last lift, as the polynomials have more and more terms as we lift.

5.8.1 A "Good Reduction" Complexity Result

Of course, it is possible that v is a "bad reduction" for f, in that $f(x, v, x_1, \ldots, x_n)$ factors more than $f(x, y, x_1, \ldots, x_n)$ does. Unlike the modulo p case, where we saw in Example 28 that this might always happen, in practice[11] these cases

[11]This can be proved for reductions *to* two or more variables, in the sense that the "bad reduction" values satisfy polynomials of degree as most $d^2 - 1$ (characteristic 0, [Rup86]) or $12d^6$ (characteristic p, [Kal95]), where d is the degree of the polynomial in the variables not being reduced.

are rare. Indeed, if $m \geq 1$, i.e. we are factoring trivariates or beyond, we can efficiently pick a good reduction, by the following result.

Theorem 44 ([vzG85, Theorem 4.5]) *Let f be a representation of a multivariate polynomial. Then there is a reduction from the problem of finding the factorisation shape of f (Definition 92) to that of finding the factorisation shape of bivariates, which is polynomial in the size and degree of f, for any of the sizes in (2.8).*

5.8.2 A Sparsity Result

We have already seen (section 2.3.7) that the complexity questions associated to sparse polynomials are challenging, and indeed

$$x^p - 1 = (x-1)(x^{p-1} + x^{p-2} + \cdots + x + 1) \tag{5.13}$$

shows that a sparse univariate polynomial can have completely dense factors. There are two possible generalisations of (5.13) to multivariates:

$$\begin{gathered}(x_1 \dots x_n)^p - 1 = \\ (x_1 \dots x_n - 1)\left((x_1 \dots x_n)^{p-1} + (x_1 \dots x_n)^{p-2} + \cdots + x_1 \dots x_n + 1\right);\end{gathered} \tag{5.14}$$

$$\prod_{i=1}^n (x_i^p - 1) = \left(\prod_{i=1}^n (x_i - 1)\right)\left(\prod_{i=1}^n (x_i^{p-1} + x_i^{p-2} + \cdots + x_i + 1)\right), \tag{5.15}$$

If we write these equations as $f = gh$, and if we write t_f for the number of monomials in the expanded[12] representation of f, then (5.14) has $t_f = 2$ and $t_h = p$, as in the univariate case, and (5.15) has $t_f = 2^n$ and $t_h = p^n$ — in particular if $n = p$, t_h is not polynomial in t_f, being $t_f^{\log_2 \log_2 t_f}$. Might there be still worse examples, say where t_f was polynomial in n and the degree, but t_h was exponential. It has recently been shown that not.

Theorem 45 ([DdO14, Theorem 1]) *If f, a polynomial in n variables, of degree at most d in each variable, factors as $f = gh$ then*

$$t_g = O\left(\max\left(t_f^{O(\log t_f \log\log t_f)}, d^{O(\log d)}\right)\right). \tag{5.16}$$

5.8.3 The Leading Coefficient Problem

It seems too good to be true, that the multivariate generalization of Hensel lifting needed no new concepts. While this is true as it stands, there is a major snag in practice with Algorithm 37, and that is the cost of imposing leading coefficients. In the univariate case, all this meant was that we had to lift very slightly further, but in the multivariate case we are lifting factors of F, which may well have many more terms than f. Solving this problem was (one of) the major contributions of the EEZ algorithm proposed in [Wan78], which can best

Figure 5.13: Algorithm 38

Algorithm 38 (Wang's EEZ Hensel Lifting)
Input: $f \in \mathbf{Z}[x_1, \ldots, x_n, x]$ *squarefree and primitive (no integer common factor)*
Output: *Factorization of* f *over* $\mathbf{Z}$

```
Write $f = \sum_{i=0}^{d} c_i(x_1, \ldots, x_n)x^i$
Recursively, factor $c_d(x_1, \ldots, x_n)$ as $\Omega F_1^{e_1} \ldots F_k^{e_k}$
Choose integers $a_1, \ldots, a_n$ such that (see [Wan78, p. 1218] for feasibility)

  1. $c_d(a_1, \ldots, a_n) \neq 0$
  2. $f^{(0)}(x) := f(a_1, \ldots, a_n, x)$ is square-free
  3. Each $F_i(a_1, \ldots, a_n)$ is divisible by a prime $p_i$, not dividing any
       $F_j(a_1, \ldots, a_n) : j < i$, $\Omega$ or $\mathrm{cont}(f^{(0)})$.

Factor $f^{(0)}(x) = c \prod_{i=1}^{m} f_i^{(0)}(x)$
[In practice, repeat the previous steps a few times, to ensure $m$ minimal, and
     hence reduce the possibility of bad reduction]
#Then $f = \prod_{i=1}^{m} f_i$ where $f_i(a_1, \ldots, a_n, x) = f_i^{(0)}(x)$.
Use the $p_i$ to distribute the $F_i$ among the leading coefficients of the $f_i$,
     to get $h_i$ but with the correct leading coefficient
For $k := 1 \ldots n$
     #$f(x_1, \ldots, x_{k-1}, a_k, \ldots, a_n, x) = \prod h_i(x_1, \ldots, x_{k-1}, a_k, \ldots, a_n, x)$
     #We know what the leading coefficients are
     Use a variant of Algorithm 37 to lift this to
          $f(x_1, \ldots, x_k, a_{k+1}, \ldots, a_n, x) = \prod h_i(x_1, \ldots, x_k, a_{k+1}, \ldots, a_n, x)$
```

be described as "discovered leading coefficients" rather than "imposed leading coefficients".

Example 30 ([Wan78, p. 1218]) *Consider*[13]

$$f(x,y,z) = (4z^2y^4 + 4z^3y^3 - 4z^4y^2 - 4z^5y)x^6 + \cdots .$$

The leading coefficient factors as $4yz^2(y+z)^2(y-z)$*. A suitable evaluation is* $(y=-14, z=3)$ *when*

$$F_1 = y \to \underbrace{-14}_{p_1=7}; F_2 = z \to \underbrace{3}_{p_2=3}; F_3 = y+z \to \underbrace{-11}_{p_3=11}; F_4 = y-z \to \underbrace{-17}_{p_4=17} .$$

$f(x,-14,3) = f_1^{(0)}f_2^{(0)}f_3^{(0)} = (187x^2-23)(44x^2+42x+1)(126x2-9x+28)$*, so* $f_1^{(0)}$ *has* p_3p_4*,* $f_2^{(0)}$ *has* $4p_3$ *and* $f_3^{(0)}$ *has* $p_1p_2^2$*. Hence the leading coefficients, are in fact* $(y+z)(y-z)$*,* $-4(y+z)$ *and* $-yz^2$*. The lift, with these leading coefficients, rather than the whole of* $4z^2y^4 + 4z^3y^3 - 4z^4y^2 - 4z^5y$*, imposed, works fine. The polynomial* f *has 88 terms when expanded, whereas if we imposed leading coefficients in the style of algorithm 37, we would be lifting against an* F *with 530 terms.*

5.9 Other Applications

5.9.1 Factoring Straight-LIne Programs

This is discussed in [Kal89b], with Monte Carlo algorithms. For $\mathbf{Q}[x_1,\ldots,x_n]$, his algorithm is polynomial in the degree, the size of the straight-line program determining the input, the size of the numerators and common denominator of the coefficients and the logarithm of the error probability. Th eoutputs are straight-line programs for the factors. These can then be converted into the standard sparse representation by means of algorithms such as Proposition 9. Forthermore, since these algorithms can be set to fail if asked to convert polynomials with more than T terms, we have a polynomial-time algorithm that returns the usual sparse format factors, or straight0lkine prorams if these would have more than T terms in sparse format.

5.9.2 p-adic Greatest Common Divisors

The entire p-adic/Hensel construction is applicable to the computation of greatest common divisors as well.

[12]The theory and implementations of this chapter have all been in terms of recursive representations, but the theory appears to be better expressed in terms of expanded representations.

[13]There is a typographical error: see `http://staff.bath.ac.uk/masjhd/JHD-CA/Wangp1220.html`.

5.9.2.1 Univariate Greatest Common Divisors

Since it is practically as cheap to use the largest possible single word prime as it is to use a small one, and the probability that p is bad seems to be proportional to $1/p$ (Observation 7), we take one such large p (possibly a second just to check that we don't have bad reduction), and compute $h_p = \gcd(f_p, g_p)$. This gives us a factorization $f_p = h_p \overline{f}_p$. Since we are not sure of the leading coefficient, we will impose $\gcd(\mathrm{lc}(f), \mathrm{lc}(g))$ as the leading coefficient of h_p and $\mathrm{lc}(f)$ as that of $\overline{f}_p$. We then lift this (Algorithm 31 or 33, or hybrid: section 5.5.4) to a factorization of $\gcd(\mathrm{lc}(f), \mathrm{lc}(g))f = h_p^{(k)} \overline{f}_p^{(k)}$ for a suitable k (as in section 4.2.1), interpret $h_p^{(k)}$ as a polynomial over the integers, make it primitive, and check that it divides f and g.

5.9.2.2 Multivariate Greatest Common Divisors

Assume we have two polynomials $f, g \in R[y][x]$ (where typically $R = \mathbf{Z}[_1, \ldots, x_n]$) and we wish to compute their greatest common divisor. We may as well (Theorem 6) assume that they are primitive with respect to x. We take a value v (normally in $\mathbf{Z}$ or the base ring of R) for y (possibly a second just to check that we don't have bad reduction), and compute $h_{y=v} = \gcd(f_{y=v}, g_{y=v})$. This gives us a factorization $f_{y=v} = h_{y=v} \overline{f}_{y=v}$. Since we are not sure of the leading coefficient, we will impose $\gcd(\mathrm{lc}_x(f), \mathrm{lc}_x(g))$ as the leading coefficient of $h_{y=v}$ and $\mathrm{lc}_x(f)$ as that of $\overline{f}_{y=v}$. We then lift this (generally Algorithm 37) to a factorization of $\gcd(\mathrm{lc}_x(f), \mathrm{lc}(g))f = h_{y=v}^{(k)} \overline{f}_{y=v}^{(k)}$ for a suitable k, interpret $h_{y=v}^{(k)}$ as a polynomial in $R[y][x]$, make it primitive, and check that it divides f and g.

5.9.3 p-adic Gröbner Bases

This section is largely taken from [Win88]. The importance of, and difficulties with, moving from computations over $\mathbf{Q}$ to finite domains are described in section 4.6. In particular we note the definition of bad reduction and the fact (Lemma 11) that there are only finitely many primes of bad reduction.

Suppose we are given F as input, and wish to compute its Gröbner base G. We can choose a prime p (it would certainly be a good idea to choose a p not dividing any leading coefficient of F), and compute a Gröbner base $G^{(1)}$ modulo $p = p^1$. In the matrix formulation of Gröbner base theory (section 3.3.5), it follows that there exist matrices X, Y and R such that

$$\begin{array}{rcl} X^{(1)}.\mathbf{F} & = & \mathbf{G}^{(1)} \\ Y^{(1)}.\mathbf{G}^{(1)} & = & \mathbf{F} \qquad \pmod p \\ R^{(1)}.\mathbf{G}^{(1)} & = & 0 \end{array} \tag{3.37$'$}$$

where $\mathbf{F}$ and $\mathbf{G}^{(1)}$ are the matrix versions of F and $G^{(1)}$. We might then be tempted to lift these equations, by analogy with (5.9), but there is a snag.

Since $G^{(1)}$ is a Gröbner base, it reduces F to zero, and hence the monomials needed to write $Y^{(1)}.\mathbf{G}^{(1)} = \mathbf{F}$ are at most those of F (or possibly lower ones that don't actually occur in a sparse representation of F). Similarly $R^{(1)}.\mathbf{G}^{(1)} = 0$ is the statement that all S-polynomials reduce to 0, and hence the highest degree that any x_i can need is its highest degree in $G^{(1)}$. However, there is no such tidy bound for $X^{(1)}$. Fortunately [Win88, Theorem 2] shows that we only need to lift the other two, whose degrees *are* bounded.

So we consider

$$\left.\begin{array}{rcl} Y^{(2)}.\mathbf{G}^{(2)} & = & \mathbf{F} \\ R^{(2)}.\mathbf{G}^{(2)} & = & 0 \end{array}\right. \pmod{p^2} \tag{5.17}$$

write $\mathbf{G}^{(2)} = \mathbf{G}^{(1)} + p\hat{\mathbf{G}}^{(2)}$ etc. as in (5.9), divide through by p and get

$$\left.\begin{array}{rcl} Y^{(1)}.\hat{\mathbf{G}}^{(2)} + \hat{Y}^{(2)}.\mathbf{G}^{(1)} & = & \frac{\mathbf{F}-Y^{(1)}.\mathbf{G}^{(1)}}{p} \\ \hat{R}^{(2)}.\mathbf{G}^{(1)} + R^{(1)}.\hat{\mathbf{G}}^{(2)} & = & 0 \end{array}\right. \pmod{p} \tag{5.18}$$

Just as (3.37) did not guarantee uniqueness, these equations don't necessarily have a unique solution. Indeed G is not unique, since we can multiply by any rational (without a p part), so we need to impose that G and every $G^{(k)}$ is monic. Then the $\hat{\mathbf{G}}^{(2)}$ part *is* unique [Win88, Theorem 4] as long as p is lucky.

This lifting process can be continued until we reach a sufficiently high power of p, at which point we can convert the coefficients from being modulo p^k to $\mathbf{Q}$ by the techniques of Algorithm 25. But what is a sufficiently high power? This is somewhat of a conundrum, unsolved in [Win88]. We can try taking a leaf out of section 4.2.5, and say that, if the solution stabilises, we will verify if it is correct.

Stabilisation This is harder to check than in section 4.2.5, since we are looking for rational not integer coefficients. Hence it is not correct to check that $\hat{\mathbf{G}}^{(k)} = 0$: generally it never will be. Instead, we have to check that the reconstitutions by Algorithm 25 are equal. Doing this efficiently, rather than reconstructing completely and discovering that the two are not equal, is an interesting programming task.

Verification We need to check two things: $F \overset{*}{\rightarrow}^{G} 0$ and that G is a Gröbner base, i.e. that every $S(g_i, g_j) \overset{*}{\rightarrow}^{G} 0$.

Unluckiness of the prime p can manifest itself in one of three ways.

1. Failure to lift. [Win88, Theorem 4] only guarantees a lift (unique for every $\hat{\mathbf{G}}^{(k)}$) if p is lucky. In some sense, this is the best case, since we at least *know* that p is unlucky.

2. Failure to stabilise — we just compute forever.

3. Failure to verify — if we terminate in a genuine Gröbner base G over the rationals, but F does not reduce to 0, then indeed p must have been unlucky. However, if the reconstructed G is not a Gröbner base over the rationals, we do not know whether p was unlucky, or the stabilisation was an accident and computing further will give the right result.

Open Problem 27 (p-adic Gröbner bases) *Convert the above, and [Win88], into a genuine algorithm for computing Gröbner bases p-adically. In particular, is it possible to use the "majority voting" idea of [IPS11] (footnote 15 on page 201) to improve the chances of having good reduction.*

5.9.4 p-adic determinants

[ABM99] describes a method for computing determinants which is "largely p-adic", in the sense that "most of" $\det(M)$ is computed via a p-adic solution to some linear systems, the l.c.m. of whose denominators is typically $\det(M)$, and generally a large factor D of $\det(M)$, and then computing $\det(M)/D$ by a Chinese Remainder approach. The expected complexity is

$$O(n^4 + n^3(\log n + s)^2),$$

where s is the length of the entries.

5.10 Conclusions

In this and the previous chapter, we have seen two different methods for computing in "large" domains via "small" ones: the modular method (Figure 4.1) and the p-adic/Hensel method (Figure 5.1). Both suffer from "bad reduction" in that the calculation in the small domain may not be a faithful representation of the calculation in the large domain. Although the p-adic/Hensel method only needs one small domain, we may compute more than one to minimise the probability of bad reduction. How do the two fare on various examples?

Linear Algebra Section 4.1 presented the modular approach for computing the determinant and other tasks of linear algebra. [Dix82] presents a p-adic approach for solving $Ax = b$ which he claims to be somewhat faster — $O(n^3 \log^2 n)$ rather than $O(n^4)$. However he admits that the modular approach is probably better for determinants themselves: a belief contradicted by [ABM99]. **TO BE COMPLETED**

gcd **computation TO BE COMPLETED**

Factorization Here p-adic methods are basically the only choice.

Gröbner bases Here the p-adic methods of Section 5.9.3 have so many unsolved issues that the modular methods of Section 4.6 seem the only realistic possibility.

In sum, both have their unique strengths, and can also compete in several areas.

Chapter 6

Algebraic Numbers and Functions

Definition 97 *An* algebraic number *is a root of a polynomial with integer coefficients, i.e. α such that $f(\alpha) = 0 : f \in \mathbf{Z}[t]$. The set of all algebraic numbers is denoted by $\mathbf{A}$. An $\alpha \in \mathbf{A} \setminus \mathbf{Q}$ will be referred to as a* non-trivial algebraic number*. If f is monic, we say that α is an* algebraic integer*.*

Allowing $f \in \mathbf{Q}[t]$ gives us no advantage, since we can clear denominators without affecting whether $f(\alpha) = 0$. Of course, "monic" only makes sense over $\mathbf{Z}[t]$.

Notation 30 *Let $\mathbf{A}$ stand for the set of* all *algebraic numbers, so that $\mathbf{Q} \subset \mathbf{A} \subset \mathbf{C}$.*

Definition 98 *An* algebraic function *is a root of a polynomial with polynomial coefficients, i.e. α such that $f(\alpha) = 0 : f \in \mathbf{Z}[x_1, \ldots, x_n][t]$. If $\alpha \notin \mathbf{Q}(x_1, \ldots, x_n)$, we will say that α is a non-trivial algebraic function.*

As above, allowing $f \in \mathbf{Q}(x_1, \ldots, x_n)[t]$ gives us no advantage, since we can clear denominators without affecting whether $f(\alpha) = 0$.

In either case, if f is irreducible, there is no *algebraic* way of distinguishing one root of f from another. For example, $\sqrt{2} \approx 1.4142$ and $-\sqrt{2} \approx -1.4142$ are both roots of $t^2 - 2 = 0$, but distinguishing them involves operations outside the language of fields, e.g. by saying that $\sqrt{2} \in [1, 2]$ but $-\sqrt{2} \in [-2, -1]$: see Section 6.5.

Notation 31 *Let $f(t)$ be the polynomial defining α, and write $f = \sum_{i=0}^{n} a_i t^i$ with $a_n \neq 0$. Until Section 6.4, we will assume that f is irreducible.*

Definition 99 *If f in Definition 97 or 98 is irreducible and primitive, we say that f is* the minimal polynomial *of α.* Strictly speaking, we have only defined f up to associates, but this is usually ignored in theory, though it can be tedious

Figure 6.1: Non-candidness of algebraics

```
> (sqrt(2)+1)*(sqrt(2)-1);
                                  1/2         1/2
                                (2    - 1) (2    + 1)
> expand(%);
                                         1
```

in practice: see the description of the `canonicalUnitNormal` property in [DT90] for a pragmatic solution.

It is perfectly possible to work with α as if it were another variable, but replacing α^n by $\left(\sum_{i=0}^{n-1} a_i\alpha^i\right)/a_n$. A simple example of this is shown in the Maple session in Figure 6.1, which also demonstrates the lack of candidness (i.e. the expression appears to contain $\sqrt{2}$, but doesn't really) if we don't expand.

Observation 17 *Introduction of denominators of a_n can be very tedious in practice, so several systems will work internally with monic f. For example, Maple itself does not, but its high-performance algorithms [vHM02, vHM04] so.*

However, mere use of a polynomial-style `expand` command, which gave us canonical forms in the case of polynomials, is insufficient to give us even normal forms when we allow algebraic numbers, as seen in Figure 6.2. Here, `simplify` has

Figure 6.2: Algebraic numbers in the denominator

```
> (sqrt(2)+1)-1/(sqrt(2)-1);
                                  1/2
                                 2    + 1 -     1
                                             --------
                                               1/2
                                              2    - 1
> expand(%);
                                  1/2
                                 2    + 1 -     1
                                             --------
                                               1/2
                                              2    - 1
> simplify(%);
                                        0
```

to "clear denominators". This process, which may have seemed quite obscure when it was introduced at school, is, from our present point of view, quite easy to understand. Consider an expression

$$E := \frac{p(\alpha)}{q(\alpha)} \tag{6.1}$$

where p and q are polynomials in α, of degree less than n. If we apply the extended Euclidean algorithm (Algorithm 5) to $q(t)$ and $f(t)$, we get $h(t) = \gcd(f, q)$ and polynomials c, d such that $cf + dq = h$. Assuming $h = 1$ (as is bound to be the case if f is irreducible), we see that

$$\begin{aligned} E &= \frac{p(\alpha)}{q(\alpha)} \\ &= \frac{d(\alpha)p(\alpha)}{d(\alpha)q(\alpha)} \\ &= \frac{d(\alpha)p(\alpha)}{c(\alpha)f(\alpha) + d(\alpha)q(\alpha)} \end{aligned}$$

(since $f(\alpha) = 0$)

$$= d(\alpha)p(\alpha),$$

and this is purely a polynomial in α.

In the case of Figure 6.2, $f(t) = t^2 - 1$ and $p(t) = 1$, $q(t) = t - 1$. The gcd is indeed 1, with $f = -1$, $d = t + 1$. Hence $\frac{1}{\sqrt{2}-1} = \frac{\sqrt{2}+1}{(\sqrt{2}+1)(\sqrt{2}-1)} = \sqrt{2} + 1$. Such is the simplicity of this approach that it is usual to insist that f is irreducible, however in section 6.4 we will explain an alternative approach.

OOnce we have agreed to clear denominators, we are representing algebraic numbers or functions as elements of $\mathbf{Q}(x_1, \ldots, x_n)[\alpha - 1, \ldots, \alpha_m]$ where the degrees in α_i are less than the degree of the corresponding minimal polynomial. In practice it may be more convenient to use a *common denominator* approach, i.e. store an algebraic number as n/d where $n \in \mathbf{Z}[x_1, \ldots, x_n][\alpha - 1, \ldots, \alpha_m]$ and $d \in \mathbf{Z}[x_1, \ldots, x_n]$ with no common factor between d and all the coefficients in n.

6.1 Representations of Finite Fields

If we are working in a finite field F, and the user explicitly calls for an algebraic extension, e.g. $\alpha := \mathrm{RootOf}(x^2 + x + 1)$, then we are almost certainly going to represent α as above, noting that there is actually no need for a denominator as we are working over a field.

But often, as in Figure 6.3, the user will just call for a finite field $K := \mathbf{F}_{p^n}$ without specifying the generator, since all finite fields of the same size p^n are isomorphic: in other words there is only one abstract field.

6.1.1 Additive Representation

The obvious thing to do is to proceed as Maple has done, and pick an irreducible polynomial f of the right degree (assuring ourselves that it is irreducible modulo

Figure 6.3: An unspecified field in Maple

```
> K:=GF(2,4);
                                         4
                        K := Z[2] [T] / <T  + T + 1>
> g:=K:-random();
                                                3
                                g := 1 + T + T
> K:-`*`(g,g);
                                          3
                                     1 + T
```

p, and not just over the integers, of course).

Maple picks the polynomial at random, but it may be more efficient to choose a polynomial that is sparse, and also one where the terms other than x^n are of as low a degree as possible, to reduce the impact of carries. The last time the author implemented finite fields, he essentially[1] started $x^n + 1$, $x^n + 2$, …, $x^n + (p-1)$, $x^n + x + 1$, $x^n + x + 2$, … until he found an irreducible polynomial.

In this representation, addition takes n coefficient operations, and multiplication, at least if done naïvely, $O(n^2)$. Karatsuba-style multiplication (Excursus B.3) may well be worthwhile if n is at all large[2]

If n is not prime, we might build the field in stages, e.g. if $n = n_1 n_2$ (with the n_i relatively prime) we can write $K := \mathbf{F}_p(\alpha, \beta)$ where α and β have minimal polynomials of degree n_1, n_2 respectively. This is generally only done if the user asks for it, though there is no reason why a system couldn't do it automatically.

6.1.2 Multiplicative representation

An alternative is to note that the multiplicative groups of a finite field (i.e. all elements except 0) is always cyclic, and to pick, as well as f, a generator g of that group, i.e. such that, while $g^{p^n-1} = 1$, no previous power of g is 1. See Figure 6.4, where we have verified that g^3 and g^5 are not 1 (which is in fact sufficient)

0 is then a special case, but all other elements β are just stored as the integer m such that $\beta = g^m$. Multiplication is then just one integer addition $(g^m) \cdot (g^{m'}) = g^{m+m'}$, with reduction if $m + m' \geq p_n - 1$.

The downside of this representation is that addition is more complicated, and in general requires conversion to additive form (expensive) and back (very expensive, known as the "discrete logarithm problem"). For large fields, the cost

[1]There were some optimisations to skip patently reducible polynomials

[2]A general consensus for integer multiplication is $n \geq 16$, so one would expect the same to hold here, but the author knows of no study here. Since, once the field is built, one is probably going to do many operations with the same n, it might be worth the field-building operation doing some work to determine the best multiplication, since if n is not an exact power of 2, the Karatsuba method needs padding or other adaptations.

Figure 6.4: Primitive elements in Maple

```
> g:=K:-PrimitiveElement();
                                                        3
                                       g := 1 + T + T
> K:-`^`(g,3);
                                           2    3
                                          T  + T
> K:-`^`(g,5);
                                                2
                                          T + T
```

of addition, if it is required, therefore rules out the multiplicative representation. For smaller fields, though, there is a useful trick.

We first note that $g^m + g^{m'} = g^m \left(1 + g^{m'-m}\right)$, so the general problem of addition reduces to the problem of adding 1.

Definition 100 *Define $\mathcal{Z}_g$, generally known as the* Zech logarithm[3] *as the function such that $g^{\mathcal{Z}_g(m)} = 1 + g^m$, with $\mathcal{Z}_g(m)$ being a special symbol $-\infty$ if $g^m = -1$.*

Then $g^m + g^{m'} = g^{m+\mathcal{Z}_g(m'-m)}$: a subtraction, a table look up, and an addition (with range reduction as in the case of multiplication).

6.2 Representations of Algebraic Numbers

The calculations in Figure 6.2 would have been valid whether $\sqrt{2} = 1.4142\ldots$ or $\sqrt{2} = -1.4142\ldots$, but often we care about *which* specific root of a polynomial we mean. For quadratics with real roots there is no problem, we (generally implicitly) use $\sqrt{n}$ to mean α such that $\alpha^2 - n$ and $\alpha \geq 0$. Once we move beyond quadratics, life is more complicated: as pointed out at (3.10), the three real roots of $x^3 - x$ can only be computed via complex numbers. A slight perturbation makes thinsg worse: the roots of $x^3 - x - \frac{1}{1000}$, numerically

$$1.000499625, -0.001000001000, -.9994996245 \tag{6.2}$$

[3]Named after Julius Zech, author of [Zec49].

Figure 6.5: An evaluation of Maple's RootOf construct

```
Digits := 100;
                                        100
evalf(%);
1.000499625499181184573370153751604066644419260719616471226765912\
  7631800401921686516946533704564198509
```

are given algebraically by

$$\left\{\begin{array}{l} \frac{1}{60}\sqrt[3]{108+12\,i\sqrt{11999919}}+20\,\frac{1}{\sqrt[3]{108+12\,i\sqrt{11999919}}}, \\ -\frac{1}{120}\sqrt[3]{108+12\,i\sqrt{11999919}}-10\,\frac{1}{\sqrt[3]{108+12\,i\sqrt{11999919}}}- \\ \frac{i\sqrt{3}}{20}\left(\frac{1}{6}\sqrt[3]{108+12\,i\sqrt{11999919}}-200\,\frac{1}{\sqrt[3]{108+12\,i\sqrt{11999919}}}\right), \\ -\frac{1}{120}\sqrt[3]{108+12\,i\sqrt{11999919}}-10\,\frac{1}{\sqrt[3]{108+12\,i\sqrt{11999919}}}+ \\ \frac{i\sqrt{3}}{20}\left(\frac{1}{6}\sqrt[3]{108+12\,i\sqrt{11999919}}-200\,\frac{1}{\sqrt[3]{108+12\,i\sqrt{11999919}}}\right) \end{array}\right\}, \tag{6.3}$$

and these cannot be expressed in terms of real radicals, i.e. the i is intrinsic. We can avoid this by means of a representation in terms of RootOf constructs:

$$\mathrm{RootOf}(f(z),\mathrm{index{=}1}),\ldots,\mathrm{RootOf}(f(z),\mathrm{index{=}3}), \tag{6.4}$$

or by saying "near" which value these roots are, as well as the polynomial:

$$\mathrm{RootOf}(f(z),1.0),\ldots,\mathrm{RootOf}(f(z),-1.0). \tag{6.5}$$

We should be careful of the difference between (6.2) and (6.5): the first specifies three floating-point numbers, but the second specifies three precise algebraic numbers, which can in theory (and in practice) be evaluated to arbitrary precision, as shown in Figure 6.5.

6.3 Factorisation with Algebraic Numbers

If we insist, as we do in this section, that the f in Definition 97 or 98 be irreducible, the obvious question is "how do we ensure this?" If $f \in \mathbf{Z}[x_1,\ldots,x_n,t]$, then the theory of Chapter 5 will factor f. However, suppose we have defined α by f, and then wish to define β by $g \in \mathbf{Z}[x_1,\ldots,x_n,\alpha,t]$. How do we ensure g is irreducible? Indeed, even if g does not involve α as such, how do we ensure that g is irreducible in $\mathbf{Z}[x_1,\ldots,x_n,\alpha,t]$, even if it is irreducible in $\mathbf{Z}[x_1,\ldots,x_n,t]$?

This is not just an abstract conundrum. Consider $f = t^2-2$, $g = t^2-8$. Both are irreducible in $\mathbf{Z}[t]$. But if α is defined to be a root of f, then g factors as $(t-2\alpha)(t+2\alpha)$, corresponding to the fact that $\sqrt{8}=2\sqrt{2}$.

For the rest of this section we will consider the case of factoring over algebraic *number* fields, i.e. polynomials in $\mathbf{Q}[\alpha_1,\ldots,\alpha_m,x_1,\ldots,x_n]$, where each α_i is defined by a minimal polynomial in $\mathbf{Z}[\alpha_1,\ldots,\alpha_{i-1}][t]$. The more general case of algebraic function fields is discussed in [DT81].

The first remark is that the methods of Section 5.3 extend to factoring over algebraic extensions of the integers modulo p.

[Wan76, Len87]

TO BE COMPLETED

Open Problem 28 (Algebraic Numbers Reviewed) *Reconsider the standard approach to factoring polynomials with algebraic number/function coefficients, as described above, in the light of recent progress in factoring polynomials with integer coefficients, notably [vH02].*

6.4 The D5 approach to algebraic numbers

The techniques of the previous section, while they do guarantee irreducibility of the defining polynomials, are extremely expensive, and often overkill.

In fact, we have already seen[4] a technique that does *not* require factoring, and that is the Gianni–Kalkbrener algorithm (Algorithms 10, 11). If we consider again the example at equation (3.48), we see that x satisfies $6-3x^2-2x^3+x^5=0$. Over $\mathbf{Z}$, this already factors as $(x^2-2)(x^3-3)$, and x^3-3 is irreducible[5] over $\mathbf{Z}[\sqrt{2}]$, so the five possible values of x are α with $\alpha^2-2=0$, $-\alpha$, β with $\beta^3-3=0$, γ with $\gamma^2+\beta\gamma+\beta^2=0$ and $-\gamma-\beta$. For each of these there is corresponding value of y, but it is hard to believe that this is simpler than

$$\left\langle x^2-2, y^2-x\right\rangle \cup \left\langle x^3-3, y-x\right\rangle. \tag{3.49}$$

[DDDD85]

For g.c.d. computations, see [KM17], who regard their setting as being a triangular set.

TO BE COMPLETEDTO BE COMPLETED[vHM16]

6.5 Distinguishing roots

In Section 3.5.5 we saw that there are essentially two ways of distinguishing the *real* roots of a polynomial f.

interval As well as asserting that $f(\alpha)=0$, quote an interval (a,b) such that it is known that f has precisely one real root in (a,b). Thus $\sqrt{2}\approx 1.4142$ would be described as $(x^2-2,(1,2))$ for example. Of course, it could equally well be $(x^2-2,(0,2))$, or many other choices. But $(x^2-2,(-2,2))$

[4]This is not the historical development of the subject, for which see [DDDD85], but is more pedagogic from our current point of view.

[5]Given that x^3-3 is irreducible over $\mathbf{Z}$, its irreducibility over $\mathbf{Z}[\sqrt{2}]$ follows from the fact that the exponent, 2 and 3, are relatively prime. But this is a very *ad hoc* argument.

would be invalid (more than one root) and $(x^2 - 2, (2, 3))$ would also be invalid (no roots).

Thom As well as asserting that $f(\alpha) = 0$, quote the signs (positive or negative) of $f'(\alpha)$, $f''(\alpha)$ etc. Here $\sqrt{2} \approx 1.4142$ would be described as $(x^2 - 2, f' > 0, f'' > 0)$. Note that $(x^2 - 2, f' > 0, f'' < 0)$ does not describe a root.

If we want to describe the *complex* roots, the only known ways are variants of the interval method.

box As well as asserting that $f(\alpha) = 0$, quote intervals (a, b) and (c, d) such that f has ony one root in the box whose corners are $a + ci$, $b + ci$, $b + di$ and $a + di$.

circle As well as asserting that $f(\alpha) = 0$, quote a point $a + ci$ and a radius r such that α is the only root of f with $|\alpha - (a + ci)| < r$.

The two representations are to some extend exchangeable, as every box is contained in a circle and *vice versa*.

Chapter 7

Calculus

Throughout this chapter we shall assume that we are in characteristic zero, and therefore all rings contains $\mathbf{Z}$, and all fields contain $\mathbf{Q}$. The emphasis in this chapter will be on *algorithms* for integration. Historically, the earliest attempts at integration in computer algebra [Sla61] were based on rules, and attempts to "do it like a human would". These were rapidly replaced by algorithmic methods, based on the systematisation, in [Ris69b], of work going back to Liouville. Recently, attempts to get 'neat' answers have revived interest in rule-based approaches, which can be surprisingly powerful on *known* integrals — see [JR10] for one recent approach. In general, though, only the algorithmic approach is capable of *proving* that an expression in unintegrable.

7.1 Introduction

We defined (Definition 37) the *formal derivative* of a polynomial purely algebraically, and observed (Proposition 2.3.6) that it satisfied the sum and product laws. We can actually make the whole theory of differentiation algebraic as follows.

Definition 101 *A* differential ring *is a ring (Definition 8) equipped with an additional unary operation, referred to as differentiation and normally written with a postfix $'$, which satisfies two additional laws:*

1. $(f+g)' = f' + g'$;
2. $(fg)' = fg' + f'g$.

A differential ring which is also a field (Definition 15) is referred to as a differential field*.*

Definition 37 and Proposition 2.3.6 can then be restated as the following result.

Proposition 72 *If R is any ring, we can make $R[x]$ into a differential ring by defining $r' = 0 \quad \forall r \in R$ and $x' = 1$.*

Definition 102 *In any differential ring, an element α with $\alpha' = 0$ is referred to as a* constant. *For any ring R, we write R_{const} for the constants of R, i.e.*

$$R_{\text{const}} = \{r \in R \mid r' = 0\}.$$

We will use $\mathcal{C}$ to stand for any appropriate field of constants.

We will revisit this definition in section 8.4.

Proposition 73 *In any differential field*

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}. \tag{7.1}$$

The proof is by differentiating $f = g\left(\frac{f}{g}\right)$ to get

$$f' = g'\left(\frac{f}{g}\right) + g\left(\frac{f}{g}\right)'$$

and solving for $\left(\frac{f}{g}\right)'$. (7.1) is generally referred to as the *quotient rule* and can be given a fairly tedious analytic proof in terms of ϵ/δ, but from our present point of view it is an algebraic corollary of the product rule. It therefore follows that $K(x)$ can be made into a differential field.

Notation 32 (Fundamental Theorem of Calculus) *(Indefinite) integration is the inverse of differentiation, i.e.*

$$F = \int f \Leftrightarrow F' = f. \tag{7.2}$$

The reader may be surprised to see a "Fundamental Theorem" reduced to the status of a piece of notation, but from the present point of view, that is what it is. We shall return to this point in section 8.3. The reader may also wonder "where did $\mathrm{d}x$ go?", but x is, from this point of view, merely that object such that $x' = 1$, i.e. $x = \int 1$.

We should also note that we have no choice over the derivative of algebraic expressions[1].

Proposition 74 *Let K be a differential field, and θ be algebraic over K with $p(\theta) = 0$ for some polynomial $p = \sum_{i=0}^n a_i z^i \in K[z]$. Then $K(\theta)$ can be made into a differential field in only one way: by defining*

$$\theta' = -\frac{\sum_{i=0}^n a_i' \theta^i}{\sum_{i=0}^n i a_i \theta^{i-1}}. \tag{7.3}$$

In particular, if the coefficients of p are all constants, so is θ.

The proof is by formal differentiation of $p(\theta) = 0$.

[1] It would be more normal to write "algebraic functions" instead of "algebraic expressions", but for reasons described in section 8.1 we reserve 'function' for specific mappings (e.g. $\mathbf{C} \to \mathbf{C}$), and the proposition is a property of differentiation applied to formulae.

7.2 Integration of Rational Expressions

The integration of polynomials is trivial:

$$\int \sum_{i=0}^{n} a_i x^i = \sum_{i=0}^{n} \frac{1}{i+1} a_i x^{i+1}. \tag{7.4}$$

Since any rational expression $f(x) \in K(x)$ can be written as

$$f = p + \frac{q}{r} \text{ with } \left\{ \begin{array}{l} p, q, r \in K[x] \\ \deg(q) < \deg(r) \end{array} \right. , \tag{7.5}$$

and p is always integrable by (7.4), we have proved the following (trivial) result: we will see later that its generalisations, Lemmas 12 and 13, are not quite so trivial.

Proposition 75 (Decomposition Lemma (rational expressions)) *In the notation of (7.5), f is integrable if, and only if, q/r is.*

q/r with $\deg(q) < \deg(r)$ is generally termed a *proper rational function*, but, since we are concerned with the algebraic form of expressions in this chapter, we will say "*proper rational expression.*".

7.2.1 Integration of Proper Rational Expressions

In fact, the integration of proper rational expressions is conceptually trivial (we may as well assume r is monic, absorbing any constant factor in q):

1. perform a square-free decomposition (Definition 38) of $r = \prod_{i=1}^{n} r_i^i$;
2. factorize each r_i completely, as $r_i(x) = \prod_{j=1}^{n_i} (x - \alpha_{i,j})$;
3. perform a partial fraction decomposition (Section 2.3.4) of q/r as

$$\frac{q}{r} = \frac{q}{\prod_{i=1}^{n} r_i^i} = \sum_{i=1}^{n} \frac{q_i}{r_i^i} = \sum_{i=1}^{n} \sum_{j=i}^{n_i} \sum_{k=1}^{i} \frac{\beta_{i,j,k}}{(x-\alpha_{i,j})^k}; \tag{7.6}$$

4. integrate this term-by-term, obtaining

$$\int \frac{q}{r} = \sum_{i=1}^{n} \sum_{j=i}^{n_i} \sum_{k=2}^{i} \frac{-\beta_{i,j,k}}{(k-1)(x-\alpha_{i,j})^{k-1}} + \sum_{i=1}^{n} \sum_{j=i}^{n_i} \beta_{i,j,1} \log(x - \alpha_{i,j}). \tag{7.7}$$

From a practical point of view, this approach has several snags:

1. we have to factor r, and even the best algorithms from the previous chapter can be expensive;
2. we have to factor each r_i *into linear factors*, which might necessitate the introduction of algebraic numbers to represent the roots of polynomials;

3. These steps might result in a complicated expression of what is otherwise a simple answer.

To illustrate these points, consider the following examples.

$$\begin{aligned}\int \frac{5x^4+60x^3+255x^2+450x+274}{x^5+15x^4+85x^3+225x^2+274x+120}\mathrm{d}x \\ = \log(x^5+15x^4+85x^3+225x^2+274x+120) \\ = \log(x+1)+\log(x+2)+\log(x+3)+\log(x+4)+\log(x+5)\end{aligned} \tag{7.8}$$

is pretty straightforward, but adding 1 to the numerator gives

$$\begin{aligned}\int \frac{5x^4+60x^3+255x^2+450x+275}{x^5+15x^4+85x^3+225x^2+274x+120}\mathrm{d}x \\ = \tfrac{5}{24}\log(x^{24}+72x^{23}+\cdots+102643200000x+9331200000) \\ = \tfrac{25}{24}\log(x+1)+\tfrac{5}{6}\log(x+2)+\tfrac{5}{4}\log(x+3)+\tfrac{5}{6}\log(x+4)+\tfrac{25}{24}\log(x+5)\end{aligned} \tag{7.9}$$

Adding 1 to the denominator is pretty straightforward,

$$\begin{aligned}\int \frac{5x^4+60x^3+255x^2+450x+274}{x^5+15x^4+85x^3+225x^2+274x+121}\mathrm{d}x \\ = \log(x^5+15x^4+85x^3+225x^2+274x+121),\end{aligned} \tag{7.10}$$

but adding 1 to both gives

$$\begin{aligned}\int \frac{5x^4+60x^3+255x^2+450x+275}{x^5+15x^4+85x^3+225x^2+274x+121}\mathrm{d}x \\ = 5\sum_{\alpha}\alpha\ln\Big(x+\frac{2632025698}{289}\alpha^4-\frac{2086891452}{289}\alpha^3+ \\ \frac{608708804}{289}\alpha^2-\frac{4556915}{17}\alpha+\frac{3632420}{289}\Big),\end{aligned} \tag{7.11}$$

where

$$\alpha = \mathrm{RootOf}\left(38569\,z^5-38569\,z^4+15251\,z^3-2981\,z^2+288\,z-11\right). \tag{7.12}$$

Hence the challenge is to produce an algorithm that achieves (7.8) and (7.10) simply, preferably gives us the second form of the answer in (7.9), *but* is still capable of solving (7.11). We might also wonder where (7.11) came from: the answer is at (7.26).

7.2.2 Hermite's Algorithm

The key to the method (usually attributed to Hermite [Her72], though informally known earlier) is to rewrite equation (7.7) as

$$\int \frac{q}{r} = \frac{s_1}{t_1} + \int \frac{s_2}{t_2}, \tag{7.13}$$

where the integral on the right-hand resolves itself as *purely* a sum of logarithms, i.e. is the $\sum_{i=1}^{n}\sum_{j=i}^{n_i}\beta_{i,j,1}\log(x-\alpha_{i,j})$ term. Then a standard argument from

Galois theory shows that s_1, t_1, s_2 and t_2 do not involve any of the α_i, i.e. that the decomposition (7.13) can be written without introducing any algebraic numbers. If we could actually obtain this decomposition without introducing these algebraic numbers, we would have gone a long way to solving objection 2 above.

We can perform a square-free decomposition (Definition 38) of r as $\prod r_i^i$, and then a partial fraction decomposition to write

$$\frac{q}{r} = \sum \frac{q_i}{r_i^i} \tag{7.14}$$

and, since each term is a rational expression and therefore integrable, it suffices to integrate (7.14) term-by-term.

Now r_i and r_i' are relatively prime, so, by Bezout's Identity (2.13), there are polynomials a and b satisfying $ar_i + br_i' = 1$. Therefore

$$\begin{aligned}
\int \frac{q_i}{r_i^i} &= \int \frac{q_i(ar_i + br_i')}{r_i^i} && (7.15)\\
&= \int \frac{q_i a}{r_i^{i-1}} + \int \frac{q_i b r_i'}{r_i^i} && (7.16)\\
&= \int \frac{q_i a}{r_i^{i-1}} + \int \frac{(q_i b/(i-1))'}{r_i^{i-1}} - \left(\frac{q_i b/(i-1)}{r_i^{i-1}}\right)' && (7.17)\\
&= -\left(\frac{q_i b/(i-1)}{r_i^{i-1}}\right) + \int \frac{q_i a + (q_i b/(i-1))'}{r_i^{i-1}}, && (7.18)
\end{aligned}$$

and we have reduced the exponent of r_i by one. When programming this method one may need to take care of the fact that, while $\frac{q_i}{r_i^i}$ is a proper rational expression, $\frac{q_i b}{r_i^{i-1}}$ may not be, but the excess is precisely compensated for by the other term in (7.18).

Hence, at the cost of a square-free decomposition and a partial fraction decomposition, but *not* a factorization, we have found the rational part of the integral, i.e. performed the decomposition of (7.13). In fact, we have done somewhat better, since the $\int \frac{s_2}{t_2}$ term will have been split into summands corresponding to the different r_i.

7.2.3 The Ostrogradski–Horowitz Algorithm

Although quite simple, Hermite's method still needs square-free decomposition and partial fractions. Horowitz [Hor69, Hor71] therefore proposed to computer algebraists the following method, which was in fact already known [Ost45], but largely forgotten[2] in the west. It follows from (7.18) that, in the notation of (7.14) $t_1 = \prod r_i^{i-1}$. Furthermore every factor of t_2 arises from the r_i, and is not

[2]The author had found no references to it, but apparently it had been taught under that name.

repeated. Hence we can choose

$$t_1 = \gcd(r, r') \text{ and } t_2 = r/t_1. \tag{7.19}$$

Having done this, we can solve for the coefficients in s_1 and s_2, and the resulting equations are linear in the unknown coefficients. More precisely, the equations become

$$q = s_1' \frac{r}{t_1} - s_1 \frac{t_1' t_2}{t_1} + s_2 t_1, \tag{7.20}$$

where the polynomial divisions are exact, and the linearity is now obvious. The programmer should note that s_1/t_1 is guaranteed to be in lowest terms, but s_2/t_2 is not (and indeed will be 0 if there is no logarithmic term).

7.2.4 The Trager–Rothstein Algorithm

Whether we use the method of section 7.2.2 or 7.2.3, we have to integrate the logarithmic part. (7.8)–(7.11) shows that this may, but need not, require algebraic numbers. How do we tell? The answer is provided by the following observation[3] [Rot76, Tra76]: if we write the integral of the logarithmic part as $\sum c_i \log v_i$, we can determine the equation satisfied by the c_i, i.e. the analogue of (7.12), by purely rational computations.

So write

$$\int \frac{s_2}{t_2} = \sum c_i \log v_i, \tag{7.21}$$

where we can assume[4]:

1. $\frac{s_2}{t_2}$ is in lowest terms;
2. the v_i are polynomials (using $\log \frac{f}{g} = \log f - \log g$);
3. the v_i are square-free (using $\log \prod f_i^i = \sum i \log f_i$);
4. the v_i are relatively prime (using $c \log pq + d \log pr = (c+d)\log p + c \log q + d \log r$);
5. the c_i are all different (using $c \log p + c \log q = c \log pq$);
6. the c_i generate the smallest possible extension of the original field of coefficients.

(7.21) can be rewritten as

$$\frac{s_2}{t_2} = \sum c_i \frac{v_i'}{v_i}. \tag{7.22}$$

[3]As happens surprisingly often in computer algebra, this was a case of simultaneous discovery.

[4]We are using "standard" properties of the log operator here without explicit justification: they are justified in Section 7.3, and revisited in Section 8.6.

Hence $t_2 = \prod v_i$ and, writing $u_j = \prod_{i \neq j} v_i$, we can write (7.22) as

$$s_2 = \sum c_i v_i' u_i. \tag{7.23}$$

Furthermore, since $t_2 = \prod v_i$, $t_2' = \sum v_i' u_i$. Hence

$$\begin{aligned}
v_k &= \gcd(0, v_k) \\
&= \gcd\left(s_2 - \sum c_i v_i' u_i, v_k\right) \\
&= \gcd\left(s_2 - c_k v_k' u_k, v_k\right) \\
&\qquad \text{since all the other } u_i \text{ are divisible by } v_k \\
&= \gcd\left(s_2 - c_k \sum v_i' u_i, v_k\right) \\
&\qquad \text{for the same reason} \\
&= \gcd\left(s_2 - c_k t_2', v_k\right).
\end{aligned}$$

But if $l \neq k$,

$$\begin{aligned}
\gcd\left(s_2 - c_k t_2', v_l\right) &= \gcd\left(\sum c_i v_i' u_i - c_k \sum v_i' u_i, v_l\right) \\
&= \gcd\left(c_l v_l' u_l - c_k v_l' u_l, v_l\right) \\
&\qquad \text{since all the other } u_i \text{ are divisible by } v_l \\
&= 1.
\end{aligned}$$

Since $t_2 = \prod v_l$, we can put these together to deduce that

$$v_k = \gcd(s_2 - c_k t_2', t_2). \tag{7.24}$$

Given c_k, this will tell us v_k. But we can deduce more from this: the c_k are precisely those numbers λ such that $\gcd(s_2 - \lambda t_2', t_2)$ is non-trivial. Hence we can appeal to Proposition 78 (page 305), and say that λ must be such that

$$P(\lambda) := \mathrm{Res}_x(s_2 - \lambda t_2', t_2) = 0. \tag{7.25}$$

If t_2 has degree n, $P(\lambda)$ is the determinant of a $2n - 1$ square matrix, n of whose rows depend linearly on λ, and thus is a polynomial of degree n in λ (strictly speaking, this argument only shows that it has degree at most n, but the coefficient of λ^n is $\mathrm{Res}(t_2', t_2) \neq 0$).

Algorithm 39 (Trager–Rothstein)
Input: $s_2, t_2 \in K[x]$ *relatively prime with* $\deg_x(s_2) < \deg_x(t_2)$, t_2 *squarefree*
Output: *A candid*[5] *expression for* $\int s_2/t_2 \mathrm{d}x$

[5]In the sense of Definition 6: every algebraic extension occurring in the expression is necessary.

```
$P(\lambda) := \mathrm{Res}(s_2 - \lambda t_2', t_2, x)$
Write $P(\lambda) = \prod_i Q_i(\lambda)^i$ (square-free decomposition)
$R := 0$
**for** $i$ in $1\ldots$ such that $Q_i \neq 1$
    $v_i := \gcd(s_2 - \lambda t_2', t_2)$ where $Q_i(\lambda) = 0$      #deg$(v_i) = i$
    $R := R + \sum_{\lambda \text{ a root of } Q_i} \lambda \log v_i(x)$
**return** $R$
```

An alternative formulation is given in [LR90], with an important clarification in [Mul97]. In practice roots of a Q_i which are rational, or possibly even quadratic over the rationals, are normally made explicit rather than represented as `RootOf` constructs. This accounts for answers such as the following:

$$\int \tfrac{3\,x^3-2\,x^2}{(x^2-2)(x^3-x-3)}\mathrm{d}x =$$
$$-\ln\left(x^2-2\right) + \sum_{r=\mathrm{RootOf}(z^3-z-3)} \frac{r\ (3+2\,r)\ln(x-r)}{3\,r^2-1}.$$

The integrands in (7.8–7.11) are all special cases of

$$\int \frac{5\,x^4-60\,x^3+255\,x^2-450\,x+a}{x^5-15\,x^4+85\,x^3-225\,x^2+274\,x-b}dx \tag{7.26}$$

whose integral is, by Algorithm 39,

$$\sum_\alpha \frac{\left(5\,\alpha^4-60\,\alpha^3+255\,\alpha^2-450\,\alpha+a\right)\ln(x-\alpha)}{5\,\alpha^4-60\,\alpha^3+255\,\alpha^2-450\,\alpha+274} \tag{7.27}$$

where $\alpha = \mathrm{RootOf}\left(z^5-15\,z^4+85\,z^3-225\,z^2+274\,z-b\right)$. (7.11) is of this form, but (7.8–7.10) show how Algorithm 39 can produce more candid expressions where available, and indeed *guarantee* not to involve any unnecessary RootOf constructs.

The same process also leads to

$$\int \frac{1}{x^2+1}\mathrm{d}x = \frac{i}{2}\left(\ln(1-ix) - \ln(1+ix)\right), \tag{7.28}$$

at which point the reader might complain "I asked to integrate a *real* function, but the answer is coming back in terms of *complex* numbers". The answer is, of course, *formally correct*: differentiating the right-hand side of (7.28) yields

$$\frac{i}{2}\left(\frac{-i}{1-ix} - \frac{i}{1+ix}\right) = \frac{i}{2}\left(\frac{-i(1+ix}{1+x^2} - \frac{i(1-ix)}{1+x^2}\right) = \frac{1}{1+x^2}:$$

the issue is that the reader, *interpreting* the symbols log etc. as the usual functions of calculus, is surprised. This question of *interpretation* as functions $\mathbf{R} \to \mathbf{R}$ or $\mathbf{C} \to \mathbf{C}$, will be taken up in section 8.5. In the next section we will give a totally algebraic definition of log etc.

7.3 Theory: Liouville's Theorem

Definition 103 *Let K be a field of expressions. The expression θ is an* elementary generator *over K if one of the following is satisfied:*

(a) *θ is algebraic over K, i.e. θ satisfies a polynomial equation with coefficients in K;*

(b) *θ (assumed to be nonzero)*[6] *is an* exponential *over K, i.e. there is an η in K such that $\theta' = \eta'\theta$, which is only an algebraic way of saying that $\theta = \exp\eta$;*

(c) *θ is a* logarithm *over K, i.e. there is an η in K such that $\theta' = \eta'/\eta$, which is only an algebraic way of saying that $\theta = \log\eta$.*

It is conceivable that more than one of these might hold. In line with the constructive ethos of this subject, when an elementary generator is specified, an explicit choice of (a)/(b)/(c) and the necessary η or defining polynomial is assumed to have been stated. See the "N.B." on page 261.

In the light of this definition, (7.28) would be interpreted as saying

$$\int \frac{1}{x^2+1}\mathrm{d}x = \frac{i}{2}\left(\theta_1 - \theta_2\right), \tag{7.28 restated}$$

where $\theta_1' = \frac{-i}{1-ix}$ and $\theta_2' = \frac{i}{1+ix}$.

We should note that, if θ is a logarithm of η, then so is $\theta + c$ for any constant (Definition 102) c. Similarly, if θ is an exponential of η, so is $c\theta$ for any constant c, including the case $c = 0$, which explains the stipulation of nonzeroness in Definition 103(b).

7.3.0.1 Laws relating elementary functions

A consequence of these definitions is that log and exp satisfy the usual laws "up to constants".

log Suppose θ_i is a logarithm of η_i. Then

$$\begin{aligned}(\theta_1+\theta_2)' &= \theta_1' + \theta_2' \\ &= \frac{\eta_1'}{\eta_1} + \frac{\eta_2'}{\eta_2} \\ &= \frac{\eta_1'\eta_2 + \eta_1\eta_2'}{\eta_1\eta_2} \\ &= \frac{(\eta_1\eta_2)'}{\eta_1\eta_2},\end{aligned}$$

[6] This clause is not normally stated, but is important in practice: see the statement about "up to a constant" later.

and hence $\theta_1 + \theta_2$ is *a* logarithm of $\eta_1\eta_2$, a rule normally expressed as (but see the discussion on page 289 for what happens when we try to interpret log as a function $\mathbf{C} \to \mathbf{C}$)

$$\log \eta_1 + \log \eta_2 = \log(\eta_1\eta_2). \tag{7.29}$$

Similarly $\theta_1 - \theta_2$ is *a* logarithm of η_1/η_2 and $n\theta_1$ is *a* logarithm of η_1^n (for $n \in \mathbf{Z}$: we have attached no *algebraic* meaning to arbitrary powers).

exp Suppose now that θ_i is an exponential of η_i. Then

$$\begin{aligned} (\theta_1\theta_2)' &= \theta_1'\theta_2 + \theta_1\theta_2' \\ &= \eta_1'\theta_1\theta_2 + \theta_1\eta_2'\theta_2 \\ &= (\eta_1 + \eta_2)'(\theta_1\theta_2) \end{aligned}$$

and hence $\theta_1\theta_2$ is *an* exponential of $\eta_1 + \eta_2$, a rule normally expressed as

$$\exp \eta_1 \exp \eta_2 = \exp(\eta_1 + \eta_2). \tag{7.30}$$

Similarly θ_1/θ_2 is *an* exponential of $\eta_1 - \eta_2$ and θ_1^n is *an* exponential of $n\eta_1$ (for $n \in \mathbf{Z}$: we have attached no *algebraic* meaning to arbitrary powers).

(1) Suppose θ is a logarithm of η, and ϕ is an exponential of θ. Then

$$\begin{aligned} \phi' &= \theta'\phi = \frac{\eta'}{\eta}\phi, \quad \text{so} \\ \frac{\phi'}{\phi} &= \frac{\eta'}{\eta} = \theta', \end{aligned}$$

and θ is *a* logarithm of ϕ, as well as ϕ being an exponential of θ.

(2) Suppose now that θ is an exponential of η, and ϕ is a logarithm of θ. Then

$$\begin{aligned} \phi' &= \frac{\theta'}{\theta} \\ &= \frac{\eta'\theta}{\theta} = \eta', \end{aligned}$$

so η and ϕ differ by a constant. But ϕ, being a logarithm, is only defined up to a constant.

(1)+(2) These can be summarised by saying that, up to constants, log and exp are inverses of each other.

Definition 104 *Let K be a field of expressions. An overfield $K(\theta_1, \ldots, \theta_n)$ of K is called a* field of elementary expressions over K *if every θ_i is an elementary generator over $K(\theta_1, \ldots, \theta_{i-1})$. A expression is* elementary over K *if it belongs to a field of elementary expressions over K.*
If K is omitted, we understand $\mathbf{C}(x)$: the field of rational expressions.

For example, the expression $\exp(\exp x)$ can be written as elementary over $K = \mathbf{Q}(x)$ by writing it as $\theta_2 \in K(\theta_1, \theta_2)$ where $\theta_1' = \theta_1$, so θ_1 is elementary over K, and $\theta_2' = \theta_1'\theta_2$, and so θ_2 is elementary over $K(\theta_1)$.

Observation 18 *Other functions can be written this way as well. For example, if $\theta' = i\theta$ (where $i^2 = -1$), then $\phi = \frac{1}{2i}(\theta - 1/\theta)$ is a suitable model for* $\sin(x)$, *as in the traditional* $\sin x = \frac{1}{2i}\left(e^{ix} - e^{-ix}\right)$. *Note that $\phi'' = -\phi$, as we would hope.*

From this point of view, the problem of integration, at least of elementary expressions, can be seen as an exercise in the following paradigm.

Algorithm 40 (Integration Paradigm)
Input: *an elementary expression f in x*
Output: *An elementary g with $g' = f$, or failure*

```
Find fields $\mathcal{C}$ of constants, $L$ of elementary expressions over $\mathcal{C}(x)$ with $f \in L$
if this fails
  then error "integral not elementary"
Find an elementary overfield $M$ of $L$, and $g \in M$ with $g' = f$
if this fails
  then error "integral not elementary"
  else return $g$
```

This looks very open-ended, and much of the rest of this chapter will be devoted to turning this paradigm into an algorithm: one that when it returns `"integral not elementary"` has actually proved this fact. We will see later on that the poly-logarithm function and the erf function (defined as $\operatorname{erf} x = \int e^{-x^2}$ up to a constant multiple) are not elementary, and can be proved so this way. Other examples of non-elementary functions are:

W the Lambert W function, the solution of $W(z)\exp(W(z)) = z$ [CGH+96], proved non-elementary in [BCDJ08];

ω the Wright ω function, the solution of $\omega(z) + \ln \omega(z) = z$ [CJ02], proved non-elementary in [BCDJ08].

The next section addresses the apparent openendedness of the search for M. Another problem, addressed in sections 7.3.2 and 7.3.3, is that, for the rest to be algorithmic, L has to be "sufficiently computable": in particular we will want to tell if elements of L are zero or not, i.e. we want L to have normal (Definition 3) representations.

7.3.1 Liouville's Principle

The first question that might come to mind is that the search for M looks pretty open-ended. Here we are helped by the following result.

Theorem 46 (Liouville's Principle) *Let f be a expression from some expression field L. If f has an elementary integral over L, it has an integral of the following form:*

$$\int f = v_0 + \sum_{i=1}^{n} c_i \log v_i, \tag{7.31}$$

where v_0 belongs to L, the v_i belong to $\hat{L}$, an extension of L by a finite number of constants algebraic over L_{const}, and the c_i belong to $\hat{L}$ and are constant.

The proof of this theorem (see, for example, [Rit48]), while quite subtle in places, is basically a statement that the only new expression in g which can disappear on differentiation is a logarithm with a constant coefficient.

In terms of the integration paradigm (Algorithm 40), this says that we can restrict our search for M to M of the form $L(c_1, \ldots, c_k, v_1, \ldots, v_k)$ where the c_i are algebraic over L_{const} and the v_i are logarithmic over $L(c_1, \ldots, c_k)$.

Another way of putting this is to say that, if f has an elementary integral over L, then f has the following form:

$$f = v_0' + \sum_{i=1}^{n} \frac{c_i v_i'}{v_i}. \tag{7.32}$$

7.3.2 Finding L

A question that might seem trivial is the opening line of Algorithm 40: "Find a field $\mathcal{C}$ of constants, and a field L of elementary expressions over $\mathcal{C}(x)$ with $f \in L$". From an algorithmic point of view, this is not so trivial, and indeed is where many of the theoretical difficulties lie. We can distinguish three major difficulties.

1. Hidden constants. It is possible to start from a field $\mathcal{C}$, make various elementary extensions (Definition 103) of $\mathcal{C}(x)$ to create L, but have L_{const} be strictly larger than $\mathcal{C}$. Consider, for example, $L = \mathbf{Q}(x, \theta_1, \theta_2, \theta_3)$ where $\theta_1' = \theta_1$, $\theta_2' = 2x\theta_2$ and $\theta_3' = (2x+1)\theta_3$. Then

$$\begin{aligned}\left(\frac{\theta_1\theta_2}{\theta_3}\right)' &= \frac{\theta_3\left(\theta_1\theta_2\right)' - \theta_1\theta_2\theta_3'}{\theta_3^2}\\ &= \frac{\theta_3\theta_1'\theta_2 + \theta_3\theta_1\theta_2' - \theta_1\theta_2\theta_3'}{\theta_3^2}\\ &= \frac{\theta_1\theta_2\theta_3 + 2x\theta_1\theta_2\theta_3 - (2x+1)\theta_1\theta_2\theta_3}{\theta_3^2}\\ &= 0,\end{aligned}$$

so $\frac{\theta_1\theta_2}{\theta_3}$ is, unexpectedly, a constant c, and we should be considering $\mathbf{Q}(c)(x, \theta_1, \theta_2)$, with $\theta_3 = c\theta_1\theta_2$. This is perhaps not so surprising if we give the θ_i their conventional meanings as $\exp(x)$ etc., and write $L = \mathbf{Q}(x, \exp(x), \exp(x^2), \exp(x^2 + x))$, where we can clearly write $\exp(x^2 +$

$x) = \exp(x^2)\exp(x)$. Of course, θ_1 might equally well be $100\exp(x)$, etc., so all we can deduce (in the language of differential algebra) is that the ratio is a constant c, not necessarily that $c = 1$.

Equally, we could consider $L = \mathbf{Q}(x, \theta_1, \theta_2, \theta_3)$ where $\theta_1' = \frac{1}{x-1}$, $\theta_2' = \frac{1}{x-1}$ and $\theta_3' = \frac{2x}{x^2-1}$. Then

$$\begin{aligned}(\theta_1 + \theta_2 - \theta_3)' &= \theta_1' + \theta_2' - \theta_3' \\ &= \frac{1}{x-1} + \frac{1}{x-1} - \frac{2x}{x^2-1} \\ &= 0,\end{aligned}$$

and again we have a hidden constant $c = \theta_1 + \theta_2 - \theta_3$, and we should be considering $\mathbf{Q}(c)(x, \theta_1, \theta_2)$, with $\theta_3 = \theta_1 + \theta_2 - c$. Again, this is not so surprising if we give the θ_i their conventional meanings as $\log(x-1)$ etc., where we can clearly write $\log(x^2-1) = \log(x-1) + \log(x+1)$. Of course, θ_1 might equally well be $100 + \log(x-1)$, etc., so all we can deduce (in the language of differential algebra) is that $\theta_1 + \theta_2 - \theta_3$ is a constant c, not necessarily that $c = 0$.

2. Hidden algebraics. It is possible to start from a field $\mathcal{C}$, make k exponential and logarithmic extensions (Definition 103) of $\mathcal{C}(x)$ to create L, but have the transcendence degree of L over $\mathcal{C}(x)$ be *less* than k, i.e. for there to be unexpected algebraic elements of L, where we had thought they were transcendental. The obvious example is that $\sqrt{x} = \exp(\frac{1}{2}\log x)$, but there are more subtle ones, such as the following variant of the exponential example from the previous item. Consider $L = \mathbf{Q}(x, \theta_1, \theta_2, \theta_3)$ where $\theta_1' = \theta_1$, $\theta_2' = 2x\theta_2$ and $\theta_3' = (2x + \frac{1}{2})\theta_3$. Then

$$\begin{aligned}\left(\frac{\theta_1\theta_2^2}{\theta_3^2}\right)' &= \frac{\theta_3\left(\theta_1\theta_2^2\right)' - 2\theta_1\theta_2^2\theta_3'}{\theta_3^3} \\ &= \frac{\theta_3\theta_1'\theta_2^2 + \theta_3\theta_1\theta_2'\theta^2 - 2\theta_1\theta_2\theta_3'}{\theta_3^3} \\ &= \frac{\theta_1\theta_2^2\theta_3 + 4x\theta_1\theta_2^2\theta_3 - 2(2x + \frac{1}{2})\theta_1\theta_2^2\theta_3}{\theta_3^3} \\ &= 0,\end{aligned}$$

so again we have an unexpected constant c. The correct rewriting now is $\mathbf{Q}(c)(x, \theta_1, \theta_2, \theta_3)$, with $\theta_3 = \sqrt{c\theta_1\theta_2^2}$.

3. Ineffective constants. The previous two difficulties, have led to the introduction of "new" constants: what are these? Their values arise from the translation of the language of functions $\mathbf{R} \to \mathbf{R}$ (or $\mathbf{C} \to \mathbf{C}$) to the language of differential algebra. We deduced a constant that might be 1, or might be 100, but equally it might be e, as when the user has both e^x and e^{x+1} in an expression, when we have to transform $e^{x+1} \to e \cdot e^x$ (or

$e^x \to \frac{1}{e} \cdot e^{x+1}$). This doesn't seem to be a problem: e is well-known to be transcendental, so we can effectively regard it as a new indeterminate. However, there are issues, pointed out in [Ric68] and [Ax71].

The last difficulty is inherent in mathematics, as we see in the following sub-subsections, while the first two can be seen as failures of candidness (Definition 6), and are addressed in Section 7.3.3.

7.3.2.1 Richardson's Result

This relates to the interpretation of expressions A, defined as in Definition 103, as functions.

Notation 33 (Richardson) *Let E be a set of expressions representing real, single-valued, partially-defined, functions of one real valriable, and $E*$ the set of functions represented by elements of E. If $A \in E$, $A(x)$ is the function represented by A.*

We use $A(x) \equiv B(x)$ to mean that A and B are defined at the same points, and equal wherever they are defined.

Theorem 47 ([Ric68, Theorem Two]) *If $E*$ contains $\log 2$, π, e^x, $\sin(x)$ and $\sqrt[+]{x^2}$, then it is impossible to decide, for $A \in E$, whether $A(x) \equiv 0$.*

If we take such an undecidable $A(x)$, and consider $A(x)e^{x^2}$, the integration problem then becomes undecidable.

7.3.2.2 Transcendental Number Theory

We stated above "e is well-known to be transcendental, so we can effectively regard it as a new indeterminate", and indeed this is true. equally, π is known to be transcendental. *But* is it legitimate to take e and π to be *two* new indeterminates? Otherwise stated, can there be a nontrivial polynomial P such that $P(e, \pi) = 0$? We don't know: see Section 8.7.

7.3.3 Risch Structure Theorem

This result, [Ris69a, Ris79] roughly speaking, says that the laws in Section 7.3.0.1 are the only relationships between non-constant elementary expressions.

Notation 34 *However, to state it precisely, we need some notation.*

$\mathcal{C}$ *Let $\mathcal{C}$ be an algebraically closed field of characteristic 0. (*Typically we will think of $\mathcal{C}$ as being $\mathbf{C}$, but it might have other parameters in it.)

D *Let D be a differential field extending $\mathcal{C}(x)$, with $x' = 1$.*

D_1 *Let D_1 be an extension of D by elementary (Definition 103) generators $\theta_1, \ldots, \theta_m$.*

(y_i, z_i) *Let* $(y_i, z_i)_{1\le i\le r}$ *with* $z_i' = y_i' z_i$ *be the (logarithm,exponential) pairs among the* θ_j*, so that if* $\theta_j' = \eta_j'\theta_j$, $z_i = \theta_j$ *and* $y_i = \eta_j$*, while if* $\theta_j' = \eta_j'/\eta_j$, $y_i = \theta_j$ *and* $z_i = \eta_j$*.* If θ_j is algebraic, then there is no corresponding pair.

N.B. *[Ris79] has a caution that there might be multiple options here, but from our point of view, Definition 103 has specified* how θ_i *is an elementary extension.*

E *Let* $E = D_1(\theta)$ *be an extension of* D_1 *by an elementary generator* θ*.*

$\overline{D}$ *The relative algebraic closure of* D *in* E*, i.e. those elements of* E *algebraic over* D*.*

Theorem 48 (Risch Structure Theorem) *Our statement is simplified from [Ris69a, Ris79]. Suppose* θ *is algebraic over* D_1*. If* θ *was specified to be algebraic (case (a) of Definition 103), there is nothing to prove. Otherwise suppose* θ *is defined by* $z = y'/y$*, so that in the exponential case (b)* $\theta = z$ *with* $y \in D_1$*, and in the logarithmic case (c)* $\theta = y$ *with* $z \in D_1$*. Then there are* $c_i \in \mathcal{C}$*,* $f \in \overline{D}$ *such that*

$$y = f + \sum c_i y_i \tag{7.33}$$

and, after renumbering so that $1, c_{s+1}, \ldots, c_r$ *are a maximal* $\mathbf{Q}$*-linear subset of* $1, c_1, \ldots, c_r$*, there are* $n \neq 0, n_i \in \mathbf{Z}$ *and* $g \in D$ *such that*

$$z^n = g\prod_{i=1}^{s} z_i^{n_i}. \tag{7.34}$$

The proof is in [Ris79], and relies heavily on the technology of differential algebra. What it means, though, is that, if $\theta = y$, ostensibly a new logarithm, is actually algebraic, then it is something in $\overline{D}$ plus a sum of logarithms with constant coefficients, and if $\theta = z$, ostensibly a new exponential, is actually algebraic, then it is an nth root of an nth root of element of D times a product of rational powers of exponentials.

This means that working out whether θ is a genuinely new logarithm/exponential or is actually algebraic over D_1 reduces to problems in D: unfortunately section 7.3.2.2 shows that the base case has, at least in principle, serious decidability issues.

7.3.4 Overview of Integration

Notation 35 *Throughout sections 7.4–7.7, we assume that we are given an integrand* $f \in L = \mathcal{C}(x, \theta_1, \ldots, \theta_n)$*, where* $L_{\text{const}} = \mathcal{C}$ *is an effective field of constants, each* θ_i *is an elementary generator (Definition 103) over* $\mathcal{C}(x, \theta_1, \ldots, \theta_{i-1})$ and*, if* θ_i *in given as an exponential or logarithmic generator, then* θ_i *is actually transcendental over* $\mathcal{C}(x, \theta_1, \ldots, \theta_{i-1})$*.* Theorem 48 can be used to verify this last hypothesis. We can therefore regard the transcendental θ_i as new variables, and any normal/canonical form for rational functions (see Section 2.2.1) in the θ_i will actually be normal/canonical for this differential-algebraic interpretation of the θ_i.

Ideally, the solution of the integration problem would then proceed by induction on n: we assume we can integrate in $K = \mathcal{C}(x, \theta_1, \ldots, \theta_{n-1})$, and reduce integration in L to problems of integration in K. It was the genius of Risch [Ris69b] to realise that a more sophisticated approach is necessary.

Definition 105 *For a differential field L, the* elementary integration problem *for L is to find an algorithm which, given an element f of L, finds an elementary overfield M of L, and $g \in M$ with $g' = f$, or proves that such M and g do not exist.*

Definition 106 *For a differential field L, the* elementary Risch differential equation problem *for L is to find an algorithm which, given elements f and g of L (with f such that* $\exp(\int f)$ *is a genuinely new expression, i.e., for any non-zero F with $F' = fF$, $M = L(F)$ is transcendental over L and has $M_{\text{const}} = L_{\text{const}}$), finds $y \in L$ with $y' + fy = g$, or proves that such a y does not exist. We write* $\text{RDE}(f, g)$ *for the solution to this problem.*

The reader might object "but I can solve this by integrating factors!". Indeed, if we write $y = z\exp(\int -f)$, we get

$$\begin{aligned} \left(z\exp(\int -f)\right)' + fz\exp(\int -f) &= g \\ z'\exp(\int -f) - fz\exp(\int -f) + fz\exp(\int -f) &= g \end{aligned}$$

which simplifies to

$$z'\exp(\int -f) = g$$

and hence $z' = g\exp(\int f)$, $z = \int g\exp(\int f)$ and

$$y = \exp(\int -f)\int_* g\exp(\int f). \tag{7.35}$$

However, if we come to apply the theory of section 7.5 to the principal integral (marked $\int_*$) of (7.35), we find the integration problem reduces to solving the original $y' + fy = g$. Hence this problem must be attacked in its own right, which we do in section 7.7 for the base case $f, g \in \mathcal{C}(x)$.

Theorem 49 (Risch Integration Theorem) *Let $L = \mathcal{C}(x, \theta_1, \ldots, \theta_n)$, where $L_{\text{const}} = \mathcal{C}$ is an effective field of constants, each θ_i is an elementary generator (Definition 103) over $\mathcal{C}(x, \theta_1, \ldots, \theta_{i-1})$* and, *if θ_i is given as an exponential or logarithmic generator, then θ_i is actually transcendental over $\mathcal{C}(x, \theta_1, \ldots, \theta_{i-1})$. Then:*

(a) *we can solve the elementary integration problem for L;*

(b) *we can solve the elementary Risch differential equation problem for L.*

Here the proof genuinely is by induction on n, and, when $n \neq 0$, the case satisfied by θ_n in Definition 103.

(a) $n = 0$ This was treated in section 7.2.

(b) $n = 0$ This will be treated in section 7.7.

The **Risch induction hypothesis** then is that *both parts* hold for $\mathcal{C}(x, \theta_1, \ldots, \theta_{n-1})$, and we prove them for $\mathcal{C}(x, \theta_1, \ldots, \theta_n)$.

(a) θ_n **logarithmic** This will be treated in section 7.4.

(b) θ_n **logarithmic** See [Dav85a].

(a) θ_n **exponential** This will be treated in section 7.5.

(b) θ_n **exponential** See [Dav85a].

(a) θ_n **algebraic** This will be treated in section 7.6.

(b) θ_n **algebraic** See [Bro90, Bro91]. The case $n = 1$, i.e. algebraic expressions in $\mathcal{C}(x, y)$ with y algebraic over $\mathcal{C}(x)$, was solved in [Dav84].

7.4 Integration of Logarithmic Expressions

Let $\theta = \theta_n$ be a (transcendental) logarithm over $K = \mathcal{C}(x, \theta_1, \ldots, \theta_{n-1})$.

Lemma 12 (Decomposition Lemma (logarithmic)) *$f \in K(\theta)$ can be written uniquely as $p + q/r$, where p, q and r are polynomials of $K[\theta]$, q and r are relatively prime, and the degree of q is less than that of r. If f has an elementary integral over K, then p and q/r each possess an elementary integral over K.*

Proof. By Liouville's Principle (Theorem 46), if f is integrable, it is of the form

$$f = v_0' + \sum_{i=1}^{n} \frac{c_i v_i'}{v_i}, \tag{7.32 bis}$$

where $v_0 \in K(\theta)$, $c_i \in \overline{\mathcal{C}}$, and $v_i \in K(c_1, \ldots, c_n)[\theta]$. Write $v_0 = p_0 + \frac{q_0}{r_0}$, where $p_0, q_0, r_0 \in K[\theta]$ with $\deg(q_0) < \deg(r_0)$, and re-arrange the v_i such that $v_1, \ldots, v_k \in K(c_1, \ldots, c_n)$, but $v_{k+1}, \ldots, v_n$ genuinely involve θ, and are monic.

Then we can re-arrange (7.32 bis) as

$$p + \frac{q}{r} = \underbrace{p_0' + \sum_{i=1}^{k} \frac{c_i v_i'}{v_i}}_{\text{in } K(c_1, \ldots, c_n)[\theta]} + \underbrace{\left(\frac{q_0}{r_0}\right)' + \sum_{i=k+1}^{n} \frac{c_i v_i'}{v_i}}_{\text{proper rational expression}}, \tag{7.36}$$

and the decomposition of the right-hand side proves the result.

This means that it is sufficient to integrate the polynomial part (which we will do in Algorithm 41) and the rational part (which we will do in Algorithm 42) separately, and that failure in either part indicates that the whole expression does not have an elementary integral. In other words, there is no cross-cancellation between these parts.

7.4.1 The Polynomial Part

Let us turn first to the polynomial part. Assume that $p = \sum_{i=0}^{n} a_i\theta^i$ and $p_0 = \sum_{i=0}^{m} b_i\theta^i$. The polynomial parts of equation (7.36) then say that

$$\sum_{i=0}^{n} a_i\theta^i = \sum_{i=0}^{m} b_i'\theta^i + \sum_{i=0}^{m} ib_i\theta'\theta^{i-1} + \underbrace{\sum_{i=1}^{k} \frac{c_iv_i'}{v_i}}_{\text{independent of }\theta} . \tag{7.37}$$

Hence $n = m$, except for the special case $n = m - 1$ and b_m constant. If we consider coefficients of θ^n (assuming $n > 0$) we have

$$a_n = b_n' + (n+1)b_{n+1}\theta'.$$

We can integrate this formally (recalling that b_{n+1} is constant) to get

$$\int a_n = b_n + (n+1)b_{n+1}\theta. \tag{7.38}$$

But $a_n \in K$, and, by the Risch induction hypothesis (page 263), we have an integration algorithm for K. In fact, not any integral will do: we want an answer in K itself, apart possibly from a new logarithmic term of θ, which will determine b_{n+1}. If $\int a_n$ contained any other logarithms, then multiplying them by θ^n would give us a new logarithm with a non-constant coefficient, which is not allowed by Liouville's Principle (Theorem 46).

Hence the contribution to $\int p$ is $b_n\theta^n + \overline{b}_{n+1}\theta^{n+1}$. However,

$$\left(b_n\theta^n + \overline{b}_{n+1}\theta^{n+1}\right)' = a_n\theta^n + nb_n\theta'\theta^{n-1}, \tag{7.39}$$

so we should subtract $nb_n\theta'$ from a_{n-1}. Of course, b_n is only determined "up to a constant of integration $\overline{b}_n\theta$". When we come to integrate a_{n-1}, we may get a term $n\overline{b}_n\theta$, which determines this. The process proceeds until we come to integrate a_0, when any new logarithms are allowed, and the constant of integration here is that of the whole integration. The corresponding algorithm is given in Figure 7.1.

7.4.2 The Rational Expression Part

Now for the rational expression part, where we have to integrate a proper rational expression, and the integral will be a proper rational expression plus a sum of logarithms *with constant coefficients* — see (*) in Algorithm 42. The proper rational expression part is determined by an analogue of Hermite's algorithm, and (7.18) is still valid.

The Trager–Rothstein algorithm is still valid, with that additional clause that the roots of $P(\lambda)$, which don't depend on θ since $P(\lambda)$ is a resultant, actually be constants.

Figure 7.1: Algorithm 41: IntLog–Polynomial

Algorithm 41 (IntLog–Polynomial)
Input: $p = \sum_{i=0}^{n} a_i\theta^i \in K[\theta]$.
Output: *An expression for* $\int p\mathrm{d}x$*, or* `failed` *if not elementary*

```
Ans:=0
for $i := n, n-1, \ldots, 1$ do
    $c_i := \int a_i \mathrm{d}x$            # integration in $K$: (7.38)
    if $c_i$ =failed or $c_i \notin K[\theta]$
        return failed
    Write $c_i = b_i + (i+1)\overline{b}_{i+1}\theta$: $b_i \in K$ and $\overline{b}_{i+1}$ constant
    Ans:=Ans+$b_i\theta^i + \overline{b}_{i+1}\theta^{i+1}$
    $a_{i-1} := a_{i-1} - i\theta' b_i$       # correction from (7.39)
$c_0 := \int a_0 \mathrm{d}x$                 # integration in $K$
if $c_0$ =failed
    return failed
Ans:=Ans+$c_0$
```

Example 31 *To see why this is necessary, consider* $\int \frac{1}{\log x}\mathrm{d}x$*. Here* $q_1 = 1$*,* $r_1 = \theta$*, and* $P(\lambda) = \mathrm{Res}(1-\lambda/x, \theta, \theta) = 1-\lambda/x$ *(made monic, i.e.* $\lambda - x$*). This would suggest a contribution to the integral of* $x \log\log x$*, which indeed gives* $\log x$ *as one term on differentiation, but also a term of* $\log\log x$*, which is not allowed, since it is not present in the integrand.*

Note that $\int \frac{1}{x\log x}\mathrm{d}x$ gives $P = \lambda - 1$ and an integral of $\log\log x$, which is correct.

7.4.3 Conclusion of Logarithmic Integration

There are essentially three ways in which an expression whose "main variable" is a logarithmic θ, i.e. which is being written in $K(\theta)$, can fail to have an elementary integral.

1. In Algorithm 42, some $P(\lambda)$, whose roots should be the coefficients of the logarithms, turns out to have non-constant roots. Again, this would violate Liouville's principle. Example 31 is a classic case of this.

   Notation 36 *It is usual to define* $\int \frac{1}{\log x}$ *to be the* logarithmic integral *function* $\mathrm{li}(x)$*, but this is* not *an elementary function in the sense of Definition 104.*

2. In Algorithm 41, some $a_i \in K$ $(i > 0)$ may have an elementary integral, but one that involves 'new' logarithms other than θ, which would violate Liouville's Principle.

Figure 7.2: Algorithm 42: IntLog–Rational Expression

Algorithm 42 (IntLog–Rational Expression)
Input: $s_2, t_2 \in K[\theta]$ *relatively prime,* $\deg s_2 < \deg t_2$.
Output: *An expression for* $\int s_2/t_2 \mathrm{d}x$*, or* `failed` *if not elementary*

```
Ans:=0
Square-free decompose $t_2$ as $\prod_{i=1}^k r_i^i$
Write $\frac{s_2}{t_2} = \sum_{i=1}^k \frac{q_i}{r_i^i}$
for $i := 1, \ldots, k$ do
    Find $a_i, b_i$ such that $a_i r_i + b_i r_i' = 1$
    for $j := i, i-1, \ldots, 2$ do
        Ans:=Ans$-\frac{q_i b_i}{(j-1) r_i^{j-1}}$        #(7.18)
        $q_i := q_i a_i + (q_i b_i/(j-1))'$
    $P(\lambda) := \mathrm{Res}(q_i - \lambda r_i', r_i, \theta)$ (made monic)
    if $P$ has non-constant coefficients        (*)
        return failed
    Write $P(\lambda) = \prod_j Q_j(\lambda)^j$ (square-free decomposition)
    for $j$ in $1 \ldots$ such that $Q_j \neq 1$
        $v_j := \gcd(q_i - \lambda r_i', r_i)$ where $Q_j(\lambda) = 0$        #deg$(v_j) = j$
        Ans:=Ans+$\sum_{\lambda \text{ a root of } Q(j)} \lambda \log v_j(x)$
```

Example 32 *Consider $\int \frac{1}{x}\log(x+1)$ with θ being $\log(x+1)$. Then $a_1 = \frac{1}{x}$, and $\int a_1 = \log(x)$, which is not in the original field. If we allow this, and suggest that the integral has a term of $\log(x)\log(x+1)$, its derivative also has a term of $\frac{1}{x+1}\log(x)$, and we seem to be trapped in a vicious cycle.*

3. In Algorithm 41, some $a_i \in K$ may itself fail to have an elementary integral: this failure cannot be compensated for elsewhere, by Lemma 12.

   Example 33 *Consider $\int \frac{1}{\log x}\log(x+1)$ with θ being $\log(x+1)$. Then $a_1 = \frac{1}{\log x}$, and, as we have seen in Example 31, this does not have an elementary integral.*

   Even if we use the logarithmic integral $\mathrm{li}(x)$ here, and claim that $\int \frac{1}{\log x} = \mathrm{li}(x)$, we get a term in the integral of $\mathrm{li}(x)\log(x+1)$, whose derivative also has a term of $\frac{\mathrm{li}\,x}{x+1}$, and we now need a theory of integrating li-involving functions, which is beyond the scope of this book and is, as far as the author knows, unsolved (note that [Che86] only considers elementary integrands*).*

7.5 Integration of Exponential Expressions

Before we begin the general theory, let us consider the often-made statement "e^{-x^2} has no integral". From the point of view of calculus, this is clearly nonsense: we have a continuous function, so it has an integral. So what is meant is "there is no formula f which differentiates to e^{-x^2}". But again this is not true: [AS64, (7.1.1)] says $\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\mathrm{d}t$, and hence $\frac{\sqrt{\pi}}{2}\mathrm{erf}(x)$ is the formula we are lookng for. But, of course, the real problem is that [AS64, (7.1.1)] is the *definition*, not a useful identity. Armed with the vocabulary of Section 7.3, we now know the correct statement: "'e^{-x^2} has no elementary integral". This we now proceed to prove.

Example 34 ("'e^{-x^2} has no elementary integral") *Following our Paradigm (Algorithm 40), we choose $\mathbf{Q}$ as our field of constants, and L as $\mathbf{Q}(x,\theta)$ with $\theta' = -2x\theta$. The integrand is then just θ, Then Liouville's Principle (Theorem 46), in the form of (7.32) says that*

$$\theta = v_0' + \sum_{i=1}^{n} \frac{c_i v_i'}{v_i}, \tag{7.40}$$

*where v_0 belongs to L, the v_i belong to $\hat{L}$, an extension of L by a finite number of constants algebraic over L_{const}, and the c_i belong to $\hat{L}$ and are constant. Furthermore, we can assume that v_i $(i>0)$ are actually in $\mathbf{Q}(x)[\theta]$ and are square-free and relatively prime (making use of property "*log*" of Section 7.3.0.1).*

1. *Write $v_0 = \sum_{i=-m}^{n} a_i\theta^i + \widehat{v}_0$, where $\widehat{v}_0$ is a proper rational fraction in $\mathbf{Q}(x)(\theta)$ whose denominator is not divisible by θ (as all such factors are*

covered in the first summation). We can then perform a square-free decomposition of the denominator of $\widehat{v}_0$, and, as in (7.14), write

$$\widehat{v}_0 = \sum \frac{q_i}{r_i^i}, \tag{7.41}$$

where the r_i are square-free and relatively prime, and the q_i are relatively prime with the r_i. (7.40) then becomes

$$\theta = \sum_{i=-m}^{n} (a_i' - 2ixa_i)\theta^i + \sum \frac{q_i'}{r_i^i} + \sum \frac{-iq_i r_i'}{r_i^{i+1}} + \sum_{i=1}^{n} \frac{c_i v_i'}{v_i}. \tag{7.42}$$

But there is an r_i^{i+1} in the denominator of the right-hand side, and nothing elsewhere to cancel it. This is a contradiction unless there are no r_i, i.e. $\widehat{v}_0 = 0$.

2. *(7.42) then becomes*

$$\theta = \sum_{i=-m}^{n} (a_i' - 2xia_i)\theta^i + \sum_{i=1}^{n} \frac{c_i v_i'}{v_i}. \tag{7.43}$$

We can now observe that there is a v_i in the denominator of the right-hand side, and nothing elsewhere to cancel it. This is a contradiction unless there are no v_i.

3. *So $\theta = \sum_{i=-m}^{n}(a_i' - 2ixa_i)\theta^i$, and equating coefficients of θ shows that the only value of i is 1, and we have*

$$1 = a_1' - 2xa_1, \tag{7.44}$$

with $a_i \in \mathbf{Q}(x)$.

N.B. *The steps so far are an example of the working of this section. (7.44) is a Risch differential equation, and its solution properly belongs in Section 7.7: nevertheless, having come this far, we may as well finish the example.*

4. *Write $a_1 = p + \sum \frac{q_i}{r_i^i}$, where $p \in \mathbf{Q}[x]$, the r_i are square-free and relatively prime in $\mathbf{Q}[x]$, and the q_i are relatively prime with the r_i. (7.44) then becomes*

$$1 = p' + \sum \frac{q_i'}{r_i^i} + \sum \frac{-ir_i' q_i}{r_i^{i+1}} - 2xp - \sum \frac{2xq_i}{r_i^i}. \tag{7.45}$$

But there is an r_i^{i+1} in the denominator of the right-hand side, and nothing elsewhere to cancel it. This is a contradiction unless there are no r_i, i.e. $a_1 = p$.

5. *Write $p = \sum_{i=0}^{n} c_i x^i$, with $c_i \in \mathbf{Q}$. Then*

$$1 = p' - 2xp = \underbrace{\sum_{i=0}^{n} ic_i x^{i-1}}_{\text{degree } n-1} -2 \underbrace{\sum_{i=0}^{n} c_i x^{i+1}}_{\text{degree } n+1}, \tag{7.46}$$

and this is impossible.

Hence θ has no elementary integral.

Now for the general theory. Throughout this section, we let $\theta = \theta_n$ be a (transcendental) exponential over $K = \mathcal{C}(x, \theta_1, \ldots, \theta_{n-1})$, so that $\theta' = \eta'\theta$. We should note that this choice is somewhat arbitrary, since $\theta^{-1} = \overline{\theta}$ satisfies $\overline{\theta}' = -\eta'\overline{\theta}$ and $K(\overline{\theta}) \equiv K(\theta)$. Hence negative powers of θ are just as legitimate as positive powers, and this translates into a difference in the next result: rather than writing expressions as "polynomial + proper rational expressions", we will make use of the following concept.

Definition 107 *A* generalised (or Laurent) polynomial *in θ over K is a sum $\sum_{i=-m}^{n} a_i\theta^i$ with $a_i \in K$.*

Lemma 13 (Decomposition Lemma (exponential)) *$f \in K(\theta)$ can be written uniquely as $p+q/r$, where p is a Laurent polynomial, q and r are polynomials of $K[\theta]$ such that θ does not divide r, q and r are relatively prime, and the degree of q is less than that of r. If f has an elementary integral over K, then each of the terms of p, and also q/r, have an elementary integral over K.*

Proof. By Liouville's Principle (Theorem 46), if f is integrable, it is of the form

$$f = v_0' + \sum_{i=1}^{n} \frac{c_i v_i'}{v_i}, \tag{7.32 ter}$$

where $v_0 \in K(\theta)$, $c_i \in \overline{\mathcal{C}}$, and $v_i \in K(c_1, \ldots, c_n)[\theta]$. Write $v_0 = p_0 + \frac{q_0}{r_0}$, where $p_0 \in K[\theta, 1/\theta]$ is a Laurent polynomial, $q_0, r_0 \in K[\theta]$ with $\deg(q_0) < \deg(r_0)$ and $\theta \nmid r_0$, and re-arrange the v_i such that $v_1, \ldots, v_k \in K(c_1, \ldots, c_n)$, but $v_{k+1}, \ldots, v_n$ genuinely involve θ, and are monic. Furthermore, we can suppose that θ does not divide any of these v_i, since $\log\theta = \eta$ (up to constants). Unlike Lemma 12, though, it is no longer the case that $(\log v_i)'$ $(i > k)$ is a proper rational expression. Let n_i be the degree (in θ) of v_i, then, recalling that we have supposed that the v_i are monic, $v_i' = n_i\eta'\theta^{n_i} +$ lower terms, and $(v_i' - n_i\eta' v_i)/v_i$ *is* a proper rational expression

Then we can re-arrange (7.32 ter) as

$$p + \frac{q}{r} = \underbrace{p_0' + \sum_{i=1}^{k} \frac{c_i v_i'}{v_i} + \sum_{i=k+1}^{n} c_i n_i \eta' +}_{\text{in } K(c_1,\ldots,c_n)[\theta,1/\theta]} \underbrace{\left(\frac{q_0}{r_0}\right)' + \sum_{i=k+1}^{n} \frac{c_i(v_i' - n_i\eta' v_i)}{v_i}}_{\substack{\text{proper rational expression}\\ \theta \text{ not dividing the denominator}}}, \tag{7.47}$$

and the decomposition of the right-hand side proves the result.

This means that it is sufficient to integrate the polynomial part (which we will do in Algorithm 43) and the rational part (which we will do in Algorithm 44) separately, and that failure in either part indicates that the whole expression does not have an elementary integral. In other words, there is no cross-cancellation between these parts.

Figure 7.3: Algorithm 43: IntExp–Polynomial

Algorithm 43 (IntExp–Polynomial)
Input: $p = \sum_{i=-m}^{n} a_i\theta^i \in K[\theta]$, *where* $\theta = \exp\eta$.
Output: *An expression for* $\int p\mathrm{d}x$, *or* `failed` *if not elementary*

```
Ans:=0
for $i := -m, \ldots, -1, 1, \ldots, n$ do
    $b_i := \mathrm{RDE}(i\eta', a_i)$
    if $b_i$ =failed
        return failed
        else Ans:=Ans+$b_i\theta^i$
$c_0 := \int a_0 \mathrm{d}x$                # integration in $K$
if $c_0$ =failed
    return failed
Ans:=Ans+$c_0$
```

7.5.1 The Polynomial Part

In fact, this is simpler than the logarithmic case, since Lemma 13 says that each summand $a_i\theta^i$ has to be integrable separately.The case $i = 0$ is just integration in K, and all the others are cases of the Risch differential equation problem (Definition 106).

This translates straightforwardly into Algorithm 43.

7.5.2 The Rational Expression Part

The last equation (7.47) of Lemma 13 says that

$$\int \frac{q}{r} = \underbrace{\left(\sum_{i=k+1}^{n} c_i n_i\right)\eta}_{\text{“correction”}} + \underbrace{\left(\frac{q_0}{r_0}\right) + \sum_{i=k+1}^{n} c_i(\log v_i - n_i\eta)}_{\text{proper rational expression}}, \tag{7.48}$$

where the v_i are monic polynomials (not divisible by θ) of degree n_i. The proper rational expression part is determined by an analogue of Hermite's algorithm, and (7.18) is still valid, though we should point out that the justification involved stating that $\gcd(r_i, r_i') = 1$, where the r_i were the square-free factors of r_0. Since $\theta' = \eta\theta$, this is no longer true if $\theta|r_i$, but this is excluded since such factors were moved into the Laurent polynomial (Definition 107) part.

Hence the Hermite part of Algorithm 44 is identical to the rational and logarithmic cases. The Trager–Rothstein part is slightly more complicated, since v_i'/v_i is no longer a proper rational expression, which is the cause of the term marked 'correction" in (7.48). Suppose we have a term s_2/t_2 (in lowest terms: the programmer must not forget this check) left after Hermite's algorithm, and

Figure 7.4: Algorithm 44: IntExp–Rational Expression

Algorithm 44 (IntExp–Rational Expression)
Input: $q, r \in K[\theta]$ *relatively prime,* $\deg q < \deg r$.
Output: *An expression for* $\int q/r \mathrm{d}x$*, or* `failed` *if not elementary*

```
Ans:=0
Square-free decompose $r$ as $\prod_{i=1}^{k} r_i^i$
Write $\frac{q}{r} = \sum_{i=1}^{k} \frac{q_i}{r_i^i}$
**for** $i := 1, \ldots, k$ **do**
    Find $a_i, b_i$ such that $a_i r_i + b_i r_i' = 1$
    **for** $j := i, i-1, \ldots, 2$ **do**
        Ans:=Ans$-\frac{q_i b_i}{(j-1) r_i^{j-1}}$        #(7.18)
        $q_i := q_i a_i + (q_i b_i/(j-1))'$
    Cancel any common factor between $q_i$ and $r_i$
    $P(\lambda) := \mathrm{Res}(q_i - \lambda(r_i' - (\deg_\theta r_i)\eta' r_i), r_i, \theta)$ (made monic)
    **if** $P$ has non-constant coefficients              (*)
        **return** failed
    Write $P(\lambda) = \prod_j Q_j(\lambda)^j$ (square-free decomposition)
    **for** $j$ in $1 \ldots$ such that $Q_j \neq 1$
        $v_j := \gcd(q_i - \lambda r_i', r_i)$ where $Q_j(\lambda) = 0$        #deg$(v_j) = j$
        Ans:=Ans+$\sum_{\lambda \text{ a root of } Q(j)} \lambda \log v_j(x)$
```

write

$$\int \frac{s_2}{t_2} = \sum c_i(\log v_i - n_i\eta), \tag{7.49}$$

(7.49) can be differentiated to

$$\frac{s_2}{t_2} = \sum c_i \frac{v_i' - n_i\eta' v_i}{v_i}. \tag{7.50}$$

Hence $t_2 = \prod v_i$, which means $N := \deg_\theta t_2 = \sum n_i$, and, writing $u_j = \prod_{i \neq j} v_i$, we can write (7.50) as

$$s_2 = \sum c_i(v_i' - n_i\eta' v_i)u_i. \tag{7.51}$$

Furthermore, since $t_2 = \prod v_i$, $t_2' = \sum v_i' u_i$. Hence

$$\begin{aligned}
v_k &= \gcd(0, v_k) \\
&= \gcd\left(s_2 - \sum c_i(v_i' - n_i\eta' v_i)u_i, v_k\right) \\
&= \gcd\left(s_2 - c_k(v_k' - n_k\eta' v_k)u_k, v_k\right) \\
&\qquad \text{since all the other } u_i \text{ are divisible by } v_k \\
&= \gcd\left(s_2 - c_k \sum (v_i' - n_i\eta' v_i)u_i, v_k\right) \\
&\qquad \text{for the same reason} \\
&= \gcd\left(s_2 - c_k(t_2' - t_2 N\eta'), v_k\right) \\
&\qquad \text{since } t_2 = v_i u_i \text{ and } N = \sum n_i
\end{aligned}$$

But if $l \neq k$,

$$\begin{aligned}
\gcd\left(s_2 - c_k(t_2' - t_2 N\eta'), v_l\right) &= \gcd\left(\sum c_i v_i' u_i - c_k \sum v_i' u_i, v_l\right) \\
&\qquad t_2 N\eta' \text{ disappears as } v_l | t_2 \\
&= \gcd\left(c_l v_l' u_l - c_k v_l' u_l, v_l\right) \\
&\qquad \text{since all the other } u_i \text{ are divisible by } v_l \\
&= 1.
\end{aligned}$$

Since $t_2 = \prod v_l$, we can put these together to deduce that

$$v_k = \gcd(s_2 - c_k(t_2' - N\eta' t_2), t_2). \tag{7.52}$$

Given c_k, this will tell us v_k. But we can deduce more from this: the c_k are precisely those numbers λ such that $\gcd(s_2 - \lambda(t_2 - N\eta' t_2)', t_2)$ is non-trivial. Hence we can appeal to Proposition 78 (page 305), and say that λ must be such that

$$P(\lambda) := \mathrm{Res}_\theta(s_2 - \lambda(t_2' - N\eta' t_2), t_2) = 0. \tag{7.53}$$

As t_2 has degree N and $t_2' - N\eta' t_2$ has degree $N - 1$, $P(\lambda)$ is the determinant of a $2N - 1$ square matrix, N of whose rows depend linearly on λ, and thus is a polynomial of degree N in λ (strictly speaking, this argument only shows that it has degree at most N, but the coefficient of λ^N is $\mathrm{Res}(t_2' - N\eta' t_2, t_2) \neq 0$).

Hence, while we needed to re-prove the result, the application of Trager–Rothstein is little different from the logarithmic, and indeed rational expression, case. As in the logarithmic case, we have the caveat (*) that the roots of P must be constant. An analogous example to that of section 7.4.2 is that of $\int \frac{x}{1+e^x}$. Here $s_2 = x$, $t_2 = 1 + \theta$ and

$$\operatorname{Res}_\theta(s_2 - \lambda(t_2' - N\eta' t_2), t_2) = \operatorname{Res}_\theta(x - \lambda(\theta - (1+\theta)), 1+\theta) = x + \lambda.$$

This would suggest a contribution of $-x\log(1+e^x)$, but this also leaves $-\log(1+e^x)$ on differentiating, which contradicts Liouville's Principle. Computer algebra systems will give answers such as Maple's

$$\int \frac{x}{1+e^x}\mathrm{d}x = \frac{1}{2}x^2 - x\ln\left(1+\mathrm{e}^x\right) - \operatorname{polylog}\left(2, -\mathrm{e}^x\right), \tag{7.54}$$

but we have, essentially, just proved that $\operatorname{polylog}\left(2, -\mathrm{e}^x\right)$ is not elementary.

7.6 Integration of Algebraic Expressions

The integration of algebraic expressions is normally taught as a bunch of, apparently *ad hoc* tricks. A typical calculation would be

$$\begin{aligned}
\int \frac{1}{\sqrt{1-x^2}}\mathrm{d}x &= \int \frac{1}{\sqrt{1-\sin^2 t}}\frac{\mathrm{d}\sin t}{\mathrm{d}t}\mathrm{d}t && (7.55)\\
&\qquad\qquad\text{substituting } x = \sin t\\
&= \int \frac{1}{\cos t}\cos t\mathrm{d}t && (7.56)\\
&= t = \arccos(x), && (7.57)
\end{aligned}$$

which we can write as $\frac{\pi}{2} + i\log\left(\sqrt{1-x^2} + ix\right)$ to show that it really is elementary, and furthermore does not violate Liouville's Principle, since the only new expression is now seem to be a logarithm with a constant coefficient. The $\frac{\pi}{2}$ is, of course, just a particular choice of constant of integration. This leaves numerous questions.

1. Why $x = \sin t$? $x = \cos t$ seems to work, but gives a different result.
2. How do I know which substitutions will work?
3. How can I tell when *no* substitutions work?
4. What about cases when it only partially works, such as

$$\int \frac{1}{\sqrt{1-x^2}\sqrt{1-2x^2}}\mathrm{d}x = \int \frac{1}{\sqrt{1-2\sin^2 t}}\mathrm{d}t = ?$$

5. These inverse functions have branch cuts — what does that mean for the result?

The last question will be taken up in Chapter 8. The others are material for this section, and indeed seem difficult, to the point where the great 20th century number theorist Hardy [Har16] even conjectured that there was no general method. Integration theory in the 1960s, typified by the SAINT program [Sla61] concentrated on "doing it like a human", i.e. by guessing and substitution.

Risch [Ris70] observed that this wasn't necessary, and [Dav81, Tra84][7] converted his observations into algorithms, and (partial) implementations.

We may suppose[8] that we are integrating $f(x, y)$ where y is algebraic over $\mathcal{C}(x)$, but not just algebraic over $\mathcal{C}$. Then Liouville's Principle (Theorem 46) says that, if f has an elementary integral

$$\int f = v_0 + \sum_{i=1}^{n} c_i \log v_i, \tag{7.58}$$

where $v_0 \in \mathcal{C}(x, y)$, $c_i \in \overline{\mathcal{C}}$ and $v_i \in \mathcal{C}(c_1, \ldots, c_n)(x, y)$. The difficult part is computing the c_i and the v_i: once we have these computed, and their derivatives subtracted from f, solving $\int f = v_0$ can be done by the method of undetermined coefficients — essentially as in Section 7.2.3.

TO BE COMPLETED

7.7 The Risch Differential Equation Problem

In this section we solve case (b), $n = 0$ of Theorem 49: viz.

> to find an algorithm which, given elements f and g of $C(x)$ (with f such that $\exp(\int f)$ is a genuinely new expression, i.e., for any non-zero F with $F' = fF$, $M = C(x, F)$ is transcendental over $C(x)$ and has $M_{\text{const}} = C$), finds $y \in C(x)$ with $y' + fy = g$, or proves that such a y does not exist. This is written as RDE(f, g).

These conditions on f mean that it is not a constant, and its integral is not purely a sum of logarithms with rational number coefficients.

We will first consider the denominator of $y \in C(x)$. We could assume that C is algebraically closed, so that all polynomials factor into linears, but in fact we need not. We will assume that we can factor polynomials, though we will see afterwards that this is algorithmically unnecessary.

Let p be an irreducible polynomial. Let α be the largest integer such that p^α divides the denominator of y, which we can write as $p^\alpha \parallel \text{den}(y)$. Let β and γ be such that $p^\beta \parallel \text{den}(f)$ and $p^\gamma \parallel \text{den}(g)$. So we can calculate the powers of p which divide the terms of the equation to be solved:

$$\underbrace{y'}_{\alpha+1} + \underbrace{fy}_{\alpha+\beta} = \underbrace{g}_{\gamma}.$$

[7]Another case of simultaneous discovery.

[8]We do this for ease of exposition. In practice it's a bad idea to use the Primitive Element construction to reduce multiple algebraic functions to a single one. The implementation in [Dav81], for example, would work for multiple square rots, but only for these.

There are then three possibilities.

1. $\underline{\beta > 1}$. In this case the terms in $p^{\alpha+\beta}$ and p^{γ} have to cancel, that is we must have $\alpha = \gamma - \beta$.

2. $\underline{\beta < 1}$ (in other words, $\beta = 0$). In this case the terms in $p^{\alpha+1}$ and p^{γ} must cancel, that is, we must have $\alpha = \gamma - 1$.

3. $\underline{\beta = 1}$. In this case, it is possible that the terms on the left-hand side cancel and that the power of p which divides the denominator of $y' + fy$ is less than $\alpha+1$. If there is no cancellation, the result is indeed $\alpha = \gamma - 1 = \gamma - \beta$. So let us suppose that there is a cancellation. We express f and y in partial fractions with respect to p: $f = F/p^{\beta} + \hat{f}$ and $y = Y/p^{\alpha} + \hat{y}$, where the powers of p which divide the denominators of $\hat{f}$ and $\hat{y}$ are at most $\beta - 1 = 0$ and $\alpha - 1$, and F and Y have degree less than that of p.

   $$y' + fy = \frac{-\alpha p' Y}{p^{\alpha+1}} + \frac{Y'}{p^{\alpha}} + \hat{y}' + \frac{FY}{p^{\alpha+1}} + \frac{\hat{f}Y}{p^{\alpha}} + \frac{F\hat{y}}{p} + \hat{f}\hat{y}. \qquad (7.59)$$

   For there to be a cancellation in this equation, p must divide $-\alpha p' Y + FY$. But p is irreducible and Y is of degree less than that of p, therefore p and Y are relatively prime. This implies that p divides $\alpha p' - F$. But p' and F are of degree less than that of p, and the only polynomial of degree less than that of p and divisible by p is zero. Therefore $\alpha = F/p'$.

Putting these together proves the following result.

Lemma 14 ([Ris69b]) $\alpha \le \max(\min(\gamma - 1, \gamma - \beta), F/p')$, *where the last term only applies when* $\beta = 1$, *and when it gives rise to a positive integer.*

In fact, it is not necessary to factorise the denominators into irreducible polynomials. It is enough to find square-free polynomials p_i, relatively prime in pairs, and non-negative integers β_i and γ_i such that $\mathrm{den}(f) = \prod p_i^{\beta_i}$ and $\mathrm{den}(g) = \prod p_i^{\gamma_i}$. When $\beta = 1$, we have, in theory, to factorise p completely, but it is enough to find the integral roots of $\mathrm{Res}_x(F - zp', p)$, by an argument similar to Trager's algorithm for calculating the logarithmic part of the integral of a rational expression.

We have, therefore, been able to bound the denominator of y by $D = \prod p_i^{\alpha_i}$, so that $y = Y/D$ with Y polynomial. So it is possible to suppress the denominators in our equation, and to find an equation

$$RY' + SY = T. \qquad (7.60)$$

Let α, β, γ and δ be the degrees of Y, R, S and T. Then (7.60) becomes

$$\underbrace{RY'}_{\alpha+\beta-1} + \underbrace{SY}_{\alpha+\gamma} = \underbrace{T}_{\delta}. \qquad (7.61)$$

There are again three possibilities[9].

[9]This is not an accidental similarity, but we refer to [Dav84] for a unifying exposition.

1. $\underline{\beta - 1 > \gamma}$. In this case, the terms of degree $\alpha + \beta - 1$ must cancel out the terms of degree δ, therefore $\alpha = \delta + 1 - \beta$.
2. $\underline{\beta - 1 < \gamma}$. In this case, the terms of degree $\alpha + \gamma$ must cancel out the terms of degree δ, therefore $\alpha = \delta - \gamma$.
3. $\underline{\beta - 1 = \gamma}$. In this case, the terms of degree $\alpha + \beta - 1$ on the left may cancel. If not, the previous analysis still holds, and $\alpha = \delta + 1 - \beta$. To analyse the cancellation, we write $Y = \sum_{i=0}^{\alpha} y_i x^i$, $R = \sum_{i=0}^{\beta} r_i x^i$ and $S = \sum_{i=0}^{\gamma} s_i x^i$. The coefficients of the terms of degree $\alpha+\beta-1$ are $\alpha r_\beta y_\alpha$ and $s_\gamma y_\alpha$. The cancellation is equivalent to $\alpha = -s_\gamma / r_\beta$.

Lemma 15 ([Ris69b]) *$\alpha \le \max(\min(\delta - \gamma, \delta + 1 - \beta), -s_\gamma/r_\beta)$, where the last term is included only when $\beta = \gamma + 1$, and only when it gives rise to a positive integer.*

Determining the coefficients y_i of Y is a problem of linear algebra. In fact, the system of equations is triangular, and is easily solved.

Example 35 *Let us reconsider (7.44), which in our notation is $y' - 2xy = 1$, i.e. $f = -2x$, $g = 1$. Since neither f nor g have any denominator, y does not, i.e. it is purely polynomial, of degree α, and $R = 1$, $S = -2x$ and $T = 1$ in (7.60). Comparing degrees gives us*

$$\underbrace{y'}_{\alpha} + \underbrace{-2xy}_{\alpha+1} = \underbrace{1}_{0}, \tag{7.62}$$

i.e. $\alpha = -1$, as predicted by Lemma 15. But this is impossible.

The case when K is an algebraic extension of $C(x)$ is treated in [Dav84], and the more general cases in [Ris69b, Dav85b]. The principles are the same, but the treatment of the cancellation case gets more tedious, both in theory and in practice.

7.8 Worked Examples

It is possible that there are functions which are not purely logarithmic or purely exponential. But the hypotheses on K we have stated enable us to integrate mixed functions, by considering the function as a member of $K(\theta)$, where K is a field of functions and θ is a logarithm or an exponential.

7.8.1 First example

Let us consider the function

$$\frac{-e^x \log^2 x + \log x \left(\dfrac{2(e^x + 1)}{x} \right) + e^x + e^{2x}}{1 + 2e^x + e^{2x}}. \tag{7.63}$$

7.8.1.1 The problem

According to Algorithm 40, we must first find our field of definition.

This function belongs to the field of functions $\mathbf{Q}(x, e^x, \log x)$. Therefore we can write $K = \mathbf{Q}(x, e^x)$, $\theta = \log x$ and apply the theory of Section 7.4.

Observation 19 *We could equally well write $K = \mathbf{Q}(x, \log x)$, $\theta = e^x$ and apply the theory of Section 7.5. The author, given the choice, either by hand or when implementing this algorithm, has tended to prefer to put the exponentials as close to x, i.e. the base of the recursion, as possible. This is due to a desire to solve the Risch Differential Equations in as small a field as possible. However, the author knows of no experimental evidence backing up this prejudice.*

Open Problem 29 *What is the "right" (probably measured in computer time) ordering for the various θ_i in Theorem 49?*

7.8.1.2 The logarithmic integral

As an element of $K(\theta)$, this function is a polynomial in θ:

$$\theta^2 \left(\frac{-e^x}{1 + 2e^x + e^{2x}} \right) + \theta \left(\frac{2}{x(1 + e^x)} \right) + \frac{e^x}{1 + e^x}.$$

Following the method of Algorithm 41, we must integrate the coefficient of the leading power of θ, that is $-e^x/(1 + e^x)^2$. This integration takes place in K.

7.8.1.3 Sub-problem 1

This integration takes place in the field $L(\phi)$, where $\phi = e^x$ and $L = \mathbf{Q}(x)$. The function to be integrated is a proper rational function, and, moreover, ϕ does not divide the denominator. Therefore the theory of Algorithm 44 applies. Square-free decomposition is quite easy, and we only have to apply Hermite's method to q/r^2 where $q = -\phi$ and $r = 1 + \phi$. We find that $r' = \phi$ (the symbol $'$ always denotes differentiation with respect to x). The Bézout identity has to be applied to r and r', which is quite easy in the present case:

$$(1)r + (-1)r' = 1.$$

By substituting these values in Hermite's method, we find an integral of $-q(-1)/(1+\phi)$, and a remainder which cancels completely. Thus the solution of this sub-problem is $-\phi/(1 + \phi)$.

7.8.1.4 Back to the main problem

In the original problem, this gives us a term of the integral, that is $-\theta^2 e^x/(1 + e^x)$. But the derivative of this term gives us also terms in θ^1. The coefficient of θ^1 to be integrated, according to Algorithm 41 is given by $a_1 - 2\theta' b_2$, where a_1 is the original coefficient and b_2 is the solution of sub-problem 1. This calculation gives

$$\frac{2}{x(1+e^x)} - \frac{2}{x}\frac{-e^x}{1+e^x} = \frac{2}{x}.$$

The integral of this function ought to give the coefficient of θ in the integral (and, possibly, determine the constant of integration in the previous sub-problem).

7.8.1.5 Sub-problem 2

In theory this integration takes place in the field $L(\phi)$, where $\phi = e^x$ and $L = \mathbf{Q}(x)$, but in reality it is quite easy to see that the answer is $2\log x$.

7.8.1.6 Back to the main problem (2)

In general, the integrals given by the sub-problems in Algorithm 41 must belong to K, but, as we have seen, they may be allowed to contain θ. That is the case here, for the integral is 2θ. This implies that the choice of the constant of integration in the last sub-problem was bad, and that it must be increased by $2/2 = 1$, as $\left(\theta^2\right)' = 2\theta/x$.

So the present state of this problem is that we have integrated the coefficients of θ^2 and of θ^1, and we have found the integral to be $\theta^2(1 - e^x/(1+e^x))$, which simplifies into $\theta^2/(1+e^x)$. We still have to integrate the coefficient of θ^0, that is $e^x/(1+e^x)$.

7.8.1.7 Sub-problem 3

This integration takes place in the field $L(\phi)$, where $\phi = e^x$ and $L = \mathbf{Q}(x)$. The function to be integrated is not a proper rational function, and has to be rewritten in the form $1 - 1/(1+e^x)$. Integration of 1 gives x (plus a constant of integration, which is the integration constant of the problem). The other part is a proper rational function q/r, where $q = -1$ and $r = 1 + e^x$. r is not divisible by e^x and has no multiple factors, therefore its integral must be a sum of logarithms. By Algorithm 44, we have to calculate

$$\mathrm{Res}_\phi(q - \lambda(r' - N\eta' r), r),$$

where $N = 1$ (the degree of r), and $\eta = x$. This simplifies into $\mathrm{Res}_\phi(-1+\lambda, 1+\phi)$, that is $-1+\lambda$. This polynomial has a root, $\lambda = 1$, which is indeed a constant. Therefore the integral of the rational part is $\log(1+e^x) - x$. The $-x$ cancels with the x of the other part, and we have $\log(1+e^x)$.

7.8.1.8 Conclusion

Thus the solution of the problem is

$$\log^2 x \left(\frac{1}{1+e^x}\right) + \log(1+e^x),$$

which is an element of the original field (possibly extended by constants, but we haven't needed to) plus a logarithm with a constant coefficient.

7.8.2 Second example

$$e^{x+10\ \log(x)} + 2\,\frac{(\log(x))^3}{\left(-1+(\log(x))^2\right)^2 x} \tag{7.64}$$

Here, unlike section 7.8.1, we have[10] no choice over the formulation, since the exponential depends on $\log x$. Hence we write $K = \mathbf{Q}(x, \log x)$ and $\theta = \exp(x + 10\log(x))$. In theory this falls in the scope of "case a: θ_n exponential" of Theorem 49, but if we look at θ in more detail, we see that it satisfies

$$\theta' = \left(1 + \frac{10}{x}\right)\theta \tag{7.65}$$

(the $\log(x)$ part has disappeared), so can be treated by the theory of section 7.7. Our integral is $\theta + \phi$, where ϕ does not depend on θ. Hence, by the analogy[11] of Lemma 13, we need to integrate the two parts separately.

7.8.2.1 Integration of θ

Here we are trying to integrate θ satisfying (7.65). The answer with be $y\theta$ where

$$y' + \left(1 + \frac{10}{x}\right)y = 1. \tag{7.66}$$

In the notation of Lemma 14, the only interesting polynomial in the denominator is $p = x$, when $\beta = 1$. This is the anomalous case (7.59). F, the numerator of p in the coefficient of y, $= 10$ and $p' = 1$, so we deduce that α, the exponent of p in the denominator of y, is (at most) 10. So we write $y = Y/x^{10}$ where $Y \in \mathbf{Q}(x)$, and substituting this into (7.66) gives

$$\frac{Y'}{x^{10}} - \frac{10Y}{x^{11}} + \left(1 + \frac{10}{x}\right)\frac{Y}{x^{10}} = 1,$$

or

$$xY' - 10Y + xY + 10Y = x^{11} \tag{7.67}$$

after clearing denominators.

xY has degree $\deg_x(Y) + 1$ and all the other terms on the left have degree at most $\deg_x(Y)$, so $\deg_x(Y) = 10$, and $Y = x^{10} + Y_1$, where $\deg_x(Y_1) < \deg_x(Y)$. Substituting this into (7.67) gives

$$xY_1' + xY_1 = x^{11} - x^{11} - 10x^{10}.$$

[10] At least if we are going to treat the problem as posed.

[11] If we thought the answer was "unintegrable", we would need to prove this version, but as we find an integral, we don't need to.

So $Y_1 = -10x^9 + Y_2$ where $\deg_x(Y_2) < \deg_x(Y_1)$. Continuing in this way gives

$$\begin{aligned} Y &= x^{10} - 10\,x^9 + 90\,x^8 - 720\,x^7 + 5040\,x^6 - 30240\,x^5 + 151200\,x^4 - \\ &\quad 604800\,x^3 + 1814400\,x^2 - 3628800\,x + 3628800, \end{aligned}$$

and it is not difficult (it *is* tedious) to verify that $\frac{Y}{x^{10}}e^{x+10\,\log(x)}$ differentiates to $e^{x+10\,\log(x)}$

7.8.2.2 Integration of ϕ

We remind the reader that $\phi = 2\,\frac{(\log(x))^3}{\left(-1+(\log(x))^2\right)^2 x}$ and the theory of section 7.4 applies. In particular, we want Algorithm 42. The partial faction decomposition is already done, and we are integrating q_2/r_2^2 with $q_2 = \frac{2}{x}(\log(x))^3$ and $r_2 = (\log(x))^2 - 1$. We find a_2 and b_2 with $a_2r_2 + b_2r_2' = 1$, which in our case are $a_2 = -1$ and $b_2 = \frac{-x}{2}(\log(x))$. This gives us a contribution of

$$-\frac{q_2b_2}{r_2} = -\frac{(\log(x))^4}{-1 + (\log(x))^2} \tag{7.68}$$

to the answer, and leaves an unintegrated term of

$$\frac{q_2a_2 + (q_2b_2)'}{r_2} = \frac{-2(\log(x))^3 - 3(\log(x))^2}{x(-1 + (\log(x))^2)}. \tag{7.69}$$

At this point the reader might notice that neither (7.68) nor (7.69) are proper rational functions, but we are saved by the remark after (7.18), that the excesses cancel. The fractional part of (7.68) is $\frac{-1}{(-1+(\log(x))^2)}$, and of (7.69) is $\frac{\frac{2}{x}\log(x)}{(-1+(\log(x))^2)}$.

This last has to be integrated by the extension to logarithms of the Trager–Rothstein algorithm. So let $P(\lambda)$ be

$$\mathrm{Res}_{\log(x)}(\frac{2}{x}\log(x) - \lambda\frac{2}{x}\log(x), -1 + (\log(x))^2) = -\frac{(1-\lambda)^2}{x^2},$$

which has a root $\lambda = 1$ (of multiplicity 2). Hence

$$\begin{aligned} v_1 &= \gcd\left(\frac{2}{x}\log(x) - \lambda\frac{2}{x}\log(x), -1 + (\log(x))^2\right) \\ &= \gcd(0, -1 + (\log(x))^2) \\ &= -1 + (\log(x))^2. \end{aligned}$$

Hence this comtribution to the integral is

$$\lambda\log(v_1) = \log\left(-1 + (\log(x))^2\right). \tag{7.70}$$

Combining (7.70) with the fractional part of (7.68) and the result of the previous section gives us

$$\frac{\left(\begin{array}{c} x^{10} - 10\,x^9 + 90\,x^8 - 720\,x^7 + 5040\,x^6 - 30240\,x^5 \\ +151200\,x^4 - 604800\,x^3 + 1814400\,x^2 - 3628800\,x + 3628800 \end{array}\right) \mathrm{e}^{x+10\,\ln(x)}}{x^{10}} + \frac{1}{-1+(\ln(x))^2} + \ln\left(-1+(\ln(x))^2\right).$$

as the final answer.

7.9 The Parallel Approach

The methods described so far this chapter have essentially taken a recursive approach to the problem: if the integrand lives in $K(x, \theta_1, \ldots, \theta_n)$, we have regarded this as $K(x)(\theta_1)(\ldots)(\theta_n)$, and the results of the previous sections all justify an "induction on n" algorithm. In this section we will adopt a different approach, regarding $K(x, \theta_1, \ldots, \theta_n)$ as the field of fractions of $K[x, \theta_1, \ldots, \theta_n]$, and taking a *distributed* (page 50), also called *parallel* as we are treating all the θ_i in parallel, view of $K[x = \theta_0, \theta_1, \ldots, \theta_n]$.

We note that Liouville's Principle (Theorem 46) can be applied to this setting.

Theorem 50 (Liouville's Principle, Parallel version) *Let f be an expression from some expression field $L = K(x, \theta_1, \ldots, \theta_n)$. If f has an elementary integral over L, it has an integral of the following form:*

$$\int f = v_0 + \sum_{i=1}^{m} c_i \log v_i, \tag{7.71}$$

where v_0 belongs to L, the v_i belong to $\hat{K}[x, \theta_1, \ldots, \theta_n]$, an extension of $K[x, \theta_1, \ldots, \theta_n]$ by a finite number of constants algebraic over K, and the c_i belong to $\hat{K}$.

Other that being more explicit about the domains objects belong to, we are asserting that the v_i are polynomial, which can be assumed since $\log(p/q) = \log(p) - \log(q)$, or in the language of differential algebra,

$$\frac{(p/q)'}{p/q} = \frac{p'}{p} - \frac{q'}{q}.$$

Notation 37 *In (7.71), we write $v_0 = \frac{p}{q}$, where $p, q \in K[x, \theta_1, \ldots, \theta_n]$, but are not necessarily relatively prime.*

From now on, until section 7.9.2, we will assume that the θ_i are transcendental. We can then infer the following algorithm for integrating expressions in this setting.

Pseudo-Algorithm 45 (Parallel Risch [NM77, Bro07])
Input: *$L = K(x, \theta_1, \ldots, \theta_n)$ a purely transcendental differential field with constants K, $f \in L$*
Output: *g an elementary integral of f, or a proof that there is no such one satisfying the assumptions of steps (1)–(3).*

(1) Decide candidate $v_1, \ldots, v_m$ (we may have too many)

(2) Decide a candidate q (which may be a multiple of the true value)

The obvious choice is $\prod r_i^{i-1}$ where $\text{den}(f) = \prod r_i^i$ as a square-free decomposition, but monomial factors of exponentials ought not to have their multiplicity decreased. However, this is insufficient in certain cases, e.g. [Dav82a, DT85, Dav82b].

(3) Decide d̄egree bounds for p (which may be too large), i.e. $n_0 \ldots, n_n$ such that

$$p = \sum_{i_0=0}^{n_0} \sum_{i_1=0}^{n_1} \cdots \sum_{i_n=0}^{n_n} c_{i_0,i_1,\ldots,i_n} x^{i_0} \prod_{j=1}^{n} \theta_j^{i_j}$$

(4) Clear denominators in the derivative of (7.71)

N.B. In practice we need to do step (4) symbolically before we can do step(3): see the example on page 283.

(5) Solve the resulting linear equations for the c_i and $c_{i_0,i_1,\ldots,i_n}$, but $c_{0,\ldots,0}$ is the the constant of integration, and is never determined

(6) **if** there's a solution
then reduce p/q to lowest terms and return the result
else "integral not found"

As explained in [Bro07], it is the decisions taken in steps (1)–(3) which mean that this is not a guaranteed "integrate or prove unintegrable" algorithm. The work of the previous sections allows partial solutions:

(1) Those $v_1, \ldots, v_m$ which depend on θ_n
(2) A candidate q (which may be a multiple of the true value)
But the multiple is in $K(x, \theta_1, \ldots, \theta_{n-1})[\theta_n]$, not in $K[x, \theta_1, \ldots, \theta_n]$
(3) A degree bound n_n for p as a polynomial in θ_n

As pointed out in Observation 19, we may have some choice over which θ we take as θ_n. If we had complete freedom (i.e. the θ_i involve only x in their definition), then we could determine all the n_i $(i > 0)$ and that part of q which depended on the θ_i. *In this case*, we would then have a complete algorithm if we did linear algebra over $K(x)$ rather than K. But the author has never seen this implemented.

7.9.1 An example

Example (7.63) was written as

$$\frac{-e^x\log^2 x+\log x\left(\frac{2(e^x+1)}{x}\right)+e^x+e^{2x}}{1+2e^x+e^{2x}},$$

but for this approach we require a distributed format, and a common denominator with a square-free decomposition, so we write

$$\frac{-xe^x\log^2 x+2e^x\log x+2\log x+xe^x+xe^{2x}}{x\left(1+e^x\right)^2}.$$

(1) Decide candidate $v_1,\ldots,v_m$.

Here The factors of the denominator are x and $(1+e^x)$, hence we might want to add both $\log x$ and $\log(1+e^x)$, but in fact $\log x$ is already in the field, so we just add $v_1=\log(1+e^x)$.

(2) Decide a candidate q (which may be a multiple of the true value).

Here The "obvious" answer is $(1+e^x)$.

(4) Clear denominators in the derivative of (7.71), i.e.

$$\frac{-xe^x\log^2 x+2e^x\log x+2\log x+xe^x+xe^{2x}}{x\left(1+e^x\right)^2}=$$

$$\left(\frac{\sum_{i=0}^{n_0}\sum_{j=0}^{n_1}\sum_{k=0}^{n_2}c_{i,j,k}x^i(\log x)^j\left(e^x\right)^k}{1+e^x}\right)'+\frac{c_1e^x}{1+e^x}=$$

$$\frac{c_1x(e^x+e^{2x})+\sum_{i,j,k}c_{i,j,k}\left(\begin{array}{c}\left((\log x)^{j-1}j+(\log x)^j i\right)(1+\mathrm{e}^x)x^i+\\(\log x)^j\left((k-1)\mathrm{e}^x+k\right)x^{i+1}\end{array}\right)\mathrm{e}^{kx}}{x\left(1+\mathrm{e}^x\right)^2}$$

(3) Decide degree bounds for p (which may be too large), i.e. $n_0\ldots,n_2$ such that

$$p=\sum_{i=0}^{n_0}\sum_{j=0}^{n_1}\sum_{k=0}^{n_2}c_{i,j,k}x^i(\log x)^j\left(e^x\right)^k.$$

Here the leading terms in the numerator of the right-hand side are c_1xe^{2x} and $c_{i,j,k}(k-1)x^{i+1}(\log x)^je^{(k+1)x}$ (generically, i.e. when $k\neq 1$). Since the highest power of e^x on the left-hand side is e^{2x}, we deduce that $k\leq 1$. Separating out $K=0$ and $k=1$ gives us a right-hand side numerator of

$$c_1x(e^x+e^{2x})+\sum_{i,j}c_{i,j,0}\left(\begin{array}{c}\left((\log x)^{j-1}j+(\log x)^j i\right)(1+\mathrm{e}^x)x^i-\\(\log x)^j\mathrm{e}^x x^{i+1}\end{array}\right)+$$

$$\sum_{i,j} c_{i,j,1} \left(\begin{array}{c} \left((\log x)^{j-1} j + (\log x)^j i \right) (1 + \mathrm{e}^x) x^i + \\ (\log x)^j x^{i+1} \end{array} \right) \mathrm{e}^x$$

Since the left-hand side has at most x (to the power 1) and $\log^2 x$, we can deduce that $i \le 1$ and $j \le 2$.

(5) Solve the resulting linear equations for the c_i and $c_{i_0,i_1,\ldots,i_n}$. The coefficient of xe^{2x} on the right-hand side is $c_1 + c_{1,0,1}$, so we deduce that $c_1 + c_{1,0,1} = 1$. Taking the c_1 and $c_{1,0,1}$ terms over to the left-hand side, and using this equality, gives

$$\frac{\left(-x (\log(x))^2 + 2 \ln(x) - x^2 (1 - c_1) \right) \mathrm{e}^x + 2 \log(x)}{x (1 + \mathrm{e}^x)^2}.$$

This now has no e^{2x}, so we can discard the possibility that $k = 1$. Hence we have to solve

$$\frac{\left(-x (\log(x))^2 + 2 \ln(x) - x^2 (1 - c_1) \right) \mathrm{e}^x + 2 \log(x)}{x (1 + \mathrm{e}^x)^2} =$$
$$\sum_{i,j} c_{i,j,0} \left(\left((\log x)^{j-1} j + (\log x)^j i \right) (1 + \mathrm{e}^x) x^i - (\log x)^j \mathrm{e}^x x^{i+1} \right).$$

The leading term[12] is $-c_{i,j,0} x^{i+1} (\log x)^j e^x$. To match the leading term on the left-hand side, we therefore need $c_{0,2,0} = 1$.

After subtracting this off, we are left with

$$\frac{\mathrm{e}^x x (c_1 - 1)}{(1 + \mathrm{e}^x)^2},$$

so we set $c_1 = 1$.

7.9.2 The Parallel Approach: Algebraic Expressions

TO BE COMPLETED

7.10 Definite Integration

We have shown (Example 35) that $\int \exp(-x^2)$ has no elementary expression, i.e. that it is a new expression, which we called erf. However, $\int_{-\infty}^{\infty} e^{-x^2}$ (where we have attached a precise numerical meaning e^x to $\exp(x)$) is well-known to be π, which is essentially another way of saying that $\mathrm{erf}(\pm\infty) = \pm 1$, and is therefore a mater of giving numerical values to functions — see Chapter 8.

TO BE COMPLETED Meijer etc.

[12]Inthis case, under any admissible ordering of the monomials $x, \log x, e^x$ — life can get more complicated when this isn't unambiguous.

7.11 Other Calculus Problems

7.11.1 Indefinite summation

The initial paper in this area was [Gos78]. Although not expressed that way in this paper, the key idea [Kar81] is the following, akin to Definition 101.

Definition 108 *A* difference ring *is a ring (Definition 8) equipped with an additional unary operation, referred to as differentiation and written*[13] *with a prefix δ, which satisfies three additional laws:*

1. $\delta(f+g) = \delta f + \delta g$;
2. $\delta(fg) = f(\delta g) + (\delta f)g + (\delta f)(\delta g)$;
3. $\delta(1) = 0$.

A difference ring which is also a field (Defintion 15) is referred to as a difference field.

One thinks of $\delta(f)$ as being $f(n+1)-f(n)$. It is worth remarking the differences with Definition 101: clause 3 is necessary, whereas previously it could be inferred from the others, and clause 2 is different, essentially because $(x^2)' = 2x$ but $\delta(n^2) = (n+1)^2 - n^2 = 2n+1$.

The process of formal (indefinite) summation $\sum$ is then that of inverting δ, as $\int$ is the prcess of inverting $'$. Just as the rôle of $\mathrm{d}x$ in integration theory is explained by the fact that $x' = 1$, i.e. $\int 1 = x$, so the traditional rôle of n in summation theory is explained by the fact that $\delta n = 1$, equivalently that $\sum 1 = n$. Note that the two procedures of integration and summation are more similar than normal notation would suggest: we are happy with $\int x = \frac{1}{2}x^2$, but less so with $\sum n = \frac{1}{2}n(n+1)$, yet the two are essentially equivalent.

To further the analogy, we know that $\int \frac{1}{x}$ cannot be expressed as a rational function of x, but is a new expression, known conventionally as $\log x$. Similarly, $\sum \frac{1}{n}$ cannot be expressed as a rational function of n [Kar81, Example 16], but is a new expression, known conventionally as H_n, the n-th harmonic number. Again, $\int \log x = x \log x - x$, and $\sum H_n = nH_n - n$.

7.11.2 Definite Symbolic Summation

Definite summation might be thought to be related to indefinite summation in the way that definite integration (Section 7.10) is to indefinite. There is certainly the same problem of evaluating expressions from difference algebra at numerical points. However, in practice we are more concerned with a subtly different class of problems, where the limits of summation also figure in the summand.

[13]In this text: notations differ.

Example 36 ([GKP94, Exercise 6.69], [Sch00b])

$$\sum_{k=1}^{n} k^2 H_{n+k} = \frac{1}{3} n \left(n + \frac{1}{2} \right) (n+1)(2H_{2n} - H_n) - \frac{1}{36} n(10n^2 + 9n - 1). \quad (7.72)$$

What are the key references here? [Sch04]?

7.11.3

7.11.4

Chapter 8

Algebra versus Analysis

We have seen in the previous chapter how we can construct an *algebraic* theory of mathematical objects such as 'exp' and 'log', and possibly others. From an algebraic point of view, they seem to behave like the mathematical objects we are familiar with from analysis. Are they the same? If not, what are the differences? This is perhaps one of the less-discussed topics[1] in computer algebra, and indeed possibly in mathematics more generally.

Notation 38 *Throughout this chapter, we use the notation* $\stackrel{?}{=}$ *to denote an equation that might or might not be true, or partially true, depending on the interpretations, from algebra or from analysis, that one places on the symbols either side of* $\stackrel{?}{=}$.

8.1 Functions and Formulae

This question turns out to be related to the difference between *functions* and *formulae* (which we have also called *expressions*). Consider the two-dimensional formula $\frac{x^2-1}{x-1}$, or `(x^2-1)/(x-1)` if we prefer one-dimensional expressions. It has potentially many rôles.

formula There are several options here: strings of characters, or a parse tree. Whichever we choose, `(x^2-1)/(x-1)` is a different formula from `x+1`.

$\in \mathbf{Q}(x)$ (see section 2.2.) This is therefore mathematically equivalent to $x+1$, and an algebra system may or may not transform one into the other: a system that aims for candidness in this context (section 2.2.2) should transform `(x^2-1)/(x-1)` into `x+1`.

$\in K(x)$ Of course, it is only convention that chooses $\mathbf{Q}$ for the ground field. It could be any extension of $\mathbf{Q}$, or, more challengingly, a finite field of positive characteristic.

[1]But see [Dav10].

rule This is what a computer scientist would think of as $\lambda x.\frac{x^2-1}{x-1}$: that rule which, given an x, computes the corresponding value of the formula

None of these are, as such, a function in the sense of Notation 3, though the last is the nearest. However, trying to express $\frac{x^2-1}{x-1}$ in this notation exposes our problems.

$$\left(\left\{\left(x, \frac{x^2-1}{x-1}\right) \mid x \in \mathbf{Q}\right\}, \mathbf{Q}, \mathbf{Q}\right)_{\mathcal{B}} \tag{8.1}$$

is illegal, because of the case $x = 1$. Ruling this out gives us

$$\left(\left\{\left(x, \frac{x^2-1}{x-1}\right) \mid x \in \mathbf{Q} \setminus \{1\}\right\}, \mathbf{Q} \setminus \{1\}, \mathbf{Q}\right)_{\mathcal{B}}. \tag{8.2}$$

If we regard $\frac{0}{0}$ as $\perp$, then we can have

$$\left(\left\{\left(x, \frac{x^2-1}{x-1}\right) \mid x \in \mathbf{Q}\right\}, \mathbf{Q}, \mathbf{Q} \cup \{\perp\}\right)_{\mathcal{B}}. \tag{8.3}$$

Making the $\perp$ explicit gives us

$$\left(\left\{\left(x, \frac{x^2-1}{x-1}\right) \mid x \in \mathbf{Q} \setminus \{1\}\right\} \cup \{(1, \perp)\}, \mathbf{Q}, \mathbf{Q} \cup \{\perp\}\right)_{\mathcal{B}}. \tag{8.4}$$

If we're going to single out a special case, we might as well (since $2 = \lim_{x\to 1} \frac{x^2-1}{x-1}$, i.e. this is a *removable singularity*, in the sense that we can give our function a value at the apparently singular point whch makes it continuous) write

$$\left(\left\{\left(x, \frac{x^2-1}{x-1}\right) \mid x \in \mathbf{Q} \setminus \{1\}\right\} \cup \{(1, 2)\}, \mathbf{Q}, \mathbf{Q}\right)_{\mathcal{B}}, \tag{8.5}$$

dropping the $\perp$, and this *is* equal to

$$\left(\{(x, x+1) \mid x \in \mathbf{Q}\}, \mathbf{Q}, \mathbf{Q}\right)_{\mathcal{B}}. \tag{8.6}$$

The case of polynomials is simpler (we restrict consideration to one variable, but this isn't necessary). Over a ring R of characteristic zero[2], equality of abstract polynomials is the same as equality of the corresponding functions:

$$p = q \text{ in } R[x] \Leftrightarrow \left(\{(x, p(x)) \mid x \in R\}, R, R\right)_{\mathcal{B}} = \left(\{(x, q(x)) \mid x \in R\}, R, R\right)_{\mathcal{B}} \tag{8.7}$$

This is a consequence of the fact that a non-zero polynomial $p - q$ has only a finite number of zeros.

If we attach a meaning to elements of $R(x)$ by analogy with (8.2), omitting dubious points in the domain, then equality in the sense of Definition 16 is related to the equality of Bourbakist functions in the following way

$$f = g \text{ in } R(x) \Leftrightarrow \left(\{(x, f(x)) \mid x \in S\}, S, R\right)_{\mathcal{B}} = \left(\{(x, g(x)) \mid x \in S\}, S, R\right)_{\mathcal{B}}, \tag{8.8}$$

where S is the intersection of the domains of f and g.

[2]See example 5 on page 25 to explain this limitation.

8.2 Branch Cuts

In the previous section, we observed that there was a difference between "expressions", whether concrete formulae or abstract expressions in $K(x)$, and functions $\mathbf{R} \to \mathbf{R}$ or $\mathbf{C} \to \mathbf{C}$. If we go beyond $K(x)$, the problems get more serious.

Example 37 *Consider the formula* `sqrt(z)`*, or* $\sqrt{z}$*, which we can formalise as* $\theta \in K(z, \theta | \theta^2 = z)$*. What happens if we try to interpret* θ *as a function* $\theta(z) : \mathbf{C} \to \mathbf{C}$*? Presumably we choose* $\theta(1) = 1$*, and, by continuity,* $\theta(1+\epsilon) = 1 + \frac{1}{2}\epsilon - \frac{1}{8}\epsilon^2 + \cdots \approx 1 + \frac{1}{2}\epsilon$*. Similarly,* $\theta(1+\epsilon i) \approx 1 + \frac{1}{2}\epsilon i$*, and so on. If we continue to track* $\theta(z)$ *around the unit circle* $\{z = x + iy : x^2 + y^2 = 1\}$*, we see that* $\theta(i) = \frac{1+i}{\sqrt{2}}$*, and* $\theta(-1) = i$*. Continuing,* $\theta(-i) = \frac{-1+i}{\sqrt{2}}$ *and* $\theta(1) = -1$*.*

It would be tempting to dismiss this as "the usual square root ambiguity", but in fact the problem is not so easily dismissed.

Example 38 *Similarly, consider* `log(z)`*, which we can formalise as* $\theta \in K(z, \theta | \exp(\theta(z)) = z\}$*. What happens if we try to interpret* θ *as a function* $\theta(z) : \mathbf{C} \to \mathbf{C}$*? We chose* $\theta(1) = 0$*, and, since* $\exp(\epsilon i) \approx \epsilon i$*,* $\theta(\epsilon i) \approx \epsilon i$*. As we track* $\theta(z)$ *around the unit circle* $\{z = x + iy : x^2 y^2 = 1\}$*, we see that* $\theta(i) = i\pi/2$*,* $\theta(-1) = i\pi$*, and ultimately* $\theta(1) = 2\pi i$*.*

8.2.1 Some Unpleasant Facts

The blunt facts[3] are the following (and many analogues).

Proposition 76 (No square root function) *There is no continuous function* $f : \mathbf{C} \to \mathbf{C}$ *(or* $\mathbf{C} \setminus \{0\} \to \mathbf{C} \setminus \{0\}$*) with the property that* $\forall z : f(z)^2 = z$*.*

Proposition 77 (No logarithm function) *There is no continuous function* $f : \mathbf{C} \setminus \{0\} \to \mathbf{C} \setminus \{0\}$ *with the property that* $\forall z : \exp(f(z)) = z$ *(note that* $f(\exp(z)) = z$ *is clearly impossible as* exp *is many:one).*

Observation 20 *This statement, about actual functions* $\mathbf{C} \to \mathbf{C}$*, might seem to be in contradiction with the statement labelled "(1)+(2)" on page 256. In fact, it is not so much a contradiction as a statement that the world of functions* $\mathbf{C} \to \mathbf{C}$ *is not as neat as the algebraic world of the previous chapter.*

In particular, the common statements

$$\log z_1 + \log z_2 \overset{?}{=} \log z_1 z_2. \tag{8.9}$$

$$\log 1/z \overset{?}{=} -\log z. \tag{8.10}$$

are false[4] for (any interpretation of) the function $\log : \mathbf{C} \setminus \{0\} \to \mathbf{C} \setminus \{0\}$, even though they are true for the usual $\log : \mathbf{R}^+ \to \mathbf{R}$. On page 256 we proved a

[3] "No continuous argument", from which the others follow, is proved in [Pri03, §9.2].

[4] Consider $z_1 = z_2 = -1$ and $z = -1$ for counter-examples.

result which we said was "normally expressed as" (8.9). What we in fact proved was that $\phi := \log \eta_1 + \log \eta_2 - \log(\eta_1\eta_2)$ was a constant, in the sense that it differentiated to 0: in fact it is

$$\phi = \begin{cases} 2\pi i & \arg\eta_1 + \arg\eta_2 > \pi \\ 0 & -\pi < \arg\eta_1 + \arg\eta_2 \leq \pi \\ -2\pi i & \arg\eta_1 + \arg\eta_2 \leq -\pi \end{cases} . \tag{8.11}$$

Similarly the discrepancy in (8.10), i.e. $\psi := \log(1/z) + \log z$, is a differential constant whose value is

$$\psi = \begin{cases} 2\pi i & z = x + iy : x < 0 \wedge y = 0 \\ 0 & \text{otherwise} \end{cases} . \tag{8.12}$$

8.2.2 The Problem with Square Roots

These difficulties arise even with the square root function, and without needing to consider differential algebra.

Example 39 ([BCD$^+$02]) *Consider two apparently similar statements:*

$$\sqrt{1-z}\sqrt{1+z} \overset{?}{=} \sqrt{1-z^2} \tag{8.13}$$
$$\sqrt{z-1}\sqrt{z+1} \overset{?}{=} \sqrt{z^2-1}. \tag{8.14}$$

Naïvely, we might consider both to be true: square both sides and they reduce to $(1-z)(1+z) = 1-z^2$ and $(z-1)(z+1) = z^2-1$, both of which are true. But in fact, while (8.13) is true [BCD$^+$02, Lemma 2], (8.14) is false, as evaluation at $z = -2$ gives $\sqrt{-3}\sqrt{-1}\overset{?}{=}\sqrt{3}$,and in fact the left-hand side is $-\sqrt{3}$, rather than $\sqrt{3}$.

8.2.3 Possible Solutions

These facts have, of course, been known since the beginnings of complex analysis, and there are various approaches to the problem, discussed in [Dav10].

Multivalued Functions (taking values not on $\mathbf{C}$, but in the set of subsets of $\mathbf{C}$, i.e. $\mathbf{P}(\mathbf{C})$). Here we think of the logarithm of 1 as being, not 0, but any of $\{\ldots, -4\pi i, -2\pi i, 0, 2\pi i, 4\pi i, \ldots\}$. Our exemplar problem (8.9) is then treated as follows.

> The equation merely states that the sum of one of the (infinitely many) logarithms of z_1 and one of the (infinitely many) logarithms of z_2 can be found among the (infinitely many) logarithms of z_1z_2, and conversely every logarithm of z_1z_2 can be represented as a sum of this kind (with a suitable choice of $\log z_1$ and $\log z_2$).
>
> [Car58, pp. 259–260] (our notation)

Since log is to be the inverse of exp, we essentially transpose the graph and have $\left(\mathrm{graph}(\exp)^T, \mathbf{C}, \mathbf{P}(\mathbf{C})\right)_{\mathcal{B}}$ as the Bourbaki formulation of logarithm.

Notation 39 *We will use function names beginning with capitals, e.g.* Log *or* Sqrt *(so* $\mathrm{Sqrt}(4) = \{-2, 2\}$*) for such multivalued functions.*

(8.13) and (8.14), then both become true, when rewritten as

$$\mathrm{Sqrt}(1-z)\,\mathrm{Sqrt}(1+z) \overset{?}{=} \mathrm{Sqrt}(1-z^2) \tag{8.13$'$}$$

$$\mathrm{Sqrt}(z-1)\,\mathrm{Sqrt}(z+1) \overset{?}{=} \mathrm{Sqrt}(z^2-1). \tag{8.14$'$}$$

! Note, however, that what appears to be a special case of (8.9), viz.

$$2\log z \overset{?}{=} \log z^2, \tag{8.15}$$

is not true in this setting, since when $z = 1$, $\mathrm{Log}\, z^2 = \mathrm{Log}\, 1 = \{2k\pi i\}$, while $2\,\mathrm{Log}\, z = 2\,\mathrm{Log}\, 1 = \{4k\pi i\}$. The problem is, in fact, that $\mathrm{Log}\, z + \mathrm{Log}\, z \neq 2\,\mathrm{Log}\, z$.

Riemann Surfaces Instead of saying that $\log 1$ has multiple values, we say that there are multiple versions of '1', each of which has a well-defined logarithm. In terms of Example 37, the '1' that we reach after going round the circle once is different from the '1' we started from. In terms of Notation 3, we have $\left(\mathrm{graph}(\exp)^T, \mathcal{R}_{\log z}, \mathbf{C}\right)_{\mathcal{B}}$, where $\mathcal{R}_{\log z}$ signifies the Riemann surface corresponding to the function $\log z$, shown in Figure 8.1. The Riemann surface view is discussed in [BCD$^+$02, Section 2.4], which concludes

> Riemann surfaces are a beautiful conceptual scheme, but at the moment they are not computational schemes.

Note that the standard counterexample to (8.10), *viz.* $z = -1$, so that $\log(1/-1) = \pi i \neq -\log(-1)$ is now solved by the fact that $1/-1$ is a -1 *on a different branch*, and in (8.9) $z_1 + z_2$ is on the appropriate branch to make (8.9) valid.

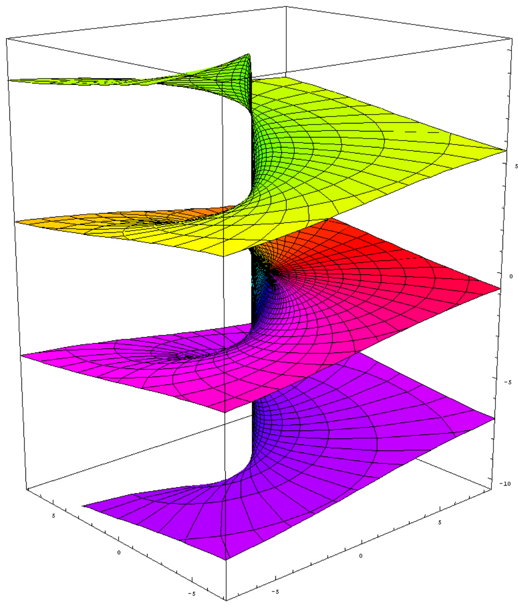
* Both the previous solutions have preserved continuity, and the identities, at the cost of not having a function $\mathbf{C} \to \mathbf{C}$.

Branches Of course, we could also consider subsets of $\mathbf{C}$ which do not contain the whole of the troublesome circle of Example 37. If we do that, then we can have a continuous version of, say, $\log z$ as in

> $\log z$ has a branch in any simply connected[5] open set which does not contain 0.
>
> [Car73, p. 61]

[5]Any two points can be connected bya path in the set, and any two such paths can be continuous transformed into each other without leaving the set.

Figure 8.1: A Riemann surface example: log



So any given branch would be $(G, D, I)_{\mathcal{B}}$, where D is a simply connected open set which does not contain 0, G is a graph obtained from one element of the graph (i.e. a pair $(z, \log(z))$ for some $z \in D$) by analytic continuation, and I is the relevant image set. While this approach preserves continuity, it has three drawbacks.

1. We now have as many definitions of log, say $\log_D$, as we have choices of D, and these do not necessarily agree, even if we insist that $1 \in D$ and $\log_D 1 = 0$. For example, if $D_1 = \mathbf{C} \setminus \{iy : y \in [0, \infty)\}$ and $D_2 = \mathbf{C} \setminus \{-iy : y \in [0, \infty)\}$, then $\log_{D_1}(-1) = -i\pi$ (since $\log(1 - \epsilon i) \approx -\epsilon i$ and we get from 1 to -1 *within* D_1 by going clockwise round the unit circle from 1 via $-i$, whose logarithm is $-i\pi/2$) but $\log_{D_2}(-1) = i\pi$, since now we go anticlockwise round the unit circle, from 1 to -1 via i.

2. $\log_D$ is not defined outside D.

3. Identities such as (8.9) are not valid, even if z_1, z_2 and $z_1 z_2$ are all in D: consider D_1 as above, with $z_1 = z_2 = -1$. This is inevitable: no assignment of a nonzero value to $\log(-1)$ can satisfy (8.10), for example.

This solution preserves continuity *within* D, at the cost of losing identities, and the uncertainty caused by the multiple options for D.

Branch Cuts Here we associate with each potentially multi-valued operator f the following.

- An "initial value" $(z_0, f(z_0))$.
- The "branch cut", which is a curve, or set of curves, B in $\mathbf{C}$ joining the singularities of f, such that $\mathbf{C} \setminus B$ is simply-connected. $f(z)$ for any $z \notin B$ is then obtained by analytic continuation from $(z_0, f(z_0))$ by analytic continuation along any path not meeting B.
- A rule for computing $f(z)$ on B, which is generally arranged to make $f(z)$ continuous with one side or the other of B.

A typical example would be log, where it is usual these days to have:

- Initial value $(1, 0)$;
- Branch cut $\{x + 0i : x \in (-\infty, 0]\}$;
- Rule for $\log x : x < 0$: $\log(x+0i) = \lim_{\epsilon \to 0^+} \log(x+\epsilon i)$ $(= i\pi + \log|x|)$.

We should note that there are many possible choices: the author was in fact initially taught

- Initial value $(1, 0)$;
- Branch cut $\{x + 0i : x \in [0, \infty)\}$;
- Rule for $\log x : x > 0$: $\log(x + 0i) = \lim_{\epsilon \to 0^+} \log(x + \epsilon i)$.

Despite the apparent arbitrariness, the world has tended to converge on the branch cuts as defined[6] in [AS64, Nat10].

8.2.4 Removable Branch Cuts

This term is introduced in [DF94, §4.2], presumably by analogy with "removable singularity". The example they give is illustrative.

Example 40 ([DF94, §4.2], our phrasing) *Consider $f(z) = g(z) - h(z)$, where $g(z) = \log(z + 1)$ and $h(z) = \log(z - 1)$. Then g has a branch cut along $(-\infty, -1]$ and h has one along $(-\infty, 1]$, and hence we would expect a branch cut for f along $(-\infty, 1]$. In fact, the contributions from g and h exactly cancel along $(-\infty, -1)$, and the actual branch cut is only $[-1, 1]$. We should note that this is only true of $g - h$, and $(1 + \epsilon)g - h$ genuinely has a branch cut along $(-\infty, 1]$ for all $\epsilon \neq 0$.*

This is, in fact, analogous to the "removable singularity" behaviour we saw at (8.1): $\frac{x^2-1-\epsilon}{x-1}$ has a genuine singularity at $x = 1$ for all $\epsilon \neq 0$, but when $\epsilon = 0$ the singularity is in fact removable.

[6]The careful reader should note that one should use printing 9 or later of [AS64]: the branch cut for arccot moved!

8.3 Fundamental Theorem of Calculus Revisited

In the previous chapter we reduced the Fundamental Theorem of Calculus to the status of Notation 32, saying that integration is the inverse of differentiation. From the algebraic point of view, that is correct. From the analytic point of view, where the following definitions hold, there is indeed something to prove.

Definition 109 (Analysis) *Given a function $f : \mathbf{R} \to \mathbf{R}$, we define, where the right-hand sides exist,*

$$\begin{array}{rcl} f'(x) & = & \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ F(x) = \int_a^x f(x)\mathrm{d}x & = & \lim_{|\Delta|\to 0} S^{\Delta}(f) = \lim_{|\Delta|\to 0} S_{\Delta}(f) \\ & & \text{provided both limits exist and are equal} \end{array} \tag{8.16}$$

where $\Delta = [x_0 = a < x_1 < \cdots < x_n = x]$ is a dissection of $[a, x]$, $|\Delta| = \max_i(x_i - x_{i-1})$,and $S^{\Delta}(f) = \sum_{i=1}^n (x_i - x_{i-1}) \max_{y\in[x_{i-1},x_i]} f(y)$ and $S_{\Delta}(f) = \sum_{i=1}^n (x_i - x_{i-1}) \min_{y\in[x_{i-1},x_i]} f(y)$ are the upper and lower approximations of "the area under the curve" of f.

Theorem 51 (Fundamental Theorem of Calculus [Apo67, §5.3]) *Let f and F be functions defined on a closed interval $[a, b]$ such that $F' = f$. If f is Riemann-integrable on $[a, b]$, then*

$$\int_a^b f(x)\mathrm{d}x = F(b) - F(a).$$

Though this is the classical statement, we must emphasise that $F' = f$ must hold *throughout* $[a, b]$, and therefore F is differentiable, hence continuous, thoughout this interval.

8.4 Constants Revisited

In the previous chapter we defined (Definition 102) a constant as being an element whose derivative was zero. How well does this compare with our usual intuition, which is of a *constant function*, i.e. one whose value is independent of the argument?

It is a truism of calculus that a constant function has to have derivative zero, and a theorem of analysis that a function whose derivative is *defined and zero* everywhere is indeed constant, but that italicised phrase is important. The classic example is the *Heaviside function*:

$$H(x) = \begin{cases} 0 & x \le 0 \\ 1 & x > 0 \end{cases}$$

which is clearly not constant, but whose derivative is zero *except at* $x = 0$, where it is undefined.

8.4.1 Constants can be useful

We can make constructive use of the concept of a differential constant, however. Mathematica, for example, writes [Lic11] $\sqrt{x^2}$ in the form $x\,\mathrm{sign}(x)$, and then treats $\mathrm{sign}(x)$ as a differential constant internally, replacing it by $\frac{1}{x}\sqrt{x^2}$ at the end.

8.4.2 Constants are often troubling

One sometimes sees[7] (but stated as an equality, rather than with our $\overset{?}{=}$)

$$\arctan(x)+\arctan(y)\overset{?}{=}\arctan\left(\frac{x+y}{1-xy}\right). \tag{8.17}$$

If we let

$$C=\arctan(x)+\arctan(y)-\arctan\left(\frac{x+y}{1-xy}\right), \tag{8.18}$$

then

$$\frac{\partial C}{\partial x}=\frac{1}{1+x^2}-\frac{\frac{(1-xy)+y(x+y)}{(1-xy)^2}}{1+\left(\frac{x+y}{1-xy}\right)^2}=\frac{1}{1+x^2}-\frac{(1-xy)+y(x+y)}{(1-xy)^2+(x+y)^2}=0, \tag{8.19}$$

and similarly $\frac{\partial C}{\partial y}=0$, so C would seem to be a constant. Differentially, it is, but in fact

$$C(x,y)=\left\{\begin{array}{ll}-\pi & xy>1; x<0\\ 0 & <1\\ \pi & xy>1; x>0\end{array}\right. : \tag{8.20}$$

see Figure 8.2.

8.5 Integrating ‘real’ Functions

In the previous chapter we saw

$$\int\frac{1}{x^2+1}\mathrm{d}x=\frac{i}{2}\left(\ln\left(1-ix\right)-\ln\left(1+ix\right)\right), \qquad 7.28bis$$

and said that the reader might complain “I asked to integrate a *real* function, but the answer is coming back in terms of *complex* numbers”.

A calculus text would normally write

$$\int\frac{1}{x^2+1}\mathrm{d}x=\arctan(x), \qquad 7.28ter$$

[7]For an example, see `http://www.mathamazement.com/Lessons/Pre-Calculus/05_Analytic-Trigonometry/sum-and-difference-formulas.html`.

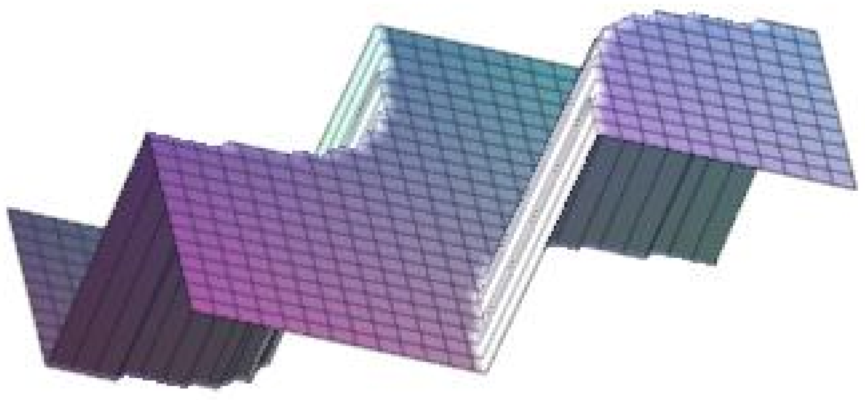


Figure 8.2: `plot3d(C, x =-4..4, y=-4..4)`: C from (8.20)

where arctan is the inverse[8] of the tan function, defined by

$$\tan(x) = \frac{\sin(x)}{\cos(x)} = \frac{1}{i}\frac{e^{ix} - e^{-ix}}{e^{ix} + e^{-ix}} = \frac{1}{i}\frac{\theta - \theta^{-1}}{\theta + \theta^{-1}} = \frac{1}{i}\frac{\theta^2 - 1}{\theta^2 + 1}, \tag{8.21}$$

where $\theta' = i\theta$ is an exponential in the sense of Definition 103.

If we write $\phi = \frac{1}{i}\frac{\theta^2-1}{\theta^2+1}$, we can deduce that $\phi' = \phi^2 + 1$, and treat this as a *definition* of "arctangents" in the same way that Definition 103 is a definition of "exponentials", by adding a clause

(d) θ (assumed to be nonzero) is a *tangent* over K, i.e. there is an η in K such that $\theta' = \eta'(\theta^2 + 1)$: θ is a tan of η.

Similarly, we could add

(e) θ is an *inverse tangent* over K, i.e. there is an η in K such that $\theta' = \eta'/(\eta^2 + 1)$: θ is an arctan of η.

We could then deduce an equivalent of statement "(1)+(2)" on page 256, that tan are (differential) inverses of each other.

Lemma 16 ((8.17) restated) *If θ is an* arctan *of α, ϕ is an* arctan *of β and ψ is an* arctan *of $\frac{\alpha+\beta}{1-\alpha\beta}$, then $\theta + \phi - \psi$ is a constant, in the sense of Definition 102.*

[8]It is common to write $\tan^{-1}(x)$ rather than $\arctan(x)$, but the author finds this open to confusion with the other abuse of notation that writes $\tan^2(x)$ for $(\tan(x))^2$. What would $\tan^{-2}(x)$ mean?

The proof is equivalent to (8.19):

$$\begin{aligned}(\theta+\phi-\psi)' &= \frac{\alpha'}{1+\alpha^2}+\frac{\beta'}{1+\beta^2}-\frac{\left(\frac{\alpha+\beta}{1-\alpha\beta}\right)'}{1+\left(\frac{\alpha+\beta}{1-\alpha\beta}\right)^2}\\ &= 0.\end{aligned}$$

However, the fact that this "constant" is given by (8.20) can trip us up: consider

$$\int \frac{1}{x^2+1}+\frac{2}{x^2+4}dx \overset{?}{=} \arctan\left(\frac{3x}{2-x^2}\right). \tag{8.22}$$

This is differentially valid, but apparently violates numerical evaluation

$$\int_1^2 \frac{1}{x^2+1}+\frac{2}{x^2+4}dx \overset{?}{=} \left[\arctan\left(\frac{3x}{2-x^2}\right)\right]_{x=1}^2 = -2\arctan(3)\approx -2.498, \tag{8.23}$$

and we have an integral of a positive function apparently being negative. In fact the correct answer is $\pi-2\arctan(3)\approx 0.644$.

One way of explaining this problem to some-one not familiar with the area is to point out that an integral should always be continuous, whereas $\arctan\left(\frac{3x}{2-x^2}\right)$ is not continuous in the interval $[1,2]$, since there's a discontinuity at $x=\sqrt{2}$ (see Figure 8.3:

$$\begin{aligned}\lim_{x\to\sqrt{2}^-}\arctan\left(\frac{3x}{2-x^2}\right) &= \arctan(+\infty)=\frac{\pi}{2} \text{ but}\\ \lim_{x\to\sqrt{2}^+}\arctan\left(\frac{3x}{2-x^2}\right) &= \arctan(-\infty)=-\frac{\pi}{2},\end{aligned}$$

and it is this discontinuity of π that needs to be accounted for.

8.6 Logarithms revisited

TO BE COMPLETED

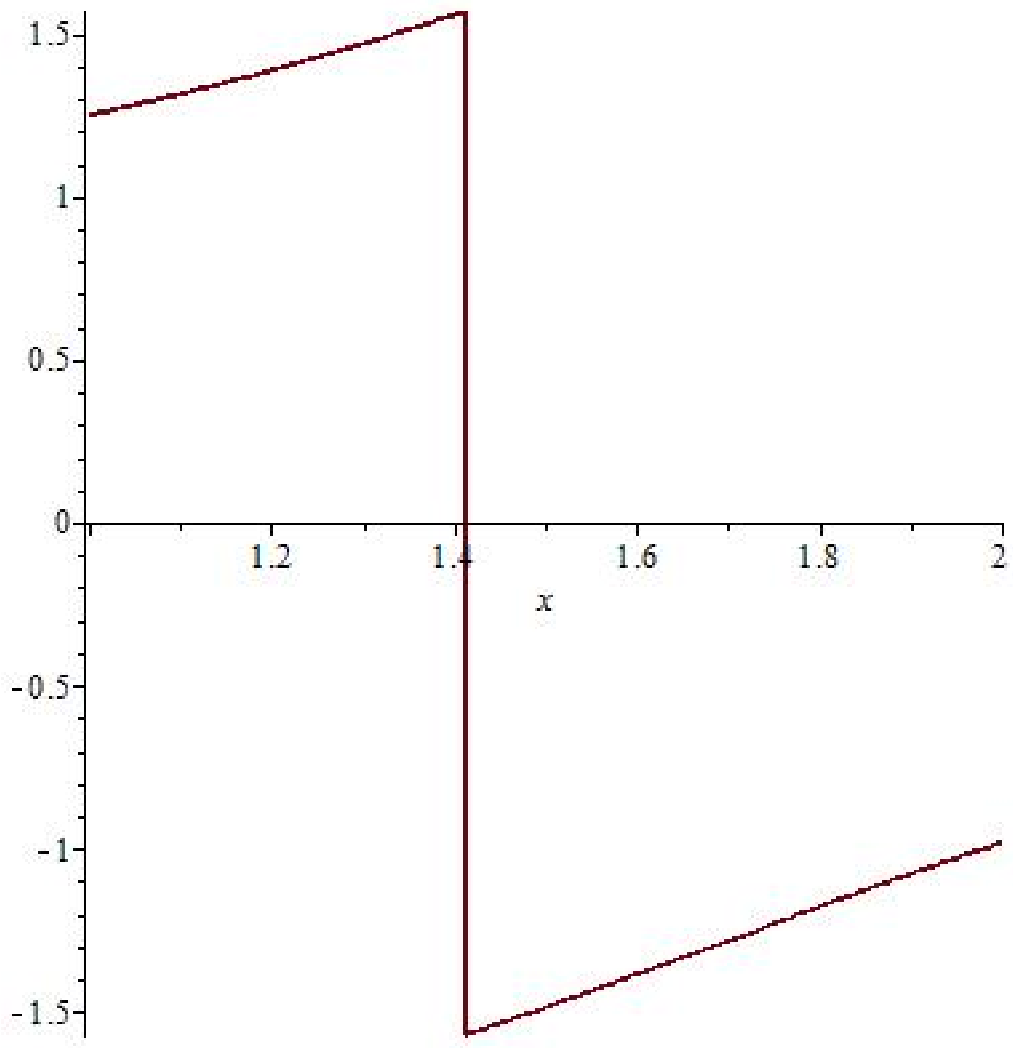
8.7 Other decision questions

We saw the theory of polynomial quantifier elimination in section 3.5.3. There, given a formula

$$Q_1x_1,\ldots,Q_kx_k\phi(x_1,\ldots,x_n)$$

(where each Q_i is either $\exists$ or $\forall$, and ϕ is a Boolean combination of equalities and inequalities between polynomials), this produces an equivalent formula $\psi(x_{k+1},\ldots,x_n)$. In particular, if $k=n$, we get either 'true' or 'false'. Expressions like $\sqrt{x^2-1}$ can be handled by writing them as y and adding

Figure 8.3: Graph of apparent integral in (8.22)



```
plot(arctan(3/(2-x^2)), x = 1 .. 2)
```

$\exists y : y^2 = x^2 - 1 \wedge$ in the appropriate place, for example. How can we add transcendental functions? This, of course, assumes that we know about the *numerical* values of the functions as well as their differential properties. It is usual to assume Schanuel's conjectures [Ax71], which roughly speaking state that there are no unexpected identities in exponentials and logarithms of complex numbers.

Conjecture 2 (Schanuel [Ax71][9]) *Given any n complex numbers $z_1, \ldots, z_n$ which are linearly independent over the rational numbers $\mathbf{Q}$, the extension field $\mathbf{Q}(z_1, \ldots, z_n, \exp(z_1), \ldots, \exp(z_n))$ has transcendence degree of at least n over $\mathbf{Q}$.*

See also [Wal14].

[AW00] take a different approach.

Definition 110 *A real or complex valued function f defined in some open domain of $\mathbf{R}$ or $\mathbf{C}$, respectively, is called strongly transcendental (with exceptional point ξ) if for all numbers x in the domain of f excluding ξ not both x and $f(x)$ are algebraic.*

It follows from Lindemann's Theorem that exp is transcendental with exceptional point 0, and log is transcendental with exceptional point 1. Similarly sin, cos and tan and their inverses, with exceptional points 0 except for arccos, which has 1.

[MW12] first consider problems of the form

$$Q_1 x_1 \Phi(x_1, \mathrm{trans}(x_1)),$$

where trans is a strongly transcendental function with

$$\mathrm{trans}'(x) = \frac{a(x) + b(x)\,\mathrm{trans}(x)}{d(x)} : a, b, d \in \mathbf{Z}[x].$$

log, exp and arctan all satisfy these requirements. Here they produce an unconditional algorithm for reducing this to 'true' or 'false', where 'unconditional' means not relying on Conjecture 2 or equivalent hard problems in number theory, and it is this unconditionality that is the most surprising part of the work. This will, for example, decide

$$\forall x_1 (\exp(x_1)(1 - x_1) \le 1 \vee x_1 \ge 1,$$

and in fact does so in less than 1 second [Ach06].

They then take problems of the form

$$Q_1 x_1 Q_2 x_2 \ldots, Q_n x_n \phi(x_1, \mathrm{trans}(x_1), x_2, \ldots, x_n),$$

apply the Collins-style quantifier elimination to $Q_2 x_2 \ldots, Q_n x_n \phi(x_1, y, x_2, \ldots, x_n)$ to get $\Phi(x_1, y)$, and apply their previous method to $Q_1 x_1 \Phi(x, \mathrm{trans}(x_1))$. Again, the process is unconditional. They also show how various other problems, e.g.

$$\forall x_1 x_1 > 7 \Rightarrow \cosh(x_1) > x_1^3 - 4x_1,$$

can be transformed into this form, and decided (17 seconds).

While this is a significant step forward, we should note that we are limited to one occurrence of trans, and this has to correspond to the first quantifier. Also, this is purely a decision procedure, and will not do more general quantifier elimination.

8.8 Limits

While we can, and have, algebraicised differentiation and integration, the process of computing limits is more fundamentally analytic. The first serious exploration of limits was in Macsyma [Wan71b, Wan71a]. This process essentially added four "constants" to Macsyma's language:

`infinity` which is the infinity of the (one-point compactification of the) complex plane;

`inf` which is the positive one of the two-point compactification of the reals ("plus infinity");

`minf` which is the negative one of the two-point compactification of the reals ("minus infinity")

`ind` which is "indeterminate".

Computer algebra systems are in general schizophrenic about whether they are computing over the reals or the complexes, but we should note the confusing fact that the reals are a subset of the complexes, but the usual (two-point, $+\infty$ and $-\infty$) compactification of the reals is *not* a subset of the compactification of the complexes.

8.8.1 A Definite Integral

With the conventional choice of branch cuts, we can write the fractional part of x as

$$\mathrm{fra}(x) = x - \lfloor x \rfloor = \frac{1}{2} + \frac{i}{2\pi}\log(-\exp(-2\pi i x)). \tag{8.24}$$

If we ask Maple to `simplify` the above, nothing happens, but with the `symbolic` option (i.e. ignoring the branch cuts of log)[10], we just get x. We note that $\mathrm{fra}'(x) = 1$ since $\mathrm{fra}(x)$ and x "differ by a constant", that (differential) constant being $\lfloor x \rfloor$.

If we ask for

$$\int_{1/2}^{1} \mathrm{fra}(1/x), \mathrm{d}x = \int_{1/2}^{1} \left(\frac{1}{x} - 1\right) \mathrm{d}x = \ln 2 - \frac{1}{2} \approx 0.193, \tag{8.25}$$

[10]Sage's `full_simplify` apparently does the same.

many systems[11] will get this right. The fun comes when we try, say, $\int_{1/3}^{1} \mathrm{fra}(1/x), \mathrm{d}x$. Maple returns $\ln 3 - \frac{1}{3} \approx 0.765$, which is absurd for integrating a function whose value is between 0 and 1 on an interval of length 2/3. It gets

$$\int_{1/3}^{1/2} \mathrm{fra}(1/x), \mathrm{d}x = \int_{1/3}^{1/2} \left(\frac{1}{x} - 2\right) \mathrm{d}x = \ln 3 - \ln 2 - \frac{2}{6} \approx 0.072 \qquad (8.26)$$

correct, but has failed to spot the singularity in the middle of the integrand in

$$\int_{1/3}^{1} \mathrm{fra}(1/x), \mathrm{d}x = \int_{1/3}^{1} \left(\frac{1}{x} - \begin{cases} 2 & x < 1/2 \\ 1 & x \geq 1/2 \end{cases}\right) \mathrm{d}x = \ln 3 - \frac{1}{2} - \frac{2}{6} \approx 0.265. \quad (8.27)$$

If we split the integral by hand, as in $\int_{1/3}^{1/2} \mathrm{fra}(1/x), \mathrm{d}x + \int_{1/2}^{1} \mathrm{fra}(1/x), \mathrm{d}x$, we get the right answer, as we do when we use Maple's preferred notation, `frac(1/x)` rather than (8.24).

[11]Sage apparently returns $\log 2$ — `https://groups.google.com/forum/#!topic/sage-support/oUkERaidET0`.

Appendix A

Algebraic Background

We are quite often concerned with estimating the sizes of things, either to compute the running time or to be sure that we have exhausted all possibilities of failure. The following notations are used.

Notation 40 *Let*

$$f(x) = \sum_{i=0}^{n} a_i x^i = a_n \prod_{i=1}^{n} (x - \alpha_i)$$

be a polynomial of degree n *(i.e.* $a_n \neq 0$*). Define the following measures of the size of the polynomial* f*:*

$H(f)$ *(often written* $||f||_\infty$*), the height or 1-norm, is* $\max_{i=0}^{n} |a_i|$;

$||f||$ *(often written* $||f||_2$*), the 2-norm, is* $\sqrt{\sum_{i=0}^{n} |a_i|^2}$;

$L(f)$ *(often written* $||f||_1$*), the length or 1-norm, is* $\sum_{i=0}^{n} |a_i|$;

$M(f)$ *, the Mahler measure of* f*, is* $|a_n| \prod_{|\alpha_i|>1} |\alpha_i|$.

A.1 The resultant and friends

A.1.1 Resultant

It quite often happens that we have to consider whether two polynomials, which are usually relatively prime, can have a common factor in certain special cases. The basic algebraic tool for solving this problem is called the resultant. In this section we shall define this object and we shall give some properties of it. We take the case of two polynomials f and g in one variable x and with coefficients in a ring R.

We write $f = \sum_{i=0}^{n} a_i x^i$ and $g = \sum_{i=0}^{m} b_i x^i$.

Definition 111 *The* Sylvester matrix *of f and g is the matrix*

$$\mathrm{Syl}(f,g) = \begin{pmatrix} a_n & a_{n-1} & \dots & a_1 & a_0 & 0 & 0 & \dots & 0 \\ 0 & a_n & a_{n-1} & \dots & a_1 & a_0 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \dots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & a_n & a_{n-1} & \dots & a_1 & a_0 & 0 \\ 0 & \dots & 0 & 0 & a_n & a_{n-1} & \dots & a_1 & a_0 \\ b_m & b_{m-1} & \dots & b_1 & b_0 & 0 & 0 & \dots & 0 \\ 0 & b_m & b_{m-1} & \dots & b_1 & b_0 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \dots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & b_m & b_{m-1} & \dots & b_1 & b_0 & 0 \\ 0 & \dots & 0 & 0 & b_m & b_{m-1} & \dots & b_1 & b_0 \end{pmatrix}$$

where there are m lines constructed with the a_i, n lines constructed with the b_i.

Definition 112 *The* resultant *of f and g, written* $\mathrm{Res}(f,g)$, *or* $\mathrm{Res}_x(f,g)$ *if there is doubt about the variable, is the determinant of this matrix. The j-*th principal subresultant coefficient, $\mathrm{psc}_j(f,g)$ *is the determinant of the matrix obtained by deleting the last j rows of f coefficients, the last j rows of g coefficients and the last $2j$ columns, i.e. the resultant of the quotients of dividing f and g by x^j.*

Well-known properties of determinants imply that the resultant belongs to R, and that $\mathrm{Res}(f,g)$ and $\mathrm{Res}(g,f)$ are equal, to within a sign. We must note that, although the resultant is defined by a determinant, this is not the best way of calculating it. Because of the special structure of the Sylvester matrix, we can consider Euclid's algorithm as Gaussian elimination in this matrix (hence the connection betwen the resultant and the g.c.d.). One can also consider the sub-resultant method as an application of the Sylvester identity (theorem 15) to this elimination. It is not very difficult to adapt advanced methods (such as the method of sub-resultants described in section 2.3.2, or the modular[1] and p-adic methods described in chapters 4 and 5) to the calculation of the resultant. Collins [1971] and Loos [1982] discuss this problem. We now give a version of Euclid's algorithm for calculating the resultant. We denote by $lc(p)$ the leading coefficient of the polynomial $p(x)$,by $degree(p)$ its degree, and by $remainder(p,q)$ the remainder from the division of $p(x)$ by $q(x)$. We give the algorithm in a recursive form.

Algorithm 46 (resultant)
Input: f,g;
Output: $r = \mathrm{Res}(f,g)$.

```
$n := degree(f)$;
$m := degree(g)$;
**if** $n > m$ **then** $r := (-1)^{nm} resultant(g,f)$
```

[1]The modular method is described in section 4.5.1.

```
        else $a_n := lc(f)$;
            if $n = 0$ then $r := a_n^m$
                else $h := remainder(g, f)$;
                    if $h = 0$ then $r := 0$
                        else $p := degree(h)$;
                            $r := a_n^{m-p} resultant(f, h)$;
return $r$;
```

It is worth noting that this returns a partially factored form of the resultant, if the underlying arithmetic supports this (e.g. the factored representation of section 2.1.3).

We write $h = \sum_{i=0}^{p} c_i x^i$ and $c_i = 0$ for $i > p$. This algorithm does indeed give the resultant of f and g for, when $n \le m$ and $n \ne 0$, the polynomials $x^i g - x^i h$ (for $0 \le i < n$) are linear combinations of the $x^j f$ (for $0 \le j < m$), and therefore we are not changing the determinant of the Sylvester matrix of f and g by replacing b_i by c_i for $0 \le i < m$. Now this new matrix has the form $\begin{pmatrix} A & * \\ 0 & B \end{pmatrix}$ where A is a triangular matrix with determinant a_n^{m-p} and B is the Sylvester matrix of f and h. From this algorithm we immediately get

Proposition 78 $\operatorname{Res}(f, g) = 0$ *if and only if f and g have a factor in common.*

It is now easy to prove the following propositions:

Proposition 79 *If the α_i are the roots of f, then*

$$\operatorname{Res}(f, g) = a_n^m \prod_{i=1}^{n} g(\alpha_i).$$

Proposition 80 *If the β_i are the roots of g, then*

$$\operatorname{Res}(f, g) = (-1)^{mn} b_m^n \prod_{i=1}^{m} f(\beta_i).$$

Proposition 81 $\operatorname{Res}(f, g) = a_n^m b_m^n \prod_{i=1}^{n} \prod_{j=1}^{m} (\alpha_i - \beta_j)$.

Proof [Duv87]. We write the right hand sides of the three propositions as

$$\begin{aligned} R_2(f, g) &= a_n^m \prod_{i=1}^{n} g(\alpha_i), \\ R_3(f, g) &= (-1)^{mn} b_m^n \prod_{i=1}^{m} f(\beta_i), \\ R_4(f, g) &= a_n^m b_m^n \prod_{i=1}^{n} \prod_{j=1}^{m} (\alpha_i - \beta_j). \end{aligned}$$

It is clear that $R_2(f,g)$ and $R_3(f,g)$ are equal to $R_4(f,g)$. The three propositions are proved simultaneously, by induction on the integer $\min(n,m)$. If f and g are swapped, their resultant is multiplied by $(-1)^{nm}$, and gives $R_4(f,g) = (-1)^{nm}R_4(g,f)$, while $R_2(f,g) = (-1)^{nm}R_3(g,f)$. We can therefore suppose that $n \leq m$. Moreover $R_2(f,g)$ is equal to a_n^m when $n = 0$, as is the resultant of f and g, and $R_4(f,g)$ is zero when $n > 0$ and $h = 0$, as is the resultant. It only remains to consider the case when $m \geq n > 0$ and $h \neq 0$. But then $R_2(f,g) = a_n^{m-p}R_2(f,h)$ for $g(\alpha_i) = h(\alpha_i)$ for each root α_i of f, and the algorithm shows that $\operatorname{Res}(f,g) = a_n^{m-p}\operatorname{Res}(f,h)$, from which we get the desired result.

Corollary 21 $\operatorname{Res}(fg,h) = \operatorname{Res}(f,h)\operatorname{Res}(g,h)$.

Lemma 17 *If f and g are polynomials in y and other variables, of total degrees d_f and d_g, then $\operatorname{Res}_y(f,g)$ is a polynomial of total degree at most $d_f d_g$.*

A.1.2 Discriminants

Definition 113 *The* discriminant *of f, $\operatorname{Disc}(f)$ or $\operatorname{Disc}_x(f)$, is*

$$a_n^{2n-2}\prod_{i=1}^{n}\prod_{\substack{j=1\\ j\neq i}}^{n}(\alpha_i-\alpha_j).$$

Proposition 82 $\operatorname{Disc}(f) = 0$ *if, and only if, f has a repeated root, i.e. is not square-free.*

Proposition 83 $\operatorname{Res}(f,f') = a_n\operatorname{Disc}(f)$. *Moreover* $\operatorname{Disc}(f) \in R$.

Corollary 22 $\operatorname{Disc}(fg) = \operatorname{Disc}(f)\operatorname{Disc}(g)\operatorname{Res}(f,g)^2$.

Whichever way they are calculated, the resultants are often quite large. For example, if the a_i and b_i are integers, bounded by A and B respectively, the resultant is less than $(n+1)^{m/2}(m+1)^{n/2}A^mB^n$, but it is very often of this order of magnitude (see section A.2). Similarly, if the a_i and b_i are polynomials of degree α and β respectively, the degree of the resultant is bounded by $m\alpha+n\beta$. A case in which this swell often matters is the use of resultants to calculate primitive elements, which uses the following result.

Proposition 84 *If α is a root of $p(x) = 0$, and β is a root of $q(x,\alpha) = 0$, then β is a root of $\operatorname{Res}_y(y-p(x), q(x,y))$.*

A.1.3 Iterated Operations

Suppose we wish to eliminate y and z from the three polynomials $f_i(\mathbf{x},y,z)$, each with total degree d. We could compute

$$R_{123} := \operatorname{Res}_y(\operatorname{Res}_z(f_1,f_2),\operatorname{Res}_z(f_1,f_3)). \tag{A.1}$$

By Lemma 17, each inner resultant has degree at most d^2, so R_{123} has total degree at most d^4.

TO BE COMPLETED[BM09]

A.2 Useful Estimates

Estimates of the sizes of various things crop up throughout computer algebra. They can be essentially of three kinds.

- **Upper bounds:** $X \leq B$.
- **Lower bounds:** $X \geq B$.
- **Average sizes:** X "is typically" B. This may be a definitive average result (but even here we have to be careful what we are averaging over: the average number of factors of a polynomial is one, for example, but this does not mean we can ignore factorisation), or some heuristic "typically we find".

Estimates are used throughout complexity theory, of course. "Worst case" complexity is driven by upper bound results, while "average case" complexity is driven by average results. They are also used in algorithms: most algorithms of the 'Chinese Remainder' variety (section 4) rely on upper bounds to tell when they have used enough primes/evaluation points. In this case, a better upper bound generally translates into a faster algorithm *in practice*.

A.2.1 Matrices

How big is the determinant $|M|$ of an $n \times n$ matrix M?

Notation 41 *If* $\mathbf{v}$ *is a vector, then* $||\mathbf{v}||_2$ *(sometimes also written* $|v|$*) denotes the* Euclidean norm *of* $\mathbf{v}$, $\sqrt{\sum |v_i^2|}$. *If* f *is a polynomial,* $||f||_2$ *denotes the Euclidean norm of its vector of ceofficients.*

Proposition 85 *If* M *is an* $n \times n$ *matrix with entries* $\leq B$, $|M| \leq n! B^n$.

This is true because the determinant is the sum of $n!$ summands, each the product of n elements, therefore bounded by B^n.

This bound is frequently used in algorithm analysis, but we can do better.

Proposition 86 *[Hadamard bound* H_r*] If* M *is an* $n \times n$ *matrix whose rows are the vectors* $\mathbf{v}_i$, *then* $|M| \leq H_r = \prod ||\mathbf{v}_i||_2$, *which in turn is* $\leq n^{n/2} B^n$.

Corollary 23 *If* f *and* g *are polynomials of degrees* n *and* m *respectively, then* $\operatorname{Res}(f,g) \leq (m+n)^{(m+n)/2} ||f||_2^m ||g||_2^n$.

Corollary 24 *If* f *is a polynomial of degree* n, *then* $\operatorname{Disc}(f) \leq n^{n-1} ||f||_2^{2n-1}$.

In practice, especially if f is not monic, it is worth taking rather more care over this estimation.

Proposition 87 (Hadamard bound (columns)) *If M is an $n \times n$ matrix whose columns are the vectors $\mathbf{v}_i$, then $|M| \le H_c = \prod ||\mathbf{v}_i||_2 \le n^{n/2}B^n$.*

In practice, for general matrices one computes $\min(H_r, H_c)$. While there are clearly bad cases (e.g. matrices of determinant 0), the Hadamard bounds are "not too bad". As pointed out in [AM01], $\log(\min(H_r, H_c)/|M|)$ is a measure of the "wasted effort" in a modular algorithm for computing the determinant, and "on average" this is $O(n)$, with a variance of $O(\log n)$. It is worth noting that this is independent of the size of the entries.

Proposition 87 has a useful consequence.

Corollary 25 *If x is the solution to $M.x = a$, where M and a have integer entries bounded by B and A respectively, then the denominators of x are bounded by $\min(H_r, H_c)$ and the numerators of x are bounded by $n^{n/2}AB^{n-1}$.*

Proof. This follows from the linear algebra result that $x = \frac{1}{|M|}\mathrm{adj}(M).a$, where $\mathrm{adj}(M)$, the adjoint of M, is the matrix whose (i, j)th entry is the determinant of the matrix obtained by striking row i and column j from M. The ith entry of $\mathrm{adj}(M).a$ is then the determinant of the matrix obtained by replacing the ith column of M by a, and we can apply Proposition 87 to this matrix.

A.2.2 Coefficients of a polynomial

Here we are working implicitly with polynomials with complex coefficients, though the bounds will be most useful in the case of integer coefficients.

Proposition 88 *In terms of Notation 40,*

$$H(f) \le ||f|| \le L(f) \le (n + 1)H(f),$$

where the first inequality is strict unless f is a monomial, the second is strict unless all the a_i are equal or zero, and the third is strict unless all the a_i are equal.

Observation 21 *The last inequality could be replaced by $\le cH(f)$, where c is the number of nonzero monomials in f, but this seems not to be much exploited.*

Proposition 89 (Landau's Inequality [Lan05], [Mig89, §4.3]) .

$$M(f) \le ||f||$$

and the inequality is strict unless f is a monomial.

Corollary 26 $|a_{n-i}| \le \binom{n}{i}M(f) \le \binom{n}{i}||f||$.

The first part of this follows from the fact that a_{n-i} is $\pm a_n$ times a sum of $\binom{n}{i}$ products of roots, and products of roots are bounded by Proposition 89. For some applications, e.g. Theorem 36, we often bound $\binom{n}{i}$ by 2^n, but for others, such as Proposition 57, the more precise value is needed. 2^n might seem like overkill, but in fact, both in general and in the application to factoring [Mig81], 2 cannot be replaced by any smaller number.

It should be noted that the Landau–Mignotte bound is not the only way to bound the coefficients of a polynomial: [Abb09] gives four methods that depend on knowing the degree of the factor being searched for, and two others that will bound the *least* height of a factor. Depressingly, he gives a family of examples that shows that no bound is superior to any other, and indeed [Abb09, 3.3.5] it may be necessary to "mix-and-match" using different bounds for different coefficients.

These results can be generalised to polynomials in several variables [Mig89, Chapter 4.4]. The definition of $M(f)$ is more complicated.

Definition 114 *The* Mahler measure *of $f \in \mathbf{C}[x_1, \ldots, x_n]$ is defined inductively on n, using Notation 40 for $n = 1$ and more generally*

$$\log M(f) = \int_0^1 \log \bigl(M(f(e^{2\pi it}, x_2, \ldots, x_n))\bigr) dt.$$

Proposition 90 ([Mig89, Proposition 4.8]) *Let*

$$f(x_1, \ldots, x_n) = \sum_{i_1=0}^{d_1} \sum_{i_2=0}^{d_2} \cdots \sum_{i_n=0}^{d_n} a_{i_1,\ldots,i_n} x_1^{i_1} x_2^{i_2} \ldots x_n^{i_n}.$$

Then

$$|a_{i_1,\ldots,i_n}| \le \binom{d_1}{i_1}\binom{d_2}{i_2} \cdots \binom{d_n}{i_n} M(f).$$

Proposition 89 is still valid.

Corollary 27 *Hence*

$$|a_{i_1,\ldots,i_n}| \le \binom{d_1}{i_1}\binom{d_2}{i_2} \cdots \binom{d_n}{i_n} ||f||.$$

A.2.3 Roots of a polynomial

Several questions can be asked about the roots of a univariate polynomial. The most obvious ones are how large/small can they be, but one is often interested in how close they can be. These questions are often asked of the *real* roots (section 3.5.5), but we actually need to study all roots, real and complex. The distinction between real and complex roots is pretty fine, as shown in the following example.

Example 41 (Wilkinson Polynomial [Wil59]) *Let W_{20} have roots at -1 , -2 , ..., -20, so that $W_{20} = (x+1)(x+2)\dots(x+20) = x^{20}+210x^{19}+\dots+20!$. Consider now the polynomial $W_{20}(x)+2^{-23}x^{19}$. One might expect this to have twenty real roots close to the original ones, but in fact it has ten real roots, at approximately -1, -2, $\dots -7$, -8.007, -8.917 and -20.847, and five pairs of complex conjugate roots, $-10.095 \pm 0.6435i$, $-11.794 \pm 1.652i$, $-13.992 \pm 2.519i$, $-16.731 \pm 2.813i$ and $-19.502 \pm 1.940i$.*

The discriminant of W_{20} is 2.74×10^{275}, which would seem well away from zero. However, the largest coefficient of W_{20} is 1.38×10^{19}, and of W'_{20} is -3.86×10^{19}. The norms of W_{20} and W'_{20} are 2.27×10^{19} and 6.11×10^{19}, so corollary 24 gives a bound of 4.43×10^{779} for $\mathrm{Disc}(W_{20})$, and a direct application of corollary 23 gives 3.31×10^{763}. Hence the discriminant of W_{20} is "much smaller than it ought to be", and W_{20} is "nearly not square-free". Put another way, the Sylvester matrix for the discriminant is very illconditioned (this was in fact Wilkinson's original motivation for constructing W_{20}): the discrepancy between the actual determinant and corollary 23 is 489 decimal digits, whereas [AM01] would lead us to expect about 17.

Notation 42 *Let $f \in \mathbf{C}[x] = \sum_{i=0}^n a_i x^i$, and let the roots of f be $\alpha_1, \dots, \alpha_n$, and define*

$$\mathrm{rb}(f) = \max_{1 \le i \le n} |\alpha_i|,$$

$$\mathrm{sep}(f) = \min_{1 \le i < j \le n} |\alpha_i - \alpha_j|.$$

$\mathrm{sep}(f)$ *is zero if, and only if, f has a repeated factor.*

Proposition 91 [Cau29, p. 122] $\mathrm{rb}(f) \le 1 + \max(|a_i|)/|a_n|$.

Corollary 28 *If the polynomial f does not take the value 0 at $x = 0$, then every root of f has absolute value at least $|a_0|/(|a_0| + \max(|a_i|)$.*

Proposition 92 [Cau29, p. 123]

$$\mathrm{rb}(f) \le \max\left(\frac{n|a_{n-1}|}{a_n}, \sqrt{\frac{n|a_{n-2}|}{a_n}}, \dots, \sqrt[n-1]{\frac{n|a_1|}{a_n}}, \sqrt[n]{\frac{n|a_0|}{a_n}}\right).$$

Proposition 93 [Knu98, 4.6.2 exercise 20]

$$\mathrm{rb}(f) \le B = 2\max\left(\frac{|a_{n-1}|}{a_n}, \sqrt{\frac{|a_{n-2}|}{a_n}}, \dots, \sqrt[n-1]{\frac{|a_1|}{a_n}}, \sqrt[n]{\frac{|a_0|}{a_n}}\right).$$

Furthermore, there is at least one root of absolute value $\ge B/(2n)$.

Applied to W_{20}, these propositions give respectively 1.38×10^{19}, 4200 and 420. If we centre the roots, to be $-9.5,\dots,9.5$, the three propositions give respectively 2.17×10^{12}, 400 and 40 (and hence the roots of W_{20} are bounded by 2.17×10^{12}, 409.5 and 49.5). While the advantages of centrality are most prominent in the case of proposition 91, they are present for all of them. There are in fact improved bounds available in this ($a_{n-1}=0$) case [Mig00].

If instead we re-balance W_{20} so that the leading and trailing coefficients are both 1, by replacing x by $\sqrt[20]{20!}x$, then the bounds for this new polynomial are 6961419, 505.76 and 50.58 (and hence the roots of W_{20} are bounded by 838284.7, 4200 and 420).

Proposition 94 ([Gra37]) *If $p(x)=p_e(x^2)+xp_o(x^2)$, i.e. $p_e(x^2)$ is the even terms of $p(x)$, then the roots of $q(x)=p_e^2(x)-xp_o^2(x)$ are the squares of the roots of p.*[2]

Applying this process to W_{20}, then computing the three bounds, and square-rooting the results, gives us bounds of 3.07×10^{18}, 239.58 and 75.76. Repeating the process gives 2.48×10^{18}, 61.66 and 34.67. On the centred polynomial, we get 2.73×10^{12}, 117.05 and 37.01, and a second application gives 1.83×10^{18}, 30.57 and 17.19 (and hence root bounds for W_{20} as 1.83×10^{18}, 40.07 and 26.69). This last figure is reasonably close to the true value of 20 [DM90].

A.2.4 Root separation

The key result is the following.

Proposition 95 ([Mah64]) $\operatorname{sep}(f) > \sqrt{3|\operatorname{Disc}(f)|}n^{-(n+2)/2}|f|^{1-n}$.

We note that the bound is zero if, and only if, the discriminant is zero, as it should be, and this bound is unchanged if we multiply the polynomial by a constant. The bound for W_{20} is 7.27×10^{-245}, but for the centred variant it becomes 1.38×10^{-113}. Should we ever need a root separation bound in practice, centring the polynomial first is almost always a good idea. Similarly re-balancing changes the separation bound to 5.42×10^{-112}, which means 6.52×10^{-113} for the original polynomial.

These bounds seem very far away from reality (i.e. 1 for W_n), but in fact are almost optimal.

Proposition 96 ([ESY06, Theorem 3.6]) *Let $a\ge3$ be an L-bit integer, and $n\ge4$ be an even integer. Then $Q(x)=\left(x^n-2(ax-1)^2\right)\left(x^n-(ax-1)^2\right)$ has degree $2n$ and coefficients of size $O(L)$, but any sub-division method requires $\Omega(nl)$ subdivisions to isolate the roots, of which there are three in $(a^{-1}-h,a^{-1}+h)$ with $h=a^{-n/2-1}$.*

[2]For the history of the attribution to Graeffe, see [Hou59].

A.2.5 Developments

The monic polynomials of degree 7 with maximum root separation and all roots in the unit circle are:

six complex $x^7 - 1$, with discriminant -823543, norm $\sqrt{2}$ and root bounds

$$[2.0, 1.383087554, 2.0]$$

and root separation bound (proposition 95) $\frac{\sqrt{3}}{56} \approx 3.09 \times 10^{-2}$ (true value 0.868);

four complex $x^7 - x$, with discriminant $6^6 = 46656$, norm $\sqrt{2}$ and root bounds

$$[2.0, 1.383087554, 2.0]$$

and root separation bound (proposition 95) $\frac{27\sqrt{21}}{18607} \approx 7.361786385 \times 10^{-3}$ (true value 1);

two complex $x^7 - \frac{1}{4}x^5 - x^3 + \frac{1}{4}x$, with discriminant $\frac{-50625}{4096} \approx -12.36$, norm $\frac{1}{4}\sqrt{34} \approx 1.457737974$ and root bounds

$$[2, 1.626576562, 2.0]$$

and root separation bound (proposition 95) $\frac{1880\sqrt{21}}{82572791} \approx 9.99 \times 10^{-5}$ (true value 1/2);

all real $x^7 - \frac{14}{9}x^5 + \frac{49}{81}x^3 - \frac{4}{81}x$, with discriminant[3]

$$\frac{10485760000}{1853020188851841} \approx 5.66 \times 10^{-6}, \tag{A.2}$$

norm $\frac{17}{81}\sqrt{86} \approx 1.946314993$ and root bounds

$$[\frac{23}{9}, 3.299831645, 2.494438258]$$

and root separation bound (proposition 95) $\frac{83980800}{322544094744037 81}\sqrt{3}\sqrt{7} \approx 1.19 \times 10^{-8}$ (true value 1/3).

If there are $n+1$ equally-spaced real roots (call the spacing 1 for the time being), then the first root contributes $n!$ to $\prod_i \prod_{j \neq i}(\alpha_i - \alpha_j)$, the second root $1!(n-1)!$ and so on, so we have a total product of

$$\prod_{i=0}^{n} i!(n-i)! = \prod_{i=0}^{n} i!^2 = \left(\prod_{i=0}^{n} i!\right)^2. \tag{A.3}$$

This is sequence `A055209` [Slo07], which is the square of `A000178`.

[3]We note how the constraint that all the roots be real forces the discriminant to be small.

If we now assume that the roots are equally-spaced in $[-1,1]$, then the spacing is $2/n$, we need to correct equation (A.3) by dividing by $(n/2)^{n(n-1)}$: call the result C_n. C is initially greater than one, with $C_3 = \frac{65536}{59049} \approx 1.11$, but $C_4 = \frac{81}{1024}$, $C_5 = \frac{51298814505517056}{37252902984619140625} \approx 0.001377042066$, and C_6 as in (A.2).

While assuming equal spacing might seem natural, it does not, in fact, lead to the largest values of the discriminant. Consider polynomials with all real roots $\in [-1,1]$, so that we may assume the extreme roots are at ± 1.

degree 4 Equally spaced roots, at $\pm\frac{1}{3}$, give a discriminant of $\frac{65536}{59049} \approx 1.11$, whereas $\pm\frac{1}{\sqrt{5}}$ gives $\frac{4096}{3125} \approx 1.31$, the optimum. The norms are respectively $\frac{\sqrt{182}}{9} \approx 1.4999$ and $\frac{\sqrt{62}}{5} \approx 1.575$.

degree 5 Equally spaced roots, at $\pm\frac{1}{2}$ and 0, give a discriminant of $\frac{81}{1024} \approx 0.079$, whereas $\pm\sqrt{\frac{3}{7}}$ and 0 gives $\frac{12288}{16807} \approx 0.73$, the optimum. The norms are respectively $\frac{\sqrt{42}}{4} \approx 1.62$ and $\frac{\sqrt{158}}{7} \approx 1.796$.

degree 6 Equally spaced roots, at $\pm\frac{3}{5}$ and $\pm\frac{1}{5}$, give a discriminant of $\frac{51298814505517056}{37252902984619140625} \approx 0.00138$. Unconstrained solving for the maximum of the discriminant, using Maple's `Groebner,Solve`, starts becoming expensive, but if we assume symmetry, we are led to choose roots at $\frac{\pm\sqrt{147\pm 42\sqrt{7}}}{21}$, with a discriminant of $\frac{67108864}{16209796869} \approx 0.0041$. The norms are respectively $\frac{2\sqrt{305853}}{625} \approx 1.77$ and $\frac{2\sqrt{473}}{21} \approx 2.07$.

degree 7 Equally spaced roots, at $\pm\frac{2}{3}$, $\pm\frac{1}{3}$ and 0, give a discriminant of $\frac{209715200000}{56157896126 36313} \approx 5.66 \cdot 10^{-6}$. Again assuming symmetry, we are led to choose roots at $\frac{\pm\sqrt{495\pm 66\sqrt{15}}}{33}$ and 0, which gives $\frac{209715200000}{5615789612636313} \approx 3.73 \cdot 10^{-5}$. The norms are respectively $\frac{17\sqrt{86}}{81} \approx 1.95$ and $\frac{2\sqrt{1577}}{33} \approx 2.41$,

degree 8 Equally spaced roots, at $\pm\frac{5}{7}$, $\pm\frac{3}{7}$ and $\pm\frac{1}{7}$, give a discriminant of $\approx 5.37 \cdot 10^{-9}$. Assuming symmetry, we get roots at $\pm \approx 0.87$, $\pm \approx 0.59$ and $\pm \approx 0.21$, with a discriminant of $\approx 9.65 \cdot 10^{-8}$. The norms are respectively ≈ 2.15 and $\frac{\sqrt{2}\sqrt{727171}}{429} \approx 2.81$.

degree 9 Equally spaced roots, at $\pm\frac{3}{4}$, $\pm\frac{1}{2}$, $\pm\frac{1}{4}$ and 0, give a discriminant of $1.15 \cdot 10^{-12}$. Assuming symmetry, we get roots at ± 0.8998, ± 0.677, ± 0.363 and zero, with a discriminant of $\approx 7.03 \cdot 10^{-11}$. The norms are respectively $\frac{\sqrt{5969546}}{1024} \approx 2.39$ and ≈ 3.296.

If we now consider the case with two complex root, which may as well be at $x = \pm i$, we have the following behaviour.

degree 4 The maximal polynomial is $x^4 - 1$, with discriminant -256 and norm $\sqrt{2}$. The bound is $\frac{\sqrt{6}}{216} \approx 0.153$.

degree 5 The maximal polynomial is $x^5 - x$, with discriminant -256 and norm $\sqrt{2}$. The bound is $\frac{4\sqrt{15}}{625} \approx 0.0248$.

degree 6 Equally spaced roots, at $\frac{\pm 1}{3}$, gives a discriminant of -108.26 and a norm of $\frac{2\sqrt{41}}{9}$. The bound is $\frac{400\sqrt{132}}{1860867} \approx 2.38 \cdot 10^{-3}$. The maximal discriminant is attained with roots at $\frac{\pm 1}{\sqrt{3}}$, with discriminant $\frac{-4194304}{19683} \approx -213.09$ and norm $\frac{2\sqrt{5}}{3}$. The bound is $\frac{4\sqrt{5}}{3375} \approx 2.65 \cdot 10^{-3}$.

degree 7 Equally spaced roots, at $\frac{\pm 1}{2}$ and 0, gives a discriminant of ≈ -12.36 and norm of $\frac{\sqrt{34}}{4} \approx 1.46$. The bound is $\frac{1800\sqrt{21}}{82572791} \approx 9.99 \cdot 10^{-5}$. The maximal discriminant is attained with roots at $\sqrt[4]{\frac{3}{11}}$, with discriminant ≈ 40.8 and norm of $\frac{2\sqrt{77}}{11} \approx 1.596$. The bound is $\approx 1.06 \cdot 10^{-4}$.

A.3 Chinese Remainder Theorem

In this section we review the result of the title, which is key to the methods in section 4.2.4, and hence to much of computer algebra.

Theorem 52 (Chinese Remainder Theorem (coprime form)) *Two congruences*

$$X \equiv a \pmod{M} \tag{A.4}$$

and

$$X \equiv b \pmod{N}, \tag{A.5}$$

where M and N are relatively prime, are precisely equivalent to one congruence

$$X \equiv c \pmod{MN}. \tag{A.6}$$

By this we mean that, given any a, b, M and N, we can find such a c that satisfaction of (A.4) and (A.5) is precisely equivalent to satisfying (A.6). The converse direction, finding (A.4) and (A.5) given (A.6), is trivial: one takes a to be $c \pmod{M}$ and b to be $d \pmod{N}$.

Algorithm 47 (Chinese Remainder)
Input: *a, b, M and N (with* $\gcd(M,N)=1$*).*
Output: *c satisfying Theorem 52.*

```
Compute $\lambda$, $\mu$ such that $\lambda M + \mu N = 1$;
        #The process is analogous to Lemma 1 (page 63)
$c$:=$a + \lambda M(b-a)$;
```

Clearly $c \equiv a \pmod{M}$, so satisfying (A.6) means that (A.4) is satisfied. What about (A.5)?

$$\begin{aligned} c &= a + \lambda M(b-a) \\ &= a + (1-\mu N)(b-a) \\ &\equiv a + (b-a) \pmod{N} \end{aligned}$$

so, despite the apparent asymmetry of the construction, $c \equiv b \pmod{N}$ as well.

In fact, we need not restrict ourselves to X being an integer: X, a and b may as well be polynomials (but M and N are still integers).

Algorithm 48 (Chinese Remainder (Polynomial form))
Input: *Polynomials* $a = \sum_{i=0}^{n} a_i x^i$, $b = \sum_{i=0}^{n} b_i x^i$, *and integers* M *and* N *(with* $\gcd(M, N) = 1$*).*
Output: *A polynomial* $= \sum_{i=0}^{n} c_i x^i$ *satisfying Theorem 52.*

```
Compute $\lambda$, $\mu$ such that $\lambda M + \mu N = 1$;
          #The process is analogous to Lemma 1 (page 63)
for $i := 0$ to $n$ do
         $c_i := a_i + \lambda M(b_i - a_i)$;
```

In some applications, e.g. Section 4.1, we will want to apply Algorithm 47 repeatedly to many primes p_i known in advance, to deduce a value c modulo $N := \prod p_i$ from many values $a_i \pmod{p_i}$. While the obvious way to do this is "one prime at a time", applying Algorithm 47 to p_1 and p_2, then to p_1p_2 and p_3, and so on, the correct way to do this is "balanced combination", first combining the p_i in pairs, then the pairs to form quadruples and so on.

Proposition 97 *In this case, the dominant cost is that of the last step, and with classical arithmetic this is* $O((\log N)^2)$*.*

A.4 Chinese Remainder Theorem for Polynomials

The theory of the previous section has an obvious generalisation if we replace $\mathbf{Z}$ by $K[y]$ for a field K, and an equivalent application to sections 4.3–4.4.

Theorem 53 (Chinese Remainder Theorem for polynomials) *Two congruences*

$$X \equiv a \pmod{M} \tag{A.7}$$

and

$$X \equiv b \pmod{N}, \tag{A.8}$$

where M *and* N *are relatively prime polynomials in* $K[y]$*, are precisely equivalent to one congruence*

$$X \equiv c \pmod{MN}. \tag{A.9}$$

By this we mean that, given any a, b, M and N, we can find such a c that satisfaction of (A.7) and (A.8) is precisely equivalent to satisfying (A.9). The converse direction, finding (A.7) and (A.8) given (A.9), is trivial: one takes a to be $c \pmod{M}$ and b to be $d \pmod{N}$.

Algorithm 49 (Chinese Remainder for Polynomials)
Input: *a, b, M and $N \in K[y]$ (with* $\gcd(M,N)=1$*).*
Output: *c satisfying Theorem 53.*

```
Compute $\lambda$, $\mu$ such that $\lambda M + \mu N = 1$;
        #As in Lemma 1 (page 63). Note $\mu$ isn't needed in practice.
$c := a + \lambda M(b-a)$;
```

Clearly $c \equiv a \pmod M$, so satisfying (A.9) means that (A.7) is satisfied. What about (A.8)?

$$\begin{aligned} c &= a + \lambda M(b-a) \\ &= a + (1-\mu N)(b-a) \\ &\equiv a + (b-a) \pmod N \end{aligned}$$

so, despite the apparent asymmetry of the construction, $c \equiv b \pmod N$ as well.

As in proposition 97, the correct way to to combine values modulo many (say d) evaluations is "balanced combination", first combining the $x - v_i$ in pairs, then the pairs to form quadruples and so on.

Proposition 98 *In this case, the dominant cost is that of the last step, and with classical arithmetic this is $O(d^2)$.*

As in Algorithm 48, X, a and b may as well be polynomials in x whose coefficients are polynomials in y (but M and N are still in y only).

Algorithm 50 (Chinese Remainder (Multivariate))
Input: *Polynomials $a = \sum_{i=0}^n a_i x^i$, $b = \sum_{i=0}^n b_i x^i \in K[y][x]$, and M and $N \in K[y]$ (with* $\gcd(M,N)=1$*).*
Output: *A polynomial $= \sum_{i=0}^n c_i x^i$ satisfying Theorem 53.*

```
Compute $\lambda$, $\mu$ such that $\lambda M + \mu N = 1$;
        #As in Lemma 1 (page 63). Note $\mu$ isn't needed in practice.
for $i := 0$ to $n$ do
        $c_i := a_i + \lambda M(b_i - a_i)$;
```

It is even possible for x (and i) to represent numerous indeterminates, as we are basically just doing coefficient-by-coefficient reconstruction.

Observation 22 *We have explicitly considered $K[y]$ (e.g. $\mathbf{Q}[y]$), but in practice we will often wish to consider* $\mathbf{Z}[y]$. Even if all the initial values are in $\mathbf{Z}[x]$, λ and μ may not be, as in $M = (y-1)$ and $N = (y+1)$, when $\lambda = \frac{-1}{2}$ and $\mu = \frac{1}{2}$. This may not be an obstacle to such reconstruction from values (often called interpolation, by analogy with interpolation over $\mathbf{R}$). Interpolation over $\mathbf{Z}[y]$ may still be possible: consider reconstructing X with $X \equiv 1 \pmod M$ and $X \equiv -1 \pmod N$, which is $1 + \frac{-1}{2}M(-1-1) = y$. But interpolation over $\mathbf{Z}[y]$ is not *always* possible: consider reconstructing X with $X \equiv 1 \pmod M$ and $X \equiv 0 \pmod N$, which gives $\frac{1}{2}(y-1)$.

A.5 Vandermonde Systems

Definition 115 *The* Vandermonde matrix[4] *generated by* $k_1,\ldots,k_n$ *is*

$$V(k_1,\ldots,k_n)=\begin{pmatrix} 1 & k_1 & k_1^2 & \ldots & k_1^{n-1}\\ 1 & k_2 & k_2^2 & \ldots & k_2^{n-1}\\ \vdots & \vdots & \vdots & & \vdots\\ 1 & k_n & k_n^2 & \ldots & k_n^{n-1}\end{pmatrix}.$$

Notation 43 *In the context of* $V(k_1,\ldots,k_n)$*, let* $P(z)=\prod_{i=1}^{n}(z-k_i)$ *and* $P_j(z)=\prod_{\substack{i=1\\ i\neq j}}^{n}(z-k_i)$.

Proposition 99 *The inverse of a Vandermonde matrix has a particularly simple form:* $V(k_1,\ldots,k_n)^{-1}=(m_{i,j})$ *with*

$$m_{i,j}=\frac{\texttt{coeff}(P_j(z),z^i)}{\prod_{k\neq j}(k_j-k_k)}=\frac{\texttt{coeff}(P_j(z),z^i)}{P_j(k_j)}. \tag{A.10}$$

For example

$$V(k_1,\ldots,k_3)^{-1}=\begin{pmatrix} \frac{k_2k_3}{(-k_2+k_1)(-k_3+k_1)} & -\frac{k_1k_3}{(-k_2+k_1)(k_2-k_3)} & \frac{k_1k_2}{(k_2-k_3)(-k_3+k_1)}\\ -\frac{k_2+k_3}{(-k_2+k_1)(-k_3+k_1)} & \frac{k_1+k_3}{(-k_2+k_1)(k_2-k_3)} & -\frac{k_1+k_2}{(k_2-k_3)(-k_3+k_1)}\\ \frac{1}{(-k_2+k_1)(-k_3+k_1)} & -\frac{1}{(-k_2+k_1)(k_2-k_3)} & \frac{1}{(k_2-k_3)(-k_3+k_1)}\end{pmatrix}$$

Corollary 29 *If all the* k_i *are distinct, then* $V(k_1,\ldots,k_n)$ *is invertible.*

Equation (A.10) and the fact that the P_j can be rapidly computed form p suggest a rapid way of computing the elements of the inverse of a $n\times n$ Vandermonde matrix in $O(n^2)$ time. In fact, we can solve a system of Vandermonde linear equations in $O(n^2)$ time and $O(n)$ space.

Algorithm 51 (Vandermonde solver)
Input: *Vandermonde matrix* $V(k_1,\ldots,k_n)$*, right-hand side* $\mathbf{w}$.
Output: *Solution* $\mathbf{x}$ *to* $V(k_1,\ldots,k_n)\mathbf{x}=\mathbf{w}$

```
$\mathbf{x} := 0$
$P := \prod_{i=1}^{n}(z-k_i)$        $\#O(n^2)$
**for** $i := 1$ **to** $n$
    $Q := P/(z-k_i)$        $\#nO(n)$
    $D := P(k_i)$           $\#nO(n)$ by Horner's rule
    **for** $j := 1$ **to** $n$
        $x_j := x_j + w_i\frac{\texttt{coeff}(Q,z^{j-1})}{D}$
```

[4]This section is based on [Zip93, §13.1]. Our algorithm 51 is his `SolveVanderMonde`, and our algorithm 52 is the one at the top of his p. 214.

In section 4.4.2 we will want to solve a slightly different system. Algorithm 51 solves a system of the form

$$\begin{array}{ccccccccccc} x_1 & + & k_1x_2 & + & k_1^2x_3 & + & \cdots & + & k_1^{n-1}x_n & = & w_1 \\ x_1 & + & k_2x_2 & + & k_2^2x_3 & + & \cdots & + & k_2^{n-1}x_n & = & w_1 \\ & & & & & & & & & \vdots & \\ x_1 & + & k_nx_2 & + & k_n^2x_3 & + & \cdots & + & k_n^{n-1}x_n & = & w_1 \end{array} \qquad \text{(A.11)}$$

whereas we need to solve a system of the form

$$\begin{array}{ccccccccc} k_1x_1 & + & k_2x_2 & + & \cdots & + & k_nx_n & = & w_1 \\ k_1^2x_1 & + & k_2^2x_2 & + & \cdots & + & k_n^2x_n & = & w_2 \\ & & & & & & & \vdots & \\ k_1^nx_1 & + & k_2^nx_2 & + & \cdots & + & k_n^nx_n & = & w_n \end{array} \quad . \qquad \text{(A.12)}$$

By comparison with (A.11), we have transposed the matrix (which is not a problem, since the inverse of the transpose is the traspose of the inverse), and multiplied column i by an extra factor of k_i. From Corollary 29, we can deduce the criteria for this system to be soluble.

Corollary 30 *If all the k_i are distinct and non-zero, then the system (A.12) is soluble.*

The following variant of Algorithm 51 will solve the system.

Algorithm 52 (Vandermonde variant solver)
Input: *Vandermonde style data $(k_1, \ldots, k_n)$, right-hand side* $\mathbf{w}$.
Output: *Solution* $\mathbf{x}$ *to (A.12)*

```
$\mathbf{x} := 0$
$P := \prod_{i=1}^{n}(z - k_i)$        #$O(n^2)$
for $i := 1$ to $n$
    $Q := P/(z - k_i)$        #$nO(n)$
    $D := k_iP(k_i)$          #$nO(n)$ by Horner's rule
    for $j := 1$ to $n$
        $x_i := x_i + w_j\frac{\texttt{coeff}(Q, z^{j-1})}{D}$
```

A.6 More matrix theory

See, for the time being, then entry `Minor_(linear_algebra)` in Wikipedia until JHD has written this. **TO BE COMPLETED**

A.7 Algebraic Structures

In this section we gather together many of the definitions we have seen throughout the book in one table. The definitive references, as far as computer algebra (rather than abstract algebra) is concerned are [DT90, DGT91], which explain why we say "g.c.d. domain" rather than "unique factorisation domain": the argument is summarised in note 30 (page 62).

Table A.1: Algebraic Structures

Name (↓What's added↓)	Reference	Example(s)
Ring $(+, -, 0, *)$	Definition 8	2×2 matrices
↓commutative *↓		
Commutative ring		$\{0, 2, 4, 6, 8, 10\} \pmod{12}$
↓multiplicative identity↓		
(Comm.) ring with 1		$\{0, 1, 2, \ldots, 11\} \pmod{12} = \mathbf{Z}_{12}$
↓No zero divisors↓		
Integral Domain	Definition 11	$\{a + b\sqrt{-5}\}$ (Example 3)
↓gcd operation↓		
g.c.d. domain	Definition 32	$\mathbf{Z}[x, y]$
↓All ideals principal↓		
principal ideal domain	Definition 14	$\mathbf{Z}$; $\mathbf{Q}[x]$
↓division↓		
field	Definition 15	$\mathbf{Q}$; $\mathbf{Z}_p$
↓Roots of polynomials↓		(see Proposition 53)
Algebraically closed field	Definition 18	$\mathbf{C}$

Note that any integral domain can be extended into its field of fractions (Definition 16), and any field can be extended into its algebraic closure (Definition 19).

Appendix B

Excursus

This appendix includes various topics on computer algebra that do not seem to be well-treated in the literature.

B.1 The Budan–Fourier Theorem

Definition 116 *Given a sequence A of non-zero numbers $a_0, \ldots, a_n$, the* number of sign variations *of A, witten $V(A)$, is the number of times two consecutive elements have different signs, i.e. $a_i a_{i+1} < 0$. If A does contain zeros, we erase them before doing this computation, or equivalent we count the number of (i, j) with $a_i a_j < 0$ and all intermediate $a_{i+1}, \ldots, a_{j-1} = 0$. If f is the polynomial $a_n x^n + \cdots a_0$, we write $V(f)$ rather than $V(A)$ where A is the sequence of coefficients of f.*

Proposition 100 $V(f) = V(f(0), f'(0), \ldots, f^{(n)}(0)$ *for a polynomial f of degree n.*

The reader should note that the definition of variation is numerically unstable. $V(1) = 0$, and therefore (by the erasure rule) $V(1, 0) = 0$. For positive ϵ, $V(1, \epsilon) = 0$, but $V(1, -\epsilon) = 1$. This is related to the fact that $x + \epsilon$ has no positive real roots, but $x - \epsilon$ has one, as seen in the following result.

Theorem 54 (Descartes' rule of signs [CA76]) *(the number of roots of f in $(0, \infty)$ is less than or equal to, by an even number, $V(f)$.*

Corollary 31 *The number of roots of f in (a, ∞) is less than or equal to, by an even number, $V(f(x - a))$.*

Corollary 32 *The number of roots of f in (a, ∞) is less than or equal to, by an even number, $V((f(a), f'(a), \ldots, f^{(n)}(a))$.*

For dense f, there is not a great deal to choose between these two formulations, but, since the derivatives of a sparse polynomial are sparse but its translate is not, corollary 32 is greatly to be preferred to corollary 31 in the sparse case.

We can also deduce some results about the number of roots of sparse polynomials. If f has n non-zero terms, $V(f) \leq n-1$. We note that $V(ax^k)=0$, and this polynomial indeed has no roots in $(0,\infty)$.

Corollary 33 *A polynomial in x with n terms, not divisible by x, has at most $2(n-1)$ roots in $\mathbf{R}$. If it is divisible by x, then the answer is at most $2n-1$.*

The example of x^3-x, which has three real roots $(\pm 1, 0)$ shows that the special case is necessary.

For the sake of simplicity, we will consider only square-free polynomials in the rest of this section: the results generalise fairly easily to non square-free ones.

Let us fix f, and consider $V(y) := V(f(x-y))$ and $N(y) := |\{x > y : f(x) = 0\}|$ as functions of y. For large enough y, both are zero. As y decreases, $N(y)$ increases monotonically, by 1 at each root of f. In fact, the same monotonic behaviour is true of $V(y)$, increasing by 1 at roots of f and by 2 at certain other points. This allows us to compute the number of roots in an interval, a result known as the Budan–Fourier Theorem[1].

Corollary 34 (Budin–Fourier Theorem) *The number of roots of f in (a,b) is less than or equal to, by an even number, $V(f(x-a)) - V(f(x-b))$.*

Corollary 35 (Budin–Fourier Theorem [Hur12]) *The number of roots of f in (a,b) is less than or equal to, by an even number, $V((f(a), f'(a), \ldots, f^{(n)}(a)) - V((f(b), f'(b), \ldots, f^{(n)}(b))$.*

For the same reasons as above, corollary 35 is to be preferred in the case of sparse polynomials.

An almost complete generalisation of Corollary 33, in the sense that the bounds depend on the number of monomials rather than the degrees, to sparse polynomials in n variables is given in [BHNS15, §2].

B.2 Equality of factored polynomials

This section treats the following problem.

Problem 8 *Given two polynomials in factored form (section 2.1.3), are they equal? More precisely, if*

$$f = \prod_{i=1}^{n} f_i^{a_i} \qquad g = \prod_{j=1}^{m} g_j^{b_j},$$

with the f_i and g_j square-free and relatively prime, i.e.:w $\gcd(f_i, f_{i'}) = 1$, $\gcd(g_j, g_{j'}) = 1$), is $f \overset{?}{=} g$.

[1]See [BdB22, Fou31]. The question of precedence was hotly disputed at the time: see [Akr82] and `http://www-history.mcs.st-andrews.ac.uk/Biographies/Budan_de_Boislaurent.html`.

The obvious solution is to expand f and g, and check that the expanded forms (which *are* canonical) are equal. Can we do better?

One important preliminary remark is that the square-free representation of a polynomial is unique. This leads to the following result.

Proposition 101 *If $f = g$, then every a_i has to be a b_j, and vice versa. For each such occurring value k, we must verify*

$$f_k = \prod_{\substack{i=1\\a_i=k}}^{n} f_i \overset{?}{=} g_k = \prod_{\substack{j=1\\b_j=k}}^{m} g_j. \tag{B.1}$$

In particular, f_k and g_k must have the same degree, i.e.

$$\sum_{\substack{i=1\\a_i=k}}^{n} \deg(f_i) = \sum_{\substack{j=1\\b_j=k}}^{m} \deg(g_j). \tag{B.2}$$

Again, we could check $f_k \overset{?}{=} g_k$ by expansion, but there is a better way.

Example 42 *Let $f = x^{2^l} - 1$ and $g = (x-1)(x+1)(x^2+1)\cdots(x^{2^{l-1}}+1)$, where both are square-free, so proposition 101 does not help. f is already expanded, but expansion of g can give rise to $x^{2^l-1} + x^{2^l-2} + \cdots + x + 1$, which has length $O(2^l)$, whereas g has length $O(l)$.*

From now on we will assume that we are working over a domain that includes the integers.

Lemma 18 *If f and g have degree at most N, and agree at $N+1$ different values, then $f = g$.*

Proof. $f - g$ is a polynomial of degree at most N, but has $N+1$ zeros, which contradicts proposition 5 if it is non-zero.

Hence it suffices to evaluate f and g at $N+1$ points x_i and check that the values agree. It is not necessary to construct any polynomial, and the integers involved are bounded (if we choose $|x_i| < N$) by BN^N, where B is a function of the coefficients of the f_i, g_j. Furthermore, we can evaluate at all these points in parallel. However, it does seem that we need to evaluate at $N+1$ points, or very nearly so, even if f and g are very small.

Open Problem 30 (Crossings of factored polynomials) *Produce some non-trivial bounds on the maximum number of zeros of $f - g$, where f and g have small factored representations. See [RR90].*

The best we can say is as follows. Suppose, in the notation of (B.1), each f_i has k_i non-zero terms, and g_j has l_j non-zero terms, and no f_i or g_j is x (if either was, then trivially $f \neq g$, since a common factor of x would have been detected). Then, by Corollary 33, $f - g$, if it is not identically zero, has at most $2\left(\sum_{\substack{i=1\\a_i=k}}^{n} k_i + \sum_{\substack{j=1\\b_j=k}}^{m} l_j - 1\right)$ roots in $\mathbf{R}$, and hence, if it is zero when evaluated at more than this number of integers, is identically zero. The factor of 2 can be dropped if we use only positive evaluation points, and rely on Theorem 54.

B.3 Karatsuba's method

This method was originally introduced in [KO63] for multiplying large integers: however, it is easier to explain in the (dense) polynomial context, where issues of carrying do not arise. Consider the product of two linear polynomials

$$(aX+b)(cX+d) = acX^2 + (ad+bc)X + bd. \tag{B.3}$$

This method so patently requires four multiplications of the coefficients that the question of its optimality was never posed. However, [KO63] rewrote it as follows:

$$(aX+b)(cX+d) = acX^2 + \left[(a+b)(c+d) - ac - bd\right]X + bd, \tag{B.4}$$

which only requires three *distinct* multiplications, ac and bd each being used twice. However, it requires four coefficients additions rather than one, so one might question the practical utility of it. For future reference, we will also express[2] equation (B.4) as

$$(aX+b)(cX+d) = ac(X^2 - X) + (a+b)(c+d)X + bd(1-X), \tag{B.5}$$

which makes the three coefficient multiplications explicit.

However, it can be cascaded. Consider a product of two polynomials of degree three (four terms):

$$(a_3Y^3 + a_2Y^2 + a_1Y + a_0)(b_3Y^3 + b_2Y^2 + b_1Y + b_0). \tag{B.6}$$

If we write $X = Y^2$, $a = a_3Y + a_2$ etc., this product looks like the left-hand side of equation (B.4), and so can be computed with three multiplications of linear polynomials, each of which can be done in three multiplications of coefficients, thus making nine such multiplications in all, rather than the classic method's 16.

If the multiplicands have 2^k terms, then this method requires $3^k = \left(2^k\right)^{\log_2 3}$ multiplications rather than the classic $\left(2^k\right)^2$. For arbitrary numbers n of terms, not necessarily a power of two, the cost of "rounding up" to a power of two is subsumed in the O notation, and we see a cost of $O(n^{\log_2 3})$ rather than the classic $O(n^2)$ coefficient multiplications. We note that $\log_2 3 \approx 1.585$, and the number of extra coefficient additions required is also $O(n^{\log_2 3})$, being three extra additions at each step. While the "rounding up" is not important in O-theory, it matters in practice, and [Mon05] shows various other formulae, e.g.

$$\begin{gathered}(aX^2+bX+c)(dX^2+eX+f) = \\ cf(1-X) + be(-X+2X^2-X^3) + ad(-X^3+X^4) + (b+c)(e+f)(X-X^2) \\ +(a+b)(d+e)(X^3-X^2) + (a+b+c)(d+e+f)X^2,\end{gathered}$$

[2]This formulation is due to [Mon05].

requiring six coefficient multiplications rather than the obvious nine, or eight if we write it as

$$\begin{gathered}\left(aX^2+(bX+c)\right)\left(dX^2+(eX+f)\right)= \\ adX^4+aX^2(eX+f)+dX^2(bX+c)+(bX+c)(eX+f)\end{gathered}$$

and use (B.4) on the last summand (asymptotics would predict $3^{\log_2 3}\approx 5.7$, so we are much nearer the asymptoic behaviour). Cascading this formula rather than (B.4) gives $O(n^{\log_3 6})$, which as $\log_3 6\approx 1.63$ is not as favorable asymptotically. His most impressive formula describes the product of two six-term polynomials in 17 coefficient multiplications, and $\log_6 17\approx 1.581$, a slight improvement. We refer the reader to the table in [Mon05], which shows that his armoury of formulae can get us very close to the asymptotic costs.

Many of these formulae can be explained as instances of the general formula [Sco15, (1)]

$$\begin{gathered}\left(\sum_{i=0}^{n-1}x_ib^i\right)\left(\sum_{i=0}^{n-1}y_ib^i\right)= \\ \left(\sum_{i=i}^{n-1}\sum_{j=0}^{i-1}(x_i-x_j)(y_j-y_i)b^{i+j}\right)+\left(\sum_{i=0}^{n-1}b^i\right)\left(\sum_{j=0}^{n-1}x_jy_jb^j\right).\end{gathered}$$

Theorem 55 *We can multiply two (dense) polynomials with m and n terms respectively in $O\left(\max(m,n)\min(m,n)^{(\log_2 3)-1}\right)$ coefficient operations.*

Let the polynomials be $f=a_{m-1}Y^{m-1}+\cdots$ and $f=b_{n-1}Y^{n-1}+\cdots$ Without loss of generality, we may assume $m\geq n$ (so we have to prove $O(mn^{\log_2 3-1})$, and write $k=\lceil\frac{m}{n}\rceil$. We can then divide f into blocks with n terms (possibly fewer for the most significant one) each: $f=\sum_{i=1}^{k-1}f_iY^{in}$. Then

$$fg=\sum_{i=1}^{k-1}(f_ig)\,Y^{in}.$$

Each f_ig can be computed in time $O(n^{\log_2 3})$, and the addition merely takes time $O(m-n)$ (since there is one addition to be performed for each power of Y where overlap occurs). Hence the total time is $O(kn^{\log_2 3})$, and the constant factor implied by the O-notation allows us to replace $k=\lceil\frac{m}{n}\rceil$ by $\frac{m}{n}$, which gives the desired result.

B.3.1 Karatsuba's method in practice

When it comes to multiplying numbers of n digits (in whatever base we are actually using, generally a power of 2), the received wisdom is to use $O(n^2)$ methods for small n (say $n<16$), Karatsuba-style methods for intermediate n (say $16\leq n<4096$ and Fast Fourier Transform methods (section B.3.4 or

[SS71]) for larger n. However, it is possible for the Fast Fourier Transform to require too much memory, and [Tak10] was forced to use these methods on numbers one quarter the required size, and then nested Karatsuba to combine the results.

B.3.2 Karatsuba's method and sparse polynomials

The use of the Karatsuba formula and its variants for sparse polynomials is less obvious. One preliminary remark is in order: in the dense case we split the multiplicands in equation (B.6) or its equivalent in two (the same point for each), and we were multiplying components of half the size. This is no longer the case for sparse polynomials, e.g. every splitting point of

$$(a_7x^7 + a_6x^6 + a_5x^5 + a_0)(b^7 + b_2x^2 + b_1x + b_0) \tag{B.7}$$

gives a 3–1 split of one or the other: indeed possibly both, as when we use x^4 as the splitting point.

Worse, in equation (B.5), the component multiplications were on 'smaller' polynomials, whereas, measured in number of terms, this is no longer the case. If we split equation (B.7) at x^4, the sub-problem corresponding to $(a+b)*(c+d)$ in equation (B.5) is

$$(a_7x^3 + a_6x^2 + a_5x^1 + a_0)(b^3 + b_2x^2 + b_1x + b_0)$$

which has as many terms as the original (B.7). This difficulty led [Fat03] to conclude that Karatsuba-based methods did not apply to sparse polynomials. It is clear that they cannot *always* apply, since the product of a polynomial with m terms by one with n terms may have mn terms, but it is probaby the difficulty of deciding *when* they apply that has led system designers to shun them.

B.3.3 Karatsuba's method and multivariate polynomials

TO BE COMPLETED

B.3.4 Faster still

It is possible to do still better than $3^k = \left(2^k\right)^{\log_2 3}$ for multiplying dense polynomials f and g with $n = 2^k$ coefficients. Let $\alpha_0, \ldots, \alpha_{2n-1}$ be distinct values. If $h = fg$ is the desired product, then $h(\alpha_i) = f(\alpha_i)g(\alpha_i)$. Hence if we write $\widetilde{f}$ for the vector $f(\alpha_0), f(\alpha_1), \ldots, f(\alpha_{2n-1})$, then $\widetilde{h} = \widetilde{f} * \widetilde{g}$, where $*$ denoted element-by-element multiplication of vectors, an $O(n)$ operation on vectors of length $2n$. This gives rise to three questions.

1. What should the α_i be?
2. How do we efficiently compute $\widetilde{f}$ from f (and $\widetilde{g}$ from g)?

3. How do we efficiently compute h from $\widetilde{h}$?

The answer to the first question is that the α_j should be the $2n$-th roots of unity in order, i.e. over $\mathbf{C}$ we would have $\alpha_i = e^{2\pi ij/2n}$. In practice, though, we work over a finite field in which these roots of unity exist. If we do this, then the answers to the next two questions are given by the Fast Fourier Transform (FFT) [SS71], and we can compute $\widetilde{f}$ from f (or h from $\widetilde{h}$) in time $O(k2^k) = O(n \log n)$ (note that we are still assuming n is a power of 2). Putting these together gives

$$\underbrace{O(n\log n)}_{\tilde{f}\text{ from }f} + \underbrace{O(n\log n)}_{\tilde{g}\text{ from }g} + \underbrace{O(n)}_{\tilde{h}\text{ from }\tilde{f},\tilde{g}} + \underbrace{O(n\log n)}_{h\text{ from }\tilde{h}} = O(n\log n). \tag{B.8}$$

While this method works fine for multiplying polynomials over a finite field K with an appropriate root of unity, or over $\mathbf{C}$, it requires substantial attention to detail to make it work over the integers. Nevertheless it is practicable, and the point at which this FFT-based method is faster than ones based on Theorem 55 depends greatly on the coefficient domain.

Notation 44 *We will denote $M_{\mathbf{Z}}(n)$ the asymptotic $O(M_{\mathbf{Z}}(n))$ time taken to multiply two integers of length n, and $M_{K[x]}(n)$ the number of operations in K or its associated structures needed to multiply two polynomials with n terms (or of degree n, since in O-speak the two are the same). Most authors omit the subscripts, and speak of $M(n)$, when the context is clear.*

For integers, the point at which this FFT-based method is faster than ones based on Theorem 55 is typically when the integers are of length a few thousand *words*. There are also complications due to carries, and naïvely this makes the complexity $O(n \log n \log\log n)$. Much theoretical effort has gone into improving this: see [HvdH16] for the latest efforts.

B.3.5 Faster division

TO BE COMPLETED

B.3.6 Faster g.c.d. computation

In order to use this method to compute g.c.d.s faster, we need to solve a slightly more general problem: the extended Euclidean one[3]. If we look at Algorithm 5, the key updating operation is (2.16), which updates the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ based only on q_i, which depends only on the leading entries of a_i and a_{i-1}. Call this matrix, immediately after q_i has been computed and used, $R^{i,0}$. Then (2.14) defines $R^{0,0}$ and (2.16) becomes

$$R^{i,0} = \begin{pmatrix} 0 & 1 \\ 1 & -q_i \end{pmatrix} R^{i-1,0}. \tag{B.9}$$

TO BE COMPLETED

[3]This section is base don [Moe73], generalising [Sch71].

B.4 Strassen's method

Just as Karatsuba's method (Excursus B.3) lets us multiply dense polynomials of degree $n-1$ (n terms) in $O(n^{\log_2 3} \approx n^{1.585})$ operations rather than $O(n^2)$, so Strassen's method, introduced in [Str69], allows us to multiply dense $n \times n$ matrices in $O(n^{\log_2 7} \approx n^{2.807})$ operations rather than $O(n^3)$ operations. Again like Karatsuba's method, it is based on "divide and conquer" and an ingenious base case for $n = 2$, analogous to (B.4).

$$\begin{pmatrix} c_{1,1} & c_{1,2} \\ c_{2,1} & c_{2,2} \end{pmatrix} = \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \begin{pmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{pmatrix} \tag{B.10}$$

is usually computed as

$$\begin{pmatrix} c_{1,1} & c_{1,2} \\ c_{2,1} & c_{2,2} \end{pmatrix} = \begin{pmatrix} a_{1,1}b_{1,1} + a_{1,2}b_{2,1} & a_{1,1}b_{1,2} + a_{1,2}b_{2,2} \\ a_{2,1}b_{1,1} + a_{2,2}b_{2,1} & a_{2,1}b_{1,2} + a_{2,2}b_{2,2} \end{pmatrix} \tag{B.11}$$

This method so patently requires eight multiplications of the coefficients that the question of its optimality was never posed. However, [Str69] rewrote it as follows:

$$\left.\begin{array}{rcl} M_1 & := & (a_{1,1} + a_{2,2})(b_{1,1} + b_{2,2}) \\ M_2 & := & (a_{2,1} + a_{2,2})b_{1,1} \\ M_3 & := & a_{1,1}(b_{1,2} - b_{2,2}) \\ M_4 & := & a_{2,2}(b_{2,1} - b_{1,1}) \\ M_5 & := & (a_{1,1} + a_{1,2})b_{2,2} \\ M_6 & := & (a_{2,1} - a_{1,1}(b_{1,1} + b_{1,2}) \\ M_7 & := & (a_{1,2} - a_{2,2})(b_{2,1} + b_{2,2}) \\ c_{1,1} & = & M_1 + M_4 - M_5 + M_7 \\ c_{1,2} & = & M_3 + M_5 \\ c_{2,1} & = & M_2 + M_4 \\ c_{2,2} & = & M_1 - M_2 + M_3 + M_6 \end{array}\right\} \tag{B.12}$$

requiring seven multiplications (rather than eight), though eighteen additions rather than four, so one might question the practical utility of it. [Win71] has a variant requiring fifteen rather than eighteen additions.

However, it can be cascaded. We first note that we do *not* require the multiplication operation to be commutative[4]. Hence if we have two 4×4 matrices to multiply, say

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} \end{pmatrix} \begin{pmatrix} b_{,1,1} & b_{,1,2} & b_{,1,3} & b_{,1,4} \\ b_{,2,1} & b_{,2,2} & b_{,2,3} & b_{,2,4} \\ b_{,3,1} & b_{,3,2} & b_{,3,3} & b_{,3,4} \\ b_{,4,1} & b_{,4,2} & b_{,4,3} & b_{,4,4} \end{pmatrix} \tag{B.13}$$

[4]A point often glossed over. For example, we can multiply 3×3 matrices in 22 multiplications [Mak86], but this assumes commutativity and hence cannot be cascaded. The best *non-commutative* results are 23 multiplications [Lad76, CBH11]. Asymptotically, both are in any case worse than Strassen, since $\log_2 7 \approx 2.807$, but $\log_3 23 \approx 2.854$ and $\log_3 22 \approx 2.813$.

we can write this as

$$\begin{pmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{pmatrix} \begin{pmatrix} B_{1,1} & B_{1,2} \\ B_{2,1} & B_{2,2} \end{pmatrix}, \tag{B.14}$$

where $A_{1,1} = \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix}$ etc., and apply (B.12) *both* to the multiplcations in (B.14), needing seven multiplications (and 18 additions) of 2×2 matrices, *and* to those multiplications themselves, thus needing $49 = 7 \times 7$ multiplications of entries, at the price of $198 = \underbrace{7 \times 18}_{\text{recursion}} + \underbrace{18 \times 4}_{2 \times 2 \text{ adds}}$ additions, rather than 64 multiplications and 48 additions.

Similarly, multiplying 8×8 matrices can be regarded as (B.14), where now $A_{1,1} = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} \\ a_{4,1} & a_{4,2} & a_{4,3} & a_{4,4} \end{pmatrix}$ etc., needing seven multiplications (and 18 additions) of 4×4 matrices, hence $343 = 7 \times 49$ multiplications of entries, at the price of $1674 = \underbrace{7 \times 198}_{\text{recursion}} + \underbrace{18 \times 16}_{4 \times 4 \text{ adds}}$ additions, rather than 512 multiplications and 448 additions.

If the matrices have 2^k rows/columns, then this method requires $7^k = \left(2^k\right)^{\log_2 7}$ multiplications rather than the classic $\left(2^k\right)^3$. For arbitrary sizes n, not necessarily a power of two, the cost of "rounding up" to a power of two is subsumed in the O notation[5], and we see a cost of $O(n^{\log_2 7})$ rather than the classic $O(n^3)$ coefficient multiplications.

We note that $\log_2 7 \approx 2.8074$, and the number of extra coefficient additions required is also $O(n^{\log_2 7})$, being $18(n/2)^2$ additions at each step.

Theorem 56 (Strassen) *We can multiply two (dense) $n \times n$ matrices in*

$$O(n^{\log_2 7}) \approx n^{2.8074}$$

multiplications, and the same order of additions, of matrix entries.

B.4.1 Strassen's method in practice

Strassen's method, with floating-point numbers, has break-even points between 400 and 2000 rows (160,000 to 4 million elements) [DN07]. This is achieved with one or two (occasionally three) levels of (B.12), followed by classical $O(n^3)$ multiplication from then on. The pragmatist also notes that modern chips have instructions for manipulating four-vectors of floating-point numbers in a single instruction, so there is no advantage in applying (B.12) all the way.

[5]In principle: substantial ingenuity is required to get good performance in practice.

B.4.2 Further developments

Seven is in fact minimal for 2×2 matrices [Lan06]. 3×3 matrices can be multiplied in 23 multiplications rather than the obvious 27 (but $\log_3 23 \approx 2.854 > \log_2 7$), and there is an approximation algorithm with 21 multiplications [Sch71], giving a $O(n^{\log_3 21 \approx 2.77})$ general method, but for 3×3 matrices the best known lower bound is 15 [Lic84, LO11]. In general the best known lower bound is $3n^2 + O(n^{3/2})$ [Lan12, MR12].

In theory one can do better than Theorem 56, $O(n^{2.376})$, [CW90][6], but these methods require unfeasably large n to be cost-effective. The complexity of matrix multiplication has to be at least $O(n^2)$ since there are that many entries.

Notation 45 *It is customary to assume that the cost of $n \times n$ matrix multiplication is $O(n^\omega)$, where $2 \leq \omega \leq 3$.*

B.4.3 Matrix Inversion

Lemma 19 (Frobenius) *Let A be a square matrix divided into blocks $\begin{pmatrix} P & Q \\ R & S \end{pmatrix}$, with P square and nonsingular. Assume that $U = S - R(P^{-1}Q)$ is also nonsingular. Then*

$$A^{-1} = \begin{pmatrix} P^{-1} + (P^{-1}Q)(U^{-1}RP^{-1}) & -(P^{-1}Q)U^{-1} \\ -(U^{-1}RP^{-1}) & U^{-1} \end{pmatrix}. \tag{B.15}$$

The proof is by direct verification. Note that the computation of the right-hand side of (B.15) requires two matrix inverses (P and U) and six matrix multiplications (viz. $P^{-1}Q$, $R(P^{-1}Q)$, RP^{-1}, $U(RP^{-1})$, $(P^{-1}Q)(U^{-1}RP^{-1})$ and $(P^{-1}Q)U^{-1}$). If the conditions of Lemma 19 were always satisfied, and we took P to be precisely one quarter the size of A (half as many rows and half as many columns), this would show that the complexity of matrix inverse is also $O(n^\omega)$ if $\omega > 2$ (if $\omega = 2$ we get an extra factor of $\log n$).

Lemma 20 *If B is nonsingular then $A := B^T B$ is symmetric and positive definite (for any non-zero $\mathbf{x}$, $\mathbf{x}A\mathbf{x}^T > 0$).*

Lemma 21 *If A is symmetric and positive definite then, in the notation of Lemma 19, so are P and U.*

Theorem 57 *If B is an invertible square matrix, B^{-1} can be computed in time $O(n^\omega)$ if $\omega > 2$ (if $\omega = 2$ we get an extra factor of $\log n$).*

Proof. We compute $A := B^T B$ in time $O(n^\omega)$, its inverse by the process after Lemma 19, and then $B^{-1} := A^{-1}A^T$ (another $O(n^\omega)$).

[6]Recently improved to $O(n^{2.373})$ [Wil12]. This reference actually claimed 2.2327, but this has been corrected to 2.2329 in [Wil14]. The most recent seems to be 2.3728639[LG14, §6.3].

B.5 Cyclotomic Polynomials

Definition 117 *A polynomial is said to be* cyclotomic *if it divides $x^n - 1$ for sme n.*

These innocent-looking polynomials are in fact a great nuisance in computer algebra, as they defy most useful beliefs. We first saw them in equation 2.1, where

$$\frac{x^n - 1}{x - 1} = x^{n-1} + \cdots + x + 1$$

showed that the quotient of two two-term polynomials could have an arbitrary number of terms.

In the complex plane, the roots of $x^n - 1$ are the nth roots of unity, i.e. $e^{2\pi ik/n}$ for $0 \leq k < n$. $k = 0$ gives us the factor $x - 1$. If k and n have a common factor l, then $e^{2\pi ik/n} = e^{2\pi i(k/l)/(n/l)}$ and is a root of $x^{n/l} - 1$.

Definition 118 *The nth* cyclotomic *polynomial, Φ_n, has as roots all the roots of $x^n - 1$ that divide no earlier $x^{n'} - 1$, i.e.*

$$\Phi_n(x) = \prod_{\substack{k=1 \\ \gcd(k,n)=1}}^{n-1} \left(x - e^{2\pi ik/n}\right).$$

It is a consequence of Galois theory that this polynomial is in $\mathbf{Q}[x]$, and in fact it is in $\mathbf{Z}[x]$. In particular, if p is prime, $\Phi_p = x^{p-1} + \cdots + x + 1$.

Proposition 102

$$x^n - 1 = \prod_{d|n} \Phi_d(x), \tag{B.16}$$

and therefore, by a trick well-known to number-theorists,

$$\Phi_n(x) = \prod_{d|n} (x^d - 1)^{\mu(n/d)}, \tag{B.17}$$

where μ is the Möbius function*:*

$$\mu(n) = \begin{cases} +1 & n \text{ squarefree with an even number of prime factors} \\ 0 & n \text{ not squarefree} \\ -1 & n \text{ squarefree with an odd number of prime factors} \end{cases}.$$

The first few factorisations of $x^n - 1$ are pretty innocuous:

$$\begin{array}{rcl}
x^2-1 &=& (x-1)(x+1)\\
x^3-1 &=& (x-1)(x^2+x+1)\\
x^4-1 &=& (x-1)(x+1)\underbrace{(x^2+1)}_{\Phi_4}\\
x^5-1 &=& (x-1)(x^4+x^3+x^2+x+1)\\
x^6-1 &=& (x-1)(x+1)(x^2+x+1)\underbrace{(x^2-x+1)}_{\Phi_6}\\
x^7-1 &=& (x-1)(x^6+x^5+x^4+x^3+x^2+x+1)\\
x^8-1 &=& (x-1)(x+1)(x^2+1)\underbrace{(x^4+1)}_{\Phi_8}\\
x^9-1 &=& (x-1)(x^2+x+1)\underbrace{(x^6+x^3+1)}_{\Phi_9(x)=\Phi_3(x^3)}
\end{array}$$

It follows from (B.16) that

$$x^{105}-1=(x-1)\Phi_3\Phi_5\Phi_7\Phi_{15}\Phi_{21}\Phi_{35}\Phi_{105},$$

but what is not so obvious is that

$$\begin{array}{rcl}
\Phi_{105} &=& x^{48}+x^{47}+x^{46}-x^{43}-x^{42}-2\,x^{41}-x^{40}-x^{39}+x^{36}+x^{35}+x^{34}\\
&& +x^{33}+x^{32}+x^{31}-x^{28}-x^{26}-x^{24}-x^{22}-x^{20}+x^{17}+x^{16}+x^{15}\\
&& +x^{14}+x^{13}+x^{12}-x^9-x^8-2\,x^7-x^6-x^5+x^2+x+1,
\end{array}$$

where the coefficients of x^{41} and x^7 are -2. The pattern continues: Φ_{385} has coefficients of ± 3, Φ_{1385} has coefficients of ± 4, and the growth continues, albeit apparently modestly. This is deceptive: $\Phi_{1181895}$ has coefficients[7] of ± 14102773, $\Phi(43730115)$ has coefficients[8] of ± 862550638890874931, and the current record [Arn11, p. 89] is $n = 99660932085$ with $\Phi(n)$ having coefficients almost as large as n^8. If we let $A^*(n)$ be the largest (in absolute value) coefficient of $\Phi(m) : m \le n$, then we now know [Bat49, Erd49] that there are coefficients c_1, c_2 such that

$$e^{c_2/\log\log n} < A^*(n) < e^{c_1/\log\log n}. \tag{B.18}$$

This slightly obscure growth formula, which can be written (but see Notation 10) as $A^*(n) = n^{\Theta(1/\log\log n)}$, means that $A^*(n)$ grows faster than any power of n, but not quite as fast as e^n.

Other places that cyclotomic polynomials occur as special cases include Open Problem 26.

[7] 1181895 is the least n with $\Phi(n)$ having coefficents larger than n [Arn11, p. 89].

[8] 43730115 is the least n with $\Phi(n)$ having coefficents larger than n^2 [Arn11, p. 89].

Appendix C

Systems

This appendix discusses various computer algebra systems, especially from the point of view of their internal data structures and algorithms, and how this relates to the ideas expressed in the body of this book. We do *not* discuss the user interfaces as such, nor is this intended to replace any manuals, or specialist books.

Moses [Mos71] described systems as falling into various categories, depending on the extent of automatic transformation. While most systems today are what Moses would call 'new left', the extent of automatic transformation still varies dramatically.

C.1 Axiom

C.1.1 Overview

See Figure C.1.

We note that the polynomials have been expanded and the greatest common divisor cancelled (in the terminology of Definition 4 we have *canonical forms* for this data type). We are told the data type of the result, `Fraction Polynomial Integer`, i.e. the field of fractions of $\mathbf{Z}$[variables]. In fact, this answer happens to have denominator 1, so lies in `Polynomial Integer`, but the system does not automatically check for such retractions.

C.1.2 History

This system [JS92], with which the author has been heavily involved, can be seen as the first of the 'Bourbakist'[1] systems — since followed, in particular, by Magma [BCM94] and SAGE. It was developed at IBM Yorktown Heights,

[1]Nicolas Bourbaki was the pseudonym of a group of mathematicians, largely French, who wrote very influential abstract mathematical texts from the 1930s on. See `http://www-history.mcs.st-andrews.ac.uk/HistTopics/Bourbaki_1.html`.

Figure C.1: Axiom output

```
(1) -> (x^2-1)^10/(x-1)^10

   (1)
    10      9      8       7       6       5       4       3      2
   x   + 10x  + 45x  + 120x  + 210x  + 252x  + 210x  + 120x  + 45x  + 10x + 1
                                            Type: Fraction Polynomial Integer
(2) -> %^2

   (2)
      20      19       18        17        16         15         14         13
     x   + 20x   + 190x   + 1140x   + 4845x   + 15504x   + 38760x   + 77520x
   +
            12          11          10          9          8         7         6
     125970x   + 167960x   + 184756x   + 167960x  + 125970x  + 77520x  + 38760x
   +
           5        4        3       2
     15504x  + 4845x  + 1140x  + 190x  + 20x + 1
                                            Type: Fraction Polynomial Integer
```

originally under name of Scratchpad II, with its implementation language known as Modlisp [Jen79, DJ80].

It was commercialised under the name Axiom by NAG Ltd in the 1990s, but never achieved the necessary commercial success, and is now open-source — `www.axiom-developer.org`.

C.1.3 Structure

All Axiom objects are typed, as seen in Figures C.1 and C.2. In the second one, we see that `a` and `b` are objects of appropriate, but different, types. The system 'knows' (for details, see [Doy99]) that an appropriate common type is `Polynomial Integer`, which is what `c` becomes.

C.2 Macsyma

C.2.1 Overview

See Figure C.3. We note that the g.c.d. is only cancelled on demand (`radcan` stands for 'radical canonicalizer'). The equation at line 3 is treated as such, but if we force the system (line 4) to regard it as a Boolean, the result is `false`, despite the fact that the comparands are *mathematically* equal.

Figure C.2: Axiom type system

```
(1) -> a:=1

   (1)  1
                                                        Type: PositiveInteger
(2) -> b:=x

   (2)  x
                                                             Type: Variable x
(3) -> c:=a+b

   (3)  x + 1
                                                     Type: Polynomial Integer
```

Figure C.3: Macsyma output

```
(%i1)  (x^2-1)^10/(x+1)^10;

                                     2     10
                                   (x  - 1)
 (%o1)                             ----------
                                          10
                                   (x + 1)

(%i2) radcan(%);

        10       9       8        7        6        5        4        3      2
(%o2)  x   - 10 x  + 45 x  - 120 x  + 210 x  - 252 x  + 210 x  - 120 x + 45 x
                                                                      - 10 x + 1
(%i3) %= (x^2-1)^10/(x+1)^10;

        10       9       8        7        6        5        4        3      2
(%o4)  x   - 10 x  + 45 x  - 120 x  + 210 x  - 252 x  + 210 x  - 120 x + 45 x
                                                                       2     10
                                                                     (x  - 1)
                                                        - 10 x + 1 = ----------
                                                                            10
                                                                     (x + 1)
(%i5) if % then 1 else 2;
(%05)                                          2
```

C.2.2 History

C.3 Maple

C.3.1 Overview

See Figure C.4. Note that cancelling the g.c.d., and expansion, did not take place until asked for. Also, while the exponents in the expanded form of $(x+1)^{10}$ were in order, the same was not true of its square.

C.3.2 History

This system started in the early 1980s, a period when computer power, and particularly memory, were much less than they are today. It was designed to support multiple users, particularly classes of students, on what were, by the standards of the time, relatively modest university resources. Two important early references are [CGGG83, CFG$^+$84]. These circumstances led to three design principles.

1. The system had to be small — early hosts systems had limitations of, for example, 110K words of memory. In particular, users must not pay, in terms of memory occupancy, for features they were not using, which led to a 'kernel plus loadable modules' design, where the kernel knew basic algebraic features, and the rest of the system was in loadable modules written in the Maple langauge itself, and therefore interpreted rather than compiled. The precise definition of 'basic' has changed over time — see section C.3.4.

2. The system had to be portable — early versions ran on both 32-bit VAX computers and 36-bit Honeywell computers. Its kernel, originally some 5500 lines of code, was macro-processed into 'languages of the BCPL family', of which the most successful, and the one used today, is C.

3. Memory was scarce, and hence re-use had to be a priority.

C.3.3 Data structures

These considerations led to an "expression DAG" design (see page 52). The internal form of an expression is a node of a certain type followed by an arbitrary number (hence we talk about an n-ary tree) of operands. If `A` is a Maple expression, the construct `op(0,A)` gives the type of the node, and `op(`i`,A)` gives the i-th operand. This is demonstrated by the session in table C.1, which builds the DAG shown in figure C.5. It might be assumed from table C.1 that Maple had some 'preferred' order which `A` matched but `B` did not. However, if we look at table Maple:code2 (run in a fresh session of Maple), we see that `B` is now the preferred instance. The point is that, since Maple's internalising process[2] con-

Figure C.4: Maple output

```
> (x^2-1)^10/(x+1)^10;

                                2     10
                              (x  - 1)
                              ----------
                                     10
                              (x + 1)

> simplify(%);

                                     10
                              (x - 1)

> expand(%);

   10       9       8        7        6        5        4        3
  x   - 10 x  + 45 x  - 120 x  + 210 x  - 252 x  + 210 x  - 120 x

               2
         + 45 x  - 10 x + 1

> %^2;

    10       9       8        7        6        5        4        3
  (x   - 10 x  + 45 x  - 120 x  + 210 x  - 252 x  + 210 x  - 120 x

               2            2
         + 45 x  - 10 x + 1)

> expand(%);

           2                3          5          6          7
  1 + 190 x  - 20 x - 1140 x  - 15504 x  + 38760 x  - 77520 x

                   8           9           10    20       19
         + 125970 x  - 167960 x  + 184756 x   + x   - 20 x

                18         17         16          15          14
         + 190 x   - 1140 x   + 4845 x   - 15504 x   + 38760 x

                  13           12           11         4
         - 77520 x   + 125970 x   - 167960 x   + 4845 x
```

Table C.1: A small Maple session

```
> A:=x+y+a*b;
                   A := x + y + a b
> op(0,A);
                           +
> op(3,A);
                          a b
> op(0,op(3,A));
                           *
> B:=y+b*a+x;
                   B := x + y + a b
```

Figure C.5: Tree for `A`, `B` corresponding to table C.1

$$\begin{tikzcd}
 & + \arrow[dl] \arrow[d] \arrow[dr] & & \\
x & y & * \arrow[d] \arrow[dr] & \\
 & & a & b
\end{tikzcd}$$

structs these trees, in which `+` nodes cannot be children of `+` nodes, or `+` nodes of `*` nodes (hence implicitly enforcing associativity of these operators), and the children are unordered (hence implicitly enforcing commutativity[3]), once the subtree corresponding to `a*b` has been built, as in the first line of table C.1, an equivalent tree, such as that corresponding to `b*a`, is stored as the same tree. If it is fundamentall unordered, which way does it print? The answer is given in [CFG$^+$84, p. 7]

> if the expression is found, the one in the table is used, and the [new] one is discarded.

Hence in table C.1, the presence of `a*b` means that `b*a` automatically becomes `a*b`. The converse behaviour occurs in table C.2.

In terms of the 10 algebraic rules on pp. 41–42, this structure automatically follows all except (8), which is implemented only in the weaker form (8').

The Maple command `expand` implements (8) fully, therefore producing, for

[2]Referred to as the 'simplifier' in [CGGG83, CFG$^+$84], but we do not use that word to avoid confusion with Maple's `simplify` command.

[3]We have found an example where this is not the case, but this is explicitly described as a bug by Maple's Senior Architect.

> Two simpl'd PROD DAGs containing the same entries but in a different order is a kernel bug by definition. [Vor10]

Table C.2: Another small Maple session

```
> B:=y+b*a+x;
                   B := y + b a + x
> op(0,B);
                           +
> op(2,B);
                          b a
> op(0,op(2,B));
                           *
> A:=x+y+a*b;
                   A := y + b a + x
```

Figure C.6: Tree for `A`, `B` corresponding to table C.2

$$\begin{tikzcd}
 & + \arrow[dl] \arrow[d] \arrow[dr] & \\
y & * \arrow[d] \arrow[dr] & x \\
 & b & a
\end{tikzcd}$$

polynomials, what we referred to (page 50) as a distributed representation[4]. This is therefore canonical (definition 4), but in a limited sense: it is canonical *within* a given Maple session, but may vary between sessions. This means that operations such as `op(`i`,A)` ($i \neq 0$) are not necessarily consistent between sessions.

C.3.4 Heuristic GCD

[CGG84, CGG89].

C.3.5 Conclusion

There are many books written on Maple, particularly in the educational context. A comprehensive list would be out-of-date before it was printed, but we should mention [Cor02].

[4]But we should note that there are no guaranteed mathematical properties of the ordering. Good properties are provided by the `MonomialOrders` of the `Groebner` package.

Figure C.7: MuPAD output

```
>> (x^2+1)^10/(x-1)^10

ans =

(x^2 + 1)^10/(x - 1)^10

>> (x^2-1)^10/(x+1)^10

ans =

(x^2 - 1)^10/(x + 1)^10

>> simplify(ans)

ans =

(x - 1)^10

>> expand(ans)

ans =

x^10 - 10*x^9 + 45*x^8 - 120*x^7 + 210*x^6 - 252*x^5 + 210*x^4 - 120*x^3 + 45*x^2 - 10

>> ans == (x^2 + 1)^10/(x - 1)^10

ans =

     0
```

C.4 MuPAD

C.4.1 Overview

See Figure C.7. We note that the g.c.d. is only cancelled on demand, and that the result of the equality test is `0`, i.e. false.

C.4.2 History

This system [FGK$^+$94] could easily have been written off as "yet another computer algebra system", until MathWorks Inc. bought the company, and made MuPAD into the "symbolic toolbox" of MatLab (replacing Maple). The session

Figure C.8: Reduce output

```
1: (x^2-1)^10/(x+1)^10;
```

$$x^{10} - 10x^9 + 45x^8 - 120x^7 + 210x^6 - 252x^5 + 210x^4 - 120x^3 + 45x^2 - 10x + 1$$

```
2: off exp;
3: (x^2-1)^10/(x+1)^10;
```

$$(x-1)^{10}$$

```
4: off gcd;
5: (x^2-1)^10/(x+1)^10;
```

$$(x-1)^{10}$$

```
6: (x^2-1)^10/(x^2+2*x+1)^10;
```

$$\frac{\left(x^2-1\right)^{10}}{(x^2+2x+1)^{10}}$$

```
7: on gcd;
8: ws;
```

$$\frac{(x-1)^{10}}{(x+1)^{10}}$$

above is taken from MatLab.

C.5 Reduce

C.5.1 Overview

Note that Reduce produces output that cuts/pastes as TEX, as in Figure C.8.

By default, the output is canonical (Definition 4), but this can be changed via the `exp` (short for **exp**and) and `gcd` (compute them) switches. Note that exact division tests are done even when `gcd` is off, as in line 5, and we need a more challenging example, as in line 6.

C.5.2 History

Appendix D

Index of Notation

Notation	Meaning	Reference	Page
(S)	The ideal generated by S	Notation 5	29
$(F, A, B)_{\mathcal{B}}$	A Bourbakist function definition	Notation 3	23
$(F, A, B)_{\mathcal{B}}\|_C$	$(F, A, B)_{\mathcal{B}}$ restricted to C	Notation 3	23
$\overset{?}{=}$	An equation whose validity depends on interpretations	Notation 38	287
$=_{\text{CA}}$	Equality of a computer algebra system	Notation 4	28
$=_O$	Equality of mathematical objects	Notation 4	28
$=_R$	Equality of representations	Notation 4	28
$\|\|f\|\|_k$	The k-norm of $f(x)$	Notation 40	303
$p \wedge q$	The logical and of p and q	Definition 77	139
$p \vee q$	The logical (inclusive) or of p and q	Definition 77	139
$\neg p$	The logical negation of p	Definition 77	139
$\mathbf{A}$	The algebraic numbers	Notation 30	239
$\mathbf{C}$	The complex numbers		
$\mathcal{C}$	Any field of constants	Definition 102	248
con(F)	The cone of F	Theorem 35	156
R_{const}	The constants of R	Definition 102	248
den(f)	The denominator of f	Proposition 12	59
$\underline{\epsilon}$	The relaxation of ϵ	Notation 21	90
$H(f)$	The height of $f(x)$, $\|\|f\|\|_\infty$	Notation 40	303
$L(f)$	The length of $f(x)$, $\|\|f\|\|_1$	Notation 40	303
$\text{lc}_x(f)$	leading coefficient of f (regarded as a polynomial in x)	Notation 13	43
li	The logarithmic integral	Notation 36	265
Ln	The multivalued logarithm	Notation 39	291

Notation	Meaning	Reference	Page
(m,d)	McCallum's size measure	Definition 85	150
$M(f)$	The Mahler measure of $f(x)$	Notation 40	303
$M(f)$	The Mahler measure of $f(x_1,\ldots,x_n)$	Definition 114	309
$M(n)$	The time needed to multiply two objects of length n	Notation 44	327
$\mathrm{mon}(H)$	The (multiplicative) monoid of H	Theorem 35	156
$\mu(n)$	The Möbius function	Proposition 102	331
$\mathbf{N}$	The nonnegative integers (natural numbers)		
$\mathrm{Newton}(p)$	The Newton series	Notation 17	55
$\mathrm{num}(f)$	The numerator of f	Proposition 12	59
$\mathcal{P}\nabla\wr\|_C(\mathcal{A})$	The Collins projection	Notation 25	149
$\mathcal{P}\nabla\wr\|_M(\mathcal{A})$	The McCallum projection	Notation 26	150
$\mathbf{Q}$	The rational numbers		
$\mathbf{R}$	The real numbers		
$\mathrm{red}(f)$	reductum of f $(f-\mathrm{lc}(f))$	Notation 13	43
$S(f,g)$	The S-polynomial of f and g	Definition 49	103
S_f	The “sugar’ of the polynomial f	Definition 56	113
$S_{(f,g)}$	The “sugar’ of the polynomial pair (f,g)	Definition 56	113
$\mathrm{Sk}(f)$	The skeleton of the polynomial f	Definition 88	191
Sqrt	The multivalued square root	Notation 39	291
$W(z)$	The Lambert W function		257
$\mathbf{Z}$	The integers		
$\mathbf{Z}_N$	The integers modulo N		
ω	The exponent of matrix multiplication	Notation 45	330
$\omega(z)$	The Wright ω function		257

Bibliography

[Abb88] J.A. Abbott. *Factorisation of Polynomials over Algebraic Number Fields*. PhD thesis, University of Bath, 1988.

[Abb02] J.A. Abbott. Sparse Squares of Polynomials. *Math. Comp.*, 71:407–413, 2002.

[Abb04] J.A. Abbott. CoCoA: a laboratory for computations in commutative algebra. *ACM SIGSAM Bulletin 1*, 38:18–19, 2004.

[Abb09] J.A. Abbott. Bounds on Factors in $\mathbf{Z}[x]$. `http://arxiv.org/abs/0904.3057`, 2009.

[Abb15] J.A. Abbott. Geobuckets in CoCoA. *Personal Communication*, 2015.

[ABD85] J.A. Abbott, R.J. Bradford, and J.H. Davenport. A Remark on Factorisation. *SIGSAM Bulletin 2*, 19:31–33, 1985.

[ABD88] J.A. Abbott, R.J. Bradford, and J.H. Davenport. Factorisation of Polynomials: Old Ideas and Recent Results. In R. Janssen, editor, *Proceedings "Trends in Computer Algebra"*, pages 81–91, 1988.

[ABM99] J.A. Abbott, M. Bronstein, and T. Mulders. Fast Deterministic Computations of Determinants of Dense Matrices. In S. Dooley, editor, *Proceedings ISSAC '99*, pages 197–204, 1999.

[Ach06] M. Achatz. Deciding polynomial-exponential problems. Master's thesis, (Diploma) Diplomarbeit Uiversität Passau, 2006.

[Ada43] Ada Augusta Countess of Lovelace. Sketch of the Analytical Engine invented by Charles Babbage, by L.F. Menabrea of Turin, with notes on the memoir by the translator. *Taylor's Scientific Memoirs 29*, 3:666–731, 1843.

[AGK97] B. Amrhein, O. Gloor, and W. Küchlin. On the Walk. *Theor. Comp. Sci.*, 187:179–202, 1997.

[AGR14a] A. Arnold, M. Giesbrecht, and D.S Roche. Faster Sparse Multivariate Polynomial Interpolation of Straight-Line Programs. `http://arxiv.org/abs/1412.4088`, 2014.

[AGR14b] A. Arnold, M. Giesbrecht, and D.S Roche. Sparse interpolation over finite fields via low-order roots of unity. In K. Nabeshima, editor, *Proceedings ISSAC 2014*, pages 27–34, 2014.

[AHU74] A.V. Aho, J.E. Hopcroft, and J.D. Ullman. The Design and Analysis of Computer Algorithms. *Addison-Wesley*, 1974.

[AHU83] A.V. Aho, J.E. Hopcroft, and J.D. Ullman. Data Structures and Algorithms. *Addison-Wesley*, 1983.

[AIR14] A. Abdesselam, C. Ikenmeyer, and G. Royle. 16,051 formulas for Ottaviani's invariant of cubic threefolds. `http://arxiv.org/abs/1402.2669`, 2014.

[Akr82] A.G. Akritas. Reflections on a Pair of Theorems by Budan and Fourier. *Mathematics Magazine*, 55:292–298, 1982.

[AKS04] M. Agrawal, N. Kayal, and N. Saxena. Primes is in P. *Ann. Math. (2)*, 160:781–793, 2004.

[AL94] W.W. Adams and P. Loustaunau. An introduction to Gröbner bases. *Amer. Math. Soc.*, 1994.

[ALM99] P. Aubry, D. Lazard, and M. Moreno Maza. On the Theories of Triangular Sets. *J. Symbolic Comp.*, 28:105–124, 1999.

[ALSU07] A.V. Aho, M.S. Lam, R. Sethi, and J.D. Ullman. Compilers: Principles, Techniques and Tools. *Pearson Addison-Wesley*, 2007.

[AM99] P. Aubry and M. Moreno Maza. Triangular Sets for Solving Polynomial Systems: A Comparison of Four Methods. *J. Symbolic Comp.*, 28:125–154, 1999.

[AM01] J.A. Abbott and T. Mulders. How Tight is Hadamard's Bound? *Experimental Math.*, 10:331–336, 2001.

[Ano07] Anonymous. MACSYMA .... `http://web.archive.org/web/20140103015032/http://www.symbolicnet.org/systems/macsyma.html`, 2007.

[AP10] B. Akbarpour and L.C. Paulson. MetiTarski: An Automatic Theorem Prover for Real-Valued Special Functions. *J. Automated Reasoning*, 44:175–205, 2010.

[Apo67] T.M. Apostol. *Calculus, Volume I, 2nd edition.* Blaisdell, 1967.

[AR15] A. Arnold and D.S Roche. Output-Sensitive Algorithms for Sumset and Sparse Polynomial Multiplication. In D. Robertz, editor, *Proceedings ISSAC 2015*, pages 29–36, 2015.

[Arn03] E.A. Arnold. Modular algorithms for computing Gröbner bases. *J. Symbolic Comp.*, 35:403–419, 2003.

[Arn11] A.D. Arnold. Algorithms for computing cyclotomic polynomials. *Ms.C. Thesis Department of Mathematics Simon Fraser University*, 2011.

[ARS+13] C. Andradas, T. Recio, J.R. Sendra, L. Tabera, and C. Villarino. Reparametrizing Rational Revolution Surfaces over the Reals. *Submitted*, 2013.

[Arw18] A. Arwin. Über die Kongruenzen von dem fünften und höheren Graden nach einem Primzahlmodulus. *Arkiv før matematik*, 14:1–48, 1918.

[AS64] M. Abramowitz and I. Stegun. Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing. *US Government Printing Office*, 1964.

[ASZ00] J.A. Abbott, V. Shoup, and P. Zimmermann. Factorization in $\mathbf{Z}[x]$: The Searching Phase. In C. Traverso, editor, *Proceedings ISSAC 2000*, pages 1–7, 2000.

[AW00] H. Anai and V. Weispfenning. Deciding Linear-Trigonometric Problems. In C. Traverso, editor, *Proceedings ISSAC 2000*, pages 14–22, 2000.

[Ax71] J. Ax. On Schanuel's Conjectures. *Ann. Math.*, 93:252–268, 1971.

[B́79] E. Bézout. Théorie générale des équations algébriques. *Ph.-D. Pierres*, 1779.

[Bac94] P. Bachmann. *Die analytische Zahlentheorie.* Teubner, 1894.

[Bak75] A. Baker. *Transcendental Number Theory.* Cambridge University Press, 1975.

[Bar68] E.H. Bareiss. Sylvester's Identity and Multistep Integer-preserving Gaussian Elimination. *Math. Comp.*, 22:565–578, 1968.

[Bas99] S. Basu. New results on quantifier elimination over real closed fields and applications to constraint databases. *J. ACM*, 46:537–555, 1999.

[Bat49] P.T. Bateman. Note on the coefficients of the cyclotomic polynomial. *Bull. AMS*, 55:1180–1181, 1949.

[BBDP07] J.C. Beaumont, R.J. Bradford, J.H. Davenport, and N. Phisanbut. Testing Elementary Function Identities Using CAD. *AAECC*, 18:513–543, 2007.

[BC90] G. Butler and J. Cannon. The Design of Cayley — A Language for Modern Algebra. In *Proceedings DISCO '90*, 1990.

[BCD$^+$02] R.J. Bradford, R.M. Corless, J.H. Davenport, D.J. Jeffrey, and S.M. Watt. Reasoning about the Elementary Functions of Complex Analysis. *Annals of Mathematics and Artificial Intelligence*, 36:303–318, 2002.

[BCDJ08] M. Bronstein, R. Corless, J.H. Davenport, and D.J. Jeffrey. Algebraic properties of the Lambert W function from a result of Rosenstein and Liouville. *J. Integral Transforms and Special Functions*, 18:709–712, 2008.

[BCM94] W. Bosma, J. Cannon, and G. Matthews. Programming with algebraic structures: design of the Magma language. In *Proceedings ISSAC 1994*, pages 52–57, 1994.

[BCR98] J. Bochnak, M. Coste, and M.-F. Roy. Real algebraic geometry. *Ergebnisse der Mathematik 38 (translated from the French and revised by the authors)*, 1998.

[BCR11] A. Bigatti, M. Caboara, and L. Robbiano. Computing inhomogeneous Grbner bases. *J. Symbolic Comp.*, 46:498–510, 2011.

[BD89] R.J. Bradford and J.H. Davenport. Effective Tests for Cyclotomic Polynomials. In P. Gianni, editor, *Proceedings ISSAC 1988*, pages 244–251, 1989.

[BD02] R.J. Bradford and J.H. Davenport. Towards Better Simplification of Elementary Functions. In T. Mora, editor, *Proceedings ISSAC 2002*, pages 15–22, 2002.

[BD07] C.W. Brown and J.H. Davenport. The Complexity of Quantifier Elimination and Cylindrical Algebraic Decomposition. In C.W. Brown, editor, *Proceedings ISSAC 2007*, pages 54–60, 2007.

[BdB22] F.F.D. Budan de BoisLaurent. Nouvelle méthode pour la résolution des équations numériques d'un degré quelconque. *Dondey-Dupré*, 1822.

[BDE$^+$13] R.J. Bradford, J.H. Davenport, M. England, S. McCallum, and D.J. Wilson. Cylindrical Algebraic Decompositions for Boolean Combinations. In *Proceedings ISSAC 2013*, pages 125–132, 2013.

[BDE$^+$14] R.J. Bradford, J.H. Davenport, M. England, S. McCallum, and D.J. Wilson. Truth Table Invariant Cylindrical Algebraic Decomposition. *To appear in J. Symbolic Computation*, 2014.

[BDS09] R.J. Bradford, J.H. Davenport, and C.J. Sangwin. A Comparison of Equality in Computer Algebra and Correctness in Mathematical Pedagogy. In J. Carette *et al.*, editor, *Proceedings Intelligent Computer Mathematics*, pages 75–89, 2009.

[BDS10] R.J. Bradford, J.H. Davenport, and C.J. Sangwin. A Comparison of Equality in Computer Algebra and Correctness in Mathematical Pedagogy (II). *International Journal of Technology in Mathematical Education 2*, 17:93–98, 2010.

[Ber67] E.R. Berlekamp. Factoring Polynomials over Finite Fields. *Bell System Tech. J.*, 46:1853–1859, 1967.

[Ber70] E.R. Berlekamp. Factoring Polynomials over Large Finite Fields. *Math. Comp.*, 24:713–735, 1970.

[BF91] J. Backelin and R. Fröberg. How we proved that there are exactly 924 cyclic 7-roots. In S.M. Watt, editor, *Proceedings ISSAC 1991*, pages 103–111, 1991.

[BF93] M. Bartolozzi and R. Franci. La regola dei segni dall' enunciato di R. Descartes (1637) alla dimostrazione di C.F. Gauss (1828). *Archive for History of Exact Sciences 335-374*, 45, 1993.

[BFSS06] A. Bostan, P. Flajolet, B. Salvy, and É. Schost. Fast computation of special resultants. *J. Symbolic Comp.*, 41:1–29, 2006.

[BHNS15] D.J. Bates, J.D. Hauenstein, M.E. Niemerg, and F. Sottile. Software for the Gale transform of fewnomial systems and a Descartes rule for fewnomials. `http://arxiv.org/abs/1505.05241`, 2015.

[BHPR11] O. Bastani, C.J. Hillar, P. Popov, and J.M. Rojas. Randomization, Sums of Squares, and Faster Real Root Counting for Tetranomials and Beyond. `http://arxiv.org/abs/1101.2642`, 2011.

[Big15] A. Bigatti. An example of Hilbert-badness. *Personal Communication*, 2015.

[BK12] M. Burr and F. Krahmer. SqFreeEVAL: An (almost) optimal real-root isolation algorithm. *J. Symbolic Comp.*, 47:153–166, 2012.

[BL98] T. Breuer and S.A. Linton. The GAP4 Type System Organising Algebraic Algorithms. In O.Gloor, editor, *Proceedings ISSAC '98*, pages 38–45, 1998.

[BM09] L. Busé and B. Mourrain. Explicit factors of some iterated resultants and discriminants. *Math. Comp.*, 78:345–386, 2009.

[BMMT94] E. Becker, M.G. Marinari, T. Mora, and C. Traverso. The shape of the shape lemma. In *Proceedings ISSAC 1994*, pages 129–133, 1994.

[Bou61] N. Bourbaki. Algèbre Commutative, chapter 2. *Hermann*, 1961.

[Bou70] N. Bourbaki. Théorie des Ensembles. *Diffusion C.C.L.S.*, 1970.

[BPR06] S. Basu, R. Pollack, and M.-F. Roy. Algorithms in Real Algebraic Geometry, 2nd ed. *Springer*, 2006.

[Bro69] W.S. Brown. Rational Exponential Expressions, and a conjecture concerning π and e. *Amer. Math. Monthly*, 76:28–34, 1969.

[Bro71a] W.S. Brown. On Euclid's Algorithm and the Computation of Polynomial Greatest Common Divisors. In *Proceedings SYMSAC 1971*, pages 195–211, 1971.

[Bro71b] W.S. Brown. On Euclid's Algorithm and the Computation of Polynomial Greatest Common Divisors. *J. ACM*, 18:478–504, 1971.

[Bro90] M. Bronstein. Integration of elementary function. *J. Symbolic Comp.*, 9:117–173, 1990.

[Bro91] M. Bronstein. The Algebraic Risch Differential Equation. In *Proceedings ISSAC 91*, pages 241–246, 1991.

[Bro03] C.W. Brown. QEPCAD B: a program for computing with semi-algebraic sets using CADs. *ACM SIGSAM Bulletin 4*, 37:97–108, 2003.

[Bro07] M. Bronstein. Structure theorems for parallel integration (Paper completed by Manuel Kauers). *J. Symbolic Comp.*, 42:757–769, 2007.

[BS86] D. Bayer and M. Stillman. The Design of Macaulay: A System for Computing in Algebraic Geometry and Commutative Algebra. In *Proceedings SYMSAC 86*, pages 157–162, 1986.

[BS93] E. Bach and J. Sorenson. Sieve Algorithms for Perfect Power Testing. *Algorithmica*, 9:313–328, 1993.

[Buc70] B. Buchberger. Ein algorithmisches Kriterium für die Lösbarkeit eines algebraischen Gleichungssystem (English translation in [Buc98]). *Aequationes Mathematicae*, 4:374–383, 1970.

[Buc79] B. Buchberger. A Criterion for Detecting Unnecessary Reductions in the Construction of Groebner Bases. In *Proceedings EUROSAM 79*, pages 3–21, 1979.

[Buc84] B. Buchberger. A critical pair/completion algorithm for finitely generated ideals in rings. *Logic and Machines: Decision Problems and Complexity*, pages 137–155, 1984.

[Buc98] B. Buchberger. An Algorithmic Criterion for the Solvability of a System of Algebraic Equations. *In Gröbner Bases and Applications*, pages 535–545, 1998.

[Bur13] M.A. Burr. Applications of Continuous Amortization to Bisection-based Root Isolation. `http://arxiv.org/abs/1309.5991`, 2013.

[BW93] T. Becker and V. (with H. Kredel) Weispfenning. Groebner Bases. A Computational Approach to Commutative Algebra. *Springer Verlag*, 1993.

[CA76] G.E. Collins and A.V. Akritas. Polynomial Real Root Isolation Using Descartes' Rule of Signs. In R.D. Jenks, editor, *Proceedings SYMSAC 76*, pages 272–275, 1976.

[Car58] C. Carathéodory. Theory of functions of a complex variable. *Chelsea Publ.*, 1958.

[Car73] H. Cartan. Elementary Theory of Analytic Functions of One or Several Complex Variables. *Addison-Wesley*, 1973.

[Car04] J. Carette. Understanding Expression Simplification. In J. Gutierrez, editor, *Proceedings ISSAC 2004*, pages 72–79, 2004.

[Cau29] A.-L. Cauchy. *Exercices de Mathématiques Quatrième Année.* De Bure Frères, Paris, 1829.

[CBH11] N.T. Courtois, G.V. Bard, and D. Hulme. A New General-Purpose Method to Multiply 3×3 Matrices Using Only 23 Multiplications. `http://arxiv.org/abs/1108.2830`, 2011.

[CD85] D. Coppersmith and J.H. Davenport. An Application of Factoring. *J. Symbolic Comp.*, 1:241–243, 1985.

[CD91] D. Coppersmith and J.H. Davenport. Polynomials whose Powers are Sparse. *Acta Arithmetica*, 58:79–87, 1991.

[CDJW00] R.M. Corless, J.H. Davenport, D.J. Jeffrey, and S.M. Watt. According to Abramowitz and Stegun, or arccoth needn't be uncouth. *SIGSAM Bulletin 2*, 34:58–65, 2000.

[CE95] G.E. Collins and M.J. Encarnación. Efficient rational number reconstruction. *J. Symbolic Comp.*, 20:287–297, 1995.

[CFG$^+$84] B.W. Char, G.J. Fee, K.O. Geddes, G.H. Gonnet, M.B. Monagan, and S.M. Watt. On the Design and Performance of the Maple System. Technical Report CS-84-13 University of Waterloo, 1984.

[CGG84] B.W. Char, K.O. Geddes, and G.H. Gonnet. GCDHEU: Heuristic Polynomial GCD Algorithm Based on Integer GCD Computation. In J.P. Fitch, editor, *Proceedings EUROSAM 84*, pages 285–296, 1984.

[CGG89] B.W. Char, K.O. Geddes, and G.H. Gonnet. GCDHEU: Heuristic Polynomial GCD Algorithm Based on Integer GCD Computations. *J. Symbolic Comp.*, 7:31–48, 1989.

[CGGG83] B.W. Char, K.O. Geddes, M.W. Gentleman, and G.H. Gonnet. The Design of MAPLE: A Compact, Portable and Powerful Computer Algebra System. In *Proceedings EUROCAL 83*, pages 101–115, 1983.

[CGH88] L. Caniglia, A. Galligo, and J. Heintz. Some New Effectivity Bounds in Computational Geometry. In *Proceedings AAECC-6*, pages 131–152, 1988.

[CGH$^+$96] R.M. Corless, G.H. Gonnet, D.E.G. Hare, D.J. Jeffrey, and D.E. Knuth. On the Lambert W Function. *Advances in Computational Mathematics*, 5:329–359, 1996.

[CGH$^+$03] D. Castro, M. Giusti, J. Heintz, G. Matera, and L.M. Pardo. The Hardness of Polynomial Equation Solving. *Foundations of Computational Mathematics*, 3:347–420, 2003.

[CH91] G.E. Collins and H. Hong. Partial Cylindrical Algebraic Decomposition for Quantifier Elimination. *J. Symbolic Comp.*, 12:299–328, 1991.

[Che86] G.W. Cherry. Integration in Finite Terms with Special Functions: the Logarithmic Integral. *SIAM J. Computing*, 15:1–21, 1986.

[Chi53] F. Chiò. Mémoire sur les Fonctions Connues sous le nom des Résultants ou des Déterminants. *A Pons & Cie*, 1853.

[CJ02] R.M. Corless and D.J. Jeffrey. The Wright ω Function. *Artificial Intelligence*, pages 76–89, 2002.

[CKM97] S. Collart, M. Kalkbrener, and D. Mall. Converting Bases with the Gröbner Walk. *J. Symbolic Comp.*, 24:465–469, 1997.

[CLO06] D.A. Cox, J.B. Little, and D.B. O'Shea. Ideals, Varieties and Algorithms. *Springer–Verlag*, 2006.

[CMXY09] C. Chen, M. Moreno Maza, B. Xia, and L. Yang. Computing Cylindrical Algebraic Decomposition via Triangular Decomposition. In J. May, editor, *Proceedings ISSAC 2009*, pages 95–102, 2009.

[Coh03] P.M. Cohn. Further Algebra and Applications. *Springer*, 2003.

[Col71] G.E. Collins. The SAC-1 System: An Introduction and Survey. In *Proceedings SYMSAC 1971*, 1971.

[Col75] G.E. Collins. Quantifier Elimination for Real Closed Fields by Cylindrical Algebraic Decomposition. In *Proceedings 2nd. GI Conference Automata Theory & Formal Languages*, pages 134–183, 1975.

[Col79] G.E. Collins. Factoring univariate integral polynomials in polynomial average time. In *Proceedings EUROSAM 79*, pages 317–329, 1979.

[Col85] G.E. Collins. The SAC-2 Computer Algebra System. In *Proceedings EUROCAL 85*, pages 34–35, 1985.

[Col98] G.E. Collins. Quantifier elimination by cylindrical algebraic decomposition — twenty years of progess. In B.F. Caviness and J.R. Johnson, editors, *Quantifier Elimination and Cylindrical Algebraic Decomposition*, pages 8–23. Springer Verlag, Wien, 1998.

[Cor02] R.M. Corless. Essential Maple 7 : an introduction for scientific programmers. *Springer-Verlag*, 2002.

[CR88] M. Coste and M.-F. Roy. Thom's Lemma, the Coding of Real Algebraic Numbers and the Computation of the Topology of Semi-Algebraic Sets. *J. Symbolic Comp.*, 5:121–129, 1988.

[CW90] D. Coppersmith and S. Winograd. Matrix Multiplication via Arithmetic Progressions. *J. Symbolic Comp.*, 9:251–280, 1990.

[CZ81] D.G. Cantor and H. Zassenhaus. A New Algorithm for Factoring Polynomials over Finite Fields. *Math. Comp.*, 36:587–592, 1981.

[Dah09] X. Dahan. Size of coefficients of lexicographical Gröbner bases: the zero-dimensional, radical and bivariate case. In J. May, editor, *Proceedings ISSAC 2009*, pages 119–126, 2009.

[Dav81] J.H. Davenport. *On the Integration of Algebraic Functions*, volume 102 of *Springer Lecture Notes in Computer Science*. Springer Berlin Heidelberg New York (Russian ed. MIR Moscow 1985), 1981.

[Dav82a] J.H. Davenport. On the Parallel Risch Algorithm (I). In *Proceedings EUROCAM '82 [Springer Lecture Notes in Computer Science 144*, pages 144–157, 1982.

[Dav82b] J.H. Davenport. On the Parallel Risch Algorithm (III): Use of Tangents. *SIGSAM Bulletin 3*, 16:3–6, 1982.

[Dav84] J.H. Davenport. Intégration Algorithmique des fonctions élémentairement transcendantes sur une courbe algébrique. *Annales de l'Institut Fourier*, 34:271–276, 1984.

[Dav85a] J.H. Davenport. Computer Algebra for Cylindrical Algebraic Decomposition. Technical Report TRITA-NA-8511 NADA KTH Stockholm (Reissued as Bath Computer Science Technical report 88-10), 1985.

[Dav85b] J.H. Davenport. The LISP/VM Foundations of Scratchpad II. *Scratchpad II Newsletter 1*, 1:4–5, 1985.

[Dav87] J.H. Davenport. Looking at a set of equations (Technical Report 87-06, University of Bath Computer Science). `http://staff.bath.ac.uk/masjhd/TR87-06.pdf`, 1987.

[Dav02] J.H. Davenport. Equality in computer algebra and beyond. *J. Symbolic Comp.*, 34:259–270, 2002.

[Dav10] J.H. Davenport. The Challenges of Multivalued "Functions". In S. Autexier *et al.*, editor, *Proceedings AISC/Calculemus/MKM 2010*, pages 1–12, 2010.

[DC10] J.H. Davenport and J. Carette. The Sparsity Challenges. In S. Watt *et al.*, editor, *Proceedings SYNASC 2009*, pages 3–7, 2010.

[DDDD85] J. Della Dora, C. DiCrescenzo, and D. Duval. About a new Method for Computing in Algebraic Number Fields. In *Proceedings EUROCAL 85*, pages 289–290, 1985.

[DdO14] Z. Dvir and R.M. de Oliveira. Factors of Sparse Polynomials are Sparse. `http://arxiv.org/abs/1404.4834`, 2014.

[DF94] A. Dingle and R.J. Fateman. Branch cuts in computer algebra. In *Proceedings ISSAC 1994*, pages 250–257, 1994.

[DGT91] J.H. Davenport, P. Gianni, and B.M. Trager. Scratchpad's View of Algebra II: A Categorical View of Factorization. In S.M. Watt, editor, *Proceedings ISSAC 1991*, pages 32–38, 1991.

[DH88] J.H. Davenport and J. Heintz. Real Quantifier Elimination is Doubly Exponential. *J. Symbolic Comp.*, 5:29–35, 1988.

[Dic13] L.E. Dickson. Finiteness of the odd perfect and primitive abundant numbers with n prime factors. *Amer. J. Math.*, 35:413–422, 1913.

[Dix82] J.D. Dixon. Exact Solutions of Linear Equations Using p-adic Methods. *Numer. Math.*, 40:137–141, 1982.

[DJ80] J.H. Davenport and R.D. Jenks. MODLISP — an Introduction. In *Proceedings LISP80*, 1980.

[DL08] J.H. Davenport and P. Libbrecht. The Freedom to Extend OpenMath and its Utility. *Mathematics in Computer Science 2(2008/9)*, pages 379–398, 2008.

[DM90] J.H. Davenport and M. Mignotte. On Finding the Largest Root of a Polynomial. *Modélisation Mathématique et Analyse Numérique*, 24:693–696, 1990.

[DN07] P. D'Alberto and A. Nicolau. Adaptive Strassen's matrix multiplication. In *Proceedings Supercomputing 2007*, pages 284–292, 2007.

[Dod66] C.L. Dodgson. Condensation of determinants, being a new and brief method for computing their algebraic value. *Proc. Roy. Soc. Ser. A*, 15:150–155, 1866.

[Doy99] N.J. Doye. Automated Coercion for Axiom. In S. Dooley, editor, *Proceedings ISSAC '99*, pages 229–235, 1999.

[DS97] A. Dolzmann and Th. Sturm. Redlog: Computer Algebra Meets Computer Logic. *ACM SIGSAM Bull. 2*, 31:2–9, 1997.

[DS00] J.H. Davenport and G.C. Smith. Fast recognition of alternating and symmetric groups. *J. Pure Appl. Algebra*, 153:17–25, 2000.

[DT81] J.H. Davenport and B.M. Trager. Factorization over finitely generated fields. In *Proceedings SYMSAC 81*, pages 200–205, 1981.

[DT85] J.H. Davenport and B.M. Trager. On the Parallel Risch Algorithm (II). *ACM TOMS*, 11:356–362, 1985.

[DT90] J.H. Davenport and B.M. Trager. Scratchpad's View of Algebra I: Basic Commutative Algebra. In *Proceedings DISCO '90*, 1990.

[Dub90] T.W. Dubé. The structure of polynomial ideals and Gröbner Bases. *SIAM J. Comp.*, 19:750–753, 1990.

[Duv87] D. Duval. Diverses Questions relatives au Calcul Formel avec les Nombres Algébriques. *Thèse d'Etat*, 1987.

[EBD15] M. England, R. Bradford, and J.H. Davenport. Improving the Use of Equational Constraints in Cylindrical Algebraic Decomposition. In D. Robertz, editor, *Proceedings ISSAC 2015*, pages 165–172, 2015.

[Ebe83] G.L. Ebert. Some comments on the modular approach to Grobnerbases. *ACM SIGSAM Bull.*, 17:28–32, 1983.

[EFG15] S. Eberhard, K. Ford, and B. Green. Invariable Generation of the Symmetric Group. `http://arxiv.org/pdf/1508.01870.pdf`, 2015.

[Erd49] P. Erdős. On the coefficients of the cyclotomic polynomial. *Portugaliae Mathematica*, 8:63–71, 1949.

[ESY06] A. Eigenwillig, V. Sharma, and C.K. Yap. Almost tight recursion tree bounds for the Descartes method. In *Proceedings ISSAC 2006*, pages 71–78, 2006.

[Fat03] R.J. Fateman. Comparing the speed of programs for sparse polynomial multiplication. *SIGSAM Bulletin 1*, 37:4–15, 2003.

[Fau02] J.-C. Faugère. A New Efficient Algorithm for Computing Gröbner Bases Without Reduction to Zero (F_5). In T. Mora, editor, *Proceedings ISSAC 2002*, pages 75–83, 2002.

[FGHR13] J.-C. Faugère, P. Gaudry, L. Huot, and G. Renault. Polynomial Systems Solving by Fast Linear Algebra. `http://arxiv.org/abs/1304.6039`, 2013.

[FGK$^+$94] B. Fuchssteiner, K. Gottheil, A. Kemper, O. Kluge, K. Morisse, H. Naundorf, G. Oevel, T. Schulze, and W. Wiwianka. MuPAD Multi Processing Algebra Data Tool Tutorial (version 1.2). *Birkhäuser Verlag*, 1994.

[FGLM93] J.C. Faugère, P. Gianni, D. Lazard, and T. Mora. Efficient Computation of Zero-Dimensional Gröbner Bases by Change of Ordering. *J. Symbolic Comp.*, 16:329–344, 1993.

[FGT01] E. Fortuna, P. Gianni, and B. Trager. Degree reduction under specialization. *J. Pure Appl. Algebra*, 164:153–163, 2001.

[fHN76] J.P. ffitch, P. Herbert, and A.C. Norman. Design Features of COBALG. In R.D. Jenks, editor, *Proceedings SYMSAC 76*, pages 185–188, 1976.

[FIS15] R. Fukasaku, H. Iwane, and Y. Sato. Real Quantifier Elimination by Computation of Comprehensive Gröbner Systems. In D. Robertz, editor, *Proceedings ISSAC 2015*, pages 173–180, 2015.

[FJLT07] K. Fukuda, A.N. Jensen, N. Lauritzen, and R. Thomas. The generic Gröbner walk. *Journal of Symbolic Computation*, 42(3):298 – 312, 2007.

[FM89] D.J. Ford and J. McKay. Computation of Galois Groups from Polynomials over the Rationals. *Computer Algebra (Lecture Notes in Pure and Applied Mathematics 113*, 1989.

[FM13] J.-C. Faugère and C. Mou. Sparse FGLM algorithms. `http://arxiv.org/abs/1304.1238`, 2013.

[Fou31] J. Fourier. Analyse des équations déterminées. *Didot*, 1831.

[FS56] A. Fröhlich and J.C. Shepherdson. Effective Procedures in Field Theory. *Phil. Trans. Roy. Soc. Ser. A 248(1955-6)*, pages 407–432, 1956.

[FSEDT14] J.-C. Faugère, M. Safey El Din, and V. Thibaut. On the complexity of computing Gröbner bases for weighted homogeneous systems. `http://arxiv.org/abs/1412.7547`, 2014.

[Ful69] W. Fulton. Algebraic Curves, An Introduction to Algebraic Geometry. *W.A. Benjamin Inc*, 1969.

[Gal79] É. Galois. *Œuvres mathématiques*. Gauthier-Villars (sous l'auspices de la SMF), 1879.

[Gia89] P. Gianni. Properties of Gröbner bases under specializations. In *Proceedings EUROCAL 87*, pages 293–297, 1989.

[GJ76] W.M. Gentleman and S.C. Johnson. Analysis of Algorithms, A Case Study: Determinants of Matrices. *ACM TOMS*, 2:232–241, 1976.

[GJY75] J.H. Griesmer, R.D. Jenks, and D.Y.Y. Yun. SCRATCHPAD User's Manual. *IBM Research Publication RA70*, 1975.

[GKP94] R.L. Graham, D.E. Knuth, and O. Patashnik. Concrete Mathematics (2nd edition). *Addison-Wesley*, 1994.

[GKS15] A. Grabowski, A. Korniłowicz, and C. Schwarzweller. Equality in Computer Proof-Assistants. In *Proceedings 2015 Federated Conference on Computer Science and Information Systems*, pages 45–54, 2015.

[GLL09] M. Giesbrecht, G. Labahn, and W. Lee. Symbolic-numeric sparse interpolation of multivariate polynomials. *J. Symbolic Comp.*, 44:943–959, 2009.

[GMN$^+$91] A. Giovini, T. Mora, G. Niesi, L. Robbiano, and C. Traverso. One sugar cube, please, or selection strategies in the Buchberger algorithm. In S.M. Watt, editor, *Proceedings ISSAC 1991*, pages 49–54, 1991.

[GN90] A. Giovini and G. Niesi. CoCoA: A User-Friendly System for Commutative Algebra. In *Proceedings DISCO '90*, 1990.

[Gon84] G.H. Gonnet. Determining Equivalence of Expressions in Random Polynomial Time. In *Proceedings 16th ACM Symp. Theory of Computing*, pages 334–341, 1984.

[Gos78] R.W. Gosper Jr. A Decision Procedure for Indefinite Hypergeometric Summation. *Proc. Nat. Acad. Sci.*, 75:40–42, 1978.

[Gra37] C.H. Graeffe. Die Auflösulg der höheren numerischen Gleichungen. *F. Schulthess*, 1837.

[Gre15] B. Grenet. Lacunaryx: Computing bounded-degree factors of lacunary polynomials. `http://arxiv.org/abs/1506.03726`, 2015.

[HA10] A. Hashemi and G. Ars. Extended F5 criteria. *J. Symbolic Comp.*, 45:1330–1340, 2010.

[Ham07] S. Hammarling. Life as a developer of numerical software. *Talk at NAG Ltd AGM*, 2007.

[Har16] G.H. Hardy. The Integration of Functions of a Single Variable (2nd. ed.). Cambridge Tract 2, C.U.P., 1916. *Jbuch. 46*, 1916.

[Har11] M.C. Harrison. Explicit Solution By Radicals of Algebraic Curves of Genus 5 or 6. `http://arxiv.org/abs/1103.4946`, 2011.

[Has53] C.B. Haselgrove. Implementations of the Todd-Coxeter Algorithm on EDSAC-1. *Unpublished*, 1953.

[HB78] D.R. Heath-Brown. Almost-primes in arithmetic progressions and short intervals. *Math. Proc. Camb. Phil. Soc.*, 83:357–375, 1978.

[HD03] A.J. Holt and J.H. Davenport. Resolving Large Prime(s) Variants for Discrete Logarithm Computation. In P.G. Farrell, editor, *Proceedings 9th IMA Conf. Coding and Cryptography*, pages 207–222, 2003.

[Hea05] A.C. Hearn. REDUCE: The First Forty Years. In T. Sturm A. Dolzmann, A. Seidl, editor, *Proceedings A3L*, pages 19–24. Books on Demand GmbH, 2005.

[Her72] E. Hermite. Sur l'intégration des fractions rationelles. *Nouvelles Annales de Mathématiques*, 11:145–148, 1872.

[Hie92] J. Hietarinta. Solving the constant quantum Yang-Baxter equation in 2 dimensions with massive use of factorizing Gröbner basis computations. In *Proceedings ISSAC 1992*, pages 350–357, 1992.

[Hie93] J. Hietarinta. Solving the two-dimensional constant quantum Yang-Baxter equation J. Math. Phys. 34(1993) pp. 1725-1756. *doi: 10.1063/1.530185*, 1993.

[Hig02] N.J. Higham. Accuracy and Stability of Numerical Algorithms, 2nd ed. *SIAM*, 2002.

[HKL16] C.J. Hillar, R. Krone, and A. Leykin. Equivariant Gröbner bases. `https://arxiv.org/abs/1610.02075`, 2016.

[HM13] J. Hu and M. Monagan. A parallel algorithm to compute the greatest common divisor of sparse multivariate polynomials. *ACM Comm. Computer Algebra*, 47:108–109, 2013.

[Hon90] H. Hong. *Improvements in CAD-Based Quantifier Elimination.* PhD thesis, OSU-CISRC-10/90-TR29 Ohio State University, 1990.

[Hor69] E. Horowitz. *Algorithm for Symbolic Integration of Rational Functions.* PhD thesis, Univ. of Wisconsin, 1969.

[Hor71] E. Horowitz. Algorithms for Partial Fraction Decomposition and Rational Function Integration. In *Proceedings Second Symposium on Symbolic and Algebraic Manipulation*, pages 441–457, 1971.

[Hou59] A.S. Householder. Dandelin, Lobačevskiĭor Graeffe? *Amer. Math. Monthly*, 66:464–466, 1959.

[HP07] H. Hong and J. Perry. Are Buchberger's criteria necessary for the chain condition? *J. Symbolic Comp.*, 42:717–732, 2007.

[HTZ$^+$09] M. Hemmer, E.P. Tsigaridas, Z. Zafeirakopoulos, I.Z. Emiris, M.I. Karavelas, and B. Mourrain. Experimental evaluation and cross-benchmarking of univariate real solvers. In *Proceedings 3rd conference on Symbolic numeric computation*, pages 45–54, 2009.

[Hua49] K. Hua, L. On the automorphisms of a sfield. *Proc. Nat. Acad. Sci. U. S. A.*, 35:386–389, 1949.

[Hur12] A. Hurwitz. Über den Satz von Budan-Fourier. *Math. Annalen*, 71:584–591, 1912.

[HvdH16] D. Harvey and J. van der Hoeven. Faster integer multiplication using plain vanilla FFT primes. `https://arxiv.org/abs/1611.07144`, 2016.

[HW79] G.H. Hardy and E.M. Wright. An Introduction to the Theory of Numbers (5th. ed.). *Clarendon Press*, 1979.

[IEE85] IEEE. IEEE Standard 754 for Binary Floating-Point Arithmetic. *IEEE*, 1985.

[IL80] O.H. Ibarra and B.S. Leininger. The Complexity of the Equivalence Problem for Straight-line Programs. In *Proceedings ACM STOC 1980*, pages 273–280, 1980.

[IPS10] I. Idrees, G. Pfister, and S. Steidel. Parallelization of Modular Algorithms. `http://arxiv.org/abs/1005.5663v1`, 2010.

[IPS11] I. Idrees, G. Pfister, and S. Steidel. Parallelization of Modular Algorithms. *J. Symbolic Comp.*, 46:672–684, 2011.

[Isa85] I.M. Isaacs. Solution of polynomials by real radicals. *Amer. Math. Monthly*, 92:571–575, 1985.

[Jef10] D.J. Jeffrey. LU Factoring of Non-Invertible Matrices. *Communications in Computer Algebra 1*, 45:1–8, 2010.

[Jen79] Jenks.R.D. MODLISP. In *Proceedings EUROSAM 79*, pages 466–480, 1979.

[JHM11] R. Jones, A. Hosking, and E. Moss. The garbage collection handbook: The Art of Automatic Memory Management (1st ed.). *Chapman & Hall/CRC*, 2011.

[Joh71] S.C. Johnson. On the Problem of Recognizing Zero. *J. ACM*, 18:559–565, 1971.

[Joh74] S.C. Johnson. Sparse Polynomial Arithmetic. In *Proceedings EUROSAM 74*, pages 63–71, 1974.

[JR10] D.J. Jeffrey and A.D. Rich. Reducing Expression Size Using Rule-Based Integration. In S. Autexier *et al.*, editor, *Proceedings CICM 2010*, pages 234–246, 2010.

[JS92] R.D. Jenks and R.S. Sutor. *AXIOM: The Scientific Computation System.* Springer-Verlag, 1992.

[Kac43] M. Kac. On the Average Number of Real Roots of a Random Algebraic Equation. *Bull. A.M.S.*, 49:314–320, 1943.

[Kah53] H.G. Kahrimanian. Analytic differentiation by a digital computer. Master's thesis, Temple U Philadelphia, 1953.

[Kal88] E. Kaltofen. Greatest Common Divisors of Polynomials given by Straight-line Programs. *J. ACM*, 35:231–264, 1988.

[Kal89a] M. Kalkbrener. Solving systems of algebraic equations by using Gröbner bases. In *Proceedings EUROCAL 87*, pages 282–292, 1989.

[Kal89b] E. Kaltofen. Factorization of Polynomials given by Straight-line Programs. *Randomness and Computation*, pages 375–412, 1989.

[Kal95] E. Kaltofen. Effective Noether irreducibility forms and applications. *J. Computer System Sci.*, 50:274–295, 1995.

[Kal10] E. Kaltofen. Fifteen years after DSC and WLSS2 what parallel computations I do today: invited lecture at PASCO 2010. In *Proceedings 4th International Workshop on Parallel and Symbolic Computation*, pages 10–17, 2010.

[Kar81] M. Karr. Summation in Finite Terms. *J. ACM*, 28:305–350, 1981.

[Kar84] N.K. Karmarkar. A New Polynomial-Time Algorithm for Linear Programming. *Combinatorica*, 4:373–395, 1984.

[Kha79] L.G. Khachian. A polynomial algorithm in linear programming. *Doklay Akad. Nauk SSSR*, 224:1093–1096, 1979.

[KM17] J. Kluesner and M. Monagan. Computing GCDs of polynomials modulo triangular sets. `http://www.cecm.sfu.ca/CAG/papers/john11.pdf`, 2017.

[KMS83] E. Kaltofen, D.R. Musser, and B.D. Saunders. A Generalized Class of Polynomials That are Hard to Factor. *SIAM J. Comp.*, 12:473–483, 1983.

[KMS13] H. Khalil, B. Mourrain, and M. Schatzman. Superfast solution of Toeplitz systems based on syzygy reduction. `http://arxiv.org/abs/1301.5798`, 2013.

[Knu74] D.E. Knuth. Big Omicron and big Omega and big Theta. *ACM SIGACT News 2*, 8:18–24, 1974.

[Knu81] D.E. Knuth. The Art of Computer Programming, Vol. II, Seminumerical Algorithms. *Second Edition*, 1981.

[Knu98] D.E. Knuth. The Art of Computer Programming, Vol. II, Seminumerical Algorithms (Third Edition). *Addison-Wesley*, 1998.

[KO63] A. Karatsuba and J. Ofman. Multiplication of multidigit numbers on automata. *Sov. Phys. Dokl.*, 7:595–596, 1963.

[Kol88] J. Kollár. Sharp effective nullstellensatz. *J.A.M.S.*, 1:963–975, 1988.

[KPT12] P. Koiran, N. Portier, and S. Tavenas. A Wronskian approach to the real τ-conjecture. `http://arxiv.org/abs/1205.1015`, 2012.

[KRW90] A. Kandri-Rody and V. Weispfenning. Non-commutative Gröbner bases in algebras of solvable type. *J. Symbolic Comp.*, 9:1–26, 1990.

[KS11] M. Kerber and M. Sagraloff. Root Refinement for Real Polynomials. `http://arxiv.org/abs/1104.1362`, 2011.

[KSD16] M. Košta, T. Sturm, and A. Dolzmann. Better answers to real questions. *J. Symbolic Comp.*, 74:255–275, 2016.

[KU08] K. Kedlaya and C. Umans. Fast modular composition in any characteristic. In *Proceedings of the 49th Annual IEEE Symposium on Foundations of Computer Science*, pages 146–155, 2008.

[Lad76] J.D. Laderman. A Non-Commutative Algorithm for Multiplying 3×3 Matrices Using 23 Multiplications. *Bull. Amer. Math. Soc.*, 82:126–128, 1976.

[Lan05] E. Landau. Sur Quelques Théorèmes de M. Petrovic Relatif aux Zéros des Fonctions Analytiques. *Bull. Soc. Math. France*, 33:251–261, 1905.

[Lan66] S. Lang. Introduction to Transcendental Numbers. *Addison-Wesley*, 1966.

[Lan06] J.M. Landsberg. The border rank of the multiplication of 2×2 matrices is seven. *J. Amer. Math. Soc.*, 19:447–459, 2006.

[Lan12] J.M. Landsberg. New lower bounds for the rank of matrix multiplication. `http://arxiv.org/abs/1206.1530`, 2012.

[Lau82] M. Lauer. Computing by Homomorphic Images. *Symbolic and Algebraic Computation (Computing Supplementum 4) Springer-Verlag*, pages 139–168, 1982.

[Laz83] D. Lazard. Gröbner Bases, Gaussian Elimination and Resolution of Systems of Algebraic Equations. In *Proceedings EUROCAL 83*, pages 146–157, 1983.

[Laz88] D. Lazard. Quantifier Elimination: Optimal Solution for Two Classical Problems. *J. Symbolic Comp.*, 5:261–266, 1988.

[Laz91] D. Lazard. A New Method for Solving Algebraic Systems of Positive Dimension. *Discrete Appl. Math.*, 33:147–160, 1991.

[Laz92] D. Lazard. Solving Zero-dimensional Algebraic Systems. *J. Symbolic Comp.*, 13:117–131, 1992.

[Laz94] D. Lazard. An Improved Projection Operator for Cylindrical Algebraic Decomposition. In *Proceedings Algebraic Geometry and its Applications*, 1994.

[Laz09] D. Lazard. Thirty years of Polynomial System Solving, and now? *J. Symbolic Comp.*, 44:222–231, 2009.

[Lec08] G. Lecerf. Fast separable factorization and applications. *AAECC*, 19:135–160, 2008.

[Len87] A.K. Lenstra. Factoring Multivariate Polynomials over Algebraic Number Fields. *SIAM J. Comp.*, 16:591–598, 1987.

[Len99a] H.W. Lenstra Jr. Finding small degree factors of lacunary polynomials. *Number theory in progress*, pages 267–276, 1999.

[Len99b] H.W. Lenstra Jr. On the factorization of lacunary polynomials. *Number theory in progress*, pages 277–291, 1999.

[LG14] F. Le Gall. Algebraic Complexity Theory and Matrix Multiplication. In K. Nabeshima, editor, *Proceedings ISSAC 2014*, pages 23–23, 2014.

[Lic84] T. Lickteig. A note on border rank. *Inform. Process. Lett.*, 18:173–178, 1984.

[Lic11] D. Lichtblau. Mathematica's $\sqrt{x^2}$. *Private Communication*, 2011.

[LL12] A. Lerario and E. Lundberg. Statistics on Hilbert's Sixteenth Problem. `http://arxiv.org/abs/1212.3823`, 2012.

[LLL82] A.K. Lenstra, H.W. Lenstra Jun., and L. Lovász. Factoring Polynomials with Rational Coefficients. *Math. Ann.*, 261:515–534, 1982.

[LN97] R. Lidl and H. Niederreiter. Finite Fields: volume 20 of Encyclopedia of Mathematics and its Applications. *Cambridge University Press*, 1997.

[LO11] J. Landsberg and G. Ottaviani. New lower bounds for the border rank of matrix multiplication. `http://arxiv.org/abs/1112.6007`, 2011.

[Loo82] R. Loos. Generalized Polynomial Remainder Sequences. *Symbolic and Algebraic Computation (Computing Supplementum 4) Springer-Verlag*, pages 115–137, 1982.

[LR90] D. Lazard and R. Rioboo. Integration of Rational Functions — Rational Computation of the Logarithmic Part. *J. Symbolic Comp.*, 9:113–115, 1990.

[LR01] T. Lickteig and M.-F. Roy. Sylvester-Habicht Sequences and Fast Cauchy Index Computation. *J. Symbolic Comp.*, 31:315–341, 2001.

[LV40] U. Le Verrier. Sur les variations séculaires des éléments elliptiques des sept planètes principales: Mercurce, Vénus, La Terre, Mars, Jupiter, Saturne et Uranus. *J. Math. Pure Appl.*, 4:220–254, 1840.

[Mah64] K. Mahler. An Inequality for the Discriminant of a Polynomial. *Michigan Math. J.*, 11:257–262, 1964.

[Mak86] O.M. Makarov. An algorithm for multiplication of 3×3 matrices. *USSR Computational Mathematics and Mathematical Physics*, 26:179–180, 1986.

[McC84] S. McCallum. *An Improved Projection Operation for Cylindrical Algebraic Decomposition.* PhD thesis, University of Wisconsin-Madison Computer Science, 1984.

[McC88] S. McCallum. An Improved Projection Operation for Cylindrical Algebraic Decomposition of Three-dimensional Space. *J. Symbolic Comp.*, 5:141–161, 1988.

[McC99] S. McCallum. On Projection in CAD-Based Quantifier Elimination with Equational Constraints. In S. Dooley, editor, *Proceedings ISSAC '99*, pages 145–149, 1999.

[McC01] S. McCallum. On Propagation of Equational Constraints in CAD-Based Quantifier Elimination. In B. Mourrain, editor, *Proceedings ISSAC 2001*, pages 223–230, 2001.

[MF71] W.A. Martin and R.J. Fateman. The MACSYMA System. In *Proceedings Second Symposium on Symbolic and Algebraic Manipulation*, pages 59–75, 1971.

[MGH$^+$03] M.B. Monagan, K.O. Geddes, K.M. Heal, G. Labahn, S.M. Vorkoetter, J. McCarron, and P. DeMarco. Maple 9 : introductory programming guide. *Maplesoft*, 2003.

[Mig74] M. Mignotte. An Inequality about Factors of Polynomials. *Math. Comp.*, 28:1153–1157, 1974.

[Mig81] M. Mignotte. Some Inequalities About Univariate Polynomials. In *Proceedings SYMSAC 81*, pages 195–199, 1981.

[Mig82] M. Mignotte. Some Useful Bounds. *Symbolic and Algebraic Computation (Computing Supplementum 4) Springer-Verlag*, pages 259–263, 1982.

[Mig89] M. Mignotte. Mathématiques pour le Calcul Formel. *PUF*, 1989.

[Mig00] M. Mignotte. Bounds for the roots of lacunary polynomials. *J. Symbolic Comp.*, 30:325–327, 2000.

[MM82] E. Mayr and A. Meyer. The Complexity of the Word Problem for Commutative Semi-groups and Polynomial Ideals. *Adv. in Math.*, 46:305–329, 1982.

[MM84] H.M. Möller and F. Mora. Upper and Lower Bounds for the Degree of Groebner Bases. In J.P. Fitch, editor, *Proceedings EUROSAM 84*, pages 172–183, 1984.

[MMN89] H. Melenk, H.M. Möller, and W. Neun. Symbolic Solution of Large Stationary Chemical Kinetics Problems. *Impact of Computing in Sci. and Eng.*, 1:138–167, 1989.

[Moe73] R. Moenck. Fast Computation of GCDs. In *Proceedings Fifth Annual ACM Symposium of Theory of Computing [ACM*, pages 142–151, 1973.

[Mon05] P.L. Montgomery. Five, Six, and Seven-Term Karatsuba-Like Formulae. *IEEE Trans. Computers*, 54:362–369, 2005.

[Mon09] D. Monniaux. Fatal Degeneracy in the Semidefinite Programming Approach to the Decision of Polynomial Inequalities. `http://arxiv.org/abs/0901.4907`, 2009.

[Mor86] F. Mora. Groebner Bases for Non-commutative Polynomial Rings. In *Proceedings AAECC-3*, pages 353–362, 1986.

[Mos71] J. Moses. Algebraic Simplification — A Guide for the Perplexed. *Comm. ACM*, 14:527–537, 1971.

[MP08] M. Monagan and R. Pearce. Parallel sparse polynomial multiplication using heaps. In D.J.Jeffrey, editor, *Proceedings ISSAC 2008*, pages 263–270, 2008.

[MP11] M. Monagan and R. Pearce. Sparse polynomial division using a heap. *J. Symbolic Comp.*, 46:807–822, 2011.

[MP12] M. Monagan and R. Pearce. POLY : A new polynomial data structure for Maple 17. *Comm. Computer Algebra*, 46:164–167, 2012.

[MP13] M. Monagan and R. Pearce. POLY : A new polynomial data structure for Maple 17. *To appear in Proc. ASCM 2012*, 2013.

[MP14] M. Monagan and R. Pearce. The design of Maple's sum-of-products and POLY data structures for representing mathematical objects. *ACM Comm. Computer Algebra*, 48:166–186, 2014.

[MR88] F. Mora and L. Robbiano. The Gröbner fan of an ideal. *J. Symbolic Comp.*, 6:183–208, 1988.

[MR10] E.W. Mayr and S. Ritscher. Degree Bounds for Gröbner Bases of Low-Dimensional Polynomial Ideals. In S.M. Watt, editor, *Proceedings ISSAC 2010*, pages 21–28, 2010.

[MR11] E.W. Mayr and S. Ritscher. Space efficient Gröbner basis computation without degree bounds. In *Proceedings ISAAC 2011*, pages 257–264, 2011.

[MR12] A. Massarenti and E. Raviolo. The Rank of $n \times n$ Matrix Multiplication is at least $3n^2 - 2\sqrt{2}n^{\frac{3}{2}} - 3n$. `http://arxiv.org/abs/1211.6320`, 2012.

[Mul97] T. Mulders. A note on subresultants and the Lazard/Rioboo/Trager formula in rational function integration. *J. Symbolic Comp.*, 24:45–50, 1997.

[Mus78] D.R. Musser. On the efficiency of a polynomial irreducibility test. *J. ACM*, 25:271–282, 1978.

[MW51] J.C.P. Miller and D.J. Wheeler. Missing Title. *Nature p. 838*, 168, 1951.

[MW12] S. McCallum and V. Weispfenning. Deciding polynomial-transcendental problems. *J. Symbolic Comp.*, 47:16–31, 2012.

[Nat10] National Institute for Standards and Technology. The NIST Digital Library of Mathematical Functions. `http://dlmf.nist.gov`, 2010.

[Neu95] J. Neubüser. Re: Origins of GAP. *Message m0t5WVW-00075GC.951018.121802@astoria.math.rwth-aachen.de to GAP-Forum on 18.10.95*, 1995.

[NM77] A.C. Norman and P.M.A. Moore. Implementing the New Risch Integration Algorithm. In *Proceedings 4th. Int. Colloquium on Advanced Computing Methods in Theoretical Physics*, pages 99–110, 1977.

[NM92] W. Neun and H. Melenk. Very large Gröbner basis calculations. *Computer Algebra and Parallelism*, pages 89–99, 1992.

[Nol53] J. Nolan. Analytic differentiation on a digital computer. Master's thesis, Math. Dept. M.I.T., 1953.

[Ost45] M.W. Ostrogradski. De l'intégration des fractions rationelles. *Bull. Acad. Imp. Sci. St. Petersburg (Class Phys.-Math.)*, 4:145–167, 1845.

[Pan02] V.Y. Pan. Univariate Polynomials: Nearly Optimal Algorithms for Numerical Factorization and Root-finding. *J. Symbolic Comp.*, 33:701–733, 2002.

[Pau07] F. Pauer. Gröbner bases with coefficients in rings. *J. Symbolic Computation*, 42:1003–1011, 2007.

[Per09] J. Perry. An extension of Buchberger's criteria for Groebner basis decision. `http://arxiv.org/abs/0906.4358`, 2009.

[Pla77] D.A. Plaisted. Sparse Complex Polynomials and Irreducibility. *J. Comp. Syst. Sci.*, 14:210–221, 1977.

[PPR14] R. Pemantle, Y. Peres, and I. Rivin. Four random permutations conjugated by an adversary generate S_n with high probability. `http://arxiv.org/abs/1412.3781`, 2014.

[PPR15] R. Pemantle, Y. Peres, and I. Rivin. Four random permutations conjugated by an adversary generate S_n with high probability. *Random Structures & Algorithms*, 49:409–428, 2015.

[PQR09] A. Platzer, J.-D. Quesel, and P. Rümmer. Real World Verificatiom. In R.A. Schmidt, editor, *Proceedings CADE 2009*, pages 485–501, 2009.

[Pri03] H.A. Priestley. Introduction to Complex Analysis (second edition). *Oxford University Press*, 2003.

[PW85] R. Pavelle and P.S. Wang. MACSYMA from F to G. *J. Symbolic Comp.*, 1:69–100, 1985.

[Rab80] M.O. Rabin. Probabilistic Algorithm for Testing Primality. *J. Number Theory*, 12:128–138, 1980.

[RESW14] S. Ruggieri, P. Eirinakis, K. Subramani, and P. Wojciechowski. On the complexity of quantified linear systems. *Theoretical Computer Science*, 518:128–135, 2014.

[Reu96] C. Reutenauer. Inversion height in free fields. *Selecta Mathematica New Series*, 2:93–109, 1996.

[Ric68] D. Richardson. Some Unsolvable Problems Involving Elementary Functions of a Real Variable. *Journal of Symbolic Logic*, 33:514–520, 1968.

[Ric97] D.S. Richardson. How to Recognize Zero. *J. Symbolic Comp.*, 24:627–645, 1997.

[Ris69a] R.H. Risch. Further Results on Elementary Functions. Technical Report RC 2402 IBM Yorktown Heights, 1969.

[Ris69b] R.H. Risch. The Problem of Integration in Finite Terms. *Trans. A.M.S.*, 139:167–189, 1969.

[Ris70] R.H. Risch. The Solution of the Problem of Integration in Finite Terms. *Bulletin A.M.S.*, 76:605–608, 1970.

[Ris79] R.H. Risch. Algebraic Properties of the Elementary Functions of Analysis. *Amer. J. Math.*, 101:743–759, 1979.

[Ris85] J.-J. Risler. Additive Complexity of Real Polynomials. *SIAM J. Comp.*, 14:178–183, 1985.

[Ris88] J.-J. Risler. Some Aspects of Complexity in Real Algebraic Geometry. *J. Symbolic Comp.*, 5:109–119, 1988.

[Rit32] J.F. Ritt. *Differential Equations from an Algebraic Standpoint.* Volume 14. American Mathematical Society, 1932.

[Rit48] J.F. Ritt. *Integration in Finite Terms, Liouville's Theory of Elementary Methods.* Columbia University Press, 1948.

[Rob85] L. Robbiano. Term orderings on the Polynomial Ring. In *Proceedings EUROCAL 85*, pages 513–525, 1985.

[Roc14] D. Roche. Polynomial interpolation. *Private Communication*, 2014.

[Ron11] X. Rong. Generating Sets. *Private Communication*, 2011.

[Rot76] M. Rothstein. *Aspects of Symbolic Integration and Simplification of Exponential and Primitive Functions.* PhD thesis, Univ. of Wisconsin, 1976.

[RR90] J.-J. Risler and F. Ronga. Testing Polynomials. *J. Symbolic Comp.*, 10:1–5, 1990.

[RS79] A.D. Rich and D.R. Stoutemyer. Capabilities of the MUMATH-79 Computer Algebra System for the INTEL-8080 Microprocessor. In *Proceedings EUROSAM 79*, pages 241–248, 1979.

[RS92] A.D. Rich and D.R. Stoutemyer. DERIVE Reference Manual. *SoftWarehouse*, 1992.

[RS13] A.D. Rich and D.R. Stoutemyer. Representation, simplification and display of fractional powers of rational numbers in computer algebra. `http://arxiv.org/abs/1302.2169`, 2013.

[Rup86] W.M. Ruppert. Reduzibilität ebener Kurven. *J. Reine Angew. Math.*, 369:167–191, 1986.

[Sag14] M. Sagraloff. A Near-Optimal Algorithm for Computing Real Roots of Sparse Polynomials. In K. Nabeshima, editor, *Proceedings ISSAC 2014*, pages 359–366, 2014.

[Sch71] A. Schönhage. Partial and total matrix multiplication. *SIAM J. Comp.*, 10:434–455, 1971.

[Sch82] A. Schönhage. The Fundamental theorem of Algebra in Terms of Computational Complexity. *Tech. Rep. U. Tübingen*, 1982.

[Sch00a] A. Schinzel. *Polynomials with Special Regard to Irreducibility.* C.U.P., 2000.

[Sch00b] C. Schneider. An implementation of Karr's summation algorithm in Mathematica. *Sém. Lothar. Combin.*, S43b:1–20, 2000.

[Sch03a] A. Schinzel. On the greatest common divisor of two univariate polynomials, I. In *A Panorama of number theory or the view from Baker's garden*, pages 337–352. C.U.P., 2003.

[Sch03b] H. Schönemann. Singular in a Framework for Polynomial Computations. *Algebra Geometry and Software Systems*, pages 163–176, 2003.

[Sch04] C. Schneider. Symbolic Summation with Single-Sum Extensions. In J. Gutierrez, editor, *Proceedings ISSAC 2004*, pages 282–289, 2004.

[Sch15] H. Schönemann. Geobuckets in SINGULAR. *Personal Communication*, 2015.

[Sco15] M. Scott. Missing a trick: Karatsuba variations. `http://eprint.iacr.org/2015/1247.pdf`, 2015.

[SD70] H.P.F. Swinnerton-Dyer. Letter to E.H. Berlekamp. *Mentioned in [Ber70]*, 1970.

[Sei54] A. Seidenberg. A new decision method for elementary algebra. *Ann. Math.*, 60:365–374, 1954.

[Sen08] J.R. Sendra. Algebraic Curves Soluble by Radicals. `http://arxiv.org/abs/0805.3214`, 2008.

[SGV94] A. Schönhage, A.F.W. Grotefeld, and E. Vetter. Fast Algorithms: A Multitape Turing Machine Implementation. *BI Wissenschaftsverlag*, 1994.

[Sla61] J. Slagle. *A Heuristic Program that Solves Symbolic Integration Problems in Freshman Calculus.* PhD thesis, Harvard U., 1961.

[Slo07] N.J.A. Sloane. The Online Encyclopedia of Integer Sequences. `http://www.research.att.com/~njas/sequences`, 2007.

[Smi76] J. Smit. The Efficient Calculation of Symbolic Determinants. In R.D. Jenks, editor, *Proceedings SYMSAC 76*, pages 105–113, 1976.

[Smi79] J. Smit. New Recursive Minor Expansion Algorithms, a Presentation in a Comparative Context. In *Proceedings EUROSAM 79*, pages 74–87, 1979.

[SS71] A. Schönhage and V. Strassen. Schnelle Multiplikation großer Zahlen. *Computing*, 7:282–292, 1971.

[SS06] A.P. Sexton and V. Sorge. Abstract matrices in symbolic computation. In *Proceedings ISSAC 2006*, pages 318–325, 2006.

[SS11] J. Schicho and D. Sevilla. Effective radical parametrization of trigonal curves. `http://arxiv.org/abs/1104.2470`, 2011.

[ST92] J.H. Silverman and J. Tate. Rational Points on Elliptic Curves. *Springer-Verlag*, 1992.

[Ste74] G. Stengle. A Nullstellensatz and a Positivstellensatz in Semialgebraic Geometry. *Mathematische Annalen*, 207:87–97, 1974.

[Sto77] D.R. Stoutemyer. sin(x)**2 + cos(x)**2 = 1. In *Proceedings 1977 MACSYMA Users' Conference*, pages 425–433, 1977.

[Sto11a] D. Stoutemyer. Ten commandments for good default expression simplification. *J. Symbolic Comp.*, 46:859–887, 2011.

[Sto11b] D. Stoutemyer. Ways to implement computer algebra compactly. *Comm. Computer Algrebra*, 178:199–224, 2011.

[Sto13] D.R. Stoutemyer. Can the Eureqa Symbolic Regression Program, Computer Algebra, and Numerical Analysis Help Each Other? *Notices A.M.S.*, 60:713–724, 2013.

[Str69] V. Strassen. Gaussian Elimination is not Optimal. *Numer. Math.*, 13:354–356, 1969.

[Stu96] T. Sturm. Real quadratic quantifier elimination in RISA/ASIR. Technical Report Memorandum ISIS-RM-5E ISIS Fujitsu Laboratories Limited, 1996.

[Tak10] D. Takahashi. Parallel implementation of multiple-precision arithmetic and 2,576,980,370,000 decimal digits of π calculation. *Parallel Computing*, 36:439–448, 2010.

[Tar51] A. Tarski. *A Decision Method for Elementary Algebra and Geometry.* 2nd ed., Univ. Cal. Press. Reprinted in *Quantifier Elimination and Cylindrical Algebraic Decomposition* (ed. B.F. Caviness & J.R. Johnson), Springer-Verlag, Wein-New York, 1998, pp. 24–84., 1951.

[TE07] E.P. Tsigaridas and I.Z. Emiris. Univariate polynomial real root isolation: Continued Fractions revisited. *Proc. 14th European Symp. Algorithms, Springer Lecture Notes in Computer Science*, 4168:817–828, 2007.

[Tra76] B.M. Trager. Algebraic Factoring and Rational Function Integration. In R.D. Jenks, editor, *Proceedings SYMSAC 76*, pages 219–226, 1976.

[Tra84] B.M. Trager. *Integration of Algebraic Functions.* PhD thesis, M.I.T. Dept. of Electrical Engineering and Computer Science, 1984.

[Tra88] C. Traverso. Gröbner trace algorithms. In P. Gianni, editor, *Proceedings ISSAC 1988*, pages 125–138, 1988.

[Tra00] Q.-N. Tran. A Fast Algorithm for Gröbner Basis Conversion and its Applications. *J. Symbolic Comp.*, 30:451–467, 2000.

[Tri78] W. Trinks. Über B. Buchbergers Verfahren, Systeme algebraischer Gleichungen zu lösen. *J. Number Theory*, 10:475–488, 1978.

[vdW34] B.L. van der Waerden. Die Seltenheit der Gleichungen mit Affekt. *Mathematische Annalen*, 109:13–16, 1934.

[vH02] M. van Hoeij. Factoring polynomials and the knapsack problem. *J. Number Theory*, 95:167–189, 2002.

[vH15] M. van Hoeij. Groebner basis in Boolean rings is not polynomial-space. `http://arxiv.org/abs/1502.07220`, 2015.

[vHM02] M. van Hoeij and M. Monagan. A Modular GCD Algorithm over Number Fields Presented with Multiple Extensions. In T. Mora, editor, *Proceedings ISSAC 2002*, pages 109–116, 2002.

[vHM04] M. van Hoeij and M. Monagan. Algorithms for Polynomial GCD Computation over Algebraic Function Fields. In J Gutierrez, editor, *Proceedings ISSAC 2004*, pages 297–304, 2004.

[vHM16] M. van Hoeij and M. Monagan. A Modular Algorithm for Computing Polynomial GCDs over Number Fields presented with Multiple Extensions. `http://arxiv.org/abs/1601.01038`, 2016.

[Vor10] S. Vorkoetter. Maple kernel. *E-mail 4BF15DC8.7030207@maplesoft.com*, 2010.

[vT83] E.W. von Tschirnhaus. Methodus auferendi omnes terminos intermedios ex data aeqvatione. *Acta Eruditorium*, ?:204–207, 1683.

[vzG85] J. von zur Gathen. Irreducibility of multivariate polynomials. *J. Computer Syst. Sci.*, 31:225–264, 1985.

[vzGG99] J. von zur Gathen and J. Gerhard. Modern Computer Algebra. *C.U.P.*, 1999.

[vzGP01] J. von zur Gathen and D. Panario. Factoring Polynomials Over Finite Fields: A Survey. *J. Symbolic Comp.*, 31:3–17, 2001.

[vzGS92] J. von zur Gathen and V. Shoup. Computing Frobenius maps and Factoring Polynomials. *Computing Complexity*, 2:187–224, 1992.

[Wal14] M. Waldschmidt. Schanuel's Conjecture: algebraic independence of transcendental numbers. `https://webusers.imj-prg.fr/~michel.waldschmidt/articles/pdf/ColloquiumDeGiorgiSchanuelConjecture2014.pdf`, 2014.

[Wan71a] P.S. Wang. Automatic Computation of Limits. In *Proceedings Second Symposium on Symbolic and Algebraic Manipulation*, pages 458–464, 1971.

[Wan71b] P.S. Wang. *Evaluation of Definite Integrals by Symbolic Manipulation.* PhD thesis, M.I.T & Project MAC TR-92, 1971.

[Wan76] P.S. Wang. Factoring Multivariate Polynomials over Algebraic Number Fields. *Math. Comp.*, 30:324–336, 1976.

[Wan78] P.S. Wang. An Improved Multivariable Polynomial Factorising Algorithm. *Math. Comp.*, 32:1215–1231, 1978.

[Wan81] P.S. Wang. A p-adic Algorithm for Univariate Partial Fractions. In *Proceedings SYMSAC 81*, pages 212–217, 1981.

[Wei88] V. Weispfenning. The Complexity of Linear Problems in Fields. *J. Symbolic Comp.*, 5:3–27, 1988.

[Wei92] V. Weispfenning. Comprehensive Gröbner Bases. *J. Symbolic Comp.*, 14:1–29, 1992.

[Wei94] V. Weispfenning. Quantifier elimination for real algebra — the cubic case. In *Proceedings ISSAC 1994*, pages 258–263, 1994.

[Wei98] V. Weispfenning. A New Approach to Quantifier Elimination for Real Algebra. *Quantifier Elimination and Cylindrical Algebraic Decomposition*, pages 376–392, 1998.

[Wei03] V. Weispfenning. Canonical Comprehensive Gröbner Bases. *J. Symbolic Comp.*, 36:669–683, 2003.

[WG93] T. Weibel and G.H. Gonnet. An Assume Facility for CAS with a Sample Implementation for Maple. In *Proceedings DISCO '92*, pages 95–103, 1993.

[WGD82] P.S. Wang, M.J.T. Guy, and J.H. Davenport. p-adic Reconstruction of Rational Numbers. *SIGSAM Bulletin 2*, 16:2–3, 1982.

[Wil59] J.H. Wilkinson. The Evaluation of the Zeros of Ill-conditioned Polynomials. *Num. Math.*, 1:150–166, 1959.

[Wil12] V.V. Williams. "multiplying matrices faster than coppersmith-winograd". In *Proceedings of the 44th symposium on Theory of Computing*, STOC '12, pages 887–898, New York, NY, USA, 2012. ACM.

[Wil14] V.V. Williams. Multiplying matrices faster than Coppersmith-Winograd. `http://theory.stanford.edu/~virgi/matrixmult-f.pdf`, 2014.

[Win71] S. Winograd. On multiplication of 2 x 2 matrices. *Linear algebra and its applications*, 4:381–388, 1971.

[Win88] F. Winkler. A p-adic Approach to the Computation of Gröbner Bases. *J. Symbolic Comp.*, 6:207–334, 1988.

[Wu86] W.-T. Wu. On zeros of algebraic equations — an application of Ritt principle. *Kexue Tongbao*, 31:1–5, 1986.

[Yan98] T. Yan. The geobucket data structure for polynomials. *J. Symbolic Comp.*, 25:285–294, 1998.

[Yap91] C.K. Yap. A new lower bound construction for commutative Thue systems with applications. *J. Symbolic Comp.*, 12:1–27, 1991.

[Yun76] D.Y.Y. Yun. On Square-free Decomposition Algorithms. In R.D. Jenks, editor, *Proceedings SYMSAC 76*, pages 26–35, 1976.

[Zar26] O. Zariski. Sull'impossibilità di risolvere parametricamente per radicali un'equazione algebrica $f(x, y) = 0$ di genere $p > 6$ a moduli generali. *Atti Accad. Naz. Lincei Rend. Cl. Sc. Fis. Mat. Natur. serie VI*, 3:660–666, 1926.

[Zas69] H. Zassenhaus. On Hensel Factorization I. *J. Number Theory*, 1:291–311, 1969.

[Zec49] J. Zech. Tafeln des Additions- und Subtractions- Logarithmen. *Weidmann*, 1849.

[Zim07] P. Zimmerman. We recommend students never to use simplify inside programs. *Personal communication*, 2007.

[Zip79a] R.E. Zippel. Probabilistic Algorithms for Sparse Polynomials. In *Proceedings EUROSAM 79*, pages 216–226, 1979.

[Zip79b] R.E. Zippel. *Probabilistic Algorithms for Sparse Polynomials.* PhD thesis, M.I.T. Dept. of EE&CS, 1979.

[Zip93] R.E. Zippel. Effective Polynomial Computation. *Kluwer Academic Publishers*, 1993.

Index

The total-degree Gröbner base from page 121.
$\{42\,c^4+555+293\,a^3+1153\,a^2b-3054\,ab^2+323\,b^3+2035\,a^2c+1642\,abc-1211\,b^2c+$
$2253\,ac^2-1252\,bc^2-31\,c^3+1347\,a^2-3495\,ab+1544\,b^2-1100\,ac+2574\,bc-368\,c^2+$
$2849\,a+281\,b+4799\,c, 21\,bc^3-57-10\,a^3-41\,a^2b+183\,ab^2-25\,b^3-104\,a^2c-$
$65\,abc+91\,b^2c-129\,ac^2+59\,bc^2-16\,c^3-33\,a^2+225\,ab-142\,b^2+67\,ac-174\,bc-$
$5\,c^2-154\,a-46\,b-310\,c, 21\,ac^3-75-29\,a^3-121\,a^2b+369\,ab^2-41\,b^3-226\,a^2c-$
$178\,abc+161\,b^2c-267\,ac^2+148\,bc^2+4\,c^3-123\,a^2+411\,ab-206\,b^2+146\,ac-$
$324\,bc+38\,c^2-329\,a-62\,b-584\,c, 14\,b^2c^2+5+a^3+9\,a^2b+2\,ab^2-b^3+23\,a^2c+$
$10\,abc-21\,b^2c+15\,ac^2-8\,bc^2+3\,c^3-3\,a^2-19\,ab-4\,b^2-6\,ac-26\,bc-10\,c^2+7\,a-b+$
$3\,c, 21\,abc^2+30-a^3-2\,a^2b-51\,ab^2+8\,b^3+40\,a^2c+25\,abc-56\,b^2c+69\,ac^2-13\,bc^2+$
$11\,c^3-18\,a^2-72\,ab+53\,b^2-8\,ac+54\,bc+10\,c^2+56\,a+29\,b+116\,c, 14\,a^2c^2+11+$
$5\,a^3+31\,a^2b-74\,ab^2+9\,b^3+45\,a^2c+22\,abc-35\,b^2c+61\,ac^2-40\,bc^2+c^3+13\,a^2-$
$95\,ab+50\,b^2-44\,ac+66\,bc+6\,c^2+63\,a+23\,b+127\,c, 21\,b^3c-6-4\,a^3+13\,a^2b+$
$48\,ab^2-10\,b^3-8\,a^2c-5\,abc-14\,b^2c+3\,ac^2-10\,bc^2+2\,c^3-30\,a^2+27\,ab-40\,b^2+$
$10\,ac-57\,bc-2\,c^2-7\,a-10\,b-40\,c, 6\,ab^2c-3-a^3-5\,a^2b+12\,ab^2-b^3-11\,a^2c-$
$2\,abc+7\,b^2c-9\,ac^2+2\,bc^2-c^3-3\,a^2+15\,ab-4\,b^2+4\,ac-6\,bc-2\,c^2-13\,a-b-$
$19\,c, 14\,a^2bc-13+3\,a^3+13\,a^2b+6\,ab^2-3\,b^3-a^2c+2\,abc+21\,b^2c-25\,ac^2+4\,bc^2-$
$5\,c^3+19\,a^2+27\,ab-26\,b^2+10\,ac-8\,bc-2\,c^2-7\,a-17\,b-33\,c, 7\,a^3c+3-5\,a^3-$
$24\,a^2b+25\,ab^2-2\,b^3-17\,a^2c-8\,abc+7\,b^2c-19\,ac^2+12\,bc^2-c^3-6\,a^2+32\,ab-$
$8\,b^2+23\,ac-17\,bc-6\,c^2-21\,a-2\,b-36\,c, 42\,b^4-3-121\,a^3-557\,a^2b+570\,ab^2+$
$275\,b^3-515\,a^2c-608\,abc-77\,b^2c-555\,ac^2+2\,bc^2-55\,c^3-645\,a^2+633\,ab-160\,b^2+$
$82\,ac-690\,bc-302\,c^2-679\,a+65\,b-1147\,c, 42\,ab^3+15-11\,a^3-85\,a^2b+6\,ab^2+$
$25\,b^3-43\,a^2c-40\,abc-49\,b^2c-39\,ac^2+4\,bc^2-5\,c^3-51\,a^2-15\,ab+58\,b^2-4\,ac+$
$6\,bc-16\,c^2-35\,a+25\,b-5\,c, 21\,a^2b^2+3+2\,a^3+25\,a^2b-45\,ab^2+5\,b^3+25\,a^2c-$
$29\,abc-14\,b^2c+9\,ac^2-16\,bc^2-c^3-6\,a^2-45\,ab+20\,b^2-47\,ac+18\,bc+c^2+14\,a+$
$5\,b+41\,c, 21\,a^3b+18-16\,a^3-74\,a^2b+24\,ab^2+2\,b^3-53\,a^2c+abc-35\,b^2c+12\,ac^2+$
$23\,bc^2+8\,c^3-36\,a^2+24\,ab+29\,b^2+40\,ac+3\,bc-8\,c^2-28\,a+23\,b-13\,c, 42\,a^4-57+$
$431\,a^3+757\,a^2b-804\,ab^2+59\,b^3+799\,a^2c-2\,abc-119\,b^2c+417\,ac^2-340\,bc^2+$
$5\,c^3+303\,a^2-1203\,ab+194\,b^2-752\,ac+246\,bc+184\,c^2+581\,a-67\,b+1013\,c\}.$

The equivalent lexicographic base.
$\{1-6\,c-41\,c^2+71\,c^3-41\,c^{18}+197\,c^{14}+106\,c^{16}-6\,c^{19}+106\,c^4+71\,c^{17}+$
$c^{20}+92\,c^5+197\,c^6+145\,c^7+257\,c^8+278\,c^9+201\,c^{10}+278\,c^{11}+257\,c^{12}+$
$145\,c^{13}+92\,c^{15}, 9741532+39671190\,c-96977172\,c^2-140671876\,c^3+7007106\,c^{18}-$
$120781728\,c^{14}-79900378\,c^{16}-1184459\,c^{19}-131742078\,c^4+49142070\,c^{17}-253979379\,c^5-$
$204237390\,c^6-337505020\,c^7-356354619\,c^8-271667666\,c^9-358660781\,c^{10}-$
$323810244\,c^{11}-193381378\,c^{12}-244307826\,c^{13}-131861787\,c^{15}+1645371\,b, -487915\,c^{18}+$
$705159\,a+4406102\,c+16292173\,c^{14}+17206178\,c^2+3830961\,c^{16}+91729\,c^{19}-$
$377534+21640797\,c^3+26686318\,c^4-4114333\,c^{17}+34893715\,c^5+37340389\,c^6+$
$47961810\,c^7+46227230\,c^8+42839310\,c^9+46349985\,c^{10}+37678046\,c^{11}+28185846\,c^{12}+$
$26536060\,c^{13}+13243117\,c^{15}\}.$