Stokes Equations

Computer Session B6

Strong Form

The motion of a highly viscous incompressible fluid enclosed within a domain $\Omega \subset \mathbf{R}^2$ with boundary $\partial\Omega$ and outward unit normal $\boldsymbol{n}$ is governed by the Stokes equations: find the velocity $\boldsymbol{u} : \Omega \to \mathbf{R}^2$ and the pressure $p : \Omega \to \mathbf{R}$ such that

$$-\Delta \boldsymbol{u} + \nabla p = \boldsymbol{f}, \quad \text{in } \Omega \tag{1a}$$

$$\nabla \cdot \boldsymbol{u} = 0, \quad \text{in } \Omega \tag{1b}$$

$$\boldsymbol{u} = \boldsymbol{0}, \quad \text{on } \partial\Omega \tag{1c}$$

where $\boldsymbol{f}$ is a given volume force. We assume that p satisfies the normalization constraint

$$(p, 1) = 0 \tag{2}$$

Problem 1. ✰ Verify that $\boldsymbol{u} = [20xy^3,\ 5x^4 - 5y^4]$ and $p = 60x^2y - 20y^3 + 1$ satisfies (1a) and (1b).

Problem 2. ✰ Show that (1a) and (1b) can be equivalently written

$$\begin{aligned} -\nabla \cdot \boldsymbol{\sigma} &= \boldsymbol{f} \\ \boldsymbol{\sigma} &= 2\boldsymbol{\varepsilon}(\boldsymbol{u}) - p\boldsymbol{I} \\ \nabla \cdot \boldsymbol{u} &= 0 \end{aligned}$$

where $\boldsymbol{\varepsilon}(\boldsymbol{v}) = \frac{1}{2}(\nabla \boldsymbol{v} + (\nabla \boldsymbol{v})^T)$.

Weak Form

To derive the weak form of (1) we introduce the spaces

$$\boldsymbol{V} = \{\boldsymbol{v} \in [H^1(\Omega)]^2 : \boldsymbol{v}|_{\partial\Omega} = \boldsymbol{0}\} \tag{3}$$
$$Q = \{q \in L^2(\Omega) : (q, 1) = 0\} \tag{4}$$

for the velocity and the pressure, respectively.

Multiplying (1a) by a function $\boldsymbol{v} \in \boldsymbol{V}$ and integrating by parts gives

$$(\boldsymbol{f}, \boldsymbol{v}) = (-\Delta \boldsymbol{u}, \boldsymbol{v}) + (\nabla p, \boldsymbol{v}) \tag{5}$$
$$= (-\boldsymbol{n} \cdot \nabla \boldsymbol{u}, \boldsymbol{v})_{\partial\Omega} + (\nabla \boldsymbol{u} : \nabla \boldsymbol{v}) + (p, \boldsymbol{n} \cdot \boldsymbol{v})_{\partial\Omega} - (p, \nabla \cdot \boldsymbol{v}) \tag{6}$$

which, since $\boldsymbol{v} = \boldsymbol{0}$ on $\partial\Omega$, simplifies to

$$(\boldsymbol{f}, \boldsymbol{v}) = (\nabla \boldsymbol{u} : \nabla \boldsymbol{v}) - (p, \nabla \cdot \boldsymbol{v}) \tag{7}$$

Similarly, multiplying (1b) by a function $q \in Q$ immediately yields

$$-(\nabla \cdot \boldsymbol{u}, q) = 0 \tag{8}$$

where we have also changed the sign on the equation $\nabla \cdot \boldsymbol{u} = 0$.

One might ask why Q is the appropriate test space for the compressibility constraint $\nabla \cdot \boldsymbol{u} = 0$. After all, the functions in Q are somewhat peculiar since they all have a zero mean value. The reason is that it suffice to test $\nabla \cdot \boldsymbol{u} = 0$ against the functions in Q, since the weak form (8) is zero anyway for q equal to a constant. To see this let c be a constant and recall that $\boldsymbol{u} = \boldsymbol{0}$ on the boundary. Using partial integration we have

$$(\nabla \cdot \boldsymbol{u}, c) = (\boldsymbol{u} \cdot \boldsymbol{n}, c)_{\partial\Omega} + (\boldsymbol{u}, \nabla c) = 0 \tag{9}$$

since $\nabla c = 0$.

Hence, the weak form of (1) reads: Find $\boldsymbol{u} \in \boldsymbol{V}$ and $p \in Q$ such that

$$a(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) = (\boldsymbol{f}, \boldsymbol{v}), \quad \forall \boldsymbol{v} \in \boldsymbol{V} \tag{10a}$$
$$b(\boldsymbol{u}, q) = 0, \qquad \forall q \in Q \tag{10b}$$

where the bilinear forms $a(\cdot, \cdot)$ and $b(\cdot, \cdot)$ are defined by

$$a(\boldsymbol{u}, \boldsymbol{v}) = (\nabla \boldsymbol{u} : \nabla \boldsymbol{v}) \tag{11}$$
$$b(\boldsymbol{u}, q) = -(\nabla \cdot \boldsymbol{u}, q) \tag{12}$$

The existence and uniqueness of a solution $(\boldsymbol{u}, p)$ to the variational equation (10) follows from the coercivity of the bilinear form $a(\cdot,\cdot)$ on $\boldsymbol{V}$ and the fact that there is a constant $\beta > 0$ such that

$$\beta\|q\| \leq \sup_{\boldsymbol{v}\in\boldsymbol{V},\, \boldsymbol{v}\neq\boldsymbol{0}} \frac{(q, \nabla\cdot\boldsymbol{v})}{\|\nabla\boldsymbol{v}\|}, \quad \forall q \in Q \tag{13}$$

This result is called the *inf-sup* condition.

Finite Element Approximation

To approximate the velocity and pressure let $\mathcal{K} = \{K\}$ be a triangle mesh of Ω, and let $Q_h \subset Q$ and $V_h \subset V$ be finite dimensional subspaces on $\mathcal{K}$. The finite element approximation to (10) takes the form: find $\boldsymbol{u}_h \in \boldsymbol{V}_h$ and $p_h \in Q_h$ such that

$$a(\boldsymbol{u}_h, \boldsymbol{v}) + b(p_h, \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}), \quad \forall \boldsymbol{v} \in \boldsymbol{V}_h \tag{14a}$$

$$b(\boldsymbol{u}_h, q) = 0, \qquad \forall q \in Q_h \tag{14b}$$

In practice the velocity components are always approximated by a single finite element space V_h, and the velocity space $\boldsymbol{V}_h$ takes the form $\boldsymbol{V}_h = V_h \times V_h$. Thus, if $\{\varphi_i\}_{i=1}^n$ is a basis for V_h, then the discrete velocity components are given by

$$u_{1,h} = \sum_{i=1}^{n} \xi_i\varphi_i, \qquad u_{2,h} = \sum_{i=1}^{n} \eta_i\varphi_i \tag{15}$$

with unknown coefficients ξ_i and η_i. A basis for $\boldsymbol{V}_h$ is given by the set

$$\left\{ \begin{bmatrix}\varphi_1\\0\end{bmatrix}, \begin{bmatrix}\varphi_2\\0\end{bmatrix}, \dots, \begin{bmatrix}\varphi_n\\0\end{bmatrix}, \begin{bmatrix}0\\\varphi_1\end{bmatrix}, \begin{bmatrix}0\\\varphi_2\end{bmatrix}, \dots, \begin{bmatrix}0\\\varphi_n\end{bmatrix} \right\} \tag{16}$$

The discrete pressure is constructed similarly. That is, if $\{\psi_i\}_{i=1}^m$ is a basis for Q_h, then

$$p_h = \sum_{i=1}^{m} \theta_i\psi_i \tag{17}$$

with unknown coefficients θ_i.

The particular ordering of the basis functions in the definition (16) implies that the linear system arising from the finite element method can be written in block form as

$$\begin{bmatrix} \boldsymbol{A} & \boldsymbol{0} & \boldsymbol{B}_1 \\ \boldsymbol{0} & \boldsymbol{A} & \boldsymbol{B}_2 \\ \boldsymbol{B}_1^T & \boldsymbol{B}_2^T & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{\xi} \\ \boldsymbol{\eta} \\ \boldsymbol{\theta} \end{bmatrix} = \begin{bmatrix} \boldsymbol{F}_1 \\ \boldsymbol{F}_2 \\ \boldsymbol{0} \end{bmatrix} \tag{18}$$

where

$$A_{ij} = (\nabla\varphi_j, \nabla\varphi_i), \quad i, j = 1, \dots, n \tag{19}$$

$$B_{d,ij} = -(\psi_j, \partial_{x_d}\varphi_i), \quad i = 1, \dots, n,\ j = 1, \dots, m,\ d = 1, 2 \tag{20}$$

$$F_{d,i} = (f_d, \varphi_i), \quad i = 1, \dots, n,\ d = 1, 2 \tag{21}$$

The Discrete Inf-Sup Condition

The linear system (18) is often written as

$$\begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{B}^T & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{U} \\ \boldsymbol{P} \end{bmatrix} = \begin{bmatrix} \boldsymbol{F} \\ \boldsymbol{0} \end{bmatrix} \tag{22}$$

with the obvious block notation. The solvability of this linear system depends on if one can establish the inf-sup condition on the subspaces $\boldsymbol{V}_h$ and Q_h. In other words there must exist a constant $\beta_h > 0$ such that

$$\beta_h \|q\| \leq \sup_{\boldsymbol{v} \in \boldsymbol{V}_h,\, \boldsymbol{v} \neq \boldsymbol{0}} \frac{(q, \nabla \cdot \boldsymbol{v})}{\|\nabla \boldsymbol{v}\|}, \quad \forall q \in Q_h \tag{23}$$

Further, β_h must not depend on the mesh size h such that it vanishes in the limit $h \to 0$. The problem is that even if the inf-sup condition holds on $\boldsymbol{V}$ and Q it need not hold on the subspaces $\boldsymbol{V}_h$ and Q_h. If it does not then the discrete solution $(\boldsymbol{u}_h, p_h)$ is not unique. This manifests itself through the appearance of artifact pressure solutions, called *spurious pressure modes*, within p_h. Spurious pressure modes are intimately related to the matrix B and occurs whenever the null space of B has a dimension larger than one. Indeed these modes are the basis vectors for the null space. The conclusion is that one must be careful when choosing the discrete spaces $\boldsymbol{V}_h$ and Q_h so that they are compatible in the sense of (23). Unfortunately, this is not easy to decide without a lot of analysis. For example, choosing both $\boldsymbol{V}_h$ and Q_h as spaces of continuous piecewise polynomials of the same degree k will violate the inf-sup condition, whereas the choice Q_h the space of piecewise constants and $\boldsymbol{V}_h$ the space of continuous piecewise quadratics will not. Pairs of velocity-pressure spaces on which the inf-sup condition do hold are called inf-sup stable.

Problem 3. ☆ Show that the linear system (22) can be solved in two steps:

1. Solve $\boldsymbol{B}^T\boldsymbol{A}^{-1}\boldsymbol{B}\boldsymbol{P} = \boldsymbol{B}^T\boldsymbol{A}^{-1}\boldsymbol{F}$ for $\boldsymbol{P}$.
2. Solve $\boldsymbol{A}\boldsymbol{U} = \boldsymbol{F} - \boldsymbol{B}\boldsymbol{P}$ for $\boldsymbol{U}$.

Is this solution strategy good from a computational point of view?

The MINI Element

The so-called MINI element is the simplest inf-sup stable element. It amounts to approximating both the pressure and each velocity component u_i with continuous piecewise linears. However, on each element K the velocity space is enriched with the cubic polynomial

$$\varphi_4 = \varphi_1\varphi_2\varphi_3 \tag{24}$$

where φ_i, $i = 1, 2, 3$, are the hat functions on K. Because of its shape φ_4 is called a bubble function.

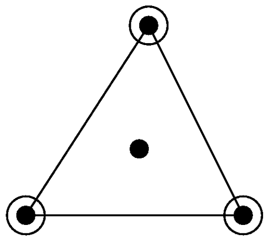

Figure 1: Location of the velocity • and pressure ◯ nodes.

Thus on each element K we make the ansatz

$$u_{h,1}|_K = \xi_1\varphi_1 + \xi_2\varphi_2 + \xi_3\varphi_3 + \xi_4\varphi_4^b \tag{25}$$
$$u_{h,2}|_K = \eta_1\varphi_1 + \eta_2\varphi_2 + \eta_3\varphi_3 + \eta_4\varphi_4^b \tag{26}$$
$$p_h|_K = \theta_1\varphi_1 + \theta_2\varphi_2 + \theta_3\varphi_3 \tag{27}$$

Element Stiffness and Gradient Matrices. The stiffness matrix $\boldsymbol{A}$ and the gradient matrices $\boldsymbol{B}_d$, $d = 1, 2$, are assembled by summing elemental contributions as usual. On triangle K we recall that the hat functions take the form $\varphi_i = a_i + b_i x_1 + c_i x_2$, $i = 1, 2, 3$, where the coefficients a_i, b_i and c_i are determined by the condition $\varphi_i(v_j) = \delta_{ij}$ with v_j triangle vertex $j = 1, 2, 3$. We also recall the integration formula

$$\int_K \varphi_1^m \varphi_2^n \varphi_3^p \, dx = \frac{m!n!p!}{(2+m+n+p)!} 2|K|, \quad m, n, p \geq 0 \tag{28}$$

Further, the gradient of the bubble function φ_4 is given by

$$\nabla\varphi_4 = (\nabla\varphi_1)\varphi_2\varphi_3 + \varphi_1(\nabla\varphi_2)\varphi_3 + \varphi_1\varphi_2(\nabla\varphi_3) \tag{29}$$

Now, straight forward calculation reveals that the 4×4 element stiffness matrix $\boldsymbol{A}^K$ has the entries

$$A_{ij}^K = (\nabla\varphi_j, \nabla\varphi_i)_K = (b_i b_j + c_i c_j)|K|, \quad i, j = 1, 2, 3 \tag{30}$$

$$A_{4,j}^K = 0, \quad j = 1, 2, 3 \tag{31}$$

$$A_{4,4}^K = (\nabla\varphi_4, \nabla\varphi_4)_K \tag{32}$$

$$\begin{aligned} &= (2(b_1^2 + c_1^2) + (b_1 b_2 + c_1 c_2) + (b_1 b_3 + c_1 c_3) \\ &\quad + (b_2 b_1 + c_2 c_1) + 2(b_2^2 + c_2^2) + (b_2 b_3 + c_2 c_3) \\ &\quad + (b_3 b_1 + c_3 c_1) + (b_3 b_2 + c_3 c_2) + 2(b_3^2 + c_3^2))|K|/180 \end{aligned} \tag{33}$$

The entries of the 4×3 element gradient matrix $\boldsymbol{B}_1^K$ are also found to be

$$B_{1,ij}^K = -(\varphi_j, \partial_x \varphi_i)_K = -b_i|K|/3, \quad i = 1, \ldots, 4, \ j = 1, 2, 3 \tag{34}$$

$$B_{1,4j}^K = -(\varphi_j, \partial_x \varphi_4)_K = -(2b_1 + 2b_2 + 2b_3 - b_j)|K|/60 \tag{35}$$

and similarly for $\boldsymbol{B}_2^K$ with b_i replaced by c_i.

Problem 4. ✰ Derive an explicit formula for the bubble function φ_4 on the triangle with vertices at $(-1, -1)$, $(0, -1)$, and $(0, 0)$. Draw a picture of φ_4.

Problem 5. ✰ Verify by direct calculation the formulas for A_{44}^K and $B_{1,11}^K$.

Enforcing a Zero Mean Value on the Pressure

In the discrete setting, a zero mean value for the pressure p_h means that

$$(p, 1) = \sum_{i=1}^{n_p} \theta_i(\varphi_i, 1) = \sum_{i=1}^{n_p} \theta_i a_i = \boldsymbol{a}^T \boldsymbol{\theta} = 0 \tag{36}$$

where $\boldsymbol{a}$ is a $n_p \times 1$ vector with entries $a_i = (\varphi_i, 1)$. To enforce this constraint we augment the linear system (18) with this equation together with a Lagrangian multiplier λ, viz.,

$$\begin{bmatrix} \boldsymbol{A} & \boldsymbol{0} & \boldsymbol{B}_1 & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{A} & \boldsymbol{B}_2 & \boldsymbol{0} \\ \boldsymbol{B}_1^T & \boldsymbol{B}_2^T & \boldsymbol{0} & \boldsymbol{a} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{a}^T & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{\xi} \\ \boldsymbol{\eta} \\ \boldsymbol{\theta} \\ \lambda \end{bmatrix} = \begin{bmatrix} \boldsymbol{F}_1 \\ \boldsymbol{F}_2 \\ \boldsymbol{0} \\ 0 \end{bmatrix} \tag{37}$$

We may think of $\lambda\boldsymbol{a}$ as a force acting on the velocities so that the pressure obtains a zero mean value.

Matlab Simulation of a Lid-Driven Cavity

Below we list the main routine for a finite element Stokes solver based on the MINI element. The code is set up to solve a classical benchmark problem called the *Lid-driven cavity*. The problem consists of simulating the motion of a fluid enclosed within the unitsquare $\Omega = [0, 1]^2$. The velocity on the lid of of the cavity has a constant speed of unity while it is zero on the other three walls, cf. Figure 2.

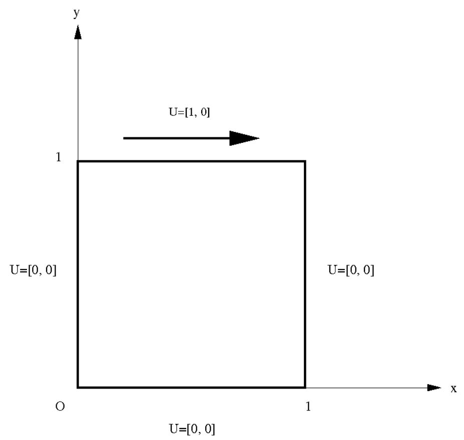

Figure 2: Problem set up of the Lid-driven cavity.

```
clear all, close all
```

```
geom = square(0,1,0,1);
[p,e,t]=initmesh(geom,'hmax',0.05);
np=size(p,2);  nt=size(t,2);
z=zeros(np+nt,1);
[A,Bx,By,a] = assemble(p,e,t);
A_tot=[1*A 0*A Bx z;
       0*A 1*A By z;
       Bx' By' sparse(np,np) a;
       z' z' a' 0];
b_tot = zeros(size(A_tot,1),1);
[A_tot,b_tot]=setbc(p,e,t,A_tot,b_tot);
x_tot=A_tot\b_tot;
xi=x_tot(1:np); eta=x_tot(np+nt+1:2*np+nt);
figure(1), pdeplot(p,e,t,'flowdata',[xi eta])
theta=x_tot(2*(np+nt)+1:end-1);
figure(2), pdesurf(p,t,theta)
```

The assembly of the involved matrices and vectors are done with the following subroutine.

```
function [A,Bx,By,a] = assemble(p,e,t)
np=size(p,2);  nt=size(t,2);
A=sparse(np+nt,np+nt);
Bx=sparse(np+nt,np);
By=sparse(np+nt,np);
a=zeros(np,1);
for i=1:nt
  % nodes, node coordinates, triangle area
  nodes=t(1:3,i);
  x=p(1,nodes);  y=p(2,nodes);
  dx=polyarea(x,y);
  % velocity degrees of freedom
  dofs=[nodes; np+i];
  % hat function gradients
  b=[y(2)-y(3); y(3)-y(1); y(1)-y(2)]/2/dx;
  c=[x(3)-x(2); x(1)-x(3); x(2)-x(1)]/2/dx;
  % element stiffness matrix
  AK=zeros(4,4);
  AK(1:3,1:3)=(b*b'+c*c')*dx;
  AK(4,4)=sum(sum((b*b'+c*c').*[2 1 1; 1 2 1; 1 1 2]*dx/180));
```

```
  A(dofs,dofs)=A(dofs,dofs)+AK;
  % element divergence matrix
  BK=zeros(4,3);
  BK(1:3,1:3)=-b*ones(1,3)*dx/3;
  BK(4,:)=(-[1 2 2; 2 1 2; 2 2 1]/60*dx*b)';
  Bx(dofs,nodes)=Bx(dofs,nodes)+BK;
  BK(1:3,1:3)=-c*ones(1,3)*dx/3;
  BK(4,:)=(-[1 2 2; 2 1 2; 2 2 1]/60*dx*c)';
  By(dofs,nodes)=By(dofs,nodes)+BK;
  % constraints to enforce mean value zero for the pressure
  a(nodes)=a(nodes)+ones(3,1)*dx/3;
end
```

The Lid-driven cavity problem involves other boundary conditions than just homogeneous ones. This is of minor concern to use as it is easy to formulate a finite element method involving inhomogeneous boundary conditions of the form $\boldsymbol{u} = \boldsymbol{g}$. In doing so, the only real problem that needs to be addressed is that $\boldsymbol{g}$ must satisfy the compatibility condition $(\boldsymbol{g}, \boldsymbol{n})_{\partial\Omega} = 0$. This is to assert that equal amounts of fluid flows into and out of the domain Ω. From the practical point of view we enforce $\boldsymbol{u} = \boldsymbol{g}$ by modifying rows of the linear system (18) corresponding to nodes on the boundary. The next lines of code show how this is done for the cavity.

```
function [A,b]=setbc(p,e,t,A,b)
np=size(p,2);  nt=size(t,2);
for i=1:np
  x=p(1,i);  y=p(2,i);
  if (x<0.001 || x>0.999 || y<0.001 || y>0.999) % cavity wall
    A(i,:)=0;
    A(i,i)=1;  b(i)=0; % ux=0
    A(np+nt+i,:)=0;
    A(np+nt+i,np+nt+i)=1;  b(np+nt+i)=0; % uy=0
  end
  if (y>0.999) % the lid of the cavity
    b(i)=1; % ux=1
  end
end
```

The geometry of the cavity is conveniently defined by the next subroutine.

```
function geom = square(xmin,xmax,ymin,ymax)
```

```
geom = [2 xmin xmax ymin ymin 1 0;
        2 xmax xmax ymin ymax 1 0;
        2 xmax xmin ymax ymax 1 0;
        2 xmin xmin ymax ymin 1 0]';
```

Problem 6. ✔ Implement the code outlined above and simulate the lid-driven cavity. Make plots of the velocity field and the pressure. Verify that the pressure has zero mean value by computing $\boldsymbol{a}^T\boldsymbol{\theta}$.

Problem 7. ✔ Verify that the null space for the matrix

$$B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$$

consists of a single constant vector. This shows that there are no spurious pressure modes except the hydrostatic mode $p_h = c$, c a constant, for this type of finite element and that the inf-sup condition is satisfied. *Hint:* See the help for the routine `null` and `full`. A quick way of obtaining B is to type
`B=A_tot(1:2*(np+nt),2*(np+nt)+1:end-1)`.

Problem 8. ✔ Simulate the double driven cavity with boundary conditions $\boldsymbol{u} = 0$ on the lines $x = 0$ and $x = 1$, and $\boldsymbol{u} = [1,\ 0]$ on $y = 0$ and $y = 1$.