Laboration 5: The Wave Equation

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Physical background

Many physical phenomena exhibit wave characteristics. For instance light, which is an electromagnetic wave, have the ability to disperse and create diffraction patterns, which are typical of waves. In our setting the electromagnetic waves propagate in a non-magnetic material with the relative permittivity $\epsilon(x, y) \geq 1$ independent of the third coordinate z. The z- independent electromagnetic wave (E, H) is decomposed into transverse electric (TE) polarized waves $(E_x, E_y, 0, 0, 0, H_z)$ and transverse magnetic (TM) polarized waves $(0, 0, E_z, H_x, H_y, 0)$. This reduces the Maxwell's equations to one scalar problem in H_z and one scalar problem problem for E_z. The other field components can then be determined from Maxwell's equations.

Let u denote the magnetic component H_z and consider TE-polarized waves in the bounded domain Ω with Neumann boundary conditions on $\partial\Omega$. From Maxwell's equations follow that $u = H_z$ satisfies

$$
\begin{aligned}
\ddot{u} - \nabla \cdot \left(\tfrac{1}{\epsilon} \nabla u\right) &= f, \quad x \in \Omega, \quad t > 0, && \text{(1a)}\\
n \cdot \nabla u &= 0, \quad x \in \partial\Omega, && \text{(1b)}\\
u(\cdot, 0) &= u_0, && \text{(1c)}\\
\dot{u}(\cdot, 0) &= v_0. && \text{(1d)}
\end{aligned}
$$

where f is a given load, u_0 and v_0 are a prescribed initial conditions, and n it the outward unit normal.

Problem 1. State for fixed t a weak (variational) formulation of the wave equation (1).

Problem 2. Derive fully discrete schemes using the standard piecewise linear approximation in space and the Forward Euler, the Backward Euler, and the Crank-Nicholson method in time.

Matlab Implementation

Let Ω be the L-shaped region $\{[-1,1] \times [-1,1] \setminus [-1,0] \times [0,1]\}$.

The command

```
>> [p,e,t]=initmesh('lshapeg','hmax',0.05);
```

triangulate the geometry. To assemble the mass and the stiffness matrices M and A you can use the code from Laboration 4 but note that the boundary condition is different.

Problem 3. Implement the above described method for the wave equation (1), using $\epsilon = 1$, $f = 0$, $u_0 = 0.1e^{-10\sqrt{(x-0.2)^2+(y-0.8)^2}}$, and $v_0 = 0$. Using $k = h = 0.05$ how long will it take before the magnetic field strength $u = H_z$ reaches 0.01 at the point $(1, -1)$? Use the Crank-Nicholson time stepping scheme and compare results with the Forward and Backward Euler time stepping schemes. Plot the amplitude of the wave at the point $(1, -1)$ versus time and comment upon the observed differences.

Hint: The command `find` can in Matlab be used to determine the integer index s such that $p(1, s) = 1$ and $p(2, s) = -1$.

Problem 4. Plot the energy of the wave (see Theorem 5.3 or the notes form the lectures) as a function of time for the Crank-Nicholson, the Forward Euler, and the Backward Euler schemes and comment upon the observed differences.