# SPECTRAL SEQUENCES

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Abstract. These notes are intended to compile general facts about spectral sequences. Specific examples are discussed in other notes. If you notice any typos or errors, please feel free to send corrections to junaid.aftab1994@gmail.com.

Contents



## 1. Motivation

In algebra, we are often interested in computing a graded object, $H^*$, which could, for example, be any of the following:

- Graded $R$-module for a ring $R$,
- Graded $\mathbb{K}$-vector space for a field $\mathbb{K}$,
- Graded $\mathbb{K}$-algebra for a field $\mathbb{K}$.

The computation of $H^*$ is often nontrivial. Progress can often be made if $H^*$ can be computed via an *approximation argument*. Such an approach is typically viable when $H^*$ is equipped with additional structure that allows for successive approximations. For instance, assume that $H^*$ is equipped with a filtration by a possibly unbounded descending sequence of sub-objects:

$$\cdots \supseteq H_n \supseteq H_{n+1} \supseteq \cdots \tag{1}$$

such that

$$\bigcup_{n=0}^{\infty} H_n = H^*, \qquad \bigcap_{n=0}^{\infty} H_n = 0$$

**Remark 1.1.** *We may also consider a filtration given by a possibly unbounded increasing sequence of sub-objects. In general, such sequences may be unbounded in both directions.*

Let's consider an example:

**Example 1.2.** Let $H^*$ be a possibly infinite-dimensional $\mathbb{K}$-vector space. For instance, consider $H^* = \mathbb{K}^\infty$, the countably infinite-dimensional $\mathbb{K}$-vector space with basis $\{e_0, e_1, \dots\}$. Define

$$H_n := \operatorname{span}\{e_p \mid p \geq n\}$$

Then $\{H_n\}_{n\in\mathbb{N}}$ defines a filtration as described in Equation (1).

In fact, Example 1.2 possesses additional structure, in the sense that $\mathbb{K}^\infty$ can be recovered from its filtration as follows. The filtration of $\mathbb{K}^\infty$ gives rise to a new graded $\mathbb{K}$-vector space known as the associated graded $\mathbb{K}$-vector space defined by

$$M_n := H_n/H_{n+1}$$

One can recover $\mathbb{K}^\infty$ up to isomorphism from its associated $\mathbb{K}$-vector space by taking direct sums:

$$\mathbb{K}^\infty \cong \bigoplus_{n=0}^{\infty} M_n \cong \bigoplus_{n=0}^{\infty} H_n/H_{n+1} \cong \bigoplus_{n=0}^{\infty} \frac{\operatorname{span}\{e_p \mid p \geq n\}}{\operatorname{span}\{e_p \mid p \geq n+1\}} \cong \bigoplus_{n=0}^{\infty} \operatorname{span}\{e_n\}$$

Generally, it might not be possible to compute an arbitrary graded object, $H^*$, in this manner. For instance, if $H^*$ is an arbitrary graded $R$-module, there may be extension problems that prevent the reconstruction of $H^*$ from the associated graded $R$-module. However, we can take the associated graded $R$-module of a filtration of $H^*$ as the first approximation to $H^*$ and hope that $H^*$ this first approximation can be refined through a limiting argument. This is the underlying philosophy behind spectral sequences:

*A spectral sequence is an algorithm for computing a graded object by taking successive approximations.*

Spectral sequences emerge as a natural computational and conceptual framework when studying filtered complexes in homological algebra and algebraic topology. These tools are particularly useful in situations where a direct computation of homology is infeasible, but a filtration imposes a manageable structure on the problem. Let us examine this case informally in action, and we will see how it naturally motivates the definition of a spectral sequence.

**Example 1.3.** (**Filtered Chain Complexes**) Let $C^\bullet$ be a co-chain complex equipped with a decreasing filtration[1] by sub-complexes:

$$\cdots \supseteq F_pC^\bullet \supseteq F_{p+1}C^\bullet \supseteq \cdots$$

Such a filtration provides a decomposition of the complex into progressively more refined components. The central question is how the cohomology of $C^\bullet$ can be reconstructed from the data of the filtration. A natural first step is to consider the associated graded complex:

$$\mathrm{G}_p\, C^\bullet := F_pC^\bullet/F_{p+1}C^\bullet. \tag{2}$$

For each $p \in \mathbb{Z}$, one may compute the cohomology of $\mathrm{G}_p\, C^\bullet$, yielding an approximation to the cohomology of the co-chain complex. However, this information may be insufficient to fully determine the cohomology of $C^\bullet$. To overcome this limitation, one introduces a spectral sequence: for $r \in \mathbb{N}$, a sequence of pages $\{E_r^{p,q}\}_{(p,q)\in\mathbb{Z}^2}$ such that $E_r^{p,q}$ is a bi-graded group. The first page is derived from the cohomology of the associated graded

[1] We could also consider an increasing filtration; however, to remain consistent with the filtration introduced earlier in Equation (1), we choose to work with a decreasing filtration. Later on, we will work with both decreasing and increasing filtrations.

complex in Equation (2). Subsequent pages refine this approximation as each subsequent page is defined as the cohomology of the preceding page.

**Remark 1.4.** *Under appropriate convergence conditions, the spectral sequence is expected to stabilize at a terminal page, $E_\infty$, which captures the associated graded components of the homology of the chain complex. We will see how the exact definition of convergence of a spectral sequence naturally arises in explicit constructions.*

## 2. Definition

Based on the discussion in Section 1, we introduce the definition of a spectral sequence in this section and comment on some details surrounding the definition. We conclude by presenting some basic formal examples.

**Definition 2.1.** Let $r_0 \in \mathbb{N}$. A cohomological spectral sequence, $E$, of $R$-modules consists of the following data:

(1) A collection of $R$-modules, $E_r^{p,q}$, with integers $p, q \geq 0$ and $r \geq r_0$,
(2) A collection of differentials

$$d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}$$

such that $d_r^{p,q} \circ d_r^{p+r,q-r+1} = 0$ and $E_{r+1}^{p,q}$ is the homology at $E_r^{p,q}$, i.e.

$$E_{r+1}^{p,q} \cong \frac{\ker d_r^{p,q}}{\operatorname{im} d_r^{p-r,q+r-1}}$$

The collection $E_r = \{(E_r^{p,q}, d_r^{p,q}) : p, q \geq 0\}$ for a fixed $r$ is called the $r$-th page. We say that the differential $d_r^{p,q}$ has bi-degree $(r, 1-r)$.

$$\begin{tikzcd}
E_0^{-1,1} & E_0^{0,1} & E_0^{1,1} \\
E_0^{-1,0} \arrow[u] & E_0^{0,0} \arrow[u] & E_0^{1,0} \arrow[u] \\
E_0^{-1,-1} \arrow[u] & E_0^{0,-1} \arrow[u] & E_0^{1,-1} \arrow[u]
\end{tikzcd}$$

(A) Zeroth page

$$\begin{tikzcd}
E_1^{-1,1} \arrow[r] & E_1^{0,1} \arrow[r] & E_1^{1,1} \\
E_1^{-1,0} \arrow[r] & E_1^{0,0} \arrow[r] & E_1^{1,0} \\
E_1^{-1,-1} \arrow[r] & E_1^{0,-1} \arrow[r] & E_1^{1,-1}
\end{tikzcd}$$

(B) First page

$$\begin{tikzcd}
{} & E_2^{-1,1} & E_2^{0,1} & E_2^{1,1} & {} \\
{} & E_2^{-1,0} & E_2^{0,0} & E_2^{1,0} & {} \\
 & E_2^{-1,-1} & E_2^{0,-1} & E_2^{1,-1} & {}
\arrow[from=1-1, to=2-3]
\arrow[from=1-2, to=2-4]
\arrow[from=1-3, to=2-5]
\arrow[from=2-1, to=3-3]
\arrow[from=2-2, to=3-4]
\arrow[from=2-3, to=3-5]
\end{tikzcd}$$

(C) Second page

Snapshots of first three pages of a cohomological spectral sequence

What is the intuition behind the definition of the differential maps? Since the differential maps $d_r^{p,q}$ compute cohomology, we expect the total degree of the map to increase by 1. So if the domain is $E_r^{p,q}$, it makes sense for the codomain of $d_r^{p,q}$ to be $E_r^{p+r,q-r+1}$. We say that the total degree of the map is 1.

**Remark 2.2.** *A homological spectral sequence is defined similarly. We omit the formal definition. The notable differences in the definition are as follows:*

(1) *We write the $r$-th page of the spectral sequence as $E^r$ instead of $E_r$.*
(2) *The differential is defined as*

$$d^r_{p,q} : E^r_{p,q} \to E^r_{p-r,q+r-1}.$$

(3) *We say that* $d^r_{p,q}$ *has bi-degree* $(r, 1-r)$ *and total degree* $-1$.
*We will invoke both notions as appropriate going forward.*

$$\begin{tikzcd}
E^0_{-1,1} \arrow[d] & E^0_{0,1} \arrow[d] & E^0_{1,1} \arrow[d] \\
E^0_{-1,0} \arrow[d] & E^0_{0,0} \arrow[d] & E^0_{1,0} \arrow[d] \\
E^0_{-1,-1} & E^0_{0,-1} & E^0_{1,-1}
\end{tikzcd}$$

(A) Zeroth page

$$\begin{tikzcd}
E^0_{-1,1} \arrow[r] & E^0_{0,1} \arrow[r] & E^0_{1,1} \\
E^0_{-1,0} \arrow[r] & E^0_{0,0} \arrow[r] & E^0_{1,0} \\
E^0_{-1,-1} \arrow[r] & E^0_{0,-1} \arrow[r] & E^0_{1,-1}
\end{tikzcd}$$

(B) First page

$$\begin{tikzcd}
{} & E^0_{-1,1} & E^0_{0,1} & E^0_{1,1} & {} \\
{} & E^0_{-1,0} & E^0_{0,0} \arrow[ull] & E^0_{1,0} \arrow[ull] & {} \arrow[ull] \\
{} & E^0_{-1,-1} & E^0_{0,-1} \arrow[ull] & E^0_{1,-1} \arrow[ull] & {} \arrow[ull]
\end{tikzcd}$$

(C) Second page

Snapshots of first three pages of a homological spectral sequence

One way to look at a cohomological spectral sequence is to imagine an infinite book, where each page is a Cartesian plane with the integral lattice points $(p, q)$ consisting of objects in the category of $R$-modules and differentials between the objects forming a co-chain complex. The cohomology groups of these co-chain complexes are precisely the groups which appear on the next page. The customary picture is shown below.

**Remark 2.3.** *Spectral sequences refine the process of calculating (co)homology in the sense that the computation of (co)homology groups at the* $r$*-th page not only yields the (co)homology groups at the* $(r+1)$*-st page, but also determines the differential maps between the (co)homology groups on the* $(r+1)$*-st page. Hence, a spectral sequence encodes a significant amount of additional information. However, the differentials can be difficult to compute explicitly. In practice, educated guesswork and ad hoc techniques are often required to determine the differential maps.*

Since cohomology is defined as a subquotient (i.e., the quotient of a $R$-submodule), we expect the modules appearing on the $(r+1)$-th page to be, in some sense, "smaller" or more refined than those on the $r$-th page. Fixing a position $(p, q) \in \mathbb{Z}^2$, we consider the sequence of $R$-modules $E_r^{p,q}$ as $r \to \infty$. This motivates the definition of the limiting page of the spectral sequence, referred to as the $E_\infty$ page, as well as the notion of convergence of spectral sequences. The definition will be provided in a later section, as it may seem somewhat arcane at first. The best way to understand how this definition arises is by examining a concrete construction where we can explicitly determine the ingredients that determine the definition of convergence of a spectral sequence. For the time being, we consider some formal constructions as examples. These examples are also important special cases in which issues of convergence do not arise.

**Example 2.4.** (**First Quadrant Spectral Sequence**) First-quadrant cohomological spectral sequences are significantly more tractable than general cohomological spectral sequences, both computationally and conceptually. A cohomological spectral sequence is a first quadrant cohomological spectral sequence if $E_r^{p,q} = 0$ for $p < 0$ or $q < 0$. Fix $(p, q) \in (\mathbb{N} \cup \{0\})^2$. In a first quadrant cohomological spectral sequence, for $r$ (as a function of $(p, q)$) large enough, the differential with co-domain $E_r^{p,q}$ has domain 0 and the differential with domain $E_r^{p,q}$ has co-domain 0. Therefore, we get

$$E_{r+1}^{p,q} \cong \frac{\ker d_r^{p,q}}{\operatorname{im} d_r^{p-r,q+r-1}} \cong \frac{E_r^{p,q}}{0} \cong E_r^{p,q}.$$

The stable value $E_r^{p,q} = E_k^{p,q}$ for $k \geq r(p,q)$ is named $E_\infty^{p,q}$. In this case, we can determine the entries on the $E_\infty$ page in a finite number of steps, and there are no issues of convergence.

$$\begin{array}{ccc}
E_0^{0,2} & E_0^{1,2} & E_0^{2,2} \\
\uparrow & \uparrow & \uparrow \\
E_0^{0,1} & E_0^{1,1} & E_0^{2,1} \\
\uparrow & \uparrow & \uparrow \\
E_0^{0,0} & E_0^{1,0} & E_0^{2,0}
\end{array}$$

(A) Zeroth page

$$\begin{array}{ccccc}
E_1^{0,2} & \longrightarrow & E_1^{1,2} & \longrightarrow & E_1^{2,2} \\
E_1^{0,1} & \longrightarrow & E_1^{1,1} & \longrightarrow & E_1^{2,1} \\
E_1^{0,0} & \longrightarrow & E_1^{1,0} & \longrightarrow & E_1^{2,0}
\end{array}$$

(B) First page

$$\begin{tikzcd}
{} \arrow[drr] & E_2^{0,2} \arrow[drr] & E_2^{1,2} \arrow[drr] & E_2^{2,2} & {} \\
{} \arrow[drr] & E_2^{0,1} \arrow[drr] & E_2^{1,1} \arrow[drr] & E_2^{2,1} & {} \\
{} & E_2^{0,0} & E_2^{1,0} & E_2^{2,0} & {}
\end{tikzcd}$$

(C) Second page

Snapshots of first three pages of a first quadrant cohomological spectral sequence

## 3. Filtered Complexes

A very common and important type of spectral sequence arises from a filtered co-chain complex. Spectral sequences associated to filtered complexes provide a powerful tool for analyzing the (co)homology of the co-chain complex by examining the simpler associated graded pieces. This approach often allows complicated computations to be broken into more manageable stages, each reflecting a piece of the overall structure.

**Definition 3.1.** Let $C^\bullet = \{C^n, \partial^n\}_{n\in\mathbb{Z}}$ be a co-chain complex of $R$-modules. A decreasing filtration of $C^\bullet$ is a sequence

$$\cdots \supseteq F_pC^\bullet \supseteq F_{p+1}C^\bullet \supseteq \cdots$$

such that each $F_pC^\bullet$ is a sub-complex of $C^\bullet$ and the differential $\partial^n$ restricts to a map

$$F_pC^n \to F_pC^{n+1}$$

for all $n \in \mathbb{Z}$ that is compatible with the filtration.

**Remark 3.2.** *We write the $j$-th entry of $F_iC^\bullet$ as $C_i^j$ for $i, j \in \mathbb{Z}$. Note that we have $\partial_i^j(C_i^j) \subseteq C_i^{j+1}$ for each $i, j \in \mathbb{Z}$. We can visualize the filtered co-chain complex as follows:*

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \cup\! | & & \cup\! | & & \cup\! | & & \\
\cdots & \longrightarrow & C_{-1}^{-1} & \xrightarrow{\partial_{-1}^{-1}} & C_{-1}^{0} & \xrightarrow{\partial_{-1}^{0}} & C_{-1}^{1} & \longrightarrow & \cdots \\
 & & \cup\! | & & \cup\! | & & \cup\! | & & \\
\cdots & \longrightarrow & C_{0}^{-1} & \xrightarrow{\partial_{0}^{-1}} & C_{0}^{0} & \xrightarrow{\partial_{0}^{0}} & C_{0}^{1} & \longrightarrow & \cdots \\
 & & \cup\! | & & \cup\! | & & \cup\! | & & \\
\cdots & \longrightarrow & C_{1}^{-1} & \xrightarrow{\partial_{1}^{-1}} & C_{1}^{0} & \xrightarrow{\partial_{1}^{0}} & C_{1}^{1} & \longrightarrow & \cdots \\
 & & \cup\! | & & \cup\! | & & \cup\! | & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

*We say that the filtration is exhaustive and separated if for each $j \in \mathbb{Z}$, we have*

$$\bigcap_{i\in\mathbb{Z}} C_i^j = 0 \quad \textit{and} \quad \bigcup_{i\in\mathbb{Z}} C_i^j = C^j.$$

*The first condition is the exhaustive condition, and the second condition is the separated condition. In other words, the filtration must eventually become arbitrarily small and arbitrarily large at each degree.*

From the data of a filtration on a co-chain complex, one constructs a spectral sequence that approximates the cohomology of $C^\bullet$ through successive approximations. Let's first discussion the motivation behind the construction. We let the $E_0$ page of the spectral sequence be the associated graded co-chain complex. That is[2],

$$E_0^{p,q} \cong G_pC^{p+q} \cong F_pC^{p+q}/F_{p+1}C^{p+q}$$

with induced differential

$$d_0^{p,q} : \frac{F_pC^{p+q}}{F_{p+1}C^{p+q}} \cong E_0^{p,q} \to E_0^{p,q+1} \cong \frac{F_pC^{p+q+1}}{F_{p+1}C^{p+q+1}}$$

induced by the map $\partial_p^{p+q} : F_pC^{p+q} \to F_pC^{p+q+1}$. The map is well-defined because $\partial_p^{p+q}(F_{p+1}C^{p+q}) \subseteq F_{p+1}C^{p+q+1}$. It is clear that these maps compose to zero. We then let the $E_1$ page denote the cohomology of the associated graded co-chain complex. That is,

$$\begin{aligned} E_1^{p,q} &\cong H^{p+q}(G_pC^\bullet) \\ &= \frac{\ker\left(d_0^{p,q} : E_0^{p,q} \to E_0^{p,q+1}\right)}{\operatorname{im}\left(d_0^{p,q-1} : E_0^{p,q-1} \to E_0^{p,q}\right)} \\ &= \frac{\ker\left(d_0^{p,q} : G_pC^{p+q} \to G_pC^{p+q+1}\right)}{\operatorname{im}\left(d_0^{p,q-1} : G_pC^{p+q-1} \to G_pC^{p+q}\right)}. \end{aligned}$$

We think of $E_1^{p,q}$ as a 'first-order approximation' to $H^{p+q}(C^\bullet)$. The question now is how to construct the differential $d_1^{p,q}$? Let's construct $d_1^{p,q}$. Note that a cohomology class $[\alpha] \in E_1^{p,q}$ represents a chain $c \in F_pC^{p+q}$ with differential $\partial_p^{p+q}c \in F_{p+1}C^{p+q+1}$. With this in mind, we define

$$\begin{aligned} d_1^{p,q} : E_1^{p,q} &\to E_1^{p+1,q} \\ [\alpha] &\mapsto [\partial_p^{p+q}c]. \end{aligned}$$

One easily sees that $d_1^{p+1,q} \circ d_1^{p,q} = 0$. So we are justified in defining

$$E_2^{p,q} := \frac{\ker(d_1^{p,q} : E_1^{p,q} \to E_1^{p+1,q})}{\operatorname{im}(d_1^{p-1,q} : E_1^{p-1,q} \to E_1^{p,q})}.$$

We can continue to construct higher-order approximations. Note that a cohomology class $[\alpha] \in E_2^{p,q}$ can be represented by some $[x] \in E_1^{p,q}$ with differential $d_1^{p,q}[x] = 0 \in E_1^{p+1,q}$. Since $d_1^{p,q}[x] = [\partial_p^{p+q}c]$, where $c \in F_pC^{p+q}$ is any chain representing $c$, we can choose $\partial_p^{p+q}c$

[2]Although the choice of $C^{p+q}$ instead of $C^q$ may initially appear unusual, for fixed $p$ the index $p+q$ is merely a shift of $q$ by a constant, and thus poses no problem. The necessity of this choice will become apparent when the spectral sequence is constructed in detail.

to be the zero element in $\ker(d_0^{p+1,q})$, meaning that $\partial_p^{p+q} c \in F_{p+2}C^{p+q+1}$. This suggests that we can define a map

$$d_2^{p,q} : E_2^{p,q} \to E_2^{p+2,q-1}.$$

Based on what we've seen so far, it seems that elements of an $r$th-order approximation $E_r^{p,q}$ should ultimately be represented by co-cycles $x \in F_pC^{p+q}$ such that $dx \in F_{p+r}C^{p+q+1}$. This turns out to be exactly the case. For each $n \in \mathbb{Z}$, we have a filtration

$$\cdots \supseteq F_{p-1}C^n \supseteq F_pC^n \supseteq F_{p+1}C^n \supseteq \cdots$$

of the object $C_n$. We think of elements of $C_n$ further down the filtration as being "closer to zero." The idea of a cohomological spectral sequence of a filtered co-chain complex is to asymptotically approximate the cohomology of $C^\bullet$ by refining co-cycles and co-boundaries through their $r$-approximations.

(1) Specifically, an $r$-almost co-cycle is a co-chain whose differential vanishes modulo terms that are $r$ steps lower in the filtration.
(2) An $r$-almost co-boundary in filtration degree $p$ is a co-cycle that is the differential of a co-chain which may be up to $r$ steps higher in filtration degree.

We now state and prove the desired result.

**Proposition 3.3.** *Every decreasing filtration of a co-chain complex $C^\bullet$ determines a cohomological spectral sequence.*

**Remark 3.4.** *We will see in the proof that the zeroth page of the spectral sequence is the associated graded co-chain complex*

$$G_pC^\bullet = F_pC^\bullet / F_{p+1}C^\bullet,$$

*and that the first page is the cohomology of this co-chain complex. Hence, the construction is consistent with the remarks made in Section 1.*

*Proof.* Choose the $E_0$ page of the spectral sequence such that

$$E_0^{p,q} = F_pC^{p+q}/F_{p+1}C^{p+q} := G_pC^{p+q}$$

For $r \geq 0$, we define $r$-almost $(p,q)$-co-cycles and $r$-almost $(p,q)$-co-boundaries as the following $R$-modules:

(1) The $R$-module of $r$-almost $(p,q)$-co-cycles is defined as

$$\begin{aligned} Z_r^{p,q} &= \left\{ c \in F_pC^{p+q} \mid \partial_p^{p+q}(c) \in F_{p+r}C^{p+q+1} \right\} / F_{p+1}C^{p+q} \\ &= \frac{F_pC^{p+q} \cap (\partial_p^{p+q})^{-1}(F_{p+r}C^{p+q+1}) + F_{p+1}C^{p+q}}{F_{p+1}C^{p+q}} := \frac{K_r^{p,q} + F_{p+1}C^{p+q}}{F_{p+1}C^{p+q}} \end{aligned}$$

In other words, $Z_r^{p,q}$ consists of co-chains in $F_pC^{p+q}$ whose co-boundaries lie in $F_{p+r}C^{p+q+1}$ modulo $F_{p+1}C^{p+q}$.

(2) The $R$-module of $r$-almost $(p,q)$-co-boundaries is defined as

$$\begin{aligned} B_r^{p,q} &= \partial_{p-r+1}^{p+q-1}(F_{p-r+1}C^{p+q-1}) \cap F_pC^{p+q} \\ &= \frac{\partial_{p-r+1}^{p+q-1}(F_{p-r+1}C^{p+q-1}) \cap F_pC^{p+q} + F_{p+1}C^{p+q}}{F_{p+1}C^{p+q}} \\ &= \frac{\partial_{p-r+1}^{p+q-1}(K_{r-1}^{p-r+1,1+r-2}) + F_{p+1}C^{p+q}}{F_{p+1}C^{p+q}} := \frac{I_r^{p,q} + F_{p+1}C^{p+q}}{F_{p+1}C^{p+q}} \end{aligned}$$

In other words, $B_r^{p,q}$ consists of co-chains in $F_pC^{p+q}$ that are in the image of $F_{p-r+1}C^{p+q-1}$ modulo $F_{p+1}C^{p+q}$.

Note that the reason we quotient out by $F^{p+1}C^{p+q}$ in the definitions of $Z_r^{p,q}$ and $B_r^{p,q}$ is that we want to localize our attention to the $p$-th graded piece of the filtered complex, and avoid interference from deeper levels of the filtration. This allows us to consider approximate co-cycles and approximate co-boundaries in the associated graded co-chain complex. Since the differentials in the co-chain complex compose to zero, note that we have,

$$B_r^{p,q} \subseteq Z_r^{p,q}$$

We can therefore define the $r$-almost $(p,q)$-cohomology by

$$E_r^{p,q} = \frac{Z_r^{p,q}}{B_r^{p,q}} \cong \frac{K_r^{p,q} + F_{p+1}C^{p+q}}{I_r^{p,q} + F_{p+1}C^{p+q}}$$

Note that we have a canonical surjective homomorphism:

$$\eta_r^{p,q} : K_r^{p,q} \longrightarrow K_r^{p,q} + F_{p+1}C^{p+q} \longrightarrow \frac{K_r^{p,q} + F_{p+1}C^{p+q}}{I_r^{p,q} + F_{p+1}C^{p+q}} \cong E_r^{p,q}.$$

mapping $x \in K_r^{p,q}$ to $[x+0]$. Note that the kernel can be identified with $I_r^{p,q} \subseteq K_r^{p,q}$. Moreover, note that $\partial_p^{p+q}$ restricts to a map from $K_r^{p,q}$ to $K_r^{p+r,q-r+1}$. Since $\partial_p^{p+q}(I_r^{p,q}) = 0$, we have a commutative diagram:

$$\begin{tikzcd}
K_r^{p,q} \arrow[r, "\partial_p^{p+q}"] \arrow[d, "\eta_r^{p,q}"'] & K_r^{p+r,q-r+1} \arrow[d, "\eta_r^{p+r,q-r+q}"] \\
E_r^{p,q} \arrow[r, dashed, "d_r^{p,q}"'] & E_r^{p+r,q-r+1}
\end{tikzcd}$$

It is clear by construction that $d_r^{p+r,q-r+1} \circ d_r^{p,q} = 0$. We now show that

$$E_{r+1}^{p,q} \cong \frac{\ker\left(d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}\right)}{\operatorname{im}\left(d_r^{p-r,q+r-1} : E_r^{p-r,q+r-1} \to E_r^{p,q}\right)}.$$

A quick computation shows that

$$\ker(d_r^{p,q}) \cong \frac{K_{r+1}^{p,q} + F^{p+1}C^{p+q}}{I_r^{p,q} + F^{p+1}C^{p+q}},$$
$$\operatorname{im}(d_r^{p-r,q+r-1}) \cong \frac{I_{r+1}^{p,q} + F^{p+1}C^{p+q}}{I_r^{p,q} + F^{p+1}C^{p+q}}.$$

Therefore, we have

$$\frac{\ker(d_r^{p,q})}{\operatorname{im}(d_r^{p-r,q+r-1})} \cong \frac{(K_{r+1}^{p,q} + F^{p+1}C^{p+q})/(I_r^{p,q} + F^{p+1}C^{p+q})}{(I_{r+1}^{p,q} + F^{p+1}C^{p+q}/(I_r^{p,q} + F^{p+1}C^{p+q})} \cong \frac{K_{r+1}^{p,q} + F_{p+1}C^{p+q}}{I_{r+1}^{p,q} + F_{p+1}C^{p+q}} \cong E_{r+1}^{p,q}$$

It is clear from the definitions that

$$Z_1^{p,q} = \ker\left(G_pC^{p+q} \to G_pC^{p+q+1}\right),$$
$$B_1^{p,q} = \operatorname{im}\left(G_pC^{p+q-1} \to G_pC^{p+q}\right).$$

Hence, $E_1^{p,q} = H^{p+q}(G_pC^\bullet)$. This completes the proof. $\square$

**Remark 3.5.** *The idea behind the construction of the spectral sequence is that as $r$ becomes large, the approximate co-cycles and co-boundaries of degree $r$ approach the actual co-cycles and co-boundaries. Therefore, we expect $E_r^{p,q}$ to approach something related to the cohomology of the co-chain complex. There are subtle issues of convergence involved, but we can attempt to identify the 'limiting page' in the special case where the filtration is bounded. For a bounded, exhaustive and separated filtration, for each $l \in \mathbb{Z}$ there exist $m(l) > n(l) \in \mathbb{Z}$ such that*

$$F_n C^l = C^l,$$
$$F_m C^l = 0.$$

*Fix any $p, q \in \mathbb{Z}$, and choose any $r > \max\{m(p+q+1) - p, p - n(p+q+1) + 1, 0\}$. Then,*

$$F_{p+r} C^{p+q+1} \subseteq F_m C^{p+q+1} = 0,$$
$$F_{p-r+1} C^{p+q-1} \supseteq F_n C^{p+q-1} = C^{p+q-1}.$$

*Therefore, we have*

$$Z_r^{p,q} = \frac{F_p C^{p+q} \cap \ker \partial^{p+q} + F_{p+1} C^{p+q}}{F_{p+1} C^{p+q}}, \quad B_r^{p,q} = \frac{F_p C^{p+q} \cap \operatorname{im} \partial^{p+q-1} + F_{p+1} C^{p+q}}{F_{p+1} C^{p+q}}.$$

*With these descriptions stated, we obviously have a surjective map*

$$F_p H^{p+q}(C^\bullet) \twoheadrightarrow E_r^{p,q}.$$

*The kernel of this map will be the cohomology classes $\alpha \in F_p H^{p+q}(C^\bullet)$ represented by a cycle $x \in F_{p+1} C^{p+q}$. That is, the kernel is exactly $F_{p+1} C^{p+q}$. Hence, for $r > \max\{m(p+q+1) - p, p - n(p+q+1) + 1, 0\}$ we have the isomorphism*

$$G_p H^{p+q}(C^\bullet) \cong E_r^{p,q}.$$

*Hence, we see that if the filtration is bounded, then for sufficiently large $r$, the $r$-almost $(p,q)$ cohomology coincides with the associated graded cohomology groups. Hence, for the case of a bounded filtration, we say that the "limiting page" of a cohomological spectral sequence, denoted $E_\infty^{p,q}$, satisfies*

$$E_\infty^{p,q} \cong G_p H^{p+q}(C^\bullet).$$

*We write*

$$E_r^{p,q} \Rightarrow G_p(H^{p+q}) = E_\infty^{p,q}$$

*and say that the spectral sequence converges weakly. The general case is dealt with by definition through the notion of a convergence of spectral sequences.*

## 4. Exact Couples

Another method to produce spectral sequences is through the use of exact couples. Exact couples offer a systematic framework for generating spectral sequences. By understanding spectral sequences through exact couples, we gain insight into their origin and emphasize the relationship between homological algebra and the iterative structure of spectral sequences.

**Definition 4.1.** An exact couple is a pair of $R$-modules $(A, E)$ along with $R$-module homomorphisms $i, j$, and $k$ such that the diagram

$$\begin{tikzcd} A \arrow[rr, "i"] & & A \arrow[dl, "j"] \\ & E \arrow[ul, "k"] & \end{tikzcd}$$

is exact at each vertex.

**Example 4.2.** Given an exact sequence

$$0 \to A \xrightarrow{i} A \xrightarrow{j} E \to 0,$$

we obtain an exact couple by taking $k = 0$.

Since $d := j \circ k$ implies

$$d \circ d = (j \circ k) \circ (j \circ k) = j \circ (k \circ j) \circ k = j \circ 0 \circ k = 0$$

we can apply the cohomology functor at $E$ and get the derived exact couple.

$$\begin{tikzcd} A' \arrow[rr, "i'"] & & A' \arrow[dl, "j'"] \\ & E' \arrow[ul, "k'"] & \end{tikzcd}$$

The data of the derived exact couple is as follows:

(1) $E' = \ker d / \operatorname{im} d$. This is simply the cohomology of $E$ with respect to the map $d$.
(2) $A' = i(A) \subseteq A$. Note that $i(A) = \operatorname{im} i = \ker j$.
(3) $i' = i|_{A'}$.
(4) $j'(i(a)) = [j(a)] \in E'$. We claim that this map is well-defined. First note that $j(a) \in \ker d$ since
$$d(j(a)) = (j \circ k \circ j)a = 0$$
Moreover, if $i(a_1) = i(a_2)$ then $a_1 - a_2 \in \ker i = \operatorname{im} k$. Hence, $j(a_1) - j(a_2) \in \operatorname{im} j \circ k = \operatorname{im} d$.
(5) $k'[e] = k(e)$. We claim that this map is well-defined. Note that $k(e) \in A' = \operatorname{im} i = \ker j$ since $e \in \ker d$ implies
$$(j \circ k)e = d(e) = 0$$
Further, $k'$ is well-defined since $e \in \operatorname{im} d$ implies that $e \in \operatorname{im} j = \ker k$.

We now claim that the derived couple of an exact couple is itself exact. This important property allows us to iterate the construction and thereby generate the successive pages of a spectral sequence. In the next step, we will formalize this idea carefully.

**Lemma 4.3.** *The derived couple of an exact couple is exact.*

*Proof.* The exactness of the derived couple follows by a straightforward diagram chase:

(1) We first show that $j' \circ i' = 0$. Indeed, if $a' \in A'$ then $a' = i(a)$. This implies that
$$(j' \circ i')a' = j'(i(a')) = [j(a')] = [(j \circ i)a] = 0$$
(2) We now show that $k' \circ j' = 0$. If $a' = i(a)$, then
$$(k' \circ j')a' = (k \circ j)a = 0$$

(3) We now show that $i' \circ k' = 0$. Note that we have

$$(i' \circ k')[e] = i'(k(e)) = (i \circ k)e = 0$$

(4) The above computations show that

$$\operatorname{im} i' \subseteq \ker j'$$
$$\operatorname{im} j' \subseteq \ker k'$$
$$\operatorname{im} k' \subseteq \ker i'$$

We now show reverse inclusions:

(a) We first show that $\ker j' \subseteq \operatorname{im} i'$. Assume that $j'(a') = 0$. Since $a' = i(a)$ for some $a \in A$, we have that $[j(a)] = 0$. This implies that $j(a) \in \operatorname{im} d$. In other words, we have $j(a) = j(k(e))$ for some $e \in E$. Hence, we have $a - k(e) \in \ker j = \operatorname{im} i$. That is, $a - k(e) = i(b)$ for some $b \in A$.

$$a' = i(a) = i(a - k(e)) = (i \circ i)b$$

Hence, $a' \in \operatorname{im} i^2 = \operatorname{im} i'$.

(b) We now show that $\ker k' \subseteq \operatorname{im} j'$. If $k'[e] = k(e) = 0$, then $e = j(a)$ for some $a \in A$. We have

$$[e] = [j(a)] = j'(i(a)) = j'(a')$$

(c) We now show that $\ker i' \subseteq \operatorname{im} k'$. Note that if $i'(a') = i(a') = 0$, then

$$a' = k(e) = k'[e]$$

This completes the proof. □

Lemma 4.3 naturally suggests iterating the process of computing the derived exact couple. To formalize this relationship, we now introduce a bi-grading structure. We also now replace the notation $A$ with $E$.

**Proposition 4.4.** *Suppose $E = \{A^{p,q}\}_{(p,q)\in\mathbb{Z}^2}$ and $E = \{E^{p,q}\}_{(p,q)\in\mathbb{Z}^2}$ are bigraded $R$-modules equipped with homomorphisms $i_1 : A \to A$, $j_1 : A \to E$ and $k_1 : E \to A$ of bi-degree $(-1,1), (0,0)$ and $(1,0)$ respectively. These data determine a cohomological spectral sequence $\{E_r, d_r\}_{r\geq 1}$ for $r = 1, 2, \ldots$ where $E_r$ is the $(r-1)$-st derived exact couple, and the differential $d_r$ is defined as*

$$d_r = j_r \circ k_r.$$

*Proof.* It suffices to check that the differentials $d_r$ have the correct bidegree, $(r, 1-r)$. By construction, $d_1$ has bi-degree $(1,0) + (0,0) = (1,0)$. Now assume that $j_{r-1}$ and $k_{r-1}$ have bidegrees $(r-2, 2-r)$ and $(1,0)$, respectively. Since $j_r(i_{r-1}(x)) = j_{r-1}(x) + d_{r-1}E_{r-1}$, $j_r$ has bidegree $(r-1, 1-r)$ since

$$\deg(j_r) = \deg(j) - (r-1)\deg(i) = (0,0) - (r-1)(-1,1) = (r-1, -(r-1)) = (r-1, 1-r)$$

Since $k_r(e + d_{r-1}E_{r-1}) = k_{r-1}(e)$, and since $k_{r-1}$ has bidegree $(1,0)$, it follows that $k_r$ also has bidegree $(1,0)$. Combining this with the inductive hypothesis, we conclude that $d_r$ has bidegree $(r, 1-r)$, as required. □

**Remark 4.5.** *It can be shown that the cohomology spectral sequences arising from the exact couple of filtered co-chain complexes are, in fact, equivalent. We do not discuss this result further here.*

## 5. Applications

We explore various applications of the theoretical framework developed earlier within the context of algebra and topology.

**Remark 5.1.** *We focus on applications where no convergence issues arise.*

### 5.1. Spectral Sequence of a Double Complex.

We discuss the construction of a spectral sequence arising from a double complex.

**Definition 5.2.** A first quadrant cohomological double complex, $C^{\bullet,\bullet}$, of $R$-modules consists of a collection of $R$-modules $\{C^{p,q}\}_{(p,q)\in\mathbb{N}^2}$ arranged in a bi-graded grid, together with two differentials:

$$d^{\mathrm{H}}_{p,q} : C^{p,q} \to C^{p+1,q}, \quad d^{\mathrm{V}}_{p,q} : C^{p,q} \to C^{p,q+1}$$

such that the following conditions hold:

$$\begin{aligned} &d^H_{p+1,q} \circ d^H_{p,q} = 0, \\ &d^V_{p,q+1} \circ d^V_{p,q} = 0, \\ &d^H_{p,q+1} \circ d^V_{p,q} + d^V_{p+1,q} \circ d^H_{p,q} = 0, \end{aligned}$$

for all $p, q \in \mathbb{N}$.

A first quadrant cohomological double complex can be visualized as a grid of $R$-modules arranged in the first quadrant, with horizontal and vertical differentials.

$$\begin{tikzcd}
\vdots & \vdots & \vdots & \vdots & \\
C^{0,2} \arrow[r, "d^H_{0,2}"] \arrow[u, "d^V_{0,2}"] & C^{1,2} \arrow[r, "d^H_{1,2}"] \arrow[u, "d^V_{1,2}"] & C^{2,2} \arrow[r, "d^H_{2,2}"] \arrow[u, "d^V_{2,2}"] & C^{3,2} \arrow[r, "d^H_{3,2}"] \arrow[u, "d^V_{3,2}"] & \cdots \\
C^{0,1} \arrow[r, "d^H_{0,1}"] \arrow[u, "d^V_{0,1}"] & C^{1,1} \arrow[r, "d^H_{1,1}"] \arrow[u, "d^V_{1,1}"] & C^{2,1} \arrow[r, "d^H_{2,1}"] \arrow[u, "d^V_{2,1}"] & C^{3,1} \arrow[r, "d^H_{3,1}"] \arrow[u, "d^V_{3,1}"] & \cdots \\
C^{0,0} \arrow[r, "d^H_{0,0}"] \arrow[u, "d^V_{0,0}"] & C^{1,0} \arrow[r, "d^H_{1,0}"] \arrow[u, "d^V_{1,0}"] & C^{2,0} \arrow[r, "d^H_{2,0}"] \arrow[u, "d^V_{2,0}"] & C^{3,0} \arrow[r, "d^H_{3,0}"] \arrow[u, "d^V_{3,0}"] & \cdots
\end{tikzcd}$$

**Remark 5.3.** *The total differential* $d = d^V + d^H$ *on the associated total complex* $\mathrm{Tot}^\bullet(C^{\bullet,\bullet})$*, defined by* $\mathrm{Tot}^n(C^\bullet) = \bigoplus_{p+q=n} C^{p,q}$ *satisfies* $d \circ d = 0$*, making* $\mathrm{Tot}(C^\bullet)$ *a co-chain complex. Each element in* $C^{p,q} \subseteq \mathrm{Tot}^n(C^\bullet)$ *is mapped, via both the horizontal and vertical differentials of the double complex to the corresponding summands in* $(\mathrm{Tot}\, C)^{p+q+1}$.

We construct a cohomological spectral sequence will do this by filtering our double complex in two different ways. We first consider the following filtration:

$$(C^{i,j}_{\mathrm{I}})_p = \begin{cases} 0 & \text{if } i < p, \\ C^{i,j} & \text{if } i \geq p. \end{cases}$$

Note that we have a decreasing filtration by columns. The total complexes of these truncations of $C^{\bullet,\bullet}$ give rise to a decreasing, exhaustive, seperated and bounded filtration on the total complex of $C^{\bullet,\bullet}$.

$$F_p \,\mathrm{Tot}^I_n(C^{\bullet,\bullet}) = \bigoplus_{i\geq p} C^{i,n-i}$$

Using Proposition 3.3 and Remark 3.5, we have the following result:

**Proposition 5.4.** *Consider a first quadrant cohomological double complex, $C^{\bullet,\bullet}$, of $R$-modules. There exists a cohomological spectral sequence $E_r^{p,q}$ for $r \geq 0$ such that:*

(1) *The zeroth page is given by the original double complex:*

$$E_0^{p,q} = \frac{F_p \operatorname{Tot}^I_{p+q}(C^{\bullet,\bullet})}{F_{p+1} \operatorname{Tot}^I_{p+q}(C^{\bullet,\bullet})} = C^{p,q}$$

*and the differentials $d_0^{p,q} \colon E_0^{p,q} \to E_0^{p,q+1}$ are the vertical differentials $d^V$ of the double complex*

(2) *The first page is given by the cohomology groups computed from the zeroth page and the differentials $d_1^{p,q} \colon E_1^{p,q} \to E_1^{p+1,q}$ are naturally induced by the horizontal differentials $d^H$.*

*Moreover, for each $(p,q) \in \mathbb{N}^2$ there exists a $R(p,q)$ such that for $r > R(p,q)$ we have*

$$E_r^{p,q} = E_\infty^{p,q} = G_p H^{p+q}(\operatorname{Tot} C^{\bullet,\bullet}).$$

We could easily have used the vertical truncations of the double complex.

$$(C_{\mathrm{I}}^{i,j})_p = \begin{cases} 0 & \text{if } j < p, \\ C^{i,j} & \text{if } j \geq p. \end{cases}$$

Note that we have a decreasing filtration by rows. The total complexes of these truncations of $C^{\bullet,\bullet}$ give rise to a decreasing, exhaustive, seperated and bounded filtration on the total complex of $C^{\bullet,\bullet}$.

$$F_p \operatorname{Tot}^{II}_n(C^{\bullet,\bullet}) = \bigoplus_{j \geq p} C^{n-j,j}$$

Using Proposition 3.3 and Remark 3.5, we have the following result:

**Proposition 5.5.** *Consider a first quadrant cohomological double complex, $C^{\bullet,\bullet}$, of $R$-modules. There exists a cohomological spectral sequence $E_r^{p,q}$ for $r \geq 0$ such that:*

(1) *The zeroth page is given by the 'transposed' original double complex:*

$$E_0^{p,q} = \frac{F_p \operatorname{Tot}^{II}_{p+q}(C^{\bullet,\bullet})}{F_{p+1} \operatorname{Tot}^{II}_{p+q}(C^{\bullet,\bullet})} = C^{q,p}$$

*and the differentials $d_0^{p,q} \colon E_0^{p,q} \to E_0^{p,q+1}$ are the induced by the horizontal differentials $d^H$ of the double complex*

(2) *The first page is given by the cohomology groups computed from the zeroth page and the differentials $d_1^{p,q} \colon E_1^{p,q} \to E_1^{p+1,q}$ are naturally induced by the vertical differentials $d^V$ of the double complex.*

*Moreover, for each $(p,q) \in \mathbb{N}^2$ there exists a $R(p,q)$ such that for $r > R(p,q)$ we have*

$$E_r^{p,q} = E_\infty^{p,q} = G_p H^{p+q}(\operatorname{Tot} C^{\bullet,\bullet}).$$

**Remark 5.6.** *Of course, we could have derived a homological spectral sequence associated to first-quadrant homological double complexes. We obtain two types of spectral sequences, which we described in words:*

(1) *The first spectral sequence is obtained by filtering columns. The zeroth page is the double complex, and the differentials are the vertical (downward facing) maps from the double complex. The first page is the homology of the first page and the maps are the horizontal (rightward facing) maps induced from the double complex.*
(2) *The first spectral sequence is obtained by filtering rows. The zeroth page is the 'transposed' double complex, and the differentials induced by the horizontal differentials from the double complex. The first page is the homology of the first page and the differentials are the vertical maps induced from the double complex.*

*We will freely use the analogous results below.*

5.2. **Five Lemma.** As a first example, we aim to prove the Five Lemma using a homological spectral sequence.

**Proposition 5.7.** ***(Five Lemma)*** *Consider the the diagram below:*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\alpha"'] & B \arrow[r] \arrow[d, "\beta"'] & C \arrow[r] \arrow[d, "\gamma"'] & D \arrow[r, "g"] \arrow[d, "\delta"'] & E \arrow[d, "\epsilon"'] \\ A' \arrow[r, "f'"] & B' \arrow[r] & C' \arrow[r] & D' \arrow[r, "g'"] & E' \end{tikzcd}$$

*be a commutative diagram with exact rows of $R$-modules. We have the following:*
(1) *If $\alpha$ is a surjective homomorphism and $\beta, \delta$ are injective homomorphisms, then $\gamma$ is an injective homomorphism.*
(2) *If $\epsilon$ is an injective homomorphism and $\beta, \delta$ are surjective homomorphisms, then $\gamma$ is a surjective homomorphism.*

**Remark 5.8.** *We use the homological spectral sequence associated with a first-quadrant homological double complex in the argument below.*

*Proof.* We only prove (1) since (2) follows by a similar argument. To construct a double complex, we take the given diagram, reflect it appropriately, adjoin kernels and cokernels on the left and right, and assign zero objects to all remaining entries, thereby obtaining a first quadrant homological double complex.

$$\begin{tikzcd} \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] \\ 0 \arrow[d] & 0 \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] \\ \operatorname{coker} g' \arrow[d] & E \arrow[l] \arrow[d] & D \arrow[l] \arrow[d] & C \arrow[l] \arrow[d] & B \arrow[l] \arrow[d] & A \arrow[l] \arrow[d] & \ker f \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] \\ \operatorname{coker} g'' & E' \arrow[l] & D' \arrow[l] & C' \arrow[l] & B' \arrow[l] & A' \arrow[l] & \ker f' \arrow[l] & 0 \arrow[l] \end{tikzcd}$$

If we consider the homological spectral sequence filters by rows, the first page is effectively computed by taking the homology of double complex in the horizontal direction. Since the double complex is exact in the horizontal direction, we observe that the first page consist only of 0's. Therefore, we conclude that this spectral sequence converges weakly to 0. Similarly, the homological spectral sequence obtained by filtering by columns also converges weakly to 0. For this spectral sequence, the $E^1$ page is obtained by taking the the homology of the double complex in the vertical direction:

$$\begin{tikzcd}
* & \ker\varepsilon \arrow[l] & \ker\delta \arrow[l] & \ker\gamma \arrow[l] & \ker\beta \arrow[l] & \ker\alpha \arrow[l] & * \arrow[l] \\
* & \operatorname{coker}\varepsilon \arrow[l] & \operatorname{coker}\delta \arrow[l] & \operatorname{coker}\gamma \arrow[l] & \operatorname{coker}\beta \arrow[l] & \operatorname{coker}\alpha \arrow[l] & * \arrow[l]
\end{tikzcd}$$

By assumption $\ker\delta = \ker\beta = \operatorname{coker}\alpha = 0$. By taking homology once more, we arrive at the $E^2$ page:

$$\begin{tikzcd}
* & * & * & \ker\gamma & * & * & * \\
* & * & * \arrow[ull] & * \arrow[ull] & * \arrow[ull] & 0 \arrow[ull] & * \arrow[ull]
\end{tikzcd}$$

As the spectral sequence converges weakly to 0, we know that on the $E^\infty$ page, no entry can remain. But this means that $\ker\gamma$ must vanish, since otherwise it could never disappear on the subsequent pages. This proves the claim. $\square$

5.3. **Snake Lemma.** As a second example, we aim to prove the Snake Lemma using a homological spectral sequence.

**Proposition 5.9.** ***(Snake Lemma)*** *Consider the diagram below:*

$$\begin{tikzcd}
 & A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[r, "g"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' &
\end{tikzcd}$$

*Then there is an exact sequence of $R$-modules:*

$$\ker\alpha \longrightarrow \ker\beta \longrightarrow \ker\gamma \overset{\tau}{\dashrightarrow} \operatorname{coker}\alpha \longrightarrow \operatorname{coker}\beta \longrightarrow \operatorname{coker}\gamma$$

*The map $\tau$ is called the connecting homomorphism.*

*Proof.* The proof is similar to Proposition 5.7. As before, we insert kernels, cokernels, and zeros to obtain a first quadrant homological double complex.

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] \\
0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
0 \arrow[d] & 0 \arrow[l] \arrow[d] & C \arrow[l] \arrow[d] & B \arrow[l] \arrow[d] & A \arrow[l] \arrow[d] & \ker f \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] \\
0 & \operatorname{coker} g' \arrow[l] & C' \arrow[l] & B' \arrow[l] & A' \arrow[l] & 0 \arrow[l] & 0 \arrow[l]
\end{tikzcd}$$

If we consider the homological spectral sequence filters by rows, the first page is effectively computed by taking the homology of double complex in the horizontal direction. Since the double complex is exact in the horizontal direction, we observe that the first page consist only of 0's. Therefore, we conclude that this spectral sequence converges weakly to 0. Similarly, the homological spectral sequence obtained by filtering by columns also converges weakly to 0. For this spectral sequence, the $E^1$ page is obtained by taking the the homology of the double complex in the vertical direction:

$$\begin{tikzcd} 0 & \ker\gamma \arrow[l] & \ker\beta \arrow[l, "\varphi"'] & \ker\alpha \arrow[l] & \ker f \arrow[l] & 0 \arrow[l] \end{tikzcd}$$

$$\begin{tikzcd} 0 & \operatorname{coker} g' \arrow[l] & \operatorname{coker}\gamma \arrow[l] & \operatorname{coker}\beta \arrow[l] & \operatorname{coker}\alpha \arrow[l, "\psi"'] & 0 \arrow[l] \end{tikzcd}$$

Here we observe a portion of the Snake Lemma sequence represented across the two rows, and standard arguments ensure that these sequences are exact. Finally, we take homology once more to examine the $E^2$ page.

$$\begin{tikzcd} 0 & \operatorname{coker}\varphi & 0 & 0 & 0 \\ 0 & 0 & 0 \arrow[ull] & \ker\psi \arrow[ull] & 0 \arrow[ull] \end{tikzcd}$$

Since all entries must vanish on the $E^\infty$ page, the one remaining map must necessarily be an isomorphism. By inverting this isomorphism, we obtain a connecting homomorphism:

$$\begin{tikzcd} \ker\alpha \arrow[r] & \ker\beta \arrow[r] & \ker\gamma \arrow[r, dashed, "\tau"] \arrow[d, two heads] & \operatorname{coker}\alpha \arrow[r] & \operatorname{coker}\beta \arrow[r] & \operatorname{coker}\gamma \\ & & \operatorname{coker}\varphi \arrow[r, "\cong"] & \ker\psi \arrow[u, hook] & & \end{tikzcd}$$

By simple arguments one can prove that this is indeed exact. $\square$

5.4. **Equivalence of Singular & Cellular Cohomologies.** We now turn to a topological application: proving the equivalence of singular and cellular cohomology.

**Proposition 5.10.** *Let $X$ be a finite-dimensional CW-complex. Then*

$$H^\bullet(X) = H^\bullet_{\text{cell}}(X)$$

*Proof.* Let $X_k$ denote its $k$-skeleton. Let $C^\bullet(X)$ denote its singular co-chain complex, so $C^n(X) = \operatorname{Hom}_{\mathbb{Z}}(C_n(X), \mathbb{Z})$, where $C_n(X)$ is the free abelian group generated by maps $\Delta^n \to X$. We filter this by setting

$$F_pC^n(X) = \{\varphi \in C^n(X) \mid \varphi|_{C_n(X_p)} = 0\} = \ker(C^n(X) \to C^n(X_p)),$$

where the map $C^n(X) \to C^n(X_p)$ is the natural restriction map. This is indeed a decreasing filtration. By Proposition 3.3 and Remark 3.5, we get a cohomological spectral sequence such that

$$E_0^{p,q} = \mathrm{G}_p\, C^{p+q}(X) \Rightarrow H^{p+q}(X).$$

We claim that $E_0^{p,q} \cong C^{p+q}(X^{p+1}, X^p)$, the group of relative cochains. Note that we have a homomorphism of short exact sequences:

$$\begin{tikzcd} 0 \arrow[r] & F_{p+1}C^{p+q}(X) \arrow[r] \arrow[d] & C^{p+q}(X) \arrow[r] \arrow[d, "\mathrm{Id}_{C^{p+q}(X)}"] & C^{p+q}(X^{p+1}) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & F_pC^{p+q}(X) \arrow[r] & C^{p+q}(X) \arrow[r] & C^{p+q}(X^p) \arrow[r] & 0 \end{tikzcd}$$

Since the middle map is an isomorphism, the snake lemma tells us that the left map is injective, and that its cokernel is isomorphic to $\ker(C^{p+q}(X^{p+1}) \to C^{p+q}(X^p))$. Hence, we have

$$E_0^{p,q} \cong \frac{F_p C^{p+q}(X)}{F_{p+1} C^{p+q}(X)} \cong \ker(C^{p+q}(X^{p+1}) \to C^{p+q}(X^p)) = C^{p+q}(X^{p+1}, X^p)$$

By definition, the cohomology of this page gives relative cohomology, so

$$E_1^{p,q} = H^{p+q}(X^{p+1}, X^p)$$

Recall that $H^{p+q}(X^{p+1}, X^p) = 0$ if $q \neq 1$, so the only nontrivial differentials on the $E_1$ page are

$$d_1^{p,1} : H^{p+1}(X^{p+1}, X^p) \to H^{p+2}(X^{p+2}, X^{p+1}).$$

One easily checks that these agree with the differentials defining cellular cohomology, so the $E_2$ page is given by

$$E_2^{p,q} = \begin{cases} H^{p+1}_{\text{cell}}(X) & \text{if } q = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Since there are no further non-trivial differentials in the spectral sequence, we have $E_2^{p,q} = E_\infty^{p,q}$. Moreover, because each diagonal $p + q = n$ contains at most one nonzero term, it follows that the associated graded pieces stabilize, and we obtain an isomorphism

$$H^p(X) \cong E_\infty^{p-1,1} \cong H^p_{\text{cell}}(X).$$

Therefore, the singular and cellular cohomology groups of $X$ are isomorphic. □

## 6. Convergence

We have seen in Section 3 that a bounded descending filtration naturally induces a *convergent* cohomological spectral sequence, in the sense that the entries $E_r^{p,q}$ stabilize for sufficiently large $r$, as a function of $(p, q)$, allowing us to define the limiting page of a spectral sequence. We now turn to the question of convergence for a general spectral sequence. We first need to define the notion of the limiting page of a general spectral sequence. If $\{E_r^{p,q}, d_r^{p,q}\}_{r \in \mathbb{N}}$ is a a cohomological spectral sequence, we have a tower of $R$-submodules

$$B_0^{p,q} \subseteq B_1^{p,q} \subseteq B_2^{p,q} \subseteq \cdots \subseteq \cdots \subseteq Z_2^{p,q} \subseteq Z_1^{p,q} \subseteq Z_0^{p,q} \tag{3}$$

Here $Z_r^{p,q}, B_r^{p,q}$ are defined as in Section 3. Define

$$Z_\infty^{p,q} = \bigcap_{r=1}^{\infty} Z_r^{p,q}, \quad B_\infty^{p,q} = \bigcup_{r=1}^{\infty} B_r^{p,q}$$

Note that $B_r^{p,q} \subseteq Z_\infty^{p,q}$ for each $(p, q) \in \mathbb{Z}^2$. Clearly, the construction generalizes to the case where we have a cohomological spectral sequence starting on the $r_0$-th page, for some $r_0 \in \mathbb{N}$. This allows us to define a potential candidate for the limit of a cohomological spectral sequence.

**Definition 6.1.** Let $r_0 \in \mathbb{N}$, and let $\{E_r^{p,q}, d_r^{p,q}\}_{r \geq r_0}$ be a cohomological spectral sequence of $R$-modules. The $E_\infty$ page of $\{E_r^{p,q}, d_r^{p,q}\}_{r \geq r_0}$ is defined such that

$$E^\infty_{p,q} = \frac{Z^\infty_{p,q}}{B^\infty_{p,q}}$$

In the specific instances examined in Section 3, the spectral sequence was shown to converge to the associated graded cohomology of a co-chain complex. This observation motivates the following definition.

**Definition 6.2.** Let $r_0 \in \mathbb{N}$. Let $\{M_n\}_{n\in\mathbb{Z}}$ be a family of $R$-modules, and let $\{E_r^{p,q}, d_r^{p,q}\}_{r\geq r_0}$ be a cohomological spectral sequence of $R$-modules.

(1) We say that $\{E_r^{p,q}, d_r^{p,q}\}_{r\geq r_0}$ converges weakly to $\{M_n\}_{n\in\mathbb{Z}}$ if there exists a decreasing exhaustive filtration

$$\cdots \supseteq F_{p-1}M_n \supseteq F_pM_n \supseteq F_{p+1}M_n \supseteq \cdots$$

for each $n \in \mathbb{Z}$ and furthermore, there exist isomorphisms

$$E_\infty^{p,q} \cong G_p(M_{p+q}) := \frac{F_pM_{p+q}}{F_{p+1}M_{p+q}}.$$

We write

$$E_r^{p,q} \Rightarrow G_p(M_{p+q})$$

(2) We say that $\{E_r^{p,q}, d_r^{p,q}\}_{r\geq r_0}$ approaches $\{M_n\}_{n\in\mathbb{Z}}$ if the filtration in (1) is exhaustive and separated.

(3) We say that $\{E_r^{p,q}, d_r^{p,q}\}_{r\geq r_0}$ converges strongly to $\{M_n\}_{n\in\mathbb{Z}}$ if it approaches $\{M_n\}_{n\in\mathbb{Z}}$ and

$$M_n = \varprojlim_{p\in\mathbb{Z}} \left(M_n/F_pM_n\right).$$

We have the following result:

**Proposition 6.3.** *The cohomological spectral sequence associated to a decreasing, exhaustive and bounded below filtration of a co-chain complex converges weakly.*

*Proof.* The proof is skipped. $\square$

## 7. More Applications

Let's look at some more applications.

7.1. **Two Column Spectral Sequence.** Let $\{E_r^{p,q}\}_{r\geq 1}$ be a cohomological spectral sequence associated to a decreasing, exhaustive and separated filtration. Assume that $E_2^{p,q} = 0$ unless $p = 0, 1$. Hence, $E_2$ pages looks like the following:

$$\begin{tikzcd}
0 \arrow[drr] & 0 \arrow[drr] & E_2^{0,1} \arrow[drr] & E_2^{1,1} \arrow[drr] & 0 & 0 \\
0 \arrow[drr] & 0 \arrow[drr] & E_2^{0,0} \arrow[drr] & E_2^{1,0} \arrow[drr] & 0 & 0 \\
0 & 0 & E_2^{0,-1} & E_2^{1,-1} & 0 & 0
\end{tikzcd}$$

Hence, we see that $E_2^{p,q} = E_\infty^{p,q}$. Assume that the spectral sequence converges weakly to $\{M_n\}_{n\in\mathbb{Z}}$. Hence, we have

$$E_2^{p,q} = E_\infty^{p,q} \cong \frac{F_pM_{p+q}}{F_{p+1}M_{p+q}}$$

If $p \neq 0, 1$, we get

$$0 = E_2^{p,q} = \frac{F_p M_{p+q}}{F_{p+1} M_{p+q}},$$

which tells us $F_p H_{p+q} = F_{p+1} H_{p+q}$ for all $q \in \mathbb{Z}$ such that $p \neq 0, 1$. Therefore the filtration looks like

$$\cdots = F_{-2} M_n = F_{-1} M_n \supseteq F_0 M_n \supseteq F_1 M_n = F_2 M_n = \cdots$$

Since the filtration is assumed to be exhaustive and separated, we have that

$$\begin{aligned} F_{-1} M_n = F_{-2} M_n = \cdots = M_n, \\ F_1 M_n = F_2 M_n = \cdots = 0. \end{aligned}$$

For $p = 0$, we notice that

$$E_2^{0,n} = E_\infty^{0,n} \cong \frac{F_0 M_n}{F_1 M_n} = F_0 M_n.$$

For $p = 1$, we get

$$E_2^{-1,n+1} = E_\infty^{1,n-1} \cong \frac{F_{-1} M_n}{F_0 H_n} = \frac{M_n}{F_0 M_n}.$$

Hence, the short exact sequence

$$0 \longrightarrow F_0 M_n \longrightarrow M_n \longrightarrow \frac{M_n}{F_0 M_n} \longrightarrow 0$$

turns into the short exact sequence

$$0 \longrightarrow E_2^{0,n} \longrightarrow M_n \longrightarrow E_2^{-1,n+1} \longrightarrow 0.$$

for each $n \in \mathbb{Z}$.

**Remark 7.1.** *If we had a homological spectral sequence, the analogous statement would be that there is a short exact sequence:*

$$0 \longrightarrow E_{0,n}^2 \longrightarrow M_n \longrightarrow E_{1,n-1}^2 \longrightarrow 0$$

*for each $n \in \mathbb{Z}$.*

7.2. **Balancing Tor.** We now prove the result that the Tor functor is a 'balanced' functor.

**Proposition 7.2.** *Let $R$ be a ring, $A$ a right $R$-module, and $B$ a left $R$-module. Denote by $\mathrm{Tor}_*^R(A, -)$ the left-derived functors of the tensor product functor $A \otimes_R -$, and by $\mathrm{Tor}_*^R(-, B)$ the left-derived functors of $- \otimes_R B$. We have that*

$$\mathrm{Tor}_*^R(A, B) \cong \mathrm{Tor}_*^R(B, A)$$

*Proof.* We choose projective resolutions

$$\begin{aligned} \cdots \to P_2 \to P_1 \to P_0 \to A \to 0, \\ \cdots \to Q_2 \to Q_1 \to Q_0 \to B \to 0 \end{aligned}$$

of $A$ and $B$, respectively. We define a first quadrant homological double complex $C_{\bullet,\bullet}$ by $C_{p,q} = P_p \otimes Q_q$, where the maps are the induced ones coming from the maps in the projective resolutions. The double complex can be visualized as follows:

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
P_0 \otimes Q_2 \arrow[d] & P_1 \otimes Q_2 \arrow[l] \arrow[d] & P_2 \otimes Q_2 \arrow[l] \arrow[d] & \cdots \arrow[l] \\
P_0 \otimes Q_1 \arrow[d] & P_1 \otimes Q_1 \arrow[l] \arrow[d] & P_2 \otimes Q_1 \arrow[l] \arrow[d] & \cdots \arrow[l] \\
P_0 \otimes Q_0 & P_1 \otimes Q_0 \arrow[l] & P_2 \otimes Q_0 \arrow[l] & \cdots \arrow[l]
\end{tikzcd}$$

Since projective modules are flat modules, the rows and columns of this double complex are indeed complexes, and the squares in the double complex are commutative. We first filter this double complex by columns. Note that the homology in the vertical direction determines the $E^1$ page such that $E^1_{p,q} = \mathrm{Tor}_q^R(P_p, B)$. Since $P_p$ are projective and hence flat modules, we have $\mathrm{Tor}_q^R(P_p, B) = 0$ for $q > 1$. Since $\mathrm{Tor}_0^R(P_p, B) = P_p \otimes B$, the $E^1$ page is as follows:

$$\begin{tikzcd}
\vdots & \vdots & \vdots & \\
0 & 0 \arrow[l] & 0 \arrow[l] & \cdots \arrow[l] \\
P_0 \otimes B & P_1 \otimes B \arrow[l] & P_2 \otimes B \arrow[l] & \cdots \arrow[l]
\end{tikzcd}$$

Taking homology of the $E^1$ page yields the $E^2$ page:

$$\begin{tikzcd}
0 & 0 & 0 & 0 & 0 & 0 \\
\mathrm{Tor}_0^R(A,B) & \mathrm{Tor}_1^R(A,B) & \mathrm{Tor}_2^R(A,B) \arrow[ull] & \mathrm{Tor}_3^R(A,B) \arrow[ull] & \mathrm{Tor}_4^R(A,B) \arrow[ull] & \mathrm{Tor}_5^R(A,B) \arrow[ull]
\end{tikzcd}$$

In a similar manner, we can filter the double complex by rows, and we obtain a spectral sequence whose $E^2$-page looks like:

$$\begin{tikzcd}
0 & 0 & 0 & 0 & 0 & 0 \\
\mathrm{Tor}_0^R(B,A) & \mathrm{Tor}_1^R(B,A) & \mathrm{Tor}_2^R(B,A) \arrow[ull] & \mathrm{Tor}_3^R(B,A) \arrow[ull] & \mathrm{Tor}_4^R(B,A) \arrow[ull] & \mathrm{Tor}_5^R(B,A) \arrow[ull]
\end{tikzcd}$$

For both spectral sequences, we have $E_2^{p,q} = E_\infty^{p,q}$. Since both spectral sequences converge to the associated graded object, we can conclude that

$$\mathrm{Tor}_*^R(A,B) \cong \mathrm{Tor}_*^R(B,A).$$

$\square$

7.3. **Universal Coefficient Theorem.** Let's prove the universal coefficient theorem in homology. Let $(C_\bullet, d_\bullet)$ be a chain complex consisting of free abelian groups, and let $A$ be any abelian group. The universal coefficient theorem in homology relates $H_\bullet(C_\bullet)$ and $H_\bullet(C_\bullet \otimes A)$.

**Proposition 7.3.** *Let $(C_\bullet, d_\bullet)$ be a chain complex consisting of free abelian groups, and let $A$ be any abelian group. We have the following short exact sequence*

$$0 \longrightarrow A \otimes H_n(C_\bullet) \longrightarrow M_n \longrightarrow Tor_1^{\mathbb{Z}}(A, H_{n-1}(C_\bullet)) \longrightarrow 0$$

*Proof.* Choose a projective resolution for $A$:

$$\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow A \longrightarrow 0.$$

We define a first quadrant homological double complex $C_{\bullet,\bullet}$ by $C_{p,q} = P_p \otimes C_q$, where the maps are the induced ones coming from the maps in the projective resolutions. The double complex can be visualized as follows:

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
P_0 \otimes C_2 \arrow[d] & P_1 \otimes C_2 \arrow[l] \arrow[d] & P_2 \otimes C_2 \arrow[l] \arrow[d] & \cdots \arrow[l] \\
P_0 \otimes C_1 \arrow[d] & P_1 \otimes C_1 \arrow[l] \arrow[d] & P_2 \otimes C_1 \arrow[l] \arrow[d] & \cdots \arrow[l] \\
P_0 \otimes C_0 & P_1 \otimes C_0 \arrow[l] & P_2 \otimes C_0 \arrow[l] & \cdots \arrow[l]
\end{tikzcd}$$

We first filter this double complex by columns. Taking the homology in the vertical direction, we obtain the $E^1$ page:

$$\begin{tikzcd}
\vdots & \vdots & \vdots & \\
P_0 \otimes H_2(C) & P_1 \otimes H_2(C) \arrow[l] & P_2 \otimes H_2(C) \arrow[l] & \cdots \arrow[l] \\
P_0 \otimes H_1(C) & P_1 \otimes H_1(C) \arrow[l] & P_2 \otimes H_1(C) \arrow[l] & \cdots \arrow[l] \\
P_0 \otimes H_0(C) & P_1 \otimes H_0(C) \arrow[l] & P_2 \otimes H_0(C) \arrow[l] & \cdots \arrow[l]
\end{tikzcd}$$

This follows because $P_p \otimes -$ in an exact functor. The rows here correspond to the complexes used to calculate $\mathrm{Tor}_*^{\mathbb{Z}}(A, -)$, so the $(p,q)$-th entry on the $E^2$ page is $\mathrm{Tor}_q^{\mathbb{Z}}(A, H_p(C))$. Let's examine this more closely. Applying $A \otimes -$ to the short exact sequence of chain complexes

$$0 \longrightarrow \operatorname{im} d_\bullet \longrightarrow \ker d_\bullet \longrightarrow H_\bullet(C_\bullet) \longrightarrow 0$$

and deriving gives a long exact sequence.

$$\cdots \to \mathrm{Tor}_2^{\mathbb{Z}}(A, H_n(C)) \to \mathrm{Tor}_1^{\mathbb{Z}}(A, \operatorname{im} d_{n-1}) \to \mathrm{Tor}_1^{\mathbb{Z}}(A, \ker d_n) \to \mathrm{Tor}_1^{\mathbb{Z}}(A, H_n(C)) \to \mathrm{Tor}_0^{\mathbb{Z}}(A, \operatorname{im} d_{n-1}) \to \cdots$$

But since $\operatorname{im} d_{n-1}$ and $\ker d_n$ are subgroups of the free abelian group $C_n$, they are themselves free. Therefore, the higher Tor groups vanish. By the long exact sequence, it follows that $\operatorname{Tor}_q^{\mathbb{Z}}(A, H_p(C)) = 0$ for all $q \geq 2$. Hence, the $E^2$ pages looks as follows:

$$\begin{tikzcd}
\operatorname{Tor}_0^{\mathbb{Z}}(A, H_2(C_\bullet)) & \operatorname{Tor}_1^{\mathbb{Z}}(A, H_2(C_\bullet)) & 0 & 0 \\
\operatorname{Tor}_0^{\mathbb{Z}}(A, H_1(C_\bullet)) & \operatorname{Tor}_1^{\mathbb{Z}}(A, H_1(C_\bullet)) & 0 \arrow[llu] & 0 \arrow[llu] \\
\operatorname{Tor}_0^{\mathbb{Z}}(A, H_0(C_\bullet)) & \operatorname{Tor}_1^{\mathbb{Z}}(A, H_0(C_\bullet)) & 0 \arrow[llu] & 0 \arrow[llu]
\end{tikzcd}$$

By Remark 7.1, we have

$$0 \longrightarrow E^2_{0,n} \longrightarrow M_n \longrightarrow E^2_{1,n-1} \longrightarrow 0$$

Note that we have $E^2_{0,n} = \operatorname{Tor}_0^{\mathbb{Z}}(A, H_n(C_\bullet)) = A \otimes H_n(C_\bullet)$ and that $E^2_{1,n-1} = \operatorname{Tor}_1^{\mathbb{Z}}(A, H_{n-1}(C_\bullet))$. Hence, the above exact sequence reads

$$0 \longrightarrow A \otimes H_n(C_\bullet) \longrightarrow M_n \longrightarrow \operatorname{Tor}_1^{\mathbb{Z}}(A, H_{n-1}(C_\bullet)) \longrightarrow 0$$

To identify $M_n$, we now filter the double complex by rows. This amounts to considering a spectral sequence whose $E^0$ page is the transposed double complex:

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
P_2 \otimes C_0 \arrow[d] & P_2 \otimes C_1 \arrow[l] \arrow[d] & P_2 \otimes C_2 \arrow[l] \arrow[d] & \cdots \arrow[l] \\
P_1 \otimes C_0 \arrow[d] & P_1 \otimes C_1 \arrow[l] \arrow[d] & P_1 \otimes C_2 \arrow[l] \arrow[d] & \cdots \arrow[l] \\
P_0 \otimes C_0 & P_0 \otimes C_1 \arrow[l] & P_0 \otimes C_2 \arrow[l] & \cdots \arrow[l]
\end{tikzcd}$$

Here the vertical maps are induced by the horizontal maps of the double complex. Hence, taking the homology in the in the vertical direction of the transposed double complex is equivalent to taking the homology of the double complex in the horizontal direction. Hence, the $(p, q)$-th entry of the $E^1$ page is $\operatorname{Tor}_p(A, C_q)$. For $p \geq 1$, this vanishes because each $C_q$ is a free abelian group. Moreover, we have $\operatorname{Tor}_0(A, C_q) = A \otimes C_q$. Hence, the $E^1$ page is given by:

$$\begin{tikzcd}
\vdots & \vdots & \vdots & \\
0 & 0 \arrow[l] & 0 \arrow[l] & \cdots \\
A \otimes C_0 & A \otimes C_1 \arrow[l] & A \otimes C_2 \arrow[l] & \cdots
\end{tikzcd}$$

Here the horizontal differentials are induced by the vertical differentials of the double complex. Taking homology, on the $E^2$ page, everything except the bottom row is zero. In the bottom row, we have:

$$H_0(A \otimes C_\bullet) \quad H_1(A \otimes C_\bullet) \quad H_2(A \otimes C_\bullet) \quad \cdots$$

This shows that $M_n = H_n(A \otimes C_\bullet)$. Therefore, we have

$$0 \longrightarrow A \otimes H_n(C_\bullet) \longrightarrow M_n \longrightarrow \mathrm{Tor}_1^{\mathbb{Z}}(A, H_{n-1}(C_\bullet)) \longrightarrow 0$$

$\square$