Hw8/2

Thursday, April 2, 2020 10:10 AM

For second-order normal coefficient linear non-homogenized equation (with the structure $\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = f(x)$)

here we have $p=0$, $q = -\frac{1}{\varepsilon}$ $f(x) = -\frac{x}{\varepsilon}$

i.e. $-\varepsilon u'' + u = x$ ... (1)

we need to solve the homogenized equation respectively.

i.e. $-\varepsilon u'' + u = 0$ ... (2)

To solve the characteristic equation

$$-\varepsilon r^2 + 1 = 0 \Rightarrow r_1 = \frac{1}{\sqrt{\varepsilon}},\ r_2 = -\frac{1}{\sqrt{\varepsilon}}$$

then the general solution of (2) is

$$c_1 \cdot e^{r_1 x} + c_2 \cdot e^{r_2 x}$$

$$= c_1 \cdot e^{\frac{x}{\sqrt{\varepsilon}}} + c_2 \cdot e^{-\frac{x}{\sqrt{\varepsilon}}} \quad \cdots (3)$$

then solve the particular solution for (1)

assuming that $\hat{u} = ax + b$

then we have $ax + b = x \Rightarrow a = 1,\ b = 0 \Rightarrow \hat{u} = x$ ... (4)

thus for (1), the general solution should be (3) + (4)

i.e. $u = c_1 \cdot e^{\frac{x}{\sqrt{\varepsilon}}} + c_2 \cdot e^{-\frac{x}{\sqrt{\varepsilon}}} + x$

with the condition $u(0) = u(1) = 0$

$$\begin{cases} c_1 + c_2 = 0 \\ c_1 \cdot e^{\frac{1}{\sqrt{\varepsilon}}} + c_2 \cdot e^{-\frac{1}{\sqrt{\varepsilon}}} + 1 = 0 \end{cases} \Rightarrow \begin{cases} c_1 = \dfrac{e^{1/\sqrt{\varepsilon}}}{1 - e^{2/\sqrt{\varepsilon}}} = \dfrac{e^{-\frac{1}{\sqrt{\varepsilon}}}}{e^{-2/\sqrt{\varepsilon}} - 1} \\ c_2 = \dfrac{-e^{1/\sqrt{\varepsilon}}}{1 - e^{2/\sqrt{\varepsilon}}} = -\dfrac{e^{-\frac{1}{\sqrt{\varepsilon}}}}{e^{-2/\sqrt{\varepsilon}} - 1} \end{cases}$$

then $u = \frac{1}{e^{-2/\sqrt{\varepsilon}} - 1}\left(e^{\frac{x-1}{\sqrt{\varepsilon}}} - e^{-\frac{x+1}{\sqrt{\varepsilon}}}\right) + x$ is the unique solution #

$x-1$ $x+1$

then $u = \frac{\quad}{e^{-2/\sqrt{\varepsilon}} - 1} \left(e^{\quad} - e^{\quad} \right) + x$ is the unique solution #

$$= x - \frac{\exp\left(\frac{x-1}{\sqrt{\varepsilon}}\right) - \exp\left(-\frac{x+1}{\sqrt{\varepsilon}}\right)}{1 - \exp\left(-\frac{2}{\sqrt{\varepsilon}}\right)}$$