Solutions Manual

Zill

A First Course in Complex Analysis with Applications 2E

Preface

This student study guide is designed to accompany the text *A First Course in Complex Analysis with Applications, Second Edition* (Jones and Bartlett Publishers, 2009) by Dennis G. Zill and Patrick D. Shanahan. It consists of seven chapters which correspond to the seven chapters of the text. Each chapter has the following features.

Review Topics

Many sections of the study guide are preceded by a review of selected topics from calculus and differential equations that are required for that section. These reviews provide concise summaries of prerequisite notation, terminology, and concepts. For additional review, students are encouraged to consult appropriate mathematics texts. Two excellent sources that were used repeatedly for the review topics are *Calculus: Early Transcendentals, Fourth Edition* (Jones and Bartlett Publishers, 2010) by Dennis G. Zill and Warren S. Wright and *Advanced Engineering Mathematics, Third Edition* (Jones and Bartlett Publishers, 2006) by Dennis G. Zill and Michael R. Cullen.

Summaries

A summary of every section of the text is provided. The summary reviews all of the key ideas of the section including all terminology, formulas, theorems, and concepts. Figures with two colors are included to aid in geometric understanding.

Exercises

Following the summary, complete solutions are given for every other odd exercise in the section (eg. problems 3, 7, 11, etc.). These are full solutions, supported by figures with two colors, that supply all of the pertinent details of the problem and incorporate the same techniques and writing style used in the text. The solutions also include references to equations, definitions, theorems, and figures in the text. The answer to each problem is typeset in color for easy reference.

Focus on Concepts

The focus on concepts problems from the text consist of conceptual word, proof, and geometrical problems. Since they are often used as topics for classroom discussion or independent study we have included detailed hints rather than full solutions for these problems. As with the exercises, only every other odd problem is included.

Final Note to Students

The most effective way to learn mathematics is to work many, many problems. You should not review a solution in this study guide before first working or, at the very least, attempting to work the problem yourself. Learning advanced mathematical topics takes significant time and effort. It may be quicker to look at the solutions, then try to work problems, but ultimately this approach will not lead to an independent understanding of concepts and problem solving strategies that are required for success.

Chapter 1

Complex Numbers and the Complex Plane

1.1 Complex Numbers and Their Properties

1.1 Summary

imaginary unit: The imaginary unit i denotes $\sqrt{-1}$, thus is defined by the property $i^2 = -1$.

complex number: A complex number is any number of the form $z = a + ib$ where a and b are real numbers. The set of all complex numbers is denoted by **C**.

pure imaginary: A complex number of the form $z = ib$ is called pure imaginary.

Re(z) and Im(z): If $z = a + ib$ then the real part of z is $\text{Re}(z) = a$ and the imaginary part of the z is $\text{Im}(z) = b$. Note $\text{Im}(z) = b$ NOT ib.

equality: Two complex numbers $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$ are equal, $z_1 = z_2$, if $a_1 = a_2$ and $b_1 = b_2$.

arithmetic operations: If $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$ then

$$\begin{aligned}
\textit{Addition}: \quad & z_1 + z_2 &=& \ (a_1 + a_2) + i(b_1 + b_2), \\
\textit{Subtraction}: \quad & z_1 - z_2 &=& \ (a_1 - a_2) + i(b_1 - b_2), \\
\textit{Multiplication}: \quad & z_1 \cdot z_2 &=& \ (a_1a_2 - b_1b_2) + i(b_1a_2 + a_1b_2), \\
\textit{Division}: \quad & \frac{z_1}{z_2} &=& \ \frac{a_1a_2 + b_1b_2}{a_2^2 + b_2^2} + i\frac{b_1a_2 - a_1b_2}{a_2^2 + b_2^2}.
\end{aligned}$$

arithmetic laws: The familiar commutative, associative, and distributive laws hold for complex numbers.

$$\begin{array}{llllll} \textit{Commutative Laws}: & z_1 + z_2 & = & z_2 + z_1, & z_1 z_2 & = & z_2 z_1 \\ \textit{Associative Laws}: & z_1 + (z_2 + z_3) & = & (z_1 + z_2) + z_3, & z_1(z_2 z_3) & = & (z_1 z_2) z_3 \\ \textit{Distributive Law}: & z_1(z_2 + z_3) & = & z_1 z_2 + z_1 z_3 & & & \end{array}$$

complex conjugate: If $z = a + ib$ then the complex conjugate of z is $\overline{z} = a - ib$. The following properties are used repeatedly:

$$\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}, \quad \overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}, \quad \overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}, \quad \overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}.$$

multiplication: To multiply two complex numbers, use the distributive law and the property $i^2 = -1$. For example,

$$(2 + 3i)(1 - 4i) = 2 - 8i + 3i - 12i^2 = 2 - 5i + 12 = 14 - 5i.$$

division: To divide two complex numbers, multiply the numerator and the denominator of z_1/z_2 by $\overline{z_2}$, then simplify. For example,

$$\frac{2+3i}{1-4i} = \frac{2+3i}{1-4i}\,\frac{1+4i}{1+4i} = \frac{2+8i+3i+12i^2}{1+4i-4i-16i^2} = \frac{-10+11i}{17} = -\frac{10}{17} + \frac{11}{17}i.$$

Exercises 1.1

3. $(5 - 9i) + (2 - 4i) = (5 + 2) + (-9 - 4)i = 7 - 13i$

7. $(2 - 3i)(4 + i) = 8 - 12i + 2i - 3i^2 = 8 - 10i + 3 = 11 - 10i$

11. $\dfrac{2-4i}{3+5i} = \dfrac{2-4i}{3+5i}\,\dfrac{3-5i}{3-5i} = \dfrac{6-10i-12i+20i^2}{9-15i+15i-25i^2} = \dfrac{-14-22i}{34} = -\dfrac{7}{17} - \dfrac{11}{17}i$

15.

$$\begin{aligned} \frac{(5-4i)-(3+7i)}{(4+2i)+(2-3i)} &= \frac{(5-3)+(-4-7)i}{(4+2)+(2-3)i} \\ &= \frac{2-11i}{6-i} \\ &= \frac{2-11i}{6-i}\,\frac{6+i}{6+i} \\ &= \frac{12+2i-66i-11i^2}{36+6i-6i-i^2} \\ &= \frac{23-64i}{37} \\ &= \frac{23}{37} - \frac{64}{37}i \end{aligned}$$

19.

$$\begin{aligned}
(3+6i)+(4-i)(3+5i)+\frac{1}{2-i} &= (3+6i)+(12+20i-3i-5i^2)+\frac{1}{2-i}\,\frac{2+i}{2+i}\\
&= (3+6i)+(17+17i)+\frac{2+i}{4+2i-2i-i^2}\\
&= (3+6i)\,\frac{5}{5}+(17+17i)\,\frac{5}{5}+\frac{2+i}{5}\\
&= \frac{15+30i}{5}+\frac{85+85i}{5}+\frac{2+i}{5}\\
&= \frac{102}{5}+\frac{116}{5}i
\end{aligned}$$

23.

$$\begin{aligned}
(-2+2i)^5 &= (-2)^5+\frac{5}{1}(-2)^4(2i)^1+\frac{20}{2}(-2)^3(2i)^2+\frac{60}{6}(-2)^2(2i)^3+\\
&\quad \frac{120}{24}(-2)^1(2i)^4+(2i)^5\\
&= -32+(5)(16)(2i)+(10)(-8)(-4)+(10)(4)(-8i)+(5)(-2)(16)+32i\\
&= -32+160i+320-320i-160+32i\\
&= 128-128i
\end{aligned}$$

27.

$$\begin{aligned}
\mathrm{Re}\left(\frac{1}{z}\right) &= \mathrm{Re}\left(\frac{1}{x+iy}\right)\\
&= \mathrm{Re}\left(\frac{1}{x+iy}\,\frac{x-iy}{x-iy}\right)\\
&= \mathrm{Re}\left(\frac{x-iy}{x^2-ixy+ixy+i^2y^2}\right)\\
&= \mathrm{Re}\left(\frac{x-iy}{x^2+y^2}\right)\\
&= \mathrm{Re}\left(\frac{x}{x^2+y^2}-i\frac{y}{x^2+y^2}\right)\\
&= \frac{x}{x^2+y^2}
\end{aligned}$$

31.

$$\begin{aligned}
\mathrm{Re}(iz) &= \mathrm{Re}(i(x+iy))\\
&= \mathrm{Re}(ix+i^2y)\\
&= \mathrm{Re}(ix-y)\\
&= -y\\
&= -\mathrm{Im}(z)
\end{aligned}$$

35. If $z_1 = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$, then

$$\begin{aligned} z_1^2 + i &= \left(-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)^2 + i \\ &= \frac{2}{4} - \frac{2}{4}i - \frac{2}{4}i + \frac{2}{4}i^2 + i \\ &= \frac{1}{2} - i - \frac{1}{2} + i \\ &= 0. \end{aligned}$$

An additional solution is $z_2 = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$.

39. Let $z = a + ib$, then $z^2 = (a+ib)^2 = a^2 + iab + iab + i^2b^2 = a^2 - b^2 + 2abi$. By Definition 1.1.2, $z^2 = i$ if and only if $\text{Re}\left(z^2\right) = \text{Re}(i)$ and $\text{Im}\left(z^2\right) = \text{Im}(i)$. This gives the following two equations:

$$\begin{aligned} a^2 - b^2 &= 0 \\ 2ab &= 1. \end{aligned}$$

From the second equation, $b = \frac{1}{2a}$. Substituting this into the first equation we get

$$a^2 - \frac{1}{4a^2} = 0 \quad \text{or} \quad a^4 = \frac{1}{4}.$$

Solving for a we find that $a = \pm\frac{\sqrt{2}}{2}$. Now since $b = \frac{1}{2a}$, we have two solutions, $a = \frac{\sqrt{2}}{2}$, $b = \frac{\sqrt{2}}{2}$ and $a = -\frac{\sqrt{2}}{2}$, $b = -\frac{\sqrt{2}}{2}$. Therefore, $z = \pm\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right)$.

43. To solve we multiply the first equation by $-i$ then sum the two equations.

$$\begin{array}{rcrcl} -i^2z_1 & - & -i^2z_2 & = & -i(2+10i) \\ -z_1 & + & (1-i)z_2 & = & 3-5i \end{array} \quad \Longrightarrow \quad \begin{array}{rcrcr} z_1 & - & z_2 & = & 10-2i \\ -z_1 & + & (1-i)z_2 & = & 3-5i \\ \hline & & -iz_2 & = & 13-7i \end{array}$$

Thus, $-iz_2 = 13 - 7i$ or $z_2 = 7 + 13i$. Now substitute this value of z_2 into the first equation and solve for z_1.

$$\begin{aligned} iz_1 - i(7+13i) &= 2 + 10i \\ iz_1 + 13 - 7i &= 2 + 10i \\ iz_1 &= -11 + 17i \\ z_1 &= 17 + 11i \end{aligned}$$

Focus on Concepts

47. (a) Since $i^4 = 1$, it follows that $i^{4k} = \left(i^4\right)^k = 1^k = 1$. So, $n = 4k$.

51. Suppose $z_1 z_2 = r$ where r is a real number. Then $z_1 = r/z_2$. Now multiply the numerator and denominator of r/z_2 by $\overline{z_2}$ and explain why the result is of the form $k\overline{z_2}$ where k is a real constant.

55. Assume that such a subset P exists and that i is in P. By the second property (applied two times) i^3 is in P. Explain why this leads to a contradiction.

1.2 Complex Plane

1.2 Review Topic: Vectors

scalar/vector: A scalar is a real number that has only a magnitude. A vector is a quantity that has both a magnitude and a direction. Geometrically, a vector is represented by a directed line segment (an arrow). Two-dimensional vectors are vectors in a coordinate plane.

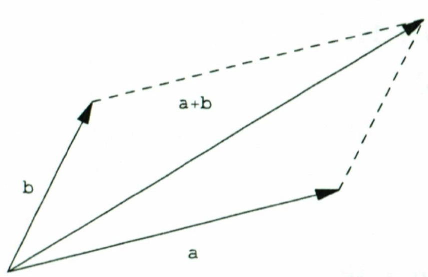
$\langle a, b \rangle$ **:** Vectors are usually denoted by boldfaced symbols such as $\mathbf{v}$ and $\mathbf{a}$ and are described analytically by giving the displacement a of the vector in the x-direction and displacement b of the vector in the y-direction using the notation $\langle a, b \rangle$. The numbers a and b are called the components of the vector $\langle a, b \rangle$.

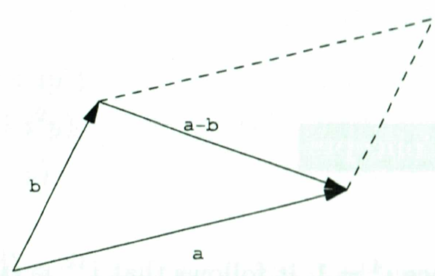
addition/subtraction/scalar multiplication: If k is a scalar and $\mathbf{a} = \langle a_1, a_2 \rangle$ and $\mathbf{b} = \langle b_1, b_2 \rangle$ are vectors, then vector addition, vector subtraction, and scalar multiplication are defined as follows:

Addition:	$\mathbf{a}+\mathbf{b}$	$=$	$\langle a_1+b_1, a_2+b_2 \rangle$
Subtraction:	$\mathbf{a}-\mathbf{b}$	$=$	$\langle a_1-b_1, a_2-b_2 \rangle$
Scalar Multiplication:	$k\mathbf{a}$	$=$	$\langle ka_1, ka_2 \rangle$

i, j vectors: The vectors $\mathbf{i}$ and $\mathbf{j}$ are defined to by $\mathbf{i} = \langle 1, 0 \rangle$ and $\mathbf{j} = \langle 0, 1 \rangle$. Notice that any vector $\langle a_1, a_2 \rangle$ can be written as $\langle a_1, a_2 \rangle = a_1\,\mathbf{i} + a_2\,\mathbf{j}$.

addition/subtraction geometrically: If $\mathbf{a}$ and $\mathbf{b}$ are vectors, then the vector sum $\mathbf{a}+\mathbf{b}$ and the vector difference $\mathbf{a}-\mathbf{b}$ can be viewed geometrically as shown in the following figures.





length: The length or magnitude of the vector $\mathbf{a} = a_1\,\mathbf{i} + a_2\,\mathbf{j}$ is $\|\mathbf{a}\| = \sqrt{a_1^2 + a_2^2}$.

dot product: The dot product of vectors $\mathbf{a} = a_1\,\mathbf{i} + a_2\,\mathbf{j}$ and $\mathbf{b} = b_1\,\mathbf{i} + b_2\,\mathbf{j}$ is defined to be the scalar

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 = \|\mathbf{a}\|\,\|\mathbf{b}\| \cos\theta$$

where θ is the angle between the vectors.

1.2 Summary

points/complex plane: The complex number $z = x + iy$ can be viewed as the point (x, y) in a coordinate plane. The set of all complex points z is called the complex plane or the z-plane. The x- and y-axes are called the real and imaginary axes, respectively.

vectors: The complex number $z = x + iy$ can also be viewed as two-dimensional position vector whose initial point is $(0, 0)$ and whose terminal point is (x, y), or as the two-dimensional vector $x\,\mathbf{i} + y\,\mathbf{j}$ which has no specified initial point. With this interpretation, complex addition can be viewed geometrically as vector sum and complex subtraction can be viewed as vector difference.

modulus: The modulus of $z = x + iy$ is $|z| = \sqrt{z\bar{z}} = \sqrt{x^2 + y^2}$. The modulus is the length of the vector representing z. The following properties are used repeatedly:

$$|z|^2 = z\bar{z}, \quad |z_1 z_2| = |z_1|\,|z_2|, \quad \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}.$$

distance: The distance between two points z_1 and z_2 in the complex plane is given by $|z_2 - z_1|$.

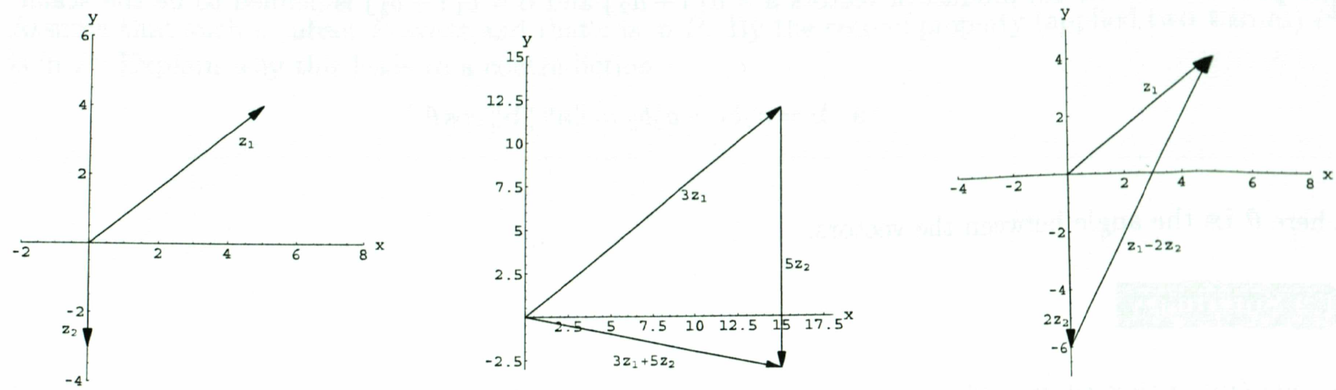
triangle inequalities: For any complex numbers z_1, z_2, and z_3 the following inequalities hold:

$$\big||z_1| - |z_2|\big| \;\leq\; |z_1 + z_2| \;\leq\; |z_1| + |z_2|$$

$$\big||z_1| - |z_2|\big| \;\leq\; |z_1 - z_2| \;\leq\; |z_1| + |z_2|$$

Exercises 1.2

3.



7. Two adjacent sides of the triangle are

$$\begin{aligned} z_1 - z_3 &= (-2-8i) - (-6-5i) &&= 4-3i, \text{ and} \\ z_2 - z_3 &= (3i) - (-6-5i) &&= 6+8i. \end{aligned}$$

These sides are represented by the two vectors $4\,\mathbf{i} - 3\,\mathbf{j}$ and $6\,\mathbf{i} + 8\,\mathbf{j}$. The angle θ between these vectors can be determined using the dot product (see Review Topic: Vectors):

$$(4\,\mathbf{i} - 3\,\mathbf{j}) \cdot (6\,\mathbf{i} + 8\,\mathbf{j}) = 24 - 24 = 0.$$

This implies that $\cos\theta = 0$, and so $\theta = \pi/2$. Therefore, z_1, z_2, and z_3 are the vertices of a right triangle.

11.

$$\begin{aligned} \left|\frac{2i}{3-4i}\right| &= \frac{|2i|}{|3-4i|} \\ &= \frac{\sqrt{(0)^2+(2)^2}}{\sqrt{(3)^2+(-4)^2}} \\ &= \frac{\sqrt{4}}{\sqrt{25}} \\ &= \frac{2}{5} \end{aligned}$$

15. In order to determine which complex number is closest to the origin we compute the distance from z_1 to 0 and the distance from z_2 to 0.

$$\begin{aligned} |z_1 - 0| &= |10+8i| &&= \sqrt{(10)^2+(8)^2} &&= \sqrt{164} \\ |z_2 - 0| &= |11-6i| &&= \sqrt{(11)^2+(-6)^2} &&= \sqrt{157} \end{aligned}$$

Therefore, z_2 is closest to the origin. On the other hand, the distance from z_1 to $1+i$ and the distance from z_2 to $1+i$ is:

$$\begin{aligned} |z_1 - (1+i)| &= |9+7i| &&= \sqrt{(9)^2+(7)^2} &&= \sqrt{130} \\ |z_2 - (1+i)| &= |10-7i| &&= \sqrt{(10)^2+(-7)^2} &&= \sqrt{149} \end{aligned}$$

Therefore, z_1 is closest to $1+i$.

19. If $|z-i| = |z-1|$, then $|z-i|^2 = |z-1|^2$. Let $z = x+iy$, then:

$$\begin{aligned}
|(x+iy)-i|^2 &= |(x+iy)-1|^2 \\
|x+i(y-1)|^2 &= |(x-1)+iy|^2 \\
x^2+(y-1)^2 &= (x-1)^2+y^2 \\
\cancel{x^2}+\cancel{y^2}-2y+\cancel{1} &= \cancel{x^2}-2x+\cancel{1}+\cancel{y^2} \\
-2y &= -2x \\
y &= x.
\end{aligned}$$

The set of points satisfying the equation $|z-i| = |z-1|$ lie on the line $y = x$ in the plane.

23. If $|z-1| = 1$, then $|z-1|^2 = 1$. Let $z = x+iy$, then:

$$\begin{aligned}
|(x+iy)-1|^2 &= 1 \\
|(x-1)+iy|^2 &= 1 \\
(x-1)^2+y^2 &= 1.
\end{aligned}$$

The set of points satisfying the equation $|z-1| = 1$ lie on the circle $(x-1)^2+y^2 = 1$ with center $(1,0)$ and radius 1.

27. We use the triangle inequalities:

$$\Big||z|-|6+8i|\Big| \le |z+6+8i| \le |z|+|6+8i|.$$

Making the substitutions $|z| = 2$ and $|6+8i| = \sqrt{6^2+8^2} = 10$, we obtain

$$|2-10| \le |z+6+8i| \le 2+10.$$

Thus, $8 \le |z+6+8i| \le 12$.

31. Let $z = x+iy$. From Definition 1.1.2, $|z|-z = 2+i$ if and only if $\text{Re}\,(|z|-z) = \text{Re}(2+i) = 2$ and $\text{Im}\,(|z|+z) = \text{Re}(2+i) = 1$. This gives the following two equations:

$$\begin{aligned}
\text{Re}\left(\sqrt{x^2+y^2}-(x+iy)\right) &= \sqrt{x^2+y^2}-x &= 2 \\
\text{Im}\left(\sqrt{x^2+y^2}-(x+iy)\right) &= -y &= 1.
\end{aligned}$$

Since $y = -1$ from the second equation, the first equation becomes

$$\begin{aligned}
\sqrt{x^2+1}-x &= 2 \\
\sqrt{x^2+1} &= 2+x \\
x^2+1 &= (2+x)^2 \\
\cancel{x^2}+1 &= 4+4x+\cancel{x^2} \\
-4x &= 3 \\
x &= -\frac{3}{4}
\end{aligned}$$

The complex number $z = -\frac{3}{4} - i$ satisfies the given equation.

Focus on Concepts

35. In each case consider the angle between z and iz.

39. Consider $|z|$.

43. Rewrite the equation as $(z_3 - z_2) = -k(z_1 - z_2)$. Now both $z_3 - z_2$ and $z_1 - z_2$ can be viewed as vectors with the same initial point z_2. Since k is a real number, $-k(z_1 - z_2)$ is just a scalar multiple of $z_1 - z_2$. Use the equality above to make a statement regarding the directions of these two vectors.

47. For both parts, set $z = x + iy$ and use the definition of modulus.

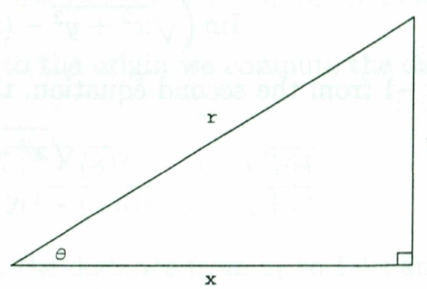
1.3 Polar Form of Complex Numbers

1.3 Review Topic: Polar Coordinates

polar coordinates: A point $P(x, y)$ in the coordinate plane with Cartesian coordinates (x, y) has polar coordinates (r, θ) where r is the length of the line segment $\overline{OP}$ from the origin to P and θ is the angle the line segment $\overline{OP}$ makes with the positive x-axis.

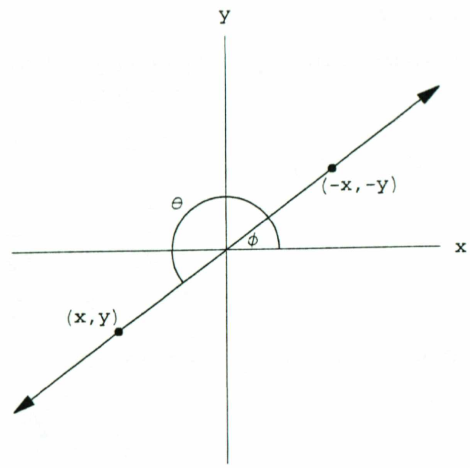
relation to cartesian coordinates: Cartesian and polar coordinates are related by trigonometric properties of a right triangle.

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ x^2 + y^2 &= r^2 \\ \frac{y}{x} &= \tan\theta \end{aligned}$$



using arctangent: Care should be taken in using arctangent to determine θ from x and y. In general, $\theta \neq \tan^{-1}\frac{y}{x}$. The reason for this is illustrated in the diagram below.

Arctangent is defined to return value in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. If (x, y) is in, say, the third quadrant, then $\tan^{-1}\frac{y}{x}$ returns the angle ϕ between the positive x-axis and the ray from the origin through $(-x, -y)$ (which is a point in the first quadrant), instead of the angle θ between the positive x-axis and the ray from the origin through (x, y). Because these rays are in opposite directions, the difference between ϕ and θ is just a multiple of π.



The following table indicates how to correctly use arctangent to find θ when $x \neq 0$. If $x = 0$, then $\theta = \frac{\pi}{2}$ when $y > 0$ and $\theta = -\frac{\pi}{2}$ when $y < 0$.

Quadrant II $\theta = \tan^{-1}(y/x) + \pi$	Quadrant I $\theta = \tan^{-1}(y/x)$
Quadrant III $\theta = \tan^{-1}(y/x) - \pi$	Quadrant IV $\theta = \tan^{-1}(y/x)$

1.3 Summary

polar form: The polar form of a nonzero complex number $z = x + iy$ is $z = r(\cos\theta + i\sin\theta)$ where (r, θ) are the polar coordinates of the point $P(x, y)$. Thus, $x = r\cos\theta$ and $y = r\sin\theta$. In order to find r and θ from x and y we use the formulas $r^2 = x^2 + y^2$ and $\tan\theta = y/x$.

modulus and argument: If $z = r(\cos\theta + i\sin\theta)$ is a polar form of a complex number z then $|z| = r$ and an argument of z is the angle θ. Note that there are infinitely many different choices for θ that give the same complex number. For example, $-2 = 2(\cos\pi + i\sin\pi) = 2(\cos 3\pi + i\sin 3\pi) = 2(\cos 5\pi + i\sin 5\pi)$ and so on. In general, if θ_0 is some argument of z, then $\arg(z) = \theta_0 + 2\pi k$, $k = 0, \pm1, \pm2, ...$ Notice that both θ and r are *real* numbers.

principal argument: If z is a nonzero complex number, then the argument θ of z that lies in the interval $(-\pi, \pi]$ is called the principal argument and is denoted by $\mathrm{Arg}(z) = \theta$. There is only one choice for the principal argument; to find it use arctangent appropriately (see the table in Review Topic: Polar Coordinates). Don't forget that $\mathrm{Arg}(z)$ is a *real* number. Any other argument for z can be expressed as $\mathrm{Arg}(z) + 2\pi k$ for some integer k.

multiplication/division: It is easy to multiply and divide complex numbers in polar form. If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, then

$$\begin{aligned} z_1 z_2 &= r_1 r_2\Big(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\Big) \\ \frac{z_1}{z_2} &= \frac{r_1}{r_2}\Big(\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)\Big) \end{aligned}$$

Integer powers of z are easy to compute in polar form as well. If $z = (\cos\theta + i\sin\theta)$, then

$$z^n = r^n(\cos n\theta + i\sin n\theta).$$

de Moivre's formula: A special case of the formula for integer powers given above occurs when $|z| = r = 1$. This formula is called de Moivre's formula:

$$(\cos\theta + i\sin\theta)^n = (\cos n\theta + i\sin n\theta).$$

Exercises 1.3

3. Since $-3i = 0 + (-3)i$, we identify $x = 0$ and $y = -3$. Then $r = \sqrt{0^2 + (-3)^2} = 3$. Since $x = 0$ and $y = -3 < 0$, $\operatorname{Arg}(-3i) = -\frac{\pi}{2}$. A different argument for $-3i$ is given by $\theta = \operatorname{Arg}(-3i) + 2\pi = \frac{3\pi}{2}$. Therefore,

$$\begin{aligned} -3i &= 3\left(\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right), && \text{using } \theta = \operatorname{Arg}(-3i) \\ -3i &= 3\left(\cos\left(\frac{3\pi}{2}\right) + i\sin\left(\frac{3\pi}{2}\right)\right), && \text{using } \theta \neq \operatorname{Arg}(-3i). \end{aligned}$$

7. For $-\sqrt{3} + i$, we identify $x = -\sqrt{3}$ and $y = 1$. Then $r = \sqrt{(\sqrt{3})^2 + 1^2} = 2$. Since $z = -\sqrt{3} + i$ is in the second quadrant,

$$\operatorname{Arg}(-\sqrt{3} + i) = \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) + \pi = -\frac{\pi}{6} + \pi = \frac{5\pi}{6}.$$

A different argument for $-\sqrt{3} + i$ is given by $\theta = \operatorname{Arg}(-\sqrt{3} + i) + 2\pi = \frac{17\pi}{6}$. Therefore,

$$\begin{aligned} -\sqrt{3} + i &= 2\left(\cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right)\right), && \text{using } \theta = \operatorname{Arg}(-\sqrt{3} + i) \\ -\sqrt{3} + i &= 2\left(\cos\left(\frac{17\pi}{6}\right) + i\sin\left(\frac{17\pi}{6}\right)\right), && \text{using } \theta \neq \operatorname{Arg}(-\sqrt{3} + i). \end{aligned}$$

11. For $-\sqrt{2} + \sqrt{7}i$, we identify $x = -\sqrt{2}$ and $y = \sqrt{7}$. Then $r = \sqrt{\left(-\sqrt{2}\right)^2 + \left(\sqrt{7}\right)^2} = 3$. Since $z = -\sqrt{2} + \sqrt{7}i$ is in the second quadrant,

$$\operatorname{Arg}\left(-\sqrt{2} + \sqrt{7}i\right) = \tan^{-1}\left(-\frac{\sqrt{7}}{\sqrt{2}}\right) + \pi \approx 2.06168.$$

A different argument for $-\sqrt{2}+\sqrt{7}i$ is given by $\theta = \text{Arg}(-\sqrt{2}+\sqrt{7}i)+2\pi \approx 8.34486$. Therefore,

$$\begin{aligned} -\sqrt{2}+\sqrt{7}i &= 3\left(\cos 2.06168 + i\sin 2.06168\right), &&\text{using } \theta = \text{Arg}(-\sqrt{2}+\sqrt{7}i) \\ -\sqrt{2}+\sqrt{7}i &= 3\left(\cos 8.34486 + i\sin 8.34486\right), &&\text{using } \theta \neq \text{Arg}(-\sqrt{2}+\sqrt{7}i). \end{aligned}$$

15.

$$\begin{aligned} 5\left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right) &= 5\left(-\frac{\sqrt{3}}{2} - i\frac{1}{2}\right) \\ &= -\frac{5\sqrt{3}}{2} - \frac{5}{2}i \end{aligned}$$

19. By formula (6) of Section 1.3

$$\begin{aligned} z_1z_2 &= 2\left(\cos\frac{\pi}{8} + i\sin\frac{\pi}{8}\right)4\left(\cos\frac{3\pi}{8} + i\sin\frac{3\pi}{8}\right) \\ &= 8\left(\cos\left(\frac{\pi}{8}+\frac{3\pi}{8}\right) + i\sin\left(\frac{\pi}{8}+\frac{3\pi}{8}\right)\right) \\ &= 8\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) \\ &= 8\left(0 + i(1)\right) \\ &= 8i. \end{aligned}$$

By formula (7) of Section 1.3

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{2\left(\cos\frac{\pi}{8} + i\sin\frac{\pi}{8}\right)}{4\left(\cos\frac{3\pi}{8} + i\sin\frac{3\pi}{8}\right)} \\ &= \frac{1}{2}\left(\cos\left(\frac{\pi}{8}-\frac{3\pi}{8}\right) + i\sin\left(\frac{\pi}{8}-\frac{3\pi}{8}\right)\right) \\ &= \frac{1}{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right) \\ &= \frac{1}{2}\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right) \\ &= \frac{\sqrt{2}}{4} - \frac{\sqrt{2}}{4}i. \end{aligned}$$

23. Since $|-i| = 1$ and $\text{Arg}(-i) = -\frac{\pi}{2}$, $-i = 1\left(\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right)$. On the other hand, $|1+i| = \sqrt{2}$ and $\text{Arg}(1+i) = \frac{\pi}{4}$, so $1+i = \sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right)$. Now by formula (7) of Section 1.3

$$\begin{aligned} \frac{-i}{1+i} &= \frac{1\left(\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right)}{\sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right)} \\ &= \frac{1}{\sqrt{2}}\left(\cos\left(-\frac{\pi}{2}-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{2}-\frac{\pi}{4}\right)\right) \\ &= \frac{\sqrt{2}}{2}\left(\cos\left(-\frac{3\pi}{4}\right) + i\sin\left(-\frac{3\pi}{4}\right)\right) &&\leftarrow \text{polar form} \end{aligned}$$

$$= \frac{\sqrt{2}}{2}\left(-\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right)$$
$$= -\frac{1}{2} - \frac{1}{2}i$$

27. Since $\left|\frac{1}{2} + \frac{1}{2}i\right| = \frac{\sqrt{2}}{2}$ and $\text{Arg}\left|\frac{1}{2} + \frac{1}{2}i\right| = \frac{\pi}{4}$,

$$\frac{1}{2} + \frac{1}{2}i = \frac{\sqrt{2}}{2}\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right).$$

Therefore, by formula (9) of Section 1.3

$$\begin{aligned}\left(\frac{1}{2} + \frac{1}{2}i\right)^{10} &= \left[\frac{\sqrt{2}}{2}\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right)\right]^{10} \\ &= \left(\frac{\sqrt{2}}{2}\right)^{10}\left(\cos\left(\frac{10\pi}{4}\right) + i\sin\left(\frac{10\pi}{4}\right)\right) \\ &= \frac{2^5}{2^{10}}\left(0 + i(1)\right) \\ &= \frac{1}{32}i\end{aligned}$$

31. By formula (9) of Section 1.3

$$\begin{aligned}&\left(\cos\left(\frac{\pi}{9}\right) + i\sin\left(\frac{\pi}{9}\right)\right)^{12}\left[2\left(\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right)\right]^5 \\ &\qquad = \left[\cos\left(\frac{12\pi}{9}\right) + i\sin\left(\frac{12\pi}{9}\right)\right]\left[2^5\left(\cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right)\right)\right] \\ &\qquad = \left[\cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right)\right]\left[32\left(\cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right)\right)\right].\end{aligned}$$

By formula (6) of Section 1.3

$$\begin{aligned}&\left[\cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right)\right]\left[32\left(\cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right)\right)\right] \\ &\qquad = 32\left(\cos\left(\frac{4\pi}{3} + \frac{5\pi}{6}\right) + i\sin\left(\frac{4\pi}{3} + \frac{5\pi}{6}\right)\right) \\ &\qquad = 32\left(\cos\left(\frac{13\pi}{6}\right) + i\sin\left(\frac{13\pi}{6}\right)\right) \qquad \leftarrow \text{polar form} \\ &\qquad = 32\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) \\ &\qquad = 16\sqrt{3} + 16i.\end{aligned}$$

35. Since $\left|\frac{\sqrt{3}}{2} + \frac{1}{2}i\right| = 1$ and $\text{Arg}\left|\frac{\sqrt{3}}{2} + \frac{1}{2}i\right| = \frac{\pi}{6}$,

$$\frac{\sqrt{3}}{2} + \frac{1}{2}i = \cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right).$$

By formula (9) of Section 1.3

$$\begin{aligned}\left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)^n &= \left(\cos\left(\frac{\pi}{6}\right)+i\sin\left(\frac{\pi}{6}\right)\right)^n \\ &= \cos\left(\frac{n\pi}{6}\right)+i\sin\left(\frac{n\pi}{6}\right).\end{aligned}$$

Therefore, in order to have $\left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)^n=-1$, we must have

$$\cos\left(\frac{n\pi}{6}\right)=-1 \quad \text{and} \quad \sin\left(\frac{n\pi}{6}\right)=0.$$

This will occur when $\frac{n\pi}{6}=\pi$, that is, when $n=6$.

Focus on Concepts

39. Use formula (6) of Section 1.3 and notice how the modulus and argument of z is related to the modulus and argument of $z_1 z$. Interpret your observation geometrically.

43. Describe the relationship geometrically in terms of a ray emanating from the origin.

47. Use the fact that if $z_1 = z_2$, then $|z_1| = |z_2|$ and $\arg z_1 = \arg z_2$.

1.4 Powers and Roots

1.4 Summary

nth roots: The n nth roots of a nonzero complex number $z = r(\cos\theta + i\sin\theta)$ are given by

$$w_k = \sqrt[n]{r}\left[\cos\left(\frac{\theta+2k\pi}{n}\right)+i\sin\left(\frac{\theta+2k\pi}{n}\right)\right], \qquad \text{where } k=0,1,2,...,n-1.$$

principal nth root: The principal nth root of a nonzero complex number $z = r(\cos\theta + i\sin\theta)$ is

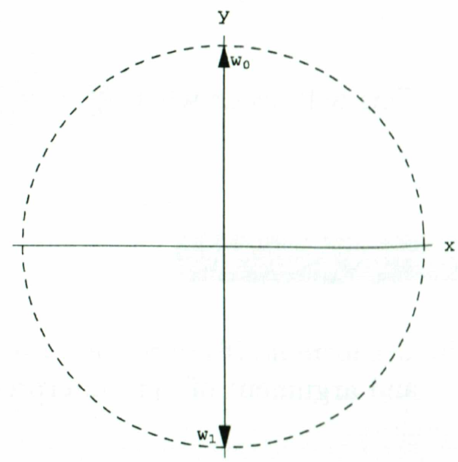
$$\sqrt[n]{r}\left[\cos\left(\frac{\operatorname{Arg}(z)}{n}\right)+i\sin\left(\frac{\operatorname{Arg}(z)}{n}\right)\right].$$

location of roots: The n nth roots of a nonzero complex number $z = r(\cos\theta + i\sin\theta)$ all lie on a circle of radius $\sqrt[n]{r}$ centered at the origin and are spaced at equal angular intervals of $2\pi/n$.

Exercises 1.4

3. Converting to polar form $-9 = 9\,(\cos\pi + i\sin\pi)$. Now identifying $r = 9$, $\theta = \operatorname{Arg}(-9) = \pi$, and $n = 2$ in formula (4) of Section 1.4 with $k = 0,\ 1$ gives

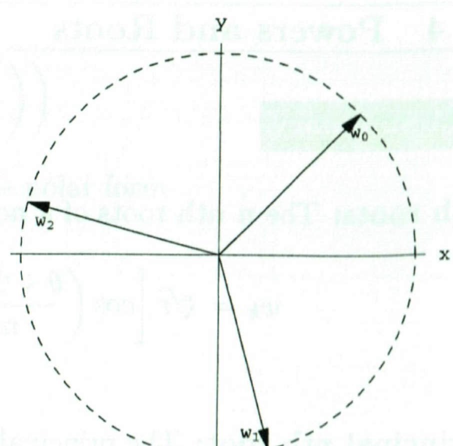
$$\begin{aligned} w_0 &= \sqrt{9}\left[\cos\left(\frac{\pi + 2(0)\pi}{2}\right) + i\sin\left(\frac{\pi + 2(0)\pi}{2}\right)\right] \\ &= 3\left[\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right] \\ &= 3\,[0 + i] \\ &= 3i \qquad \leftarrow \text{principal root} \end{aligned}$$



$$\begin{aligned} w_1 &= \sqrt{9}\left[\cos\left(\frac{\pi + 2(1)\pi}{2}\right) + i\sin\left(\frac{\pi + 2(1)\pi}{2}\right)\right] \\ &= 3\left[\cos\left(\frac{3\pi}{2}\right) + i\sin\left(\frac{3\pi}{2}\right)\right] \\ &= 3\,[0 - i] \\ &= -3i. \end{aligned}$$

7. Converting to polar form $-1 + i = \sqrt{2}\left(\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right)$. Now identifying $r = \sqrt{2}$, $\theta = \operatorname{Arg}(-1+i) = \frac{3\pi}{4}$, and $n = 3$ in formula (4) of Section 1.4 with $k = 0,\ 1,\ 2$ gives

$$\begin{aligned} w_0 &= \sqrt[3]{\sqrt{2}}\left[\cos\left(\frac{3\pi/4 + 2(0)\pi}{3}\right) + i\sin\left(\frac{3\pi/4 + 2(0)\pi}{3}\right)\right] \\ &= \sqrt[6]{2}\left[\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right] \\ &= \sqrt[6]{2}\left[\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right] \\ &= 2^{-1/3} + 2^{-1/3}i \quad \leftarrow \text{principal root} \\ &\approx 1.25992 + 1.25992i \end{aligned}$$



$$\begin{aligned} w_1 &= \sqrt[3]{\sqrt{2}}\left[\cos\left(\frac{3\pi/4 + 2(1)\pi}{3}\right) + i\sin\left(\frac{3\pi/4 + 2(1)\pi}{3}\right)\right] \\ &= \sqrt[6]{2}\left[\cos\left(\frac{11\pi}{12}\right) + i\sin\left(\frac{11\pi}{12}\right)\right] \\ &\approx -1.08422 + 0.29052i \end{aligned}$$

$$\begin{aligned} w_2 &= \sqrt[3]{\sqrt{2}}\left[\cos\left(\frac{3\pi/4+2(2)\pi}{3}\right)+i\sin\left(\frac{3\pi/4+2(2)\pi}{3}\right)\right] \\ &= \sqrt[6]{2}\left[\cos\left(\frac{19\pi}{12}\right)+i\sin\left(\frac{19\pi}{12}\right)\right] \\ &\approx 0.29052-1.08422i. \end{aligned}$$

11. Converting to polar form $3+4i=5\,(\cos(\theta))+i\sin(\theta))$, where $\theta=\arctan(4/3)$. Now identifying $r=5$ and $n=2$ in formula (4) of Section 1.4 with $k=0,\ 1$ gives

$$\begin{aligned} w_0 &= \sqrt{5}\left[\cos\left(\frac{\theta+2(0)\pi}{2}\right)+i\sin\left(\frac{\theta+2(0)\pi}{2}\right)\right] \\ &= \sqrt{5}\left[\cos\left(\frac{\theta}{2}\right)+i\sin\left(\frac{\theta}{2}\right)\right] \\ w_1 &= \sqrt{5}\left[\cos\left(\frac{\theta+2(1)\pi}{2}\right)+i\sin\left(\frac{\theta+2(1)\pi}{2}\right)\right] \\ &= \sqrt{5}\left[\cos\left(\frac{\theta}{2}+\pi\right)+i\sin\left(\frac{\theta}{2}+\pi\right)\right] \\ &= \sqrt{5}\left[-\cos\left(\frac{\theta}{2}\right)-i\sin\left(\frac{\theta}{2}\right)\right]. \end{aligned}$$

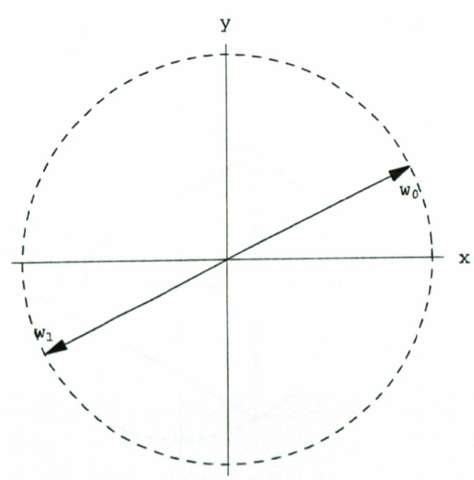
Since $\cos\theta=\frac{3}{5}$, the half-angle formulas give

$$\begin{aligned} \cos\left(\frac{\theta}{2}\right) &= \sqrt{\frac{1+\cos\theta}{2}} & \text{and} \quad \sin\left(\frac{\theta}{2}\right) &= \sqrt{\frac{1-\cos\theta}{2}} \\ &= \sqrt{\frac{1+\frac{3}{5}}{2}} & &= \sqrt{\frac{1-\frac{3}{5}}{2}} \\ &= \frac{2}{\sqrt{5}} & &= \frac{1}{\sqrt{5}}. \end{aligned}$$

Therefore,

$$\begin{aligned} w_0 &= \sqrt{5}\left[\cos\left(\frac{\theta}{2}\right)+i\sin\left(\frac{\theta}{2}\right)\right] \\ &= \sqrt{5}\left[\frac{2}{\sqrt{5}}+\frac{1}{\sqrt{5}}i\right] \\ &= 2+i \quad \leftarrow \text{principal root} \end{aligned}$$

$$\begin{aligned} w_1 &= \sqrt{5}\left[-\cos\left(\frac{\theta}{2}\right)-i\sin\left(\frac{\theta}{2}\right)\right] \\ &= \sqrt{5}\left[-\frac{2}{\sqrt{5}}-\frac{1}{\sqrt{5}}i\right] \\ &= -2-i. \end{aligned}$$



15. (a)

$$\begin{aligned}(4+3i)^2 &= 16+12i+12i+9i^2\\ &= 16+24i-9\\ &= 7+24i.\end{aligned}$$

(b) From formula (4) of Section 1.4, the two square roots have the same modulus but arguments that differ by $2\pi/2 = \pi$. But from equation (6) of Section 2.3, adding π to the argument of a complex number is equal to multiplying the complex number by $1(\cos\pi + i\sin\pi) = -1$. Therefore, the two roots of $7+24i$ are $4+3i$ and $-4-3i$.

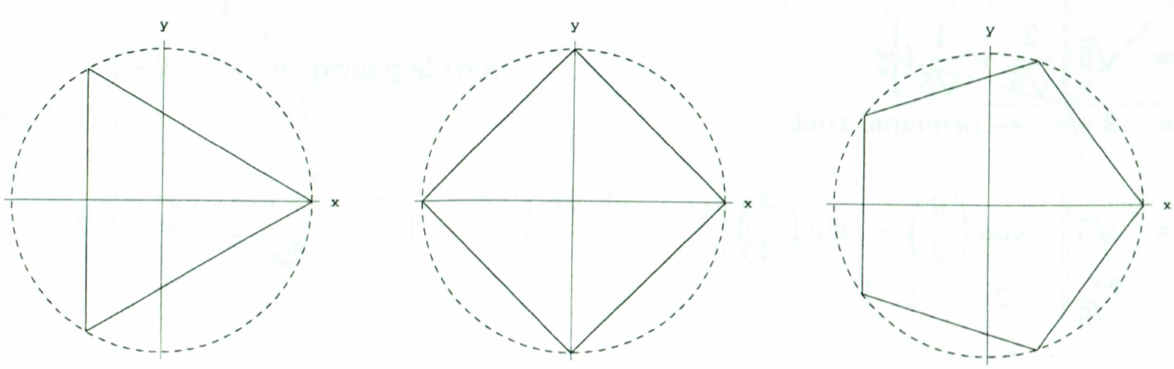
19. (a) Since $1 = 1(\cos 0 + i\sin 0)$, formula (4) of Section 1.4 with $r=1$ and $\theta = 0$ gives

$$\begin{aligned}1^{1/n} &= \sqrt[n]{1}\left[\cos\left(\frac{0+2k\pi}{n}\right) + i\sin\left(\frac{0+2k\pi}{n}\right)\right]\\ &= \cos\left(\frac{2k\pi}{n}\right) + i\sin\left(\frac{2k\pi}{n}\right), \qquad k = 0,\ 1,\ 2, ..., n-1\end{aligned}$$

(b)

$$\begin{aligned}1^{1/3} &= \left\{\cos\left(\frac{2k\pi}{3}\right) + i\sin\left(\frac{2k\pi}{3}\right) \,\middle|\, k=0,\ 1,\ 2\right\}\\ &= \left\{1,\ -\frac{1}{2}+\frac{\sqrt{3}}{2}i,\ -\frac{1}{2}-\frac{\sqrt{3}}{2}i\right\}\\ 1^{1/4} &= \left\{\cos\left(\frac{2k\pi}{4}\right) + i\sin\left(\frac{2k\pi}{4}\right) \,\middle|\, k=0,\ 1,\ 2,\ 3\right\}\\ &= \{1,\ i,\ -1,\ -i\}\\ 1^{1/5} &= \left\{\cos\left(\frac{2k\pi}{5}\right) + i\sin\left(\frac{2k\pi}{5}\right) \,\middle|\, k=0,\ 1,\ 2,\ 3,\ 4\right\}\\ &= \{1,\ 0.30902+0.95106i,\ -0.80902+0.58779i,\ -0.80902-0.58779i,\ 0.30902-0.95106i\}\end{aligned}$$

(c)



23. **(a)** If $(z+2)^5 + z^5 = 0$, then

$$\left(\frac{z+2}{-z}\right)^5 = 1.$$

Let $w = -(z+2)/z$, then

$$\begin{aligned} wz &= -z-2 \\ wz+z &= -2 \\ z &= -\frac{2}{w+1}. \end{aligned}$$

Since, $w^5 = 1$, part (a) of Problem 19 gives

$$w = \left[\cos\left(\frac{2k\pi}{5}\right) + i\sin\left(\frac{2k\pi}{5}\right)\right] \qquad k = 0, 1, 2, 3, 4.$$

Notice then that $w\overline{w} = 1$ and $w + \overline{w} = 2\cos(2k\pi/5)$. We use these values to determine z.

$$\begin{aligned} z &= -\frac{2}{w+1} \\ &= -\frac{2}{(w+1)}\frac{(\overline{w}+1)}{(\overline{w}+1)} \\ &= -\frac{2\overline{w}+2}{w\overline{w}+(w+\overline{w})+1} \\ &= -\frac{2\overline{w}+2}{1+2\cos(2k\pi/5)+1} \\ &= -\frac{(2\cos(2k\pi/5)+2) - i2\sin(2k\pi/5)}{2+2\cos(2k\pi/5)} \\ &= -1 + i\frac{\sin(2k\pi/5)}{1+\cos(2k\pi/5)}, \qquad k = 0, 1, 2, 3, 4. \end{aligned}$$

Therefore, $z = -1,\ -1 \pm 3.07768i,\ -1 \pm 0.726543i$.

(b) Conjecture: All solutions of $(z+2)^n + z^n = 0$ lie on the line $x = -1$.

Focus on Concepts

27. Use the fact that the n nth roots all lie on the same circle and are spaced at equal angular intervals of $2\pi/n$.

31. From formula (4) of Section 1.4 any nth root w_k of z has the form

$$w_k = \sqrt[n]{r}\left[\cos\left(\frac{\theta+2k\pi}{n}\right) + i\sin\left(\frac{\theta+2k\pi}{n}\right)\right].$$

If w_k is real, then

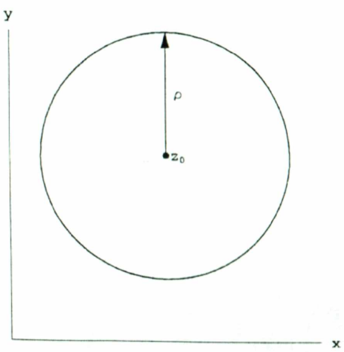
$$\sin\left(\frac{\theta+2k\pi}{n}\right) = 0.$$

Now what does this say about θ (and, consequently, z)?

1.5 Sets of Points in the Complex Plane

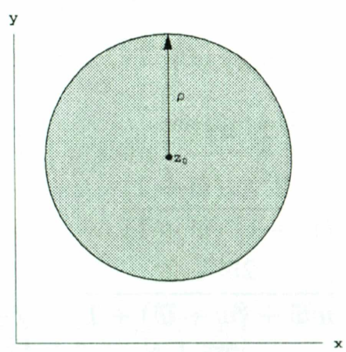
1.5 Summary

disks and neighborhoods: The following sets of points in the complex plane are used frequently.



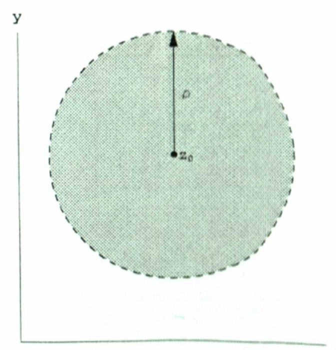
$|z - z_0| = \rho$

circle



$|z - z_0| \leq \rho$

disk

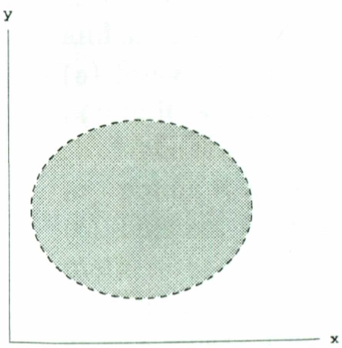


$|z - z_0| < \rho$

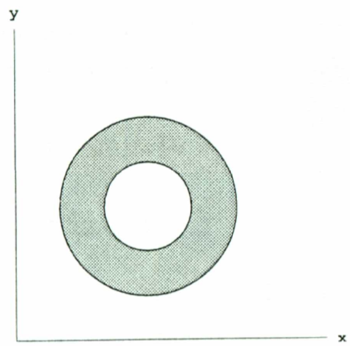
neighborhood of z_0

interior, exterior, and boundary points: A point z_0 is called an interior point of a set S if there exists some neighborhood of z_0 that lies entirely within S. If z_0 is an interior point of S, then it must be true that z_0 is in S, but it need not be true that every point of S is an interior point. A point z_1 is called an exterior point of a set S if there exists some neighborhood of z_1 that contains no points of S. If z_1 is an exterior point of S, then z_1 cannot be in S, but it need not be true that every point that is not in S is an exterior point. Points which are neither interior nor exterior to S are called boundary points of S. Every neighborhood of a boundary point of S must contain at least one point of S and at least one point not in S. It is possible that some boundary points are in S while others are not in S.

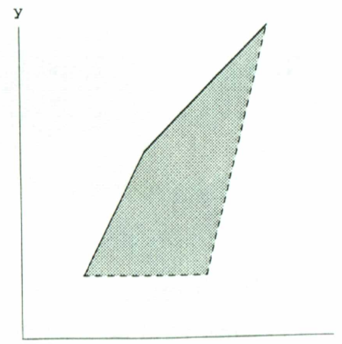
open/closed sets: A set S in the complex plane is called an open set if for every point z_0 of S there is a neighborhood of z_0 that lies entirely within S. That is, every point of an open set S is an interior point of S. An equivalent characterization of an open set S is a set which contains none of its boundary points. A set S which contains all of its boundary points is called closed. A set which contains some, but not all, of its boundary points is neither open nor closed. Therefore, "not open" does not mean "closed."



S is open

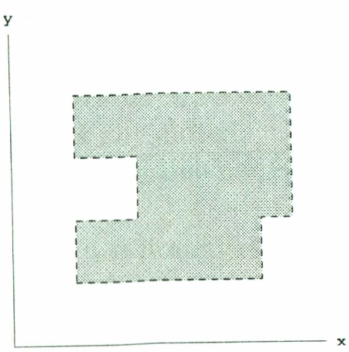


S is closed

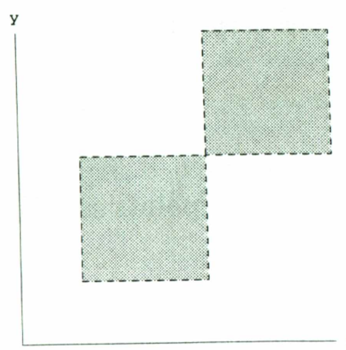


S is neither open nor closed

connected: A set S is called connected if every pair of points z_1 and z_2 in S can be connected by a polygonal path that consists of a finite number of line segments joined end to end where each line segment is contained entirely inside of S.



S is connected



S is not connected

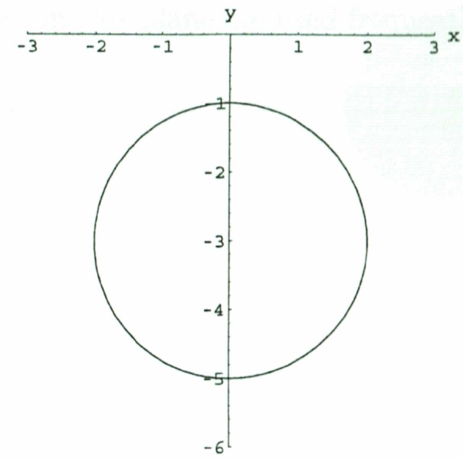
domains/regions: An open and connected subset of the complex plane is called a domain. A region consists of a domain together with all, some, or none of its boundary points.

bounded: A set S in the complex plane is bounded if there is a real number $R > 0$ such that $|z| < R$ for every point z in S. In other words, a set S is bounded if there exists a neighborhood of 0 that has S completely inside of it. If S is not bounded, then call S unbounded.

Exercises 1.5

3.

By identifying $z_0 = -3i$ and $\rho = 2$ in (1) we see that this equation defines a circle of radius 2 centered at $-3i$.

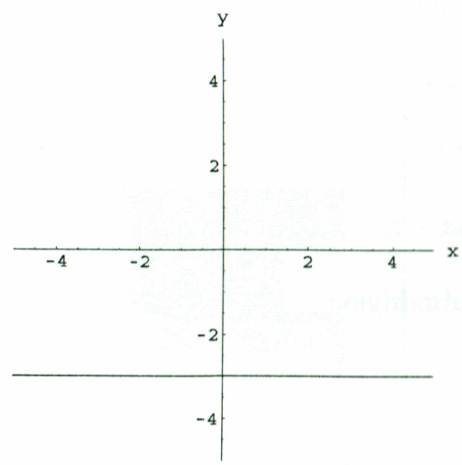


7.

If we set $z = x + iy$, then

$$\operatorname{Im}(\bar{z} + 3i) = \operatorname{Im}(x - iy + 3i) = 3 - y.$$

Thus, $\operatorname{Im}(\bar{z} + 3i) = 6$ if $3 - y = 6$, or $y = -3$. Therefore, the set of points satisfying $\operatorname{Im}(\bar{z}+3i) = 6$ is the line $y = -3$.

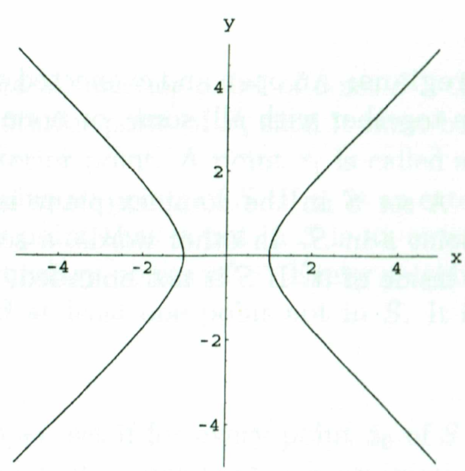


11.

If we set $z = x + iy$, then

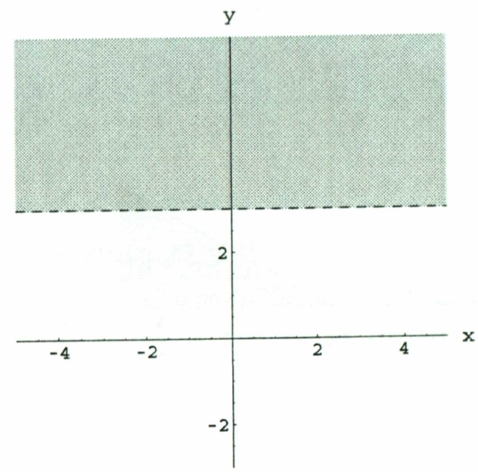
$$\begin{aligned} \operatorname{Re}\left(z^2\right) &= \operatorname{Re}\left((x + iy)^2\right) \\ &= \operatorname{Re}\left((x^2 - y^2) + 2xyi\right) \\ &= x^2 - y^2. \end{aligned}$$

Therefore, the set of points satisfying $\operatorname{Re}\left(z^2\right) = 1$ is the hyperbola $x^2 - y^2 = 1$.



15.

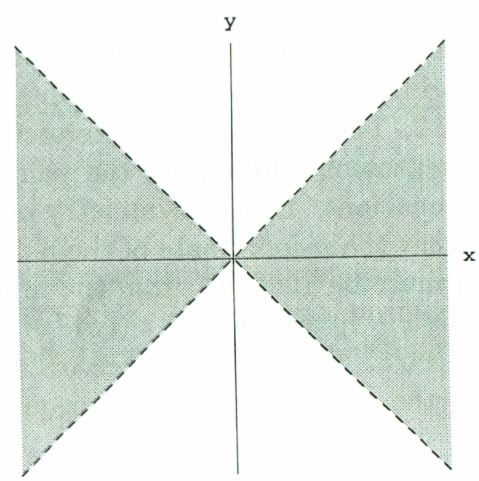
If we set $z = x + iy$, then $\text{Im}(z) = \text{Im}(x + iy) = y$, and so, the set $\text{Im}(z) > 3$ is the half-plane $y > 3$.



(a) Since this set contains none of its boundary points it is open.
(b) Since this set does not contain all of its boundary points it is not closed.
(c) This set is open and connected, so it is a domain.
(d) This set is not bounded because there does not exist a neighborhood of 0 that contains it.
(e) This set is connected since any pair of points in it can be joined by a polygonal path completely inside of S.

19.

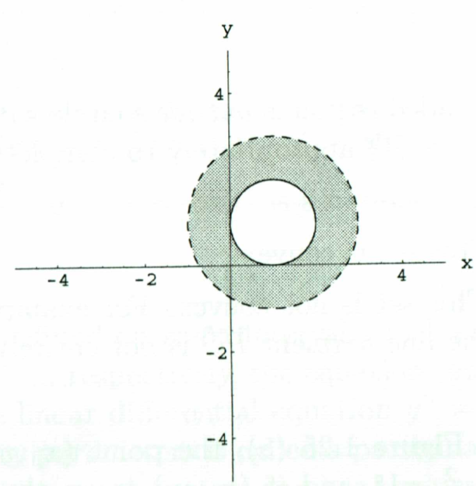
If we set $z = x + iy$, then $\text{Re}\left(z^2\right) = x^2 - y^2$. It follows that the set of points satisfying $\text{Re}\left(z^2\right) > 0$ is the set $x^2 - y^2 > 0$. That is, $(x + y)(x - y) > 0$. This inequality holds when $x > -y$ and $x > y$, or when $x < -y$ and $x < y$.



(a) Since this set contains none of its boundary points it is open.
(b) Since this set does not contain all of its boundary points it is not closed.
(c) This set is open but not connected, so it is not a domain.
(d) This set is not bounded because there does not exist a neighborhood of 0 that contains it.
(e) This set is not connected since any polygonal path joining $z_1 = 1$ to $z_2 = -1$ must leave S.

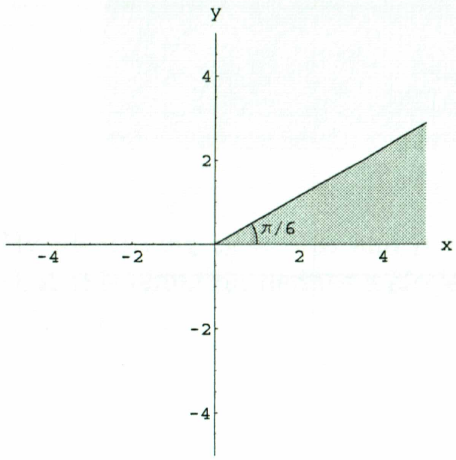
23.

Identifying $\rho_1 = 1$, $\rho_2 = 2$, and $z_0 = 1 + i$ in (2) we see that this set is an annulus. However, since we have $1 \leq |z - 1 - i|$, the inner circle of boundary points *is* included in the set.

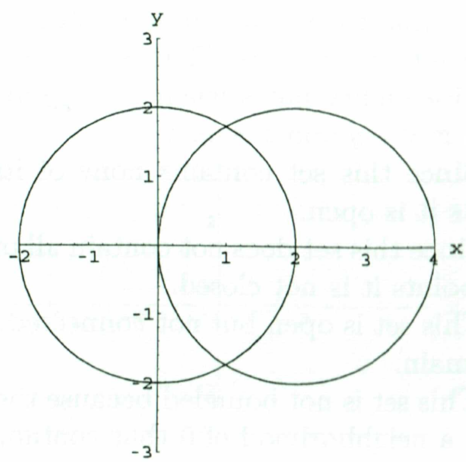


(a) Since this set contains some of its boundary points it is not open.
(b) Since this set does not contain all of its boundary points it is not closed.
(c) This set is not open, so it is not a domain.
(d) This set is bounded because $|z| < 5$ is a neighborhood of 0 that contains S.
(e) This set is connected since any pair of points in it can be joined by a polygonal path completely inside of S.

27.



31.



From (1), $|z| = 2$ is a circle with radius 2 centered at 0, and $|z - 2| = 2$ is a circle with radius 2 centered at 2. The two points of intersection of these circles represent solutions to the pair of simultaneous equations. By the symmetry of the figure, notice that the x-coordinate of the points of intersection must be 1. Thus, from the first equation, with $z = 1 + iy$ we have $|z| = \sqrt{1^2 + y^2} = 2$, and so, $y = \pm\sqrt{3}$. Therefore, $z = 1 \pm \sqrt{3}$ are the solutions to the given pair of simultaneous equations.

Focus on Concepts

35. **(a)** Disks are examples of connected sets.

(b) Unions of disjoint disks are examples of sets that are not connected.

39. The shaded region is *outside* a circle with radius 4 centered at $4i$ *or* inside a circle with radius 3 centered at 1. Use (1) appropriately to complete the description.

43. **(a)** This set is convex.

(d) This set is not convex. For example, if $P = 2 + i$ and $Q = -2 + i$, then both P and Q are in S but the line segment $\overline{PQ}$ is not entirely in S.

47. From Figure 1.25 (a), the point (x_0, y_0) on the unit circle will correspond to the real number $\frac{1}{4}$ if $x_0^2 + y_0^2 = 1$ and if (x_0, y_0) is on the line with slope -4 and y-intercept 1. Thus, we must have

$y_0 = -4x_0 + 1$. Substituting this expression for y_0 in the first equation gives:

$$
\begin{aligned}
x_0^2 + y_0^2 &= 1 \\
x_0^2 + (-4x_0 + 1)^2 &= 1 \\
x_0^2 + 16x_0^2 - 8x_0 + \cancel{1} &= \cancel{1} \\
x_0(17x_0 - 8) &= 0 \\
x_0 &= \frac{8}{17}. \qquad \leftarrow \text{since } x_0 \neq 0
\end{aligned}
$$

This implies that $y_0 = -4x_0 + 1 = -\frac{15}{17}$. Therefore, $z = \frac{8}{17} - i\frac{15}{17}$ is the point on the unit circle corresponding to the real number $\frac{1}{4}$.

1.6 Applications

1.6 Review Topic: Differential Equations

differential equation: An equation containing the derivatives of one or more unknown functions with respect to one more independent variables is called a differential equation. If the equation involves only the derivatives of functions with respect to a single independent variable, then it is called an ordinary differential equation.

order: The order of the highest derivative in a differential equation is called the order of the equation. For example, $y'' - 3ty' + t^5y = 0$ is a second-order ordinary differential equation and $e^{3t}\dfrac{d^3y}{dt^3} - y^7 + 5 = \cos(t)$ is a third-order differential equation.

linear: An ordinary differential equation is called linear if it has the form

$$a_n(t)\frac{d^ny}{dt^n} + a_{n-1}(t)\frac{d^{n-1}y}{dt^{n-1}} + ... + a_1(t)\frac{dy}{dt} + a_0(t)y = g(t).$$

If $g(t) = 0$, then the linear differential equation is called homogeneous. Therefore, $y'' - 3ty' + t^5y = 0$ is a 2nd-order, linear, homogeneous differential equation, but $e^{3t}\dfrac{d^3y}{dt^3} - y^7 + 5 = \cos(t)$ is a nonlinear differential equation.

solution: A solution of a differential equation is a function f defined on some interval which satisfies the equation. That is, when f, f', f'', ... are substituted for y, y', y'', ... respectively, the equation reduces to an identity. For example, both $7e^{2t}$ and $3e^{-2t}$ are solutions of the linear differential equation $y'' = 4y$ on the interval $(-\infty, \infty)$. A solution that does not contain any arbitrary parameters is called a particular solution. If a solution of an nth-order differential equation contains n arbitrary parameters and has the property that every solution can be obtained by using appropriate choices of the parameters, then we call it a general solution. Thus, $7e^{2t}$ is a particular solution and $c_1e^{2t} + c_2e^{-2t}$ is a general solution of $y'' = 4y$.

1.6 Summary

quadratic formula: The quadratic formula for finding the roots of the quadratic equation $az^2 + bz + c = 0$ is still valid when the coefficients a, b and c are complex numbers

$$z = \frac{-b + (b^2 - 4ac)^{1/2}}{2a}.$$

When $b^2 - 4ac$ is a complex number, use formula (4) of Section 1.4 with $n = 2$ to find the two values of $(b^2 - 4ac)^{1/2}$.

Euler's formlula: If θ is a real number, then define $e^{i\theta}$ by the following formula:

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

differential equations: Given a linear, 2nd-order, homogeneous differential equation with constant coefficients

$$ay'' + by' + cy = 0,$$

the function $y = e^{mx}$ is a solution whenever m is a solution to the auxiliary equation $am^2 + bm + c = 0$. If the roots of the equation $am^2 + bm + c = 0$ are complex, that is, if $m = \alpha + i\beta$, then by Euler's formula:

$$\begin{aligned} y_1 &= \tfrac{1}{2}\left[e^{(\alpha+i\beta)x} + e^{(\alpha+i\beta)x}\right] &&= e^{\alpha x}\cos\beta x \text{ and} \\ y_2 &= \tfrac{1}{2i}\left[e^{(\alpha+i\beta)x} - e^{(\alpha+i\beta)x}\right] &&= e^{\alpha x}\sin\beta x \end{aligned}$$

are real solutions to the differential equation.

exponential form: If z is a complex number with polar form $z = r(\cos\theta + i\sin\theta)$, then

$$z = re^{i\theta}$$

is called the exponential form of z.

electrical engineering: The linear, 2nd-order differential equation

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = E_0\sin\gamma t$$

is important in electrical engineering because it describes the charge $q(t)$ on a capacitor in an LRC-series circuit. One can find a particular solution to this differential equation using complex analysis. Since $i(t) = q'(t)$ is used by electrical engineers to denote the current in the circuit, we now use the symbol $j = \sqrt{-1}$ to denote the imaginary unit. The complex impedance of the circuit is

$$Z_C = R + j\left(L\gamma - \frac{1}{C\gamma}\right).$$

The steady-state current can be written in terms of the complex impedance:

$$i_p(t) = \mathrm{Im}\left(\frac{E_0}{Z_C} e^{j\gamma t}\right).$$

To find the charge $q_p(t)$ you integrate $i_p(t)$ with respect to t. In a similar manner, if we have the differential equation

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = E_0 \cos\gamma t,$$

then the steady-state current is given by:

$$i_p(t) = \mathrm{Re}\left(\frac{E_0}{Z_C} e^{j\gamma t}\right).$$

Exercises 1.6

3. We identify $a = 1$, $b = -(1+i)$, and $c = 6 - 17i$ in formula (3) of Section 1.6 to obtain

$$\begin{aligned} z &= \frac{1 + i + \left((-1-i)^2 - 4(1)(6-17i)\right)^{1/2}}{2(1)} \\ &= \frac{1 + i\,(2i - 24 + 68i)^{1/2}}{2} \\ &= \frac{1 + i + (-24+70i)^{1/2}}{2}. \end{aligned}$$

Now we apply formula (4) of Section 1.4 to find the two square roots $(-24+70i)^{1/2}$. Since

$$|-24+70i| = \sqrt{(-24)^2 + (70)^2} = 74,$$

if $\theta = \mathrm{Arg}(-24+70i)$, then

$$(-24+70i)^{1/2} = \pm\sqrt{74}\left[\cos\left(\frac{\theta}{2}\right) + i\sin\left(\frac{\theta}{2}\right)\right].$$

Since $\cos\theta = -12/37$, the half-angle formulas give

$$\begin{aligned} \cos\frac{\theta}{2} &= \sqrt{\frac{1+\cos\theta}{2}} & \text{and} \quad \sin\frac{\theta}{2} &= \sqrt{\frac{1-\cos\theta}{2}} \\ &= \sqrt{\frac{1-\frac{12}{37}}{2}} & &= \sqrt{\frac{1+\frac{12}{37}}{2}} \\ &= \frac{5}{\sqrt{74}} & &= \frac{7}{\sqrt{74}}. \end{aligned}$$

Thus,

$$(-24+70i)^{1/2} = \pm\sqrt{74}\left[\frac{5}{\sqrt{74}} + i\frac{7}{\sqrt{74}}\right] = \pm(5+7i).$$

Finally, using these two square roots we have

$$\begin{aligned} z &= \frac{1+i+(-24+70i)^{1/2}}{2} \\ &= \frac{1+i\pm(5+7i)}{2} \\ &= \frac{6+8i}{2},\ \frac{-4-6i}{2} \\ &= 3+4i,\ -2-3i. \end{aligned}$$

Identifying $a=1$, $z_1=3+4i$ and $z_2=-2-3i$ in (5) we obtain

$$z^2-(1+i)z+6-17i=(z-3-4i)(z+2+3i).$$

7. Since $|-10|=10$ and $\operatorname{Arg}(-10)=\pi$, a polar form of -10 is $10(\cos\pi+i\sin\pi)$. Therefore, an exponential form of -10 is

$$-10=10e^{\pi i}.$$

11. First note that $(3-i)^2=8-6i$. Now since $|8-6i|=10$ and $\operatorname{Arg}(8-6i)=\arctan(-3/4)$, a polar form of $(3-i)^2$ is $10\left[\cos\left(\tan^{-1}\left(-\frac{3}{4}\right)\right)+i\sin\left(\tan^{-1}\left(-\frac{3}{4}\right)\right)\right]$. Therefore, an exponential form of $(3-i)^2$ is

$$(3-i)^2=10e^{i\tan^{-1}\left(-\frac{3}{4}\right)}.$$

15. The auxiliary equation for this differential equation is $m^2+m+1=0$ which has solutions

$$m=\frac{-1\pm\sqrt{1-4}}{2}=-\frac{1}{2}\pm\frac{\sqrt{3}}{2}i.$$

Now identifying $\alpha=-\frac{1}{2}$ and $\beta=\frac{\sqrt{3}}{2}$ in formula (8) of Section 1.6 gives the solutions

$$y_1=e^{-\frac{1}{2}x}\cos\frac{\sqrt{3}}{2}x \quad\text{and}\quad y_2=e^{-\frac{1}{2}x}\sin\frac{\sqrt{3}}{2}x.$$

19. With the identifications $L=1$, $R=1$, $C=1/2$, and $\gamma=1$, the complex impedance is given by

$$\begin{aligned} Z_C &= R+j\left(L\gamma-\frac{1}{C\gamma}\right) \\ &= 1+j\,(1-2) \\ &= 1-j \end{aligned}$$

Therefore, the impedance is $Z = |Z_C| = |1 - j| = \sqrt{2}$. Now from (18) with the identification $E_0 = 100$, $\gamma = 1$, and $Z_C = 1 - j$, the steady-state current is

$$\begin{aligned} i_p(t) &= \operatorname{Im}\left(\frac{E_0}{Z_C}\,e^{jt}\right) \\ &= \operatorname{Im}\left(\frac{100}{1-j}\,e^{jt}\right) \\ &= \operatorname{Im}\left(\frac{100}{1-j}\,\frac{1+j}{1+j}\,e^{jt}\right) \\ &= \operatorname{Im}\left(\frac{100}{2}(1+j)\,(\cos t + j\sin t)\right) \\ &= 50\operatorname{Im}\left(\cos t - \sin t + j\,(\cos t + \sin t)\right) \\ &= 50(\cos t + \sin t). \end{aligned}$$

In order to find the steady-state charge we integrate the current $i_p(t)$ with respect to t:

$$q_p(t) = \int i_p(t)\,dt = \int 50(\cos t + \sin t)dt = 50(\sin t - \cos t).$$

Focus on Concepts

23. Use formula (5) of Section 1.6 with $a = 4$, $z_1 = -\frac{3}{2} + \frac{5}{2}i$, and $z_2 = \overline{z}_1$.

27. Use long division.

31. Use (5) of Section 1.6 and Problem 22 to determine an auxiliary equation which has $\alpha + i\beta = -5 + 2i$ as a root.

Preface

This student study guide is designed to accompany the text *A First Course in Complex Analysis with Applications, Second Edition* (Jones and Bartlett Publishers, 2009) by Dennis G. Zill and Patrick D. Shanahan. It consists of seven chapters which correspond to the seven chapters of the text. Each chapter has the following features.

Review Topics

Many sections of the study guide are preceded by a review of selected topics from calculus and differential equations that are required for that section. These reviews provide concise summaries of prerequisite notation, terminology, and concepts. For additional review, students are encouraged to consult appropriate mathematics texts. Two excellent sources that were used repeatedly for the review topics are *Calculus: Early Transcendentals, Fourth Edition* (Jones and Bartlett Publishers, 2010) by Dennis G. Zill and Warren S. Wright and *Advanced Engineering Mathematics, Third Edition* (Jones and Bartlett Publishers, 2006) by Dennis G. Zill and Michael R. Cullen.

Summaries

A summary of every section of the text is provided. The summary reviews all of the key ideas of the section including all terminology, formulas, theorems, and concepts. Figures with two colors are included to aid in geometric understanding.

Exercises

Following the summary, complete solutions are given for every other odd exercise in the section (eg. problems 3, 7, 11, etc.). These are full solutions, supported by figures with two colors, that supply all of the pertinent details of the problem and incorporate the same techniques and writing style used in the text. The solutions also include references to equations, definitions, theorems, and figures in the text. The answer to each problem is typeset in color for easy reference.

Focus on Concepts

The focus on concepts problems from the text consist of conceptual word, proof, and geometrical problems. Since they are often used as topics for classroom discussion or independent study we have included detailed hints rather than full solutions for these problems. As with the exercises, only every other odd problem is included.

Final Note to Students

The most effective way to learn mathematics is to work many, many problems. You should not review a solution in this study guide before first working or, at the very least, attempting to work the problem yourself. Learning advanced mathematical topics takes significant time and effort. It may be quicker to look at the solutions, then try to work problems, but ultimately this approach will not lead to an independent understanding of concepts and problem solving strategies that are required for success.

Chapter 2

Complex Functions and Mappings

2.1 Complex Functions

Review Topic: Functions

function: A function f from a set A to a set B is a rule that assigns to each element $a \in A$ exactly one element $b \in B$. The notation $f : A \to B$ is used to denote a function from A to B.

values: If the element $b \in B$ is assigned to the element $a \in A$ by a function $f : A \to B$, then we call b the image of a under f, or the value of f at a. We write $b = f(a)$ to indicate this assignment.

domain: If $f : A \to B$, then the set A is called the domain of f, written as $\mathrm{Dom}(f)$.

range: If $f : A \to B$, then the set of all images in B is called the range of f, written as $\mathrm{Range}(f)$. Using set notation the range is defined by

$$\mathrm{Range}(f) = \{f(a) \mid a \in A\}.$$

one-to-one If $f : A \to B$, then f is called one-to-one if every element in the range of f corresponds to exactly one element in the domain of f. In other words, f is one-to-one if whenever $a_1 \neq a_2$, then $f(a_1) \neq f(a_2)$.

onto: If $f : A \to B$, then f is called onto if the range of f is equal to the entire set B. That is, f is onto if for every $b \in B$ there is some $a \in A$ so that $f(a) = b$.

2.1 Summary

complex function: A complex function, or a complex-valued function of a complex variable, is a function whose domain and range are subsets of the set $\mathbf{C}$. That is, if $f : A \to B$ is a complex function, then we must have $A \subseteq \mathbf{C}$ and $B \subseteq \mathbf{C}$.

$w = f(z)$: The notation $w = f(z)$ is used to represent a complex-valued function of a complex variable, while the notation $y = f(x)$ is used to represent a real-valued function of a real variable.

$\mathrm{Re}(f)$ and $\mathrm{Im}(f)$: Given a complex function $w = f(z)$, if we set $z = x + iy$ and express the function in terms of two real functions as:

$$f(z) = u(x, y) + iv(x, y),$$

then the functions $u(x, y)$ and $v(x, y)$ are called the real and imaginary parts of f, respectively. We use the notation $\mathrm{Re}(f) = u(x, y)$ and $\mathrm{Im}(f) = v(x, y)$.

complex exponential function: The complex exponential function is defined by:

$$e^z = e^x \cos y + ie^x \sin y.$$

polar coordinates: Given a complex function $w = f(z)$, we can replace the complex variable z with its polar form $z = r(\cos\theta + i\sin\theta)$ and express the function in terms of two real functions as:

$$f(z) = u(r, \theta) + iv(r, \theta).$$

We still call $\mathrm{Re}(f) = u(r, \theta)$ and $\mathrm{Im}(f) = v(r, \theta)$ the real and imaginary parts of f, but these real functions are different from those obtained using Cartesian form $x + iy$ of the complex variable z.

Exercises 2.1

3. **(a)** For $z = 1$ we have $|z| = 1$ and $\mathrm{Arg}(z) = 0$. Therefore,

$$f(1) = \log_e 1 + i(0) = 0.$$

(b) For $z = 4i$ we have $|z| = 4$ and $\mathrm{Arg}(z) = \frac{\pi}{2}$. Therefore,

$$f(4i) = \log_e 4 + i\frac{\pi}{2} \approx 1.38629 + 1.57080i.$$

(c) For $z = 1 + i$ we have $|z| = \sqrt{2}$ and $\mathrm{Arg}(z) = \frac{\pi}{4}$. Therefore,

$$f(1+i) = \log_e \sqrt{2} + i\frac{\pi}{4} = \frac{1}{2}\log_e 2 + i\frac{\pi}{4} \approx 0.34657 + 0.78540i.$$

7. **(a)** For $z = 3$ we have $r = |z| = 3$ and $\theta = \mathrm{Arg}(z) = 0$. Therefore,

$$f(3) = 3 + i\cos^2 0 = 3 + i(1)^2 = 3 + i.$$

(b) For $z = -2i$ we have $r = |z| = 2$ and $\theta = \text{Arg}(z) = -\frac{\pi}{2}$. Therefore,

$$f(-2i) = 2 + i\cos^2\left(-\frac{\pi}{2}\right) = 2 + i(0)^2 = 2.$$

(c) For $z = 2 - i$ we have $r = |z| = \sqrt{5}$. Moreover, from equating the Cartesian and polar forms of the point $z = 2 - i = \sqrt{5}(\cos\theta + i\sin\theta)$ we obtain $\cos\theta = 2/\sqrt{5}$. Therefore,

$$f(2-i) = \sqrt{5} + i\left(\frac{2}{\sqrt{5}}\right)^2 = \sqrt{5} + \frac{4}{5}i \approx 2.23607 + 0.8i.$$

11. If we set $z = x + iy$, then

$$\begin{aligned} f(z) &= (x+iy)^3 - 2(x+iy) + 6 \\ &= x^3 + 3x^2yi + 3xy^2i^2 + y^3i^3 - 2x - 2yi + 6 \\ &= x^3 + 3x^2yi - 3xy^2 - y^3i - 2x - 2yi + 6 \\ &= (x^3 - 3xy^2 - 2x + 6) + (3x^2y - y^3 - 2y)i. \end{aligned}$$

Therefore, $\text{Re}(f) = x^3 - 3xy^2 - 2x + 6$ and $\text{Im}(f) = 3x^2y - y^3 - 2y$.

15. If we set $z = x + iy$, then $2(x+iy) + i = 2x + (2y+1)i$. So, from (3) of Section 2.1 we have:

$$\begin{aligned} e^{2z+i} &= e^{2x+(2y+1)i} \\ &= e^{2x}\cos(2y+1) + ie^{2x}\sin(2y+1). \end{aligned}$$

Therefore, $\text{Re}(f) = e^{2x}\cos(2y+1)$ and $\text{Im}(f) = e^{2x}\sin(2y+1)$.

19. If we set $z = r(\cos\theta + i\sin\theta)$, then

$$\begin{aligned} f(z) &= \left[r(\cos\theta + i\sin\theta)\right]^4 \\ &= r^4(\cos 4\theta + i\sin 4\theta) \qquad \leftarrow \text{see (9) in Section 1.3} \\ &= r^4\cos 4\theta + ir^4\sin 4\theta. \end{aligned}$$

Therefore, $\text{Re}(f) = r^4\cos 4\theta$ and $\text{Im}(f) = r^4\sin 4\theta$.

23. Since $\text{Re}(z)$ and z^2 are defined for all complex numbers, the natural domain of $f(z) = 2\text{Re}(z) - iz^2$ is the set **C** of all complex numbers.

Focus on Concepts

27. **(a)** Does $\arg(z)$ assign one and only one value to z?

(d) See part (a).

31. In order to determine the natural domain consider the questions: For which values of x and y is $\cos(x - y)$ defined? How about $\sin(x - y)$? In order to determine the range, consider $|f(z)|$.

35. Verify that $|e^{-z}| = e^{-x}$, then use this identity to answer the question.

2.2 Complex Functions as Mappings

Review Topic: Parametric Curves

parametric curve: If $f(t)$ and $g(t)$ are real-valued functions of a real variable t, then the set C of all points $(f(t), g(t))$, where $a \leq t \leq b$, is called a parametric curve. The equations $x = f(t)$ and $y = g(t)$ are called parametric equations for C, and the variable t is called the parameter.

eliminating the parameter: Given a set of parametric equations $x = f(t)$, $y = g(t)$ it is sometimes possible to eliminate the parameter t to obtain a single Cartesian equation of the curve. For example, if the parametric equations $x = t^2 - t + 2$, $y = -t^2 + t$ for a curve C are added together we obtain the single equation $x + y = 2$ (so, the curve C defined by these equations is a line).

smooth curves: A parametric curve given by $x = f(t)$, $y = g(t)$, $a \leq t \leq b$ is called smooth if f' and g' are continuous on $[a, b]$ and not simultaneously zero on (a, b). A piecewise smooth curve consists of a finite number of smooth curves joined end to end.

tangent lines: If C is a parametric curve given by $x = x(t)$, $y = y(t)$, $a \leq t \leq b$ and if both $f(t)$ and $g(t)$ are differentiable on (a, b), then the slope of the tangent line to C at $(f(t), g(t))$ is given by

$$\frac{dy}{dx} = \frac{g'(t)}{f'(t)}$$

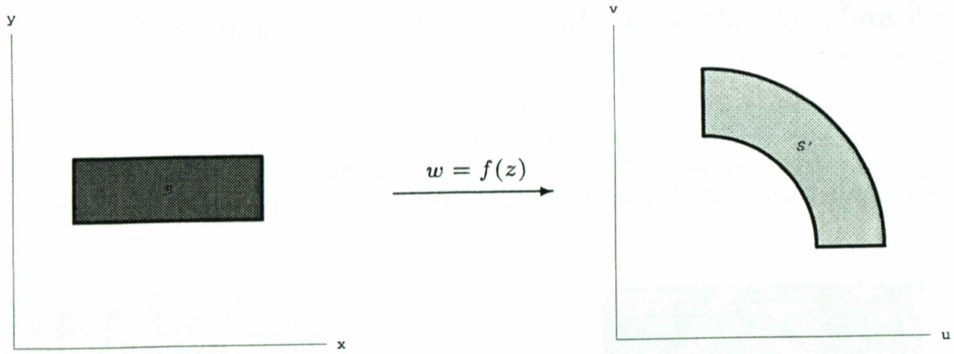
provided $f'(t) \neq 0$.

2.2 Summary

complex mapping: The term complex mapping is used to refer to the correspondence determined by a complex function $w = f(z)$ between points in the z-plane and their images in the w-plane.

image of S under f: If S is a set and $f(z)$ is a complex function, then the image of S under f is the set S' consisting of the images under the mapping $w = f(z)$ of all of the points in S.

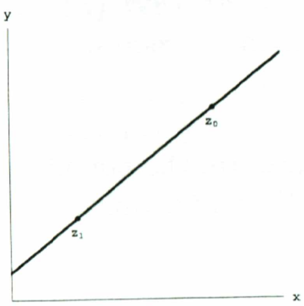
geometric representation: A geometric representation of a complex mapping consists of a set S shown in one copy of the complex plane and the image S' of S under f shown in a second copy of the complex plane. See below.



complex parametric curves: If $x(t)$ and $y(t)$ are real-valued functions of a real parameter t, then the set C consisting of all points $z(t) = x(t) + iy(t)$, $a \le t \le b$ is called a complex parametric curve. The complex-valued function of the real variable t, $z(t) = x(t) + iy(t)$ is called a parametrization of C.

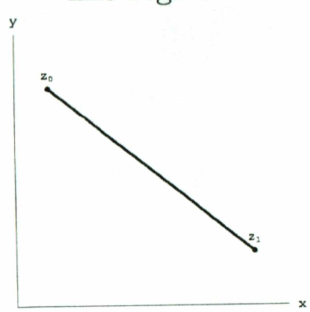
commonly used parametrizations: The following parametrizations are commonly used to help describe complex mappings.

line



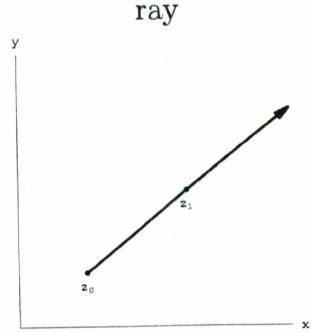
$$z(t) = z_0(1-t) + z_1 t, \quad -\infty < t < \infty$$

line segment



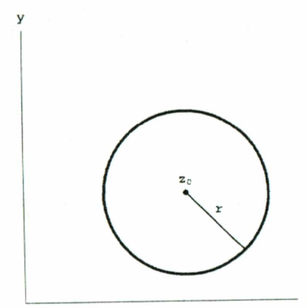
$$z(t) = z_0(1-t) + z_1 t, \quad 0 \le t \le 1$$

ray



$$z(t) = z_0(1-t) + z_1 t, \quad 0 \le t < \infty$$

circle



$$z(t) = z_0 + re^{it}, \quad 0 \le t \le 2\pi$$

image of a parametric curve: If $w = f(z)$ is a complex mapping and if C is a curve parametrized by $z(t)$, $a \le t \le b$, then $w(t) = f(z(t))$, $a \le t \le b$ is a parametrization of the image, C', of C under $w = f(z)$.

Exercises 2.2

3. The half-plane S can be described by the two simultaneous inequalities

$$-\infty < x < \infty \quad \text{and} \quad y > 2. \tag{1}$$

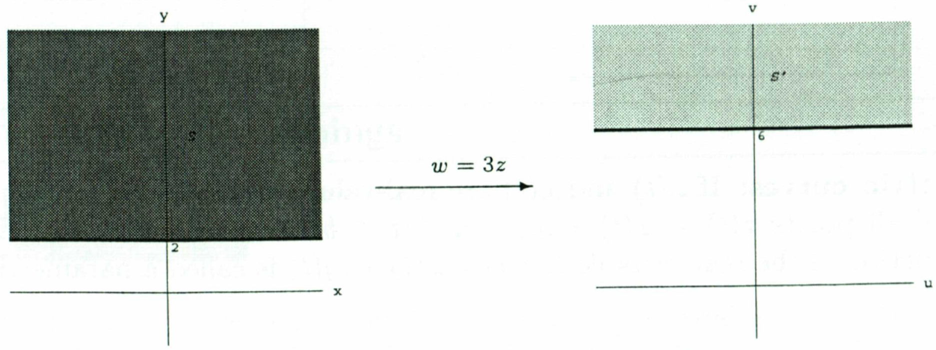
After replacing z by $x + iy$ in the mapping $w = 3z$, we obtain $w = 3(x + iy) = 3x + 3yi$, and so the real and imaginary parts of the mapping $w = 3z$ are

$$u = 3x \quad \text{and} \quad v = 3y. \tag{2}$$

The image S' of S under $w = 3z$ is found by using the equations in (2) to transform the bounds on x and y in (1) into bounds on u and v. If we substitute $x = u/3$ and $y = v/3$ in (1) and simplify, then we find that S' is given by

$$-\infty < u < \infty \quad \text{and} \quad v > 6.$$

Therefore, S' is the half-plane $\text{Im}(w) > 6$. The mapping is depicted below.



7. The half-plane S can be described by the two simultaneous inequalities

$$-\infty < x < \infty \quad \text{and} \quad y \leq 1. \tag{3}$$

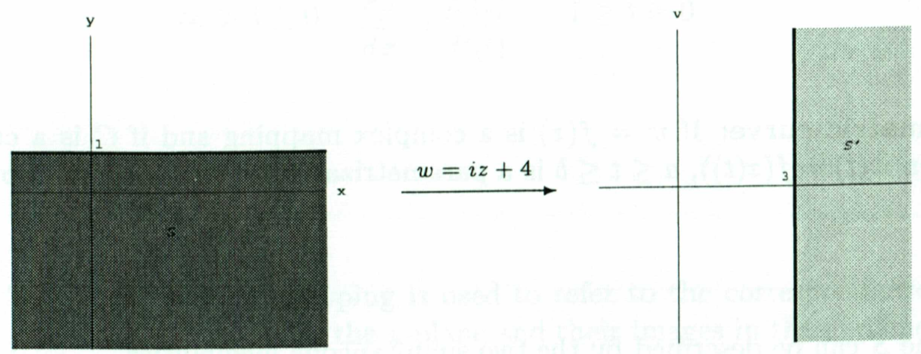
After replacing z by $x + iy$ in the mapping $w = iz + 4$, we obtain $w = i(x + iy) + 4 = (4 - y) + ix$, and so the real and imaginary parts of the mapping $w = iz + 4$ are

$$u = 4 - y \quad \text{and} \quad v = x. \tag{4}$$

The image S' of S under $w = iz + 4$ is found by using the equations in (4) to transform the bounds on x and y in (3) into bounds on u and v. If we substitute $x = v$ and $y = 4 - u$ in (3) and simplify, then we find that S' is given by

$$u \geq 3 \quad \text{and} \quad -\infty < v < \infty.$$

Therefore, S' is the half-plane $\text{Re}(w) \geq 3$. The mapping is depicted below.



11. The line $x = 0$ can be described by

$$x = 0 \quad \text{and} \quad -\infty < y < \infty. \tag{5}$$

From (1) of Section 2.1, the real and imaginary parts of $w = z^2$ are

$$u = x^2 - y^2 \quad \text{and} \quad v = 2xy. \tag{6}$$

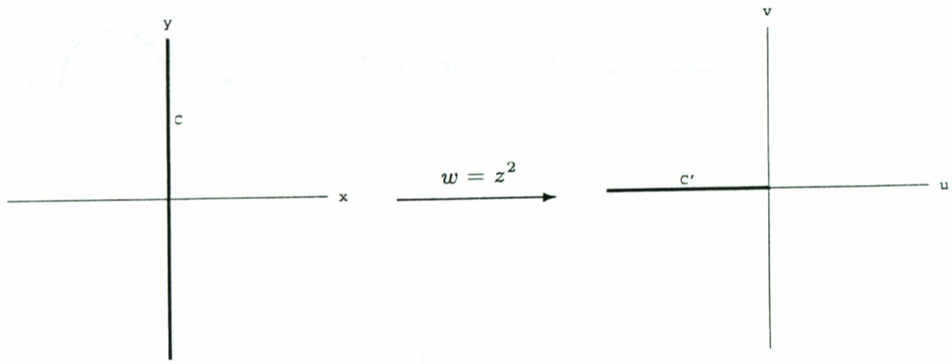
For any point $z = 0 + iy$ on the line $x = 0$, the equations describing the image in (6) become

$$u = -y^2 \quad \text{and} \quad v = 0. \tag{7}$$

Since $-\infty < y < \infty$ from (5), we have $0 \le y^2 < \infty$, and consequently, $-\infty < u \le 0$ from the first equation in (7). Therefore, the image C' of the line $x = 0$ under $w = z^2$ is the ray

$$-\infty < u \le 0 \quad \text{and} \quad v = 0.$$

The mapping is depicted below.



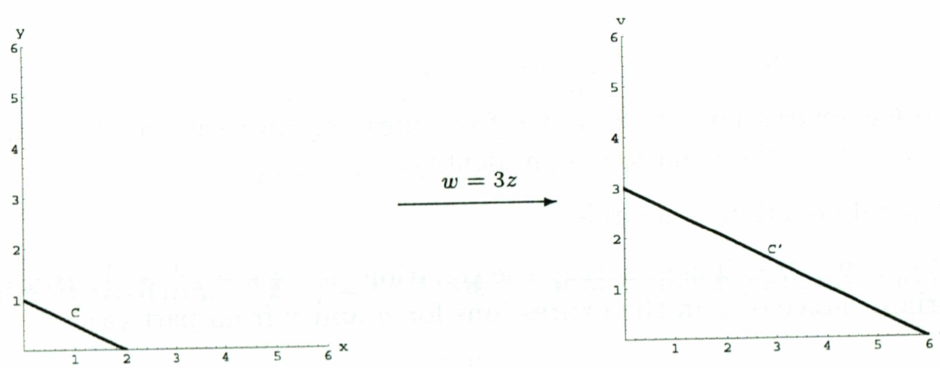
15. **(a)** With the identifications $z_0 = 2$ and $z_1 = i$ in parametrization (7) of Section 2.2, we see that the parametric curve C is the line segment from 2 to i.

(b) Using $z(t) = 2(1-t) + it$ and $f(z) = 3z$, the image of the parametric curve C is given by (11) of Section 2.2:

$$\begin{aligned} w(t) &= f(z(t)) \\ &= 3\,(2(1-t) + it) \\ &= 6(1-t) + 3it. \end{aligned}$$

Therefore, a parametrization of the image C' is $w(t) = 6(1-t) + 3it,\ 0 \le t \le 1$.

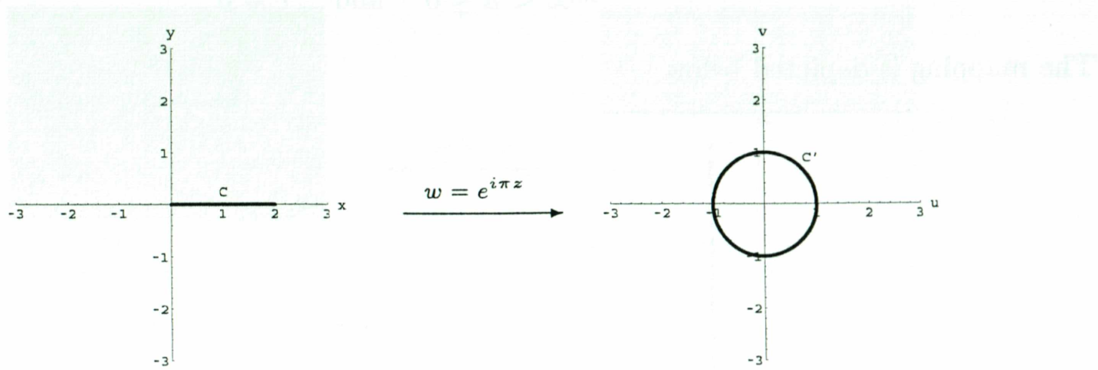
(c) With the identifications $z_0 = 6$ and $z_1 = 3i$ in parametrization (7) of Section 2.2, we see that the parametric curve C' is the line segment from 6 to $3i$. Plots for parts (a) and (c) are shown below.



19. **(a)** Replacing the parameter t in $z(t)$ with a new parameter $2s$ we obtain a different parametrization of C, $Z(s) = 2s,\ 0 \le s \le 1$. Thus, with the identifications $z_0 = 0$ and $z_1 = 2$ in parametrization (7) of Section 2.2, we see that the parametric curve C is the line segment from 0 to 2.

(b) Using $z(t) = t$ and $f(z) = e^{i\pi z}$, the image of the parametric curve C is given by (11) of Section 2.2, $w(t) = f(z(t)) = e^{i\pi t}$, $0 \le t \le 2$. Therefore, a parametrization of the image C' is $w(t) = e^{i\pi t}$, $0 \le t \le 2$.

(c) Replacing the parameter t in $w(t)$ with a new parameter s/π we obtain a different parametrization of C', $W(s) = e^{is}$, $0 \le s \le 2\pi$. Now with the identifications $z_0 = 0$ and $r = 1$ in parametrization (10) of Section 2.2, we see that the parametric curve C' is the unit circle. Plots for parts (a) and (c) are shown below.



23. The circle $|z| = 2$ has center $z_0 = 0$ and radius $r = 2$. Thus, from (10) of Section 2.2, a parametrization of C is $z(t) = 2e^{it}$, $0 \le t \le 2\pi$. The image of C under $f(z) = 1/z$ is given by (11) of Section 2.2:

$$\begin{aligned} w(t) &= f(z(t)) \\ &= \frac{1}{2e^{it}} \\ &= \frac{1}{2}e^{-it}. \end{aligned}$$

Replacing the parameter t in $w(t)$ with a new parameter $-s$ we obtain a different parametrization of C', $W(s) = \frac{1}{2}e^{is}$, $-2\pi \le s \le 0$. Now with the identifications $z_0 = 0$ and $r = \frac{1}{2}$ in parametrization (10) of Section 2.2, we see that C' is a circle centeblue at 0 with radius 1/2. That is, C' is the circle $|w| = \frac{1}{2}$.

Focus on Concepts

27. **(a)** Set $z = 1 + iy$ and then write $1/z$ in the form $u + iv$.

(b) Substitute the expressions for u and v from part (a) into the equation $(u - \frac{1}{2})^2 + v^2 = \frac{1}{4}$ and simplify to confirm that the equation is an identity.

(c) Use the general equation of a circle.

(d) The point $u + iv = 0 + i0$ does satisfy the equation $(u - \frac{1}{2})^2 + v^2 = \frac{1}{4}$. But can this point be given by an appropriate choice of y in the expressions for u and v from part (a)?

31. Consider the geometric relationship between a point $x + iy$ and its conjugate $x - iy$.

2.3 Linear Mappings

Review Topic: Composition

composition: Let f and g be functions. The composition of f and g is the function

$$(f \circ g)(z) = f(g(z)).$$

Notice that in order for $f \circ g$ to be defined at z, the value of $g(z)$ must be in the domain of f. In other words, the domain of $f \circ g$ is the subset of those z in the domain of g for which $g(z)$ is in the domain of f.

order: In general $(f \circ g)(z) \neq (g \circ f)(z)$. For example, if $f(z) = z^2$ and $g(z) = z + i$, then $(f \circ g)(z) = (z+i)^2 = z^2 + 2iz - 1$ but $(g \circ f)(z) = z^2 + i$.

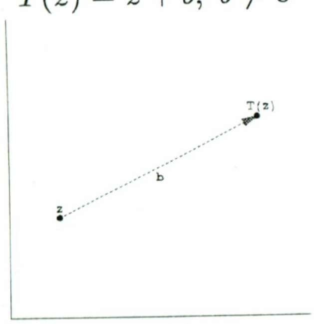
2.3 Summary

complex linear function: A complex linear function is a function of the form $f(z) = az + b$ where a and b are any complex constants.

translation/rotation/magnification: There are three special types of linear functions: a translation, a rotation, and a magnification. These functions are described below.

translation

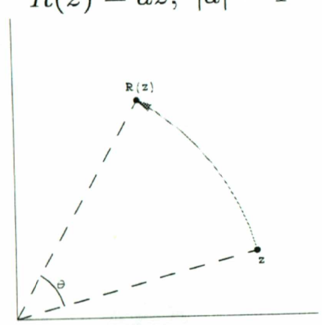
$T(z) = z + b, \; b \neq 0$



translates the point z along the vector b

rotation

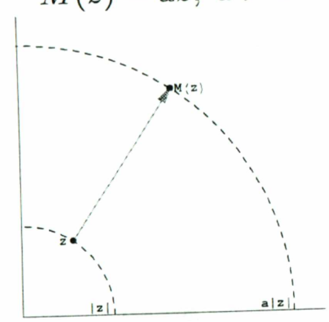
$R(z) = az, \; |a| = 1$



rotates the point z through an angle of $\theta = \text{Arg}(a)$ radians about the origin

magnification

$M(z) = az, \; a > 0$



magnifies the modulus of the point z by a factor of a

linear mapping as a composition: If $f(z) = az + b$ is a complex linear mapping and $a \neq 0$, then we can express f as:

$$f(z) = |a| \left(\frac{a}{|a|} z \right) + b.$$

This means that $f(z)$ is the composition $f(z) = (T \circ M \circ R)(z)$ where $R(z) = \frac{a}{|a|} z$ is rotation by $\text{Arg}(a)$, $M(z) = |a|z$ is magnification by $|a|$, and $T(z) = z + b$ is translation by b.

image of a point under a linear mapping: If $f(z) = az + b$ is a complex linear mapping with $a \neq 0$, and if $w_0 = f(z_0)$ is plotted in the same copy of the complex plane as z_0, then w_0 is the point obtained by

(*i*) rotating z_0 through angle $\text{Arg}(a)$ about the origin,

(*ii*) magnifying the result by $|a|$, and

(*iii*) translating the result by b.

geometry of linear mappings: Because a non-constant linear mapping is a composition of a rotation, a magnification, and a translation, the image of a geometric figure can have a different size, but its basic shape is the same. For example, the image of a circle is a circle, the image of a 5-sided polygon is a 5-sided polygon, etc.

Exercises 2.3

3. **(a)** Using (6) of Section 2.3, we express the linear mapping $f(z) = 3iz$ as

$$f(z) = |3i|\left(\frac{3i}{|3i|}z\right) = 3(iz) + 0.$$

Thus, the image of the closed disk $|z| \leq 1$ is found by

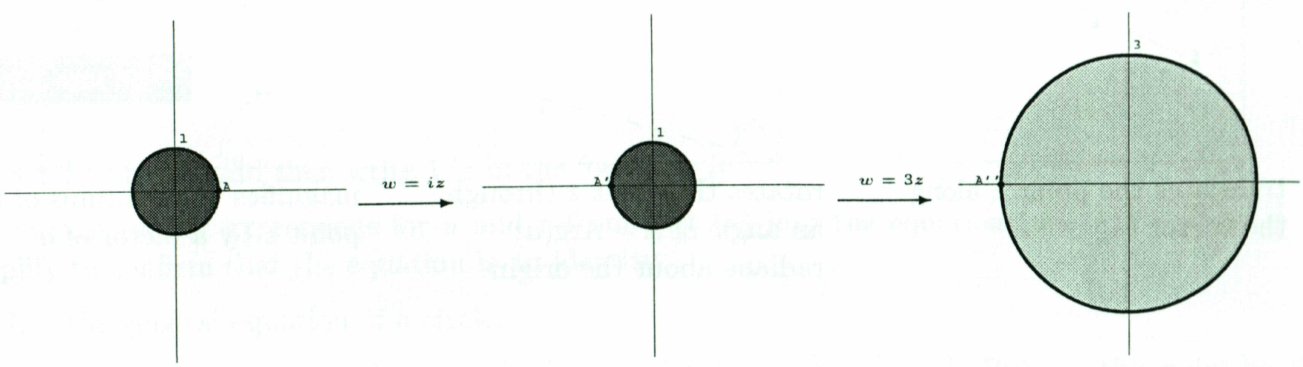
(*i*) rotating the disk by $\text{Arg}(i) = \pi$,

(*ii*) magnifying the result by $|3i| = 3$, and

(*iii*) translating the result by 0 (no translation).

Under the rotation (*i*), the disk is mapped onto itself and under the magnification (*ii*) the result is mapped onto a disk centered at the origin with radius 3. Therefore, the image is the closed disk $|w| \leq 3$.

(b)



7. **(a)** Using (6) of Section 2.3, we express the linear mapping $f(z) = z + 2i$ as

$$f(z) = |1|\left(\frac{1}{|1|}z\right) + 2i = 1(1z) + 2i.$$

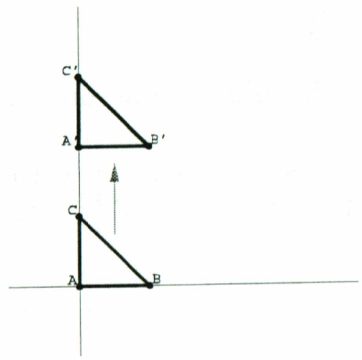
Thus, the image of the triangle is found by (*i*) rotating the disk by $\operatorname{Arg}(1) = 0$ (no rotation),

(*ii*) magnifying the result by $|1| = 1$ (no magnification), and

(*iii*) translating the result by $2i$.

Under the translation (*iii*), the vertices 0, 1, and i are mapped to $2i$, $1 + 2i$, and $3i$, respectively. Therefore, the image is the triangle with vertices $2i$, $1 + 2i$, and $3i$.

(b)



11. **(a)** Using (6) of Section 2.3, we express the linear mapping $f(z) = -3z + i$ as

$$f(z) = |-3|\left(\frac{-3}{|-3|}z\right) + i = 3(-1z) + i.$$

Thus, the image of the triangle is found by

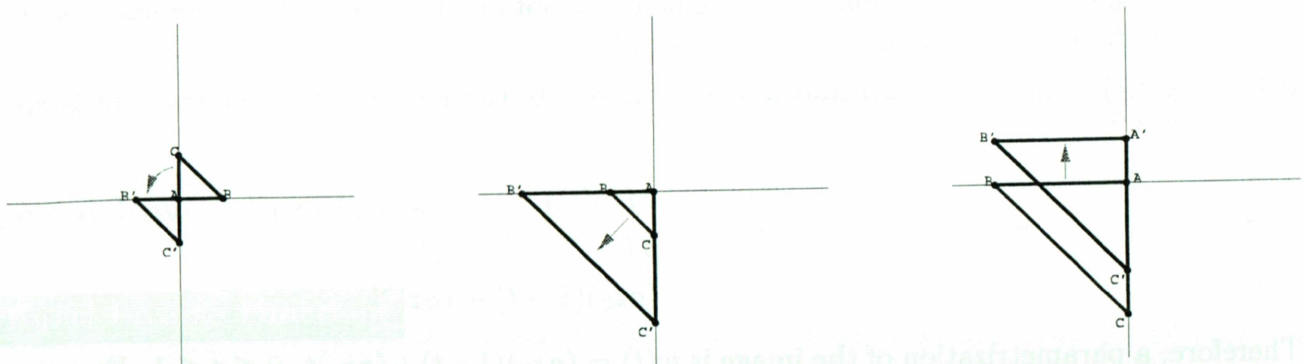
(*i*) rotating the disk by $\operatorname{Arg}(-1) = \pi$,

(*ii*) magnifying the result by $|-3| = 3$, and

(*iii*) translating the result by i.

Under the rotation (*i*), the vertices 0, 1, and i are mapped to 0, -1, and $-i$, respectively. Under the magnification (*ii*), these vertices are then mapped to 0, -3, and $-3i$. Finally, these vertices are mapped to i, $-3 + i$, and $-2i$ under the translation (*iii*). Therefore, the image is the triangle with vertices i, $-3 + i$, and $-2i$.

(b)



rotate by π magnify by 3 translate by i

15. Using (6) of Section 2.3, we express the linear mapping $f(z) = -\frac{1}{2}z + 1 - \sqrt{3}i$ as

$$f(z) = \left|-\frac{1}{2}\right| \left(\frac{-1/2}{|-1/2|}z\right) + 1 - \sqrt{3}i = \frac{1}{2}(-1z) + 1 - \sqrt{3}i.$$

Thus, the image of the triangle is found by

(i) rotating the disk by $\text{Arg}(-1) = \pi$,

(ii) magnifying the result by $\left|-\frac{1}{2}\right| = \frac{1}{2}$, and

(iii) translating the result by $1 - \sqrt{3}i$.

19. There are many ways to map the imaginary axis onto the line containing i and $1 + 2i$. One approach consists of first rotating the imaginary axis onto the line $y = x$, then translating this line 1 unit in the y-direction. In other words, one solution to the problem is a linear mapping that consists of a rotation clockwise through $\pi/4$ radians followed by a translation by i. This gives $f(z) = e^{-\pi i/4}z + i$.

23. **(a)** Using $z(t) = z_0(1-t) + z_1 t$ and $T(z) = z + b$, the image of the line segment is given by (11) of Section 2.2:

$$\begin{aligned} w(t) &= T(z(t)) \\ &= z_0(1-t) + z_1 t + b \\ &= z_0(1-t) + z_1 t + b(1-t) + bt \\ &= (z_0 + b)(1-t) + (z_1 + b)t. \end{aligned}$$

Therefore, a parametrization of the image is $w(t) = (z_0 + b)(1-t) + (z_1 + b)t$, $0 \le t \le 1$. From parametrization (7) of Section 2.2, we see that the image is a line segment from $z_0 + b$ to $z_1 + b$.

(b) Using $z(t) = z_0(1-t) + z_1 t$ and $R(z) = az$, $|a| = 1$, the image of the line segment is given by (11) of Section 2.2:

$$\begin{aligned} w(t) &= R(z(t)) \\ &= a\left(z_0(1-t) + z_1 t\right) \\ &= (az_0)(1-t) + (az_1)t. \end{aligned}$$

Therefore, a parametrization of the image is $w(t) = (az_0)(1-t) + (az_1)t$, $0 \le t \le 1$. From parametrization (7) of Section 2.2, we see that the image is a line segment from az_0 to az_1. Since $|a| = 1$, the endpoints az_0 and az_1 of the image line segment are obtained by rotating the points z_0 and z_1, respectively, though an angle of $\text{Arg}(a)$ about the origin.

(c) Using $z(t) = z_0(1-t) + z_1 t$ and $M(z) = az$, $a > 0$, the image of the line segment is given by (11) of Section 2.2:

$$\begin{aligned} w(t) &= M(z(t)) \\ &= a\left(z_0(1-t) + z_1 t\right) \\ &= (az_0)(1-t) + (az_1)t. \end{aligned}$$

Therefore, a parametrization of the image is $w(t) = (az_0)(1-t) + (az_1)t$, $0 \le t \le 1$. From parametrization (7) of Section 2.2, we see that the image is a line segment from az_0 to az_1. Since $a > 0$, the endpoints az_0 and az_1 of the image line segment are obtained by magnifying the points z_0 and z_1, respectively, by a factor of a.

Focus on Concepts

27. **(a)** Given translations $T_1(z) = z + b_1$ and $T_2(z) = z + b_2$, their composition is

$$T_1 \circ T_2(z) = T_1(z + b_2) = (z + b_2) + b_1 = z + (b_1 + b_2).$$

If we set $b = b_1 + b_2$, then the composition can be written as $T_1 \circ T_2(z) = z + b$. Therefore, $T_1 \circ T_2$ is a translation if $b_1 + b_2 \neq 0$ and the identity if $b_1 + b_2 = 0$. Since $T_2 \circ T_1(z) = z + (b_1 + b_2)$, the order of composition does not matter.

(b) Modify the argument in part (a).

(c) Modify the argument in part (a).

31. Let $f(z) = az + b$ for some complex constants a and b. Consider the identity $|z| = |az + b|$ for the values $z = 0$ and $z = 1$. What do the two resulting equations tell you about the coefficients a and b, and, consequently, the linear mapping $w = f(z)$?

35. **(a)** Solve the system of equations:

$$\begin{aligned} az_1 + b &= w_1 \\ az_2 + b &= w_2 \end{aligned}$$

for a and b.

(b) Consider f_1 a rotation and f_2 a magnification. What is the image of $z = 0$?

39. **(a)** Rotate, magnify, and translate the annulus in order to determine its image, then determine the least and greatest distance a point in the image can be from the origin.

(b) Determine the points in the original annulus that map onto the points that are closest and farthest from the origin identified in part (a).

(c) Notice that the closer a point $f(z)$ is to the origin, the farther $1/f(z)$ is from the origin and the farther $f(z)$ is from the origin, the closer $1/f(z)$ is to the origin.

2.4 Special Power Functions

Review Topic: Inverse Functions

one-to-one functions: A function f is one-to-one if whenever $f(a_1) = f(a_2)$, then $a_1 = a_2$. In other words, f is one-to-one if whenever $a_1 \neq a_2$, then $f(a_1) \neq f(a_2)$. Put yet another way, f is one-to-one if every element b in the range of f corresponds to exactly one element a in the domain of f.

inverse function: If f is a one-to-one function with domain A and range B, then the inverse function f^{-1} is the function with domain B and range A defined by $f^{-1}(b) = a$ if $f(a) = b$.

composition properties of inverse functions: If f is a one-to-one function with domain A and range B and if f^{-1} is its inverse, then

$$\begin{aligned}
\left(f \circ f^{-1}\right)(b) &= f\left(f^{-1}(b)\right) = b \qquad \text{for all } b \text{ in } B, \text{ and} \\
\left(f^{-1} \circ f\right)(a) &= f^{-1}\left(f(a)\right) = a \qquad \text{for all } a \text{ in } A.
\end{aligned}$$

inverse mappings: If a one-to-one function f maps a set S onto a set T, then the inverse function f^{-1} maps the set T onto the set S.

finding inverses algebraically: In order to find the inverse of a one-to-one function f algebraically we solve the equation $f(a) = b$ for the independent variable a, then relabel the independent and dependent variables appropriately. For example, the complex function $f(z) = 5z - 2i$ is one-to-one. If we solve the equation $w = 5z - 2i$ for the independent variable z we obtain $z = (w + 2i)/5$. Relabeling the independent and dependent variables gives the inverse function $f^{-1}(z) = (z + 2i)/5$.

restricting domains: If f is a function that is not one-to-one, then it does not have an inverse function. However, it may be possible to restrict the domain of f to a subset on which f is one-to-one and, consequently, use the restricted domain to determine an inverse function. For example, the real function $f(x) = x^2$ is not one-to-one on its natural domain $(-\infty, \infty)$. The function $f(x) = x^2$ is one-to-one, however, on the restricted domain $[0, \infty)$. On this restricted domain we have the inverse function $f^{-1}(x) = \sqrt{x}$.

2.4 Summary

complex polynomial function: A complex polynomial function is a function of the form

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \ldots + a_1 z + a_0$$

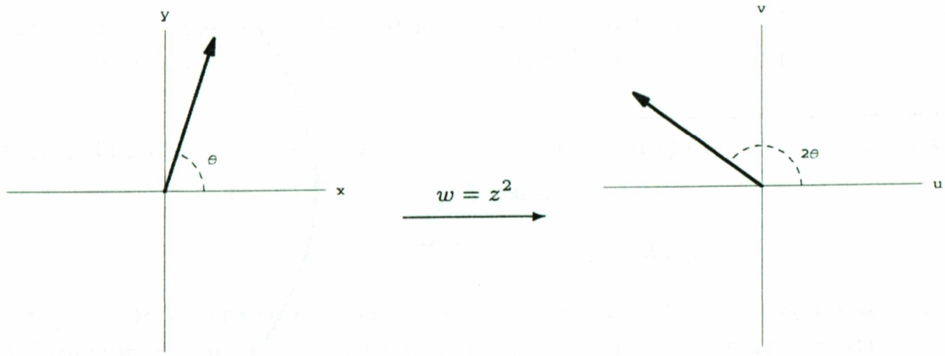
where n is a positive integer and a_n, a_{n-1}, ... a_1, and a_0 are complex constants. If $a_n \neq 0$, then n is called the degree of the polynomial $p(z)$.

complex power functions: A complex power function is a function of the form $f(z) = z^\alpha$ where α is a complex constant.

complex squaring function $f(z) = z^2$: The complex squaring function $f(z) = z^2$ can be expressed in terms of it real and imaginary parts using either a Cartesian, polar, or exponential form of the variable z. Each is given below:

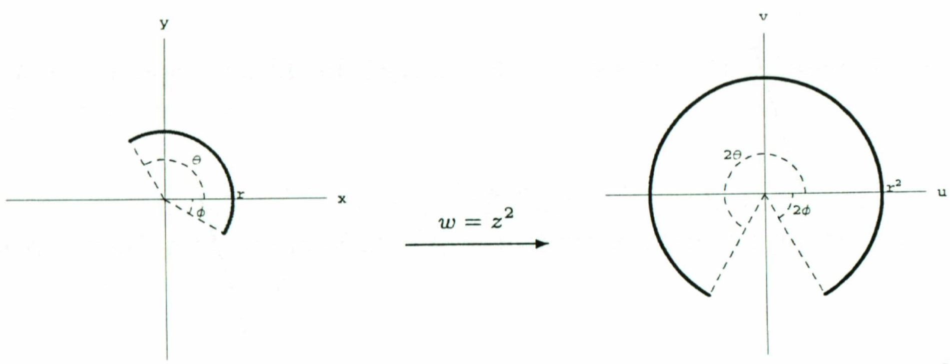
$$f(z) = z^2 = \overbrace{(x^2 - y^2) + 2xyi}^{\text{Cartesian}} = \overbrace{r^2(\cos 2\theta + i \sin 2\theta)}^{\text{polar}} = \overbrace{r^2 e^{i2\theta}}^{\text{exponential}}$$

the mapping $w = z^2$: As a mapping, $w = z^2$ squares the modulus of a point and doubles its argument. The following summarizes some properties of the mapping $w = z^2$



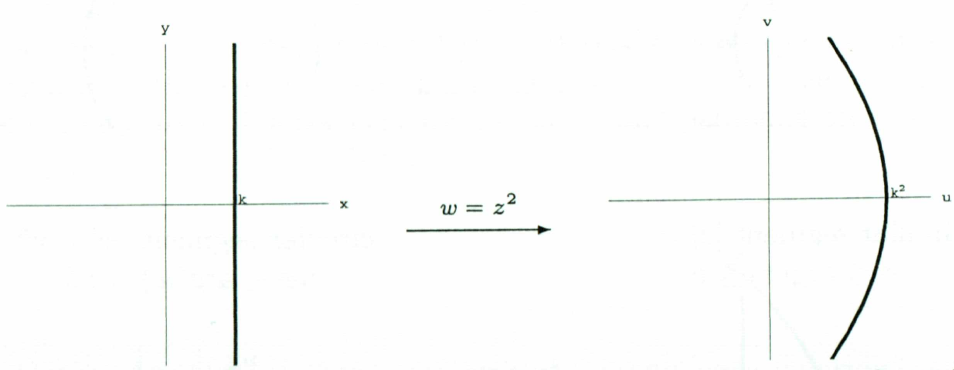
ray $\arg(z) = \theta$

ray $\arg(w) = 2\theta$



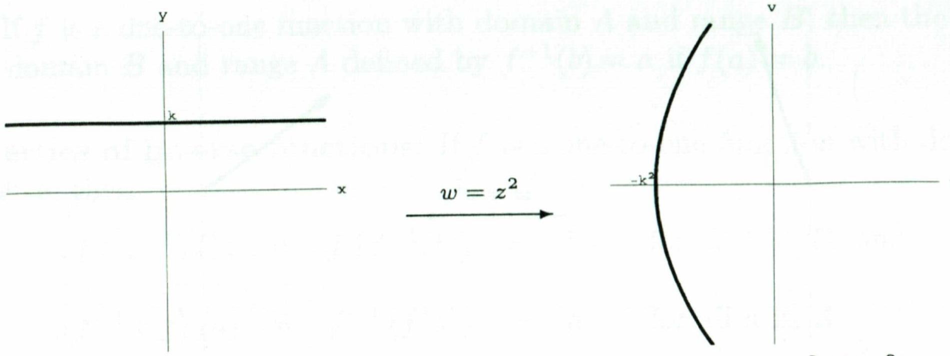
circular segment $|z| = r$,
$\phi \leq \arg(z) \leq \theta$

circular segment $|w| = r^2$,
$2\phi \leq \arg(w) \leq 2\theta$



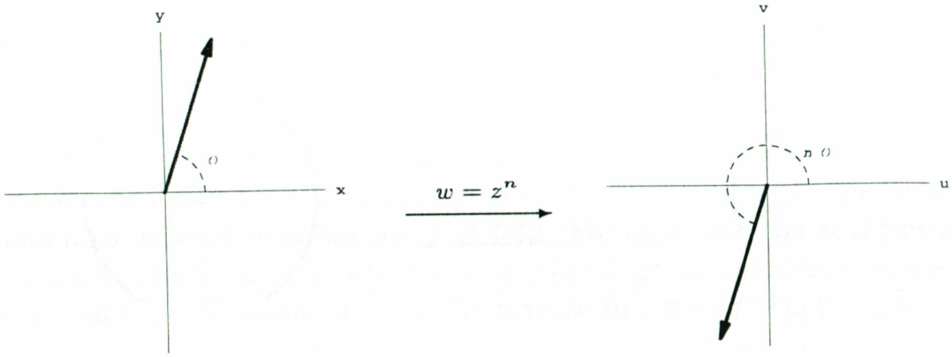
vertical line $\mathrm{Re}(z) = k$

parabola $u = k^2 - v^2/(4k^2)$

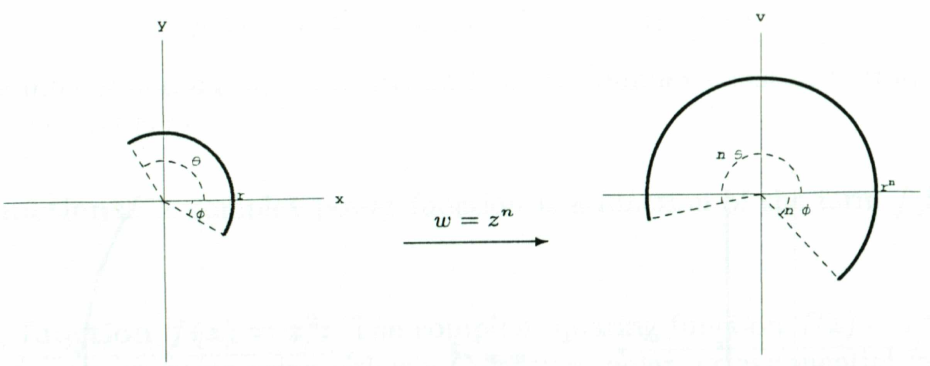


horizontal line $\mathrm{Im}(z) = k$ | parabola $u = v^2/(4k^2) - k^2$

the mapping $w = z^n$: As a mapping, $w = z^n = r^n e^{in\theta}$ raises the modulus of a point to the nth power and increases its argument by a factor of n. The following summarizes some properties of the mapping $w = z^n$



ray $\arg(z) = \theta$ | ray $\arg(w) = n\theta$



circular segment $|z| = r$, $\phi \le \arg(z) \le \theta$ | circular segment $|w| = r^n$, $n\phi \le \arg(w) \le n\theta$

principal square root function $z^{1/2}$: The principal square root function is defined by

$$
\begin{aligned}
z^{1/2} &= \sqrt{|z|}\, e^{i\mathrm{Arg}(z)/2} \\
&= \sqrt{r} e^{i\theta/2}, \qquad \theta = \mathrm{Arg}(z).
\end{aligned}
$$

If the domain of $f(z) = z^2$ is restricted to the set $-\pi/2 < \arg(z) \leq \pi/2$, then f is one-to-one and the principal square root function is the inverse of the function $f(z) = z^2$ on this restricted domain.

principal nth root function $z^{1/n}$: For integers $n \geq 2$, the principal nth root function is defined by

$$\begin{aligned} z^{1/n} &= \sqrt[n]{|z|}\, e^{i\mathrm{Arg}(z)/n} \\ &= \sqrt[n]{r} e^{i\theta/n}, \qquad \theta = \mathrm{Arg}(z). \end{aligned}$$

If the domain of $f(z) = z^n$ is restricted to the set $-\pi/n < \arg(z) \leq \pi/n$, then f is one-to-one and the principal square root function is the inverse of the function $f(z) = z^n$ on this restricted domain.

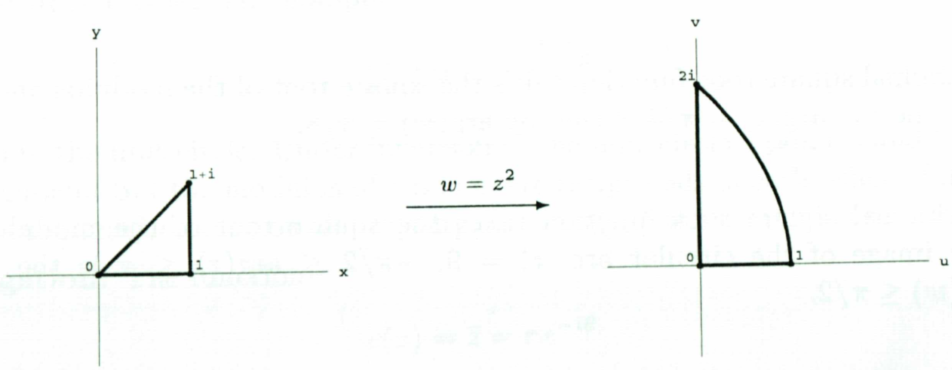
multiple-valued functions: A rule F that assigns a set of one or more complex numbers to each complex number in a subset of $\mathbf{C}$ is called a multiple-valued function. For example, $G(z) = \arg(z)$ is a multiple-valued function that assigns an infinite set of numbers (all differing by a multiple of 2π) to a nonzero complex number z. Multiple-valued functions are not functions (as defined in Section 2.1).

Exercises 2.4.1

3. By identifying $k = 3$ in equation (3) of Section 2.4.1 we see that the image of the line $x = 3$ under the mapping $w = z^2$ is the parabola $u = 9 - v^2/36$.

7. Notice that the positive imaginary axis is the ray $\arg(z) = \pi/2$. From the 2.4 Summary, the image of this ray is a ray making an angle of $2(\pi/2) = \pi$ radians with the positive real axis. Therefore, the image is the ray $\arg(w) = \pi$, the negative real aixs.

11. The triangle with vertices 0, 1, and $1+i$ consists of three line segments. We treat each of these segments separately. The first segment from 0 to 1 lies in the ray $\arg(z) = 0$. So, by the discussion on pages 73-74 of Section 2.4, its image under $w = z^2$ lies in the ray $\arg(w) = 2 \cdot 0 = 0$. Since the endpoints $z = 0$ and $z = 1$ map to $w = 0$ and $w = 1$, respectively, the image is the line segment from 0 to 1. Similarly, the line segment from 0 to $1 + i$ lies in the ray $\arg(z) = \frac{\pi}{4}$ and so its image lies in the ray $\arg(w) = 2 \cdot \frac{\pi}{4} = \frac{\pi}{2}$. Since the endpoints $z = 0$ and $z = 1 + i$ map to $w = 0$ and $w = 2i$, respectively, the image is the line segment from 0 to $2i$. The last line segment from 1 to $1 + i$ lies in the vertical line $x = 1$, not a ray emanating from the origin, thus its image will lie in a parabola. Identifying $k = 1$ in equation (3) of Section 2.4.1 we see that the image will lie in the parabola $u = 1 - v^2/4$. Since the endpoints $z = 1$ and $z = 1 + i$ map to $w = 1$ $(v = 0)$ and $w = 2i$ $(v = 2)$, respectively, the image of this last line segment is the parabolic arc $u = 1 - v^2/4$, $0 \leq v \leq 2$. Therefore, the image consists of the line segment from 0 to 1, line segment from 0 to $2i$, and parabolic arc $u = 1 - v^2/4$, $0 \leq v \leq 2$. The mapping is depicted in the figure below.



15. The function $f(z) = 2z^2 + 1 - i$ can be expressed as a composition $f(z) = (h \circ g)(z)$ of the linear function $h(z) = 2z + 1 - i$ and the squaring function $g(z) = z^2$. Under the squaring function, the ray $\arg(z) = \pi/3$ is mapped onto the ray $\arg(w) = 2\pi/3$. Next we consider the action of the linear function on this image. By (6) of Section 2.3, the linear mapping $h(z) = 2z + 1 - i$ consists of a rotation by $\operatorname{Arg}(2/|2|) = 0$ (no rotation), followed by a magnification by $|2| = 2$, and a translation by $1 - i$. The ray $\arg(w) = 2\pi/3$ is mapped onto itself by the magnification. After the translation by $1 - i$, the image is a ray emanating from $1 - i$ making an angle of $2\pi/3$ with the line $y = 1$. Since 0 is not in the set $\arg(z) = \pi/3$, the point $f(0) = 1 - i$ is not included in the image. Therefore, the image can be described as the ray emanating from $1-i$ and containing $(\sqrt{3}-1)i$ excluding the point $1-i$.

19. The function $f(z) = \frac{1}{4}e^{i\pi/4}z^2$ can be expressed as a composition $f(z) = (h \circ g)(z)$ of the linear function $h(z) = \frac{1}{4}e^{i\pi/4}z$ and the squaring function $g(z) = z^2$. Because the squaring function squares moduli and doubles arguments, the circular arc $|z| = 2$, $0 \le \arg(z) \le \pi/2$ is mapped onto the circular arc $|w| = 4$, $0 \le \arg(w) \le \pi$. Next we consider the action of the linear function on this image. By (6) of Section 2.3, the linear mapping $h(z) = \frac{1}{4}e^{i\pi/4}z$ consists of a rotation by $\pi/4$, followed by a magnification by $1/4$, and a translation by 0 (no translation). Under the rotation, the circular arc $|w| = 4$, $0 \le \arg(w) \le \pi$ maps onto the circular arc $|w| = 4$, $\pi/4 \le \arg(w) \le 5\pi/4$. Finally, under the magnification, this image is mapped onto the circular arc $|w| = 1$, $\pi/4 \le \arg(w) \le 5\pi/4$. Therefore, the image is the circular arc $|w| = 1$, $\pi/4 \le \arg(w) \le 5\pi/4$.

23. **(a)** Since the function $f(z) = z^2$ squares the modulus and doubles the argument, the image of the region $1 \le |z| \le 2$, $\pi/4 \le \arg(z) \le 3\pi/4$ is the region $1 \le |w| \le 4$, $\pi/2 \le \arg(w) \le 3\pi/2$.

(b) Since the function $f(z) = z^3$ cubes the modulus and triples the argument, the image of the region $1 \le |z| \le 2$, $\pi/4 \le \arg(z) \le 3\pi/4$ is the region $1 \le |w| \le 8$, $3\pi/4 \le \arg(w) \le 9\pi/4$.

(c) Since the function $f(z) = z^4$ raises the modulus to the 4th power and quadruples the argument, the image of the region $1 \le |z| \le 2$, $\pi/4 \le \arg(z) \le 3\pi/4$ is the region $1 \le |w| \le 16$, $\pi \le \arg(w) \le 3\pi$. That is, the image is the annulus $1 \le |w| \le 16$.

Exercises 2.4.2

27. From (14) of Section 2.4.2, since $|-1| = 1$ and $\operatorname{Arg}(-1) = \pi$, we have that

$$(-1)^{1/3} = \sqrt[3]{1}e^{i\pi/3} = \frac{1}{2} + \frac{\sqrt{3}}{2}i.$$

31. Since the principal square root function takes the square root of the modulus and halves the argument, the image of the ray $\arg(z) = \pi/4$ is the ray $\arg(w) = \pi/8$.

35. Since the principal square root function takes the square root of the modulus and halves the argument, the image of the circular arc $|z| = 9$, $-\pi/2 \le \arg(z) \le \pi$ is the circular arc $|w| = 3$, $-\pi/4 \le \arg(w) \le \pi/2$.

39. We consider the image of the boundary of the region first. The boundary of the region lies in the union of the ray $\arg(z) = \pi/2$ and the parabola $x = 4 - \frac{1}{16}y^2$. Since the principal square root function takes the square root of the modulus and halves the argument, the image of the ray $\arg(z) = \pi/2$ is the ray $\arg(w) = \pi/4$. In order to determine the image of the parabola we use the fact that $f(z) = z^{1/2}$ is the inverse function of $f(z) = z^2$ defined on the restricted domain $-\pi/2 < \arg(z) \leq \pi/2$. By (3) of Section 2.4.1 with the identification $k^2 = 4$, the image of the lines $x = \pm 2$ map onto the parabola $u = 4 - \frac{1}{16}v^2$ under $w = z^2$. Only the line $x = 2$ is in the restricted domain of $f(z) = z^2$, and so we conclude that the image of the parabola $x = 4 - \frac{1}{16}y^2$ under $w = z^{1/2}$ is the vertical line $u = 2$. Thus, the image is bounded by the ray $\arg(w) = \pi/4$ (or, equivalently, the line $u = v$) and the line $u = 2$. Since the point $-7 + 24i$ is in the region and $(-7 + 24i)^{1/2} = 3 + 4i$, the image of the region must be the set bounded by the lines $u = v$ and $u = 2$ containing the point $3 + 4i$.

Focus on Concepts

43. Use the fact that $\arg(w) = \pi/2$ and $\arg(w) = -3\pi/2$ are the same ray.

47. Consider different subarcs of the circle $|z| = \sqrt{2}$.

51. Remember that the range of the principal nth root function is $-\dfrac{\pi}{n} < \arg(w) \leq \dfrac{\pi}{n}$.

55. The mapping $w = 2iz^2 - i$ is a composition of the mappings $w = z^2$ and $w = 2iz - i$. First, find the image of the quarter disk S under the mapping $w = z^2$. Second, determine the image of this set under the linear mapping $w = 2iz - i$. The point in the image of this composition that is farthest from the origin realizes the maximum modulus M and the point closest to the origin realizes the minimum modulus L.

2.5 Reciprocal Function

2.5 Summary

reciprocal function: The function $f(z) = 1/z$ is called the reciprocal function.

inversion in the unit circle: The mapping

$$g(z) = \frac{1}{z} = \frac{1}{r}e^{i\theta}$$

is called inversion in the unit circle. Under inversion in the unit circle a point z and its image $w = g(z)$ have the same argument but the modulus of z and w are reciprocals of each other. Therefore, if $z_0 \neq 0$, then $w_0 = 1/z_0$ is the unique point on the ray $\arg(z) = \arg(z_0)$ with modulus $|w_0| = 1/|z_0|$.

complex conjugation: The function

$$c(z) = \bar{z} = re^{-i\theta}$$

is called the complex conjugation function. As a mapping, complex conjugation is a reflection across the real axis. Therefore, given z_0, $w_0 = \overline{z_0}$ is the unique point with the same modulus as z_0 and $\arg(w_0) = -\arg(z_0)$.

reciprocal mapping: The complex reciprocal function $f(z) = 1/z$ can be written as the composition

$$f(z) = \frac{1}{z} = \frac{1}{re^{i\theta}} = \frac{1}{r}e^{-i\theta} = \overline{\left(\frac{1}{r}e^{i\theta}\right)} = c(g(z))$$

where $g(z) = \frac{1}{r}e^{i\theta}$ is inversion in the unit circle and $c(z) = \overline{z}$ is complex conjugation. Therefore, if $z_0 \neq 0$, then $w_0 = 1/z_0$ is the point obtained by

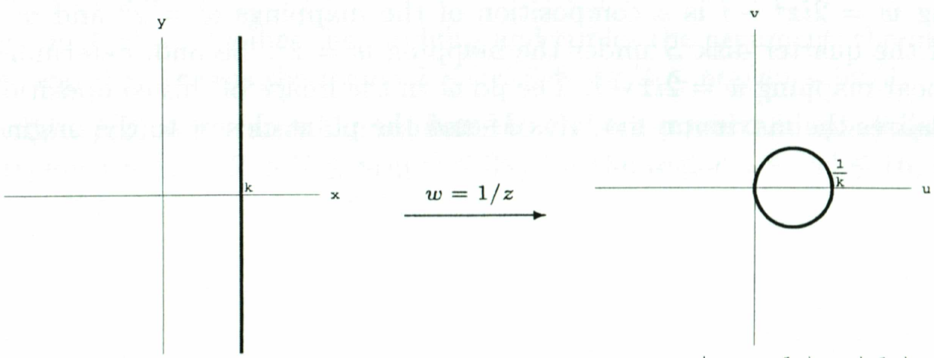
(*i*) inverting z_0 in the unit circle, then

(*ii*) reflecting the result across the real axis.

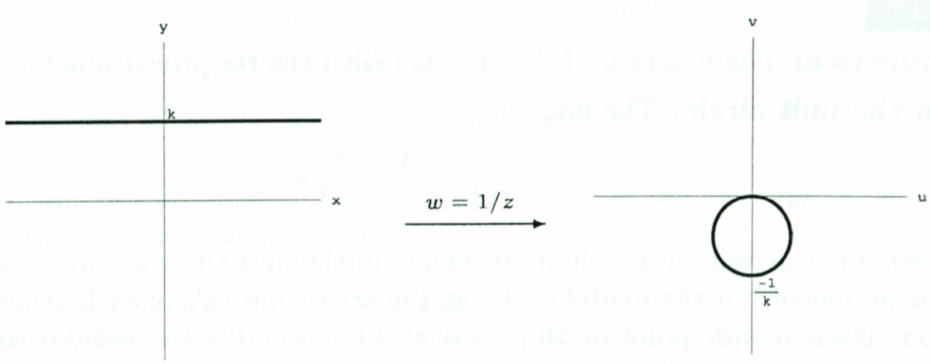
$1/z$ on the extended complex plane: The reciprocal function on the extended complex plane is defined by

$$f(z) = \begin{cases} 1/z, & \text{if } z \neq 0 \text{ or } \infty \\ \infty, & \text{if } z = 0 \\ 0, & \text{if } z = \infty. \end{cases}$$

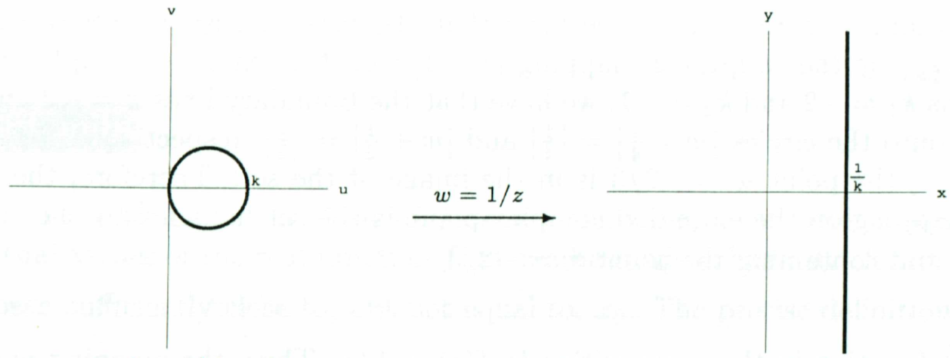
mapping properties of $w = 1/z$: The following summarizes some properties of the mapping $w = 1/z$.



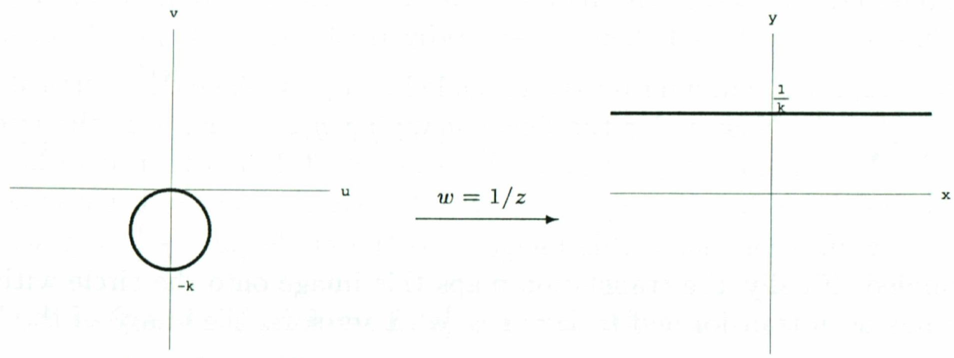
vertical line $x = k$ circle $\left|w - \frac{1}{2k}\right| = \left|\frac{1}{2k}\right|$



horizontal line $y = k$ circle $\left|w + \frac{1}{2k}i\right| = \left|\frac{1}{2k}\right|$



circle $\left|z - \frac{k}{2}\right| = \left|\frac{k}{2}\right|$ vertical line $u = 1/k$



circle $\left|z + \frac{k}{2}i\right| = \left|\frac{k}{2}\right|$ horizontal line $v = 1/k$

Exercises 2.5

3. In order to find the image of the set $|z| = 3$, $-\pi/4 \leq \arg(z) \leq 3\pi/4$ under the mapping $w = 1/z$, we first invert this set in the unit circle, then reflect the result across the real axis. Under inversion in the unit circle, points with modulus 3 have images with modulus 1/3, and the arguments are unchanged. Hence, the image under inversion in the unit circle is the set $|w| = 1/3$, $-\pi/4 \leq \arg(w) \leq 3\pi/4$. Reflecting this set across the real axis negates arguments and does not change the modulus. Therefore, the image is the semicircle $|w| = 1/3$, $-3\pi/4 \leq \arg(w) \leq \pi/4$.

7. In order to find the image of the set $\arg(z) = \pi/4$ under the mapping $w = 1/z$, we first invert this set in the unit circle, then reflect the result across the real axis. If we set $z = re^{i\theta}$, then this set is given by $0 < r$, $\theta = \pi/4$. Under inversion in the unit circle, points with modulus $r > 0$ have images with modulus $1/r > 0$, , and the arguments are unchanged. Hence, the image under inversion in the unit circle is the set $\arg(w) = \pi/4$. Reflecting this set across the real axis negates arguments and does not change the modulus. Therefore, the image is the ray $\arg(w) = -\pi/4$.

11. From the summary of Section 2.5 we have that the circle $\left|z + \frac{k}{2}i\right| = \left|\frac{k}{2}\right|$ is mapped onto the horizontal line $v = 1/k$ by the reciprocal mapping $w = 1/z$ on the extended complex plane. By identifying $k = 2$, we have that the image of the circle $|z + i| = 1$ is the horizontal line $v = 1/2$.

15. From the summary of Section 2.5 we have that the vertical line $x = k$ is mapped onto the circle $\left|w - \frac{1}{2k}\right| = \left|\frac{1}{2k}\right|$ by the reciprocal mapping $w = 1/z$ on the extended complex plane. By making the identifications $k_1 = -2$ and $k_2 = -1$, we have that the boundary lines $x = -2$ and $x = -1$ of the set S are mapped onto the circles $\left|w + \frac{1}{4}\right| = \left|\frac{1}{4}\right|$ and $\left|w + \frac{1}{2}\right| = \left|\frac{1}{2}\right|$, respectively. Since the point $z = -3/2$ is in the set S, the point $w = -2/3$ is in the image of the set. Therefore, the image of S under the reciprocal mapping on the extended complex plane is the set bounded by the circles $\left|w + \frac{1}{4}\right| = \frac{1}{4}$ and $\left|w + \frac{1}{2}\right| = \frac{1}{2}$ and containing the point $w = -2/3$.

19. **(a)** The first function in the composition is $f(z) = 1/z$. Thus, the mapping $w = g(f(z))$ first inverts in the unit circle, then reflects across the real axis. From the discussion on page 65 of Section 2.3, the second function in the composition, $g(z) = 2iz + 1$, rotates through an angle of $\text{Arg}(2i) = \pi/2$ about the origin, magnifies by $|2i| = 2$, and then translates by 1. Therefore, the mapping $w = f(z)$ inverts in the unit circle, reflects across the real axis, rotates through an angle of $\text{Arg}(2i) = \pi/2$ about the origin, magnifies by $|2i| = 2$, and then translates by 1.

(b) Under the reciprocal mapping on the extended complex plane, the vertical line $x = 4$ maps onto the circle $\left|w - \frac{1}{8}\right| = \frac{1}{8}$. Now under the linear mapping $g(z) = 2iz + 1$, the circle is rotated through an angle of $\pi/2$ about the origin, magnified by a factor of 2, and then translated by 1. The rotation maps the circle onto the circle $\left|w - \frac{1}{8}i\right| = \frac{1}{8}$ with the same radius but whose center has been rotated by $\pi/2$. The magnification maps this image onto the circle $\left|w - \frac{1}{4}i\right| = \frac{1}{4}$ whose center and radius have been doubled. Finally, the translation maps this image onto the circle with the same radius, but whose center has been transformed to $w = 1 + \frac{1}{4}i$. Therefore, the image of the line $x = 4$ is the circle $\left|w - 1 - \frac{1}{4}i\right| = \frac{1}{4}$.

(c) Under the reciprocal mapping on the extended complex plane, the circle $|z + 2| = 2$ maps onto the vertical line $u = -\frac{1}{4}$. Now under the linear mapping $g(z) = 2iz + 1$, the line is rotated through an angle of $\pi/2$ about the origin, magnified by a factor of 2, and then translated by 1. The rotation maps the line onto the line $v = -\frac{1}{4}$. The magnification maps this image onto the line $v = -\frac{1}{2}$. Finally, since the translation is along a vector ($b = 1 + 0i$) in the same direction of this line, the line maps onto itself. Therefore, the image of the circle $|z + 2| = 2$ is the line $v = -\frac{1}{2}$.

Focus on Concepts

23. Modify the procedure used in Example 2 of Section 2.5.

27. **(a)** Notice that L is a generalized circle with $A = 0$ (see (7) in Problem 25 of Section 2.5). By Problem 26, the image of L is the generalized circle given by (8). What must be true about the coefficients in (8) in order for the image to be a line?

(b) Use part (a) and consider the coefficients of the image line.

(c) As in part (a), use (8) to determine an equation of the image circle. Complete the square in order to determine the center and radius.

2.6 Limits and Continuity

Review Topic: Real Limits

limit of a real function: Informally, the limit of f as x tends to x_0 exists and is equal to L, denoted $\lim_{x\to x_0} f(x) = L$, means that values of the real function $f(x)$ can be made arbitrarily close to the real number L if values of x are chosen sufficiently close to, but not equal to, x_0. The precise definition is:

$\lim_{x\to x_0} f(x) = L$ *if for every* $\varepsilon > 0$ *there exists a* $\delta > 0$ *such that* $|f(x) - L| < \varepsilon$ *whenever* $0 < |x - x_0| < \delta$.

left and right-hand limits: The limit of f as x tends to x_0 from the left, denoted by $\lim_{x\to x_0^-} f(x)$, and the limit from the right, denoted $\lim_{x\to x_0^+} f(x)$, are defined by:

$\lim_{x\to x_0^-} f(x) = L$ *if for every* $\varepsilon > 0$ *there exists a* $\delta > 0$ *such that* $|f(x) - L| < \varepsilon$ *whenever* $x_0 - \delta < x < x_0$*;*
for $\lim_{x\to x_0^+} f(x) = L$, *we require* $x_0 < x < x_0 + \delta$.

A real limit exists when the left and right-hand limits are equal. That is,

$$\lim_{x\to x_0} f(x) = L \quad \text{if and only if} \quad \lim_{x\to x_0^-} f(x) = \lim_{x\to x_0^+} f(x) = L.$$

properties of real limits: The following properties allow many real limits to be evaluated in a somewhat mechanical fashion. Suppose that f and g are real functions. If $\lim_{x\to x_0} f(x) = L$ and $\lim_{x\to x_0} g(x) = M$, then

(*i*) $\lim_{x\to x_0} c = c$, where c is a real constant,

(*ii*) $\lim_{x\to x_0} x = x_0$,

(*iii*) $\lim_{x\to x_0} cf(x) = cL$, where c is a real constant,

(*iv*) $\lim_{x\to x_0} (f(x) \pm g(x)) = L \pm M$,

(*v*) $\lim_{x\to x_0} f(x) \cdot g(x) = L \cdot M$,

(*vi*) $\lim_{x\to x_0} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}$, provided $M \neq 0$.

continuity of a real function: A real function f is continuous at a point x_0 if $\lim_{x\to x_0} f(x) = f(x_0)$. If follows from the properties of real limits that real polynomial and rational functions are continuous at all points in their domains.

limit of a real multivariable function: The definition of limit for a real multivariable function $F(x, y)$ is similar to that for a real function f.

$\lim\limits_{(x,y)\to(x_0,y_0)} F(x, y) = L$ *if for every* $\varepsilon > 0$ *there exists a* $\delta > 0$ *such that* $|f(x, y) - L| < \varepsilon$ *whenever* $0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta$.

continuity of a real multivariable function: A real multivariable function F is continuous at a point (x_0, y_0) if $\lim\limits_{(x,y)\to(x_0,y_0)} F(x, y) = F(x_0, y_0)$. If follows from properties of real multivariable limits that two-variable polynomial and rational functions are continuous at all points in their domains.

2.6 Summary

limit of a complex function: Informally, the limit of f as z tends to z_0 exists and is equal to L, denoted $\lim\limits_{z\to z_0} f(z) = L$, means that values of the complex function $f(z)$ can be made arbitrarily close to the complex number L if values of z are chosen sufficiently close to, but not equal to, z_0. The precise definition is similar to that of a real function:

$\lim\limits_{z\to z_0} f(z) = L$ *if for every* $\varepsilon > 0$ *there exists a* $\delta > 0$ *such that* $|f(z) - L| < \varepsilon$ *whenever* $0 < |z - z_0| < \delta$.

In this definition, both ε and δ are real numbers, while z, z_0, $f(z)$, and L are complex numbers.

criterion for the nonexistence of a limit: In the limit of a complex function, z is allowed to approach z_0 from any direction in the complex plane. This provides a means of showing that certain complex limits do not exist: if $f(z)$ approaches two complex numbers $L_1 \neq L_2$ as z approaches z_0 along two different curves, then $\lim\limits_{z\to z_0} f(z)$ does not exist.

real and imaginary parts of a limit: One method that is used to evaluate complex limits is to consider the real multivariable limits of the real and imaginary parts of f. If $f(z) = u(x, y) + iv(x, y)$, $z_0 = x_0 + iy_0$, and $L = u_0 + iv_0$, then

$$\lim_{z\to z_0} f(z) = L \quad \text{if and only if} \quad \lim_{(x,y)\to(x_0,y_0)} u(x, y) = u_0 \text{ and } \lim_{(x,y)\to(x_0,y_0)} v(x, y) = v_0.$$

properties of complex limits: The following properties allow many complex limits to be evaluated in a somewhat mechanical fashion. Suppose that f and g are complex functions. If $\lim\limits_{z\to z_0} f(z) = L$ and $\lim\limits_{z\to z_0} g(z) = M$, then

(*i*) $\lim\limits_{z\to z_0} c = c$, where c is a complex constant,

(*ii*) $\lim\limits_{z\to z_0} z = z_0$,

(*iii*) $\lim\limits_{z\to z_0} cf(z) = cL$, where c is a complex constant,

(iv) $\lim_{z \to z_0} (f(z) \pm g(z)) = L \pm M$,

(v) $\lim_{z \to z_0} f(z) \cdot g(z) = L \cdot M$,

(vi) $\lim_{z \to z_0} \dfrac{f(z)}{g(z)} = \dfrac{L}{M}$, provided $M \neq 0$.

continuity of a complex function: A complex function f is continuous at a point z_0 if $\lim_{z \to z_0} f(z) = f(z_0)$.

criteria for continuity at a point: A complex function f is continuous at a point z_0 if each of the following three conditions hold:

(i) $\lim_{z \to z_0} f(z)$ exists,

(ii) f is defined at z_0, and

(iii) $\lim_{z \to z_0} f(z) = f(z_0)$.

real and imaginary parts of a continuous function: If $f(z) = u(x, y) + iv(x, y)$ and $z_0 = x_0 + iy_0$, then the complex function f is continuous at the point z_0 if and only if both real multivariable functions u and v are continuous at the point (x_0, y_0).

properties of continuous functions: If f and g are continuous at the point z_0, then cf (where c is a complex constant), $f \pm g$, and $f \cdot g$ are continuous functions at the point z_0. If $g(z_0) \neq 0$, then f/g is also continuous at the point z_0. From this it follows that all complex polynomial functions and all complex rational functions are continuous on their domains.

a bounding property for continuous functions: If f is a continuous function defined on a closed and bounded region R, then f is bounded on R. That is, there is a real constant $M > 0$ such that $|f(z)| < M$ for all z in R.

branches of multiple-valued functions: Recall from Section 2.4 that a multiple-valued function F is a rule that assigns a set of one or more complex numbers to each complex number in a subset of **C**. A branch of a multiple-valued function F is a function f_1 that is continuous on some domain and that assigns exactly one of the multiple values of F to each point z in that domain.

branch points and branch cuts: A branch cut for a branch f_1 is a curve that is excluded from the domain of the multiple-valued function F so that f_1 is continuous on the remaining points. A branch point is a point that is on the branch cut of every branch.

limit at infinity: The limit of f as z tends to ∞ exists and is equal to L, denoted $\lim_{z \to \infty} f(z) = L$, means that values of the complex function $f(z)$ can be made arbitrarily close to the complex number L if values of z are chosen so that $|z|$ is sufficiently large. A useful result for evaluating limits at infinity is:

$$\lim_{z \to \infty} f(z) = L \quad \text{if and only if} \quad \lim_{z \to 0} f\left(\frac{1}{z}\right) = L.$$

infinite limit: The limit of f as z tends to z_0 is infinity, denoted $\lim_{z\to z_0} f(z) = \infty$, means that $|f(z)|$ can be made arbitrarily large if values of z are chosen sufficiently close to, but not equal to, z_0. A useful result for evaluating infinite limits is:

$$\lim_{z\to z_0} f(z) = \infty \quad \text{if and only if} \quad \lim_{z\to z_0} \frac{1}{f(z)} = 0.$$

Exercises 2.6.1

3. In order to use Theorem 2.6.1 we need to find the real and imaginary parts of $f(z) = |z|^2 - i\bar{z}$. Setting $z = x + iy$ we obtain

$$\begin{aligned} |z|^2 - i\bar{z} &= |x+iy|^2 - i(\overline{x+iy}) \\ &= x^2 + y^2 - ix - y \\ &= (x^2 + y^2 - y) + ix. \end{aligned}$$

Identifying $x_0 = 1$, $y_0 = -1$, $u(x,y) = x^2 + y^2 - y$, and $v(x,y) = x$, we have

$$\lim_{(x,y)\to(1,-1)} (x^2 + y^2 - y) = 3 \quad \text{and} \quad \lim_{(x,y)\to(1,-1)} x = 1.$$

Therefore, by Theorem 2.6.1, $\lim_{z\to 1-i} \left(|z|^2 - i\bar{z}\right) = 3 - i$.

7. In order to use Theorem 2.6.1 we need to find the real and imaginary parts of $f(z) = \dfrac{e^z - e^{\bar{z}}}{\operatorname{Im}(z)}$. Setting $z = x + iy$ we obtain

$$\begin{aligned} \frac{e^z - e^{\bar{z}}}{\operatorname{Im}(z)} &= \frac{e^{x+iy} - e^{x-iy}}{y} \\ &= \frac{e^x \cos y + ie^x \sin y - e^x \cos y + ie^x \sin y}{y} \\ &= 2e^x \left(\frac{\sin y}{y}\right) i. \end{aligned}$$

Identifying $x_0 = 0$, $y_0 = 0$, $u(x,y) = 0$, and $v(x,y) = 2e^x \left(\dfrac{\sin y}{y}\right)$, we have

$$\lim_{(x,y)\to(0,0)} 0 = 0 \quad \text{and} \quad \lim_{(x,y)\to(0,0)} 2e^x \left(\frac{\sin y}{y}\right) = 2\left(\lim_{x\to 0} e^x\right)\left(\lim_{y\to 0} \frac{\sin y}{y}\right) = 2.$$

In the limit of $v(x,y)$ we made use of the fundamental trigonometric limit $\lim_{y\to 0} \dfrac{\sin y}{y} = 1$. Therefore, by Theorem 2.6.1, $\lim_{z\to 1-i} \left(|z|^2 - i\bar{z}\right) = 2i$.

11. By (15), we have

$$\lim_{z\to e^{i\pi/4}} z = e^{i\pi/4}.$$

Using this limit, Theorem 2.6.2(ii), and Theorem 2.6.2(iv), we obtain

$$\begin{aligned}\lim_{z\to e^{i\pi/4}}\left(z+\frac{1}{z}\right) &= e^{i\pi/4}+\frac{1}{e^{i\pi/4}}\\ &= e^{i\pi/4}+e^{-i\pi/4}\\ &= \cos(\pi/4)+i\sin(\pi/4)+\cos(-\pi/4)+i\sin(-\pi/4)\\ &= \sqrt{2}.\end{aligned}$$

15. By (15), (16), and Theorem 2.6.2(ii), we have

$$\lim_{z\to z_0}(z-z_0)=0.$$

Thus, we cannot apply Theorem 2.6.2(iv) without simplifying the rational function in the limit. Notice that:

$$\begin{aligned}\frac{(az+b)-(az_0+b)}{z-z_0} &= \frac{az+\cancel{b}-az_0-\cancel{b}}{z-z_0}\\ &= \frac{a(z-z_0)}{z-z_0}.\end{aligned}$$

Becuase z is not allowed to take on the value z_0 in the limit we can cancel the common factor in the limit:

$$\begin{aligned}\lim_{z\to z_0}\frac{(az+b)-(az_0+b)}{z-z_0} &= \lim_{z\to z_0}\frac{a(z-z_0)}{z-z_0}\\ &= \lim_{z\to z_0}a\\ &= a.\end{aligned}$$

19. **(a)** If z approaches 0 along the real axis, then $z=x+0i$ where the real number x is approaching 0. For this approach we have

$$\lim_{z\to 0}\left(\frac{z}{\bar{z}}\right)^2=\lim_{x\to 0}\left(\frac{x+0i}{x-0i}\right)^2=\lim_{x\to 0}\left(\frac{x}{x}\right)^2=\lim_{x\to 0}(1)^2=1.$$

(b) If z approaches 0 along the imaginary axis, then $z=0+yi$ where the real number y is approaching 0. For this approach we have

$$\lim_{z\to 0}\left(\frac{z}{\bar{z}}\right)^2=\lim_{y\to 0}\left(\frac{0+yi}{0-yi}\right)^2=\lim_{y\to 0}\left(\frac{yi}{-yix}\right)^2=\lim_{y\to 0}(-1)^2=1.$$

(c) No, the two limits in parts (a) and (b) do not imply that the limit is 1. In order for the limit to exist and be equal to one, the function must approach 1 along every possible curve through 0. This has only been verified for two such curves.

(d) If z approaches 0 along the line $y = x$, then $z = x + xi$ where the real number x is approaching 0. For this approach we have

$$\lim_{z\to 0}\left(\frac{z}{\bar{z}}\right)^2 = \lim_{x\to 0}\left(\frac{x+x}{x-xi}\right)^2 = \lim_{x\to 0}\left(\frac{x(1+i)}{x(1-i)}\right)^2 = \lim_{x\to 0}\left(\frac{1+i}{1-i}\right)^2 = \frac{2i}{-2i} = -1.$$

(e) By the criterion for the nonexistence of a limit, since the limits in parts (a) and (d) are not the same, the limit $\lim_{z\to 0}\left(\frac{z}{\bar{z}}\right)^2$ does not exist.

23. Since the numerator of this rational function is approaching a finite number while the denominator is approaching 0, we expect the limit to be infinite. We use (26) of Section 2.6 to establish this. By (15), (16), and Theorem 2.6.2, we have

$$\lim_{z\to i}\frac{z^2+1}{z^2-1} = \frac{i^2+1}{i^2-1} = \frac{0}{-2} = 0.$$

Therefore, by (26) we obtain

$$\lim_{z\to i}\frac{z^2-1}{z^2+1} = \infty.$$

Exercises 2.6.2

27. In order to show that f is continuous, we use Definition 2.6.2. By (15), (16), and Theorem 2.6.2, we have

$$\lim_{z\to 2-i}\left(z^2 - iz + 3 - 2i\right) = (2-i)^2 - i(2-i) + 3 - 2i = 4 - 4i - 1 - 2i - 1 + 3 - 2i = 5 - 8i.$$

In addition,

$$f(2-i) = (2-i)^2 - i(2-i) + 3 - 2i = 5 - 8i.$$

Therefore, f is continuous at $z_0 = 2 - i$, since $\lim_{z\to 2-i}\left(z^2 - iz + 3 - 2i\right) = f(2-i)$.

31. In order to show that f is continuous, we use Definition 2.6.2. By (15), (16), and Theorem 2.6.2, we have

$$\lim_{z\to 1}\frac{z^3-1}{z-1} = \lim_{z\to 1}\frac{(z-1)(z^2+z+1)}{z-1} = \lim_{z\to 1}\left(z^2+z+1\right) = 1^2+1+1 = 3.$$

In addition, $f(1) = 3$. Therefore, f is continuous at $z_0 = 1$, since $\lim_{z\to 1} f(z) = f(1)$.

35. In order to show that f is not continuous, we use the criteria for continuity at a point. Notice that $f(-i)$ is undefined because the denominator is 0. Therefore, f is not continuous at $z_0 = -i$ because criterion for continuity (ii) does not hold.

39. In order to show that f is not continuous, we use the criteria for continuity at a point. Consider the limit $\lim_{z \to i} f(z)$. If z approaches i along the imaginary axis, then $|z| \neq 1$. For this approach we have

$$\lim_{z \to i} f(z) = \lim_{z \to i} \frac{z^3 - 1}{z - 1} = \frac{i^3 - 1}{i - 1} = i.$$

On the other hand, if z approaches i along the unit circle, then $|z| = 1$. For this approach we have

$$\lim_{z \to i} f(z) = \lim_{z \to i} 3 = 3.$$

Since $i \neq 3$, the criterion for the nonexistence of a limit tells us that $\lim_{z \to i} f(z)$ does not exist. Therefore, f is not continuous at $z_0 = i$ because criterion for continuity (i) does not hold.

43. If we let $z = x + iy$, then

$$f(z) = \frac{z-1}{z\bar{z} - 4} = \frac{x + iy - 1}{(x + iy)(x - iy) - 4} = \frac{x - 1 + iy}{x^2 + y^2 - 4} = \frac{x-1}{x^2 + y^2 - 4} + i\frac{y}{x^2 + y^2 - 4}.$$

Thus, the real and imaginary parts of f are

$$u(x, y) = \frac{x-1}{x^2 + y^2 - 4} \quad \text{and} \quad v(x, y) = \frac{y}{x^2 + y^2 - 4},$$

respectively. By equation (14) of Section 2.6.1, two-variable rational functions are continuous on their domains, and so, both u and v are continuous for all (x, y) such that $x^2 + y^2 \neq 4$. Therefore, it follows from Theorem 2.6.3 that f is continuous for all z such that $|z| \neq 2$.

Focus on Concepts

47. **(a)** Use the fact that $\text{Re}(z) = \dfrac{z + \bar{z}}{2}$.

(b) Use the fact that $\text{Im}(z) = \dfrac{z - \bar{z}}{2i}$.

(c) Use the fact that $|z| = \sqrt{z\bar{z}}$.

51. **(a)** Identify $z_0 = 1 + i$, $f(z) = (1 - i)z + 2i$, and $L = 2 + 2i$ in Definition 2.6.1.

(b) Notice that after simplification

$$|f(z) - L| = |(1 - i)z - 2| = |1 - i| \cdot |z - (1 + i)| = \sqrt{2} \cdot |z - (1 + i)|.$$

55. **(a)** To show $f(z) = \text{Arg}(z)$ is discontinuous at $z = -r$ on the negative real axis, use the criterion for the nonexistence of a limit. First consider letting z approach $-r$ along the quarter of the circle $|z| = r$ lying in the second quadrant, then consider letting z approach $-r$ along the quarter of the circle $|z| = r$ lying in the third quadrant.

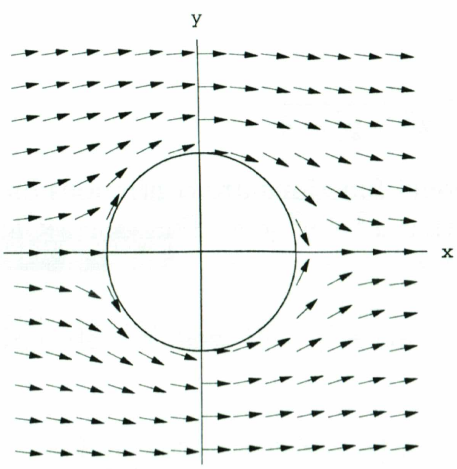
2.7 Applications

Review Topic: Vector Fields

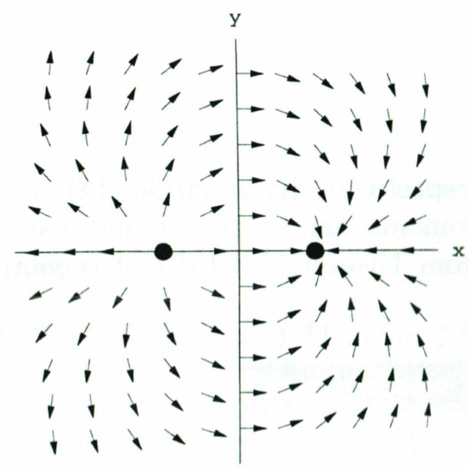
vector fields: A vector-valued function $\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ is called a two-dimensional vector field.

graphical representation: A graphical representation of a vector field $\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ is obtained by plotting the vector $\mathbf{F}(x, y)$ based at the initial point (x, y) in the plane.

applications: Vector fields are used widely in applications to science and engineering. For example, the motion of a fluid can be described by a velocity field in which each vector $\mathbf{F}(x, y)$ represents the velocity of a particle at the point (x, y). Another common application is a force field in which each vector $\mathbf{F}(x, y)$ represents the force on a particle at the point (x, y).



velocity field



force field

2.7 Summary

complex representation of a vector field: The complex function $f(z) = P(x, y) + iQ(x, y)$ is called the complex representation of the vector field $\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$. In general, both $\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ and its complex representation $f(z) = P(x, y) + iQ(x, y)$ are called vector fields.

graphical representation: A graphical representation of a vector field $f(z) = P(x, y) + iQ(x, y)$ is obtained by plotting the complex number $f(z)$ as a vector based at the initial point z in the complex plane.

fluid flow: A planar flow is the flow of a fluid in which the motion and physical traits of the fluid are the same in all planes parallel to the $xy-$plane. If $f(z)$ represents the velocity of a particle of the fluid located at the point z in the complex plane, then $f(z)$ is called a velocity field.

streamlines: If $z(t) = x(t) + iy(t)$ is a parametrization of a path that a particle follows in a planar fluid flow, then the derivative $z'(t) = x'(t) + iy'(t)$ represents the velocity of the particle at point $z(t)$. Therefore, if $f(z) = P(x,y) + iQ(x,y)$ is the velocity field of the fluid, then the following equations must hold:

$$\begin{aligned} \frac{dx}{dt} &= P(x,y) \\ \frac{dy}{dt} &= Q(x,y). \end{aligned}$$

The family of solutions to this system of differential equations is called the streamlines of the planar fluid flow with velocity field $f(z)$.

finding streamlines: One method for solving the system of differential equations

$$\begin{aligned} \frac{dx}{dt} &= P(x,y) \\ \frac{dy}{dt} &= Q(x,y). \end{aligned}$$

is to convert it to an ordinary differential equation using the chain rule

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} \quad \text{or} \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt}.$$

This gives the ordinary differential equation

$$\frac{dy}{dx} = \frac{Q(x,y)}{P(x,y)}.$$

Separation of variables and exact equations are two methods that can sometimes be used to solve this equation.

separation of variables: An ordinary differential equation of the form

$$\frac{dy}{dx} = g(x)h(y)$$

is called separable and can be solved by integration as follows:

$$\begin{aligned} \frac{dy}{dx} &= g(x)h(y) \\ \int \frac{dy}{h(y)} &= \int g(x)\, dx. \end{aligned}$$

If $H(y)$ and $G(x)$ are antiderivatives of $1/h(y)$ and $g(x)$, respectively, then the equation

$$H(y) = G(x) + c$$

gives solutions of the differential equation.

exact equations: An ordinary differential equation of the form

$$M(x,y)dx + N(x,y)dy = 0$$

is called exact if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

If M, N, and their first partials are continuous, then this condition ensures that there is a function $F(x, y)$ such that $\partial F/\partial x = M$ and $\partial F/\partial y = N$. We find F using partial integration:

$$\begin{aligned} \frac{\partial F}{\partial x} &= M(x, y) \\ F(x, y) &= \int M(, x, y)\, dx + h(y) \end{aligned}$$

where h is found by differentiating

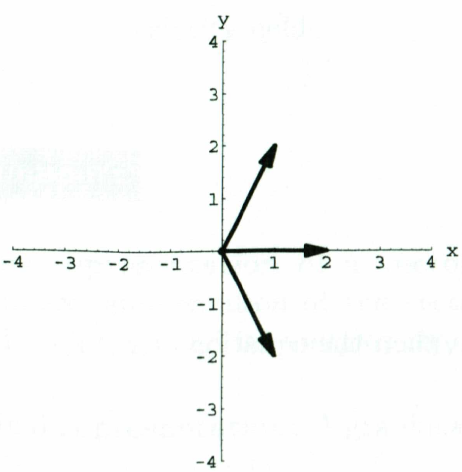
$$h'(y) = \frac{\partial F}{\partial y} - \frac{\partial}{\partial y}\left[\int M(, x, y)\, dx\right] = N(x, y) - \frac{\partial}{\partial y}\left[\int M(, x, y)\, dx\right].$$

Exercises 2.7

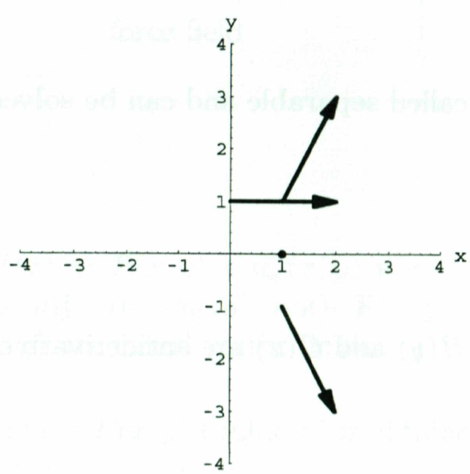
3. By evaluating f we find that

$$\begin{aligned} f(1) &= \overline{1 - 1^2} = 0, \\ f(1+i) &= \overline{1 - (1+i)^2} = \overline{1 - 2i} = 1 + 2i, \\ f(1-i) &= \overline{1 - (1-i)^2} = \overline{1 + 2i} = 1 - 2i, \text{ and} \\ f(i) &= \overline{1 - (i)^2} = \overline{1 + 1} = 2. \end{aligned}$$

Plotting these values as vectors is shown below.



(a) plotted as position vectors

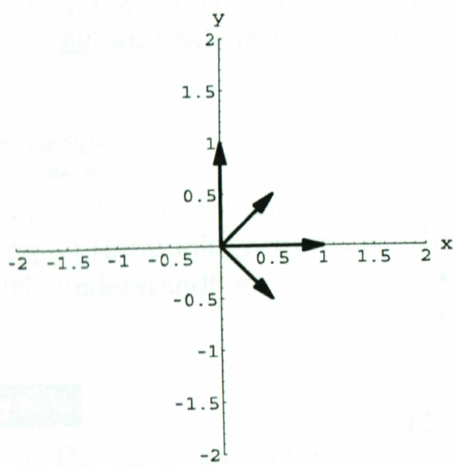


(b) plotted as vectors in the vector field

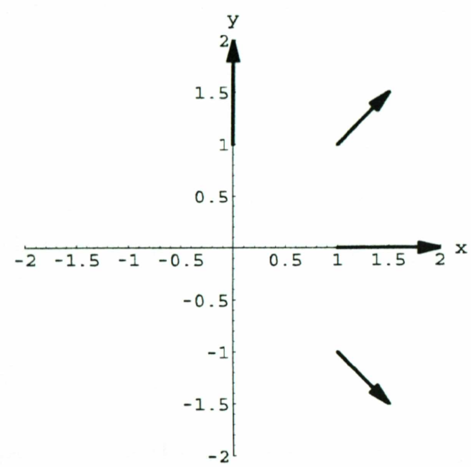
7. By evaluating f we find that

$$\begin{aligned} f(1) &= \frac{1}{\bar{1}} = 1, \\ f(1+i) &= \frac{1}{\overline{1+i}} = \frac{1}{1-i} = \frac{1}{2} + \frac{1}{2}i, \\ f(1-i) &= \frac{1}{\overline{1-i}} = \frac{1}{1+i} = \frac{1}{2} - \frac{1}{2}i, \text{ and} \\ f(i) &= \frac{1}{\bar{i}} = \frac{1}{-i} = i. \end{aligned}$$

Plotting these values as vectors is shown below.



(a) plotted as position vectors



(b) plotted as vectors in the vector field

11. **(a)** Letting $z = x+iy$ we have $f(z) = i(x+iy) = -y+ix$. By identifying $P(x,y) = -y$ and $Q(x,y) = x$ in (3) of Section 2.7, we see that the streamlines are solutions to the system of differential equations

$$\begin{aligned} \frac{dx}{dt} &= -y \\ \frac{dy}{dt} &= x. \end{aligned}$$

This system can be transformed into a single ordinary differential equation by using the chain rule

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} \quad \text{or} \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt}.$$

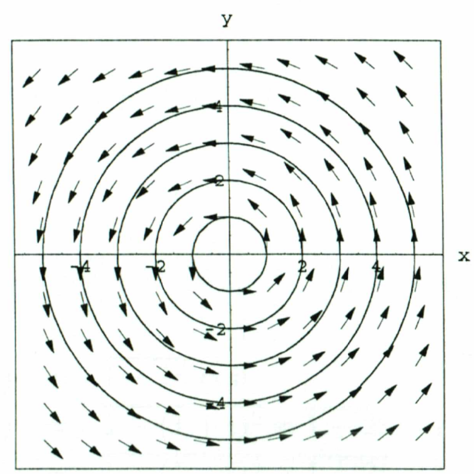
Substituting $dy/dt = x$ and $dx/dt = -y$ into the second equation yields

$$\frac{dy}{dx} = -\frac{x}{y}.$$

This differential equation can be solved by separation of variables:

$$\begin{aligned} \frac{dy}{dx} &= -\frac{x}{y} \\ \int y\, dy &= -\int x\, dx \\ \frac{1}{2}y^2 &= -\frac{1}{2}x^2 + c \\ x^2 + y^2 &= c. \end{aligned}$$

Therefore, streamlines are circles $x^2 + y^2 = c$.



(b) Streamlines for Problem 11

Focus on Concepts

15. (a) Let $c = a + bi$, then solve the system

$$\begin{aligned} \frac{dx}{dt} &= a \\ \frac{dy}{dt} &= b, \end{aligned}$$

which can be converted into the ordinary differential equation:

$$\frac{dy}{dx} = \frac{b}{a}.$$

(b) Consider the magnitude and direction of the velocity field at an arbitrary point z. Do these quantities vary across the complex plane?

Preface

This student study guide is designed to accompany the text *A First Course in Complex Analysis with Applications, Second Edition* (Jones and Bartlett Publishers, 2009) by Dennis G. Zill and Patrick D. Shanahan. It consists of seven chapters which correspond to the seven chapters of the text. Each chapter has the following features.

Review Topics

Many sections of the study guide are preceded by a review of selected topics from calculus and differential equations that are required for that section. These reviews provide concise summaries of prerequisite notation, terminology, and concepts. For additional review, students are encouraged to consult appropriate mathematics texts. Two excellent sources that were used repeatedly for the review topics are *Calculus: Early Transcendentals, Fourth Edition* (Jones and Bartlett Publishers, 2010) by Dennis G. Zill and Warren S. Wright and *Advanced Engineering Mathematics, Third Edition* (Jones and Bartlett Publishers, 2006) by Dennis G. Zill and Michael R. Cullen.

Summaries

A summary of every section of the text is provided. The summary reviews all of the key ideas of the section including all terminology, formulas, theorems, and concepts. Figures with two colors are included to aid in geometric understanding.

Exercises

Following the summary, complete solutions are given for every other odd exercise in the section (eg. problems 3, 7, 11, etc.). These are full solutions, supported by figures with two colors, that supply all of the pertinent details of the problem and incorporate the same techniques and writing style used in the text. The solutions also include references to equations, definitions, theorems, and figures in the text. The answer to each problem is typeset in color for easy reference.

Focus on Concepts

The focus on concepts problems from the text consist of conceptual word, proof, and geometrical problems. Since they are often used as topics for classroom discussion or independent study we have included detailed hints rather than full solutions for these problems. As with the exercises, only every other odd problem is included.

Final Note to Students

The most effective way to learn mathematics is to work many, many problems. You should not review a solution in this study guide before first working or, at the very least, attempting to work the problem yourself. Learning advanced mathematical topics takes significant time and effort. It may be quicker to look at the solutions, then try to work problems, but ultimately this approach will not lead to an independent understanding of concepts and problem solving strategies that are required for success.

Chapter 3

Analytic Functions

3.1 Differentiability and Analyticity

3.1 Summary

derivative: If a complex function f is defined in a neighborhood of a point z_0, then the derivative of f at z_0 is

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta) - f(z_0)}{\Delta z}$$

provided this limit exists. If $f'(z_0)$ exists, then f is said to be differentiable at z_0.

differentiation rules: The familiar rules of differentiation in calculus also apply to complex functions.

Constant Rules: $\dfrac{d}{dx}c = 0$ and $\dfrac{d}{dx}cf(z) = cf'(z)$

Sum Rule: $\dfrac{d}{dx}[f(z) \pm g(z)] = f'(z) \pm g'(z)$

Product Rule: $\dfrac{d}{dx}[f(z)g(z)] = f(z)g'(z) + f'(z)g(z)$

Quotient Rule: $\dfrac{d}{dx}\left[\dfrac{f(z)}{g(z)}\right] = \dfrac{g(z)f'(z) - f(z)g'(z)}{[g(z)]^2}$

$$\textit{Chain Rule:} \qquad \frac{d}{dx} f(g(z)) = f'(g(z))g'(z)$$

$$\textit{Power Rule:} \qquad \frac{d}{dx} z^n = nz^{n-1}, \text{ } n \text{ an integer}$$

analytic: A complex function f is said to be analytic at a point z_0 if f is differentiable at z_0 and at every point in some neighborhood of z_0.

analytic is differentiable: If f is analytic at z_0, then f is also differentiable at z_0, but the converse need not be true. That is, there do exist complex functions f such that f is differentiable at z_0 but not analytic at z_0.

entire: A function f is analytic in a domain D if f is analytic at every point in D. If f is analytic at every point in the complex plane, then f is called an entire function.

polynomial and rational functions: Complex polynomial functions are entire, complex rational functions are analytic on their domains.

analyticity rules: If f and g are analytic in a domain D, then the sum $f(z)+g(z)$, the difference $f(z)-g(z)$, and the product $f(z)g(z)$ are analytic in D. The quotient $f(z)/g(z)$ is analytic at all points where $g(z) \neq 0$.

differentiable is continuous: If f is differentiable at z_0, then f is also continuous at z_0, but the converse need not be true. That is, there do exist complex functions f such that f is continuous at z_0 but not differentiable at z_0.

L'Hôspital's rule: If f and g are analytic at z_0, $f(z_0) = 0$, $g(z_0) = 0$, and $g'(z_0) \neq 0$, then

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \lim_{z \to z_0} \frac{f'(z)}{g'(z)}.$$

Exercises 3.1

3. To find $f'(z)$, we replace z_0 with z in (1) of Section 3.1. Since $f(z) = iz^3 - 7z^2$,

$$\begin{aligned} f(z+\Delta z) - f(z) &= i(z+\Delta z)^3 - 7(z+\Delta z)^2 - (iz^3 - 7z^2) \\ &= i(z^3 + 3z^2\Delta z + 3z(\Delta z)^2 + (\Delta z)^3) - 7(z^2 + 2z\Delta z + (\Delta z)^2) - iz^3 + 7z^2 \\ &= \cancel{iz^3} + 3iz^2\Delta z + 3iz(\Delta z)^2 + i(\Delta z)^3 - \cancel{7z^2} - 14z\Delta z - 7(\Delta z)^2 - \cancel{iz^3} + \cancel{7z^2} \\ &= 3iz^2\Delta z + 3iz(\Delta z)^2 + i(\Delta z)^3 - 14z\Delta z - 7(\Delta z)^2. \end{aligned}$$

So, (1) gives

$$\begin{aligned} f'(z) &= \lim_{\Delta z \to 0} \frac{3iz^2\Delta z + 3iz(\Delta z)^2 + i(\Delta z)^3 - 14z\Delta z - 7(\Delta z)^2}{\Delta z} \\ &= \lim_{\Delta z \to 0} \frac{\Delta z(3iz^2 + 3iz\Delta z + i(\Delta z)^2 - 14z - 7\Delta z)}{\Delta z} \end{aligned}$$

$$
\begin{aligned}
&= \lim_{\Delta z \to 0} (3iz^2 + 3iz\Delta z + i(\Delta z)^2 - 14z - 7\Delta z) \\
&= 3iz^2 - 14z.
\end{aligned}
$$

Therefore, $f'(z) = 3iz^2 - 14z$.

7. Since $f(z) = 5z^2 - 10z + 8$,

$$
\begin{aligned}
f(z) - f(z_0) &= 5z^2 - 10z + 8 - (5z_0^2 - 10z_0 + 8) \\
&= 5z^2 - 10z + \cancel{8} - 5z_0^2 + 10z_0 - \cancel{8} \\
&= 5z^2 - 10z - 5z_0^2 + 10z_0.
\end{aligned}
$$

So, (12) gives

$$
\begin{aligned}
f'(z) &= \lim_{z \to z_0} \frac{5z^2 - 10z - 5z_0^2 + 10z_0}{z - z_0} \\
&= \lim_{z \to z_0} \frac{5(z^2 - z_0^2) - 10(z - z_0)}{z - z_0} \\
&= \lim_{z \to z_0} \frac{(z - z_0)(5(z + z_0) - 10)}{z - z_0} \\
&= \lim_{z \to z_0} (5(z + z_0) - 10) \\
&= 5(z_0 + z_0) - 10 \\
&= 10z_0 - 10.
\end{aligned}
$$

Therefore, after replacing z_0 with the symbol z, $f'(z) = 10z - 10$.

11. If $f(z) = (2 - i)z^5 + iz^4 - 3z^2 + i^6$, then

$$
\begin{aligned}
f'(z) &= \frac{d}{dz}\left[(2 - i)z^5\right] + \frac{d}{dz}\left[iz^4\right] - \frac{d}{dz}\left[3z^2\right] + \frac{d}{dz}\left[i^6\right] && \leftarrow \text{sum rule} \\
&= (2 - i)\frac{d}{dz}z^5 + i\frac{d}{dz}z^4 - 3\frac{d}{dz}z^2 + 0 && \leftarrow \text{constant rules} \\
&= (2 - i) \cdot 5z^4 + i \cdot 4z^3 - 3 \cdot 2z && \leftarrow \text{power rule} \\
&= (10 - 5i)z^4 + 4iz^3 - 6z.
\end{aligned}
$$

Therefore, $f'(z) = (10 - 5i)z^4 + 4iz^3 - 6z$.

15. If $(z) = \dfrac{iz^2 - 2z}{3z + 1 - i}$, then

$$\begin{aligned} f'(z) &= \frac{(3z+1-i)\frac{d}{dz}\left[iz^2 - 2z\right] - (iz^2 - 2z)\frac{d}{dz}\left[3z + 1 - i\right]}{(3z+1-i)^2} && \leftarrow \text{quotient rule} \\ &= \frac{(3z+1-i)(2iz-2) - (iz^2-2z)(3)}{(3z+1-i)^2} && \leftarrow \text{constant, sum, power rules} \\ &= \frac{6iz^2 + (-4+2i)z - 2 + 2i - 3iz^2 + 6z}{(3z+1-i)^2} \\ &= \frac{3iz^2 + (2+2i)z - 2 + 2i}{(3z+1-i)^2}. \end{aligned}$$

Therefore, $f'(z) = \dfrac{3iz^2 + (2+2i)z - 2 + 2i}{(3z+1-i)^2}$.

19. **(a)** Identifying $f(z) = |z|^2$ and $z_0 = 0$ in (1) of Section 3.1 gives

$$\begin{aligned} f'(0) &= \lim_{\Delta z \to 0} \frac{|0 + \Delta z|^2 - |0|^2}{\Delta z} \\ &= \lim_{\Delta z \to 0} \frac{|\Delta z|^2}{\Delta z} \\ &= \lim_{\Delta z \to 0} \frac{\Delta z \cdot \overline{\Delta z}}{\Delta z} \\ &= \lim_{\Delta z \to 0} \left(\overline{\Delta z}\right) \\ &= 0. \end{aligned}$$

Since $f'(0) = 0$, f is differentiable at the origin.

(b) In order to show that f is not differentiable at any point $z \neq 0$ we use (1) of Section 3.1 and the criterion for the nonexistence of a limit from Section 2.6.1. Let $z_0 = x_0 + iy_0$ with $x_0 \neq 0$ or $y_0 \neq 0$. By (1) of Section 3.1 we have

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{|z_0 + \Delta z|^2 - |z_0|^2}{\Delta z}.$$

If Δz approaches 0 along the real axis, then $\Delta z = \Delta x + 0i$ where the real number Δx is approaching 0. For this approach we have

$$\begin{aligned} f'(z_0) &= \lim_{\Delta z \to 0} \frac{|z_0 + \Delta z|^2 - |z_0|^2}{\Delta z} \\ &= \lim_{\Delta x \to 0} \frac{|x_0 + iy_0 + \Delta x|^2 - |x_0 + iy_0|^2}{\Delta x} \end{aligned}$$

$$
\begin{aligned}
&= \lim_{\Delta x \to 0} \frac{(x_0 + \Delta x)^2 + \cancel{y_0^2} - x_0^2 - \cancel{y_0^2}}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\cancel{x_0^2} + 2x_0 \Delta x + \Delta x^2 - \cancel{x_0^2}}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\Delta x(2x_0 + \Delta x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} (2x_0 + \Delta x) \\
&= 2x_0.
\end{aligned}
$$

On the other hand, if If Δz approaches 0 along the imaginary axis, then $\Delta z = 0 + i\Delta y$ where the real number Δy is approaching 0. For this approach we have

$$
\begin{aligned}
f'(z_0) &= \lim_{\Delta z \to 0} \frac{|z_0 + \Delta z|^2 - |z_0|^2}{\Delta z} \\
&= \lim_{\Delta y \to 0} \frac{|x_0 + iy_0 + i\Delta y|^2 - |x_0 + iy_0|^2}{i\Delta y} \\
&= \lim_{\Delta x \to 0} \frac{\cancel{x_0^2} + (y_0 + \Delta y)^2 - \cancel{x_0^2} - y_0^2}{i\Delta y} \\
&= \lim_{\Delta x \to 0} \frac{\cancel{y_0^2} + 2y_0 \Delta y + \Delta y^2 - \cancel{y_0^2}}{i\Delta y} \\
&= \lim_{\Delta x \to 0} \frac{\Delta y(2y_0 + \Delta y)}{i\Delta y} \\
&= \lim_{\Delta x \to 0} \frac{2y_0 + \Delta y}{i} \\
&= -2iy_0.
\end{aligned}
$$

Since both x_0 and y_0 are real numbers, $2x_0 = -2iy_0$ if and only if $x_0 = y_0 = 0$. However, we started with the assumption that either x_0 or y_0 is nonzero. Therefore, the limits along the two curves above are not equal, and consequently, $f'(z_0)$ does not exist.

23. If we identify $f(z) = z^7 + i$ and $g(z) = z^{14} + 1$, then

$$
f(i) = i^7 + i = -i + i = 0 \quad \text{and} \quad g(i) = i^{14} + 1 = -1 + 1 = 0.
$$

So the limit $\lim\limits_{z \to i} \dfrac{z^7 + i}{z^{14} + 1}$ has indeterminate form $0/0$. Since f and g are polynomial functions, both are necessarily analytic at $z_0 = i$. Using

$$
f'(z) = 7z^6, \quad g'(z) = 14z^{13}, \quad f'(i) = 7i^6 = -7, \quad g'(i) = 14i^{13} = 14i,
$$

we see that (13) of Section 3.1 gives

$$\lim_{z\to i}\frac{z^7+i}{z^{14}+1}=\lim_{z\to i}\frac{7z^6}{14z^{13}}=\frac{-7}{14i}=\frac{1}{2}i.$$

27. Since $f(z)=\dfrac{iz^2-2z}{3z+1-i}$ is a complex rational function, it is analytic on its domain. So f fails to be analytic when $3z+1-i=0$, that is, when $z=-\frac{1}{3}+\frac{1}{3}i$. Therefore, f is analytic for all complex numbers z such that $z\neq-\frac{1}{3}+\frac{1}{3}i$.

Focus on Concepts

31. Replacing z_0 with the symbol z in Definition 3.1.1 gives

$$\frac{d}{dz}cf(z)=\lim_{\Delta z\to 0}\frac{cf(z+\Delta z)-cf(z)}{\Delta z}=\lim_{\Delta z\to 0}c\left[\frac{f(z+\Delta z)-f(z)}{\Delta z}\right].$$

Now evaluate this limit using Theorem 2.6.2(i) and Definition 3.1.1.

35. **(a)** If Δz approaches 0, then $r\to 0$, but θ need not approach a limit.

(b) Use the fact that $\overline{z+\Delta z}=\bar{z}+\overline{\Delta z}$ to help simplify the limit.

(c) Consider different limiting values of θ and use the criterion for the nonexistence of a limit.

3.2 Cauchy-Riemann Equations

Review Topic: Partial Derivatives

partial derivatives: If $F(x,y)$ is a function of two real variables, then the partial derivative of F with respect to x is

$$\frac{\partial F}{\partial x}=\lim_{\Delta x\to 0}\frac{F(x+\Delta x,y)-F(x,y)}{\Delta x}.$$

Similarly, the partial derivative of F with respect to y is

$$\frac{\partial F}{\partial y}=\lim_{\Delta y\to 0}\frac{F(x,y+\Delta y)-F(x,y)}{\Delta y}.$$

The partial derivatives $\partial F/\partial x$ and $\partial F/\partial y$ are often represented by the alternative notation F_x and F_y, respectively.

computing partial derivatives: To compute $\partial F/\partial x$, use the laws of ordinary differentiation while treating y as a constant. To compute $\partial F/\partial y$, use the laws of ordinary differentiation while treating x as a constant.

second-order partial derivatives: Since $\partial F/\partial x$ and $\partial F/\partial y$ are functions of two variables, we can compute partial derivatives of these functions. This gives four, second-order partial derivatives defined by

$$\frac{\partial^2 F}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial x}\right) \qquad \frac{\partial^2 F}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y}\right)$$

$$\frac{\partial^2 F}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial x}\right) \qquad \frac{\partial^2 F}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial y}\right)$$

mixed second-order partial derivatives: If a function $F(x, y)$ has continuous second-order partial derivatives, then the mixed second-order partial derivatives are equal, that is,

$$\frac{\partial^2 F}{\partial x \partial y} = \frac{\partial^2 F}{\partial y \partial x}.$$

chain rule: If $F(u, v)$ is differentiable and if $u = g(x, y)$ and $v = f(x, y)$ have continuous first-order partial derivatives, then

$$\frac{dF}{dx} = \frac{\partial F}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial x} \quad \text{and} \quad \frac{dF}{dy} = \frac{\partial F}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial y}.$$

3.2 Summary

Cauchy-Riemann equations: Suppose that $f(z) = u(x, y) + iv(x, y)$ is differentiable at a point $z = x + iy$. Then at (x, y) the first-order partial derivatives of u and v exist and satisfy the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

criterion for non-analyticity: If the Cauchy-Riemann equations fail be satisfied at one or more points z in a domain D, then f is not analytic in D.

criteria for analyticity: If $u(x, y)$ and $v(x, y)$ are continuous and have continuous first-order partial derivatives in a domain D, and if u and v satisfy the Cauchy-Riemann at all points of D, then $f(z) = u(x, y) + iv(x, y)$ is analytic in D. Notice that this statement tells us that the Cauchy-Riemann equations alone are not sufficient to ensure analyticity.

formula for $f'(z)$: If $u(x, y)$ and $v(x, y)$ are continuous and have continuous first-order partial derivatives in some neighborhood of a point z, and if u and v satisfy the Cauchy-Riemann at z, then $f(z) = u(x, y) + iv(x, y)$ is differentiable at z and

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}.$$

constant functions: The Cauchy-Riemann equations can be used to establish the following results regarding constant functions. Suppose $f(z)$ is analytic in a domain D.

(*i*) If $|f(z)|$ is constant in D, then $f(z) = c$ in D, where c is a complex constant.

(*ii*) If $f'(z) = 0$ in D, then $f(z) = c$ in D, where c is a complex constant.

polar form: Suppose $f(z) = u(r,\theta) + iv(r,\theta)$ is a complex function expressed in terms of a polar form $z = r(\cos\theta + i\sin\theta)$ of the variable z. The polar form of the Cauchy-Riemann equations is

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}.$$

A polar form of the derivative of f is

$$f'(z) = e^{-i\theta}\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right) = \frac{1}{r}e^{-i\theta}\left(\frac{\partial v}{\partial \theta} - i\frac{\partial u}{\partial \theta}\right).$$

Exercises 3.2

3. If we set $z = x + iy$, then $f(z) = \text{Re}(z) = \text{Re}(x+iy) = x$. We now identify $u(x,y) = x$ and $v(x,y) = 0$. Computing the first-order partial derivatives gives

$$\frac{\partial u}{\partial x} = 1 \qquad \frac{\partial v}{\partial x} = 0$$

$$\frac{\partial u}{\partial y} = 0 \qquad \frac{\partial v}{\partial y} = 0.$$

Since the first Cauchy-Riemann equation

$$\frac{\partial u}{\partial x} = 1 \neq 0 = \frac{\partial v}{\partial y}$$

is satisfied for no points in the plane, we conclude that $f(z) = \text{Re}(z)$ is nowhere analytic.

7. From $f(z) = x^2 + y^2$ we identify $u(x,y) = x^2 + y^2$ and $v(x,y) = 0$. Computing the first-order partial derivatives gives

$$\frac{\partial u}{\partial x} = 2x \qquad \frac{\partial v}{\partial x} = 0$$

$$\frac{\partial u}{\partial y} = 2y \qquad \frac{\partial v}{\partial y} = 0.$$

The first Cauchy-Riemann equation

$$\frac{\partial u}{\partial x} = 2x = 0 = \frac{\partial v}{\partial y}$$

is satisfied if and only if $x = 0$. On the other hand, the second Cauchy-Riemann equation

$$\frac{\partial u}{\partial y} = 2y = 0 = -\frac{\partial v}{\partial x}$$

is satisfied if and only if $y = 0$. Therefore, the Cauchy-Riemann equations are satisfied only at the point $z = 0$. Consequently, in any domain D there exists points in the domain where the Cauchy-Riemann equations fail to be satisfied. By the criterion for non-analyticity, this implies that $f(z) = x^2 + y^2$ is nowhere analytic.

11. **(a)** From $f(z) = e^{x^2-y^2}\cos 2xy + ie^{x^2-y^2}\sin 2xy$ we identify $u(x, y) = e^{x^2-y^2}\cos 2xy$ and $v(x, y) = e^{x^2-y^2}\sin 2xy$. Computing the first-order partial derivatives gives

$$\frac{\partial u}{\partial x} = 2e^{x^2-y^2}(x\cos 2xy - y\sin 2xy) \qquad \frac{\partial v}{\partial x} = 2e^{x^2-y^2}(y\cos 2xy + x\sin 2xy)$$

$$\frac{\partial u}{\partial y} = -2e^{x^2-y^2}(y\cos 2xy + x\sin 2xy) \qquad \frac{\partial v}{\partial y} = 2e^{x^2-y^2}(x\cos 2xy - y\sin 2xy).$$

The Cauchy-Riemann equations

$$\begin{aligned}\frac{\partial u}{\partial x} &= 2e^{x^2-y^2}(x\cos 2xy - y\sin 2xy) &= \frac{\partial v}{\partial y}\\ \frac{\partial u}{\partial y} &= -2e^{x^2-y^2}(y\cos 2xy + x\sin 2xy) &= -\frac{\partial v}{\partial x}\end{aligned}$$

are satisfied at every point in the complex plane. Since u, v, and their first-order partial derivatives are continuous at all points in the plane, it follows that $f(z) = e^{x^2-y^2}\cos 2xy + ie^{x^2-y^2}\sin 2xy$ is an entire function.

(b) By formula (9) in Section 3.2,

$$f'(z) = 2e^{x^2-y^2}(x\cos 2xy - y\sin 2xy) + i2e^{x^2-y^2}(y\cos 2xy + x\sin 2xy)$$

for all z in the complex plane.

15. **(a)** From

$$f(z) = \frac{\cos\theta}{r} - i\frac{\sin\theta}{r}$$

we identify

$$u(x, y) = \frac{\cos\theta}{r} \quad \text{and} \quad v(x, y) = \frac{\sin\theta}{r}.$$

Computing the first-order partial derivatives gives

$$\frac{\partial u}{\partial r} = -\frac{\cos\theta}{r^2} \qquad \frac{\partial v}{\partial r} = \frac{\sin\theta}{r^2}$$

$$\frac{\partial u}{\partial \theta} = -\frac{\sin\theta}{r} \qquad \frac{\partial v}{\partial \theta} = -\frac{\cos\theta}{r}.$$

The polar form of the Cauchy-Riemann equations

$$\begin{aligned}\frac{\partial u}{\partial r} &= -\frac{\cos\theta}{r^2} &= \frac{1}{r}\frac{\partial v}{\partial \theta}\\ \frac{\partial v}{\partial r} &= \frac{\sin\theta}{r^2} &= -\frac{1}{r}\frac{\partial u}{\partial \theta}\end{aligned}$$

are satisfied at all points $z = r(\cos\theta + i\sin\theta)$ such that $r \neq 0$. Since u, v, and their first-order partial derivatives are continuous at all points z such that $r \neq 0$, it follows that $f(z) = \dfrac{\cos\theta}{r} - i\dfrac{\sin\theta}{r}$ is analytic in the punctured complex plane, $z \neq 0$.

(b) By formula (11) in Section 3.2,

$$f'(z) = e^{-i\theta}\left(-\frac{\cos\theta}{r^2} + i\frac{\sin\theta}{r^2}\right)$$

for all z in the complex plane.

19. (a) From $f(z) = x^2 + y^2 + 2ixy$ we identify $u(x,y) = x^2 + y^2$ and $v(x,y) = 2xy$. Computing the first-order partial derivatives gives

$$\begin{aligned} \frac{\partial u}{\partial x} &= 2x & \frac{\partial v}{\partial x} &= 2y \\ \frac{\partial u}{\partial y} &= 2y & \frac{\partial v}{\partial y} &= 2x. \end{aligned}$$

The first Cauchy-Riemann equation

$$\frac{\partial u}{\partial x} = 2x = \frac{\partial v}{\partial y}$$

is satisfied at all points in the complex plane. On the other hand, the second Cauchy-Riemann equation

$$\frac{\partial u}{\partial y} = 2y = -2y = -\frac{\partial v}{\partial x}$$

is satisfied if and only if $y = 0$. Therefore, the Cauchy-Riemann equations are satisfied at all points such that $y = 0$, that is, at all points on the $x-$axis. Since there is no neighborhood in which the Cauchy-Riemann equations are satisfied, $f(z) = x^2 + y^2 + 2ixy$ is nowhere analytic. However, since the u, v, and their first-order partial derivatives are continuous at all points in the complex plane, and since u and v satisfy the Cauchy-Riemann equations at every point on the $x-$axis, $f(z) = x^2 + y^2 + 2ixy$ is differentiable on the $x-$axis.

(b) By formula (9) in Section 3.2, on the $x-$axis we have

$$\begin{aligned} f'(z) &= 2x + i2y \\ &= 2x. \qquad \leftarrow y = 0 \text{ on the } x\text{−axis} \end{aligned}$$

23. (a) From (3) in Section 2.1, $f(z) = e^z = e^x\cos y + ie^x\sin y$, and so we identify $u(x,y) = e^x\cos y$ and $v(x,y) = e^x\sin y$. Computing the first-order partial derivatives gives

$$\begin{aligned} \frac{\partial u}{\partial x} &= e^x\cos y & \frac{\partial v}{\partial x} &= e^x\sin y \\ \frac{\partial u}{\partial y} &= -e^x\sin y & \frac{\partial v}{\partial y} &= e^x\cos y. \end{aligned}$$

The Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = e^x \cos y = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = -e^x \sin y = -\frac{\partial v}{\partial x}$$

are satisfied at every point in the complex plane. Since u, v, and their first-order partial derivatives are continuous at all points in the plane, it follows that $f(z) = e^z$ is an entire function.

(b) By formula (9) in Section 3.2, $f'(z) = e^x \cos y + ie^x \sin y$, therefore, $f'(z) = f(z)$.

Focus on Concepts

27. Let $f(z) = u(x,y) + iv(x,y)$, then $g(z) = \overline{f(z)} = u(x,y) - iv(x,y)$. Since f is analytic, u and v satisfy the Cauchy-Riemann equations. Can $U(x,y) = u(x,y)$ and $V(x,y) = -v(x,y)$ also satisfy the Cauchy-Riemann equations?

31. If $f(z) = u(x,y) + iv(x,y)$ is analytic and $f'(z) = a + ib$ where a and b are real constants, then by formula (9) of Section 3.2 we have

$$f'(z) = a + ib = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}.$$

Partially integrate the equations

$$a = \frac{\partial u}{\partial x} \quad \text{and} \quad b = \frac{\partial v}{\partial x}$$

and then use the Cauchy-Riemann equations to determine formulas for $u(x,y)$ and $v(x,y)$.

35. To solve for u_x in (12), multiply the first equation by $\cos\theta$, multiply the second equation by $-\frac{1}{r}\sin\theta$, and then sum the resulting equations. Modify this procedure to solve for v_x in (13). Finally, substitute these expressions in for u_x and v_x in formula (9) in Section 3.2

3.3 Harmonic Functions

3.3 Summary

Laplace's equation: Let $\phi(x,y)$ be a function of two real variables. The second-order partial differential equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

is called Laplace's equation. This equation appears in many applications in science and engineering. The sum $\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}$ is also denoted by the symbol $\nabla^2\phi$.

harmonic function: A function $\phi(x, y)$ is called harmonic in a domain D if it has continuous first and second-order partial derivatives and satisfies Laplace's equation at all points in D.

real and imaginary parts of an analytic function: If $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain D, then the functions $u(x, y)$ and $v(x, y)$ are both harmonic functions in D.

harmonic conjugate: If two functions $u(x, y)$ and $v(x, y)$ are both harmonic in a domain D, and if u and v satisfy the Cauchy-Riemann equations in D, then $v(x, y)$ is called a harmonic conjugate of $u(x, y)$ in D. Notice that order is important; v is called a harmonic conjugate of u, but u is not called a harmonic conjugate of v.

corresponding analytic function: If $u(x, y)$ is a harmonic function in a domain D and if $v(x, y)$ is a harmonic conjugate of $u(x, y)$ in D, then the corresponding complex function $f(z) = u(x, y) + iv(x, y)$ is analytic in D.

Exercises 3.3

3. (a) From the partial derivatives

$$\frac{\partial u}{\partial x} = 2x, \quad \frac{\partial^2 u}{\partial x^2} = 2, \quad \frac{\partial u}{\partial y} = -2y, \quad \frac{\partial^2 u}{\partial y^2} = -2$$

we see that u satisfies Laplace's equation at every point in the complex plane

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2 - 2 = 0.$$

Therefore, u is harmonic in $\mathbf{C}$.

(b) If $v(x, y)$ is a harmonic conjugate, then u and v must satisfy the Cauchy-Riemann equations, and so we must have

$$\begin{aligned} \frac{\partial u}{\partial x} &= 2x = \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} &= -2y = -\frac{\partial v}{\partial x}. \end{aligned}$$

Partial integration of the first equation with respect to the variable y gives

$$v(x, y) = \int 2x\, dy = 2xy + h(x).$$

The partial derivative with respect to x of this last equation is

$$\frac{\partial v}{\partial x} = 2y + h'(x).$$

When this result is substituted into the second Cauchy-Riemann equation we obtain

$$\begin{aligned} -2y &= -2y - h'(x) \\ h'(x) &= 0 \\ h(x) &= C \end{aligned}$$

where C is a real constant. Therefore, $v(x, y) = 2xy + C$ is a harmonic conjugate of $u(x, y)$.
(c) The corresponding analytic function is $f(z) = x^2 - y^2 + i(2xy + C)$.

7. (a) From the partial derivatives

$$\frac{\partial u}{\partial x} = e^x \cos y + e^x(x \cos y - y \sin y), \qquad \frac{\partial^2 u}{\partial x^2} = 2e^x \cos y + e^x(x \cos y - y \sin y),$$

$$\frac{\partial u}{\partial y} = e^x(-y \cos y - x \sin y - \sin y), \qquad \frac{\partial^2 u}{\partial y^2} = e^x(-x \cos y - 2 \cos y + y \sin y)$$

we see that u satisfies Laplace's equation at every point in the complex plane

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} &= 2e^x \cos y + e^x(x \cos y - y \sin y) + e^x(-x \cos y - 2 \cos y + y \sin y) \\ &= 2e^x \cos y + xe^x \cos y - ye^x \sin y - xe^x \cos y - 2e^x \cos y + ye^x \sin y \\ &= 0. \end{aligned}$$

Therefore, u is harmonic in $\mathbf{C}$.
(b) If $v(x, y)$ is a harmonic conjugate, then u and v must satisfy the Cauchy-Riemann equations, and so we must have

$$\begin{aligned} \frac{\partial u}{\partial x} &= e^x \cos y + e^x(x \cos y - y \sin y) = \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} &= e^x(-y \cos y - x \sin y - \sin y) = -\frac{\partial v}{\partial x}. \end{aligned}$$

Partial integration of the first equation with respect to the variable y gives

$$\begin{aligned} v(x, y) &= \int (e^x \cos y + e^x(x \cos y - y \sin y))\, dy \\ &= e^x \int \cos y\, dy + xe^x \int \cos y\, dy - e^x \int y \sin y\, dy \quad \leftarrow x \text{ and } e^x \text{ treated as constants} \\ &= e^x \sin y + xe^x \sin y - e^x \left(-y \cos y + \int \cos y\, dy\right) \quad \leftarrow \text{integration by parts for 3rd integral} \\ &= e^x \sin y + xe^x \sin y + ye^x \cos y - e^x \sin y + h(x) \\ &= xe^x \sin y + ye^x \cos y + h(x). \end{aligned}$$

The partial derivative with respect to x of this last equation is

$$\frac{\partial v}{\partial x} = xe^x \sin y + e^x \sin y + ye^x \cos y + h'(x).$$

When this result is substituted into the second Cauchy-Riemann equation we obtain

$$\begin{aligned} e^x(-y\cos y - x\sin y - \sin y) &= -(xe^x \sin y + e^x \sin y + ye^x \cos y + h'(x)) \\ -ye^x \cos y - xe^x \sin y - e^x \sin y &= -xe^x \sin y - e^x \sin y - ye^x \cos y - h'(x) \\ h'(x) &= 0 \\ h(x) &= C \end{aligned}$$

where C is a real constant. Therefore, $v(x,y) = xe^x \sin y + ye^x \cos y + C$ is a harmonic conjugate of $u(x,y)$.

(c) The corresponding analytic function is $f(z) = e^x(x\cos y - y\sin y) + i(e^x(x\sin y + y\cos y) + C)$.

11. From the partial derivatives

$$\frac{\partial u}{\partial x} = y+1, \quad \frac{\partial^2 u}{\partial x^2} = 0, \quad \frac{\partial u}{\partial y} = x+2, \quad \frac{\partial^2 u}{\partial y^2} = 0$$

we see that u satisfies Laplace's equation at every point in the complex plane

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 + 0 = 0.$$

Therefore, u is harmonic in **C**.

If $v(x,y)$ is a harmonic conjugate, then u and v must satisfy the Cauchy-Riemann equations, and so we must have

$$\begin{aligned} \frac{\partial u}{\partial x} &= y+1 = \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} &= x+2 = -\frac{\partial v}{\partial x}. \end{aligned}$$

Partial integration of the first equation with respect to the variable y gives

$$v(x,y) = \int (y+1)dy = \frac{1}{2}y^2 + y + h(x).$$

The partial derivative with respect to x of this last equation is

$$\frac{\partial v}{\partial x} = h'(x).$$

When this result is substituted into the second Cauchy-Riemann equation we obtain

$$\begin{aligned} x+2 &= -h'(x) \\ h'(x) &= -x-2 \\ h(x) &= \int(-x-2)dx \\ h(x) &= -\frac{1}{2}x^2 - 2x + C \end{aligned}$$

where C is a real constant. Therefore, $v(x, y) = \frac{1}{2}y^2 + y - \frac{1}{2}x^2 - 2x + C$ is a harmonic conjugate of $u(x, y)$.

(c) The corresponding analytic function is $f(z) = xy + x + 2y - 5 + i\left(\frac{1}{2}y^2 + y - \frac{1}{2}x^2 - 2x + C\right)$. If $f(2i) = -1 + 5i$, then

$$\begin{aligned}
(0)(2) + 0 + 2(2) - 5 + i\left(\frac{1}{2}(2)^2 + 2 - \frac{1}{2}(0)^2 - 2(0) + C\right) &= -1 + 5i \quad \leftarrow \text{since } z = 2i,\ x = 0 \text{ and } y = 2 \\
-\cancel{1} + i(4 + C) &= -\cancel{1} + 5i \\
4 + C &= 5 \\
C &= 1.
\end{aligned}$$

Therefore, the analytic function satisfying the condition $f(2i) = -1 + 5i$ is

$$f(z) = xy + x + 2y - 5 + i\left(\frac{1}{2}y^2 + y - \frac{1}{2}x^2 - 2x + 1\right).$$

15. From the partial derivatives

$$\frac{\partial u}{\partial r} = 3r^2\cos 3\theta, \quad \frac{\partial^2 u}{\partial r^2} = 6r\cos 3\theta, \quad \frac{\partial u}{\partial \theta} = -3r^3\sin 3\theta, \quad \frac{\partial^2 u}{\partial \theta^2} = -9r^3\cos 3\theta$$

we see that u satisfies the Laplace's equation in polar coordinates at every point in the complex plane excluding the origin

$$\begin{aligned}
r^2\frac{\partial^2 u}{\partial r^2} + r\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2} &= r^2\left(6r\cos 3\theta\right) + r\left(3r^2\cos 3\theta\right) - 9r^3\cos 3\theta \\
&= 6r^3\cos 3\theta + 3r^3\cos 3\theta - 9r^3\cos 3\theta \\
&= 0.
\end{aligned}$$

Therefore, u is harmonic in the punctured complex plane $z \neq 0$.

Focus on Concepts

19. (a) By the chain rule we have

$$\frac{\partial \phi}{\partial x} = -x(x^2 + y^2 + z^2)^{-3/2}.$$

Next, by the product and chain rules

$$\begin{aligned}
\frac{\partial^2 \phi}{\partial x^2} &= 3x^2(x^2 + y^2 + z^2)^{-5/2} - (x^2 + y^2 + z^2)^{-3/2} \\
&= \frac{3x^2}{(x^2 + y^2 + z^2)^{5/2}} - \frac{1}{(x^2 + y^2 + z^2)^{3/2}} \cdot \frac{x^2 + y^2 + z^2}{x^2 + y^2 + z^2} \\
&= \frac{3x^2 - x^2 - y^2 - z^2}{(x^2 + y^2 + z^2)^{5/2}} \\
&= \frac{2x^2 - y^2 - z^2}{(x^2 + y^2 + z^2)^{5/2}}.
\end{aligned}$$

Now repeat for $\partial^2\phi/\partial y^2$ and $\partial^2\phi/\partial z^2$ and sum the resulting second-order partial derivatives.

(b) Modify the procedure in part (a).

3.4 Applications

Review Topic: Vector Calculus

gradient: If $f(x, y)$ is a differentiable function of two real variables, then the vector-valued function

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

is called the gradient of f. The symbols $\mathbf{i}$ and $\mathbf{j}$ here represent the standard basis vectors $\langle 1, 0\rangle$ and $\langle 0, 1\rangle$, respectively.

del differential operator: The gradient vector can also be interpreted as the result of applying the del (short for "delta") differential operator

$$\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j}$$

to the function $f(x, y)$.

level curves: For a function $f(x, y)$, the curves defined by $f(x, y) = c$, where c is a constant, are called the level curves of f.

orthogonality of the gradient: The gradient vector $\nabla f(x_0, y_0)$ at a point (x_0, y_0) is perpendicular (or, orthogonal) to the level curve of f passing through (x_0, y_0).

divergence: If $\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ is a vector field, then the divergence of $\mathbf{F}$ is the scalar

$$\text{div } \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}.$$

This quantity can also be represented as the dot product of the del operator with the vector field

$$\text{div } \mathbf{F} = \nabla \cdot \mathbf{F} = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j}\right) \cdot \left(P\mathbf{i} + Q\mathbf{j}\right) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}.$$

If the vector field $\mathbf{F}$ is the velocity field of a fluid, then the divergence represents the net volume of fluid flowing through an element of surface area per unit time. Therefore, div $\mathbf{F}(x_0, y_0) > 0$ indicates the presence of a source of fluid flow at the point (x_0, y_0), whereas, div $\mathbf{F}(x_0, y_0) < 0$ indicates the presence of a sink of fluid flow at the point (x_0, y_0).

curl: If $\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ is a vector field, then the curl of $\mathbf{F}$ is the vector

$$\text{curl } \mathbf{F} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}.$$

This vector can also be represented as cross product of the three-dimensional del operator with the three-dimensional vector field $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + 0\mathbf{k}$ as follows

$$\begin{aligned}
\text{curl } \mathbf{F} &= \nabla \times \mathbf{F} \\
&= \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right) \times (P\mathbf{i} + Q\mathbf{j} + 0\mathbf{k}) \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & 0 \end{vmatrix} \\
&= \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}.
\end{aligned}$$

If the vector field $\mathbf{F}$ is the velocity field of a fluid, then the curl represents the tendency of the fluid to turn a paddlewheel inserted into the flow perpendicular to the xy-plane.

gradient field and potential function: A vector field $\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ is called a gradient field if there exists a function $\phi(x, y)$ such that $\mathbf{F} = \nabla\phi$. If $\mathbf{F} = \nabla\phi$ is a gradient field, then $\phi(x, y)$ is called a potential function for $\mathbf{F}$.

criterion for a gradient field: Let $\mathbf{F} = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ be a vector field defined in a simply connected domain D. If P and Q have continuous first-order partial derivatives in D then $\mathbf{F}$ is a gradient field if and only if $\dfrac{\partial Q}{\partial x} = \dfrac{\partial P}{\partial y}$ throughout D (this is equivalent to $\nabla \times \mathbf{F} = \mathbf{0}$ in D).

3.4 Summary

orthogonal families: If $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain D, then level curves $u(x, y) = c_1$ and $v(x, y) = c_2$ are orthogonal families in D. This means that at any point $z_0 = x_0 + iy_0$ in D, the tangent line L_1 to the level curve $u(x, y) = u_0 = u(x_0, y_0)$ is perpendicular to the tangent line L_2 to the level curve $v(x, y) = v_0 = v(x_0, y_0)$.

gradient fields and Laplace's equation: Laplace's equation occurs naturally in several applications involving gradient fields. For example, if $\mathbf{F}$ represents the electric field intensity in a suitable domain D due to a collection of charges on the boundary of D, then Faraday's law implies that $\mathbf{F} = -\nabla\phi$ is a gradient field. (With the hypotheses of the criterion for a gradient field, this is equivalent to curl $\mathbf{F} = \mathbf{0}$ in D.) If there are no charges in the domain, then Gauss' law asserts that div $\mathbf{F} = 0$. Thus, we have

$$0 = \text{div } \mathbf{F} = \nabla \cdot \mathbf{F} = -\nabla \cdot \nabla\phi = -\left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j}\right) \cdot \left(\frac{\partial \phi}{\partial x}\mathbf{i} + \frac{\partial \phi}{\partial y}\mathbf{j}\right) = -\frac{\partial^2 \phi}{\partial x^2} - \frac{\partial^2 \phi}{\partial y^2}.$$

This means that the electric field potential ϕ satisfies Laplace's equation $\nabla^2\phi = 0$ in D. For a gravitational force field $\mathbf{F}$ the situation is similar except it is the law of conservation of mechanical energy that ensures $\mathbf{F} = \nabla\phi$. If $\mathbf{F}$ represents the velocity field of a planar fluid flow then the flow is called incompressible if div $\mathbf{F} = 0$ and irrotational if curl $\mathbf{F} = \mathbf{0}$. As above this implies that there is a velocity potential ϕ which satisfies Laplace's equation. Finally, if ϕ represents the temperature at point (x, y) at time t, then the heat equation states

$$K\nabla^2\phi + \nabla K \cdot \nabla\phi = \sigma\rho\frac{\partial\phi}{\partial t}$$

where K is the thermal conductivity, σ is specific heat, and ρ is the density of the medium. When K is a constant and the temperature is independent of time (called the steady-state temperature), then $\nabla K = \mathbf{0}$ and $\partial\phi/\partial t = 0$, and so the the heat equation reduces to Laplace's equation $\nabla^2\phi = 0$.

complex potential: If a real potential function $\phi(x, y)$ satisfies Laplace's equation $\nabla^2\phi = 0$ in a domain D and if $\psi(x, y)$ is a harmonic conjugate of $\phi(x, y)$ in D, then the function

$$\boldsymbol{\Omega}(z) = \phi(x, y) + i\psi(x, y)$$

is analytic in D and is called the complex potential function corresponding to ϕ.

equipotential curves: If $\boldsymbol{\Omega}(z) = \phi(x, y) + i\psi(x, y)$ is a complex potential function in a domain D, then the level curves $\phi(x, y) = c_1$ are called equipotential curves. The level curves $\phi(x, y) = c_1$ and $\psi(x, y) = c_2$ form an orthogonal family in D.

applications of complex potentials: The following table summarizes some common applications of complex potential functions $\boldsymbol{\Omega}(z) = \phi(x, y) + i\psi(x, y)$.

Application	**Potential Function ϕ**	**Level Curves $\phi = c_1$**	**Level Curves $\psi = c_2$**
electrostatics	electrostatic potential	equipotential curves	lines of force
gravitation	gravitational potential	equipotential curves	lines of force
heat flow	steady-state temperature	isotherms	lines of heat flux
fluid flow	velocity potential	equipotential curves	streamlines of flow

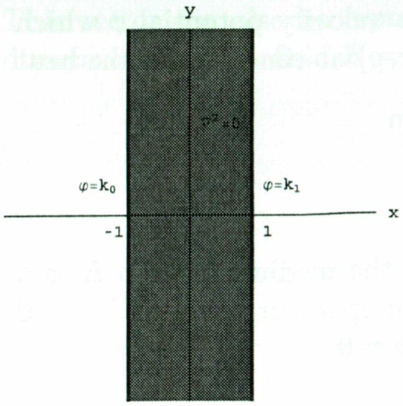
Dirichlet Problem: If D is a domain in the plane and g is a function defined on the boundary C of D, then the problem of finding a function $\phi(x, y)$ which

(i) satisfies Laplace's equation $\nabla^2\phi = 0$ in D, and

(ii) equals g on the boundary C of D

is called a Dirichlet problem.

Dirichlet problem in an infinite strip: The Dirichlet problem illustrated below



$$\textit{Solve:} \qquad \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial x^2} = 0, \quad -1 < x < 1, \quad -\infty < y < \infty$$

$$\textit{Subject to:} \quad \phi(-1, y) = k_0, \quad \phi(1, y) = k_1, \quad -\infty < y < \infty.$$

has the solution

$$\phi(x, y) = \frac{k_1 - k_0}{2} x + \frac{k_1 + k_0}{2}.$$

A complex potential function for ϕ is

$$\mathbf{\Omega}(z) = \frac{k_1 - k_0}{2} z + \frac{k_1 + k_0}{2}.$$

Exercises 3.4

3. If we set $z = x + iy$, then

$$f(z) = \frac{1}{z} = \frac{1}{x + iy} \cdot \frac{x - iy}{x - iy} = \frac{x}{x^2 + y^2} - i\frac{y}{x^2 + y^2}.$$

This gives the families of level curves

$$\frac{x}{x^2 + y^2} = c_1 \quad \text{and} \quad -\frac{y}{x^2 + y^2} = c_2.$$

For the first family, if $c_1 \neq 0$, then we complete the square in the variable x.

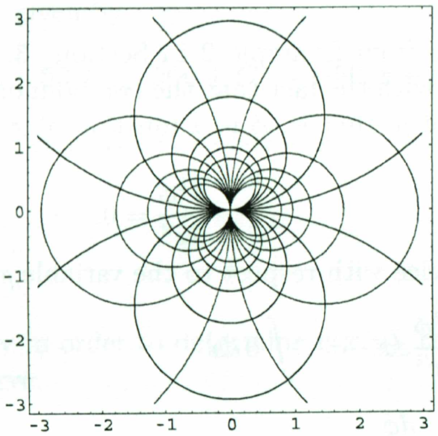
$$\begin{aligned}
\frac{x}{x^2 + y^2} &= c_1 \\
\frac{x}{c_1} &= x^2 + y^2 \\
\frac{1}{4c_1^2} &= x^2 - \frac{1}{c_1}x + \frac{1}{4c_1^2} + y^2 \\
\left(\frac{1}{2c_1}\right)^2 &= \left(x - \frac{1}{2c_1}\right)^2 + y^2.
\end{aligned}$$

Therefore, the level curves $x/(x^2 + y^2) = c_1$, $c_1 \neq 0$ are circles centered at $z = \frac{1}{2c_1}$ with radius $\left|\frac{1}{2c_1}\right|$. If $c_1 = 0$, then the level curve consists of the single point $z = 0$. For the second family, if $c_2 \neq 0$, then we complete the square in the variable y.

$$-\frac{y}{x^2 + y^2} = c_2$$

$$
\begin{aligned}
-\frac{y}{c_2} &= x^2 + y^2 \\
\frac{1}{4c_2^2} &= x^2 + y^2 + \frac{1}{c_2}y + \frac{1}{4c_2^2} \\
\left(\frac{1}{2c_1}\right)^2 &= x^2 + \left(y + \frac{1}{2c_2}\right)^2.
\end{aligned}
$$

Therefore, the level curves $-y/(x^2+y^2) = c_2$, $c_2 \neq 0$ are circles centered at $z = -\frac{1}{2c_2}i$ with radius $\left|\frac{1}{2c_2}\right|$. If $c_2 = 0$, then the level curve consists of the single point $z = 0$. Since $f(z) = 1/z$ is analytic for all $z \neq 0$, these families of circles are orthogonal at all points $z \neq 0$. In the following figure, the level curves $x/(x^2+y^2) = c_1$ are shown in color and the level curves $-y/(x^2+y^2) = c_2$ are shown in black.



Orthogonal families for $f(z) = 1/z$.

7. We identify $u(x,y) = e^{-x}\cos y$ and $v(x,y) = -e^{-x}\sin y$. First we use implicit differentiation to find the slope of the level curve $e^{-x}\cos y = c_1$.

$$
\begin{aligned}
\frac{d}{dx}\left(e^{-x}\cos y\right) &= \frac{d}{dx}c_1 \\
-e^{-x}\cos y + e^{-x}\frac{d}{dx}\cos y &= 0 \quad \leftarrow \text{product rule} \\
-e^{-x}\cos y - e^{-x}\sin y\,\frac{dy}{dx} &= 0 \quad \leftarrow \text{chain rule.}
\end{aligned}
$$

Solving for the derivative in the last equation yields

$$\frac{dy}{dx} = -\frac{\cos y}{\sin y}.$$

Now we use implicit differentiation to find the slope of the level curve $-e^{-x}\sin y = c_2$.

$$\frac{d}{dx}\left(-e^{-x}\sin y\right) = \frac{d}{dx}c_2$$

$$\begin{aligned}
-e^{-x}\sin y + e^{-x}\frac{d}{dx}\sin y &= 0 && \leftarrow \text{product rule} \\
-e^{-x}\sin y + e^{-x}\cos y\,\frac{dy}{dx} &= 0 && \leftarrow \text{chain rule.}
\end{aligned}$$

Solving for the derivative in the last equation yields

$$\frac{dy}{dx} = \frac{\sin y}{\cos y}.$$

The product of the two slopes is

$$-\frac{\cos y}{\sin y}\cdot\frac{\sin y}{\cos y} = -1.$$

Therefore, the level curves are orthogonal.

11. **(a)** We repeat the procedure from Example 2 in Section 3.4 with appropriate modifications. The shape of the domain D along with the fact that the two boundary conditions are constant suggest that a solution ϕ is independent of y, that is, ϕ is a function of x alone. With this assumption Laplace's equation becomes

$$\frac{d^2\phi}{dx^2} = 0.$$

We integrate this equation twice with respect to the variable x.

$$\begin{aligned}
\int \frac{d^2\phi}{dx^2}dx &= \int 0\,dx \\
\frac{d\phi}{dx} &= a && \leftarrow \text{after first integration} \\
\int \frac{d\phi}{dx}dx &= \int a\,dx \\
\phi(x) &= ax + b. && \leftarrow \text{after second integration}
\end{aligned}$$

The boundary conditions $\phi(0, y) = 50$ and $\phi(1, y) = 0$ allow us to solve for the coefficients a and b. In particular

$$\phi(0, y) = a(0) + b = 50 \quad \Longrightarrow \quad b = 50.$$

Given $b = 50$ we then have

$$\phi(1, y) = a(1) + 50 = 0 \quad \Longrightarrow \quad a = -50.$$

Therefore, $\phi(x, y) = -50x + 50$.

(b) If $\psi(x, y)$ is a harmonic conjugate, then ϕ and ψ must satisfy the Cauchy-Riemann equations, and so we must have

$$\begin{aligned}
\frac{\partial\phi}{\partial x} &= -50 = \frac{\partial\psi}{\partial y} \\
\frac{\partial\phi}{\partial y} &= 0 = -\frac{\partial\psi}{\partial x}.
\end{aligned}$$

Partial integration of the first equation with respect to the variable y gives

$$\psi(x, y) = -\int 50\, dy = -50y + h(x).$$

The partial derivative with respect to x of this last equation is

$$\frac{\partial \psi}{\partial x} = h'(x).$$

When this result is substituted into the second Cauchy-Riemann equation we obtain

$$\begin{aligned} 0 &= -h'(x) \\ h(x) &= C \end{aligned}$$

where C is a real constant. Therefore, $\psi(x, y) = -50y + C$ is a harmonic conjugate of $\phi(x, y)$. A complex potential function is then given by

$$\boldsymbol{\Omega}(z) = -50x + 50 + i(-50y + C) = -50(x + iy) + 50 + iC.$$

Replacing $x + iy$ with the symbol z gives $\boldsymbol{\Omega}(z) = -50z + 50 + iC$.

Focus on Concepts

15. Set $z = x + iy$ in $f(z)$ and simplify in order to determine $v(x, y)$, then factor the equation $v(x, y) = 0$ in order to determine the level curve.