

Description of nonTrivialIndicialOperators

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Basic definitions and notation in this document follow SST([1]).

definition 1. ([1] p.68) For a vector $\mathbf{u} \in \mathbb{N}^n$, define an element $[\theta]_{\mathbf{u}} \in \mathbb{C}[\theta]$ by

$$[\theta]_{\mathbf{u}} = \prod_{i=1}^n \prod_{j=0}^{u_i-1} (\theta_i - j).$$

definition 2. ([1] p.158) Let $A \in \mathbb{Z}^{d \times n}$ be a homogeneous matrix and let $\beta \in \mathbb{C}^d$ be a parameter vector. Choose a weight vector $w \in \mathbb{R}^n$ that is generic for I_A . Let $V = V(\text{find}_w(H_A(\beta)))$. Then V is the set of all fake exponents of $H_A(\beta)$ with respect to w . For each $p \in V$, define a subset $S_p(\text{in}_w(I_A))$ of $S(\text{in}_w(I_A))$ by

$$S_p(\text{in}_w(I_A)) = \{(\mathbf{a}, \sigma) \in S(\text{in}_w(I_A)) \mid \langle \theta_i - a_i \mid i \notin \sigma \rangle + \langle A\theta - \beta \rangle = \langle \theta_1 - p_1, \dots, \theta_n - p_n \rangle\}.$$

Furthermore, define a subset $S_\beta(\text{in}_w(I_A))$ of $S(\text{in}_w(I_A))$ by

$$S_\beta(\text{in}_w(I_A)) = \bigcup_{p \in V} S_p(\text{in}_w(I_A)).$$

Finally, define

$$E_\beta(\text{in}_w(I_A)) = E(\text{in}_w(I_A)) \cap S_\beta(\text{in}_w(I_A)).$$

definition 3. Let $A \in \mathbb{Z}^{d \times n}$ be a homogeneous matrix, and take a weight vector $w \in \mathbb{R}^n$ that is generic for I_A . Let $\beta \in \mathbb{C}^d$ be a parameter vector satisfying $E_\beta(\text{in}_w(I_A)) \neq \emptyset$, and take any $(\mathbf{a}, \sigma) \in E_\beta(\text{in}_w(I_A))$. Then, for any $f_{(\mathbf{a}, \sigma)} \in (\mathbb{C}^d)^* \setminus \{0\}$ which vanishes on the subspace $\mathbb{C} \cdot \langle \mathbf{a}_j \mid j \in \sigma \rangle$, we define

$$g_{(\mathbf{a}, \sigma)} := f_{(\mathbf{a}, \sigma)}\left(\sum_{i=1}^n \mathbf{a}_i \theta_i - \beta\right) := \sum_{i=1}^n f_{(\mathbf{a}, \sigma)}(\mathbf{a}_i) \theta_i - f_{(\mathbf{a}, \sigma)}(\beta) \in \mathbb{C}[\theta].$$

Then $g_{(\mathbf{a}, \sigma)}$ vanishes on (\mathbf{a}, σ) .

definition 4. Let $A \in \mathbb{Z}^{d \times n}$ be a homogeneous matrix, and take a weight vector $w \in \mathbb{R}^n$ that is generic for I_A . Let $\beta \in \mathbb{C}^d$ be a parameter vector satisfying $E_\beta(\text{in}_w(I_A)) \neq \emptyset$, and take any $(\mathbf{a}, \sigma) \in E_\beta(\text{in}_w(I_A))$. In what follows, we fix an element

$$q_{(\mathbf{a}, \sigma)} \in (\widetilde{\text{in}}_w(I_A) : [\theta]_{\mathbf{a}} g_{(\mathbf{a}, \sigma)}) \setminus (\widetilde{\text{in}}_w(I_A) : [\theta]_{\mathbf{a}}),$$

which is uniquely determined by A , w , β , and (\mathbf{a}, σ) .

Let

$$V_{\mathbf{a}} = \bigcup_{i=1}^n \bigcup_{j=0}^{a_i-1} \{v \in \mathbb{C}^n \mid v_i = j\}.$$

Then $\langle [\theta]_{\mathbf{a}} \rangle = \mathbb{I}(V_{\mathbf{a}})$. For any standard pair $(\mathbf{b}, \tau) \in S(\text{in}_w(I_A))$, we identify (\mathbf{b}, τ) with the set of vectors

$$(\mathbf{b}, \tau) = \{\mathbf{b}' \in \mathbb{C}^n \mid b'_i = b_i \ (\forall i \notin \tau)\} \subseteq \mathbb{C}^n.$$

We define

$$U_{\mathbf{a}, g(\mathbf{a}, \sigma)} = \{u \in S(\text{in}_w(I_A)) \setminus ((\mathbf{a}, \sigma) \cup V_{\mathbf{a}}) \subseteq \mathbb{C}^n \mid g_{(\mathbf{a}, \sigma)}(u) \neq 0\},$$

and

$$S_{U_{\mathbf{a}, g(\mathbf{a}, \sigma)}} = \{(\mathbf{b}, \tau) \in S(\text{in}_w(I_A)) \setminus \{(\mathbf{a}, \sigma)\} \mid \exists u \in U_{\mathbf{a}, g(\mathbf{a}, \sigma)} \text{ such that } u \in (\mathbf{b}, \tau)\}.$$

If $S_{U_{\mathbf{a}, g(\mathbf{a}, \sigma)}} = \emptyset$, then we define $q_{(\mathbf{a}, \sigma)} = 1$.

If $U_{\mathbf{a}, g(\mathbf{a}, \sigma)} \neq \emptyset$, define

$$S_{U_{\mathbf{a}, g(\mathbf{a}, \sigma)}} = \{(\mathbf{b}, \tau) \in S(\text{in}_w(I_A)) \setminus \{(\mathbf{a}, \sigma)\} \mid \exists u \in U_{\mathbf{a}, g(\mathbf{a}, \sigma)} \text{ such that } u \in (\mathbf{b}, \tau)\}.$$

For each $(\mathbf{b}, \tau) \in S_{U_{\mathbf{a}, g(\mathbf{a}, \sigma)}}$, set

$$j = \begin{cases} \min(\sigma \setminus \tau), & \text{if } \sigma \not\subseteq \tau, \\ \min\{k \in (\{1, \dots, n\} \setminus \tau) \mid a_k \neq b_k\}, & \text{if } \sigma \subseteq \tau, \end{cases}$$

and define $l_{(\mathbf{b}, \tau)} = \theta_j - b_j$. Now let

$$\mathcal{L} = \{l_{(\mathbf{b}, \tau)} \in \mathbb{C}[\theta] \mid (\mathbf{b}, \tau) \in S_{U_{\mathbf{a}, g(\mathbf{a}, \sigma)}}\},$$

and define

$$q_{(\mathbf{a}, \sigma)} = \prod_{l_{(\mathbf{b}, \tau)} \in \mathcal{L}} l_{(\mathbf{b}, \tau)}.$$

Then $q_{(\mathbf{a}, \sigma)}$ is uniquely determined by A , w , β , and (\mathbf{a}, σ) , and it satisfies

$$q_{(\mathbf{a}, \sigma)} \in (\widetilde{\text{in}}_w(I_A) : [\theta]_{\mathbf{a}} g_{(\mathbf{a}, \sigma)}) \setminus (\widetilde{\text{in}}_w(I_A) : [\theta]_{\mathbf{a}}).$$

definition 5. Let $A \in \mathbb{Z}^{d \times n}$ be a homogeneous matrix and let $\beta \in \mathbb{C}^d$ be a parameter vector. Choose a weight vector $w \in \mathbb{R}^n$ that is generic for I_A . For a monomial $m \in D \cdot \text{in}_w(I_A)$, define the operation $(*)$ as follows:

$(*)$ replace m by m' of minimal w -weight among all monomials satisfying $m \equiv m' \pmod{D \cdot I_A}$.

Algorithm 6.

Input: $A = (\mathbf{a}_1, \dots, \mathbf{a}_n) \in \mathbb{Z}^{d \times n}$: a homogeneous matrix, $w \in \mathbb{R}^n$: a weight vector generic for I_A ,

$\beta \in \mathbb{C}^d$: a parameter vector satisfying $E_{\beta}(\text{in}_w(I_A)) \neq \emptyset$.

(If w is not generic for $H_A(\beta)$, replace w with another weight vector w' such that

$\text{in}_{w'}(I_A) = \text{in}_w(I_A)$ and w' is generic for $H_A(\beta)$.)

Output: the distraction \tilde{I} of a D -ideal I generated by some elements of $\text{in}_{(-w,w)}(H_A(\beta))$.

Let I be the zero ideal in D .

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while( $E_\beta(\text{in}_w(I_A)) \neq \emptyset$ ) {
  Take any element  $(\mathbf{a}, \sigma)$  from  $E_\beta(\text{in}_w I_A)$ .
   $i := 0$ 
   $T := \emptyset$ 
  Take any linear form  $f_{(\mathbf{a}, \sigma)}^{(i)} \in (\mathbb{C}^d)^*$  such that  $f_{(\mathbf{a}, \sigma)}^{(i)}(\mathbf{a}_j) = 0$  for all  $j \in \sigma$  and  $f_{(\mathbf{a}, \sigma)}^{(i)}(\mathbf{a}_k) \neq 0$  for
  every  $k \notin \sigma$ .
  while( $i \neq d - |\sigma|$ ) {
    Apply the operation  $(*)$  to each term of  $q_{(\mathbf{a}, \sigma)}^{(i)}[\theta] \mathbf{a} g_{(\mathbf{a}, \sigma)}^{(i)}$ , and denote the resulting polynomial by
     $h_{(\mathbf{a}, \sigma)}$ .
     $T := T \cup \{t \in \{1, \dots, n\} \setminus \sigma \mid f_{(\mathbf{a}, \sigma)}^{(i)}(\mathbf{a}_t) q_{(\mathbf{a}, \sigma)}^{(i)}[\theta] \mathbf{a} + 1_t \equiv \text{in}_{(-w,w)}(h_{(\mathbf{a}, \sigma)}) \pmod{D \cdot I_A}\}$ 
    while(there exists  $c \in \mathbb{C}$  such that  $(\text{in}_{(-w,w)}(h_{(\mathbf{a}, \sigma)})) g_{(\mathbf{a}, \sigma)}^{(i)} + c \cdot \text{in}_{(-w,w)}(h_{(\mathbf{a}, \sigma)}) \in D \cdot \text{in}_w I_A$ ) {
      Apply the operation  $(*)$  to each term of  $(\text{in}_{(-w,w)}(h_{(\mathbf{a}, \sigma)})) g_{(\mathbf{a}, \sigma)}^{(i)}$  that are not scalar multiples
      of  $\text{in}_{(-w,w)}(h_{(\mathbf{a}, \sigma)})$ , and denote the resulting polynomial by  $h'_{(\mathbf{a}, \sigma)}$ .
       $h_{(\mathbf{a}, \sigma)} := h'_{(\mathbf{a}, \sigma)} + c \cdot h_{(\mathbf{a}, \sigma)}$ 
    }
     $I := I + \langle \text{in}_{(-w,w)}(h_{(\mathbf{a}, \sigma)}) \rangle$ 
     $i := i + 1$ 
    if( $i \neq d - |\sigma|$ ) {
      Take any linear form  $f_{(\mathbf{a}, \sigma)}^{(i)} \in (\mathbb{C}^d)^*$  such that  $f_{(\mathbf{a}, \sigma)}^{(i)}(\mathbf{a}_j) = 0$  for all  $j \in \sigma \cup T$  and  $f_{(\mathbf{a}, \sigma)}^{(i)}(\mathbf{a}_k) \neq 0$ 
      for every  $k \notin \sigma \cup T$ .
    }
  }
   $E_\beta(\text{in}_w(I_A)) := E_\beta(\text{in}_w(I_A)) \setminus \{(\mathbf{a}, \sigma)\}$ 
}
return( $\tilde{I}$ )

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The function `nonTrivialIndicialOperators` returns a list of generators of the output ideal \tilde{I} in Algorithm 6.

Reference

- [1] Mutsumi Saito, Bernd Sturmfels, Nobuki Takayama, “*Gröbner Deformations of Hypergeometric Differential Equations*”, Springer, Algorithms and Computation in Mathematics, Volume 6, 2000.