TOPOLOGICAL K-THEORY

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ABSTRACT. These are notes on topological K-theory. I compiled them during my graduate studies at the University of Maryland while participating in a reading course under the guidance of Dr. Jonathan Rosenberg. If you notice any typos or errors, please feel free to send corrections to junaid.aftab1994@gmail.com.

CONTENTS

1. WHY K-THEORY?

Let X be a topological space. The premise behind K-theory is that the global topological properties of X can be studied by studying vector bundles over X. K-theory is a generalized cohomology theory that formalizes this idea. K-theory can be used to address the following questions:

(1) Which spheres $\mathbb{S}^n$ are parallelizable? Adams' proof that only $\mathbb{S}^0, \mathbb{S}^1, \mathbb{S}^3, \mathbb{S}^7$ are parallelizable[1] used K-theory. The solution to this problem is related to the classification of division algebras.

(2) How many linearly independent vector fields are there on $\mathbb{S}^n$, or more generally on any smooth manifold M?

[1] A smooth manifold is parallelizable if its tangent bundle is trivial.

K-theory also provides the framework for other related ideas in mathematics, such as the Atiyah-Singer index theorem. More recently, K-theory has also been applied to physics, both in high-energy theory and condensed matter physics. Specific applications include the classification of topological insulators and topological phases of matter.

Part 1. Vector Bundles

In what follows, let $\mathbb{K}$ be either $\mathbb{R}$ or $\mathbb{C}$. We work in the category of topological spaces, **Top**.

2. Definitions & Examples

Definition 2.1. Let $E, X \in \mathbf{Top}$. A **$\mathbb{K}$-vector bundle** is a triple (E, X, π), where $\pi : E \longrightarrow X$ is a continuous surjective map such that:

(1) For every $x \in X$, the fiber $\pi^{-1}(x) := \{v \in E \mid \pi(v) = x\} := E_{\pi^{-1}(x)}$ is a $\mathbb{K}$-vector space.
(2) For every $x \in X$, there exists an open neighborhood U_x of x in X and a homeomorphism $\varphi_x : \pi^{-1}(U_x) \longrightarrow U \times \mathbb{K}^{n_x}$ for some integer $n_x \in \mathbb{N}$ (possibly depending on x) such that the following diagram commutes:

$$\begin{tikzcd} \pi^{-1}(U_x) \arrow[rr, "\varphi_x"] \arrow[dr, "\pi"'] & & U_x \times k^{n_x} \arrow[dl, "\mathrm{proj}_1"] \\ & U & \end{tikzcd}$$

Here proj_1 denotes the projection onto the first factor. The map φ is called a local trivialization.

X is called the base space and E is called the total space.

If the map $x \mapsto n_x$ in the definition of a vector bundle is constant, we say that the vector bundle is of rank n if the image of this constant map is $n \in \mathbb{N}$. We shall mostly be concerned with this case. If the constant dimension is n, we say that the vector bundle is of rank n which we write as $\dim X$.

Remark 2.2. *It can be argued that the dimension of a fibre is locally constant. In particular, it is constant on each connected component of X and if X is connected then the dimension is constant. From now on we implicitly assume that all topological spaces are connected in order for the rank of a vector bundle to be well-defined.*

Remark 2.3. *The pair (U_x, φ_x) is called a locally trivializing cover. In what follows, we shall occasionally write a locally trivializing cover as $\{(U_\alpha, \varphi_\alpha)\}_\alpha$ or simply as $\{U_\alpha\}_\alpha$. If we need to emphasize the choice of $x \in X$, we will replace α with x.*

Remark 2.4. *In what follows, we simply use the phrase vector bundle to refer to a generic $\mathbb{K}$-vector bundle, assuming the base field $\mathbb{K}$ is understood. Moreover, if (E, X, π) is a vector bundle, we will from time to time write that $E \longrightarrow X$ is a vector bundle, or simply that E is a vector bundle.*

Example 2.5. Let $X \in \mathbf{Top}$. The trivial rank n vector bundle is $X \times \mathbb{K}^n \longrightarrow X$ with the map being the the usual projection map. It is clear that this is a vector bundle since $X \times \mathbb{K}^n \longrightarrow X$ is a continuous surjection such that the following diagram commutes:

$$\begin{tikzcd} X \times \mathbb{K}^n \arrow[r, "\mathrm{Id}"] \arrow[d] & X \times \mathbb{K}^n \arrow[dl, "\mathrm{proj}_1"] \\ X & \end{tikzcd}$$

Remark 2.6. *For $n = 0$, we identify $X \cong X \times \mathbb{K}^0$. The trivial vector bundle $X \times \mathbb{K}^n$ is sometimes written as ε^n for $n \in \mathbb{N} \cup \{0\}$.*

If it is possible to choose $U_x = X$ for some $x \in X$, then the vector bundle is called a trivial bundle. Every vector bundle locally resembles a trivial bundle, although perhaps not globally. Hence, a vector bundle can be thought of as a *twisted product space*.

Example 2.7. Let's consider two examples derived from the category of smooth manifolds which give examples of smooth vector bundles. However, we will not define smooth vector bundles here.

(1) Let M be a smooth manifold. The tangent bundle, (TM, M, π), of a smooth manifold is a (smooth) vector bundle. The map π is the projection map and the vector space structure on the fibers is the usual one. The local trivialization are defined as follows. Given any smooth chart (U, φ) for M, define a map $\varphi : \pi^{-1}(U) \to U \times \mathbb{R}^n$ by
$$\varphi(p, v^i \partial_i) = (p, v^1, \cdots, v^n).$$
The composite map
$$\pi^{-1}(U) \xrightarrow{\varphi} U \times \mathbb{R}^n \xrightarrow{\varphi \times \mathrm{Id}} \varphi(U) \times \mathbb{R}^n$$
is equal to the coordinate map that makes the TM into a smooth manifold. Since both the coordinate map and $\varphi \times \mathrm{Id}$ are diffeomorphisms, so is φ.

(2) (Sketch) Similarly, the normal bundle to a smooth manifold, denoted as NM, is an example of a (smooth) vector bundle.

Remark 2.8. *A number of examples given below are also instances of smooth vector bundles. However, we will not discuss the nuanced details in these notes.*

Example 2.9. Let $\mathbb{RP}^n \cong \mathbb{S}^n/\mathbb{Z}_2$. The rank one bundle over $\mathbb{RP}^n$ is a vector bundle with total space
$$\gamma^1_{n+1} = \{([x], v) \in \mathbb{RP}^n \times \mathbb{R}^{n+1} \mid v = tx \text{ for some } t \in \mathbb{R}^\times\}$$
endowed with the subspace topology. The map $\pi : \gamma^1_{n+1} \to \mathbb{RP}^n$ is just the projection. For $[x] \in \mathbb{RP}^n$, let $x \in U \subseteq \mathbb{S}^n$ be any open set such that $U \cap a(U) = \emptyset$, where $a : \mathbb{S}^n \to \mathbb{S}^n$ is the antipodal map. Let U_x denote the image of U in $\mathbb{RP}^n$. A homeomorphism $\varphi_x : U_x \times \mathbb{R} \to \pi^{-1}(U_x)$ is defined by the requirement that
$$\varphi_x([y], t) = ([y], ty)$$
for each $(y, t) \in U \times \mathbb{R}$. The pair (U_x, φ_x) is a local trivialization of γ^1_{n+1}.

Remark 2.10. *Rank one vector bundles are called line bundles. The rank one vector bundle over $\mathbb{RP}^n$ is called the canonical line bundle.*

Definition 2.11. Let $X_i, E_i \in$ **Top** and let (E_1, X_1, π_1) and (E_2, X_2, π_2) be two vector bundles. A **morphism of vector bundles** is given by a pair of continuous functions $f : E_1 \to E_2$ and $g : X_1 \to X_2$ such that the following diagram commutes:
$$\begin{tikzcd} E_1 \arrow[r, "f"] \arrow[d, "\pi_1"'] & E_2 \arrow[d, "\pi_2"] \\ X_1 \arrow[r, "g"'] & X_2 \end{tikzcd}$$

Remark 2.12. *The commutativity of the diagram implies that for each $x \in X$, the map f in a morphism of vector bundles maps each vector space $E_{1,\pi_1^{-1}(x_1)}$ linearly onto the corresponding vector space $E_{2,\pi_2^{-1}(g(x_1))}$*

An important special case we will consider is when $X_1 = X_2 = X$. In this case, $g = \mathrm{Id}_X$.
$$\begin{tikzcd} E_1 \arrow[rr, "f"] \arrow[dr, "\pi_1"'] & & E_2 \arrow[dl, "\pi_2"] \\ & X & \end{tikzcd}$$

Two vector bundles (E_1, X, π_1) and (E_2, X, π_2) (of the same rank) are isomorphic if the map f from E_1 to E_2 is a fiber preserving homeomorphism.

Example 2.13. Let $X \in \mathbf{Top}$, and let $E = X \times \mathbb{K}^n$ and $F = X \times \mathbb{K}^m$ be two trivial bundles over X. Any morphism $\varphi : E \to F$ determines a map

$$\varphi : X \to \mathrm{Hom}(\mathbb{K}^n, \mathbb{K}^m)$$

by the formula

$$\varphi(x)(v) = \mathrm{proj}_2(\varphi(x))v$$

If we give $\mathrm{Hom}(\mathbb{K}^n, \mathbb{K}^m) \cong \mathbb{K}^{nm}$ its usual topology, then φ is continuous. Conversely, any such continuous map $\varphi : X \to \mathrm{Hom}(\mathbb{K}^n, \mathbb{K}^m)$ determines a homomorphism $\varphi : E \to F$.

Note that a the total space of a vector bundle is a topological space such that the fibers are vector spaces. This begs the question: is it possible to infer properties about a morphism of vector bundles if only partial information about the action of the morphism on the fibers is provided. The answer is yes, and here is a sample proposition along these lines:

Lemma 2.14. *Let $X \in$ **Top**, and let (E_1, X, π_1) and (E_1, X, π_2) be two vector bundles. Let $f : E_1 \to E_2$ be a morphism of vector bundles. For each $x \in X$, if $f_x := f|_{\pi_1^{-1}(x)}$ is a linear isomorphism for each fiber $\pi_1^{-1}(x)$, then f is an isomorphism of vector bundles.*

Proof. The map f is one-to-one and onto, since each f_x is a linear isomorphism and f takes each fiber in E_1 to the corresponding fiber in E_2. Since continuity is only a local condition, we may WLOG assume that $E_1 = X \times \mathbb{K}^n$ and $E_2 = X \times \mathbb{K}^n$ are trivial vector bundles. By Example 2.13, a continuous function f from E_1 to E_2 yields a continuous maps of the form

$$\varphi : X \to \mathrm{Hom}(\mathbb{K}^n, \mathbb{K}^n)$$

By assumption, $\varphi(X) \subseteq \mathrm{Isom}(\mathbb{K}^n, \mathbb{K}^n)$, the set of isomorphism from from $\mathbb{K}^n$ to $\mathbb{K}^n$. This allows us to construct

$$\varphi' : X \to \mathrm{Hom}(\mathbb{K}^n, \mathbb{K}^n),$$

which yields a continuous map $f' : E_2 \to E_1$ which is a continuous inverse of f. □

Example 2.15. The following is a list of some examples of isomorphisms of vector bundles.

(1) Let $\mathbb{S}^1 \subseteq \mathbb{C}$. The map

$$f : \mathbb{S}^1 \times \mathbb{R} \to T\mathbb{S}^1$$
$$\varphi(e^{i\theta}, t) = (e^{i\theta}, tie^{i\theta})$$

is an isomorphism of vector bundles since $f_x : \mathbb{R} \to T_x\mathbb{S}^1$ is a linear isomorphism for each $x \in \mathbb{S}^1$.

(2) Similarly, the normal bundle $N\mathbb{S}^n$ is isomorphic to the trivial vector bundle $\mathbb{S}^n \times \mathbb{R}$ by the map

$$f : N\mathbb{S}^n \to \mathbb{S}^n \times \mathbb{R}$$
$$(x, tx) \mapsto (x, t)$$

3. Transition Functions

Any vector bundle that is not a trivial (product) bundle requires more than one local trivialization. Lemma 3.1 shows that the composition of two local trivializations has a simple form where they overlap:

Lemma 3.1. *Let $X, E \in$ **Top** and let $\pi : E \to X$ be a rank n vector bundle. Suppose*

$$\varphi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{K}^n$$
$$\varphi_\beta : \pi^{-1}(U_\beta) \to U_\beta \times \mathbb{K}^n$$

are two local trivializations of E with $U_\alpha \cap U_\beta \neq \emptyset$. There exists a continuous map $\theta_{\alpha\beta} : U \cap V \to \mathrm{GL}(n, \mathbb{K})$ called a transition function such that the composition $g_{\alpha\beta} := \varphi_\alpha \circ \varphi_\beta^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{K}^n \to (U_\alpha \cap U_\beta) \times \mathbb{K}^n$ has the form

$$\varphi_\alpha \circ \varphi_\beta^{-1}(x, v) = (x, g_{\alpha\beta}(x)v),$$

Proof. The following diagram commutes:

$$\begin{tikzcd}
(U_\alpha \cap U_\beta) \times \mathbb{K}^n \arrow[dr, "\mathrm{proj}_1"'] & \pi^{-1}(U_\alpha \cap U_\beta) \arrow[l, "\varphi_\beta"'] \arrow[r, "\varphi_\alpha"] \arrow[d, "\pi"] & (U_\alpha \cap U_\beta) \times \mathbb{K}^n \arrow[dl, "\mathrm{proj}_1"] \\
& U_\alpha \cap U_\beta &
\end{tikzcd}$$

Hence,

$$\varphi_\alpha \circ \varphi_\beta^{-1}(x, v) = (x, \eta_{\alpha\beta}(x, v))$$

for some continuous map $\eta_{\alpha\beta} : U \cap V \times \mathbb{K}^n \to \mathbb{K}^n$. For each fixed $x \in U \cap V$, we have:

$$(x, v) = \mathrm{Id}_{U_\alpha \cap U_\beta} = (\varphi_\beta \circ \varphi_\alpha^{-1}) \circ (\varphi_\alpha \circ \varphi_\beta^{-1})(x, v) = (x, \eta_{\beta\alpha} \circ \eta_{\alpha\beta}(x, v))$$

Hence, $(\eta_{\beta\alpha} \circ \eta_{\alpha\beta})(x, \cdot) = \mathrm{Id}_{\mathbb{K}^n}$ for each fixed $x \in X$. Moreover, each $\eta_{\alpha\beta}(x, \cdot)$ is $\mathbb{K}$-linear since

$$\varphi_\alpha \circ \varphi_\beta^{-1}(x, c_1 v_1 + c_2 v_2) = c_1(\varphi_\alpha \circ \varphi_\beta^{-1})(x, v_1) + c_2(\varphi_\alpha \circ \varphi_\beta^{-1})(x, v_2)$$

for each fixed $x \in X$. So there is a non-singular $n \times n$ matrix $g_{\alpha\beta}(x)$ such that $\eta_{\alpha\beta}(x, v) = g_{\alpha\beta}(x)v$. Hence, we have a map $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(n, \mathbb{K})$. It can be checked that this map is continuous. □

The maps $g_{\alpha\beta}$ are called transition function. The transition functions satisfy the following properties:

Lemma 3.2. *Let $E, X \in$ **Top** and let $\pi : E \to X$ be a rank n vector bundle with transition function $g_{\alpha\beta}$'s. The transition functions satisfy the following properties:*

(1) $g_{\gamma\beta} g_{\beta\alpha}(x)(x) = g_{\gamma\alpha}(x)$, *for all $x \in U_\alpha \cap U_\beta \cap U_\gamma$.*
(2) $g_{\alpha\alpha}(x) = \mathrm{Id}_{\mathbb{K}^n}$, *for all $x \in U_\alpha$.*
(3) $g_{\beta\alpha}(x) = g_{\alpha\beta}^{-1}(x)$, *for all $x \in U_\alpha \cap U_\beta$.*

Proof. It suffices to prove (1). On $U_\alpha \cap U_\beta \cap U_\gamma$, we have:

$$(\varphi_\alpha \circ \varphi_\beta^{-1}) \circ (\varphi_\beta \circ \varphi_\gamma^{-1}) = \varphi_\alpha \circ \varphi_\gamma^{-1}.$$

Hence,

$$g_{\gamma\beta} g_{\beta\alpha}(x)(x) = g_{\gamma\alpha}(x)$$

for all $x \in U_\alpha \cap U_\beta \cap U_\gamma$. (2) follows by letting $\beta, \gamma = \alpha$ and using the fact that $g_{\alpha\alpha}(x)$ admits a smooth inverse. (3) follows from (1) and (2). □

Using Lemma 3.1 as motivation, we can also reverse the reasoning and start with an open cover $\{U_\alpha\}_\alpha$ of X and then patching together different $U_\alpha \times \mathbb{K}^n$ to form a total space E using the consistency demanded by the transition functions as in Lemma 3.2. Considering this, we construct a vector bundle as follows:

Proposition 3.3. *Let $X \in$ **Top**. Assume we are given an open cover $\{U_\alpha\}_\alpha$ of X, and a family of continuous functions*

$$\{g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(n, \mathbb{K})\}$$

satisfying the conditions in Lemma 3.2. There is a rank n vector bundle $\pi : E \to X$ with local trivializations $\varphi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{K}^n$ whose transition functions are the given maps $g_{\alpha\beta}$'s.

Proof. Let

$$E' = \coprod_\alpha \{\alpha\} \times U_\alpha \times \mathbb{K}^n$$

Define an equivalence relation $\sim$ on E as follows:

$$(\alpha, x, v) \sim (\beta, y, w) \quad \iff \quad x = y \quad v = g_{\alpha\beta}(x)w$$

Let us check that $\sim$ is indeed an equivalence relation.

- The relation is reflexive since
$$(\alpha, x, v) \sim (\alpha, x, g_{\alpha\alpha}(x)v) = (\alpha, x, \mathrm{Id}_{\mathbb{K}^n} \cdot v) = (\alpha, x, v).$$
- It is also symmetric since if $(\alpha, x, v) \sim (\beta, x, w)$, then it follows
$$w = g_{\alpha\beta}^{-1}(x)v = g_{\beta\alpha}(x)v$$
Hence, $(\beta, x, w) \sim (\alpha, x, v)$ and $\sim$ is symmetric.
- If $(\alpha, x, u) \sim (\beta, x, v)$ and $(\beta, x, v) \sim (\gamma, x, w)$, then $u = g_{\alpha\beta}(x)v$ and $v = g_{\beta\gamma}(x)w$. Therefore,
$$u = g_{\alpha\beta}(x)v = g_{\alpha\beta}(x)g_{\beta\gamma}(x)w = g_{\alpha\gamma}(x)w$$
Hence $(\alpha, x, u) \sim (\gamma, x, w)$. Hence $\sim$ is transitive.

Let $E = E'/\sim$ and $\pi : E \to X$ that maps $[\alpha, x, v]$ to x. If W is a subset of $U_\beta \times \mathbb{K}^n$, then

$$q^{-1}\left(q(\beta \times W)\right) = \coprod_\alpha \alpha \times h_{\alpha\beta}(W),$$

where $h_{\alpha\beta} : U_\alpha \cap U_\beta \times \mathbb{K}^n \to U_\alpha \cap U_\beta \times \mathbb{K}^n$ is defined by

$$h_{\alpha\beta}(x, v) = (x, g_{\alpha\beta}(x)v).$$

In particular, if $\{\beta\} \times W$ is an open subset of $\{\beta\} \times U_\beta \times \mathbb{K}^n$, then

$$q^{-1}\left(q(\{\beta\} \times W)\right)$$

is an open subset of $\coprod_\alpha \{\alpha\} \times U_\alpha \times \mathbb{K}^n$. Thus, q is an open continuous map. Since its restriction $q_\alpha \equiv q|_{\{\alpha\} \times U_\alpha \times \mathbb{K}^n}$ is injective,

$$(q_\alpha(\{\alpha\} \times U_\alpha \times \mathbb{K}^n), q_\alpha^{-1})$$

is an atlas for E. Hence E is a topological manifold. Note that for $x \in X$, the fiber $E_{\pi^{-1}(x)} := \pi^{-1}(x)$ is the set of all equivalence classes of the form $[(\alpha, x, v)]$ for arbitrary v and α such that $x \in U_\alpha$. We can define a vector space structure on $E_{\pi^{-1}(p)}$ by choosing a fixed U_α containing x and setting

$$c_1[(\alpha, x, v_1)] + c_2[(\alpha, x, v_2)] = [(\alpha, x, c_1 v_1 + c_2 v_2)]$$

for $c_1, c_2 \in \mathbb{K}$. The fact that the maps $v \mapsto g_{\alpha\beta}(x)v$ are all linear isomorphisms guarantees that this is independent of the choice of α. The projection map $\pi : E \to X$ is continuous since it induces projection maps on the charts. The local trivialization condition is met by construction. We conclude that $\pi : E \to X$ is a vector bundle. $\square$

Remark 3.4. *It can be showed that the vector bundle constructed above is unique (up to isomorphism). We skip the details.*

Given a vector bundle (E_1, π_1, X), recall that can we find transition functions. Is the vector bundle (E_2, π_2, X) constructed with these transition functions is isomorphic to E_1? Yes:

Proposition 3.5. *Let (E_1, X, π_1) be a vector bundle with local trivializations*

$$\varphi_\alpha : \pi_1^{-1}(U_\alpha) \to U_\alpha \times \mathbb{K}^n$$

The vector bundle (E_2, X, π_2), constructed using the gluing functions

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(n, \mathbb{K}), \qquad g_{\alpha\beta}(x, v) = (x, \varphi_{\alpha\beta}(x)v)$$

is isomorphic to (E_1, X, π_1).

Proof. Invoking Lemma 2.14, we must ensure we can find a homeomorphism $f : E_1 \to E_2$ restricting to a linear isomorphism in each fiber. This function f is given by

$$f : E_1 \to E_2, \quad (x, v) \mapsto [\alpha, \varphi_\alpha(x, v)],$$

where α is chosen such that $x \in U_\alpha$. To see that f is well-defined, consider $x \in U_\alpha \cap U_\beta$, then

$$[\beta, \varphi_\beta(x, v)] = [\alpha, \varphi_{\alpha\beta}\varphi_\beta(x, v)] = [\alpha, \varphi_\alpha(x, v)]$$

We also need to check that f is a homeomorphism. To verify that f is continuous, consider the composition

$$U_\alpha \times \mathbb{K}^n \xrightarrow{(\varphi_\alpha^2)^{-1}} \pi_2^{-1}(U_\alpha) \xrightarrow{f} \pi_1^{-1}(U_\alpha) \xrightarrow{\varphi_\alpha^1} U_\alpha \times \mathbb{K}^n,$$

which is the identity and hence continuous. Henc, f is continuous. We construct an inverse:

$$f^{-1} : E_2 \to E_1, \quad [\alpha, b, v] \mapsto (b, \varphi_\alpha^{-1}(v)).$$

By a similar check as before, we see that f^{-1} is well-defined and continuous. The last thing to check is that the functions restrict to a linear isomorphism on each fiber. Fix $x \in X$, then

$$f|_{\pi_1^{-1}(x)} : (x, v) \mapsto [\alpha, \varphi_\alpha(x, v)],$$

for $x \in U_\alpha$ is a linear isomorphism since φ_α restricts to a linear isomorphism on $\pi_1^{-1}(x)$. □

4. Sections of Vector Bundles

How does one distinguish between two non-isomorphic vector bundles? This, in general, is a difficult topological problem. Studying sections on vector bundles can help us with this task.

Definition 4.1. Let $E, X \in$ **Top** and let (E, X, π) be a vector bundle. A **local section** is a continuous map $s : U \to E$ for some open set $U \subseteq X$ such that $\pi \circ s = \mathrm{Id}_U$. In other words, a section $s : U \to E$ is such that for each $x \in U$, $s(x) \in \pi^{-1}(x)$. The space of local sections is denoted as $\Gamma(U, E)$.

Remark 4.2. *If $U = X$, then a local section is called a global section. We simply use the phrase section in this case.*

Remark 4.3. *A section is called the zero section if $s(x)$ is the zero vector of $\pi^{-1}(x)$ for each $x \in X$. A section is called nowhere zero if $s(x)$ is a non-zero vector of $\pi^{-1}(x)$ for each $x \in X$.*

Example 4.4. A section of the trivial bundle $X \times \mathbb{K}^n \to X$ is a continuous function $f : X \to \mathbb{K}^n$.

Note that every vector bundle has a zero section, and a trivial vector bundle has a nowhere zero section. Thus, if vector bundle has no nowhere zero sections, then the vector bundle is not isomorphic to the trivial bundle.

Example 4.5. Let $s : \mathbb{RP}^n \to \gamma_{n+1}^1$ be any section, and consider the composition

$$\mathbb{S}^n \to \mathbb{RP}^n \xrightarrow{s} \gamma_{n+1}^1$$

which carries each $x \in \mathbb{S}^n$ to some pair $(\{\pm x\}, t(x)x) \in \gamma_{n+1}^1$. The map $x \mapsto t(x)$ is a continuous map $\mathbb{S}^n \to \mathbb{R}$. Since the composition defined above agrees on antipodal points, we have $t(-x) = -t(x)$. This follows from the computation:

$$([x], t(x)x) = ([-x], t(-x)(-x)) = ([x], -t(-x)x)$$

Since $\mathbb{S}^n$ is connected, it follows from the intermediate value theorem that $t(x_0) = 0$ for some $x_0 \in \mathbb{S}^n$. Hence, s cannot be everywhere non-zero. Thus, γ^1_{n+1} is not a trivial vector bundle.

Example 4.6. Consider $T\mathbb{S}^n$. A section is just a vector field on $\mathbb{S}^n$. By the Hairy Ball Theorem, $\mathbb{S}^n$ has a non-vanishing vector field if and only if n is odd. From this, it follows that the tangent bundle of $T\mathbb{S}^n$ is not isomorphic to the trivial bundle if n is even and nonzero.

Remark 4.7. *Let $\pi : E \to X$ be a vector bundle. We show that π is a homotopy equivalence. Let s be the zero section, we have $\pi \circ s = \mathrm{Id}_X$. Let $x \in X$, $v \in E_x$, the map $\pi_t(v) = tv$ is well defined $\pi_1 = \mathrm{Id}_E$, $s \circ \pi = \pi_0$. This is equivalent to saying that $s \circ \pi$ is homotopic to Id_E.*

Let's push the idea further of studying the classification of vector bundles by studying sections on a vector bundle. We do this by studying families of sections on a vector bundle.

Definition 4.8. Let $E, X \in$ **Top** and let (E, X, π) be a vector bundle. Let $\{s_1, \dots, s_n\}$ be a collection of sections. The sections $s_1, \dots, s_n$ are called **nowhere linearly dependent** if, for each $x \in X$, the vectors $\{s_1(x), \dots, s_n(x)\}$ are linearly independent.

The existence of nowhere dependent sections is rather special:

Proposition 4.9. *Let $E, X \in$ **Top** and let (E, X, π) be a rank n vector bundle. Then (E, X, π) is a trivial vector bundle if and only if it admits n global sections $s_1, \dots, s_n$ which are nowhere linearly dependent.*

Proof. A n-dimensional trivial vector bundle clearly admits n nowhere linearly dependent global sections. Conversely, let $s_1, \dots, s_n$ be global sections of the vector bundle which are nowhere linearly dependent. Define $f : X \times \mathbb{K}^n \to E$ by

$$f(x, x) = x_1 s_1(x) + \dots + x_n s_n(x).$$

Evidently, f is continuous and maps each fiber of the trivial bundle is mapped isomorphically onto the corresponding fiber of $E \to X$. Lemma 2.14 implies that f is an isomorphism of bundles, and the vector bundle is trivial. □

Remark 4.10. *In fact, the argument in Proposition 4.9 can be adapted to show that there if $E \to X$ is a vector bundle, we have a bijection*

$$\{\textit{Local Trivializations}\} \longleftrightarrow \{\textit{Existence of Linearly Independent Local Sections}\}$$

Example 4.11. $T\mathbb{S}^1$ admits one nowhere zero section

$$s(x_1, x_2) = ((x_1, x_2), (-x_2, x_1)).$$

We can rewrite this in terms of complex numbers. If we set $z = x_1 + ix_2$, then the section s is given by

$$z \mapsto iz.$$

Hence, $T\mathbb{S}^1$ is the trivial vector bundle.

Example 4.12. The tangent bundle to the 3-sphere $\mathbb{S}^3 \subseteq \mathbb{R}^4$ admits three nowhere linearly dependent sections $s_i(x) = (x, \overline{s}_i(x))$ where

$$\overline{s}_1(x) = (-x_2, x_1, -x_4, x_3),$$
$$\overline{s}_2(x) = (-x_3, x_4, x_1, -x_2),$$
$$\overline{s}_3(x) = (-x_4, -x_3, x_2, x_1)$$

It is easy to check that the three vectors $\overline{s}_1(x)$, $\overline{s}_2(x)$, and $\overline{s}_3(x)$ are orthogonal to each other and to $x = (x_1, x_2, x_3, x_4)$. Hence, s_1, s_2, and s_3 are nowhere linearly dependent sections of the tangent bundle of $\mathbb{S}^3$ in $\mathbb{R}^4$ [2]. Hence, $T\mathbb{S}^3$ is the trivial vector bundle.

[2]The above formulas come in fact from the quaternion multiplication in $\mathbb{R}^4$. If we identify $\mathbb{H}$ with $\mathbb{R}^4$ via the coordinates (x_1, x_2, x_3, x_4), then we can describe the three sections s_1, s_2, and s_3 of the tangent bundle of S^3 in $\mathbb{H}$ by the formulas $\overline{s}_1(z) = iz$, $\overline{s}_2(z) = jz$, and $\overline{s}_3(z) = kz$.

5. Operations on Vector Bundles

We discuss some constructions allowing us to construct new vector bundles bundles out of known vector bundles. Most of the ensuing constructions will assume that we are working over the same base space. Before discussing some elaborate constructions, we note some easy constructions that allow us to construct new vector bundles. Let $\pi : E \to X$ be a rank n vector bundle.

(1) (**Restriction**) If $A \subseteq X$, then

$$\pi_{|A} : \pi^{-1}(A) \to A$$

is clearly a vector bundle of rank n. Indeed, if $\{(U_\alpha, \varphi_\alpha)\}_\alpha$ be a locally trivializing cover for $\pi : E \to X$, the sets $V_\alpha = U_\alpha \cap A$ form an open covering of A, and

$$\varphi_\alpha|_{\pi^{-1}(V_\alpha)} : \pi^{-1}(V_\alpha) \to V_\alpha \times \mathbb{K}^n$$

are the required locally trivializing maps. We call this vector bundle the restriction of E over A.

(2) (**Subbundle**) (Sketch) If $F \subseteq E$ is subspace with the subspace topology such that $F_{\pi^{-1}(x)} \cap E_{\pi^{-1}(x)}$ is a vector subspace of $E_{\pi^{-1}(x)}$ of fixed dimension for each $x \in X$, then the restriction

$$\pi^F : F \to X$$

is a vector bundle. As in (1), it can be easily checked that this is a vector bundle. We call this a vector sub-bundle of E.

(3) **(Products)** If $\pi_1 : E_1 \to X$ and $\pi_2 : E_2 \to X$ are two vector bundles of ranks n_1 and n_2, respective, then

$$\pi_1 \times \pi_2 : E_1 \times E_2 \to X \times X$$

is a vector bundle of rank $n_1 + n_2$. This is called a product vector bundle. Indeed, if $\{(U_\alpha, \varphi_\alpha)\}_\alpha$ be a locally trivializing cover for $\pi_1 : E_1 \to X$, and let $\{(V_\beta, \psi_\beta)\}_\beta$ be a locally trivializing cover for $\pi_2 : E_2 \to X$. Consider the maps

$$\varphi_\alpha \times \psi_\beta : (\pi_1 \times \pi_2)^{-1}(U_\alpha \times V_\beta) \to (U_\alpha \times V_\beta) \times \mathbb{K}^{n_1+n_2}$$

Then $\{(U_\alpha \times V_\beta, \varphi_\alpha \times \psi_\beta)\}_{\alpha,\beta}$ is the required local trivialization.

We now discuss some other interesting constructions of vector bundles.

(1) (**Quotient Bundle**) Let $\pi : E \to X$ be a rank n vector bundle, and let $E' \subseteq E$ be a rank n' subbundle. We can form a quotient bundle, $E/E' \to X$, such that

$$(E/E')_{\pi^{-1}(x)} = E_{\pi^{-1}(x)}/E'_{\pi^{-1}(x)}$$

for each $x \in X$. Since E' is a subbundle, we can choose a system of trivializations $\{(U_\alpha, \varphi_\alpha)\}_\alpha$ such that

$$\varphi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times (\mathbb{K}^{n'} \times \{0\}) \subseteq U_\alpha \times \mathbb{K}^n$$

if a diffeomorphism. Let $q_{n'} : \mathbb{K}^n \to \mathbb{K}^{n-n'}$ be the projection onto the last $(n - n')$ coordinates. Then, the trivializations for E/E' are given by $\{(U_\alpha, \{\mathrm{Id} \times q_{n'}\} \circ \varphi_\alpha)\}$.

(2) (**Dual Bundle**) Let $\pi : E \to X$ be a rank vector bundle with local trivialization $\{(U_\alpha, \varphi_\alpha)\}_\alpha$. Let

$$\{g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(n, \mathbb{K})\}$$

be the gluing functions. Consider the gluing functions:

$$\{g^*_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(n, \mathbb{K})\}$$

where $g^*_{\alpha\beta}$ is $(g^{-1})^T_{\alpha\beta}$ if $\mathbb{K} = \mathbb{R}$ and $(g^{-1})^\dagger_{\alpha\beta}$ if $\mathbb{K} = \mathbb{C}$. It is easy to see that these gluing functions satisfy the condition of Proposition 3.3. The corresponding vector bundle $E^* \to X$ is the dual bundle of $E \to X$.

(3) (**Whitney Sum**) Let $\pi_1 : E_1 \to X$ and $\pi_2 : E_2 \to X$ be two vector bundles of ranks n_1 and n_2 respectively. Let $\{(U_\alpha, \varphi^1_\alpha)\}_\alpha$ and $\{(U_\beta, \varphi^2_\beta)\}_\beta$ be local trivializations for E_1 and E_2 respectively. Then $\{U_\alpha \cap U_\beta\}_{\alpha,\beta}$ is an open cover for X. We write $W_{\alpha,\beta} = U_\alpha \cap U_\beta$. Let
$$\{g^i_{\alpha\beta\alpha'\beta'} : W_{\alpha\beta} \cap W_{\alpha'\beta'} \to \mathrm{GL}(n_i, \mathbb{K})\}$$
be the gluing functions of E_i for $i = 1, 2$. Consider the gluing functions given by
$$\{g^1_{\alpha\beta\alpha'\beta'} \oplus g^2_{\alpha\beta\alpha'\beta'} : W_{\alpha\beta} \cap W_{\alpha'\beta'} \to \mathrm{GL}(n_1 + n_2, \mathbb{K})\}.$$
Utilizing the isomorphism $\mathbb{K}^{n_1} \oplus \mathbb{K}^{n_2} \cong \mathbb{K}^{n_1+n_2}$, we can express $g^1_{\alpha\beta\alpha'\beta'} \oplus g^2_{\alpha\beta\alpha'\beta'}$ in matrix form:
$$g^1_{\alpha\beta\alpha'\beta'} \oplus g^2_{\alpha\beta\alpha'\beta'} = \begin{pmatrix} g^1_{\alpha\beta\alpha'\beta'} & 0 \\ 0 & g^2_{\alpha\beta\alpha'\beta'} \end{pmatrix}.$$
It is easy to see that these gluing functions satisfy the condition of Proposition 3.3. The corresponding vector bundle $E_1 \oplus E_2 \to X$ the direct sum or Whitney sum of E_1 and E_2.

(4) (**Tensor Bundle**) This is the same as (2). Just replace $\oplus$ with $\otimes$ and $n_1 + n_2$ by $n_1 n_2$.

(5) (**Hom Bundle**) If $\pi_1 : E_1 \to X$ and $\pi_2 : E_2 \to X$ are two vector bundles, we can define the vector bundle $\mathrm{Hom}(E_1, E_2) \to X$ by
$$\mathrm{Hom}(E_1, E_2) := E_1^* \otimes E_2,$$

Remark 5.1. *We have a bijection*
$$\{\text{Sections of } \mathrm{Hom}(E_1, E_2) \to X\} \longleftrightarrow \{\text{Vector Bundle Morphisms } E_1 \to E_2\}$$
Indeed, if $f : E_1 \to E_2$ is a vector bundle morphism, then there is an associated section
$$\begin{aligned} \sigma : X &\to \mathrm{Hom}(E_1, E_2) \\ x &\mapsto f_x. \end{aligned}$$
Conversely, if $\sigma : X \to \mathrm{Hom}(E_1, E_2)$ is a section of π, then we obtain a vector bundle morphism
$$\begin{aligned} f : E_1 &\to E_2 \\ e_1 &\mapsto \sigma(\pi_1(e_1))(e_1). \end{aligned}$$
It is clear that this defines a bijection.

Remark 5.2. *Here is another way to consider the Whitney sum of two vector bundles. Let $\pi_1 : E_1 \to X$ and $\pi_2 : E_2 \to X$ be two vector bundles of ranks n_1 and n_2 respectively. The Whitney sum is the restriction of the product $E_1 \times E_2$ over the diagonal $\Delta = \{(x, x) \in X \times X\}$ is exactly $E_1 \oplus E_2$.*

Example 5.3. The following is a basic list of examples of the Whitney sum construction:

(1) The direct sum of two trivial bundles is again a trivial bundle.
(2) The direct sum of nontrivial bundles can also be trivial. For example, we have,
$$T\mathbb{S}^n \oplus N\mathbb{S}^n \cong \mathbb{S}^n \times \mathbb{K}^{n+1}$$
The map yield the desired isomorphism is simply given by $(x, v, tx) \mapsto (x, v + tx)$.

Example 5.4. If $E \to X$ is a line bundle, then $\mathrm{Hom}(E, E) \cong E^* \otimes E$ is a trivial bundle. Indeed, it suffices to show that there exists a non-vanishing global section and the global section
$$\begin{aligned} \sigma : X &\to \mathrm{Hom}(E, E) \\ x &\mapsto \mathrm{Id}_{E_x} \end{aligned}$$
does the job.

We can also invoke categorical constructions two construct new vector bundles out of previously known vector bundles. We now discuss one such construction, the pullback of a vector bundle.

Proposition 5.5. *(**Pullback**) Let X, Y be topological spaces, and let $\pi : E \to X$ be a vector bundle. Given a continuous map $f : X \to Y$, there exists a vector bundle $f^*\pi : f^*E \to X$ with a map $f^* : f^*E \to E$ taking the fibers of f^*E isomorphically onto the corresponding fibers of E.*

$$\begin{tikzcd} f^*E \arrow[r, "f^*"] \arrow[d, "f^*\pi"'] & E \arrow[d, "\pi"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

Proof. Consider

$$f^*E := \{(x, e) \in X \times E \mid f(x) = \pi(e)\},$$

Let $f^*\pi$ and f^* be projections onto the first and second coordinates, respectively. Consider the product vector bundle

$$\mathrm{Id}_X \times \pi : X \times E \to X \times Y$$

Consider the graph of f:

$$\Gamma_f = \{(x, y) \in X \times Y \mid y = f(x)\}$$

Note that we have

$$(x, e) \in (\mathrm{Id}_X \times \pi)^{-1}(\Gamma_f) \iff f(x) = \pi(e).$$

Hence, the inverse image of Γ_f is f^*E. Uniqueness follows from categorical nonsense. Indeed, the pullback bundle is nothing other than the fibered product in a category-theoretic sense. □

Example 5.6. The following is a basic list of examples of the pullback:

(1) The restriction of a vector bundle $\pi : E \to X$ over a subspace $A \subseteq X$ can be viewed as a pullback with respect to the inclusion map $A \hookrightarrow X$.
(2) Let f be a constant map, having an image as a single point $y \in Y$. Then $f^*(E)$ is just the product $X \times \pi^{-1}(y)$, a trivial bundle.
(3) The tangent bundle $T\mathbb{S}^n$ is the pullback of the tangent bundle $T\mathbb{RP}^n$ via the quotient map $\mathbb{S}^n \to \mathbb{RP}^n$. Indeed, we define a map

$$\begin{tikzcd} T\mathbb{S}^n \arrow[r, "dF"] \arrow[d] & T\mathbb{RP}^n \arrow[d] \\ \mathbb{S}^n \arrow[r, "F"] & \mathbb{RP}^n \end{tikzcd}$$

Here dF is the differential of the quotient map F. Note that dF sends $(p, v) \in T\mathbb{S}^n$ to the corresponding equivalence class $([p], v)$ in $T\mathbb{RP}^n$. The diagram clearly commutes. The claim then follows by uniqueness of the pullback bundle construction.

Remark 5.7. *As anticipated, the pullback construction behaves as expected with respect to the composition of functions, direct sum, and tensor product:*

$$\begin{aligned} (f \circ g)^*(E) &\cong g^*(f^*(E)), \\ 1^*(E) &\cong E, \\ f^*(E_1 \oplus E_2) &\cong f^*(E_1) \oplus f^*(E_2), \\ f^*(E_1 \otimes E_2) &\cong f^*(E_1) \otimes f^*(E_2). \end{aligned}$$

In each case, it is necessary to verify that the vector bundle on the right satisfies the characteristic property of a pullback. For instance, in the last case there exists a natural map from $f^*(E_1) \otimes f^*(E_2)$ to $E_1 \otimes E_2$, which is an isomorphism on each fiber. Therefore, $f^*(E_1) \otimes f^*(E_2)$ satisfies the condition to be the pullback $f^*(E_1 \otimes E_2)$.

6. Vector Bundles Over Paracompact Spaces

The purpose of this technical section is to discuss various properties of vector bundles over compact Hausdorff spaces. First, let's use the direct sum construction to argue that over a compact Hausdorff base space, every sub-bundle of a vector bundle is a direct summand of the vector bundle.

Proposition 6.1. *Let X be a paracompact Hausdorff topological space[3], $\pi : E \to X$ be a rank n vector bundle. Let $F \subseteq E$ be a subspace such that $\dim F_{\pi^{-1}(x)} \cap E_{\pi^{-1}}(x) = m \leq n$ for each $x \in X$. Let π^F be the associated sub-bundle. Then there exists a sub-bundle $\pi^{F^\perp}$ defined by $F^\perp \subseteq E$ such that*

$$\pi^F \oplus \pi^{F^\perp} \cong \pi$$

Proof. Let $\{(U_\alpha, \varphi_\alpha)\}_\alpha$ be a locally trivial cover. The local trivializations,

$$\varphi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{K}^n,$$

induce an inner product, $\langle , \rangle_\alpha$, on $\pi^{-1}(U_\alpha)$ by pulling back the standard inner product on $\mathbb{K}^n$. Since E is paracompact and Hausdorff, a partition of unity $\{\rho_\alpha\}_\alpha$ exists that is subordinate to the open cover $\{U_\alpha\}_\alpha$. An inner product on all of E is then obtained by setting

$$\langle v, w\rangle_E = \sum_\alpha \rho_\alpha(\pi(v))\langle v, w\rangle_\alpha$$

Note that here v, w are assumed to be in the same fiber. Define $F^\perp$ such that,

$$F^\perp_{\pi^{-1}(x)} := (F_{\pi^{-1}(x)})^\perp \subseteq E_{\pi^{-1}(x)}$$

for each $x \in X$. We have

$$E_{\pi^{-1}(x)} = F^\perp_{\pi^{-1}(x)} \oplus F_{\pi^{-1}(x)}$$

for each $x \in X$. We show that the natural projection

$$\pi^{F^\perp} : F^\perp \to X$$

is a vector bundle. Note that $\pi^F : F \to X$ admits m independent local sections, $\{s_1, \cdots, s_m\}$ over each U_α. We can enlarge this set of m independent local sections of to a set of n independent local sections[4]. We can now apply the Gram-Schmidt orthogonalization process to $\{s_1, \ldots, s_m, s_{m+1}, \ldots, s_n\}$ in each fiber, using the given inner product, to obtain new sections $\{s'_1, \ldots, s'_m, s'_{m+1}, \ldots, s_n\}$. We have a new local trivialization for for U_α:

$$\psi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{K}^n$$

This ψ_α carries $F_{\pi^{-1}(x)}$ to $U \times \mathbb{K}^m$ and $F^\perp_{\pi^{-1}(x)}$ to $U \times \mathbb{K}^{n-m}$. So $(\psi_\alpha)_{F^\perp}$ is a local trivialization of $F^\perp$. □

Remark 6.2. *It can be checked that specifying a inner product on a vector bundle $E \to X$ is equivalent to choosing a section of the bundle $E^* \otimes E^* \to X$ whose value at each point $x \in X$ gives a positive-definite inner product on E_x.*

One might wonder if given an rank n vector bundle $\pi : E \to X$, can one provide an embedding of E into a trivial bundle? Here is an answer:

Proposition 6.3. *Let X be a compact Hausdorff space, and let $\pi : E \to X$ be a vector bundle. There exists a vector bundle $\pi' : E' \to X$ such that $\pi \oplus \pi'$ is isomorphic to a trivial bundle.*

---

[3]We need the topological space to be both paracompact and Hausdorff for it to admit a partition of unity.

[4]First choose $s_{m+1}, \ldots, s_n$ in the fiber $\pi^{-1}(x)$, and then take the same vectors for all *nearby* fibers. This is sufficient since if $s_1, \ldots, s_m, s_{m+1}, \ldots, s'_n$ will remain independent for *points near x* by continuity of the determinant function.

Remark 6.4. *By Proposition 6.1 it suffices to show that an arbitrary vector bundle over a compact, Hausdorff space is a sub-bundle of a trivial vector bundle. The idea is to find, using Urysohn's lemma, a finite (assuming compactness) cover of the base space X and bump functions, and use the functions given to us by Urysohn's lemma to construct projections of the local trivializations to build an isomorphism between E and a subbundle of $X \times \mathbb{K}^n$ for some $N \in \mathbb{N}$.*

Proof. Let $\{(U_i, \varphi_i)\}_{\alpha=1}^m$ be a finite locally trivializing cover and let $\{\rho_i\}_{i=1}^m$ be a partition of unity subordinate to the finite open cover. Since $\sum_{i=1}^m \rho_i \equiv 1$, the sets

$$\{\rho_i^{-1}((0,1])\}_{i=1}^m$$

is a cover of X. Define $\psi_i : \pi^{-1}(U_i) \to \mathbb{K}^n$ by

$$\psi_i(v) = (\rho_i(\pi(v))) \cdot (\mathrm{proj}_2 \circ \varphi_i(v)),$$

Then ψ_i is a linear injection on each fiber over $\rho_i^{-1}(0,1]$. Let

$$\psi := (\psi_1, \ldots, \psi_m) : E \to \mathbb{K}^N$$

for some $N \in \mathbb{N}$. Now, ψ is a linear injection on each fiber. Finally, consider

$$f := (\pi, \psi) : E \to X \times \mathbb{K}^N$$

The image of f is a sub-bundle of $X \times \mathbb{K}^N$ Thus we have E isomorphic to a sub-bundle of $X \times \mathbb{K}^n$. The claim now follows from Proposition 6.1. □

We now show that over a paracompact Hausdorff space, two pullback bundles induced by two homotopic maps are isomorphic. To prove this claim, we need a few preliminary results.

Lemma 6.5. *Let E, X be topological spaces, and let $\pi : E \to X \times [a,b]$ be a vector bundle which restricts to trivial bundles onto $E|_{X\times[a,c]}$ and $E|_{X\times[c,b]}$ for some $c \in (a,b)$. Then $\pi : E \to X \times [a,b]$ is a trivial vector bundle.*

Proof. The proof of can be found in [Hat03]. □

Lemma 6.6. *Let E, X be topological spaces and let $\pi : E \to X \times I$ be a vector bundle. Then there exists an open cover $\{U_\alpha\}_\alpha$ of X such that each restriction*

$$\pi|_{\pi^{-1}(U_\alpha\times I)} : \pi^{-1}(U_\alpha \times I) \to U_\alpha \times I$$

is a trivial vector bundle.

Proof. Let $\{W_\alpha\}_\alpha$ be a locally trivial cover for E. Fix $x_0 \in X$. The open cover $\{W_\alpha\}_\alpha$ also covers $\{x_0\} \times I$. By compactness of I, we can extract a finite subcover of $\{W_\alpha\}_\alpha$ covering $\{x_0\} \times I$, which we label as $\{W_i\}_{i\in J(x_0)}$ where $J(x_0)$ is a finite set. Using the Lebesgue number lemma, let

$$0 = t_0 < t_1, \cdots, t_k < 1$$

be a partition of I such that each $\{x_0\} \times [t_j, t_{j+1}]$ is contained in one of $W_{i(j)}$. Now each $\{x_0\} \times [t_j, t_{j+1}]$ is an open subset of $W_{i(j)}$. By the Tube lemma, there exist open sets $x_0 \in U_{x_0,j}$ such that $U_{x_0,j} \times [t_j, t_{j+1}]$ is contained in $W_{i(j)}$. By construction, E is trivial over each $U_{x_0,j} \times [t_j, t_{j+1}]$. Hence, if we set

$$U_{x_0} = U_{x_0,1} \cap \cdots \cap U_{x_0,k}$$

then E is trivial over $U_{x_0} \cap [t_j, t_{j+1}]$. Lemma 6.5 implies that E is trivial over $U_{x_0} \times I$. Repeat this construction for all $x \in X$ to obtain the desired open cover. □

Having these two technical lemmas we can go on and prove the following result.

Proposition 6.7. *Let X be a paracompact Hausdorff topological space, and let $\pi : E \to X \times I$ be a vector bundle. Its restrictions over $X \times \{0\}$ and $X \times \{1\}$ are isomorphic vector bundles.*

Remark 6.8. *The idea is to "push along" the restricted bundle over $X \times \{1\}$ to the restricted bundle over $X \times \{0\}$. By Lemma 6.6, we can choose an open cover $\{U_\alpha\}_\alpha$ of X such that E is trivial over $\{U_\alpha \times I\}_\alpha$. If X is compact, we can find a finite subcover and relabel this cover as $\{U_i\}$. Using Urysohn's Lemma, we find a partition of unity $\{\rho_i\}$ subordinate to $\{U_i\}$ which makes the "push along" argument to work. This argument can be generalized to a paracompact Hausdorff space.*

Proof. First assume that X is compact. Let the cardinality of $\{U_i\}_{i=1}^m$ be as in Remark 6.8. For each i, we define functions

$$\begin{aligned} \psi_i : X &\to \mathbb{R} \\ x &\mapsto \gamma_1(x) + \ldots + \rho_i(x) \end{aligned}$$

In particular, $\psi_0 = 0$ and $\psi_m = 1$. Define

$$X_i = \{(x, \psi_i(x)) \mid x \in X\} \subseteq X \times I$$

Notice $X_0 = X \times \{0\}$ and $X_m = X \times \{1\}$. Let E_i denote the vector bundle restricted to X_i. Now we define isomorphisms $f_i : E_i \to E_{i-1}$ between the restricted vector bundles. The isomorphisms f_i are given by

$$f_i(x, \psi_i(x), v) = (x, \psi_{i-1}(x), v).$$

Essentially, f_i is the identity outside $\pi^{-1}(U_i \times I) \cap E_i$ and on $\pi^{-1}(U_i \times I) \cap E_i$, it projects each fiber $\pi^{-1}(x, \psi_i(x))$ to the fiber $\pi^{-1}(x, \psi_{i-1}(x))$. This can be seen by considering a point outside U_i and computing

$$\psi_i(x) = \psi_{i-1}(x) + \rho_i(x) = \psi_{i-1}(x),$$

which holds since $\operatorname{supp}(\rho_i) \subseteq U_i$. For f_i to be an isomorphism of vector bundles, we need to check it is a homeomorphism and a linear isomorphism on each fiber. For continuity, we remark that f_i is a composition of continuous functions. The inverse of f_i is given by

$$f_i^{-1}(x, \psi_{i-1}(x), v) = (x, \psi_i(x), v),$$

which is continuous by the same reasoning. Outside $\pi^{-1}(U_i \times I) \cap E_i$, the function f_i is the identity and thus maps fibers isomorphically to each other. On $\pi^{-1}(U_i \times I) \cap E_i$, we can use the fact that E trivializes over $U_i \times I$, which yields the trivialization $h_i : \pi^{-1}(U_i \times I) \to U_i \times I \times \mathbb{K}^n$. The composition

$$\begin{aligned} h_i \circ f_i \circ h_i^{-1} : (U_i \times I) \cap X_i \times \mathbb{K}^n &\to (U_i \times I) \cap X_{i-1} \times \mathbb{K}^n \\ (x, \psi_i(x), v) &\mapsto (x, \psi_{i-1}(x), v) \end{aligned}$$

is a linear isomorphism on each fiber, and thus, f_i must be as well. Since f_i is a homeomorphism and a linear isomorphism on each fiber, f_i is an isomorphism of vector bundles. The composition

$$f := f_1 \circ \ldots \circ f_m$$

is an isomorphism of vector bundles. In particular, it is an isomorphism between the restrictions of E over $X_m = X \times \{1\}$ and $X_0 = X \times \{0\}$. This argument can be generalized to the case where X is paracompact and Hausdorff. See [Hat03, Proposition 1.7.]. □

Corollary 6.9. *Let X be a paracompact Hausdorff space, and let $\pi : E \to X$ be a vector bundle. Given homotopic maps $g_0, g_1 : A \to X$, where A is compact, the pullback bundles $g_0^*(E)$ and $g_1^*(E)$ are isomorphic.*

Proof. Let $G : A \times I \to X$ be the homotopy from g_0 to g_1. If we consider the pullback bundle $G^*(E)$, then the vector bundles $g_0^*(E)$ and $g_1^*(E)$ are isomorphic to the restrictions of $G^*(E)$ over $A \times \{0\}$ and $A \times \{1\}$. By Proposition 6.7, these vector bundles are isomorphic. □

Corollary 6.10. *Every vector bundle over a contractible topological space is homotopic to a trivial vector bundle.*

Proof. This follows at once from Proposition 6.7. □

7. Vector Bundles Over Spheres

In this section, we will develop tools to classify vector bundles over spheres. We will see that we can use results from from algebraic topology, in particular homotopy theory, to study vector bundles. This will be an instructive exercise before we deal with the general case in the next section.

Let $X = \mathbb{S}^k$ for some $k \geq 0$. Note that $\mathbb{S}^k$ can be covered by two contractible open sets[5], $U_\pm$ such that $U_+ \cap U_-$ is homotopic to the equator $\mathbb{S}^{k-1}$. Since $U_\pm$ is contractible, Corollary 6.10 implies that any vector bundle over $U_\pm$ is trivial, which means any vector bundle over $\mathbb{S}^k$ can be identified with a single transition function

$$f : U_+ \cap U_- \to \mathrm{GL}(n, \mathbb{K})$$

Since $U_+ \cap U_-$ is homotopic to $\mathbb{S}^{k-1}$, we need to only consider gluing functions f of the form

$$f : \mathbb{S}^{k-1} \to \mathrm{GL}(n, \mathbb{K})$$

Definition 7.1. A map $f : \mathbb{S}^{k-1} \to \mathrm{GL}(n, \mathbb{K})$ is called a **clutching function** for the vector bundle E_f constructed using f as a transition function.

Complex vector bundles over spheres turn out to have slightly better behavior than in the real case, so we will prove the following basic result about the complex case before dealing with the real case.

Proposition 7.2. *Let* $\mathbf{Vect}_n^{\mathbb{C}}(\mathbb{S}^k)$ *denote the set of isomorphism classes of rank n complex vector bundles over* $\mathbb{S}^k$. *The map*

$$\Phi : [\mathbb{S}^{k-1}, \mathrm{GL}(n, \mathbb{C})] \to \mathbf{Vect}_n^{\mathbb{C}}(\mathbb{S}^k),$$
$$[f] \mapsto [E_f]$$

is a bijection. Here $[E_f]$ *is the isomorphism class of the vector bundle* E_f *as in Proposition 3.3 and* $[\mathbb{S}^{k-1}, \mathrm{GL}(n, \mathbb{C})]$ *is the set of homotopy class of continuous maps between* $\mathbb{S}^{k-1}$ *and* $\mathrm{GL}(n, \mathbb{C})$.

Proof. We first prove Φ is well-defined. Given two homotopic maps $f_0, f_1 : \mathbb{S}^{k-1} \to \mathrm{GL}(n, \mathbb{C})$, there exists a homotopy

$$F : \mathbb{S}^{k-1} \times I \to \mathrm{GL}(n, \mathbb{C})$$

We can use F to construct a vector bundle $p : E_F \to \mathbb{S}^k \times I$. The vector bundle E_F will restrict to E_{f_0} over $\mathbb{S}^k \times \{0\}$ and to E_{f_1} over $\mathbb{S}^k \times \{1\}$. By Corollary 6.10, the bundles E_{f_0} and E_{f_1} are isomorphic since $\mathbb{S}^k$ is compact, and we conclude that Φ is well-defined. We now show that Φ is a bijection. We construct an inverse Ψ. Given a vector bundle $p : E \to \mathbb{S}^k$, its restrictions E_+ and E_- over the upper and lower hemispheres respectively are trivial by contractibility of $\mathbb{S}^k_\pm \cong \mathbb{D}^k_\pm$. Choosing trivializations

$$h_\pm : E_\pm \to \mathbb{D}^k_\pm \times \mathbb{C}^n,$$

the composition $h_+ \circ h_-^{-1}$ induces a function $f : \mathbb{S}^{k-1} \to \mathrm{GL}(n, \mathbb{C})$. We define $\Psi(E)$ to be the homotopy class of f. We must check that $\Psi(E)$ is independent of the choice of trivializations and hence well-defined. Two trivializations, $h^\pm_{0,1} : E_\pm \to \mathbb{D}^k_\pm \times \mathbb{C}^n$, differ by a map

$$\widetilde{h}_\pm : \mathbb{D}^k_\pm \to \mathrm{GL}(n, \mathbb{C})$$

Since $\mathbb{D}^k_\pm$ is contractible, $\widetilde{h}_\pm$ is homotopic to a constant map

$$c_\pm : \mathbb{D}^k_\pm \to \mathrm{GL}(n, \mathbb{C})$$
$$x \mapsto A_\pm$$

[5]Take the northern and southern hemispheres $\mathbb{S}^k_+$ and $\mathbb{S}^k_-$ respective and enlarge them slightly to open balls U_+ and U_-.

By path-connectedness of $\mathrm{GL}(n,\mathbb{C})$, any constant map is homotopy equivalent to the map that sends everything to the identity in $\mathrm{GL}(n,\mathbb{C})$ by composing with a path going from $A_\pm \in \mathrm{GL}(n,\mathbb{C})$ to $\mathrm{Id}_n \in \mathrm{GL}(n,\mathbb{C})$. We obtain

$$[h_0^\pm] = [\widetilde{h}_\pm \circ h_1^\pm] = [c_\pm \circ h_1^\pm] = [\mathrm{Id} \circ h_1^\pm] = [h_1^\pm]$$

Since $h_0^\pm$ and $h_1^\pm$ are homotopy equivalent, the compositions

$$h_0^+ \circ (h_0^-)^{-1},\ h_1^+ \circ (h_1^-)^{-1}$$

are homotopy equivalent as well and induce homotopy equivalent clutching functions f_0 and f_1. We conclude that Ψ is well-defined. Clearly, Φ and Ψ are inverses to each other. □

Recall that $\mathrm{GL}(n,\mathbb{C})$ deformation retracts onto $\mathrm{U}(n)$. Therefore, we have that

$$[\mathbb{S}^{k-1}, \mathrm{GL}(n,\mathbb{C})] \cong [\mathbb{S}^{k-1}, \mathrm{U}(n)] \cong \pi_{k-1}(\mathrm{U}(n)).$$

for $k \geq 1$ as sets. Thus, the classification of complex vector bundles over spheres is inherently linked to the classification of homotopy groups of unitary groups.

Remark 7.3. *Since $\pi_{k-1}(\mathrm{U}(n))$ is a group, this suggest that $\mathbf{Vect}_n^{\mathbb{C}}(\mathbb{S}^k)$ or an associated object should have the structure of a group. This will be taken up in Section 9 when we get to K-theory.*

Example 7.4. We have the following examples:

(1) Since $\mathrm{U}(n)$ is path-connected, $\pi_0(\mathrm{U}(n)) = 0$ is the trivial group. Hence, $\mathbf{Vect}_n^{\mathbb{C}}(\mathbb{S}^1)$ is a set with one element. Therefore, each rank n complex vector bundle over $\mathbb{S}^1$ is homotopy equivalent to the trivial bundle $\mathbb{S}^1 \times \mathbb{C}^n$.
(2) Consider 1-dimensional complex vector bundles over $\mathbb{S}^2$. Since $\mathrm{U}(1) \cong \mathbb{S}^1$, we have $\pi_1(\mathrm{U}(1)) = \mathbb{Z}$. Hence, $\mathbf{Vect}_1^{\mathbb{C}}(\mathbb{S}^2) \cong \mathbb{Z}$ as a set.
(3) Consider 1-dimensional complex vector bundles over $\mathbb{S}^k$ for $k \geq 3$. Since $\pi_{k-1}(\mathbb{S}^1) = 0$ for $k \geq 3$, we have Hence, $\mathbf{Vect}_1^{\mathbb{C}}(\mathbb{S}^3)$ is a set with one element. Hence, every rank 1 complex bundle over $\mathbb{S}^k$ for $k \geq 3$ is homotopy equivalent to the trivial line bundle.

The preceding analysis does not quite work for real vector bundles since $\mathrm{GL}(n,\mathbb{R})$ is not path-connected. $\mathrm{GL}(n,\mathbb{R})$ has exactly two path components, $\mathrm{GL}_n^\pm(\mathbb{R})$. The closest analogy with the complex case is obtained by considering oriented real vector bundles.

Definition 7.5. Let $E, X \in \mathbf{Top}$ and let be a rank n real vector bundle. The vector is called **orientable** if each fiber can be given an orientation such that there exists an open cover $\{U_\alpha\}_\alpha$ of X such that the local trivializations $\varphi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{R}^n$ carry the orientation of the fibers $\pi^{-1}(b)$ to the standard orientation of $\mathbb{R}^n$.

The set of isomorphism classes of rank n real oriented vector bundles over a base space X is denoted $\mathbf{Vect}_n^{\mathbb{R},+}(X)$. Note that the morphisms in this category are required to preserve orientations. We now consider real oriented vector bundles over spheres. Since $\mathbb{S}^k$ is connected for $k \geq 1$, all fibers must have the same orientation and the clutching function can be taken to map only into $\mathrm{GL}(n,\mathbb{R})^+$ if $k \geq 2$.

Proposition 7.6. *For $k \geq 2$, there exists a bijection of sets*

$$[\mathbb{S}^{k-1}, \mathrm{GL}(n,\mathbb{R})^+] \cong \mathbf{Vect}_n^{\mathbb{R},+}(\mathbb{S}^k)$$

Proof. The proof is analogous to Proposition 7.2. □

We can deal with the case $k = 1$ separately:

Example 7.7. Since $\mathrm{GL}(n,\mathbb{R})$ has two path-components, we have $\pi_0(\mathrm{GL}(n,\mathbb{R})) = \mathbb{Z}_2$. Hence, $\mathbf{Vect}_n^{\mathbb{R}}(\mathbb{S}^1)$ is a set with two elements. When $n = 1$, the corresponding bundles are the trivial bundle and the canonical line bundle over $\mathbb{RP}^1$. When $n > 1$, the Mobius bundle is replaced by direct sum of the canonical bundle over $\mathbb{RP}^1$ with $n - 1$ trivial bundles.

Let $k \geq 2$. Recall that $\mathrm{GL}^{\pm}(n, \mathbb{R})$ deformation retracts onto $\mathrm{SO}(n)$. Therefore, we have that

$$[\mathbb{S}^{k-1}, \mathrm{GL}^{\pm}(n, \mathbb{R})] \cong [\mathbb{S}^{k-1}, \mathrm{SO}(n)] \cong \pi_{k-1}(\mathrm{SO}(n)).$$

for $k \geq 2$ as sets. Thus, the classification of real oriented vector bundles over spheres is inherently linked to the classification of homotopy groups of the special orthogonal group.

Remark 7.8. *Since $\pi_{k-1}(\mathrm{SO}(n))$ is a group for $k \geq 2$, this suggest that $\mathbf{Vect}_n^{\mathbb{R},+}(\mathbb{S}^k)$ or an associated object should have the structure of a group. This will be taken up in Section 9 when we get to K-theory.*

Example 7.9. Consider 2-dimensional real vector bundles over $\mathbb{S}^2$. Since $\mathrm{SO}(2) \cong \mathbb{S}^1$, we have $\pi_1(\mathrm{SO}(2)) = \mathbb{Z}$. Hence, $\mathbf{Vect}_2^{\mathbb{R},+}(\mathbb{S}^2) \cong \mathbb{Z}$ as a set.

8. Classification of Vector Bundles

We would like to classify rank n-vector bundles of over a fixed topological space up to isomorphism. We have made partial progress toward this goal by relating the classification of vector bundles over spheres to questions in homotopy theory. The purpose of this section is to extend the discussion to an arbitrary compact topological space.

Remark 8.1. *We assume that let $n \in \mathbb{N}$ and $m \in \mathbb{N}$ is such that $m \geq n$.*

8.1. Grassmannian.

We briefly discuss the Grassmannian in this section. Later, we will see that the Grassmannian plays a crucial role in the classification problem for vector bundles.

Definition 8.2. Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$. The (n, m)-**Stiefel set**, $V_n(\mathbb{K}^m)$, consists of the set of n-frames. In other words, the set of ordered orthonormal set of vectors $\{v_1, \ldots, v_n\} \subseteq \mathbb{K}^m$.

Note that an element of $V_n(\mathbb{K}^m)$ can be thought of as a $m \times n$ matrix by writing a n-frame as a matrix of n column vectors in $\mathbb{K}^m$. We then have

$$V_n(\mathbb{K}^m) = \left\{A \in \mathbb{K}^{m \times n} : A^* A = I_n\right\}.$$

We endow $V_n(\mathbb{K}^m)$ with the subspace topology inherited from $\mathbb{K}^{m \times n}$. Note that $V_k(\mathbb{K}^m)$ is a compact topological space since it is a closed subspace of $(\mathbb{S}^{m-1})^n$.

Remark 8.3. *It can then be shown that $V_k(\mathbb{K}^m)$ is a (smooth) manifold using submanifold theory. It customary to refer to $V_n(\mathbb{K}^m)$ as a (n, m)-Stiefel manifold. We simply abbreviate a (n, m)-Stiefel manifold as simply a Stiefel manifold.*

Definition 8.4. Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$. The (n, m)-**Grassmannian**, $G_n(\mathbb{K}^m)$, is the set of all n-dimensional subspaces of $\mathbb{K}^m$,

Remark 8.5. *We refer to a (n, m)-Grassmannian as simply the Grassmannian.*

Let's now verify that $G_n(\mathbb{K}^m)$ is indeed a topological space.

Proposition 8.6. *The Grassmannian, $G_n(\mathbb{K}^m)$, is a compact, Hausdorff topological space.*

Proof. There is a natural surjection

$$p : V_n(\mathbb{K}^m) \to G_n(\mathbb{K}^m)$$

sending an n-frame to the subspace it spans. Hence, $G_n(\mathbb{K}^m)$ can be topologized by giving it the quotient topology with respect to this surjection. So, $G_n(\mathbb{K}^m)$ is a compact topological space since $V_n(\mathbb{K}^m)$ is compact. To show $\mathrm{Gr}_n(\mathbb{K}^m)$ is Hausdorff, it suffices to show that for any two n-planes P_1, P_2 in $\mathrm{Gr}_n(\mathbb{K}^m)$, there is a linear functional

$$\varphi_{P_1, P_2} : \mathrm{Gr}_n(\mathbb{K}^m) \to \mathbb{K}$$

that assumes different values on P_1 and P_2. Fix $x_0 \in P_1 \setminus P_2$ and let φ_{P_1,P_2} be the Euclidean distance function from a n-plane to x_0. Clearly, φ_{P_1,P_2} is a well-defined continuous function on such that $\varphi_{P_1,P_2}(P_1) = 0$ and $\varphi_{P_1,P_2}(P_2) > 0$. □

Remark 8.7. *It can be checked that $Gr_n(\mathbb{K}^m)$ has the structure of a (smooth) manifold.*

Example 8.8. We check that $\mathrm{Gr}_n(\mathbb{K}^m) \cong \mathrm{Gr}_{m-n}(\mathbb{K}^m)$. Consider the map

$$\begin{aligned} f : \mathrm{Gr}_n(\mathbb{K}^m) &\to \mathrm{Gr}_{m-n}(\mathbb{K}^m) \\ P &\mapsto P^{\perp} \end{aligned}$$

The orthogonal complement is taken with respect to, for example, the standard inner product on $\mathbb{K}^m$. It is clear that f is a bijection. Consider the commutative diagram:

$$\begin{tikzcd} \mathrm{V}_n(\mathbb{K}^m) \arrow[r, "f'"] \arrow[d] & \mathrm{V}_{m-n}(\mathbb{V}^m) \arrow[d] \\ \mathrm{Gr}_n(\mathbb{K}^m) \arrow[r, "f"] & \mathrm{Gr}_{m-n}(\mathbb{K}^m) \end{tikzcd}$$

It is clear that f is continuous if f' is continuous. The map f' corresponds to computing the orthogonal complement of an ordered orthonormal basis of vectors by, say, the Gram-Schmidt process. This process is clearly continuous. Hence, f is continuous. Since $\mathrm{Gr}_n(\mathbb{K}^m)$ is compact and $\mathrm{Gr}_{m-n}(\mathbb{K}^m)$ is Hausdorff, f is a homeomorphism.

Since a Grassmannian is a space encoding information about vector subspaces it comes with natural vector bundle.

Definition 8.9. The **universal/tautological bundle** vector bundle over $G_n(\mathbb{K}^m)$ is set

$$\gamma_m^n := \{(\omega, v) \in G_n(\mathbb{K}^m) \times \mathbb{K}^m \mid v \in \omega\}$$

along with a map which is a projection onto the first coordinate.

Example 8.10. Let $n = 1$. Then $G_1(\mathbb{K}^{m+1})$ is the m-dimensional projective space $\mathbb{KP}^m$ and γ_{m+1}^1 is the canonical line bundle over $\mathbb{KP}^m$.

Proposition 8.11. *Let $n \in \mathbb{N}$ and $m \geq n$. The projection*

$$\begin{aligned} \pi : \gamma_m^n &\to G_n(\mathbb{K}^m) \\ (\omega, v) &\mapsto \omega \end{aligned}$$

is a vector bundle.

Proof. For $\omega \in \mathrm{Gr}_n(\mathbb{K}^m)$, let $\pi_\omega : \mathbb{K}^m \to \omega$ be the orthogonal projection, and let

$$U_\omega = \{\omega' \in \mathrm{Gr}_n(\mathbb{K}^m) \mid \dim \pi_\omega(\omega') = n\}$$

In particular, $\omega \in U_\omega$. We show that U_ω is open in $\mathrm{Gr}_n(\mathbb{K}^m)$. Note that U_ω is open if and only if its pre-image in $V_n(\mathbb{K}^m)$ is an open set. The pre-image is the set:

$$p^{-1}(U_\omega) = \{\{v_1, \ldots, v_n\} \in V_n(\mathbb{K}^m) \mid \pi_\omega(v_1), \ldots, \pi_\omega(v_n) \text{ are linearly independent}\}$$

If $[\pi_\omega]$ denotes the matrix representation of π_ω, then $p^{-1}(U_\omega)$ consists of $\{v_1, \ldots, v_n\} \in V_n(\mathbb{K}^m)$ such that the $n \times n$ matrix with columns

$$[\pi_\omega]v_1, \ldots, [\pi_\omega]v_n$$

has a non-zero determinant. Since the value of this determinant is a continuous function, it follows that U_ω is an open set. Define the map

$$\begin{aligned} \varphi_\omega : \pi^{-1}(U_\omega) &\to U_\omega \times \mathbb{K}^n \\ (\omega', v) &\mapsto (\omega', \pi_\omega(v)) \end{aligned}$$

It is clear that φ_ω is a bijection that is a linear isomorphism on each fiber. Furthermore, φ_ω is continuous since its two coordinate functions are continuous. One can check that φ_ω^{-1} is continuous. This shows that $\{(U_\omega, \varphi_\omega)\}_{\omega \in \mathrm{Gr}_n(\mathbb{K}^m)}$ is a local trivialization. □

Remark 8.12. *We can also define other vector bundles over the Grassmannian. For instance, the universal/tautological quotient bundle over* $\mathrm{Gr}_n(\mathbb{K}^m)$*, denoted as* β_m^{m-n}*, is such that the fiber of the universal/tautological quotient bundle* $\omega \in \mathrm{Gr}_n(\mathbb{K}^m)$ *is the quotient* $\mathbb{K}^m/\omega$*. It can be checked that this is a well-defined vector bundle.*

8.2. **Grassmanian as a Classifying Space.** A vector bundle $\pi : E \longrightarrow X$ is a family of $\mathbb{K}$-vector spaces parametrized by the points of X. A natural question arises: does there exist a universal such family? More precisely, does there exist a vector bundle $E_{\text{univ}} \longrightarrow Y$ such that every vector bundle $E \longrightarrow X$ arises as the pullback of $E_{\text{univ}} \longrightarrow Y$ along a suitable map? If such a universal bundle exists, what is the corresponding parameter space Y for vector bundles? This question exemplifies a moduli problem, and the space Y that provides a universal solution is referred to as the classifying space for vector bundles. We see that the Grassmannian will solve this moduli problem. Indeed, Suppose we have a rank n vector bundle $E \longrightarrow X$. For each $x \in X$, the fiber E_x is isomorphic to $\mathbb{K}^n$, which can be identified with an element of $G_n(\mathbb{K}^m)$ for some $m \geq n$. This suggests that the Grassmannian naturally arises whenever we have a vector bundle in sight. In fact, we can show that any vector bundle over a compact base is isomorphic to the pullback of a vector bundle over a Grassmannian.

Proposition 8.13. *Let* X *be a compact topological space and let* $\pi : E \longrightarrow X$ *be a rank-*n *vector bundle. Then there exists a* $M \geq n$ *and smooth maps* f, f' *which express* π *as the pullback.*

$$\begin{tikzcd} E \arrow[r, "f'"] \arrow[d, "\pi"] & \beta_M^{M-n} \arrow[d] \\ X \arrow[r, "f"] & \mathrm{Gr}_n(\mathbb{K}^M) \end{tikzcd}$$

Proof. There exists a finite locally trivial open cover $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$ of X and a collection of local sections $s_{\alpha,1}, \ldots, s_{\alpha,n} : U_\alpha \longrightarrow E$ forming a basis over each U_α. Let $\{\rho_\alpha\}_{\alpha \in A}$ be a partition of unity subordinate to the open cover. For each $\alpha \in A$ and $i = 1, \cdots, n$, the sections

$$\tilde{s}_{\alpha,i} = \rho_\alpha s_{\alpha,i}$$

define global sections that vanish outside U_α. Define

$$V = \mathrm{Span}\{\tilde{s}_{\alpha,i} \mid \alpha \in A,\ i = 1, \cdots, n\}$$

Then $V \cong \mathbb{K}^M$ for some $M \in \mathbb{N}$. For each $x \in X$, consider the evaluation map

$$\begin{aligned} \mathrm{ev}_x : \mathbb{K}^M &\longrightarrow E_x, \\ \tilde{s}_{\alpha,i} &\mapsto \tilde{s}_{\alpha,i}(x). \end{aligned}$$

This map is surjective and induces an isomorphism

$$\mathbb{K}^M/\ker \mathrm{ev}_x \cong E_x.$$

The inverses of these isomorphisms fit together to form the map f', where f is defined by $x \mapsto \ker \mathrm{ev}_x$. It is clear that the diagram commutes. □

Proposition 8.13 shows that every vector bundle $\pi : E \longrightarrow X$ over a compact topological space is pulled back from the Grassmannian, but it does not provide a single classifying space for all vector bundles since the choice of $\mathbb{K}^M$ depends on π. Furthermore, we might like to drop the assumption that X is compact. This can be done by considering the infinite Grassmannian as a model space for a classifying space. For fixed $n \in \mathbb{N}$ and $m \geq n$, the definition of the infinite Grassmannian is based on the observation that the inclusions

$$\mathbb{K}^m \subseteq \mathbb{K}^{m+1} \subseteq \ldots$$

give inclusions

$$G_n(\mathbb{K}^m) \subseteq G_n(\mathbb{K}^{m+1}) \subseteq \ldots$$

Definition 8.14. Let $n \in \mathbb{N}$. The **infinite Grassmannian** is defined as the colimit:

$$G_n(\mathbb{K}^\infty) := \varinjlim_{m \geq n} G_n(\mathbb{K}^m) = \bigcup_{m=1}^{\infty} G_n(\mathbb{K}^m)$$

As a set, $G_n(\mathbb{K}^\infty)$ is the set n-dimensional subspaces of the vector space $\mathbb{K}^m$ for some $m \geq n$. Note that $G_n(\mathbb{K}^\infty)$ is endowed with the weak topology, so a set in $G_n(\mathbb{K}^\infty)$ is open if and only if its intersects with $G_n(\mathbb{K}^m)$ is an open set for all $m \geq n$.

Remark 8.15. *We will abbreviate $G_n(\mathbb{K}^\infty)$ as simply G_n.*

For $n \in \mathbb{N}$ and $m \geq n$, we similarly have the inclusions

$$\gamma_m^n \subseteq \gamma_{m+1}^n \subseteq \cdots$$

This yields the following definition:

Definition 8.16. Let $n \in \mathbb{N}$. The **infinite universal/tautological bundle** is defined over G_n as the colimit:

$$\gamma^n := \gamma_\infty^n := \varinjlim_{m \geq n} \gamma_m^n = \bigcup_{m=1}^{\infty} \gamma_m^n$$

endowed with with the weak topology.

We can now state and prove that the infinite Grassmannian is the classifying space for vector bundles.

Proposition 8.17. *Let X be a paracompact Hausdorff topological space. There is a bijection of sets*

$$\mathbf{Vect}_n^{\mathbb{K}}(X) \cong [B, G_n]$$

Proposition 8.17 is a fundamental result, establishing that isomorphism classes of vector bundles correspond bijectively to homotopy classes of maps into Grassmannians. This provides a crucial insight that homotopical invariants contain significant geometric information. We begin by proving a lemma.

Lemma 8.18. *Let X be a topological pace and let $\pi : E \to X$ be a rank n-vector bundle. The data of a continuous function $f : X \to G_n$ such that $E \cong f^*\gamma^n$ is equivalent to the data of a continuous function $g : E \to \mathbb{K}^\infty$ which is a linear injection on each fiber.*

Proof. Assume we are given a continuous function $f : B \to G_n$ and an isomorphism $E \cong f^*\gamma^n$. We have the data of this diagram below. The map g is the composite along the top row of this diagram.

$$\begin{tikzcd} E \arrow[r, "\cong"] \arrow[dr, "\pi"'] & f^*\gamma^n \arrow[r] \arrow[d] & \gamma^n \arrow[r] \arrow[d] & \mathbb{K}^\infty \\ & X \arrow[r, "f"] & G_n & \end{tikzcd}$$

Conversely, given a continuous function $g : E \to \mathbb{K}^\infty$ that is a linear isomorphism on each fiber, define

$$f(x) := g(\pi^{-1}(x)) \in G_n.$$

The map is well-defined since $\pi^{-1}(x)$ is an rank n vector space, and applying g we get an rank n subspace of $\mathbb{K}^\infty$. The vector bundle isomorphism

$$E \to f^*\gamma^n,$$
$$e \mapsto (\pi(e), g(e)).$$

Note that $g(e) \in g(\pi^{-1}(\pi(e)))$ since $e \in \pi^{-1}(\pi(e))$. This is an isomorphism because g is a linear injection and hence bijection on the fibers. □

We now prove Proposition 8.17.

Proof. (Proposition 8.17)

(1) (Injectivity). Suppose that $E \sim f^*\gamma^n \cong (f')^*\gamma^n$. Let g and $g' : E \to \mathbb{K}^\infty$ be the corresponding maps as in Lemma 8.18. It suffices to show that $g \cong g'$. Suppose that
$$H : E \times I \to \mathbb{K}^\infty$$
is a homotopy from g to g' such that H_t is a linear injection on each fiber. Then we may define a homotopy between f and f' given by
$$F : X \times I \to G_n$$
$$F(x,t) = H(\pi^{-1}(x), t)$$
We construct a homotopy between g to g'. Consider the homotopy
$$A : \mathbb{K}^\infty \times I \to \mathbb{K}^\infty$$
$$((x_1, x_2, \dots), t) \mapsto (1-t)(x_1, x_2, \dots) + t(x_1, 0, x_2, 0, \dots).$$
Similarly, consider the homotopy
$$B : \mathbb{K}^\infty \times I \to \mathbb{K}^\infty$$
$$((x_1, x_2, \dots), t) \mapsto (1-t)(x_1, x_2, \dots) + t(0, x_1, 0, x_2, \dots).$$
For each $t \in I$, $L_t, B_t : \mathbb{K}^\infty \to \mathbb{K}^\infty$, are injective linear maps. We have $g_0 \cong A_1 \circ g_0 =: H_0$ (putting g_0 into the odd coordinates), $g_1 \cong B_1 \circ g_1 =: H_1$ (putting g_1 into the even coordinates), and $H_0 \cong H_1$ via $(1-t)H_0 + tH_1$ (all through maps that are linear injections on each fiber).

(2) (Surjectivity) Suppose $\pi : E \to X$ is an rank-n vector bundle. Let $\{U_\alpha\}_\alpha$ be an open cover of X such that E is trivial over each U_α for each α. Since X is a paracompact Hausdorff topological space, we can assume WLOG that the open cover is countable [Hat03, Lemma 1.21]. Since X is paracompact Hausdorff, we can find a partition of unity $\{\rho_\alpha\}_\alpha$ with ρ_α supported in U_α. Let $g_\alpha : \pi^{-1}(U_\alpha) \to \mathbb{K}^n$ be the composition of a trivialization $\pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{K}^n$ with projection onto $\mathbb{K}^n$. The map
$$(\phi_\alpha \circ \pi) g_\alpha : E \to \mathbb{K}^n,$$
$$e \mapsto \phi_\alpha(\pi(e)) g_\alpha(e).$$
extends g_α to a map on $E \to \mathbb{K}^n$ that is zero outside $\pi^{-1}(U_\alpha)$. Then we can define
$$g : E \to \mathbb{K}^\infty \cong (\mathbb{K}^n)^\infty,$$
$$e \mapsto (g_1(e), g_2(e), \dots).$$
By Lemma 8.18 this corresponds to the required map $f : E \to G_n$. This shows surjectivity.

This completes the proof. □

Remark 8.19. *An explicit calculation of $[X, G_n]$ is usually beyond the reach of what is possible technically, so Proposition 8.17 is of limited use for computational purposes. Its main importance is due more to its theoretical implication that topological K-theory is representable.*

Example 8.20. Consider $\pi : T\mathbb{S}^n \to \mathbb{S}^n$. Each fiber $\pi^{-1}(x)$ is a point in $G_1(\mathbb{R}^{n+1})$, so we have a map
$$\mathbb{S}^n \to G_n(\mathbb{R}^{n+1})$$
$$x \mapsto \pi^{-1}(x)$$
Via the inclusion $\mathbb{R}^{n+1} \hookrightarrow \mathbb{R}^\infty$, we can view this as a map $f : \mathbb{S}^n \to G_n(\mathbb{R}^\infty)$, and $\pi : T\mathbb{S}^n$ is the pullback of γ^n.

Remark 8.21. *There is also a version of Proposition 8.17 for oriented vector bundles. Let $\widetilde{G_n}(\mathbb{K}^m)$ be the space of oriented n-planes in $\mathbb{K}^m$, the quotient space of $V_n(\mathbb{K}^m)$ obtained by identifying two n frames when they determine the same oriented subspace of $\mathbb{K}^m$. We can define $\widetilde{G_n}$ as above. The universal oriented bundle*

$$\widetilde{\gamma}_n \to \widetilde{G_n}$$

consists of pairs $(\omega, v) \in \widetilde{G_n} \times \mathbb{K}^\infty$ with $v \in \omega$. Note that $\widetilde{\gamma^n}$ is the pullback of γ^n via the natural projection map:

$$\widetilde{G_n} \to G_n$$

One can show that we have

$$\mathbf{Vect}_n^{+,\mathbb{K}}(\mathbb{K}, X) \cong [X, \widetilde{G_n}]$$

There are several important points to consider:

(1) *Both $\widetilde{G_n}(\mathbb{K}^m)$ and $\widetilde{G_n}$ are path-connected since $\mathbf{Vect}_n^{\mathbb{K}}(X)$ and $\mathbf{Vect}^{+,\mathbb{K}}(X)$ have a single element when X is a point.*

(2) *A vector bundle $E \approx f^*\gamma^n$ is orientable if and only if its classifying map $f : X \to G_n$ lifts to a map $\widetilde{f} : X \to \widetilde{G_n}$. Orientations of $E \to X$ correspond bijectively with lifts $\widetilde{f}$.*

(3) *The natural projection*

$$\widetilde{G_n} \to G_n$$

yields a 2-1 covering map. In fact, $\widetilde{G_n}$ is the universal covering space of G_n since $\widetilde{G_n}(\mathbb{K}^\infty)$ is simply connected because of the triviality of

$$\mathbf{Vect}_n^{+,\mathbb{K}}(\mathbb{S}^1) \cong [\mathbb{S}^1, \widetilde{G_n}]$$

Part 2. K-Theory

All topological spaces are implicitly assumed to be connected, paracompact Hausdorff topological spaces from now on. We denote the category of such spaces as **ParaHaus**. We will additionally assume that a topological space is compact if we wish to invoke Proposition 6.3. We denote the category of such spaces as **CHaus**. Pointed and homotopy versions of these categories are written appropriately.

9. Unreduced and Reduced K-Theory

9.1. **Grothendieck Group of a Commutative Semigroup.** A semigroup is a nonempty set equipped with an associative binary operation. e adopt the convention that the semigroup contains an identity element, denoted by e. In other words, it is an algebraic structure resembling a group but without the requirement of inverses. There exists a universal construction that associates to any commutative semigroup an abelian group, known as its Grothendieck group.

Proposition 9.1. *Let A be a commutative semigroup. Then there exists an abelian group $G(A)$ and a semigroup homomorphism $i : A \to G(A)$ satisfying the following universal property: if B is an abelian group and $\phi : A \to B$ a semigroup homomorphism, then there is a unique group homomorphism $\bar{\phi} : G(A) \to B$ such that the diagram*

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "\phi"'] & G(A) \arrow[dl, dashed, "\bar{\phi}"] \\ B & \end{tikzcd}$$

Proof. Define $G(A) = A \times A/\sim$ where $\sim$ is the equivalence relation[6]:

$$(a, b) \sim (a_0, b_0) \iff \text{there exists } c \in A \text{ such that } a + b_0 + c = a_0 + b + c.$$

It can be checked that $\sim$ is an equivalence relation. Denote the equivalence class of (a, b) by $[a, b]$. We can define the addition on $G(A)$ by

$$[a, b] + [a_0, b_0] = [a + a_0, b + b_0].$$

It can be checked that addition is well-defined. Note that $G(A)$ also has the structure of an abelian group: (e, e) is the identity, and the inverse of $[a, b]$ is $[b, a]$. Let

$$\begin{aligned} i : A &\to G(A) \\ a &\mapsto [a, e], \end{aligned}$$

If $\phi : A \to B$ is a semigroup homomorphism, then define

$$\begin{aligned} \bar{\phi} : G(A) &\to B \\ \bar{\phi}[a, b] &= \phi(a) - \phi(b) \end{aligned}$$

$\bar{\phi}$ is well-defined on equivalence classes because ϕ is a semigroup homomorphism. Moreover, $\bar{\phi}$ is also a group homomorphism. Moreover, $\bar{\phi}$ is unique because the requirement $\bar{\phi}[a, e] = \phi(a)$, but this automatically determines $\bar{\phi}$ on all of $G(A)$, since

$$\begin{aligned} \bar{\phi}[a, b] &= \bar{\phi}([a, e] + [e, b]) \\ &= \bar{\phi}([a, e]) + \bar{\phi}([e, b]) \\ &= \bar{\phi}([a, e]) - \bar{\phi}([b, e]) \\ &= \phi(a, 0) - \phi(b, 0) \end{aligned}$$

This completes the proof. $\square$

[6]We need to add c because A might not be cancellative: $a + c = a + c'$ does not imply $c = c'$. Thus, just as with localization of rings, we need to add an extra term to actually get an equivalence relation.

Remark 9.2. *It can be checked that G is a covariant functor from the category of commutative semigroups to the category of abelian groups. In fact, G is left-adjoint to the forgetful functor from the category of abelian groups to the category of commutative semigroups.*

Remark 9.3. *If A is a commutative semiring, then $G(A)$ is a commutative ring with multiplication*

$$[a, b] \cdot [a_0, b_0] = [a \cdot a_0 + b \cdot b_0, a \cdot b_0 + b \cdot a_0].$$

Example 9.4. Let $A = (\mathbb{N}, +)$. Then $A = \mathbb{N}$ is semigroup under addition. Then $(\mathbb{Z}, +)$ together with the inclusion map $\iota : \mathbb{N} \to \mathbb{Z}$ satisfies the universal property in Proposition 9.1. Hence,

$$G(\mathbb{N}) \cong \mathbb{Z}$$

9.2. K^0**: Unreduced K-Theory.** We are now prepared to define the (unreduced) K^0 group associated with a topological space. Throughout, we assume that the underlying topological space is compact and Hausdorff. For $\mathbb{K} = \mathbb{R}, \mathbb{C}$, let

$$\mathbf{Vect}^{\mathbb{K}}(X) = \left(\bigsqcup_{n \in \mathbb{N}} \mathbf{Vect}_n^{\mathbb{K}}(X) \right) \cup \{\varepsilon^0\}$$

be the set of isomorphism classes of vector bundles over X. Here ε^0 is the rank-0 vector bundle $X \cong X \times \mathbb{K}^0$.

Remark 9.5. *By abuse of notation, we write an isomorphism class of a vector bundle $E \to X$ in $\mathbf{Vect}^{\mathbb{K}}(X)$ as simply E.*

The set $\mathbf{Vect}^{\mathbb{K}}(X)$ is endowed with the structure of a commutative semigroup under the operation of direct sum of vector bundles. The identity is given by ε^0. Moreover, it forms a commutative semiring when equipped with the additional operation of tensor product of vector bundles. The unit is given by ε^1 is the rank-1 vector bundle $X \times \mathbb{K}$.

Definition 9.6. Let X be a topological space. The K**-theory** of X, denoted as $K^0_{\mathbb{K}}(X)$, is the commutative ring

$$K^0_{\mathbb{K}}(X) = G(\mathbf{Vect}^{\mathbb{K}}(X))$$

We can explicitly describe elements of $K^0_{\mathbb{K}}(X)$. Every element of $K^0_{\mathbb{K}}(X)$ is of the form

$$\begin{aligned} [E, F] &= [E, \varepsilon^0] + [\varepsilon^0, F] \\ &= [E, \varepsilon^0] - [F, \varepsilon^0] \\ &:= [E] - [F], \end{aligned}$$

where $E, F \to X$ are (isomorphism classes of) vector bundles over X. An element of $K^0_{\mathbb{K}}(X)$ is called a virtual vector bundle. If X is a compact topological space, we can say a bit more.

Proposition 9.7. *Let be X is in fact compact Hausdorff topological space.*

1. *Every element of $K^0_{\mathbb{K}}(X)$ can be represented as $[H] - [\varepsilon^n]$, where $H \to X$ is a (isomorphism class of) vector bundle and $n \in \mathbb{N}$.*
2. *We have*

$$[E] - [\varepsilon^n] = [F] - [\varepsilon^m] \iff E \oplus \varepsilon^{m+k} = E \oplus \varepsilon^{n+k}$$

for some $k \in \mathbb{N}$.

Proof. The proof is given below:

(1) Let $[E, F] \in K^0_{\mathbb{K}}(X)$. By Proposition 6.3 we can always find a vector bundle $G \to X$ such that $F \oplus G \cong \varepsilon^n$ for some $n \in \mathbb{N}$. Then we have

$$\begin{aligned}[E, F] &= [E \oplus G, F \oplus G] \\ &= [E \oplus G, \varepsilon^n] \\ &= [E \oplus G, \varepsilon^0] + [\varepsilon^0, \varepsilon^n] \\ &= [E \oplus G, \varepsilon^0] - [\varepsilon^n, \varepsilon^0] \\ &= [E \oplus G] - [\varepsilon^n] := [H] - [\varepsilon^n]\end{aligned}$$

(2) We have $[E] - [\varepsilon^n] = [F] - [\varepsilon^m]$ if and only if there exists a vector bundle $G \to X$ such that $E \oplus \varepsilon^m \oplus G \cong F \oplus \varepsilon^n \oplus G$. Let $G' \to X$ be a vector bundle such that $G \oplus G' \cong \varepsilon^k$ for some $k \in \mathbb{N}$. Then, $E \oplus \varepsilon^m \oplus G \cong F \oplus \varepsilon^n \oplus G$ implies that

$$E \oplus \varepsilon^m \oplus G \oplus G' \cong F \oplus \varepsilon^n \cong G \oplus G' \iff E \oplus \varepsilon^m \oplus \varepsilon^k \cong F \oplus \varepsilon^n \oplus \varepsilon^k$$

This completes the proof. □

Proposition 9.7(2) motivates us to introduce the definition of stable equivalence of vector bundles; that is, two vector bundles E and F are stably equivalent if and only if

$$E \oplus \varepsilon^k \cong F \oplus \varepsilon^k$$

for some $k \in \mathbb{N}$. Let $\mathrm{Vect}^{\mathbb{K}}_{\mathrm{Stable}}(X)$ denote the equivalence class of stable vector bundles over X. We write an equivalence class in $\mathrm{Vect}^{\mathbb{K}}_{\mathrm{Stable}}(X)$ as $[E]_s$. Note that $\mathrm{Vect}^{\mathbb{K}}_{\mathrm{Stable}}(X)$ is a commutative semigroup. We have the following result.

Corollary 9.8. *Let X be a paracompact Hausdorff topological space. We have a homomorphism of commutative semigroups:*

$$\begin{aligned}\mathrm{Vect}^{\mathbb{K}}_{\mathrm{Stable}}(X) &\longrightarrow K^0_{\mathbb{K}}(X) \\ [E]_s &\mapsto [E, \varepsilon^0]\end{aligned}$$

If X is a compact Hausdorff space, then the map is an isomorphism on its image.

Proof. This is clear. □

Example 9.9. The following is a basic list of computations of $K^0_{\mathbb{K}}(X)$.

(1) Let $X = \{*\}$. A rank n vector bundle is the trivial vector bundle $\{*\} \times \mathbb{K}^n$. Hence, $\mathbf{Vect}^{\mathbb{K}}(X) \cong \mathbb{N}$. Therefore,

$$K^0_{\mathbb{K}}(\{*\}) = G(\mathbb{N}) \cong \mathbb{Z}$$

(2) Let $X = \coprod_{i=1}^n X_i$, where each X_i is a paracompact Hausdorff (second countable) space. A vector bundle on X is a choice of a vector bundle on each $X_1, \cdots, X_n$. The same is true for isomorphism classes of vector bundles on X. Therefore,

$$\mathbf{Vect}^{\mathbb{K}}(X) \cong \bigoplus_{i=1}^n \mathbf{Vect}^{\mathbb{K}}(X_i)$$

Since G is a left adjoint functor and $\oplus$ is a coproduct, we have that G commutes with direct sums. Hence, we obtain an isomorphism

$$\begin{aligned}K^0_{\mathbb{K}}(X) &= G\left(\bigoplus_{i=1}^n \mathbf{Vect}^{\mathbb{K}}(X_i)\right) \\ &= \bigoplus_{i=1}^n G(\mathbf{Vect}^{\mathbb{K}}(X_i)) = \bigoplus_{i=1}^n K^0_{\mathbb{K}}(X_i).\end{aligned}$$

In particular, if X is a discrete set consisting n points, then $K^0_{\mathbb{K}}(X) \cong \mathbb{Z}^n$

We now argue that the construction of K-theory is functorial. Indeed, note that

$$K^0_{\mathbb{K}} : \mathbf{ParaHaus}^{\mathbf{Op}} \longrightarrow \mathbf{CRing}$$
$$X \mapsto K^0_{\mathbb{K}}(X).$$

is a functor because if $f : X \longrightarrow Y$ is continuous map between two paracompact Hausdorff space, then pullback operation induces a map

$$f^* : \mathbf{Vect}^{\mathbb{K}}(Y) \longrightarrow \mathbf{Vect}^{\mathbb{K}}(X)$$
$$[E] \mapsto [f^*E]$$

Remark 5.7 verifies that f^* is a map between commutative semi-rings. By functoriality of G (Proposition 9.1), we get a map

$$f^* : K^0_{\mathbb{K}}(X) \longrightarrow K^0_{\mathbb{K}}(Y),$$

which we also denote by f^*. Since the induced homomorphism between $K^0_{\mathbb{K}}(\cdot)$ groups depends only on the homotopy class of f (Corollary 6.9) , the functor $K^0_{\mathbb{K}} : \mathbf{ParaHaus}^{\mathbf{Op}} \longrightarrow \mathbf{CRing}$ descends to a contravariant functor

$$K^0_{\mathbb{K}} : \mathbf{hParaHaus}^{\mathbf{Op}} \longrightarrow \mathbf{CRing}$$

9.3. $\widetilde{K}^0$**: Reduced** K **Theory.** We now discuss the reduced $K^0_{\mathbb{K}}$ group that are associated with pointed topological spaces. By way of motivation, if (X, x) is a pointed paracompact, Hausdorff topological space, then we have a sequence of maps

$$x \xrightarrow{i} X \xrightarrow{p} x$$

These maps induce maps

$$\mathbb{Z} \cong K^0_{\mathbb{K}}(x) \xrightarrow{p^*} K^0_{\mathbb{K}}(X) \xrightarrow{i^*} K^0_{\mathbb{K}}(x) \cong \mathbb{Z}$$

Since $p \circ i = \mathrm{Id}_{\{x\}}$, we have that $i^* \circ p^* = \mathrm{Id}_{\mathbb{Z}}$. This shows that $\mathbb{Z}$ is a direct summand of $K^0_{\mathbb{K}}(X)$. We attempt to analyze the complement in the following manner:

Definition 9.10. Let (X, x) be a pointed paracompact, Hausdorff topological space. Let $i : \{x\} \longrightarrow X$ denote the inclusion map. The **reduced** K**-theory** of X, denoted as $\widetilde{K}^0_{\mathbb{K}}(X)$ is the kernel

$$\widetilde{K}^0_{\mathbb{K}}(X) := \ker(i^* : K^0_{\mathbb{K}}(X) \longrightarrow K^0_{\mathbb{K}}(x) \cong \mathbb{Z}),$$

Remark 9.11. *Since $K^0_{\mathbb{K}}(X)$ is a commutative ring, $\widetilde{K}^0_{\mathbb{K}}(X)$ is a proper ideal of $K^0_{\mathbb{K}}(X)$*

Definition 9.10 implies that we have a short exact sequence of abelian groups:

$$0 \longrightarrow \widetilde{K}^0_{\mathbb{K}}(X) \longrightarrow K^0_{\mathbb{K}}(X) \xrightarrow{i^*} \mathbb{Z} \longrightarrow 0,$$

Since $\mathbb{Z}$ is a free abelian group, we have an isomorphism:

$$K^0_{\mathbb{K}}(X) \cong \widetilde{K}^0_{\mathbb{K}}(X) \oplus \mathbb{Z}.$$

of abelian groups. Hence, the complement of $\mathbb{Z}$ in $K^0_{\mathbb{K}}(X)$ is precisely the reduced K-theory of a pointed paracompact, Hausdorff topological space. Additionally, note that the map

$$i^* : K^0_{\mathbb{K}}(X) \longrightarrow K^0_{\mathbb{K}}(x) \cong \mathbb{Z}$$

sends $[E] := [E, \varepsilon^0]$ to $\dim E$. Hence, a $[E] - [F]$ is sent to $\dim E - \dim F$, called the virtual rank.

Corollary 9.12. *$\widetilde{K}^0_{\mathbb{K}}$ defines a contravariant functor*

$$\widetilde{K}^0_{\mathbb{K}} : \mathbf{hParaHaus}_*^{\mathbf{Op}} \longrightarrow \mathbf{CRng},$$

where **CRng** *is the category of non-unital commutative rings*

Proof. Functorility of $K^0_{\mathbb{K}}$ and the kernel implies the functorality of $\widetilde{K}^0_{\mathbb{K}}$. □

We now specialize to the case of compact, pointed topological space. If (X, x) is pointed compact Hausdorff topological space a general element of $K^0_{\mathbb{K}}(X)$ can be written as $[E] - [\varepsilon^n]$ for some $n \in \mathbb{N}$ and is mapped to $\dim E - n$. Hence, $\widetilde{K}^0_{\mathbb{K}}(X)$ consists of elements of the form $[E] - [\varepsilon^{\dim E}]$. In fact, we have

$$[E] - [\varepsilon^{\dim E}] = [F] - [\varepsilon^{\dim F}] \iff E \oplus \varepsilon^{\dim F + k} = F \oplus \varepsilon^{\dim E + k}$$

for some $k \in \mathbb{N}$. We would like to Grothenedieck completion of the image of the map in **??** to be isomorphism to $\widetilde{K}^0_{\mathbb{K}}$. This motivates us to introduce a more general definition of stable vector bundles. We say two vector bundles E and F are eventually stably equivalent if and only if

$$E \oplus \varepsilon^k \cong F \oplus \varepsilon^{k'}$$

for some $k, k \in \mathbb{N}$. Let $\mathrm{Vect}^{\mathbb{K}}_{\mathrm{eStable}}(X)$ denote the equivalence class of eventually stable vector bundles over X. We write an equivalence class in $\mathrm{Vect}^{\mathbb{K}}_{\mathrm{eStable}}(X)$ as $[E]_{es}$. Note that $\mathrm{Vect}^{\mathbb{K}}_{\mathrm{eStable}}(X)$ is a commutative semigroup. We have the following result.

Proposition 9.13. *Let (X, x) is a pointed compact Hausdorff topological space. There is a map of commutative semigroups*

$$\phi : \mathrm{Vect}^{\mathbb{K}}_{\mathrm{eStable}}(X) \longrightarrow K^0_{\mathbb{K}}(X)$$
$$[E]_s \mapsto [E, \varepsilon^{\dim E}]$$

such that

$$G(\phi(\mathrm{Vect}^{\mathbb{K}}_{\mathrm{eStable}}(X))) \cong \widetilde{K}^0_{\mathbb{K}}(X)$$

Proof. It is straightfoward to verify that ϕ is a homomorophism of commutative semigroups onto $\widetilde{K}^0_{\mathbb{K}}(X)$. It is easy to verify using the definition of eventually stably isomorphic that this map is injective. The claim now follows by the universal property of Grothendieck completion. □

Remark 9.14. *Note that $K^0_{\mathbb{K}}(X)$ can be recovered from $\widetilde{K}^0_{\mathbb{K}}(X)$ because:*

$$K^0_{\mathbb{K}}(X) \cong \widetilde{K}^0_{\mathbb{K}}(X_+),$$

where $X_+ = X \coprod \{*\}$. Hence, $\widetilde{K}^0_{\mathbb{K}}$ of compact Hausdorff space can be thought of as the $K^0_{\mathbb{K}}$ group of locally compact Hausdorff spaces.

10. Representability of K-Theory

We have constructed functors

$$K^0_{\mathbb{K}} : \mathbf{hCHaus}^{\mathbf{Op}} \to \mathbf{CRing},$$
$$\widetilde{K}^0_{\mathbb{K}} : \mathbf{hCHaus}^{\mathbf{Op}}_{*} \to \mathbf{CRng}.$$

These functors can be thought of as set-valued functors. We now argue that these set-valued functor are representable. For $k \in \mathbb{N}$, we know from Proposition 8.17 that there is a bijection of sets:

$$\mathbf{Vect}^{\mathbb{K}}_n(X) \cong [X, G_n(\mathbb{K}^\infty)]$$

Recall the following facts:

1. If $\mathbb{K} = \mathbb{R}$, then $G_n(\mathbb{K}^\infty) \cong BO(n)$, where $BO(n)$ is the classifying space of principal $O(n)$-bundles. Hence, we interchangeably write $BO(n)$ for $G_n(\mathbb{R}^\infty)$ from now on.
2. If $\mathbb{K} = \mathbb{C}$, then $G_n(\mathbb{C}^\infty) \cong BU(n)$, where $BU(n)$ is the classifying space of principal $U(n)$-bundles. Hence, we interchangeably write $BU(n)$ for $G_n(\mathbb{C}^\infty)$ from now on.

Note that we have the inclusions

$$G_n(\mathbb{K}^m) \subseteq G_{n+1}(\mathbb{K}^{m+1})$$

for $m, n \in \mathbb{N}$ such that $n \geq m$. Therefore, we have the inclusions

$$G_n(\mathbb{K}^\infty) = \varinjlim_{m \geq n} G_n(\mathbb{K}^m) \subseteq \varinjlim_{m \geq n} G_{n+1}(\mathbb{K}^{m+1}) = G_{n+1}(\mathbb{K}^\infty)$$

If $\mathbb{K} = \mathbb{R}$, we write $BO := \varinjlim_{n \in \mathbb{N}} G_n(\mathbb{R}^\infty)$, and $\mathbb{K} = \mathbb{C}$, $BU := \varinjlim_{n \in \mathbb{N}} G_n(\mathbb{C}^\infty)$.

Remark 10.1. *The notation above is suggested as follows. If $K = \mathbb{R}$, we have inclusions*

$$O(n) \subseteq O(n+1) \subseteq \cdots$$

Hence, we can consider the colimit:

$$O := O(\infty) = \varinjlim_{n \in \mathbb{N}} O(n)$$

Here $O := O(\infty)$ is the infinite orthogonal group. It can be shown that $G_n(\mathbb{R}^\infty)$ is the classifying space for principal-$O(\infty)$ bundles. This motivates the notation above. Similar remarks apply if $K = \mathbb{C}$. The infinite unitary group is denoted as $U := U(\infty)$.

We have the following result:

Proposition 10.2. *Let (X, x) be a pointed compact Hausdorff topological space. If $\mathbb{K} = \mathbb{R}$, we have:*

$$\widetilde{K}^0_{\mathbb{R}}(X) \cong [X, BO]$$
$$K^0_{\mathbb{R}}(X) \cong [X_+, \mathbb{Z} \times BO]$$

as sets. If $\mathbb{K} = \mathbb{C}$, we have:

$$\widetilde{K}^0_{\mathbb{C}}(X) \cong [X, BU]$$
$$K^0_{\mathbb{C}}(X) \cong [X_+, \mathbb{Z} \times BU]$$

as sets.

Proof. We only prove the case $\mathbb{K} = \mathbb{R}$. Let $\{\varepsilon^0 := \mathbf{Vect}^{\mathbb{K}}_0(X)$. Using the definition of eventually stably equivalent vector bundles, note that $\mathbf{Vect}^{\mathbb{K}}(X)$ can be regarded as a filtered colimit.

$$\mathbf{Vect}^{\mathbb{K}}(X) = \varinjlim_{n \in \mathbb{N} \cup \{0\}} \mathbf{Vect}^{\mathbb{K}}_n(X)$$

Identifying $\widetilde{K}^0_{\mathbb{R}}(X)$ with the Grothendieck completion of eventually stably equivalent vector bundles, we have

$$\begin{aligned}
\widetilde{K}^0_{\mathbb{R}}(X) &= \varinjlim_{n \in \mathbb{N} \cup \{0\}} G(\mathbf{Vect}^{\mathbb{K}}_n(X)) \\
&= G(\varinjlim_{n \in \mathbb{N} \cup \{0\}} [X, G_n(\mathbb{K}^\infty)]) \\
&= G(\varinjlim_{n \in \mathbb{N}} [X, G_n(\mathbb{K}^\infty)]) \\
&= G([X, \varinjlim_{n \in \mathbb{N}} G_n(\mathbb{K}^\infty)]) = G([X, BO]) = [X, BO]
\end{aligned}$$

Note that $\varinjlim_{n \in \mathbb{N}}$ commutes with $[X, G_n(\mathbb{K}^\infty)]$ because X is compact. Moreover, $G([X, BO]) = [X, BO]$ since $[X, BO]$ is a because BO is a topological group. We also have

$$K^0_{\mathbb{K}}(X) \cong \mathbb{Z} \oplus \widetilde{K}^0_{\mathbb{K}}(X) \cong [X, Z] \oplus [X, BO] \cong [X, \mathbb{Z} \times BO]$$

The case $\mathbb{K} = \mathbb{C}$ is similar. $\square$

If $\mathbb{K} = \mathbb{R}$, we write $K^0_{\mathbb{R}}(X)$ as $KO(X)$ and $\widetilde{K}^0_{\mathbb{R}}(X)$ as $\widetilde{KO}(X)$. Similarly, if $\mathbb{K} = \mathbb{C}$, we write $K^0_{\mathbb{K}}(X)$ as $KU(X)$ and $\widetilde{K}^0_{\mathbb{R}}(X)$ as $\widetilde{KU}(X)$. We use this notation from now on when we specialize to $\mathbb{K} = \mathbb{R}, \mathbb{C}$.

11. Fredholm Operators

We discuss an alternative model for K-theory. One such model is given by the space of Fredholm operators. This perspective allows the use of techniques from functional analysis, such as operator theory and Banach or Hilbert space methods, to study vector bundles and their classification. Moreover, the space of Fredholm operators naturally arises in index theory, where the Fredholm index serves as an invariant capturing topological information. This connection plays a crucial role in the Atiyah-Singer index theorem, which bridges topology, geometry, and analysis.

11.1. **Fredholm Operators.** Let V, W be $\mathbb{K}$-vector spaces, and let $T : V \to W$ is a linear transformation. In linear algebra, we are interested determining the solution space of the equation $Tv = w$. The solution space can be better understood by measuring the surjectivity and injectivity of the operator T.

(1) T is surjective if and only if $Tv = w$ has a solution for each $w \in W$. Note that
$$T \text{ is surjective} \iff \operatorname{coker} T = W/R(T) = 0.$$
Hence, if $\operatorname{coker} T \neq 0$, then $\operatorname{coker} T$ can be thought of as measuring the extent to which T fails to be surjective. This can be made precise as follows. If $\operatorname{coker} T$ is finite-dimensional, we can find finitely many linear functionals $\varphi_1, \dots, \varphi_k$ such that
$$w \in \operatorname{Im}(T) \iff \varphi_1(w) = \cdots = \varphi_k(w) = 0$$

(2) T is injective if and only if $\ker T = 0$. In other words,
$$Tv = w \text{ have either no or unique solutions} \iff \ker T = 0$$
Hence, if $\ker T \neq 0$, then $\ker T$ can be thought of as measuring the extent to which T fails to admit unique solutions. This can be made precise as follows. If $\ker T$ is finite-dimensional, we can find a basis $v_1, \dots, v_k$ for $\ker T$. such that
$$Tv = w \text{ has a solution } v_0 \iff v_0 + \ker T \text{ is the full solution space}$$

Hence if both $\ker T$ and $\operatorname{coker} T$ are finite-dimensional, the existence and uniquess of solutions of $Tv = w$ can be determined by *finitely many pieces of data*. This motivates the definition of Fredholm operators. We specialize to the case of Hilbert spaces.

Definition 11.1. Let $\mathscr{H}_1, \mathscr{H}_2$ be Hilbert spaces. An operator $T : \mathscr{H}_1 \to \mathscr{H}_2$ is a **Fredholm operator** if both $\ker T$ and coker T are both finite dimensional. The space of Fredholm operators on X in denoted as $\mathfrak{F}(\mathscr{H}_1, \mathscr{H}_2)$.

We now provide an alternative characterization of Fredholm operators. In particular, the following result asserts that an operator is a Fredholm operator exactly when it is invertible modulo a compact operator.

Proposition 11.2. *Let $\mathscr{H}_1, \mathscr{H}_2$ be Hilbert spaces, and let $T \in \mathscr{L}(\mathscr{H}_1, \mathscr{H}_2)$.*

(1) *T is a Fredholm operator if and only if both $\ker T$ and $\ker T^*$ finite dimensional and $R(T)$ is closed. In particular,*
$$T \text{ is a Fredholm operator} \iff T^* \text{ is a Fredholm operator}$$

(2) ***(Atkinson's Theorem)*** *T is a Fredholm operator if and only if there exists an operator $S : \mathscr{H}_2 \to \mathscr{H}_1$ such that*

$$ST = I_{\mathscr{H}_1} + K_1$$

$$TS = I_{\mathscr{H}_2} + K_2$$

for some compact operators $K_1 \in \mathscr{K}(\mathscr{H}_1)$ and $K_2 \in \mathscr{K}(\mathscr{H}_2)$.

Proof. The proof is given below:

(1) Suppose that T is a Fredholm operator. By definition, $\ker T$ is finite-dimensional. We show that $R(T)$ is closed. Since T is Fredholm, we have $\mathscr{H}_2 = R(T) + Z$ as vector spaces, where Z is finite-dimensional and hence closed subspace of $\mathscr{H}_2$. We can assume that T is injective. Otherwise since $\ker(T)$ is a closed subspace, $\mathscr{H}_1/\ker(T) \cong \ker(T)^\perp$ is a Hilbert space and we can replace T by the restricted map $\ker(T)^\perp \to \mathscr{H}_2$. Now consider the map $S : \mathscr{H}_1 \oplus Z \to \mathscr{H}_2$ defined by

$$S(x, z) = T(x) + z.$$

The map S is a bounded linear isomorphism, and hence by the open mapping theorem S is a homeomorphism. Thus, $R(T) = S(\mathscr{H}_1 \oplus \{0\})$ is closed since $\mathscr{H}_1 \cong \mathscr{H}_1 \oplus \{0\}$ is closed. Now recall that $\ker T^* = R(T)^\perp$. Since $R(T)$ is closed and $\mathscr{H}_2 = R(T) \oplus R(T)^\perp$, it follows that

$$\ker T^* = R(T)^\perp \cong \mathscr{H}_2/R(T) = \operatorname{coker} T$$

Hence, $\ker T^*$ is finite-dimensional as well. Conversely, assume that both $\ker T, \ker T^*$ finite dimensional and $R(T)$ is closed. Since $\overline{R(T)} = R(T)$, we have $\mathscr{H}_2 = R(T) \oplus R(T)^\perp$. Since $\ker T^* = R(T)^\perp$ is finite-dimensional, we have

$$\ker T^* = R(T)^\perp \cong \mathscr{H}_2/R(T) = \operatorname{coker} T$$

is finite-dimensional. The last claim follows from the result from Hilbert space theory that if $R(T)$ is closed then $R(T^*)$ is also closed.

(2) Suppose T is a Fredholm operator. By (1), $R(T)$ is closed. We have the following decompositions:

$$\mathscr{H}_1 = \ker T \oplus \ker T^\perp$$

$$\mathscr{H}_2 = R(T) \oplus R(T)^\perp$$

Since is the restricted operator

$$T|_{(\ker T)^\perp} : (\ker T)^\perp \to R(T)$$

is invertible, let

$$S : R(T) \to (\ker T)^\perp$$

be the inverse of $T|_{(\ker T)^\perp}$. Since $R(T)^\perp \cong \operatorname{coker} T$ is finite dimensional, we can extend to all of $\mathscr{H}_2$ by defining $S \equiv 0$ on $R(T)^\perp$. Let P be the orthogonal projection onto $\ker T$, and let Q be the orthogonal projection onto $R(T)^\perp \cong \operatorname{coker} T$. It follows that

$$ST + P = \mathrm{Id}_{\mathscr{H}_1}$$

$$TS + Q = \mathrm{Id}_{\mathscr{H}_2}$$

Since $\ker T$ and $R(T)^\perp \cong \operatorname{coker} T$ are finite dimensional, P and Q are finite rank operators and hence are compact operators.

Conversely, suppose that there exists an $S : \mathscr{H}_2 \to \mathscr{H}_1$ such that

$$ST = I_{\mathscr{H}_1} + K_1$$

$$TS = I_{\mathscr{H}_2} + K_2$$

for some compact operators $K_1 \in \mathcal{K}(\mathcal{H}_1)$ and $K_2 \in \mathcal{K}(\mathcal{H}_2)$. Hence, $ST - I_{\mathcal{H}_1}$ is compact. Let $f \in \ker T$.

$$f = (I_{\mathcal{H}_1} - ST)f + STf = (I_{\mathcal{H}_1} - ST)f.$$

Hence, $\ker T \subseteq R(I_{\mathcal{H}_1} - ST)$. In particular, the closed unit ball in $\ker T$ is contained in the closed unit ball of $R(I_{\mathcal{H}_1} - ST)$. Since $I_{\mathcal{H}_1} - ST$ is compact, the closed unit ball of $\ker T$ is compact. Thus, $\ker T$ is finite dimensional. Similarly, the equation $TS = I_{\mathcal{H}_2} + K_2$ implies that $I_{\mathcal{H}_2} - TS$ is compact. Taking the adjoint of this equation gives

$$(I_{\mathcal{H}_2} - TS)^* = I_{\mathcal{H}_2} - S^*T^*$$

Since the adjoint of a compact linear map is a compact linear map, we have that $I_{\mathcal{H}_2} - S^*T^*$ is a compact operator. We can repeat the argument above to show that $\ker T^*$ is finite dimensional. We now show that $R(T)$ is closed. Since finite rank operators are dense in $\mathcal{K}(\mathcal{H}_1)$, for $\varepsilon > 0$ let F_ε be a finite-rank operator such that $\|F_\varepsilon - K_1\| < \varepsilon$. Then, for every $x \in \ker(F_\varepsilon)$, we have

$$\begin{aligned}
\|S\|\|Tx\| &\geq \|STx\| \\
&= \|x + K_1 x\| \\
&= \|x + F_\varepsilon x + K_1 x - F_\varepsilon x\| \\
&\geq \|x\| - \|K_1 - F_\varepsilon\|\|x\| \\
&\geq (1 - \varepsilon)\|x\|
\end{aligned}$$

Thus, T is bounded below on $\ker(F_\varepsilon)$, which implies that $T(\ker(F_\varepsilon))$ is closed. On the other hand, $T(\ker(F_\varepsilon)^\perp)$ is finite-dimensional since $\ker(F_\varepsilon)^\perp = R(F_\varepsilon^*)$ is finite-dimensional. Therefore,

$$R(T) = T(\ker(F_\varepsilon)) + T(\ker(F_\varepsilon)^\perp)$$

is closed. Hence, T is a Fredholm operator by (1).

This completes the proof. □

If T is a Fredholm operator, we have argued that $\dim \ker T$ can be thought of as measuring the extent to which $Tx = w$ fails to admit a unique solution, and $\dim \operatorname{coker} T$ can be thought of as measuring the extent to which $Tx = w$ fails to admit a solution. Hence, the expression

$$\dim \ker T - \dim \operatorname{coker} T$$

is a measurement that captures this information in a single numerical quantity. Despite being defined algebraically, the index of an operator T turns out to be a topological invariant. This is the beginning of a much deeper connection between the functional analysis and topology.

Definition 11.3. Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces, and let $T \in \mathfrak{F}(\mathcal{H}_1, \mathcal{H}_2)$. The **Fredholm** index of T is

$$\operatorname{Ind}(T) := \dim \ker T - \dim \operatorname{coker} T.$$

Remark 11.4. *If T is a Fredholm operator, note that*

$$\operatorname{Ind}(T) = \dim \ker T - \dim \ker T^*$$

This is because $R(T) = (\ker T^*)^\perp$. Hence, $\operatorname{coker} T = \ker T^*$ since $R(T)$ is closed.

Remark 11.5. *The Fredholm index can be thought of as a measure of how far an operator is from being invertible. Indeed, if T is invertible, then T is Fredholm with index 0.*

Example 11.6. Let $T : \mathscr{H}_1 \to \mathscr{H}_2$ be a linear operator between finite-dimensional Hilbert spaces. Then:

$$\begin{aligned} \operatorname{ind} T &= \dim \ker T - \dim \operatorname{coker} T \\ &= \dim \ker T + \dim R(T) - \dim(\mathscr{H}_2) \\ &= \dim \mathscr{H}_1 - \dim \mathscr{H}_2 \end{aligned}$$

by the rank-nullity theorem.

Remark 11.7. *Example 11.6 implies that the index on in finite dimensional spaces is 'almost useless' since it is always constant. Thus the analysis of index only becomes useful in the case of infinite dimensional Hilbert spaces.*

Example 11.8. **(Shift Operators)** Let $\mathscr{H} = \ell^2(\mathbb{K})$. We compute the index of the left and right shoft operators.

(1) Consider the left shift operator

$$L : \mathscr{H} \to \mathscr{H},$$
$$(x_1, x_2, x_3, \ldots) \mapsto (x_2, x_3, x_4, \ldots).$$

Then $\dim \ker L = 1$ and $\dim \operatorname{coker} L = 0$. Hence,

$$\operatorname{Ind}(L) = 1 - 0 = 1$$

(2) Consider the right shift operator

$$R : \mathscr{H} \to \mathscr{H},$$
$$(x_1, x_2, x_3, \ldots) \mapsto (0, x_1, x_2, x_3, \ldots).$$

Then $\dim(\ker R) = 0$ and $\dim(\operatorname{coker} R) = 1$, so

$$\operatorname{Ind}(R) = 0 - 1 = -1$$

We now discuss some basic algebraic properties of the Fredholm index.

Proposition 11.9. *Let $\mathscr{H}$ be a Hilbert space. Then:*

(1) *If $T, S \in \mathfrak{F}(\mathscr{H})$, then $S \circ T$ is a Fredholm operator and*

$$\operatorname{Ind}(S \circ T) = \operatorname{Ind}(S) + \operatorname{Ind}(T)$$

(2) *If $T \in \mathfrak{F}(\mathscr{H})$, then*

$$\operatorname{Ind}(T^*) = -\operatorname{Ind}(T)$$

(3) *If $\mathscr{H}$ is seperable, the map*

$$\operatorname{Ind} : \mathfrak{F}(\mathscr{H}) \to \mathbb{Z}$$

is surjective

Proof. The proof is given below:

(1) For every composition ST of linear maps S and T there is an exact sequence of vector spaces[7]

$$0 \to \ker T \to \ker ST \to \ker S \to \operatorname{coker} T \to \operatorname{coker} ST \to \operatorname{coker} S \to 0.$$

Recalling that an exact complex has zero Euler characteristic we get

$$0 = \dim \ker T - \dim \ker ST + \dim \ker S - \dim \operatorname{coker} T + \dim \operatorname{coker} ST - \dim \operatorname{coker} S$$

[7] This follows from the snake lemma.

which is immediately[8] rearranged to give

$$\operatorname{Ind}(ST) = \operatorname{Ind}(S) + \operatorname{Ind}(T).$$

(2) A calculation gives:

$$\begin{aligned}\operatorname{Ind}(T^*) &= \dim(\ker T^*) - \dim(\ker T)\\ &= -(\dim(\ker T) - \dim(\ker T^*)) = -\operatorname{Ind}(T).\end{aligned}$$

(3) We have $\mathscr{H} \cong \ell^2(\mathbb{K})$. Let L, R be the shift operators as in Example 11.8. By (1), we have $\operatorname{Ind}(L^m) = m$ and $\operatorname{Ind}(R^m) = -m$ for $m \in \mathbb{N}$. This implies the claim.

This completes the proof. □

Let's now discuss topological properties of the space of Fredholm operators.

Proposition 11.10. *Let $\mathscr{H}$ be a seperable Hilbert space.*

(1) *$\mathfrak{F}(\mathscr{H})$ is an open subset of $\mathscr{L}(\mathscr{H})$. In particular, $\mathfrak{F}(\mathscr{H})$ can be topologized with the subsapce topology in $\mathscr{L}(\mathscr{H})$.*
(2) *The map* Ind *is continuous, where $\mathbb{Z}$ is equipped with the discrete topology.*

Proof. The proof is given below:

(1) Let $\mathscr{K}(\mathscr{H})$ denote the two sided ideal of compact operators of in $\mathscr{L}(\mathscr{H})$. Let $\pi : \mathscr{L}(\mathscr{H}) \to \mathscr{K}(\mathscr{H})$ be the canonical projection. By Proposition 11.2, we have that:

$$\mathfrak{F}(\mathscr{H}) \cong \pi^{-1}(\mathscr{L}(\mathscr{H})/\mathscr{K}(\mathscr{H}))^{\times},$$

where $(\mathscr{L}(\mathscr{H})/\mathscr{K}(\mathscr{H}))^{\times}$ is the set of all invertible elements in $\mathscr{L}(\mathscr{H})/\mathscr{K}(\mathscr{H})$. Since $(\mathscr{L}(\mathscr{H})/\mathscr{K}(\mathscr{H}))^{\times}$ is an open set $\mathfrak{F}(\mathscr{H})$ is an open subspace of $\mathscr{L}(\mathscr{H})$.

(2) It suffices to show that Ind is locally constant. Let $T \in \mathfrak{F}(\mathscr{H})$. Let $J : (\ker T)^{\perp} \to \mathscr{H}$ denote the inclusion map and $Q : \mathscr{H} \to R(T)$ denote the projection map. We have:

$$\operatorname{Ind} J = -\dim\ker T$$
$$\operatorname{Ind} Q = \dim\operatorname{coker} T$$

Thus,

$$\operatorname{Ind}(QTJ) = \operatorname{Ind} Q + \operatorname{Ind} T + \operatorname{Ind} J = 0.$$

Clearly, we have that

$$QTJ : (\ker T)^{\perp} \to R(T)$$

is invertible. Let $\varepsilon := 1/\|(QTJ)^{-1}\|$, and pick $T' \in \mathscr{H}$ such that $\|T - T'\| < \varepsilon\|Q\|\|J\|$. It thus follows that

$$\|QTJ - QT'J\| = \|Q(T - T')J\| \le \|Q\| \cdot \|T - T'\| \cdot \|J\| < \varepsilon.$$

Hence, $QT'J$ is invertible, and so

$$\operatorname{Ind} QT'J = \operatorname{Ind} Q + \operatorname{Ind} T' + \operatorname{Ind} J = 0.$$

Hence, $\operatorname{Ind} T = \operatorname{Ind} T'$ for all $T' \in \mathfrak{F}(\mathscr{H})$ such that $\|T - T'\| < \varepsilon\|Q\|\|J\|$, implying that the index map is locally constant.

This completes the proof. □

An immediate consequence of Proposition 11.10 is that the index is constant on the path-connected components of $\mathfrak{F}(\mathscr{H})$. In fact, the converse is also true:

[8]Since $\ker T$, $\ker S$, $\operatorname{coker} T$, and $\operatorname{coker} S$ are all finite dimensional, $\ker(ST)$ and $\operatorname{coker}(ST)$ are also finite dimensional. This immediately follows from the result that if $U \to V \to W$ is an exact sequence of vector spaces such that U and W are finite-dimensional, then V is finite-dimensional as well.

Proposition 11.11. *Let $\mathscr{H}$ be a infinite dimensional separable Hilbert space. The Fredholm index induces a bijection:*

$$\varphi : \{\text{path-connected components of } \mathfrak{F}(\mathscr{H})\} \to \mathbb{Z}.$$

Remark 11.12. *In the proof of Proposition 11.11, we shall use the well-known fact that* $\mathrm{GL}(\mathscr{H})$*, the group of invertible operators on a Hilbert space,* $\mathscr{H}$*, is path connected.*

Proof. We have shown that φ is both well-defined and surjective. For each $n \in \mathbb{Z}$, consider the set

$$F_n = \{T \in \mathfrak{F}(\mathscr{H}) : \mathrm{Ind}(T) = n\} = \varphi^{-1}\{n\}$$

Injectivity follows after we show that each F_n is path-connected. First, consider $n = 0$. Clearly, $I \in F_0$. In fact, $\mathrm{GL}(\mathscr{H}) \subseteq F_0$. We show that each $T \in F_0$ can be connected to I by a path. Let $T \in F_0$. Then $\dim\ker T = \dim\ker T^*$. Let $\{v_i\}_{i=1}^k$ and $\{w_i\}_{i=1}^k$ be orthonormal bases for $\ker T$ and $\ker T^*$ respectively. Define an operator

$$\psi : \mathscr{H} \to \mathscr{H}$$

as follows: for $x = x_0 + \sum_{i=1}^k \lambda_i v_i \in H$ where $x_0 \in (\ker T)^\perp$, set

$$\psi(x) = \sum_{i=1}^k \lambda_i w_i$$

Then $\ker\psi = \ker T^\perp$ and $R(\psi) = \ker T^*$. Note that $T + \psi$ is invertible, since if $x \in \ker(T + \psi)$, then $T(x) = -\psi(x)$, and hence $x = 0$, and $R(T + \psi) = \mathscr{H}$. In fact, the same reasoning shows that $T + t\psi$ is invertible for all $t > 0$. Hence,

$$\gamma : [0,1] \to \mathscr{L}(\mathscr{H}),$$
$$\gamma(t) = T + t\psi.$$

is a path contained in $\mathfrak{F}(\mathscr{H})$ that connects T with an invertible operator. The invertible operator, $T + \psi$, can the be connected to I. Thus, F_0 is path-connected. For $n > 0$, consider R^n, where R is the forward shift operator. Recall that $\mathrm{Ind}(R^n) = -n$. Let $T \in F_n$. By Proposition 11.9, TR^n is Fredholm and

$$\mathrm{Ind}(TR^n) = \mathrm{Ind}(T) + \mathrm{Ind}(R^n) = 0,$$

so $TR^n \in F_0$. Since $R^n L^n = I$, we have

$$T = TR^n L^n \in F_0 L^n$$

Hence, $F_n \subseteq F_0 L^n$. But Proposition 11.9 implies that $F_0 L^n \subseteq F_n$. Therefore, $F_n = F_0 L^n$, and it follows that F_n is path-connected. For $n < 0$, it follows from taking adjoints that $F_n = F_{-n}^*$. So F_n is path-connected for all $n \in \mathbb{Z}$. □

12. Atiyah-Janich Theorem (Incomplete)

We can interpret Proposition 11.11 from a topological point of view. Firstly, we note that $K^0_{\mathbb{K}}(*) \cong \mathbb{Z}$. Next, if X is a topological space, let $[X, \mathfrak{F}(\mathscr{H})]$ denote the homotopy classes of maps $X \to F$. Then

$$[*, \mathfrak{F}(\mathscr{H})] \simeq \{\text{path-connected components of } \mathfrak{F}(\mathscr{H})\}$$

Hence, we can write Proposition 11.11 as

$$[*, \mathfrak{F}(\mathscr{H})] \cong K^0_{\mathbb{K}}(*)$$

The Atiyah-Janich Theorem is a far-reaching generalization of this fact:

Proposition 12.1. *Let $\mathscr{H}$ be an infinite dimensional separable Hilbert space, and suppose that X is a compact Hausdorff topological space. Then there is a group isomorphism*

$$[X, \mathfrak{F}(\mathscr{H})] \cong K^0_{\mathbb{K}}(X)$$

Remark 12.2. *What is the intuition behind Proposition 12.1? For each $T \in [X, \mathfrak{F}(\mathscr{H})]$, $T_x := T(x)$ is a linear map $\mathscr{H} \to \mathscr{H}$. A natural idea is to form*

$$\coprod_{x\in X} \ker T_x \quad \textit{and} \quad \coprod_{x\in X} \operatorname{coker} T_x$$

One expects that these are vector bundles over X. Then

$$\left[\coprod_{x\in X} \ker T_x\right] - \left[\coprod_{x\in X} \operatorname{coker} T_x\right] \in K^0_{\mathbb{K}}(X).$$

However, this construction will not in general give us a vector bundle. For instance, let $X = [-1, 1]$ and $\mathscr{H} = \mathbb{C}$. Suppose for each $x \in X$ we have a linear operator

$$T_x : \mathscr{H} \to \mathscr{H},$$
$$h \mapsto xh.$$

Then

$$\dim\ker T_x = \begin{cases} 1, & \text{if } x = 0, \\ 0, & \text{if } x \neq 0. \end{cases} \qquad \dim\operatorname{coker} T_x = \begin{cases} 1, & \text{if } x = 0, \\ 0, & \text{if } x \neq 0. \end{cases}$$

But since $[-1, 1]$ is connected, any vector bundle over $[-1, 1]$ must have constant rank. Hence, we must be careful to turn this intuition into a genuine proof.

Proof. (Proposition 12.1) □

13. Clifford Algebras

In this section, we discuss the fundamental properties of Clifford algebras. Clifford algebras offer a framework for understanding the Bott periodicity phenomenon observed in K-theory.

13.1. **Motivation.** It is a well-known fact that $SU(2) \cong \mathbb{S}^3 \cong \mathbb{H}_u$, where $\mathbb{H}$ is the the non-commutative algebra of quaraternions and $\mathbb{H}_u$ is the subalgebra of unit quarternions. Identifying $\mathbb{R}^3$ with *imaginary* quarternions[9], we define the action of $SU(2)$ over $\mathbb{R}^3$ by

$$Z \cdot X = ZXZ^{-1},$$

where $Z \in SU(2)$ and X is any *imaginary* quaternion. It turns out that the map ρ_Z (where $\rho_Z(X) = ZXZ^{-1}$) is indeed a rotation, and that the map $\rho : Z \mapsto \rho_Z$ is a surjective homomorphism $\rho : SU(2) \to SO(3)$ whose kernel is $\{\pm \mathrm{Id}\}$.

What about rotations in $\mathbb{R}^n$? The relevant group here is $SO(n)$. For $n = 3$, we see that the action of $SO(3)$ on $\mathbb{R}^3$ can be realized by the action of $SU(2)$ on $\mathbb{R}^3$. Moreover, it is also the case that the action of $SU(2) \cong \mathbb{H}_u$ can be described in terms of multiplication in the $\mathbb{H}$ which obviously contains $\mathbb{H}_u$. The group that plays the role of $SU(2)$ are the spin groups (which are double covers of $SO(n)$). Spin groups can be realized as subgroups of an algebra called the Clifford algebra such that rotation in $\mathbb{R}^n$ can be described in terms of multiplication in Clifford algebras.

13.2. **Definition.** We define Clifford algebras, and then and move to some particular Clifford algebras we are most interested in. We start by reviewing the definition of a tensor algebra generated by a vector space, V.

Definition 13.1. Let V be a $\mathbb{K}$-vector space. The **tensor algebra** generated by V

$$T(V) = \bigoplus_{n\geq 0} V^{\otimes^n},$$

where $V^{\otimes^n}$ is the n-fold tensor product of V with itself.

[9] Of the form $a\mathbf{i} + b\mathbf{j} + c\mathbf{j}$ for $a, b, c \in \mathbb{R}$

Remark 13.2. *For every $n \geq 0$, there is a natural injection $i_n : V^{\otimes^n} \to T(V)$. The multiplicative unit 1 of $T(V)$ is the image $i_0(1)$ in $T(V)$ of the unit*

Remark 13.3. *It can be easily checked that the tensor algebra satisfies the following universal property. Given any $\mathbb{K}$-algebra A, for any $\mathbb{K}$-linear map $f : V \to A$, there is a unique K-algebra homomorphism $g : T(V) \to A$ so that*

$$f = g \circ i_1,$$

such that the following diagram commutes:

$$\begin{tikzcd} V \arrow[r, "i_1"] \arrow[dr, "f"'] & T(V) \arrow[d, "g"] \\ & A \end{tikzcd}$$

Indeed, any such map extends uniquely to an algebra morphism $g : T(V) \to A$ defined by $g(\lambda) = \lambda \cdot 1_A$ for $\lambda \in \mathbb{K}$, $g(x) = f(x)$ for $x \in V$, and more generally

$$g(x_1 \otimes \ldots \otimes x_p) = f(x_1) \cdots f(x_p).$$

Most algebras of interest arise as well-chosen quotients of the tensor algebra $T(V)$. This is true for the exterior algebra $\bigwedge V$, where we take the quotient of $T(V)$ modulo the ideal generated by all elements of the form $v \otimes v$, where $v \in V$. A Clifford algebra may be viewed as a refinement of the exterior algebra.

Definition 13.4. Let V be a $\mathbb{K}$-vector space, and let Φ be a quadratic form associated with a symmetric bilinear form $\varphi : V \times V \to \mathbb{K}$. The **Clifford algebra** $C(V, \Phi)$ is defined by the quotient

$$C(V, \Phi) = \frac{T(V)}{\langle i_1(v) \otimes i_1(v) - \Phi(v) i_0(1) \rangle}.$$

where we quotient by the ideal generated by elements of the form $i_1(v) \otimes i_1(v) - \Phi(v) i_0(1)$ for $v \in V$.

Remark 13.5. *It is easy to check that the $C(V, \Phi)$ satisfies the following universal property: for every $\mathbb{K}$-algebra A and every $\mathbb{K}$-linear map $f : V \to A$ with $(f(v))^2 = \Phi(v) \cdot 1_A$ for all $v \in V$, there is a unique algebra homomorphism $g : \mathrm{C}(V, \Phi) \to A$ such that the following diagram commutes:*

$$\begin{tikzcd} V \arrow[r, "i_\Phi"] \arrow[dr, "f"'] & \mathrm{C}(V, \Phi) \arrow[d, "g"] \\ & A \end{tikzcd}$$

Remark 13.6. *There is a $\mathbb{K}$-linear map $i_\Phi : V \to C(V, \Phi)$ defined as the composition:*

$$V \xrightarrow{i_0} T(V) \twoheadrightarrow C(V, \Phi)$$

where the second map is the canonical projection map. Multiplication of $u, v \in V$ in $C(V, \Phi)$ is written as $i_\Phi(u) \cdot i_\Phi(v)$. Hence, $i_\Phi(v)^2 = \Phi(v) i_\Phi(1) := \Phi(v)$.

Remark 13.7. *Observe that when $\Phi \equiv 0$, then the Clifford algebra $\mathrm{C}(V, 0)$ is just the exterior algebra $\bigwedge V$.*

Note that we have

$$i_\Phi(u+v)^2 = i_\Phi(u)^2 + i_\Phi(v)^2 + i_\Phi(u) i_\Phi(v) + i_\Phi(v) i_\Phi(u)$$

Using the fact that $i_\Phi(u) = \Phi(u)$ and $\Phi(u+v) - \Phi(u) - \Phi(v) = 2\varphi(u, v)$, we get

$$i_\Phi(u) i_\Phi(v) + i_\Phi(v) i_\Phi(u) = 2\varphi(u, v) i_\Phi(1)^{10}.$$

[10] We assume that characteristic of $\mathbb{K}$ is $\neq 2$.

As a consequence, if V is of finite dimension and $\{e_1, \cdots, e_n\}$ is an orthogonal basis with respect to φ, we have, in particular,

$$i_\Phi(e_j)i_\Phi(e_k) + i_\Phi(e_k)i_\Phi(e_j) = 0 \text{ for all } j \neq k. \tag{1}$$

13.3. $\mathbb{Z}_2$ **Graded Structure on Clifford Algebras.** We show that every Clifford algebra admits a $\mathbb{Z}_2$ graded structure. We begin this subsection by reviewing preliminary details about $\mathbb{Z}_2$ graded algebras.

Definition 13.8. A $\mathbb{Z}_2$**-graded algebra** is a $\mathbb{K}$-algebra A together with a decomposition into a direct sum of the form

$$A = A_0 \oplus A_1$$

such that multiplication operation in the algebra, A, satisfies:

$$A_{[i]} \cdot A_{[i]} \subseteq A_{[i+j]}$$

Here $[i], [j]$ denote the $\mathbb{Z}_2$ equivalence class of the integer. i, j The elements of V_i are called **homogeneous elements** of parity i. The parity of element, x, is denoted as $|x|$. A $\mathbb{Z}_2$-graded algebra is a $\mathbb{Z}_2$ **graded commutative algebra** if

$$xy = (-1)^{|x||y|}yx$$

for all homogeneous elements $x, y \in A$.

We now claim that Clifford algebras admit a $\mathbb{Z}_2$ grading. Let V be a finite dimensional $\mathbb{K}$-vector space endowed with a symmetric bilinear form, Φ. If $\{e_1, \ldots, e_n\}$ is a basis of V, then the Clifford algebra $\mathrm{C}(V, \Phi)$ consists of certain kinds of "polynomials," linear combinations of monomials of the form

$$\sum_{J=\{j_1, j_2, \ldots, j_k\}} \lambda_J \underbrace{i_\Phi(e_{j_1}) \cdots i_\Phi(e_{j_k})}_{i_\Phi(e_J)},$$

where J is any subset (possibly empty) of $\{1, \ldots, n\}$. Equation (1) implies that we can always reoder the basis in e_J such that $1 \leq i_1 < i_2 \ldots < i_k \leq n$. In other words, we have a $\mathbb{K}$-linear map defines on the 'monomials' as

$$\alpha(i_\Phi(e_{\sigma(j_1)}) \cdots i_\Phi(e_{\sigma(j_k)})) = (-1)^{\mathrm{sgn}(\sigma)} i_\Phi(e_{j_1}) \cdots i_\Phi(e_{j_k}),$$

where $\sigma\{j_1, \cdots, j_k\} = \{i_1, \cdots, i_k\}$ such that $1 \leq i_1 < i_2 \ldots < i_k \leq n$. The map α allows us to introduced a $\mathbb{Z}_2$ graded structure on $\mathrm{C}(V, \Phi)$.

Proposition 13.9. *Let V be a finite dimensional $\mathbb{K}$-vector space endowed with a quadratic form, Φ. The Clifford algebra, $\mathrm{C}(V, \Phi)$, is a $\mathbb{Z}_2$ graded algebra such that:*

$$\mathrm{C}(V, \Phi) = \mathrm{C}_0(V, \Phi) \oplus \mathrm{C}_1(V, \Phi),$$

where

$$\mathrm{C}_i(V, \Phi) = \{x \in \mathrm{C}(V, \Phi) \mid \alpha(x) = (-1)^i x\},$$

Proof. Since every element of $\mathrm{C}(V, \Phi)$ is a linear combination of the form $\sum_J \lambda_J i_\Phi(e_J)$, in view of the description of α given above, we see that the elements of $\mathrm{C}_0(V, \Phi)$ are those for which the monomials $i_\Phi(e_J)$ are products of an even number of factors, and the elements of $\mathrm{C}_1(\Phi)$ are those for which the monomials $i_\Phi(e_J)$ are products of an odd number of factors. This yields the desired $\mathbb{Z}_2$ graded decomposition. □

We now discuss $\mathbb{Z}_2$ graded tensor products of algebras and argue that tensor products of Clifford algebras over direct sums of vector spaces is $\mathbb{Z}_2$ graded tensor products of algebras.

Definition 13.10. Let $A = A_0 \oplus A_1$ and $B = B_0 \oplus B_1$ be two $\mathbb{Z}_2$ graded algebras. The $\mathbb{Z}_2$**-graded tensor product**, $A\widehat{\otimes}B$ is a $\mathbb{Z}_2$ graded algebra such that

$$(A\widehat{\otimes}B)_0 = (A_0 \otimes B_0) \oplus (A_1 \otimes B_1)$$
$$(A\widehat{\otimes}B)_1 = (A_0 \otimes B_1) \oplus (A_1 \otimes B_0)$$

such that

$$(a_1 \otimes b_1)(a_2 \otimes b_2) = (-1)^{|a_2||b_1|} a_1 a_2 \otimes b_1 b_2,$$

where a_i and b_i are homogeneous elements of parity $|a_i|$, $|b_i|$, respectively.

Proposition 13.11. *Let (V, Φ_V) and (W, Φ_W) be vector spaces endowed with quadratic forms. Let $(V \oplus W, \Phi_V \oplus \Phi_W)$ be their orthogonal direct sum. Then there is an isomorphism of $\mathbb{Z}_2$-graded associative algebras:*

$$\mathrm{C}(V \oplus W, \Phi_V \oplus \Phi_W) \cong \mathrm{C}(V, \Phi_V)\widehat{\otimes}\mathrm{C}(W, \Phi_W),$$

where $\widehat{\otimes}$ denotes the $\mathbb{Z}_2$-graded tensor product.

Proof. Define a $\mathbb{K}$-linear map

$$\phi : V \oplus W \longrightarrow \mathrm{C}(V, \Phi_V)\widehat{\otimes}\mathrm{C}(W, \Phi_W)$$
$$v + w \longmapsto v \otimes 1 + 1 \otimes w$$

Identify V with its image in $\mathrm{C}(V, \Phi_V)$. Similarly for W. Note that:

$$\begin{aligned}
\phi(v + w)^2 &= (v \otimes 1 + 1 \otimes w)^2 \\
&= (v \otimes 1)(v \otimes 1) + (v \otimes 1)(1 \otimes w) + (1 \otimes w)(v \otimes 1) + (1 \otimes w)(1 \otimes w) \\
&= v^2 \otimes 1 + v \otimes w + (-1)^{|v||w|} v \otimes w + 1 \otimes w^2 \\
&= (\Phi_V(v) + \Phi_W(w))1 \otimes 1.
\end{aligned}$$

The last equality follows since $|v||w| = 1$ as elements of V_1 and V_2 always have degree one. By universality, ϕ extends to a homomorphism of $\mathbb{Z}_2$-graded associative algebras

$$\Phi : \mathrm{C}(V \oplus W, \Phi_V \oplus \Phi_W) \rightarrow \mathrm{C}(V, \Phi_V)\widehat{\otimes}\mathrm{C}(W, \Phi_W),$$

which is injective since it is one-to-one on generators and is surjective because the image contains $\mathrm{C}(V, \Phi_V)\widehat{\otimes}1$ and $1\widehat{\otimes}\mathrm{C}(W, \Phi_W)$ which generate $\mathrm{C}(V, \Phi_V)\widehat{\otimes}\mathrm{C}(W, \Phi_W)$. □

As a corollary, we obtain the following result:

Proposition 13.12. *Let (V, Φ) be a finite-dimensional $\mathbb{K}$-vector space endowed with a quadratic form Φ. The map $i_\Phi : V_1 \longrightarrow \mathrm{C}(V, \Phi)$ is injective. Given an* orthogonal *basis $(e_1, \ldots, e_n)$ of V, the $2^n - 1$ products*

$$\{i_\Phi(e_{i_1}) \cdots i_\Phi(e_{i_k}) \mid 1 \leq i_1 < i_2 \ldots < i_k \leq n\}$$

and 1 form a basis of $\mathrm{C}(V, \Phi)$. Thus, $\mathrm{C}(V, \Phi)$ has dimension 2^n.

Proof. The proof is by induction on $n = \dim(V)$. For $n = 1$, the tensor algebra $T(V)$ is just the polynomial ring $\mathbb{K}[X]$, where $i_\Phi(e_1) = X$. Thus, $\mathrm{C}(V, \Phi) = \mathbb{K}[X]/(X^2 \pm 1)$, and the result is obvious. For $n > 1$, Equation (1) implies that

$$\{i_\Phi(e_{i_1}) \cdots i_\Phi(e_{i_k}) \mid 1 \leq i_1 < i_2 \ldots < i_k \leq n\}$$

and 1 generate $C(V, \Phi)$. Thus, we have a splitting

$$(V, \Phi) = \bigoplus_{k=1}^{n} (V_k, \Phi_k),$$

where V_k has dimension 1. Choosing a basis so that $e_k \in V_k$, the theorem follows by induction from Proposition 13.11. □

Remark 13.13. *Since i_Φ is injective, for simplicity of notation we often write v for $i_\Phi(v)$. Hence, the expression $i_\Phi(v)i_\Phi(v) = \Phi(v)i_0(1)$ is simply written as $v^2 = \Phi(v)$. In particular, we write $e_je_k + e_ke_j = 0$ for all $j \neq k$.*

13.4. **Classification of Real Clifford Algebras.** We classify all Clifford algebra $C_p^q(\mathbb{R}^n)$ where $n = p + q$ for $p, q \geq 0$, which we now define:

Definition 13.14. Let $\mathbb{K} = \mathbb{R}$ and $n = p + q$. Consider the quadratic form $Q_p^q : \mathbb{R}^n \to \mathbb{R}$,

$$(x_1, \ldots, x_n) \mapsto -\sum_{i=1}^{p} x_i^2 + \sum_{i=p+1}^{q} x_i^2.$$

The **real Clifford algebra** is the Clifford algebra $C(\mathbb{R}^n, Q_p^q)$ and it is denoted as C_p^q.

Consider the following special cases:

(1) Let $q = 0$. Then the quadratic forms reduces to:

$$(x_1, \ldots, x_n) \mapsto -\sum_{i=1}^{n} x_i^2.$$

We denote the Clifford algebra C_n^0 as C_n.

(2) Let $p = 0$. Then the quadratic forms reduces to:

$$(x_1, \ldots, x_n) \mapsto \sum_{i=1}^{n} x_i^2.$$

We denote the Clifford algebra C_0^n as C^n.

Let's start by pinning down the structure of the Clifford algebras C_p^q for small values of p, q.

Example 13.15. The following list of examples classify C_p^q for small values of p, q.

(1) Clearly, $C_0 \cong C^0 \cong C_0^0 \cong \mathbb{R}$.

(2) Let $V = \mathbb{R}^1$, $e_1 = 1$, and assume that $Q_1(x_1e_1) = -x_1^2$. We have the relation $e_1^2 = -1$. Since $T(V) \cong \mathbb{K}[X]$, we see that

$$C_1 \cong \mathbb{R}[X]/(X^2 + 1) \cong \mathbb{C}.$$

Alternatively, note that C_1 is spanned by the basis $\{1, e_1\}$. Under the bijection $e_1 \mapsto i$, C_1 is isomorphic to the algebra of complex numbers $\mathbb{C}$.

(3) Let $V = \mathbb{R}^1$, $e_1 = 1$, and assume that $Q_1(x_1e_1) = x_1^2$. We have the relation $e_1^2 = 1$. Since $T(V) \cong K[X]$, we see that

$$\begin{aligned} C^1 &\cong \mathbb{R}[x]/(X^2 - 1) \\ &\cong \mathbb{R}[x]/(X - 1) \oplus \mathbb{R}[x]/(X + 1) \\ &\cong \mathbb{R} \oplus \mathbb{R} \end{aligned}$$

(4) Let $V = \mathbb{R}^2$, $\{e_1, e_2\}$ be the canonical basis, and assume that $Q_2(x_1e_1 + x_2e_2) = -(x_1^2 + x_2^2)$. Then, C_2 is spanned by the basis $\{1, e_1, e_2, e_1e_2\}$. Furthermore, we have

$$e_1^2 = -1, \quad e_2^2 = -1, \quad (e_1e_2)^2 = -1, \quad e_2e_1 = -e_1e_2.$$

Under the bijection

$$e_1 \mapsto \mathbf{i} \quad e_2 \mapsto \mathbf{j} \quad e_1e_2 \mapsto \mathbf{k},$$

it is easily checked that the quaternion identities Thus C_2, is isomorphic to the algebra of quaternions. That is,

$$C_2 \cong \mathbb{H}.$$

(5) Let $V = \mathbb{R}^2$, $\{e_1, e_2\}$ be the canonical basis, and assume that $Q_2(x_1e_1+x_2e_2) = (x_1^2+x_2^2)$. Then, C^2 is spanned by the basis $\{1, e_1, e_2, e_1e_2\}$. Furthermore, we have

$$e_1^2 = 1, \quad e_2^2 = 1, \quad (e_1e_2)^2 = 1, \quad e_2e_1 = -e_1e_2.$$

We get a homomorphism

$$C^2 \to M_2(\mathbb{R}),$$

given by

$$e_1 \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad e_2 \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

The homomorphism is onto because the two given matrices generate $M_2(\mathbb{R})$ as an algebra. The dimension of both $M_2(\mathbb{R})$ and C_0^2 is 4. So we have that

$$C^2 \cong M_2(\mathbb{R}).$$

(6) Let $V = \mathbb{R}^2$, $\{e_1, e_2\}$ be the canonical basis, and assume that $Q_2(x_1e_1+x_2e_2) = -x_1^2+x_2^2$. Then, C_1^1 is spanned by the basis $\{1, e_1, e_2, e_1e_2\}$. Furthermore, we have

$$e_1^2 = -1 \qquad e_2^2 = 1 \qquad e_1e_2 = -e_2e_1$$

Again, we get an isomorphism with $M_2(\mathbb{R})$, given by

$$e_1 \mapsto \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad e_2 \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Thus we have the following classification of Clifford algebras for now:

q/p	0	1	2
0	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$
1	$\mathbb{R} \oplus \mathbb{R}$	$M_2(\mathbb{R})$	
2	$M_2(\mathbb{R})$		

While graded tensor products are the "morally correct" things to consider to classify Clifford algebras, they are not computationally helproof ul. Instead, we want to rephrase everything in terms of ordinary tensor products. We have the following result:

Proposition 13.16. *For all $n, s, t \geq 0$, we have the following isomorphisms:*

$$C_0^n \otimes_{\mathbb{R}} C_2^0 \cong C_{n+2}^0$$
$$C_n^0 \otimes_{\mathbb{R}} C_0^2 \cong C_0^{n+2}$$
$$C_s^t \otimes_{\mathbb{R}} C_1^1 \cong C_{s+1}^{t+1}$$

with $\otimes$ denoting the ungraded tensor product.

Remark 13.17. *All tensors products above actually represent the tensor product over $\mathbb{R}$, $\otimes_{\mathbb{R}}$. In what follows, we contiue to suppress write $\otimes_{\mathbb{R}}$ as $\otimes$.*

Proof. Letting $\{\varepsilon_1, \ldots, \varepsilon_{n+2}\}$ denote the standard basis for $\mathbb{R}^{n+2,0}$, $\{e_1', \ldots, e_n'\}$ the usual basis for $\mathbb{R}^{0,n}$, and $\{\varepsilon_1'', \varepsilon_2''\}$ the usual basis for $\mathbb{R}^{2,0}$. We define $f : \mathbb{R}^{n+2,0} \to C_0^n \otimes C_2^0$ by:

$$f(\varepsilon_i) = \begin{cases} e_i' \otimes \varepsilon_1''\varepsilon_2'', & \text{if } 1 \leq i \leq n, \\ 1 \otimes \varepsilon_{i-n}, & \text{if } n+1 \leq i \leq n+2. \end{cases}$$

It can be easily checked that:

$$f(\varepsilon_i)\cdot f(\varepsilon_j)+f(\varepsilon_i)\cdot f(\varepsilon_j) = -2\delta_{i,j}1\otimes 1 \qquad 1\le i,j\le n$$
$$f(\varepsilon_i)\cdot f(\varepsilon_j)+f(\varepsilon_i)\cdot f(\varepsilon_j) = -2\delta_{i,j}1\otimes 1 \qquad n+1\le i,j\le n+2$$
$$f(\varepsilon_i)\cdot f(\varepsilon_j)+f(\varepsilon_i)\cdot f(\varepsilon_j) = 0 \qquad 1\le i\le n \quad n+1\le j\le n+2$$

But then for $x = x_1\varepsilon_1+\ldots+x_{n+2}\varepsilon_{n+2}$, we have, by linearity of f, that

$$f(x)^2 = -(x_1^2+\ldots+x_{n+2}^2)1\otimes 1 = Q^0_{n+2}(x)\cdot 1\otimes 1$$

Therefore, f factorizes uniquely through $\widehat{f}: C^0_{n+2}\longrightarrow C^n_0\otimes C^0_2$. Clearly, $\widehat{f}$ is surjective. This shows that:

$$C^0_{n+2}\cong C^n_0\otimes C^0_2$$

The second isomorphism is proved in exactly the same way. The proof of the third isomorphism is essentially the same as the first two. □

In order to use the above proposition to identity more the isomorphism classes, we need the following proposition:

Proposition 13.18. *The following are isomorphisms of real associative algebras:*

(1) $M_r(\mathbb{R})\otimes M_s(\mathbb{R})\cong M_{rs}(\mathbb{R})$
(2) $\mathbb{C}\otimes\mathbb{H}\cong M_2(\mathbb{C})$
(3) $\mathbb{H}\otimes\mathbb{H}\cong M_4(\mathbb{C})$
(4) $\mathbb{C}\otimes\mathbb{C}\cong\mathbb{C}\oplus\mathbb{C}$

Proof. Skipped. □

The last isomorphism in Proposition 13.16 allows us to move diagonally. Moving diagonally (one step) is the same as tensoring with $C^1_1\cong M_2(\mathbb{R})$. We get:

q/p	0	1	2	3	4	5	6	7
0	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$					
1	$\mathbb{R}\oplus\mathbb{R}$	$M_2(\mathbb{R})$	$M_2(\mathbb{C})$	$M_2(\mathbb{H})$				
2	$M_2(\mathbb{R})$	$M_2(\mathbb{R})\oplus M_2(\mathbb{R})$	$M_4(\mathbb{R})$	$M_4(\mathbb{C})$	$M_4(\mathbb{H})$			
3		$M_4(\mathbb{R})$	$M_4(\mathbb{R})\oplus M_4(\mathbb{R})$	$M_8(\mathbb{R})$	$M_8(\mathbb{C})$	$M_8(\mathbb{H})$		
4			$M_8(\mathbb{R})$	$M_8(\mathbb{R})\oplus M_8(\mathbb{R})$	$M_{16}(\mathbb{R})$	$M_{16}(\mathbb{C})$	$M_{16}(\mathbb{H})$	
5				$M_{16}(\mathbb{R})$	$M_{16}(\mathbb{R})\oplus M_{16}(\mathbb{R})$	$M_{32}(\mathbb{R})$	$M_{32}(\mathbb{C})$	$M_{32}(\mathbb{H})$
6					$M_{32}(\mathbb{R})$	$M_{32}(\mathbb{R})\oplus M_{32}(\mathbb{R})$	$M_{64}(\mathbb{R})$	$M_{64}(\mathbb{C})$
7						$M_{64}(\mathbb{R})$	$M_{64}(\mathbb{R})\oplus M_{64}(\mathbb{R})$	$M_{128}(\mathbb{R})$

To continue it is necessary to extend the top row and the left column. For example, let us continue with the top row. From the first isomorphism in Proposition 13.16, we get:

$$C^0_3\cong C^1_0\otimes C^0_2\cong(\mathbb{R}\oplus\mathbb{R})\otimes\mathbb{H}\cong\mathbb{H}\oplus\mathbb{H}$$
$$C^0_4\cong C^2_0\otimes C^0_2\cong M_2(\mathbb{R})\otimes\mathbb{H}\cong M_2(\mathbb{H})$$

Moving diagonally, we get:

We cannot continue along the top row without first extending the left column. Using the second isomorphism Proposition 13.16, we get:

$$C^3_0\cong C^0_1\otimes C^2_0\cong\mathbb{C}\otimes M_2(\mathbb{R})\cong M_2(\mathbb{C})$$
$$C^4_0\cong C^0_2\otimes C^2_0\cong\mathbb{H}\otimes M_2(\mathbb{R})\cong M_2(\mathbb{H})$$
$$C^5_0\cong C^0_3\otimes C^2_0\cong(\mathbb{H}\oplus\mathbb{H})\otimes M_2(\mathbb{R})\cong M_2(\mathbb{H})\oplus M_2(\mathbb{H})$$
$$C^6_0\cong C^0_4\otimes C^2_0\cong M_2(\mathbb{H})\otimes M_2(\mathbb{R})\cong M_4(\mathbb{H})$$

q/p	0	1	2	3	4	5	6	7
0	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H}\oplus\mathbb{H}$	$M_2(\mathbb{H})$			
1	$\mathbb{R}\oplus\mathbb{R}$	$M_2(\mathbb{R})$	$M_2(\mathbb{C})$	$M_2(\mathbb{H})$	$M_2(\mathbb{H})\oplus M_2(\mathbb{H})$	$M_4(\mathbb{H})$		
2	$M_2(\mathbb{R})$	$M_2(\mathbb{R})\oplus M_2(\mathbb{R})$	$M_2(\mathbb{R})$	$M_4(\mathbb{C})$	$M_4(\mathbb{H})$	$M_4(\mathbb{H})\oplus M_4(\mathbb{H})$	$M_8(\mathbb{H})$	
3		$M_4(\mathbb{R})$	$M_4(\mathbb{R})\oplus M_4(\mathbb{R})$	$M_8(\mathbb{R})$	$M_8(\mathbb{C})$	$M_8(\mathbb{H})$	$M_8(\mathbb{H})\oplus M_8(\mathbb{H})$	$M_{16}(\mathbb{H})$
4			$M_8(\mathbb{R})$	$M_8(\mathbb{R})\oplus M_8(\mathbb{R})$	$M_{16}(\mathbb{R})$	$M_{16}(\mathbb{C})$	$M_{16}(\mathbb{H})$	$M_{16}(\mathbb{H})\oplus M_{16}(\mathbb{H})$
5				$M_{16}(\mathbb{R})$	$M_{16}(\mathbb{R})\oplus M_{16}(\mathbb{R})$	$M_{32}(\mathbb{R})$	$M_{32}(\mathbb{C})$	$M_{32}(\mathbb{H})$
6					$M_{32}(\mathbb{R})$	$M_{32}(\mathbb{R})\oplus M_{32}(\mathbb{R})$	$M_{64}(\mathbb{R})$	$M_{64}(\mathbb{C})$
7						$M_{64}(\mathbb{R})$	$M_{64}(\mathbb{R})\oplus M_{64}(\mathbb{R})$	$M_{128}(\mathbb{R})$

q/p	0	1	2	3	4	5	6	7
0	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H}\oplus\mathbb{H}$	$M_2(\mathbb{H})$			
1	$\mathbb{R}\oplus\mathbb{R}$	$M_2(\mathbb{R})$	$M_2(\mathbb{C})$	$M_2(\mathbb{H})$	$M_2(\mathbb{H})\oplus M_2(\mathbb{H})$	$M_4(\mathbb{H})$		
2	$M_2(\mathbb{R})$	$M_2(\mathbb{R})\oplus M_2(\mathbb{R})$	$M_2(\mathbb{R})$	$M_4(\mathbb{C})$	$M_4(\mathbb{H})$	$M_4(\mathbb{H})\oplus M_4(\mathbb{H})$	$M_8(\mathbb{H})$	
3	$M_2(\mathbb{C})$	$M_4(\mathbb{R})$	$M_4(\mathbb{R})\oplus M_4(\mathbb{R})$	$M_8(\mathbb{R})$	$M_8(\mathbb{C})$	$M_8(\mathbb{H})$	$M_8(\mathbb{H})\oplus M_8(\mathbb{H})$	$M_{16}(\mathbb{H})$
4	$M_2(\mathbb{H})$	$M_4(\mathbb{C})$	$M_8(\mathbb{R})$	$M_8(\mathbb{R})\oplus M_8(\mathbb{R})$	$M_{16}(\mathbb{R})$	$M_{16}(\mathbb{C})$	$M_{16}(\mathbb{H})$	$M_{16}(\mathbb{H})\oplus M_{16}(\mathbb{H})$
5	$M_2(\mathbb{H})\oplus M_2(\mathbb{H})$	$M_4(\mathbb{H})$	$M_8(\mathbb{C})$	$M_{16}(\mathbb{R})$	$M_{16}(\mathbb{R})\oplus M_{16}(\mathbb{R})$	$M_{32}(\mathbb{R})$	$M_{32}(\mathbb{C})$	$M_{32}(\mathbb{H})$
6	$M_4(\mathbb{H})$	$M_4(\mathbb{H})\oplus M_4(\mathbb{H})$	$M_8(\mathbb{H})$	$M_{16}(\mathbb{C})$	$M_{32}(\mathbb{R})$	$M_{32}(\mathbb{R})\oplus M_{32}(\mathbb{R})$	$M_{64}(\mathbb{R})$	$M_{64}(\mathbb{C})$
7		$M_8(\mathbb{H})$	$M_8(\mathbb{H})\oplus M_8(\mathbb{H})$	$M_{16}(\mathbb{H})$	$M_{32}(\mathbb{C})$	$M_{64}(\mathbb{R})$	$M_{64}(\mathbb{R})\oplus M_{64}(\mathbb{R})$	$M_{128}(\mathbb{R})$

Moving diagonally, we get:
We can now complete the top row:

$$C_5^0 \cong C_0^3 \otimes C_2^0 \cong M_2(\mathbb{C}) \otimes \mathbb{H} \cong M_2(\mathbb{R}) \otimes (\mathbb{C} \otimes \mathbb{H}) \cong M_2(\mathbb{R}) \otimes M_2(\mathbb{C}) \cong M_4(\mathbb{C})$$
$$C_6^0 \cong C_0^4 \otimes C_2^0 \cong M_2(\mathbb{H}) \otimes \mathbb{H} \cong M_2(\mathbb{R}) \otimes (\mathbb{H} \otimes \mathbb{H}) \cong M_2(\mathbb{R}) \otimes M_4(\mathbb{R}) \cong M_8(\mathbb{R})$$
$$C_7^0 \cong C_0^5 \otimes C_2^0 \cong (M_2(\mathbb{H}) \oplus M_2(\mathbb{H})) \otimes \mathbb{H} \cong (M_2(\mathbb{H}) \otimes \mathbb{H}) \oplus (M_2(\mathbb{H}) \otimes \mathbb{H}) \cong M_8(\mathbb{R}) \oplus M_8(\mathbb{R})$$

Similarly, we can also complete the first column by noting that:

$$C_0^7 \cong C_5^0 \otimes C_0^2 \cong M_4(\mathbb{C}) \otimes M_2(\mathbb{R}) \cong M_8(\mathbb{C})$$

Moving diagonally, we get:

q/p	0	1	2	3	4	5	6	7
0	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H}\oplus\mathbb{H}$	$M_2(\mathbb{H})$	$M_4(\mathbb{C})$	$M_8(\mathbb{R})$	$M_8(\mathbb{R})\oplus M_8(\mathbb{R})$
1	$\mathbb{R}\oplus\mathbb{R}$	$M_2(\mathbb{R})$	$M_2(\mathbb{C})$	$M_2(\mathbb{H})$	$M_2(\mathbb{H})\oplus M_2(\mathbb{H})$	$M_4(\mathbb{H})$	$M_8(\mathbb{C})$	$M_{16}(\mathbb{R})$
2	$M_2(\mathbb{R})$	$M_2(\mathbb{R})\oplus M_2(\mathbb{R})$	$M_2(\mathbb{R})$	$M_4(\mathbb{C})$	$M_4(\mathbb{H})$	$M_4(\mathbb{H})\oplus M_4(\mathbb{H})$	$M_8(\mathbb{H})$	$M_{16}(\mathbb{C})$
3	$M_2(\mathbb{C})$	$M_4(\mathbb{R})$	$M_4(\mathbb{R})\oplus M_4(\mathbb{R})$	$M_8(\mathbb{R})$	$M_8(\mathbb{C})$	$M_8(\mathbb{H})$	$M_8(\mathbb{H})\oplus M_8(\mathbb{H})$	$M_{16}(\mathbb{H})$
4	$M_2(\mathbb{H})$	$M_4(\mathbb{C})$	$M_8(\mathbb{R})$	$M_8(\mathbb{R})\oplus M_8(\mathbb{R})$	$M_{16}(\mathbb{R})$	$M_{16}(\mathbb{C})$	$M_{16}(\mathbb{H})$	$M_{16}(\mathbb{H})\oplus M_{16}(\mathbb{H})$
5	$M_2(\mathbb{H})\oplus M_2(\mathbb{H})$	$M_4(\mathbb{H})$	$M_8(\mathbb{C})$	$M_{16}(\mathbb{R})$	$M_{16}(\mathbb{R})\oplus M_{16}(\mathbb{R})$	$M_{32}(\mathbb{R})$	$M_{32}(\mathbb{C})$	$M_{32}(\mathbb{H})$
6	$M_4(\mathbb{H})$	$M_4(\mathbb{H})\oplus M_4(\mathbb{H})$	$M_8(\mathbb{H})$	$M_{16}(\mathbb{C})$	$M_{32}(\mathbb{R})$	$M_{32}(\mathbb{R})\oplus M_{32}(\mathbb{R})$	$M_{64}(\mathbb{R})$	$M_{64}(\mathbb{C})$
7	$M_8(\mathbb{C})$	$M_8(\mathbb{H})$	$M_8(\mathbb{H})\oplus M_8(\mathbb{H})$	$M_{16}(\mathbb{H})$	$M_{32}(\mathbb{C})$	$M_{64}(\mathbb{R})$	$M_{64}(\mathbb{R})\oplus M_{64}(\mathbb{R})$	$M_{128}(\mathbb{R})$

It turns out, it suffices to complete the above 8×8 table since real Clifford algebras are 8-periodic.

Proposition 13.19. *(8-**Periodicity of Real Clifford Algebras**) For all $n, s, t \geq 0$, the following are isomorphisms of real algebras:*

(1) $C_{n+8}^0 \cong C_n^0 \otimes M_{16}(\mathbb{R})$,
(2) $C_0^{n+8} \cong C_0^n \otimes M_{16}(\mathbb{R})$, *and*
(3) $C_{t+4}^{s+4} \cong C_t^s \otimes M_{16}(\mathbb{R})$.

Proof. This follows directly from repeated application of Proposition 13.16 and the following isomorphisms:

$$\otimes_{i=1}^{4} C_1^1 \cong M_2(\mathbb{R}) \otimes M_2(\mathbb{R}) \otimes M_2(\mathbb{R}) \otimes M_2(\mathbb{R}) \cong M_{16}(\mathbb{R}),$$
$$(\otimes_{i=1}^{2} C_2^0) \otimes (\otimes_{i=1}^{2} C_0^2) \cong M_4(\mathbb{R}) \otimes (\mathbb{H} \otimes \mathbb{H}) \cong M_4(\mathbb{R}) \otimes M_4(\mathbb{R}) \cong M_{16}(\mathbb{R}).$$

This completes the proof. □

13.5. **Classification of Complex Clifford Algebras.** We provide the definition and the classification of complex Clifford algebras.

Definition 13.20. Let $\mathbb{K} = \mathbb{C}$ and $n = p + q$. Consider the quadratic form $\Phi_n : \mathbb{C}^n \to \mathbb{C}$,

$$(z_1, \dots, z_n) \mapsto \sum_{i=1}^{p+1} z_i^2.$$

The **complex Clifford algebra** is the Clifford algebra $C(\mathbb{C}^n, \Phi_n)$ and it is denoted as E_n.

It easier to derive the 2-periodicity of complex Clifford algebras.

Proposition 13.21. *(2-**Periodicity of Complex Clifford Algebras**) For all $n \geq 0$, the following are isomorphisms of complex algebras:*

$$E_{n+2} \cong E_n \otimes_{\mathbb{C}} M_2(\mathbb{C})$$

Proof. (Sketch) Let $E_{n+2} = E_n \oplus \mathbb{C}e_1 \oplus \mathbb{C}e_2$, and define a complex linear map $\phi : E_{n+2} \to E_n \otimes M_2(\mathbb{C})$ by

$$\phi(x) = x \otimes \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix},$$
$$\phi(e_1) = 1 \otimes \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix},$$
$$\phi(e_2) = 1 \otimes \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix},$$

for all $\mathbf{x} \in E_n$. One checks that ϕ is a Clifford map, and that the induced map

$$\phi : E_{n+2} \to E_n \otimes M_2(\mathbb{C}),$$

being injective on generators and mapping between equidimensional spaces, is an isomorphism. □

14. Clifford Modules (Incomplete)

Part 3. References

References

[Hat03] Allen Hatcher. "Vector bundles and K-theory". In: (2003) (cit. on pp. 14, 15, 22).