

Discription of nonTrivialIndicialOperators

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5 February 2026

Basic definitions and notation in this document follow SST([1]).

definition 1. ([1] p.68) For a vector $\mathbf{u} \in \mathbb{N}^n$, define an element $[\theta]_{\mathbf{u}} \in \mathbb{C}[\theta]$ by

$$[\theta]_{\mathbf{u}} = \prod_{i=1}^n \prod_{j=0}^{u_i-1} (\theta_i - j).$$

definition 2. ([1] p.158) Let $A \in \mathbb{Z}^{d \times n}$ be a homogeneous matrix and let $\beta \in \mathbb{C}^d$ be a parameter vector. Choose a weight vector $w \in \mathbb{R}^n$ that is generic for I_A . Let $V = V(\text{find}_w(H_A(\beta)))$. Then V is the set of all fake exponents of $H_A(\beta)$ with respect to w . For each $p \in V$, define a subset $S_p(\text{in}_w(I_A))$ of $S(\text{in}_w(I_A))$ by

$$S_p(\text{in}_w(I_A)) = \{(\mathbf{a}, \sigma) \in S(\text{in}_w(I_A)) \mid \langle \theta_i - a_i \mid i \notin \sigma \rangle + \langle A\theta - \beta \rangle = \langle \theta_1 - p_1, \dots, \theta_n - p_n \rangle\}.$$

Furthermore, define a subset $S_\beta(\text{in}_w(I_A))$ of $S(\text{in}_w(I_A))$ by

$$S_\beta(\text{in}_w(I_A)) = \bigcup_{p \in V} S_p(\text{in}_w(I_A)).$$

Finally, define

$$E_\beta(\text{in}_w(I_A)) = E(\text{in}_w(I_A)) \cap S_\beta(\text{in}_w(I_A)).$$

definition 3. Let $A \in \mathbb{Z}^{d \times n}$ be a homogeneous matrix and let $\beta \in \mathbb{C}^d$ be a parameter vector. Choose a weight vector $w \in \mathbb{R}^n$ that is generic for I_A . For a monomial $m \in D \cdot \text{in}_w(I_A)$, define the operation $(*)$ as follows:

$(*)$ replace m by m' of minimal w -weight among all monomials satisfying $m \equiv m' \pmod{D \cdot I_A}$.

Algorithm 4.

Input: $A = (\mathbf{a}_1, \dots, \mathbf{a}_n) \in \mathbb{Z}^{d \times n}$: a homogeneous matrix, $w \in \mathbb{R}^n$: a weight vector generic for I_A ,

$\beta \in \mathbb{C}^d$: a parameter vector satisfying $E_\beta(\text{in}_w(I_A)) \neq \emptyset$.

(If w is not generic for $H_A(\beta)$, replace w with another weight vector w' such that

$\text{in}_{w'}(I_A) = \text{in}_w(I_A)$ and w' is generic for $H_A(\beta)$.)

Output: the distraction \tilde{I} of a D -ideal I generated by some elements of $\text{in}_{(-w, w)}(H_A(\beta))$.

Let I be the zero ideal in D .

while($E_\beta(\text{in}_w(I_A)) \neq \emptyset$) {

 Take any element (\mathbf{a}, σ) from $E_\beta(\text{in}_w I_A)$.

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 $i := 0$ 
 $T := \emptyset$ 
Take any linear form  $f_{(\mathbf{a}, \sigma)}^{(i)} \in (\mathbb{C}^d)^*$  such that  $f_{(\mathbf{a}, \sigma)}^{(i)}(\mathbf{a}_j) = 0$  for all  $j \in \sigma$  and  $f_{(\mathbf{a}, \sigma)}^{(i)}(\mathbf{a}_k) \neq 0$  for every  $k \notin \sigma$ .
while( $i \neq d - |\sigma|$ ){
  Apply the operation  $(*)$  to each term of  $q_{(\mathbf{a}, \sigma)}^{(i)}[\theta] \mathbf{a} g_{(\mathbf{a}, \sigma)}^{(i)}$ , and denote the resulting polynomial by  $h_{(\mathbf{a}, \sigma)}$ .
   $T := T \cup \{t \in \{1, \dots, n\} \setminus \sigma \mid f_{(\mathbf{a}, \sigma)}^{(i)}(\mathbf{a}_t) q_{(\mathbf{a}, \sigma)}^{(i)}[\theta] \mathbf{a}_{+1_t} \equiv \text{in}_{(-w, w)}(h_{(\mathbf{a}, \sigma)}) \pmod{D \cdot I_A}\}$ 
  while(there exists  $c \in \mathbb{C}$  such that  $(\text{in}_{(-w, w)}(h_{(\mathbf{a}, \sigma)})) g_{(\mathbf{a}, \sigma)}^{(i)} + c \cdot \text{in}_{(-w, w)}(h_{(\mathbf{a}, \sigma)}) \in D \cdot \text{in}_w I_A$ ){
    Apply the operation  $(*)$  to each term of  $(\text{in}_{(-w, w)}(h_{(\mathbf{a}, \sigma)})) g_{(\mathbf{a}, \sigma)}^{(i)}$  that are not scalar multiples of  $\text{in}_{(-w, w)}(h_{(\mathbf{a}, \sigma)})$ , and denote the resulting polynomial by  $h'_{(\mathbf{a}, \sigma)}$ .
     $h_{(\mathbf{a}, \sigma)} := h'_{(\mathbf{a}, \sigma)} + c \cdot h_{(\mathbf{a}, \sigma)}$ 
  }
   $I := I + \langle \text{in}_{(-w, w)}(h_{(\mathbf{a}, \sigma)}) \rangle$ 
   $i := i + 1$ 
  if( $i \neq d - |\sigma|$ ){
    Take any linear form  $f_{(\mathbf{a}, \sigma)}^{(i)} \in (\mathbb{C}^d)^*$  such that  $f_{(\mathbf{a}, \sigma)}^{(i)}(\mathbf{a}_j) = 0$  for all  $j \in \sigma \cup T$  and  $f_{(\mathbf{a}, \sigma)}^{(i)}(\mathbf{a}_k) \neq 0$  for every  $k \notin \sigma \cup T$ .
  }
}
 $E_\beta(\text{in}_w(I_A)) := E_\beta(\text{in}_w(I_A)) \setminus \{(\mathbf{a}, \sigma)\}$ 
}
return( $\tilde{I}$ )

```

The function `nonTrivialIndicialOperators` returns a list of generators of the output ideal \tilde{I} in Algorithm 4.

Reference

- [1] Mutsumi Saito, Bernd Sturmfels, Nobuki Takayama, “*Gröbner Deformations of Hypergeometric Differential Equations*”, Springer, Algorithms and Computation in Mathematics, Volume 6, 2000.