1 Solution Method for EI

We want to solve

$$\mathop{\mathbb{E}}_{S\subseteq E:|S|\leq k}\left[(b-f^*(S))^+\right]$$

1.1 objective function is submodular

1.2 greedy method

Suppose we have chosen $S=\{x_1,x_2,\ldots,x_n\}$ as a batch of points we are going to evaluate next, and if we want to incorporate one more point e, which is distinct from $x_1,x_2,\ldots,x_n$, such that the expected improvement increases most, we use the following criterion to find e:

$$\mathop{\arg\max}_{e\in E\setminus S}\mathrm{EI}(S\cup\{e\}) \tag{1}$$

We define $f^*(S)=\min\limits_{x\in S:y(x)=1}f(x)$, and write expected improvement as

$$\begin{aligned}\mathrm{EI}(S\cup\{e\}) &= \mathbb{E}[b-\min_{x\in S\cup\{e\}:y(x)=1}f(x)]\\ &= \begin{cases}\mathbb{E}[b-f^*(S)] & \text{if } y(e)=0\ ,\\ \mathbb{E}[b-\min\{f(e),f^*(S)\}] & \text{if } y(e)=1\ ,\end{cases}\\ &= \mathbb{E}[b-f^*(S)+\mathbb{1}_{\{y(e)=1,f(e)<f^*(S)\}}[f^*(S)-f(e)]]\end{aligned}$$

Thus equation (1) becomes

$$\begin{aligned}&\mathop{\arg\max}_{e\in E\setminus S}\mathbb{E}[\mathbb{1}_{\{y(e)=1,f(e)<f^*(S)\}}[f^*(S)-f(e)]]\\ &=\mathop{\arg\max}_{e\in E\setminus S}\mathbb{E}[\mathbb{E}[\mathbb{1}_{\{y(e)=1,f(e)<f^*(S)\}}[f^*(S)-f(e)]|f^*(S)]\\ &=\mathop{\arg\max}_{e\in E\setminus S}\mathbb{E}[\mathbb{1}_{\{f(e)<f^*(S)\}}\mathbb{P}(y(e)=1|f^*(S))[f^*(S)-f(e)]]\\ &=\mathop{\arg\max}_{e\in E\setminus S}\sum_{i=1}^{|S|}\mathbb{P}(y(e)=1,y(x_i)=1,y(x_j)=0,\forall j<i)[f(x_i)-f(e)]^+\\ &+\mathbb{P}(y(e)=1,y(x_j)=0,\forall j)[b-f(e)]^+\end{aligned}$$