DERIVED FUNCTORS

JUNAID AFTAB

ABSTRACT. These are notes on derived functors. There may be typos; please send corrections to junaid.aftab1994@gmail.com.

CONTENTS

Throughout, let $\mathscr{A}$ be a locally small abelian category to ensure that the Hom functors are set-valued.

1. Hom Functors

We briefly review the Hom functors.

Definition 1.1. Let $A \in \mathscr{A}$. The Hom functor $\operatorname{Hom}(A, -) : \mathscr{A} \to \mathbf{Ab}$, is defined by

$$\operatorname{Hom}(A, -)(B) = \operatorname{Hom}(A, B),$$

for all $B \in \mathscr{A}$.

Let's verify that $\operatorname{Hom}(A, -)$ is indeed a functor.

Lemma 1.2. *For $A \in \mathscr{A}$, $\operatorname{Hom}(A, -)$ is a covariant functor.*

Proof. If $f : B \to B'$ is a morphism $\mathscr{A}$, then $\operatorname{Hom}(A, -)(f) : \operatorname{Hom}(A, B) \to \operatorname{Hom}(A, B')$ is given by $h \mapsto f \circ h$. Note that the composite $f \circ h$ makes sense:

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[rr, bend right, "f\circ h"'] & B \arrow[r, "f"] & B' \end{tikzcd}$$

We call $\operatorname{Hom}(A, -)(f)$ the induced map, and we denote it by f_*. If f is the identity map $1_B : B \to B$, then

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[rr, bend right, "h"'] & B \arrow[r, "1_B"] & B \end{tikzcd}$$

Hence so that $(1_B)_* = 1_{\mathrm{Hom}(A,B)}$. Suppose now that $g : B' \to B''$. We have the following diagram:

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[rrr, bend left, "(g\circ f)\circ h"] \arrow[rr, bend right, "f\circ h"'] & B \arrow[r, "f"] & B' \arrow[r, "g"] \arrow[r, bend right, "g\circ(f\circ h)"'] & B'' \end{tikzcd}$$

Clearly, $g \circ (f \circ h) = (g \circ f) \circ h$ Therefore, we have $(g \circ f)_* = g_* \circ f_*$. $\square$

We now discuss the contravariant Hom functor.

Definition 1.3. Let $B \in \mathscr{A}$. The contravariant Hom functor $\mathrm{Hom}(A, -) : \mathscr{A} \to \mathbf{Ab}$, is defined by

$$\mathrm{Hom}(-, A)(B) = \mathrm{Hom}(A, B),$$

for all $A \in \mathscr{A}$.

Remark 1.4. *It can be verified, similarly to Lemma 1.2, that the contravariant Hom functor is indeed a well-defined contravariant functor.*

We now show that the Hom functors are are also left and right exact depending on the choice of the Hom functor.

Proposition 1.5. *Let $A \in \mathscr{A}$.*

1. *The functor $\mathrm{Hom}(A, -) : \mathscr{A} \to \mathbf{Ab}$ is a left exact functor.*
2. *The functor $\mathrm{Hom}(-, A) : \mathscr{A} \to \mathbf{Ab}$ is a left exact functor.*

Proof. The proof is as follows:

1. Let
$$0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$$
be an exact sequence in $\mathscr{A}$. Applying $\mathrm{Hom}(A, -)$, which we denote as h_A in the rest of the proof, we obtain homomorphisms
$$0 \to \mathrm{Hom}_{\mathscr{A}}(A, X) \xrightarrow{h_A(f)} \mathrm{Hom}_{\mathscr{A}}(A, Y) \xrightarrow{h_A(g)} \mathrm{Hom}_{\mathscr{A}}(A, Z)$$
of abelian groups. We claim that this sequence is exact. If $h_A(f)(\alpha) = 0$, then $f \circ \alpha = 0$, but f is a monomorphism, so $\alpha = 0$.
$$\begin{tikzcd} & A \arrow[d, "\alpha"] \arrow[dr, "0"] & \\ 0 \arrow[r] & X \arrow[r, "f"] & Y \end{tikzcd}$$
Since h_A is a functor, we have $h_A(g) \circ h_A(f) = 0$. If $\beta \in \ker h_A(g)$, then $g \circ \beta = 0$. The universal property of the kernel implies that β factors through a morphism $X \to \ker g$.
$$\begin{tikzcd} & & A \arrow[d, "\beta"] \arrow[dr, "0"] & & \\ 0 \arrow[r] & X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r] & 0 \end{tikzcd}$$
But we have canonical isomorphisms
$$X \xrightarrow{\sim} \mathrm{coim}\ f \xrightarrow{\sim} \mathrm{im}\ f \xrightarrow{\sim} \ker\ g$$

the first as f is a monomorphism, the second by the first isomorphism theorem in a small abelian category and the third because the sequence is exact at Y. The composite of the composite of these with the canonical morphism $\ker g \to B$ is g.

$$\begin{tikzcd} & A \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ X \arrow[rr, "f"] & & Y \end{tikzcd}$$

Therefore, we obtain a morphism $\alpha : X \to A$ satisfying $f \circ \alpha = \beta$.

(2) The statement in (2) is the dual of the statement in (1).

This completes the proof. □

In fact, as the next lemma shows, exactness of a sequence can be checked by studying all possible Hom functors. More precisely:

Proposition 1.6. *Let $\mathscr{A}$ be a small abelian category. A sequence*

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

is exact if every sequence

$$\mathrm{Hom}(A, X) \xrightarrow{h_A(f)} \mathrm{Hom}(A, Y) \xrightarrow{h_A(g)} \mathrm{Hom}_{\mathscr{A}}(A, Z)$$

is exact for each $A \in \mathscr{A}$.

Proof. For $A = X$, we get

$$g \circ f = h_X(g) \circ h_X(f)(\mathrm{id}_A) = 0,$$

so we have a monomorphism $s : \mathrm{im}\, f \to \ker g$.

$$\begin{tikzcd} X \arrow[d, "\mathrm{Id}_X"'] & & \\ X \arrow[r, "f"] \arrow[rr, bend right, "0"'] & Y \arrow[r, "g"] & Z \end{tikzcd}$$

For $A = \ker g$ and $\iota : \ker g \hookrightarrow Y$, we have $h_X(g)(\iota) = g \circ \iota = 0$, so there exists $\alpha : \ker g \to X$ with $f \circ \alpha = \iota$.

$$\begin{tikzcd} & \ker g \arrow[dl, "\alpha"'] \arrow[d, hook, "\iota"] & \\ X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \end{tikzcd}$$

Then ι factors as a morphism $t : \ker g \to \mathrm{im}\, f$ which is the inverse to s. □

Corollary 1.7. *Let $\mathscr{A}$ be a small abelian category. A sequence*

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

is exact if every sequence

$$\mathrm{Hom}(Z, A) \xrightarrow{h_A(g)} \mathrm{Hom}(Y, A) \xrightarrow{h_A(f)} \mathrm{Hom}_{\mathscr{A}}(X, Z)$$

is exact for each $A \in \mathscr{A}$.

Proof. The statement is dual to the statement in Proposition 1.6. □

Example 1.8. The functor $\mathrm{Hom}(A,-)$ need not be right exact. To see this, let $\mathscr{A} = \mathbf{Ab}$ be the category of abelian groups and let $A = \mathbb{Z}/2\mathbb{Z}$. Consider the short exact sequence:

$$0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/2\mathbb{Z} \to 0$$

Applying $\mathrm{Hom}(\mathbb{Z}/2\mathbb{Z},-)$ and noting that

$$\mathrm{Hom}(\mathbb{Z}/2\mathbb{Z},\mathbb{Z}) = 0$$
$$\mathrm{Hom}(\mathbb{Z}/2\mathbb{Z},\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z},$$

we obtain the sequence:

$$0 \to 0 \to 0 \to \mathbb{Z}/2\mathbb{Z} \to 0$$

This sequence is not right exact since $0 \to \mathbb{Z}/2\mathbb{Z}$ is not a surjective map.

2. Tensor Product Functor

Let's now introduce the tensor product functor. We assume the construction of the tensor product functor is known. Note that the tensor product functor is only defined in $\mathbf{Mod}_R$, the category of left R-modules. In what follows, we assume that R is a commutative ring, so we do not need to distinguish between left and right R-modules. Using a clever argument exploiting the adjunction between the Hom and tensor product functors, we can show the following:

Proposition 2.1. *Let R be a ring and let $\mathbf{Mod}_R$ be the category of left R-modules. Let M be a right R-module. The functor $M \otimes_R -$ is a right exact functor.*

Proof. Let $0 \to A \to B \to C \to 0$ be an exact sequence in $\mathbf{Mod}_R$. We show that

$$M \otimes_R A \to M \otimes_R B \to M \otimes_R C \to 0$$

is an exact sequence. Proposition 1.5 and Corollary 1.7 imply that

$$M \otimes_R A \to M \otimes_R B \to M \otimes_R C \to 0$$

is an exact sequence if and only if

$$0 \to \mathrm{Hom}(M \otimes_R C, X) \to \mathrm{Hom}(M \otimes_R B, X) \to \mathrm{Hom}(M \otimes_R A, X)$$

is an exact sequence for each left R-module X. We have

$$\mathrm{Hom}(M \otimes_R N, X) = \mathrm{Hom}(N, \mathrm{Hom}(M, X)),$$

for all R-modules N. Hence, the sequence above can be written as

$$0 \to \mathrm{Hom}(C, \mathrm{Hom}(M,X)) \to \mathrm{Hom}(B, \mathrm{Hom}(M,X)) \to \mathrm{Hom}(A, \mathrm{Hom}(M,X))$$

which is indeed exact by Proposition 1.5. $\square$

Example 2.2. The functor $M \otimes_R -$ need not be left exact functor. To see this, take $R = \mathbb{Z}$. Consider the sequence:

$$0 \to \mathbb{Z} \hookrightarrow \mathbb{Q}$$

Letting $M = \mathbb{Z}$ and noting that,

$$\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q} \cong 0$$
$$\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z} \cong \mathbb{Z},$$

we obtain the sequence:

$$0 \to \mathbb{Z} \to 0$$

which is not left exact since the map $\mathbb{Z} \to 0$ is not a surjective map.

3. Projective & Injective Objects

We now introduce special objects that can rectify the failure of the exactness of the Hom and tensor product functors.

3.1. **Projective Objects.** We first define the notion of projective objects.

Definition 3.1. An object $P \in \mathscr{A}$ is called projective if the functor $\operatorname{Hom}(P, -)$ is an exact functor.

Remark 3.2. *An object P is projective if and only if for every morphism $Y \to Z \to 0$ and $P \to Z$, there exists a morphism $P \to Y$ such that the diagram*

$$\begin{tikzcd}
 & P \arrow[dl, dashed] \arrow[d] & \\
Y \arrow[r] & Z \arrow[r] & 0
\end{tikzcd}$$

commutes.

Example 3.3. The following are examples of projective objects:

(1) The zero object in a small abelian category is projective.
(2) In $\mathbf{Mod}_R$, the object R is projective: indeed, the functor

$$\begin{aligned} \operatorname{Hom}(R, -) : \operatorname{Mod} R &\to \mathbf{Ab} \\ M &\mapsto M \end{aligned}$$

is just the forgetful functor, and hence clearly is exact.

Proposition 3.4. *An object P in $\mathscr{A}$ is projective if and only if every exact sequence*

$$0 \to X \xrightarrow{f} Y \xrightarrow{p} P \to 0$$

in $\mathscr{A}$ splits.

Proof. Skipped. ☐

Proposition 3.5. *A direct summand of a projective object is a projective object. Moreover, an arbitrary direct sum of projective objects is a projective object*

Proof. Let $P_1, P_2 \in \mathscr{A}$ such that $P_1 \oplus P_2$ such that $P_1 \oplus P_2$ is a projective object. Consider an epimorphism $f : Y \to Z \to 0$ and a morphism $\beta : P_1 \to Z$. Along with the zero morphism from P_2 to Z, the universal property of the co-product implies that there is a unique morphism $\gamma : P_1 \oplus P_2 \to Z$. Since $P_1 \oplus P_2$ is a projective object, there is a morphism $\gamma' : P_1 \oplus P_2 \to Y$ such that the diagram

$$\begin{tikzcd}
 & P \arrow[d, "\iota_{P_1}"] \arrow[ddl, dashed] \arrow[dd, bend left, "\beta"] & \\
 & P \oplus Q \arrow[dl, dashed, "\gamma'"] \arrow[d, dashed, "\gamma"'] & Q \arrow[l, "\iota_{P_2}"'] \arrow[dl, "0"] \\
Y \arrow[r, "f"] & Z \arrow[r] & 0
\end{tikzcd}$$

commutes. The required morphism is then $\gamma' \circ \iota_{P_1}$. A similar argument as above shows that a direct sum of projective objects is a projective object. ☐

Projective objects in $\mathbf{Mod}_R$ can be easily characterized in terms of free R-modules, which we now define:

Definition 3.6. A left R-module, F, is a **free module** if it is isomorphic to an arbitrary direct sum of copies of R as a left R-module. That is,

$$F \cong \bigoplus_{i \in I} R := R^I$$

Remark 3.7. *Any free R-module F has a basis B in bijection with its indexing set, and therefore a map $F \to A$ for some left R-module A is prescribed uniquely by its (arbitrary) values on B.*

$$\mathrm{Hom}_{\mathbf{Mod}_R}(F, A) = \mathrm{Hom}_{\mathbf{Sets}}(B, A)$$

Proposition 3.8. *A free R-module, F, is a projective module.*

Proof. Consider $\beta : F \to Z$ and a surjective R-module homomorphism[1] $f : Y \to Z \to 0$. Let B be a basis for F.

$$\begin{tikzcd} & F \arrow[dl, dashed, "g"'] \arrow[d, "\beta"] & \\ Y \arrow[r, "f"] & Z \arrow[r] & 0 \end{tikzcd}$$

For each $b \in B$, the element $\beta(b) \in Z$ has the form $f(b) = p(a_b)$ for some $a_b \in A$, because f is surjective. By the Axiom of Choice, there is a function $u : B \to Y$ with $u(b) = a_b$ for all $b \in B$. By the remark above, we have an R-homomorphism $g : F \to Y$ with $g(b) = a_b$ for all $b \in B$. Clearly, g is the required morphism. □

Proposition 3.9. *The following statements are equivalent:*

(1) *P is projective in $\mathbf{Mod}_R$.*
(2) *There is a module Q such that $P \oplus Q \cong R^I$ for some set I. The module R^I is a called a free module.*

Proof. Assume that P is a projective object and let I be the set of generators of P and let R^I denote a free module on the set of generators of P. Consider the natural map $\pi : R^I \to P$. It clearly is a surjective, and, since P is projective, it splits. Therefore,

$$P \oplus \ker \pi \cong R^I$$

The converse follows since a free module is projective and a direct summand of projective module is a projective module by Proposition 3.5. □

Remark 3.10. *Every projective module need not be free. For example, consider*

$$R = \mathbb{Z}/6\mathbb{Z} = \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$$

$\mathbb{Z}/3\mathbb{Z}$ is a projective $\mathbb{Z}/6\mathbb{Z}$-module since $\mathbb{Z}/6\mathbb{Z}$ is a projective $\mathbb{Z}/6\mathbb{Z}$-module. However, $\mathbb{Z}/3\mathbb{Z}$ is not a free $\mathbb{Z}/6\mathbb{Z}$-module: a (finitely generated) free $\mathbb{Z}/6\mathbb{Z}$-module F is a direct sum of, say, n copies of $\mathbb{Z}/6\mathbb{Z}$, and so F has 6^n elements. Therefore, $\mathbb{Z}/3\mathbb{Z}$ is not a free $\mathbb{Z}/6\mathbb{Z}$ since it has only three elements.

Example 3.11. Let $\mathscr{A} = \mathbf{Ab}$. The functor $\mathrm{Hom}(\mathbb{Z}, -)$ is an exact functor. This is because $\mathbb{Z}$ is a free object in $\mathbf{Ab}$.

[1] Epimorphisms and surjective R-module homomorphisms coincide in the category of R-modules.

Example 3.12. Let $\mathscr{A} = \mathbf{Ab}$. The functor $\mathrm{Hom}(\mathbb{Q}, -)$ is not an exact functor. This is because $\mathbb{Q}$ is not a projective object in **Ab** since $\mathbb{Q}$ cannot be a summand of a free $\mathbb{Z}$-module because a free $\mathbb{Z}$-module is not divisible but $\mathbb{Q}$ is a divisible group.

3.2. **Injective Objects.** We now define the notion of injective objects.

Definition 3.13. Let $\mathscr{A}$ be a small abelian category. An object $I \in \mathscr{A}$ is called injective if the functor $\mathrm{Hom}(-, I)$ is an exact functor.

Remark 3.14. *Injective objects in $\mathscr{A}$ are just projective objects in $\mathscr{A}^{op}$.*

Remark 3.15. *An object I is injective if and only if for every morphisms $0 \to X \to Y$ and $X \to Y$, there exists a unique morphism $Y \to I$ such that the diagram*

$$\begin{tikzcd} & I & \\ 0 \arrow[r] & X \arrow[u] \arrow[r] & Y \arrow[ul, dashed] \end{tikzcd}$$

commutes.

Proposition 3.16. *Let $\mathscr{A}$ be a small abelian category. An object I in $\mathscr{A}$ is injective if and only if every exact sequence*

$$0 \to I \xrightarrow{i} Y \xrightarrow{f} Z \to 0$$

in $\mathscr{A}$ splits.

Proof. Skipped. □

Proposition 3.17. *A direct summand of an injective object is an injective object. Moreover, an arbitrary product of injective objects is an injective object.*

Proof. Let $I_1, \oplus I_2 \in \mathscr{A}$ such that $I_1 \oplus I_2$ is an injective object. Consider a monomorphism $f : 0 \to X \to Y$ and a morphism $\gamma : X \to I_1$. Note that $\iota_1 \circ \gamma$ is a morphism from X to $E_1 \oplus E_2$, where ι_1 is the canonical inlcusion map. Since $E_1 \oplus E_2$ is injective, there is a morphism $\gamma' : Y \to E_1 \oplus E_2$ such that the diagram

$$\begin{tikzcd} & E_1 \arrow[rr, bend left, "\iota_1"] & & E_1 \oplus E_2 \arrow[ll, bend left, "\pi_1"] \\ 0 \arrow[r] & X \arrow[u, "\gamma"] \arrow[rr, "f"] & & Y \arrow[u, dashed, "\gamma'"'] \end{tikzcd}$$

commutes. Then $\pi_1 \circ \gamma'$ is the required morphism, where π_1 is the canonical projection map. A similar argument as above shows that a product of injective objects is an injective object. □

We now characterize injective objects in $\mathbf{Mod}_R$.

Proposition 3.18. ***(Baer's criterion)*** *An R-module, I, is injective if and only if for every left ideal $J \subseteq R$ and every R-module homomorphism $g : J \to I$, there exists $g' : R \to I$ such that the following diagram*

$$\begin{tikzcd} 0 \arrow[r] & J \arrow[r] \arrow[d, "g"] & R \arrow[dl, dashed, "g'"] \\ & I & \end{tikzcd}$$

commutes.

Proof. The forward implication is clear. For the reverse implication, consider the diagram:

$$\begin{tikzcd} 0 \arrow[r] & J \arrow[r] \arrow[d, "g"] & R \\ & I & \end{tikzcd}$$

Consider the set of all intermediate extensions:

$$S = \{(C, h) \mid J \subseteq C \subseteq R \text{ submodule, } h \in \mathrm{Hom}(C, I) \text{ and } h|_J = g\}$$

Set $(C, h) \leq (C', h')$ if and only if $C \subseteq C'$ and $h'|_C = h$. Note that $S \neq \emptyset$ because we can choose $C = J$. Suppose $\{(C_x, h_x)\}_{x \in I}$ is a chain for an index set I such that for any $x, y \in I$, $(C_x, h_x) \leq (C_y, h_y)$. Let

$$C = \bigcup_{x \in I} C_x$$

and define $h : C \to I$ by setting $h(a) = h_x(a)$ if $a \in C_x$ for some $x \in I$. This is well-defined by assumption, and $h|_{C_x} = h_x$ for any $x \in I$. Hence $(C_x, h_x) \leq (C, h)$ for any $x \in I$, showing that (C, h) is an upper bound. By Zorn's lemma, the chain has a maximal element, (C, h). If $C = R$, we are done. Otherwise, let $b \in R \setminus C$. Consider the sequence:

$$0 \to J \xrightarrow{f_1} R \oplus C \xrightarrow{f_2} Rb \oplus C \to 0 \qquad f_2(r, c) = rb + c \quad f_1(r) = (r, -rb)$$

where $J = \{a \in R \mid ab \in C\}$. Let $g : J \to I$, $g(a) = h(ab)$ and hence there exists a g' such that the diagram

$$\begin{tikzcd} 0 \arrow[r] & J \arrow[r] \arrow[d, "g"] & R \arrow[dl, dashed, "g'"] \\ & I & \end{tikzcd}$$

commutes. Consider a morphism:

$$\hat{h} : Rb \oplus C \to I$$
$$rb + c \mapsto h(c) + rg'(1)$$

We show that $\hat{h}$ is well-defined. If $rb + c = r'b + c'$, then $(r - r')b = c' - c \in C$. It follows that $(r - r') \in J$. Therefore, $h((r - r')b)$ and $g(r - r')$ are defined. Moreover,

$$h(c' - c) = h((r - r')b) = g(r - r') = g'(r - r') = (r - r')g'(1).$$

Thus,

$$h(c') - h(c) = rg'(1) - r'g'(1),$$

which implies that

$$h(c') + r'g'(1) = h(c) + rg'(1)$$

Clearly, $\hat{h}(c) = h(c)$ so $\hat{h}'$ extends h. With $\hat{C} = Rb + c$, we have that $(C, h) \leq (\hat{C}, \hat{h})$, so $(C, h) = (\hat{C}, \hat{h})$. Hence $b \in C$, a contradiction. This completes the proof. $\square$

Example 3.19. The following are examples of injective objects as can be easily deduced from Proposition 3.18.

(1) $\mathbb{Z}/n\mathbb{Z}$ is an injective $\mathbb{Z}/n\mathbb{Z}$-module for any $n \geq 1$.
(2) $\mathbb{Z}/3\mathbb{Z}$ is an injective $\mathbb{Z}/6\mathbb{Z}$-module, but not an injective $\mathbb{Z}/9\mathbb{Z}$-module.
(3) $\mathbb{Q}$ is an injective $\mathbb{Z}$-module. A homomorphism $f \colon n\mathbb{Z} \to \mathbb{Q}$ extends to a homomorphism $g \colon \mathbb{Z} \to \mathbb{Q}$. Just take $y \in \mathbb{Q}$ such that $ny = f(n)$ and define $g(z) = zy$.

Corollary 3.20. *Let $A \in \mathbf{Ab}$. A is an injective $\mathbb{Z}$-module if and only if A is a divisible group.*

Proof. Assume that A is an injective $\mathbb{Z}$-module. Let $a \in A$ and $n \in \mathbb{Z}$. Consider the group homomorphism

$$\begin{aligned} f\colon n\mathbb{Z} &\to \mathbb{Z} \\ n &\mapsto a \end{aligned}$$

By assumption, f extends to a group homomorphism

$$\hat{f} : \mathbb{Z} \to I$$

such that $\hat{f}(nk) = f(nk)$ for each $k \in \mathbb{Z}$. Note that we have

$$a = f(n) = \hat{f}(n \cdot 1) = n\hat{f}(1)$$

Hence, A is a divisible group. Conversely, assume that A is a divisible group. We show that the criterion in Proposition 3.18 is satisfied. Let $J \subseteq \mathbb{Z}$ be an abelian subgroup and let $g : J \to A$ be a group homomorphism. Let $\{((K, g')\}$ be the set of pairs (K, g') such that $J \subseteq K \subseteq \mathbb{Z}$ and $g' : K \to \mathbb{Z}$ is a homomorphism with $g'|_U = g$. The set is non-empty since as it contains (J, J), and it is partially ordered by

$$(K_1, g_1') \leq (K_2, g_2') \quad \Leftrightarrow \quad K_1 \subseteq K_2 \text{ and } g_2'|_{K_1} = g_1'.$$

It is clear that any ascending chain has an upper bound. By Zorn's Lemma, the set contains a maximal element (K, g'). We claim that $K = \mathbb{Z}$. Suppose not. Let $k \in \mathbb{Z} \setminus K$. If

$$\langle k \rangle \cap K = \{0\},$$

the sum $K + \langle k \rangle$ is in fact a direct sum, and we can extend g' to $K + \langle k \rangle$ by choosing an arbitrary image of k in $\mathbb{Z}$ and extending linearly. This is a contradiction. Hence, assume that

$$nk \in \langle k \rangle \cap K$$

for some $n \neq 0$. Choose n_0 such that n_0 is minimal. Since $n_0 \in K$, and g' is defined on K, $g'(nk)$ is well-defined. Since A is divisible, there exists $a \in A$ such that

$$na = g'(nk).$$

It is now easy to see that we can extend g' to $K + \langle k \rangle$ by defining $g'(k) = a$. This is also a contradiction. $\square$

Example 3.21. Let $\mathscr{A} = \mathbf{Ab}$ and let k be a field of characteristic zero. The functor $\mathrm{Hom}(-, k)$ is an exact functor. This is because k is a divisible group since for any $g \in k$ and $n \in \mathbb{Z}$, there exists an $h \in k$ such that $hn = g$. since $\mathbb{Q} \subseteq k$.

4. Resolutions & Derived Functors

An arbitrary R-module, M, might be quite complicated to study; however, one can always find a set of (possibly infinite) generator for M[2]. In other words, one can always find a surjective morphism $F^0 \to M \to 0$, where F^0 is a free R-module. Since M is not a free R-module, the morphism

$$F^0 \to M \to 0$$

[2]A fact we used in a proof in the previous section.

is in general not injective; indeed, the any non-trivial relationship between generators of M will force the kernel to be non-zero. However, we can repeat the construction as above: if we take a generating set for the kernel of the morphism $F^0 \to M \to 0$, one can always find a morphism $F^1 \to F^0$, which is surjective onto the kernel of the morphism $F^0 \to M \to 0$, and where F^1 is a free R-module on the generating set of the kernel of the morphism $F^0 \to M \to 0$. We have the following sequence:

$$F^1 \to F^0 \to M \to 0$$

We can repeat the above process unless it terminates, which only happens when there is no non-trivial relationship among elements generating the free module at the left end of the sequence

$$F^i \to \cdots \to F^1 \to F^0 \to M \to 0$$

This motivates the idea of taking a resolution of an object in a category by special types of objects (free R-modules in the case considered above) in order to study the structure of the original object in the category.

4.1. **Projective & Injective Resolutions.** For a object $X \in \mathscr{A}$, we will first discuss taking a resolution of X in $\mathscr{A}$ by projective objects in $\mathscr{A}$. An arbitrary category may not have projective objects, though.

Example 4.1. Let $\mathscr{A} = \mathbf{Ab_{Fin}}$ be the category of finite abelian groups. $\mathscr{A}$ has no projective objects except for the trivial abelian group. Indeed, the exact sequence

$$0 \to \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$$

is non-split, since $\mathbb{Z}/2n\mathbb{Z}$ is not isomorphic to $\mathbb{Z}/n\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. Hence, $\mathbb{Z}/n\mathbb{Z}$ is not projective. But every other non-zero finite abelian group has a direct summand $\mathbb{Z}/n$ and the direct summand of a projective object is a projective object.

This motivates the following definition:

Definition 4.2. $\mathscr{A}$ has enough projectives if for every $X \in \mathscr{A}$ there exists an epimorphism $f : P \to X \to 0$ where P is a projective object.

Example 4.3. Clearly, the category of R-modules has enough projective objects. Indeed, free modules are projective objects and free module exist in abdundance in the category of R-modules.

Definition 4.4. A projective resolution of $X \in \mathscr{A}$ is a nonnegative complex $P^\bullet$ together with a morphism $\epsilon : P^0 \to M$ such that

$$\cdots \to P^3 \to P^2 \to P^1 \to P^0 \xrightarrow{\varepsilon} M \to 0$$

is exact and the P^i's are projective objects.

Example 4.5. In **Ab**, the abelian group $\mathbb{Z}/n\mathbb{Z}$ has a projective resolution

$$0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0.$$

Proposition 4.6. *If $\mathscr{A}$ has enough projectives, then every object has a projective resolution.*

Proof. Take any $X \in \mathscr{A}$. There is an epimorphism $P^0 \to X \to 0$ from a projective object. P^0. Taking the kernel $K^0 \to P^0$, we have a projective P^1 with an epimorphism

$P^1 \to K^0 \to 0$. We take its kernel $K^1 \to P^1$ and again get a projective $P^2 \to K^1 \to$. This way, we get that the diagram:

$$\begin{tikzcd}
 & 0 \arrow[dr] & & 0 & & & & \\
 & & K^1 \arrow[ur] \arrow[dr] & & & & & \\
\cdots \arrow[r] & P^2 \arrow[ur] \arrow[rr] & & P^1 \arrow[rr] \arrow[dr] & & P^0 \arrow[r] & X \arrow[r] & 0 \\
 & & & & K^0 \arrow[ur] \arrow[dr] & & & \\
 & & & 0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

Continuing, this gives a projective resolution of X. □

We can similarly define injective resolutions.

Definition 4.7. $\mathscr{A}$ has enough injectives if for every $X \in \mathscr{A}$ there exists a monomorphism $f : 0 \to X \to I$ where I is an injective object.

Proposition 4.8. *The category of R-modules has enough injective objects.*

Proof. See [Rot09, Theorem 3.38]. □

Definition 4.9. An injective resolution of $X \in \mathscr{A}$ is a non-negative complex $I_\bullet$ together with a morphism $\epsilon : 0 \to X \to I_0$ such that

$$0 \to X \xrightarrow{\varepsilon} I_0 \to I_1 \to \cdots$$

is exact and the I_i's are injective objects.

Proposition 4.10. *If $\mathscr{A}$ has enough injectives, then every object has a injective resolution.*

Proof. The statement is the dual of the statement in Proposition 4.6, so it is clearly true. □

Example 4.11. In **Ab**, an injective resolution of $\mathbb{Z}$ is

$$0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$$

and an injective resolution of $\mathbb{Z}/n\mathbb{Z}$ is

$$0 \to \mathbb{Z}/n\mathbb{Z} \to \mathbb{Q}/\mathbb{Z} \to \mathbb{Q}/\mathbb{Z} \to 0$$

4.2. **Derived functors.** Derived functors provide us with a tool to quantitatively measure the failure of a functor to be an exact functor. The philosophy behind derived functors is the following: if $\mathscr{F} : \mathscr{A} \to \mathscr{D}$ is a left exact functor between two abelian categories, then any short exact sequence in $\mathscr{A}$,

$$0 \to A \to B \to C \to 0,$$

gets transformed to a left exact sequence in $\mathscr{D}$:

$$0 \to \mathscr{F}(A) \to \mathscr{F}(B) \to \mathscr{F}(C),$$

A right derived functor is a a sequence of functors $R^i\mathscr{F} : \mathscr{A} \to \mathscr{D}$ for all $i \geq 0$ and a functorial isomorphism $R^0\mathscr{F} \cong \mathscr{F}$ such that that for any short exact sequence,

$$0 \to A \to B \to C \to 0,$$

in $\mathscr{A}$ there is a long exact sequence,

$$0 \to R^0\mathscr{F}(A) \to R^0\mathscr{F}(B) \to R^0\mathscr{F}(C) \to R^1\mathscr{F}(A) \to R^1\mathscr{F}(B) \to R^1\mathscr{F}(C) \to R^2\mathscr{F}(A) \to \cdots,$$

for all $i \geq 0$. We expect that $R^1\mathscr{F}(A)$ to quantitatively measure the failure of $\mathscr{F}$ to be a right exact functor since $R^1\mathscr{F}(A) = 0$ if and only if the sequence,

$$0 \to \mathscr{F}(A) \to \mathscr{F}(B) \to \mathscr{F}(C),$$

is a right exact sequence.

On the other hand, if $\mathscr{F} : \mathscr{A} \to \mathscr{D}$ is a right exact sequence, a left derived functor is a sequence of functors $L^i\mathscr{F} : \mathscr{A} \to \mathscr{D}$ along with a functorial isomorphism $L^0\mathscr{F} \cong \mathscr{F}$ yieldsing a long exact sequence

$$\cdots \to L^1\mathscr{F}(A) \to L^1\mathscr{F}(B) \to L^1\mathscr{F}(C) \to L^0\mathscr{F}(A) \to L^0\mathscr{F}(B) \to L^0\mathscr{F}(C) \to 0.$$

The theory of left and right derived functors is quite similar. Therefore, in what follows we shall only focus on left derived functors of covariant functors. The theory of left derived functors of contravariant functors is similar to the theory of left derived functors of covariant functors, which we now describe. Left derived functors are constructed by means of projective resolutions.

Definition 4.12. Let $\mathscr{A}$ be a locally small abelian category with enough projectives, $\mathscr{D}$ be an abelian category, and $\mathscr{F} : \mathscr{A} \to \mathscr{D}$ be a right exact functor. Given $X \in \mathscr{A}$, choose a projective resolution of X:

$$\cdots \to P^3 \to P^2 \to P^1 \to P^0 \xrightarrow{\varepsilon} X \to 0.$$

Apply $\mathscr{F}$ to the above complex to obtain (the truncated) complex:

$$\cdots \to \mathscr{F}(P^3) \to \mathscr{F}(P^2) \to \mathscr{F}(P^1) \to \mathscr{F}(P^0).$$

The i-th left derived functor of $\mathscr{F}$ is defined as:

$$L^i(\mathscr{F}(X)) = H_i(\mathscr{F}(P^\bullet)).$$

Here $H_i(\mathscr{F}(P^\bullet))$ is the i-th homology (defined similarly to cohomology) of $P^\bullet$,

Remark 4.13. *If $\mathscr{F}$ is a right exact* contravariant *functor, then the left derived functor is defined by taking an* injective resolution*.*

The above definition naturally begs the question: is the definition of a left-derived functor well-defined? If this is the case, the definition of a left-derived functor should be independent of the projective resolution chosen. We show that this is indeed the case.

Proposition 4.14. ***(Comparison Theorem)*** *Let $\mathscr{A}$ be a locally small abelian category with enough projectives and let $f : X \to Y$ be a morphism in $\mathscr{A}$. Let*

$$\cdots \xrightarrow{d_2^X} P^1 \xrightarrow{d_1^X} P^0 \xrightarrow{d_0^X} X \longrightarrow 0$$

and

$$\cdots \xrightarrow{d_2^Y} Q^1 \xrightarrow{d_1^Y} Q^0 \xrightarrow{d_0^Y} Y \longrightarrow 0$$

be projective resolutions for X and Y. Then there is a sequence of homomorphisms $f^i : P^i \to Q^i$ such that the following diagram commutes:

$$\begin{tikzcd}
\cdots \arrow[r, "d_2^X"] & P^1 \arrow[r, "d_1^X"] \arrow[d, "f^1"] & P^0 \arrow[r, "d_0^X"] \arrow[d, "f^0"] & X \arrow[r] \arrow[d, "f"] & 0 \\
\cdots \arrow[r, "d_2^Y"] & Q^1 \arrow[r, "d_1^Y"] & Q^0 \arrow[r, "d_0^Y"] & Y \arrow[r] & 0
\end{tikzcd}$$

Furthermore, any two such extensions of f are chain homotopic.

Proof. (Existence) We proceed by induction on i. For the base case, note that since P^0 is projective, the morphism $f \circ d_0^X$ lifts to a unique morphism f^0 such that the right most square in the diagram above commutes. Assume that $f^i : P^i \to Q^i$ has been constructed. Denote by $\ker d_i^X$ and $\ker d_i^Y$ denote the kernels of d_i^X and d_i^Y, respectively. Since d_{i+1}^X factors through $\mathrm{im} d_{i+1}^X$ which is isomorphicm to $\ker d_i^X$, we can think of d_{i+1}^X as mapping into $\ker d_i^X$. Moreover, f^i factors into $\ker d_i^Y$ since $f^{i-1} d_i^X = d_i^Y f^i$. Thus, consider the diagram:

$$\begin{tikzcd}
P^{i+1} \arrow[r, "d_{i+1}^X"] \arrow[d, dashed, "f^{i+1}"] & \ker d_i^X \arrow[r] \arrow[d, "f^i"] & 0 \\
Q^{i+1} \arrow[r, "d_{i+1}^Y"] & \ker d_i^Y \arrow[r] & 0
\end{tikzcd}$$

The composition $f^i d_{i+1}^X$ gives a map from P^{i+1} to $\ker d_i^Y$, onto which d_{i+1}^Y surjects. Thus, the map f^{i+1} is furnished by the defining property of the projective object P^{i+1}, completing the induction.

(Uniqueness) To show that two extensions $\{f^i\}$ and $\{g^i\}$ are chain homotopic, we consider the difference $h^i := f^i - g^i$ and construct a chain homotopy $s^i : P^i \to Q^{i+1}$ such that

$$h^i = d_{i+1}^Y s^i + s^{i-1} d_i^X$$

We proceed by induction. Observe that $h^{-1} = f - f \equiv 0$, so that h^0 maps P^0 $\ker d_0^Y$ by the universal property of kernels, and therefore lifts to a map $s^0 : P^0 \to Q^1$ as in the following diagram:

$$\begin{tikzcd}
 & P^0 \arrow[r, "d_0^X"] \arrow[dl, dashed, "s^0"'] \arrow[d, "h^0"] & A \arrow[d, "h^{-1}"] \arrow[dl, dashed, "0"'] \\
Q^1 \arrow[r, "d_1^Y"'] & \ker d_0^Y \arrow[r] & 0
\end{tikzcd}$$

This gives the base case for the induction. Suppose that $s^i : P^i \to Q^{i+1}$ has been constructed such that $h^i = d_{i+1}^Y s^i + s^{i-1} d_i^X$. It follows that the map $h^{i+1} - s^i d_{i+1}^X$ maps P^{i+1} into $\ker d_{i+1}^Y$ since

$$d_{i+1}^Y (h^{i+1} - s^i d_{i+1}^X) = h^i d_{i+1}^X - (h^i - s^{i-1} d_i^X) d_{i+1}^X = h^i d_{i+1}^X - h^i d_{i+1}^X = 0.$$

$$\begin{tikzcd} & P^{i+1} \arrow[dl, dashed, "s^{i+1}"'] \arrow[d, "h^{i+1}-s^i d^X_{i+1}"] & \\ Q^{i+2} \arrow[r, "d^Y_{i+2}"'] & \ker d^Y_{i+1} \arrow[r] & 0 \end{tikzcd}$$

Thus we have the diagram above and projectivity furnishes the map s^{i+1} such that $d^Y_{i+2}s^{i+1} = h^{i+1} - s^i d^X_{i+1}$. $\square$

As a consequence of the comparison theorem, if $P^\bullet$ is a projective resolution for X and $Q^\bullet$ is a projective resolution for Y such that there is a morphism $f : X \to Y$, we get a well-defined map

$$H_i(F(P^\bullet)) \longrightarrow H_i(F(Q^\bullet)),$$

which is an isomorphism by the chain homotopy conclusion in the comparison theorem. Similarly, we have:

Corollary 4.15. *Let $\mathscr{A}, \mathscr{D}$ be abelian categories. The following are corollaries of Proposition 4.14:*

(1) *Suppose that $P^\bullet$ and $Q^\bullet$ are projective resolutions of $X \in \mathscr{A}$. Then there is a canonical isomorphism between $H_i(\mathscr{F}(P^\bullet))$ and $H_i(\mathscr{F}(Q^\bullet))$ for each $i \geq 0$.*
(2) *let $\mathscr{F} : \mathscr{A} \to \mathscr{D}$ be a right exact functor. For any $X \in \mathscr{A}$, $L^0\mathscr{F}(X) \cong \mathscr{F}(X)$.*
(3) *Let $\mathscr{F} : \mathscr{A} \to \mathscr{D}$ be a functor. If P is a projective object in $\mathscr{A}$, then $L^i\mathscr{F}(P) = 0$ for $i \geq 1$.*

Proof. The proof proceeds as follows:

(1) If $P^\bullet$ and $Q^\bullet$ are two projective resolutions of $X \in \mathscr{A}$, then $\mathrm{Id}_X : X \to X$ gives rise to unique maps (up to homotopy) by the Proposition 4.14 such that the diagram

$$\begin{tikzcd} \cdots \arrow[r] & P^1 \arrow[r] \arrow[d, bend left, "f^1"] & P^0 \arrow[r] \arrow[d, bend left, "f^0"] & X \arrow[d, bend left, "\mathrm{Id}_X"] \\ \cdots \arrow[r] & Q^1 \arrow[r] \arrow[u, bend left, "g^1"] & Q^0 \arrow[r] \arrow[u, bend left, "g^1"] & X \arrow[u, bend left, "\mathrm{Id}_X"] \end{tikzcd}$$

commutes. Hence, there are two chain homotopies

$$s : H_\bullet(\mathscr{F}(P^\bullet)) \to H^*(\mathscr{F}(Q^\bullet))$$
$$q : H_\bullet(\mathscr{F}(Q^\bullet)) \to H^*(\mathscr{F}(P^\bullet))$$

such that both sq and qs compose to the identity (by uniqueness up to homotopy). Hence the derived functor is well-defined: for two choices of projective resolutions of objects, the construction yields isomorphic derived functors.

(2) Choose a projective resolution of X:

$$\cdots \to P^3 \to P^2 \to P^1 \to P^0 \xrightarrow{\varepsilon} X \to 0.$$

Since $\mathscr{F}$ is right exact, the sequence

$$\mathscr{F}(P^1) \xrightarrow{\varphi} \mathscr{F}(P^0) \xrightarrow{\psi} \mathscr{F}(X) \to 0$$

is exact. Hence, ψ is an epimorphism and ψ is the cokernel of φ. By the first isomorphism theorem,

$$\mathscr{F}(X) \cong \frac{\mathscr{F}(P^0)}{\ker\psi} \cong \frac{\mathscr{F}(P^0)}{\mathrm{im}\varphi} \cong \mathrm{coker}\varphi$$

Hence, we have

$$L^0\mathscr{F}(X) = H_0(\mathscr{F}(P^\bullet)) \cong \mathrm{coker}\varphi \cong \mathscr{F}(X).$$

(3) Consider the projective resolution:

$$\cdots \to 0 \cdots \to 0 \to P \xrightarrow{\mathrm{Id}_P} P \to 0$$

Hence, we consider the homology of the complex:

$$\cdots \to 0 \cdots \to 0 \to \mathscr{F}(P)$$

and it is clear that

$$L^i\mathscr{F}(P) \cong H_i(\mathscr{F}(P^\bullet)) = 0$$

for $i \geq 1$.

□

We now prove the horseshoe lemma (Lemma 4.16). The horsehoe lemma allows us to construct a short exact sequence of projective resolutions given a short exact sequence of objects in an abelian category. In the statement and the proof of the horseshoe lemma, for ease of notation we use subscripts instead of superscipts to label indices of all projective objects.

Lemma 4.16. ***(Horseshoe Lemma)*** *Let $\mathscr{A}$ be an abelian category and let*

$$0 \to A \to A' \to A'' \to 0$$

be a short exact sequence in $\mathscr{A}$. Assume that there are projective resolutions $P^\bullet$ and $(P'')^\bullet$ of A and A'' respectively. Then there is projective resolution $(P')^\bullet$ of A' such that the following diagram commutes.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & A & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & P'_1 & \longrightarrow & P'_0 & \longrightarrow & A' & \longrightarrow & 0 \\
 & & \downarrow j & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & P''_1 & \longrightarrow & P''_0 & \longrightarrow & A'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Proof. Composition gives a map $P_0 \to A'$, and a map $P_0^{''} \to A'$ is furnished by projectivity. Using the universal property of the co-product, these combine to give a map $P_0 \oplus P_0^{''} \to A'$, and we set $P'_0 := P_0 \oplus P_0^{''}$. The sequence $P_0 \to P_0 \oplus P''_0 \to P_0^{''}$ is obviously split exact, we will show that the morphism $P_0 \oplus P''_0 \to A'$ is an epimorphism. This follows by applying the snake lemma to the two right most exact columns, yielding a morphism $\ker\varepsilon''_0 \to 0 \to \mathrm{coker}\,\varepsilon'_0$.

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\ker \epsilon_0 \arrow[r] \arrow[d, dashed] & P_0 \arrow[r, "\varepsilon_0"] \arrow[d] & A \arrow[r] \arrow[d] & 0 \arrow[d, dashed] \\
\ker \epsilon_0' \arrow[r] \arrow[d, dashed] & P_0 \oplus P_0'' \arrow[r, "\varepsilon_0'"] \arrow[d] & A' \arrow[r] \arrow[d] & \mathrm{coker}\varepsilon_0' \arrow[d, dashed] \\
\ker \epsilon_0'' \arrow[r] & P_0'' \arrow[r, "\varepsilon_0''"] \arrow[d] & A'' \arrow[r] \arrow[d] & 0” \arrow[d] \\
 & 0 & 0 & 0
\end{tikzcd}$$

By the snake lemma (**??**), the left most column is exact and the connecting morphism yields a sequence which has a subsequence of the form

$$\cdots \to 0 \to \mathrm{coker}\, \varepsilon_0' \to 0 \to \cdots$$

Hence, $\mathrm{coker}\, \varepsilon_0' = 0$ and the morphism $P_0 \oplus P_0'' \to A'$ is an epimorphism. We then apply the same procedure to the diagram with kernels to construct $P_1 \oplus P_1'' \to \ker\varepsilon_0'$, where the product is projective and the map is an epimorphism onto the kernel.

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
P_1 \arrow[r] \arrow[d] \arrow[dr, dashed] & \ker \epsilon_0 \arrow[r] \arrow[d] & P_0 \arrow[r, "\varepsilon_0"] \arrow[d] & A \arrow[r] \arrow[d] & 0 \\
P_1 \oplus P_1'' \arrow[r] \arrow[d] & \ker \epsilon_0' \arrow[r] \arrow[d] & P_0 \oplus P_0'' \arrow[r, "\varepsilon_0'"] \arrow[d] & A' \arrow[r] \arrow[d] & 0 \\
P_1'' \arrow[r] \arrow[d] \arrow[ur, dashed] & \ker \epsilon_0'' \arrow[r] \arrow[d] & P_0'' \arrow[r, "\varepsilon_0''"] \arrow[d] & A'' \arrow[r] \arrow[d] & 0 \\
0 & 0 & 0 & 0 &
\end{tikzcd}$$

We continue this way iteratively and construct $P_n = P_n \oplus P_n''$ at the n-th step with the desired properties. $\square$

References

[Rot09] Joseph J. Rotman. *An introduction to homological algebra*. Vol. 2. Springer, 2009 (cit. on p. 11).