Homework 3

- Show that the expected value of a log-normal variable $S(t)$

$$f(x) = \frac{\exp\left(\frac{-(\log(x/S_0) - (\mu - \sigma^2/2)t)^2}{2\sigma^2 t}\right)}{x\sigma\sqrt{2\pi t}}, \qquad \text{for } x > 0.$$

$$f(x) = 0, \qquad \text{for } x \le 0$$

is given as

$$E[S(t)] = S_0 e^{\mu t}$$

Definition A random variable X is said to have a lognormal distribution if its probability density function is given by

$$f(x) = \begin{cases} \frac{1}{x\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\ln(x) - \mu}{\sigma}\right)^2} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

$\Rightarrow$ $X \sim \text{Lognormal}(\mu, \sigma^2)$ <Notation>

$\log(X) \sim \text{Normal}(\mu, \sigma^2)$

THEOREM. If $X \sim \text{Lognormal}(\mu, \sigma^2)$ then

$$E[X] = e^{\mu + \frac{1}{2}\sigma^2}$$

Proof. Let t be the positive integer.

We compute the n th moment of X

$$E[X^n] = \int_0^\infty x^n f(x)\, dx$$

$$= \int_0^\infty x^n \frac{1}{x\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\ln(x)-\mu}{\sigma}\right)^2} dx$$

Substituting $z = \ln x \iff x = e^z$ and $dx = e^z dz$

$$E[X^n] = \int_{-\infty}^\infty e^{nz} \frac{1}{\cancel{e^z}\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{z-\mu}{\sigma}\right)^2\right) \cancel{e^z}\, dz$$

$$E[X^n] = \int_{-\infty}^\infty e^{zn} \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{z-\mu}{\sigma}\right)^2\right) dz$$

$$E[X^n] = E\left[e^{n(\log X)}\right]$$

$\because$ $\log X \sim \text{Normal}(\mu, \sigma^2)$ and

$$\frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{z-\mu}{\sigma}\right)^2\right)$$ is according p.d.f.

That means $E[X^n] = M_{\log X}(n)$ where $M_{\log X}(n)$ denotes the moment generating function of the random variable $\log X \sim \text{Normal}(\mu, \sigma^2)$

Therefore, $M_{\log X}(n) = e^{\mu n + \frac{1}{2}\sigma^2 n^2}$

Thus, letting $n=1$, we get

$$E[X] = e^{\mu + \frac{1}{2}\sigma^2}$$ which completes the proof ■

In our model,

- A stock price $S(t)$ follows lognormal distribution

$$S(t) = S_0 e^{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma\sqrt{t}Z}, \qquad Z \sim N(0,1)$$

Define a random variable Y as follow

$$Y := \frac{S(t)}{S_0} = \exp\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma\sqrt{t}\,Z$$

Then the following distribution of random variable Y is

$$f(x) = \frac{1}{x\,(\sigma\sqrt{t})\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{\ln x - (\mu - \sigma^2/2)t}{(\sigma\sqrt{t})}\right)^2\right)$$

substituting $\sigma' = \sigma\sqrt{t}$ and $\mu' = \left(\mu - \frac{\sigma^2}{2}\right)t$

we can see that $Y \sim \text{Lognormal}(\mu', \sigma'^2)$

Then $E[Y] = \dfrac{E[S(t)]}{S_0} = e^{\mu' + \frac{1}{2}\sigma'^2}$ by previous theorem since $\mu' + \frac{1}{2}\sigma'^2 = \left(\mu - \cancel{\frac{\sigma^2}{2}}\right)t + \cancel{\frac{1}{2}\sigma^2 t} = \mu t$

$E[S(t)] = S_0 e^{\mu t}$