ON RATIONALITY, IS THE SERIES 1+$\frac{1}{2^5}$+$\frac{1}{3^5}$+... AN IRRATIONAL NUMBER

A RESEARCH PROJECT SUBMITTED

BY

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DECLARATION

I hereby declare that this research project is my original work and has not been submitted to any other university for a degree or any other award.

Signature........................ Date........................

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This research project has been written and submitted for review with approval of my supervision.

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DEDICATION

This work is solely dedicated to my brother for the unwavering support, encouragement, and belief in me which have been my driving force throughout this journey. He has always been there to lend a helping hand, offer words of wisdom, and cheer me on, even during the toughest times from the start and completion of my project.

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Lastly, I would like to take a moment to acknowledge the hard work and dedication I have invested in this project. The countless hours of research, analysis, and writing have truly paid off, and I am proud of the outcome. This project has been a valuable learning experience, and I am grateful for the opportunity to push myself and expand my knowledge.

ABSTRACT

This research project embarks on checking whether the sum of the series $1+\frac{1}{2^5}+\frac{1}{3^5}+...$ is irrational.The research commences by defining a series,an irrational number and its properties.Drawing from formulas used to solve the sum of the series $1+\frac{1}{2^2}+\frac{1}{3^2}+...$(sum of reciprocals of squares) and $1+\frac{1}{2^4}+\frac{1}{3^4}+...$, I have proceeded to use the same to compute the approximate value of the sum of our given series.To achieve my objective,I use python to compute the summation of the series approximated to the different nth values and in my case I have computed upto the value $\frac{1}{10^5}$.Moreover ,I analyze the behaviours of the solutions; i.e whether they converge or diverge.

Keywords: Irrational numbers,convergence,divergence,rational numbers

Chapter 1

INTRODUCTION

1.1 Introduction

The series $1+\frac{1}{2^5}+\frac{1}{3^5}+$...is a specific type of infinite series known as a p-series(A p-series in mathematics refers to an infinite series of the form $\sum_{n=1}^{\infty}\frac{1}{n^p}$ where p is a positive real number. The p-series is named because of its dependence on the exponent p.). In this case, it is a p-series with p=5.The series $1+\frac{1}{2^5}+\frac{1}{3^5}+$... can be represented using sigma notation as $\sum_{n=1}^{\infty}\frac{1}{n^5}$

Series

Let a_1,a_2, a_3,... be the sequence,then the expression a_1+a_2+a_3+...is called the series **S** associated with given sequence. It can be finite or infinite according to the given sequence.

Definition 1. An **irrational number** is a real number that cannot be expressed as a ratio of integers i.e if p and q are integers, they cannot be expressed as p/q,q $\neq 0$

Example

square root of primes - $\sqrt{2}$,$\sqrt{5}$

Definition 2. A series is said to **converge** if and only if there exists a number l such that for every arbitrarily small positive number ε , there is integer N such that for all n $\geq$ N, Ł< ε

Examples

1.To determine whether the series $1+\frac{1}{2^4}+\frac{1}{3^4}+$... converges, we can use the p-series test. The p-series test states that a series of the form $\sum\frac{1}{n^p}$ converges if $p>1$. in this case the series can be represented as $\sum\frac{1}{n^4}$ where p=4 .since $p>1$,p= $4>1$, the series $\sum\frac{1}{n^4}$ converges and therefore $1+\frac{1}{2^4}+\frac{1}{3^4}+$...converges.

2.The series 1- $\frac{1}{2}$ +$\frac{1}{3}$ -$\frac{1}{4}$ + $\frac{1}{5}$ coverges to ln 2

Definition 3. A series is said to **diverge** if it keeps on either increasing or decreasing as the terms of series tends to ∞

Divergence tests

1. integral test Compare the series to an improper integral to check for divergence.If the improper interval associated with the series diverges then the series diverges.

2. Ratio test Check the limit of ratio of consecutive terms of the series.If the Limit is greater than 1, then the series diverges.

Example $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+$.... ∞

1.2 Background Information

History of irrational numbers

They were discovered by Hippasus of Metapontum, a greek philosopher and early follower of Pythagoras in the 5th Century BC.According to a famous anecdote, the Greek mathematician was drowned at sea for challenging the ratio doctrine of numbers

Background:
Euler proved in the 1700's that the infinite series $1+\frac{1}{2^2}+\frac{1}{3^2}+...$ adds up to $\frac{\pi^2}{6}$ and the infinite series $1+\frac{1}{2^4}+\frac{1}{3^4}+...$adds upto to $\frac{\pi^4}{90}$.Infact he found a method fo finding similar similar formulas for any sum of the form $1+\frac{1}{2^k}+\frac{1}{3^k}+...$provided k is an even number.No such simple formula has yet been found for the sum of the series where k is an odd number.However,for k=3 has been proven to be an irrational number.

1.3 Statement of the problem

In the 1700's, Euler studied and concluded that $\sum_{n=1}^{\infty}\frac{1}{n^4}=\frac{\pi^4}{90}$ and $\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$.The concept of a series forming an irrational number has been proved by many authors.It was proved by Euler not too long ago that the sum of the series $1+\frac{1}{2^3}+\frac{1}{3^3}+...$ is an irrational number and this raised the question is the sum of the series $1+\frac{1}{2^5}+\frac{1}{3^5}+...$an irrational number.It was not proved and that is why I study the series in this project.

1.4 Objectives

1.4.1 General obejective

To apply mathematical techniques such as Comparison Test to analyze and understand the convergence properties of the series $1+\frac{1}{2^5}+\frac{1}{3^5}+...$ within the broader context of infinite series analysis.

1.4.2 Specific Objectives

1. Compute the exact sum of the series $1+\frac{1}{2^5}+\frac{1}{3^5}+...$upto $\frac{1}{10^5}$

2. Determine whether the series $1+\frac{1}{2^5}+\frac{1}{3^5}+...$ converges or diverges with random nth values using the P-test and comparison test.

3. Analyze the error when truncating the series $1+\frac{1}{2^5}+\frac{1}{3^5}+...$ to a finite number of terms

1.5 Significance of study

The study of the sum of the series $1+\frac{1}{2^5}+\frac{1}{3^5}+...$ has various practical implications.

- Engineering and construction - in construction, precise measurement involving square roots and other irrational values are necessary fo designing structures like arches,bridges and buldings.

- Computer science - used in various algorithms and calculations .In numerical analysis,algorithms for solving equations or optimizing function often involve irrational numbers.

Chapter 2

LITERATURE REVIEW

The study of the series $1+\frac{1}{2^5}+\frac{1}{3^5}+...$ was imposed by Leonhard Euler in the 1700's after he worked on $1+\frac{1}{2^3}+\frac{1}{3^3}+...$ and found out that its sum was an irrational number. So then the next series with n as an odd number was $...+\frac{1}{n^5}$ and its irrationability remains unproven.

Ivan Niven (1958) in his book [Irrational Numbers] gives out a theorem;

Theorem 1.1: Almost all real numbers are irrational

PROOF: Firstly, we need to prove that the positive rational numbers in the unit interval (0,1)have measure zero.These numbers are the rationals h/k with $1 \leqq h \leqq k$, and each such rational can be enclosed with an interval $(h/k\text{-}\varepsilon/k^3)$.

To avoid unnecessary duplication of the irrational,a restriction can be imposed that h and k be relatively prime but the present proof is correct with or without this restriction. The total length of these intervals is
$\leqq \sum_{k=1}^{\infty}\sum_{h=1}^{k}\frac{2\varepsilon}{k^3}=\sum_{k=1}^{\infty}\frac{2\varepsilon}{k^2}\leq 2\varepsilon+\sum_{k=2}^{\infty}\frac{2\varepsilon}{k(k-1)}$
$=2\varepsilon+\sum_{k=2}^{\infty}2\varepsilon(\frac{1}{k-1}-\frac{1}{k})=4\varepsilon$ which can be made arbitrarily small by suitable choice of ε.
This covering procedure can be extended to all positive rationals by observing that the reciprocal of any $h/k \leqq 1$ is $k/h \geqq 1$ and can be enclosed in an interval $(k/h\text{-}\varepsilon/k^3,\ k/h+\varepsilon/k^3)$ The estimate of the total lengths of these intervals is identical with that above, and so can be made arbitrarily small. The proof of the theorem is completed by the obvious extension to the negative rational numbers and zero.

Purav Patel AND Sashank Varma in their book "UNDERSTANDING IRRATIONAL NUMBERS" states that; It is with the irrational numbers, which include $\sqrt{2}$ and π, that mathematicians discovered a number system lacking material referents or models that build on intuition (Struik, 1987). Such abstraction is associated with many surprising properties. For instance, there are more irrational numbers than natural numbers, integers, or rational numbers. The set of all irrationals is uncountably infinite in cardinality, whereas the latter sets are each countably finite (Struik, 1987). Mathematics teachers, in an attempt to facilitate understanding, encourage students to think of irrational numbers like π by using rational number approximations like 3.14.

The uncountability of irrational numbers is further supported by Ivan Niven with a theorem in his book.

Theorem: The irrational numbers are not countable.A fortiori the real numbers are not countable

PROOF

Suppose the irrational numbers were countable , say *y1,y2,y3*....But the rationals are countable by say *r1,r2,r3*....Thus we could intersperse these two sequences to get a single sequence *r1,y1,r2,y2,r3,y3*....Thus the reals would be countable and so would have measure zero.But this would say that the whole real line can be covered by a set of intervals of arbitrarily small total length, a contradiction from which the theorem follows.

Chapter 3

METHODOLOGY

The methodology begins with a review of what other authors have established about the topic. Computational tools, (in this case Python), have been utilized to compute the exact sum of series $1+\frac{1}{2^5}+\frac{1}{3^5}+...$ upto $\frac{1}{10^5}$ values.

3.1 Python Code

”sum = 1
limit = 10
for i in range(2,limit+1):
sum = sum+(1/i**5)
print(”Adding upto 1/”+str(i)+”^{5}: ” + str(sum))
if(type(sum) == ’integer’):
print(”The rational sum is: ”+str(sum)+”when adding upto 1/”+str(i)+”^{5}”)”

3.2 Code Output

Adding upto $1/2^5$: 1.03125

Adding upto $1/3^5$: 1.0353652263374487

Adding upto $1/4^5$: 1.0363417888374487

Adding upto $1/5^5$: 1.0366617888374487

Adding upto $1/6^5$: 1.036790389660494

Adding upto $1/7^5$: 1.0368498886787603

Adding upto $1/8^5$: 1.0368804062568853

Adding upto $1/9^5$: 1.0368973413446938

Adding upto $1/10^5$: 1.0369073413446939

Chapter 4

ERROR

4.1 Computation of approximate sums using a calculator

$1+\frac{1}{2^5}$

$1+\frac{1}{2^5}+\frac{1}{3^5}$

$1+\frac{1}{2^5}+\frac{1}{3^5}+\frac{1}{4^5}$

$1+\frac{1}{2^5}+\frac{1}{3^5}+\frac{1}{4^5}+\frac{1}{5^5}$

$1+\frac{1}{2^5}+\frac{1}{3^5}+\frac{1}{4^5}+\frac{1}{5^5}+\frac{1}{6^5}$

$1+\frac{1}{2^5}+\frac{1}{3^5}+\frac{1}{4^5}+\frac{1}{5^5}+\frac{1}{6^5}+\frac{1}{7^5}$

$1+\frac{1}{2^5}+\frac{1}{3^5}+\frac{1}{4^5}+\frac{1}{5^5}+\frac{1}{6^5}+\frac{1}{7^5}+\frac{1}{8^5}$

$1+\frac{1}{2^5}+\frac{1}{3^5}+\frac{1}{4^5}+\frac{1}{5^5}+\frac{1}{6^5}+\frac{1}{7^5}+\frac{1}{8^5}+\frac{1}{9^5}$

$1+\frac{1}{2^5}+\frac{1}{3^5}+\frac{1}{4^5}+\frac{1}{5^5}+\frac{1}{6^5}+\frac{1}{7^5}+\frac{1}{8^5}+\frac{1}{9^5}+\frac{1}{10^5}$

All this summation can we written in simplified fraction form as;

$1+\frac{1}{32}=1.03125$

$1+\frac{1}{32}+\frac{1}{243}=1.035365226$

$1+\frac{1}{32}+\frac{1}{243}+\frac{1}{1024}=1.036661789$

$1+\frac{1}{32}+\frac{1}{243}+\frac{1}{1024}+\frac{1}{3125}=1.036661789$

$1+\frac{1}{32}+\frac{1}{243}+\frac{1}{1024}+\frac{1}{3125}+\frac{1}{7776}=1.03679039$

$1+\frac{1}{32}+\frac{1}{243}+\frac{1}{1024}+\frac{1}{3125}+\frac{1}{7776}+\frac{1}{16807}=1.036849889$

$1+\frac{1}{32}+\frac{1}{243}+\frac{1}{1024}+\frac{1}{3125}+\frac{1}{7776}+\frac{1}{16807}+\frac{1}{32768}=1.036880406$

$1+\frac{1}{32}+\frac{1}{243}+\frac{1}{1024}+\frac{1}{3125}+\frac{1}{7776}+\frac{1}{16807}+\frac{1}{32768}+\frac{1}{59049}=1.036897341$

$1+\frac{1}{32}+\frac{1}{243}+\frac{1}{1024}+\frac{1}{3125}+\frac{1}{7776}+\frac{1}{16807}+\frac{1}{32768}+\frac{1}{59049}+\frac{1}{100000}=1.036907341$

4.2 ERROR=EXACT-APPROXIMATE VALUE

$1+\frac{1}{2^5}$ [1.03125-1.03125]=0

$1+\frac{1}{2^5}+\frac{1}{3^5}$ [1.0353652263374487-1.035365226]=0.000000000337448

$1+\frac{1}{2^5}+\frac{1}{3^5}+\frac{1}{4^5}$ [1.0363417888374487-1.036341789]=-0.000000000162552

$1+\frac{1}{2^5}+\frac{1}{3^5}+\frac{1}{4^5}+\frac{1}{5^5}$ [1.0366617888374487-1.036661789]=-0.000000000162552

$1+\frac{1}{2^5}+\frac{1}{3^5}+\frac{1}{4^5}+\frac{1}{5^5}+\frac{1}{6^5}$ [1.036790389660494-1.03679039]=-0.000000000339506

$1+\frac{1}{2^5}+\frac{1}{3^5}+\frac{1}{4^5}+\frac{1}{5^5}+\frac{1}{6^5}+\frac{1}{7^5}$ [1.0368498886787603-1.036849889]=-0.00000000032124 $1+\frac{1}{2^5}+\frac{1}{3^5}+\frac{1}{4^5}+\frac{1}{5^5}+\frac{1}{6^5}+\frac{1}{7^5}+\frac{1}{8^5}$ [1.03688040625
1.036880406]= 0.000000000256885

$1+\frac{1}{2^5}+\frac{1}{3^5}+\frac{1}{4^5}+\frac{1}{5^5}+\frac{1}{6^5}+\frac{1}{7^5}+\frac{1}{8^5}+\frac{1}{9^5}$ [1.0368973413446938-1.036897341]=0.000000000344693829

$1+\frac{1}{2^5}+\frac{1}{3^5}+\frac{1}{4^5}+\frac{1}{5^5}+\frac{1}{6^5}+\frac{1}{7^5}+\frac{1}{8^5}+\frac{1}{9^5}+\frac{1}{10^5}$ [1.0369073413446939-1.036907341]=0.000000000344693829

Chapter 5

Testing For Convergence

Series of the type $\sum_{n=a}^{\infty} \frac{1}{n^p}$(where a $\geq$ 1) which is a p series converges or diverges by comparing it to an improper integral

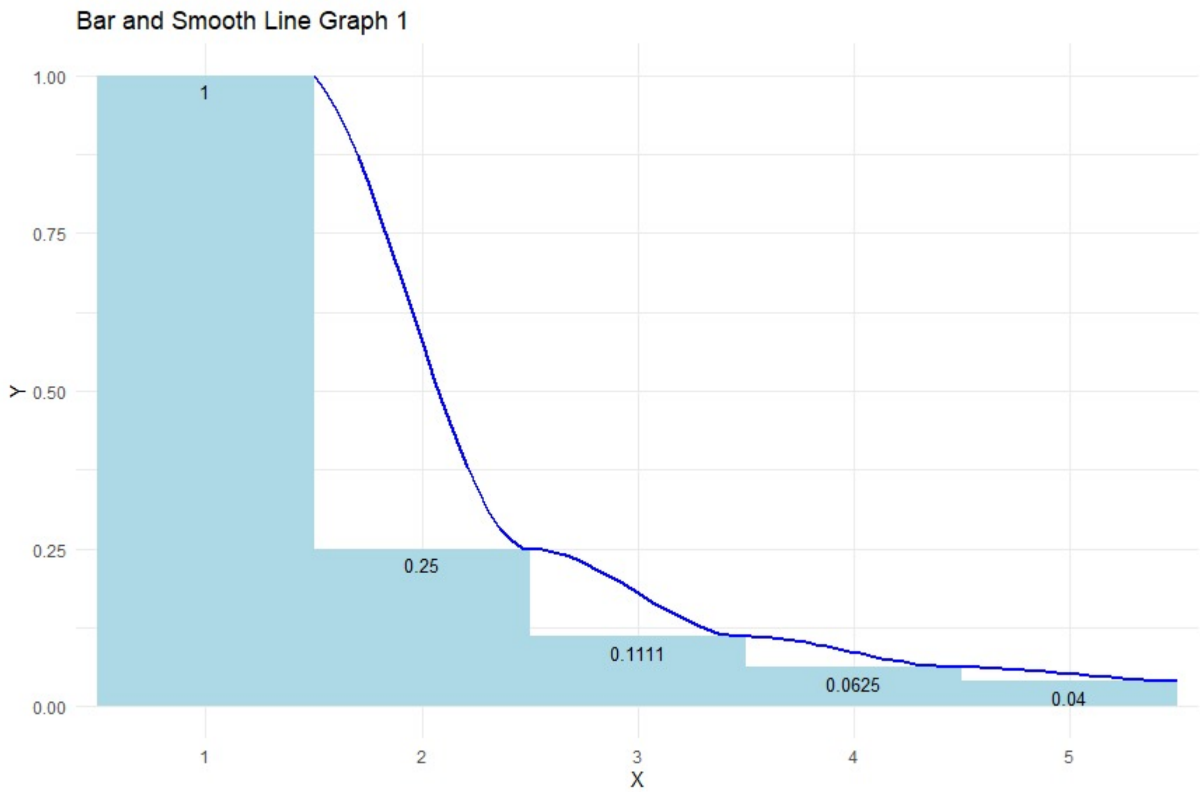
Integral Test

Suppose f(x) is a positive decreasing continous function on the interval (1,∞) with f(n) =a_n . Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if $\int_1^{\infty}$ f(x)dx converges that is:
If $\int_1^{\infty}$ f(x)dx is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.
If $\int_1^{\infty}$ f(x)dx is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.

$\int_1^{\infty} \frac{1}{x^p}$ dx converges if p >1 and diverges if p $\leq$1

Let's compare the series

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ to the improper integral $\int_1^{\infty} \frac{1}{x^2}$ dx $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+...=\sum_{n=1}^{\infty} \frac{1}{n^2}$



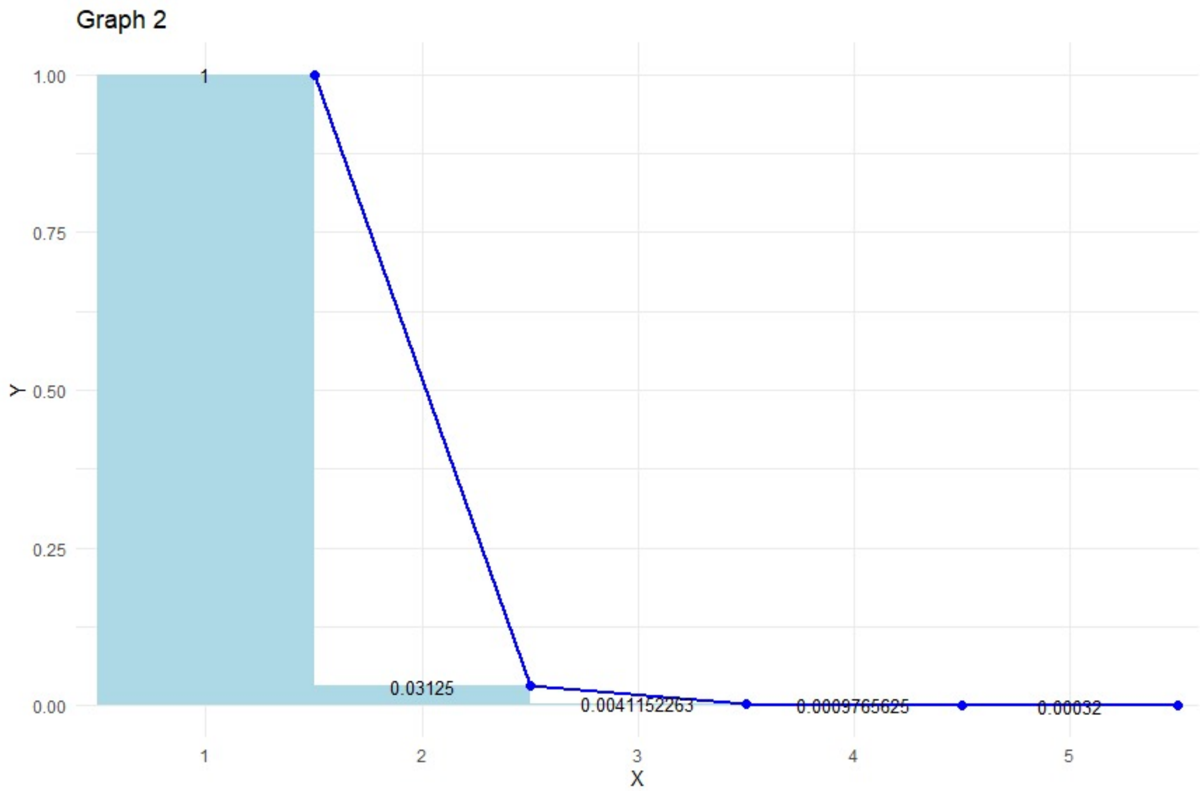
By comparing areas in the graph we see that ;

S_n=[1+$\sum_{n=2}^{\infty} \frac{1}{n^2}$ =$\frac{\pi^2}{6}$]+[$\int_1^{\infty} \frac{1}{x^2}$dx=1]

Since the sequence S_n is increasing(because each a_n) and bounded so we concluded that the sequence of partial sums converges and hence the series $\sum_{i=1}^{\infty} \frac{1}{n^2}$ converge.

So using our series $1+\frac{1}{2^5}+\frac{1}{3^5}+\frac{1}{4^5+\ldots}$ we shall compare it with the improper integral $\int_1^{\infty} \frac{1}{x^5}$dx

$1+\frac{1}{2^5}+\frac{1}{3^5}+\frac{1}{4^5}+\ldots=\sum_{n=1}^{\infty} \frac{1}{n^5}$



Having $\int_1^{\infty} \frac{1}{x^5}$dx makes p>1 and so the series converges

Following that if $\int_1^{\infty}$f(x)dx is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent makes $\sum_{n=1}^{\infty} \frac{1}{n^5}$ convergent.

Chapter 6

Findings

- The series $\frac{1}{1^5}+\frac{1}{2^5}+\frac{1}{3^5}+\frac{1}{4^5}+\frac{1}{5^5}+...$ converges as proved by the tests of convergence.
- The series $\frac{1}{1^5}+\frac{1}{2^5}+\frac{1}{3^5}+\frac{1}{4^5}+\frac{1}{5^5}+...$also possesses the absolute convergence property.
- Having its summation as $\sum_{n=1}^{\infty}\frac{1}{n^5}$ with p=5 in which 5>1 confirms its absolute convergence.

  This is the series in absolute form $|1|+|\frac{1}{32}|+|\frac{1}{243}|+|\frac{1}{1024}|+...$
- For even positive integers like 2,4,6... ,the sum of the series are generally transcendental numbers and irrational

  $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+...=\frac{\pi^2}{6}$

  $\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\frac{1}{5^4}+...=\frac{\pi^4}{90}$

  $\frac{1}{1^6}+\frac{1}{2^6}+\frac{1}{3^6}+\frac{1}{4^6}+\frac{1}{5^6}+...=\frac{\pi^6}{945}$
- For odd integers like 3,5,7...the situation is more complex.

  Roger Apéry's theorem from 1978 has proven for the irrationality of the series $\frac{1}{1^3}+\frac{1}{2^3}+\frac{1}{3^3}+\frac{1}{4^3}+\frac{1}{5^3}+...$It states that the value of the Riemann zeta function at S=3 denoted as ζ (3) is irrational.
- **Error**

  The error obtained from computing the exact and approximate summmation of the series $\frac{1}{1^5}+\frac{1}{2^5}+\frac{1}{3^5}+\frac{1}{4^5}+\frac{1}{5^5}+...$ keeps on fluctuating between -ve and +ve values due to the rounding off of the exact sum values obtained.

Chapter 7

Conclusion and Recommendation

7.1 Conclusion

Based on this study ,it is observed that:

The number 1.0369073413446939,which is the exact sum of the series truncated to $...+\frac{1}{10^5}$ is not an irrational number because it has a finite decimal expansion, which can be expressed as a ratio of two integers as $\frac{10369073413446939}{10000000000000000}$

7.2 Recommendation

In this research,I focused on the series $\frac{1}{1^5}+\frac{1}{2^5}+\frac{1}{3^5}+\frac{1}{4^5}+\frac{1}{5^5}+...$ and its properties. I therefore recommend the following;

1.The study can be extended to compute the sum of a series with k as 7 since it is the next odd number after 5.

2.Study the behaviour of the errors obtained when comparing the approximate and exact summations.

3. Evaluate if the sum of the series to infinity without truncation is irrational.

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