Discription of nonTrivialIndicialOperators

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Basic definitions and notation in this document follow SST([1]).

definition 1. ([1] p.68) For a vector $\boldsymbol{u} \in \mathbb{N}^n$, define an element $[\theta]_{\boldsymbol{u}} \in \mathbb{C}[\theta]$ by

$$[\theta]_{\boldsymbol{u}} = \prod_{i=1}^{n} \prod_{j=0}^{u_i - 1} (\theta_i - j).$$

definition 2. ([1] p.158) Let $A \in \mathbb{Z}^{d \times n}$ be a homogeneous matrix and let $\beta \in \mathbb{C}^d$ be a parameter vector. Choose a weight vector $w \in \mathbb{R}^n$ that is generic for I_A. Let $V = V(\mathrm{find}_w(H_A(\beta)))$. Then V is the set of all fake exponents of $H_A(\beta)$ with respect to w. For each $p \in V$, define a subset $S_p(\mathrm{in}_w(I_A))$ of $S(\mathrm{in}_w(I_A))$ by

$$S_p(\mathrm{in}_w(I_A)) = \{(\boldsymbol{a}, \sigma) \in S(\mathrm{in}_w(I_A)) \mid \langle \theta_i - a_i \mid i \notin \sigma \rangle + \langle A\theta - \beta \rangle = \langle \theta_1 - p_1, ..., \theta_n - p_n \rangle\}.$$

Furthermore, define a subset $S_\beta(\mathrm{in}_w(I_A))$ of $S(\mathrm{in}_w(I_A))$ by

$$S_\beta(\mathrm{in}_w(I_A)) = \bigcup_{p \in V} S_p(\mathrm{in}_w(I_A)).$$

Finally, define

$$E_\beta(\mathrm{in}_w(I_A)) = E(\mathrm{in}_w(I_A)) \cap S_\beta(\mathrm{in}_w(I_A)).$$

definition 3. Let $A \in \mathbb{Z}^{d \times n}$be a homogeneous matrix and let $\beta \in \mathbb{C}^d$ be a parameter vector. Choose a weight vector $w \in \mathbb{R}^n$ that is generic for I_A. For a monomial $m \in D \cdot \mathrm{in}_w(I_A)$, define the operation $(*)$ as follows:

$(*)$ replace m by m' of minimal w-weight among all monomials satisfying $m \equiv m' \pmod{D \cdot I_A}$.

Algorithm 4.

Input: $A = (\boldsymbol{a}_1, ..., \boldsymbol{a}_n) \in \mathbb{Z}^{d \times n}$: a homogeneous matrix, $w \in \mathbb{R}^n$: a weight vector generic for I_A,
$\beta \in \mathbb{C}^d$: a parameter vector satisfying $E_\beta(\mathrm{in}_w(I_A)) \neq \emptyset$.
(If w is not generic for $H_A(\beta)$, replace w with another weight vector w' such that
$\mathrm{in}_{w'}(I_A) = \mathrm{in}_w(I_A)$ and w' is generic for $H_A(\beta)$.)

Output: the distraction $\widetilde{I}$ of a D-ideal I generated by some elements of $\mathrm{in}_{(-w,w)}(H_A(\beta))$.

Let I be the zero ideal in D.
while($E_\beta(\mathrm{in}_w(I_A)) \neq \emptyset$){
  Take any element $(\boldsymbol{a}, \sigma)$ from $E_\beta(\mathrm{in}_w I_A)$.

$i := 0$

$T := \emptyset$

Take any linear form $f^{(i)}_{(\boldsymbol{a},\sigma)} \in (\mathbb{C}^d)^*$ such that $f^{(i)}_{(\boldsymbol{a},\sigma)}(\boldsymbol{a}_j) = 0$ for all $j \in \sigma$ and $f^{(i)}_{(\boldsymbol{a},\sigma)}(\boldsymbol{a}_k) \neq 0$ for every $k \notin \sigma$.

while($i \neq d - |\sigma|$){

Apply the operation $(*)$ to each term of $q^{(i)}_{(\boldsymbol{a},\sigma)}[\theta]_{\boldsymbol{a}} g^{(i)}_{(\boldsymbol{a},\sigma)}$, and denote the resulting polynomial by $h_{(\boldsymbol{a},\sigma)}$.

$T := T \cup \{t \in \{1, ..., n\} \setminus \sigma \mid f^{(i)}_{(\boldsymbol{a},\sigma)}(\boldsymbol{a}_t) q^{(i)}_{(\boldsymbol{a},\sigma)}[\theta]_{\boldsymbol{a}+1_t} \equiv \mathrm{in}_{(-w,w)}(h_{(\boldsymbol{a},\sigma)}) \pmod{D \cdot I_A}\}$

while(there exists $c \in \mathbb{C}$ such that $(\mathrm{in}_{(-w,w)}(h_{(\boldsymbol{a},\sigma)})) g^{(i)}_{(\boldsymbol{a},\sigma)} + c \cdot \mathrm{in}_{(-w,w)}(h_{(\boldsymbol{a},\sigma)}) \in D \cdot \mathrm{in}_w I_A$){

Apply the operation $(*)$ to each term of $(\mathrm{in}_{(-w,w)}(h_{(\boldsymbol{a},\sigma)})) g^{(i)}_{(\boldsymbol{a},\sigma)}$ that are not scalar multiples of $\mathrm{in}_{(-w,w)}(h_{(\boldsymbol{a},\sigma)})$, and denote the resulting polynomial by $h'_{(\boldsymbol{a},\sigma)}$.

$h_{(\boldsymbol{a},\sigma)} := h'_{(\boldsymbol{a},\sigma)} + c \cdot h_{(\boldsymbol{a},\sigma)}$

}

$I := I + \langle \mathrm{in}_{(-w,w)}(h_{(\boldsymbol{a},\sigma)}) \rangle$

$i := i + 1$

if($i \neq d - |\sigma|$){

Take any linear form $f^{(i)}_{(\boldsymbol{a},\sigma)} \in (\mathbb{C}^d)^*$ such that $f^{(i)}_{(\boldsymbol{a},\sigma)}(\boldsymbol{a}_j) = 0$ for all $j \in \sigma \cup T$ and $f^{(i)}_{(\boldsymbol{a},\sigma)}(\boldsymbol{a}_k) \neq 0$ for every $k \notin \sigma \cup T$.

}

}

$E_\beta(\mathrm{in}_w(I_A)) := E_\beta(\mathrm{in}_w(I_A)) \setminus \{(\mathbf{a}, \sigma)\}$

}

return($\widetilde{I}$)

The function `nonTrivialIndicialOperators` returns a list of generators of the output ideal $\widetilde{I}$ in Algorithm4.

Reference

[1] Mutsumi Saito, Bernd Sturmfels, Nobuki Takayama, "*Gröbner Deformations of Hypergeometric Differential Equations*", Springer, Algorithms and Computation in Mathematics, Volume 6, 2000.