HOMOLOGY

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Abstract. These are assorted notes on singular homology in the context of algebraic topology. I took these notes during graduate school. There may be typos; please send corrections to junaida@umd.edu or junaid.aftab1994@gmail.com.

Contents

Part 1. Computational Tools

1. Mayer-Vietoris Sequence

In addition to the long exact sequence of homology groups for a pair (X, A), there is another sort of long exact sequence, known as a Mayer–Vietoris sequence, which is equally powerful but is sometimes more convenient to use. The Mayer–Vietoris sequence is also applied frequently in induction arguments, where one might know that a certain statement is true for A, B, and $A \cap B$ by induction and then deduce that it is true for $A \cup B$ by the exact sequence[1].

Lemma 1.1. *(**Barrett-Whitehead Lemma**) Suppose we have the following commutative diagram of abelian groups, where the two rows are exact:*

$$\begin{tikzcd}
\cdots \arrow[r] & A_n \arrow[r, "i_n"] \arrow[d, "f_n"] & B_n \arrow[r, "j_n"] \arrow[d, "g_n"] & C_n \arrow[r, "k_n"] \arrow[d, "h_n"] & A_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \cdots \\
\cdots \arrow[r] & A'_n \arrow[r, "i'_n"] & B'_n \arrow[r, "j'_n"] & C'_n \arrow[r, "k'_n"] & A'_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

Assume each map $h_n : C_n \to C_{n-1}$ is an isomorphism. Then there is a long exact sequence

$$\cdots \longrightarrow A_n \xrightarrow{(i_n, f_n)} B_n \oplus A'_n \xrightarrow{g_n - i'_n} B'_n \xrightarrow{k_n h_n^{-1} j'_n} A_{n-1} \longrightarrow \cdots$$

[1] Mayer–Vietoris sequence can also be thought of as an abelianization of the Seifert Van Kampen Theorem.

Proof. The proof is by a diagram chase. We omit details. $\square$

Proposition 1.2. *(**Mayer-Viertoris Sequence**) Let $X_1, X_2 \subseteq X$ be open sets such that $X = X_1 \cup X_2$. Let*

$$i_1 : X_0 \hookrightarrow X_1, \qquad i_2 : X_1 \hookrightarrow X$$

denote inclusions for $i = 1, 2$. Then there is a long exact sequence

$$\cdots \to H_n(X_1 \cap X_2; \mathbb{Z}) \to H_n(X_1; \mathbb{Z}) \oplus H_n(X_2; \mathbb{Z}) \to H_n(X; \mathbb{Z}) \to H_{n-1}(X_1 \cap X_2; \mathbb{Z}) \to \cdots$$

Proof. We have the following diagram:

$$\begin{tikzcd}
(X_1 \cap X_2, \emptyset) \arrow[r, "i_1"] \arrow[d, "i_2"] & (X_1, \emptyset) \arrow[r, "f"] \arrow[d, "j_1"] & (X_1, X_1 \cap X_2) \arrow[d, "h"] \\
(X_2, \emptyset) \arrow[r, "j_2"] & (X, \emptyset) \arrow[r, "g"] & (X, X_2)
\end{tikzcd}$$

Applying **??** yields the following diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & H_n(X_1 \cap X_2; \mathbb{Z}) \arrow[r, "H_n(i_n)"] \arrow[d, "H_n(f_n)"'] & H_n(X_1; \mathbb{Z}) \arrow[r, "H_n(j_n)"] \arrow[d, "H_n(g_n)"'] & H_n(X_1, X_1 \cap X_2; \mathbb{Z}) \arrow[r, "\delta_n"] \arrow[d, "H_n(h_n)"'] & H_n(X_1 \cap X_2; \mathbb{Z}) \arrow[r] \arrow[d, "H_n(i_2)"] & \cdots \\
\cdots \arrow[r] & H_n(X_2; \mathbb{Z}) \arrow[r, "H_n(i'_n)"] & H_n(X; \mathbb{Z}) \arrow[r, "H_n(j'_n)"] & H_n(X, X_2; \mathbb{Z}) \arrow[r, "\delta'_n"] & H_n(X_2; \mathbb{Z}) \arrow[r] & \cdots
\end{tikzcd}$$

The excision axioms implies that $H_n(h_n)$ is an isomorphism for each $n \geq 0$. Lemma 1.1 then implies the existence of the desired long exact sequence. $\square$

Remark 1.3. *By using augmented chain complexes, we also obtain a corresponding Mayer-Vietoris sequence for the reduced homology groups. We omit details.*

Remark 1.4. *We can also use the Mayer-Viertoris sequence to compute the homology groups of sphere. Indeed, consider the following argument. Let $X = \mathbb{S}^n$, $A = \mathbb{S}^n \setminus \{S\}$, and $B = \mathbb{S}^n \setminus \{N\}$, where S and N are the south pole and north pole, respectively. Then*

$$A \simeq \mathbb{R}^n \quad B \simeq \mathbb{R}^n \quad A \cap B \simeq \mathbb{S}^{n-1}$$

From the Mayer-Vietoris sequence for reduced homology groups, we get $\widetilde{H}_k(\mathbb{S}^n) \simeq \widetilde{H}_{k-1}(\mathbb{S}^{n-1})$ for all i. By induction, we find as before:

$$\widetilde{H}_k(\mathbb{S}^n; \mathbb{Z}) \simeq \begin{cases} \mathbb{Z}, & k = n \\ 0, & k \neq n. \end{cases}$$

Proposition 1.5. *(**Suspension Theorem**) Let X be a topological space and let SX be its suspension. We have*

$$\widetilde{H}_{n-1}(X; \mathbb{Z}) \cong \widetilde{H}_n(SX; \mathbb{Z})$$

for $n \geq 1$.

Proof. Let P, Q denote the collapsed spaces $X \times \{0\}$ and $X \times \{1\}$ respectively. Let $A = SX - \{P\}$ and let $B = SX - \{Q\}$. Each of A and B are homeomorphic to the cone space

$$CX = (X \times I)/(X \times \{0\})$$

By the Mayer-Vietoris sequence for reduced homology, since $A \cap B = X \times (0, 1)$, we obtain the exact sequence

$$\cdots \to \widetilde{H}_n(A; \mathbb{Z}) \oplus \widetilde{H}_n(B; \mathbb{Z}) \to \widetilde{H}_n(SX; \mathbb{Z}) \to \widetilde{H}_{n-1}(A \cap B; \mathbb{Z}) \to \widetilde{H}_{n-1}(A; \mathbb{Z}) \oplus \widetilde{H}_{n-1}(B; \mathbb{Z}) \to \cdots$$

for all n. Note that CX is contractible[2]. Moreover, $X \times (0,1)$ deformation retracts down to X. Hence, the sequence simplifies to:

$$\cdots \to 0 \to \widetilde{H}_n(SX;\mathbb{Z}) \to \widetilde{H}_{n-1}(X;\mathbb{Z}) \to 0 \to \cdots$$

This proves the claim. □

2. Degree Theory

We focus on the application of homology theory to degree theory (and fixed point theory) in this section. We shall see that we can use homology theory to study continuous maps between spheres. This allows us to introduce and study the notion of degree of a map between spheres. Moreover, degree of maps between spheres will play a fundamental role in our being able to compute cellular homology, which we discuss in the next section.

Definition 2.1. The degree of a continuous map $f : \mathbb{S}^n \to \mathbb{S}^n$ is defined as:

$$\deg f := H_n(f)(1)$$

where

$$H_n(f) : H_n(\mathbb{S}^n;\mathbb{Z}) \cong \mathbb{Z} \to H_n(\mathbb{S}^n;\mathbb{Z}) \cong \mathbb{Z}$$

is the homomorphism induced by f in homology, and $1 \in \mathbb{Z}$ denotes the generator.

Remark 2.2. *In what follows, we write $H_n(f)$ as f_n.*

Proposition 2.3. *Here are some basic properties of degree:*

1. $\deg(\mathrm{Id}_{\mathbb{S}^n}) = 1$.
2. *If f is not surjective, then* $\deg(f) = 0$.
3. *If $f \simeq g$ are homotopic maps, then* $\deg(f) = \deg(g)$.
4. $\deg(g \circ f) = \deg(g) \cdot \deg(f)$.
5. *If f is a homotopy equivalence, then* $\deg(f) = \pm 1$.

Proof. The proof is given below:

1. This follows because $(\mathrm{Id}_{\mathbb{S}^n})_n = \mathrm{Id}_{\mathbb{Z}}$ which is multiplication by the integer 1.
2. Indeed, if f is not surjective, there is some $y \notin \mathrm{Im}(f)$. Then we can factor f in the following way:
$$\mathbb{S}^n \to \mathbb{S}^n \setminus \{y\} \to \mathbb{S}^n$$
Since $\mathbb{S}^n \setminus \{y\} \simeq \mathbb{R}^n$ is contractible, $H_n(\mathbb{S}^n \setminus \{y\}) = 0$. Therefore, $f_n = 0$, so $\deg(f) = 0$.
3. This follows because because if f and g are homotopic, then $f_n = g_n$ for $n \geq 0$.
4. This is clear.
5. By definition, there exists a map $g : \mathbb{S}^n \to \mathbb{S}^n$ so that $g \circ f \simeq \mathrm{Id}_{\mathbb{S}^n}$ and $f \circ g \simeq \mathrm{Id}_{\mathbb{S}^n}$. The claim follows directly from previous results, since $f \circ g \simeq \mathrm{id}_{\mathbb{S}^n}$ implies that $\deg(f) \cdot \deg(g) = \deg(\mathrm{Id}_{\mathbb{S}^n}) = 1$.

This completes the proof. □

We now prove a far less obvious result:

Proposition 2.4. *Let $n \geq 1$ and let $A \in O(n+1)$ denote an orthogonal linear transformation. Set $f := A|_{\mathbb{S}^n}$. Then* $\deg(f) = \det(A) = \pm 1$.

[2]Indeed, the homotopy $h_t(x,s) = (x,(1-t)s)$ continuously shrinks CX down to its vertex point.

Proof. The group $O(n+1)$ has two connected components, $O^+(n+1)$ and $O^-(n+1)$, distinguished by det: $O(n+1) \to \{+1, -1\}$. By homotopy invariance, it suffices to check the result for one such A in each component. Note that $I_{n+1} \in O^+(n+1)$ has degree 1. Hence, every $A \in O^+(n+1)$ has degree one. Consider $O^-(n+1)$. We take A to be reflection in a hyperplane $H \subseteq \mathbb{R}^{n+1}$. WLOG, we can assume the hyperplane is $\mathbb{R}^n \times \{0\} \subset \mathbb{R}^{n+1}$ and

$$A(x_0, x_1, \ldots, x_n) = (x_0, x_1, \ldots, -x_n).$$

The upper and lower hemispheres U and L of $\mathbb{S}^n$ can be regarded as singular n-simplices, via their standard homeomorphisms with Δ^n [3]. Then the generator of $H_n(\mathbb{S}^n)$ is $[U - L]$. The reflection map r maps the cycle $U - L$ to $L - U = -(U - L)$. So

$$H_n(r)([U-L]) = [L-U] = [-(U-L)] = -1 \cdot [U-L],$$

so $\deg(r) = -1$. □

Corollary 2.5. *If $a : \mathbb{S}^n \to \mathbb{S}^n$ is the antipodal map, then* $\deg(a) = (-1)^{n+1}$.

Proof. Note that a is a composition of $n+1$ reflections since there are $n+1$ coordinates in x, each changing sign by an individual reflection. From above, we know that the composition of maps leads to multiplication of degrees. □

This immediately gives a proof of the following famous result.

Proposition 2.6. ***(Hairy Ball Theorem)*** *$\mathbb{S}^n$ has a continuous vector field of nonzero tangent vectors if and only if n is odd.*

Proof. Suppose $x \mapsto v(x)$ is a tangent vector field on $\mathbb{S}^n$, assigning to a vector $x \in \mathbb{S}^n$ the vector $v(x)$ tangent to $\mathbb{S}^n$ at x. Regarding $v(x)$ as a vector at the origin instead of at x, we have that $x \perp v(x)$ for each $x \in \mathbb{S}^n$. If $v(x) \neq 0$ for all x, we may normalize so that $|v(x)| = 1$ for all x by replacing $v(x)$ by $v(x)/|v(x)|$. Consider

$$W : \mathbb{S}^n \times I \times \mathbb{S}^n \qquad W(x,t) = (\cos t\pi)x + (\sin t\pi)v(x)$$

Note that W indeed takes values in $\mathbb{S}^n$ since $x \perp v(x)$ for each $x \in \mathbb{S}^n$. Hence, we obtain a homotopy from the identity map on $\mathbb{S}^n$ to the antipodal map on $\mathbb{S}^n$. This implies that $(-1)^{n+1} = 1$ and n must be odd.

Conversely, if n is odd, say $n = 2k-1$, we can define $v : \mathbb{R}^{2k} \to \mathbb{R}^{2k}$ by

$$v(x_1, x_2, \ldots, x_{2k-1}, x_{2k}) = (-x_2, x_1, \ldots, -x_{2k}, x_{2k-1})$$

Then $v(x) \perp x$ for each $x \in \mathbb{S}^n$. Hence, v is a vector field on $\mathbb{S}^n$, and it is non-vanishing since $|v(x)| = 1$ for all $x \in \mathbb{S}^n$. □

2.1. **Applications to Fixed Point Theory.** Let's note some results in fixed point theory that can be easily proved using degree theory.

(1) If $f : \mathbb{S}^n \to \mathbb{S}^n$ has no fixed points, then $\deg(f) = (-1)^{n+1}$. Since $f(x) \neq x$, the segment

$$(1-t)f(x) + t(-x)$$

[3]Here we are implicitly using the result that if $C \subseteq \mathbb{R}^n$ is a compact convex subset with nonempty interior, then C is a closed n-cell (and its interior is an open n-cell). That is, $C \cong \mathbb{D}^n$.

from $-x$ to $f(x)$ does not pass through the origin in $\mathbb{R}^{n+1}$. So we can normalize to obtain a homotopy:

$$g_t(x) := \frac{(1-t)f(x) + t(-x)}{\|(1-t)f(x) + t(-x)\|} : \mathbb{S}^n \to \mathbb{S}^n.$$

Note that this homotopy is well defined since $(1-t)f(x) - tx \neq 0$ for any $x \in \mathbb{S}^n$ and $t \in [0,1]$, because $f(x) \neq x$ for all x. Then g_t is a homotopy from f to a, the antipodal map, and the result follows.

(2) (**Brouwer's Fixed Point Theorem**) Let $n \geq 1$ and suppose that a continuous map $f : \mathbb{D}^n \to \mathbb{D}^n$ has no fixed points. Since $\mathbb{D}^n \cong \mathbb{S}^n_+$, where $\mathbb{S}^n_+$ is the northern hemisphere, f can be thought of a map from $\mathbb{S}^n_+$ to $\mathbb{S}^n_+$. Now we can extend f to a map on $\mathbb{S}^n$ as follows. We define

$$g(x) = \begin{cases} f(x), & \text{if } x \in \mathbb{S}^n_+, \\ f \circ r(x), & \text{if } x \in \mathbb{S}^n_- \end{cases}$$

where $r(x)$ is reflection about the plane through the equator and $\mathbb{S}^n_+$ is the southern hemisphere. It is clear that $g(x)$ is a continuous function; furthermore $g(x)$ has no fixed points. By (1), g is homotopic to the antipodal map on $\mathbb{S}^n$ that has degree $(-1)^{n+1}$. Clearly, g is not surjective[4]. It follows that $\deg g = 0$. But $(-1)^{n+1} \neq 0$ for $n \geq 1$. Hence, every continuous map $f : \mathbb{D}^n \to \mathbb{D}^n$ has a fixed point.

(3) Consider a continuous map $f : \mathbb{S}^{2n} \to \mathbb{S}^{2n}$. Then either f or $-f$ must have a fixed point. If f and $-f$ don't have fixed points, then f and $-f$ are homotopic to the antipodal map. Thus f and $-f$ have degree -1. But both f and $-f$ both cannot have the same degree. Hence there is a point $x \in \mathbb{S}^{2n}$ such that $f(x) = \pm x$.

(4) Consider a continuous map $f : \mathbb{S}^n \to \mathbb{S}^n$ of degree zero. Then there exist points $x, y \in \mathbb{S}^n$ for such that $f(x) = x$ and $f(y) = -y$. If not, then f and $-f$ are homotopic to the antipodal map. But then degree of $\pm f$ is $0 \neq \pm(-1)^{n+1}$. Hence, there are points $x, y \in \mathbb{S}^n$ for such that $f(x) = x$ and $f(y) = -y$.

Remark 2.7. *Brouwer's fixed point thereom can be derived without resorting to degree theory. First, note that for $n \geq 1$ there does not exist a continuous map $r : \mathbb{D}^n \to \partial\mathbb{D}^n$ such that $r(x) = x$ for $x \in \partial\mathbb{D}^n$. Assume by contradiction that there exists a retraction $r : \mathbb{D}^n \to \partial\mathbb{D}^n = \mathbb{S}^{n-1}$. Then, if $i : \mathbb{S}^{n-1} \to \mathbb{D}^n$ is the inclusion, we have $r \circ i = Id_{\mathbb{S}^{n-1}}$.*

$$\begin{tikzcd} \mathbb{S}^{n-1} \arrow[rr, "Id"] \arrow[dr, "i"'] & & \mathbb{S}^{n-1} \\ & \mathbb{D}^n \arrow[ur, "r"'] & \end{tikzcd}$$

If $n > 1$, we then have:

[4]Because no point in the southern hemisphere is in the image.

$$\begin{tikzcd}
\mathbb{Z} \cong H_{n-1}(\mathbb{S}^{n-1};\mathbb{Z}) \arrow[rr, "Id"] \arrow[dr, "0"'] & & \mathbb{H}_{n-1}(\mathbb{S}^{n-1};\mathbb{Z}) \cong \mathbb{Z} \\
& H_{n-1}(\mathbb{D}^n;\mathbb{Z}) = 0 \arrow[ur, "0"'] &
\end{tikzcd}$$

This is a contradiction. A similar argument can be made in the case $n = 0$. We can now prove Brouwer's fixed point thereom: let $f : \mathbb{D}^n \to \mathbb{D}^n$ be a continuous map. Assume by contradiction that $f(x) \neq x$ for all $x \in \mathbb{D}^n$. Then, we may define a function $r : \mathbb{D}^n \to \mathbb{S}^{n-1}$ in the following way. Let $x \in \mathbb{D}^n$ and let $[f(x), x)$ denote the (unique) ray based at $f(x)$ passing through x. Define $r(x)$ to be the unique element in $([f(x), x)) \cap \partial\mathbb{D}^n \setminus \{f(x)\}$. Then, r is continuous and is a retraction $\mathbb{D}^n \to \partial\mathbb{D}^n$, a contradiction.

Remark 2.8. *Here is a cute application of (3) above. Consider $g : \mathbb{RP}^{2n} \to \mathbb{RP}^{2n}$. By covering space theory, the map g lifts to a map $\hat{g} : \mathbb{RP}^{2n} \to \mathbb{S}^{2n}$. Define f as in the diagram below:*

$$\begin{tikzcd}
\mathbb{S}^{2n} \arrow[r, dotted, "f"] \arrow[d] & \mathbb{S}^{2n} \arrow[d] \\
\mathbb{RP}^{2n} \arrow[r, "g"'] \arrow[ur, dotted, "\tilde{g}"] & \mathbb{RP}^{2n}
\end{tikzcd}$$

Choose $x \in \mathbb{S}^{2n}$ such that $f(x) = \pm x$. Then the point $[x] \in \mathbb{RP}^{2n}$ is a fixed point for g[5]. We have just seen that any linear transformation of $\mathbb{R}^{2n+1}$ has a real eigenvalue! The analogous result is not necessarily true for $\mathbb{R}^{2n}$ for $n \geq 1$. Consider the following matrix:

$$R = \begin{pmatrix} R_{\theta_1} & 0 & \dots & 0 \\ 0 & R_{\theta_1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & R_{\theta_n} \end{pmatrix}$$

Here $\mathbb{R}_{\theta_i}$ is a 2-by-2 matrix that represents a rotation through angle θ_i. Clearly, R maps $\mathbb{S}^{2n-1}$ to $\mathbb{S}^{2n-1}$, but it has no fixed point since each R_{θ_i} has no (real) eigenvector.

Remark 2.9. *Here is a cute application of (4) above. Let $F : \mathbb{D}^n \to \mathbb{D}^n$ be a continuous function such that $F(x) \neq 0$ for $x \in \mathbb{D}^n$. We can then consider the map*

$$G(x) = \frac{F(x)}{\|F(x)\|}$$

Note that G maps $\mathbb{S}^{n-1}$ to $\mathbb{S}^{n-1}$. Moreover, G has degree zero since $\mathbb{D}^n$ has trivial homology. Hence, there are points $x, y \in \mathbb{S}^{n-1}$ such that $G(x) = x$ and $G(y) = -y$. If we think of F defining a vector field on $\mathbb{D}^n$, then this meas that there exists a point on$\mathbb{S}^{n-1}$ where the vector field points radially outward and another point on $\mathbb{S}^{n-1}$ where where the vector field points radially inward.

[5]For $m \geq$, recall that $\mathbb{RP}^m = \mathbb{S}^{m+1}/\sim$ where we identify antipodal points in $\mathbb{S}^m$. Covering space theory tells us that $\mathbb{S}^m$ is the universal covering space of $\mathbb{RP}^m$. Hence, the map g lifts since $\mathbb{S}^{m+1}$ is simply-connected.

Remark 2.10. *Consider a continuous map $f : \mathbb{S}^n \to \mathbb{S}^n$ for $n \geq 1$. If f doesn't have a fixed point, then f is homotopic to the antipodal map. If n is odd, then the linear transformation represented by the matrix*

$$R_t = \begin{pmatrix} R_{t\pi} & 0 & \dots & 0 \\ 0 & R_{t\pi} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & R_{t\pi} \end{pmatrix}$$

is homotopy between the antipodal map and the identity map. If n is even, then we can apply the argument above to the "equatorial sphere" $\mathbb{S}^{n-1} \subset \mathbb{S}^n$. In any case, we see that if f is homotopic to continuous map with a fixed point.

2.2. **Local Degrees.** How to compute degrees, though? We describe a technique for computing degrees which can be applied to most maps that arise in practice. Assume $f : \mathbb{S}^n \to \mathbb{S}^n$ is surjective, and that f has the property that there exists some $y \in \mathrm{Im}(\mathbb{S}^n)$ so that $f^{-1}(y)$ is a finite number of points, say $f^{-1}(y) = \{x_1, x_2, \dots, x_m\}$. Let U_i be a neighborhood of x_i such that all U_i's get mapped to some neighborhood V of y. So

$$f(U_i \setminus \{x_i\}) \subseteq V \setminus \{y\}$$

$$\begin{tikzcd}
 & \mathbb{Z} \cong H_n(U_i, U_i \setminus x_i; \mathbb{Z}) \arrow[r, "f_n|_{x_i}"] \arrow[dl, "\cong"'] \arrow[d, "k_i"] & \mathbb{Z} \cong H_n(V, V \setminus y; \mathbb{Z}) \arrow[d, "\cong"] \\
\mathbb{Z} \cong H_n(\mathbb{S}^n, \mathbb{S}^n \setminus x_i; \mathbb{Z}) & H_n(\mathbb{S}^n, \mathbb{S}^n - f^{-1}(y); \mathbb{Z}) \arrow[l, "p_i"'] \arrow[r, "f_n"] & H_n(\mathbb{S}^n, \mathbb{S}^n \setminus y; \mathbb{Z}) \\
 & \mathbb{Z} \cong \widetilde{H}_n(\mathbb{S}^n; \mathbb{Z}) \arrow[r, "f_n"] \arrow[u, "j"'] \arrow[ul, "\cong"] & \mathbb{Z} \cong \widetilde{H}_n(\mathbb{S}^n; \mathbb{Z}) \arrow[u, "\cong"']
\end{tikzcd}$$

As f is continuous, we can choose the U_i's to be disjoint. Let $f|_{x_i} : U_i \to V$ be the restriction of f to U_i, with the induced homomorphism

$$f_n|_{x_i} : H_n(U_i, U_i \setminus \{x_i\}; \mathbb{Z}) \to H_n(V, V \setminus \{y\}; \mathbb{Z}).$$

Define the *local* degree of f at x_i, $\deg(f|_{x_i})$, to be the degree of $f_n|_{x_i}$. We then have the following result:

Lemma 2.11. *The degree of f equals the sum of local degrees at points in a generic finite fiber, that is,*

$$\deg(f) = \sum_{i=1}^{m} \deg(f|_{x_i}).$$

Proof. Consider the commutative diagram above. The two isomorphisms in the upper half of the diagram come from excision, while the lower two isomorphisms come from exact sequences of pairs. The maps k_i and p_i are induced by inclusions, so the triangles and squares commute. The map j comes from the long exact sequence in homology. By excision,

$$H_n(\mathbb{S}^n, \mathbb{S}^n - f^{-1}(y); \mathbb{Z}) \cong \bigoplus_{i=1}^{m} H_n(U_i, U_i \setminus x_i; \mathbb{Z}) \cong \mathbb{Z}^m$$

The map p_i is the projection onto the i-th summand. We have:

$$k_i(1) = (0, \dots, 0, 1, 0, \dots, 0)$$

where the entry 1 is in the ith place. Also, $p_i \circ j(1) = 1$ for all i, so

$$j(1) = (1, 1, \dots, 1) = \sum_{i=1}^{m} k_i(1).$$

Commutativity of the upper square says that the middle f_n takes $k_i(1)$ to $\deg f\,|_{x_i}$, hence the sum $\sum_i k_i(1) = j(1)$ is taken to $\sum_i \deg f|_{x_i}$. The commutativity of the lower square gives:

$$\deg f = f_n j(1) = f_n \left(\sum_{i=1}^{m} k_i(1) \right) = \sum_{i=1}^{m} \deg f|_{x_i}.$$

□

3. Cellular Homology

We define the cellular homology of a CW complex X in terms of a given cell structure, then we show that it coincides with the singular homology, so it is in fact independent on the cell structure. Cellular homology is very useful for computations. Before discussing cellular homology, we compute the relative homology groups of a topological space, X, that can be given the structure of a CW complex.

Lemma 3.1. *Let X be a topological space that can be endowed with the structure of a CW complex. Then:*

(1) *The relative romology $H_k(X^n, X^{n-1}; \mathbb{Z})$ is given by:*

$$H_k(X^n, X^{n-1}; \mathbb{Z}) = \begin{cases} 0, & \text{if } k \neq n \\ \mathbb{Z}^{\#n\text{-cells}}, & \text{if } k = n. \end{cases}$$

for $k \geq 1$.

(2) *$H_k(X^n; \mathbb{Z}) = 0$ if $k > n \geq 1$. In particular, if X is finite dimensional, then $H_k(X; \mathbb{Z}) = 0$ if $k > \dim(X)$.*

(3) *The inclusion $i : X^n \hookrightarrow X$ induces an isomorphism $H_k(X^n; \mathbb{Z}) \cong H_k(X)$ if $k < n$.*

Proof. The proof is given below:

(1) Since (X^n, X^{n-1}) is a good pair, we have:

$$H_k(X^n, X^{n-1}; \mathbb{Z}) \cong \widetilde{H}_k(X^n/X^{n-1}; \mathbb{Z}) = H_k(X^n/X^{n-1}; \mathbb{Z}) \cong \bigvee_{i=1}^{\#n\text{-cells}} \mathbb{S}^n \cong \begin{cases} 0, & \text{if } k \neq n \\ \mathbb{Z}^{\#n\text{-cells}}, & \text{if } k = n. \end{cases}$$

(2) Since (X^n, X^{n-1}) is a good pair for each $n \geq 1$, we can consider the following portion of the long exact sequence:

$$H_{k+1}(X^n, X^{n-1}; \mathbb{Z}) \longrightarrow H_k(X^{n-1}; \mathbb{Z}) \longrightarrow H_k(X^n; \mathbb{Z}) \longrightarrow H_k(X^n, X^{n-1}; \mathbb{Z})$$

; $\mathbb{Z}$If $k+1 \neq n$ and $k \neq n$, we have from part (1) that $H_{k+1}(X^n, X^{n-1}; \mathbb{Z}) = 0$ and $H_k(X^n, X^{n-1}) = 0$. Thus $H_k(X^{n-1}; \mathbb{Z}) \cong H_k(X^n; \mathbb{Z})$. Hence, if $k > n$ (so in particular, $n \neq k+1$ and $n \neq k$), we get by iteration that

$$H_k(X^n; \mathbb{Z}) \xrightarrow{\cong} H_k(X^{n-1}; \mathbb{Z}) \xrightarrow{\cong} \cdots \xrightarrow{\cong} H_k(X^0; \mathbb{Z})$$

Note that X^0 is just a collection of points, so $H_k(X^0;\mathbb{Z}) = 0$. Thus, when $k > n \geq 1$, we have $H_k(X^n;\mathbb{Z}) = 0$ as desired.

(3) We only prove the statement for finite-dimensional CW complexes. Let $k < n$, and consider the following portion of the long exact sequence:

$$\cdots \to H_{k+1}(X^{n+1}, X^n;\mathbb{Z}) \to H_k(X^n;\mathbb{Z}) \to H_k(X^{n+1};\mathbb{Z}) \to H_k(X^{n+1}, X^n;\mathbb{Z}) \to \cdots$$

Since $k < n$, we have $k+1 \neq n+1$ and $k \neq n+1$, so by part (1), we get that $H_{k+1}(X^{n+1}, X^n;\mathbb{Z}) = 0$ and $H_k(X^{n+1}, X^n;\mathbb{Z}) = 0$. Thus, $H_k(X^n) \cong H_k(X^{n+1};\mathbb{Z})$. By repeated iteration, we obtain:

$$H_k(X^n;\mathbb{Z}) \cong H_k(X^{n+1};\mathbb{Z}) \cong H_k(X^{n+2};\mathbb{Z}) \cong \cdots \cong H_k(X^{n+l};\mathbb{Z}) = H_k(X;\mathbb{Z}),$$

where l is such that $X^{n+l} = X$ (since we assumed X is finite dimensional). See [Hat02] for the case when X is infinite-dimensional.

□

In what follows we define the cellular homology of a CW complex, X, in terms of a given cell structure, then we show that it coincides with the singular homology.

Definition 3.2. The cellular homology $H^{\mathrm{CW}}(X)$ of a CW complex X is the homology of the cellular chain complex $(C_*(X), d_*)$ indexed by the cells of X, i.e.,

$$C_n(X) := H_n(X^n, X^{n-1};\mathbb{Z}) = \mathbb{Z}^{\#n\text{-cells}},$$

and with differentials $d_n : C_n(X) \to C_{n-1}(X)$ defined by the following diagram: d_n etc. are defined in the obvious way to make the diagram commute. It is easy to check that $d_{n+1} \circ d_n = 0$ since the composition of these two maps induces two successive maps in one of the diagonal exact sequences.

$$\begin{tikzcd}
 & & & H_n(X^{n+1}, X^n;\mathbb{Z}) = 0 & \\
0 = H_n(X^{n-1};\mathbb{Z}) \arrow[dr] & & H_n(X^{n+1};\mathbb{Z}) \cong H_n(X;\mathbb{Z}) \arrow[ur] & & \\
 & H_n(X^n;\mathbb{Z}) \arrow[ur, "i_n"] \arrow[dr, "j_n"] & & & \\
H_{n+1}(X^{n+1}, X^n;\mathbb{Z}) \arrow[ur, "\partial_{n+1}"] \arrow[rr, "d_{n+1}"] & & H_n(X^n, X^{n-1};\mathbb{Z}) \arrow[rr, "d_n"] \arrow[dr, "\partial_n"] & & H_{n-1}(X^{n-1}, X^{n-2};\mathbb{Z}) \\
 & & & H_{n-1}(X^{n-1};\mathbb{Z}) \arrow[ur, "j_{n-1}"] & \\
 & & H_{n-1}(X^{n-2};\mathbb{Z}) = 0 \arrow[ur] & &
\end{tikzcd}$$

Proposition 3.3. *We have:*

$$H_n^{\mathrm{CW}}(X) \cong H_n(X;\mathbb{Z})$$

for all n, where $H_n(X;\mathbb{Z})$ is the singular homology of X.

Proof. Since $H_n(X^{n+1}, X^n;\mathbb{Z}) = 0$ and $H_n(X;\mathbb{Z}) \cong H_n(X^{n+1};\mathbb{Z})$, we get from the diagram above that

$$H_n(X;\mathbb{Z}) \cong \frac{H_n(X^n;\mathbb{Z})}{\ker i_n} \cong \frac{H_n(X^n;\mathbb{Z})}{\operatorname{Im} \partial_{n+1}}.$$

Now, $H_n(X^n;\mathbb{Z}) \cong \operatorname{Im} j_n \cong \ker \partial_n \cong \ker d_n$. The first isomorphism comes from j_n being injective, while the second follows by exactness. Finally, $\ker \partial_n = \ker d_n$ since $d_n = j_{n-1} \circ \partial_n$ and j_{n-1} is injective. Also, we have $\operatorname{Im} \partial_{n+1} = \operatorname{Im} d_{n+1}$. Indeed, $d_{n+1} = j_n \circ \partial_{n+1}$ and j_n is injective. Altogether, we have

$$H_n(X;\mathbb{Z}) \cong \frac{\ker d_n}{\operatorname{Im} d_{n+1}} = H_n^{\mathrm{CW}}(X).$$

□

Let's make some observations which are immediate:

(1) If X has no n-cells, then $H_n(X;\mathbb{Z}) = 0$. Indeed, in this case we have $C_n = H_n(X^n, X^{n-1};\mathbb{Z}) = 0$. Therefore, $H_n^{\mathrm{CW}}(X;\mathbb{Z}) = 0$.
(2) If X is connected and has a single 0-cell, then $d_1 : C_1 \to C_0$ is the zero map. Indeed, since X contains only a single 0-cell, $C_0 = \mathbb{Z}$. Also, since X is connected, $H_0(X) = \mathbb{Z}$. So, by the above theorem, $\mathbb{Z} = H_0(X;\mathbb{Z}) = \ker d_0/\operatorname{Im} d_1 = \mathbb{Z}/\operatorname{Im} d_1$. This implies that $\operatorname{Im} d_1 = 0$, so d_1 is the zero map as desired.
(3) If X has no cells in adjacent dimensions, then $d_n = 0$ for all n, and $H_n(X;\mathbb{Z}) \cong \mathbb{Z}^{\#n\text{-cells}}$ for all n. Indeed, in this case, all maps d_n vanish. So for any n, $H_n^{\mathrm{CW}}(X) \cong C_n \cong \mathbb{Z}^{\#n\text{-cells}}$.

Example 3.4. When $n > 1$, $\mathbb{S}^n \times \mathbb{S}^n$ has one 0-cell, two n-cells, and one $2n$-cell. Since $n > 1$, these cells are not in adjacent dimensions. Hence:

$$H_k(\mathbb{S}^n \times \mathbb{S}^n;\mathbb{Z}) = \begin{cases} \mathbb{Z}, & \text{if } k = 0, 2n \\ \mathbb{Z}^2, & \text{if } k = n \\ 0, & \text{otherwise.} \end{cases}$$

We next discuss how to compute, in general, the maps

$$d_n : C_n(X) = \mathbb{Z}^{\#n\text{-cells}} \to C_{n-1}(X) = \mathbb{Z}^{\#(n-1)\text{-cells}}$$

of the cellular chain complex. Let us consider the n-cells $\{e_n^\alpha\}_\alpha$ as the basis for $C_n(X)$ and the $(n-1)$-cells $\{e_{n-1}^\beta\}_\beta$ as the basis for $C_{n-1}(X)$. In particular, we can write:

$$d_n(e_n^\alpha) = \sum_\beta d_{\alpha,\beta} \cdot e_{n-1}^\beta \qquad d_{\alpha,\beta} \in \mathbb{Z},$$

Proposition 3.5. ***(Cellular Boundary Formula)*** *The coefficient $d_{\alpha,\beta}$ is equal to the degree of the map $\Delta_{\alpha,\beta} : \mathbb{S}_\alpha^{n-1} \to \mathbb{S}_\beta^{n-1}$ defined by the composition:*

$$\mathbb{S}_\alpha^{n-1} = \partial \mathbb{D}_\alpha^n \xrightarrow{\varphi_\alpha^n} X^{n-1} = X^{n-2} \cup_\gamma \mathbb{D}_\gamma^{n-1} \xrightarrow{\textit{collapse}} X^{n-1}/(X^{n-2} \cup_{\gamma \neq \beta} \mathbb{D}_\gamma^{n-1}) = \mathbb{S}_\beta^{n-1},$$

where φ_α^n is the attaching map of $\mathbb{D}_\alpha^n$, and the collapsing map sends $X^{n-2} \cup_{\gamma\neq\beta} \mathbb{D}_\gamma^{n-1}$ to a point.

Proof. We will proceed with the proof by chasing the following diagram:

$$\begin{tikzcd}
H_n(\mathbb{D}^n_\alpha, \mathbb{S}^{n-1}_\alpha; \mathbb{Z}) \arrow[r, "\cong"] \arrow[d, "(\Phi^n_\alpha)_*"'] & \widetilde{H}_n(\mathbb{S}^{n-1}_\alpha; \mathbb{Z}) \arrow[r, "(\Delta_{\alpha,\beta})_*"] \arrow[d, "(\phi^n_\alpha)_*"'] & \widetilde{H}_n(\mathbb{S}^{n-1}_\beta; \mathbb{Z}) \\
H_n(X^n, X^{n-1}; \mathbb{Z}) \arrow[r, "\partial_n"] \arrow[dr, "d_n"'] & \widetilde{H}_{n-1}(X^{n-1}; \mathbb{Z}) \arrow[r, "q_*"] \arrow[d, "j_{n-1}"] & \widetilde{H}_{n-1}(X^{n-1}/X^{n-2}; \mathbb{Z}) \arrow[u, "q_{\beta,*}"'] \arrow[d, "\cong"'] \\
& H_{n-1}(X^{n-1}, X^{n-2}; \mathbb{Z}) \arrow[r, "\cong"'] & H_{n-1}(X^{n-1}/X^{n-2}, X^{n-2}/X^{n-2}; \mathbb{Z})
\end{tikzcd}$$

The maps are as follows:

(1) Φ^n_α is the characteristic map of the cell e^n_α, and ϕ^n_α is its attaching map.
(2) The map

$$q_* : \widetilde{H}_{n-1}(X^{n-1}; \mathbb{Z}) \to \widetilde{H}_{n-1}(X^{n-1}/X^{n-2}; \mathbb{Z}) = \bigoplus_\beta \widetilde{H}_{n-1}(\mathbb{D}^{n-1}_\beta/\partial\mathbb{D}^{n-1}_\beta; \mathbb{Z})$$

is induced by the quotient map $q : X^{n-1} \to X^{n-1}/X^{n-2}$.
(3) $q_\beta : X^{n-1}/X^{n-2} \to \mathbb{S}^{n-1}_\beta$ collapses the complement of the cell e^{n-1}_β to a point, the resulting quotient sphere being identified with $\mathbb{S}^{n-1}_\beta = \mathbb{D}^{n-1}_\beta/\partial\mathbb{D}^{n-1}_\beta$ via the characteristic map Φ^{n-1}_β.
(4) $\Delta_{\alpha\beta} : \mathbb{S}^{n-1}_\alpha \to \mathbb{S}^{n-1}_\beta$ is the composition $q_\beta \circ q \circ \phi^n_\alpha$, i.e., the attaching map of e^n_α followed by the quotient map $X^{n-1} \to \mathbb{S}^{n-1}_\beta$ collapsing the complement of $\mathbb{D}^{n-1}_\beta$ in X^{n-1} to a point.

The top left-hand square commutes by naturality of the long-exact sequence in reduced homology. The top right-hand square commutes by the definition of $\Delta_{\alpha,\beta}$. The bottom left-hand triangle commutes by definition of d_n . The bottom right-hand square commutes due to the relationship between reduced and relative homology.

The map $(\Phi^n_\alpha)_*$ takes the generator $[\mathbb{D}^n_\alpha] \in H_n(\mathbb{D}^n_\alpha, \mathbb{S}^{n-1}_\alpha)$ to a generator of the $\mathbb{Z}$-summand of $H_n(X^n, X^{n-1})$ corresponding to $\mathbb{D}^n_\alpha$, i.e.,

$$(\Phi^n_\alpha)_*([\mathbb{D}^n_\alpha]) = \mathbb{D}^n_\alpha$$

Since the top left square and the bottom left triangle both commute, this gives that

$$\sum_\beta d_{\alpha,\beta}\mathbb{D}^{n-1}_\beta = d_n(\mathbb{D}^n_\alpha) = d_n \circ (\Phi^\alpha_n)_*([\mathbb{D}^n_\alpha]) = j_{n-1} \circ (\phi^n_\alpha)_*([\mathbb{D}^n_\alpha]).$$

Here we have implicitly identified $H_n(\mathbb{D}^n_\alpha, \mathbb{S}^{n-1}_\alpha)$ with $H_n(\mathbb{S}^{n-1}_\alpha)$. Looking to the bottom right square, recall that since X is a CW complex, (X^n, X^{n-1}) is a good pair. This gives the isomorphism

$$H_{n-1}(X^{n-1}, X^{n-2}; \mathbb{Z}) \cong \widetilde{H}_{n-1}(X^{n-1}/X^{n-2}; \mathbb{Z}) \cong H_{n-1}(X^{n-1}/X^{n-2}, X^{n-2}/X^{n-2}; \mathbb{Z})$$

Notice that the map q_β, collapsing all the $n-1$ cells of X to the $n-1$ cell $\mathbb{S}^{n-1}_\beta$, induces the map $q_{\beta,*}$, which projects linear combinations of $\{\mathbb{D}^{n-1}_{\beta'}\}$ onto its summand of $\mathbb{D}^{n-1}_\beta$.

Therefore, the value of $d_n(\mathbb{D}_i^n)$ is going to be the sum of the projections $q_{\beta',*}$ on the $n-1$ dimensional cells e_β^{n-1}. In other words:

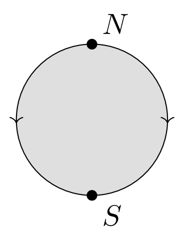
$$\sum_\beta d_{\alpha,\beta}\mathbb{D}_\beta^\beta = d_n(\mathbb{D}_n^\alpha) = \sum_\beta q_{\beta*} \circ q_* \circ (\phi_n^\alpha)_* \circ [\mathbb{D}_n^\alpha].$$

As noted before, we have defined $(\Delta_{\alpha\beta})_* = q_{\beta*} \circ q_* \circ (\phi_n^\alpha)_*$. The result now follows. $\square$

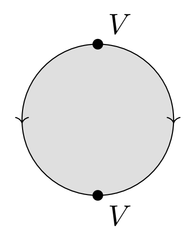
Example 3.6. Let $X = \mathbb{S}^2$. We $\mathbb{S}^2$ with $\mathbb{D}^2/\sim$ such that

$$(x,y) \sim (x',y') = x',\ y = |y'|$$

This induces a cell decomposition into one 2-cell, the image of the interior, one 1-cell, the image of $\mathbb{S}^1 \setminus \{(0,1),(0,-1)\}$, and two 0-cells, the images of $(0,1)$ and $(0,1)$ which are N and S. Let $A = \{N,S\}$. Since A is a sub-complex, X/A inherits a CW complex structure with one one 2-cell, one 1-cell and one 0-cell.



$X = \mathbb{S}^2$



$X = \mathbb{S}^2/\{N,S\}$

We have

$$0 \to \mathbb{Z} \xrightarrow{d_2} \mathbb{Z} \xrightarrow{d_1} \mathbb{Z} \to 0$$

Since X/A is connected as has a single 0-cell, $d_1 \equiv 0$. The attaching map of the two-cell in either case can be identified with the map:

$$\phi_{1,2}(e^{\phi i}) = \begin{cases} e^{i\phi} & 0 \le \phi \le \pi \\ e^{-i\phi} & \pi \le \phi \le 2\pi \end{cases}$$

The map has degree 0. Hence, $d_2 \equiv 0$. As a result, we have

$$H_n(\mathbb{S}^2/\{N,S\};\mathbb{Z}) = \begin{cases} \mathbb{Z}, & \text{if } n = 0,1,2 \\ 0, & \text{otherwise.} \end{cases}$$

Example 3.7. Recall that $\mathbb{CP}^n$ has one cell in each even dimension $0,2,4,\ldots,2n$. So $\mathbb{CP}^n$ has no two cells in adjacent dimensions. Hence:

$$H_k(\mathbb{CP}^n;\mathbb{Z}) = \begin{cases} \mathbb{Z}, & \text{if } k = 0,2,4,\ldots,2n \\ 0, & \text{otherwise.} \end{cases}$$

Example 3.8. Recall that $\mathbb{RP}^n$ has a CW structure with one k-cell $\mathbb{D}^k$ in each dimension $0 \le k \le n$. The attaching map for $\mathbb{D}^k$ is the standard 2-fold covering map $\phi : \mathbb{S}^{k-1} \to \mathbb{RP}^{k-1}$ identifying a point and its antipodal point in $\mathbb{S}^{k-1}$. To compute the boundary map d_k, we compute the degree of the composition

$$f : \mathbb{S}^{k-1} \to \mathbb{RP}^{k-1} \to \frac{\mathbb{RP}^{k-1}}{\mathbb{RP}^{k-2}} = \mathbb{S}^{k-1}$$

We consider a neighborhood V of y and the two neighborhoods U_1 and U_2 given to exist by the local homeomorphism property of f. One of the homeomorphisms is the identity map and the other homemorphisms is the anti-podal map. Then by the local degree formula implies

$$d_k = 1 + (-1)^k$$

It follows that

$$d_k = \begin{cases} 0 & \text{if } k \text{ is odd,} \\ 2 & \text{if } k \text{ is even,} \end{cases}$$

and therefore we obtain that

$$H_k(\mathbb{RP}^n; \mathbb{Z}) = \begin{cases} \mathbb{Z}_2 & \text{if } k \text{ is odd, } 0 < k < n, \\ \mathbb{Z} & \text{if } k = 0, n \text{ is odd,} \\ 0 & \text{otherwise.} \end{cases}$$

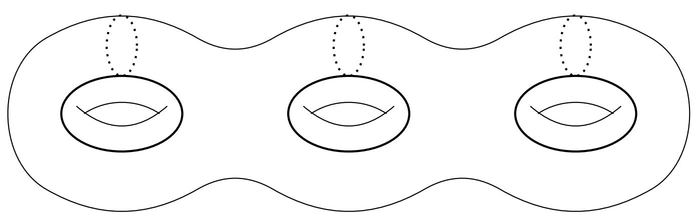
Example 3.9. Let M_g be the closed oriented surface of genus g, with its usual CW structure: one 0-cell, $2g$ 1-cells $\{a_1, b_1, \dots, a_g, b_g\}$, and one 2-cell attached by the product of commutators $[a_1, b_1] \cdot \dots \cdot [a_g, b_g]$. The associated cellular chain complex of M_g is:

$$0 \xrightarrow{d_3} \mathbb{Z} \xrightarrow{d_2} \mathbb{Z}^{2g} \xrightarrow{d_1} \mathbb{Z} \xrightarrow{d_0} 0$$

Since M_g is connected and has only one 0-cell, we get that $d_1 = 0$. We claim that d_2 is also the zero map. As the attaching map sends the generator to $a_1 b_1 a_1^{-1} b_1^{-1} \dots a_g b_g a_g^{-1} b_g^{-1}$, when we collapse all 1-cells (except a_i) to a point, the word defining the attaching map $a_1 b_1 a_1^{-1} b_1^{-1} \dots agbga_g^{-1} b_g^{-1}$ reduces to $a_i a_i^{-1}$. Hence, the coefficient$d_{ea_i} = 1 - 1 = 0$. Altogether, $d_2(e) = 0$. So the homology groups of M_g are given by

$$H_n(M_g; \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } n = 0, 2, \\ \mathbb{Z}^{2g} & \text{for } n = 1, \\ 0 & \text{otherwise.} \end{cases}$$

For $g = 3$, see the figure below to visualize the $2g = 6$ generators of $H_1(M_3)$:



4. Euler Characteristic

Definition 4.1. Let X be a finite CW complex of dimension n. The Euler characteristic of X is defined as:

$$\chi(X) = \sum_{i=0}^{n} (-1)^i \cdot \#n - \text{cells} = \sum_{i=0}^{n} (-1)^i \cdot \# \operatorname{rank}(C_i^{\mathrm{CW}})$$

Here C_i^{CW} is the i-th abelian group in the chain complex that determines cellular homology. We show that the Euler characteristic does not depend on the cell structure chosen for the space X. As we will see below, this is not the case.

Proposition 4.2. *The Euler characteristic can be computed as:*

$$\chi(X) = \sum_{i=0}^{n} (-1)^i \cdot \operatorname{rank}(H_i^{\mathrm{CW}}(X;\mathbb{Z}))$$

In particular, $\chi(X)$ is independent of the chosen cell structure on X.

Proof. We use the following notation: $B_i = \operatorname{Im}(d_{i+1})$, $Z_i = \ker(d_i)$, and $H_i^{CW} = Z_i/B_i$. The additivity of rank yields that

$$\operatorname{rank}(C_i) = \operatorname{rank}(Z_i) + \operatorname{rank}(B_{i-1})$$

and

$$\operatorname{rank}(Z_i) = \operatorname{rank}(B_i) + \operatorname{rank}(H_i^{\mathrm{CW}}).$$

Substitute the second equality into the first, multiply the resulting equality by $(-1)^i$, and sum over i to get that

$$\chi(X) = \sum_{i=0}^{n} (-1)^i \cdot \operatorname{rank}(H_i^{\mathrm{CW}})$$

Since cellular homology is isomorphic to singular homology and the latter is homotopy invariant, the result follows.. □

Proposition 4.3. *Let X, Y be finite-dimensional CW complexes and let*

$$\chi(X) = \sum_{i=0}^{n} (-1)^i a_i$$

$$\chi(Y) = \sum_{j=0}^{m} (-1)^j b_j$$

Here a_i is the number of i-cells in X. Similarly, b_j is the number of j-cells in B The Euler characteristic enjoys some nice properties:

(1) $\chi(X \times Y) = \chi(X) \times \chi(Y)$
(2) *If $X = A \cup B$ such that A, B are sub-complexes of X. Then*

$$\chi(X) = \chi(A) + \chi(B) - \chi(A \cap B)$$

(3) *If $p : \widetilde{X} \to X$ is an n-sheeted covering space, then*

$$\chi(\widetilde{X}) = n\chi(X)$$

Proof. The proof is given below:

(1) For any index k, k-cells in $X \times Y$ are created by considering products of r-cells and $k-r$ cells from X and Y respectively where $0 \leq r \leq k$. Hence the number of k-cells is

$$\sum_{r=0}^{k} a_r b_{k-r}$$

Therefore,

$$\chi(X) \times \chi(Y) = \left(\sum_{i=0}^{n} (-1)^i a_i\right) \times \left(\sum_{j=0}^{m} (-1)^j b_j\right) = \sum_{k=0}^{m+n} (-1)^k \sum_{r=0}^{k} a_r b_{k-r} = \chi(X \times Y)$$

(2) Let a_i^A denote the number of i-cells in A. Similarly, let a_i^B be the number of i-cells in B. Similarly, let $a_i^{A\cap B}$ be the number of i-cells in $A \cap B$. We have
$$a_i = a_i^A + a_i^B - a_i^{A\cap B}$$
for $i = 1, \cdots, n$. Therefore, we have,
$$\begin{aligned}\chi(X) &= \sum_{i=0}^{n}(-1)^i a_i \\ &= \sum_{i=0}^{n}(-1)^i a_i^A + \sum_{i=0}^{n}(-1)^i a_i^B - \sum_{i=0}^{n}(-1)^i a_i^{A\cap B} \\ &= \chi(A) + \chi(B) - \chi(A \cap B)\end{aligned}$$

(3) Recall that if $\mathbb{D}_\alpha^k$ is a k-cell in X, then $\widetilde{X}$ has n k-cells. Therefore, it is clear that
$$\chi(\widetilde{X}) = n\chi(X)$$

□

Example 4.4. Let M_g be the closed oriented surface of genus g. We have
$$\chi_{M_g} = 2 - 2g$$
Thus all the orientable surfaces M_g are distinguished from each other by their Euler characteristics.

5. Tor Functor

In this section, we discuss the Tor functor which will play an important role in the discussion of homology with coefficients. We first briefly review the tensor product construction.

Remark 5.1. *We work with commutative rings below. Hence, we don't make any distinction between the categories of left R-modules and right R-modules. We use the generic phrase 'R-module' to refer to a left/right R-module.*

Definition 5.2. Let R be a commutative ring and let M, N be two R-modules. The tensor product of M and N, denoted $M \otimes N$, is the R-module
$$M \otimes_R N := M \times N/\sim$$
where $\sim$ is enforces the relations
$$\begin{aligned}(m + m') \otimes_R n &= m \otimes_R n + m' \otimes_R n, \\ m \otimes_R (n + n') &= m \otimes_R n + m \otimes_R n' \\ m \otimes_R rn &= mr \otimes_R n\end{aligned}$$
for $r \in R, m \in M$ and $n \in N$.

Remark 5.3. *It is easily checked that $M \otimes_R N$ is a R-module. For example, the zero element of $M \otimes_R N$ is given by*
$$0 \otimes_R n = m \otimes_R 0 = 0 \otimes_R 0 := 0_{M\otimes N}$$
since, for example,
$$0 \otimes_R n = (0 + 0) \otimes_R n = 0 \otimes n + 0 \otimes n$$

which implies $0 \otimes n = 0_{M\otimes N}$*. Similarly, the inverse of an element* $m \otimes_R n$ *is*

$$-(m \otimes_R n) = (-m) \otimes_R n = m \otimes_R (-n)$$

Remark 5.4. *The tensor product satisfies the following universal property, which asserts that if* $\varphi : M \times N \to P$ *is any* R*-bilinear map, then there exists a unique* R*-linear map* $\overline{\varphi} : M \otimes_R N \to P$ *such that* $\varphi = \overline{\varphi} \circ i$*, where* $i : M \times N \to M \otimes_R N$ *is the natural map given by* $(m, n) \mapsto m \otimes_R n$*.*

We prove some properties of the tensor product:

Proposition 5.5. *The tensor product satisfies the following properties:*

(1) $M \otimes_R N \cong N \otimes_R M$ *via the isomorphism* $m \otimes_R n \mapsto n \otimes_R m$*.*
(2) $(\bigoplus_i M_i) \otimes_R N \cong \bigoplus_i (M_i \otimes_R N)$ *via the isomorphism* $(m_i)_i \otimes_R n \mapsto (m_i \otimes_R n)_i$*.*
(3) $M \otimes_R (N \otimes_R P) \cong (M \otimes_R N) \otimes_R P$ *via the isomorphism* $m \otimes_R (n \otimes_R p) \mapsto (m \otimes_R n) \otimes_R p$*.*
(4) $R \otimes_R M \cong M$ *via the isomorphism* $r \otimes_R m \mapsto rm$*.*

Proof. The proof is given below:

(1) The map $\varphi : M \times N \to N \otimes_R M$ defined by $(m, n) \mapsto n \otimes_R m$ is clearly R-bilinear and therefore induces a R-linear map

$$\begin{aligned} \overline{\varphi} : M \otimes_R N &\to N \otimes_R M \\ m \otimes_R n &\mapsto n \otimes_R m \end{aligned}$$

Similarly, the map $\psi : N \times M \to M \otimes_R N$ defined by $(n, m) \mapsto m \otimes_R n$, is R-bilinear and therefore induces a R-linear map

$$\begin{aligned} \overline{\psi} : N \otimes_R M &\to M \otimes_R N \\ n \otimes_R m &\mapsto m \otimes_R n \end{aligned}$$

Clearly, $\overline{\varphi}$ and $\overline{\psi}$ are inverses. The claim follows.

(2) Same as (1).
(3) Same as (1).
(4) The map $\varphi : \mathbb{Z} \times M \to M$ defined by $(n, a) \mapsto na$ is a R-bilinear map and therefore induces a R-bilinear map

$$\begin{aligned} \overline{\varphi} : \mathbb{Z} \otimes_R M &\to M \\ n \otimes a &\mapsto na. \end{aligned}$$

Suppose $\overline{\varphi}(n \otimes m) = 0$. Then $nm = 0$ and

$$n \otimes m = 1 \otimes (nm) = 1 \otimes 0 = 0_{\mathbb{Z}\otimes M}.$$

Thus, $\overline{\varphi}$ is injective. Moreover, if $m \in M$, then

$$\overline{\varphi}(1 \otimes m) = m$$

and $\overline{\varphi}$ is surjective as well.

This completes the proof. □

The tensor product, $\otimes_R$, defines a functor from the category of R-modules to itself such that if N is a R-module, then

$$- \otimes_R M(N) = N \otimes_R M$$

Moreover, if $f : N \to N'$ is a R-module morphism, then

$$- \otimes_R M(f) : N \otimes_R M \xrightarrow{f \otimes_R \mathrm{Id}_M} N' \otimes_R M$$

Using a clever argument exploiting the adjunction between the Hom and tensor product functors (which we take for granted in these notes), we can show the following:

Proposition 5.6. *Let R be a commutative ring and let M be a R-module. The functor $- \otimes_R M$ is a right exact functor.*

Remark 5.7. *We freely invoke results about the Hom functor discussed in the cohomology notes in the proof of Proposition 5.6 below.*

Proof. (Sketch) Let

$$0 \to A \to B \to C \to 0$$

be an exact sequence in the category of R-modules. We show that

$$A \otimes_R M \to B \otimes_R M \to C \otimes_R M \to 0$$

is an exact sequence. Results about the Hom functor imply that

$$A \otimes_R M \to B \otimes_R M \to C \otimes_R M \to 0$$

is an exact sequence if and only if

$$0 \to \mathrm{Hom}(C \otimes_R M, X) \to \mathrm{Hom}(B \otimes_R M, X) \to \mathrm{Hom}(A \otimes_R M, X)$$

is an exact sequence for all R-module X. We have

$$\mathrm{Hom}(N \otimes_R M, X) = \mathrm{Hom}(N, \mathrm{Hom}(M, X)),$$

for all left R-modules N. Hence, the sequence above can be written as

$$0 \to \mathrm{Hom}(C, \mathrm{Hom}(M, X)) \to \mathrm{Hom}(B, \mathrm{Hom}(M, X)) \to \mathrm{Hom}(A, \mathrm{Hom}(M, X))$$

which is indeed exact by the fact that the $\mathrm{Hom}(-, \mathrm{Hom}(M, X))$ functor is left exact. □

Example 5.8. The functor $- \otimes_R M$ need not be left exact functor. To see this, take $R = \mathbb{Z}$. Consider the sequence:

$$0 \to \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z}$$

Here $\cdot n$ is the multiplication by n map. Let $M = \mathbb{Z}/n\mathbb{Z}$ we obtain a map:

$$\mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \xrightarrow{\cdot n \otimes_{\mathbb{Z}} \mathrm{Id}_{\mathbb{Z}/n\mathbb{Z}}} \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/n\mathbb{Z}$$

However, this is the zero map since we have

$$\cdot n \otimes_{\mathbb{Z}} \mathrm{Id}_{\mathbb{Z}/n\mathbb{Z}}(1 \otimes_{\mathbb{Z}} \overline{m}) = n \otimes_{\mathbb{Z}} \overline{m} = 1 \otimes_{\mathbb{Z}} \overline{nm} = 0.$$

The zero map is not injective.

Remark 5.9. *A R-module M is called flat if $- \otimes_R M$ is a left exact functor. If M is a projective R-module, then $- \otimes_R M$ is a left exact functor. This follows because a projective R-module is a direct summand of a free R-module, a free R-module is a flat module and that a R-module is flat if and only if each summad is a flat R-module. Details skipped.*

Since the $- \otimes_R M$ functor is a right exact functor which in general is not a left exact functor, we can consider its left derived functor.

Definition 5.10. Let R be a ring and let M be a R-module. The i-th Tor functor is the i-th left derived functor of $- \otimes_R M$. It is denoted as

$$\mathrm{Tor}_i^R(-, M)$$

By definition, $\mathrm{Tor}_i^R(-, M)$ is computed as follows. If N is a R-module, take any projective resolution

$$\cdots \to P^1 \to P^0 \to N \to 0,$$

and form the chain complex:

$$\cdots \to P^2 \otimes_R M \to P^1 \otimes_R M \to P^0 \otimes_R M$$

Then $\mathrm{Tor}_i^R(N, M)$ is the homology of this complex at position i.

$$\mathrm{Tor}_i^R(N, M) = H_i((P^i \otimes_R M)_\bullet)$$

Remark 5.11. *General results about derived functors show that the homology is independent of the choice of the projective resolution.*

If R is a commutative ring and M is a R-module, we can define another functor $M \otimes_R -$. The definition is similar to that of the functor defined above. It can also be checked that $M \otimes_R -$ is right exact functor that is, in general, not left exact. Hence, we can attempt to construct a left-derived functor associated to $M \otimes_R -$ as above. We label that derived functor $\overline{\mathrm{Tor}}_i^R(M, -)$. We have the following result:

Proposition 5.12. ***(Balanacing Tor)*** *Let R be a commutative ring and let M be a R-module. Then*

$$\mathrm{Tor}_i^R(-, M) \cong \overline{\mathrm{Tor}}_i^R(M, -)$$

That is, for every R-module N, we have

$$\mathrm{Tor}_i^R(N, M) \cong \overline{\mathrm{Tor}}_i^R(M, N)$$

Proof. See [Wei94] for a proof. □

Remark 5.13. *In light of Proposition 5.12, we can identify Tor_i^R with $\overline{\mathrm{Tor}}_i^R$ for each $i \geq 0$. This allows us to compute projective resolutions of either N or M to compute $\mathrm{Tor}_i^R(N, M)$ for each $i \geq 0$.*

Proposition 5.14. *Let R be a commutative ring and let M be a R-module. The Tor functor satisfies the following properties:*

1. *$\mathrm{Tor}_0^R(N, M) \cong N \otimes_R M$ for any R-modules M, N.*
2. *If N is a projective R-module, then $\mathrm{Tor}_i^R(N, M) = 0$ for all $i \geq 1$*
3. *Any $f : N_1 \to N_2$ R-module homomorphism induces a morphism*

$$f_*^i : \mathrm{Tor}_i^R(N_1, M) \longrightarrow \mathrm{Tor}_i^R(N_2, M)$$

   for each $i \geq 0$.

4. *Any short exact sequence $0 \to N_1 \xrightarrow{\phi} N_2 \xrightarrow{\psi} N_3 \to 0$ of R-modules induces a long exact sequence:*

$$\cdots \to \mathrm{Tor}_1^R(N_1, M) \to \mathrm{Tor}_1^R(N_2, M) \to \mathrm{Tor}_1^R(N_3, M) \to N_1 \otimes_R M \to N_2 \otimes_R M \to N_3 \otimes_R M \to 0$$

Proof. (1) and (2) follow from general properties of derived functors. For (3), let $P_1^\bullet$ be a projective resolution of N_1 and $P_2^\bullet$ be a projective resolution of N_2. General properties about projective resolutions imply that that f lifts to a chain map $\varphi^\bullet : P_1^\bullet \longrightarrow P_2^\bullet$. Then, $\varphi^\bullet$ induces a morphism of chain complexes $P_1^\bullet \otimes_R M \longrightarrow P_2^\bullet \otimes_R M$ which, in turn, induces a morphism:

$$f_*^i : \mathrm{Tor}_i^R(N_1, M) \longrightarrow \mathrm{Tor}_i^R(N_2, M)$$

for each $i \geq 0$. For (4), let $P^\bullet$ be a projective resolution of M. Then there is an induced short exact sequence of chain complexes:

$$0 \to N_1 \otimes_R P^\bullet \to N_2 \otimes_R P^\bullet \to N_3 \otimes_R P^\bullet \to 0$$

because each module P^i is projective. Applying the long exact sequence in homology produces the required long exact sequence. □

The above proposition show that the Ext groups 'measure' and 'repair' the non-exactness of the tensor product functor. We now specialize to the category of $\mathbb{Z}$-modules. In what follows, we fix G to be an abelian group. We have the following result:

Lemma 5.15. *For any abelian group A, we have*

$$\mathrm{Tor}_i^{\mathbb{Z}}(A, G) = 0 \quad \textit{if } i > 1.$$

Proof. Recall that any abelian group, A, admits a two-step free resolution.

$$0 \to F_1 \to F_0 \to A \to 0$$

Thus, $\mathrm{Tor}_i^{\mathbb{Z}}(A, G) = 0$ if $n > 1$. □

Remark 5.16. *Only* $\mathrm{Tor}_1(-, G)$ *encodes any interesting information. In what follows, we adopt the notation:*

$$\mathrm{Tor}(-, G) := \mathrm{Tor}_1^{\mathbb{Z}}(-, G).$$

Proposition 5.17. *If $R = \mathbb{Z}$, the Tor functor satisfies the following properties:*

(1) $\mathrm{Tor}\left(\bigoplus_i A_i, G\right) \cong \bigoplus_i \mathrm{Tor}(A_i, G)$.
(2) *If A is a free abelian group, then* $\mathrm{Tor}(A, G) = 0$.
(3) $\mathrm{Tor}(\mathbb{Z}/n\mathbb{Z}, G) \cong \mathit{ker}(G \xrightarrow{n} G)$.
(4) *For a short exact sequence: $0 \to B \to C \to D \to 0$ of abelian groups, there is a natural exact sequence:*

$$0 \to \mathrm{Tor}(B, G) \to \mathrm{Tor}(C, G) \to \mathrm{Tor}(D, G) \to B \otimes G \to C \otimes G \to D \otimes G \to 0.$$

Proof. The proof is given below:

(1) This follows from the identity,

$$\left(\bigoplus_i A_i\right) \otimes_{\mathbb{Z}} G = \bigoplus_i (A_i \otimes_{\mathbb{Z}} G)$$

and noting that taking direct sums of projective resolutions of A_i forms a projective resolution for $\bigoplus_i A^i$, and that homology commutes with direct sums.

(2) If A is free, then

$$0 \to A \to A \to 0$$

is a projective resolution of A, so $\mathrm{Tor}(A, G) = 0$.

(3) The exact sequence
$$0 \to \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$$
is a projective resolution of $\mathbb{Z}/n\mathbb{Z}$. Tensoring with G and dropping the right-most term yields the complex:
$$G \cong \mathbb{Z} \otimes G \xrightarrow{\cdot n \otimes 1_G} G \cong \mathbb{Z} \otimes G \to 0,$$
Thus, $\mathrm{Tor}(\mathbb{Z}/n\mathbb{Z}, G) = \ker(G \xrightarrow{n} G)$.

(4) This follows from Proposition 5.14(4).

This completes the proof. □

6. Homology with Coefficients

In this section, we discuss homology with coefficients and the universal coefficient theorem in homology. Let G be an abelian group and X a topological space. We define the homology of X with G-coefficients, denoted $H_n(X; G)$ for $n \in \mathbb{N}$, as the homology of the chain complex:
$$C_\bullet(X; G) = C_\bullet(X) \otimes G$$
consisting of finite formal sums $\sum_i \eta_i \cdot \sigma_i$ with $\eta_i \in G$, and with boundary maps given by
$$\partial_n^G := \partial_n \otimes \mathrm{Id}_G.$$
Since ∂_n satisfies $\partial_n \circ \partial_{n+1} = 0$, it follows that $\partial_n^G \circ \partial_{n+1}^G = 0$. Hence,
$$(C_\bullet(X); G, \partial_\bullet^G)$$
is indeed a chain complex. We can construct versions of the usual modified homology groups (relative, reduced, etc.) in the most natural way.

(1) **(Relative homology with G-coefficients)** Consider the augmented chain complex:
$$C_1(X; G) \longrightarrow C_0(X; G) \longrightarrow G \longrightarrow 0$$
where $\epsilon(\sum_i \eta_i \sigma_i) = \sum_i \eta_i \in G$. Reduced homology with G-coefficients is defined as the homology of the augmeneted chain complex.

(2) **(Relative chain Complex with G-coefficients)** Define relative chains with G-coefficients by:
$$C_n(X, A; G) := C_n(X; G)/C_n(A; G),$$
Consider the chain complex:
$$C_1(X, A; G) \longrightarrow C_0(X, A; G) \longrightarrow 0$$
The relative homology with G-coefficients is defined as the homology of the augmeneted chain complex.

(3) **(Cellular homology with G-coefficients)** We can build cellular homology with G-coefficients by defining
$$C_n^G(X) = H_n(X_n, X_{i-1}; G) \cong G^{(\text{number of } i\text{-cells})}$$
The cellular boundary maps are given by:
$$d_n^G\,(e_n^\alpha) = \sum_\beta d_{\alpha\beta} e_{i-1}^\beta,$$

where $d_{\alpha\beta}$ is as before the degree of a map $\Delta_{\alpha\beta} : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$. As it is the case for integers, we get an isomorphism:

$$H_n^{CW}(X;G) \cong H_n(X;G)$$

Example 6.1. By studying the chain complex with G-coefficients, it follows that:

$$H_n(\{*\};G) = \begin{cases} G & \text{if } n = 0, \\ 0 & \text{if } n \neq 0. \end{cases}$$

Example 6.2. (Sketch) The homology of a sphere as before by induction and using the long exact sequence of the pair $(\mathbb{D}^n, \mathbb{S}^n)$ to be:

$$H_n(\mathbb{S}^n;G) = \begin{cases} G & \text{if } i = 0, n, \\ 0 & \text{otherwise.} \end{cases}$$

We now prove an important theorem that relates how homology with different coefficients are related.

Proposition 6.3. ***(Universal Coefficient Theorem)*** *If a chain complex $(C_\bullet, \partial_\bullet)$ of free abelian groups has homology groups $H_n(C_\bullet)$, then the homology groups $H_n(C_\bullet;G)$ are determined by the short exact sequence:*

$$0 \longrightarrow H_n(C_\bullet) \otimes_{\mathbb{Z}} G \longrightarrow H_n(C_\bullet;G) \xrightarrow{h} \operatorname{Tor}(H_{n-1}(C_\bullet), G) \longrightarrow 0$$

Proof. We have

$$B_n = \operatorname{im} \partial_{n+1} \subseteq Z_n = \ker \partial_n \subseteq C_n$$

Since $\partial_n|_{Z_n} = 0$ and $\partial_{n-1}|_{B_{n-1}} = 0$, we have the following short exact sequence of chain complexes:

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & Z_{n+1} \arrow[r, hook, "i_{n+1}"] \arrow[d, "\partial_{n+1}=0"] & C_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, "\partial_{n+1}"] & B_n \arrow[r] \arrow[d, "\partial_n=0"] & 0 \\
0 \arrow[r] & Z_n \arrow[r, hook, "i_n"] \arrow[d, "\partial_n=0"] & C_n \arrow[r, "\partial_n"] \arrow[d, "\partial_n"] & B_{n-1} \arrow[r] \arrow[d, "\partial_{n-1}=0"] & 0 \\
0 \arrow[r] & Z_{n-1} \arrow[r, hook, "i_{n-1}"] \arrow[d] & C_{n-1} \arrow[r, "\partial_{n-1}"] \arrow[d] & B_{n-2} \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

Since C_n is a free abelian group, Z_n, B_n are also free abelian groups. Hence, $\otimes_{\mathbb{Z}}$ is an exact functor if applied to each row. Hence, we have a short exact sequence of chain complexes:

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & Z_{n+1} \otimes_{\mathbb{Z}} G \arrow[r, hook, "i_{n+1}\otimes_{\mathbb{Z}}\mathrm{id}_G"] \arrow[d, "0"] & C_{n+1} \otimes_{\mathbb{Z}} G \arrow[r, "\partial_{n+1}\otimes_{\mathbb{Z}}\mathrm{id}_G"] \arrow[d, "\partial_{n+1}\otimes_{\mathbb{Z}}\mathrm{id}_G"] & B_n \otimes_{\mathbb{Z}} G \arrow[r] \arrow[d, "0"] & 0 \\
0 \arrow[r] & Z_n \otimes_{\mathbb{Z}} G \arrow[r, hook, "i_n\otimes_{\mathbb{Z}}\mathrm{id}_G"] \arrow[d, "0"] & C_n \otimes_{\mathbb{Z}} G \arrow[r, "\partial_n\otimes_{\mathbb{Z}}\mathrm{id}_G"] \arrow[d, "\partial_n\otimes_{\mathbb{Z}}\mathrm{id}_G"] & B_{n-1} \otimes_{\mathbb{Z}} G \arrow[r] \arrow[d, "0"] & 0 \\
0 \arrow[r] & Z_{n-1} \otimes_{\mathbb{Z}} G \arrow[r, hook, "i_{n-1}\otimes_{\mathbb{Z}}\mathrm{id}_G"] \arrow[d] & C_{n-1} \otimes_{\mathbb{Z}} G \arrow[r, "\partial_{n-1}\otimes_{\mathbb{Z}}\mathrm{id}_G"] \arrow[d] & B_{n-2} \otimes_{\mathbb{Z}} G \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

By the snake lemma (**??**) we have a connecting homomorphism

$$\Delta_n : B_n \otimes_{\mathbb{Z}} G \to Z_n \otimes_{\mathbb{Z}} G$$

Following the construction of the connecting homomorphism in the proof of the snake lemma, we see that $\Delta_n = j_n \otimes \mathrm{Id}_M$ where $j_n : B_n \to Z_n$ is the inclusion $B_n \subseteq Z_n$. Additionally, we get an exact sequence

$$\cdots \to H_n(B\otimes_{\mathbb{Z}}G) \xrightarrow{(j_n\otimes_{\mathbb{Z}}\mathrm{id}_G)_*} H_n(Z\otimes_{\mathbb{Z}}G) \xrightarrow{(i_n\otimes_{\mathbb{Z}}\mathrm{id}_G)_*} H_n(C\otimes_{\mathbb{Z}}G) \xrightarrow{(\partial_n\otimes_{\mathbb{Z}}\mathrm{id}_G)_*} H_{n-1}(B\otimes_{\mathbb{Z}}G) \to \cdots$$

Since the chain complexes $Z \otimes_{\mathbb{Z}} G$ and $B \otimes_{\mathbb{Z}} G$ have null boundary maps, we deduce that the sequence

$$\cdots \to B_n \otimes_{\mathbb{Z}} G \xrightarrow{j_n\otimes_{\mathbb{Z}}\mathrm{id}_G} Z_n \otimes_{\mathbb{Z}} G \xrightarrow{i_n\otimes_{\mathbb{Z}}\mathrm{id}_G} H_n(C \otimes_{\mathbb{Z}} G) \xrightarrow{(\partial_n\otimes_{\mathbb{Z}}\mathrm{id}_G)_*} B_{n-1} \otimes_{\mathbb{Z}} G \to \cdots$$

We claim that this sequence is natural. Suppose that we have another chain complex $(C'_\bullet, \partial'_\bullet)$ and a chain map $f : C_\bullet \to C'_\bullet$. We want to see that the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & B_n(C) \otimes_{\mathbb{Z}} M \arrow[r, "j_n\otimes_{\mathbb{Z}}\mathrm{id}_G"] \arrow[d, "f_n\otimes_{\mathbb{Z}}\mathrm{Id}_M"] & Z_n(C) \otimes_{\mathbb{Z}} M \arrow[r, "i_n\otimes_{\mathbb{Z}}\mathrm{id}_G"] \arrow[d, "f_n\otimes_{\mathbb{Z}}\mathrm{Id}_M"] & H_n(C \otimes_{\mathbb{Z}} M) \arrow[r, "(\partial_n\otimes_{\mathbb{Z}}\mathrm{id}_G)_*"] \arrow[d, "(f_n\otimes_{\mathbb{Z}}\mathrm{Id}_M)_*"] & B_{n-1}(C) \otimes_{\mathbb{Z}} M \arrow[r] \arrow[d, "f_n\otimes_{\mathbb{Z}}\mathrm{Id}_M"] & \cdots \\
\cdots \arrow[r] & B_n(C') \otimes_{\mathbb{Z}} M \arrow[r, "j'_n\otimes_{\mathbb{Z}}\mathrm{id}_G"] & Z_n(C') \otimes_{\mathbb{Z}} M \arrow[r, "i'_n\otimes_{\mathbb{Z}}\mathrm{id}_G"] & H_n(C' \otimes_{\mathbb{Z}} M) \arrow[r, "(\partial'_n\otimes_{\mathbb{Z}}\mathrm{id}_G)_*"] & B_{n-1}(C') \otimes_{\mathbb{Z}} M \arrow[r] & \cdots
\end{tikzcd}$$

commutes. It suffices to show that the three squares commute. For the left-most square, given an n-boundary b_n, its image $f_n(b_n)$ is also an n-boundary, hence $f_n \circ j_n(b_n) = f_n(b_n) = j'_n \circ f_n(b_n)$. Therefore, we deduce that

$$(f_n \otimes \mathrm{id}_G) \circ (j_n \otimes \mathrm{id}_G) = (j'_n \otimes \mathrm{id}_G) \circ (f_n \otimes \mathrm{id}_G).$$

The middle square commutes for similar reasons. Finally, the right most square commutes since

$$(f_n \otimes_{\mathbb{Z}} \mathrm{id}_G) \circ (\partial_n \otimes_{\mathbb{Z}} \mathrm{id}_G) = (\partial'_n \otimes_{\mathbb{Z}} \mathrm{id}_G) \circ (f_n \otimes_{\mathbb{Z}} \mathrm{id}_G).$$

From this discussion, we deduce the natural short exact sequences:

$$0 \to \mathrm{Coker}(j_n \otimes_{\mathbb{Z}} \mathrm{id}_G) \xrightarrow{i_n \otimes \mathrm{id}} H_n(C \otimes_{\mathbb{Z}} G) \xrightarrow{(\partial_n \otimes_{\mathbb{Z}} \mathrm{id}_G)_*} \mathrm{Ker}(j_{n-1} \otimes_{\mathbb{Z}} \mathrm{id}_G) \to 0.$$

Consider the projections $\pi_n : Z_n \to H_n(C)$, and the short exact sequence

$$0 \to B_n \xrightarrow{j_n} Z_n \xrightarrow{\pi_n} H_n(C) \to 0.$$

Note that these sequences are free resolutions of the homology modules $H_n(C)$. Hence, by properties of Tor, we have

$$0 \to \mathrm{Tor}(H_n(C), G) \to B_n \otimes_{\mathbb{Z}} G \xrightarrow{\iota_n \otimes \mathrm{id}_G} C_n \otimes_{\mathbb{Z}} G \xrightarrow{\pi_n \otimes \mathrm{id}} H_n(C) \otimes_{\mathbb{Z}} G \to 0.$$

We can now identify $\mathrm{Ker}(j_{n-1} \otimes_{\mathbb{Z}} \mathrm{id}_G)$ as $\mathrm{Tor}(H_{n-1}(C), G)$ and $\mathrm{Coker}(j_n \otimes_{\mathbb{Z}} \mathrm{id}_G)$ as $H_n(C) \otimes_{\mathbb{Z}} G$. The claim follows. $\square$

Remark 6.4. *The sequence in Proposition 6.3 splits. This is because in the beginning of the proof, we considered the the exact sequences*

$$0 \to Z_n \xrightarrow{i_n} C_n \xrightarrow{\partial_n} B_{n-1} \to 0,$$

This sequence splits since B_{n-1} is a free abelian group. Hence, there must exist group homomorphisms $l_n : C_n \to Z_n$ such that $l_n \circ i_n = id$: $Z_n \to Z_n$ for all $n \in \mathbb{Z}$. Hence the compositions $(\pi_n \circ l_n)_{n \in \mathbb{N}}$ induce maps in homology

$$((\pi_n \circ l_n) \otimes id_G)_* : H_n(C \otimes_{\mathbb{Z}} G) \to H_n(C) \otimes_{\mathbb{Z}} G,$$

which are such that

$$((\pi_n \circ l_n) \otimes id_G)_* \circ ((i_n)_* \otimes id_G) = (id_G \otimes id_G)_*.$$

As a consequence, the sequence splits and we obtain

$$H_n(C_\bullet \otimes_{\mathbb{Z}} G) \cong (H_n(C_\bullet) \otimes_{\mathbb{Z}} G) \oplus \mathrm{Tor}(H_{n-1}(C), G).$$

Remark 6.5. *There is also a universal coefficient theorem for homology where $\mathbb{Z}$ is replaced by a PID, R and G is a R-module. In this case, we have*

$$0 \longrightarrow H_n(C_\bullet) \otimes_R G \longrightarrow H_n(C_\bullet; G) \xrightarrow{\ h\ } \mathrm{Tor}_1^R(H_{n-1}(C_\bullet), G) \longrightarrow 0$$

This comes from first establishing that Tor_i^R vanishes for $i \geq 2$ for when R is a PID, and then going through a proof for universal coefficient theorem essentially as above. Details skipped.

Example 6.6. Suppose $X = K$ is the Klein bottle, and $G = \mathbb{Z}/4$. Recall that $H_1(K; \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}/2$, and $H_2(K; \mathbb{Z}) = 0$, so:

$$\begin{aligned} H_2(K; \mathbb{Z}/4) &= (H_2(K; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}/4) \oplus \mathrm{Tor}(H_1(K), \mathbb{Z}/4) \\ &= \mathrm{Tor}(\mathbb{Z}, \mathbb{Z}/4) \oplus \mathrm{Tor}(\mathbb{Z}/2, \mathbb{Z}/4) \\ &= 0 \oplus \mathbb{Z}/2 \\ &= \mathbb{Z}/2. \end{aligned}$$

Example 6.7. Let $X = \mathbb{RP}^n$ and $G = Z/2\mathbb{Z}$. Recall that we have

$$H_k(\mathbb{RP}^n; \mathbb{Z}) = \begin{cases} \mathbb{Z}/2\mathbb{Z} & \text{if } k \text{ is odd, } 0 < k < n, \\ \mathbb{Z} & \text{if } k = 0, n \text{ is odd,} \\ 0 & \text{otherwise.} \end{cases}$$

We compute $H_k(\mathbb{RP}^n;\mathbb{Z}/2\mathbb{Z})$. We consider multiple cases. For $k=0$, we have:

$$H_0(\mathbb{RP}^n;\mathbb{Z}/2\mathbb{Z}) \cong H_0(\mathbb{RP}^n;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} = \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z}.$$

For $k=1$, we have:

$$\begin{aligned} H_1(\mathbb{RP}^n;\mathbb{Z}/2\mathbb{Z}) &\cong H_1(\mathbb{RP}^n;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \oplus \mathrm{Tor}(H_0(\mathbb{RP}^n),\mathbb{Z}/2\mathbb{Z}) \\ &= (\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}) \oplus \mathrm{Tor}(\mathbb{Z},\mathbb{Z}/2\mathbb{Z}) \\ &= \mathbb{Z}/2\mathbb{Z} \oplus 0 = \mathbb{Z}/2\mathbb{Z}. \end{aligned}$$

For $1<k<n$, such that k is an odd integer, we have

$$\begin{aligned} H_k(\mathbb{RP}^n;\mathbb{Z}/2\mathbb{Z}) &\cong (H_k(\mathbb{RP}^n;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}) \oplus \mathrm{Tor}(H_{k-1}(\mathbb{RP}^n;\mathbb{Z}),\mathbb{Z}/2\mathbb{Z}) \\ &= (\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}) \oplus \mathrm{Tor}(0,\mathbb{Z}/2\mathbb{Z}) \\ &= \mathbb{Z}/2\mathbb{Z} \end{aligned}$$

For $1<k<n$, such that k is an even integer, we have

$$\begin{aligned} H_k(\mathbb{RP}^n;\mathbb{Z}/2\mathbb{Z}) &\cong (H_k(\mathbb{RP}^n;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}) \oplus \mathrm{Tor}(H_{k-1}(\mathbb{RP}^n;\mathbb{Z}),\mathbb{Z}/2\mathbb{Z}) \\ &= (0 \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}) \oplus \mathrm{Tor}(\mathbb{Z}/2\mathbb{Z},\mathbb{Z}/2\mathbb{Z}) \\ &= \mathbb{Z}/2\mathbb{Z} \end{aligned}$$

For $k=n$ even, we have

$$H_k(\mathbb{RP}^n;\mathbb{Z}/2\mathbb{Z}) = (0 \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}) \oplus \mathrm{Tor}(\mathbb{Z}/2\mathbb{Z},\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$$

If $k=n$ is odd, we have

$$H_k(\mathbb{RP}^n;\mathbb{Z}/2\mathbb{Z}) = (\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}) \oplus \mathrm{Tor}(0,\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$$

All in all, we have

$$H_k(\mathbb{RP}^n;\mathbb{Z}/2\mathbb{Z}) = \begin{cases} \mathbb{Z}/2\mathbb{Z} & \text{if } k=0,\cdots,n \\ 0 & \text{otherwise.} \end{cases}$$

References

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