**Technische Universität Berlin** Winter term 2025/26
**Institute of Mathematics**
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# Numerical Mathematics II for Engineers
# Tutorial 2

Topics : Method of characteristics, Heat equation, classification of PDEs.

Discussion in the tutorials of the week 03–07 November 2025

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*Disclaimers:*

- To test your code, we provide tests. This should help you to locate precisely your errors, and code more efficiently. To run the test, you need to install pytest[1].
- Exercises should be solved in **fixed groups of 3 students**.
- Hand in the solution in **one folder** labeled **hw2_group[group_number]** and containing:
  - **One pdf** for the theoretical questions and comments on the numerical results,
  - **One python file per programming exercise**: e.g. for this homework, you should have one python files: heat.py .
  - Write the group number and all names of your members **in each file**.

**Exercise 2.1: Method of characteristics**

We consider again the linear advection equation with constant velocity $a \in \mathbb{R}$

$$u_t - au_x = 0 \qquad \forall (t,x) \in \mathbb{R}_+ \times \mathbb{R} \tag{1}$$
$$u(0,x) = u_0(x) \qquad \forall x \in \mathbb{R}, \tag{2}$$

with the discontinuous initial condition (see Figure 1)

$$u_0(x) = \begin{cases} 0.5, & x < 1, \\ 1, & x = 1, \\ 2, & x > 1. \end{cases}$$

We solve this equation using the method of characteristics.

(a) Draw the characteristics in the $(x,t)$ plane for the following cases: $a = 1$, $a = 2$ and $a = -0.5$.

(b) Use your drawing to determine the solution at $t = 2$ for each case. Compare them with the theoretical solution.

[1] https://docs.pytest.org/en/stable/getting-started.html. Only the section "Install pytest" is relevant for you!

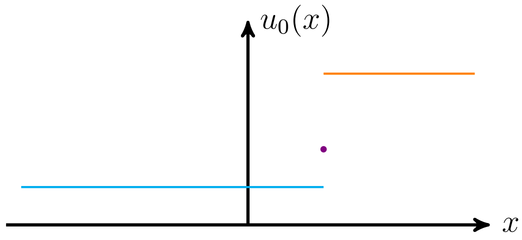


Figure 1: Initial condition $u_0(x)$.

**Exercise 2.2: Heat equation**

We consider the 1D heat equation on $(0,1) \times (a,b)$ with Dirichlet boundary conditions and no source term

$$u_t - u_{xx} = 0 \qquad \forall (t,x) \in (0,1) \times (a,b) \tag{3}$$
$$u(0,x) = u_0(x) \qquad \forall x \in (a,b), \tag{4}$$
$$u(t,a) = g_a(t), u(t,b) = g_b(t) \qquad \forall t \in (0,1). \tag{5}$$

We want to solve this equation with central FD in space and the implicit trapezoidal method in time. For this, the template `heat_template.py` and the test `heat_test.py` are provided.

(a) The fully discretized system has the form (see Lecture 04)

$$\underbrace{\left(\mathbf{I} - \frac{\Delta t}{2}\mathbf{A}\right)}_{\mathbf{A}_l} \mathbf{U}^{n+1} = \underbrace{\left(\mathbf{I} + \frac{\Delta t}{2}\mathbf{A}\right)}_{\mathbf{A}_r} \mathbf{U}^{n} + \frac{\Delta t}{2h^2}\left(g(t_n) + g(t_{n+1})\right). \tag{6}$$

Implement the computation of $\mathbf{A}_l$ and $\mathbf{A}_r$ as well as the update formula (6) in the functions `create_matrices`, `get_rhs`, and `update_boundaries`.

*Hint:* In the code, the vector `g` represents the value $g(t_n) + g(t_{n+1})$ from the lecture.

(b) We solve the equation for $a = 0, b = 3\pi/2$, $u_0(x) = \sin(x)$, $g_a(t) = 0$ and $g_b(t) = -e^{-t}$. The exact solution to (3) - (5) is then $u_{\text{exact}}(t,x) = e^{-t}\sin(x)$. Check that $u_{\text{exact}}$ is indeed the exact solution. Plot then the exact and approximated solutions at final time for $N = 40$.

(c) When $\Delta t$ is proportional to $h$, the scheme should converge with second order in space and in time. We examine the error at the final time $t = 1$. Check numerically the convergence with respect to time using a convergence table regarding the time-step size $N_T \in \{20, 40, \dots 640\}$.

*Note: You can use the parameter* ***`Nrefine`*** *for the convergence study.*

(d) Plot the solution at final time for $N_T = 80$. How does the solution look like? In particular, compare it to the solution to the advection equation studied in homework 02.

**Exercise 2.3: Classification of PDEs**

(a) Determine if the following PDEs are linear or nonlinear and their order.

$$-2u_t = 2u_{xxxx} \qquad x \in \Omega \subset \mathbb{R},\ t > 0, \tag{7}$$
$$u + u_x u_{xx} + u_t = 0 \qquad x \in \Omega \subset \mathbb{R},\ t > 0, \tag{8}$$
$$\operatorname{div}(u\nabla u) = u_t \qquad x \in \Omega \subset \mathbb{R},\ t > 0, \tag{9}$$

(b) We consider the two following PDEs:

$$
\begin{aligned}
-2u_{xx} - 2u_{yy} + 2u_{xy} &= 0 \qquad (x, y) \in \Omega \subseteq \mathbb{R}^2, && (10)\\
-u_{xx} + xu_y - u_{yy} - zu_{zz} - yu_z &= 0 \qquad (x, y, z) \in \Omega \subseteq \mathbb{R}^3. && (11)
\end{aligned}
$$

For each PDE, determine the maximal doman $\Omega$ for wich the equation is elliptic, parabolic, or hyperbolic.