

Bayesian Active Learning for Finding Maximally-valued Exemplars

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We are motivated by a problem of searching short peptides with desired property in biochemistry, and generalize it as a Bayesian optimal search problem related to active learning. In active learning, we have training data with unknown binary labels. Obtaining labels is expensive, and we wish to obtain a small number of labels, so that the statistical classifier built from them is good. We consider a variant in which each datapoint has an associated known value, and our goal is to find a datapoint with a positive label and large value. We proposed two greedy algorithms to address this problem and proved lower bound of the algorithms that guarantees their performance. We also built a prediction model for our motivated problem, and use the proposed optimal search algorithm to find desired peptides. We showed our proposed method did significant better job than typical approach in biochemistry field.

1. Introduction

In this paper we try to develop a method for helping our biochemist collaborators efficiently discover minimal peptide with particular biochemical property, which can be used as a powerful tool for biological studies as well as preparation of novel materials. Finding such peptides is very challenging, because there are very few of them exist in a huge peptide library, and laboratory constraints allow limited number of peptide candidates to be tested. We formulate this problem as an optimal search problem related to active learning.

There are two current approaches: One is to search peptides without quantitative help, and purely based on domain knowledge. This is likely to fail because the number of peptides with the desired property is small, and there is no clear direction to find minimal peptide with desired property. The other approach is to use available data to train a statistical model, and test the top N most likely hits and then stop. However, considering the small size of training set compared with the huge search space, the trained statistical model only has “local” knowledge while lack information about the rest part of the space, and this approach will only find local optimum.

In this paper we develop a Bayesian classification model, with its prior distribution designed based on knowledge from domain expert, to predict whether a peptide has desired property. We also

propose an optimal search algorithm using greedy heuristic, and prove its performance guarantee compared with the optimal solution.

There are a number of methods for optimal search problems developed in the last a couple of years, and they all aim to effectively collect information so as to make the best decisions under uncertainty. In this setting, they need to trade off the reward by sampling (i.e. exploitation) and the cost by acquiring this information (i.e. exploration). For example, in drug discovery, people search for a chemical derivative of the base molecule that best treats disease. To achieve this goal, they choose molecules to test to maximize the expected quality of the best compound discovered (Negoescu et al. 2011). Since the budget for testing is limited, they need to test the most informative and high quality molecules. To address this problem, Jones & Schonlau proposed Expected Improvement algorithm to sample points sequentially (Jones et al. 1998). Ginsbourger used constant liar heuristic to extend Expected Improvement algorithm to parallel setting (Ginsbourger et al. 2008). There are quite a few papers about parallel sampling in active learning research community (Chen et al. 2013, Hoi et al. 2006b,a), but they only aim to maximize information gain (i.e. pure exploration).

We propose a new optimal search algorithm motivated by a minimal peptide search problem in biochemistry. In section 2, we describe the biochemical problem in detail and mathematically formalize it into a more general problem. In section 3, we state the Bayesian classification model used for the problem. In section 4, we provide the formulation of the proposed optimization algorithm, prove its performance guarantee, and summarize the algorithm in pseudo code. In section 5, we use available real data to validate our statistical model, compare the performance of our proposed optimization method against other methods, and show progress of this biochemical application using our method.

2. Problem Formulation

We first describe the application that motivates our research, and then we provide mathematical formalism to address a more general problem. In the last sub-section we derive our method in solving this problem.

2.1. Motivating application.

Over the past decade, many chemistry-based tools have been developed for protein labeling, which have great use in biological studies and preparation of novel materials. In the recent years, focus has been moved to short peptide labeling techniques. Peptides are short chains of amino acid monomers, and they are much shorter than protein. Because of that, peptide labeling has minimal perturbation to native systems, which is a much desired feature. Based on prevailing short peptide labeling methods, a biochemistry research group in UC San Diego, who is also our collaborator,

takes a step further and aims to develop a novel method of selectively adding and removing the desired label. To support this goal, they need to find such peptide from an enormous peptide library ($\sim 10^{27}$ peptides) that only reacts with some specific enzymes, while stays inert to other enzymes. This is a challenging task because this kind of peptide is extremely rare in nature (less than 1 in 10^4), and experiment cost is so expensive that the project can only afford to test hundreds of peptide candidates.

To design optimal experiments for this project is critical, because with so large search space, so rare the occurrence of target and so small amount of permitted evaluations, random sampling will almost surely lead to failure. The motivation for our work lies in developing robust prediction model given a small training dataset, and optimal search algorithm that finds target with least number of evaluations.

2.2. General problem statement.

We now formalize and generalize our problem as an active learning problem, which includes but is not limited to our motivating application.

Let E be a generic search space of exemplars. In our motivating application, E is the search space of peptide candidates. Each element $x \in E$ has an unknown binary label $y(x) = \{0, 1\}$. A known deterministic function $f(x)$ measures the cost or disutility associated with x , which is length of the peptide candidate in the motivating application. Our goal is to perform experiments so as to find x such that it has positive label and its cost function $f(x)$ is as small as possible.

To obtain labels of exemplars, we can do a batch of experiments, which evaluates the labels for a subset $S \subseteq E$. We let cardinality of S be constrained by some number K (i.e. $|S| \leq K$), because budget of doing experiments is usually limited. We measure the quality of S by

$$f^*(S) = \min_{x \in S: y(x)=1} f(x), \quad (1)$$

where we assume $\min \emptyset = \infty$. $f^*(S)$ measures the smallest cost function for positive labeled elements in the set S , and in our motivating application, that means the shortest length of desired peptides in the set of peptides to test.

Let b be a target value and we wish to find $S \subseteq E$ such that $f^*(S)$ is better than b . In our motivating application, we let b be the length of the shortest previously known desired peptide. Since $f^*(S)$ is an unknown random variable, we consider the following two probabilistic measures:

$$\begin{aligned} \text{Probability of Improvement:} \quad & P^*(S) = \mathbb{P}(f^*(S) < b) \\ \text{Expected Improvement:} \quad & EI(S) = \mathbb{E}[(b - f^*(S))^+] \end{aligned} \quad (2)$$

Given constraint on the cardinality of S , we wish to find S that maximizes one of these two measures. Let $g(S)$ be either $P^*(S)$ or $EI(S)$, and we can write our goal as

$$\max_{S \subseteq E: |S| \leq K} g(S). \quad (3)$$

3. Statistical model

We use Naive Bayes as the classification method, which, despite the name, has performed quite well in many cases. Let $X = (X_1, \dots, X_n)$ be an instance with n features and Y be its label. By Bayes's Rule, we have:

$$\mathbb{P}(Y = y|X = x) = \frac{\mathbb{P}(X = x|Y = y)\mathbb{P}(Y = y)}{\mathbb{P}(X = x)} = \frac{\mathbb{P}(X = x|Y = y)\mathbb{P}(Y = y)}{\sum_{y'} \mathbb{P}(X = x|Y = y')\mathbb{P}(Y = y')}$$

The Naive Bayes classifier assumes that the presence or absence of a particular feature is unrelated to the presence or absence of any other feature, given the class variable, i.e.

$$\mathbb{P}(Y = y|X = x) = \frac{\prod_{j=1}^n \mathbb{P}(X_j = x_j|Y = y)\mathbb{P}(Y = y)}{\sum_{y'} \prod_{j=1}^n \mathbb{P}(X_j = x_j|Y = y')\mathbb{P}(Y = y')}$$

In our motivation application, we have a set of peptides, each with length less than or equal to L . Each peptide is a sequence of amino acids. We use a reduced alphabet for amino-acids, i.e., we group them into K groups. For each peptide, let A_i be the amino acid on position j , and let X_i be the class of this amino acid. For a specific enzyme, let $Y(x) = 1$ if peptide x is a substrate for that enzyme and 0 if not.

We let $\theta_{y,j}(k) = \mathbb{P}(X_i = k|Y(X) = y)$, for each $j = 1, \dots, L$, $k = 1, \dots, K$ and $y \in \{0, 1\}$. We further assume some known prior distribution $\mathbb{P}(Y(x) = y)$, $y \in \{0, 1\}$. Let θ be the full set of parameters $\theta_{y,j}(k)$, for $j = 1, \dots, L$, $k = 1, \dots, K$ and $y \in \{0, 1\}$. Then, given an unlabeled peptide, we can calculate its probability being a substrate as:

$$\mathbb{P}(Y(x) = 1|\theta) = \frac{\mathbb{P}(Y(x) = 1) \prod_j \theta_{1,j}(x_j)}{\left[\mathbb{P}(Y(x) = 1) \prod_j \theta_{1,j}(x_j) \right] + \left[\mathbb{P}(Y(x) = 0) \prod_j \theta_{0,j}(x_j) \right]} \quad (4)$$

We estimate the parameters $\theta_{y,j}(k)$ using Bayesian inference. We assume for each $j = 1, \dots, L$, $y \in \{0, 1\}$, the vector $\theta_{y,j} \sim \text{Dirichlet}(\alpha_{y,j}(1), \dots, \alpha_{y,j}(K))$. A good initial choice for the parameter vector $\alpha_{y,j} = (\alpha_{y,j}(1), \dots, \alpha_{y,j}(6))$ can be choosing $\alpha_{y,j}(k)$ to be constant across k , and y , and to only depend upon j . Since amino acids further from the serine are less likely to have a strong influence on its activity, we choose this value to be 1 in the positions next to the serine and to increase as j moves further.

We further assume two hyper parameters γ_0 and γ_1 that characterize the distribution for $y = 0$ and $y = 1$ respectively. Then, with the prior distribution and hyper parameters, our posterior distribution is also Dirichlet. In particular, it is $\text{Dirichlet}(\alpha_{y,j}(1) + \gamma_y N_{y,j}(1), \dots, \alpha_{y,j}(K) + \gamma_y N_{y,j}(K))$, where $N_{y,j}(k)$ counts how many peptides x in the training data with $Y(x) = y$ had $x_j = k$. That is, it counts how many peptides had amino acid j in class j .

4. Optimization Algorithm

(3) is a combinatorial optimization problem with huge feasible region. This problem is considered NP-hard and yet there is no known algorithm that solves it quickly. We propose an approximation algorithm, which solves (3) using greedy heuristic, that is, starting with the empty set $S = \emptyset$, iteratively find element e such that

$$\arg \max_{e \in E \setminus S} g(S \cup \{e\}), \quad (5)$$

and incorporate it into S until $|S| = K$ for some chosen K . This approach reduces the size of search space dramatically from $|E|^{|S|}$ down to $|E| \times |S|$, and we prove that if objective function g is either P^* or EI, the proposed greedy algorithm is guaranteed to be near-optimal with a lower bound. Additionally, we show that under the statistical model described in section 3, we can formulate (5) as Mixed-Integer Nonlinear Programming (MINLP) and solve it efficiently using a off-the-shelf MINLP solver.

4.1. Lower bound of greedy algorithm

In this subsection, we show that when the objective function g in (5) is either P^* or EI, the proposed greedy algorithm has performance guarantee. The result is stated in the following:

THEOREM 1. *If objective function is probability of improvement (i.e $P^*(S)$) or expected improvement (i.e EI(S)), the greedy algorithm is guaranteed to achieve a factor $(1 - 1/e)(\approx 63\%)$ of the optimal value.*

We prove the theorem using the following three lemmas. Lemma 1 is a result from the analysis of greedy heuristic in combinatorial optimization by Nemhauser, which provides lower bound of greedy heuristic given that objective function satisfies certain conditions. Lemma 2 and 3 show P^* and EI are the objective functions that satisfy conditions stated in Lemma 1, and thus greedy algorithm has a lower bound.

LEMMA 1. *Nemhauser et al. (1978) If $F(S)$ is submodular, nondecreasing and $F(\emptyset) = 0$, the greedy heuristic always produces a solution whose value is at least $1 - [(K - 1)/K]^K$ times the optimal value, where $|S| \leq K$. This bound can be achieved for each K and has a limiting value of $1 - 1/e$, where e is the base of the natural logarithm.*

LEMMA 2. *Probability of improvement $P^*(S)$ is submodular, nondecreasing and $P^*(\emptyset) = 0$.*

LEMMA 3. *Expected improvement EI(S) is submodular, nondecreasing and EI(\emptyset) = 0.*

Proof of Theorem 1. From Lemma 2 and 3, we know that P^* and EI are submodular, nondecreasing and their measure of the empty set is 0. From Lemma 1, we conclude that if the objective function g in (5) is P^* or EI, the greedy algorithm is guaranteed to achieve a factor of $(1 - 1/e)$ of the optimal value. \square

Proof of Lemma 2. First we show $P^*(\emptyset) = 0$.

$$P^*(\emptyset) = \mathbb{P}(f^*(\emptyset) < b) = \mathbb{P}(\infty < b) = 0.$$

To show $P^*(S)$ is nondecreasing, let $A \subseteq B \subseteq E$ where E is a finite set, then

$$\begin{aligned} P^*(B) &= \mathbb{P}(f^*(B) < b) \\ &= \mathbb{P}(f^*(B) < b | f^*(A) \geq b) \mathbb{P}(f^*(A) \geq b) + \mathbb{P}(f^*(B) < b | f^*(A) < b) \mathbb{P}(f^*(A) < b) \\ &= \mathbb{P}(f^*(B) < b | f^*(A) \geq b) \mathbb{P}(f^*(A) \geq b) + \mathbb{P}(f^*(A) < b) \\ &\geq \mathbb{P}(f^*(A) < b) \\ &= P^*(A) \end{aligned}$$

Lastly, we want to show $P^*(S)$ is submodular. For $e \in E \setminus B$,

$$\begin{aligned} &P^*(A \cup \{e\}) - P^*(A) \\ &= \mathbb{P}(f^*(A \cup \{e\}) < b) - \mathbb{P}(f^*(A) < b) \\ &= \mathbb{P}(f^*(A \cup \{e\}) < b | f^*(A) < b) \mathbb{P}(f^*(A) < b) + \mathbb{P}(f^*(A \cup \{e\}) < b | f^*(A) \geq b) \mathbb{P}(f^*(A) \geq b) - \mathbb{P}(f^*(A) < b) \\ &= \mathbb{P}(f^*(A) < b) + \mathbb{P}(f^*(A \cup \{e\}) < b | f^*(A) \geq b) \mathbb{P}(f^*(A) \geq b) - \mathbb{P}(f^*(A) < b) \\ &= \mathbb{P}(f^*(A \cup \{e\}) < b | f^*(A) \geq b) \mathbb{P}(f^*(A) \geq b) \\ &= \mathbb{P}(f(e) < b, y(e) = 1 | f^*(A) \geq b) \mathbb{P}(f^*(A) \geq b) \\ &= \mathbb{P}(f(e) < b, y(e) = 1, f^*(A) \geq b) \end{aligned}$$

Using similar argument,

$$\begin{aligned} &P^*(B \cup \{e\}) - P^*(B) \\ &= \mathbb{P}(f(e) < b, y(e) = 1, f^*(B) \geq b) \\ &= \mathbb{P}(f(e) < b, y(e) = 1, f^*(A) \geq b, f^*(B \setminus A) \geq b) \end{aligned}$$

Therefore, $P^*(A \cup \{e\}) - P^*(A) \geq P^*(B \cup \{e\}) - P^*(B)$, thus $P^*(S)$ is submodular. \square

Proof of Lemma 3. First we show $EI(\emptyset) = 0$.

$$EI(\emptyset) = \mathbb{E}[(b - f^*(\emptyset))^+] = \mathbb{E}[0] = 0.$$

To show $EI(S)$ is nondecreasing, let $A \subseteq B \subseteq E$ where E is a finite set. Since $f^*(B) \leq f^*(A)$, $b - f^*(B) \geq b - f^*(A)$, and $(b - f^*(B))^+ \geq (b - f^*(A))^+$, therefore, $\mathbb{E}[(b - f^*(B))^+] \geq \mathbb{E}[(b - f^*(A))^+]$.

Lastly, we want to show $P^*(S)$ is submodular. For $e \in E \setminus B$, consider $\mathbb{E}[(b - f^*(A \cup \{e\}))^+] - \mathbb{E}[(b - f^*(A))^+]$. We can write

$$(b - f^*(A \cup \{e\}))^+ = \begin{cases} (b - f^*(A))^+ & \text{if } y(e) = 0 \\ (b - \min\{f(e), f^*(A)\})^+ & \text{if } y(e) = 1 \end{cases}$$

Then

$$\begin{aligned} & \mathbb{E}[(b - f^*(A \cup \{e\}))^+] - \mathbb{E}[(b - f^*(A))^+] \\ &= \mathbb{P}(y(e) = 1) \mathbb{E}[(b - \min\{f(e), f^*(A)\})^+ - (b - f^*(A))^+ | y(e) = 1] \\ &= \mathbb{P}(y(e) = 1) \mathbb{P}(f(e) < f^*(A) | y(e) = 1) \mathbb{E}[(b - e)^+ - (b - f^*(A))^+ | y(e) = 1, f(e) < f^*(A)] \\ &= \mathbb{E}[\mathbb{1}_{y(e)=1, f(e) < f^*(A)} ((b - e)^+ - (b - f^*(A))^+)] \end{aligned}$$

Since $f^*(A) \geq f^*(B)$, $\mathbb{1}_{y(e)=1, f(e) < f^*(A)} ((b - e)^+ - (b - f^*(A))^+) \geq \mathbb{1}_{y(e)=1, f(e) < f^*(B)} ((b - e)^+ - (b - f^*(B))^+)$, thus

$$\text{EI}(A \cup \{e\}) - \text{EI}(A) \geq \text{EI}(B \cup \{e\}) - \text{EI}(B)$$

$\text{EI}(S)$ is submodular. \square

4.2. Probability of Improvement

We first show in Proposition 1 that (5) can be written into the form in which the objective function can be easily calculated using any statistical classifier. This is the general form of greedy algorithm for optimization over probability of improvement and can be used with any underlying statistical model. In addition, we show that, using the Naive Bayes classifier proposed in section 3, we can further formulate (5) as a MINLP, which can be solved efficiently using an off-the-shelf MINLP solver.

PROPOSITION 1. *If the objective function g is P^* , we can write (5) as*

$$\arg \max_{e \in E \setminus S, f(e) < b} \mathbb{P}(y(e) = 1 | y(x) = 0, \forall x \in S). \quad (6)$$

Proof of proposition 1.

$$\begin{aligned} P^*(S \cup \{e\}) &= \mathbb{P}(f^*(S \cup \{e\}) < b) \\ &= \mathbb{P}(f^*(S) < b) + \mathbb{P}(f^*(S) \geq b) \mathbb{P}(f(e) < b, y(e) = 1 | f^*(S) \geq b), \end{aligned}$$

so (5) becomes

$$\max_{e \in E \setminus S} P^*(S \cup \{e\}) = \max_{e \in E \setminus S} \mathbb{P}(f(e) < b, y(e) = 1 | f^*(S) \geq b). \quad (7)$$

Note that when $f(e) \geq b$, $\mathbb{P}(f(e) < b, y(e) = 1 | f^*(S) \geq b) = 0$, thus our algorithm will always propose e such that $f(e) < b$. Therefore, it is reasonable to assume that $f(x) < b$ for $\forall x \in S$, and $f^*(S) \geq b$ is equivalent to $y(x) = 0$ for $\forall x \in S$. Now we can write (7) as

$$\max_{e \in E \setminus S, f(e) < b} \mathbb{P}(y(e) = 1 | y(x) = 0, \forall x \in S). \quad \square$$

In our motivating application, we use the Naive Bayes model described in section 3, and we want to formulate (6) as a MINLP. First write equation (4) as

$$\mathbb{P}(Y(x) = 1|\theta) = \frac{\prod_j \eta_j(x_j)}{\prod_j \eta_j(x_j) + \frac{\mathbb{P}(Y(x)=0)}{\mathbb{P}(Y(x)=1)}}, \quad (8)$$

where

$$\eta_j(x_j) = \frac{\theta_{1,j}(x_j)}{\theta_{0,j}(x_j)} \text{ for } \forall j \in \{1, \dots, L\}.$$

Then we can write equation (6) as

$$\arg \max_{e \in E \setminus S, f(e) < b} \frac{\prod_j \eta_j(e_j)}{\prod_j \eta_j(e_j) + \frac{\mathbb{P}(Y(e)=0)}{\mathbb{P}(Y(e)=1)}}, \quad (9)$$

where

$$\eta_j(e_j) = \frac{\mathbb{P}(e_j | Y(e) = 1, Y(x) = 0, \forall x \in S)}{\mathbb{P}(e_j | Y(e) = 0, Y(x) = 0, \forall x \in S)}.$$

Now we can formulate equation (9) as a MINLP,

$$\begin{aligned} \max \quad & \frac{\prod_j \Sigma_k x_j(k) \eta_j(k)}{\prod_j \Sigma_k x_j(k) \eta_j(k) + \frac{\mathbb{P}(Y(x)=0)}{\mathbb{P}(Y(x)=1)}} \\ \text{s.t} \quad & k \in \{1, \dots, K\} \\ & x_j(k) \in \{0, 1\} \\ & \Sigma_k x_j(k) = 1, \end{aligned} \quad (10)$$

where

$$x_j(k) = \begin{cases} 1 & \text{if } e_j = k \\ 0 & \text{else.} \end{cases}$$

We summarize the algorithm in Algorithm 1.

Algorithm 1 (*Probability of Improvement*)

Require: Inputs M, J, K , data set D and prior distribution of $\theta_y \sim \text{Dirichlet}(\alpha_y), y \in \{1, 0\}$

- 1: $S \leftarrow \emptyset$
- 2: Calculate posterior distribution of $\theta_1 \sim \text{Dirichlet}(\alpha_1 | \{x | x \in D, y(x) = 1\})$.
- 3: **for** $m = 1$ **to** M **do**
- 4: $\text{COUNT} \leftarrow 0$
- 5: Calculate posterior distribution of $\theta_0 \sim \text{Dirichlet}(\alpha_0 | \{x | x \in D, y(x) = 0\} \cup S)$.
- 6: **loop**
- 7: Sample θ_1 from $\text{Dirichlet}(\alpha_1 | \{x | x \in D, y(x) = 1\})$ and θ_0 from $\text{Dirichlet}(\alpha_0 | \{x | x \in D, y(x) = 0\} \cup S)$.
- 8: $\eta \leftarrow \frac{\theta_1}{\theta_0}$


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9:      Solve MINLP in equation (10) to find  $x$ .
10:      $COUNT \leftarrow COUNT + x$ .
11:   end loop
12:   for  $j = 1$  to  $J$  do
13:      $e_j \leftarrow \arg \max_{k \in \{1, \dots, K\}} COUNT_{kj}$ 
14:   end for
15:    $S \leftarrow (S, e)$ 
16: end for

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4.3. Expected Improvement

The formulation for the case of expected improvement is similar to that of probability of improvement, and for our motivating application, we can also formulate (5) as a MINLP.

PROPOSITION 2. *If the objective function g is EI, we can write (5) as*

$$\arg \max_{e \in E \setminus S} c_0 \mathbb{P}_0(e)(b - f(e))^+ + \sum_{i=1}^{|S|} c_i \mathbb{P}_i(e)(f(x_i) - f(e))^+, \quad (11)$$

where

$$\mathbb{P}_0(e) = \mathbb{P}(y(e) = 1 | y(x) = 0, \forall x \in S),$$

$$\mathbb{P}_i(e) = \mathbb{P}(y(e) = 1 | y(x_i) = 1, y(x_j) = 0, \forall j < i, x_i, x_j \in S),$$

and $c_i (i = 0, \dots, |S|)$ are known coefficients.

Proof of proposition 2. Since choosing e such that $f(e) \geq b$ has no contribution to the objective function, by using similar argument as dealing with probability of improvement, we argue that $f(x) < b$ for $\forall x \in S$. Thus

$$f^*(S) \begin{cases} = \infty & \text{if } y(x) = 0 \text{ for } \forall x \in S, \\ < b & \text{else.} \end{cases}$$

Now objective function we want to maximize becomes

$$\begin{aligned} & \mathbb{E}[(b - f^*(S \cup \{e\}))^+] \\ &= \mathbb{E}[(b - f(e))^+ \mathbb{1}_{f^*(S) = \infty, y(e)=1}] + \mathbb{E}[(b - f^*(S \cup \{e\}))^+ \mathbb{1}_{f^*(S) < b}] \\ &= \mathbb{E}[(b - f(e))^+ \mathbb{1}_{f^*(S) = \infty, y(e)=1}] + \mathbb{E}[(b - f^*(S)) \mathbb{1}_{f^*(S) < b}] + \mathbb{E}[(f^*(S) - f(e)) \mathbb{1}_{y(e)=1, f(e) < f^*(S) < b}]. \end{aligned}$$

so

$$\begin{aligned} & \max_{e \in E \setminus S} \mathbb{E}[(b - f^*(S \cup \{e\}))^+] , \\ &= \max_{e \in E \setminus S, f(e) < b} \mathbb{E}[(b - f(e)) \mathbb{1}_{f^*(S) = \infty, y(e)=1}] + \mathbb{E}[(f^*(S) - f(e)) \mathbb{1}_{y(e)=1, f(e) < f^*(S) < b}]. \end{aligned} \quad (12)$$

For $e \in E \setminus S, f(e) < b$,

$$\mathbb{E}[(b - f(e)) \mathbb{1}_{f^*(S)=\infty, y(e)=1}] = \mathbb{P}(y(e) = 1, y(x) = 0, \forall x \in S)(b - f(e)), \quad (13)$$

$$\begin{aligned} & \mathbb{E}[(f^*(S) - f(e)) \mathbb{1}_{y(e)=1, f(e) < f^*(S) < b}], \\ &= \mathbb{E}[\mathbb{E}[(f^*(S) - f(e)) \mathbb{1}_{y(e)=1, f(e) < f^*(S) < b} | f^*(S) = l], \\ &= \sum_{l \in L, f(e) < l} \mathbb{P}(y(e) = 1 | f^*(S) = l)(l - f(e)) \mathbb{P}(f^*(S) = l), \end{aligned}$$

where $L = \{f(x) : x \in S\}$. If we rank elements in S such that $f(x_i) \leq f(x_j), \forall i < j, x_i, x_j \in S$, we can write equation above as

$$\sum_{i=1}^{|S|} \mathbb{P}(y(e) = 1, y(x_i) = 1, y(x_j) = 0, \forall j < i, x_i, x_j \in S)(f(x_i) - f(e))^+. \quad (14)$$

Substitute (13) (14) into (12), and note that $\mathbb{P}(y(e) = 1, \mathcal{F}(x_1, \dots, x_{|S|}) \propto \mathbb{P}(y(e) = 1 | \mathcal{F}(x_1, \dots, x_{|S|}))$ with known coefficient given S , we get (11) in proposition 2. \square

Note that each term in the summation of (11) has a similar structure as (6). Let $S = \{p^1, \dots, p^{|S|}\}$, we can write (11) as a MINLP:

$$\begin{aligned} \max \quad & \sum_{i=0}^{|S|} c_i \frac{\prod_j \Sigma_k x_j(k) \eta_j^i(k)}{\prod_j \Sigma_k x_j(k) \eta_j^i(k) + \frac{\mathbb{P}(Y(x)=0)}{\mathbb{P}(Y(x)=1)}} (f_i - f(e))^+ \\ \text{s.t} \quad & k \in \{1, \dots, K\} \\ & x_j(k) \in \{0, 1\} \\ & \Sigma_k x_j(k) = 1, \end{aligned} \quad (15)$$

where

$$\begin{aligned} x_j(k) &= \begin{cases} 1 & \text{if } e_j = k \\ 0 & \text{else,} \end{cases} \\ f_i &= \begin{cases} b & \text{if } i = 0 \\ f(p^i) & \text{else,} \end{cases} \end{aligned}$$

and c_i 's are known coefficients. We summarize it in Algorithm 2.

Algorithm 2 (*Expected Improvement*)

Require: Inputs M, J, K , data set D and prior distribution of $\theta_y \sim \text{Dirichlet}(\alpha_y), y \in \{1, 0\}$

- 1: $S \leftarrow \emptyset$
- 2: **for** $m = 1$ to M **do**
- 3: $\text{COUNT} \leftarrow 0$
- 4: **if** S is not empty **then**
- 5: Sort elements in S as $\{p^1, \dots, p^{|S|}\}$ such that $f(p^i) \leq f(p^j), \forall i < j$.

```

6:   end if
7:   Calculate posterior distribution of  $\theta_1^0 \sim \text{Dirichlet}(\alpha_1 | \{x | x \in D, y(x) = 1\})$  and  $\theta_0^0 \sim \text{Dirichlet}(\alpha_0 | \{x | x \in D, y(x) = 0\} \cup S)$ .
8:   for  $i = 1$  to  $|S|$  do
9:     Calculate posterior distribution of  $\theta_1^i \sim \text{Dirichlet}(\alpha_1 | \{x | x \in D, y(x) = 1\} \cup \{p^i\})$  and  $\theta_0^i \sim \text{Dirichlet}(\alpha_0 | \{x | x \in D, y(x) = 0\} \cup \{p^j | j < i\})$ .
10:  end for
11:  loop
12:    Sample  $\theta_1^{i=0:|S|}$  and  $\theta_0^{i=0:|S|}$  from posterior distribution.
13:     $\eta^{i=0:|S|} \leftarrow \frac{\theta_1^{i=0:|S|}}{\theta_0^{i=0:|S|}}$ 
14:    Solve MINLP in equation (15) to find  $x$ .
15:     $COUNT \leftarrow COUNT + x$ .
16:  end loop
17:  for  $j = 1$  to  $J$  do
18:     $e_j \leftarrow \arg \max_{k \in \{1, \dots, K\}} COUNT_{kj}$ 
19:  end for
20:   $S \leftarrow (S, e)$ 
21: end for

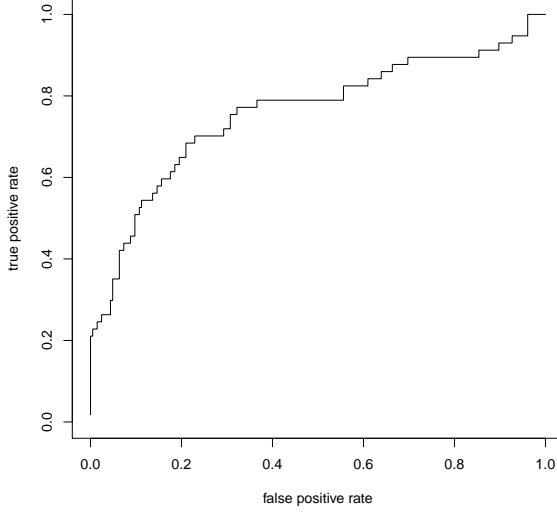
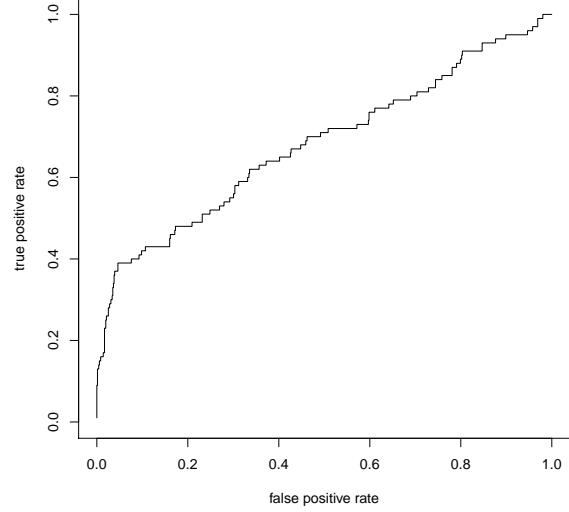
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5. Model Validation and Performance

In this section we validate our statistical model and evaluate the performance of our proposed optimization methods. For model validation, we use real experimental data as training data, and use the leave-one-out cross validation to produce receiver operating characteristic(ROC) curve. We show the plots in Section 5.1. To evaluate the performance of our proposed algorithm, we compare it with two methods introduced in Section 5.2, using either P^* or EI as performance metrics. We show the performance results in Section 5.3. In Section 5.4 we show that after performing two rounds of experiments using the proposed algorithm to generate peptide candidates, we successfully found more short peptides with desired property.

5.1. Naive Bayes Classifier

Since our training data is expensive and highly skewed, we use the leave-one-out cross validation procedure to choose the optimal hyper parameters. For each setting of the hyper parameters, we obtain an receiver operating characteristic(ROC) curve using the result of the leave-one out procedure and choose the setting with highest AUC(area under curve).

**Figure 0** using data set #0 & #1**Figure 0** using data set #0, #1 & #2**Figure 1** ROC curve using leave-one-out cross validation

In Figure 1, note ROC curve to the left is better than the one to the right. This is because data set #2 was generated by our algorithm based on the previous two data sets, and due to the exploration manner of our algorithm, data set #2 should lie in the region that is more challenging for the classifier. Thus it is reasonable that our classifier performs worse using all available data sets.

5.2. Comparison Methods

We compare our proposed algorithm to the two methods that our biochemist collaborators originally considered to use: one is a typical biological evolution approach, that is, mutate known peptides with positive label, and take N peptide candidates with highest predicted probability being positive for testing. The other method is to use the trained classifier to predict probability being positive for all peptides in the library, and select N most probable ones for testing. We show their performance in Section 5.3.

5.3. Performance Comparison

The benchmark was performed on real experiment data. We use data set #0 and #1 to train the statistical model, and based on this model, we generate peptides for testing using our proposed algorithm and the other two comparison methods. Then we train the statistical model using data set #0, #1 and #2, and treat it as oracle to evaluate the performance of the methods.

In Figure 2 we show the benchmark using P^* as performance metric. We can see Probability of Improvement algorithm is significantly better than the other two methods. Part of the reason

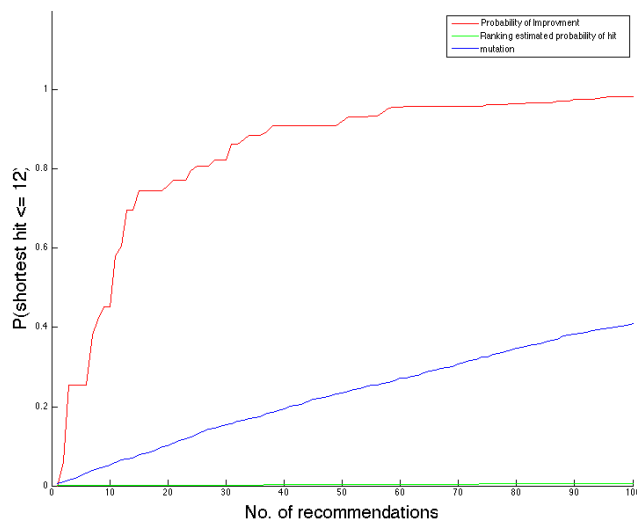


Figure 2 Benchmark of Probability of Improvement algorithm

is that the Probability of Improvement algorithm is designed to make the best decisions under uncertainty and choose the best balance between exploitation and exploration. We can see the effect of uncertainty of the current model by noting that the ranking method performs the worst, because in this method it assumes the underlying statistical model is accurate and does pure exploitation.

We also plan to show the benchmark using EI as performance metric in the future.

5.4. Progress on Real Experiments

Given the initial training data with size, we have performed two rounds of experiments using recommendations generated by Probability of Improvement algorithm. We show the histogram of length distribution of desired peptides found in Figure 3.

From the histogram we conclude that the usage of Probability of Improvement algorithm did a great job in finding short desired peptides.

6. Conclusion

We proposed an optimal search algorithm based on greedy heuristic for solving the active learning problem motivated by a biochemistry problem, which is described in Section 2, and proved that the proposed algorithm guarantees to achieve at least a factor $(1-1/e)$ of the optimal value. From benchmark results, we further showed that the proposed algorithm outperformed the other two heuristic search methods. In addition to theoretic results, We demonstrated effectiveness of our methods by applying them to the motivating problem, in which we have found a significant number of short peptides with desired property.

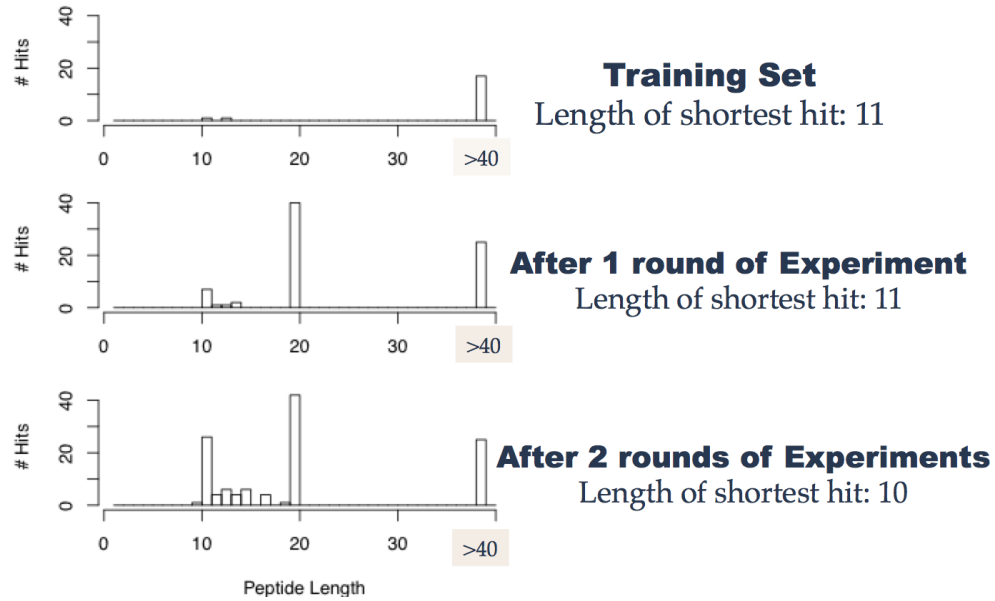


Figure 3 Length distribution of target peptides.

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