Description of nonTrivialIndicialOperators

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Basic definitions and notation in this document follow SST([1]).

definition 1. ([1] p.68) For a vector $\boldsymbol{u} \in \mathbb{N}^n$, define an element $[\theta]_{\boldsymbol{u}} \in \mathbb{C}[\theta]$ by

$$[\theta]_{\boldsymbol{u}} = \prod_{i=1}^{n} \prod_{j=0}^{u_i-1} (\theta_i - j).$$

definition 2. ([1] p.158) Let $A \in \mathbb{Z}^{d \times n}$ be a homogeneous matrix and let $\beta \in \mathbb{C}^d$ be a parameter vector. Choose a weight vector $w \in \mathbb{R}^n$ that is generic for I_A. Let $V = V(\text{find}_w(H_A(\beta)))$. Then V is the set of all fake exponents of $H_A(\beta)$ with respect to w. For each $p \in V$, define a subset $S_p(\text{in}_w(I_A))$ of $S(\text{in}_w(I_A))$ by

$$S_p(\text{in}_w(I_A)) = \{(\boldsymbol{a}, \sigma) \in S(\text{in}_w(I_A)) \mid \langle \theta_i - a_i \mid i \notin \sigma \rangle + \langle A\theta - \beta \rangle = \langle \theta_1 - p_1, ..., \theta_n - p_n \rangle\}.$$

Furthermore, define a subset $S_\beta(\text{in}_w(I_A))$ of $S(\text{in}_w(I_A))$ by

$$S_\beta(\text{in}_w(I_A)) = \bigcup_{p \in V} S_p(\text{in}_w(I_A)).$$

Finally, define

$$E_\beta(\text{in}_w(I_A)) = E(\text{in}_w(I_A)) \cap S_\beta(\text{in}_w(I_A)).$$

definition 3. Let $A \in \mathbb{Z}^{d \times n}$ be a homogeneous matrix, and take a weight vector $w \in \mathbb{R}^n$ that is generic for I_A. Let $\beta \in \mathbb{C}^d$ be a parameter vector satisfying $E_\beta(\text{in}_w(I_A)) \neq \emptyset$, and take any $(\boldsymbol{a}, \sigma) \in E_\beta(\text{in}_w(I_A))$. Then, for any $f_{(\boldsymbol{a}, \sigma)} \in (\mathbb{C}^d)^* \setminus \{0\}$ which vanishes on the subspace $\mathbb{C} \cdot \langle \boldsymbol{a}_j \mid j \in \sigma \rangle$, we define

$$g_{(\boldsymbol{a}, \sigma)} := f_{(\boldsymbol{a}, \sigma)}(\sum_{i=1}^{n} \boldsymbol{a}_i \theta_i - \beta) := \sum_{i=1}^{n} f_{(\boldsymbol{a}, \sigma)}(\boldsymbol{a}_i)\theta_i - f_{(\boldsymbol{a}, \sigma)}(\beta) \in \mathbb{C}[\theta].$$

Then $g_{(\boldsymbol{a}, \sigma)}$ vanishes on $(\boldsymbol{a}, \sigma)$.

definition 4. Let $A \in \mathbb{Z}^{d \times n}$ be a homogeneous matrix, and take a weight vector $w \in \mathbb{R}^n$ that is generic for I_A. Let $\beta \in \mathbb{C}^d$ be a parameter vector satisfying $E_\beta(\text{in}_w(I_A)) \neq \emptyset$, and take any $(\boldsymbol{a}, \sigma) \in E_\beta(\text{in}_w(I_A))$. In what follows, we fix an element

$$q_{(\boldsymbol{a}, \sigma)} \in (\widetilde{\text{in}}_w(I_A) : [\theta]_{\boldsymbol{a}} g_{(\boldsymbol{a}, \sigma)}) \setminus (\widetilde{\text{in}}_w(I_A) : [\theta]_{\boldsymbol{a}}),$$

which is uniquely determined by A, w, β, and $(\boldsymbol{a}, \sigma)$.

Let
$$V_{\boldsymbol{a}} = \bigcup_{i=1}^{n} \bigcup_{j=0}^{a_i-1} \{v \in \mathbb{C}^n \mid v_i = j\}.$$

Then $\langle [\theta]_{\boldsymbol{a}} \rangle = \mathbb{I}(V_{\boldsymbol{a}})$. For any standard pair $(\boldsymbol{b}, \tau) \in S(\mathrm{in}_w(I_A))$, we identify $(\boldsymbol{b}, \tau)$ with the set of vectors

$$(\boldsymbol{b}, \tau) = \{\boldsymbol{b}' \in \mathbb{C}^n \mid b'_i = b_i \ (\forall i \notin \tau)\} \subseteq \mathbb{C}^n.$$

We define

$$U_{\boldsymbol{a}, g_{(\boldsymbol{a}, \sigma)}} = \{u \in S(\mathrm{in}_w(I_A)) \setminus ((\boldsymbol{a}, \sigma) \cup V_{\boldsymbol{a}}) \subseteq \mathbb{C}^n \mid g_{(\boldsymbol{a}, \sigma)}(u) \neq 0\},$$

and

$$S_{U_{\boldsymbol{a}, g_{(\boldsymbol{a}, \sigma)}}} = \{(\boldsymbol{b}, \tau) \in S(\mathrm{in}_w(I_A)) \setminus \{(\boldsymbol{a}, \sigma)\} \mid \exists u \in U_{\boldsymbol{a}, g_{(\boldsymbol{a}, \sigma)}} \text{ such that } u \in (\boldsymbol{b}, \tau)\}.$$

If $S_{U_{\boldsymbol{a}, g_{(\boldsymbol{a}, \sigma)}}} = \emptyset$, then we define $q_{(\boldsymbol{a}, \sigma)} = 1$.

If $U_{\boldsymbol{a}, g_{(\boldsymbol{a}, \sigma)}} \neq \emptyset$, define

$$S_{U_{\boldsymbol{a}, g_{(\boldsymbol{a}, \sigma)}}} = \{(\boldsymbol{b}, \tau) \in S(\mathrm{in}_w(I_A)) \setminus \{(\boldsymbol{a}, \sigma)\} \mid \exists u \in U_{\boldsymbol{a}, g_{(\boldsymbol{a}, \sigma)}} \text{ such that } u \in (\boldsymbol{b}, \tau)\}.$$

For each $(\boldsymbol{b}, \tau) \in S_{U_{\boldsymbol{a}, g_{(\boldsymbol{a}, \sigma)}}}$, set

$$j = \begin{cases} \min(\sigma \setminus \tau), & \text{if } \sigma \nsubseteq \tau, \\ \min\{k \in (\{1, \ldots, n\} \setminus \tau) \mid a_k \neq b_k\}, & \text{if } \sigma \subseteq \tau, \end{cases}$$

and define $l_{(\boldsymbol{b}, \tau)} = \theta_j - b_j$. Now let

$$\mathcal{L} = \{l_{(\boldsymbol{b}, \tau)} \in \mathbb{C}[\theta] \mid (\boldsymbol{b}, \tau) \in S_{U_{\boldsymbol{a}, g_{(\boldsymbol{a}, \sigma)}}}\},$$

and define

$$q_{(\boldsymbol{a}, \sigma)} = \prod_{l_{(\boldsymbol{b}, \tau)} \in \mathcal{L}} l_{(\boldsymbol{b}, \tau)}.$$

Then $q_{(\boldsymbol{a}, \sigma)}$ is uniquely determined by A, w, β, and $(\boldsymbol{a}, \sigma)$, and it satisfies

$$q_{(\boldsymbol{a}, \sigma)} \in (\widetilde{\mathrm{in}}_w(I_A) : [\theta]_{\boldsymbol{a}} g_{(\boldsymbol{a}, \sigma)}) \setminus (\widetilde{\mathrm{in}}_w(I_A) : [\theta]_{\boldsymbol{a}}).$$

definition 5. Let $A \in \mathbb{Z}^{d \times n}$ be a homogeneous matrix and let $\beta \in \mathbb{C}^d$ be a parameter vector. Choose a weight vector $w \in \mathbb{R}^n$ that is generic for I_A. For a monomial $m \in D \cdot \mathrm{in}_w(I_A)$, define the operation $(*)$ as follows:

$(*)$ replace m by m' of minimal w-weight among all monomials satisfying $m \equiv m' \pmod{D \cdot I_A}$.

Algorithm 6.
Input: $A = (\boldsymbol{a}_1, ..., \boldsymbol{a}_n) \in \mathbb{Z}^{d \times n}$: a homogeneous matrix, $w \in \mathbb{R}^n$: a weight vector generic for I_A,
$\quad \beta \in \mathbb{C}^d$: a parameter vector satisfying $E_\beta(\mathrm{in}_w(I_A)) \neq \emptyset$.
$\quad$ (If w is not generic for $H_A(\beta)$, replace w with another weight vector w' such that
$\quad \mathrm{in}_{w'}(I_A) = \mathrm{in}_w(I_A)$ and w' is generic for $H_A(\beta)$.)

Output: the distraction $\widetilde{I}$ of a D-ideal I generated by some elements of $\text{in}_{(-w,w)}(H_A(\beta))$.

Let I be the zero ideal in D.

while($E_\beta(\text{in}_w(I_A)) \neq \emptyset$){

    Take any element $(\boldsymbol{a}, \sigma)$ from $E_\beta(\text{in}_w I_A)$.

    $i := 0$

    $T := \emptyset$

    Take any linear form $f_{(\boldsymbol{a},\sigma)}^{(i)} \in (\mathbb{C}^d)^*$ such that $f_{(\boldsymbol{a},\sigma)}^{(i)}(\boldsymbol{a}_j) = 0$ for all $j \in \sigma$ and $f_{(\boldsymbol{a},\sigma)}^{(i)}(\boldsymbol{a}_k) \neq 0$ for every $k \notin \sigma$.

    while($i \neq d - |\sigma|$){

        Apply the operation $(*)$ to each term of $q_{(\boldsymbol{a},\sigma)}^{(i)}[\theta]_{\boldsymbol{a}} g_{(\boldsymbol{a},\sigma)}^{(i)}$, and denote the resulting polynomial by $h_{(\boldsymbol{a},\sigma)}$.

        $T := T \cup \{t \in \{1, ..., n\} \setminus \sigma \mid f_{(\boldsymbol{a},\sigma)}^{(i)}(\boldsymbol{a}_t) q_{(\boldsymbol{a},\sigma)}^{(i)}[\theta]_{\boldsymbol{a}+1_t} \equiv \text{in}_{(-w,w)}(h_{(\boldsymbol{a},\sigma)}) \pmod{D \cdot I_A}\}$

        while(there exists $c \in \mathbb{C}$ such that $(\text{in}_{(-w,w)}(h_{(\boldsymbol{a},\sigma)})) g_{(\boldsymbol{a},\sigma)}^{(i)} + c \cdot \text{in}_{(-w,w)}(h_{(\boldsymbol{a},\sigma)}) \in D \cdot \text{in}_w I_A$){

            Apply the operation $(*)$ to each term of $(\text{in}_{(-w,w)}(h_{(\boldsymbol{a},\sigma)})) g_{(\boldsymbol{a},\sigma)}^{(i)}$ that are not scalar multiples of $\text{in}_{(-w,w)}(h_{(\boldsymbol{a},\sigma)})$, and denote the resulting polynomial by $h'_{(\boldsymbol{a},\sigma)}$.

          $h_{(\boldsymbol{a},\sigma)} := h'_{(\boldsymbol{a},\sigma)} + c \cdot h_{(\boldsymbol{a},\sigma)}$

        }

        $I := I + \langle \text{in}_{(-w,w)}(h_{(\boldsymbol{a},\sigma)}) \rangle$

        $i := i + 1$

        if($i \neq d - |\sigma|$){

            Take any linear form $f_{(\boldsymbol{a},\sigma)}^{(i)} \in (\mathbb{C}^d)^*$ such that $f_{(\boldsymbol{a},\sigma)}^{(i)}(\boldsymbol{a}_j) = 0$ for all $j \in \sigma \cup T$ and $f_{(\boldsymbol{a},\sigma)}^{(i)}(\boldsymbol{a}_k) \neq 0$ for every $k \notin \sigma \cup T$.

        }

    }

    $E_\beta(\text{in}_w(I_A)) := E_\beta(\text{in}_w(I_A)) \setminus \{(\mathbf{a}, \sigma)\}$

}

return($\widetilde{I}$)

The function `nonTrivialIndicialOperators` returns a list of generators of the output ideal $\widetilde{I}$ in Algorithm 6.

Reference

[1] Mutsumi Saito, Bernd Sturmfels, Nobuki Takayama, "*Gröbner Deformations of Hypergeometric Differential Equations*", Springer, Algorithms and Computation in Mathematics, Volume 6, 2000.