공학수학 1 과제 2

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1.3 #7 $xy' = y + 2x^3 \sin^2\frac{y}{x}$ $\quad \frac{y}{x} = u \quad y = ux,\ y' = u'x + u \cdots ①$

$x=0$ 일때는 $\frac{y}{x}$ 정의 안되므로 $x \neq 0$ ∴ 양변 x로 나누면

$y' = \frac{y}{x} + 2x^2 \sin^2\frac{y}{x}$ ① 대입하면

$u'x + u = u + 2x^2 \sin^2 u \rightarrow \frac{u'}{2\sin^2 u} = x \Rightarrow \frac{\partial u}{2\sin^2 u} = x\,\partial x$

$\int \frac{1}{2\sin^2 u}\,\partial u = \int x\,\partial x \rightarrow -\frac{1}{2\tan u} = \frac{1}{2}x^2 + C$ $\quad \therefore \frac{1}{\tan(\frac{y}{x})} = -x^2 + C$

1.3 #13 $y' \cosh^2 x = \sin^2 y \rightarrow \frac{y'}{\sin^2 y} = \frac{1}{\cos^2 hx} \Rightarrow \frac{\partial y}{\sin^2 y} = \frac{\partial x}{\cos^2 hx}$

$\int \frac{1}{\sin^2 y}\,\partial y = \int \frac{1}{\cos^2 hx}\,\partial x \quad \frac{1}{\tan y} = \tanh(x) + C \rightarrow x = 0\ \ y = \frac{1}{2}\pi$ 대입

$0 = 0 + C \quad \therefore C = 0 \qquad \therefore \frac{1}{\tan y} = \tanh(x)$

1.3 #19 시간 t에서 박테리아의 개수를 $x(t)$라 하면

$x'(t) = k \cdot x(t) \rightarrow \frac{x'(t)}{x(t)} = k \quad \int \frac{x'(t)}{x(t)}\,\partial t = \int k\,\partial t$ 에서

$\log x(t) = kt + C \rightarrow x(t) = e^{kt} \cdot C$ 7일에 2배가 는다 했으므로

$e^{k \cdot 7일} = 2 \quad \therefore e^{k \cdot 14일} \cdot C = 4,\ e^{k \cdot 28일} \cdot C = 16$ $\quad$ ∴ 4배, 16배

1.4 #3 $\sin x \cos y\,\partial x + \cos x \sin y\,\partial y = 0$

$\frac{\partial \sin x \cos y}{\partial y} = -\sin x \sin y \quad \frac{\partial \cos x \sin y}{\partial x} = -\sin x \sin y$ $\quad$ ∴ 완전미분 방정식이다

$\int \sin x \cos y\,\partial x = -\cos x \cos y + k(y) \quad \frac{\partial(-\cos x \cos y + k(y))}{\partial y} = \cos x \sin y + k'(y) = \cos x \sin y$

$k'(y) = 0 \quad \therefore k(y) = C \qquad \therefore -\cos x \cos y + C$

1.4 #7 $2x \tan y\,\partial x + \sec^2 y\,\partial y = 0$

$\frac{\partial(2x\tan y)}{\partial y} = 2x\sec^2 y \quad \frac{\partial \sec^2 y}{\partial x} = 0$ $\quad$ ∴ 완전미분 방정식이 아니다.

뒤에서 계속

$P(x,y)\partial x + Q(x,y)\partial y = 0$ ---① 이라 할 때,

i) $R = \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$ 일 때,

$R = \frac{1}{\sec^2 y}\left(2x\sec^2 y - 0\right) = 2x$ (x만의 함수으로 성립.)

$F(x) = e^{\int R(x)\partial x} = e^{x^2}$

$\therefore F(x)(①) = e^{x^2}\cdot 2x\tan y\,\partial x + e^{x^2}\cdot\sec^2 y\,\partial y = 0$

$\int e^{x^2}\cdot 2x\tan y\,\partial x = e^{x^2}\cdot\tan y + k(y)$ $\quad \frac{\partial(e^{x^2}\cdot\tan y + k(y))}{\partial y} = e^{x^2}\cdot\sec^2 y + k'(y) = e^{x^2}\cdot\sec^2 y$

$k'(y) = 0 \rightarrow k(y) = C$ $\quad \therefore e^{x^2}\cdot\tan y + C$

1.4 # 11 $\quad \underbrace{2\cosh x\cos y}_{\rightarrow P(x,y)}\partial x - \underbrace{\sinh x\sin y}_{\rightarrow Q(x,y)}\partial y = 0$ ---① 라 하자.

$\frac{\partial P}{\partial y} = -2\cosh x\sin y$ $\quad \frac{\partial Q}{\partial x} = -\cosh x\sin y$ $\quad \boxed{\therefore \text{완전미분방정식이 아니다.}}$

i) $R = \frac{1}{Q}\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$ 일 때,

$R = \frac{1}{-\sinh x\sin y}\left(-2\cosh x\sin y + \cosh x\sin y\right) = \frac{\cosh x}{\sinh x}$

$F(x) = e^{\int R(x)\partial x} = e^{\log(\sinh x)} = \sinh x$

$\therefore F(x)(①) = 2\sinh x\cosh x\cos y\,\partial x - \sin^2 hx\sin y\,\partial y = 0$

$\int 2\sinh x\cosh x\cos y\,\partial x = \sin^2 hx\cos y + k(y)$ $\quad \frac{\partial(\sin^2 hx\cos y + k(y))}{\partial y} = -\sin^2 hx\sin y + k'(y) = -\sin^2 hx\sin y$

$k'(y) = 0 \rightarrow k(y) = C$

$\therefore \sin^2 hx\cos y + C$

1.5 #8 $y' + y\tan x = e^{-0.01x}\cos x \quad y(0)=0$

양변에 $F(x)$를 곱하면 $Fy' + F\cdot\tan x\cdot y = F\cdot e^{-0.01x}\cos x$

만약, $F' = F\tan x$ 이면 좌변은 $(Fy)'$ 가 된다.

$\therefore \frac{F'}{F} = \tan x$ 에서 $\log F = -\log(\cos x) \rightarrow F = \frac{1}{\cos x}$

$\therefore \underset{ⓐ}{\frac{1}{\cos x}\cdot y' + \frac{\sin x}{\cos^2 x}\cdot y} = \underset{ⓑ}{e^{-0.01x}}$ $\quad \int ⓐ\,\partial x = \int ⓑ\,\partial x$ 에서

$\frac{1}{\cos x}\cdot y(x) = -100e^{-0.01x} + C$ $\quad x=0, y=0$ 대입하면

$C = 100$ $\quad \therefore y = 100\cos x(e^{-0.01x} - 1)$

1.5 #13 $y' = 6(y - 2.5)\tanh 1.5x$

$y' - 6y = -15\tanh 1.5x$

양변에 e^{-6x}을 곱하면

$\underset{ⓐ}{e^{-6x}\cdot y' - 6y\cdot e^{-6x}} = \underset{ⓑ}{-15\cdot e^{-6x}\tan 1.5x}$ $\quad \int ⓐ\,\partial x = \int ⓑ\,\partial x$ 에서

$\therefore e^{-6x}\cdot y(x) = -15\int e^{-6x}\cdot\tanh 1.5x\,\partial x$ $\quad u = \frac{3}{2}x$ 라 하면 $\partial u = \frac{3}{2}\partial x$

$= -10\int e^{-4u}\cdot\tanh u\,\partial u = -10\int \frac{(e^{2u}-1)}{e^{2u}+1}\cdot e^{-4u}$ $\quad v = e^{2u}$ 라하면 $\partial v = 2e^{2u}\partial u$

$= -5\int \frac{v-1}{v^3(v+1)}\partial v = -5\left(\int\left(\frac{2}{v+1} - \frac{2}{v} + \frac{2}{v^2} - \frac{1}{v^3}\right)\partial v\right) = -5\left(2\log(v+1) - 2\log v - \frac{2}{v} + \frac{1}{2v^2} + C\right)$

$= -5\left(2\log(e^{2u}+1) - 4u - \frac{2}{e^{2u}} + \frac{1}{2e^{4u}} + C\right) = -10\log(e^{2u}+1) + 20u + \frac{10}{e^{2u}} - \frac{5}{2e^{4u}} + C$

$= -10\log(e^{3x}+1) + 30x + \frac{10}{e^{3x}} - \frac{5}{2e^{6x}} + C$

1.5 #26 $y' = \frac{\tan y}{x-1}$ $y(0) = \frac{1}{2}\pi$

$\frac{y'}{\tan y} = \frac{1}{x-1}$ 에서 $\int \frac{y'}{\tan y} dy = \int \frac{1}{x-1} dx$

$\log(\sin y) = \log(x-1) + C$ $\sin y = (x-1)C$ $x=0$ $y=\frac{1}{2}\pi$ 대입

$1 = -C$ $C = -1$ $\therefore \sin y = -(x-1)$

2.1 #7 $y'' + y'^3 \sin y = 0$

$y' = z$ 라 하면 $z = y' = \frac{\partial y}{\partial x}$

$y'' = \frac{\partial y'}{\partial x} = \frac{\partial y'}{\partial y} \cdot \frac{\partial y}{\partial x} = \frac{\partial z}{\partial y} \cdot z = -z^3 \sin y$ ...①

①의 양변을 $-z^3$로 나누면 $-\frac{1}{z^2} \cdot \frac{\partial z}{\partial y} = \sin y \longrightarrow \int -\frac{\partial z}{z^2} = \int \sin y \, \partial y$

$\frac{1}{z} = -\cos y + C_1$ $z = \frac{\partial y}{\partial x}$를 대입하면

$\frac{\partial x}{\partial y} = -\cos y + C_1 \rightarrow \int \partial x = \int (-\cos y + C_1) \partial y$

$x = -\sin y + C_1 y + C_2$

2.1 #10 $y'' + (1 + \frac{1}{y}) y'^2 = 0$

$y' = z$라 하면 $z = y' = \frac{\partial y}{\partial x}$

$y'' = \frac{\partial y'}{\partial x} = \frac{\partial y'}{\partial y} \cdot \frac{\partial y}{\partial x} = \frac{\partial z}{\partial y} \cdot z = -z^2 (1 + \frac{1}{y})$ ...①

①의 양변을 $-z^2$로 나누면 $-\frac{1}{z} \cdot \frac{\partial z}{\partial y} = (1 + \frac{1}{y}) \longrightarrow \int -\frac{\partial z}{z} = \int 1 + \frac{1}{y} \, \partial y$

$-\log z = y + \log y + \log C_1 \rightarrow \frac{1}{z} = e^y \cdot y \cdot C_1$ $z = \frac{\partial y}{\partial x}$를 대입하면

$\frac{\partial x}{\partial y} = e^y \cdot y \cdot C_1 \rightarrow \int \partial x = \int C_1 \cdot y \cdot e^y \partial y$ $\therefore x = C_1 (y-1) e^y + C_2$

2.2 #22 $y''+4y'+(\pi^2+4)y=0$ ---① $y(\frac{1}{2})=1$, $y'(\frac{1}{2})=-2$

$y=e^{kx}$ 라 하면 $y'=ke^{kx}$ $y''=k^2e^{kx}$

$\therefore$ ①: $e^{kx}(k^2+4k+\pi^2+4)=0$ $k=-2\pm\sqrt{-\pi^2}=-2\pm\pi i$

$\therefore y_1=e^{(-2+\pi i)x}=e^{-2x}\cdot\cos\pi x$, $y_2=e^{(-2-\pi i)x}=e^{-2x}\cdot\sin\pi x$

$\therefore y=e^{-2x}(A\cos\pi x+B\sin\pi x)$

$y(\frac{1}{2})=1$ 에서 $e^{-1}(B)=1$ $B=e$

$y'=-2e^{-2x}(A\cos\pi x+B\sin\pi x)+\pi e^{-2x}(-A\sin\pi x+B\cos\pi x)$

$y'(\frac{1}{2})=-2$ 에서 $-2e^{-1}\cdot(B)+\pi\cdot e^{-1}(-A)=-2$

$A=0$

$\therefore y=e^{-2x}(e\sin\pi x)$

#26 $y''-k^2y=0$ ---① $(k\neq0)$ $y(0)=1$, $y'(0)=1$

$y=e^{\lambda x}$ 라 하면 $y'=\lambda e^{\lambda x}$ $y''=\lambda^2e^{\lambda x}$

$\therefore$ ①: $e^{\lambda x}(\lambda^2-k^2)=0$ $\lambda=\pm k$

$\therefore y=Ae^{kx}+Be^{-kx}$

$y(0)=1$ 에서 $A+B=1$

$y'=k(Ae^{kx}-Be^{-kx})$

$y'(0)=1$ 에서 $k(A-B)=1$ $\therefore A=\frac{k+1}{2k}$ $B=\frac{k-1}{2k}$

$\therefore y=\frac{1}{2k}\left((k+1)e^{kx}-(k-1)e^{-kx}\right)$

2.2 #27 $\underline{y''+2\pi y'+\pi^2 y=0}\cdots①$ $y(0)=4.5$ $y'(0)=-4.5\pi-1=13.137$

$y=e^{kx}$ 라 하면 $y'=ke^{kx}$ $y''=k^2e^{kx}$

$\therefore ①: e^{kx}(k^2+2\pi k+\pi^2)=0 \rightarrow k=-\pi$

$\therefore y=(A+Bx)e^{-\pi x}$

$y(0)=4.5$ 에서 $A=4.5$

$y'=(B-A\pi-B\pi x)e^{-\pi x}$

$y'(0)=-4.5\pi-1$ 에서 $B-4.5\pi=-4.5\pi-1$ $B=-1$

$\therefore y=(4.5-x)e^{-\pi x}$

2.2 #29 $\underline{y''+0.54y'+(0.0729+\pi)y=0}\cdots①$ $y(0)=0$, $y'(0)=1$

$y=e^{kx}$ 라 하면 $y'=ke^{kx}$ $y''=k^2e^{kx}$

$\therefore ①: e^{kx}(k^2+0.54k+0.0729+\pi)=0 \rightarrow k=-0.27\pm\sqrt{\pi}\,i$

$\therefore y=e^{-0.27x}(A\cos\sqrt{\pi}x+B\sin\sqrt{\pi}x)$

$y(0)=0$ 에서 $A=0$

$y'=-0.27\cdot e^{-0.27x}(A\cos\sqrt{\pi}x+B\sin\sqrt{\pi}x)+\sqrt{\pi}\cdot e^{-0.27x}(-A\sin\sqrt{\pi}x+B\cos\sqrt{\pi}x)$

$y'(0)=1$ 에서 $-0.27\cdot(A)+\sqrt{\pi}\cdot(B)=1$ $B=\frac{1}{\sqrt{\pi}}$

$\therefore y=e^{-0.27x}\left(\frac{1}{\sqrt{\pi}}\sin\sqrt{\pi}x\right)$

2.4 #11 $y(t) = C_1 e^{-(\alpha-\beta)t} + C_2 e^{-(\alpha+\beta)t}$ $y(0)=y_0$ $v(0)=v_0$

$y(0)=y_0$ 에서 $C_1 + C_2 = y_0 \cdots ①$

$$v(t) = \frac{\partial(y(t))}{\partial t} = -(\alpha-\beta)\cdot C_1 \cdot e^{-(\alpha-\beta)t} - (\alpha+\beta)\cdot C_2 \cdot e^{-(\alpha+\beta)t}$$

$v(0)=v_0$ 에서 $-(\alpha-\beta)\cdot C_1 - (\alpha+\beta)\cdot C_2 = v_0$

$$\begin{array}{rl} & (\alpha-\beta)\cdot C_1 + (\alpha+\beta) C_2 = -v_0 \\ - & (\alpha-\beta) C_1 + (\alpha-\beta) C_2 = (\alpha-\beta) y_0 \quad ① \times (\alpha-\beta) \\ \hline & 2\beta \cdot C_2 = -v_0 + (\beta-\alpha) y_0 \end{array} \qquad \therefore C_2 = \frac{(\beta-\alpha)y_0 - v_0}{2\beta}$$

$$= \frac{\left(\left(1-\frac{\alpha}{\beta}\right)y_0 - \frac{v_0}{\beta}\right)}{2}$$

$$\begin{array}{rl} & (\alpha-\beta) C_1 + (\alpha+\beta) C_2 = -v_0 \\ - & (\alpha+\beta) C_1 + (\alpha+\beta) C_2 = (\alpha+\beta) y_0 \quad ① \times (\alpha+\beta) \\ \hline & -2\beta \cdot C_1 = -v_0 - (\alpha+\beta) y_0 \end{array} \qquad \therefore C_1 = \frac{(\alpha+\beta)y_0 + v_0}{2\beta}$$

$$= \frac{\left(1+\frac{\alpha}{\beta}\right)y_0 + \frac{v_0}{\beta}}{2}$$

$\therefore$ Q.E.D