A Proof of the Pythagorean Theorem via Heron's Formula and Circle Geometry:2nd Revision

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Introduction

While searching for a way to newly prove the Pythagorean Theorem, we have both decided to try to find a new way to prove it using this diagram. Then we came up with this idea.

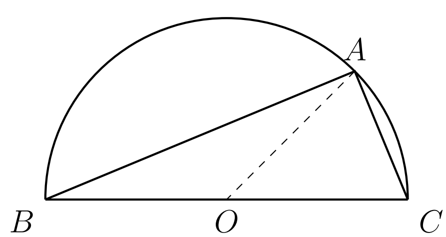
Overview

Heron's formula, proven by Heron in the first century, tells us how to calculate the area of a triangle using only three sides of it. Although the most famous proofs of this formula uses the Pythagorean Theorem to prove it, there are proofs, such as a proof from Professor Paul Yiu [1] which makes this formula able to use without causing circular reasoning.

We had proven the Pythagorean Theorem using Heron's formula and properties of a triangle inscribed in a semicircle. The proof is divided into two exhaustive cases: $a \neq b$ and $a = b$, with $a, b > 0$ and c as hypotenuse. However, after finishing our initial report and submitting it(to the public), we have found numerous errors in our report. Thus, we have written a revised version of the proof.

Case I: $a \neq b$

Let the triangle ABC be a circle inscribed so that $\overline{BC}$ is the diameter. Let O be the center of the circle. Draw lines $\overline{AO}$, dividing $\triangle ABC$ into triangles $\triangle ABO$ and $\triangle AOC$



[1]http://users.math.uoc.gr/ pamfilos/Yiu.pdf

Using Heron's Formula

Let $a = AB$, $b = AC$, and $c = BC$. Let p be the semi-perimeter of triangle ABO, i.e.:

$$p = \frac{a+c}{2}$$

Then Heron's formula gives the area of triangle ABO as:

$$\Delta_{ABO} = \sqrt{p(p-a)(p-c)(p-a)} = \sqrt{\frac{b^2(c-b)}{16}}$$

Similarly, for triangle AOC:

$$\Delta_{AOC} = \sqrt{\frac{a^2(c-a)}{16}}$$

Since triangle AOC = triangle ABO, we get:

$$\Delta_{ABO} = \Delta_{AOC}$$

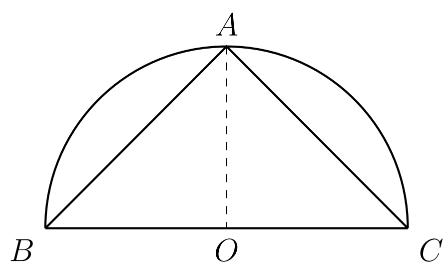
Squaring both sides and simplifying:

$$\sqrt{\frac{b^2(c-b)}{16}} = \sqrt{\frac{a^2(c-a)}{16}}$$

After algebraic manipulation, this leads to the following.

$$c^2 = a^2 + b^2$$

Case II: $a = b$



Assume $\overline{AB} = \overline{AC}$, and again $\overline{BC}$ is the diameter. Then the triangle ABC is isosceles and inscribed in a semicircle.

By Thales' Theorem, the angle $\angle A$ is a right angle:

$$\angle ABC = \angle ACB = 45^\circ \Rightarrow \angle BAC = 90^\circ$$

And since $\overline{AB} = \overline{AC}$,

$$a^2 = b^2$$

Using trigonometry or basic geometry:

$$\text{Area} = \frac{1}{2} \cdot a \cdot b = \frac{1}{2}a^2$$

And the area can also be found like this:

$$\text{Area} = \frac{1}{2} \cdot \frac{c}{2} \cdot c = \frac{1}{4}c^2$$

Thus:

$$c^2 = 2a^2$$

$$c^2 = a^2 + b^2$$

Conclusion

Dividing into the cases $a = b$ and $a \neq b$, we have shown that:

$$c^2 = a^2 + b^2$$

for all real numbers $a, b > 0$. This completes the proof of the Pythagorean Theorem.