we have:

$$E_{\text{tot}} = \sigma T^4$$

Here,

$$\sigma = \frac{8\pi^5}{15}\frac{k^4}{h^3c^3} \tag{3.39}$$

Thus the total energy emitted per unit area of the source per second at a given temperature is $\int_0^\infty E_\lambda \, d\lambda$ and is proportional to the fourth power of the absolute temperature. This is called *Stefan's law*.

Schrodinger Equation

To obtain an equation satisfied by ψ, we can start with the well-known and the simplest type of a plane monochromatic wave described by:

$$\psi(x, t) = A_0 e^{i(k_0 x - \omega t)} \tag{3.40}$$

where the amplitude of disturbance is A_0 and wave vector $k = 2\pi/\lambda$. If the wave is propagating along the x-axis, then:

$$k = k_0 \hat{x}$$

Hence

$$\psi(x, t) = A_0 e^{i(px - Et)/\hbar} \tag{3.41}$$

where $E = \hbar\omega,\ p = \hbar k_0$. Differentiation of ψ gives:

$$i\hbar\frac{\partial\psi}{\partial t} = E\psi(x, t) \tag{3.42}$$

or

$$-i\hbar\frac{\partial\psi}{\partial x} = p\psi(x, t) \tag{3.43}$$

or

$$\frac{-\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} = \frac{p^2}{2m}\psi(x, t) \tag{3.44}$$

For a free non-relativistic particle:

$$E = \frac{p^2}{2m} \tag{3.45}$$

and

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi(x, t)}{\partial x^2} \tag{3.46}$$

This is *one-dimensional time-dependent Schrodinger equation* for a free particle. It can also be learnt from Eq. (3.45) that E and p can be represented by operators:

$$E \to i\hbar\frac{\partial}{\partial t}, \quad p \to -i\hbar\frac{\partial}{\partial x}$$

Equation (3.46) can also be written as:

$$E\psi = \frac{p^2}{2m}\psi(x, t) \quad (3.4$$

The derivation, in Eq. (3.47), can be generalized in three dimensions. Hence

$$\psi(r, t) = A_0 \exp\left[\frac{i}{\hbar}(pr - Et)\right] = A_0 \exp\left[\frac{i}{\hbar}(p_x x + p_y y + p_z z - Et)\right] \quad (3.4$$

We have

$$i\hbar\frac{\partial\psi}{\partial t} = E\psi$$

$$-i\hbar\frac{\partial\psi}{\partial x} = p_x\psi; \quad -i\hbar\frac{\partial\psi}{\partial y} = p_y\psi, \quad -i\hbar\frac{\partial\psi}{\partial z} = p_z\psi$$

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} = \frac{p_x^2}{2m}\psi; \quad -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial y^2} = \frac{p_y^2}{2m}\psi; \quad -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial z^2} = \frac{p_z^2\psi}{2m}$$

Since,

$$E = \frac{p^2}{2m} = \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2)$$

we have,

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2}\right) = -\frac{\hbar^2}{2m}\nabla^2\psi \quad (3.49$$

Equation (3.49) is the *three-dimensional Schrodinger equation* for a free particle. Again, E and p can be represented by operators as follows:

$$E \to i\hbar\frac{\partial}{\partial t}, \quad p \to -i\hbar\nabla$$

Then

$$E\psi = \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2) = \frac{p^2}{2m}\psi \quad (3.50)$$

Let us now consider the particle in a potential energy function $V(r, t)$. According to classical mechanics, the total energy would be given by:

$$E = \frac{p^2}{2m} + V(r, t) \quad (3.51)$$

Since the potential energy function does not depend on E and p, the wave function should satisfy:

$$i\hbar\frac{\partial\psi}{\partial t} = \left[\frac{p^2}{2m} + V(r, t)\right]\psi = -\frac{\hbar^2}{2m}\nabla^2\psi + V\ldots$$

Equation (3.52) is the *time-dependent Schrodinger equation* for a particle in potential field of $V(r, t)$. It can also be written as:

$$i\hbar\frac{\partial\psi}{\partial t} = H\psi \tag{3.53}$$

Here

$$H = \frac{p^2}{2m} + V = -\frac{\hbar^2}{2m}\nabla^2 + V \tag{3.54}$$

H is known as the *Hamiltonian operator.*

Time-Independent Schrodinger Equation

The time-dependent Schrodinger equation is written as:

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V(x)\psi(x, t) \tag{3.55}$$

If the energy of a system is independent of time, the general solution of the Schrodinger equation can be expressed as a function of spatial coordinates and a function of time by using the method of separation of variables. That is,

$$\psi(x, t) = \psi(x)T(t) \tag{3.56}$$

Substituting Eq. (3.55) in Eq. (3.56) gives:

$$\frac{i\hbar}{T(t)}\frac{dT}{dt} = \frac{1}{\psi}\left[-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x)\psi\right] \tag{3.57}$$

Equation (3.57) can be separated in space and time, because the left-hand side has only terms of time, whereas the right-hand side has only spatial variables. Hence

$$\frac{i\hbar}{T}\frac{dT}{dt} = \frac{1}{\psi}\left[-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x)\psi\right] = E \tag{3.58}$$

where

$$T(t) = e^{-\frac{i}{\hbar}Et} \tag{3.59}$$

Hence

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x)\psi = E\psi \tag{3.60}$$

Thus the solution can be written as:

$$\psi(x, t) = \psi(x)e^{-\frac{i}{\hbar}Et} \tag{3.61}$$

The probability density is given by:

$$\rho = |\psi(x, t)|^2 = |\psi(x)|^2 \tag{3.62}$$

Equation (3.60) can be written as:

$$\boxed{H\psi(x) = E\psi(x)} \tag{3.63}$$

Here

$$H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x) = \frac{p^2}{2m} + V(x) \tag{3.64}$$

Physical Interpretation of ψ and the Probability Current Density

The Schrodinger equation can be written as:

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + V\psi \tag{3.65}$$

Taking complex conjugate of Eq. (3.65):

$$-i\hbar\left(\frac{\partial\psi^*}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi^* + V\psi^*\right) \tag{3.66}$$

Now multiplying Eq. (3.65) with ψ^* and Eq. (3.66) with ψ and subtracting, we obtain:

$$i\hbar\left(\psi^*\frac{\partial\psi}{\partial t} + \psi\frac{\partial\psi^*}{\partial t}\right) = -\frac{\hbar^2}{2m}(\psi^*\nabla^2\psi - \psi\nabla^2\psi^*) \tag{3.67}$$

or

$$i\hbar\frac{\partial(\psi\psi^*)}{\partial t} = -\frac{\hbar^2}{2m}\left[\psi^*\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\psi - \psi\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\psi^*\right] \tag{3.68}$$

Equation (3.68) can be written as:

$$\frac{\partial\psi\psi^*}{\partial t} + \frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z} = 0$$

where

$$J_x = \frac{i\hbar}{2m}\left(\psi\frac{\partial\psi^*}{\partial x} - \psi^*\frac{\partial\psi}{\partial x}\right), \quad J_y = \frac{i\hbar}{2m}\left(\psi\frac{\partial\psi^*}{\partial y} - \psi^*\frac{\partial\psi}{\partial y}\right), \quad J_z = \frac{i\hbar}{2m}\left(\psi\frac{\partial\psi^*}{\partial z} - \psi^*\frac{\partial\psi}{\partial z}\right)$$

Thus Eq. (3.68) can be written as:

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\mathbf{J} = 0 \tag{3.69}$$

Here

$$\rho = \psi^*\psi \quad \text{and} \quad \mathbf{J} = \frac{i\hbar}{2m}(\psi\nabla\psi^* - \psi^*\nabla\psi) \tag{3.70}$$

Equation (3.70) is the *equation of continuity for microscopic particles*, that is, the spatial outward flux of particle is equal to rate of flow with time. Here $\psi^*\psi$ and $\mathbf{J}$ are interpreted respectively as probability density and probability current density. $|\psi|^2\,d\tau = 1$ represents the probability of finding the particle in $d\tau$ volume element. $\iiint|\psi|^2 d\tau = 1$ represents the probability of finding the particle in the whole space which, of course, will be unity.