# Numerical Methods in Physics

## Lecture 4 - Discrete and Fast Fourier Transforms

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# Discrete Fourier Transform

**Theorem 1 (Shannon's Sampling Theorem)** *Let $h$ denote a function which satisfies $\int_{-\infty}^{\infty} |h(t)|^2 \, dt < \infty$, i.e., a signal with finite energy. This signal is assumed to be sampled at a rate $1/\Delta$. If $h$ is bandwidth limited in its continuous frequency spectrum by the Nyquist frequency $\nu_\mathrm{c} = 1/(2\Delta)$, i.e., if for the Fourier transform $H$ of the relation $H(\nu) = 0$ holds for all $\nu$ with $|\nu| \geq \nu_\mathrm{c}$, then the function $h$ can be recovered without any errors from its sample values using the interpolation function*

$$g(t) = \frac{\sin(2\pi\nu_\mathrm{c} t)}{2\pi\nu_\mathrm{c} t}.$$

*Thus, $h$ may be expressed as*

$$h(t) = \sum_{k=-\infty}^{\infty} h_k g(t - k\Delta),$$

*where $h_k \equiv h(k\Delta)$ are the samples of $h$.*

If $h$ is *not* bandwith limited, all frequency components outside of $[-\nu_\mathrm{c}, \nu_\mathrm{c}]$ are moved into this range (aliasing).

# Aliasing

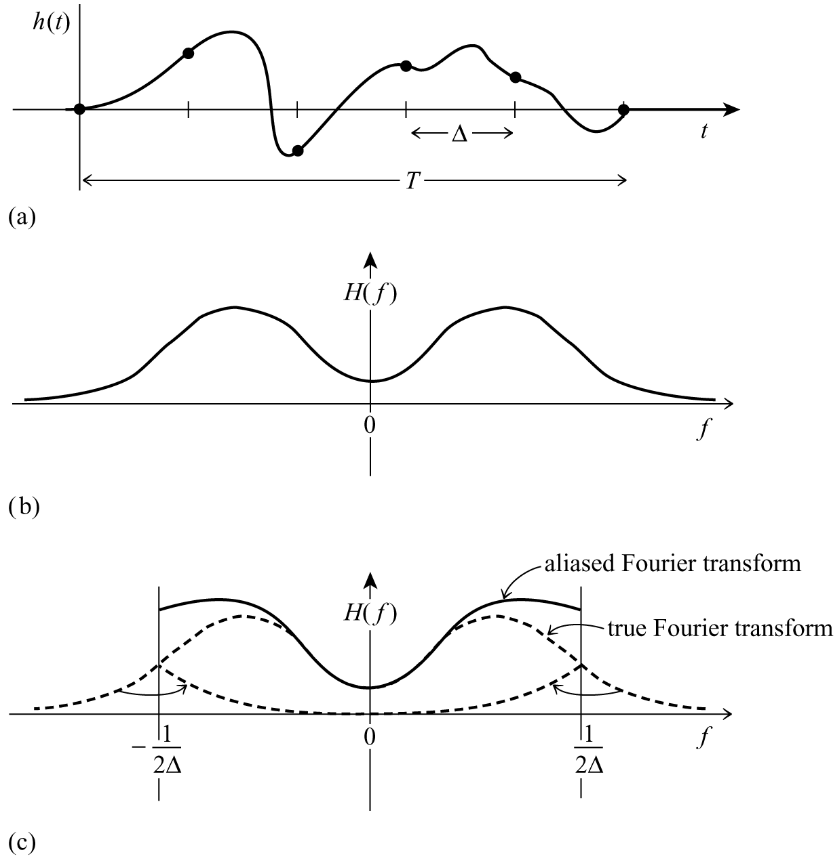
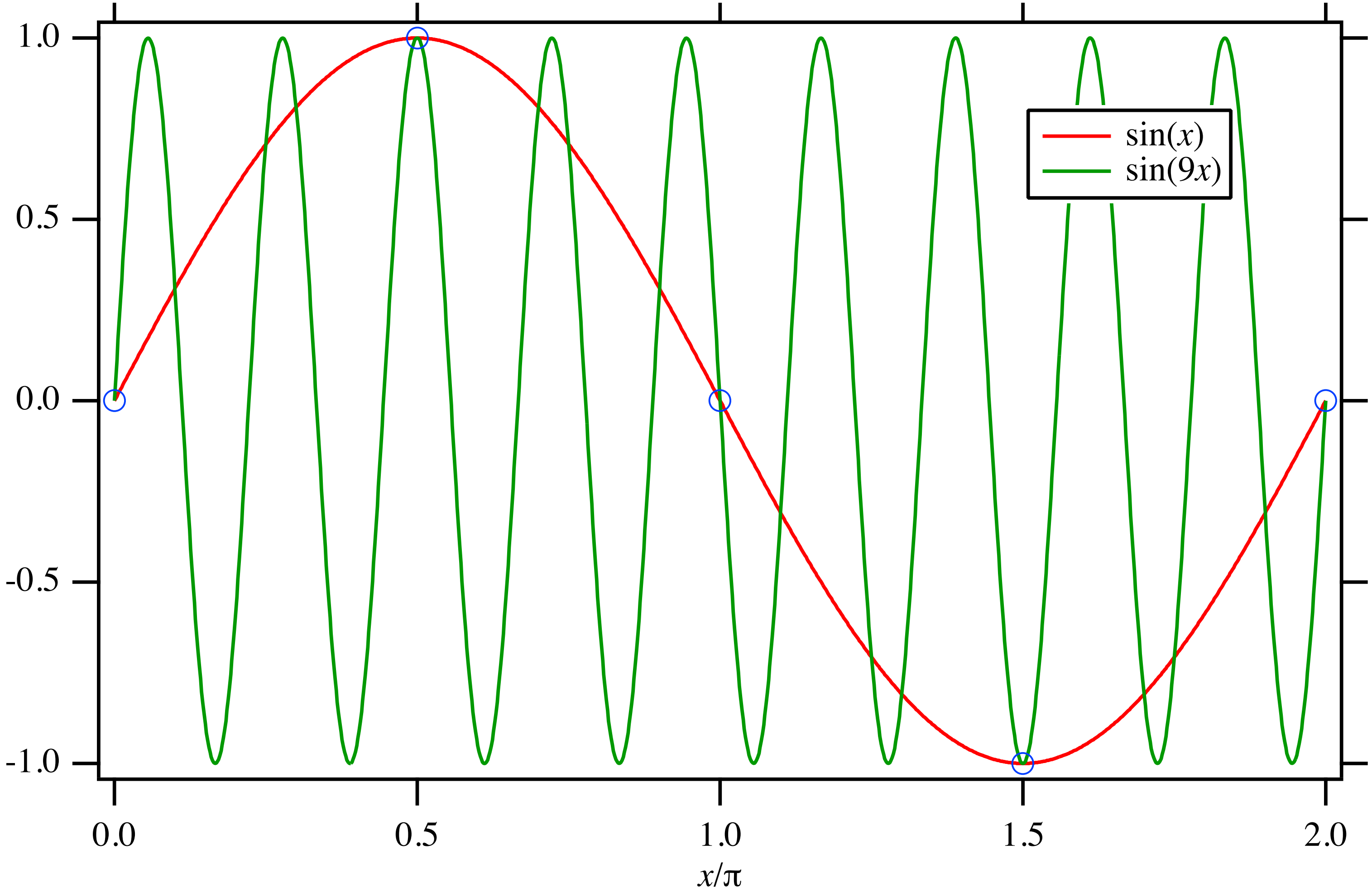
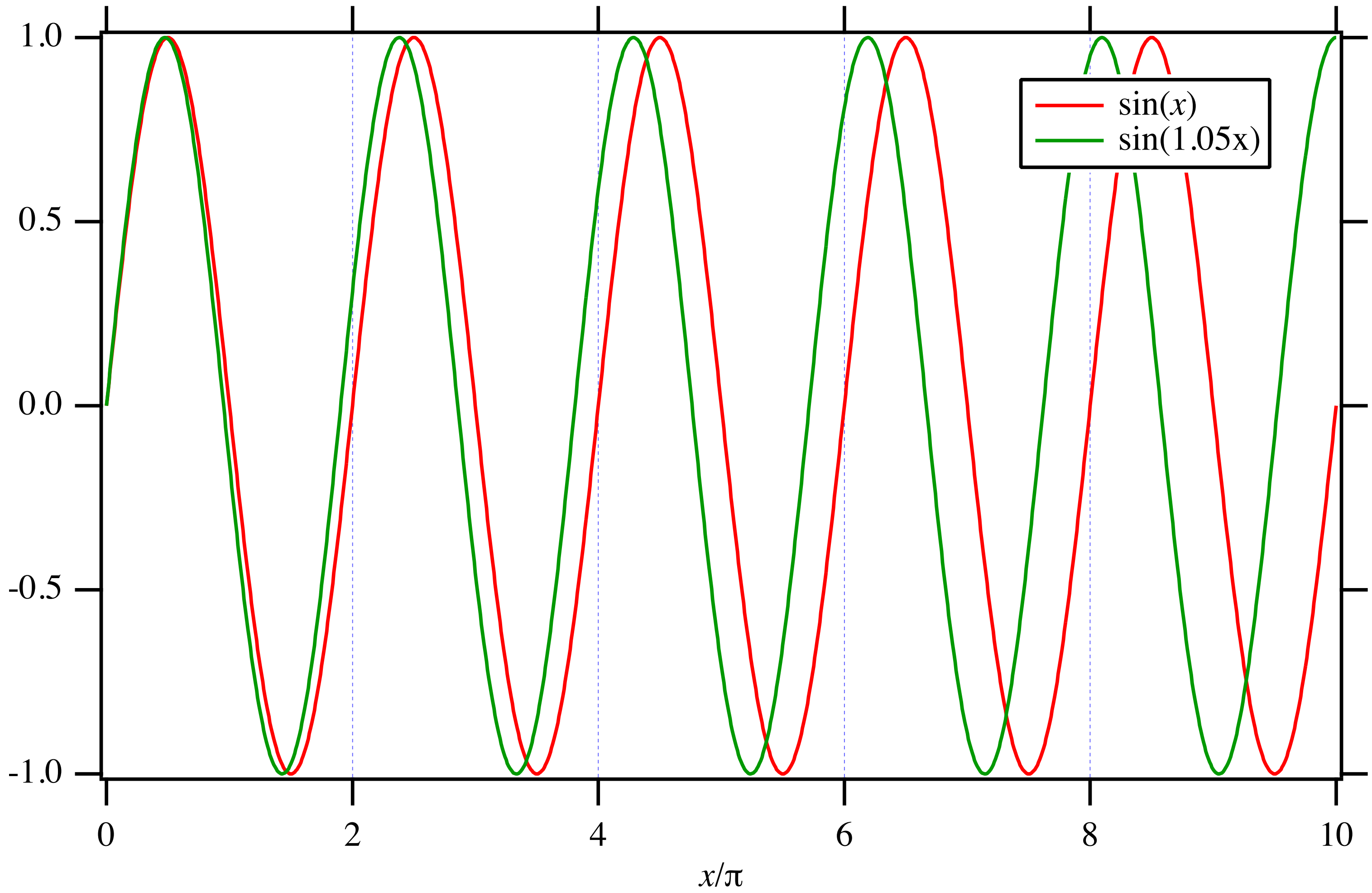



Figure 12.1.1. The continuous function shown in (a) is nonzero only for a finite interval of time $T$. It follows that its Fourier transform, whose modulus is shown schematically in (b), is not bandwidth limited but has finite amplitude for all frequencies. If the original function is sampled with a sampling interval $\Delta$, as in (a), then the Fourier transform (c) is defined only between plus and minus the Nyquist critical frequency. Power outside that range is folded over or "aliased" into the range. The effect can be eliminated only by low-pass filtering the original function *before sampling*.

# Aliasing

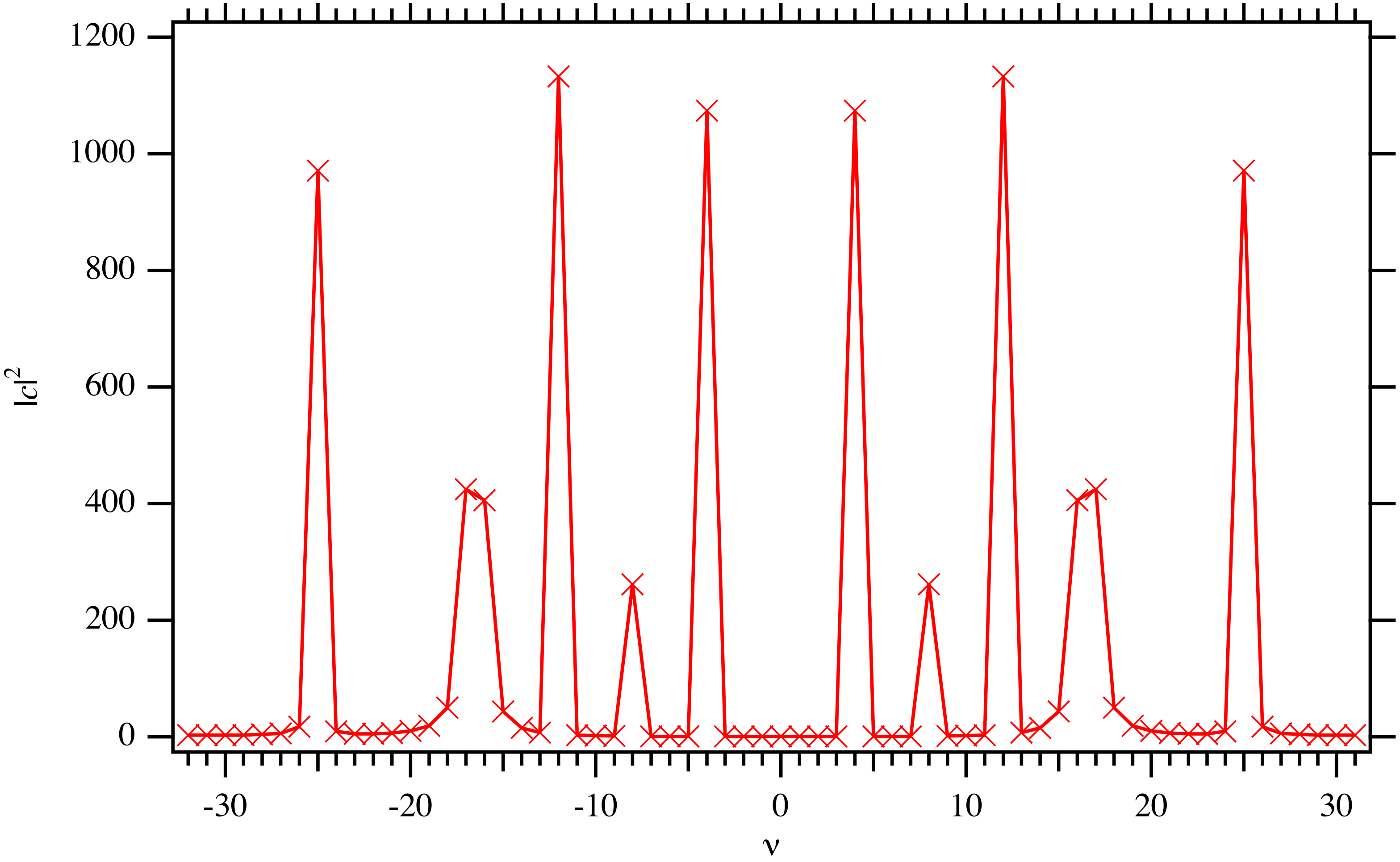


# Sampling time *vs* sampling rate



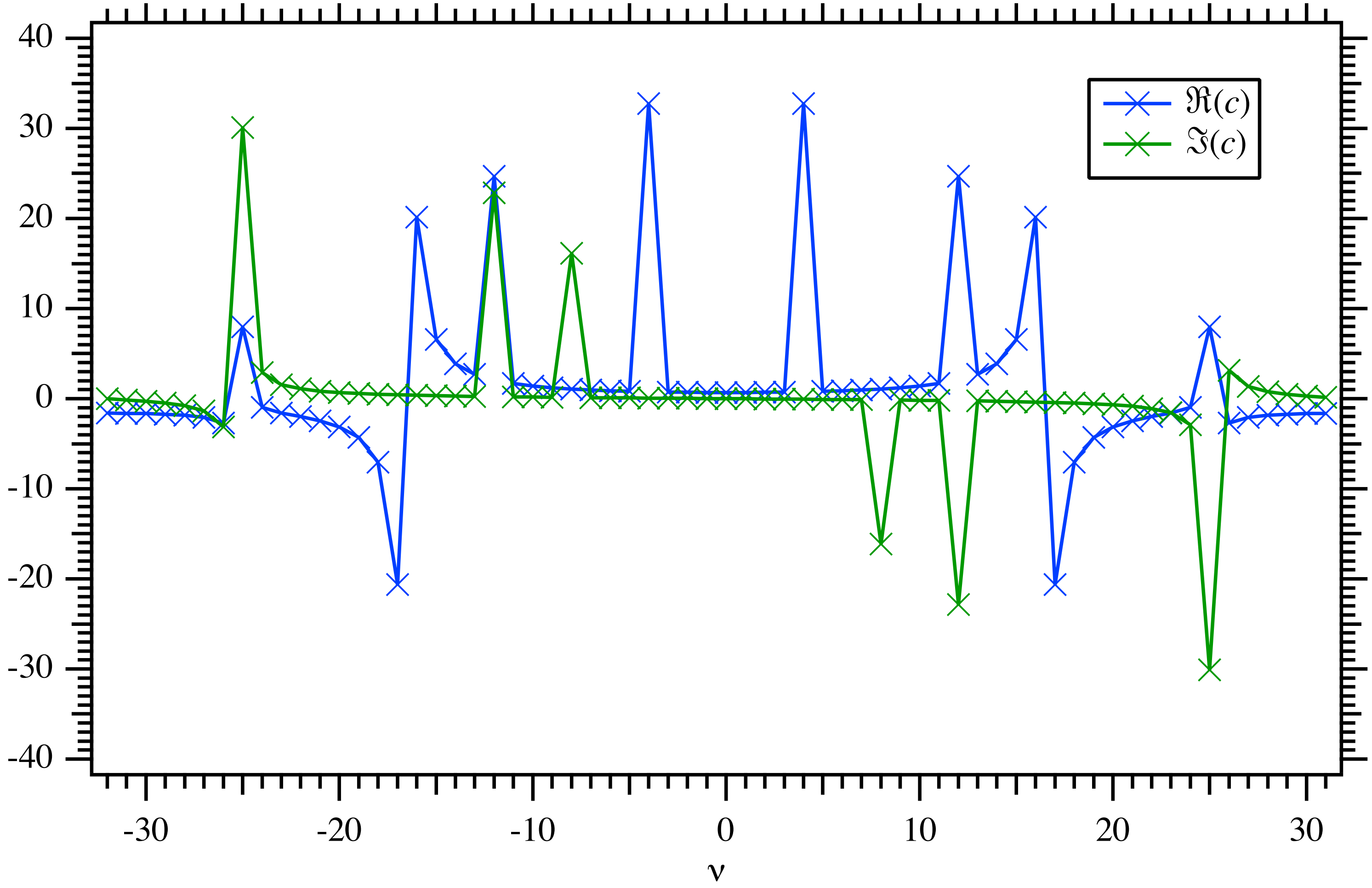
# Leakage

$$h(t) = \cos(2\pi\times 4t)+\frac{1}{2}\sin(2\pi\times 8t)+\sin(2\pi\times 12t+\frac{\pi}{4})+\sin(2\pi\times 16.5t)+\sin(2\pi\times 25.1t)$$

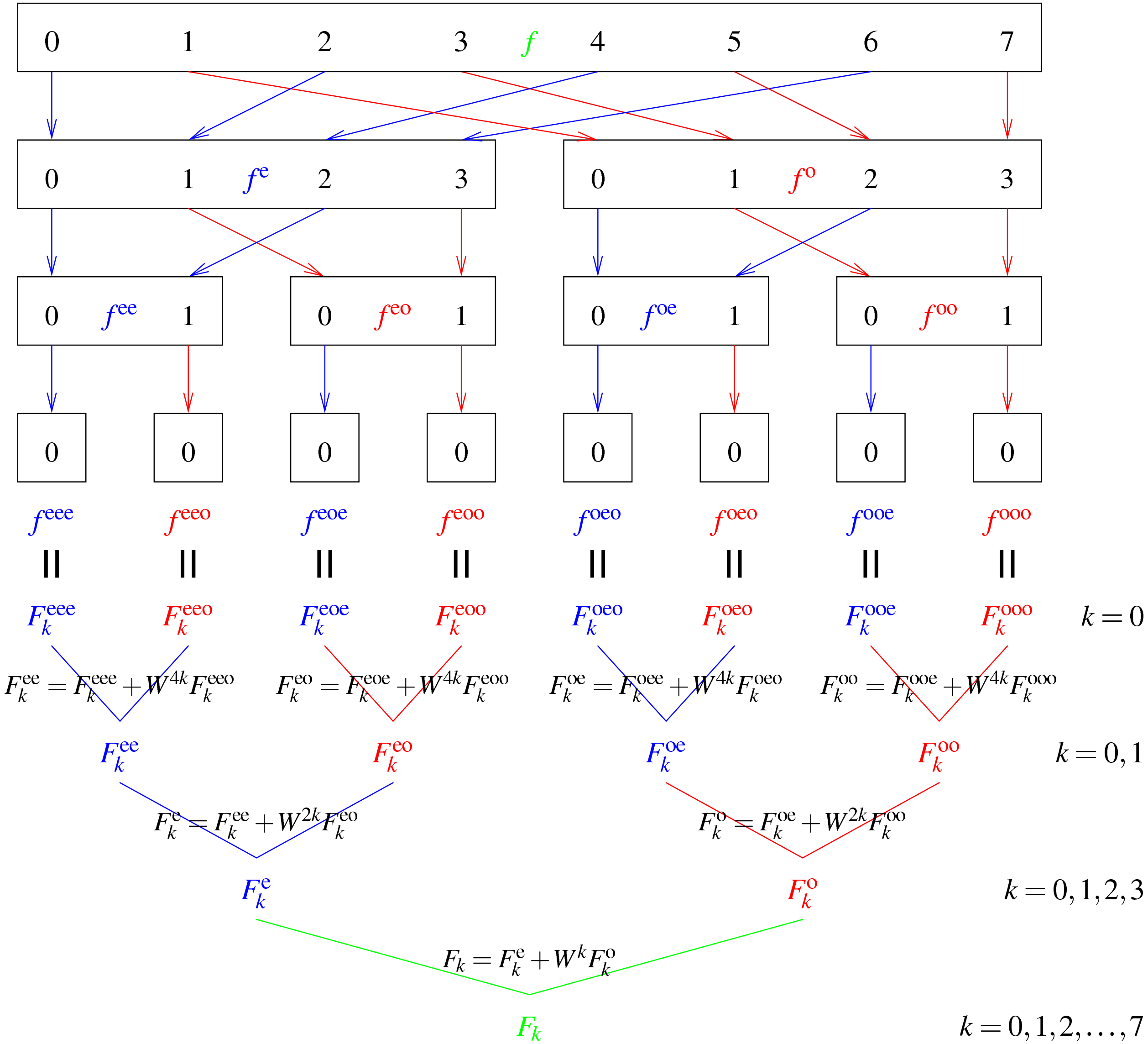


# Leakage

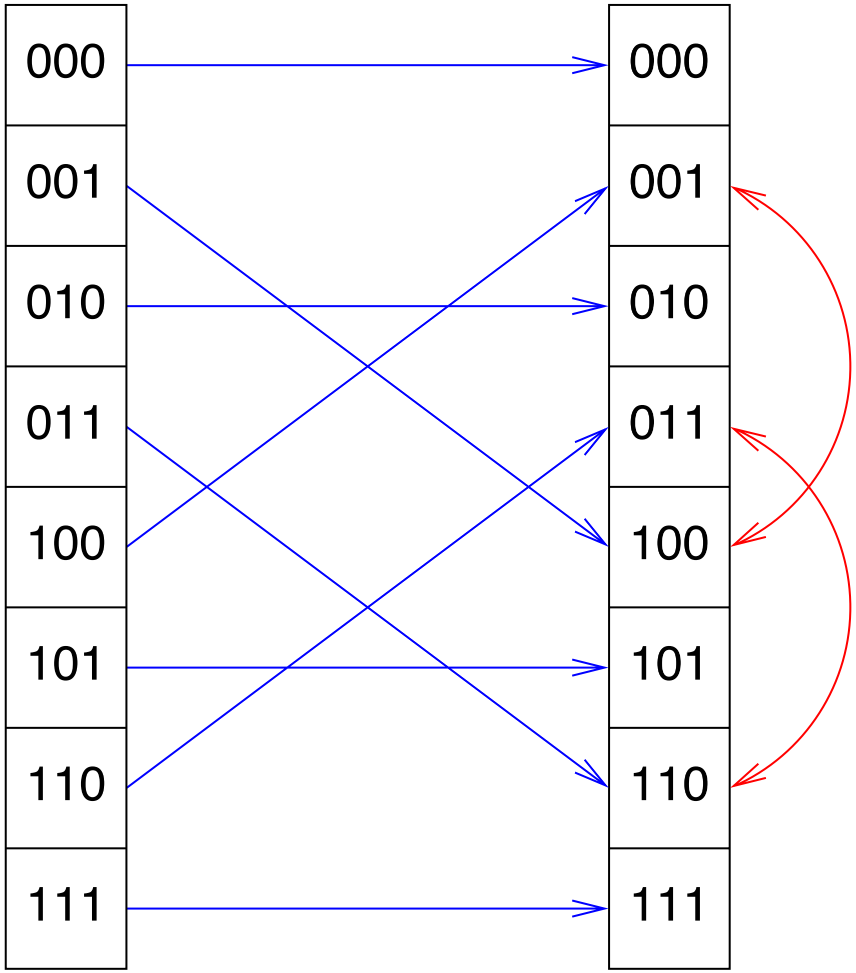
$$h(t) = \cos(2\pi\times 4t) + \frac{1}{2}\sin(2\pi\times 8t) + \sin(2\pi\times 12t + \frac{\pi}{4}) + \sin(2\pi\times 16.5t) + \sin(2\pi\times 25.1t)$$



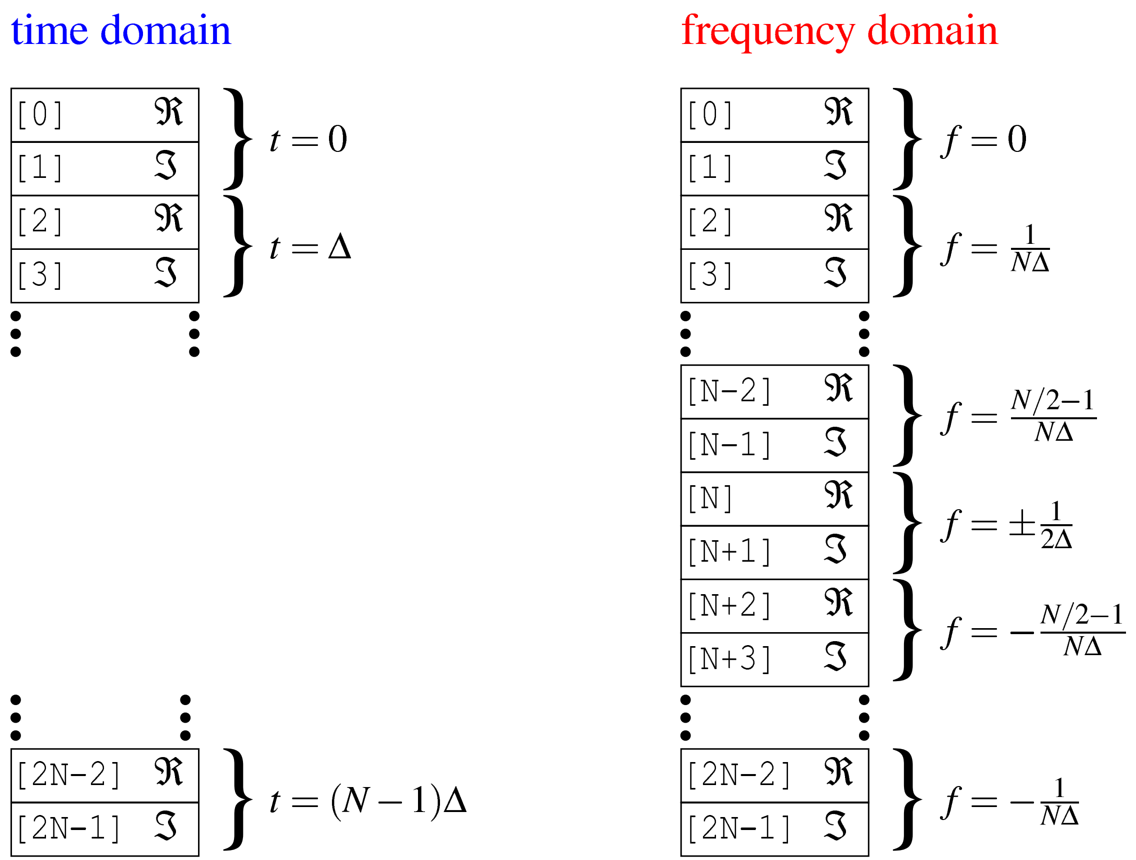
# Cooley-Tukey algorithm exemplified on $N = 8$



# Bit reversal



# Order of FFT output




Ordering of the complex arrays containing the data in the time domain and in the frequency domain. They are declared as real arrays of length $2N$.

# Actual implementations

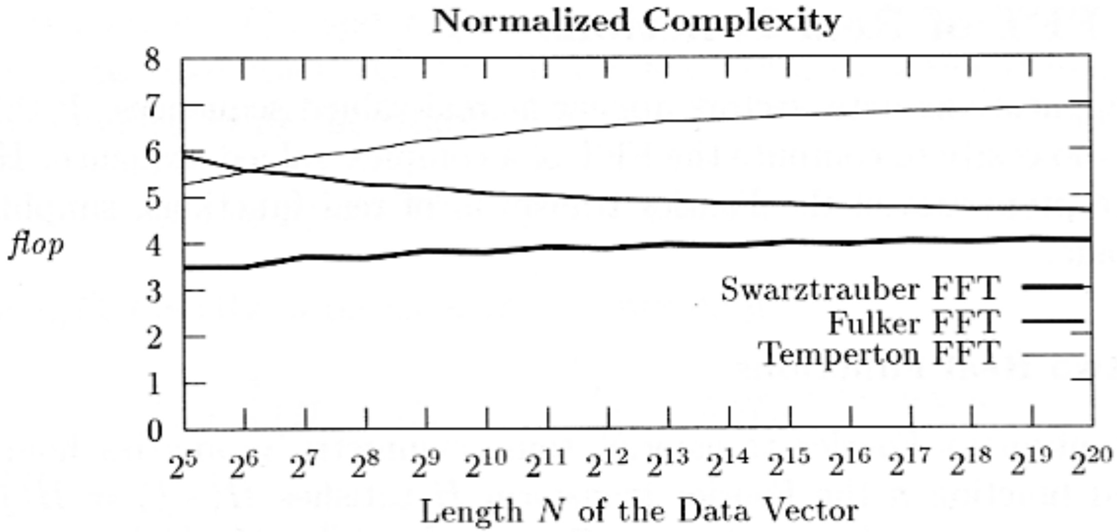



**Figure 11.10:** The normalized number $a(N)/(N \log_2 N)$ of floating-point operations [*flop*] of three FFT programs: `FFTPACK/cfftf` by P. N. Swarztrauber, one program by D. Fulker and one by C. Temperton.

# Actual implementations

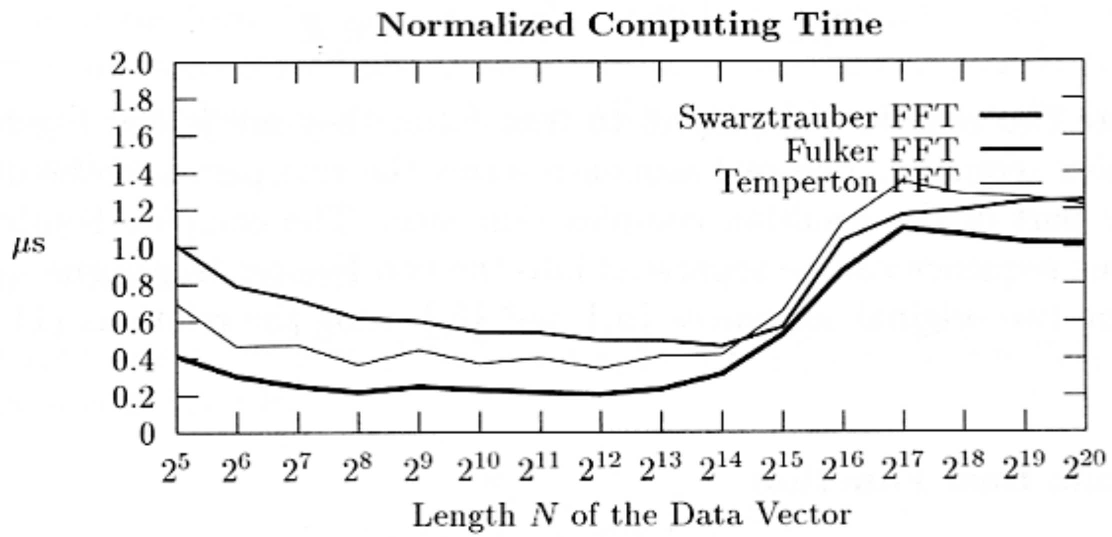



**Figure 11.11:** Normalized computing time $T(N)/(N \log_2 N)$ in microseconds.

# Actual implementations

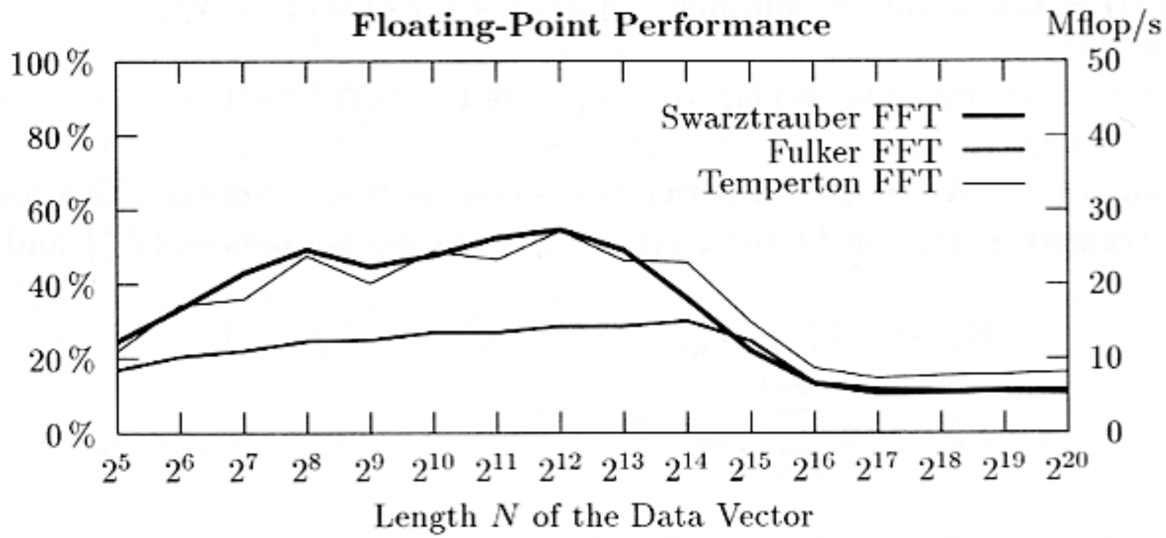



**Figure 11.12:** Floating-point performance (Mflop/s) and efficiency (%).