Hw 12

April 26, 2020

(1) Prove that $u(x)$ is the unique solution.

For second-order normal coefficient linear non-homogenized equation, with the structure

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + p\frac{\mathrm{d}y}{\mathrm{d}x} + qy = f(x),$$

here we have $p = 0, q = -1, f(x) = -1$, i.e., $-u'' + u = x$ ... (1).

We need to solve the homogenized equation respectively, i.e., $-u'' + u = 0 \quad (2)$.

To solve the characteristic equation:

$$-r^2 + 1 = 0. \;\Rightarrow\; r_1 = 1,\; r_2 = -1.$$

Then the general solution of (2) is

$$c_1 \cdot e^{r_1 x} + c_2 \cdot e^{r_2 x} = c_1 \cdot e^x + c_2 \cdot e^{-x}.$$

Then solve the particular solution for (1).

Assuming that $\tilde{u} = ax + b$. Then we have $ax + b = x \;\Rightarrow\; \tilde{u} = x$.

Thus for (1), the general solution should be $u = c_1 \cdot e^x + c_2 \cdot e^{-x} + x$, with the condition $u(0) = u(1) = 0$, we have

$$\begin{cases} c_1 + c_2 = 0 \\ c_1 \cdot e + c_2 \cdot e^{-1} + 1 = 0 \end{cases}$$

Then $\frac{e^{-1}}{e^{-2}-1}(e^{x-1} - e^{-x-1}) + x = x - \frac{exp(x-1)-exp(-x-1)}{1-exp(-2)}$ is the unique solution.

(2) Using upwind finite difference scheme, find out the matrix L^h and vector $R^h f$, such that the numerical solution satisfies $L^h u^h = R^h f$.

Compare with the original one, we have $b = 0, c = 1, f(x) = x$, then we have

$$\begin{cases} r = \frac{1}{h^2} + \frac{b^+}{h} = \frac{1}{h^2}, \\ s = \frac{2}{h^2} + \frac{b^+}{h} + \frac{b^-}{h} + c = \frac{2}{h^2} + 1. \\ t = \frac{1}{h^2} + \frac{b^-}{h} = \frac{1}{h^2}. \end{cases}$$

Thus,

$$L^h = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -r & s & -t & 0 & \cdots & 0 & 0 & 0 \\ 0 & -r & s & -t & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -r & s & -t \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{pmatrix}$$

and

$$R^h f = (0, \frac{1}{N}, \frac{2}{N}, ..., \frac{N-1}{N}, 0)^T.$$

(3) Convert $L^h u^h = R^h f$ to Markov Reward Process.

Applying the UFD scheme for the above ODE, for $i = 1, 2, \ldots, N-1$, we have

$$-u_{i-1}^h \cdot \frac{1}{h^2} + u_i^h \cdot (\frac{2}{h^2} + 1) - u_{i+1}^h \cdot \frac{1}{h^2} = f_i,$$

where $f_i = x_i$. Thus we have

$$u_i^h = \frac{2}{2+h^2}(\frac{h^2}{2}x_i + \frac{1}{2}(u_{i-1}^h + u_{i+1}^h)).$$

Denote $\gamma = \frac{2}{2+h^2}$, $\ell^h(x) = \frac{h^2}{2}x$ and $p^h(x \pm he_i|x) = \frac{1}{2}$, then we can get the **Markov Reward Process** as follows:

$$u(x) = \gamma\{\ell^h(x) + p^h(x \pm he_i|x)u(x \pm he_i)\}.$$

(4) Write a pseudo code for value iteration.

Algo: 1 Value Iteration for $-u'' + u = x$

Input: $\hat{\epsilon}$: tolerance, $\hat{n}$: max iteration
Output: $v(x)$

function $VI(\hat{n}, \hat{\epsilon})$
    Initialize guess $\{v(x) : x \in O^h\}$
    $flag \leftarrow 1, n \leftarrow 0$
    while $flag$ **do**
        $\epsilon \leftarrow 0,\ n \leftarrow n + 1$
        for $x \in O^h$ **do**
            $u(x) \leftarrow v(x)$
        end for
        for $x \in O^h$ **do**
            $v(x) \leftarrow F^h u(x)$
            $\epsilon \leftarrow \max\{\epsilon, |u(x) - v(X)|\}$
        end for
        if $\epsilon < \hat{\epsilon}$ **then**
            $flag = 0$
        end if
    end while
    return $v(x) : x \in O^h$

(5) Write a pseudo code for first visit Monte-Carlo Method.

Algo: 2 First Visit Monte-Carlo Method for $-u'' + u = x$

Input: n: number of trials
Output: $v(x)$

function $MC1(n)$
  Initialize $Tot = 0$
  for $x \in O^h$ **do**
    for $i = 1, 2, ..., n$ **do**
      Generate $\omega_i = \{S_0, R_1, \ldots, S_T, R_{T+1}\}$
      Compute $G \leftarrow R_1 + \gamma R_2 + \cdots + \gamma^T R_{T+1}$
      $Tot \leftarrow Tot + G$
    end for
    return $v(x) = \frac{Tot}{n}$
  end for
  return $v(x) : x \in O^h$

(6) Prove the consistency.
We have already known that

$$Lu = \begin{cases} -u'' + u & 0 < x < 1 \\ u & x = 0, 1 \end{cases}$$

We need to show that there exist a constant K and α, such that

$$\|L^h R^h v - R^h L v\|_\infty \leq K h^\alpha, \ \forall v \in \mathbb{C}^2([0,1]),$$

where $\|v\|_\infty = \max_i |v_i|, \ \forall v \in \mathbb{R}^N$.

(i) $(L^h R^h v)_0 = v_0 = Lv(0) = Lv(x_0) = (R^h L v)_0$, then

$$|(L^h R^h v)_0 - (R^h L v)_0| = 0.$$

(ii) Similarly,

$$|(L^h R^h v)_N - (R^h L v)_N| = 0.$$

(iii) For $1 \leq i \leq N-1$,

$$\begin{aligned} (L^h R^h v)_i &= -\delta_h \delta_{-h} v_i + v_i, \\ (R^h L v)_i &= Lv(x_i) = -v''(x_i) + v(x_i). \end{aligned} \tag{1}$$

which means that $|(L^h R^h v)_i - (R^h L v)_i| = O(h^2)$.

Therefore, L^h is consistent with order $\alpha = 2$.

(7) Prove the stability.
We need to show that $\|v\|_\infty \leq K \|L^h v\|_\infty, \ \forall v \in \mathbb{E}^{N+1}$.

(i) If $|v_0| = \|v\|_\infty$, then

$$\|L^h v\|_\infty \geq |(L^h v)_0| = |v_0| = \|v\|_\infty.$$

(ii) If $|v_N| = \|v\|_\infty$, then

$$\|L^h v\|_\infty \geq |(L^h v)_N| = |v_N| = \|v\|_\infty.$$

(iii) If $|v_i| = \|v\|_\infty$ for $1 \leq i \leq N-1$, then

$$\begin{aligned}
\|L^h v\|_\infty &\geq |(L^h v)_i| \\
&= |-rv_{i-1} + sv_i - tv_{i+1}| \\
&= |r(v_i - v_{i-1}) + t(v_i - v_{i+1}) + (s - r - t)v_i| \\
&= |r(v_i - v_{i-1}) + t(v_i - v_{i+1}) + v_i| \\
&\geq |v_i| = \|v\|_\infty.
\end{aligned} \tag{2}$$

Therefore we have

$$\|v\|_\infty \leq \|L^h v\|_\infty, \forall v \in \mathbb{R}^{N+1}.$$

(8) Find the convergent rate.
Suppose h is small enough, then we will have

$$\begin{aligned}
\|u^h - R^h u\|_\infty &\leq \|L^h(u^h - R^h u)\|_\infty \\
&= \|L^h u^h - L^h R^h u\|_\infty \\
&= \|R^h f - L^h R^h u\|_\infty \\
&= \|R^h Lu - L^h R^h u\|_\infty \\
&= O(h^2) \ \textit{by consistency}.
\end{aligned} \tag{3}$$

Therefore, the convergence rate is 2.