A Proof of the Pythagorean Theorem via Heron's Formula and Circle Geometry

Jun Ohg, Jungwoo Lee

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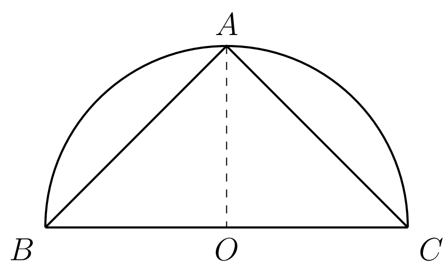
Overview

While searching for a way to newly prove the Pythagorean Theorem, we have both decided to try to find a new way to prove it using this diagram. Then we came up with this idea. We prove the Pythagorean Theorem using Heron's formula and properties of a triangle inscribed in a semicircle. The proof is divided into two exhaustive cases: $a \neq b$ and $a = b$, with $a, b > 0$ and c as hypotenuse.

Case I: $a \neq b$

Let the triangle ABC be a circle inscribed so that $\overline{BC}$ is the diameter. Let O be the center of the circle. Draw lines $\overline{AO}$, dividing $\triangle ABC$ into triangles $\triangle ABO$ and $\triangle AOC$



Using Heron's Formula

Let $a = AB$, $b = AC$, and $c = BC$. Let p be the semi-perimeter of triangle ABO, i.e.:

$$p = \frac{a+c}{2}$$

Then Heron's formula gives the area of triangle ABO as:

$$\Delta_{ABO} = \sqrt{p(p-a)(p-c)(p-a)} = \sqrt{\frac{b^2(c-b)}{4}}$$

Similarly, for triangle AOC:

$$\Delta_{AOC} = \sqrt{\frac{a^2(c-a)}{4}}$$

Adding both triangle areas, we get:

$$\Delta_{ABC} = \Delta_{ABO} + \Delta_{AOC}$$

Squaring both sides and simplifying:

$$\sqrt{\frac{b^2(c-b)}{4}} + \sqrt{\frac{a^2(c-a)}{4}} = \text{Area}$$

After algebraic manipulation, this leads to the following.

$$c^2 = a^2 + b^2$$

Case II: $a = b$

Assume $AB = AC$, and again $\overline{BC}$ is the diameter. Then the triangle ABC is isosceles and inscribed in a semicircle.

By Thales' Theorem, the angle $\angle A$ is a right angle:

$$\angle ABC = \angle ACB = 45^\circ \Rightarrow \angle BAC = 90^\circ$$

Using trigonometry or basic geometry:

$$\text{Area} = \frac{1}{2} \cdot a \cdot b = \frac{1}{2} a^2$$

And the hypotenuse is:

$$c = \sqrt{a^2 + b^2} = \sqrt{2a^2} = a\sqrt{2}$$

Thus:

$$c^2 = a^2 + b^2$$

Conclusion

Dividing into the cases $a = b$ and $a \neq b$, we have shown that:

$$c^2 = a^2 + b^2$$

for all real numbers $a, b > 0$. This completes the proof of the Pythagorean Theorem.