OPERATOR ALGEBRAS: C^*-ALGEBRAS

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Abstract. This document comprises notes on Banach algebras and C^*-algebras, with a focus on their applications to spectral theory and representation theory. A portion of these notes was taken during my participation in the Groundwork for Operator Algebras Lecture Series (GOALS) workshop at IPAM, UCLA. If you come across any typos, please send corrections to junaid.aftab1994@gmail.com.

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Remark 0.1. *Unless otherwise specified, we work over $\mathbb{K} = \mathbb{C}$, the field of complex numbers in these notes.*

Part 1. Banach Algebras

1. Why Banach Algebras?

Banach algebras are perhaps the most general type of operator algebra. They also provide a natural setting for studying classical topics in functional analysis, such as spectral theory. Key concepts like the spectrum, resolvent sets, and spectral radius can be examined within this framework, facilitating the analysis of operator behavior on Hilbert spaces and beyond. Additionally, C^*-algebras and von Neumann algebras form special classes of Banach algebras. Therefore, studying Banach algebras is a fundamental step in the broader study of operator algebras.

2. Definitions & Examples

We begin with a detailed discussion of definitions and examples in Banach algebra theory. If $\mathscr{H}$ is a Hilbert space, note that the Banach space, $\mathscr{B}(\mathscr{H})$, of bounded linear operators on $\mathscr{H}$ has more structure than that of a Banach space. Indeed, if $T, S \in \mathscr{B}(\mathscr{H})$, then $T \circ S \in \mathscr{B}(\mathscr{H})$ such that

$$\|T \circ S\| \leq \|T\|\|S\|$$

This observation motivates the definition of a Banach algebra. We will proceed in multiple steps.

Definition 2.1. Let V be a complex vector space.

- V is an **algebra** if the underlying abelian group admits a multiplication operation
$$\cdot : V \times V \to V \qquad (x, y) \mapsto x \cdot y$$
endowing V with a ring structure compatible with the given scalar multiplication.
- V is a **normed algebra** if it is equipped with a submultiplicative norm.

Remark 2.2. *If V is a (normed) algebra, we say V is a unital (normed) algebra if it admits an identity element. That is, there exists a $e \in V$ such that $e \cdot x = x \cdot e = x$ for all $x \in V$. It is a simple exercise to check that the identity in a unital algebra is unique. The proof is identical from group theory.*

Remark 2.3. *If is V a (normed) algebra, we say V is an abelian (normed) algebra if $x \cdot y = y \cdot x$ for all $x, y \in V$*

Remark 2.4. *From now on, we abbreviate the ring multiplication operator $x \cdot y$ as simply xy for x, y in an algebra.*

If V is a normed algebra, then the norm induces a metric on V which in turn induces a topology on V called the norm topology. Here is a sample proposition:

Lemma 2.5. *Let V be a normed algebra. Addition, scalar multiplication and multiplication are are continuous in the norm topology on V.*

Proof. Let's consider the multiplication operation. Let $x_n \to x$ and $y_n \to y$. The submultiplicativity of the norm implies that

$$\|xy - x_n y_n\| \leq \|xy - xy_n\| + \|xy_n - x_n y_n\| \leq \|x\|\|y - y_n\| + \|x - x_n\|\|y_n\| \to 0.$$

It is clear that addition and scalar multiplication are continuous. □

Definition 2.6. A **Banach algebra**, A, is a normed algebra that is complete in the metric topology induced by the norm.

In other words, a Banach algebra, A, is a Banach space endowed with a sub-multiplicative operation making X into a normed algebra.

Remark 2.7. *If A is unital Banach algebra with identity element $e \in A$, then note that we have*

$$\|e\| = \|ee\| \leq \|e\|\|e\|$$

Hence, $\|e\| \geq 1$. In a C^*-algebra, we have $\|e\| = 1$. Observe that for any $r \in \mathbb{R}$ with $r \geq 1$, $(A, r\|\cdot\|)$ remains a Banach algebra. Moreover, note that $\|\cdot\|$ and $r\|\cdot\|$ are equivalent norms. Therefore, it is possible to modify the norm of any Banach algebra such that the unit has a norm of 1 with respect to this new norm.

Notice that a subalgebra is itself an algebra. A subalgebra of a normed algebra is a normed algebra. The closure of a subalgebra of a normed algebra is a normed algebra. Therefore the closure of any subalgebra of a Banach algebra is again a Banach algebra. This observation generates a long list of examples:

Example 2.8. The following is a list of some basic examples of Banach algebras:

(1) Let S be a set, and let $\ell^\infty(S)$ be the collection of all bounded complex-valued functions on S. Then $\ell^\infty(S)$ is a Banach algebra with respect to the usual pointwise operations defined as follows:

$$\begin{aligned}(f+g)(s) &:= f(s) + g(s),\\ (fg)(s) &:= f(s)g(s),\\ (\lambda f)(s) &:= \lambda f(s)\end{aligned}$$

for all $s \in S$. It is a commutative unital Banach algebra with identity given by the constant function $x \mapsto 1$. The norm is given by

$$\|f\|_\infty = \sup_{s \in S} |f(s)|.$$

(2) Let X be a locally compact Hausdorff space, X. Let $C_b(X)$ denote the space of bounded continuous complex-valued functions on X. It can be checked that $C_b(X)$ is a closed subalgebra of $\ell^\infty(X)$. Hence, $C_b(X)$ is a unital commutative Banach algebra.

(3) Let X be a locally compact Hausdorff space, X. Let $C_0(X)$ be the the space of continuous complex-valued functions on X that vanish at infinity. It can be checked that $C_0(X)$ is a closed subalgebra of $C_b(X)$. Hence, $C_0(X)$ is a commutative Banach algebra. It is unital if and only if X is compact.

(4) Let $\mathscr{H}$ be a Hilbert space. Then $\mathscr{B}(\mathscr{H})$ is a (generally non-commutative) unital Banach algebra with multiplication operation given by composition, identity given by the identity operator, and norm given by the operator norm. In particular, if $\mathscr{H} = \mathbb{C}^n$ then $M_n(\mathbb{C})$ is a Banach algebra under matrix multiplication and the operator norm.

Remark 2.9. *If X is locally compact and Hausdorff, then $C_b(X)$ contains many functions due to Urysohn's Lemma. If X is locally compact but not compact, then $C_b(X)$ is likely non-separable. Let $X = \mathbb{R}$. Consider the subset $K \subseteq C_b(\mathbb{R})$ consisting of functions that are either 0 or 1 at the integers. There is an uncountable subset S of K such that:*

$$\|f - g\| \geq 1, \quad \textit{whenever } f, g \in S \textit{ with } f \neq g.$$

Given a countable subset B of $C_b(\mathbb{R})$, it follows that there is an $s \in S$ that is at least $\frac{1}{2}$ distance away from every element of B. Thus, B is not dense in $C_b(\mathbb{R})$.

Remark 2.10. *If X is a locally compact Hausdorff space, we have that $C(X)$ is also a Banach algebra. In general, we have the following inclusions of Banach algebras:*

$$C_0(X) \subsetneq C_b(X) \subsetneq C(X)$$

If X is a compact Hausdorff space, then

$$C_0(X) = C_b(X) = C(X)$$

Definition 2.11. Let A and B be Banach algebras. A morphism $\phi : A \to B$ is a continuous $\mathbb{C}$-linear, multiplicative map. Moreover, ϕ is an isometric isomorphism if it has a continuous inverse and ϕ is an isometry.

Remark 2.12. *If A, B are unital Banach algebras, then a morphism $\phi : A \to B$ is unital morphism if $\phi(e_A) = e_B$.*

Example 2.13. Let X be a compact topological space and let $Y \subseteq X$ be a compact subspace. Then the restriction of functions is a momorphism of Banach algebras from $C(X)$ to $C(Y)$. This includes the special case when $Y = \{x\}$, consisting of a single element. In this case, $C(Y) \cong \mathbb{C}$, and the restriction is the evaluation homomorphism $\delta_x : C(X) \to \mathbb{C}$ mapping f to $f(x)$.

Commutative Banach algebras are much easier to study. In spite of the fact that we are most interested in algebras modeling $\mathscr{B}(\mathscr{H})$ for some Hilbert space $\mathscr{H}$ — which are not commutative — the theory of commutative Banach algebras plays a very important role in the sequel. We discuss an important example of a commutative Banach algebras to end this section:

Example 2.14. Consider $L^1(\mathbb{R})$, the Banach space of Lebesgue integrable complex-valued functions on $\mathbb{R}$. The convolution of two functions f and g in $L^1(\mathbb{R})$ is defined as

$$(f * g)(x) := \int_{\mathbb{R}} f(x-y)g(y)\,dy.$$

$L^1(\mathbb{R})$ is a commutative Banach algebra. It is easy to check that the $+$ and $*$ operations on $L^1(\mathbb{R})$ satisfy the axioms of an abelian algebra. That is, we have,

$$\begin{aligned}(f * g) * h &= f * (g * h)\\ f * (g + h) &= f * g + f * h\\ (f + g) * h &= f * h + g * h\\ f * g &= g * f\end{aligned}$$

Moreover, $*$ is sub-multiplicative. Indeed, we have,

$$\begin{aligned}\|f * g\|_1 &= \int_{\mathbb{R}}\int_{\mathbb{R}} |f(x-y)||g(y)|\,dy\,dx\\ &\leq \int_{\mathbb{R}}\int_{\mathbb{R}} |f(x-y)||g(y)|\,dx\,dy\\ &= \int_{\mathbb{R}}\left[\int_{\mathbb{R}} |f(x-y)|\,dx\right]|g(y)|\,dy\\ &= \|f\|_1 \int_{\mathbb{R}} |g(y)|\,dy = \|f\|_1\,\|g\|_1\end{aligned}$$

$L^1(\mathbb{R})$ is a non-unital Banach algebra. Assume that there is a $f \in L^1(\mathbb{R})$ is a unit. For each $g \in L^1(\mathbb{R})$, we have by the convolution theorem,

$$\hat{g} = \widehat{f * g} = \hat{f}\hat{g}$$

But if $g \in L^1(\mathbb{R})$ is a Gaussian, then $\hat{g}$ doesn't vanish. Hence, $\hat{f} \equiv 1$. But the Riemann Lebesgue lemma implies that $\hat{f} \in C_0(\mathbb{R})$. This is a contradiction.

Remark 2.15. *The convolution on $L^1(\mathbb{R})$ is first defined on the dense subspace $C_c(\mathbb{R})$, and subsequently be extended by continuity to $L^1(\mathbb{R})$.*

Example 2.16. Let $\mathbb{D} = \{z \in \mathbb{C} : |z| \leq 1\}$ be the open unit disk. Let $\mathscr{O}(\mathbb{D}) \subseteq C(\mathbb{D})$ be the subalgebra of holomorphic functions on $\mathbb{D}^\circ$. By Morera's Theorem, the uniform limit of holomorphic functions is holomorphic, which implies that $\mathscr{O}(\mathbb{D})$ is a closed subalgebra of $C(\mathbb{D})$. Let $\mathscr{A}(\mathbb{D})$ be the image of $\mathscr{O}(\mathbb{D})$ in $C(\mathbb{S}^1)$ via the restriction map. Since the map $g \mapsto g|_{\mathbb{S}^1}$ is isometric by the Maximum Modulus Principle, $\mathscr{A}(\mathbb{D})$ is complete and therefore closed in $\mathscr{C}(\mathbb{S}^1)$. Hence, $\mathscr{A}(\mathbb{D})$ is a closed subalgebra of $\mathscr{C}(\mathbb{S}^1)$ and forms a Banach algebra, commonly referred to as the disk algebra. It consists of those functions in $C(\mathbb{S}^1)$ that have holomorphic extensions to $\mathbb{D}$.

3. Unitization

When a Banach algebra, A, does not contain a unit, we can always add one, as follows. Form the vector space

$$A' := A \oplus \mathbb{C},$$

and make this into an algebra by means of

$$(a, \lambda)(b, \mu) := (ab + \lambda b + \mu a, \lambda\mu)$$

for each $a, b \in A$ and $\lambda, \mu \in \mathbb{C}$. In other words, $(0, 1) \in A'$, which can be identified with $1 \in \mathbb{C}$, is the identity, e', in A'. Furthermore, we can define a norm on A' by

$$\|(a, \lambda)\|_{A'} := \|a\|_A + |\lambda|$$

In particular, $\|e'\| = |1| = 1$. We have,

$$\begin{aligned} \|(a, \lambda)(b, \mu)\|_{A'} &\leq \|a\|_A\|b\|_A + |\lambda|\|b\|_A + |\mu|\|a\|_A + |\lambda||\mu| \\ &= \|(a, \lambda)\|_{A'}\|(b, \mu)\|_{A'} \end{aligned}$$

Hence, A' is a unital Banach algebra containing A.

If A already has a unit e, then the algebra A' is isomorphic to the direct sum $A \oplus \mathbb{C}$ of the algebras A and $\mathbb{C}$, where we define multiplication component-wise. The isomorphism from A' to $A \oplus \mathbb{C}$ is given by

$$(a, \lambda) \mapsto (a + \lambda e) \oplus \lambda$$

As a crucial example, we compute the unitization $C_0(X)'$. We recall the one-point compactification X_∞ of the space X. Let ∞ denote a new point, and define $X_\infty = X \cup \{\infty\}$, where X_∞ is X with an additional point. A set $U \subseteq X_\infty$ is open if it is either open in X or contains ∞ and the set $X \setminus U$ is compact in X. Every continuous function in $C(X_\infty)$ restricts to a continuous function on X, so $C_0(X)$ can be identified with the subspace of continuous functions on X_∞ that vanish at ∞.

Proposition 3.1. *Let X be a locally compact non-compact Hausdorff space. There is a topological (non-isometric) isomorphism of Banach algebras $C(X_\infty) \cong C_0(X)'$.*

Proof. Extending every $f \in C_0(X)$ by zero to X_∞, we consider $C_0(X)$ as a subspace of $C(X_\infty)$. Define $\psi : C_0(X)' \to C(X_\infty)$ by $\psi(f,\lambda) = f + \lambda 1_X$, where $1(x) = 1$ for all $x \in X_\infty$. Moreover, define $\phi : C(X_\infty) \to C_0(X)'$ by $\phi(f) = (f|_{C(X)}, 0)$. Then ψ is an isomorphism of algebras with inverse ϕ as can be easily checked. For the norms, we have

$$\|\psi(f,\lambda)\|_{C(X_\infty)} = \sup_{x\in X_\infty} |f(x)+\lambda| \leq \sup_{x\in X} |f(x)| + |\lambda| = \|(f,\lambda)\|_{C(X)'}.$$

This shows that ψ is continuous. It can also be checked that ϕ is continuous. This proves the claim. □

4. Ideals

We can define the notion of an ideal in a Banach algebra.

Definition 4.1. Let A be a Banach algebra. A subspace $I \subseteq A$ of a commutative Banach algebra A is called a left ideal if for any $a \in I$, it follows that $ab \in I$ for all $b \in A$. A right ideal is defined similarly.

Proposition 4.2. *Let A be a Banach algebra and let $I \subseteq A$ be a closed two-sided left proper ideal. The quotient space A/I is a Banach algebra with respect to the quotient norm.*

Proof. It is a standard fact from Banach space theory that A/I is Banach space with the quotient norm. We only show that the quotient norm is sub-multiplicative. For given $a, b \in A$ and for every $\epsilon > 0$, by the definition of the quotient norm, there exist $m, n \in I$ such that

$$\|a+m\| \leq \|\pi(a)\| + \epsilon, \qquad \|b+n\| \leq \|\pi(b)\| + \epsilon$$

Since $(a+m)(b+n) \in ab + I$, we have

$$\begin{aligned}\|\pi(a)\pi(b)\| &= \|\pi((a+m)(b+n))\| \\ &\leq \|(a+m)(b+n)\| \\ &\leq \|a+m\|\|b+n\| \\ &\leq \|\pi(a)\|\|\pi(b)\| + \epsilon(\|\pi(a)\| + \|\pi(b)\| + \epsilon).\end{aligned}$$

This holds for every ϵ, and so it implies the desired claim. □

Example 4.3. Let A be a Banach algebra.

(1) There are two trivial ideals, the one consisting of the zero element and the one consisting of A itself.
(2) Any ideal, I, that contains the unit element e is equal to A.
(3) If A is a non-unital algebra, is clear that the embedding of A into A' is linear and isometric and that A sits inside of A' as a closed ideal[1]. Hence, we can in a way think of A' as the smallest unital Banach algebra in which A sits as an ideal.

Definition 4.4. Let A be a Banach algebra. An ideal $I \subseteq A$ is a left (resp. right) maximal ideal if it is not contained in any other non-left (resp. right) trivial ideal.

Zorn's lemma implies that maximal ideals always exist.

Proposition 4.5. *Let A be a Banach algebra. Any non-trivial left (resp. right) ideal is a subset of a left (resp. right) maximal ideal.*

[1] In fact, as a maximal ideal, given that its co-dimension is equal to 1.

Proof. Denote the ideal by I. A partial order among left (resp. right) ideals containing I is established through inclusion. Consider any chain of such non-trivial left (resp. right) ideals I_α, i.e., for any $\alpha \neq \beta$, either $I_\alpha \subseteq I_\beta$ or $I_\beta \subseteq I_\alpha$. We claim that $U = \bigcup_\alpha I_\alpha$ contains I, is a left (resp. right) ideal, and hence an upper bound. Clearly, U is a subspace since all I_β are subspaces. Any $x \in U$ is in some I_α, and if $y \in A$ is any element, we have that $xy \in I_\alpha$ and thus in U. Since e is not in any of the I_β, it is not in U, and hence U is non-trivial. That $I \subseteq U$ is evident. By Zorn's lemma, there exists a maximal element M, i.e., M is an ideal such that whenever V is an ideal that contains I and M, then $V = M$. Hence, M is a maximal ideal. $\square$

Corollary 4.6. *Let A be a Banach algebra and let $I \subseteq A$ be a left (resp. right) ideal. Then $\overline{I}$ is a left (resp. right) ideal. In particular any maximal ideal is closed.*

Proof. This follows from the continuity of multiplication and the fact that the closure of a non-trivial ideal is non-trivial. $\square$

We can extend the notion of a maximal ideal to a non-unital Banach algebra.

Definition 4.7. Let A be a non-unital Banach algebra. A left (resp. right) ideal $I \subseteq A$ is called regular if A/I contains a unit.

Remark 4.8. *A Zorn's lemma argument can be used to show that a non-unital Banach algebra contains a maximal regular ideal.*

Remark 4.9. *Let A be a Banach algebra. The following standard facts from the theory of ideals of rings in abstract algebra carry over to the theory of ideals in a Banach algebra:*

- *An element $a \in A$ is invertible if and only if it is not contained in a left (resp. right) maximal ideal.*
- *A two-sided ideal $I \subseteq A$ is maximal if and only if A/I is a field.*

Details omitted.

5. Spectrum

In this section, we consider the notion of the spectrum of an element in a Banach algebra. The spectrum of an element in a Banach algebra generalizes the notion of eigenvalues of a matrix. For each element in a Banach algebra, we show that its spectrum is a non-empty, compact set. We then define the spectral radius of an element of a Banach algebra, and we prove the spectral radius formula.

Definition 5.1. Let A be a unital Banach algebra with unit e. An element $x \in A$ is called **invertible** if there exists $y \in A$ such that

$$xy = yx = e$$

The element y is called the inverse of x, denoted x^{-1}.

Remark 5.2. *It can be easily checked that the inverse of an element is unique. Let $\mathrm{GL}(A)$ denote the set of invertible elements of A. Then $\mathrm{GL}(A)$ forms a group, and $(xy)^{-1} = y^{-1}x^{-1}$ for $x, y \in \mathrm{GL}(A)$.*

Clearly, we need A to have a unit in order to define the notion of invertibility.

Definition 5.3. Let A be a unital Banach algebra with unit e. The **resolvent of a in A**, denoted as $\rho_A(a)$, is the set,

$$\rho_A(a) = \{z \in \mathbb{C} \mid a - ze \text{ such that } a - ze \text{ is invertible}\}$$

The **spectrum of a in A**, denoted as $\rho_A(a)$, is the set,

$$\sigma_A(a) = \{z \in \mathbb{C} \mid a - ze \text{ such that } a - ze \text{ is not invertible}\}$$

Remark 5.4. *Clearly, we have, $\sigma_A(a) = \rho_A(a)^c$ for each $a \in A$.*

When A has no unit, the resolvent and the spectrum are defined through the embedding of A in its unitization $A' = A \oplus \mathbb{C}$. Hence, we define $\sigma_A(a) = \sigma_{A'}(a)$ for each $a \in A$.

Lemma 5.5. *Let A be a non-unital Banach algebra and let A' be its unitization. Then*

$$\sigma_{A'}(a) = \sigma_A(a) \cup \{0\}$$

Proof. Let e' denote the identity in A'. Then $0 \in \sigma_A(x)$ because otherwise $e' = x^{-1}x \in A$, a contradiction. Thus, $\sigma_A(x) \cup \{0\} = \sigma_A(x) := \sigma_{A'}(x)$. □

Example 5.6. Here is a list of basic examples of the spectrum of some concrete Banach algebras.

(1) When $A = M_n(\mathbb{C})$ is the algebra of $n \times n$ matrices, the spectrum of $a \in A$ is just the set of eigenvalues of a.
(2) When $A = C(X)$ for some compact topological space,the spectrum of $f \in A$ is just the range of f. Indeed,

$$\begin{aligned}\sigma_A(f) &= \{z \in \mathbb{C} \mid f - z1 \text{ is not invertible } \}\\ &= \{z \in \mathbb{C} \mid f - z1 = 0 \text{ for some } x \in X \ \}\\ &= \{z \in \mathbb{C} \mid z = f(x) \text{ for some } x \in X \ \} = \mathrm{Range}(f)\end{aligned}$$

(3) Let X be a locally compact, non-compact topological space and let $A = C_0(X)$. We have that $\mathrm{Range}(f) \cup \{0\} \subseteq \sigma_{A'}(f)$. If $\lambda \neq 0$ such that $\lambda \in f(X)$, then consider

$$g(x) = \frac{f(x)}{\lambda(\lambda - f(x))}$$

$g(x)$ is continuous on X. Moreover, because $f \in C_(X)$ implies $g \in C_0(X)$. It is easily verified that

$$\left(\frac{1}{\lambda} + g(x)\right)(\lambda - f(x)) = 1$$

Hence,

$$\sigma_{A'}(f) = \mathrm{Range}(f) \cup \{0\}$$

This is consistent with the result we would get if were to apply the characterization of the unitization of $C_0(X)$.

We can establish numerous algebraic properties of the spectrum. Here is a sample proposition:

Proposition 5.7. *Let A, B be unital Banach algebras.*

(1) *If $a, b \in A$, then*

$$\sigma_A(ab) \cup \{0\} = \sigma_A(ba) \cup \{0\}.$$

(2) *If $a, u \in A$, and u is invertible, then*

$$\sigma_A(uau^{-1}) = \sigma_A(a).$$

(3) *If $\phi : A \to B$ is a unital morphism, then*

$$\sigma_B(\phi(a)) \subseteq \sigma_A(a)$$

Proof. The proof is given below:

(1) Indeed, let $0 \neq \lambda \in \rho_A(ab)$ and set $u := (ab - \lambda)^{-1}$. Hence $abu = uab = 1 + \lambda u$, and from this we obtain

$$\begin{aligned}(ba - \lambda)(bua - 1) &= \lambda \\ (bua - 1)(ba - \lambda) &= \lambda.\end{aligned}$$

Thus $ba - \lambda$ is invertible, and so $\lambda \in \rho_A(ba)$.

(2) Note that we have

$$uau^{-1} - \lambda e = uau^{-1} - \lambda ueu^{-1} = u(a - \lambda e)u^{-1}$$

It is then clear that $uau^{-1} - \lambda e$ is invertible if and only if $u(a - \lambda e)u^{-1}$ is invertible. The claim follows.

(3) Let $\lambda \in \sigma_B(\phi(a))$. Then

$$\phi(a) - \lambda e_B = \phi(a - \lambda e_A)$$

is not invertible. Since ϕ preserves invertibility, $a - \lambda e_A$ is not invertible. Hence, $\lambda \in \sigma_A(a)$.

This completes the proof. □

Remark 5.8. *Invertibility sometimes depends on the algebra in which one allows the inverse to exist. For instance, if $A \subseteq B$ is a unital subalgebra of a unital Banach algebra B, then by Proposition 5.7(3), $\rho_A(a) \subseteq \rho_B(a)$, but the containment may be strict. For example, consider the Banach subalgebra $\mathscr{A}(\mathbb{D}) \subseteq C(\mathbb{T})$. Here $\mathscr{A}(\mathbb{D})$ is the disk algebra. The function $f(z) = z$ in $C(\mathbb{T})$ is invertible in $C(\mathbb{T})$ but not in $A(\mathbb{D})$, since its inverse would have to be $\frac{1}{z}$, which has a singularity at the origin.*

The rest of the section is devoted to proving some crucial properties of the spectrum of an element in a Banach algebra. We first prove some important properties about invertible elements in a Banach algebra.

Lemma 5.9. *Let A be a unital Banach algebra, and let $a \in A$.*

(1) *If $\|a\| < 1$, then $a - e$ is invertible with inverse $\sum_{n=0}^{\infty} a^n$.*
(2) *If $\|a - e\| < 1$, then $a - e$ is invertible with inverse $\sum_{n=0}^{\infty} (1 - a)^n$*
(3) *For each $z \in \mathbb{C} \setminus \{0\}$, $(a - ze)^{-1}$ always exists when $|z| > \|a\|$.*
(4) *The group $\mathrm{GL}(A)$ is an open set.*
(5) *The inversion map $a \mapsto a^{-1}$ of $\mathrm{GL}(A)$ is a homeomorphism.*
(6) *If $a \in \mathrm{GL}(A)$, then*

$$\sigma_A(a^{-1}) = \{\lambda^{-1} \mid \lambda \in \sigma_A(a)\}$$

Proof. The proof is given below:

(1) We first show that the sum is a Cauchy sequence. Indeed, for $n > m$, we have

$$\left\|\sum_{k=0}^{n} a^k - \sum_{k=0}^{m} a^k\right\| = \left\|\sum_{k=m+1}^{n} a^k\right\| \leq \sum_{k=m+1}^{n} \|a^k\| \leq \sum_{k=m+1}^{n} \|a\|^k$$

The sum on the right converges to zero by the theory of the geometric series. Since A is complete, the sum $\sum_{k=0}^{\infty} a^k$ is a well-defined element of A. Now compute

$$\sum_{k=0}^{n} a^k(a-e) = \sum_{k=0}^{n}(a^k - a^{k+1}) = a - e^{n+1}.$$

Hence

$$\|e - \sum_{k=0}^{n} a^k(a-e)\| = \|a^{n+1}\| \leq \|a\|^{n+1}.$$

which goes to 0 for $n \to \infty$, as $\|a\| < 1$ by assumption. Thus

$$\lim_{n\to\infty} \sum_{k=0}^{n} a^k(a-e) = e.$$

By a similar argument,

$$\lim_{n\to\infty} (a-e) \sum_{k=0}^{n} a^k = e.$$

So that, by continuity of multiplication in a Banach algebra, one finally has

$$\lim_{n\to\infty} \sum_{k=0}^{n} a^k = (a-e)^{-1}.$$

(2) This follows from (1).
(3) Note that

$$(a - ze) = z(z^{-1}a - e)$$

Since $|z| > \|a\|$, we have that $|z^{-1}a| < 1$. The claim follows from (1).
(4) Given $a \in \mathrm{GL}(A)$, let $b \in A$ for which $\|b\| < \|a^{-1}\|^{-1}$. Observe that we have,

$$\|a^{-1}b\| \leq \|a^{-1}\|\|b\| < 1$$

Hence

$$a + b = a(e + a^{-1}b)$$

has an inverse, namely

$$(e + a^{-1}b)^{-1}a^{-1}$$

which exists by (1). It follows that

$$\{y \in A \mid \|x - y\| < \|a^{-1}\|^{-1}\| \subseteq \mathrm{GL}(A).$$

Indeed, if $\|y - x\| \leq \|a^{-1}\|^{-1}$, then if $b = y - x$, then $y = x + b$ is invertible from above. This shows that $\mathrm{GL}(A)$ is open in A.

(5) Let $a \in \mathrm{GL}(A)$. Let b such that $\|a-b\| < \frac{1}{2}\|a^{-1}\|^{-1}$. Note that,

$$\|b^{-1}\| - \|a^{-1}\| \leq \|b^{-1} - a^{-1}\| = \|b^{-1}(a-b)a^{-1}\| \leq \frac{1}{2}\|b^{-1}\|,$$

Hence, $\|b^{-1}\| \leq 2\|a^{-1}\|$. If $\varepsilon \in (0,1)$, we can now choose b such that $\|a-b\| < \frac{\varepsilon}{2}\|a^{-1}\|^{-2}$. Since the estimate above continues to hold, we have

$$\|b^{-1} - a^{-1}\| \leq \|b^{-1}(a-b)a^{-1}\| \leq 2\|a^{-1}\|^2\|b-a\| \leq \varepsilon$$

This shows that the inversion map is continuous. Since the inversion map is its own inverse, the claim follows.

(6) If $\lambda \neq 0$, then note that

$$a - \lambda e = a\lambda(e\lambda^{-1} - a^{-1}) = -a\lambda(a^{-1} - e\lambda^{-1})$$

Hence, if a is invertible and $\lambda \neq 0$, then $a - \lambda e$ is invertible if and only if $a^{-1} - \lambda^{-1}e$ is invertible. The claim follows from this observation.

This completes the proof. □

Example 5.10. Let A be a Banach algebra. Here is a list of basic examples of the topological group $\mathrm{GL}(A)$:

(1) Let $A = M_n(\mathbb{C})$. Then the unit group $\mathrm{GL}(A)$ is the group of invertible matrices. The continuity of the determinant function in this case gives another proof that $A^\times$ is open.

(2) Let $A = C(X)$ for a compact Hausdorff space X. Then the unit group $\mathrm{GL}(A)$ consists of all $f \in C(X)$ with $f(x) \neq 0$ for every $x \in X$.

Proposition 5.11. *Let A be a unital Banach algebra, and let $a \in A$. The spectrum of a is a compact set contained in the unit ball of radius $\|a\|$ in $\mathbb{C}$.*

Proof. Lemma 5.9(2) implies that

$$\sigma_A(a) \subseteq \{z \in \mathbb{C} \mid |z| \leq |a|\}$$

We show that $\sigma_A(a)$ is compact by showing that $\sigma_A(a)$ is a closed set. Given $a \in A$, we now define a function $f : \mathbb{C} \to A$ by

$$f(z) := a - ze.$$

Clearly, f is a continuous function. Because $\mathrm{GL}(A)$ is open in A by Lemma 5.9(3), it follows that $f^{-1}(\mathrm{GL}(A))$ is open in $\mathbb{C}$. But

$$f^{-1}(\mathrm{GL}(A)) = \{z \in \mathbb{C} \mid \text{such that } a - ze \text{ is invertible}\} = \rho_A(a)$$

Hence, $\sigma_A(a) = \rho_A(a)^c$ is a closed set. This shows that $\sigma_A(a)$ is a compact set. □

Is the spectrum of an element non-empty? The answer is yes, and we now prove it. Since the spectrum is a generalization of the study of eigenvalues of a complex-valued matrix, it is expected that some complex analysis will be required to prove the claim.

Definition 5.12. Let A be a Banach algebra, and let $W \subseteq \mathbb{C}$. A function $f : W \to A$ is a **Banach-algebra valued holomorphic function**

$$\frac{\partial f}{\partial z}(z_0) := \lim_{z \in W, z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists for each $z_0 \in W$.

Remark 5.13. *Major results from single-valued complex analysis continue to hold, such as Cauchy's integral formula or Liouville's theorem. However, we don't include these results here.*

Proposition 5.14. *Let A be a Banach algebra and let $a \in A$. Then $\sigma_A(a) \neq \emptyset$.*

Proof. WLOG, assume that $a \neq 0$ because if $a = 0$, then $\sigma_A(0) = \{0\}$. Consider the function,

$$f : \rho_A(a) \to A \qquad z \mapsto (a - ze)^{-1}$$

We show that f is a Banach space-valued holomorphic function. For $\lambda \neq \mu \in \rho_A(a) = \mathbb{C}$, we have

$$\begin{aligned}(\lambda a - e)^{-1} &= (\lambda a - e)^{-1}(\mu a - e)(\mu a - e)^{-1} \\ &= (\lambda a - e)^{-1}((\mu - \lambda)e + \lambda a - e)(\mu a - e)^{-1} \\ &= ((\mu - \lambda)(\lambda a - e)^{-1} + e)(\mu a - e)^{-1} = (\mu - \lambda)(\lambda a - e)^{-1}(\mu a - e)^{-1} + (\mu a - e)^{-1}\end{aligned}$$

Therefore, we have,

$$\frac{f(\lambda) - f(\mu)}{\lambda - \mu} = -(\lambda a - e)^{-1}(\mu a - e)^{-1}$$

Therefore,

$$\lim_{\mu \to \lambda} = \frac{f(\lambda) - f(\mu)}{\lambda - \mu} = -(\lambda a - e)^{-2} = -f(\lambda)^2$$

This shows that f is a Banach space-valued holomorphic function. If $|\lambda| > \|a\|$, we have,

$$\begin{aligned}f(\lambda) &= (\lambda a - e)^{-1} \\ &= \lambda^{-1}\left(e - \frac{a}{\lambda}\right)^{-1} = \frac{1}{\lambda}\sum_{n=0}^{\infty}\left(\frac{a}{\lambda}\right)^n = \frac{1}{\lambda}e + \frac{1}{\lambda^2}a + \cdots .\end{aligned}$$

For $r > \|a\|$, let Γ_r denote a contour that is a circle of radius r. Since

$$\frac{1}{2\pi i}\int_{\Gamma_r} \lambda^m d\lambda = \delta_{m,-1},$$

we have,

$$a^n = \frac{1}{2\pi i}\int_{\Gamma_r} \lambda^n f(\lambda) d\lambda$$

for each $n \geq 0$. If $\sigma_A(a) = \emptyset$, then $\rho_A(a) = \mathbb{C}$ and f is an entire function. Moreover, f is bounded. Indeed, if $|\lambda| > 2\|a\|$, then

$$\|f(\lambda)\| \leq \frac{1}{|\lambda|}\frac{1}{1 - \frac{\|a\|}{|\lambda|}} \leq \frac{1}{|\lambda| - \|a\|} \leq \frac{1}{\|a\|}$$

Hence, f is constant. But then we would have[2]

$$0 \leq \|e\| = \|a^0\| = \frac{1}{2\pi}\int_{\Gamma_r} \|f(\lambda)\| d\lambda \leq \frac{M_r}{2\pi}\int_{\Gamma_r} d\lambda = 0$$

where $M_r = \max_{\lambda \in \Gamma_r} \|f(\lambda)\|$. Hence, $e = 0$, a contradiction. Hence, $\sigma_A(a)$ is non-empty for each $a \neq 0$. ☐

[2]Here we assume that a Banach space valued integral exists and is well-defined.

Definition 5.15. Let A be a unital Banach algebra and let $a \in A$. The **spectral radius** of $a \in A$ is defined as

$$r(a) := \sup\{|z| \mid z \in \sigma_A(a)\}.$$

Example 5.16. Let X be a compact Hausdorff space and let $A = C(X)$. Then $r(f) = \|f\|_\infty$.

Remark 5.17. *Let $A = M_2(\mathbb{C})$. Consider the family of matrices $\{A_t \mid t \in \mathbb{R}^+\}$ such that*

$$A_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

Note that $\sigma_A(A_t) = \{1\}$ and $r(A_t) = \{1\}$. A is a C^*-algebra, and we can compute the norm of A_t as,

$$\|A_t\|^2 = r(A_t^T A^T)$$

Note that,

$$A_t^T A_t = \begin{pmatrix} 1 & t \\ t & 1+t^2 \end{pmatrix}$$

Note that $r(A_t^T A_t) = 1/2(2 + t^2 + t\sqrt{4+t^2})$. We see that thin this case, $r(A_t) < \|A_t\|$ for $t > 0$. Hence, we have that the spectral radius can be less than the norm of an element.

We end with a non-example of a Banach algebra.

Example 5.18. Let $A = \mathbb{C}[x]$, the algebra of complex polynomials in one variable, x. Let $w \in \mathbb{C}$ and $p \in \mathbb{C}[x]$ be a non-constant polynomial. Then $p-w$ is a non-constant polynomial, and so has a zero z_w by the fundamental theorem of algebra. This means that $p(z_w) = w$. Hence, p is surjective. That is, $\sigma_A(p) = \mathbb{C}$. Hence, $A = \mathbb{C}[x]$ is not a Banach algebra since $\sigma_A(a)$ is a non-compact fact.

Remark 5.19. *Similarly, $B = \mathbb{C}(x)$, the field field of quotients for $A = \mathbb{C}[x]$ is not a Banach algebra.*

6. Polynomial Functional Calculus

We discuss polynomial functional calculus.

Proposition 6.1. *Let A be a unital Banach algebra, and let $a \in A$.*

(1) ***(Polynomial Spectral Mapping Theorem)*** *For a polynomial $p(z)$ on $\mathbb{C}[z]$, define $p(\sigma_A(a))$ as $\{p(z) \mid z \in \sigma_A(a)\}$. Then*

$$p(\sigma_A(a)) = \sigma_A(p(a)).$$

(2) ***(Spectral Radius Formula)*** *We have,*

$$r(a) = \lim_{n\to\infty} \|a^n\|^{1/n}$$

Proof. The proof is given below:

(1) We may suppose that p is not constant. If $\mu \subseteq \mathbb{C}$, there are elements $\lambda_0, \ldots, \lambda_n \in \mathbb{C}$, where $\lambda_0 \neq 0$, such that

$$p(z) - \mu = \lambda_0(z - \lambda_1 e)\cdots(z - \lambda_n e),$$

This follows because $\mathbb{C}$ is algebraically closed. Therefore,

$$p(a) - \mu = \lambda_0(a - \lambda_1 e)\cdots(a - \lambda_n e).$$

Note that $p(a) - \mu$ is invertible if and only if all $a - \lambda_1, \dots, a - \lambda_n$ are invertible. Therefore, we have

$$\begin{aligned} p(a) - \mu \text{ is invertible} &\iff \text{at least one of } a - \lambda_i e \text{ is not invertible} \\ &\iff \lambda_i \in \sigma_A(a) \text{ for some } i \\ &\iff \mu = p(\lambda_i) \text{ for some } \lambda_i \in \sigma_A(a) \end{aligned}$$

The last statement follows since $p(\lambda) = \mu$ for each $\lambda = \lambda_1, \cdots, \lambda_n$. The claim now follows.

(2) If $\lambda \in \sigma_A(a)$ and $n \in \mathbb{N}$, (1) implies that we have $\lambda^n \in \sigma_A(a^n)$. Therefore,

$$|\lambda^n| \leq r(a^n) \leq \|a^n\|$$

Thus $|\lambda| \leq \|a^n\|^{1/n}$. Taking supremum as λ ranges over $\sigma_A(a)$ yields

$$r(a) \leq \liminf_{n\to\infty} \|a^n\|^{1/n}$$

Let M_r as in Proposition 5.11. Using the formula for a^n as in Proposition 5.11, we have $\|a^n\| \leq r^{n+1} M_r$. Thus,

$$\limsup_{n\to\infty} \|a^n\|^{\frac{1}{n}} \lim_{n\to\infty} r^{\frac{n+1}{n}} M_r^{\frac{1}{n}} = r$$

for $r > \|a\|$. Since $|\lambda| > r(a)$ implies that $\lambda \in \rho_A(a)$, it follows that if $r, r' > r(a)$, then Γ_r and $\Gamma_{r'}$ are homotopic in $\rho_A(a)$. Thus, the formula for a^n in Proposition 5.11 holds for all $r > r(a)$. It follows that

$$\limsup_{n\to\infty} \|a^n\|^{\frac{1}{n}} \leq r(a).$$

This completes the proof. $\square$

Remark 6.2. *Consider the algebra homomorphism*

$$\pi : \mathbb{C}[z] \to A$$

such that $\pi(1) = e$ *and* $\pi(z) = a$ *for some* $a \in A$. *We then have,*

$$\pi\Big(\sum_{i=0}^{n} c_i z^i\Big) = \sum_{i=0}^{n} c_i a^i$$

This homomorphism is called the polynomial functional calculus for a. *Hence, Proposition 5.11(1) is a statement about the spectrum of the polynomial functional calculus for* a.

Remark 6.3. *There is also holomorphic functional calculus that is not discussed in the notes.*

Corollary 6.4. ***(Gelfand-Mazur)*** *Let* A *be a unital Banach algebra in which every non-zero element is invertible. Then* A *is isometrically isomorphic to* $\mathbb{C}$.

Proof. Let $a \in A$. Since $\sigma_A(a) \neq \emptyset$, there is some $\lambda_a \in \mathbb{C}$ such that $a - \lambda_a e = 0$ is not invertible. By assumption, $a - \lambda_a e$. Hence, $a = \lambda_a \cdot e$. Define the map,

$$h : A \to \mathbb{C} \qquad h(a) = \lambda_a$$

It is easy to check that h is a linear map. Moreover, we have,

$$ab - \lambda_a \lambda_b \cdot e = ab - \lambda_a \cdot b + \lambda_a \cdot b - \lambda_a \lambda_b \cdot e = (a - \lambda_a \cdot e)b + \lambda_a(b - \lambda_b \cdot e) = 0$$

This shows that $h(ab) = h(a)h(b)$. Hence, h is an algebra morphism. Clearly, h is a bijective. Moreover, $a - \lambda_a \cdot e = 0$ implies that $|\lambda_a| = \|a\|$. Hence, h is an isometric algebra isomorphism. ☐

Remark 6.5. *Most of the results presented above hold for a non-unital Banach algebra by passing to its unitization. For example, if A is a non-unital algebra, A' is its unitization, $a \in A$ and $p \in \mathbb{C}[z]$, then*

$$\sigma_A(p(x)) := \sigma_{A'}(p(x)) = p(\sigma_{A'}(x)) := p(\sigma_A(x))$$

by the proof of the polynomial spectral mapping theorem.

7. Gelfand Transform

The Gelfand transform is a fundamental tool in the theory of Banach algebras, providing a powerful method for analyzing the structure of commutative algebras. It is crucial in the study of the spectrum of operators and in representation theory. The Gelfand transform also plays a central role in understanding the duality between a Banach algebra and its space of continuous functions. The Gelfand transform is also the first step in establishing that the category of C^*-algebras is equivalent to the category of locally compact Hausdorff spaces.

How should one study an arbitrary unital commutative Banach algebra? A key tenet of modern mathematics is that an abstract object should be studied by examining the algebra of continuous functions on it.

Definition 7.1. Let A be a Banach algebra. The **character space**, denoted as $\widehat{A}$, is the set of all non-zero linear maps,

$$\omega : A \to \mathbb{C},$$

that are also group homomorphisms. Each such ω is called a **character**.

Lemma 7.2. *Let A be a unital Banach algebra, and let $\widehat{A}$ be its character space. The following statements are true:*

1. *If $\omega \in \widehat{A}$, then $\omega(e) = 1$.*
2. *If $x \in \mathrm{GL}(A)$, then $\omega(x) \neq 0$.*
3. *If $\omega \in \widehat{A}$, then*
$$|\omega(a)| \leq \|a\|$$
for each $a \in A$. In particular, ω is continuous.
4. *If A is commutative, then there is a bijective correspondence between $\widehat{A}$ and the set of all two-sided maximal ideals in A (which are closed in A).*
5. *For every $\lambda \in \sigma_A(a)$, there is a character $\omega \in \widehat{A}$ such that $\omega_\lambda(a) = \lambda$.*

Proof. The proof is given below:

1. Let $a \in A$ such that $\omega(a) \neq 0$. Then,
$$\omega(a) = \omega(ae) = \omega(a)\omega(e)$$
This implies that
$$\omega(a)(1 - \omega(e)) = 0$$
Since $\omega(a) \neq 0$, we must have that $\omega(e) = 1$.

(2) We have
$$\omega(x^{-1})\omega(x) = \omega(x^{-1}x) = \omega(e) = 1$$
Hence $\omega(x) \neq 0$.

(3) For each $a \in A$, we have that $\sigma_A(a) \subseteq B(0, \|a\|)$. Therefore, if $|z| > \|a\|$, then $a - ze$ is invertible. Hence,
$$\omega(a) - z = \omega(a) - z\omega(e) = \omega(a - ze) \neq 0$$
Therefore, if $|z| > \|a\|$, then we cannot have that $\omega(a) = z$. Hence,
$$|\omega(a)| \leq \|a\|$$
In particular, this implies that ω is continuous.

(4) Let $\omega \in \widehat{A}$. Clearly, $\ker \omega$ is a closed subsapce since ω is continuous. Since ω is multiplicative, $\ker \omega$ is a two-sided ideal. Since the kernel of every linear map into $\mathbb{C}$ has codimension one, $\ker \omega$ is a maximal ideal.

Conversely, let I be a two-sided maximal ideal of A. Proposition 4.2 implies that A/I is a Banach algebra. It is a standard algebraic fact that every element in A/I is invertible. Hence, Corollary 6.4 implies that $A/I \cong \mathbb{C}$. Hence, there is a homomorphism $\psi : A/I \to \mathbb{C}$. We can define a map $\omega : A \to \mathbb{C}$ by $\omega = \psi \circ \tau$, where τ is the canonical projection map. This map is clearly linear, since τ and ψ are. Also,
$$\begin{aligned}\omega(a)\omega(b) &= \psi(\tau(a))\psi(\tau(b)) \\ &= \psi(\tau(a)\tau(b)) \\ &= \psi(\tau(ab)) \\ &= \omega(ab),\end{aligned}$$
Therefore, ω is multiplicative; it is nonzero because $\omega(b) \neq 0$ for each $b \notin I$. Hence $\omega \in \widehat{A}$. Finally, $I \subseteq \ker(\omega)$ since $I = \ker(\tau)$; but if $b \notin I$ we know that $\omega(b) \neq 0$. Hence, $I = \ker \omega$. This also shows that I is closed.

(5) Since $a - \lambda e$ is not invertible, it generates a proper ideal $I = \langle a - \lambda e \rangle$ in A. I is contained in a maximal ideal, M, which is the kernel of some character $\omega_\lambda \in \widehat{A}$. Moreover,
$$\omega_\lambda(a) = \lambda \iff \lambda - ae \in \ker \omega_\lambda \iff \lambda - ae \in M$$
The last condition is true. Hence, $\omega_\lambda(a) = \lambda$.

This completes the proof. □

Remark 7.3. *Let A be a non-unital Banach algebra. In this case, the proof of Lemma 7.2 can be modified. For instance, we can still show that for each $\omega \in \widehat{A}$, we have that $\|\omega\| \leq 1$. For every $a \in A$ and $n \in \mathbb{N}$, we have*
$$|\omega(a)| = |\omega(a^n)|^{1/n} \leq \|\omega\|^{1/n}\|a^n\|^{1/n}$$
Thus
$$|\omega(a)| \leq \limsup_{n\to\infty} \|\omega\|^{1/n}\|a^n\|^{1/n} = r(a) \leq \|a\|^{3}.$$

[3]Note that the spectral radius is defined by passing to the unitzation of A.

Therefore, $\|\omega\| \leq 1$. Recall that a left (resp. right) ideal $I \subseteq A$ is called regular if A/I contains a unit. We can show that there is bijection between $\widehat{A}$ and the set of all regular two-sided maximal ideals. The proof in Proposition 5.11(3) now goes through since we can apply the Gelfand-Mazur theorem under the assumption that I is a maximal regular ideal.

Remark 7.4. *If A is a commutative Banach algebra, Lemma 7.2 implies that $\widehat{A} \neq \emptyset$ since Zorn's lemma guarantees the existence of at least one non-zero maximal ideal if A is unital, and the existence of a non-zero maximal regular ideal if A is non-unital. In order to produce a non-trivial maximal ideal, we have to assume that A is a commutative Banach algebra Indeed, the statement above does not hold if $A = M(2, \mathbb{C})$. This follows because $M(2, \mathbb{C})$ is a simple ring.*

If A is a (unital) commutative Banach algebra, Lemma 7.2 also implies that

$$\widehat{A} \subseteq A^*,$$

where A^* is the dual space of A. Recall that A^* is usually considered endowed with the weak-$*$ topology. Hence, the topology on $\widehat{A}$ is the relative weak-$*$ topology. In fact, we can say a bit more about the topological properties of $\widehat{A}$.

Proposition 7.5. *Let A be a unital commutative Banach algebra. Then $\widehat{A}$ is a compact Hausdorff subspace of A^* in the relative weak-$*$ topology on $\widehat{A}$.*

Proof. Clearly, $\widehat{A}$ is a weak-$*$ Hausdorff space since A^* is a weak-$*$ Hausdorff space. We now show that $\widehat{A}$ is weak-$*$ closed. Let $(\omega_n)_{n\in\mathbb{N}} \subseteq \widehat{A}$ such that $\omega_n \to \omega$ in the weak-$*$ topology for some $\omega \in A$. That is, $\omega_n(a) \to \omega(a)$ for all $a \in A$. We have,

$$\begin{aligned}
|\omega(ab) - \omega(a)\omega(b)| &= |\omega(ab) - \omega_n(ab) + \omega_n(a)\omega_n(b) - \omega(a)\omega(b)| \\
&\leq |\omega(ab) - \omega_n(ab)| + |\omega_n(a)\omega_n(b) - \omega(a)\omega(b)| \\
&\leq |\omega(ab) - \omega_n(ab)| + |(\omega_n(a) - \omega(a))\omega_n(b) + \omega(a)(\omega_n(b) - \omega(b))| \\
&\leq |\omega(ab) - \omega_n(ab)| + |\omega_n(a) - \omega(a)|\|b\| + \|a\||\omega_n(b) - \omega(b)|.
\end{aligned}$$

Since $\omega_n \to \omega$ in the weak-$*$ topology, we obtain that

$$|\omega(ab) - \omega(a)\omega(b)| = 0$$

Hence, $\omega \in \widehat{A}$. Hence, $\widehat{A}$ is weak-$*$ closed. By Lemma 7.2(2), we have $\widehat{A}$ is contained in the unit ball in A^*. By the Banach-Alaoglu theorem, the unit ball in A^* is weak-$*$ compact. Since, $\widehat{A}$ is weak-$*$ closed set of a weak-$*$ compact set, $\widehat{A}$ is a weak-$*$ compact set since A^* is Hausdorff. □

The motivation behind the Gelfand transform is that a Banach algebra, A, should be studied by invoking the principle of duality: elements in a Banach algebra can be studied can by studying the collection of evaluation maps. More precisely, we consider the map

$$\widetilde{\Gamma} : A \to A^{**} \qquad \Gamma(a)(\phi) = \phi(a)$$

When $\omega \in \widehat{A}$, this defines $\widetilde{\Gamma}(a)$ as a function on $\widehat{A}$ for each $a \in A$. The definition of the weak-$*$ topology implies that $\widetilde{\Gamma}(a) \in C(\widehat{A})$.

Definition 7.6. Let A be a unital commutative Banach algebra. The map Γ defined as,

$$\Gamma : A \to C(\widehat{A}) \qquad \Gamma(a)(\omega) = \omega(a),$$

is the **Gelfand transform**.

Proposition 7.7. *Let A be a unital commutative Banach algebra, and let $\Gamma : A \to C(\widehat{A})$ denote the Gelfand transform.*

(1) *The Gelfand transform is an algebra homomorphism.*
(2) *The spectrum of $a \in A$ is:*
$$\sigma_A(a) = \{\omega(a) \mid \omega \in \widehat{A}\}$$
(3) *The Gelfand transform is a contraction, that is,*
$$\|\Gamma(a)\|_\infty \leq \|a\|.$$
for each $a \in A$.

Proof. The proof is given below:

(1) This is clear.
(2) For $a \in A$ and $z \in \mathbb{C}$, consider the element $a - ze$. If $a - ze$ is invertible, then $\omega(a - ze) \neq 0$ for each $\omega \in \widehat{A}$. If $a - ze$ is not invertible, then $a - ze$ is contained in a proper maximal ideal of A. Invoking Lemma 7.2(3), we have that there is a $\omega \in \widehat{A}$ such that $\omega(a - ze) = 0$. Hence $a - ze$ is invertible if and only if $\omega(a - ze) \neq 0$ for all $\omega \in \widehat{A}$ if and only if $\omega(a) \neq \omega(z)$ for all $\omega \in \widehat{A}$. Hence,
$$\rho_A(a) = \{z \in \mathbb{C} \mid z \neq \omega(a) \text{ for all } \omega \in \widehat{A}\}$$
Taking the complement, we have,
$$\begin{aligned}\sigma_A(a) &= \{z \in \mathbb{C} \mid z = \omega(a) \text{ for some } \omega \in \widehat{A}\} \\ &= \{\omega(a) \mid \omega \in \widehat{A}\}\end{aligned}$$
(3) We have,
$$\|\Gamma(a)\|_\infty = \sup_{\omega \in \widehat{A}} |\omega(a)| = r(a) \leq \|a\|$$

This completes the proof. □

Remark 7.8. *Note that $x \in \mathrm{GL}(A)$ if and only if $\Gamma(a)$ never vanishes. Observe that*
$$\begin{aligned}a \text{ is not invertible} &\iff I = \langle a \rangle \text{ is proper} \\ &\iff a \text{ is contained in a maximal ideal} \\ &\iff \omega(a) = 0 \text{ for some } \omega \in \widehat{A} \\ &\iff \Gamma(a) \text{ has a zero.}\end{aligned}$$

There are examples non-unital Banach algebras. It turns out that we can extend the results discussed to the case of non-unital Banach algebras.

Proposition 7.9. *Let A be a non-unital commutative Banach algebra and let A' denote its unitization. Let Γ denote the Gelfand transform as in Proposition 7.7.*

(1) *We have,*
$$\widehat{A'} = \widehat{A} \cup \{\phi_0\},$$
where we have define $\phi_0((a, \lambda)) = \lambda$
(2) *$\widehat{A}$ is a locally compact, Hausdorff space.*
(3) *The Gelfand transform is a contractive, algebra homomorphism into $C_0(\widehat{A})$.*

(4) *The spectrum of* $a \in A$ *is*

$$\sigma_A(a) = \{\omega(a) \mid \omega \in \widehat{A}\} \cup \{0\}$$

for each $a \in A$.

Proof. (Sketch) The proof is given below:

(1) It is clear that there can be no other elements of $\widehat{A'}$. If $\phi \in \widehat{A'}$, then the restriction of ϕ to A has a unique multiplicative extension to A' (see Proposition 11.6), unless it identically vanishes on A. In the latter case, ϕ_0 is clearly the only possibility.
(2) Since $\widehat{A'}$ is a compact Hausdorff space, $\widehat{A} = \widehat{A'} \setminus \{\phi_0\}$ is a locally compact, Hausdorff space by the characterization of locally compact, Hausdorff spaces.
(3) Note that $\Gamma(a)(\phi_0) = \phi_0(a) = 0$ This means that $\Gamma(A) \subseteq C_0(\hat{A})$.
(4) We have,

$$\sigma_A(a) := \sigma_{A'}(a) = \{\omega(a) \mid \omega \in \widehat{A'}\} = \{\omega(a) \mid \omega \in \widehat{A}\} \cup \{0\}$$

This completes the proof. $\square$

Remark 7.10. *If* A *is a (unital) commutative Banach algebra, we have proved that the Gelfand transform is a contractive (hence injective) algebra homomorphism. The Gelfand transform becomes an isometric* $*$*-isomorphism if* A *is a commutative* C^**-algebra.*

We end with some examples. We first compute $\widehat{C(X)}$ when X is a compact Hausdorff space.

Example 7.11. Let X be a compact Hausdorff space. We find $\widehat{C(X)}$. By Lemma 7.2(3), it suffices to compute the set of maximal ideals of $C(X)$. For each $x \in X$, define I_x by

$$I_x = \{f \in C(X) : f(x) = 0\}$$

Let I be a *proper* ideal in $C(X)$. We claim that $I \subseteq I_x$ for some $x \in X$. Assume this is not the case. Then for each $x \in X$, we can find a $f_x \in I$ such that $f_x \notin I_x$. That is, $f_x(x) \neq 0$. Since f_x is continuous there is a open neighbourhood $x \in U_x$ that $f_x|_{U_x} \neq 0$. Since X is compact, the open cover $(U_x)_{x \in X}$ admits a finite sub-cover. Hence, we can find $x_1, \ldots, x_n \in X$ such that $X = \cup_{i=1}^n U_{x_i}$. Consider the function

$$f(x) = \sum_{i=1}^n |f_{x_i}(x)|^2 = \sum_{i=1}^n \overline{f_{x_i}(x)} f_{x_i}(x)$$

Clearly, $f \in I$. But by construction $f > 0$ on X. Hence, f is invertible, so that I contains an invertible element, contradicting that I is a proper ideal. Hence, we have that for every proper ideal, I, there exists a $x_I \in I$ such that $I \subseteq I_{x_I}$. Moreover, let $x \neq y$. Since X is compact and Hausdorff, X is a normal space. By Urysohn's lemma, there exists $f, g \in C(X)$ such that $f|_{\{x\}} = 0, f|_{\{y\}} = 1$, and $g|_{\{y\}} = 1, g|_{\{y\}} = 0$. This shows that $I_x \subsetneq I_y$ and $I_y \subsetneq I_x$. Hence, we can conclude that the set of maximal ideas, $\mathscr{M}$. is of the form

$$\mathscr{M} = \{I_x \mid x \in X\}$$

In particular, we can conclude that $\widehat{C(X)} = X$ as sets. In fact the map

$$\begin{aligned} \varphi : X &\to \widehat{C(X)} \\ x &\mapsto \mathrm{Ev}_x \end{aligned}$$

is a homemorphism. It is clear that each $\mathrm{Ev}_x \in \widehat{C(X)}$, and $\mathrm{Ev}_x \neq \mathrm{Ev}_x$ for $x \neq y$ as above. If $x_\alpha \to x$, then $f(x_\alpha) \to f(x)$ for each $f \in C(X)$, which implies that $\mathrm{Ev}_{x_\alpha} \to \mathrm{Ev}_x$ in the weak-$*$topology. This shows that φ is a continuous injection. It is also surjective by the discussion above. Since these spaces are compact Hausdorff, the claim follows.

In fact, one can generalize Example 7.11.

Example 7.12. Let X be a locally compact Hausdorff space. We claim that

$$\begin{aligned} \varphi : X &\to \widehat{C_0(X)} \\ x &\mapsto \mathrm{Ev}_x \end{aligned}$$

is a homeomorphism. Continuity follows as in Example 7.11. Injectivity follows as in Example 7.11 by applying Uryhson's lemma to X_∞. We show φ is onto. Let $\omega \in \widehat{C_0(X)}$. By the Riesz representation theorem , there exists a positive Radon measure μ on X such that

$$\omega(f) = \int_X f(x)\, d\mu(x) \quad \text{for all } f \in C_0(X).$$

Thus, we have

$$0 = \omega\left(\overline{(f - \omega(f))}(f - \omega(f))\right) = \int_X |f(x) - \omega(f)|^2 d\mu(x).$$

This means that, for every $f \in C_0(X)$, f equals the constant function $\omega(f)$ μ-almost everywhere. Hence, there is a point $x_0 \in X$ such that

$$\omega(f) = f(x_0) \quad \text{for all } f \in C_0(X).$$

In other words, $\omega = \mathrm{Ev}_{x_0}$. One can φ to a continuous and bijective map from X_∞ onto $\widehat{A'}$. This is a homemorphism as in Example 7.11. Hence, so is φ.

Classical Mathematics	Quantum Mathematics
Groups	Quantum groups
Cohomology	Quantum Cohomology
Topology	C^*-algebras
Differential Geometry	Non Commutative Geometry
Probability Theory	Free Probability
Information Theory	Quantum Information Theory

Correspondence of some topics in *classical mathematics* and *quantum mathematics.*

Part 2. C^*-Algebras I

8. Why C^*-Algebras?

Linear algebra studies linear operators which are linear maps $T : \mathbb{C}^n \to \mathbb{C}^n$. If $T, S : \mathbb{C}^n \to \mathbb{C}^n$ are two such linear operators, then $T \circ S \neq S \circ T$ in general. This is perhaps the first instance where one encounters the phenomenon of 'non-commutativity' in mathematics. More generally, functional analysis studies infinite-dimensional linear spaces with additional analytic structures. A key example is that of a Hilbert spaces. If $\mathscr{H}$ is an infinite-dimensional Hilbert space, we consider the Hilbert space of bounded/continuous linear operator on $\mathscr{H}$:

$$\mathscr{B}(\mathscr{H}) := \{T : H \to H \mid T \text{ is linear and bounded}\},$$

Once again, elements of $\mathscr{B}(\mathscr{H})$ are non-commutative, in general. The phenomenon of non-commutativity is prevalent in different topics of mathematics and physics. Examples include quantum physics, linear algebra, representation theory of groups, etc. The theory of operator algebras (including C^*-algebras) attempts to capture the essence of non-commutativity. The pioneers of operator algebras, Francis Murray and John von Neumann, wrote in their very first article on operator algebras[4] in 1936 that

"various aspects of the quantum mechanical formalism suggest strongly the elucidation of this subject."

Hence, the study of operator algebras can be considered an essential part of *non-commutative mathematics*, which is also called *quantum mathematics*[5] more colloquially and popularly.

Why C^*-algebras, though? C^*-algebras can be thought of as a non-commutative or quantum version of topology. This is the content of a result of Gelfand and Naimark (to be proved later):

Theorem 8.1. *(Gelfand & Naimark) Let A be a C^*-algebra. Then A is commutative if and only if A is isometrically $*$-isomorphic to $C_0(X)$ for some locally compact Hausdorff topological space, X.*

Corollary 8.2. *Let A be a unital C^*-algebra. Then A is commutative if and only if A is isometrically $*$-isomorphic to $C(X)$ for some compact Hausdorff topological space, X.*

[4] It was an article on von Neumann algebras

[5] Thus, phrases such as *quantum groups*, *quantum cohomology*, etc., are used to describe objects studied in *quantum mathematics.*

Hence, any locally compact Hausdorff topological space gives rise to a commutative C^*-algebra. On the other hand, any commutative C^*-algebra is exactly of this form. In this sense, commutative C^*-algebras correspond to "commutative-topology," and we may view the theory of non-commutative C^*-algebras as a kind of "non-commutative topology." This duality is also the basis for other topics in non-commutative/quantum mathematics. Refer to Table 1.

9. $*$-Algebras

We first define $*$-algebras. The $*$-operation on a algebra is analogous to taking adjoints in complex matrix algebras.

Definition 9.1. Let V be an $\mathbb{C}$-algebra. Then V is a **$*$-algebra** if there is a map $* : A \to A$ is an antiautomorphism and an involution. More precisely, $*$ is required to satisfy the following properties:

(1) $(x + y)^* = x^* + y^*$,
(2) $(xy)^* = y^*x^*$,
(3) $(x^*)^* = x$,
(4) $(kx)^* = \overline{k}x^*$

for all $x, y \in V$ and $k \in \mathbb{C}$.

A Banach $*$-algebra is a Banach algebra with a $*$ map.

Definition 9.2. A **Banach $*$-algebra** is a Banach algebra that is a $*$-algebra. A morphism $\phi : A \to B$ between two Banach $*$-algebra A and B is a morphism of the underlying Banach algebras that that is $*$-preserving. That is,

$$\phi(a^*) = \phi(a)^*$$

for each $a \in A$.

We end with some properties of Banach $*$-algebra.

Proposition 9.3. *Let A be a unital Banach $*$-algebra.*

(1) $e = e^*$.
(2) *If $x \in \mathrm{GL}(A)$, then $x^* \in \mathrm{GL}(A)$, and $(x^*)^{-1} = (x^{-1})^*$.*
(3) *Then*
$$\sigma_A(a^*) = \overline{\sigma_A(a)}$$
for each $a \in A$.

Proof. The proof is given below:

(1) By definition, we have $e^*e = e^*$. Applying $*$, this implies $e^* = e^*e = e$.
(2) This is clear.
(3) By (1)
$$a^* - \lambda e = a^* - (\overline{\lambda}e)^* = (a - \overline{\lambda}e)^*$$
The claim follows by (2) now.

This completes the proof. ∎

10. C^*-Algebras

We have seen that the canonical example of a non-commutative Banach algebra if $\mathscr{B}(\mathscr{H})$, where $\mathscr{H}$ is some Hilbert space. It turns out that $\mathscr{B}(\mathscr{H})$ has even more structure. Recall that the if $T \in \mathscr{B}(\mathscr{H})$, the adjoint of T, denoted as T^*, is in $\mathscr{B}(\mathscr{H})$ defined by the property

$$\langle \Psi, T^*\Phi \rangle_{\mathscr{H}} := \langle T\Psi, \Phi \rangle_{\mathscr{H}}$$

for all $\Psi, \Phi \in \mathscr{H}$. For $T, S \in \mathscr{B}(\mathscr{H})$ and $\lambda \in \mathbb{C}$, it is well-known that the adjoint operator satisfies the following properties:

(1) $(T + S)^* = T^* + S^*$;
(2) $T^{**} = T$;
(3) $(TS)^* = S^*T^*$;
(4) $(\lambda T)^* = \overline{\lambda} T^*$.

Remark 10.1. *Since the adjoint operation, denoted as $*$, is idempotent ($T^{**} = T$), we say that $*$ is an involution.*

This makes $\mathscr{B}(\mathscr{H})$ into a Banach $*$-algebra. How does the adjoint operation interact with the norm? Pick $\Psi \in H$, and use the Cauchy-Schwarz inequality to estimate

$$\|T\Psi\|^2 = \langle T\Psi, T\Psi \rangle = \langle \Psi, T^*T\Psi \rangle \le \|\Psi\| \|T^*T\Psi\| \le \|T^*T\| \|\Psi\|^2.$$

Using the definition of the operator norm and the sub-multiplicative nature of the operator norm, we infer that

$$\|T\|^2 \le \|T^*T\| \le \|T^*\| \|T\|. \tag{1}$$

This implies to $\|T\| \le \|T^*\|$. Replacing T by T^* and the fact that $T^{**} = T$ implies $\|T^*\| \le \|T\|$. Hence,

$$\|T^*\| = \|T\|,$$

for each $T \in \mathscr{B}(\mathscr{H})$. Substituting this in Equation (1), we derive the crucial property

$$\|T^*T\| = \|T\|^2 \tag{2}$$

for each $T \in \mathscr{B}(\mathscr{H})$. This is called that C^* identity. This discussion motivates the definition of a C^*-algebra.

Definition 10.2. A **C^*-algebra**, A, is a Banach $*$-algebra that is a $*$-algebra that satisfies the C^* identity:

$$\|a^*a\| = \|a\|^2$$

for each $a \in A$. A morphism $\phi : A \to B$ between two C^* algebras A and B is a morphism of the underlying Banach $*$-algebras.

Example 10.3. The following is a list of some basics examples of C^* algebras:

(1) As discussed above, $\mathscr{B}(\mathscr{H})$ is a C^* algebra for any Hilbert space, $\mathscr{H}$.
(2) If X is a locally compact Hausdorff topological space, then $C_0(X)$ is a C^* algebra with involution given by complex conjugation. This is an example of a non-unital C^* algebra.
(3) If X is a locally compact Hausdorff topological space, $C_b(X)$ is also a C^* algebra with involution given by complex conjugation.

Remark 10.4. *More interesting examples and constructions will be discussed later on.*

Analogously with the algebraic characterization of operators in $\mathscr{B}(\mathscr{H})$, we have the following special class of elements a in a C^*-algebra, A

- a is normal if $a^*a = aa^*$,
- a is self-adjoint if $a = a^*$,
- a is a projection if $a = a^* = a^2$,
- a is a unitary if $a^*a = aa^* = 1$,
- a is an isometry if $a^*a = 1$,

Proposition 10.5. *Let A be a non-zero C^*-algebra. The following is a list of some elementary properties of a C^*-algebra.*

(1) *$\|a\| = \|a^*\|$ for each $a \in A$,*
(2) *A Banach $*$-algebra is a C^*-algebra if and only if $\|a\|^2 \leq \|a^*a\|$ for each $a \in A$*
(3) *If A is unital and e is the identity element, then $\|e\| = 1$.*
(4) *Any element a in a C^*-algebra is the sum of two self-adjoint operators.*
(5) *If A is unital and $a \in A$ is unitary, then $\sigma_A(a) \subseteq \mathbb{S}^1$.*
(6) *If A is unital, and $a \in A$ is a self-adjoint element, then $r(a) = \|a\|$.*
(7) *If A is unital, and $a \in A$ is a normal element, then $r(a) = \|a\|$.*
(8) *If A is unital, then the norm on A is unique.*
(9) *A linear map between C^*-algebras is $*$-preserving if and only if it maps self-adjoint elements to self-adjoint elements.*
(10) *Let A, B be two unital C^* algebras. A $*$-homomorphism $\pi : A \to B$ is contractive (i.e. $\|\pi(a)\| \leq \|a\|$ for each $a \in A$) and hence continuous. Moreover, a $*$-isomorphism between C^* algebras is isometric.*
(11) *If $\phi \in \widehat{A}$ and $a \in A$ is self-adjoint, then $\phi(a) \in \mathbb{R}$.*
(12) *If $\phi \in \widehat{A}$, then ϕ is $*$-preserving with $\|\phi\| = 1$.*
(13) *If $\phi \in \widehat{A}$, then $\phi(a^*a) \geq 0$.*
(14) *If A is unital, and $\phi \in \widehat{A}$, and a is a unitary, then $|\phi(a)| = 1$.*

Proof. The proof is given below:

(1) We use the C^*-identity. We have,
$$\|a\|^2 = \|aa^*\| \leq \|a\|\|a^*\|$$
Hence, we have $\|a\| \leq \|a^*\|$. Similarly, we have $\|a^*\| \leq \|a^{**}\| = \|a\|$. The result now follows.
(2) If A is a Banach $*$-algebra satisfying the given assumption, we have,
$$\|a\|^2 \leq \|a^*a\| \leq \|a^*\|\|a\| \leq \|a\|\|a\|$$
The last inequality follows since (1) holds in a Banach $*$-algebra. Hence, we have an equality above, which implies that the C^* identity holds. The converse is clear.
(3) The C^* identity implies that
$$\|e\| = \|ee\| = \|e^*e\| = \|e\|^2$$
Hence, $\|e\| = 1$[6].
(4) Simply note that, $a = \mathrm{Re}(a) + i\,\mathrm{Im}(a)$, where
$$\mathrm{Re}(a) = \frac{1}{2}(a + a^*) \qquad \mathrm{Im}(a) = \frac{1}{2i}(a - a^*)$$
It is clear that $\mathrm{Re}(a)$ and $\mathrm{Im}(a)$ are self-adjoint elements.

[6] Note that $\|e\| \neq 0$ since $e \neq 0$ since A is a non-zero C^* algebra.

(5) Recall that for any invertible element, a, in a Banach algebra,we have

$$\sigma_A(a^{-1}) = \{\lambda^{-1} \mid \lambda \in \sigma_A(a)\}$$

Since a and a^* are unitary, we have,

$$\|a\|^2 = \|a^*a\| = \|e\| = 1 = \|e\| = \|aa^*\| = \|a^*\|^2,$$

Hence, $\|a\| = \|a^*\| = 1$. Therefore, if $\lambda \in \sigma_A(a)$, we have $|\lambda| \leq 1$. Similarly, for any $\lambda \in \sigma_A(a)$, we have that $\lambda^{-1} \in \sigma_A(a^{-1}) = \sigma_A(a^*)$. Hence, $|\lambda^{-1}| \leq 1$. Hence, $|\lambda| = 1$.

(6) The C^*-identity implies that

$$\|a\|^2 = \|a^*a\| = \|a^2\|$$

Repeated use of the C^* identity implies that,

$$\|a\|^{2^n} = \|a^{2^n}\|$$

Hence,

$$r(a) = \lim_{n\to\infty} \|a^n\|^{\frac{1}{n}} = \lim_{n\to\infty} \|a^{2^n}\|^{\frac{1}{2^n}} = \lim_{n\to\infty} \|a\|^{\frac{2^n}{2^n}} = \|a\|.$$

(7) The C^* identity implies that,

$$\|a^2\|^2 = \|(a^2)(a^2)^*\| = \|(a^*a)^*(a^*a)\| = \|a^*a\|^2 = (\|a\|^2)^2$$

holds. The remaining argument is same as in (8).

(8) If $a \in A$ is any element, then,

$$\|a\|^2 = \|a^*a\| = r(a^*a)$$

Hence, we see that the spectral radius *intrinsically* determines the norm of any element in a C^* algebra. Hence, the norm is uniquely defined.

(9) A $*$-preserving map clearly sends self-adjoint elements to self-adjoint elements. Conversely, let $\phi : A \to B$ be a linear map that maps self-adjoint elements to self-adjoint elements. Using (5), we have,

$$\phi(a) = \phi(\mathrm{Re}(a)) + i\phi(\mathrm{Im}(a))$$

$$\phi(a^*) = \phi(\mathrm{Re}(a)) - i\phi(\mathrm{Im}(a)).$$

Since $\mathrm{Re}(a)$ and $\mathrm{Im}(a)$ are self-adjoint, $\phi(\mathrm{Re}(a))$ and $\phi(\mathrm{Im}(a))$ are self-adjoint by assumption. So

$$\begin{aligned}
\phi(a)^* &= \phi(\mathrm{Re}(a) + i\,\mathrm{Im}(a))^* \\
&= (\phi(\mathrm{Re}(a)) + i\phi(\mathrm{Im}(a)))^* \\
&= \phi(\mathrm{Re}(a)) - i\phi(\mathrm{Im}(a)) \\
&= \phi(a^*).
\end{aligned}$$

(10) Let $a \in A$. Then a^*a is a normal element in A, which means $\|a^*a\| = r(a^*a)$ by (9). By Proposition 5.7(3) , we have, $r(\pi(a^*a)) \leq r(a^*a)$. Hence,

$$\begin{aligned}\|a\|^2 &= \|a^*a\| \\ &= r(a^*a) \\ &\geq r(\pi(a^*a)) \\ &= r(\pi(a^*)\pi(a)) \\ &= \|\pi(a)^*\pi(a)\| = \|\pi(a)\|^2.\end{aligned}$$

If π is a $*$-isomorphism, then a symmetric argument shows the inequality above is an equality.

(11) Let $a = a^*$, and let $\phi(a) = \alpha + i\beta \in \mathbb{C}$. Note that $\phi(a) + i\lambda = \phi(a + i\lambda)$ and $|\phi(x) + i\lambda| \leq \|a + i\lambda\|$ for all $\lambda \in \mathbb{R}$. Hence:

$$\begin{aligned}\alpha^2 + (\lambda + \beta)^2 &= |\phi(a) + i\lambda|^2 \\ &\leq \|a + i\lambda\|^2 \\ &= \|(a + i\lambda)^*(a + i\lambda)\| \\ &= \|a^2 + \lambda^2\| \\ &\leq \|a\|^2 + \lambda^2\end{aligned}$$

Thus, $\alpha^2 + 2\lambda\beta + \beta^2 \leq \|a\|^2$, for all $\lambda \in \mathbb{R}$, which implies $\beta = 0$.

(12) We already know that $\|\phi\| \leq 1$. But $\phi(e) = 1$ implies that $\|\phi\| = 1$. The $*$-preserving condition follows from (10) and (12).

(13) We have,

$$\phi(a^*a) = \phi(a^*)\phi(a) = \overline{\phi(a)}\phi(a) \geq 0$$

The last equality follows by (13).

(14) We have,

$$|\phi(a)|^2 = \overline{\phi(a)}\phi(a) = \phi(a^*)\phi(a) = \phi(a^*a) = \phi(e) = 1$$

This completes the proof. □

Let's end with a non-example of a C^*-algebra.

Example 10.6. Let $\mathscr{A}(\mathbb{D})$ be the disk algebra. It is clear that $\mathscr{A}(\mathbb{D})$ is a Banach $*$-algebra with the $*$-operation given by

$$f^*(z) := \overline{f(\bar{z})}$$

We show that $\mathscr{A}(\mathbb{D})$ is not a C^*-algebra. Let $f(z) = e^{iz} \in \mathscr{A}(\mathbb{D})$. We have $f^*(z) = e^{-iz}$. For $z \in \mathbb{D}$

$$f^*f(z) := f^*(z)f(z) = e^{-iz}e^{iz} = e^{i(z-\bar{z})} = 1$$

Therefore $\|f^*f\| = 1$. On the other hand, we have

$$\begin{aligned}\|f\|^2 &= \sup\{|e^{iz}|^2 : z \in \mathbb{D}\} \\ &= \sup\{e^{-i\bar{z}+iz} : z \in \mathbb{D}\} \\ &= \sup\{e^{-2\operatorname{Im} z} : z \in \mathbb{D}\} \\ &= \sup\{e^{-2b} : b \in [-1, 1]\} \\ &= e^2.\end{aligned}$$

Hence, the C^*-identity is not satisfied.

11. Unitization

Not all C^*-algebras have units. The primary example is $C_0(\mathbb{R})$. In this case, we can consider the unitization of a non-unital C^* algebra.

Definition 11.1. Let A be a non-unital C^*-algebra. The (smallest) unital C^*-algebra containing A is called its **unitization**, A'. defined A' as follows:

$$A' := A \oplus \mathbb{C}$$

with algebraic operations given by

$$\begin{aligned}(a,\alpha)\cdot(b,\beta) &= (ab+\alpha b+\beta a, \alpha\beta)\\ (a,\alpha)^* &= (a^*,\overline{\alpha})\\ \|(a,\alpha)\| &= \sup_{b\in A, \|b\|\leq 1} \|ab+\alpha b\|\end{aligned}$$

Remark 11.2. *Consider the map,*

$$\Phi : A \to \mathscr{B}(A) \qquad a \mapsto L_a,$$

where L_a is the left-multiplication operator on A. It is easy to check that Φ is a $*$-homomorphism. We have that $\|L_a\| = \|a\|$. Indeed, if $x \in A$ such that $\|x\| \leq 1$, then

$$\|ax\| \leq \|a\|\|x\|$$

Hence,

$$\|L_a\| = \sup_{\|x\|\leq 1} \|ax\| \leq \sup_{\|x\|\leq 1} \|a\|\|x\| = \|a\|$$

If $x = a^*/\|a\|$, then $\|x\| = 1$, and

$$L_a(x) = \|aa^*/a\| = \|a\|$$

This shows that Φ is an isometric $*$-homomorophism. Hence, A can be identified with a $*$-subalgebra of $\mathscr{B}(A)$. If we identify $a \in A$ with the left multiplication operator $L_a \in \mathscr{B}(A)$, and we identify (a,α) with the operator $L_a + \alpha\,\mathrm{Id}_A$, then the norm on A' is the norm induced from $\mathscr{B}(A)$ on the $*$-subalgebra $\langle L_a, Id \mid a \in A\rangle$.

Remark 11.3. *We have already seen that the norm on a C^* algebra is unique. Thus, the norm defined above is the 'right' choice.*

Let's verify that A' is indeed a unital C^*-algebra. The unit is $(0,1)$, and clearly A' is a$*$-algebra. It is a norm by the remark made above. Moreover, note that the identification $a \mapsto L_a$ is isometric. Indeed, using the C^*-identity in A, we have for any nonzero $a \in A$,

$$\|a\| = \left\|a\Big(\frac{a^*}{\|a\|}\Big)\right\| \leq \sup_{\|b\|\leq 1} \|ab\| \leq \|a\| \sup_{\|b\|\leq 1} \|b\| = \|a\|.$$

So, $\|(a,0)\|_{A'} = \|a\|_A$, and the embedding of A into A' is isometric. Since $\mathscr{B}(A)$ is complete

$$\{L_a + \alpha\,\mathrm{id}_A : a \in A, \alpha \in \mathbb{C}\}$$

is complete since it is a closed subspace of $\mathscr{B}(A)$. Hence, A' is a Banach algebra.

It remains to show that the given norm satisfies the C^*-identity. To that end, we compute for $a \in A$ and $\alpha \in \mathbb{C}$, oh

$$\begin{aligned}\|(a,\alpha)\|^2 &= \sup_{\|b\|\le 1} \|ab+\alpha b\|^2 \\ &= \sup_{\|b\|\le 1} \|b^*(a^*a+\alpha a^*+\overline{\alpha}a+|\alpha|^2\,\mathrm{Id}_A)b\| \\ &\le \sup_{\|b\|\le 1} \|a^*a+\alpha a^*+\overline{\alpha}a+|\alpha|^2\,\mathrm{Id}_A\,\| \\ &= \|(a,\alpha)^*(a,\alpha)\| \le \|(a,\alpha)^*\|\|(a,\alpha)\|.\end{aligned}$$

So $\|(a,\alpha)\| \le \|(a,\alpha)^*\|$, and a symmetric argument yields $\|(a,\alpha)^*\| = \|(a,\alpha)\|$. Then the above inequality gives

$$\|(a,\alpha)\|^2 \le \|(a,\alpha)^*(a,\alpha)\| \le \|(a,\alpha)\|^2.$$

This proves the C^* identity in A'.

Remark 11.4. *We denote the identity element in A' as e'.*

Remark 11.5. *Properties of unital C^* algebras discussed in Proposition 10.5 can be extended to the case of non-unital C^* algebras by passing to the unitization of non-unital C^* algebras.*

One thing that makes unitizations nice to work with is that a $*$-homomorphism always has a unique and natural extension to the unitization.

Proposition 11.6. *Let A, B be C^*-algebras with B unital and A non-unital and $\pi : A \to B$ a $*$-homomorphism. Then there is a unique extension of π to a unital $*$-homomorphism $\tilde{\pi} : A' \to B$ given by*

$$\tilde{\pi}(a+\lambda e') = \pi(a) + \lambda e_B$$

Proof. We just need to check that the given formula is a $*$-homomorphism. Linearity and $*$-preserving are immediate. For $a, b \in A$ and $\lambda, \eta \in \mathbb{C}$, we compute

$$\begin{aligned}\tilde{\pi}(a+\lambda e')\tilde{\pi}(b+\eta e') &= (\pi(a)+\lambda e_B)(\pi(b)+\eta e_B) \\ &= \pi(ab)+\lambda\pi(b)+\eta\pi(a)+\lambda\eta e_B \\ &= \tilde{\pi}(ab+\lambda b+\eta a+\lambda\eta e').\end{aligned}$$

The uniqueness is forced by the fact that we require $\tilde{\pi}$ to be linear and $e' \mapsto e_B$. □

Remark 11.7. *Note that the proof of Proposition 11.6 works also when we have $\pi : A \to B$ with B non-unital. Moreover, we did not actually use the fact that π was $*$-preserving in the proof. Indeed, it suffices to assume that π is linear and multiplicative map. Moreover, essentially the same proof works if A and B are assumed to be Banach algebras.*

12. Gelfand-Naimark Theorem

We prove the Gelfand-Naimark Theorem for commutative C^*-algebras.

Theorem 12.1. *(Gelfand & Naimark) Let A be a C^*-algebra. Then A is commutative if and only if A is isometrically $*$-isomorphic to $C_0(X)$ for some locally compact Hausdorff topological space, X.*

We have already observed that if A is considered as a Banach $*$-algebra, the Gelfand transform,

$$\Gamma : A \to C_0(\widehat{A})$$

defines a contractive (and hence continuous) algebra homomorphism. The purpose of the remainder of the section is to show that Γ is $*$-preserving surjective isometry.

Proof. We first show that Γ is an isometry. Since A is commutative, every element in A is normal. Hence,

$$\|\Gamma(a)\|_\infty = \sup_{\omega \in \widehat{A}} |\omega(a)| = r(a) = \|a\|$$

The last equality follows from Proposition 10.5(8). Hence, Γ is isometric and injective. We now show that Γ is $*$-preserving. By Proposition 10.5(10), it suffices to show that Γ maps self-adjoint elements to self-adjoint elements. But if $a \in A$ is any self-adjoint element, we have that

$$\sigma_A(a) = \{\omega(a) \mid \omega \in \widehat{A}\} \subseteq \mathbb{R}$$

by Proposition 10.5(12). Hence, range$(\Gamma(a)) \subseteq \mathbb{R}$, which means $\Gamma(a) = \overline{\Gamma(a)}$ is self-adjoint. This shows that Γ is $*$-preserving. We now show that Γ is surjective. We have that $\Gamma(A)$ is a $*$-subalgebra of $C_0(\widehat{A})$. $\Gamma(A)$ separates points. Indeed, if $\omega_1 \neq \omega_2 \in \widehat{A}$, there is some $a \in A$ such that $\omega_1(a) \neq \omega_2(a)$. Hence, $\Gamma(a)(\omega_1) \neq \Gamma(a)(\omega_2)$. Moreover, $\Gamma(A)$ vanishes no-where. Indeed, if $\omega \in \widehat{A}$, since ω is non-zero, then there is a $a \in A$ such that $\Gamma(a)(\omega) \neq 0$. The Stone–Weierstrass theorem[7] now implies that $\overline{\Gamma(A)} = C_0(\widehat{A})$. But since A is a closed set of itse,f Γ is a linear isometry and $C_0(\widehat{A})$ is a normed space, general Banach space theory shows that $\Gamma(A)$ is in fact closed[8]. Hence, $\Gamma(A) = C_0(\widehat{A})$. ☐

13. Continuous Functional Calculus

Recall that Proposition 6.1 we characterized the spectrum of an element of a Banach algebra obtained by applying a polynomial function to an element of a Banach algebra. This is an example of polynomial functional calculus, which deals with the study of polynomials of functions of elements in a Banach algebra. Note that polynomial functional calculus can be extended to holomorphic functional calculus in a Banach algebra. In this section, we establish a more general functional calculus for C^*-algebras: continuous functional calculus, which is a functional calculus which allows the application of a continuous function to normal elements of a C^*-algebra.

13.1. **Motivation.** We already know how to apply polynomial functional calculus for polynomials defined on the spectrum of any element a of a Banach algebra. If we wish to extend polynomial functional calculus to continuous functions defined on spectrum, it seems obvious to approximate a continuous function by polynomials according to the Stone-Weierstrass theorem, and insert the element into these polynomials and to show that this sequence of elements converges in A. In particular, we shall approximate continuous functions on the spectrum of an element by Laurent polynomials, i.e., by polynomials of the form

$$p(z, \overline{z}) = \sum_{k,l=0}^{N} c_{k,l} z^k \overline{z}^l \qquad c_{k,l} \in \mathbb{C}$$

[7]The complex version for locally compact Hausdorff spaces.

[8]The claim is that the image of a closed set under a linear isometry from a Banach space to a normed vector space is closed.

Here, $\overline{z}$ denotes the complex conjugation, which is an involution on the complex numbers. To be able to insert a in place of z in this kind of polynomial, Banach $*$-algebras are considered, i.e., Banach algebras that also have an involution $*$, and a^* is inserted in place of $\overline{z}$. In order to obtain a homomorphism

$$\mathbb{C}[z, \overline{z}] \to A,$$

a restriction to normal elements, i.e., elements with $a^*a = aa^*$, is necessary, as the polynomial ring $\mathbb{C}[z, \overline{z}]$ is commutative. If

$$(p_n(z, \overline{z}))_n$$

is a sequence of polynomials that converges uniformly on the spectrum of a to a continuous function f, the convergence of the sequence

$$(p_n(a, a^*))_{n\in\mathbb{N}}$$

in A to an element $f(a)$ must be ensured. A detailed analysis of this convergence problem shows that it is necessary to resort to C^*-algebras. These considerations lead to the so-called continuous functional calculus.

13.2. **Construction.** Let A be a C^* algebra. For any $a \in A$ that is a normal element, we write $C^*(a)$ for the C^*-algebra generated by a. This can be identified as the norm closure of the set of all polynomials in a, a^* with zero constant term, i.e.,

$$C^*(a) = \overline{\{p(a, a^*) \mid p \in \mathbb{C}[z_1, z_2], p(0, 0) = 0\}}.$$

When A is unital with unit e, $C^*(a, e)$ can be identified with the closure of the set of all polynomials on a, a^*

$$C^*(a, e) = \overline{\{p(a, a^*) \mid p \in \mathbb{C}[z_1, z_2]\}}.$$

Moreover, we denote as $C(\sigma_A(a))$ the C^*-algebra of continuous functions on $\sigma_A(a)$, the spectrum of a. Note that all of these C^* algebras are commutative.

Proposition 13.1. *Let A be a unital C^* algebra and let a be a normal element of A. Then there exists a unique $*$-isometric algebra isomomorphism*

$$\Phi_a \colon C(\sigma_A(a)) \to C^*(e, a)$$

with $\Phi_a(1_{\sigma_A(a)}) = e$ for $1_{\sigma_A(a)}(z) = 1$ and $\Phi_a(\mathrm{Id}_{\sigma_A(a)}) = a$ for the identity. The mapping Φ_a is called the continuous functional calculus of the normal element a. Usually it is suggestively set $f(a) := \Phi_a(f)$.

We first need to prove the following lemma.

Lemma 13.2. *Let A be a unital C^*-algebra, and let B be a C^*-subalgebra containing the identity of A. Then for all $b \in B$, we have*

$$\sigma_B(b) = \sigma_A(b).$$

Proof. Clearly, we have that have

$$\sigma_A(b) \subseteq \sigma_B(b),$$

and the reverse inclusion will follow if we can show that b is invertible in A, implies that it is already invertible in B. First assume that $b = b^*$. Let,

$$E = C^*(b, b^{-1}) \subseteq A,$$
$$D = C^*(e, b) \subseteq E \cap B.$$

We show that $E = D$. This readily implies that $b^{-1} \in B$. Since $(b^{-1})^* = b^{-1}$, E is the closure of the algebra generated by $\{b, b^{-1}\}$. In particular, E is a unital commutative C^*-algebra. Hence,

$$E \cong C(\widehat{E})$$

Let $D' \subseteq C(\widehat{E})$ denote the image of D under the Gelfand transform. D' is a closed[9] $*$-subalgebra of $C(\widehat{E})$. Furthermore, if $\phi, \psi \in \widehat{E}$ are such that $\phi \neq \psi$, we must have $\phi(a) \neq \psi(a)$ since otherwise ϕ, ψ agree on a, a^{-1} and hence everywhere. Since $\mathrm{Ev}_a \in D'$, D' separates the points of $\widehat{E}$. By Stone-Weierstrass, $\overline{D'} = C(\widehat{E})$. Since D' is also closed, it follows that $D' = C(\widehat{E})$. Hence,

$$C(\widehat{D}) \cong D' = C(\widehat{E})$$

Hence, $\widehat{D} = \widehat{E}$, which in turn implies that $D = E$. If b is not necessarily self-adjoint, note that b^*b is invertible in A since b is invertible in A. Then

$$(b^*b)^{-1}b^*b = e,$$

and

$$b^{-1} = (b^*b)^{-1}b^*.$$

By the argument above, $(b^*b)^{-1} \in B$. Hence, $b^{-1} \in B$. □

Proof. (Proposition 13.1) (Existence) Let $B = C^*(a, e)$. Since B is a unital commutative C^*-algebra, the Gelfand transform gives us an isometric $*$-isomorphism

$$\Psi : B \to C(\widehat{B}).$$

The key argument is to explicitly identity $\widehat{B}$ for $B = C^*(a, e)$. Note that any non-zero, linear and multiplicative map on B is uniquely defined by its action on a. Hence, the map

$$\tau : \widehat{B} \to \sigma_A(a) \qquad \tau(\omega) := \omega(a)$$

is a continuous bijection. Since $\widehat{B}$ and $\sigma_A(a)$ are compact and Hausdorff, τ is a homeomorphism. Then we get an isometric *-isomorphism

$$\Theta : C(\sigma_A(a)) \to C(\widehat{B})$$

by

$$\Theta(f)(\omega) = f(\tau(\omega)) = f(\omega(a)) \qquad f \in C(\sigma_A(a)),\ \omega \in \widehat{B}$$

The desired conclusion follows by letting $\Phi = \Psi^{-1} \circ \Theta$. Indeed, note that $\Theta(\mathrm{Id}_{\sigma_A(a)})(\omega) = \tau(\omega) = \omega(a)$. Therefore, $\Theta(\mathrm{Id}_{\sigma_A(a)}) = \mathrm{Ev}_a$. Hence, $\Psi^{-1} \circ \Theta(\mathrm{Id}_{\sigma_A(a)}) = a$ as required.

(Uniqueness) Since $\Phi_a(1_{\sigma_A(a)})$ and $\Phi_a(\mathrm{Id}_{\sigma_A(a)})$ are fixed, Φ_a is already uniquely defined for all Laurent polynomials since Φ_a is a *-homomorphism. These polynomials form a dense subalgebra of $C(\sigma_A(a))$ by the Stone-Weierstrass theorem. Thus Φ_a is unique. □

Proposition 13.1 shows that continuous functional calculus can be used to reduce some abstract problems involving normal elements of a (not necessarily commutative!) C^*-algebra to problems about function algebras.

Remark 13.3. *In what follows, we shall write $f(a)$ for $\Psi_a(f)$ from time to time.*

Corollary 13.4. *Let A be a unital C^*-algebra A, and let a be a normal element of A. Let $f \in C(\sigma_A(a))$. We have the following:*

[9][Junaid:Why is it closed? Is D a $*$-closed subalgebra of E?]

(1) *$f(a) \in A$ is normal*
(2) *(**Spectral Mapping Theorem**) $f(\sigma_A(a)) = \sigma_A(f(a))$*

Proof. The proof is given below:

(1) We have,
$$f(a)^* f(a) = \Phi_a(f)^* \Phi_a(f) = \Phi_a(\overline{f} f) = \Phi_a(f\overline{f}) = \Phi_a(f)\Phi_a(f)^* = f(a)f(a)^*$$
(2) Since $f(a) \in C^*(a,e)$ and Ψ_a is a $*$-isometric isomorphism, we have
$$\sigma_A(f(a)) := \sigma_A(\Phi_a(f)) = \sigma_A(f) = f(\sigma_A(a)).$$
The last equality follow since the spectrum of f is simply its range.

This completes the proof. □

We now present some applications of Proposition 13.1 and Corollary 13.4. We first present a complete characterization of self-adjoint, unitary and projection elements in a unital C^*-algebra.

Corollary 13.5. *Let A be a unital C^*-algebra, and let $a \in A$ be a normal element. Then:*

(1) *a is self-adjoint if and only if $\sigma_A(a) \subseteq \mathbb{R}$.*
(2) *a is unitary if and only if $\sigma_A(a) \subseteq \mathbb{S}^1$.*
(3) *a is a projection if and only if $\sigma_A(a) \subseteq \{0,1\}$.*

Proof. The proof is given below:

(1) The forward direction was proved in **??**. Conversely, assume that $\sigma_A(a) \in \mathbb{R}$. Then,
$$a^* = \Phi_a(\mathrm{Id}_{\sigma_A(a)})^* = \Phi_a(\overline{\mathrm{Id}_{\sigma_A(a)}}) = \Phi_a(\mathrm{Id}_{\sigma_A(a)}) = a.$$
Hence, a is self-adjoint.
(2) The forward direction was proved in Proposition 10.5. Conversely, assume that $\sigma_A(a) \subseteq \mathbb{S}^1$. Then,
$$a^*a = \Phi_a(\mathrm{Id}_{\sigma_A(a)})^* \Phi_a(\mathrm{Id}_{\sigma_A(a)}) = \Phi_a(\overline{\mathrm{Id}_{\sigma_A(a)}}\, \mathrm{Id}_{\sigma_A(a)}) = \Phi_a(1_{\sigma_A(a)}) = e$$
Similarly, $aa^* = e$. Hence, a is a unitary.
(3) If a is a projection, then $a^2 = a = a^*$. Hence, $\sigma_A(a) \subseteq \mathbb{R}$. Since $a^2 = a$, we have that,
$$\Phi_a(\mathrm{Id}_{\sigma_A(a)}) = a = a^2 = \Phi_a(\mathrm{Id}_{\sigma_A(a)})^2 = \Phi_a(\mathrm{Id}^2_{\sigma_A(a)})$$
Hence, $\mathrm{Id}^2_{\sigma_A(a)} = \mathrm{Id}_{\sigma_A(a)}$, implying that $\sigma_A \subseteq \{0,1\}$. The converse follows via a similar argument.

This completes the proof. □

Here is another application:

Corollary 13.6. *Let A be a unital C^*-algebra. If $a \in A$ is unitary such that $\sigma_A(a) \subsetneq \mathbb{S}^1$, then there exists a self-adjoint $b \in A$ such that $a = e^{ib}$.*

Proof. WLOG, we can assume that $-1 \notin \sigma_A(a)$. Let $\ln : \mathbb{C} \setminus (-\infty, 0] \to \mathbb{C}$ denote the principal branch of the logarithm function. Note that we have $e^{\ln(\cdot)} = \mathrm{Id}_{\sigma_A(a)}(\cdot)$. Since $|\mathrm{Id}_{\sigma_A(a)}| = 1$, the real part of ln restricted to $\sigma_A(a)$ vanishes. Hence, $\ln|_{\sigma_A(a)} = ih$ for some real-valued function $h \in C(\sigma_A(a))$. Let $b = \Phi_a(h)$. Since h is real-valued, b is self-adjoint. Moreover, we have
$$a = \Phi_a(\mathrm{Id}_{\sigma_A(a)}) = \Phi_a(e^{ih}) = e^{i\Phi_a(h)} = e^{ib}.$$
□

Here is another application:

Corollary 13.7. *Let A, B be unital C^*-algebras and let $\varphi : A \to B$ be a unital $*$-homomorphism. If $a \in A$ is normal element, then for every $f \in C(\sigma_A(a))$, we have $\varphi(f(a)) = f(\varphi(a))$.*

Proof. Note that $\varphi(a)$ is normal and $\sigma_B(\varphi(a)) \subseteq \sigma_A(a)$, and so the restriction of f to $\sigma_B(a)$ is continuous. Define two unital $*$-homomorphisms $\Phi_1, \Phi_2 : C(\sigma_A(a)) \to B$ by

$$\Phi_1(f) := \varphi(\Phi_a(f)) = \varphi(f(a)),$$

and

$$\Phi_2(f) := \Phi_{\varphi(a)}(f|_{\sigma_B(\varphi(a))}) = f|_{\sigma_B(\varphi(a))}(\varphi(a)).$$

It is easy to check that they both map $1_{\sigma_A(a)}$ and $\mathrm{Id}_{\sigma_A(a)}$ to 1_B and $\varphi(a)$, respectively. Thus they agree on all polynomials of two variables z and $\overline{z}$ over $\sigma_A(a)$ and since they are continuous, they agree on all of $C(\sigma_A(a))$. $\square$

14. Positive Elements

Continuous Functional Calculus (CFC) is a powerful tool for manipulating normal elements of a C^*-algebra. Granted, every element of a non-commutative C^*-algebra is not normal. Nonetheless, we can associate a self-adjoint element for every element in a non-commutative C^*-algebra, allowing us to spread the influence of the functional calculus to an entire non-commutative C^*-algebra. In this section, we discuss how positive elements can be defined via CFC.

Definition 14.1. Let A be a C^*-algebra. A self-adjoint element $a \in A$ is **positive** if $\sigma_A(a) \subseteq \mathbb{R}^+$.

Remark 14.2. *If A is a C^*-algebra, the subset of positive elements is denoted by A_+, and if $a \in A_+$ we write $a \geq 0$.*

Example 14.3. Let $A = C_0(X)$ for some locally compact topological space. Positive elements in are non-negative real-valued functions.

We now discuss applications of CFC to produce new elements in a C^*-algebra.

Proposition 14.4. *Let A be a unital C^*-algebra.*

(1) *If $a \in A$ is a self-adjoint element, then a can be written uniquely as*

$$a = a_+ - a_- \qquad a_\mp a_\pm = 0$$

for $a_\pm \in A_+$. The elements $a_\pm$ are called the positive/negative parts of a.

(2) *Every $a \in A_+$ and $n \geq 1$, there exists a unique element $b \in A_+$ such that $a = b^n$. The element b is called the n-th root of a.*

Proof. The proof is given below:

(1) Consider the functions

$$f^+(t) = \max(t, 0), \qquad f^-(t) = -\min(t, 0)$$

We have that $f^\pm \in C(\sigma_A(a))$ and $f^+(t) - f^-(t) = t$[10]. By CFC, we have that,

$$a = a_+ - a_-$$

[10]Since a is self-adjoint, we can assume that $t \in \mathbb{R}$.

such that $a_\pm = \Psi_a(f^\pm)$. Since $f^\pm$ are non-negative real-valued functions, we have that $\sigma_A(a_\pm) \subseteq \mathbb{R}^+$ by the spectral mapping theorem, Hence, $a_\pm$ are positive elements. Moreover, note that $f^\mp \cdot f^\pm = 0$. Hence, $a^\mp a^\pm = 0$ by CFC.

(2) For each $n \in \mathbb{N}$, consider the function

$$f_n : \mathbb{R}^+ \to \mathbb{R}^+, x \mapsto \sqrt[n]{x},$$

which is a continuous function on $\sigma_A(a) \subseteq \mathbb{R}^+$. Define, $b = \Phi_a(f_n)$. Then

$$b^n = \Phi_a(f_n)^n = \Phi_a(f_n^n) = \Phi_a(\mathrm{Id}_{\sigma_A(a)}) = a$$

by CFC. By the spectral mapping theorem. we have

$$\sigma_A(b) = \sigma_A(f_n(a)) = f_n(\sigma_A(a)) \subseteq \mathbb{R}^+$$

i.e., b is positive. If $c \in A_+$ is another positive element such that $c^n = a = b^n$, then

$$c = \Phi_a(f_n) = \Phi_a(f_n) = b$$

by uniqueness of the inverse of the square root function defined on $\sigma_A(a)$. This proves uniqueness.

This completes the proof. □

If X is compact and $f \in C(X)_+$, then notice that $|f(x) - t| \leq t$ for every real number $t \geq \|f\|$. Conversely, if $|f(x)-t| \leq t$ for some $t \geq \|f\|$, then $f(x) \geq 0$ for all x and so $f \geq 0$. These observations are behind some of the statements in the next result.

Lemma 14.5. *Let A be a unital C^*-algebra, and let a be a self-adjoint element. Then the following are equivalent.*

(1) $a \geq 0$
(2) $\|\alpha e - a\| \leq \alpha$ *for all* $\alpha \geq \|a\|$
(3) $\|\alpha e - a\| \leq \alpha$ *for some* $\alpha \geq \|a\|$.

Proof. We first prove (1) implies (2). Since $a \geq 0$, $a = a^*$. Hence, $C^*(a)$ is abelian. Recall that $C(\sigma_A(a)) \cong C^*(a)$, such that the identity function in $C(\sigma_A(a))$ corresponds to a. Since $a \geq 0$, we have that the identity function is in $C(\sigma_A(a))_+$. Since $\sigma_A(a)$ is compact, the discussion preceding the statement of the proposition then implies that (2) is true if we take f to be the identity function. Clearly, (2) implies (3). (3) implies (1) follows from an argument similar to that that implies (1) implies (2). □

Lemma 14.5 gives us a nice characterization of positive elements. We have the following corollaries.

Corollary 14.6. *Let A be a unital C^*-algebra.*

(1) *Then A_+ is closed.*
(2) *If $a, b \in A_+$, then $a + b \in A_+$.*
(3) *If $a, b \in A_+$ and a and b commute, then $ab \in A_+$.*
(4) *a is positive if and only if $a = b^*b$ for some $b \in A$.*

Proof. The proof is given below:

(1) Suppose $(a_n) \in A_+$ converges to $a \in A$. Then

$$\|a_n^* - a^*\| = \|a_n - a\| \to 0,$$

and so $(a_n^*) = (a_n) \in A_+$ converges to a^*. Hence $a^* = a$. Moreover, we have that $(\|a_n\|)$ converges to $\|a\|$. By Lemma 14.5

$$\|\|a_n\|e - a_n\| \leq \|a_n\|$$

for each $n \in \mathbb{N}$. Hence,

$$\|\|a\|e - a\| \leq \|a\|$$

for each $n \in \mathbb{N}$. By Lemma 14.5, we have that $a \geq 0$.

(2) Clearly, $a + b$ is self-adjoint. It suffices to assume that $\|a\|, \|b\| \leq 1$[11]. But

$$\|1 - \frac{1}{2}(a+b)\| = \frac{1}{2}\|(1-a) + (1-b)\| \leq 1$$

by Lemma 14.5. Hence, $\frac{1}{2}(a+b) \geq 0$ by Lemma 14.5, which implies that $a + b \geq 0$.

(3) Note that,

$$(a+b)^2 = a^2 + 2ab + b^2$$

By the spectral mapping theorem and (2) above, $a^2, b^2, (a+b)^2 \in A_+$. Hence, $2ab \in A_+$ which in turn implies that $ab \in A_+$.

(4) The forward implication follows from Proposition 14.4(2) by simply taking the square root of a. Conversely, clearly a is self-adjoint. We show that $a = b^*b$ implies that $\sigma_A(a) \subseteq \mathbb{R}^+$. Note that $a = b^*b$. Hence, we shall apply CFC to $a = b^*b$. Define

$$f(t) = \begin{cases} \sqrt{t} & \text{if } t \geq 0, \\ 0 & \text{otherwise} \end{cases} \qquad g(t) = \begin{cases} 0 & \text{if } t \geq 0, \\ \sqrt{-t} & \text{otherwise.} \end{cases}$$

Then for all $t \in \mathbb{R}$, we have $f(t)g(t) = 0$ and $f(t)^2 - g(t)^2 = t$. Since f and g both vanish at 0, we get self-adjoint elements of $u = \Phi_{b^*b}(f)$ and $v = \Phi_{b^*b}(g)$ of A such that

$$\Phi_{b^*b}(g)\Phi_{b^*b}(f) = \Phi_{b^*b}(g)\Phi_{b^*b}(f) = 0, \quad \Phi_{b^*b}(f)^2 - \Phi_{b^*b}(g)^2 = b^*b$$

But then

$$\Phi_{b^*b}(g)(\Phi_{b^*b}(f)^2 - \Phi_{b^*b}(g)^2)\Phi_{b^*b}(g) = -\Phi_{b^*b}(g)^4.$$

Thus[12]

$$\sigma_A((bv)^*bv) = \sigma_A(-\Phi_{b^*b}(g)^4) \subseteq (-\infty, 0].$$

Thus $-\Phi_{b^*b}(g)^4 = 0$. Since v is self-adjoint, this means $\Phi_{b^*b}(g) = 0$. But then $b^*b = \Phi^2_{b^*b}(f)$, and $\Phi^2_{b^*b}(f)$ is positive by the spectral mapping theorem for normal elements.

This completes the proof. □

Remark 14.7. *Note that Corollary 14.6(3) is not true in general. Indeed, let*

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$A, B \in M_2(\mathbb{C})$ *are positive. However, AB is not positive since*

$$AB = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$$

---

[11]This because is a positive-scalar of a positive element is a positive element

[12]Here we use the result that if $a \in A$ and $\sigma_A(a^*a) \subseteq (-\infty, 0]$, then $a = 0$.

is not even self-adjoint.

Remark 14.8. *All claims about positive elements are true in a non-unital C^*-algebra. These facts can be proved by passing to the unitization of a non-unital C^*-algebra.*

Part 3. Spectral Theory in Hilbert Spaces

The spectral theory of C^*-algebras provides a powerful framework for understanding the structure of operators in functional analysis and quantum mechanics. By analyzing the spectra of elements in a C^*-algebra, one gains insight into the algebra's representations, its ideal structure, and its connection to non-commutative topology. In particular, the spectral theorem for C^*-algebras includes as special cases the classical results for normal operators on Hilbert spaces, offering a deeper perspective on the interplay between algebraic and topological properties. We explore spectral theory within the context of C^*-algebras.

15. Motivation

Spectral theory is considered with normal operators on a Hilbert space. We first collect some facts about a normal operators on a Hilbert space.

Proposition 15.1. *Let $\mathscr{H}$ be a Hilbert space and let $T \in \mathscr{B}(\mathscr{H})$ be a normal operator.*

(1) *T is normal if and only if*
$$\|Tx\| = \|T^*x\|$$
for all $x \in \mathscr{H}$.

(2) *λ is an eigenvalue of T if and only if $\overline{\lambda}$ is an eigenvalue of T^*. for each $\lambda \in \mathbb{C}$.*

(3) *If λ, μ are distinct eigenvalues of T, then $E_\lambda \perp E_\mu$ where E_λ, E_μ are eigenspaces corresponding to μ, λ respectively.*

Proof. The proof is given below:

(1) We have
$$\|T^*x\|^2 = \langle T^*x, T^*x\rangle = \langle TT^*x, x\rangle = \langle T^*Tx, x\rangle = \langle Tx, Tx\rangle = \|Tx\|^2$$
The claim follows.

(2) Since $T - \lambda I$ is a normal operator for each $\lambda \in \mathbb{C}$, the claim follows by (1).

(3) Let $x \in E_\lambda, y \in E_\mu$. Note that
$$\overline{\lambda}\langle x, y\rangle = \langle \overline{\lambda}x, y\rangle = \langle T^*x, y\rangle = \langle x, Ty\rangle = \langle x, \mu y\rangle = \overline{\mu}\langle x, y\rangle.$$
So we find that
$$\langle x, y\rangle(\overline{\mu} - \overline{\lambda}) = 0,$$
which implies that $\langle x, y\rangle = 0$ since $\overline{\mu} - \overline{\lambda} \neq 0$.

This completes the proof. □

To set the stage for what follows, we first review the spectral theorem in the finite-dimensional setting. The following statements are well-known in the finite-dimensional setting.

Proposition 15.2. *Let $\mathscr{H}$ be a finite-dimensional Hilbert space. Let $T \in \mathscr{B}(\mathscr{H})$ be a normal operator. Then H has a basis of eigenvectors for T.*

Proof. We know that $\sigma(T) \neq \emptyset$[13]. Let $\lambda \in \sigma(T)$, and $v \in E_\lambda$. By (1), v is also an eigenvector of T^*. Let $\mathscr{M} = \{v\}^\perp$. Then W is invariant under T^* and T. It follows that $T|_W = S$ is a normal operator on $\mathscr{B}(\mathscr{M})$. The result follows by induction. □

[13]This also follows from matrix-algebraic considerations

Let $\sigma_{\mathscr{B}(\mathscr{H})}(T) = \{\lambda_1, \ldots, \lambda_n\}$. Note that $\sigma_{\mathscr{B}(\mathscr{H})}(T)$ is equipped here with the subspace topology from $\mathbb{C}$, which is just the discrete topology. Thus, every function $\sigma_{\mathscr{B}(\mathscr{H})}(T) \to \mathbb{C}$ is automatically continuous. Consider the continuous functional calculus of T:

$$\Phi_T : C(\sigma_{\mathscr{B}(\mathscr{H})}(T)) \to C^*(T, I)$$

For $\lambda_i \in \sigma_{\mathscr{B}(\mathscr{H})}(T)$, let $\chi_{\lambda_i} \in C(\sigma(N))$ denote the indicator function of the singleton $\{\lambda_i\}$:

$$\chi_{\lambda_i}(x) = \begin{cases} 1 & \text{if } x = \lambda_i, \\ 0 & \text{if otherwise} \end{cases}$$

Let $P_{\lambda_i} = \Phi_T(\chi_{\lambda_i})$. Continuous functional calculus implies the following properties:

(1) Each P_{λ_i} is an orthogonal projection, since we have
$$\begin{aligned} P_{\lambda_i}^2 &= \Phi_T(\chi_\lambda)^2 = \Phi_T(\chi_\lambda^2) = \Phi_T(\chi_\lambda) = P_{\lambda_i}, \\ P_{\lambda_i}^* &= \Phi_T(\chi_\lambda)^* = \Phi_T(\chi_\lambda^*) = \Phi_T(\chi_\lambda) = P_{\lambda_i}. \end{aligned}$$

(2) For $\lambda_i, \lambda_j \in \sigma_{\mathscr{B}(T)}(R)$ with $i \neq j$, we have
$$P_{\lambda_i} P_{\lambda_j} = \Phi_T(\chi_{\lambda_i})\Phi_T(\chi_{\lambda_j}) = \Phi_T(\chi_{\lambda_i} \cdot \chi_{\lambda_j}) = \Phi_T(0) = 0.$$
It follows from this that $\mathrm{Im}(P_{\lambda_i}) \perp \mathrm{Im}(P_{\lambda_j})$.

(3) We have $1_{\sigma_{\mathscr{B}(\mathscr{H})(T)}} = \chi_{\lambda_1} + \cdots + \chi_{\lambda_n}$, where $1_{\sigma_{\mathscr{B}(\mathscr{H})(T)}}$ denotes the constant one function $x \mapsto 1$. Thus, we have
$$I = \Phi_T(1_{\sigma_{\mathscr{B}(\mathscr{H})(T)}}) = \Phi_T(\chi_{\lambda_1} + \cdots + \chi_{\lambda_m}) = P_{\lambda_1} + \cdots + P_{\lambda_n}.$$

(4) We have $\mathrm{Id}_{\sigma_{\mathscr{B}(\mathscr{H})(T)}} = \lambda_1 \chi_{\lambda_1} + \cdots + \lambda_n \chi_{\lambda_n}$. Thus, we have
$$\begin{aligned} T &= \Phi_T(\mathrm{Id}_{\sigma_{\mathscr{B}(\mathscr{H})(T)}}) \\ &= \Phi_T(\lambda_1 \chi_{\lambda_1} + \cdots + \lambda_n \chi_{\lambda_n}) \\ &= \lambda_1 P_{\lambda_1} + \cdots + \lambda_n P_{\lambda_n}. \end{aligned}$$

(5) Any $f \in C_{\mathscr{B}(\mathscr{H})}(T)$ can be written as $f_{\sigma_{\mathscr{B}(\mathscr{H})(T)}} = f(\lambda_1)\chi_{\lambda_1} + \cdots + f(\lambda_n)\chi_{\lambda_n}$ Thus, we have
$$\begin{aligned} f(T) &:= \Phi_T(a) = \Phi_T(f) \\ &= \Phi_T(f(\lambda_1)\chi_{\lambda_1} + \cdots + f(\lambda_n)\chi_{\lambda_n}) \\ &= f(\lambda_1)P_{\lambda_1} + \cdots + f(\lambda_n)P_{\lambda_n}. \end{aligned}$$

The content of the statement
$$f(T) = f(\lambda_1)P_{\lambda_1} + \cdots + f(\lambda_n)P_{\lambda_n} \tag{3}$$
for $f \in C(\sigma_{\mathscr{B}(\mathscr{H})})$ is the spectral theorem for normal operators in finite-dimensional Hilbert spaces. We introduce a different notation to represent this result. Assume that $\sigma_{\mathscr{B}(\mathscr{H})}(T)$ is endowed with an operator-valued function $P_{\mathscr{B}(\mathscr{H})}(T)$ on $\mathscr{P}(\sigma_{\mathscr{B}(\mathscr{H})}(T))$ such that

$$\begin{aligned} P_{\sigma_{\mathscr{B}(\mathscr{H})(T)}} : \mathscr{P}(\sigma_{\mathscr{B}(\mathscr{H})}(T)) &\to \mathscr{B}(\mathscr{H}) \\ A &\mapsto \sum_{\lambda_i \in A} P_i. \end{aligned}$$

for each $A \subseteq \sigma_{\mathscr{B}(\mathscr{H})}(T)$. It can be checked that $P_{\sigma_{\mathscr{B}(\mathscr{H})}(T)}(T)$ satisfies properties analogous to that of an ordinary measure. The map $P_{\sigma_{\mathscr{B}(\mathscr{H})}}$ is a special case of what will later be called a $\mathscr{H}$-projection valued measure on $\sigma_{\mathscr{B}(\mathscr{H})}(T)$. One can then write Equation (3) as

$$f(T) = \int_{\sigma(T)} f(z)\, dP_{\sigma_{\mathscr{B}(\mathscr{H})}(T)}(z) \tag{4}$$

Here the integral is of $f \in C(\sigma_{\mathscr{B}(\mathscr{H})}(T))$ with respect to the operator-valued measure $P_{\sigma_{\mathscr{B}(\mathscr{H})}(T)}$. If $\mathscr{H}$ is a general infinite-dimensional Hilbert space, the spectral theorem for normal operators asserts that each normal operator $T \in \mathscr{B}(\mathscr{H})$ is associated with a (unique) $\mathscr{H}$-projection valued measure $P_{\sigma_{\mathscr{B}(\mathscr{H})}(T)}(z)$ such that Equation (4) holds.

16. Point, Residual & Continuous Spectrum

Remark 16.1. *From now on we write $\sigma_{\mathscr{B}(\mathscr{H})}(\cdot)$ as simply $\sigma(\cdot)$.*

For $T \in \mathscr{B}(\mathscr{H})$, recall that

$$\sigma(T) = \{\lambda \in \mathbb{C} \mid T - \lambda I \text{ is not invertible}\}$$

In infinite dimensions, invertibility may fail due to a lack of surjectivity without injectivity failing. Consider the shift operator:

$$R : \ell^2(\mathbb{N};\mathbb{C}) \to \ell^2(\mathbb{N};\mathbb{C})$$
$$(a_1, a_2, a_3, \cdots) \mapsto (0, a_1, a_2, \cdots)$$

Note that R is injective, but not surjective. General Banach space theory informs us that a bounded operator T on a Banach space is invertible if and only if T is bounded below and has dense range. Hence, there are three basic ways in which $T - \lambda I$ fails to be a bijection:

(1) $T - \lambda I$ is not injective. Equivalently, $Tx = \lambda x$ for some $x \neq 0$, which means that λ is an eigenvalue of T.

(2) $T - \lambda I$ is injective but the range of $T - \lambda I$ is not dense in $\mathscr{H}$. Equivalently, there is a non-zero bounded linear functional $\varphi \in \mathscr{H}'$ such that

$$\varphi((T - \lambda I)x) = 0$$

for all $x \in \mathscr{H}$. This, in turn is equivalent to

$$(T - \lambda I)^\star \varphi^\star = (T^* - \overline{\lambda} I)(\varphi^\star) = 0$$

which means $\overline{\lambda}$ is an eigenvalue of $T^\star$. Note that the shift operator R satisfies this case.

(3) $T - \lambda I$ is injective and has dense range but $T - \lambda I$ is not bounded below. Hence, after suitable normalization we obtain a sequence of unit vectors $(x_n)_{n\in\mathbb{N}}$ such that

$$\lim_{n\to\infty} \|(T - \lambda I)x_n\| = 0$$

Hence, λ is an 'approximate eigenvalue' of T in this case. Let's consider an example. Consider the multiplication operator

$$M_x : L^2([0,1]) \to L^2([0,1])$$
$$f \mapsto xf$$

We show that M_x has no eigenvalues. If λ were an eigenvalue of M_x, then

$$(x - \lambda)f = 0$$

almost everywhere. From this, we can conclude that $f = 0$ almost everywhere on $\{\lambda\}^c$. Since a singleton set has measure 0, it follows that $f = 0$ almost everywhere, a contradiction. However, we show that $\lambda = 0$ is an 'approximate eigenvalue' of M. The functions

$$f_n = \sqrt{n}\chi_{[0,1/n]},$$

for $n \in \mathbb{N}$ have unit norm, and we have

$$\|Mf_n\|^2 = \int_0^1 x^2|f_n|^2 dx = \int_0^{1/n} nx dx = \frac{n}{2}x^2\Big|_0^{1/n} = \frac{1}{n} \to 0.$$

So 0 is an approximate eigenvalue of M.

These three cases motivate the following definition:

Definition 16.2. Let $\mathscr{H}$ be a Hilbert space and let $\lambda \in \sigma(T)$.

(1) λ is in the **point spectrum**, $\sigma_p(T)$, if $T - \lambda I$ is not injective. That is,

$$\sigma_p(T) = \{\lambda \in \sigma(T) \mid \ker(T - \lambda I) \neq \{0\}\}$$

Elements of $\sigma_p(T)$ are called eigenvalue of T.

(2) λ is in the **residual spectrum**, $\sigma_r(T)$, if $T - \lambda I$ is injective, and the closure of the image of $T - \lambda I$ is not equal to $\mathscr{H}$. That is,

$$\sigma_r(T) = \{\lambda \in \sigma(T) \mid \ker(T - \lambda I) = \{0\} \text{ and } \overline{\text{Range}(T - \lambda I)} \neq \mathscr{H}\}$$

(3) λ is in the **continuous spectrum**, $\sigma_c(T)$, if $T - \lambda I$ is injective, not surjective, but its image is dense in $\mathscr{H}$.

$$\sigma_c(T) = \{\lambda \in \sigma(T) \mid \ker(T - \lambda I) = \{0\}, \text{Range}(T - \lambda I) \neq \mathscr{H} \text{ and } \overline{\text{Range}(T - \lambda I)} = \mathscr{H}\}$$

We have the following simplification for normal operators on a Hilbert space.

Proposition 16.3. *Let $\mathscr{H}$ be a Hilbert space and let $T \in \mathscr{B}(\mathscr{H})$ be a normal operator. Then $\sigma_r(T) = \emptyset$.*

Proof. If $\lambda \in \sigma_r(T)$, then $T - \lambda I$ is injective and $\overline{\lambda}$ is an eigenvalue of T^*. By (2), $\lambda = \overline{\overline{\lambda}}$ is an eigenvalue of $T = T^{**}$ which can only happen if $T - \lambda I$ is not injective, a contradiction. Hence, $\sigma_r(T) = \emptyset$. ☐

Remark 16.4. *In view of Proposition 16.3, every element of the spectrum of a normal operator is an 'approximate eigenvalue.'*

17. Spectral Theorem for Compact Operators

We cannot expect, even for self-adjoint operators, to have a spectral theory as is the case for finite-dimensional Hilbert spaces. There is, however, one important special class of operators, compact normal operators, for which the spectral theory closely resembled the finite-dimensional case.

17.1. **Compact Operators.** We first review compact operators.

Definition 17.1. Let $\mathscr{H}$ be a Hilbert space. An operator $K \in \mathscr{B}(\mathscr{H})$ is said to be a **compact operator** if

$$K(B_1) := \{Kx \in H : x \in \mathscr{H} \text{ and } \|x\| \leq 1\}$$

has compact closure in $\mathscr{H}$. That is $\overline{K(B_1)}$ is compact.

Example 17.2. Let $T \in \mathscr{B}(\mathscr{H})$ be a finite-rank operator. This is because a $\overline{K(B_1)}$ is closed and bounded in Range(T) which is contained in a finite-dimensional space that is isomorphic to $\mathbb{C}^N$ for some $N \in \mathbb{N}$. Hence, $\overline{K(B_1)}$ is compact.

Let's discuss an alternative characterization of compact operators.

Proposition 17.3. *Let $\mathscr{H}$ be a Hilbert space, and let $T \in \mathscr{B}(\mathscr{H})$.*

(1) *T is a compact operator if and only if the image of every bounded sequence has a convergent subsequence.*
(2) *The set of compact operators, $\mathscr{K}(\mathscr{H})$ is a linear subspace of $\mathscr{B}(\mathscr{H})$.*
(3) *The set of linear space of compact operators, $\mathscr{K}(\mathscr{H})$ is a two-sided ideal of $\mathscr{B}(\mathscr{H})$.*
(4) *$\mathscr{K}(\mathscr{H})$ is a closed ideal. That is, a limit of compact operators is compact.*

Proof. The proof is given below:

(1) Suppose T is a compact operator. Let $(x_n)_{n\in\mathbb{N}}$ be a bounded sequence in $\mathscr{H}$. Then $\|x_n\|_X \leq M$ for all $n \in \mathbb{N}$ and $M > 0$. In particular, $\frac{1}{M}(x_n)_{n\in\mathbb{N}} \in B_1 \subseteq \mathscr{H}$ and $\frac{1}{M}(Tx_n)_{n\in\mathbb{N}} \in T(B_1)$ for every $n \in \mathbb{N}$. Since $\overline{T(B)} \subseteq \mathscr{H}$ is compact and hence sequentially compact, a subsequence $\left(\frac{1}{M}(Tx_{n_k})_{k\in\mathbb{N}}\right)_{k\in\mathbb{N}}$ converges in Y. Hence, $(Tx_{n_k})_{k\in\mathbb{N}}$ is also a convergent sequence.

   For the converse, it suffices to show that $\overline{T(B_1)}$ is sequentially compact. Let $(y_n)_{n\in\mathbb{N}}$ be any sequence in $\overline{T(B_1)}$. For every $n \in \mathbb{N}$, there exists $z_n \in T(B_1(0))$ such that
$$\|y_n - z_n\| \leq \frac{1}{n}$$
   and $z_n = T(x_n)$ for some $x_n \in B_1$. Since $(x_n)_{n\in\mathbb{N}} \subseteq B_1$, a subsequence $(z_{n_k})_{k\in\mathbb{N}}$ converges to some $z \in Y$ by assumption. Since
$$\begin{aligned}\limsup_{k\to\infty} \|y_{n_k} - z\| &\leq \limsup_{k\to\infty} \left[\|y_{n_k} - z_{n_k}\| + \|z_{n_k} - z\|\right] \\ &\leq \limsup_{k\to\infty} \left(\frac{1}{n_k} + \|z_{n_k} - z\|\right) = 0,\end{aligned}$$
   we conclude that a subsequence of $(y_n)_{n\in\mathbb{N}}$ converges. Being closed, $\overline{T(B_1)}$ must contain the limit z, which proves that $\overline{T(B_1)}$ is sequentially compact.
(2) It is clear that $\mathscr{K}(\mathscr{H})$ is closed under scalar multiplication. Let $K_1, K_2 \in \mathscr{K}(\mathscr{H})$, qne $(x_n)_{n\in\mathbb{N}}$ be a boueded sequence. Take a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ such that that $(K_1(x_{n_k}))_{k\in\mathbb{N}}$ converges. Since $(x_{n_k})_{k\in\mathbb{N}}$ is bounded, take a subsequence $(x_{n_{k_l}})_{l\in\mathbb{N}}$ such that $(K_2(x_{n_{k_l}}))_{l\in\mathbb{N}}$. It is clear that
$$((K_1 + K_2)(x_{n_{k_l}}))_{l\in\mathbb{N}}$$
   converges. The claim follows by (1).
(3) Let $T \in \mathscr{B}(\mathscr{H}), K \in \mathscr{K}(\mathscr{H})$. We show that $TK \in \mathscr{K}(\mathscr{H})$. Let $(x_n)_{n\in\mathbb{N}}$ be a bounded sequence. By (1), $(K(x_n))_{n\in\mathbb{N}}$ converges. Clearly, $(TK(x_n))_{n\in\mathbb{N}}$ converges as well. Hence, $TK \in \mathscr{K}(\mathscr{H})$ by (1). A similar argument shows that $KT \in \mathscr{K}(\mathscr{H})$. The claim follows.
(4) Let $\|T_n - T\| \to 0$, To show that $\overline{T(B_1)}$ is compact, it suffices to show that $T(B_1)$ is totally bounded. Choose $\epsilon > 0$, and $N \in \mathbb{N}$ such that $\|T - T_n\| < \frac{\epsilon}{2}$ for $n \geq N$. Since T_N is compact, $T_N(B_1)$ is totally bounded and hence has a finite $\frac{1}{2}\epsilon$-net
$$\{s_1, ..., s_k\} \subseteq T_N(B_1)$$

If $y \in T(B_1)$, then $y = Tx$ for some $x \in B_1$. Then $\tilde{y} = T_N x$ satisfies $\|y - \tilde{y}\| < \frac{\varepsilon}{2}$. Since $\{s_1, ..., s_k\}$ is a $\frac{\epsilon}{2}$-net for $T_N(B_1)$, we have $\|\tilde{y} - s_i\| < \frac{\varepsilon}{2}\epsilon$ for some i. Hence

$$\|y - s_i\| < \epsilon$$

Hence $T(B_1)$ is totally bounded and so T is compact.

This completes the proof. □

Proposition 17.3 implies that $\mathscr{K}(\mathscr{H})$ of compact operators is a closed, two-sided ideal of $\mathscr{B}(\mathscr{H})$. Hence, we have the following definition:

Definition 17.4. Let $\mathscr{H}$ be a Hilbert space. The quotient algebra

$$\mathscr{Q}(\mathscr{H}) = \mathscr{B}(\mathscr{H})/\mathscr{K}(\mathscr{H})$$

is a Banach algebra, known as the **Calkin algebra**.

Let's prove another characterization of compact operators for seperable Hilbert spaces.

Proposition 17.5. *Let $\mathscr{H}$ be a seperable Hilbert space. T is compact if and only if T is a limit of finite-rank operators in the operator norm topology.*

Proof. Finite-rank operators are compact by Example 17.2 and a limit of compact operators is compact by Proposition 17.3. Conversely, let T be a compact operator. Since $\mathscr{H}$ is separable, it has an orthonormal basis $\{e_k\}_{k\in\mathbb{N}}$, and by compactness, $\overline{T(B_1)}$ is compact. In particular, for every $\varepsilon > 0$, there exists some $N \in \mathbb{N}$ such that

$$\sum_{k>N} |\langle Tx, e_k\rangle|^2 < \varepsilon^2$$

for all x satisfying $\|x\| \leq 1$. We can thus define the partial sums

$$T_n(x) = \sum_{k=1}^{n} \langle Tx, e_k\rangle e_k.$$

This is a bounded linear operator because $\|T_n x\|^2 \leq \|x\|^2$ by Bessel's inequality, and the range of T_n is contained within the span of $\{e_1, \ldots, e_n\}$, so T_n is a finite rank operator for each n. It suffices to show that this choice of T_n does converge to T. Let $n > N$. For any $x \in B_1$, we have

$$\begin{aligned}
\|T_n x - Tx\|^2 &= \left\| \sum_{k=1}^{n} \langle Tx, e_k\rangle e_k - \sum_{k=1}^{\infty} \langle Tx, e_k\rangle e_k \right\|^2 \\
&= \left\| \sum_{k=n+1}^{\infty} \langle Tx, e_k\rangle e_k \right\|^2 \\
&= \sum_{k=n+1}^{\infty} \|\langle Tx, e_k\rangle e_k\|^2 \\
&\leq \sum_{k>N} \|\langle Tx, e_k\rangle e_k\|^2 < \varepsilon^2
\end{aligned}$$

The claim follows. □

Remark 17.6. *It can be checked that if $T \in \mathscr{K}(\mathscr{H})$, then $T^* \in \mathscr{K}(\mathscr{H})$. Then $\mathscr{K}(\mathscr{H})$ is a C^*-algebra because $\mathscr{K}(\mathscr{H})$ is a norm closed $*$-subalgebra of $\mathscr{B}(\mathscr{H})$.*

Proposition 17.3(4) states that compact operators are close to being finite-rank operators, as compact operators can be expressed as limits of finite-rank operators when $\mathscr{H}$ is a separable Hilbert space. This suggests that compact operators behave similarly to finite-rank operators. Let's look at an example.

Example 17.7. Consider the operator

$$K : \ell^2(\mathbb{N};\mathbb{C}) \to \ell^2(\mathbb{N};\mathbb{C})$$
$$(a_1, a_2, a_3, \dots) \mapsto \left(\frac{a_1}{1}, \frac{a_2}{2}, \frac{a_3}{3}, \dots\right).$$

Let

$$(T_n(a))_j = \begin{cases} \frac{a_j}{j} & \text{if } j \le n; \\ 0 & \text{otherwise.} \end{cases}$$

T_n is a linear operator, and T_n is a finite-rank operator since the range of T_n is generated by $e_1, \dots, e_n$, a finite dimensional space. For $a \in \ell^2(\mathbb{N};\mathbb{C})$,

$$\|T(a) - T_n(a)\|^2 = \sum_{j \ge n+1} \frac{|a_j|^2}{j^2} \le \frac{1}{(n+1)^2} \sum_{j \ge 1} |a_j|^2,$$

so $\|T - T_n\| \le \frac{1}{(n+1)^2} \to 0$. Hence, T is compact by Proposition 17.3.

We end by discussing how to define a large class of compact operators on seperable Hilbert spaces

Proposition 17.8. *Let $\mathscr{H}$ be a separable Hilbert space. Let $T \in \mathscr{B}(\mathscr{H})$ such that there exists an orthonormal basis $\{e_n\}_{n\in\mathbb{N}}$ such that*

$$\sum_{n\in\mathbb{N}} \|Te_n\|^2 < \infty.$$

Then T is a compact operator.

Remark 17.9. *An operator satisfying the condition in Proposition 17.8 is called a Hilbert-Schmidt operator.*

Proof. We use Proposition 17.5 and show that T is a limit of finite rank operators. For each $n \in \mathbb{N}$, define an operator $A_n : \mathscr{H} \to \mathscr{H}$ by

$$T_n(x) := \sum_{k=1}^{n} \langle x, e_k \rangle T(e_k).$$

Clearly, T_n is a finite-operator. For any $x \in \mathscr{H}$ such that $\|x\| \le 1$, we have the estimate

$$\begin{aligned}
\|(T - T_n)x\| &= \left\| \sum_{k=n+1}^{\infty} \langle x, e_k \rangle T(e_k) \right\| \\
&\le \sum_{k=n+1}^{\infty} |\langle x, e_k \rangle| \|Te_k\| \\
&\le \left(\sum_{k=n+1}^{\infty} |\langle x, e_k \rangle|^2 \right)^{\frac{1}{2}} \left(\sum_{k=n+1}^{\infty} \|Te_k\|^2 \right)^{\frac{1}{2}}, \\
&\le \left(\sum_{k=1}^{\infty} |\langle x, e_k \rangle|^2 \right)^{\frac{1}{2}} \left(\sum_{k=n+1}^{\infty} \|Te_k\|^2 \right)^{\frac{1}{2}}, \\
&\le \|x\| \left(\sum_{k=n+1}^{\infty} \|Te_k\|^2 \right)^{\frac{1}{2}} \to 0
\end{aligned}$$

as $n \to \infty$. This proves the claim. $\square$

Example 17.10. (**Integral Operator**) Let $L^2(X, \mu)$ be a seperable Hilbert space with orthonormal basis $\{b_n\}_{n \in \mathbb{N}}$. Let $k \in L^2(X \times X, \mu \times \mu)$, and Consider the operator

$$\begin{aligned}
T : L^2(X) &\to L^2(X) \\
f &\mapsto Tf
\end{aligned}$$

such that

$$T(f)(x) = \int_0^1 k(x, y) f(y)\, d\mu(y)$$

It can be easily checked that $T \in \mathscr{B}(\mathscr{H})$. We claim that T is a compact operator by showing that T is a Hilbert-Schmidt operator. By Fubini's theorem, for almost every $x \in X$, $k_x(y) := k(x, y) \in L^2(X, \mu)$ function. Hence, for almost every $x \in X$,

$$(Kb_n)(x) := \int_X k(x, y) b_n(y) d\mu(y) = \int_X k_x(y) b_n(y) d\mu(y) = \langle k_x, \overline{b_n} \rangle.$$

Moreover, by the dominated convergence theorem,

$$\sum_{n=1}^{\infty} \|Kb_n\|^2 = \sum_{n=1}^{\infty} \int_X |\langle k_x, \overline{b_n} \rangle|^2 d\mu(x) = \int_X \sum_{n=1}^{\infty} |\langle k_x, \overline{b_n} \rangle|^2 d\mu(x).$$

It is clear that $\{\overline{b_n}\}_{n \in \mathbb{N}}$ is also an orthonormal basis for $L^2(X, \mu)$. Thus,

$$k_x = \sum_{n=1}^{\infty} \langle k_x, \overline{b_n} \rangle \overline{b_n}, \quad \|k_x\|^2 = \sum_{n=1}^{\infty} |\langle k_x, \overline{b_n} \rangle|^2.$$

We conclude that

$$\sum_{n=1}^{\infty} \|Kb_n\|^2 = \int_X \|k_x\|^2 d\mu(x) = \int_X \left(\int_X |k(x, y)|^2 d\mu(y) \right) d\mu(x) < \infty.$$

Hence, T is a compact operator. T is an example of an integral operaotr with integrable kernel. An example of such a kernel is the two-dimensional Gaussian kernel

$$k(x,y) := e^{-\pi(x^2+y^2)}.$$

17.2. **Spectral Theorem.** We now prove the spectral theorem for compact normal operators. We first show that the spectrum of a compact operator has properties similar to those of an operator defined on a finite-dimensional Hilbert space.

Lemma 17.11. *Let $\mathscr{H}$ be a infinite-dimensional Hilbert space and let $K \in \mathscr{B}(\mathscr{H})$ be a normal operator.*

(1) *Every non-zero $\lambda \in \sigma(K)$ is contained in the point spectrum. That is, every non-zero $\lambda \in \sigma(K)$ is an eigenvalue for K.*
(2) *If $\lambda \in \sigma_p(T)$ and $\lambda \neq 0$, then the eigenspace E_λ is finite-dimensional.*
(3) *We have*
$$\sigma(K) = \sigma_p(K) \cup \{0\}$$
(4) *$\sigma(K)$ is countable. Each non-zero eigenvalue is isolated, and if the set $\sigma(K) \setminus \{0\} := (\lambda_n)_{n\in\mathbb{N}}$ is infinite, then $(\lambda_n)_{n\in\mathbb{N}}$ converges to 0.*

Proof. The proof is given below:

(1) We show that $\lambda \notin \sigma_c(K)$. It suffices to show that $\mathrm{Range}(K - \lambda I)$ is closed. Because $K - \lambda I$ is a bounded operator, its kernel E_λ is closed. Hence, we have the decomposition
$$\mathscr{H} = E_\lambda^\perp \oplus E_\lambda.$$
The key is to show that $S = (K - \lambda I)|_{E_\lambda^\perp}$ is bounded below. Suppose the contrary. Then there is a sequence
$$(x_n)_{n\in\mathbb{N}} \in E_\lambda^\perp$$
with $\|x_n\| = 1$ for all n and $z_n = Sx_n \to 0$. By the compactness of K, there is a subsequence such that $(Kx_{n_l})_{l\in\mathbb{N}}$ converges. Suppose without loss of generality that the subsequence is the entire sequence, so $y_n = Kx_n \to y$ for some $y \in \mathscr{H}$. Then,
$$\lambda x_n = Sx_n + Tx_n = z_n + y_n \to 0 + y = y.$$
Since $\lambda \neq 0$, we have $\|y\| = |\lambda| > 0$, it follows that $x_n \to \lambda^{-1} y \in E_\lambda^\perp \setminus \{0\}$ and
$$S(\lambda^{-1}y) = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} z_n = 0,$$
contradicting $E_\lambda^\perp \cap E_\lambda = \{0\}$. So there exists a $\delta > 0$ such that
$$\|Sx\| \geq \delta\|x\|$$
for all $x \in E_\lambda^\perp$. Iif $(y_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathrm{Range}(T - \lambda I)$, then $(S^{-1}y_n)$ is a Cauchy sequence in $E_\lambda^\perp$, and we have
$$\begin{aligned}\lim_{n\to\infty} y_n &= \lim_{n\to\infty} S(S^{-1}y_n)\\ &= S\left(\lim_{n\to\infty} S^{-1}y_n\right)\\ &\in \mathrm{Range}(S) = \mathrm{Range}(\lambda I - T).\end{aligned}$$
The claim follows.

(2) Note that E_λ is closed. Let $S = T|_{E_\lambda}$. Since T is compact, S is compact as well since the restriction of a compact operator to a closed subspace is a compact operator. However, $S = \lambda I$, which is only compact for finite dimensional spaces by the Riesz's lemma. Hence, E_λ is finite-dimensional for $\lambda \neq 0$.

(3) By (1), it suffices to show that $0 \in \sigma(K)$. If $0 \notin \sigma(K)$, then K is with inverse $L \in \mathscr{B}(\mathscr{H})$. Hence,

$$KL = I$$

Since $\mathscr{K}(\mathscr{H})$ is a two-sided ideal, $I \in \mathscr{B}(\mathscr{H})$ which is a contradiction since $\mathscr{H}$ is infinite-dimensional and closure of the unit ball in $\mathscr{H}$ is not compact by Riesz's lemma.

(4) It suffices to prove that for all $\varepsilon > 0$ there exist at most finitely many $\lambda_\varepsilon \in \sigma(K)$ such that that $|\lambda_\varepsilon| \geq \varepsilon$. Suppose to the contrary. Then for some $\varepsilon_0 > 0$ we may select a sequence $(\lambda_n)_{n\in\mathbb{N}}$ with $|\lambda_n| \geq \varepsilon_0$ for all $n \in \mathbb{N}$. By (2), all the corresponding eigenvectors are orthogonal. Hence, we can extract an orthonormal sequence $(x_n)_{n\in\mathbb{N}}$ such that $x_n \in E_{\lambda_n}$. for each $n \in \mathbb{N}$. Using the compactness of K, the sequence $(K(x_n))_{n\in\mathbb{N}}$ admits a convergence subsequence which we can assume to be $(K(x_n))_{n\in\mathbb{N}}$. We have,

$$\|K(x_n) - K(x_m)\|^2 = \|\lambda_n x_n - \lambda_m x_m\|^2 = \lambda_n^2 + \lambda_m^2 \geq 2\varepsilon_0^2$$

for all $m \neq n$. This contradicts that $(K(x_n))_{n\in\mathbb{N}}$ is a Cauchy sequence. This shows that $\sigma(K)$ is countable. The remaining claim follows similarly.

This completes the proof. □

Remark 17.12. *Note that* 0 *may or may not be in the point spectrum of* $K \in \mathscr{K}(\mathscr{H})$.

We can now prove the spectral theorem for compact normal operators.

Proposition 17.13. ***(Spectral Theorem for Compact Normal Operators)*** *Let* $\mathscr{H}$ *be a Hilbert space and let* $K \in \mathscr{K}(\mathscr{H})$. *Let* $\{\lambda_n\}_{n\in J}$ *be the non-zero eigenvalues of* K. *If* P_n *is the projection onto* E_{λ_n},

$$K = \sum_{n\in J} \lambda_n P_n,$$

where the sum converges in operator norm.

Proof. We use continuous functional calculus:

$$\Psi_K : C(\sigma(K)) \to C^*(K, I)$$

Since each λ_n is isolated in $\sigma(K)$ the characteristic function χ_{λ_n} is continuous on $\sigma(K)$. If $\sigma(K)$ is infinite, then $(\lambda_n)_{n\in J} \to 0$. Hence,

$$\mathrm{Id}_{C(\sigma(K))} = \sum_{n\in J} \lambda_n \chi_{\lambda_n} \Longrightarrow K = \sum_{n\in J} \lambda_n \Psi_K(\chi_{\lambda_n})$$

is well-defined. Moreover, $\Psi_T(\chi_{\lambda_n}) = P_n$ as in the discussion after Proposition 15.2. This completes the proof. □

Example 17.14. Let $H = \ell^2(\mathbb{N})$. Let $(\lambda_n)_{n\in\mathbb{N}}$ be any sequence tending to 0, and let P_n be the projection onto the space spanned by e_n. Then

$$K = \sum_{n=1}^{\infty} \lambda_n P_n$$

defines a compact operator by Proposition 17.5. We have

$$\sigma_K = (\lambda_n)_{n\in\mathbb{N}}^{\infty} \cup \{0\}$$

If $0 \notin (\lambda_n)_{n\in\mathbb{N}}^{\infty}$, then 0 is not an eigenvalue of K.

18. Projection-Valued Measures

Part 4. Commutative Harmonic Analysis

We discuss commutative harmonic analysis as an application of operator algebra theory. Classical harmonic analysis begins with the fact in Fourier theory that

$$\{e^{in\theta} : n \in \mathbb{Z}\}$$

are eigenfunctions of the Laplacian on $\mathbb{S}^1$, and they form a complete basis for $L^2(\mathbb{S}^1)$:

$$L^2(\mathbb{S}^1) = \bigoplus_{n\in\mathbb{Z}} \mathrm{Span}(e^{in\theta})$$

This is because any $f \in L^2(\mathbb{S}^1)$ can be written as

$$f(\theta) = \sum_{n\in\mathbb{Z}} \widehat{f}(n)e^{in\theta},$$

where $\widehat{f}(n)$ are the Fourier coefficients. This point of view highlights the group-theoretic nature of Fourier analysis. Indeed, the group $U(1) \cong \mathbb{S}^1$ acts on $L^2(\mathbb{S}^1)$ by translation:

$$z \cdot f(x) = f(z^{-1}x), \quad z \in \mathbb{S}^1, \ f \in L^1(\mathbb{S}^1).$$

Note that we have

$$\{e^{in\theta} : n \in \mathbb{Z}\} = \mathrm{Hom}_{\mathrm{CTS}}(U(1), \mathbb{C}^*)$$

Hence, the decomposition of $L^2(\mathbb{S}^1)$ in terms of the eigenfunctions of the Laplacian are precisely the 1-dimensional irreducible (unitary) representations of $U(1) \cong \mathbb{S}^1$. Hence, $L^2(\mathbb{S}^1)$ is a direct sum of irreducible representations of $U(1)$. This establishes a relationship between Fourier series and representation theory of $U(1)$. This also suggests that Fourier series can be generalized by studying the representation theory of more general groups. For instace, the group $\mathbb{R}$ acts on $L^1(\mathbb{R})$ by translation as above. Classical harmonic analysis also studies the Fourier transform, which states that any any $f \in L^1(\mathbb{R})$ can be written as

$$f(x) = \int_{\mathbb{R}} \widehat{f}(k)e^{ikx}dk$$

Hence, in a vague sense the two observations can be reconciled by stating that we have a *direct integral decomposition*

$$\text{“}L^1(\mathbb{R}) = \int_{\mathbb{R}} \mathrm{Span}(e^{ikx})dk\text{”}$$

Commutative harmonic analysis makes this intuition precise by generalizing Fourier theory by studying the representation theory of a locally compact groups via operator algebras.

Remark 18.1. *We focus on commutative harmonic analysis which deals with locally compact abelian groups. The representation theory of a general possibly non-abelian locally compact group requires von-Neumann algebra theory, and it is called non-commutative harmonic analysis.*

19. Topological Groups

The first step in generalizing classical harmonic analysis is to identify the appropriate spaces for analysis. To establish a meaningful structure, the space should be endowed with both a topology and a group operation. The group structure ensures that the space can act on itself in a well-defined manner, as observed in the cases of $\mathbb{R}$ and $\mathbb{S}^1$. This naturally leads to the definition of topological groups.

19.1. **Definition & Examples.** We start by recalling the notion of topological groups.

Definition 19.1. A **topological group** is a group G, together with a topology on the set G such that the multiplication, m, and inversion, i, maps are continuous.

Remark 19.2. *A group is a topological group if and only if the the map*

$$\tau : G \times G \to G$$
$$(x, y) \mapsto xy^{-1}$$

is a continuous map. Indeed, if G is a topological group then the map τ is a composition of the continuous maps

$$(x, y) \mapsto (x, y^{-1}), \quad (x, y) \mapsto xy$$

so it is continuous. Conversely, if τ is continuous, we see that $x^{-1} = e_G x^{-1}$ and hence the inversion is a composition of the two continuous maps

$$x \mapsto (e_G, x), \quad (x, y) \mapsto xy^{-1}$$

It follows that multiplication is also a composition of continuous maps

$$(x, y) \mapsto (x, y^{-1}), \quad (x, y) \mapsto xy^{-1}$$

So G is a topological group.

Example 19.3. The following is a list of examples of topological groups:

(1) Any group is a topological group when equipped with the discrete topology.
(2) $\mathbb{R}$ and $\mathbb{C}$ become topological groups with their usual topology.
(3) Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$. Then $\mathrm{GL}(n, \mathbb{K})$ is a topological group. The topology on $\mathrm{GL}(n, \mathbb{K})$ is the subspace topology inherited from $\mathbb{K}^{n^2}$. It is clear that $\mathrm{GL}(n, \mathbb{K})$ is a topological group since matrix multiplication and matrix inversion are continuous maps.
(4) The circle group
$$U(1) = \{z \in \mathbb{C} \mid |z| = 1\} \subseteq \mathbb{C}^{\times}$$
is a topological group.

Definition 19.4. Let G, H be topological groups. A group homomorphism $\varphi : G \to H$ is a **topological group morphism** if it is continuous. That is, for every $\varepsilon > 0$ and $x \in G$ there exists open sets $x \in U \subseteq G$ and $\varphi(x) \in V \subseteq H$ such that

$$y \in U \Rightarrow \varphi(y) \in V$$

Remark 19.5. *Hence, we obtain the category of topological groups.*

If G is a topological group, we can define three important maps. For every $a \in G$, the maps

$$\rho_a(x) = xa, \quad \lambda_a(x) = ax, \quad \gamma_a(x) = axa^{-1}$$

are the left translation map, the right translation map and the conjugation map respectively. Topological groups are topologically homoegnous:

Proposition 19.6. *Let G be a topological group. For $a \in G$, the maps ρ_a, λ_a and γ_a are homeomorphisms of G. In other words, the topology of G is invariant under translations and conjugation.*

Proof. The maps are continuous by the continuity of the multiplication and the inversion maps. The respective inverse maps are also continuous for similar reasons. $\square$

Remark 19.7. *Proposition 19.6 has a number of remarkable implications. For instance, it follows that every open set $x \in W \subseteq G$ can be written as*

$$W = xU = Vx,$$

where $U = \lambda_{x^{-1}}(W)$ and $V = \rho_{x^{-1}}(W)$ are open sets containing e_G.

The following local criterion for a group homomorphism to be a topological group morphism is useful and is motivated based on the observation that topological groups are topologically homoegnous.

Lemma 19.8. *Let G, H be topological groups and let $f : G \to H$ be a group homomorphism. The following are equivalent:*

1. *f is a topological group morphism.*
2. *f is continuous at $e_G \in G$.*

Proof. Clearly, (1) implies (2). Let (x_j) be a net with $x_j \to x$ in G. Then $x^{-1}x_j \to x^{-1}x = e_G$, and we have

$$f(x)^{-1}f(x_j) = f(x^{-1}x_j) \to f(e_G) = e_H,$$

which then implies $f(x_j) \to f(x)$. Thus, f is continuous. $\square$

The following observation is about Hausdorff topological groups is also quite useful.

Proposition 19.9. *Let G be a topological group. G is Hausdorff if and only if some singleton $\{x_0\} \subseteq G$ is closed.*

Proof. The forward implication is clear. Conversely, suppose that $\{x_0\} \subseteq G$ is closed for some $x_0 \in G$. The preimage of $\{x_0\}$ under the continuous map

$$(x, y) \mapsto x^{-1}yx_0$$

is the diagonal $\Delta_G = \{(x, x) \mid x \in G\} \subseteq G \times G$, which is therefore closed in $G \times G$. Thus, G is Hausdorff. $\square$

19.2. **Properties of Topological Groups.**

We can construct new topological groups by the following proposition.

Proposition 19.10. *The following statements are true.*

1. *Every subgroup of a topological group is a topological group.*
2. *The product of topological groups is a topological group.*
3. *Quotients of a topological group by a normal subgroup is a topological group.*

Proof. The proof is given below:

1. Let $H \subseteq G$ be a subgroup of a topological group. Then the map $H \times H \to H$ defined by
$$(x_1, x_2) \mapsto x_1x_2^{-1}$$
is a restriction of the continuous map $\tau : G \times G \to G$, since the subspace topology on $H \times H \subseteq G \times G$ agrees with the product topology of $H \times H$. Thus, $H \times H$ is a topological group.
2. This is clear.

(3) Let $N \lhd G$ be a normal subgroup of a topological group. We first show that the quotient map $\pi : G \to G/N$ is open. If $U \subseteq G$ is open, then

$$\pi^{-1}(\pi(U)) = \bigcup_{x \in N} Ux$$

is open in G. Hence, $\pi(U)$ is open in G/N. It follows that there is a natural homeomorphism between

$$(G/N) \times (G/N) \cong (G \times G)/(N \times N)$$

We deduce that in order to check the continuity of the map

$$G/N \times G/N \to G/N$$
$$(xN, yN) \mapsto xy^{-1}N$$

it suffices to check the continuity of $G \times G \to G/N$. Consider the following commutative diagram:

$$\begin{tikzcd} G \times G \arrow[r, "\pi"] \arrow[d] & G \arrow[d] \\ G/N \times G/N \arrow[r] & G/N \end{tikzcd}$$

This shows that

$$G \times G \to G/N \times G/N \to G/N$$

is a composition of continuous maps.

This completes the proof. □

Let's now talk about some properties of subgroups of topological groups.

Proposition 19.11. *Let G be a topological group and let $H \subseteq G$ be a topological group.*

(1) *$\overline{H}$ is a subgroup of G.*
(2) *If H is an open subgroup, then H is closed as well.*
(3) *If H is a normal subgroup, then $\overline{H}$ is also a normal subgroup.*
(4) *G/H is Hausdorff if and only if H is closed.*

Proof. The proof is given below:

(1) $\tau|_H$ is clearly continuous. Hence, H is a topological group. Moreover,

$$\tau(\overline{H} \times \overline{H}) = \tau(\overline{H \times H}) \subseteq \overline{\tau(H \times H)} \subseteq \overline{H}$$

Hence, $\overline{H}$ is a subgroup.

(2) Writing G as the union of left cosets, we get

$$G \setminus H = \bigcup_{x \in G} xH$$

As H is open, so is xH for every $x \in G$. Hence, the complement $G \setminus H$, being the union of open sets, is open, so H is closed.

(3) We have

$$\gamma_a(\overline{H}) \subseteq \overline{\gamma_a(H)} = H$$

for all $a \in G$. This shows that H is normal in G.

(4) This follows from Proposition 19.9.

This completes the proof. □

We end with some technical properties of topological groups.

Proposition 19.12. *Let G be a topological group.*

(1) *If U is an open set containing e_G, then*
$$U^{-1} = \{x^{-1} : x \in U\}$$
is also an open set containing e_G.
(2) *Every open set, U, containing e_G contains an symmetric open set, $W \subseteq U$. That is, $W = W^{-1}$.*
(3) *For a given open set U of e_G, there is an open set V of the unit such that $V^2 \subseteq U$.*
(4) *If $A, B \subseteq G$ are compact subsets, then AB is compact.*
(5) *If A, B are subsets of G and A or B is open, then so is AB.*

Proof. The proof is given below:

(1) Follows from the continuity of the inversion.
(2) Take $W = U \cap U^{-1}$.
(3) Let $A \subseteq G \times G$ be the inverse image of U under the multiplication map
$$m : G \times G \to G$$
Then A is open in the product topology of $G \times G$. Therefore, there are open sets W, X of e_G such that $(e_G, e_G) \in W \times X \subseteq A$. Let $V = W \cap X$. Then V is an open set of e_G, and $V \times V \subseteq A$. Hence, $V^2 \subseteq U$.
(4) AB is the image of the compact set $A \times B$ under the multiplication map.
(5) Assume A is open. Then
$$AB = \bigcup_{y \in B} Ay$$
is open since every set Ay is open.

This completes the proof. □

20. Locally Compact Hausdorff Groups

We now specialize our discussion to locally compact Hausdorff topological groups. We will see that such groups admit a natural measure, which enables the development of integration theory on these spaces. This, in turn, allows us to formulate an analogue of the Fourier transform.

Definition 20.1. A topological group is a **locally compact Hausdorff topological group** (LCH) if it is Hausdorff and locally compact as a topological space.

Remark 20.2. *We will write a locally compact Hausdorff topological group as simply locally compact Hausdorff group.*

Remark 20.3. *Recall that a topological space, X, is locally compact if every point of has a (closed) compact neighborhood. That is for every $x \in X$, there exists an open set U and a compact set K such that*
$$x \in U \subseteq K$$
The set K is called a compact neighborhood.

Example 20.4. Here are some examples of locally compact Hausdorff groups:

(1) All discrete groups are locally compact Hausdorff. In particular, finite groups are locally compact Hausdorff with the discrete topology.

(2) The real line $\mathbb{R}$, the complex plane $\mathbb{C}$, and its closed subset $\mathbb{S}^1$ are all locally compact Hausdorff groups.
(3) More generally, all Lie groups are locally compact Hausdorff groups.

Remark 20.5. *There are many examples of non-locally compact Hausdorff groups as well. Examples include infinite-dimensional Lie groups, the Virasoro group, Kac-Moody groups, and the group of unitary operators on a Hilbert space.*

We will make use of $C_c(G)$, the vector space of continuous and compactly supported functions on G, extensively in what follows. We prove an important lemma that every $f \in C_c(G)$ is uniformly continuous when G is a locally compact Hausdorff group.

Lemma 20.6. *Let G be a locally compact Hausdorff group. Then any $f \in C_c(G)$ is uniformly continuous.*

Remark 20.7. *If G is topological group and $f : G \to \mathbb{C}$ is a function, then f is uniformly continuous if for all $\epsilon > 0$ there exists an open neixybourhood $e_G \in U$ such that for all $x, y \in G$ satisfying $x^{-1}y \in U$ or $yx^{-1} \in U$ we have*

$$|f(x) - f(y)| < \epsilon$$

When showing that f is uniformly continuous, it suffices to only work with the condition $x^{-1}y \in U$, because the other condition can be dealt with similarly. To satisfy both conditions, one can simply intersects the two open sets for which the two conditions are satisfied.

Proof. Let $f \in C_c(G)$ and let $e_G \in U \subseteq K$ be a compact neighborhood of e_G. As f is continuous, for every $x \in G$, there exists an open set $e_G \in V_x \subseteq K$[14] such that

$$y \in xV_x \implies |f(x) - f(y)| < \epsilon/2$$

Let U_x be a symmetric open set with $U_x^2 \subseteq V_x$. The sets xU_x, for $x \in (\text{supp}(f))K$, form an open covering of the compact set $(\text{supp}(f))K$. So there are $x_1, \dots, x_n \in (\text{supp}(f))K$ such that

$$(\text{supp}(f))K \subseteq x_1U_{x_1} \cap \cdots \cap x_nU_{x_x}$$

Let $U = U_{x_1} \cap \cdots \cap U_{x_n}$. Then U is a symmetric open set of e_G. Now, let $x, y \in G$ with $x^{-1}y \in U$. If $x \notin \text{supp}(f)K$, then $y \notin \text{supp}(f)$ as

$$x \in yU^{-1} = yU \subseteq yK$$

So in this case, we conclude $f(x) = f(y) = 0$. It remains to consider the case when $x \in \text{supp}(f)K$. Then there exists j with $x \in x_jU_{x_j}$, and so $y \in x_jU_{x_j}U \subseteq x_jK$. It follows that

$$|f(x) - f(y)| \leq |f(x) - f(x_j)| + |f(x_j) - f(y)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

This completes the proof. □

21. Haar Measure

A fundamental property of locally compact Hausdorff groups is that they provide a natural setting for analysis. This is due to the existence of a unique left-translation invariant measure on every such group. Before proceeding, we first review some essential terminology from measure theory.

Definition 21.1. Let $(X, \mathscr{A}, \mu)$ be a measure space.

[14]We can always intersect a V_x with U to ensure this is the case.

(1) If $\mathscr{A}$ is generated by all open subsets of X, then $\mathscr{A}$ is called the **Borel σ-algebra** of X.
(2) μ is called a **locally finite measure** if for every point $x \in X$ there exists an open set $x \in U$ such that $\mu(U) < \infty$.
(3) If $\mathscr{A}$ is the Borel σ-algebra and μ is a locally finite measure, then μ is called a **Radon measure** if the following two conditions hold:
  (a) For all $E \in \mathscr{A}$, we have
$$\mu(E) = \inf \{\mu(U);\, U \text{ is open and } E \subseteq U\}.$$
  (b) For all $E \in \mathscr{A}$ such E is an open set, we have
$$\mu(E) = \sup \{\mu(K);\, K \text{ is compact and } K \subseteq E\}.$$

Remark 21.2. *If X is a locally compact Hausdorff space and μ is a Radon measure, then one has*
$$\mu(E) = \sup \{\mu(K);\, K \textit{ is compact and } K \subseteq E\}.$$
such that $E \in \mathscr{A}$ and E is σ-finite.

We will repeatedly make density arguments while working with locally compact Hausdorff topological spaces. The following lemma will be useful.

Lemma 21.3. *Let X be a locally compact Hausdorff space, and let μ be a σ-finite outer Radon measure on X. Then the space $C_c(X)$ is dense in $L^p(X, \mu)$ for every $1 \leq p < \infty$.*

Proof. Let $V \subseteq L^p(X, \mu)$ be the closure of $C_c(X)$ inside $L^p(X, \mu)$. We have to show $V = L^p$. By measure theory, the space of simple functions are dense in $L^p(X, \mu)$. Any such function is a linear combination of functions of the form $\mathbb{1}_A(x)$, where $A \subseteq X$ is a measurable. Since μ is σ-finite, we can assume that each A has finite measure since we make a density argument below. Hence, it suffices to show that $\mathbb{1}_A(x) \in V$. By outer regularity, there exists a sequence $A \subseteq U_n$ of open sets such that 1_{U_n} converges to
$$1_{U_n}(x) \Rightarrow 1_A(x)$$
in $L^p(X, \mu)$. Thus, it suffices to assume that A is open. By inner regularity, we can similarly reduce to the case when A is a compact set. For a given $\epsilon > 0$, there exists an open set $A \subseteq U$ such that $\mu(U \setminus A) < \epsilon$. By Urysohn's lemma, there is $g \in C_c(X)$ with $0 \leq g \leq 1$, the function vanishes outside U, and is constantly equal to 1 on A. The estimate
$$\|\mathbb{1}_A(x) - g(x)\|_p^p = \int_{U \setminus A} |g(x)|^p \, dx \leq \mu(U \setminus A) < \epsilon$$
shows the claim. □

Remark 21.4. *If X is a locally compact Hausdorff space, then $C_c(X)$ is dense in $C_0(X)$. Hence, we have that $C_0(X)$ is dense in $L^p(X, \mu)$ for every $1 \leq p < \infty$.*

We now turn to the main definition.

Definition 21.5. If $(G, \mathscr{A}, \mu)$ is a measure space such that μ is a Radon measure and G is a topological group. μ is called a **left Haar measure** on G if it is left invariant. That is,
$$\mu(xE) = \mu(E)$$
for all $E \in \mathscr{A}$ and $x \in X$.

We now state the all important result:

Proposition 21.6. *Let G be a locally compact Hausdorff group. There exists a non-zero left-invariant outer Radon measure on G that is uniquely determined up to positive multiples.*

Proof. See [Fol99; Fol16] for the proof. □

Remark 21.7. *We denote the Haar measure on G by μ_G.*

Remark 21.8. *We will also implicitly assume that all locally compact Hausdorff spaces are σ-compact. This will ensure that μ_G is σ-finite to avoid any technical difficulties with measure-theoretic nuances. See [Fol16] for more details.*

Example 21.9. Here are some examples of Haar measures:

(1) If G is a discrete group, the counting measure is a Haar measure. Hence, every Haar measure on a discrete group is a positive multiple of the counting measure.
(2) The Lebesgue measure on $\mathbb{R}$ is a Haar measure [Fol99].
(3) Consider $\mathrm{GL}(1,\mathbb{R}) = \mathbb{R}^\times$. Endow $\mathbb{R}^\times$ with the sigma algbera $\mathscr{B}_{\mathbb{R}^\times}$ given by the restriction of the Borel sigma algebra of $\mathbb{R}$ to $\mathbb{R}^\times$. We show that the Haar measure on $\mathbb{R}^\times$ is given by

$$\mu_{\mathbb{R}^\times}(dx) = \frac{dx}{|x|}$$

We only check left-invariance of $\mu_{\mathbb{R}^\times}$. Let $A \subseteq \mathscr{B}_{\mathbb{R}^\times}$. For $c \in \mathbb{R}^\times$, we have,

$$\mu_{\mathbb{R}^\times}(cA) = \int_{cA} \mu_{\mathbb{R}^\times}(dx) = \int_{cA} \frac{dx}{|x|} = \int_A \frac{cdu}{|cu|} = \int_A \frac{du}{|u|} = \mu_{\mathbb{R}^\times}(A)$$

This follows from the simple substitution $x = cu$, $dx = cdu$ and

$$x = cu \in cA \iff u \in A$$

(4) Consider $\mathrm{GL}(n,\mathbb{R})$. Endow $\mathrm{GL}(n,\mathbb{R})$ with the σ-algebra $\mathscr{B}_{\mathrm{GL}(n,\mathbb{R})}$ given by the restriction of the Borel σ-algebra of $\mathbb{R}^{n^2}$ to $\mathrm{GL}(n,\mathbb{R})$. Let

$$dx = dx_{11} \cdots dx_{n1}\, dx_{12} \cdots dx_{n2} \cdots dx_{nn}$$

be the restriction of the Lebesgue measure on $\mathbb{R}^{n^2}$ to the open subset

$$\mathrm{GL}(n,\mathbb{R}) = \{x = (x_{ij})_{i,j=1}^n \mid \det(x) \neq 0\}.$$

We show that the Haar measure on $\mathrm{GL}(n,\mathbb{R})$ is given by

$$\mu_{\mathrm{GL}(n,\mathbb{R})}(dx) = \frac{dx}{|\det(x)|}$$

Let $A \subseteq \mathscr{B}_{\mathrm{GL}(n,\mathbb{R})}$. We only check left-invariance of $\mu_{\mathrm{GL}(n,\mathbb{R})}$. For $C \in \mathrm{GL}(n,\mathbb{R})$, we have:

$$\begin{aligned}\mu_{\mathrm{GL}(n,\mathbb{R})}(CA) &= \int_{CA} \frac{1}{\det(x)^n}\,dx \\ &= \int_{T(A)} \frac{1}{\det(x)^n}\,dx \\ &= \int_A \frac{1}{\det(Cu)^n}\det(C)^n du \\ &= \int_A \frac{1}{\det(C)^n\det(u)^n}\det(C)^n du \\ &= \int_A \frac{1}{\det(u)^n} du \\ &= \mu(A)\end{aligned}$$

Consider the linear transformation $T : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ defined by $T(M) = CM$. Considering the canonical basis in $\mathbb{R}^n$ it's easy to see that $\det(T) = \det(C)^n$ (see that this is the Jacobian of T). So we have:

$$\begin{aligned}\mu_{\mathrm{GL}(n,\mathbb{R})}(CA) &= \int_{CA} \frac{1}{\det(x)^n}\,dx \\ &= \int_{T(A)} \frac{1}{\det(x)^n}\,dx \\ &= \int_A \frac{1}{\det(Cu)^n}\det(C)^n\,du \\ &= \int_A \frac{1}{\det(C)^n\det(u)^n}\det(C)^n\,du \\ &= \int_A \frac{1}{\det(u)^n}\,du = \mu_{\mathrm{GL}(n,\mathbb{R})}(A).\end{aligned}$$

Example 21.9(4) can be generalized as follows. Let $G \subseteq \mathbb{R}^n$ is an open subset, and left translations are given by affine maps:

$$xy = A(x)y + b(x)$$

where $A(x)$ is a linear transformation of $\mathbb{R}^n$ and $b(x) \in \mathbb{R}^n$. Then

$$\frac{dx}{|\det A(x)|}$$

is a left Haar measure on G, where dx denotes Lebesgue measure on $\mathbb{R}^n$. The argument is similar to that of Example 21.9(4). Here are some examples.

Example 21.10. The $ax+b$ group G is the group of all affine transformations $x \to ax+b$ of $\mathbb{R}$ with $a > 0$ and $b \in \mathbb{R}$. On G,

$$\frac{da\,db}{a^2}$$

is a left Haar measure.

Using the Haar measure on a locally compact group, we can prove the following properties:

Proposition 21.11. *Let G be a locally compact group.*

(1) *Every compact set has finite measure.*
(2) *Every non-empty open set has strictly positive measure.*
(3) $\mu_G(G) < \infty$ *if and only if* G *is compact.*

Proof. The proof is given below:

(1) Let K be a compact set. By local finiteness of μ_G, we have
$$K \subseteq \bigcup_{x \in K} U_x$$
such that U_x is open and $\mu_G(U_x) < \infty$. Since K is compact, we have
$$K \subseteq \bigcup_{x_1, \cdots, x_n \in K} U_{x_i}$$
Now K has finite measure.

(2) Assume there is a non-empty open set U of measure zero. Then every translate xU of U has measure zero by left-invariance. As every compact set can be covered by finitely many translates of U, every compact set has measure zero. Being a Radon measure, μ is zero, a contradiction.

(3) If G is compact, then $\mu_G(G) < \infty$ by (1). Conversely, suppose $\mu_G(G) < \infty$. Let K be a compact neighborhood of e_G. If G is not compact, there is an infinite sequence $(x_n)_{n \in \mathbb{N}}$ such that
$$x_n \notin \bigcup_{k<n} x_k U$$
This follows because a finite union of compact sets is compact. Let $V \subseteq G$ be an open set containing e_G such that $V^{-1}V \subseteq K$[15]. The sets $x_n V$ are pairwise disjoint since $VV^{-1} \subseteq U$. Since V has positive measure, $\mu_G(G) = \infty$, a contradiction.

This completes the proof. □

Remark 21.12. *If* X *is a locally compact Hausdorff space, and* μ *is a complex Radon measure on* X*, we can define* $I_\mu : C_0(X) \to \mathbb{C}$ *by*
$$I_\mu(f) := \int_X f \, d\mu.$$
By the Riesz representation theorem, the map $\mu \mapsto I_\mu$ *is an isomorphism between the vector space* $M(X)$ *of complex Radon measures on* G *and the dual vector space* $C_0(X)^*$*. By this correspondence, if* $X = G$ *is a locally compact Hausdorff group then a left-invariant measure* μ *is mapped to a functional that is unchanged by the left translation. In other words,* μ *is a left-invariant Radon measure if and only if*
$$I_\mu(f) = I_\mu(L_x f), \quad g \in G.$$

The following result is quite useful:

Proposition 21.13. *Let* G *be a locally compact Hausdorff group. For* $1 \leq p < \infty$*, and* $f \in L^p(G, \mu_G)$*, the maps* $y \mapsto L_y f$ *and* $y \mapsto R_y f$ *are continuous maps from* G *to* $L^p(G, \mu_G)$*.*

---

[15]The proof is similar to the statement in Proposition 19.12(4).

Proof. We only consider the left-translation map. It suffices to show continuity at $e_G \in G$. Let $\epsilon > 0$. First, take $f \in C_c(G)$. Let $\epsilon > 0$. Fix a compact identity neighbourhood $e_G \in K \subseteq G$. Since f is uniformly continuous (Lemma 20.6), there is a symmetric neighbourhood $e_G \in U \subseteq K$ such that

$$|f(yx) - f(x)|^p < \frac{\epsilon}{\mu(K \operatorname{supp} f)}$$

for all $y \in U$ and all $x \in G$. Since U is symmetric, we have

$$\|L_y f - f\|_p^p = \int_{U \operatorname{supp} f} |f(y^{-1}x) - f(x)|^p dx \leq \mu(U \operatorname{supp} f) \frac{\epsilon}{\mu(K \operatorname{supp} f)} \leq \epsilon.$$

This shows that $y \mapsto L_y f$ is continuous for $f \in C_c(G)$. If $f \in L^p(G)$. Use Lemma 21.3 to find $\widetilde{f} \in C_c(G)$ such that

$$\|f - \widetilde{f}\|_p < \varepsilon/3$$

There is some identity neighbourhood $e_G \in U \subseteq G$ such that

$$\|L_y \widetilde{f} - \widetilde{f}\|_p < \frac{\epsilon}{3}$$

for all $y \in U$. Hence,

$$\|L_y f - f\|_p \leq \|L_y f - L_y \widetilde{f}\|_p + \|L_y \widetilde{f} - \widetilde{f}\|_p + \|\widetilde{f} - f\|_p < \epsilon.$$

□

22. Modular Function

Let G be a locally compact Hausdorff topological group with left Haar measure μ_G. We wish to investigate the extent to which μ_G fails to be right-invariant. For each $x \in G$, we can define a new measure on G μ_x, defined by

$$(\mu_G)_x(A) = \mu_G(Ax)$$

for all measurale subsets. It is easy to check that μ_x is a Haar measure as for $y \in G$ one has

$$(\mu_G)_x(yA) = \mu_G(yAx) = \mu_G(Ax) = (\mu_G)_x(A)$$

Therefore, by the uniqueness of the Haar measure, there exists a number $\Delta(x) > 0$ with $(\mu_G)_x = \Delta(x)\mu_G$. In this way, one gets a map $\Delta_G : G \to \mathbb{R}^+$

Definition 22.1. Let G be a locally compact group. The **modular map** is the function $\Delta_G : G \to \mathbb{R}^+$. We say G is **unimodular** if $\Delta \equiv 1$.

If G is unimodular, every left Haar measure is right-invariant as well.

Proposition 22.2. *Let G be a locally compact Hausdorff group, and let Δ_G denote the modular function.*

(1) *For $y \in G$, if f be μ_G-integrable, then $R_y f$ is μ_G-integrable and*

$$\int_G R_y f(x)\, d\mu_G(x) := \int_G f(xy)\, d\mu_G(x) = \Delta_G(y^{-1}) \int_G f(x)\, d\mu_G(x)$$

(2) *Δ_G is a continuous group homomorphism.*
(3) *If G is abelian or compact, then $\Delta_G \equiv 1$.*

Proof. The proof is given below:

(1) It is clear if $f = \chi_A$ of a measurable set A. Indeed,
$$\begin{aligned}\int_G R_y f(x)\, d\mu_G(x) &= \int_G \chi_A(xy)\, d\mu_G(x)\\ &= \int_G \chi_{Ay^{-1}}(x)\, d\mu_G(x)\\ &= \mu_G(Ay^{-1})\\ &= \Delta_G(y^{-1})\mu_G(A)\\ &= \Delta_G(y^{-1})\int_G \chi_A(x)\, d\mu_G(x) = \Delta_G(y^{-1})\int_G f(x)\, d\mu_G(x)\end{aligned}$$
The claim now follows by an approximating a μ_G-integrable function with simple functions.

(2) For $x, y \in G$ and a measurable set $A \subseteq G$, we have
$$\begin{aligned}(\Delta_G)(xy)\mu_G(A) = (\mu_G)_{xy}(A) = \mu_G(Axy) = (\mu_G)_y(Ax)\\ = \Delta_G(y)\mu_G(Ax) = \Delta_G(y)\Delta_G(x)\mu_G(A).\end{aligned}$$
Choose A with $0 < \mu_G(A) < \infty$ to get
$$\Delta_G(xy) = \Delta_G(x)\Delta_G(y)$$
Hence, Δ_G is a group homomorphism. Choose $f \in C_c(G)$ such that $c := \int_G f(x)\, d\mu_G \neq 0$. By (1), we have
$$\Delta_G(y) = \frac{1}{c}\int_G f(xy^{-1})\, d\mu_G(x) = \frac{1}{c}\int_G R_{y^{-1}} f(x)\, d\mu_G(x).$$
Hence, it suffices to prove that if $f \in C_c(G)$, the function
$$s \mapsto \int_G f(xs)\, d\mu_G(x)$$
is continuous on G. It suffices to show that the function is continuous at e_G. Let $K = \operatorname{supp}(f)$, and let V be a compact symmetric neighborhood of e_G. For $s \in V$, one has $\operatorname{supp}(R_s f) \subseteq KV$. Let $\epsilon > 0$. As f is uniformly continuous, there is a symmetric neighborhood W of e_G such that for $s \in W$, one has
$$|f(xs) - f(x)| < \frac{\epsilon}{\mu_G(KV)}.$$
For $s \in U = W \cap V$, one therefore gets
$$\begin{aligned}\left|\int_G f(xs) - f(x)\, d\mu_G(x)\right| &\leq \int_{KV} |f(xs) - f(x)|\, d\mu_G(x)\\ &< \frac{\epsilon}{\mu_G(KV)} \cdot \mu_G(KV) = \epsilon.\end{aligned}$$
This proves the claim.

(3) If G is abelian, then every right translation is a left translation, and so every left Haar measure is right-invariant. Hence, $\Delta_G \equiv 1$. If G is compact, then so is the image of the continuous map Δ_G. As Δ is a group homomorphism, the image is also a subgroup of $\mathbb{R}^+$. But the only compact subgroup of $\mathbb{R}^+$ is the trivial group $\{1\}$, which means that $\Delta_G \equiv 1$.

This completes the proof. □

Remark 22.3. *Let G be a locally compact Hausdorff topological group. Since $\mathbb{R}^\times$ is abelian Δ_G factors through $G/[G,G]$:*

$$\widetilde{\Delta_G} : G/[G,G] \to \mathbb{R}^\times$$

It then follows as in Proposition 22.2(3) that if $G/[G,G]$ is compact then that $\Delta(G) = \{1\}$.

We also have the following result:

Proposition 22.4. *(**Change of Variables**) Let G be a locally compact Hausdorff group, and let Δ_G denote the modular function. If f is μ_G-integrable, we have*

$$\int_G f(x^{-1})\Delta_G(x^{-1})\, d\mu_G(x) = \int_G f(x)\, d\mu_G(x)$$

Proof. We define another Haar measure by using μ_G. Then we show that it is the same as μ, and as a consequence, we obtain the desired result. Assume that $f \in C_c(G)$ and let $I(f)$ denote the LHS. By Proposition 22.2(1),

$$\begin{aligned}
I(L_z f) &= \int_G f(z^{-1}x^{-1})\, \Delta_G(x^{-1})\, d\mu_G(x) \\
&= \int_G f((xz)^{-1})\, \Delta_G(x^{-1})\, d\mu_G(x) \\
&= \Delta_G(z^{-1}) \int_G f(x^{-1})\, \Delta_G((xz^{-1})^{-1})\, d\mu_G(x) \\
&= \int_G f(x^{-1})\, \Delta_G(x^{-1})\, d\mu_G(x) \\
&= I(f).
\end{aligned}$$

The conclusion can be extended to $C_0(G)$ by a density argument. By Remark 21.12, the measure associated to I is left-invariant, and consequently a Haar measure. Hence, there exists a $c > 0$ such that

$$\int_G f(x^{-1})\Delta_G(x^{-1})\, d\mu_G(x) = I(f) = c\int_G f(x)\, d\mu_G(x)$$

We show that $c = 1$. Choose a symmetric open set V of e_G with $|1 - \Delta(s)| < \epsilon$ for every $s \in V$. Choose a non-zero symmetric function $f \in C_c^+(V)$. Then,

$$\begin{aligned}
|1-c| \int_G f(g)\, d\mu_G(x) &= \left| \int_G f(g)\, d\mu_G(x) - I(f) \right| \\
&\leq \int_G \left| f(g) - f(g^{-1})\Delta(g^{-1}) \right| d\mu_G(x) \\
&= \int_V f(g) \left| 1 - \Delta(g^{-1}) \right| d\mu_G(x) \\
&< \epsilon \int_G f(g)\, d\mu_G(x)
\end{aligned}$$

Hence, $|1 - c| < \epsilon$, and as $\epsilon > 0$ was arbitrary, it follows $c = 1$. $\square$

Remark 22.5. *Proposition 22.4 can be stated as implying that*

$$d\mu_G(x^{-1}) = \Delta_G(x^{-1})\, d\mu_G(x)$$

23. Basic Representation Theory

We have argued that classical harmonic analysis can be understood from the perspective of representation theory. Since we aim to extend harmonic analysis to locally compact groups, we now review the fundamentals of representation theory in the context of locally compact groups in this section

Remark 23.1. *Let* $\mathbb{K} = \mathbb{R}, \mathbb{C}$.

Definition 23.2. Let G be a locally compact group. A **unitary representation** is a $\mathbb{K}$-Hilbert space $\mathscr{H}$ with a homomorphism

$$\pi : G \to \mathrm{U}(\mathscr{H})$$

that is continuous with respect to the the strong operator topology on $\mathrm{U}(\mathscr{H})$.

Remark 23.3. *More generally, one can consider non-unitary representations of G. That is, continuous group homomorphisms from G to $\mathrm{GL}(V)$, where V is a topological vector space. However, we only consider representations.*

Remark 23.4. *It is customary to refer to either the Hilbert space $\mathscr{H}$ or the homomorphism π as a representation. However, note that it is the pair $(\mathscr{H}, \pi)$ that constitutes the unitary representations.*

Remark 23.5. *Note strong operator topology can be replaced by the apparently less restrictive weak operator topology in Definition 23.2. This is because the weak and strong operator topologies coincide on $\mathrm{U}(\mathscr{H})$. Indeed, if $T_\alpha \to T$ is a weakly convergent net of unitary operators, then for any $u \in H_\pi$,*

$$\|(T_\alpha - T)u\|^2 = \|T_\alpha u\|^2 - 2\,\mathrm{Re}\langle T_\alpha u, Tu\rangle + \|Tu\|^2 = 2\|u\|^2 - 2\,\mathrm{Re}\langle T_\alpha u, Tu\rangle.$$

The last term converges to $2\|Tu\|^2 = 2\|u\|^2$ by weak convergence, so

$$\|(T_\alpha - T)u\| \to 0$$

Example 23.6. Let G be a locally compact group, and let S be a locally compact Hausdorff topological space. Assume that G acts on S. In this case, G also acts on functions on S, by

$$[\pi(x)f](y) = f(x^{-1}y).$$

for $x \in G$ and $y \in S$.

(1) If S has a G-invariant Radon measure μ, then π defines a unitary representation on $L^2(S, \mu)$. Indeed, we have

$$\begin{aligned}
\langle L_x g, L_x f\rangle &= \int_S g(x^{-1}y)f(x^{-1}y)\,d\mu(y) \\
&= \int_S L_x(gf)(y)\,d\mu(y) \\
&= \int_S (gf)(y)\,d\mu(y) \\
&= \int_S g(y)f(y)\,d\mu(y) = \langle g, f\rangle
\end{aligned}$$

(2) If S admits a strongly quasi-invariant measure μ; that is, a Radon measure μ such that $d\mu(xy) = \varphi(x,y)\,d\mu(y)$ for some positive continuous function φ, then $\widetilde{\pi}$ defines a unitary representation on $L^2(S, \mu_\varphi)$ by the rule:

$$[\widetilde{\pi}(x)f](y) = \varphi(x, x^{-1}y)^{-1/2} f(x^{-1}y)$$

We have $\widetilde{\pi}(xy) = \widetilde{\pi}(x)\widetilde{\pi}(y)$ because the chain rule for Radon-Nikodym derivatives gives

$$\varphi(xx', y) = \varphi(x, x'y)\varphi(x', y)$$

It can be checked that $\widetilde{\pi}$ is unitary.

The continuity of π in both cases results from properties about locally compact groups.

Example 23.7. Let G be a locally compact group. Let's consider two special case of Example 23.6.

(1) Left translations yield the left regular representation π_L of G on $L^2(G; \mu_G)$:

$$[\pi_L(x)f](y) = L_x f(y) = f(x^{-1}y)$$

for $x, y \in G$ and $f \in L^2(G; \mu_G)$. Here μ_G is the left-invariant Haar measure.

(2) Right translations yield the right regular representation π_R of G on $L^2(G; \rho_G)$:

$$[\pi_R(x)f](y) = f(yx)$$

for $x, y \in G$ and $f \in L^2(G; \rho_G)$. Here ρ_G is a right-invariant Haar measure on G.

(3) Right translations yield a representation $\widetilde{\pi}_R$ of G on $L^2(G; \mu_G)$:

$$[\pi_R(x)f](y) = \sqrt{\Delta(x)} f(yx)$$

for $x, y \in G$ and $f \in L^2(G; \mu_G)$. This follows from an argument similar to that of Example 23.6(2).

Definition 23.8. Let G be a locally compact group and let $(\pi_1, \mathscr{H}_1)$ and $(\pi_2, \mathscr{H}_2)$ be two unitary representations. An **intertwining map** for $(\pi_1, \mathscr{H}_1)$ and $(\pi_2, \mathscr{H}_2)$ is a linear map $T : \mathscr{H}_1 \to \mathscr{H}_2$ such that

$$\pi_2(x) \circ T = T \circ \pi_1(x) \quad \text{for all } x \in G.$$

The set of intertwining operators is denoted as $\mathrm{Hom}(\pi_1, \pi_2)$.

Remark 23.9. *We say $(\pi_1, \mathscr{H}_1)$ and $(\pi_2, \mathscr{H}_2)$ are unitarily equivalent if T is a unitary operator. In this case,*

$$T^* \circ \pi_2(x) \circ T = \pi_1(x)$$

for all $x \in G$.

We introduce more definitions.

Definition 23.10. Let G be a locally compact group and let $(\pi, \mathscr{H})$ be a unitary representation. An **invariant subspace** M is a closed subspace of $\mathscr{H}$ such that $\pi(x)M \subseteq M$ for all $x \in G$.

If $M \subseteq \mathscr{H}$ is an invariant subsapce and $M \neq \{0\}$, the restriction of π to M,

$$\pi_M(x) = \pi(x)|_M,$$

defines a representation of G on M, called a subrepresentation of π.

Proposition 23.11. *Let G be a locally compact group and let $(\pi, \mathscr{H})$ be a unitary representation. If M is an invariant subspace, then so is $M^\perp$.*

Proof. If $u \in M$ and $v \in M^{\perp}$, then

$$\langle \pi(x)v, u\rangle = \langle v, \pi(x^{-1})u\rangle = 0$$

so $\pi(x)v \in M^{\perp}$ for all $x \in G$. $\square$

Recalling the definition of a direct sum of representation, we get the following corollory:

Corollary 23.12. *Let G be a locally compact group and let $(\pi, \mathscr{H})$ be a unitary representation. If π has a non-trivial invariant subspace (i.e., $M \neq \{0\}, \mathscr{H}$), then*

$$(\pi, \mathscr{H}) = (\pi_M, M) \oplus (\pi_{M^{\perp}}, M^{\perp})$$

This motivates the following definition.

Definition 23.13. Let G be a locally compact group and let $(\pi, \mathscr{H})$ be a unitary representation. $(\pi, \mathscr{H})$ is an **irreducible representation** if $\mathscr{H}$ admits no non-trivial invariant subspace. Otherwise, $(\pi, \mathscr{H})$ is a **reducible representation**.

We end by giving some results regarding reducibility of representations.

Proposition 23.14. *Let G be a locally compact group and let $(\pi, \mathscr{H})$ be a unitary representation.*

(1) *Let M be a closed subspace of $\mathscr{H}$ and let P_M be the orthogonal projection onto M. Then M is an invariant subspace if and only if $P|_M \in \mathrm{Hom}(\pi, \pi)$.*
(2) ***(Schur's Lemma I)*** *$(\pi, \mathscr{H})$ is irreducible if and only if $\mathrm{Hom}(\pi, \pi)$ contains only scalar multiples of the identity*
(3) ***(Schur's Lemma II)*** *Let $(\pi_1, \mathscr{H}_1)$ and $(\pi_2, \mathscr{H}_2)$ be irreducible unitary representations of G. If $(\pi_1, \mathscr{H}_1)$ and $(\pi_2, \mathscr{H}_2)$ are unitarily inequivalent, then $\mathrm{Hom}(\pi_1, \pi_2) = \{0\}$.*
(4) ***(Schur's Lemma III)*** *Let $(\pi_1, \mathscr{H}_1)$ and $(\pi_2, \mathscr{H}_2)$ be irreducible unitary representations of G. If $(\pi_1, \mathscr{H}_1)$ and $(\pi_2, \mathscr{H}_2)$ are unitarily equivalent, then $\mathrm{Hom}(\pi_1, \pi_2)$ is 1-dimensional.*

Proof. The proof is given below:

(1) If $P|_M \in \mathrm{Hom}(\pi, \pi)$ and $v \in M$, then

$$\pi(x)v = \pi(x)P|_M v = P|_M \pi(x)v \in M$$

for ll $x \in G$, so M is invariant. Conversely, if M is invariant, we have

$$\pi(x)P|_M v = \pi(x)v = P|_M \pi(x)v$$

for all $x \in G$ and $v \in M$. Moreover,

$$\pi(x)P|_M v = 0 = P|_M \pi(x)v$$

for all $x \in G$ and $v \in M^{\perp}$. Hence, $\pi(x)P|_M = P|_M \pi(x)$ for all $x \in G$. Hence, M is invariant.

(2) If $(\pi, \mathscr{H})$ is reducible, then $\mathrm{Hom}(\pi, \pi)$ contains a non-trivial projection by (1) which is not a scalar multiple of the identity. Conversely, let $(\pi, \mathscr{H})$ be irreducible and assume $T \in \mathrm{Hom}(\pi, \pi)$ is not a scalar multiple of identity. WLOG, assume T is self-adjoint. Otherwise, write T as a sum of self-adjoint operators. Since $\pi(x)$ commutes with T for all $x \in G$, then T commutes with all the projections generated by Borel functional calculus (**??**) for T. Thus $\mathrm{Hom}(\pi, \pi)$ contains non-trivial projections, contradicting that $(\pi, \mathscr{H})$ be irreducible by (1).

(3) If $T \in \operatorname{Hom}(\pi_1, \pi_2)$, then $T^* \in \operatorname{Hom}(\pi_2, \pi_1)$. It follows that $T^*T \in \operatorname{Hom}(\pi_1, \pi_1)$, so $T^*T = cI$ for some $c \in \mathbb{K}$ by (2). Hence, $TT^* = \bar{c}I$. Hence, either $T = 0$ or $c^{-1/2}T$ is unitary. Since $(\pi_1, \mathscr{H}_1)$ and $(\pi_2, \mathscr{H}_2)$ are inequivalent, only the former can be true. Hence, $\operatorname{Hom}(\pi_1, \pi_2) = \{0\}$

(4) Since $(\pi_1, \mathscr{H}_1)$ are equivalent, the argument in (3) shows that $\operatorname{Hom}(\pi_1, \pi_2)$ consists of scalar multiples of unitary operators. If $T_1, T_2 \in \operatorname{Hom}(\pi_1, \pi_2)$, then

$$T_2^{-1}T_1 = T_2^*T_1 \in \operatorname{Hom}(\pi_1, \pi_1),$$

so $T_2^{-1}T_1 = cI$ and $T_1 = cT_2$, so $\dim \operatorname{Hom}(\pi_1, \pi_2) = 1$.

This completes the proof. □

We have the following corollary:

Corollary 23.15. *Every irreducible unitary representation of a locally compact abelian group is one-dimensional.*

Proof. Let $(\pi, \mathscr{H})$ be an irreducible unitary representation. The operators $\pi(x)$ all commute with one another and so belong to $\operatorname{Hom}(\pi, \pi)$. By Proposition 23.14, we have $\pi(x) = c_x I$ for each $x \in G$. But then every one-dimensional subspace of $\mathscr{H}$ is invariant, so $\dim \mathscr{H} = 1$. □

24. Group Algebras

24.1. **Convolution Algebra.** The existence of a Haar measure on a locally compact Hausdorff group allows for the definition of a convolution operation, a fundamental tool in classical harmonic analysis, on the space $C_c(G)$ of complex-valued continuous functions with compact support. This operation endows $C_c(G)$ with an associative algebra structure, making it a natural object of study in functional analysis.

Definition 24.1. Let G be a locally compact Hausdorff group. If $f, g \in C_c(G)$, the **convolution** is defined as

$$f * g(x) = \int_G f(y)g(y^{-1}x)\, d\mu_G(y),$$

Remark 24.2. *The following identity is sometimes useful:*

$$R_y(f * g)(x) = \int_G f(z)g(z^{-1}xy)\, d\mu_G(z) = \int_G f(z)(R_y g)(z^{-1}x)\, d\mu_G(z) = f * (R_y g)(x).$$

Similarly, we have $L_y(f * g) = (L_y f) * g$.

$C_c(G)$ can be given the structure of a $*$-algebra.

Lemma 24.3. *Let G be a locally compact group. Then $C_c(G)$ is a $*$-algebra with multiplication operation is given by convolution and involution given by*

$$f^*(x) = \Delta_G(x)\overline{f(x^{-1})}$$

Proof. (Sketch) Let $f, g \in C_c(G)$. We claim that $f * g \in C_c(G)$. We first show that $f * g$ is continuous. It suffices to show continuity at $e_G \in G$. Let $\varepsilon > 0$. If $K = \operatorname{supp}(f)$, let $M = \sup_{g \in K} |f(x)|$. Since g is continuous at e_G, then there exists an open set of $e_G \in U$ for all $x \in U$, we have $|g(x) - g(e_G)| < \frac{\varepsilon}{M\mu_G(K)}$. We have

$$\begin{aligned}|f*g(x)-f*g(e_G)| &= \left|\int_G f(y)g(y^{-1}x)\,d\mu_G(y) - \int_G f(y)g(y^{-1})\,d\mu_G(y)\right| \\ &\leq \int_G |f(y)|\left|g(y^{-1}x)-g(y^{-1})\right|\,d\mu_G(y) \\ &\leq \frac{\epsilon}{M\mu_G(K)}\int_K |f(y)|\,dy \leq \epsilon.\end{aligned}$$

Hence, $f*g$ is continuous. We now show that $f*g$ is compactly supported. Let $L = \mathrm{supp}(g)$. We have

$$f*g(x) = \int_G f(y)g(y^{-1}x)d\mu_G(y) = \int_K f(y)g(y^{-1}x)d\mu_G(y).$$

If $y^{-1}x \notin L$, then $g(y^{-1}x) = 0$. For $y \in K$, note that if $x \notin LK$, then $g(y^{-1}x) = 0$. Therefore, $f*g(x)$ is compactly supported on LK. It is easy to check that we have

$$\begin{aligned}(f*g)*h &= f*(g*h) \\ f*(g+h) &= f*g+f*h \\ (f+g)*h &= f*h+g*h \\ (f^*)^* &= f \\ (f+g)^* &= f^*+g^*\end{aligned}$$

We only verify that $(f*g)^* = g^* * f^*$ for all $f,g \in C_c(G)$. We have

$$(f*g)^*(x) = \overline{(f*g)(x^{-1})}\Delta_G(x) = \Delta_G(x)\int_G \overline{f(y)g(y^{-1}x^{-1})}\,d\mu_G(y)$$

Similarly, we have

$$\begin{aligned}g^* * f^*(x) &= \int_G g^*(y)f^*(y^{-1}x)\,d\mu_G(y) \\ &= \int_G \overline{g(y^{-1})}\Delta_G(y)\overline{f(x^{-1}y)}\Delta_G(y^{-1}x)\,d\mu_G(y) \\ &= \Delta_G(x)\int_G \overline{g(y^{-1})f(x^{-1}y)}\,d\mu_G(y) = \Delta_G(x)\int_G \overline{(y^{-1}x^{-1})f(y)}\,d\mu_G(y).\end{aligned}$$

This completes the proof. □

For a locally compact Hausdorff group, recognizing that $C_c(G)$ naturally forms a $*$-algebra is an important step in making use of functional analytic techniques. However, infinite-dimensional $*$-algebras are very hard to understand if they are not equipped with a suitable topology. Fortunately, we can consider $L^1(G)$, which can be thought of as the completion of $C_c(G)$ under the L^1 norm, and argue that the convolution and $*$-operation induce a Banach $*$-algebra structure on $L^1(G)$. We first once again state the definition of convolution.

Definition 24.4. Let G be a locally compact Hausdorff group. If $f,g \in L^1(G)$, the **convolution** is defined as

$$f*g(x) = \int_G f(y)g(y^{-1}x)\,d\mu_G(y),$$

whenever the integral exists.

We can now show that $L^1(G)$ is a Banach $*$-algebra.

Proposition 24.5. *Let G be a locally compact group. Then $L^1(G)$ is a Banach $*$-algebra with multiplication operation is given by convolution and involution given as in Lemma 24.3. $L^1(G)$ is called rhe convolution algebra.*

Proof. (Sketch) It can be checked that $L^1(G)$ is a Banach space. Let $f, g \in L^1(G)$. We first show that Then $f * g$ is well-defined almost everywhere and it satisfies

$$\|f * g\|_1 \leq \|f\|_1 \|g\|_1$$

If $f, g \in C_c(G)$, Fubini's theorem shows that the function $G \times G \ni (x, y) \mapsto f(y)g(y^{-1}x)$ satisfies

$$\begin{aligned}\int_G \int_G |f(y)g(y^{-1}x)|\, d\mu_G(y)\, d\mu_G(x) &= \int_G \int_G |f(y)g(y^{-1}x)|\, d\mu_G(x)\, d\mu_G(y) \\ &= \int_G |f(y)g(x)|\, d\mu_G(x)\, d\mu_G(y) \\ &= \|f\|_1 \|g\|_1 < \infty.\end{aligned}$$

Therefore,

$$\|f * g\|_1 \leq \|f\|_1 \|g\|_1$$

for all $f, g \in C_c(G)$. By a density argument, the inequality holds for all $f, g \in L^1(G)$ as well. Similarly, a density argument shows that involution on $C_c(G)$ extends to $L^1(G)$. All other identities that hold at the $C_c(G)$ level continue to hold at the $L^1(G)$ level. Hence, $L^1(G)$ is a Banach $*$-algebra. ☐

The integral defining $f * g(x)$ can be expressed in several different forms:

$$\begin{aligned}f * g(x) &= \int_G f(y)g(y^{-1}x)\, d\mu_G(y) \\ &= \int_G f(xy)g(y^{-1})\, d\mu_G(y) \\ &= \int_G f(y^{-1})g(yx)\Delta_G(y^{-1})\, d\mu_G(y) \\ &= \int_G f(xy^{-1})g(y)\Delta_G(y^{-1})\, d\mu_G(y)\end{aligned}$$

The equality of the integrals follows from the substitutions $y \to xy$ and $y \to y^{-1}$ according to Proposition 22.4. Let's now look at some properties of $L^1(G)$.

Proposition 24.6. *Let G be a locally compact group. $L^1(G)$ is abelian if and only if G is abelian.*

Proof. If $L^1(G)$ is abelian by Proposition 22.4 and an alternative expression for the convolution as written above, we have

$$\begin{aligned}0 &= f * g(x) - g * f(x) \\ &= \int_G f(xy)g(y^{-1})\, d\mu_G(y) - \int_G g(y)f(y^{-1}x)\, d\mu_G(y) \\ &= \int_G g(y)[\Delta_G(y^{-1})f(xy^{-1}) - f(y^{-1}x)]\, d\mu_G(y)\end{aligned}$$

for all $f, g \in L^1(G)$ and $x \in X$. If $f \in C_c^+(G)$ in particular, consider

$$g(y) = \overline{\Delta_G(y^{-1})(f(xy^{-1}) - f(y^{-1}x)})$$

We can then conclude that

$$\Delta_G(y^{-1})f(xy^{-1}) = f(y^{-1}x)$$

holds for all $f \in C_c^+(G)$. Let $x = e_G$. Then

$$\Delta_G(y)f(y) = f(y)$$

for each $y \in G$ and $f \in C_c^+(G)$. By Urysohn's lemma, for each $y \in G$ we choose $f_y \in C_c^+(G)$ such that $f_y(y) = 1$. Hence, $\Delta_G \equiv 1$. Hence,

$$f(xy^{-1}) = f(y^{-1}x) \Rightarrow f(xy) = f(yx)$$

for each $f \in C_c^+(G)$ and $x, y \in G$. If $xy \neq yx$, by Urysohn's lemma, there exists $f \in C_c^+(G)$ such that $f(xy) = 1$ and $f(yx) = 0$. Hence, we must have $xy = yx$ for each $x, y \in G$. Hence, G is abelian. The converse is clear. □

We shall shortly prove that $L^1(G)$ has a multiplicative unit if and only if G is discrete. The concept of a Dirac function and Dirac net is useful here.

Definition 24.7. Let G be a locally compact group with Haar measure μ_G. A **Dirac function** on G is a non-negative element $\varphi \in L^1(G)$ such that

$$\int_G \varphi(x)\, d\mu_G(x) = 1$$

If $\{U_i\}_{i \in I}$ is a direct set of open sets containing e_G, then a **Dirac net** is a family $(\varphi_i)_{i \in I}$ of Dirac functions.

Remark 24.8. *A Dirac net is also called an approximation to the identity.*

Lemma 24.9. *Let G be a locally compact Hausdorff group.*

(1) *If φ and ψ are Dirac functions, then so is their convolution product $\varphi * \psi$.*
(2) *If $f \in L^1(G)$ and $(\varphi_i)_{i \in I}$ is a Dirac net, then $(f * \varphi_i)_{i \in I}$ and $(\varphi_i * f)_{i \in I}$ converge to f in $L^1(G)$.*

Proof. The proof is given below:

(1) Clearly, $\varphi * \psi$ is non-negative. We have

$$\begin{aligned}
\int_G \varphi * \psi(x)\, d\mu_G(x) &= \int_G \int_G \varphi(y)\psi(y^{-1}x)\, d\mu_G(y)\, d\mu_G(x) \\
&= \int_G \int_G \varphi(y)\psi(y^{-1}x)\, d\mu_G(x)\, d\mu_G(y) \\
&= \int_G \varphi(y)\, d\mu_G(y) \int_G \psi(x)\, d\mu_G(x) = 1
\end{aligned}$$

(2) Consider $(\varphi_i * f)_{i \in I}$. Let $\varepsilon > 0$ and $g \in C_c(G)$. Fix a compact neighborhood $e_G \in K \subseteq G$. Since g is uniformly continuous (Lemma 20.6), there exists an open symmetric set $e_G \in V \subseteq K$ such that

$$|g(y^{-1}x) - g(x)| < \frac{\epsilon}{\mu_G(K \operatorname{supp} g)}$$

for $x \in G$ and $y \in V$. For every $U_i \subseteq V$ we obtain the following estimate:

$$\begin{aligned}\|\varphi_i * g - g\|_1 &\le \int_{K\,\mathrm{supp}\,g}\left(\int_G \varphi_i(y)\,|g(y^{-1}x)-g(x)|d\mu_G(y)\right)d\mu_G(x)\\ &\le \mu(K\,\mathrm{supp}\,g)\cdot\frac{\epsilon}{\mu(K\,\mathrm{supp}\,g)} = \epsilon.\end{aligned}$$

This shows that

$$(\varphi_i)_{i\in I} * g \to g$$

for all $g \in C_c(G)$. By a density argument, we have

$$(\varphi_i)_{i\in I} * f \to f$$

for all $f \in L^1(G)$ because $\|\varphi_i\|_1 = 1$ for each $i \in I$. Moreover, we have

$$(g * \varphi_i)_{i\in I} = (\varphi_i^* * g^*)_{i\in I}^* \to (g^*)^* = g$$

for all $g \in C_c(G)$ since $(\varphi_i^*)_{i\in I}$ is a Dirac net for the neighborhood basis $\{U_i^{-1} \mid i \in I\}$ of $e_G \in G$. A density argument completes the proof.

This completes the proof. □

We can now prove that $L^1(G)$ has a multiplicative unit if and only if G is discrete.

Proposition 24.10. *Let G be a locally compact Hausdorff group. Then $L^1(G)$ has a multiplicative unit if and only if G is discrete.*

Proof. If G is discrete,

$$(f*\chi_{e_G})(x) = \int_G f(y)\chi_{e_G}(y^{-1}x)\,d\mu_G(y) = f(x) = \int_G \chi_{e_G}(y)f(y^{-1}x)\,d\mu_G(y) = (\chi_{e_G}*f)(x)$$

Conversely, assume that $L^1(G)$ is unital with unit f but G is not discrete. Let $(\varphi_i)_{i\in I}$ be a Dirac net. By Lemma 24.9(2)

$$(\varphi_i)_{i\in I} = (f * \varphi_i)_{i\in I} \to f$$

in $L^1(G)$. Hence, $(\varphi_i)_{i\in I} \to f$ pointwise almost everywhere. Since G is Hausdorff, for any $x \neq e_G$, there exists i_x such that $\varphi_i(x) = 0$ for each open set U_i such that $U_i \subseteq U_{i_x}$. Since G is not discrete, then $\{e_G\}$ has measure zero[16], so $f = 0$ almost everywhere, which is a contradiction. □

We now arrive at the main result of this section.

Proposition 24.11. *Let G be a locally compact Hausdorff group. We have a bijection:*

$$\{\text{Unitary Representations of } G\} \longleftrightarrow \{\text{Non-degenerate } *\text{-representations of } L^1(G)\}$$

Remark 24.12. *Unitary representations of ($*$-)algebras are defined in a manner similar to Definition 23.2. The only difference is that we require the map π to be a ($*$-)algebra homomorphism.*

Remark 24.13. *If $(\widetilde{\pi},\mathscr{H})$ is a unitary representation of a $*$-algebra A, we say that $\widetilde{\pi}$ is non-degenerate if $\overline{\mathrm{Span}(\widetilde{\pi}(A))\mathscr{H}} = \mathscr{H}$. It can be checked that $\widetilde{\pi}$ is non-degenerate if and only if for every $u \in \mathscr{H}$ there exists an $a \in A$ such that $\widetilde{\pi}(a)u \neq 0$.*

We prove Proposition 24.11 in multiple steps.

[16]Verify this.

Proposition 24.14. *Let G be a locally compact Hausdorff group and let $(\pi, \mathscr{H})$. Then*

$$\widetilde{\pi} : L^1(G) \to \mathrm{U}(\mathscr{H})$$
$$f \mapsto \int_G f(x)\pi(x)\, d\mu_G(x)$$

defines a non-degenerate $*$-representation of $L^1(G)$.

Remark 24.15. *Here the integral if the Gelfand-Pettis (weak) integral that is to be interpreted such that if $u, v \in \mathscr{H}$, we define $\widetilde{\pi}(f)u$ by specifying the inner product*

$$\langle \widetilde{\pi}(f)u, v\rangle = \int_G f(x)\langle \pi(x)u, v\rangle\, d\mu_G(x)$$

The integral on the right is to be interpreted as an ordinary integral of the function $f(-)\langle\pi(-)u, v\rangle$ on $L^1(G)$ for fixed $u, v \in \mathscr{H}$. Note that we have

$$\begin{aligned}\langle \widetilde{\pi}(f)u, v\rangle &= \int_G f(x)\langle \pi(x)u, v\rangle\, d\mu_G(x)\\ &= \int_G \langle f(x)\pi(x)u, v\rangle\, d\mu_G(x) = \left\langle \int_G f(x)\pi(x)u\, d\mu_G(x), v\right\rangle.\end{aligned}$$

Since $v \in \mathscr{H}$ is arbitrary, this shows that

$$\widetilde{\pi}(f)u = \int_G f(x)\pi(x)u\, d\mu_G(x)$$

as elements of $\mathscr{H}$.

Proof. For $\widetilde{\pi}$ to be a $*$-algebra representation, we must check that

$$\begin{aligned}\widetilde{\pi}(f+g) &= \widetilde{\pi}(f) + \widetilde{\pi}(g)\\ \widetilde{\pi}(cf) &= c\widetilde{\pi}(f)\\ \widetilde{\pi}(f * g) &= \widetilde{\pi}(f)\widetilde{\pi}(g)\\ \widetilde{\pi}(f^*) &= \widetilde{\pi}(f)^*\end{aligned}$$

for $c \in \mathbb{K}$ and $f, g \in L^1(G)$ and $x \in G$. The first two properties are clear. We check the third and four properties. For $u, v \in \mathscr{H}$, we have,

$$\begin{aligned}\langle \widetilde{\pi}(f*g)u, v\rangle &= \int_G \langle (f*g)(x)\pi(x)u, v\rangle d\mu_G(x)\\ &= \int_G (f*g)(x)\langle \pi(x)u, v\rangle d\mu_G(x)\\ &= \int_G\int_G f(y)g(y^{-1}x)\langle \pi(x)u, v\rangle d\mu_G(y)d\mu_G(x)\\ &= \int_G\int_G f(y)g(y^{-1}x)\langle \pi(x)u, v\rangle d\mu_G(x)\, d\mu_G(y)\\ &= \int_G\int_G f(y)g(x)\langle \pi(y)\pi(x)u, v\rangle d\mu_G(x)\, d\mu_G(y)\\ &= \int_G\int_G f(y)g(x)\langle \pi(x)u, \pi(y)^*v\rangle d\mu_G(x)\, d\mu_G(y)\\ &= \int_G f(y)\langle \widetilde{\pi}(g)u, \pi(y)^*v\rangle d\mu_G(y) = \int_G f(y)\langle \pi(y)\widetilde{\pi}(g)u, v\rangle d\mu_G(y) = \langle \widetilde{\pi}(f)\widetilde{\pi}(g)u, v\rangle.\end{aligned}$$

This proves the third property. We now prove the fourth property.

$$\begin{aligned}
\langle\widetilde{\pi}(f)u, v\rangle &= \int_G f(x)\langle\pi(x)u, v\rangle \, d\mu_G(x) \\
&= \int_G f(x)\langle u, \pi(x)^*v\rangle \, d\mu_G(x) \\
&= \int_G f(x)\overline{\langle\pi(x^{-1})v, u\rangle} \, d\mu_G(x) \\
&= \overline{\int_G \Delta_G(x)\overline{f(x^{-1})}\langle\pi(x)v, u\rangle \, d\mu_G(x)} \\
&= \overline{\int_G f^*(x)\langle\pi(x)v, u\rangle \, d\mu_G(x)} = \overline{\langle\widetilde{\pi}(f^*)v, u\rangle} = \langle u, \widetilde{\pi}(f^*)v\rangle.
\end{aligned}$$

We now check that the representation is non-degenerate. Suppose $\varepsilon > 0$. Since π is strongly continuous, for any $u_1, \cdots, u_n \in \mathscr{H}$ we can choose an open set $e_G \in V$ such that

$$\|\pi(x)u_i - u_i\| < \varepsilon$$

for $x \in V$. If $f \in L^1(G)$ such that $\|f\|_1 = 1$, then

$$\begin{aligned}
|\langle\widetilde{\pi}(f)u_i - u_i, v\rangle| &= \left|\int_G f(x)\langle\pi(x)u_i, v\rangle \, d\mu_G(x) - \langle u_i, v\rangle\right| \\
&= \left|\int_G f(x)\langle\pi(x)u_i - u_i, v\rangle \, d\mu_G(x)\right| \\
&\leq \int_G f(x) \, |\langle\pi(x)u_i - u_i, v\rangle| \, d\mu_G(x) \\
&\leq \epsilon\|v\|.
\end{aligned}$$

for each $v \in V$. This shows that $\|\widetilde{\pi}(f)u_i - u_i\| \leq \varepsilon$. Hence, If $(\varphi_i)_{i\in I}$ is a Dirac net, we have

$$(\widetilde{\pi}(\varphi_i))_{i\in I} \to \mathrm{Id}_{\mathscr{H}}$$

strongly. Hence, $\operatorname{Span}\widetilde{\pi}(L^1(G))\mathscr{H} \subseteq \mathscr{H}$ is dense in $\mathscr{H}$. This shows that $\widetilde{\pi}$ is non-degenerate.

□

Remark 24.16. *The $*$-representation $\widetilde{\pi}$ constructed in Proposition 24.14 is called the integrated representation of π.*

Proposition 24.17. *Let G be a locally compact group. If $(\widetilde{\pi}, \mathscr{H})$ is a non-degenerate $*$-representation of $L^1(G)$, then unitary representation $(\pi, \mathscr{H})$ of G.*

Proof. The intuition is that $\pi(x)$ should be the limit of $\widetilde{\pi}(f)$ as f approaches the δ-function at $x \in G$. Let $(\varphi_i)_{i\in I}$ be a Dirac net. Recall that if $f \in L^1(G)$, we have

$$(\varphi_i * f)_{i\in I} \to f \implies (L_x(\varphi_i * f))_{i\in I} = ((L_x\varphi_i) * f)_{i\in I} \to L_x f$$

in $L^1(G)$ for any $x \in G$. Hence,

$$(\widetilde{\pi}(L_x\varphi_i)\widetilde{\pi}(f)v)_{i\in I} \to \widetilde{\pi}(L_x f)v \tag{5}$$

for all $v \in \mathscr{H}$. Consider

$$\mathscr{D} = \operatorname{Span}\{\widetilde{\pi}(f)v \mid f \in L^1(G) \; v \in \mathscr{H}\} \subseteq \mathscr{H}$$

Note that $\overline{\mathscr{D}} = \mathscr{H}$. Indeed, if $u \perp \mathscr{D}$, then

$$0 = \langle u, \widetilde{\pi}(f)v\rangle = \langle \widetilde{\pi}(f^*)u, v\rangle$$

for all $v \in \mathscr{H}$ and $f \in L^1(G)$. Hence $u = 0$ since π is non-degenerate. From Equation (5) $(\widetilde{\pi}(L_x\varphi_i))_{i\in I}$ converges strongly on $\mathscr{D}$ to an operator $\pi(x)$

$$\begin{aligned} \pi(x) : \mathscr{D} &\to \mathscr{D} \\ \widetilde{\pi}(f)v &\mapsto \widetilde{\pi}(L_x f)v \end{aligned}$$

for all $x \in \mathscr{D}$. We show that $\widetilde{\pi}(x)$ is a unitary operator on $\mathscr{D}$. We will use the identity that

$$g^* * (L_x f) = (L_x^{-1} g)^* * f$$

for all $f, g \in L^1(G)$. Note that we have

$$\begin{aligned} \left\| \sum_i \widetilde{\pi}(L_x f_i)v_i \right\|^2 &= \sum_{i,j} \langle \widetilde{\pi}(L_x f_i)v_i, \widetilde{\pi}(L_x f_j)v_j\rangle \\ &= \sum_{i,j} \langle \widetilde{\pi}(L_x f_j)^* \widetilde{\pi}(L_x f_i)v_i, v_j\rangle \\ &= \sum_{i,j} \langle \widetilde{\pi}((L_x f_j)^* * L_x f_i)v_i, v_j\rangle \\ &= \sum_{i,j} \langle \widetilde{\pi}((L_x^{-1} L_x f_j)^* * f_i)v_i, v_j\rangle \\ &= \sum_{i,j} \langle \widetilde{\pi}(f_j^* * f_i)v_i, v_j\rangle \\ &= \sum_{i,j} \langle \widetilde{\pi}(f_i)v_i, \widetilde{\pi}(f_j)v_j\rangle = \left\| \sum_i \widetilde{\pi}(f_i)v_i \right\|^2 \end{aligned}$$

This shows that $\widetilde{\pi}(x)$ is a unitary operator for each $x \in G$. Since $\mathscr{D}$ is dense in $\mathscr{H}$, $\widetilde{\pi}(x)$ extends to a unitary operator on $\mathscr{H}$. We can now define the candidate representation:

$$\begin{aligned} \pi : G &\to \mathrm{U}(\mathscr{H}) \\ x &\mapsto \pi(x) \end{aligned}$$

It is clear that π satisfies the required algebraic properties for π to be a representation. We show that π is strongly continuous. It suffices to verify continuity on $\mathscr{D}$. Let $\varepsilon > 0$. Recall that since left translations are continuous, we have that for $f_1, \ldots, f_n \in L^1(G)$, there is an open set $e_G \in U \subseteq G$ such that

$$\|L_x f_i - f_i\|_1 < \frac{\epsilon}{n}$$

for all $i = 1, \cdots, n$ and all $x \in U$. Moreover, if $u_1, \cdots, u_n \in \mathscr{H}$ are unit vectors we have that

$$\begin{aligned}\left\|\pi(x)\sum_i \pi(f_i)v_i - \sum_i \pi(f_i)v_i\right\|_1 &\le \sum_i \|\pi(L_x f_i - f_i)v_i\|_1 \\ &\le \sum_i \|L_x f_i - f_i\|_1 \\ &\le \frac{\epsilon}{n}\cdot n = \epsilon.\end{aligned}$$

This proves the claim. □

Remark 24.18. *The representation π constructed in Proposition 24.17 is called the disintegrated representation of $\widetilde{\pi}$.*

We can now prove Proposition 24.11.

Proof. (Proposition 24.11) It suffices to argue that the disintegrated representation π constructed in Proposition 24.17 from $\widetilde{\pi}$ is unique and it integrates to $\widetilde{\pi}$. We first show that π integrates to $\widetilde{\pi}$. Denote by $\widetilde{\pi}'$ the integrated representation of π. We show that $\widetilde{\pi}' = \widetilde{\pi}$. It suffices to show the equality on

$$\mathscr{D} = \mathrm{Span}\{\widetilde{\pi}(f)v \mid f \in L^1(G)\ v \in \mathscr{H}\} \subseteq \mathscr{H}$$

Hence, it suffices to show that

$$\widetilde{\pi}'(f)\widetilde{\pi}(g)v = \widetilde{\pi}(f)\widetilde{\pi}(g)v = \widetilde{\pi}(f * g)v$$

for $f, g \in L^1(G)$ and $v \in \mathscr{H}$. We have

$$\begin{aligned}\widetilde{\pi}'(f)\widetilde{\pi}(g)v &= \int_G f(x)\pi(x)\widetilde{\pi}(g)v\, d\mu_G(x) \\ &= \int_G f(x)\widetilde{\pi}(L_x g)v\, d\mu_G(x) \\ &= \int_G \widetilde{\pi}(f(x)L_x g)v\, d\mu_G(x) \\ &= \widetilde{\pi}\left(\int_G f(x)L_x g\, dx\right)v \\ &= \widetilde{\pi}(f * g)v.\end{aligned}$$

In the second last step we use have used the last comment made in Remark 24.15. This shows that $\widetilde{\pi}' = \widetilde{\pi}$. We now show uniqueness. Suppose $\widehat{\pi}$ is another disintegrated unitary representation of $\widetilde{\pi}$ that integrates to $\widetilde{\pi}$. That is,

$$\int_G f(x)\pi(x) d\mu_G(x) = \widetilde{\pi}(f) = \int_G f(x)\widehat{\pi}(x) d\mu_G(x)$$

for all $f \in L^1(G)$. In particular, we have

$$\int_G f(x)\langle v, \pi(x)u\rangle d\mu_G(x) = \widetilde{\pi}(f) = \int_G f(x)\langle v, \widehat{\pi}(x)v\rangle d\mu_G(x)$$

for all $u, v \in \mathscr{H}$. This implies that

$$\langle v, \pi(x)u\rangle = \langle v, \widehat{\pi}(x)v\rangle$$

for all $u, v \in \mathscr{H}$, which in turn implies that $\pi(x) = \widehat{\pi}(x)$ for all $x \in G$. □

24.2. **Group C^*-Algebra.** We would like to use the full machinery of Gelfand theory and C^*-algebras to develop abstract harmonic analysis. One obstacle to applying this framework to harmonic analysis is that $L^1(G)$ is not typically a C^*-algebra. Below is a typical example.

Example 24.19. Let $G = \mathbb{Z}$ with the counting measure as the Haar measure. Take $x \in \ell^1(\mathbb{Z})$ such that

$$x_0 = 1, x_1 = x_2 = -1, x_n = 0$$

for $n \neq 0, 1, 2$. We have

$$(x * x)_n = \begin{cases} 3, & n = 0, \\ -1, & n = \pm 2, \\ 0, & \text{otherwise.} \end{cases}$$

This gives $\|x * x\| = 5$ while $\|x\|_2^2 = 9$.

However, we will now construct a C^*-algebra into which $L^1(G)$ has a dense inclusion, thereby enabling the application of powerful techniques from C^*-algebra theory to harmonic analysis on locally compact groups. Note that the left-regular representation, π_L, (Example 23.7) is a natural representation of G onto $L^2(G, \mu_G)$. Let $\widetilde{\pi^L}$ denote the integrate form of π^L.

$$\widetilde{\pi^L} : L^1(G) \to \mathrm{U}(L^2(G))$$

We observe that $\widetilde{\pi^L}$ is injective.

Lemma 24.20. *Let G be a locally compact Hausdorff group and*

$$\widetilde{\pi^L} \colon L^1(G) \to \mathscr{B}(L^2(G))$$

the integrated form of the left-regular representation. Then $\widetilde{\pi^L}$ is injective.

Proof. Observe that by the definition of convolution and integrated representations, we have the identity

$$\widetilde{\pi^L}(f)g = f * g \in L^1(G, \mu_G) \cap L^2(G)$$

for all $f, g \in L^1(G, \mu_G)$. Take $f \in L^1(G, \mu_G)$ such that $\widetilde{\pi^L}(f) = 0$. If $(\varphi_i)_{i \in I}$ is a Dirac net in $L^1(G, \mu_G) \cap L^2(G)$, then

$$0 = (\widetilde{\pi^L}(f)\varphi_i)_{i \in I} = (f * \varphi_i)_{i \in I} \xrightarrow{\|\cdot\|_1} f,$$

This shows that $f = 0$. ☐

Since $\widetilde{\pi^L}$ is injective we can think of $L^1(G, \mu_G)$ as a $*$-subalgebra of $\mathscr{B}(L^2(G))$. The first C^*-algebra we consider is constructed by considering the completion of $\pi^L(L^1(G))$. Its definition is justified by the central role of the regular representation.

Definition 24.21. Let G be a locally compact Hausdorff group and

$$\widetilde{\pi^L} \colon L^1(G) \to \mathscr{B}(L^2(G))$$

the integrated form of the left-regular representation. Then $C^*_{\text{red}}(G) = \overline{\widetilde{\pi^L}(L^1(G))}^{\|\cdot\|}$ is the **reduced group C^*-algebra** of G.

So that we don't always have to choose a specific representation, we often work with another completion $L^1(G)$, leading to the definition of a group C^*-algebra.

Definition 24.22. Let G be a locally compact Hausdorff group. For $f \in L^1(G)$, define the universal C^*-norm by

$$\|f\|_{C^*_{\max}} = \sup\{\|\pi(f)\| \mid \pi \text{ unitary representation of } G\}.$$

The completion of $L^1(G)$ with respect to this norm is called the **group C^*-algebra** of G, and it is denoted by $C^*_{\max}(G)$.

Remark 24.23. *The group C^*-algebra satisfies a universal property.*

We have the following result:

Proposition 24.24. *Let G be a locally compact Hausdorff group. We have a bijection:*

$$\{\text{Unitary representations of } G\} \longleftrightarrow \{\text{Non-degenerate} *\text{-representations of } C^*_{\max}(G)\}$$

Proof. By Proposition 24.11, it suffices to argue that there is a one-to-one correspondence between unitary $*$-representations of $L^1(G)$ and $C^*_{\max}(G)$. This follows from the fact that for every non-degenerate $*$-representation $\pi\colon L^1(G) \to \mathscr{B}(\mathscr{H})$, we have

$$\|\pi(f)\| \leq \|f\|_{C^*_{\max}},$$

so that π uniquely extends to a non-degenerate $*$-representation of $C^*_{\max}(G)$. Vice versa, every non-degenerate $*$-representation of $C^*_{\max}(G)$ restricts to a non-degenerate $*$-representation of its dense subalgebra $L^1(G)$. $\square$

Part 5. C^*-Algebras II

25. States

We discuss states in this section. These will be useful in the GNS construction to be discussed later on.

Definition 25.1. Let A be a C^*-algebra. A **state** on A is a linear functional such that:
(1) φ is positive. That is, $\varphi(A_+) \subseteq \mathbb{R}^+$. Equivalently, $\varphi(a^*a) \geq 0$ for each $a \in A$
(2) The norm of φ is one. That is,

$$\|\varphi\| := \sup\{|\varphi(a)| : \|a\| = 1\} = 1.$$

The subset $S(A) \subseteq A^*_{\leq 1}$ consisting of states is called the **state space**.

Proposition 25.2. *Let A be a C^*-algebra, and let φ be a positive linear functional on A.*
(1) *φ is $*$-preserving. That is,*

$$\varphi(a^*) = \overline{\varphi(a)}$$

(2) *(**Cauchy-Schwartz**) For $a, b \in A$, we have,*

$$|\varphi(b^*a)|^2 \leq \varphi(b^*b)\varphi(a^*a)$$

such that we have equality if and only if $\varphi(a^*b) = \varphi(b^*a)$.
(3) *If A is unital, $\varphi(e) = \|\varphi\|$.*

Proof. The proof is given below:
(1) Note that Proposition 10.5 that it suffices to show that φ maps self-adjoint elements to self-adjoint elements. If $a \in A$ is a self-adjoint element, then $a = a_+ - a_-$ for some $a_\pm \in A_+$. Note that

$$\varphi(a) = \varphi(a_+) - \varphi(a_-) \in \mathbb{R},$$

since φ is a positive linear functional. The claim follows.
(2) Let $\lambda \in \mathbb{C}$. Since φ is positive, it follows that

$$\varphi((\lambda a + b)^*(\lambda a + b)) = |\lambda|^2\varphi(a^*a) + \overline{\lambda}\varphi(a^*b) + \lambda\varphi(b^*a) + \varphi(b^*b) \geq 0.$$

Because this expression must be real for all $\lambda \in \mathbb{C}$, it follows that $\overline{\varphi(a^*b)} = \varphi(b^*a)$. Hence, we get the inequality,

$$|\lambda|^2\varphi(a^*a) + 2\mathrm{Re}(\lambda\varphi(b^*a)) + \varphi(b^*b) \geq 0.$$

Let $\gamma \in \mathbb{S}^1$ such that $\gamma\varphi(b^*a) = |\varphi(b^*a)|$. Given $t \in \mathbb{R}$, put $\lambda = t\gamma$. Hence, we get the inequality,

$$t^2\varphi(a^*a) + 2t|\varphi(b^*a)| + \varphi(b^*b) \geq 0.$$

As we can do this for any $t \in \mathbb{R}$ and this is a real quadratic, for this to be always non-negative we need $b^2 \leq 4ac$, i.e.,

$$4|\varphi(b^*a)|^2 \leq 4\varphi(a^*a)\varphi(b * b)$$

The desired result now follows.
(3) Clearly, $\varphi(e) \leq \|\varphi\|$. Using $b = e$ in (2), we have

$$|\varphi(a)|^2 \leq \varphi(a^*a)\varphi(e) \leq \|\varphi\|\|a^*a\|\varphi(e) = \|\varphi\|\|a\|^2\varphi(e)$$

Taking the supremum over all a of norm one, we get

$$\|\varphi\|^2 \leq \|\varphi\|\varphi(e)$$

It follows that

$$\|\varphi\| \leq \varphi(e).$$

Hence, $\varphi(e) = \|\varphi\|$.

This completes the proof. □

26. Representations of C^*-Algebras

Definition 26.1. Let A be a C^*-algebra. A **representation** of A is a pair, $(\pi, \mathscr{H})$, where $\mathscr{H}$ is a Hilbert space and $\pi : A \to \mathscr{B}(\mathscr{H})$ is a $*$-homomorphism.

Definition 26.2. Let A be a C^*-algebra. Two representations of A, $(\pi_1, \mathscr{H}_1)$, $(\pi_2, \mathscr{H}_2)$, are **unitarily equivalent** if there is a unitary operator $U : \mathscr{H}_1 \to \mathscr{H}_2$ such that

$$U\pi_1(a) = \pi_2(a)U,$$

for all a in A. In this case, we write $(\pi_1, \mathscr{H}_1) \sim_U (\pi_2, \mathscr{H}_2)$ or $\pi_1 \sim_U \pi_2$.

Remark 26.3. *We usually write that π is a representation of A on $\mathscr{H}$.*

An important problem is representation theory is to understand irreducible representations of an abstract mathematical object.

Definition 26.4. Let A be a C^*-algebra and let $(\pi, \mathscr{H})$ be a representation of A. A subspace $\mathscr{N} \subseteq \mathscr{H}$ is said to be **invariant** if $\pi(a)\mathscr{N} \subseteq \mathscr{N}$, for all a in A. The representation $(\pi, \mathscr{H})$ is said to be **irreducible** if the only invariant subspaces of $\{0\}$ and $\mathscr{H}$.

An important technique in representation theory is to construct 'larger' representations from 'smaller' representations by means of algebraic operation.

Definition 26.5. Let A be a C^*-algebra and $(\pi_\iota, \mathscr{H}_\iota)$, $\iota \in I$, be a collection of representations of A. The **direct sum representation** is $(\bigoplus_{\iota\in I} \pi_\iota, \bigoplus_{\iota\in I} \mathscr{H}_\iota)$, where $\bigoplus_{\iota\in I} \mathscr{H}_\iota$ consists of tuples, $x = (x_\iota)_{\iota\in I}$ satisfying $\sum_{\iota\in I} \|x_\iota\|^2 < \infty$ and

$$(\bigoplus_{\iota\in I} \pi_\iota(a)x)_\iota = \pi_\iota(a)x_\iota, \quad \iota \in I.$$

Proposition 26.6. *Let A be a C^*-algebra, and let $(\pi, \mathscr{H})$ be a representation of A. A closed subspace $\mathscr{N} \subseteq \mathscr{H}$ is invariant if and only if $\mathscr{N}^\perp$ is invariant.*

Proof. Assume that $\mathscr{N} \subseteq \mathscr{H}$ is a closed invariant subspace. Let x in $\mathscr{N}^\perp$ $y \in N$ and $a \in A$. We have

$$\langle \pi(a)x, y\rangle = \langle x, \pi(a)^* y\rangle = \langle x, \pi(a^*)y\rangle = 0,$$

since $\pi(a^*)\mathscr{N} \subseteq \mathscr{N}$. Hence, $\pi(a)x\mathscr{N}^\perp$, showing that $\mathscr{N}^\perp$ is invariant. The converse follows from the observation that $(\mathscr{N}^\perp)^\perp = \mathscr{N}$. □

The previous proposition shows that it is possible to define two representations of A by simply restricting the operators to either $\mathscr{N}$ or $\mathscr{N}^\perp$. Moreover, the direct sum of these two representations is unitarily equivalent to the original representation. That is, we have

$$(\pi, \mathscr{H}) \sim_U (\pi|_{\mathscr{N}}, \mathscr{N}) \oplus (\pi|_{\mathscr{N}^\perp}, \mathscr{N}^\perp).$$

We now discuss a characterizion irreducible representations of a C^*-algebra.

Definition 26.7. Let A be a C^*-algebra, and let $(\pi, \mathscr{H})$ be a representation of A. A vector $x \in \mathscr{H}$ is **cyclic** if

$$\overline{\operatorname{span}\{\pi(a)x \mid a \in A\}} = \mathscr{H}.$$

Remark 26.8. *We say that the representation is a cyclic representation if each non-zero vector in $\mathscr{H}$ is a cyclic vector.*

Remark 26.9. *A representation, $(\pi, \mathscr{H})$ of a C^*-algebra A is non-degenerate if the only $x \in \mathscr{H}$ such that $\pi(a)x = 0$ for all a in A is $x = 0$. Otherwise, the representation is degenerate. It can be easily showed that a representation $(\pi, \mathscr{H})$ of a unital C^*-algebra is non-degenerate if and only if $\pi(a) = I_{\mathscr{H}}$ implies that $a = e$.*

Proposition 26.10. *Let A be a C^*-algebra, and let $(\pi, \mathscr{H})$ be a non-degenerate representation of A. $(\pi, \mathscr{H})$ is an irreducible representation if and only if the representation $(\pi, \mathscr{H})$ is cyclic.*

Proof. Assume that $(\pi, \mathscr{H})$ is irreducible. Let x be a non-zero vector, then

$$\text{span}\{\pi(a)x \mid a \in A\}$$

is an invariant subspace and its closure is a closed invariant subspace. If it is 0, then the representation is degenerate, which is impossible. Otherwise, it must be $\mathscr{H}$, meaning that x is a cyclic vector for π. Conversely, suppose that $(\pi, \mathscr{H})$ is non-degenerate, but reducible. Let $\mathscr{N}$ be a proper closed invariant subspace which is neither 0 nor $\mathscr{H}$. If x is any non-zero vector in $\mathscr{N}$, then

$$\text{Span}\{\pi(a)x \mid a \in A\} \subseteq \mathscr{N}$$

Hence, it cannot be dense in $\mathscr{H}$. This shows that $(\pi, \mathscr{H})$ is not a cyclic representation. □

We end this section give a more useful criterion for a representation to be reducible.

Proposition 26.11. *(**Schur's Lemma**) Let A be a C^*-algebra, and let $(\pi, \mathscr{H})$ be a non-degenerate representation of A. Then $(\pi, \mathscr{H})$ is irreducible if and only if the only positive operators which commute with its image are scalar multiplies of the identity operator.*

Proof. First assume that $(\pi, \mathscr{H})$ is a reducible representation. Let $\mathscr{N} \subseteq \mathscr{H}$ be a non-trivial proper closed invariant subspace of $\mathscr{H}$. Let p be the orthogonal projection onto $\mathscr{N}$. We have that $p = p^*$ and $\sigma_A(p) \in \{0, 1\}$, which means that p is positive. We check that it commutes with $\pi(a)$, for any a in A. If $x \in \mathscr{N}$, we know that $\pi(a)x \in \mathscr{N}$ and so

$$(p\pi(a))x = p(\pi(a)x) = \pi(a)x = \pi(a)(px) = (\pi(a)p)x.$$

On the other hand, if x is in $\mathscr{N}^\perp$, then so is $\pi(a)x$ and

$$(p\pi(a))x = p(\pi(a)x) = 0 = \pi(a)(0) = \pi(a)(px) = (\pi(a)p)x.$$

Since every vector in $\mathscr{H}$ is the sum of two as above, we see that

$$p\pi(a) = \pi(a)p$$

for each $a \in A$. As both $\mathscr{N}$ and $\mathscr{N}^\perp$ are non-empty, this operator is not a scalar multiple of the identity operator on $\mathscr{H}$. Conversely, suppose that p is a positive that is not a scalar multiple of the identity operator on $\mathscr{H}$, but commutes with every element of $\pi(a)$ for each $a \in A$. We must have that

$$\sigma_A(p) = \{0, 1\}$$

We may then find non-zero continuous functions[17] f, g on $\sigma_A(p)$ whose product is zero. By CFC, $f(p)$ and $g(p)$ are well-defined operators in $\mathscr{B}(\mathscr{H})$. Since f is non-zero, the operator $\Phi_p(f)$ is non-zero. Let $\mathscr{N}$ denote the closure of its range, which is a non-zero subspace of $\mathscr{H}$. On the other hand, $\Phi_p(g)$ is also a non-zero operator, but it is zero on the range of

[17]Use Uryshon's lemma or simply consider bump functions.

$\Phi_p(f)$ and hence on N. This implies that $\mathscr{N}$ is a proper subspace of H. Note that $\Phi_a(f)$ commutes with $\pi(a)$ for each $a \in A$. Let $a \in A$. Indeed, for any $\epsilon > 0$, we may find a polynomial $q(x) \in C(\sigma_A(p))$ such that

$$\|f - q\|_\infty < \epsilon$$

in $C(\sigma_A(p))$, This means that

$$\|\Phi_p(f) - \Phi_p(q)\| < \epsilon$$

It is clear that $\Phi_p(q)$ will commute with $\pi(a)$, since p commutes with $\pi(a)$. A triangle inequality type argument now shows tht $\Phi_p(f)$ must also commute with $\pi(a)$. Finally, we claim that $\mathscr{N}$ is invariant under $\pi(a)$. In fact, it suffices to check that the range of $\Phi_p(f)$ is invariant. But if $x \in \mathscr{H}$, we have

$$\pi(a)(\Phi_p(f)x) = \pi(a)\Phi_p(f)x = \Phi_p(f)\pi(a)x \in \Phi_p(f)\mathscr{H},$$

Hence, $\mathscr{N}$ is a proper, closed invariant subspace of $\mathscr{H}$. This completes the proof. □

27. GNS Construction & Theorem

27.1. **GNS Construction.** In this section, we explicitly construct a representation of a given C^*-algebra. The basic idea is that multiplication allows one to see the elements of a C^*-algebra acting as linear transformations of itself. The problem is, of course, that the C^*-algebra does not usually have the structure of a Hilbert space. To produce an inner product or bilinear form, we use the linear functionals on the C^*-algebra in a clever way, leading to the Gelfand-Naimark-Segal (GNS) construction. First consider the following example.

Example 27.1. Let A be a unital C^*-algebra, and $(\pi, \mathscr{B}(\mathscr{H}))$ be a non-degenerate representation of A. Let $x \in \mathscr{H}$ such that $\|x\| = 1$, and consider the linear functional

$$\varphi(a) := \langle \pi(a)x, x\rangle$$

on A. Note that,

$$\begin{aligned}
\varphi(a^*a) &= \langle \pi(a^*a)x, x\rangle \\
&= \langle \pi(a^*)\pi(a)x, x\rangle \\
&= \langle \pi(a)^\dagger \pi(a)x, x\rangle \\
&= \langle \pi(a)x, \pi(a)x\rangle \\
&\geq 0
\end{aligned}$$

Here † denote the Hilbert space adjoint operator. Moreover, note that,

$$\begin{aligned}
\|\varphi(a)\| &= |\langle \pi(a)x, x\rangle| \\
&\leq \|\pi(a)x\|_{\mathscr{H}}\|x\|_{\mathscr{H}} \\
&\leq \|\pi(a)\|_{\mathscr{B}(\mathscr{H})}\|x\|_{\mathscr{H}}\|x\|_{\mathscr{H}} \\
&= \|\pi(a)\|_{\mathscr{B}(\mathscr{H})}\|x\|^2_{\mathscr{H}} \\
&\leq \|a\|_A\|x\|^2_{\mathscr{H}} = \|a\|_A
\end{aligned}$$

The last inequality follows since π is a $*$-algebra which are known to be contractive. Hence, $\|\varphi\| \leq 1$. But note that,

$$\varphi(e) = \langle \pi(e)x, x\rangle = \langle I_{\mathscr{H}}x, x\rangle = \langle x, x\rangle = 1.$$

Hence, $\|\varphi\| = 1$. This shows that φ is a state.

Example 27.1 implies that we can associate a state corresponding to a non-degenerate representation of a C^*-algebra. We now show that the converse is true as well, which is the GNS construction.

Proposition 27.2. *(**GNS Construction**) Let A be a C^*-algebra. If φ is any state on A, there is a non-degenerate representation $(\pi_\varphi, \mathscr{H}_\varphi)$ and a unit vector $x_\varphi \in \mathscr{H}_\varphi$ such that*

$$\varphi(a) = \langle \pi_\varphi(a)x_\varphi, x_\varphi\rangle_\varphi$$

for any $a \in A$. Moreover, the representation is unique up to unitary equivalence. That is, if $\pi : A \to \mathscr{B}(\mathscr{H})$ is another representation with cyclic unit vector $x \in \mathscr{H}$ satisfying

$$\varphi(a) = \langle \pi x, x\rangle$$

for all $a \in A$, then there exists a unique unitary $U : \mathscr{H}_\varphi \to \mathscr{H}$ such that

$$Ux_\varphi = x$$

and

$$U\pi_\varphi(a) = \pi(a)U$$

for all $a \in A$.

Proof. If φ is a state, then φ defines a sesqui-linear form. Define the set,

$$\mathscr{N}_\varphi = \{a \in A \mid \varphi(a^*a) = 0\}$$

We first show that φ is a closed left ideal. Clearly, φ is closed since φ is continuous[18]. Moreover, $\mathscr{N}_\varphi$ is a vector subspace. Indeed, if $a, b \in \mathscr{N}_\varphi$, and $\lambda \in \mathbb{C}$, then

$$\varphi((\lambda a + b)^*(\lambda a + b)) = |\lambda|^2\varphi(a^*a) + \lambda\varphi(b^*a) + \overline{\lambda}\varphi(a^*b) + \varphi(b^*b) = 0$$

The first and fourth terms are zero by assumption. The second and third terms are zero by Proposition 25.2(2). If $a \in \mathscr{N}_\varphi$ and $b \in A$, consider the functional

$$\psi(c) = \varphi(a^*ca)$$

for any $c \in A$. This is clearly another positive linear functional, and so we have

$$\|\psi\| = \psi(1) = \varphi(a^*a) \qquad (*)$$

Then we have

$$0 \le \varphi((ba)^*ba) = \varphi(a^*b^*ba) = \psi(b^*b) \le \|\psi\|\|b^*b\| = \varphi(a^*a)\|b\|^2 = 0$$

This shows that $ba \in \mathscr{N}_\varphi$. Hence, $\mathscr{N}_\varphi$ is a closed left ideal. Consider the map,

$$\begin{aligned}(,) : A \times A &\to \mathbb{C}\\ (a, b) &\mapsto \varphi(a^*b)\end{aligned}$$

Our discussion above shows that the map defined above is a sesquilinear form. Indeed, it is easy to check that the map is linear in the second argument. Moreover, it is conjugate symmetric since,

$$(a, b) = \varphi(a^*b) = \varphi((ba^*)^*) = \overline{\varphi(ba^*)} = \overline{(b, a)}$$

But it might not be an inner product since $\mathscr{N}_\varphi \neq \emptyset$. Consider $V_\varphi = A/\mathscr{N}_\varphi$. We can now define an *honest* inner product on V_φ by the formula:

$$\langle a + \mathscr{N}_\varphi, b + \mathscr{N}_\varphi\rangle_\varphi := \varphi(b^*a).$$

[18]Note that φ is the composition of the maps $a \mapsto (a, a^*) \mapsto \varphi(a^*a)$ which is indeed continuous.

To see that this is well-defined, note that, for any $x, y \in \mathscr{N}_\varphi$ and $a, b \in A$,

$$\begin{aligned}\varphi((b+y)^*(a+x)) &= \varphi(b^*a + b^*x + y^*a + y^*x)\\ &= \varphi(b^*a) + \varphi(b^*x) + \varphi(y^*a) + \varphi(y^*x)\\ &= \varphi(b^*a).\end{aligned}$$

The last equality follows since $\mathscr{N}_\varphi$ is a left ideal. Let $\mathscr{H}_\varphi$ to be the completion of V_φ with respect to the norm induced by $\langle \cdot, \cdot \rangle_\varphi$. The action of A on A by left multiplication induces a representation $\pi_\varphi : A \to \mathscr{B}(\mathscr{H}_\varphi)$ is given by left multiplication:

$$\pi_\varphi(a)(b + \mathscr{N}_\varphi) = ab + \mathscr{N}_\varphi.$$

The fact that $\mathscr{N}_\varphi$ is an ideal ensures that $\pi_\varphi(a)$ is a well-defined map from V_φ to V_φ for each $a \in A$. Moreover, the map $\pi_\varphi(a)$ is bounded from V_φ to V_φ for each $a \in A$. Indeed,

$$\begin{aligned}\|\pi_\varphi(a)\|^2 &= \sup\{\langle \pi_\varphi(a)(x + \mathscr{N}_\varphi), \pi_\varphi(a)(x + \mathscr{N}_\varphi)\rangle_\varphi \mid x \in A, \varphi(x^*x) = 1\}\\ &= \sup\{\varphi((ax)^*(ax)) \mid x \in A, \varphi(x^*x) = 1\}\\ &= \sup\{\varphi(x^*a^*ax) \mid x \in A, \varphi(x^*x) = 1\}\\ &\leq \sup\{\varphi(x^*x)\|a^*a\| \mid x \in A, \varphi(x^*x) = 1\}\\ &= \|a^*a\| = \|a\|^2.\end{aligned}$$

We have used the information in $(*)$. It is clear that $\pi_\varphi(a)$ is linear and multiplicative from V_φ to V_φ for each $a \in A$. We check that $\pi_\varphi(a)$ is $*$-preserving from V_φ to V_φ for each $a \in A$. For $a, b, c \in A$, we have

$$\begin{aligned}\langle \pi_\varphi(a^*)b + \mathscr{N}_\varphi, c + \mathscr{N}_\varphi\rangle_\varphi &= \langle a^*b + \mathscr{N}_\varphi, c + \mathscr{N}_\varphi\rangle_\varphi\\ &= \varphi(c^*(a^*b))\\ &= \varphi((ac)^*b)\\ &= \langle b + \mathscr{N}_\varphi, ac + \mathscr{N}_\varphi\rangle_\varphi\\ &= \langle b + \mathscr{N}_\varphi, \pi_\varphi(a)c + \mathscr{N}_\varphi\rangle_\varphi\\ &= \langle \pi_\varphi(a^*)b + \mathscr{N}_\varphi, c + \mathscr{N}_\varphi\rangle_\varphi.\end{aligned}$$

Since this holds for arbitrary b and $c \in A$, we conclude that $\pi_\varphi(a)^* = \pi_\varphi(a)^*$. A standard density argument now shows that the map $\pi_\varphi : A \to \mathscr{B}(\mathscr{H}_\varphi)$ is a well-defined, linear, multiplicative and $*$-preserving map. Consider the vector $x_\varphi = 1 + \mathscr{N}_\varphi$. Note that,

$$\begin{aligned}\|x_\varphi\| &= \|1 + \mathscr{N}_\varphi\|\\ &= \varphi(1^*1)^{1/2}\\ &= \varphi(1)^{1/2}\\ &= 1.\end{aligned}$$

If b is any element of A, it is clear that

$$\pi_\varphi(b)x_\varphi = b \cdot 1 + \mathscr{N}_\varphi = b + \mathscr{N}_\varphi.$$

It follows then that $\pi_\varphi(A)x_\varphi$ contains $A/\mathscr{N}_\varphi$ and is therefore dense in H_φ. Note that we have,

$$\langle \pi_\varphi(a)x_\varphi, x_\varphi\rangle_\varphi = \langle a + \mathscr{N}_\varphi, 1 + \mathscr{N}_\varphi\rangle = \varphi(a)$$

for each $a \in A$. This proves existence. Let $\pi : A \to \mathscr{B}(\mathscr{H})$ be another representation with cyclic unit vector $x \in \mathscr{H}$ satisfying

$$\varphi(a) = \langle \pi(a)x, x \rangle$$

for all $a \in A$. Let V denote the set

$$V = \{\pi(a)x : a \in A\} \subseteq \mathscr{H}.$$

Define

$$\begin{aligned} U : V_\varphi &\to V, \\ \pi_\varphi(a)x_\varphi &\mapsto \pi(a)x \end{aligned}$$

We first show that U is well-defined. Assume that $\pi_\varphi(a)x_\varphi = \pi_\varphi(b)x_\varphi$. We must show that $\pi(a)x = \pi(b)x$. Certainly, using that π is a $*$-morphism:

$$\begin{aligned} \langle \pi(a-b)x, \pi(a-b)x \rangle &= \langle \pi((a-b)^*(a-b))x, x \rangle \\ &= \varphi((a-b)^*(a-b)) \\ &= \langle \pi_\varphi((a-b)^*(a-b))x_\varphi, x_\varphi \rangle_\varphi \\ &= \langle \pi_\varphi(a-b)x_\varphi, \pi_\varphi(a-b)x_\varphi \rangle_\varphi \\ &= \|\pi_\varphi(a-b)x\|_\varphi^2 = 0. \end{aligned}$$

Hence, we have $\pi(a)x = \pi(b)x$. This shows that U is a well-defined map. We now check that U is an injective map. Assume that $\pi(a)x = \pi(b)x$ for some $a, b \in A$. Invoking the above calculation, we have,

$$\|\pi_\varphi(a-b)x\|_\varphi^2 = \|\pi(a-b)\|^2$$

Hence, $\pi_\varphi(a)x = \pi_\varphi(b)x$, implying that U is injective. By a simple inspection, U is surjective. Hence, U is a bijection. To see that U preserves the inner product, notice that for $a, b \in A$, the following holds:

$$\begin{aligned} \langle U(\pi_\varphi(a)x_\varphi), U(\pi(b)x_\varphi) \rangle &= \langle \pi(a)x, \pi(b)x \rangle \\ &= \langle \pi(b^*a)x, x \rangle \\ &= \varphi(b^*a). \end{aligned}$$

A verbatim argument shows that

$$\langle \pi_\varphi(a)x_\varphi, \pi(b)x_\varphi \rangle = \varphi(b^*a)$$

This implies that U preserves inner products. Since U preserves inner products, it follows that U is also bounded. Moreover, it is a simple matter to check that U is a linear map. Hence, we see that

$$\begin{aligned} U : V_\varphi &\to V \\ \pi_\varphi(a)x_\varphi &\mapsto \pi(a)x \end{aligned}$$

is a is unitary map. Using the density of V_φ in $\mathscr{H}_\varphi$ (and that of V in $\mathscr{H}$), we may uniquely extend U to a bounded linear operator

$$\widetilde{U} : \mathscr{H}_\varphi \to \mathscr{H}$$

Simple density arguments can be used to show that $\widetilde{U}$ is surjective, preserves the inner product and is a unitary equivalence. The claim follows at once if we replace $\widetilde{U}$ by U. $\square$

Remark 27.3. *A state, φ, is called a pure state if φ cannot be written as a non-trivial convex combination of states. With a bit more work, one can show that the representation constructed in Proposition 27.2 is irreducible if and only if φ is a pure state.*

27.2. **GNS Theorem.** The study of C^*-algebras is motivated by the prime example of norm closed $*$-algebras of operators on Hilbert space. With this in mind, it is natural to find ways that a given abstract C^*-algebra may act as operators on Hilbert space. Such an object is called a representation of a C^*-algebra, and the study of representations of a C^*-algebra leads to the proof of the Gelfand-Naimark-Segal theorem:

Theorem 27.4. *(Gelfand, Naimark & Segal) Let A be a C^*-algebra. Then A is isometrically $*$-isomorphic to a $*$-closed subalgebra of $\mathscr{B}(\mathscr{H})$ for some Hilbert space, $\mathscr{H}$.*

In order to prove Theorem 27.4, we will take the direct sum of *a lot* of the representations given in Proposition 27.2 to produce a faithful representation of a C^*-algebra. We must take into account one caveat, though. Proposition 27.2 is proved under the assumption that there exists a state defined on A. In order to take a direct sum of *a lot* of representations given in Proposition 27.2, we need to show the existence of *a lot* of states on A.

Lemma 27.5. *Let A be a unital C^*-algebra.*

(1) *If φ is a bounded linear functional on A which satisfies*
$$1 = \|\varphi\| = \varphi(e),$$
then φ is a state.

(2) *Let a be a non-zero, self-adjoint (hence normal) element of A. Then there is a state ψ on A such that*
$$|\psi(a)| = \|a\|.$$

Proof. The proof is given below:

(1) Skipped.

(2) Let $B = C^*(a, e) \subseteq A$. Let $\lambda = r(a)$[19]. Let $\mathrm{Ev}_\lambda : C(\sigma_A(a)) \to \mathbb{C}$ be given by evaluation at λ. Since Ev_λ is a character on $C(\sigma_A(a))$, it is, in particular, a state on $C(\sigma_A(a))$. Since $B \cong C^*(a, e)$, we have furnished a state on B. Since B is a closed subspace of A, the Hahn-Banach theorem allows us to extend it to a linear functional $\psi \in A^*$ with the same norm (i.e., $\|\psi\| = 1$). As
$$\psi(1) = \mathrm{Ev}_\lambda(1) = 1,$$
(1) tells us that ψ is also a state. Since the Gelfand transform takes a to the function $f(z) = z$, it follows that
$$|\psi(a)| = |\lambda| = r(a) = \|a\|$$
The last equality follows since a is self-adjoint.

This completes the proof. □

We are now ready to prove Theorem 27.4.

[19]We know this exists since $\sigma_A(a)$ is a non-empty, closed subset.

Proof. (Theorem 27.4) Let $D \subseteq A$ be a dense subset. For each $a \in D$, let $\varphi_a \in S(A)$ be such that $|\varphi_a(a^*a)| = \|a^*a\| = \|a\|^2$. Let $(\pi_a, \mathscr{H}_a, x_{\varphi_a})$ representation constructed in Proposition 27.2. Consider the direct sum representation:

$$\pi := \bigoplus_{a \in D} \pi_a : A \to \mathscr{B}\left(\bigoplus_{a \in D} \mathscr{H}_a\right) := \mathscr{B}(\mathscr{H}).$$

Assume $a \neq 0 \in A$ such that $\pi(a) = 0$. Then

$$\|\pi_a(a)x_{\varphi_a}\|^2 = \langle \pi_a(a^*a)x_{\varphi_a}, x_{\varphi_a}\rangle_{\varphi_a} = \varphi_a(a^*a) = \|a^*a\| \neq 0$$

This shows that π is non-zero, a contradiction. Hence, π is injective. It is clear that π is a $*$-homomorphism since each π_a is a faithful $*$-homomorphism. $\square$

Remark 27.6. *With a bit more work, one can show that each state constructed in Lemma 27.5 can be taken to be a pure state. Hence, π in the proof of Theorem 27.4 can be taken to be a direct sum of irreducible representations of A.*

28. Universal C^*-Algebras

29. Crossed Product C^*-Algebras

30. Tensor Products of C^*-Algebras

31. Colimits of C^*-Algebras