HOMOLOGY

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Abstract. These are notes on algebraic topology covering singular homology. I took these notes during graduate school to better understand singular homology. The purpose of the notes is to discuss the theoretical foundations of singular homology, focusing on examples from low dimensions (≤ 2) and simple examples in arbitrary dimensions usually covered in a first year graduate course. There may be typos; please send corrections to junaid.aftab1994@gmail.com.

Contents

Part 1. Preliminaries

1. Topological Spaces

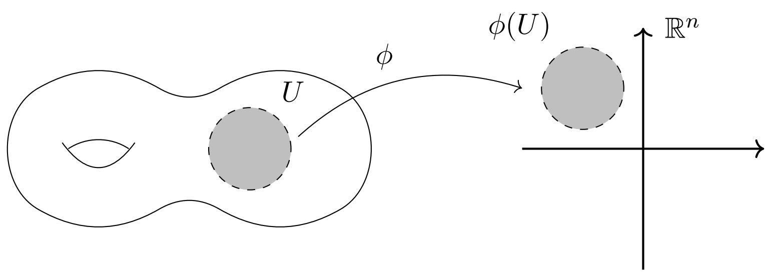
The main topological spaces of interest are topological manifolds and topological spaces that admit the structure of a CW complex. We discuss such topological spaces below.

1.1. **Topological Manifolds.** Topological manifolds are spaces that locally look like Euclidean space. Familiar examples are plane curves such as circles and parabolas etc. and surfaces such as spheres, tori etc.

Definition 1.1. Let X be a topological space. X is a topological n-manifold if X is a second-countable, Hausdorff space that is locally Euclidean. That is, each point of X is contained in a coordinate chart, which is a pair (U, ϕ), where U is an open subset of X and $\phi : U \to \phi(U) \subseteq \mathbb{R}^n$ is a homeomorphism from U to an open subset $\phi(U)$ of $\mathbb{R}^n$.

Remark 1.2. *The number n is attached to a single chart and might a priori depend on the chart itself. This turns out to be not the case. This result is called the invariance of dimension and will be proved later.*

Remark 1.3. *A collection of charts (U_α, ϕ_α) such that $\bigcup_\alpha U_\alpha = M$ is an atlas for X.*



We discuss the implication of the conditions imposed in Definition 1.1. Since a topological manifold is locally Euclidean, it is easy to see that it inherits a number of properties of Euclidean space locally. For instance, we have the following:

Proposition 1.4. *Let X be a topological n-manifold. Then X is locally compact, locally path-connected and locally contractible.*

Proof. Every point of X has a neighborhood homeomorphic to the open unit ball in $\mathbb{R}^n$. Each open ball in $\mathbb{R}^n$ is locally compact, locally compact and locally path-connected, locally contractible. The claim follows. □

The locally Euclidean condition does not impose any topological properties at the global level. The second-countablity and Hausdorff conditions account for this detail. Intuitively, Hausdorff spaces have 'enough open sets.' This ensures that familiar properties hold: for example, in a Hausdorff space, finite subsets are closed, limits of convergent sequences are unique etc. Moreover, this condition also excludes certain pathological examples like the line with two origins etc. On the other hand, second-countable spaces 'don't have too many open sets that are required to cover the space.' The following is a sample global topological property of a topological n-manifold.

Proposition 1.5. *Let X be a topological n-manifold. X has a countable basis of precompact coordinate balls.*

Proof. First consider the special case in which X can be covered by a single chart. Suppose $\varphi : M \to U \subseteq \mathbb{R}^n$ is a global coordinate chart. Let

$$\mathcal{B} = \{B_r(x) : x \in \mathbb{Q},\ x \in \mathbb{Q}^n\ B_{r'}(x) \subseteq U \text{ for some } r' < r\}$$

Each $B_r(x) \in \mathcal{B}$ is pre-compact in U, and it is easy to check that $\mathcal{B}$ is a countable basis for the topology of U. Because φ is a homeomorphism, it follows that $\varphi^{-1}(\mathcal{B})$ is a countable basis for X, consisting of pre-compact coordinate balls.

More generally, each $p \in M$ is in the domain of a coordinate chart. Since X is second-countable, X is covered by countably many coordinate charts $\{(U_i, \varphi_i)\}_{i=1}^{\infty}$. By the argument in the preceding paragraph, each U_i has a countable basis of coordinate balls that are pre-compact in U_i. If $V \subseteq U_i$ is one of these balls, then the closure of V in U_i is compact, and because X is Hausdorff, it is closed in X. It follows that the closure of V in X is the same as its closure in U_i, so V is precompact in X as well. Clearly, the union of all these countable bases is a countable basis for X. $\square$

Example 1.6. The following is a list of examples of topological manifolds.

(1) $\mathbb{R}^n$ is a topological n-manifold. $\mathbb{R}^n$ is covered by a single chart $(\mathbb{R}^n, \mathrm{Id}_{\mathbb{R}^n})$, where $\mathrm{Id}_{\mathbb{R}^n} : \mathbb{R}^n \to \mathbb{R}^n$ is the identity map.

(2) (**Spheres**) The unit n-sphere, $\mathbb{S}^n$, is Hausdorff and second-countable because it is a topological subspace of $\mathbb{R}^{n+1}$. For each $1 \leq i \leq n+1$, consider the sets:

$$U_i^+ = \{(u^1, \cdots, u^{n+1}) \in \mathbb{R}^{n+1} \mid u^i > 0\}$$
$$U_i^- = \{(u^1, \cdots, u^{n+1}) \in \mathbb{R}^{n+1} \mid u^i < 0\}$$

Let $f : \mathbb{B}^n \to \mathbb{R}$ be the continuous function defined by

$$f(x) = \sqrt{1 - \|u\|^2}$$

For each $1 \leq i \leq n$, $U_i^{\pm} \cap \mathbb{S}^n$ is the graph of the function

$$u^i = \pm f(u^1, \cdots, \widehat{u^i}, \cdots, u^{n+1}),$$

where the hat indicates that u^i is omitted. Thus, each subset $U_i^{\pm} \cap \mathbb{S}^n$ is locally Euclidean of dimension n, and the maps $\phi_i \colon U_i^{\pm} \cap \mathbb{S}^n \to \mathbb{B}^n$ given by

$$\phi_i(u^1, \cdots, u^{n+1}) = (u^1, \cdots, \widehat{u^i}, \cdots, u^{n+1})$$

defines the desired homemorphism.

(3) (**Real Projective Space**) The real projective space, $\mathbb{RP}^n$, is defined as the quotient space,

$$\mathbb{RP}^n = (\mathbb{R}^{n+1} \setminus \{0\}) / \sim$$

with the equivalence relation

$$x \sim y \text{ in } \mathbb{R}^{n+1} \setminus \{0\} \iff x = \lambda y \text{ for some } \lambda \in \mathbb{R}^{\times}$$

It is made into a topological space by giving it the quotient topology via the map

$$\pi \colon \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{RP}^n,$$

where $[x] := \pi(x) = \operatorname{span}\{x\}$. It can be easily checked that

$$\mathbb{RP}^n \cong \mathbb{S}^n / \sim$$

where $\sim$ is the equivalence relation on $\mathbb{S}^n$ such that $x \sim -x$ (i.e., antipodal points are identified). We check that $\mathbb{RP}^n$ is both second-countable and Hausdorff:

- Consider the quotient map:
$$q : \mathbb{S}^n \to \mathbb{S}^n/\sim$$
Note that q is an open map. Indeed for any open subset $V \subseteq \mathbb{S}^n$, we have:
$$q^{-1}(q(V)) = V \cup -V,$$
Since $\mathbb{S}^n$ is second-countable, $\mathbb{RP}^n \cong \mathbb{S}^n/\sim$ is also second-countable as q is an open map.
- If $[x], [y] \in \mathbb{S}^n/\sim$, then one can choose $\varepsilon > 0$ small enough that
$$U = \mathbb{B}(x, \varepsilon) \cap \mathbb{S}^n \qquad V = \mathbb{B}(y, \varepsilon) \cap \mathbb{S}^n$$
are open sets in $\mathbb{S}^n$ such that $\pm U, \pm V$ are pairwise disjoint. Since,
$$q^{-1}(q(U)) = U \cup -U \qquad q^{-1}(q(V)) = V \cup -V,$$
$q^{-1}(q(U))$ and $q^{-1}(q(V))$ are open disjoint subsets of $\mathbb{S}^n/\sim$ containing $[x]$ and $[y]$. Hence, $\mathbb{RP}^n \cong \mathbb{S}^n/\sim$ is Hausdorff.

For each $1 \leq i \leq n+1$, consider the sets:
$$\widetilde{U}_i = \{(u^1, \cdots, u^{n+1}) \in \mathbb{R}^{n+1} \mid u^i \neq 0\}$$
Let $U_i = \pi(\widetilde{U}_i)$. By properties of the quotient topology, U_i is an open subset of $\mathbb{R}^n$. Consider the map $\phi_i \colon U_i \to \mathbb{R}^n$ defined as:
$$\phi_i([u]) = \left(\frac{u^1}{u^i}, \cdots, \frac{u^{i-1}}{u^i}, 1, \frac{u^{i+1}}{u^i}, \cdots, \frac{u^{n+1}}{u^i}\right).$$
This map is well-defined because its value is unchanged by multiplying x by a nonzero constant. By properties of the quotient topology, ϕ_i is continuous. In fact, ϕ_i is a homeomorphism because it has a continuous inverse given by
$$\phi_i^{-1}(u^1, \cdots, u^n) = [u^1, \cdots, u^{i-1}, 1, u^i, \cdots, u^n];$$
This shows that $\mathbb{RP}^n$ is locally Euclidean of dimension n.

(4) (**Tori**) For a positive integer n, the n-torus is the product space $\mathbb{T}^n = \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$. It is clear that a product of topological manofolds is a topolopgical manifold. Hence, T is topological n-manifold since $\mathbb{S}^1$ is a 1-manifold.

Sets such as closed intervals in $\mathbb{R}$ and closed balls in $\mathbb{R}^n$ fail to be both topological manifolds since they 'have a boundary of sorts.' We make precise the notion of a topological manifold with boundary.

Definition 1.7. Let X be a topological space. X is a topological n-manifold with boundary if X is a second countable, Hausdorff space such that each point $x \in M$ is contained in a coordinate chart, (U, ϕ), such that:

(1) (**Interior Chart**) Either $\phi : U \to \phi(U) \subseteq \mathbb{R}^n$ is a homeomorphism from U to an open subset $\phi(U)$ of $\mathbb{R}^n$.
(2) (**Boundary Chart**) Or $\phi : U \to \phi(U) \subseteq \mathbb{H}^n$ is a homeomorphism from U to an open subset $\phi(U)$ of $\mathbb{H}^n$, the upper-half plane, such that $\phi(x) \cap \partial\mathbb{H}^n \neq \emptyset$.

A point $p \in M$ is called an interior point of X if it is in the domain of some interior chart or a boundary chart (U, ϕ) such that $\phi(U) \cap \partial\mathbb{H}^n = \emptyset$. It is a boundary point of X if it is in the domain of a boundary chart that sends p to $\partial\mathbb{H}^n$. The boundary of X (the set of all its boundary points) is denoted by ∂M; similarly, its interior, the set of all its interior points, is denoted by $\mathrm{Int}(M)$.

Remark 1.8. *A point $p \in M$ might apriori simultaneously be a boundary point and an interior point, meaning that there is one interior chart whose domain contains p, and another boundary chart that sends p to $\partial\mathbb{H}^n$. This turns out not to be the case. This result is called the invariance of boundary and will be proved later.*

Example 1.9. (Sketch) The following is a list of basic examples of a topological manifold with boundary.

(1) $\overline{\mathbb{B}^n}$ is smooth n-manifold with boundary. One can prove this by definition. We skip details.
(2) If X is a n-dimesnional manifold with boundary, then ∂M is a $(n-1)$-dimensional manifold without boundary. We skip details.

1.2. **CW Complexes.** An arbitrary topological space can be difficult to visualize and analyze. We shall focus on the subcategory of topological spaces that can be constructed inductively using 'cells.'

Definition 1.10. An (open) n-cell is a topological space that is homeomorphic to the open unit ball $\mathbb{D}^n$. A closed n-cell is a topological space homeomorphic to $\overline{\mathbb{D}^n}$.

New topological spaces can be constructed from old topological spaces by attaching an n-cell. Let X be a topological space and suppose there is a map $\varphi : \mathbb{S}^{n-1} \to X$ a map. One can form the new space, $X \coprod_\varphi \mathbb{D}^n$, from the disjoint union $X \coprod \mathbb{D}^n$ by identifying each $\varphi(x) \in \mathbb{S}^{n-1}$ with $\varphi(x) \in X$ for all $x \in \mathbb{S}^{n-1}$, and equipping the resulting set with the quotient topology. The map φ is called the characteristic map. We refer to the space $X \coprod_\varphi \mathbb{D}^n$ as being obtained from X by 'attaching an n-cell,' and call $\varphi : \mathbb{S}^{n-1} \to X$ the attaching map, and $\mathbb{D}^n \to X \coprod_\varphi \mathbb{D}^n$ the characteristic map of the n-cell $\mathbb{D}^n$. Using the the universal properties of the disjoint union and quotient topology, we have the following commutative diagram[1]:

$$\begin{tikzcd}
\mathbb{S}^{n-1} \arrow[r, "\varphi"] \arrow[d, hook, "\iota"] & X \arrow[d] \arrow[ddr, bend left, "f_1"] & \\
\mathbb{D}^n \arrow[r] \arrow[drr, bend right, "f_2"'] & X \coprod_\varphi \mathbb{D}^n \arrow[dr, dotted, "f"] & \\
& & Y
\end{tikzcd}$$

One can also attach more than one n-cell. Let $\{\mathbb{D}^n_i\}_{i \in I_n}$ be a collection of n-cells and let $\varphi^n_i : \mathbb{S}^{n-1}_i \to X$ be a collection of continuous maps. One can form the new space, $X \coprod_{\substack{i \in I_n \\ \varphi_i}} \mathbb{D}^n_i$, by attaching the aforementioned collection of n-cells using the rule prescribed above. Once again, we have a commutative diagram:

$$\begin{tikzcd}
\coprod_{\substack{i \in I_n \\ \varphi^n_i}} \mathbb{S}^{n-1}_i \arrow[r, "f"] \arrow[d, hook, "\iota"] & X \arrow[d] \\
\coprod_{\substack{i \in I_n \\ \varphi^n_i}} \mathbb{D}^n_i \arrow[r] & X \coprod_{\substack{i \in I_n \\ \varphi^n_i}} \mathbb{D}^n_i
\end{tikzcd}$$

[1]In fact, this shows that $X \coprod_\varphi \mathbb{D}^n$ is a pushout in **Top**.

Definition 1.11. Let X be a topological space. A CW decomposition of X is a sequence of subspaces
$$X^0 \subseteq X^1 \subseteq X^2 \subseteq \cdots \subseteq X^n \subseteq \cdots \quad n \in \mathbb{N},$$
of X such that the following three conditions are satisfied:

(1) The space X^0 is discrete.
(2) The space X^n is obtained from X^{n-1} by attaching a (possibly) infinite number of n-cells $\{\mathbb{D}_i^n\}_{i \in I_n}$ via attaching maps $\varphi_i : \mathbb{S}_i^{n-1} \to X^{n-1}$.
(3) The topology of X is compatible with quotient topology on X that makes the
$$\coprod_{n \in \mathbb{N}} X^n \to X$$
continuous. In other words, $A \subseteq X$ is open if and only if $A \cap X^n$ is open for all $n \in \mathbb{N}$.

Remark 1.12. *Each cell $\mathbb{D}_i^n$ has its characteristic map Φ_i^n, which is by definition the composition of continuous maps:*

$$\begin{tikzcd} \mathbb{D}_i^n \arrow[r, hook] \arrow[rrr, bend left, "\Phi_i^n"] & X^{n-1} \coprod \mathbb{D}_i^n \arrow[r, two heads] & X^n \arrow[r, hook] & X \end{tikzcd}$$

Proposition 1.13. *Let X be a topological space with a CW decomposition. $A \subseteq X$ is open if and only if $(\Phi_i^n)^{-1}(\mathbb{D}_i^n)$ is open for each $i \in I_n$ and $n \in \mathbb{N}$. In particular, X is a quotient space of $\coprod_{\substack{n \in \mathbb{N} \\ i \in I_n}} \mathbb{D}_i^n$*

Proof. The forward implication is clear. Conversely, suppose $(\Phi_i^n)^{-1}(\mathbb{D}_i^n)$ is open in $\mathbb{D}_i^n$ for each for each $i \in I_n$ and $n \in \mathbb{N}$. Suppose by induction on n that $A \cap X^{n-1}$ is open in X^{n-1}. Since $(\Phi_i^n)^{-1}(\mathbb{D}_i^n)$ is open in $\mathbb{D}_i^n$ for all $\in I_n$, then $A \cap X^n$ is open in X^n by the definition of the quotient topology on X^n. The last implication is clear by definition. □

Definition 1.14. Let X be a topological space. X is a CW complex if X admits a CW decomposition satisfying the following two properties:

(1) The closure of each cell is contained in a union of finitely many cells.
(2) The topology of X is coherent with $\{\{\overline{\mathbb{D}_i^n}\}_{i \in I_n} : n \in \mathbb{N}\}$[2].

We can now define the following categories:

(1) **CW**: The objects of **CW**, the category of CW complexes, are topological spaces that admit a CW structure and morphisms between CW complexes are cellular continuous maps. That is, $f(X^n) \subseteq Y^n$ for each $n \geq 0$.
(2) $\mathbf{CW}_*$: The category of pointed CW complexes, $\mathbf{CW}_*$, is defined analogously to $\mathbf{Top}_*$.
(3) The category of pairs of CW complexes, $\mathbf{CW^2}$, is defined analogously to $\mathbf{Top_2}$.

A CW complex is finite (or finite-dimensional) if there are only finitely many cells involved. Every finite CW decomposition is automatically a finite CW complex. In fact, every locally finite CW decomposition is automatically a CW complex.

[2]That is, $A \subseteq X$ is open/closed if and only if $A \cap \overline{\mathbb{D}_i^n}$ is open/closed for each $i \in I_n$ and $n \in \mathbb{N}$.

Proposition 1.15. *Let X be a topological space endowed with a CW decomposition. If $\{\{\overline{\mathbb{D}_i^n}\}_{i\in I_n} : n \in \mathbb{N}\}$ is a locally finite collection, then X is a CW complex.*

Proof. By assumption, every point $\overline{\mathbb{D}_i^n}$ has a neighborhood that intersects only finitely many cells. Since $\overline{\mathbb{D}_i^n}$ is compact, it is covered by finitely many such neighborhoods. This readily implies (1) in Definition 1.14. Suppose $A \subsetneq X$ is a subset such that $A \cap \overline{\mathbb{D}_i^n}$ is closed for each $i \in I_n$ and $n \in \mathbb{N}$. Given $x \in X \setminus A$, let W_x be a neighborhood of x that intersects the closures of only finitely many cells, say $\overline{\mathbb{D}_1^{n_1}}, \dots, \overline{\mathbb{D}_k^{n_k}}$. Since $A \setminus \overline{\mathbb{D}_j^{n_j}}$ is closed in $\overline{\mathbb{D}_j^{n_j}}$ and thus in X, it follows that

$$W \setminus A = W \setminus \left(A \cap \overline{\mathbb{D}_1^{n_1}}\right) \cup \ldots \cup \left(A \cap \overline{\mathbb{D}_k^{n_k}}\right)$$

is a neighborhood of x contained in $X \setminus A$. Thus $X \setminus A$ is open, so A is closed. This readily implies (2) in Definition 1.14. □

Remark 1.16. *In the examples that follows, we will not explicitly check that condition (3) in Definition 1.11 is satisfied. It should be straightforward to do verify these claims, though.*

Example 1.17. $\mathbb{S}^n$ has a CW structure with one 0-cell ($\mathbb{D}^0$) and one n-cell ($\mathbb{D}^n$). The attaching map for the n-cell is $\varphi : \mathbb{S}^{n-1} = \partial\mathbb{D}^n \to \{*\}$. Indeed, let $N = (0, \dots, 0, 1)$ in $\mathbb{S}^n$. Consider the map $\sigma_N : \mathbb{S}^n \setminus \{N\} \to \mathbb{R}^n$ by

$$\sigma_N(u^1, \dots, u^{n+1}) = \left(\frac{u^1}{1 - u^{n+1}}, \dots, \frac{u^n}{1 - u^{n+1}}\right)$$

Similarly, consider $\beta_N : \mathbb{R}^n \to \mathbb{S}^n \setminus \{N\}$

$$\beta_N(u^1, \dots, u^n) = \left(\frac{2u^1}{\|u\|^2 + 1}, \dots, \frac{2u^n}{\|u\|^2 + 1}, \frac{\|u\|^2 - 1}{\|u\|^2 + 1}\right).$$

It is easy to check that σ_N, β_N are inverses of each other. Hence, $\mathbb{R}^n \cong \mathbb{S}^n \setminus \{N\}$. The map σ_N is called the stereographic projection. $\mathbb{S}^n$ can now be given a CW structure with one 0-cell ($\mathbb{D}^0$) and one n-cell ($\mathbb{D}^n$). The attaching map for the n-cell is $\varphi : \mathbb{S}^{n-1} = \partial\mathbb{D}^n \to \{*\}$.

Example 1.18. $\mathbb{S}^n$ can be given a different CW structure with two k-cells in each dimension for $0 \le k \le n$. Let $X^0 = \mathbb{S}^0 = \{\mathbb{D}_1^0, \mathbb{D}_2^0\}$. Then $X^1 = \mathbb{S}^1$ where the two 1-cells $\mathbb{D}_1^1, \mathbb{D}_2^1$ are attached to the 0-cells by homeomorphisms on their boundary. Similarly, two 2-cells can be attached to $X^1 = \mathbb{S}^1$ by homeomorphism on their boundary, giving $X^2 = \mathbb{S}^2$. Proceed inductively.

Example 1.19. There are natural inclusions

$$\mathbb{S}^0 \subseteq \mathbb{S}^1 \subseteq \cdots \subseteq \mathbb{S}^n \subseteq \cdots \subseteq$$

We can then define

$$\mathbb{S}^\infty = \bigcup_{n \ge 0} \mathbb{S}^n$$

If $\mathbb{S}^n$ is given a CW structue as in Example 1.18 for each $n \ge 0$, then $\mathbb{S}^\infty$ is a CW complex as well.

Example 1.20. Consider $\mathbb{RP}^n$. An easy observation shows that $\mathbb{RP}^n$ is a quotient of $\mathbb{D}^n$ by the relation $x \sim -x$ on the boundary $\mathbb{S}^{n-1}$[3]. Thus, $\mathbb{RP}^n$ can be obtained from $\mathbb{RP}^{n-1}$ by

[3] It is easy to check that these identifications are consistent with out discussion of the real projective plane, which is $\mathbb{RP}^2$, in Section 4.

attaching a one cell.

$$\begin{tikzcd} \mathbb{S}^{n-1} \arrow[r, hook] \arrow[d, twoheadrightarrow] & \mathbb{D}^n \arrow[d, twoheadrightarrow] \\ \mathbb{RP}^{n-1} \arrow[r, hook] & \mathbb{RP}^n \end{tikzcd}$$

Thus $\mathbb{RP}^n$ can be built as a CW complex with a single cell in each dimension $\leq n$.

Example 1.21. The complex projective space, $\mathbb{CP}^n$, is defined as the quotient space,

$$\mathbb{CP}^n = (\mathbb{C}^{n+1} \setminus \{0\})/\sim$$

with the equivalence relation

$$x \sim y \text{ in } \mathbb{C}^{n+1} \setminus \{0\} \iff x = \lambda y \text{ for some } \lambda \in \mathbb{C}^\times$$

Note that there is a map

$$\mathbb{D}^{2n} \to \mathbb{CP}^n \qquad z = (z_0, \ldots, z_{n-1}) \mapsto [z_0, \ldots, z_{n-1}, t] \quad t = \sqrt{1 - \|z\|}$$

The boundary of $\mathbb{D}^{2n}$ (where $t = 0$) is sent to $\mathbb{CP}^{n-1}$. In this way, $\mathbb{CP}^n$ is obtained from $\mathbb{CP}^{n-1}$ by attaching one $2n$-cell. So $\mathbb{CP}^n$ has a CW structure with one cell in each even dimension $0, 2, \ldots, 2n$.

A subcomplex of X is a subspace $Y \subseteq X$ that is a union of cells of X, such that if Y contains a cell, it also contains its closure. It follows immediately that the union and the intersection of any collection of subcomplexes are themselves subcomplexes. Examples of a sub-complexes would be be the subspaces X^n for $n \geq 0$ in the definition of a CW complex.

Proposition 1.22. *Suppose X is a CW complex and Y is a subcomplex of X. Then Y is closed in X, and with the subspace topology and the cell decomposition that it inherits from X, it is a CW complex.*

Proof. Let $\mathbb{D}^n \subseteq Y$ denote such an n-cell in Y. Since $\overline{\mathbb{D}^n} \subseteq Y$, the finitely many cells of X that have nontrivial intersections with $\mathbb{D}^n$ must also be cells of Y. So condition (1) in Definition 1.14 is automatically satisfied by Y. In addition, any characteristic map $\varphi : \mathbb{D}^n \to X$ for $\mathbb{D}^n$ in X also serves as a characteristic map for $\mathbb{D}^n$ in Y.

Suppose $A \subseteq Y$ is a subset such that $A \cap \mathbb{D}^n$ is closed in $\mathbb{D}^n$ for every n-cell $\mathbb{D}^n$ contained in Y. Let $\mathbb{D}^n$ be a n-cell of X that is not contained in Y. We know that $\overline{\mathbb{D}^n} \setminus \mathbb{D}^n$ is contained in the union of finitely many cells of X; some of these, say $\mathbb{D}_1^{n_1}, \ldots, \mathbb{D}_k^{n_k}$, might be contained in Y. Then $\overline{\mathbb{D}_1^{n_1}} \cup \ldots \cup \overline{\mathbb{D}_k^{n_k}} \subseteq Y$, and

$$A \cap \overline{\mathbb{D}^n} = A \cap (\overline{\mathbb{D}_1^{n_1}} \cup \ldots \cup \overline{\mathbb{D}_k^{n_k}}) \cap \overline{\mathbb{D}^n} = ((A \cap \overline{\mathbb{D}_1^{n_1}}) \cup \cdots \cup (A \cap \overline{\mathbb{D}_k^{n_k}})) \cap \overline{\mathbb{D}^n}$$

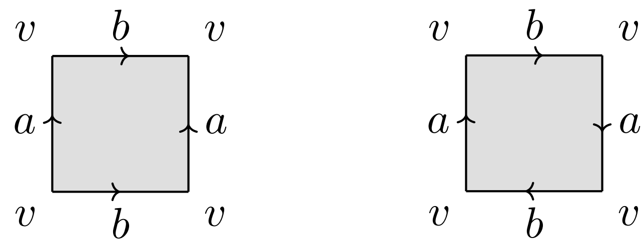
which is closed in $\overline{\mathbb{D}^n}$. It follows that A is closed in X and therefore in Y. This implies (2) in Definition 1.14. Hence Y is a CW complex. Taking $A = Y$ shows that Y is closed. $\square$

Let's discuss some 2-dimensional examples. It is well-known that compact, connected 2-dimensional manifolds are classified into the following types:

(1) $\mathbb{S}^2$,
(2) A connected sum of g-tori $\mathbb{T}^2$ (or a g-hold torus) for $g \geq 2$,
(3) A connected sum of g-projective spaces $\mathbb{RP}^2$, for $g \geq 2$.

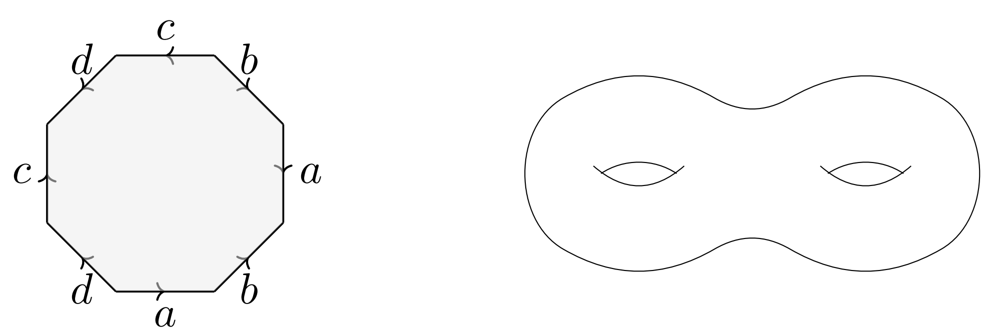
We have already discussed a CW-structure on $\mathbb{S}^2$. We discuss examples of the other 2-manifolds below:

Example 1.23. Consider $X = \mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ or $\mathbb{RP}^2$. X can be constructed as the quotient of a rectangle by identifying opposite edges as shown below. Each space has the structure of a CW complex with a single 0-cell, 2 1-cells and a single 2-cell.

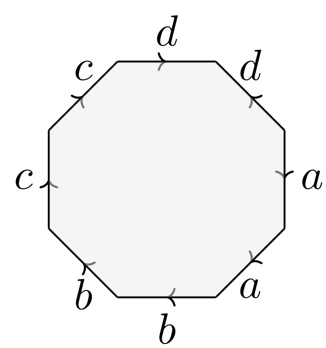


$X = \mathbb{T}$ $\qquad$ $X = \mathbb{RP}^2$

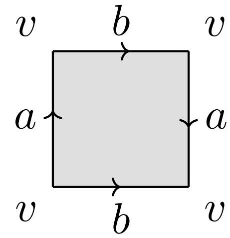
Example 1.24. For $g \geq 1$, a model for a connected sum of g-$\mathbb{T}^2$ is given by M_g, which is called a orientable surface of genus g. M_g is constructed from a polygon with $4g$ sides by identifying pairs of edges. M_g has a CW structure a single 0-cell, g 1-cells and a single 2-cell. M_g can be can be visualized as a g-holed torus. See this link for a visual argument as to why this is the case for $g = 2$.



Example 1.25. For $g \geq 2$, a model for a connected sum of g-$\mathbb{RP}^2$ is given by N_g, which is called a non-orientable surface of genus g. N_g is constructed from a polygon with g sides by identifying pairs of edges. N_g has a CW structure a single 0-cell, g 1-cells and a single 2-cell.



Remark 1.26. *N_2 is usually called a Klein bottle. Another model for the Klein bottle is given by the CW structure shown below:*



It can be checked that both models are homeomorphic.

2. Categorical Remarks

Algebraic topology employs tools from abstract algebra to study topological spaces. The essence of the subject can be succinctly captured through the lens of category theory. The key idea is to construct functors from a 'topological category' to an 'algebraic category', allowing us to attach topological invariants to topological spaces that can be computed algebraically. Fpr example, we will construct the homology functor. This will be a functor

$$H_n : \mathbf{Top} \to \mathbf{Ab}$$

for each $n \geq 0$. Here $\mathbf{Ab}$ is the category of abelian groups.

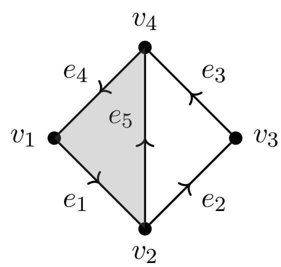
Naturally, we will use the language of category theory throughout these notes. In this section, we make some brief categorical remarks. We will primarily focus on the following categories arising in algebraic topology:

(1) $\mathbf{Top}$: The objects of $\mathbf{Top}$ are topological spaces and morphisms are continuous functions.
(2) $\mathbf{Top}_*$: The objects of $\mathbf{Top}_*$ are pointed topological spaces, (X, x_0), and morphisms between pointed topological spaces $f : (X, x_0) \to (Y, y_0)$ are continuous maps such that $f(x_0) = y_0$.
(3) $\mathbf{Top^2}$: The objects of $\mathbf{Top^2}$ are all pairs (X, A), where X is a topological space and $A \subseteq X$ is a subspace, and morphisms between pairs of topological spaces $f : (X, A) \to (Y, B)$ are continuous maps $f : X \to Y$ such that $f(A) \subseteq B$.
(4) $\mathbf{Top^3}$: The objects of $\mathbf{Top}^3$ are all triples (X, A, B), where X is a topological space and $B \subseteq A \subseteq X$ are subspaces, and morphisms between triples of topological spaces $f : (X, A, C) \to (Y, B, D)$ are continuous maps $f : X \to Y$ such that $f(A) \subseteq B$ and $f(C) \subseteq D$.
(5) $\mathbf{Htpy}$: The objects of $\mathbf{Htpy}$ are topological spaces, and morphisms between topological spaces are homotopy classes of continuous maps.
(6) $\mathbf{Htpy}_*$ The objects of $\mathbf{Htpy}$ are pointed topological spaces, and morphisms between pointed topological spaces are basepoint-preserving homotopy classes of based continuous maps.

Part 2. Simplicial & Singular Homology

3. What is Homology?

Heuristically, homology measures the existence of holes in a topological space. For instance, consider the topological space, X, shown below:



Intuitively, the boundary of an edge in the diagram can be thought of as a formal difference between the 'target' and the 'source'. So, the boundary of e_1 is given by $v_2 - v_1$. Moreover, let us define a chain of paths to be a formal sum of edges. For instance, we have the chains

$$\begin{aligned} c_1 &= e_1 + e_5 + e_4 \\ c_2 &= e_2 + e_3 + e_5^{-1} \\ c_3 &= e_1 + e_2 + e_3 + e_4 \end{aligned}$$

Here e_5^{-1} denotes the that edge e_5 is traversed in the opposite direction. In this terminology, we say there is a loop in X if the boundary of a formal sum of edges vanish. For instance, the boundary of c_1, c_2, c_3 vanishes. However, the loop c_1 can be shrunk to a point by deforming the path e_4+e_1 to e_5 by continuously moving it within the interior of c_1. In our terminology, this can be detected by the fact that c_1 is the boundary of the solid triangle v_1, v_2, v_4. On the other hand, c_2 cannot be shrunk to a point since the triangle v_2, v_3, v_4 is hollow. Hence, we expect that there is one hole in X. The first homology group shall detect the presence of such a hole.

Remark 3.1. *The above intuition can be made precise by Hurewicz theorem which states that*

$$H_1(X) \cong \frac{\pi_1(X)}{[\pi_1(X), \pi_1(X)]}$$

That is $H_1(X)$ is isomorphic to the abelianization of $\pi_1(X)$, the fundamental group of X which quite literally is a measure of holes in a topological space.

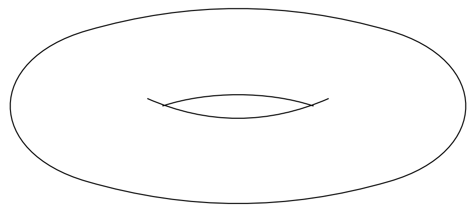
More generally, the n-th homology group measures the existence of n-dimensional holes in a topological space, X. We consider an n-dimensional loop in $L \subseteq X$ which are reasonably such that that $L \cong \mathbb{S}^n$. For some n-dimensional loops $L \subseteq X$, there might be an $(n+1)$-dimensional disc $D \subseteq X$ such that $D \cong \mathbb{B}^{n+1}$ and L is the boundary of D. A n-dimensional hole will then be a n- dimensional loop that is not the boundary of a n-dimensional disc. The n-th homology group will then be a measure of the existence of such n-dimensional holes.

Remark 3.2. *The phrase n-dimensional is only used informally in this section. Moreover, the statement above is only meant for intuition and should be taken with a grain of salt. In* *general, there is only a homomorphism*

$$h_h : \pi_n(X) \to H_n(X),$$

for $n \geq 2$ if X is path-connected.

The remarks made above precisely hold in one one important case. Consider a hollow doughnut-shaped surface, D, considered as a subset of $\mathbb{R}^3$.



Intuitively, there are two 1-dimensional holes in D. We will see that the first homology group (with $\mathbb{Z}$-coefficients) will be $\mathbb{Z} \oplus \mathbb{Z}$. The fact that there is a direct sum of two copies of $\mathbb{Z}$ is a measure of the existence of two and only two 1-dimensional holes. Note that D is hollow. Hence, we expect that there is 2-dimensional hole in D - a region in space with empty volume! We will see that the second homology group (once again with $\mathbb{Z}$-coefficients) will be $\mathbb{Z}$ since there is one and only one 2-dimensional hole.

4. Simplicial Homology

Let X be a topological space. In this section, we will define simplicial homology. Simplicial homology has the advantage of being computationally tractable since it can be used when a topological space can be triangulated. Indeed, we will define it in terms of Δ-complexes which will serve as basic building block of our triangulation.

Definition 4.1. Let $[v_0, v_1, \ldots, v_n]$ be an ordered tuple in $\mathbb{R}^m$.

- $[v_0, v_1, \ldots, v_n] \subseteq \mathbb{R}^m$ is said to be affinely independent if the set
$$\{v_1 - v_0, v_2 - v_0, \ldots, v_n - v_0\}$$
is linearly independent[4].
- Given an affinely independent ordered tuple $[v_0, v_1, \ldots, v_n] \subseteq \mathbb{R}^m$, the n-simplex generated by $[v_0, v_1, \ldots, v_n]$ is the convex span in $\mathbb{R}^m$ of the $n+1$ points $v_0, \ldots, v_n$:
$$\mathrm{Conv}[v_0, v_1, \ldots, v_n] = \left\{ x = \sum_{i=0}^{n} t_i v_i \in \mathbb{R}^m \mid t_i \geq 0, \sum_i t_i = 1 \right\},$$
We call the points v_i the vertices of the n-simplex $[v_0, v_1, \ldots, v_n]$.
- Given an n-simplex $\mathrm{Conv}[v_0, v_1, \ldots, v_n]$, the face opposite to v_i is the $(n-1)$-simplex:
$$\mathrm{Conv}[v_0, \ldots, \widehat{v}_i, \ldots, v_n] := \{x \in \mathrm{Conv}[v_0, v_1, \ldots, v_n] \mid t_i = 0\}.$$
The boundary of an n-simplex is the union of its faces.

Geometrically, one can think of an n-simplex as the smallest convex subset containing $v_0, \ldots, v_n$ such that the points do not lie in a hyperplane of dimension less than n. As an example consider the standard n-simplex:

[4]Thus necessarily $n \leq m$

Definition 4.2. The standard simplex, $\Delta^n \subseteq \mathbb{R}^{n+1}$, is

$$\Delta^n = \left\{(t_0, \dots, t_n) \in \mathbb{R}^{n+1} \mid t_i \geq 0, \sum_i t_i = 1\right\}.$$

Remark 4.3. *The standard simplex allows one to induce coordinates on all n-simplices by sending $e_i \mapsto v_i$ inducing a map of simplicies:*

$$(t_0, \dots, t_n) \mapsto \sum_{i=0}^{n} t_i v_i$$

$(t_0, \dots, t_n)$ *are are called barycentric coordinates.*

Definition 4.4. A Δ-complex structure on a topological space X is a collection of maps $\{\sigma_j^n : \Delta^n \to X\}_{n\geq 0}^{j\in J_n}$ such that:

- The restriction $\sigma_j^n|_{\mathrm{Int}(\Delta^n)}$ is injective, and each point of X is in the image of exactly one such $\sigma_j^n|_{\mathrm{Int}(\Delta^n)}$.
- Restriction of each σ_j^n to a face of Δ^n is one of the maps $\sigma_k^{n-1} : \Delta^{n-1} \to X$.
- A set $A \subseteq X$ is open if and only if $(\sigma_j^n)^{-1}(A)$ is open in X for each σ_j^n.

Remark 4.5. *In what follows, we shall identify a $\sigma_j^n : \Delta^n \to X$ with a n-simplex $[v_0, \cdots, v_n]$.*

Our goal is to define the simplicial homology groups of a Δ-complex structure on a topological space, X. Let $\Delta_n(X)$ be the free abelian group with basis the open n-simplices of X. Elements of $\Delta_n(X)$ are called n-chains. These can be written as finite formal sums

$$\sum_{j\in J_n} n_j \sigma_j^n \qquad n_j \in \mathbb{Z}$$

Definition 4.6. Let X be a topological space with a Δ-complex structure. The boundary operator

$$\partial_n^\Delta : \Delta_n(X) \to \Delta_{n-1}(X)$$

is defined on each basis element of $\Delta_n(X)$ as:

$$\partial_n^\Delta [v_0, \dots, v_n] = \sum_{i=0}^{n} (-1)^i [v_0, \dots, \widehat{v_i}, \dots, v_n]$$

We say that

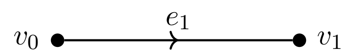
$$\partial_n^\Delta \Big(\sum_{j\in J_n} n_j \sigma_j^n \Big) \in \Delta_{n-1}(X)$$

is the boundary of $\sum_{j\in J_n} n_i \sigma_j^n$ in X.

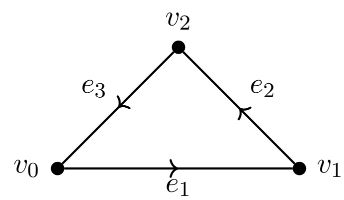
Remark 4.7. *Note that the boundary of an n-simplex in X is a $\mathbb{Z}$-linear combination (with coefficients ± 1) of $n-1$-simplices. This provides one motivation as to why we consider $\mathbb{Z}$-linear combinations of n-simplices. Moreover, heuristically the signs are inserted to take orientations into account, so that all the faces of a simplex are coherently oriented. See* *Example 4.8.*

Example 4.8. The following are examples of boundaries some standard simplexes.

(1) Consider $X = \Delta^1$. Then $\partial[v_0, v_1] = [v_1] - [v_0]$



(2) Consider $X = \Delta^2$. Then $\partial[v_0, v_1, v_2] = [v_1, v_2] - [v_0, v_2] + [v_0, v_1]$



Lemma 4.9. *Let X be a topological space with a Δ-complex structure. The map,*

$$\partial^{\Delta}_{n-1} \circ \partial^{\Delta}_{n} : \Delta_n(X) \xrightarrow{\partial^{\Delta}_{n}} \Delta_{n-1}(X) \xrightarrow{\partial^{\Delta}_{n-1}} \Delta_{n-2}(X)$$

is zero for each $n \geq 0$.

Proof. Note that:

$$\sum_{0\leq j<i\leq n} (-1)^i(-1)^j[v_0, \cdots, \widehat{v_j}, \cdots, \widehat{v_i}, \cdots, v_n] + \sum_{0\leq i<j\leq n} (-1)^i(-1)^{j-1}[v_0, \cdots, \widehat{v_i}, \cdots, \widehat{v_j}, \cdots, v_n] = 0$$

The latter two summations cancel since after switching i and j in the second sum, it becomes the negative of the first. □

Remark 4.10. *Note that $\Delta^1 \in \operatorname{Ker} \partial^{\Delta}_1$ if and only if $v_0 = v_1$. In this case, Δ^1 can be thought of as a circle or a 1-loop. Indeed, this observation motivates the the observation that n-loops in X correspond to elements of $\operatorname{Ker} \partial^{\Delta}_n$ for each $n \geq 1$. Moreover, the condition $\partial^{\Delta}_n \circ \partial^{\Delta}_{n+1} = 0$ is the observation that the boundary of a $\mathbb{Z}$-linear combination of $(n+1)$-simplicies is a n-loop.*

Let $C^{\Delta}_n(X) = \Delta_n(X)$ for each $n \geq 0$. Purely algebraically, we have a sequence of homomorphisms of abelian groups:

$$\cdots \xrightarrow{\partial^{\Delta}_{n+1}} C^{\Delta}_n(X) \xrightarrow{\partial^{\Delta}_{n}} C^{\Delta}_{n-1}(X) \xrightarrow{\partial^{\Delta}_{n-1}} C^{\Delta}_{n-2}(X) \xrightarrow{\partial^{\Delta}_{n-2}} \cdots$$

The boundary map $\partial^{\Delta}_n : C^{\Delta}_n(X) \longrightarrow C^{\Delta}_{n-1}(X)$ is such that

$$\partial^{\Delta}_{n-1} \circ \partial^{\Delta}_n = 0$$

That is,

$$\operatorname{Im}(\partial^{\Delta}_{n+1}) \subseteq \operatorname{Ker}(\partial^{\Delta}_n)$$

Elements of $\operatorname{Ker}(\partial^{\Delta}_n)$ are called n-cycles (or n-loops) and elements of $\operatorname{Im}(\partial^{\Delta}_{n+1})$ are called n-boundaries.

Definition 4.11. Let X be a topological space with a Δ-complex structure. The n-th simplicial homology group of X with $\mathbb{Z}$-coefficients of the associated chain complex $(C^{\Delta}_n(X), \partial^{\Delta}_n)_{n\in\mathbb{N}}$ is

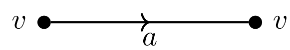
$$H^{\Delta}_n(X; \mathbb{Z}) = \frac{\operatorname{Ker}(\partial^{\Delta}_n)}{\operatorname{Im}(\partial^{\Delta}_{n+1})}$$

$H^{\Delta}_n(X; \mathbb{Z})$ is called the n-th simplicial homology group of X.

Remark 4.12. *In what follows, we will not explicitly verify Definition 4.4(3). For instance, we will not explicitly verify that the Δ-complex structure on the circle, $\mathbb{S}^1$, in Example 4.13(1) is compatible with the topology on $\mathbb{S}^1$. Similarly, Example 4.13(2)-(8) we will not explicitly verify that the Δ-complex structure is compatible with the underlying quotient topology. It should be straightforward to do verify these claims, though.*

Example 4.13. We compute simplicial homology groups of various topological spaces below.

(1) **(Circle)** Consider $X = \mathbb{S}^1$ with a Δ-complex structure with a single 1-simplex and a single 0-simplex.



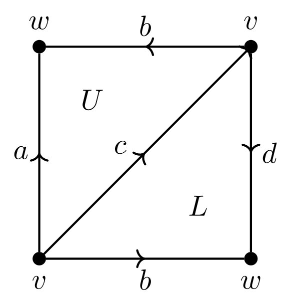
We have a chain complex of the following form:

$$\cdots \longrightarrow 0 \xrightarrow{\partial_2^\Delta} \mathbb{Z} \xrightarrow{\partial_1^\Delta} \mathbb{Z} \xrightarrow{\partial_0^\Delta} 0.$$

Here ∂_1^Δ is the zero map. Therefore, we have:

$$H_n^\Delta(\mathbb{S}^1, \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } n = 0, 1 \\ 0 & \text{otherwise} \end{cases}$$

(2) (**Mobius Band**) Consider $X = M$, the Mobius band. A Δ-complex structure on M is pictured below.



We have a complex of the following form:

$$\cdots 0 \xrightarrow{\partial_3^\Delta} \mathbb{Z}^{\oplus 2} \xrightarrow{\partial_2^\Delta} \mathbb{Z}^{\oplus 4} \xrightarrow{\partial_1^\Delta} \mathbb{Z}^{\oplus 2} \xrightarrow{\partial_0^\Delta} 0.$$

We have

$$\partial_1^\Delta a = \partial_1^\Delta b = \partial_1^\Delta d = w - v \quad \partial_1^\Delta c = 0$$

Hence $\operatorname{Im} \partial_1^\Delta \cong \mathbb{Z}$, implying that $H_0^\Delta(X) \cong \mathbb{Z}^{\oplus 2}/\mathbb{Z} \cong \mathbb{Z}$. Also

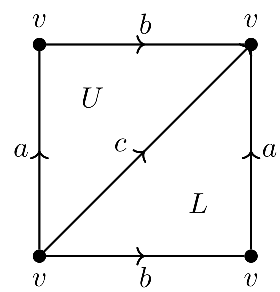
$$\partial_2^\Delta U = a - b - c \qquad \partial_2^\Delta L = b - d - c$$

This implies ∂_2^Δ is injective. Hence $H_2^\Delta(X) \cong 0$. A basis for $\operatorname{Ker} \partial_1^\Delta$ is $\{x = a - d, y = b - d, z = c\}$. Hence $\operatorname{Ker} \partial_1^\Delta \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$. A basis for $\operatorname{Im} \partial_2^\Delta$ is $\{x - y - z, y - z\}$. An equivalent basis is $\{x, y - z\}$. Hence $H_1^\Delta(X) \cong \mathbb{Z}$.

$$H_n^\Delta(M, \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{for } n = 0, 1 \\ 0 & \text{for } n \geq 2 \end{cases}$$

(3) (**Torus**) Consider the $X = T$, the torus, with the Δ-complex structure is pictured below having one vertex, three edges a, b, and c, and two 2-simplices U and L[5]. As in the previous example, $\partial_1^\Delta = 0$. Also $\partial_2^\Delta U = a + b - c = \partial_2^\Delta L$. We have a complex of the following form:

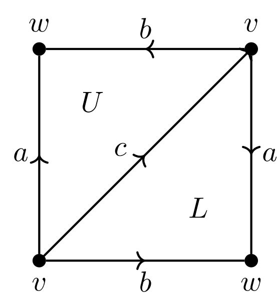
$$\cdots 0 \xrightarrow{\partial_3^\Delta} \mathbb{Z}^{\oplus 2} \xrightarrow{\partial_2^\Delta} \mathbb{Z}^{\oplus 3} \xrightarrow{\partial_1^\Delta} \mathbb{Z} \xrightarrow{\partial_0^\Delta} 0.$$



Since $\partial_1^\Delta = 0$, $H_0^\Delta(T) \cong \mathbb{Z}$ Since $\{a, b, a+b-c\}$ is a valid basis for $\mathbb{Z}^{\oplus 3}$, it follows that $H_1^\Delta(T) \cong \mathbb{Z}^2$ with basis the homology classes $[a]$ and $[b]$. Since there are no 3-simplices, $H_2^\Delta(T)$ is equal to $\operatorname{Ker} \partial_2^\Delta$, which is infinite cyclic generated by $U - L$. Thus,

$$H_n^\Delta(T, \mathbb{Z}) \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & \text{for } n = 1 \\ \mathbb{Z} & \text{for } n = 0, 2 \\ 0 & \text{for } n \geq 3 \end{cases}$$

(4) (**Real Projective Plane**) Consider $X = \mathbb{RP}^2$. The delta complex structure is pictured below.



We have a complex of the following form:

$$\cdots 0 \xrightarrow{\partial_3^\Delta} \mathbb{Z}^{\oplus 2} \xrightarrow{\partial_2^\Delta} \mathbb{Z}^{\oplus 3} \xrightarrow{\partial_1^\Delta} \mathbb{Z}^{\oplus 2} \xrightarrow{\partial_0^\Delta} 0.$$

We have

$$\partial_1^\Delta b = \partial_1^\Delta a = w - v \qquad \partial^\Delta c = 0.$$

Hence $\operatorname{Im} \partial_1^\Delta \cong \mathbb{Z}$, implying that $H_0^\Delta(X) = \mathbb{Z}^{\oplus 2}/\mathbb{Z} \cong \mathbb{Z}$. Also

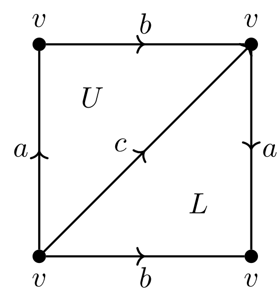
$$\partial_2^\Delta U = a - b - c \qquad \partial_2^\Delta L = b - a - c$$

[5] T is homemorphic to the donught-shaped surface in Section 3.

This implies ∂_2^Δ is injective. Hence $H_2^\Delta(X) \cong 0$. A basis for $\mathrm{Ker}\,\partial_1^\Delta$ is $\{x = a-b, y = c\}$. Hence $\mathrm{Ker}\,\partial_1^\Delta \cong \mathbb{Z} \oplus \mathbb{Z}$. A basis for $\mathrm{Im}\partial_2^\Delta$ is $\{x-y, -x-y\}$. An equivalent basis is $\{x-y, 2y\}$. Hence $H_1^\Delta(X) \cong \mathbb{Z}_2$.

$$H_n^\Delta(\mathbb{RP}^2, \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{for } n = 0 \\ \mathbb{Z}_2 & \text{for } n = 1 \\ 0 & \text{for } n \geq 2 \end{cases}$$

(5) **(Klein Bottle)** Consider $X = K$, the Klein bottle, with the Δ-complex structure is pictured below having one vertex, three edges a, b and c, and two 2-simplices U and L.



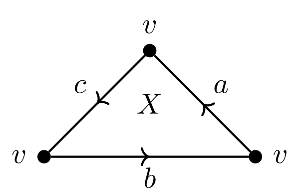
We have a complex of the following form:

$$\cdots 0 \xrightarrow{\partial_3^\Delta} \mathbb{Z}^{\oplus 2} \xrightarrow{\partial_2^\Delta} \mathbb{Z}^{\oplus 3} \xrightarrow{\partial_1^\Delta} \mathbb{Z} \xrightarrow{\partial_0^\Delta} 0.$$

Clearly, $\partial_1^\Delta = 0$. $\partial_2^\Delta U = a+b-c$ and $\partial_2^\Delta L = a-b+c$. Since $\partial_1^\Delta = 0$, $H_0^\Delta(K) \cong \mathbb{Z}$. We have $\mathrm{Im}(\partial_2^\Delta) = \mathrm{span}\{2a, a+b-c\}$. Since $\{a, a+b-c, c\}$ is a valid basis for $\mathbb{Z}^{\oplus 3}$, it follows that $H_1^\Delta(K) \cong \mathbb{Z} \oplus \mathbb{Z}_2$. Since there are no 3-simplices, $H_2^\Delta(K)$ is equal to $\mathrm{Ker}\,\partial_2^\Delta$, which is easily seen to be trivial. Thus,

$$H_n^\Delta(K, \mathbb{Z}) \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z}_2 & \text{for } n = 1 \\ \mathbb{Z} & \text{for } n = 0 \\ 0 & \text{for } n \geq 2 \end{cases}$$

(6) **(Triangular Parachute)** Let X be a triangular parachute obtained from Δ^2 by identifying its three vertices to a single point.



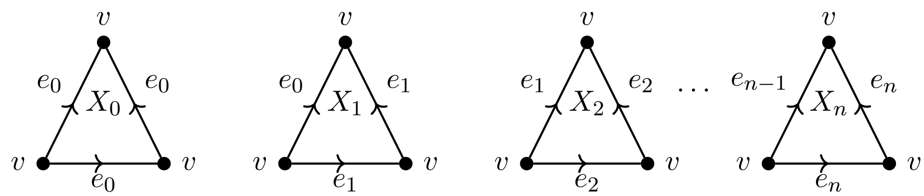
We have 1 face, 3 edges, and 1 vertex so that $\Delta^2(X)$, $\Delta^0(X) \cong \mathbb{Z}$, $\Delta^1(X) \cong \mathbb{Z}^3$. Note that

$$\partial_2^\Delta(X) = b + a - c \qquad \partial_1^\Delta(a) = \partial_1^\Delta(b) = \partial_1^\Delta(c) = \partial_0^\Delta(v) = 0$$

Hence $\ker \partial_2^\Delta = 0$, $\ker \partial_1^\Delta = \mathbb{Z}^3$, $\ker \partial_0^\Delta = \mathbb{Z}$. On the other hand, $\mathrm{Im}\,\partial_2^\Delta = \mathbb{Z}$ as the subgroup $\langle b + a - c \rangle$ is free on one generator. Hence we have,

$$H_n^{\Delta}(X,\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{for } n=0 \\ \mathbb{Z}^{\oplus 2} & \text{for } n=1 \\ 0 & \text{for } n \geq 2 \end{cases}$$

(7) Let X be the topological space obtained obtained from $n+1$ 2-simplices $\Delta_0^2, \ldots, \Delta_n^2$ by identifying all three edges of Δ_0^2 to a single edge, and for $i > 0$ identifying the edges $[v_0, v_1]$ and $[v_1, v_2]$ of Δ_i^2 to a single edge and the edge $[v_0, v_2]$ to the edge $[v_0, v_1]$ of Δ_{i-1}^2.



We have 1 vertex, $n+1$ edges, and $n+1$ faces so that $\Delta_0(X) \cong \mathbb{Z}$, $\Delta_1(X), \Delta_2(X) \cong \mathbb{Z}^{n+1}$. We have a complex of the following form:

$$\cdots 0 \xrightarrow{\partial_3^{\Delta}} \mathbb{Z}^{\oplus n+1} \xrightarrow{\partial_2^{\Delta}} \mathbb{Z}^{\oplus n+1} \xrightarrow{\partial_1^{\Delta}} \mathbb{Z}^{\oplus 1} \xrightarrow{\partial_0^{\Delta}} 0.$$

Clearly, $\partial_0^{\Delta} = 0$ and Im $\partial_1^{\Delta} = 0$. Hence $H_0^{\Delta}(X) \cong \mathbb{Z}$. Let's compute Im ∂_2. Note that:

$$\partial_2 X_i = \begin{cases} e_0 & \text{if } i=0 \\ 2e_i - e_{i-1} & \text{if } i>1 \end{cases}$$

It is clear that a basis for Im $\partial_2 = \{e_0\} \cup \{2e_i - e_{i-1} : 1 \leq i \leq n\}$. Note that in $H_1^{\Delta}(X) = \ker\partial_1/\text{Im } \partial_2$, we set $e_0 = 0$ and $2e_i - e_{i-1} = 0$ so that $e_0 = 0$, $2e_i = e_{i-1}$. This implies that

$$2e_1 = e_0 = 0 \quad 2^2 e_2 = e_0 = 0 \quad \cdots \quad 2^k e_k = e_0 = 0$$

so that Therefore:

$$H_1^{\Delta}(X) \cong \mathbb{Z}^{n+1}/(\mathbb{Z} \times 2\mathbb{Z} \times \cdots \times 2^n\mathbb{Z}) \cong \mathbb{Z}_2 \times \mathbb{Z}_4 \times \cdots \times \mathbb{Z}_{2^n}$$

It is easy to see that Ker $\partial_2^{\Delta} = 0$. Hence $H_2^{\Delta}(X) = 0$. Therefore, we have:

$$H_n^{\Delta}(X,\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{for } n=0 \\ \mathbb{Z}_2 \times \mathbb{Z}_4 \times \cdots \times \mathbb{Z}_{2^n} & \text{for } n=1 \\ 0 & \text{for } n \geq 2 \end{cases}$$

(8) Let X_n be obtained from an n-simplex by identifying all faces of the same dimension. Since there is only one k-simplex for each $k \leq n$, we see that $\Delta^k(X_n) \cong \mathbb{Z}$ for $k \leq n$.

Choose a generator σ_k for each of these. Note that the restriction of σ_k to a $(k-1)$-dimensional face will just be σ_{k-1}. Thus,

$$\partial_k \sigma_k = \sum_{i=0}^{k} (-1)^i \sigma_{k-1} = \begin{cases} 0 & \text{if } k = 0, \\ 0 & \text{if } k \leq n, \text{ and } k \text{ is odd,} \\ \sigma_{k-1} & \text{if } k \leq n, \text{ and } k \text{ is even,} \\ 0 & \text{if } k > n. \end{cases}$$

Therefore:

$$\mathrm{Ker}(\partial_k) = \begin{cases} \mathbb{Z} & \text{if } k = 0, \\ \mathbb{Z} & \text{if } k \leq n, \text{ and } k \text{ is odd,} \\ 0 & \text{if } k \leq n, \text{ and } k \text{ is even,} \\ 0 & \text{if } k > n. \end{cases} \qquad \mathrm{Im}(\partial_k) = \begin{cases} 0 & \text{if } k = 0, \\ 0 & \text{if } k \leq n, \text{ and } k \text{ is odd,} \\ \mathbb{Z} & \text{if } k \leq n, \text{ and } k \text{ is even,} \\ 0 & \text{if } k > n. \end{cases}$$

Hence:

$$H_k^{\Delta}(X_n, \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } k = 0, \\ \mathbb{Z} & \text{if } k = n, \text{ and } n \text{ is odd,} \\ 0 & \text{else.} \end{cases}$$

5. Homological Algebra

In this section we take an algebraic detour and introduce the basics of homological algebra by discussing exact sequences, chain complexes and some diagram chasing lemmas. We work in the category **Ab**, the category of abelian groups. However, the discussion applies verbatim in $\mathbf{Mod}_R$[6].

5.1. Exact Sequences.

We first discuss the fundamental concept of exact sequences.

Definition 5.1. A sequence

$$A \xrightarrow{f} B \xrightarrow{g} C$$

of two homomorphisms of abelian groups is said to be exact at B if $\mathrm{Im}\, f = \mathrm{Ker} g$. More generally, a sequence

$$\ldots \to A_{n+1} \xrightarrow{f_{n+1}} A_n \xrightarrow{f_n} A_{n-1} \to \ldots \quad n \in \mathbb{Z}$$

is said to be exact if it is exact at A_n for each $n \in \mathbb{Z}$. Such a sequence is called a long exact sequence of abelian groups.

The following is an important special case:

Definition 5.2. A short exact sequence of abelian groups is an exact sequence of the form

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0.$$

that is exact in each degree.

Example 5.3. Using the notion of exactness, we can rephrase familiar definitions from basic algebra. Suppose $f : A \to B$ is a homomorphism of abelian groups.

- f is injective if and only if $0 \to A \xrightarrow{f} B$ is exact. Indeed, the sequence is exact at A if and only if $\mathrm{Ker}\, f = 0$ if and only if f is injective.

[6]In general, in any abelian category.

- f is surjective if and only if $A \xrightarrow{f} B \to 0$ is exact. Indeed, the sequence is exact at B if and only if $\operatorname{Im} f = B$ if and only if f is surjective.
- f is an isomorphism if and only if $0 \to A \xrightarrow{f} B \to 0$ is exact. This follows from the two statements above.

5.2. **Chain Complexes & Chain Homotopy.**

Definition 5.4. A chain complex is a sequence of abelian groups and homomorphisms

$$\cdots \xrightarrow{\partial_{n+2}} C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots$$

for $n \in \mathbb{Z}$ which satisfies $\partial_n \circ \partial_{n+1} = 0$, for each $n \in \mathbb{Z}$. That is,

$$\operatorname{Im} \partial_{n+1} \subseteq \operatorname{Ker} \partial_n \quad \Longleftrightarrow \quad \partial_n \circ \partial_{n+1} = 0$$

We refer to the entire complex as $(C_\bullet, \partial_\bullet)$ or sometimes just $C_\bullet$. The maps ∂_n are called the boundary operators of the chain complex.

Example 5.5. Let X be a topological space. The chain complex

$$(C_n^\Delta(X), \partial_n^\Delta)_{n\in\mathbb{N}}$$

encountered in Section 4 is a chain complex. We call this the simplicial chain complex. Note that in this example the abelian groups are all zero for negative subscripts; this, however, is not part of the definition in general.

The fact that $\partial_n \circ \partial_{n+1} = 0$ means that

$$\operatorname{Im} \partial_{n+1} \subseteq \operatorname{Ker} \partial_n$$

Elements of $\operatorname{Ker} \partial_n$ are called n-chains and elements of $\operatorname{Im} \partial_{n+1}$ are called n-boundaries.

Definition 5.6. The n-th homology group of the chain complex $C_\bullet$ to be the quotient group

$$H_n = H_n(C_\bullet) = \frac{\operatorname{Ker} \partial_n}{\operatorname{Im} \partial_{n+1}}$$

We will use the notation $[c]$ to denote the class of an element $c \in Z_n$ in H_n.

Example 5.7. Consider the chain complex

$$C_\bullet : \cdots \to 0 \to \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\partial_2} \mathbb{Z} \xrightarrow{\partial_1} 0 \to \cdots,$$

where the chain groups are given by

$$C_1 = \mathbb{Z}, \quad C_2 = \mathbb{Z} \oplus \mathbb{Z}, \qquad C_n = 0 \text{ for } n \neq 1, 2,$$

The homomorphism ∂_2 is defined by $\partial_2(x, y) = 3x + 3y$. Note that we have the following

$$\operatorname{Ker} \partial_n \cong \begin{cases} \mathbb{Z}, & \text{if } n = 1 \text{ or } n = 2, \\ 0, & \text{if } n \neq 1, 2. \end{cases}$$

Similarly, we have

$$\operatorname{Im} \partial_n \cong \begin{cases} 3\mathbb{Z}, & \text{if } n = 2, \\ 0, & \text{if } n \neq 2. \end{cases}$$

Hence

$$H_n(C_\bullet) \cong \begin{cases} \mathbb{Z}_3, & \text{if } n = 1, \\ \mathbb{Z}, & \text{if } n = 2, \\ 0, & \text{if } n \neq 1, 2. \end{cases}$$

We now define the notion of a chain map between chain complexes.

Definition 5.8. Let $(C_\bullet, \partial_\bullet)$ and $(C'_\bullet, \partial'_\bullet)$ be chain complexes of abelian groups. A chain map between $(C_\bullet, \partial_\bullet)$ and $(C'_\bullet, \partial'_\bullet)$ is a sequence of group homomorphisms $f_n : C_n \to C'_n$ for $n \in \mathbb{Z}$ such that the diagram commutes:

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, "f_{n+1}"] & C_n \arrow[r, "\partial_n"] \arrow[d, "f_n"] & C_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \cdots \\
\cdots \arrow[r] & C'_{n+1} \arrow[r, "\partial'_{n+1}"] & C'_n \arrow[r, "\partial'_n"] & C'_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

Proposition 5.9. *Chain complexes of abelian groups form a category, denoted as* $\mathbf{Chain_{Ab}}$.

Proof. Objects in $\mathbf{Chain_{Ab}}$ are chain complexes of abelian groups and a morphism between chain complexes of abelian groups is a chain map. If $(C_\bullet, \partial_\bullet)$, $(C'_\bullet, \partial'_\bullet)$ and $(C''_\bullet, \partial''_\bullet)$ are chain complexes such that $f_\bullet : C_\bullet \to C'_\bullet$ and $g_\bullet : C'_\bullet \to C''_\bullet$ are two chain maps. Then

$$(g \circ f)_\bullet : C_\bullet \to C''_\bullet$$

is the chain map given by $(g \circ f)_n = g_n \circ f_n$. This is indeed a valid chain map as the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, "f_{n+1}"] & C_n \arrow[r, "\partial_n"] \arrow[d, "f_n"] & C_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \cdots \\
\cdots \arrow[r] & C'_{n+1} \arrow[r, "\partial'_{n+1}"] \arrow[d, "g_{n+1}"] & C'_n \arrow[r, "\partial'_n"] \arrow[d, "g_n"] & C'_{n-1} \arrow[r] \arrow[d, "g_{n-1}"] & \cdots \\
\cdots \arrow[r] & C''_{n+1} \arrow[r, "\partial''_{n+1}"] & C''_n \arrow[r, "\partial''_{n+1}"] & C''_{n-1} \arrow[r, "\partial''_{n+1}"] & \cdots
\end{tikzcd}$$

commutes essentially by construction as can be easily checked. This defines the composition of two chain maps. Moreover, the identity chain map

$$\mathrm{Id} : C_\bullet \to C_\bullet$$

is the chain map given by $\mathrm{Id}_n = \mathrm{Id}_{C_n}$ where Id_{C_n} is the identity homomorphism from C_n to C_n.

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, "\mathrm{Id}_{n+1}"] & C_n \arrow[r, "\partial_n"] \arrow[d, "\mathrm{Id}_n"] & C_{n-1} \arrow[r] \arrow[d, "\mathrm{Id}_{n-1}"] & \cdots \\
\cdots \arrow[r] & C_{n+1} \arrow[r, "\partial'_{n+1}"] & C_n \arrow[r, "\partial'_n"] & C_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

All that is required is check that composition of chain maps satisfies the associativity property and the composition of a chain map with the identity chain map yields the original chain map. All these are routine checks. $\square$

Definition 5.10. Suppose $(C_\bullet, \partial)$ and $(C'_\bullet, \partial')$ are two chain complexes with chain maps $f_\bullet, g_\bullet$. A chain homotopy between $f_\bullet, g_\bullet$ is a series of maps $T_n : C_n$ and C_{n+1} such that

$$\begin{aligned} f_n - g_n &= \partial'_{n+1} T_n + T_{n-1}\partial_n \qquad && n \geq 1 \\ T_0 \circ \partial_1 &= f_0 - g_0 && n = 0 \end{aligned}$$

That is, the following diagram commutes

$$\begin{tikzcd} \cdots \arrow[r] \arrow[dl, "T_{n+1}"'] & C_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, "f_{n+1}"', "g_{n+1}"] \arrow[dl] & C_n \arrow[r, "\partial_n"] \arrow[d, "f_n"', "g_n"] \arrow[dl, "T_n"'] & C_{n-1} \arrow[r] \arrow[d, "f_{n-1}"', "g_{n-1}"] \arrow[dl, "T_{n-1}"'] & \cdots \arrow[dl, "T_{n-2}"] \\ \cdots \arrow[r] & C'_{n+1} \arrow[r, "\partial'_{n+1}"'] & C'_n \arrow[r, "\partial'_n"'] & C'_{n-1} \arrow[r] & \cdots \end{tikzcd}$$

Remark 5.11. *In Section 7, we will provide a geometric intuition behind the definition of a chain homotopy.*

Proposition 5.12. *Let $(C_\bullet, \partial)$ and $(C'_\bullet, \partial')$ be two chain complexes. The relation of chain homotopy between these chain complexes is an equivalence relation.*

Proof. Let $f_\bullet$ be a chain map. Define $T_n : C_n \to C'_{n+1}$ to be the zero map. Then

$$T_{n-1}\partial_n + \partial'_{n+1} T_n = 0 = f - f,$$

so f is chain homotopic to itself. Assume $f_\bullet$ is chain homotopic to $g_\bullet$. That is, consider maps $f_n, g_n : C_n \to C'_n$ such that there is a $T_n : C_n \to C'_{n+1}$ such that

$$f_n - g_n = T_{n-1}\partial_n + \partial'_{n+1} T_n$$

Then

$$g_n - f_n = -(f_n - g_n) = -T_{n-1}\partial_n + \partial'_{n+1} T_n = (-T_{n-1})\partial_n + \partial'_{n+1}(-T_n)$$

Therefore $g_\bullet$ is chain homotopic to $f_\bullet$. Finally, suppose $f_\bullet$ is chain homotopic to $g_\bullet$ and that $g_\bullet$ is chain homotopic to $h_\bullet$. Then there exist maps $T_n, S_n : C_n \to C'_{n+1}$ such that

$$f_n - g_n = T_{n-1}\partial_n + \partial'_{n+1} T_n$$

$$g_n - h_n = S_{n-1}\partial_n + \partial'_{n+1} S_n$$

Adding equations, we get

$$f_n - h_n = (T_{n-1} + S_{n-1})\partial_n + \partial'_{n+1}(T_n + S_n)$$

Thus, $f_\bullet$ is chain homotopic to $h_\bullet$. This proves the claim. □

5.3. **Homology of a Chain Complex.** Given a chain complex, $(C_\bullet, \partial_\bullet)$, the condition $\partial_n \circ \partial_{n+1} = 0$ implies that

$$\operatorname{Im} \partial_{n+1} \subseteq \operatorname{Ker} \partial_n$$

This motivates the following definition:

Definition 5.13. Let $(C_\bullet, \partial_\bullet)$ be a chain complex. The n-th homology group is defined as

$$H_n(C_\bullet) := \frac{\operatorname{Ker} \partial_n}{\operatorname{im}\, \partial_{n+1}}.$$

Example 5.14. Consider the chain complex:

$$\cdots \xrightarrow{\partial} \mathbb{Z}/8\mathbb{Z} \xrightarrow{\partial} \mathbb{Z}/8\mathbb{Z} \xrightarrow{\partial} \mathbb{Z}/8\mathbb{Z} \xrightarrow{\partial} \mathbb{Z}/8\mathbb{Z} \to 0 \to 0 \to \cdots$$

where $C_n = \mathbb{Z}/8\mathbb{Z}$ for $n \leq 0$ and $C^n = 0$ for $n > 0$ and the map ∂ is given by $x \bmod 8 \mapsto 4x \bmod 8$. It is easy to see that

$$\operatorname{Ker} \partial = \{\overline{0}, \overline{2}, \overline{4}, \overline{6}\} \cong \mathbb{Z}/4\mathbb{Z}$$
$$\operatorname{Im} \partial = \{\overline{0}, \overline{4}\} \cong \mathbb{Z}/2\mathbb{Z}$$

for $n < 0$. Hence,

$$H_n(C_\bullet) \cong \frac{\mathbb{Z}/4\mathbb{Z}}{\mathbb{Z}/2\mathbb{Z}} \cong \mathbb{Z}/2\mathbb{Z}$$

Trivially, $H_n(C_\bullet) \cong 0$ for $n > 0$ and $H_0(C_\bullet) \cong \mathbb{Z}/4\mathbb{Z}$.

Proposition 5.15. *Let $(C, \partial_\bullet), (C', \partial'_\bullet)$ be chain complexes and let $f_\bullet, g_\bullet$ be chain maps between the chain complexes. If there is a chain homotopy $f_\bullet$ and $g_\bullet$, then the induced maps in homology are equal, i.e., we have:*

$$H_n(f) = H_n(g) : H_n(C_\bullet) \to H_n(C'_\bullet)$$

Proof. Let $(T_n)_{n \geq 1}$ be the sequence of maps defining a chain homotopy. Let $[c] \in H_n(C)$. If $n = 0$, we have

$$H_0(f)([c]) = [f_0(c)] = [g_0(c) + \partial_1 T_0(c)] = [g_0(c)] = H_0(g)([c])$$

For $n \geq 1$, we have:

$$\begin{aligned} H_n(f)([c]) &= [f_n(c)] \\ &= [g_n(c) + \partial'_{n+1} T_n(c) + T_{n-1}\partial_n(c)] \\ &= [g_n(c)] \\ &= H_n(g)([c]) \end{aligned}$$

The third equality uses that c is a n-cycle and that a homology class is not changed if we add a n-boundary. The claim follows. ☐

Proposition 5.16. *There is a functor H_n : $\mathbf{Chain_{Ab}} \to \mathbf{Ab}$ that associates to a chain complex over abelian groups its n-th homology group.*

Proof. Consider a chain map between chain complexes given by the following diagram:

$$\begin{tikzcd} \cdots \arrow[r] & C_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, "f_{n+1}"] & C_n \arrow[r, "\partial_n"] \arrow[d, "f_n"] & C_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \cdots \\ \cdots \arrow[r] & C'_{n+1} \arrow[r, "\partial'_{n+1}"] & C'_n \arrow[r, "\partial'_n"] & C'_{n-1} \arrow[r] & \cdots \end{tikzcd}$$

The relation $f_n \partial_{n+1} = \partial'_{n+1} f_{n+1}$ implies that f_n takes n-cycles to takes n-cycles for each $n \in \mathbb{N}$. This is because if $\partial_n c = 0$, then

$$\partial'_n(f_n(c)) = f_{n-1}(\partial_n c) = 0$$

Also, f_n takes n boundaries to n-boundaries since

$$f_n(\partial_{n+1} c) = \partial'_{n+1}(f_{n+1} c)$$

Hence f_n descends to a homomorphism

$$H_n(f) : H_n(C_\bullet) \to H_n(C'_\bullet)$$

It remains to check that $H_n(g \circ f) = H_n(g) \circ H_n(f)$ and that $H_n(\mathrm{Id}_X) = \mathrm{Id}_{H_n(X)}$. Both of these are immediate from the definitions. $\square$

5.4. **Diagram Chasing Lemmas.** In this section, we discuss some important diagram chasing lemmas in homological algebra. In particular, we discuss the short-five lemma, four-lemma, five-lemma and the snake lemma. The snake lemma allows us to construct a long exact sequence of homology[7] of pairs of topological spaces. We end our discussion by discussing the braid lemma. The braid lemma will allow us to construct a long exact sequence of homology of triples of topological spaces.

Proposition 5.17. ***(Short Five Lemma)*** *Consider the the diagram below:*

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "\alpha"'] & B \arrow[r, "g"] \arrow[d, "\beta"'] & C \arrow[r] \arrow[d, "\gamma"'] & 0 \\
0 \arrow[r] & A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' \arrow[r] & 0
\end{tikzcd}$$

be a commutative diagram with exact rows of abelian groups. We have the following:

(1) *If α and γ are injective homomorphisms, then β is an injective homomorophism.*
(2) *If α and γ are surjective homomorphisms, then β is a surjective homorophism.*
(3) *If α and γ are isomorphisms, then β is an isomorphism.*

Proof. The proof of (2) is similar to that of (1). (3) follows from (1) and (2). We prove (1). Let $b \in B$ such that $\beta(b) = 0$. Clearly, $g'(\beta(b)) = 0$. But by the commutativity of the right-most square, we also have that $\gamma(g(b)) = 0$. Since γ is injective, $g(b) = 0$. Hence $b \in \operatorname{Ker} g = \operatorname{Im} f$. Therefore, there is a $a \in A$ such that $f(a) = b$. By the commutativity of the left-most square, we must have that $f'(\alpha(a)) = 0$. Since f' and α are injective, $a = 0$. Hence, $b = 0$.

$$\begin{tikzcd}
a \arrow[r, mapsto, "f"] \arrow[d, mapsto, "\alpha"] & b \arrow[r, mapsto, "g"] \arrow[d, mapsto, "\beta"] & g(b) \arrow[d, mapsto, "\gamma"'] \\
\alpha(a) \arrow[r, mapsto, "f'"] & 0 \arrow[r, mapsto, "g'"] & 0
\end{tikzcd}$$

This completes the proof. $\square$

Proposition 5.18. ***(Four & Five Lemmas)*** *Consider the the diagram below:*

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\alpha"'] & B \arrow[r, "g"] \arrow[d, "\beta"'] & C \arrow[r, "h"] \arrow[d, "\mu"'] & D \arrow[r, "i"] \arrow[d, "\gamma"'] & E \arrow[d, "\delta"'] \\
A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' \arrow[r, "h'"] & D' \arrow[r, "i'"] & E'
\end{tikzcd}$$

be a commutative diagram with exact rows of abelian groups. We have the following:

(1) ***(Four Lemma I)*** *If α is a surjective homomorphism and β and γ are injective homomorphisms, then μ is an injective homomorphism.*
(2) ***(Four Lemma II)*** *If δ is an injective homomorphism and β and γ are surjective homomorphisms, then μ is a surjective homomorphism.*

[7] And cohomology later on.

(3) ***(Five Lemma)*** *If α is a surjective homomorphism, δ is an injective homomorphism, and β and γ are isomorphisms, then μ is an isomorphism.*

Proof. (3) follows from (1) and (2) and the proof of (2) is similar to that of (1). We only prove (1). Let $c \in C$ such that $\mu(c) = 0$. Note that: $\gamma \circ h(c) = h' \circ \mu(c) = 0$ Since γ is an injective homomorphism, $h(c) = 0$. Therefore, there exists a $b \in B$ such that $g(b) = c$. Note that $g' \circ \beta(b) = \mu \circ g(c) = 0$ Hence, there exists a $a' \in A$ such that $f'(a') = \beta(b)$. Since α is a surjective homomorphism, α, there exists a $a \in A$ such that $\alpha(a) = a'$. Since β is an injective homomorphism, we must have that $f(a) = b$. But then $c = g(b) = f \circ g(a) = 0$.

$$\begin{tikzcd}
a \arrow[r, mapsto, "f"'] \arrow[d, mapsto, "\alpha"] \arrow[rr, bend left, "0"] & b \arrow[r, mapsto, "g"'] \arrow[d, mapsto, "\beta"] & c \arrow[r, mapsto, "h"'] \arrow[d, mapsto, "\mu"] & h(c) \arrow[d, mapsto, "\gamma"] \\
a' \arrow[r, mapsto, "f'"] & \beta(b) \arrow[r, mapsto, "g'"] & 0 \arrow[r, mapsto, "h'"] & 0
\end{tikzcd}$$

This completes the proof. $\square$

Where are we headed? We would like to consider a short exact sequence of chain complexes. A short exact sequence of chain complexes is a commutative diagram of the form:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & A_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, "i_{n+1}"] & A_n \arrow[r, "\partial_n"] \arrow[d, "i_n"] & A_{n-1} \arrow[r, "\partial_{n-1}"] \arrow[d, "i_{n-1}"] & \cdots \\
\cdots \arrow[r] & B_{n+1} \arrow[r, "\partial'_{n+1}"] \arrow[d, "j_{n+1}"] & B_n \arrow[r, "\partial'_n"] \arrow[d, "j_n"] & B_{n-1} \arrow[r, "\partial'_{n-1}"] \arrow[d, "j_{n-1}"] & \cdots \\
\cdots \arrow[r] & C_{n+1} \arrow[r, "\partial''_{n+1}"] \arrow[d] & C_n \arrow[r, "\partial''_n"] \arrow[d] & C_{n-1} \arrow[r, "\partial''_{n-1}"] \arrow[d] & \cdots \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

We abbreviate the diagram as

$$0_\bullet \longrightarrow A_\bullet \xrightarrow{i_\bullet} B_\bullet \xrightarrow{j_\bullet} C_\bullet \longrightarrow 0_\bullet$$

The snake lemma will be a statement about short exact sequences in $\mathbf{Chain_{Ab}}$.

We can define the category of short exact sequence of chain complexes, $\mathbf{Chain_{Ab}^{Exact}}$. Objects in $\mathbf{Chain_{Ab}^{Exact}}$ are short exact sequences of chain complexes. A morphism between short exact sequence of chain complexes is a diagram

$$\begin{tikzcd}
0_\bullet \arrow[r] & A_\bullet \arrow[r, "i_\bullet"] \arrow[d, "f_\bullet"] & B_\bullet \arrow[r, "j_\bullet"] \arrow[d, "g_\bullet"] & C_\bullet \arrow[r] \arrow[d, "h_\bullet"] & 0_\bullet \\
0_\bullet \arrow[r] & A'_\bullet \arrow[r, "i'_\bullet"] & B'_\bullet \arrow[r, "j'_\bullet"] & C'_\bullet \arrow[r] & 0_\bullet
\end{tikzcd}$$

such that $f_\bullet, g_\bullet, h_\bullet$ are chain maps. We will not go through the pain of writing the diagram out explicitly.

Proposition 5.19. *(**Snake Lemma**) Consider a short exact sequence in* $\mathbf{Chain_{Ab}}$*:*

$$0_\bullet \longrightarrow A_\bullet \xrightarrow{i_\bullet} B_\bullet \xrightarrow{j_\bullet} C_\bullet \longrightarrow 0_\bullet$$

For each $n \geq 1$, there exist connecting morphisms

$$\delta_n : H_n(C_\bullet) \to H_{n-1}(A_\bullet)$$

such that there is a long exact sequence of homology groups:

$$\begin{tikzcd}
\cdots \arrow[r] & H_{n+1}(B_\bullet) \arrow[r] & H_{n+1}(C_\bullet) \arrow[dll, "\delta_{n+1}"] \\
H_n(A_\bullet) \arrow[r] & H_n(B_\bullet) \arrow[r] & H_n(C_\bullet) \arrow[dll, "\delta_n"] \\
H_{n-1}(A_\bullet) \arrow[r] & H_{n-1}(B_\bullet) \arrow[r] & \cdots
\end{tikzcd}$$

In fact, the above construction defines a functor from $\mathbf{Chain_{Ab}^{Exact}}$ *to* $\mathbf{Ab^{Long}}$*, the category of long exact sequences of abelian groups.*

Proof. The short exact sequence of chain complexes can be drawn more explicitly as:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & A_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, "i_{n+1}"] & A_n \arrow[r, "\partial_n"] \arrow[d, "i_n"] & A_{n-1} \arrow[r, "\partial_{n-1}"] \arrow[d, "i_{n-1}"] & \cdots \\
\cdots \arrow[r] & B_{n+1} \arrow[r, "\partial'_{n+1}"] \arrow[d, "j_{n+1}"] & B_n \arrow[r, "\partial'_n"] \arrow[d, "j_n"] & B_{n-1} \arrow[r, "\partial'_{n-1}"] \arrow[d, "j_{n-1}"] & \cdots \\
\cdots \arrow[r] & C_{n+1} \arrow[r, "\partial''_{n+1}"] \arrow[d] & C_n \arrow[r, "\partial''_n"] \arrow[d] & C_{n-1} \arrow[r, "\partial''_{n-1}"] \arrow[d] & \cdots \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Since $i_\bullet$ and $j_\bullet$ are chain maps, the homology functor maps induce maps on the homology groups which we write as $H_n(i)$ and $H_n(j)$ for each $n \in \mathbb{N}$. The connecting morphisms δ_n are constructed as follows:

Let $c \in C_n$ be a cycle representative for $[\alpha] \in H_n(C)$. Then, since j_n is surjective, there exists $b \in B_n$ such that $c = j_n(b)$. Therefore, we have that $\partial'_n(b) \in B_{n-1}$. By the commutativity of the diagram, we know that

$$j_{n-1}(\partial''_n(b)) = \partial''_n(j_{n-1}(b)) = \partial''_n(c) = 0,$$

since c is a cycle. Therefore, $\partial'_n(b) \in \operatorname{Ker} j_{n-1} = \operatorname{Im} i_{n-1}$. So, there exists a (unique, since i_{n-1} is injective) $a \in A_{n-1}$ with $\partial'_n(b) = i_{n-1}(a)$. We show that a is a cycle. Note that

$$i_{n-2}(\partial_{n-1}(a)) = \partial'_{n-1}(i_{n-1}(a)) = \partial'_{n-1}(\partial'_n(b)) = 0$$

Since i_{n-2} is injective, this implies that $\partial_{n-1}(a) = 0$. Finally, we define $\delta_n([\alpha]) = [a] \in H_{n-1}(A)$. We have to show that this assignment is independent of all choices.

(1) Suppose we choose $b' \in B_n$ such that $j_n(b') = c$. Then, $b' - b \in \ker j_n = \text{Im } i_n$. So, there exists $a' \in A_n$ such that $b' - b = i_n(a')$. Therefore,

$$\begin{aligned} \partial'_n(b') &= \partial'_n(b) + \partial'_n(i_n(a')) \\ &= \partial(b) + i_{n-1}(\partial_n(a')) \\ &= i_{n-1}(a) + i_{n-1}(\partial_n(a')) \\ &= i_{n-1}(a + \partial_n(a')). \end{aligned}$$

So we see that changing b to b' amounts to changing a by a homologous cycle $a + \partial_n(a')$.

(2) If instead of c we use $c + \partial''_{n+1}(c')$ for some $c' \in C_{n+1}$. But then, $c' = j_{n+1}(b')$ for some $b' \in B_{n+1}$. So,

$$\begin{aligned} c + \partial''_{n+1}(c') &= c + \partial''_{n+1}(j_{n+1}(b')) \\ &= c + j_n(\partial'_{n+1}(b')) \\ &= j_n(b + \partial'_{n+1}(b')) \end{aligned}$$

Then b will be replaced by $b + \partial'_{n+1}(b')$, which leaves $\partial'_n(b)$ unchanged, hence a unchanged.

We now prove that the following sequence is exact:

$$\cdots \longrightarrow H_n(A) \xrightarrow{H_n(i)} H_n(B) \xrightarrow{H_n(j)} H_n(C) \xrightarrow{\delta_n} H_{n-1}(A) \xrightarrow{H_{n-1}(i_{n-1})} H_{n-1}(B) \longrightarrow \cdots$$

(1) Im $H_n(i) \subseteq$ Ker $H_n(j)$. This is immediate since $j_n \circ i_n = 0$ implies $H_n(j) \circ H_n(i) = 0$.

(2) Im $H_n(j) \subseteq$ Ker δ_n. We have $\delta_n \circ H_n(j) = 0$ since in this case $\partial'_n b = 0$ in the definition of δ_n.

(3) Im $\delta_n \subseteq$ Ker $i_{n-1,*}$. This follows because $H_{n-1}(i_{n-1}) \circ \delta_{n-1,*}$ takes $[c]$ to $[\partial'_n b] = 0$.

(4) Ker $H_{n-1}(i_{n-1}) \subseteq$ Im δ_n. Given a cycle $a \in A_{n-1}$ such that $i_{n-1}(a) = \partial'_n b$ for some $b \in B_n$, $j_n(b)$ is a cycle since $\partial''_n j_n(b) = j_{n-1}(\partial'_n b) = j_{n-1} i_{n-1}(a) = 0$, and δ_n takes $[j_n(b)]$ to $[a]$.

(5) Ker $H_n(j) \subseteq$ Im $H_n(i)$. A homology class in Ker $H_n(j)$ is represented by a cycle $b \in B_n$ with $j_n(b)$ a boundary, so $j_n(b) = \partial''_{n+1} c'$ for some $c' \in C_{n+1}$. Since j_{n+1} is surjective, $c' = j_{n+1}(b')$ for some $b' \in B_{n+1}$. We have

$$\begin{aligned} j_n(b - \partial'_{n+1} b') &= j_n(b) - j_n(\partial'_{n+1} b') \\ &= j_n(b) - \partial''_{n+1} j_{n+1}(b') \\ &= 0 \end{aligned}$$

since $\partial''_{n+1} j_{n+1}(b') = \partial'_{n+1} c' = j_n(b)$. So $b - \partial'_{n+1} b' = i_n(a)$ for some $a \in A_n$. This a is a cycle since

$$\begin{aligned} i_{n-1}(\partial_n a) &= \partial'_n i_n(a) \\ &= \partial'_n (b - \partial'_{n+1} b') \\ &= \partial'_n b \\ &= 0 \end{aligned}$$

and i_{n-1} is injective. Thus $H_n(i)[a] = [b - \partial'_{n+1} b'] = [b]$, showing that $H_n(i)$ maps onto Ker $H_n(j)$.

(6) Ker $\delta_n \subseteq$ Im $H_n(j)$. Given a cycle $c \in C_n$ such that $\delta_n(c) = 0$, we have that $\delta_n(c) = \partial_n(a)$ for some $a \in A_n$. Let $b \in B_n$ be the element constructed in the definition of δ_n. The element $b - i_n(a)$ is a cycle since

$$\begin{aligned} \partial'_n(b - i_n(a)) &= \partial'_n b - \partial'_n i_n(a) \\ &= \partial'_n b - i_{n-1}(\partial_n a) \\ &= \partial'_n b - i_{n-1}(\delta_n(c)) \\ &= 0 \end{aligned}$$

Note that:

$$j_n(b - i_n(a)) = j_n(b) - j_n i_n(a') = j_n(b) = c$$

So $H_n(j)$ maps $[b - i_n(a)]$ to $[c]$.

We now show that the above construction defines a functor from $\mathbf{Chain}_{\mathbf{Ab}}^{\mathbf{Exact}}$ to $\mathbf{Ab}^{\mathbf{Long}}$. Consider the following diagram in $\mathbf{Chain}_{\mathbf{Ab}}^{\mathbf{Exact}}$:

$$\begin{tikzcd} 0_\bullet \arrow[r] & A_\bullet \arrow[r, "i_\bullet"] \arrow[d, "f_\bullet"] & B_\bullet \arrow[r, "j_\bullet"] \arrow[d, "g_\bullet"] & C_\bullet \arrow[r] \arrow[d, "h_\bullet"] & 0_\bullet \\ 0_\bullet \arrow[r] & A'_\bullet \arrow[r, "i'_\bullet"] & B'_\bullet \arrow[r, "j'_\bullet"] & C'_\bullet \arrow[r] & 0_\bullet \end{tikzcd}$$

We show that induces the following commutative diagram.

$$\begin{tikzcd} \cdots \arrow[r] & H_n(A) \arrow[r, "H_n(i)"] \arrow[d, "H_n(f_n)"] & H_n(B) \arrow[r, "H_n(j)"] \arrow[d, "H_n(g_n)"] & H_n(C) \arrow[r, "\delta_n"] \arrow[d, "H_n(h_n)"] & H_{n-1}(A) \arrow[r, "H_{n-1}(i_{n-1})"] \arrow[d, "H_{n-1}(f_{n-1})"] & H_{n-1}(B) \arrow[r] \arrow[d, "H_{n-1}(g_{n-1})"] & \cdots \\ \cdots \arrow[r] & H_n(A') \arrow[r, "H_n(i'_n)"] & H_n(B') \arrow[r, "H_n(j'_n)"] & H_n(C') \arrow[r, "\delta'_n"] & H_{n-1}(A') \arrow[r, "H_n(i'_{n-1})"] & H_{n-1}(B') \arrow[r] & \cdots \end{tikzcd}$$

The commutativity of the first two squares and the last square is obvious since n-th homology is a functor. It suffices to check that the the diagram

$$\begin{tikzcd} H_n(C) \arrow[r, "\delta_n"] \arrow[d, "H_n(h_n)"'] & H_{n-1}(A) \arrow[d, "H_{n-1}(f_{n-1})"] \\ H_n(C') \arrow[r, "\delta'_n"] & H_{n-1}(A') \end{tikzcd}$$

is commutative. Recall that the map $\delta_n : H_n(C) \to H_{n-1}(A)$ was defined by $\delta_n[c] = [a]$ where $c = j_n(b)$ and $i_{n-1}(a) = \partial'_n b$. Consider $h_n(c) \in C'_n$. Note that

$$h_n(c) = h_n(j_n(b)) = j'_n(g_n(b))$$
$$i'_{n-1}(f_{n-1}(a)) = g_{n-1}(i_{n-1}(a)) = g_{n-1}(\partial'_n(b)) = d'_n(g_n(b)).$$

Here d'_n is the map from B'_n to B'_{n-1}. Hence,

$$[\delta'_n h_n(c))] = [f_{n-1}(a)] = [f_{n-1}\delta_n(c)]$$

This shows that the construction defines a functor from the $\mathbf{Chain}_{\mathbf{Ab}}^{\mathbf{Exact}}$ to $\mathbf{Ab}^{\mathbf{Long}}$. $\square$

Proposition 5.20. *(**Braid Lemma**) Suppose three long exact sequences and a chain complex we have a commutative diagram. Then the chain complex is also a long exact sequence*

$$\begin{tikzcd}
\cdots \arrow[rr] \arrow[dr] & & A \arrow[rr, "g_1"] \arrow[dr, "f_1"] & & D \arrow[rr, "h_3"] \arrow[dr, "g_2"] & & G \arrow[rr, "j_4"] \arrow[dr, "h_4"] & & J \arrow[dr] & \\
\cdots & O \arrow[ur, "g_0"] \arrow[dr, "j_0"] & & C \arrow[ur, "h_2"] \arrow[dr, "f_2"] & & F \arrow[ur, "j_3"] \arrow[dr, "g_3"] & & I \arrow[ur, "f_5"] \arrow[dr, "h_5"] & & \cdots \\
\cdots \arrow[rr] \arrow[ur] & & B \arrow[rr, "j_1"] \arrow[ur, "h_1"] & & E \arrow[rr, "f_3"] \arrow[ur, "j_2"] & & H \arrow[rr, "g_4"] \arrow[ur, "f_4"] & & K \arrow[ur] &
\end{tikzcd}$$

Proof. WLOG, assume that the f maps describe the chain complex. By symmetry, it suffices to show exactness at C, E and H:

$\ker(f_2) \subseteq \operatorname{Im}(f_1)$: Let $x \in \ker(f_2)$. Then $0 = f_2(x) = j_2 f_2(x) = g_2 h_2(x)$ by commutativity. It follows that $h_2(x) \in \ker(g_2) = \operatorname{Im}(g_1)$. So $\exists x_1 \in A$ such that $g_1(x_1) = h_2(x)$. By commutativity, $g_1(x_1) = h_2 f_1(x_1)$. So we have that $0 = g_1(x_1) - h_2(x) = h_2(f_1(x_1) - x)$. Let $x_2 := f_1(x_1) - x \in \ker(h_2) = \operatorname{Im}(h_1)$. Then $\exists x_3 \in B$ such that $h_1(x_3) = x_2$. Note that

$$j_1(x_3) = f_2 h_1(x_3) = f_2(x_2) = f_2(f_1(x_1) - x) = 0,$$

where the last equality follows from $f_2 \circ f_1 = 0$ and $f_2(x) = 0$. We therefore have that $x_3 \in \ker(j_1) = \operatorname{Im}(j_0)$. So there exists $x_4 \in O$ such that $j_0(x_4) = x_3$. Consider$g_0(x_4)$. It satisfies

$$f_1 g_0(x_4) = h_1 j_0(x_4) = h_1(x_3) = x_2 = f_1(x_1) - x$$

Therefore, we have $x = f_1(x_1 - g_0(x_4))$. This shows that $x \in \operatorname{Im}(f_1)$.

$$\begin{tikzcd}
 & g_0(x_4), x_1 \arrow[rr, mapsto, "g_1"] \arrow[dr, "f_1"] & & h_2(x) \arrow[dr, mapsto, "g_2"] & \\
x_4 \arrow[ur, "g_0"] \arrow[dr, "j_0"] & & x_2, x \arrow[ur, mapsto, "h_2"] \arrow[dr, mapsto, "f_2"] & & 0 \\
 & x_3 \arrow[ur, mapsto, "h_1"] \arrow[rr, "j_1"] & & 0 \arrow[ur, mapsto, "j_2"] &
\end{tikzcd}$$

$\ker(f_3) \subseteq \operatorname{Im}(f_2)$: Let $x \in E$ be such that $f_3(x) = 0$. By commutativity, $g_3 j_2(x) = 0$, so $j_2(x) \in \ker(g_3) = \operatorname{Im}(g_2)$. Then $\exists x_1 \in D$ such that $g_2(x_1) = j_2(x)$. It satisfies $h_3(x_1) = j_3 g_2(x_1) = j_3 j_2(x) = 0$, as (j_i) is a chain complex. So $x_1 \in \ker(h_3) = \operatorname{Im}(h_2)$. Therefore, there exists $x_2 \in C$ such that $h_2(x_2) = x_1$. This element is such that $j_2 f_2(x_2) = g_2 h_2(x_2) = g_2(x_1) = j_2(x)$. We therefore have $j_2(f_2(x_2) - x) = 0$. Let $x_3 := f_2(x_2) - x$.

Then $x_3 \in \ker(j_2) = \mathrm{Im}(j_1)$. Let $x_4 \in B$ be such that $j_1(x_4) = x_3$. x_4 is such that $f_2h_1(x_4) = j_1(x_4) = x_3 = f_2(x_2) - x$. Finally, we see that $x = f_2(x_2 - h_1(x_4))$, so $x \in \mathrm{Im}(f_2)$ as required.

$$\begin{tikzcd}
 & & x_1 \arrow[rr, mapsto, "h_3"] \arrow[dr, mapsto, "g_2"] & & 0 \\
 & h_1(x_4), x_2 \arrow[ur, mapsto, "h_2"] \arrow[dr, mapsto, "f_2"] & & 0, j_2(x) \arrow[ur, mapsto, "j_3"] \arrow[dr, mapsto, "g_3"] & \\
x_4 \arrow[ur, mapsto] \arrow[rr, mapsto, "j_1"] & & x_3, x \arrow[ur, mapsto, "j_2"] \arrow[rr, mapsto, "f_3"] & & 0
\end{tikzcd}$$

$\ker(f_4) \subseteq \mathrm{Im}(f_3)$: Let $x \in H$ be such that $f_4(x) = 0$. Then $0 = h_5f_4(x) = g_4(x)$. So $x \in \ker(g_4) = \mathrm{Im}(g_3)$. Let $x_1 \in F$ be such that $g_3(x_1) = x$. Then $h_4j_3(x_1) = f_4g_3(x_1) = f_4(x) = 0$. So $j_3(x_1) \in \ker(h_4) = \mathrm{Im}(h_3)$. Let $x_2 \in D$ be such that $h_3(x_2) = j_3(x_1)$. Then $j_3(x_1) = j_2g_2(x_2)$, such that $x_3 := g_2(x_2) - x_1 \in \ker(j_3) = \mathrm{Im}(j_2)$. Let $x_4 \in E$ be such that $j_2(x_4) = x_3$. Then $f_3(x_4) = g_3j_2(x_4) = g_3(x_3) = g_3(g_2(x_2) - x_1) = -g_3(x_1) = -x$. Therefore, $x = f_3(-x_4)$, and $x \in \mathrm{Im}(f_3)$ as required.

$$\begin{tikzcd}
x_2 \arrow[rr, mapsto, "h_3"] \arrow[dr, "g_2"] & & j_3(x_1) \arrow[dr, mapsto, "h_4"] & & \\
 & x_3, x_1 \arrow[ur, mapsto, "j_3"] \arrow[dr, mapsto, "g_3"] & & 0 \arrow[dr, mapsto, "h_5"] & \\
x_4 \arrow[ur, mapsto, "j_2"] \arrow[rr, mapsto, "f_3"] & & -x, x \arrow[ur, mapsto, "f_4"] \arrow[rr, mapsto, "g_4"] & & 0
\end{tikzcd}$$

□

6. Singular Homology

In this section, we define singular homology. Singular homology is difficult to compute, but singular homology has nice theoretical properties which allows us to prove a host of properties about a homology theory. It can be checked that simplicial homology and singular homology is coincide as we will do later on. Hence, simplicial homology provides a computational tool to compute homology, and singular homology provides a theoretical tool to study homology theoretically.

Definition 6.1. Let X be a topological space. A singular n-simplex is a continuous map $\sigma : \Delta^n \to X$.

Example 6.2. Since Δ^0 is a point, a 0-simplex in X is simply a point in X. Since Δ^1 is a closed interval, a 1-simplex is a path in X.

Remark 6.3. *The phrase 'singular' is used here to express the idea that σ need not be an embedding or a homeomorphism but can have 'singularities' where its image does not look at all like a simplex. All that is required is that σ be continuous.*

Let X be a topological space and let $C_n(X)$ be the free abelian group with basis the set of singular n-simplices in X:

$$C_n(X) = \left\{ \sum_{i=0}^{n} n_i \sigma_i : n_i \in \mathbb{Z}, \sigma_i : \Delta^n \to X \text{ continuous} \right\}$$

where each formal sum $\sum_{i=0}^{n} n_i\sigma_i$ is finite, i.e., all but finitely many n_i are zero. Elements of $C_n(X)$, called n-chains.

Remark 6.4. *Let $\sigma : \Delta^n \to X$ be a singular n-simplex in X. If we restrict σ to one of the faces of Δ^n, we get a continuous map from an $n-1$-simplex into X. Is this a singular $n-1$ simplex? While any face of Δ^n is an $(n-1)$-simplex, it is not the* standard *$(n-1)$-simplex, since the domain is wrong. Thus, strictly speaking, the restriction of an n-simplex σ in X to a face is not actually a singular $(n-1)$-simplex in X, since it is not a continuous map from Δ^{n-1} into X. This issue can be avoided as follows. Consider the map:*

$$d_i^n : \Delta^{n-1} \to \Delta^n, \quad (t_0, \dots, t_{n-1}) \mapsto (t_0, \dots, t_{i-1}, 0, t_i, \dots, t_n),$$

for each $0 \leq i \leq n$. The image $d_i(\Delta^{n-1}) \subseteq \Delta^n$ can be identified with the the i-th face of Δ^n. If $\sigma : \Delta^n \to X$ is a singular n-simplex, then composition $\sigma \circ d_i$ is then a singular $(n-1)$-simplex in X. For the most part, however, we shall ignore this pedantic issue because after all, it's clear what we mean. For the most part, we shall use the notation

$$\sigma|_{[v_0,\dots,\widehat{v_i},\dots,v_n]}$$

to refer to the map d_i.

Definition 6.5. Let X be a topological space and let $C_n(X)$ be the free abelian group with basis the set of singular n-simplices in X. The n-th boundary map

$$\partial_n : C_n(X) \to C_{n-1}(X)$$

is defined on the basis of $C_n(X)$ by the formula

$$\partial_n(\sigma) = \sum_{i=0}^{n} (-1)^i \sigma \circ d_i$$

Lemma 6.6. *Let X be a topological space. The composition*

$$\partial_{n-1} \circ \partial_n : C_n(X) \xrightarrow{\partial_n} C_{n-1}(X) \xrightarrow{\partial_{n-1}} C_{n-2}(X)$$

is zero.

Proof. The crucial observation about the maps d_i's we need is that for every $n \geq 2$ and every $0 \leq j < i \leq n$, we have:

$$d_i^n \circ d_j^{n-1} = d_j^n \circ d_{i-1}^{n-1} : \Delta^{n-2} \to \Delta^n$$

Indeed, it is easy to verify that both maps are given by

$$(t_0, t_1, \dots, t_{n-2}) \mapsto (t_0, \dots, t_{j-1}, 0, t_j, \dots, t_{i-1}, 0, t_i, \dots, t_{n-2})$$

Note that

$$\begin{aligned}
\partial_{n-1} \circ \partial_n(\sigma) &= \sum_{j=0}^{n-1} \sum_{i=0}^{n} (-1)^{i+j} \sigma \circ d_i^n \circ d_j^{n-1} \\
&= \sum_{j=0}^{n-1} \sum_{i=0}^{j} (-1)^{i+j} d_i^n \circ d_j^{n-1} + \sum_{j=0}^{n-1} \sum_{i=j+1}^{n} (-1)^{i+j} d_i^n \circ d_j^{n-1} \\
&= \sum_{j=0}^{n-1} \sum_{i=0}^{j} (-1)^{i+j} d_i^n \circ d_j^{n-1} + \sum_{j=0}^{n-1} \sum_{i=j+1}^{n} (-1)^{i+j} d_j^n \circ d_{i-1}^{n-1} \\
&= \sum_{j=0}^{n-1} \sum_{i=0}^{j} (-1)^{i+j} d_i^n \circ d_j^{n-1} + \sum_{j=0}^{n-1} \sum_{i=j}^{n-1} (-1)^{i+j+1} d_j^n \circ d_i^{n-1} \\
&= \sum_{j=0}^{n-1} \sum_{i=0}^{j} (-1)^{i+j} d_i^n \circ d_j^{n-1} - \sum_{j=0}^{n-1} \sum_{i=j}^{n-1} (-1)^{i+j} d_j^n \circ d_i^{n-1}
\end{aligned}$$

The second last equality follows by a shift of the inner summation index in the second nested sum. If we now interchange the roles of i and j in the second sum, the two nested sums cancel. $\square$

Remark 6.7. *In what follows, we shall write the boundary operator as*

$$\partial_n(\sigma) = \sum_{i=0}^{n} (-1)^i \sigma|_{[v_0,\ldots,\widehat{v_i},\ldots,v_n]}$$

Note that:

$$\sum_{j<i} (-1)^i (-1)^j \sigma \mid_{[v_0,\cdots,\widehat{v_j},\cdots,\widehat{v_i},\cdots,v_n]} + \sum_{j>i} (-1)^i (-1)^{j-1} \sigma \mid_{[v_0,\cdots,\widehat{v_i},\cdots,\widehat{v_j},\cdots,v_n]} = 0.$$

The latter two summations cancel since after switching i *and* j *in the second sum, it becomes the negative of the first.*

Purely algebraically, we have a sequence of homomorphisms of abelian groups:

$$\cdots \xrightarrow{\partial_{n+1}} C_n(X) \xrightarrow{\partial_n} C_{n-1}(X) \xrightarrow{\partial_{n-1}} C_{n-2}(X) \xrightarrow{\partial_{n-2}} \cdots .$$

The boundary map $\partial_n : C_n(X) \longrightarrow C(X)_{n-1}$ is such that

$$\partial_n \circ \partial_{n+1} = 0$$

That is:

$$\mathrm{Im}(\partial_{n+1}) \subseteq \mathrm{Ker}(\partial_n)$$

A sequence $(C_n(X), \partial_n)_{n\in\mathbb{N}}$ satisfying these properties is is called a chain complex. Elements of $\mathrm{Ker}(\partial_n)$ are called (singular) n-cycles and elements of $\mathrm{Im}(\partial_{n+1})$ are called (singular) n-boundaries.

Definition 6.8. Let X be a topological space. The n-th homology of the chain complex $(C_n(X), \partial_n)_{n\in\mathbb{N}}$ is

$$H_n(X;\mathbb{Z}) = \frac{\mathrm{Ker}(\partial_n)}{\mathrm{Im}(\partial_{n+1})}$$

$H_n(X)$ is called the n-th singular homology group of X with $\mathbb{Z}$ coefficients.

Calculation with singular homology is difficult because each C_n is generally a free abelian group on uncountably many generators! Eventually, however, we will show that simplicial homology and singular homology are isomorphic.

Remark 6.9. *We will also introduce cellular homology which is isomorphic to singular homology and is amenable to computation.*

Here is a trivial computation:

Example 6.10. (**Singular Homology of a Point**) If X is a single point, then there is exactly one map $\Delta_n \to X$, and it is continuous, so $C_n(X) = \mathbb{Z}$ for all n. Moreover,

$$\partial_n(\sigma_n) = \sum_{i=0}^{n-1}(-1)^i \sigma_{n-1} = \begin{cases} 0 & \text{for } n \text{ odd} \\ \sigma_{n-1} & \text{for } n \text{ even} \end{cases}$$

We end up with:

$$\cdots \xrightarrow{\cong} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \longrightarrow 0$$

Thus, we can quotient out to get the homology:

$$H_n(X;\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{for } n = 0 \\ 0 & \text{for } n \geq 1 \end{cases}$$

On the other hand, singular homology is much nicer theoretically, because we don't have to worry about choosing a Δ-complex structure, so it provides a convenient tool to prove various properties about a homology theory. For instance:

Proposition 6.11. *Let X be a topological space.*

(1) *Let $(X_\alpha)_{\alpha \in A}$ are the path-connected components of X. Then,*

$$H_n(X;\mathbb{Z}) \cong \bigoplus_{\alpha \in A} H_n(X_\alpha;\mathbb{Z})$$

(2) *(0**-th Singular Homology Groups**) If X is nonempty and path-connected, then $H_0(X) \cong \mathbb{Z}$. Hence, for any space X, $H_0(X;\mathbb{Z})$ is a direct sum of $\mathbb{Z}$'s, one for each path-component of X.*

Proof. The proof is given below:

(1) Since Δ^n is path-connected, and an n-simplex $\sigma : \Delta^n \to X$ is a continuous map, we have that $\operatorname{Im}(\sigma) \subseteq X_\alpha$ for some α. Therefore, we get a decomposition:

$$C_n(X) \cong \bigoplus_{\alpha} C_n(X_\alpha).$$

The boundary maps preserve this decomposition, i.e., $\partial_n(C_n(X_\alpha)) \subseteq C_{n-1}(X_\alpha)$. Hence $\ker(\partial_n)$ and $\operatorname{Im}(\partial_{n+1})$ split similarly as direct sums, and the result follows.

(2) By definition, $H_0(X;\mathbb{Z}) = C_0(X)/\operatorname{Im}\partial_1$. Define a homomorphism

$$\varepsilon : C_0(X) \to \mathbb{Z} \qquad \varepsilon\left(\sum_{i=0}^{n} n_i\sigma_i\right) = \sum_{i=0}^{n} n_i$$

This is obviously surjective if X is non-empty. We claim that $\operatorname{Ker}\varepsilon = \operatorname{Im}\partial_1$ if X is path-connected. Observe first that $\operatorname{Im}\partial_1 \subseteq \operatorname{Ker}\varepsilon$ since for a singular 1-simplex $\sigma : \Delta^1 \to X$, we have

$$\varepsilon\partial_1(\sigma) = \varepsilon(\sigma\big|_{[v_1]} - \sigma\big|_{[v_0]}) = 1 - 1 = 0$$

For the reverse inclusion, $\operatorname{Ker}\varepsilon \subseteq \operatorname{Im}\partial_1$, suppose $\varepsilon\left(\sum_{i=0}^n n_i\sigma_i\right) = 0$, so $\sum_{i=0}^n n_i = 0$. The σ_i's are singular 0-simplices, which are simply points of X. Choose a path $\tau_i : I \to X$ from a basepoint, x_0, to $\sigma_i(v_0)$, and let σ_0 be the singular 0-simplex with image x_0. We can view τ_i as a singular 1-simplex, a map $\tau_i : [v_0, v_1] \to X$, and then we have $\partial\tau_i = \sigma_i - \sigma_0$. Hence,

$$\partial\left(\sum_{i=0}^n n_i\tau_i\right) = \sum_{i=0}^n n_i\sigma_i - \sum_{i=0}^n n_i\sigma_0 = \sum_{i=0}^n n_i\sigma_i,$$

since $\sum_{i=0}^n n_i = 0$. Thus, $\sum_{i=0}^n n_i\sigma_i$ is a boundary, which shows that $\operatorname{Ker}\varepsilon \subseteq \operatorname{Im}\partial_1$. Hence, ε induces an isomorphism $H_0(X;\mathbb{Z}) \cong \mathbb{Z}$.

This completes the proof. □

Remark 6.12. *It is often very convenient to have a slightly modified version of homology for which a point has trivial homology groups in all dimensions, including zero. This is done by defining the* reduced homology *groups $\widetilde{H}_n(X)$ to be the homology groups of the augmented chain complex:*

$$\cdots \to C_2(X) \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\epsilon} \mathbb{Z} \to 0$$

where

$$\epsilon\left(\sum_{i=0}^n n_i\sigma_i\right) = \sum_{i=0}^n n_i$$

Since $\epsilon \circ \partial_1 = 0$, ϵ vanishes on $\operatorname{Im}\partial_1$ and hence induces a map $H_0(X) \to \mathbb{Z}$ with kernel $\widetilde{H}_0(X)$, so

$$H_0(X;\mathbb{Z}) \cong \widetilde{H}_0(X;\mathbb{Z}) \oplus \mathbb{Z}$$

Why all the fuss about singular homology? Singular homology defines a functor from **Top** to **Ab**. Thus, singular homology yields an invariant that can distinguish spaces.

Proposition 6.13. *H_n : **Top** $\to$ **Ab** is a covariant functor for each $n \geq 0$.*

Proof. Let X, Y be topological spaces and let $f : X \to Y$ be a continuous map. Then, we have a sequence of induced homomorphisms:

$$f_n : C_n(X) \to C_n(Y) \qquad f_n(\sigma) = f \circ \sigma : \Delta^n \to X$$

Extending linearly gives a group homomorphism.

$$f_n\left(\sum_{i=0}^n n_i\sigma_i\right) = \left(\sum_{i=0}^n n_i f_n\sigma_i\right) = \left(\sum_{i=0}^n n_i f \circ \sigma_i\right)$$

The following diagram

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & C_{n+1}(X) & \xrightarrow{\partial_{n+1}} & C_n(X) & \xrightarrow{\partial_n} & C_{n-1}(X) & \longrightarrow & \cdots \\
 & & \downarrow f_{n+1} & & \downarrow f_n & & \downarrow f_{n-1} & & \\
\cdots & \longrightarrow & C_{n+1}(Y) & \xrightarrow{\partial'_{n+1}} & C_n(Y) & \xrightarrow{\partial'_n} & C_{n-1}(Y) & \longrightarrow & \cdots
\end{array}$$

commutes because:

$$\begin{aligned}
f_n(\partial_{n+1}\sigma) &= f_n\left(\sum_{i=0}^{n}(-1)^i \sigma_\alpha|_{[v_0,\ldots,\widehat{v_i},\ldots,v_n]}\right) \\
&= \sum_{i=0}^{n}(-1)^i f_n \sigma_\alpha|_{[v_0,\ldots,\widehat{v_i},\ldots,v_n]} \\
&= \sum_{i=0}^{n}(-1)^i f \circ \sigma_\alpha|_{[v_0,\ldots,\widehat{v_i},\ldots,v_n]} = \partial'_{n+1}(f_{n+1}\sigma).
\end{aligned}$$

Hence, we have a functor from **Top** to $\mathbf{Chain_{Ab}}$. By Proposition 5.16, we have a functor from $\mathbf{Chain_{Ab}}$ to **Ab**. Composing the two functors yields the desired functor. □

7. The Eilenberg-Steenrod Axioms

We have met two homology theories: simplicial homology and singular homology. Later on, we will discuss cellular homology. In fact, there are many other homology theories in mathematics. Eilenberg and Steenrod united the different homology theories by laying out a set of axioms that all homology theories satisfy.

Definition 7.1. (**Eilenberg-Steenrod Axioms**) A homology theory with $\mathbb{Z}$ coefficients consists of

(1) A family of functors $H_n : \mathbf{Top}^2 \to \mathbf{Ab}$ for $n \geq 0$, and
(2) A family of natural transformations $\delta_n : H_n \to H_{n-1} \circ p$, where p is the functor sending (X, A) to $(A, \varnothing)$ and $f : (X, A) \to (Y, B)$ to $f|_A : (A, \varnothing) \to (B, \varnothing)$.

such that the following axioms are satisfied:

(a) (Homotopy invariance) If $f, g : (X, A) \to (Y, B)$ are homotopic maps, then the induced maps

$$H_n(f), H_n(g) : H_n(X, A; \mathbb{Z}) \to H_n(Y, B; \mathbb{Z})$$

are such that $H_n(f) = H_n(g)$ for $n \geq 0$[8].

(b) (Long exact sequence) The inclusions

$$(A, \emptyset) \hookrightarrow (X, \emptyset) \hookrightarrow (X, A)$$

give rise to a long exact sequence

$$\cdots \longrightarrow H_{n+1}(X; \mathbb{Z}) \longrightarrow H_{n+1}(X, A; \mathbb{Z}) \xrightarrow{\delta_{n+1}} H_n(A; \mathbb{Z}) \longrightarrow H_n(X; \mathbb{Z}) \longrightarrow \cdots$$

[8]In other words, H_n may be regarded as a functor from **hTop** to **Ab**.

(c) (Excision) If $Z \subseteq A \subseteq X$ are topological spaces such that $\overline{Z} \subseteq \mathrm{Int}(A)$, the inclusion of pairs $(X \setminus Z, A \setminus Z) \subseteq (X, A)$ induces isomorphisms

$$H_n(X \setminus Z, A \setminus Z; \mathbb{Z}) \to H_n(X, A; \mathbb{Z})$$

for all $n \geq 0$.

(d) (Additivity) If $X = \coprod_\alpha X_\alpha$ is the disjoint union of a family of topological spaces X_α, then

$$H_n(X; \mathbb{Z}) = \bigoplus_\alpha H_n(X_\alpha; \mathbb{Z})$$

for each $n \in \mathbb{N}$.

Additionally, if a homology theory satisfies the following additional axiom

(e) (Dimension Axiom) For any one-point set $X = \{\bullet\}$,

$$H_n(X; \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } n = 0 \\ 0 & \text{otherwise,} \end{cases}$$

the the homology theory is called an ordinary homology theory with $\mathbb{Z}$ coefficients.

We first introduce the notion of relative homology functors:

$$H_n : \mathbf{Top}^2 \to \mathbf{Ab}$$

for $n \geq 0$. Given $(X, A) \in \mathbf{Top}^2$, we have $C_n(A) \subseteq C_n(X)$ such that ∂_n restricts to a map from $C_n(A)$ to $C_{n-1}(A)$. Therefore, we can consider a chain complex $(C_\bullet(A), \partial_\bullet|_A)$ which is a sub-complex [9] of the chain complex $(C_\bullet, \partial_\bullet)$ The chain complex $(C_\bullet(A), \partial_\bullet|_A)$ is usually drawn as:

$$\cdots \longrightarrow C_2(A) \xrightarrow{\partial_2|_A} C_1(A) \xrightarrow{\partial_1|_A} C_0(A)$$

Note that $C_n(A)$ is an abelian subgroup of $C_n(X)$. Hence, we e can consider quotient group

$$C_n(X, A) = \frac{C_n(X)}{C_n(A)}$$

Since the boundary map

$$\partial_n : C_n(X) \to C_{n-1}(X)$$

takes $C_n(A)$ to $C_{n-1}(A)$, it induces a quotient boundary map

$$\overline{\partial}_n : C_n(X, A) \to C_{n-1}(X, A)$$

Since $\partial_{n+1} \circ \partial_n = 0$ on $C_n(X)$, we have that $\overline{\partial}_{n+1} \circ \overline{\partial}_n = 0$ on $C_n(X, A)$. Therefore, we get a chain complex $(C_\bullet(X, A), \overline{\partial}_\bullet)$ The chain complex is usually drawn as:

$$\cdots \longrightarrow C_2(X, A) \xrightarrow{\overline{\partial}_2} C_1(X, A) \xrightarrow{\overline{\partial}_1} C_0(X, A)$$

The above discussion implies that the construction of relative singular chain complexes defines a functor from $\mathbf{Top}^2$ to $\mathbf{Chain}_{\mathbf{Ab}}$.

[9] Given a chain complex $(C_\bullet, \partial_\bullet)$, a subcomplex of $(C_\bullet, \partial_\bullet)$ is given by a family of subgroups $C'_n \subseteq C_n$ such that the boundary operator $\partial'_n : C_n \to C_{n-1}$ restricts to a homomorphism $C'_n \to C'_{n-1}$ for all n.

Definition 7.2. Let $(X, A) \in \mathbf{Top}^2$. The n-th relative homology group with $\mathbb{Z}$ coefficients, $H_n(X, A)$, is the n-th homology group of the chain complex $(C_\bullet(X, A), \overline{\partial}_\bullet)$. That is:

$$H_n(X, A; \mathbb{Z}) = \frac{\text{Ker } \overline{\partial}_n}{\text{Im } \overline{\partial}_{n+1}}$$

It is clear that the n-th relative homology group with $\mathbb{Z}$ coefficients defines a functor from $\mathbf{Top}^2$ to $\mathbf{Ab}$.

Remark 7.3. *Since the homology of the empty set is trivial for all $n \geq 0$, we have:*

$$H_n(X, \emptyset; \mathbb{Z}) = H_n(X; \mathbb{Z}), \quad \forall n \geq 0.$$

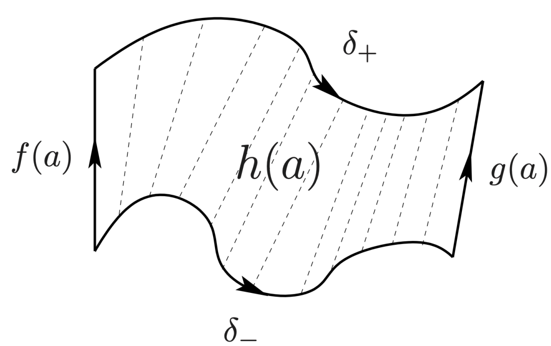
By considering the definition of the relative boundary map we see that: x

(1) Elements of $H_n(X, A; \mathbb{Z})$ are represented by relative n-cycles: n-chains $\alpha \in C_n(X)$ such that $\partial_n(\alpha) \in C_{n-1}(A)$.
(2) A relative n-cycle, α, is trivial in $H_n(X, A; \mathbb{Z})$ iff it is a relative n-boundary: $\alpha = \partial_{n+1}(\beta) + \gamma$ for some $\beta \in C_{n+1}(X)$ and $\gamma \in C_n(A)$.

In Example 6.10, we have already seen that singular homology with $\mathbb{Z}$ coefficients satisfies the dimension axiom. Moreover, an argument similar to that given in Proposition 6.11(a) shows that singular homology satisfies the additivity axiom. The purpose of the remainder of this section is to show that singular homology satisfies the homotopy invariance, long exact sequence and excision axioms. Hence, singular homology is an ordinary homology theory.

7.1. **Homotopy Invariance of Singular Homology.** We prove that singular homology groups satisfy the homotopy invariance axiom. We content ourselves to give a proof in the absolute case. The proof in the relative homology case is similar. In order to prove this statement, we we will make use of the notion of a chain homotopy between chain complexes as introduced in Section 5.

Remark 7.4. *What does this definition of a chain homotopy mean geometrically? Let h be a homotopy between maps f, g from X to Y. Consider a 1-chain, a, in X. Then $f(a), g(b)$ are 1-chains in Y. The homotopy h maps the endpoints of $f(a)$ to the endpoints of $g(a)$.*



The image is taken from [Alu21].

Let's look at the boundary of $h(a)$ in the diagram above. If we read counterclockwise starting at the bottom right, we see:

$$\partial_2 h(a) = g(a) - \delta_+ - f(a) + \delta_-$$

What is $\delta_+ - \delta_-$? It is $h(\partial_1 a)$! Hence:

$$\partial_2 h(a) = g(a) - f(a) - h(\partial_1 a)$$

Hence the definition of a chain homotopy mimics the notion of a homotopy at the level of chain complexes.

Proposition 7.5. *Let X and Y be topological spaces. If $f, g : X \to Y$ are two homotopic maps, then*

$$H_n(f) = H_n(g) : H_n(X;\mathbb{Z}) \to H_n(Y;\mathbb{Z})$$

for each $n \geq 0$.

Before providing the proof, we discuss the idea behind the proof. The essential ingredient is a procedure for subdividing $\Delta^n \times I$ into simplices. In $\Delta^n \times I$, let

$$\Delta^n \times 0 = [v_0, \dots, v_n]$$
$$\Delta^n \times 1 = [w_0, \dots, w_n]$$

where v_i and w_i have the same image under the projection $\Delta^n \times I \to \Delta^n$. We can pass from $[v_0, \dots, v_n]$ to $[w_0, \dots, w_n]$ by interpolating a sequence of n simplices, each obtained from the preceding one by moving one vertex v_i up to w_i, starting with v_n and working backwards to v_0. For instance,

$$[v_0, \dots, v_i, w_{i+1}, \dots, w_n]$$

moves up to

$$[v_0, \dots, v_{i-1}, w_i, \dots, w_n]$$

The region between these two n simplices is exactly the $(n+1)$ simplex

$$[v_0, \dots, v_i, w_i, \dots, w_n]$$

Lemma 7.6. *$\Delta^n \times I$ is the union of $n+1$ copies of Δ^{n+1}.*

Proof. For $i = -1, 0, \dots, n-1$, let $g_i : \Delta^n \to I$ denote the map

$$g_i(s_0, s_1, \dots, s_n) = \sum_{i<j} s_j.$$

Let $G_i \subseteq \Delta^n \times I$ denote the graph of g_i. Then G_i is homeomorphic to Δ^n via the projection $\Delta^n \times I \to \Delta^n$ onto the first factor. Let us now label the vertices at the "bottom" (i.e., $\Delta^n \times \{0\}$) of $\Delta^n \times I$ by $v_0, v_1, \dots, v_n$ and those at the "top" (i.e., $\Delta^n \times \{1\}$) by $w_0, w_1, \dots, w_n$. Then G_i is the n-simplex

$$G_i = [v_0, \dots, v_i, w_{i+1}, \dots, w_n].$$

Since G_i lies below G_{i-1} as $g_i \leq g_{i-1}$, it follows that the region between G_i and G_{i-1} is the $(n+1)$-simplex $[v_0, \dots, v_i, w_i, \dots, w_n]$; this is indeed an $(n+1)$-simplex as w_i is not in G_i and hence not in the n-simplex $[v_0, \dots, v_i, w_i, \dots, w_n]$. Since

$$0 = g_n \leq g_{n-1} \leq \dots \leq g_0 \leq g_{-1} = 1,$$

we see that $\Delta^n \times I$ is the union of the regions between the G_i, and hence the union of $n+1$ different $(n+1)$-simplices $[v_0, \dots, v_i, w_i, \dots, w_n]$, each intersecting the next in an n-simplex face. $\square$

Proof. (Proposition 7.5) Given a homotopy $F : X \times I \to Y$ from f to g and a singular simplex $\sigma : \Delta^n \to X$, we can form the composition

$$F \circ (\sigma \times 1) : \Delta^n \times I \to X \times I \to Y$$

Using this, we can define *prism operators* $P_n : C_n(X) \to C_{n+1}(Y)$ by the following formula:

$$P_n\sigma = \sum_{i=0}^{n+1} (-1)^i F \circ (\sigma \times 1)|_{[v_0,\cdots,v_i,w_i,\cdots,w_n]}.$$

The prism operator is our proposed chain homotopy. A simple computation shows that we have

$$f_n - g_n = \partial'_{n+1} P_n + P_{n-1}\partial_n$$

Indeed:

$$\begin{aligned}\partial'_{n+1} P_n(\sigma) &= \sum_{j \le i} (-1)^i (-1)^j F \circ (\sigma \times 1)|_{[v_0,\ldots,\widehat{v_j},\ldots,v_i,w_i,\ldots,w_n]} \\ &+ \sum_{j \ge i} (-1)^i (-1)^{j+1} F \circ (\sigma \times 1)|_{[v_0,\ldots,v_i,w_i,\ldots,\widehat{w_j},\ldots,w_n]}\end{aligned}$$

The terms with $i = j$ in the two sums cancel except for

$$F \circ (\sigma \times 1)|_{[\widehat{v_0},w_0,\ldots,w_n]} = g \circ \sigma = g_n(\sigma),$$

and

$$-F \circ (\sigma \times 1)|_{[v_0,\ldots,v_n,\widehat{w_n}]} = -f \circ \sigma = -f_n(\sigma).$$

The terms with $i \neq j$ are exactly $-P_{n-1}\partial_n(\sigma)$. Hence, the sequence of maps $(P_n)_{n \ge 0}$ defines a chain homotopy. The claim follows by invoking Proposition 5.15. □

Corollary 7.7. *If X is contractible, then $H_n(X;\mathbb{Z}) = 0$ for all $n > 0$.*

Proof. Immediate from the previous corollary and that $H_n(\{*\}) = 0$ for $n \ge 1$. □

7.2. **Long Exact Sequence in Singular Homology.** We now prove that singular homology satisfies the long exact sequence axiom. The importance of the long exact sequence axiom is that is allows us to compute homology groups of various spaces in using an 'inductive' and/or 'bottom-up' approach, as we shall see in various examples later on. We have a short exact sequence of chain complexes:

$$0_\bullet \longrightarrow (C_\bullet(A), \partial_\bullet|_A) \xrightarrow{i_\bullet} (C_\bullet, \partial_\bullet) \xrightarrow{j_\bullet} (C_\bullet(X,A), \overline{\partial}_\bullet) \longrightarrow 0_\bullet$$

Invoking the snake lemma (Proposition 5.19), we have the following long exact sequence is homology associated to the pair of spaces (X, A):

$$\cdots \longrightarrow H_{n+1}(X;\mathbb{Z}) \longrightarrow H_{n+1}(X,A;\mathbb{Z}) \xrightarrow{\delta_{n+1}} H_n(A) \longrightarrow H_n(X;\mathbb{Z}) \longrightarrow \cdots$$

Remark 7.8. *The boundary map $\delta_n : H_n(X,A;\mathbb{Z}) \to H_{n-1}(A;\mathbb{Z})$ has a very simple description: if a class $[\alpha] \in H_n(X,A;\mathbb{Z})$ is represented by a relative cycle α, then $\delta_n[\alpha]$ is the class of the cycle $\delta_n\alpha$ in $H_{n-1}(A;\mathbb{Z})$.*

Remark 7.9. *An easy generalization of the long exact sequence of a pair (X, A) is the long exact sequence of a triple $(X, A, B) \in \mathbf{Top}^3$. Indeed, we have $(X, A), (X, B), (A, B) \in \mathbf{Top}^2$. The three long exact sequences assemble in the following diagram:*

$$
\begin{tikzcd}
H_{n+2}(X;\mathbb{Z}) \arrow[rr, dotted, "g_1"] \arrow[dr, "f_1"] & & H_{n+2}(X,A;\mathbb{Z}) \arrow[rr, Rightarrow] \arrow[dr, dotted, "g_2"] & & H_{n+1}(A,B;\mathbb{Z}) \arrow[rr, dashed, "j_4"] \arrow[dr, Rightarrow] & & H_n(B;\mathbb{Z}) \\
& H_{n+2}(X,B;\mathbb{Z}) \arrow[ur, Rightarrow] \arrow[dr, "f_2"] & & H_{n+1}(A;\mathbb{Z}) \arrow[ur, dashed, "j_3"] \arrow[dr, dotted, "g_3"] & & H_{n+1}(X,B;\mathbb{Z}) \arrow[ur, "f_5"] \arrow[dr, Rightarrow] & \\
H_{n+2}(A,B;\mathbb{Z}) \arrow[rr, dashed, "j_1"] \arrow[ur, Rightarrow] & & H_{n+1}(B;\mathbb{Z}) \arrow[rr, "f_3"] \arrow[ur, dashed, "j_2"] & & H_{n+1}(X;\mathbb{Z}) \arrow[rr, dotted, "g_4"] \arrow[ur, "f_4"] & & H_{n+1}(X,A;\mathbb{Z})
\end{tikzcd}
$$

The braid lemma (Proposition 5.20) implies that the chain complex labeled with $\Rightarrow$ arrows is a chain complex. This is the desired long exact sequence in homology generated by (X, A, B).

7.3. **Excision in Singular Homology.** We now prove that singular homology satisfies the excision axiom. The important of the excision axiom is that if $A \subseteq X$ if n-chains are "sufficiently inside" of A, we can cut A out without affecting the relative homology groups $H_n(X, A; \mathbb{Z})$. Here is the formal statement we'd like to prove in this section:

Proposition 7.10. *Suppose $Z \subseteq A \subseteq X$ are topological spaces such that $\overline{Z} \subseteq \mathrm{Int}(A)$. Then there is an inclusion of the pair $(X \setminus Z, A \setminus Z) \subseteq (X, A)$, and the induced map*

$$H_n(X \setminus Z, A \setminus Z; \mathbb{Z}) \to H_n(X, A; \mathbb{Z})$$

is an isomorphism for all $n \geq 0$. Equivalently, for subspaces $A, B \subseteq X$ whose interiors cover X, the inclusion $(B, A \cap B) \hookrightarrow (X, A)$ induces isomorphisms

$$H_n(B, A \cap B; \mathbb{Z}) \cong H_n(X, A; \mathbb{Z})$$

Remark 7.11. *To see that the two statements of the Excision Theorem are equivalent, just take $B = X \setminus Z$ (or $Z = X \setminus B$). Then $A \cap B = A \setminus Z$, and the condition $\overline{Z} \subset \mathrm{int}(A)$ is equivalent to $X = \mathrm{int}(A) \cup \mathrm{int}(B)$.*

Let's provide some intuition. Assume that $X = \mathrm{int}(A) \cup \mathrm{int}(B)$. We expect $H_n(X, A)$ remains unchanged if we cut A out. This argument works when all chains are belong either to A or B. But if a chain doesn't entirely lie entirely in A or B, then we have a problem. The solution is given by the method of barycentric subdivision: replace the 'large' chains with 'small' chains by subdividing. We formalize this intuition in Proposition 7.13. We first prove a lemma.

Lemma 7.12. *Let $S = [v_0, v_1, \ldots, v_n]$ denote an n-simplex in some Euclidean space. Then if $x, y \in S$, one has*

$$\|x - y\| \leq \max_{0 \leq i \leq n} \|v_i - y\|$$

Hence[10]

$$\operatorname{diam} S = \max_{0 \leq i,j \leq n} \|v_i - v_j\|.$$

If b is the barycenter of S

$$b = \sum_{i=0}^{n} \frac{1}{n+1} v_i,$$

[10]The diameter of a simplex is the maximum Euclidean distance between any two of its points.

then

$$\|b - v_i\| \leq \frac{n}{n+1} \operatorname{diam} S.$$

Proof. Let $x, y \in S$, and write $x = \sum_{i=0}^n s_i v_i$ with $\sum_{i=0}^n s_i = 1$. Then

$$\|x - y\| \leq \sum_{i=0}^n s_i \|v_i - y\| \leq \max_{0 \leq i \leq n} \|v_i - y\|.$$

This shows in particular $\|y - v_i\| \leq \max_{0 \leq j \leq n} \|v_i - v_j\|$ for each $0 \leq i \leq n$. Hence, $\|x - y\| \leq \max_{0 \leq i \leq n} \|v_i - y\|$. If b is the barycenter, we have

$$\begin{aligned}
\|b - v_i\| &= \left\| \frac{1}{n+1} \sum_{j=0}^n v_j - v_i \right\| \\
&\leq \frac{1}{n+1} \sum_{i=0}^n \|v_j - v_i\| \\
&\leq \frac{n}{n+1} \max_{0 \leq i,j \leq n} \|v_j - v_i\| = \frac{n}{n+1} \operatorname{diam} S
\end{aligned}$$

This completes the proof. □

We now formalize the idea that subdividing a 'large' singular chains into a union of 'small' singular chains. Let $U = \{U_j\}$ be a collection of subspaces of X whose interiors form an open cover of X, and let $C_n^U(X)$ be the subgroup of $C_n(X)$ consisting of chains $\sum_i n_i \sigma_i$ such that each σ_i has image contained in some set in the cover U. The boundary map ∂_n takes $C_n^U(X)$ to $C_{n-1}^U(X)$, so the groups $C_U^n(X)$ form a chain complex. We denote the homology groups of this chain complex by $H_n^U(X)$.

Proposition 7.13. *Consider the chain map* $\iota_\bullet : C_\bullet^U(X) \hookrightarrow C_\bullet(X)$ *such that* i_n *is the inclusion map for each* $n \geq 0$*. There is a chain map* $\rho : C_\bullet(X) \to C_\bullet^U(X)$ *such that* $\iota \circ \rho$ *and* $\rho \circ \iota$ *are chain homotopic to the identity.*

Proof. See [Hat02] for the the proof. □

Proof. (Proposition 7.10) Assume that $X = A \cup B$. WLOG, assume that A and B are open sets. We have

$$C_n^U(X) = C_n(A) + C_n(B) \qquad C_n(A \cap B) = C_n(A) \cap C_n(B)$$

Therefore, we have

$$\frac{C_n(B)}{C_n(A \cap B)} = \frac{C_n(B)}{C_n(A) \cap C_n(B)} \cong \frac{C_n(A) + C_n(B)}{C_n(A)} \cong \frac{C_n^U(X)}{C_n(A)}$$

All the maps appearing in the proof of Proposition 7.13 take chains in A to chains in A. So these maps induce quotient maps when we factor out chains in A and the quotient maps satisfy all the corresponding formulas in the proof of Proposition 7.13. There, Proposition 7.13 implies that the inclusion

$$C_n^U(X)/C_n(A) \hookrightarrow C_n(X)/C_n(A)$$

induces an isomorphism on homology. Since

$$C_n^U(X)/C_n(A) = \frac{C_n(B)}{C_n(A \cap B)},$$

we have that

$$H_n(B, A \cap B; \mathbb{Z}) \cong H_n(X, A; \mathbb{Z})$$

for each $n \geq 0$ □

8. Relative Homology

We discuss relative homology in more detail in this section. We start with a useful lemma.

Lemma 8.1. *Let $A \subseteq X$ be topological spaces. Consider an exact sequence of abelian groups:*

$$A \to B \to C \to D \to E$$

(1) *$C = 0$ if and only if the map $A \to B$ is surjective and $D \to E$ is injective.*
(2) *For a pair of spaces $(X, A) \in \mathbf{Top}^2$, the inclusion $A \hookrightarrow X$ induces isomorphisms on all homology groups if and only if $H_n(X, A; \mathbb{Z}) = 0$ for all $n \geq 0$.*

Proof. The proof is as follows:

(1) Let $\alpha, \beta, \gamma, \delta$ be the corresponding maps. By exactness,

$$\mathrm{Im}(\alpha) = \ker(\beta), \quad \mathrm{Im}(\beta) = \ker(\gamma), \quad \mathrm{Im}(\gamma) = \ker(\delta).$$

Note that α is surjective iff $\ker(\beta) = B$ iff $\mathrm{Im}(\beta) = 0$, and δ is injective iff $\mathrm{Im}(\gamma) = 0$ iff $\ker(\gamma) = C$. Putting both together, α is surjective and δ is injective iff $C = 0$, since $\mathrm{Im}(\beta) = \ker(\gamma)$.

(2) Consider the following part of the the long exact sequence in homology:

$$\cdots \longrightarrow H_{n+1}(A; \mathbb{Z}) \longrightarrow H_{n+1}(X; \mathbb{Z}) \longrightarrow H_{n+1}(X, A; \mathbb{Z}) \longrightarrow H_n(A; \mathbb{Z}) \longrightarrow H_n(X; \mathbb{Z}) \longrightarrow \cdots$$

The maps $H_n(A; \mathbb{Z}) \to H_n(X; \mathbb{Z})$ are isomorphisms for all $n \geq 0$ if and only if they are both injective and surjective for all $n \geq 0$. By re-indexing, this is true if and only if the leftmost map in our five-term exact sequence is surjective and the rightmost map is injective for all $n \geq 0$. But (1), this is true if and only if the middle group vanishes for all $n \geq 0$.

This completes the proof. □

Remark 8.2. *As per Lemma 8.1, we can think of $H_n(X, A; \mathbb{Z})$ as measuring the failure of the induced morphism $H_n(A; \mathbb{Z}) \to H_n(X; \mathbb{Z})$ to be an isomorphism for each $n \geq 0$.*

Based on Lemma 8.1, we can characterize relative homology groups for $n = 0, 1$.

Proposition 8.3. *Let $A \subseteq X$ be topological spaces.*

(1) *$H_0(X, A; \mathbb{Z}) = 0$ if and only if A meets each path-component of X.*
(2) *$H_1(X, A; \mathbb{Z}) = 0$ if and only if $H_1(A; \mathbb{Z}) \to H_1(X; \mathbb{Z})$ is surjective and each path-component of X contains at most one path-component of A.*
(3) *Let (X, x) be a pointed topological space. Then*

$$H_n(X, x; \mathbb{Z}) \cong H_n(X; \mathbb{Z}) \cong \widetilde{H}_n(X; \mathbb{Z})$$

for each $n \geq 1$.

Proof. The proof is given below:

(1) We first prove the special case that if X is a non-empty *path-connected* space and $A \subseteq X$, then $H_0(X, A; \mathbb{Z}) = 0$ if and only if A is not-empty. Consider the end of the long exact sequence for the pair $(X, A; \mathbb{Z})$:

$$H_0(A;\mathbb{Z}) \to H_0(X;\mathbb{Z}) \to H_0(X, A;\mathbb{Z}) \to 0$$

Note that $H_0(X;\mathbb{Z}) \cong \mathbb{Z}$. If A is empty, the sequence is,

$$0 \to \mathbb{Z} \to H_0(X, A;\mathbb{Z}) \to 0$$

Since the map from $\mathbb{Z}$ to $H_0(X, A;\mathbb{Z})$ is injective, $H_0(X, A;\mathbb{Z})$ must be non-zero. If A is non-empty, pick a point $a \in A$ and consider the homology class $[a] \in H_0(A;\mathbb{Z})$. The image of $[a]$ under

$$H_0(A;\mathbb{Z}) \to H_0(X;\mathbb{Z})$$

is the homology class of a point, which generates the co-domain. So $H_0(A;\mathbb{Z}) \to H_0(X;\mathbb{Z})$ is onto. Hence

$$H_0(A;\mathbb{Z}) \to H_0(X, A;\mathbb{Z})$$

is onto as well implying that and $H_0(X, A;\mathbb{Z}) = 0$. More generally, suppose X has multiple connected components. Assume that A meets each path component of X. If X_i is a component of X, then $H_0(A \cap X_i;\mathbb{Z}) \to H_0(X_i;\mathbb{Z})$ is surjective. But then

$$H_0(A;\mathbb{Z}) = \bigoplus_i H_0(A \cap X_i;\mathbb{Z}) \to \bigoplus_i H_0(X_i;\mathbb{Z}) = H_0(X;\mathbb{Z})$$

is surjective. Therefore, So $H_0(X, A;\mathbb{Z}) = 0$. Conversely, if A does not meet a component of X, say X_j, then $H_0(X_j, A;\mathbb{Z}) \neq 0$. But then $H_0(X_j, A;\mathbb{Z}) \neq 0$ is a direct summand of $H_0(X, A;\mathbb{Z})$. Hence $H_0(X, A;\mathbb{Z})$ must be non-zero.

(2) If $H_1(X, A;\mathbb{Z}) = 0$, then $H_1(A;\mathbb{Z}) \to H_1(X;\mathbb{Z})$ is onto and $H_0(A;\mathbb{Z}) \to H_0(X;\mathbb{Z})$ is injective by Lemma 8.1. This last statement can't be true if some path component X_i of X contains multiple components of A because then $H_0(A \cap X_i) \cong \mathbb{Z}^n$ for some $n \geq 2$ while $H_0(X_i) = \mathbb{Z}$. So then

$$H_0(A \cap X_i) \to H_0(X_i)$$

can't be one-to-one, and the same follows for

$$H_0(A;\mathbb{Z}) \to H_0(X;\mathbb{Z})$$

If $H_1(A;\mathbb{Z}) \to H_1(X;\mathbb{Z})$ is onto, then the kernel of the map $H_1(X;\mathbb{Z}) \to H_1(X, A;\mathbb{Z})$ is $H_1(X;\mathbb{Z})$. So the map $H_1(X;\mathbb{Z}) \to H_1(X, A;\mathbb{Z})$ is the 0 map. Similarly, if each component of X contains at most one component of A, then $H_0(A;\mathbb{Z}) \to H_0(X;\mathbb{Z})$ is injective. So its kernel is 0, so the image of $H_1(X, A;\mathbb{Z}) \to H_0(A;\mathbb{Z})$ is 0. But then by exactness, $0 = H_1(X, A;\mathbb{Z})$.

(3) If $n \geq 2$, then $H_n(X;\mathbb{Z}) = 0$ and $H_{n-1}(x;\mathbb{Z}) = 0$, and thus we immediately see $H_n(X, x) \cong H_n(X;\mathbb{Z})$ by inspecting the long exact sequence in relative homology. For $n = 1$, consider the following part of the long exact sequence in relative homology:

$$0 \to H_1(X;\mathbb{Z}) \to H_1(X, x;\mathbb{Z}) \to H_0(X;\mathbb{Z}) \cong \mathbb{Z} \to H_0(X;\mathbb{Z}) \to H_0(X, x;\mathbb{Z}) \to 0$$

Proposition 8.3(1) readily implies that

$$0 \to H_1(X;\mathbb{Z}) \to H_1(X, x;\mathbb{Z})$$

is surjective if and only if

$$H_0(X;\mathbb{Z}) \cong \mathbb{Z} \to H_0(X;\mathbb{Z})$$

is injective if and only if it is not-the zero map. The last equivalence follows from the observation that $H_0(X;\mathbb{Z})$ is a free abelian group. If it were the zero map, the map $H_0(X;\mathbb{Z}) \to H_0(X,x;\mathbb{Z})$ will be injective. However, this is not the case since the point $x \in X$ defines a generator $\langle x\rangle$ of $H_0(X;\mathbb{Z})$ that is is in the kernel of the map $H_0(X;\mathbb{Z}) \to H_0(X,x)$. Therefore, the claim is true for $n = 1$ as well.

This completes the proof. □

Definition 8.4. Let (X, A) be in $\mathbf{Top^2}$. If $A \subseteq X$ is a closed subspace such that there exists a neighborhood V of X such that A is a strong deformation retract of V, we say that $(X, A;\mathbb{Z})$ is a good pair.

The next proposition allows us to identify an alternative way to think about relative homology in most cases of interest.

Proposition 8.5. *Let (X, A) be a good pair. Then*

$$H_n(X, A;\mathbb{Z}) \cong H_n(X/A, A/A;\mathbb{Z}) \cong \widetilde{H}_n(X/A;\mathbb{Z})$$

for all $n \geq 0$.

Proof. Consider the following diagram:

$$\begin{array}{ccccc}
H_n(X,A;\mathbb{Z}) & \longrightarrow & H_n(X,V;\mathbb{Z}) & \longleftarrow & H_n(X-A,V-A;\mathbb{Z}) \\
\downarrow & & \downarrow & & \downarrow \\
H_n(X/A,A/A;\mathbb{Z}) & \longrightarrow & H_n(X/A,V/A;\mathbb{Z}) & \longleftarrow & H_n(X/A-A/A,V/A-A/A;\mathbb{Z})
\end{array}$$

The upper left horizontal map is an isomorphism since in the long exact sequence of the triple (X, V, A) (Remark 7.9) the groups $H_n(V, A)$ are zero for all $n \geq 0$, because a deformation retraction of V onto A gives a homotopy equivalence of pairs $(V, A) \simeq (A, A)$, and $H_n(A, A) = 0$. The deformation retraction of V onto A induces a deformation retraction of V/A onto A/A, so the same argument shows that the lower left horizontal map is an isomorphism as well. The other two horizontal maps are isomorphisms directly from excision. The right-hand vertical map is an isomorphism. It follows that the left-hand vertical arrow also is an isomorphism. □

Corollary 8.6. *If $(X, A;\mathbb{Z})$ is a good pair, then there is an exact sequence:*

$$\cdots \to \widetilde{H}_n(A;\mathbb{Z}) \to \widetilde{H}_n(X;\mathbb{Z}) \to \widetilde{H}_n(X/A;\mathbb{Z}) \to \widetilde{H}_{n-1}(A;\mathbb{Z}) \to \widetilde{H}_{n-1}(X;\mathbb{Z}) \to \cdots$$

Proof. This is clear. □

Corollary 8.7. *Let $(X_\alpha, x_\alpha)_{\alpha\in I}$ be a collection of good pairs in $\mathbf{Top}_*$. Let $X = \bigvee_{\alpha\in I} X_\alpha$ with the basepoint $x = (x_\alpha)_{\alpha\in I}$ in $\mathbf{Top}_*$. Then*

$$\widetilde{H}_n(X;\mathbb{Z}) \cong \bigoplus_{\alpha\in I} \widetilde{H}_n(X_\alpha;\mathbb{Z})$$

for $n \geq 1$.

Proof. Since $(X_\alpha, x_\alpha)_{\alpha\in I}$ be a collection of good pairs, (X, x) is also a good pair. We have:

$$\begin{aligned}
\widetilde{H}_n(X;\mathbb{Z}) &= \widetilde{H}_n\left(\coprod_{\alpha\in I} X_\alpha \Big/ \coprod_{\alpha\in I}\{x_\alpha\};\mathbb{Z}\right)\\
&\cong H_n\left(\coprod_{\alpha\in I} X_\alpha, \coprod_{\alpha\in I}\{x_\alpha\};\mathbb{Z}\right)\\
&\cong \bigoplus_{\alpha\in I} H_n(X_\alpha, x_\alpha;\mathbb{Z})\\
&\cong \bigoplus_{\alpha\in I} \widetilde{H}_n(X_\alpha;\mathbb{Z}).
\end{aligned}$$

The first and third equivalences follow by Proposition 8.3. The second equivalence follows by observing that the additivity axiom holds in $\mathbf{Top}^2$ as can be checked. □

Example 8.8. (**Homology of Spheres**) We now are now in a position to compute the reduced homology groups of spheres. The reduced homology groups of spheres are given as:

$$\widetilde{H}_k(\mathbb{S}^n;\mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & \text{if } k = n \\ 0, & \text{if } k \neq 0 \end{cases}$$

Since $(\mathbb{D}^n, \mathbb{S}^{n-1})$ is a good pair and $\mathbb{D}^n/\mathbb{S}^{n-1} \cong \mathbb{S}^n$, the long exact sequence in relative reduced homology yields:

$$\cdots \to \widetilde{H}_k(\mathbb{S}^{n-1};\mathbb{Z}) \to \widetilde{H}_k(\mathbb{D}^n;\mathbb{Z}) \to \widetilde{H}_k(\mathbb{S}^n;\mathbb{Z}) \to \widetilde{H}_{k-1}(\mathbb{S}^{n-1};\mathbb{Z}) \to \widetilde{H}_{k-1}(\mathbb{D}^n;\mathbb{Z}) \to \cdots$$

Since $\mathbb{D}^n$ is contractible, $\widetilde{H}_k(\mathbb{D}^n;\mathbb{Z}) = 0$ for $k \geq 0$. Therefore,

$$\widetilde{H}_k(\mathbb{S}^{n-1};\mathbb{Z}) \to 0 \to \widetilde{H}_k(\mathbb{S}^n;\mathbb{Z}) \to \widetilde{H}_{k-1}(\mathbb{S}^{n-1};\mathbb{Z}) \to 0 \to \cdots$$

Hence, we have:

$$\widetilde{H}_k(\mathbb{S}^n;\mathbb{Z}) \cong \widetilde{H}_{k-1}(\mathbb{S}^n;\mathbb{Z})$$

The result now follows via induction and the observation that

$$\widetilde{H}_0(\mathbb{S}^0;\mathbb{Z}) \cong \mathbb{Z} \qquad \widetilde{H}_k(\mathbb{S}^0;\mathbb{Z}) \cong 0 \quad k > 0$$

The computation above readily implies the following:

$$H_k(\mathbb{S}^n;\mathbb{Z}) \cong \begin{cases} \mathbb{Z}\oplus\mathbb{Z}, & \text{if } k = 0, n = 0 \\ \mathbb{Z} & \text{if } k = 0, n > 1 \\ \mathbb{Z} & \text{if } k = n > 0 \\ 0 & \text{otherwise} \end{cases}$$

Corollary 8.9. *Let $m \neq n$.*

(1) *$\mathbb{S}^m$ and $\mathbb{S}^n$ are not homotopy equivalent.*
(2) *$\mathbb{R}^m$ and $\mathbb{R}^n$ are not homeomorphic.*
(3) *If $U \subseteq \mathbb{R}^m$ and $V \subseteq \mathbb{R}^n$ are non-empty homeomorphic open sets, then $m = n$.*

Proof. The proof is given below:

(1) This follows from Example 8.8 since the homology groups are not isomorphic for $m \neq n$.
(2) If m or n is zero, this is clear. So let $m, n > 0$. Assume we have a homeomorphism $f : \mathbb{R}^m \to \mathbb{R}^n$. WLOG assume that $f(0) = 0$. This restricts to a homeomorphism $\mathbb{R}^m \setminus \{0\} \to \mathbb{R}^n \setminus \{f(0)\}$. But these spaces are homotopy equivalent to spheres of different dimension, yielding a contradiction.
(3) For all $x \in U$ and for all $k \in \mathbb{Z}$, we have

$$H_k(U, U \setminus \{x\}) \cong H_k(\mathbb{R}^m, \mathbb{R}^m \setminus \{x\})$$

by the Excision Theorem. Combining this with the long exact sequence for the reduced homology of $(\mathbb{R}^m, \mathbb{R}^m \setminus \{x\})$ and the fact that $\mathbb{R}^m \setminus \{x\}$ is homotopy equivalent to $\mathbb{S}^{m-1}$, we obtain for all $x \in U$ and all $k \in \mathbb{Z}$:

$$H_k(U, U \setminus \{x\}) \cong H_k(\mathbb{R}^m, \mathbb{R}^m \setminus \{x\}) \cong \widetilde{H}_{k-1}(\mathbb{R}^m \setminus \{x\}) \cong \begin{cases} \mathbb{Z}, & k = m \\ 0, & k \neq m. \end{cases}$$

Similarly,

$$H_k(V, V \setminus \{x\}) \cong H_k(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\}) \cong \widetilde{H}_{k-1}(\mathbb{R}^n \setminus \{x\}) \cong \begin{cases} \mathbb{Z}, & k = n \\ 0, & k \neq n. \end{cases}$$

If U, V are homeomorphic via $f : U \to V$, then

$$H_k(U, U \setminus \{x\}) \cong H_k(V, V \setminus \{f(x)\})$$

The claim follows by comparing homology groups.

This completes the proof. □

Remark 8.10. *If X is a topological space, $x \in X$, and $U \subseteq X$ is an open neighborhood of x, then for all $n \in \mathbb{Z}$, the Excision Theorem yields that*

$$H_n(X, X \setminus \{x\}) \cong H_n(U, U \setminus \{x\}).$$

In particular, for all $n \in \mathbb{Z}$, the group $H_n(X, X \setminus \{x\})$ depends only on the topology of a neighborhood of x. Therefore, these homology groups are called the local homology groups of X at x.

Example 8.11. Let $A \subseteq X$ be a finite set of points in X. We compute $H_n(X, A; \mathbb{Z})$ in the two cases:

(1) Let $X = \mathbb{S}^2$. Assume $|A| = k$ for $k \geq 1$. Since A is assumed to be non-empty, Proposition 8.3 implies $H_0(\mathbb{S}^2, A; \mathbb{Z}) = 0$. The long exact sequence in relative homology implies we have:

$$\cdots \longrightarrow H_{n+1}(A; \mathbb{Z}) \longrightarrow H_{n+1}(\mathbb{S}^2; \mathbb{Z}) \longrightarrow H_{n+1}(\mathbb{S}^2, A; \mathbb{Z}) \longrightarrow H_n(A; \mathbb{Z}) \longrightarrow H_n(\mathbb{S}^2; \mathbb{Z}) \longrightarrow \cdots$$

Noting that,

$$H_1(\mathbb{S}^2; \mathbb{Z}) = 0 \qquad H_0(\mathbb{S}^2; \mathbb{Z}) = \mathbb{Z} \qquad H_0(A; \mathbb{Z}; \mathbb{Z}) = \mathbb{Z}^k,$$

the right most end of the long exact sequence becomes

$$\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow H_1(\mathbb{S}^2, A; \mathbb{Z}) \longrightarrow \mathbb{Z}^k \longrightarrow \mathbb{Z} \longrightarrow 0 \longrightarrow 0$$

Since $\mathbb{Z}$ is a free abelian group, the sequence above splits and implies that

$$\mathbb{Z}^k \cong H_1(\mathbb{S}^2, A; \mathbb{Z}) \oplus \mathbb{Z}.$$

Hence,

$$H_1(\mathbb{S}^2, A; \mathbb{Z}) \cong \mathbb{Z}^{k-1}.$$

For $n \geq 2$, $H_n(A; \mathbb{Z}) = 0$ implies that we have the sequence

$$\cdots \longrightarrow 0 \longrightarrow H_{n+1}(\mathbb{S}^2; \mathbb{Z}) \longrightarrow H_{n+1}(\mathbb{S}^2, A; \mathbb{Z}) \longrightarrow 0 \longrightarrow H_n(\mathbb{S}^2; \mathbb{Z}) \longrightarrow H_n(\mathbb{S}^2, A; \mathbb{Z}) \longrightarrow \cdots$$

By Lemma 8.1, $H_n(\mathbb{S}^2) \to H_n(\mathbb{S}^2, A)$ is surjective. But the map is also injective. Hence by exactness and the first isomorphism theorem, Therefore,

$$H_n(\mathbb{S}^2, A; \mathbb{Z}) \cong H_n(\mathbb{S}^2; \mathbb{Z}).$$

Hence, we have:

$$H_n(\mathbb{S}^2, A; \mathbb{Z}) \cong \begin{cases} 0 & \text{if } n \geq 3 \\ \mathbb{Z} & \text{if } n = 2 \\ \mathbb{Z}^{k-1} & \text{if } n = 1 \\ 0 & \text{if } n = 0 \end{cases}$$

(2) Let $X = \mathbb{S}^1 \times \mathbb{S}^1$. It can be checked that $\mathbb{S}^1 \times \mathbb{S}^1$ is homemorphic to the torus, T, considered in Section 4. As in (1), $H_0(\mathbb{S}^1 \times \mathbb{S}^1, A; \mathbb{Z}) = 0$ and $H_n(\mathbb{S}^1 \times \mathbb{S}^1, A; \mathbb{Z}) \cong H_n(\mathbb{S}^1 \times \mathbb{S}^1)$ for $n \geq 2$. For $n = 1$, noting that,

$$H_1(\mathbb{S}^1 \times \mathbb{S}^1; \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z} \qquad H_0(\mathbb{S}^1 \times \mathbb{S}^1; \mathbb{Z}) = \mathbb{Z} \qquad H_0(A; \mathbb{Z}) = \mathbb{Z}^k$$

the right most end of the long exact sequence in Homology becomes

$$\cdots \longrightarrow 0 \longrightarrow \mathbb{Z} \oplus \mathbb{Z} \longrightarrow H_1(\mathbb{S}^1 \times \mathbb{S}^1, A; \mathbb{Z}) \longrightarrow \mathbb{Z}^k \longrightarrow \mathbb{Z} \longrightarrow 0 \longrightarrow 0$$

The reduced homology version of the sequence above is

$$\cdots \longrightarrow 0 \longrightarrow \mathbb{Z} \oplus \mathbb{Z} \longrightarrow \widetilde{H}_1(\mathbb{S}^1 \times \mathbb{S}^1, A; \mathbb{Z}) \longrightarrow \mathbb{Z}^{k-1} \longrightarrow 0$$

Since $\mathbb{Z}$ is a free $\mathbb{Z}$-module, the sequence above splits and implies that

$$H_1(\mathbb{S}^1 \times \mathbb{S}^1, A; \mathbb{Z}) \cong \widetilde{H}_1(\mathbb{S}^1 \times \mathbb{S}^1, A) \cong \mathbb{Z}^{k+1}.$$

Hence, we have:

$$H_n(\mathbb{S}^1 \times \mathbb{S}^1, A; \mathbb{Z}) \cong \begin{cases} 0 & \text{if } n \geq 3 \\ \mathbb{Z} & \text{if } n = 2 \\ \mathbb{Z}^{k+1} & \text{if } n = 1 \\ 0 & \text{if } n = 0 \end{cases} x$$

Example 8.12. We compute $H_1(\mathbb{R},\mathbb{Q})$. We have the following exact sequence in homology,

$$\cdots \to H_n(\mathbb{Q}) \to H_n(\mathbb{R}) \to H_n(\mathbb{R},\mathbb{Q}) \to H_{n-1}(\mathbb{Q}) \to \cdots$$

Since $\mathbb{Q}$ is a totally disconnected set, every point $q \in \mathbb{Q}$ is a path-component. Hence, we have

$$H_n(\mathbb{Q}) = \begin{cases} \bigoplus_{\alpha\in\mathbb{Q}} \mathbb{Z} & \text{if } n = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Since $\mathbb{R}$ is contractible, the long-exact sequence on the right becomes

$$0 \to H_1(\mathbb{R},\mathbb{Q}) \to \bigoplus_{\alpha\in\mathbb{Q}} \mathbb{Z} \to \mathbb{Z} \to 0.$$

This implies that the map $H_1(\mathbb{R},\mathbb{Q}) \to \bigoplus_{\alpha\in Q} \mathbb{Z}$ is injective, and since subgroups of free groups are free, $H_1(\mathbb{R},\mathbb{Q})$ is a free abelian group. Since $\mathbb{Z}$ is a free $\mathbb{Z}$-module, we have

$$\bigoplus_{\alpha\in\mathbb{Q}} \mathbb{Z} \cong H_1(\mathbb{R},\mathbb{Q}) \oplus \mathbb{Z} \Rightarrow H_1(\mathbb{R},\mathbb{Q}) \cong \bigoplus_{\alpha\in\mathbb{Q}} \mathbb{Z}$$

If $\sigma_q : \Delta^0 \to \mathbb{Q}$, the set $\{\sigma_0 - \sigma_q \mid q \in Q\}$ is a basis for $H_1(\mathbb{R},\mathbb{Q})$.

9. Equivalence of Simplicial Homology & Singular Homology

Let X be a topological space that admits a Δ-complex structure. We say that a subspace $A \subseteq X$ admits a Δ-subcomplex structure on X if A is a union of simplicies of X. Relative simplicial homology group can be defined in the same way as relative (singular) homology groups. That is, the n-th relative simplicial homology group, $H_n^\Delta(X,A;\mathbb{Z})$, is the n-th homology group of the chain complex:

$$\cdots \longrightarrow \Delta_2(X)/\Delta_2(A) \xrightarrow{\overline{\partial}_2^\Delta} \Delta_1(X)/\Delta_1(A) \xrightarrow{\overline{\partial}_1^\Delta} \Delta_0(X;\mathbb{Z})/\Delta_0(A)$$

That is:

$$H_n^\Delta(X,A;\mathbb{Z}) = \frac{\operatorname{Ker} \overline{\partial}_n^\Delta}{\operatorname{Im} \overline{\partial}_{n+1}^\Delta}$$

As before, this yields a long exact sequence of simplicial homology groups for the pair $(X,A;\mathbb{Z})$ by the same algebraic argument as for singular homology. We now show that the simplicial homology groups of X corresponding to any Δ-complex structure on X coincides with its singular homology groups of X.

Proposition 9.1. *Let X be a topological space that admits a Δ-complex structure and let A be a Δ-subcomplex of X. The inclusion map*

$$\Delta_n(X,A) \hookrightarrow C_n(X,A)$$

induces an isomorphism

$$H_n^\Delta(X,A;\mathbb{Z}) \cong H_n(X,A;\mathbb{Z})$$

for each $n \geq 0$.

Remark 9.2. *Taking $A = \emptyset$, we obtain the equivalence of absolute singular and simplicial homology.*

Our strategy will be to proceed by induction X_k^Δ consisting of all simplices of dimension k or less.

Proof. We proceed in multiple steps:

(1) First suppose that X is finite dimensional. That is, $X_m^\Delta = \emptyset$ for $m \geq n$ for some $n \in \mathbb{N}$. Assume that $A = \emptyset$. Consider the following diagram:

$$\begin{tikzcd}
H_{n+1}^\Delta(X_k^\Delta, X_{k-1}^\Delta\ Z) \arrow[r] \arrow[d, "f_1"] & H_n^\Delta(X_{k-1}^\Delta\ Z) \arrow[r] \arrow[d, "f_2"] & H_n^\Delta(X_k^\Delta\ Z) \arrow[r] \arrow[d, "f_3"] & H_n^\Delta(X_k^\Delta, X_{k-1}^\Delta\ Z) \arrow[r] \arrow[d, "f_4"] & H_{n-1}^\Delta(X_{k-1}^\Delta\ Z) \arrow[d, "f_5"] \\
H_{n+1}(X_k^\Delta, X_{k-1}^\Delta\ Z) \arrow[r] & H_n(X_{k-1}^\Delta\ Z) \arrow[r] & H_n(X_k^\Delta\ Z) \arrow[r] & H_n(X_k^\Delta, X_{k-1}^\Delta\ Z) \arrow[r] & H_{n-1}(X_{k-1}^\Delta\ Z)
\end{tikzcd}$$

Note that $\Delta_k(X_k^\Delta, X_{k-1}^\Delta)$ is a free abelian group generated by k-simplices and $\Delta_n(X_k^\Delta, X_{k-1}^\Delta) = \emptyset$ for $n \neq k$. Therefore, we have:

$$\Delta_k(X_k^\Delta, X_{k-1}^\Delta) = \begin{cases} \text{free abelian group generated by } k\text{-simplices} & \text{if } n = k \\ \emptyset & \text{if } n \neq k \end{cases}$$

A simple calculation shows that:

$$H_n^\Delta(X_k^\Delta, X_{k-1}^\Delta\ Z) = \begin{cases} \mathbb{Z}^{\#k-\text{simplicies}} & \text{if } n = k \\ 0 & \text{if } n \neq k \end{cases}$$

It is easy to check that $(X_k^\Delta, X_{k-1}^\Delta)$ is a good pair and

$$X_k^\Delta / X_{k-1}^\Delta = \bigvee_{i=1}^{\#k-\text{simplicies}} \mathbb{S}^k$$

Therefore, Corollary 8.7 implies

$$H_n(X_k^\Delta, X_{k-1}^\Delta\ Z) = \begin{cases} \mathbb{Z}^{\#k-\text{simplicies}} & \text{if } n = k \\ 0 & \text{if } n \neq k \end{cases}$$

Therefore, both f_1 and f_4 are isomorphisms. An induction argument shows that f_2 and f_5 are isomorphisms. The five lemma (Proposition 5.18) then implies that f_3 is an isomorphism.

(2) Suppose that X is possibly infinite-dimensional. Assume that $A = \emptyset$. Note that a compact set $C \subseteq X$ can meet only finitely many open simplices of X. If not, C would contain an infinite sequence of points x_i, each lying in a different open simplex. Then the sets

$$U_i = X - \bigcup_{i \neq j} \{x_j\}$$

which are open since their pre-images under the characteristic maps of all the simplices are clearly open, form an open cover of C with no finite sub-cover. This can be applied to show the map

$$H_n^\Delta(X; \mathbb{Z}) \to H_n(X; \mathbb{Z})$$

is bijective. For surjectivity, let $[c] \in H_n^\Delta(X; \mathbb{Z})$. Choose a representative n-cycle, α, of $[c]$. Now α is a linear combination of finitely many singular simplices with compact images, meeting only finitely many open simplices of X. Hence, α in X_k^Δ for some k. We have shown that

$$H_n(X_k^\Delta\ Z) \cong H_n^\Delta(X_k^\Delta\ Z)$$

So there exists a n-cycle $v \in \Delta_n(X_k^{\Delta})$ such that $[v]$ gets mapped to $[c]$. This proves surjectivity. Injectivity is similar so we omit details.

(3) Now consider the general case where $A \neq \emptyset$. Consider the following diagram:

$$\begin{tikzcd}
H_{n+1}^{\Delta}(X,A;\mathbb{Z}) \arrow[r] \arrow[d, "f_1"] & H_n^{\Delta}(A;\mathbb{Z}) \arrow[r] \arrow[d, "f_2"] & H_n^{\Delta}(X;\mathbb{Z}) \arrow[r] \arrow[d, "f_3"] & H_n^{\Delta}(X,A;\mathbb{Z}) \arrow[r] \arrow[d, "f_4"] & H_{n-1}^{\Delta}(A;\mathbb{Z}) \arrow[d, "f_5"] \\
H_{n+1}(X,A;\mathbb{Z}) \arrow[r] & H_n(A;\mathbb{Z}) \arrow[r] & H_n(X;\mathbb{Z}) \arrow[r] & H_n(X,A;\mathbb{Z}) \arrow[r] & H_{n-1}(A;\mathbb{Z})
\end{tikzcd}$$

By (2), f_2, f_3, f_5 are isomorphisms. The claim now follows by induction and the five-lemma.

This completes the proof. $\square$

Example 9.3. Let $X = \mathbb{S}^1 \times \mathbb{S}^1$. Note that X is homemorphic to the torus, T, considered in Section 4. Hence, Proposition 9.1 implies that

$$H_n(X;\mathbb{Z}) = H_n^{\Delta}(T;\mathbb{Z}) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & \text{for } n = 1 \\ \mathbb{Z} & \text{for } n = 0, 2 \\ 0 & \text{for } n \geq 3 \end{cases}$$

Consider $Y = \mathbb{S}^1 \vee \mathbb{S}^1 \vee \mathbb{S}^2$. We have

$$H_n(Y) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & \text{for } n = 1 \\ \mathbb{Z} & \text{for } n = 0, 2 \\ 0 & \text{for } n \geq 3 \end{cases}$$

We see that

$$H_n(X;\mathbb{Z}) = H_n(Y)$$

for $n \geq 0$. It can be checked that the covering spaces of X and Y have different homology groups. Hence, X and Y are not homotopy equivalent. Therefore, homology groups might not be able to distinguish topological spaces that are not homotopy equivalent.

Part 3. **Computationals & Applications**

10. Mayer-Vietoris Sequence

In addition to the long exact sequence of homology groups for a pair (X, A), there is another sort of long exact sequence, known as a Mayer–Vietoris sequence, which is equally powerful but is sometimes more convenient to use. The Mayer–Vietoris sequence is also applied frequently in induction arguments, where one might know that a certain statement is true for A, B, and $A \cap B$ by induction and then deduce that it is true for $A \cup B$ by the exact sequence[11].

[11]Mayer–Vietoris sequence can also be thought of as an abelianization of the Seifert Van Kampen Theorem.

Lemma 10.1. *(**Barrett-Whitehead Lemma**) Suppose we have the following commutative diagram of abelian groups, where the two rows are exact:*

$$\begin{tikzcd}
\cdots \arrow[r] & A_n \arrow[r, "i_n"] \arrow[d, "f_n"] & B_n \arrow[r, "j_n"] \arrow[d, "g_n"] & C_n \arrow[r, "k_n"] \arrow[d, "h_n"] & A_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \cdots \\
\cdots \arrow[r] & A'_n \arrow[r, "i'_n"] & B'_n \arrow[r, "j'_n"] & C'_n \arrow[r, "k'_n"] & A'_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

Assume each map $h_n : C_n \to C_{n-1}$ is an isomorphism. Then there is a long exact sequence

$$\cdots \longrightarrow A_n \xrightarrow{(i_n, f_n)} B_n \oplus A'_n \xrightarrow{g_n - i'_n} B'_n \xrightarrow{k_n h_n^{-1} j'_n} A_{n-1} \longrightarrow \cdots$$

Proof. The proof is by a diagram chase. We omit details. □

Proposition 10.2. *(**Mayer-Viertoris Sequence**) Let $X_1, X_2 \subseteq X$ be open sets such that $X = X_1 \cup X_2$. Let*

$$i_1 : X_0 \hookrightarrow X_1, \qquad i_2 : X_1 \hookrightarrow X$$

denote inclusions for $i = 1, 2$. Then there is a long exact sequence

$$\cdots \to H_n(X_1 \cap X_2; \mathbb{Z}) \to H_n(X_1; \mathbb{Z}) \oplus H_n(X_2; \mathbb{Z}) \to H_n(X; \mathbb{Z}) \to H_{n-1}(X_1 \cap X_2; \mathbb{Z}) \to \cdots$$

Proof. We have the following diagram:

$$\begin{tikzcd}
(X_1 \cap X_2, \emptyset) \arrow[r, "i_1"] \arrow[d, "i_2"] & (X_1, \emptyset) \arrow[r, "f"] \arrow[d, "j_1"] & (X_1, X_1 \cap X_2) \arrow[d, "h"] \\
(X_2, \emptyset) \arrow[r, "j_2"] & (X, \emptyset) \arrow[r, "g"] & (X, X_2)
\end{tikzcd}$$

Applying Remark 7.9 yields the following diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & H_n(X_1 \cap X_2; \mathbb{Z}) \arrow[r, "H_n(i_n)"] \arrow[d, "H_n(f_n)"'] & H_n(X_1; \mathbb{Z}) \arrow[r, "H_n(j_n)"] \arrow[d, "H_n(g_n)"'] & H_n(X_1, X_1 \cap X_2; \mathbb{Z}) \arrow[r, "\delta_n"] \arrow[d, "H_n(h_n)"'] & H_n(X_1 \cap X_2; \mathbb{Z}) \arrow[r] \arrow[d, "H_n(i_2)"] & \cdots \\
\cdots \arrow[r] & H_n(X_2; \mathbb{Z}) \arrow[r, "H_n(i'_n)"] & H_n(X; \mathbb{Z}) \arrow[r, "H_n(j'_n)"] & H_n(X, X_2; \mathbb{Z}) \arrow[r, "\delta'_n"] & H_n(X_2; \mathbb{Z}) \arrow[r] & \cdots
\end{tikzcd}$$

The excision axioms implies that $H_n(h_n)$ is an isomorphism for each $n \geq 0$. Lemma 10.1 then implies the existence of the desired long exact sequence. □

Remark 10.3. *By using augmented chain complexes, we also obtain a corresponding Mayer-Vietoris sequence for the reduced homology groups. We omit details.*

Remark 10.4. *We can also use the Mayer-Viertoris sequence to compute the homology groups of sphere. Indeed, consider the following argument. Let $X = \mathbb{S}^n$, $A = \mathbb{S}^n \setminus \{S\}$, and $B = \mathbb{S}^n \setminus \{N\}$, where S and N are the south pole and north pole, respectively. Then*

$$A \simeq \mathbb{R}^n \quad B \simeq \mathbb{R}^n \quad A \cap B \simeq \mathbb{S}^{n-1}$$

From the Mayer-Vietoris sequence for reduced homology groups, we get $\widetilde{H}_k(\mathbb{S}^n) \simeq \widetilde{H}_{k-1}(\mathbb{S}^{n-1})$ for all i. By induction, we find as before:

$$\widetilde{H}_k(\mathbb{S}^n; \mathbb{Z}) \simeq \begin{cases} \mathbb{Z}, & k = n \\ 0, & k \neq n. \end{cases}$$

Proposition 10.5. ***(Suspension Theorem)*** *Let X be a topological space and let SX be its suspension. We have*

$$\widetilde{H}_n(X;\mathbb{Z}) \cong \widetilde{H}_{n+1}(SX;\mathbb{Z})$$

for $n \geq -1$.

Proof. For $n = -1$, $\widetilde{H}_{-1}(X;\mathbb{Z})$ is the trivial group. Since SX is path-connected, $\widetilde{H}_0(SX;\mathbb{Z})$ is also the trivial group. Let $n \geq 0$. Let P, Q denote the collapsed spaces $X \times \{0\}$ and $X \times \{1\}$ respectively. Let $A = SX - \{P\}$ and let $B = SX - \{Q\}$. Each of A and B are homeomorphic to the cone space

$$CX = (X \times I)/(X \times \{0\})$$

By the Mayer-Vietoris sequence for reduced homology, since $A \cap B = X \times (0,1)$, we obtain the exact sequence

$$\cdots \to \widetilde{H}_{n+1}(A;\mathbb{Z}) \oplus \widetilde{H}_{n+1}(B;\mathbb{Z}) \to \widetilde{H}_{n+1}(SX;\mathbb{Z}) \to \widetilde{H}_n(A \cap B;\mathbb{Z}) \to \widetilde{H}_n(A;\mathbb{Z}) \oplus \widetilde{H}_n(B;\mathbb{Z}) \to \cdots$$

for all n. Note that CX is contractible[12]. Moreover, $X \times (0,1)$ deformation retracts down to X. Hence, the sequence simplifies to:

$$\cdots \to 0 \to \widetilde{H}_{n+1}(SX;\mathbb{Z}) \to \widetilde{H}_n(X;\mathbb{Z}) \to 0 \to \cdots$$

This proves the claim. □

11. Degree Theory

We focus on the application of homology theory to degree theory (and fixed point theory) in this section. We shall see that we can use homology theory to study continuous maps between spheres. This allows us to introduce and study the notion of degree of a map between spheres. Moreover, degree of maps between spheres will play a fundamental role in our being able to compute cellular homology, which we discuss in the next section.

Definition 11.1. The degree of a continuous map $f : \mathbb{S}^n \to \mathbb{S}^n$ is defined as:

$$\deg f := H_n(f)(1)$$

where

$$H_n(f) : H_n(\mathbb{S}^n;\mathbb{Z}) \cong \mathbb{Z} \to H_n(\mathbb{S}^n;\mathbb{Z}) \cong \mathbb{Z}$$

is the homomorphism induced by f in homology, and $1 \in \mathbb{Z}$ denotes the generator.

In what follows, we write $H_n(f)$ as f_n.

Proposition 11.2. *Here are some basic properties of degree:*

(1) $\deg(\mathrm{Id}_{\mathbb{S}^n}) = 1$.
(2) *If f is not surjective, then* $\deg(f) = 0$.
(3) *If $f \simeq g$ are homotopic maps, then* $\deg(f) = \deg(g)$.
(4) $\deg(g \circ f) = \deg(g) \cdot \deg(f)$.
(5) *If f is a homotopy equivalence, then* $\deg(f) = \pm 1$.

Proof. The proof is given below:

(1) This follows because $(\mathrm{Id}_{\mathbb{S}^n})_n = \mathrm{Id}_{\mathbb{Z}}$ which is multiplication by the integer 1.

[12] Indeed, the homotopy $h_t(x,s) = (x,(1-t)s)$ continuously shrinks CX down to its vertex point.

(2) Indeed, if f is not surjective, there is some $y \notin \mathrm{Im}(f)$. Then we can factor f in the following way:
$$\mathbb{S}^n \to \mathbb{S}^n \setminus \{y\} \to \mathbb{S}^n$$
Since $\mathbb{S}^n \setminus \{y\} \simeq \mathbb{R}^n$ is contractible, $H_n(\mathbb{S}^n \setminus \{y\}) = 0$. Therefore, $f_n = 0$, so $\deg(f) = 0$.

(3) This follows because because if f and g are homotopic, then $f_n = g_n$ for $n \geq 0$.

(4) This is clear.

(5) By definition, there exists a map $g : \mathbb{S}^n \to \mathbb{S}^n$ so that $g \circ f \simeq \mathrm{Id}_{\mathbb{S}^n}$ and $f \circ g \simeq \mathrm{Id}_{\mathbb{S}^n}$. The claim follows directly from previous results, since $f \circ g \simeq \mathrm{id}_{\mathbb{S}^n}$ implies that $\deg(f) \cdot \deg(g) = \deg(\mathrm{Id}_{\mathbb{S}^n}) = 1$.

This completes the proof. □

We now prove a far less obvious result:

Proposition 11.3. *Let $n \geq 1$ and let $A \in O(n+1)$ denote an orthogonal linear transformation. Set $f := A|_{\mathbb{S}^n}$. Then $\deg(f) = \det(A) = \pm 1$.*

Proof. The group $O(n+1)$ has two connected components, $O^+(n+1)$ and $O^-(n+1)$, distinguished by det: $O(n+1) \to \{+1, -1\}$. By homotopy invariance, it suffices to check the result for one such A in each component. Note that $I_{n+1} \in O^+(n+1)$ has degree 1. Hence, every $A \in O^+(n+1)$ has degree one. Consider $O^-(n+1)$. We take A to be reflection in a hyperplane $H \subseteq \mathbb{R}^{n+1}$. WLOG, we can assume the hyperplane is $\mathbb{R}^n \times \{0\} \subset \mathbb{R}^{n+1}$ and
$$A(x_0, x_1, \ldots, x_n) = (x_0, x_1, \ldots, -x_n).$$

The upper and lower hemispheres U and L of $\mathbb{S}^n$ can be regarded as singular n-simplices, via their standard homeomorphisms with Δ^n [13]. Then the generator of $H_n(\mathbb{S}^n)$ is $[U - L]$. The reflection map r maps the cycle $U - L$ to $L - U = -(U - L)$. So
$$H_n(r)([U - L]) = [L - U] = [-(U - L)] = -1 \cdot [U - L],$$
so $\deg(r) = -1$. □

Corollary 11.4. *If $a : \mathbb{S}^n \to \mathbb{S}^n$ is the antipodal map, then $\deg(a) = (-1)^{n+1}$.*

Proof. Note that a is a composition of $n+1$ reflections since there are $n+1$ coordinates in x, each changing sign by an individual reflection. From above, we know that the composition of maps leads to multiplication of degrees. □

This immediately gives a proof of the following famous result.

Proposition 11.5. ***(Hairy Ball Theorem)*** *$\mathbb{S}^n$ has a continuous vector field of nonzero tangent vectors if and only if n is odd.*

Proof. Suppose $x \mapsto v(x)$ is a tangent vector field on $\mathbb{S}^n$, assigning to a vector $x \in \mathbb{S}^n$ the vector $v(x)$ tangent to $\mathbb{S}^n$ at x. Regarding $v(x)$ as a vector at the origin instead of at x, we have that $x \perp v(x)$ for each $x \in \mathbb{S}^n$. If $v(x) \neq 0$ for all x, we may normalize so that $|v(x)| = 1$ for all x by replacing $v(x)$ by $v(x)/|v(x)|$. Consider
$$\begin{aligned} W : \mathbb{S}^n \times I &\to \mathbb{S}^n, \\ (x,t) &\mapsto (\cos(\pi t))x + (\sin(\pi t))v(x). \end{aligned}$$

[13] Here we are implicitly using the result that if $C \subseteq \mathbb{R}^n$ is a compact convex subset with nonempty interior, then C is a closed n-cell (and its interior is an open n-cell). That is, $C \cong \mathbb{D}^n$.

Note that W indeed takes values in $\mathbb{S}^n$ since $x \perp v(x)$ for each $x \in \mathbb{S}^n$. Hence, we obtain a homotopy from the identity map on $\mathbb{S}^n$ to the antipodal map on $\mathbb{S}^n$. This implies that $(-1)^{n+1} = 1$ and n must be odd. Conversely, if n is odd, say $n = 2k-1$, we can define $v : \mathbb{R}^{2k} \to \mathbb{R}^{2k}$ by

$$v(x_1, x_2, \dots, x_{2k-1}, x_{2k}) = (-x_2, x_1, \dots, -x_{2k}, x_{2k-1})$$

Then $v(x) \perp x$ for each $x \in \mathbb{S}^n$. Hence, v is a vector field on $\mathbb{S}^n$, and it is non-vanishing since $|v(x)| = 1$ for all $x \in \mathbb{S}^n$. $\square$

11.1. **Applications to Fixed Point Theory.** Let's note some results in fixed point theory that can be easily proved using degree theory.

(1) If $f : \mathbb{S}^n \to \mathbb{S}^n$ has no fixed points, then $\deg(f) = (-1)^{n+1}$. Since $f(x) \neq x$, the segment

$$(1-t)f(x) + t(-x)$$

from $-x$ to $f(x)$ does not pass through the origin in $\mathbb{R}^{n+1}$. So we can normalize to obtain a homotopy:

$$\begin{aligned} g_t(x) : \mathbb{S}^n &\to \mathbb{S}^n \\ x &\mapsto \frac{(1-t)f(x) + t(-x)}{\|(1-t)f(x) + t(-x)\|} \end{aligned}$$

Note that this homotopy is well defined since $(1-t)f(x) - tx \neq 0$ for any $x \in \mathbb{S}^n$ and $t \in [0,1]$, because $f(x) \neq x$ for all x. Then g_t is a homotopy from f to a, the antipodal map, and the result follows.

(2) (**Brouwer's Fixed Point Theorem)** Let $n \geq 1$ and suppose that a continuous map $f : \mathbb{D}^n \to \mathbb{D}^n$ has no fixed points. Since $\mathbb{D}^n \cong \mathbb{S}^n_+$, where $\mathbb{S}^n_+$ is the northern hemisphere, f can be thought of a map from $\mathbb{S}^n_+$ to $\mathbb{S}^n_+$. Now we can extend f to a map on $\mathbb{S}^n$ as follows. We define

$$g(x) = \begin{cases} f(x), & \text{if } x \in \mathbb{S}^n_+, \\ f \circ r(x), & \text{if } x \in \mathbb{S}^n_- \end{cases}$$

where $r(x)$ is reflection about the plane through the equator and $\mathbb{S}^n_+$ is the southern hemisphere. It is clear that $g(x)$ is a continuous function; furthermore $g(x)$ has no fixed points. By (1), g is homotopic to the antipodal map on $\mathbb{S}^n$ that has degree $(-1)^{n+1}$. Clearly, g is not surjective[14]. It follows that $\deg g = 0$. But $(-1)^{n+1} \neq 0$ for $n \geq 1$. Hence, every continuous map $f : \mathbb{D}^n \to \mathbb{D}^n$ has a fixed point.

(3) Consider a continuous map $f : \mathbb{S}^{2n} \to \mathbb{S}^{2n}$. Then either f or $-f$ must have a fixed point. If f and $-f$ don't have fixed points, then f and $-f$ are homotopic to the antipodal map. Thus f and $-f$ have degree -1. But both f and $-f$ both cannot have the same degree. Hence there is a point $x \in \mathbb{S}^{2n}$ such that $f(x) = \pm x$.

(4) Consider a continuous map $f : \mathbb{S}^n \to \mathbb{S}^n$ of degree zero. Then there exist points $x, y \in \mathbb{S}^n$ for such that $f(x) = x$ and $f(y) = -y$. If not, then f and $-f$ are homotopic to the antipodal map. But then degree of $\pm f$ is $0 \neq \pm(-1)^{n+1}$. Hence, there are points $x, y \in \mathbb{S}^n$ for such that $f(x) = x$ and $f(y) = -y$.

[14] Because no point in the southern hemisphere is in the image.

Remark 11.6. *Brouwer's fixed point thereom can be derived without resorting to degree theory. First, note that for $n \geq 1$ there does not exist a continuous map $r : \mathbb{D}^n \to \partial\mathbb{D}^n$ such that $r(x) = x$ for $x \in \partial\mathbb{D}^n$. Assume by contradiction that there exists a retraction $r : \mathbb{D}^n \to \partial\mathbb{D}^n = \mathbb{S}^{n-1}$. Then, if $i : \mathbb{S}^{n-1} \to \mathbb{D}^n$ is the inclusion, we have $r \circ i = Id_{\mathbb{S}^{n-1}}$.*

$$\begin{tikzcd} \mathbb{S}^{n-1} \arrow[rr, "Id"] \arrow[dr, "i"'] & & \mathbb{S}^{n-1} \\ & \mathbb{D}^n \arrow[ur, "r"'] & \end{tikzcd}$$

If $n > 1$, we then have:

$$\begin{tikzcd} \mathbb{Z} \cong H_{n-1}(\mathbb{S}^{n-1};\mathbb{Z}) \arrow[rr, "Id"] \arrow[dr, "0"] & & \mathbb{H}_{n-1}(\mathbb{S}^{n-1};\mathbb{Z}) \cong \mathbb{Z} \\ & H_{n-1}(\mathbb{D}^n;\mathbb{Z}) = 0 \arrow[ur, "0"] & \end{tikzcd}$$

This is a contradiction. A similar argument can be made in the case $n = 0$. We can now prove Brouwer's fixed point thereom: let $f : \mathbb{D}^n \to \mathbb{D}^n$ be a continuous map. Assume by contradiction that $f(x) \neq x$ for all $x \in \mathbb{D}^n$. Then, we may define a function $r : \mathbb{D}^n \to \mathbb{S}^{n-1}$ in the following way. Let $x \in \mathbb{D}^n$ and let $[f(x), x)$ denote the (unique) ray based at $f(x)$ passing through x. Define $r(x)$ to be the unique element in $([f(x), x)) \cap \partial\mathbb{D}^n \setminus \{f(x)\}$. Then, r is continuous and is a retraction $\mathbb{D}^n \to \partial\mathbb{D}^n$, a contradiction.

Remark 11.7. *Here is a cute application of (3) above. Consider $g : \mathbb{RP}^{2n} \to \mathbb{RP}^{2n}$. By covering space theory, the map g lifts to a map $\hat{g} : \mathbb{RP}^{2n} \to \mathbb{S}^{2n}$. Define f as in the diagram below:*

$$\begin{tikzcd} \mathbb{S}^{2n} \arrow[r, dotted, "f"] \arrow[d] & \mathbb{S}^{2n} \arrow[d] \\ \mathbb{RP}^{2n} \arrow[r, "g"'] \arrow[ur, dotted, "\tilde{g}"] & \mathbb{RP}^{2n} \end{tikzcd}$$

Choose $x \in \mathbb{S}^{2n}$ such that $f(x) = \pm x$. Then the point $[x] \in \mathbb{RP}^{2n}$ is a fixed point for g[15]. We have just seen that any linear transformation of $\mathbb{R}^{2n+1}$ has a real eigenvalue! The analogous result is not necessarily true for $\mathbb{R}^{2n}$ for $n \geq 1$. Consider the following matrix:

$$R = \begin{pmatrix} R_{\theta_1} & 0 & \dots & 0 \\ 0 & R_{\theta_1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & R_{\theta_n} \end{pmatrix}$$

Here $\mathbb{R}_{\theta_i}$ is a 2-by-2 matrix that represents a rotation through angle θ_i. Clearly, R maps $\mathbb{S}^{2n-1}$ to $\mathbb{S}^{2n-1}$, but it has no fixed point since each R_{θ_i} has no (real) eigenvector.

[15]For $m \geq$, recall that $\mathbb{RP}^m = \mathbb{S}^{m+1}/\sim$ where we identify antipodal points in $\mathbb{S}^m$. Covering space theory tells us that $\mathbb{S}^m$ is the universal covering space of $\mathbb{RP}^m$. Hence, the map g lifts since $\mathbb{S}^{m+1}$ is simply-connected.

Remark 11.8. *Here is a cute application of (4) above. Let $F : \mathbb{D}^n \to \mathbb{D}^n$ be a continuous function such that $F(x) \neq 0$ for $x \in \mathbb{D}^n$. We can then consider the map*

$$G(x) = \frac{F(x)}{\|F(x)\|}$$

Note that G maps $\mathbb{S}^{n-1}$ to $\mathbb{S}^{n-1}$. Moreover, G has degree zero since $\mathbb{D}^n$ has trivial homology. Hence, there are points $x, y \in \mathbb{S}^{n-1}$ such that $G(x) = x$ and $G(y) = -y$. If we think of F defining a vector field on $\mathbb{D}^n$, then this meas that there exists a point on$\mathbb{S}^{n-1}$ where the vector field points radially outward and another point on $\mathbb{S}^{n-1}$ where where the vector field points radially inward.

Remark 11.9. *Consider a continuous map $f : \mathbb{S}^n \to \mathbb{S}^n$ for $n \geq 1$. If f doesn't have a fixed point, then f is homotopic to the antipodal map. If n is odd, then the linear transformation represented by the matrix*

$$R_t = \begin{pmatrix} R_{t\pi} & 0 & \dots & 0 \\ 0 & R_{t\pi} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & R_{t\pi} \end{pmatrix}$$

is homotopy between the antipodal map and the identity map. If n is even, then we can apply the argument above to the ”equatorial sphere” $\mathbb{S}^{n-1} \subset \mathbb{S}^n$. In any case, we see that if f is homotopic to continuous map with a fixed point.

11.2. **Local Degrees.** How to compute degrees, though? We describe a technique for computing degrees which can be applied to most maps that arise in practice. Assume $f : \mathbb{S}^n \to \mathbb{S}^n$ is surjective, and that f has the property that there exists some $y \in \mathrm{Im}(\mathbb{S}^n)$ so that $f^{-1}(y)$ is a finite number of points, say $f^{-1}(y) = \{x_1, x_2, \dots, x_m\}$. Let U_i be a neighborhood of x_i such that all U_i's get mapped to some neighborhood V of y. So

$$f(U_i \setminus \{x_i\}) \subseteq V \setminus \{y\}$$

$$\begin{tikzcd}
 & \mathbb{Z} \cong H_n(U_i, U_i \setminus x_i; \mathbb{Z}) \arrow[r, "f_n|_{x_i}"] \arrow[dl, "\cong"'] \arrow[d, "k_i"] & \mathbb{Z} \cong H_n(V, V \setminus y; \mathbb{Z}) \arrow[d, "\cong"] \\
\mathbb{Z} \cong H_n(\mathbb{S}^n, \mathbb{S}^n \setminus x_i; \mathbb{Z}) & H_n(\mathbb{S}^n, \mathbb{S}^n - f^{-1}(y); \mathbb{Z}) \arrow[l, "p_i"'] \arrow[r, "f_n"] & H_n(\mathbb{S}^n, \mathbb{S}^n \setminus y; \mathbb{Z}) \\
 & \mathbb{Z} \cong \widetilde{H}_n(\mathbb{S}^n; \mathbb{Z}) \arrow[r, "f_n"] \arrow[ul, "\cong"] \arrow[u, "j"'] & \mathbb{Z} \cong \widetilde{H}_n(\mathbb{S}^n; \mathbb{Z}) \arrow[u, "\cong"']
\end{tikzcd}$$

As f is continuous, we can choose the U_i's to be disjoint. Let $f|_{x_i} : U_i \to V$ be the restriction of f to U_i, with the induced homomorphism

$$f_n|_{x_i} : H_n(U_i, U_i \setminus \{x_i\}; \mathbb{Z}) \to H_n(V, V \setminus \{y\}; \mathbb{Z}).$$

Define the *local* degree of f at x_i, $\deg(f|_{x_i})$, to be the degree of $f_n|_{x_i}$. We then have the following result:

Lemma 11.10. *The degree of f equals the sum of local degrees at points in a generic finite fiber, that is,*

$$\deg(f) = \sum_{i=1}^{m} \deg(f|_{x_i}).$$

Proof. Consider the commutative diagram above. The two isomorphisms in the upper half of the diagram come from excision, while the lower two isomorphisms come from exact sequences of pairs. The maps k_i and p_i are induced by inclusions, so the triangles and squares commute. The map j comes from the long exact sequence in homology. By excision,

$$H_n(\mathbb{S}^n, \mathbb{S}^n - f^{-1}(y); \mathbb{Z}) \cong \bigoplus_{i=1}^{m} H_n(U_i, U_i \setminus x_i; \mathbb{Z}) \cong \mathbb{Z}^m$$

The map p_i is the projection onto the i-th summand. We have:

$$k_i(1) = (0, \dots, 0, 1, 0, \dots, 0)$$

where the entry 1 is in the ith place. Also, $p_i \circ j(1) = 1$ for all i, so

$$j(1) = (1, 1, \dots, 1) = \sum_{i=1}^{m} k_i(1).$$

Commutativity of the upper square says that the middle f_n takes $k_i(1)$ to $\deg f\,|_{x_i}$, hence the sum $\sum_i k_i(1) = j(1)$ is taken to $\sum_i \deg f|_{x_i}$. The commutativity of the lower square gives:

$$\deg f = f_n j(1) = f_n\left(\sum_{i=1}^{m} k_i(1)\right) = \sum_{i=1}^{m} \deg f|_{x_i}.$$

This completes the proof. $\square$

12. Cellular Homology

We define the cellular homology of a CW complex X in terms of a given cell structure, then we show that it coincides with the singular homology, so it is in fact independent on the cell structure. Cellular homology is very useful for computations. Before discussing cellular homology, we compute the relative homology groups of a topological space, X, that can be given the structure of a CW complex.

Lemma 12.1. *Let X be a topological space that can be endowed with the structure of a CW complex. Then:*

1. *The relative romology $H_k(X^n, X^{n-1}; \mathbb{Z})$ is given by:*

$$H_k(X^n, X^{n-1}; \mathbb{Z}) = \begin{cases} 0, & \text{if } k \neq n \\ \mathbb{Z}^{\#n\text{-cells}}, & \text{if } k = n. \end{cases}$$

   for $k \geq 1$.
2. *$H_k(X^n; \mathbb{Z}) = 0$ if $k > n \geq 1$. In particular, if X is finite dimensional, then $H_k(X; \mathbb{Z}) = 0$ if $k > \dim(X)$.*
3. *The inclusion $i : X^n \hookrightarrow X$ induces an isomorphism $H_k(X^n; \mathbb{Z}) \cong H_k(X)$ if $k < n$.*

Proof. The proof is given below:

(1) Since (X^n, X^{n-1}) is a good pair, we have:

$$\begin{aligned} H_k(X^n, X^{n-1}; \mathbb{Z}) &\cong \widetilde{H}_k(X^n/X^{n-1}; \mathbb{Z}) \\ &= H_k(X^n/X^{n-1}; \mathbb{Z}) \\ &\cong \bigvee_{i=1}^{\#n\text{-cells}} \mathbb{S}^n \cong \begin{cases} 0, & \text{if } k \neq n, \\ \mathbb{Z}^{\#n\text{-cells}}, & \text{if } k = n. \end{cases} \end{aligned}$$

(2) Since (X^n, X^{n-1}) is a good pair for each $n \geq 1$, we can consider the following portion of the long exact sequence:

$$H_{k+1}(X^n, X^{n-1}; \mathbb{Z}) \longrightarrow H_k(X^{n-1}; \mathbb{Z}) \longrightarrow H_k(X^n; \mathbb{Z}) \longrightarrow H_k(X^n, X^{n-1}; \mathbb{Z})$$

If $k+1 \neq n$ and $k \neq n$, we have from (1) we have that $H_{k+1}(X^n, X^{n-1}; \mathbb{Z}) = 0$ and $H_k(X^n, X^{n-1}) = 0$. Thus

$$H_k(X^{n-1}; \mathbb{Z}) \cong H_k(X^n; \mathbb{Z})$$

Hence, if $k > n$ (so in particular, $n \neq k+1$ and $n \neq k$), we get by iteration that

$$H_k(X^n; \mathbb{Z}) \xrightarrow{\cong} H_k(X^{n-1}; \mathbb{Z}) \xrightarrow{\cong} \cdots \xrightarrow{\cong} H_k(X^0; \mathbb{Z})$$

Note that X^0 is just a collection of points, so $H_k(X^0; \mathbb{Z}) = 0$. Thus, when $k > n \geq 1$, we have $H_k(X^n; \mathbb{Z}) = 0$ as desired.

(3) We only prove the statement for finite-dimensional CW complexes. Let $k < n$, and consider the following portion of the long exact sequence:

$$\cdots \to H_{k+1}(X^{n+1}, X^n; \mathbb{Z}) \to H_k(X^n; \mathbb{Z}) \to H_k(X^{n+1}; \mathbb{Z}) \to H_k(X^{n+1}, X^n; \mathbb{Z}) \to \cdots$$

Since $k < n$, we have $k+1 \neq n+1$ and $k \neq n+1$, so by part (1), we get that $H_{k+1}(X^{n+1}, X^n; \mathbb{Z}) = 0$ and $H_k(X^{n+1}, X^n; \mathbb{Z}) = 0$. Thus, $H_k(X^n) \cong H_k(X^{n+1}; \mathbb{Z})$. By repeated iteration, we obtain:

$$H_k(X^n; \mathbb{Z}) \cong H_k(X^{n+1}; \mathbb{Z}) \cong H_k(X^{n+2}; \mathbb{Z}) \cong \cdots \cong H_k(X^{n+l}; \mathbb{Z}) = H_k(X; \mathbb{Z}),$$

where l is such that $X^{n+l} = X$ since we assumed X is finite dimensional. See [Hat02] for the case when X is infinite-dimensional.

This completes the proof. □

In what follows we define the cellular homology of a CW complex, X, in terms of a given cell structure, then we show that it coincides with the singular homology.

Definition 12.2. The cellular homology $H^{\mathrm{CW}}(X)$ of a CW complex X is the homology of the cellular chain complex $(C_*(X), d_*)$ indexed by the cells of X, i.e.,

$$C_n(X) := H_n(X^n, X^{n-1}; \mathbb{Z}) = \mathbb{Z}^{\#n\text{-cells}},$$

and with differentials $d_n : C_n(X) \to C_{n-1}(X)$ defined by the following diagram: d_n etc. are defined in the obvious way to make the diagram commute. It is easy to check that $d_{n+1} \circ d_n = 0$ since the composition of these two maps induces two successive maps in one of the diagonal exact sequences.

$$\begin{tikzcd}
 & & & H_n(X^{n+1}, X^n; \mathbb{Z}) = 0 & \\
0 = H_n(X^{n-1}; \mathbb{Z}) \arrow[dr] & & H_n(X^{n+1}; \mathbb{Z}) \cong H_n(X; \mathbb{Z}) \arrow[ur] & & \\
 & H_n(X^n; \mathbb{Z}) \arrow[ur, "i_n"] \arrow[dr, "j_n"] & & & \\
H_{n+1}(X^{n+1}, X^n; \mathbb{Z}) \arrow[ur, "\partial_{n+1}"] \arrow[rr, "d_{n+1}"] & & H_n(X^n, X^{n-1}; \mathbb{Z}) \arrow[rr, "d_n"] \arrow[dr, "\partial_n"] & & H_{n-1}(X^{n-1}, X^{n-2}; \mathbb{Z}) \\
 & & & H_{n-1}(X^{n-1}; \mathbb{Z}) \arrow[ur, "j_{n-1}"] & \\
 & & H_{n-1}(X^{n-2}; \mathbb{Z}) = 0 \arrow[ur] & &
\end{tikzcd}$$

Proposition 12.3. *Let X be a topological space that admits a CW-complex structure. We have:*

$$H_n^{\mathrm{CW}}(X) \cong H_n(X; \mathbb{Z})$$

for all n, where $H_n(X; \mathbb{Z})$ is the singular homology of X.

Proof. Since $H_n(X^{n+1}, X^n; \mathbb{Z}) = 0$ and $H_n(X; \mathbb{Z}) \cong H_n(X^{n+1}; \mathbb{Z})$, we get from the diagram above that

$$H_n(X; \mathbb{Z}) \cong \frac{H_n(X^n; \mathbb{Z})}{\ker i_n} \cong \frac{H_n(X^n; \mathbb{Z})}{\operatorname{Im} \partial_{n+1}}.$$

Now, $H_n(X^n; \mathbb{Z}) \cong \operatorname{Im} j_n \cong \ker \partial_n \cong \ker d_n$. The first isomorphism comes from j_n being injective, while the second follows by exactness. Finally, $\ker \partial_n = \ker d_n$ since $d_n = j_{n-1} \circ \partial_n$ and j_{n-1} is injective. Also, we have $\operatorname{Im} \partial_{n+1} = \operatorname{Im} d_{n+1}$. Indeed, $d_{n+1} = j_n \circ \partial_{n+1}$ and j_n is injective. Altogether, we have

$$H_n(X; \mathbb{Z}) \cong \frac{\ker d_n}{\operatorname{Im} d_{n+1}} = H_n^{\mathrm{CW}}(X).$$

This completes the proof. $\square$

Let's make some observations which are immediate:

(1) If X has no n-cells, then $H_n(X; \mathbb{Z}) = 0$. Indeed, in this case we have $C_n = H_n(X^n, X^{n-1}; \mathbb{Z}) = 0$. Therefore, $H_n^{\mathrm{CW}}(X; \mathbb{Z}) = 0$.
(2) If X is connected and has a single 0-cell, then $d_1 : C_1 \to C_0$ is the zero map. Indeed, since X contains only a single 0-cell, $C_0 = \mathbb{Z}$. Also, since X is connected, $H_0(X) = \mathbb{Z}$. So, by the above theorem, $\mathbb{Z} = H_0(X; \mathbb{Z}) = \ker d_0 / \operatorname{Im} d_1 = \mathbb{Z} / \operatorname{Im} d_1$. This implies that $\operatorname{Im} d_1 = 0$, so d_1 is the zero map as desired.
(3) If X has no cells in adjacent dimensions, then $d_n = 0$ for all n, and $H_n(X; \mathbb{Z}) \cong \mathbb{Z}^{\#n\text{-cells}}$ for all n. Indeed, in this case, all maps d_n vanish. So for any n, $H_n^{\mathrm{CW}}(X) \cong C_n \cong \mathbb{Z}^{\#n\text{-cells}}$.

Example 12.4. When $n > 1$, $\mathbb{S}^n \times \mathbb{S}^n$ has one 0-cell, two n-cells, and one $2n$-cell. Since $n > 1$, these cells are not in adjacent dimensions. Hence:

$$H_k(\mathbb{S}^n \times \mathbb{S}^n; \mathbb{Z}) = \begin{cases} \mathbb{Z}, & \text{if } k = 0, 2n \\ \mathbb{Z}^2, & \text{if } k = n \\ 0, & \text{otherwise.} \end{cases}$$

Example 12.5. Recall that $\mathbb{CP}^n$ has one cell in each even dimension $0, 2, 4, \dots, 2n$. So $\mathbb{CP}^n$ has no two cells in adjacent dimensions. Hence:

$$H_k(\mathbb{CP}^n;\mathbb{Z}) = \begin{cases} \mathbb{Z}, & \text{if } k = 0, 2, 4, \dots, 2n \\ 0, & \text{otherwise.} \end{cases}$$

We next discuss how to compute, in general, the maps

$$d_n : C_n(X) = \mathbb{Z}^{\#n\text{-cells}} \to C_{n-1}(X) = \mathbb{Z}^{\#(n-1)\text{-cells}}$$

of the cellular chain complex. Let us consider the n-cells $\{e_n^\alpha\}_\alpha$ as the basis for $C_n(X)$ and the $(n-1)$-cells $\{e_{n-1}^\beta\}_\beta$ as the basis for $C_{n-1}(X)$. In particular, we can write:

$$d_n(e_n^\alpha) = \sum_\beta d_{\alpha,\beta} \cdot e_{n-1}^\beta \qquad d_{\alpha,\beta} \in \mathbb{Z},$$

Proposition 12.6. ***(Cellular Boundary Formula)*** *The coefficient $d_{\alpha,\beta}$ is equal to the degree of the map $\Delta_{\alpha,\beta} : \mathbb{S}_\alpha^{n-1} \to \mathbb{S}_\beta^{n-1}$ defined by the composition:*

$$\mathbb{S}_\alpha^{n-1} = \partial\mathbb{D}_\alpha^n \xrightarrow{\varphi_\alpha^n} X^{n-1} = X^{n-2} \cup_\gamma \mathbb{D}_\gamma^{n-1} \xrightarrow{\text{collapse}} X^{n-1}/(X^{n-2} \cup_{\gamma\neq\beta} \mathbb{D}_\gamma^{n-1}) = \mathbb{S}_\beta^{n-1},$$

where φ_α^n is the attaching map of $\mathbb{D}_\alpha^n$, and the collapsing map sends $X^{n-2} \cup_{\gamma\neq\beta} \mathbb{D}_\gamma^{n-1}$ to a point.

Proof. We will proceed with the proof by chasing the following diagram:

$$\begin{tikzcd}
H_n(\mathbb{D}_\alpha^n, \mathbb{S}_\alpha^{n-1};\mathbb{Z}) \arrow[r, "\cong"] \arrow[d, "(\Phi_\alpha^n)_*"'] & \widetilde{H}_n(\mathbb{S}_\alpha^{n-1};\mathbb{Z}) \arrow[rr, "(\Delta_{\alpha,\beta})_*"] \arrow[d, "(\phi_\alpha^n)_*"'] & & \widetilde{H}_n(\mathbb{S}_\beta^{n-1};\mathbb{Z}) \\
H_n(X^n, X^{n-1};\mathbb{Z}) \arrow[r, "\partial_n"] \arrow[dr, "d_n"'] & \widetilde{H}_{n-1}(X^{n-1};\mathbb{Z}) \arrow[rr, "q_*"] \arrow[d, "j_{n-1}"] & & \widetilde{H}_{n-1}(X^{n-1}/X^{n-2};\mathbb{Z}) \arrow[u, "q_{\beta,*}"'] \arrow[d, "\cong"'] \\
 & H_{n-1}(X^{n-1}, X^{n-2};\mathbb{Z}) \arrow[rr, "\cong"'] & & H_{n-1}(X^{n-1}/X^{n-2}, X^{n-2}/X^{n-2};\mathbb{Z})
\end{tikzcd}$$

The maps are as follows:

(1) Φ_α^n is the characteristic map of the cell e_α^n, and ϕ_α^n is its attaching map.
(2) The map

$$q_* : \widetilde{H}_{n-1}(X^{n-1};\mathbb{Z}) \to \widetilde{H}_{n-1}(X^{n-1}/X^{n-2};\mathbb{Z}) = \bigoplus_\beta \widetilde{H}_{n-1}(\mathbb{D}_\beta^{n-1}/\partial\mathbb{D}_\beta^{n-1};\mathbb{Z})$$

is induced by the quotient map $q : X^{n-1} \to X^{n-1}/X^{n-2}$.
(3) $q_\beta : X^{n-1}/X^{n-2} \to \mathbb{S}_\beta^{n-1}$ collapses the complement of the cell e_β^{n-1} to a point, the resulting quotient sphere being identified with $\mathbb{S}_\beta^{n-1} = \mathbb{D}_\beta^{n-1}/\partial\mathbb{D}_\beta^{n-1}$ via the characteristic map Φ_β^{n-1}.
(4) $\Delta_{\alpha\beta} : \mathbb{S}_\alpha^{n-1} \to \mathbb{S}_\beta^{n-1}$ is the composition $q_\beta \circ q \circ \phi_\alpha^n$, i.e., the attaching map of e_α^n followed by the quotient map $X^{n-1} \to \mathbb{S}_\beta^{n-1}$ collapsing the complement of $\mathbb{D}_\beta^{n-1}$ in X^{n-1} to a point.

The top left-hand square commutes by naturality of the long-exact sequence in reduced homology. The top right-hand square commutes by the definition of $\Delta_{\alpha,\beta}$. The bottom left-hand triangle commutes by definition of d_n . The bottom right-hand square commutes due to the relationship between reduced and relative homology.

The map $(\Phi^n_\alpha)_*$ takes the generator $[\mathbb{D}^n_\alpha] \in H_n(\mathbb{D}^n_\alpha, \mathbb{S}^{n-1}_\alpha)$ to a generator of the $\mathbb{Z}$-summand of $H_n(X^n, X^{n-1})$ corresponding to $\mathbb{D}^n_\alpha$, i.e.,

$$(\Phi^n_\alpha)_*([\mathbb{D}^n_\alpha]) = \mathbb{D}^n_\alpha$$

Since the top left square and the bottom left triangle both commute, this gives that

$$\sum_\beta d_{\alpha,\beta}\mathbb{D}^{n-1}_\beta = d_n(\mathbb{D}^n_\alpha) = d_n \circ (\Phi^\alpha_n)_*([\mathbb{D}^n_\alpha]) = j_{n-1} \circ (\phi^n_\alpha)_*([\mathbb{D}^n_\alpha]).$$

Here we have implicitly identified $H_n(\mathbb{D}^n_\alpha, \mathbb{S}^{n-1}_\alpha)$ with $H_n(\mathbb{S}^{n-1}_\alpha)$. Looking to the bottom right square, recall that since X is a CW complex, (X^n, X^{n-1}) is a good pair. This gives the isomorphism

$$\begin{aligned} H_{n-1}(X^{n-1}, X^{n-2}; \mathbb{Z}) &\cong \widetilde{H}_{n-1}(X^{n-1}/X^{n-2}; \mathbb{Z}) \\ &\cong H_{n-1}(X^{n-1}/X^{n-2}, X^{n-2}/X^{n-2}; \mathbb{Z}). \end{aligned}$$

Notice that the map q_β, collapsing all the $n-1$ cells of X to the $n-1$ cell $\mathbb{S}^{n-1}_\beta$, induces the map $q_{\beta,*}$, which projects linear combinations of $\{\mathbb{D}^{n-1}_{\beta'}\}$ onto its summand of $\mathbb{D}^{n-1}_\beta$. Therefore, the value of $d_n(\mathbb{D}^n_i)$ is going to be the sum of the projections $q_{\beta',*}$ on the $n-1$ dimensional cells e^{n-1}_β. In other words:

$$\sum_\beta d_{\alpha,\beta}\mathbb{D}^\beta_\beta = d_n(\mathbb{D}^\alpha_n) = \sum_\beta q_{\beta*} \circ q_* \circ (\phi^\alpha_n)_* \circ [\mathbb{D}^\alpha_n].$$

As noted before, we have defined $(\Delta_{\alpha\beta})_* = q_{\beta*} \circ q_* \circ (\phi^\alpha_n)_*$. The result now follows. $\square$

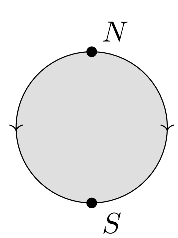
Example 12.7. Let $X = \mathbb{S}^2$. We $\mathbb{S}^2$ with $\mathbb{D}^2/\sim$ such that

$$(x, y) \sim (x', y') = x',\ y = |y'|$$

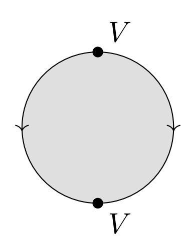
This induces a cell decomposition into one 2-cell, the image of the interior, one 1-cell, the image of $\mathbb{S}^1 \setminus \{(0,1), (0,-1)\}$, and two 0-cells, the images of $(0,1)$ and $(0,1)$ which are N and S. Let $A = \{N, S\}$. Since A is a sub-complex, X/A inherits a CW complex structure with one one 2-cell, one 1-cell and one 0-cell. We have

$$0 \to \mathbb{Z} \xrightarrow{d_2} \mathbb{Z} \xrightarrow{d_1} \mathbb{Z} \to 0$$

Since X/A is connected as has a single 0-cell, $d_1 \equiv 0$. The attaching map of the two-cell in



$X = \mathbb{S}^2$



$X = \mathbb{S}^2/\{N, S\}$

either case can be identified with the map:

$$\phi_{1,2}(e^{\phi i}) = \begin{cases} e^{i\phi} & 0 \leq \phi \leq \pi \\ e^{-i\phi} & \pi \leq \phi \leq 2\pi \end{cases}$$

The map has degree 0. Hence, $d_2 \equiv 0$. As a result, we have

$$H_n(\mathbb{S}^2/\{N,S\};\mathbb{Z}) = \begin{cases} \mathbb{Z}, & \text{if } n = 0,1,2 \\ 0, & \text{otherwise.} \end{cases}$$

Example 12.8. Recall that $\mathbb{RP}^n$ has a CW structure with one k-cell $\mathbb{D}^k$ in each dimension $0 \leq k \leq n$. The attaching map for $\mathbb{D}^k$ is the standard 2-fold covering map $\phi : \mathbb{S}^{k-1} \to \mathbb{RP}^{k-1}$ identifying a point and its antipodal point in $\mathbb{S}^{k-1}$. To compute the boundary map d_k, we compute the degree of the composition

$$f : \mathbb{S}^{k-1} \to \mathbb{RP}^{k-1} \to \frac{\mathbb{RP}^{k-1}}{\mathbb{RP}^{k-2}} = \mathbb{S}^{k-1}$$

We consider a neighborhood V of y and the two neighborhoods U_1 and U_2 given to exist by the local homeomorphism property of f. One of the homeomorphisms is the identity map and the other homemorphisms is the anti-podal map. Then by the local degree formula implies

$$d_k = 1 + (-1)^k$$

It follows that

$$d_k = \begin{cases} 0 & \text{if } k \text{ is odd,} \\ 2 & \text{if } k \text{ is even,} \end{cases}$$

and therefore we obtain that

$$H_k(\mathbb{RP}^n;\mathbb{Z}) = \begin{cases} \mathbb{Z}_2 & \text{if } k \text{ is odd, } 0 < k < n, \\ \mathbb{Z} & \text{if } k = 0, n \text{ is odd,} \\ 0 & \text{otherwise.} \end{cases}$$

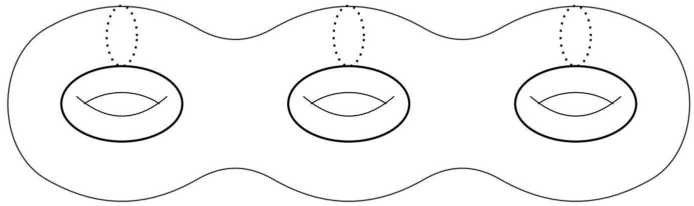
Example 12.9. Let M_g be the closed oriented surface of genus g, with its usual CW structure: one 0-cell, $2g$ 1-cells $\{a_1, b_1, \ldots, a_g, b_g\}$, and one 2-cell attached by the product of commutators $[a_1, b_1] \cdot \ldots \cdot [a_g, b_g]$. The associated cellular chain complex of M_g is:

$$0 \xrightarrow{d_3} \mathbb{Z} \xrightarrow{d_2} \mathbb{Z}^{2g} \xrightarrow{d_1} \mathbb{Z} \xrightarrow{d_0} 0$$

Since M_g is connected and has only one 0-cell, we get that $d_1 = 0$. We claim that d_2 is also the zero map. As the attaching map sends the generator to $a_1b_1a_1^{-1}b_1^{-1} \ldots a_gb_ga_g^{-1}b_g^{-1}$, when we collapse all 1-cells (except a_i) to a point, the word defining the attaching map $a_1b_1a_1^{-1}b_1^{-1} \ldots agbga_g^{-1}b_g^{-1}$ reduces to $a_ia_i^{-1}$. Hence, the coefficient$d_{ea_i} = 1 - 1 = 0$. Altogether, $d_2(e) = 0$. So the homology groups of M_g are given by

$$H_n(M_g;\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } n = 0,2, \\ \mathbb{Z}^{2g} & \text{for } n = 1, \\ 0 & \text{otherwise.} \end{cases}$$

For $g = 3$, see the figure below to visualize the $2g = 6$ generators of $H_1(M_3)$:



13. Euler Characteristic

Definition 13.1. Let X be a finite CW complex of dimension n. The Euler characteristic of X is defined as:

$$\chi(X) = \sum_{i=0}^{n} (-1)^i \cdot \# n - \text{cells} = \sum_{i=0}^{n} (-1)^i \cdot \# \operatorname{rank}(C_i^{\mathrm{CW}})$$

Here C_i^{CW} is the i-th abelian group in the chain complex that determines cellular homology. We show that the Euler characteristic does not depend on the cell structure chosen for the space X. As we will see below, this is not the case.

Proposition 13.2. *The Euler characteristic can be computed as:*

$$\chi(X) = \sum_{i=0}^{n} (-1)^i \cdot \operatorname{rank}(H_i^{\mathrm{CW}}(X; \mathbb{Z}))$$

In particular, $\chi(X)$ is independent of the chosen cell structure on X.

Proof. We use the following notation: $B_i = \operatorname{Im}(d_{i+1})$, $Z_i = \ker(d_i)$, and $H_i^{CW} = Z_i/B_i$. The additivity of rank yields that

$$\operatorname{rank}(C_i) = \operatorname{rank}(Z_i) + \operatorname{rank}(B_{i-1})$$

and

$$\operatorname{rank}(Z_i) = \operatorname{rank}(B_i) + \operatorname{rank}(H_i^{\mathrm{CW}}).$$

Substitute the second equality into the first, multiply the resulting equality by $(-1)^i$, and sum over i to get that

$$\chi(X) = \sum_{i=0}^{n} (-1)^i \cdot \operatorname{rank}(H_i^{\mathrm{CW}})$$

Since cellular homology is isomorphic to singular homology and the latter is homotopy invariant, the result follows.. $\square$

Proposition 13.3. *Let X, Y be finite-dimensional CW complexes and let*

$$\chi(X) = \sum_{i=0}^{n} (-1)^i a_i$$

$$\chi(Y) = \sum_{j=0}^{m} (-1)^j b_j$$

Here a_i is the number of i-cells in X. Similarly, b_j is the number of j-cells in B The Euler characteristic enjoys some nice properties:

(1) $\chi(X \times Y) = \chi(X) \times \chi(Y)$

(2) *If $X = A \cup B$ such that A, B are sub-complexes of X. Then*

$$\chi(X) = \chi(A) + \chi(B) - \chi(A \cap B)$$

(3) *If $p : \widetilde{X} \to X$ is an n-sheeted covering space, then*

$$\chi(\widetilde{X}) = n\chi(X)$$

Proof. The proof is given below:

(1) For any index k, k-cells in $X \times Y$ are created by considering products of r-cells and $k - r$ cells from X and Y respectively where $0 \leq r \leq k$. Hence the number of k-cells is

$$\sum_{r=0}^{k} a_r b_{k-r}$$

Therefore,

$$\chi(X) \times \chi(Y) = \left(\sum_{i=0}^{n} (-1)^i a_i\right) \times \left(\sum_{j=0}^{m} (-1)^j b_j\right) = \sum_{k=0}^{m+n} (-1)^k \sum_{r=0}^{k} a_r b_{k-r} = \chi(X \times Y)$$

(2) Let a_i^A denote the number of i-cells in A. Similarly, let a_i^B be the number of i-cells in B. Similarly, let $a_i^{A\cap B}$ be the number of i-cells in $A \cap B$. We have

$$a_i = a_i^A + a_i^B - a_i^{A\cap B}$$

for $i = 1, \cdots, n$. Therefore, we have,

$$\begin{aligned}
\chi(X) &= \sum_{i=0}^{n} (-1)^i a_i \\
&= \sum_{i=0}^{n} (-1)^i a_i^A + \sum_{i=0}^{n} (-1)^i a_i^B - \sum_{i=0}^{n} (-1)^i a_i^{A\cap B} \\
&= \chi(A) + \chi(B) - \chi(A \cap B)
\end{aligned}$$

(3) Recall that if $\mathbb{D}_\alpha^k$ is a k-cell in X, then $\widetilde{X}$ has n k-cells. Therefore, it is clear that

$$\chi(\widetilde{X}) = n\chi(X)$$

□

Example 13.4. Let M_g be the oriented surface of genus g. We have

$$\chi_{M_g} = 2 - 2g$$

Let N_g be the oriented surface of genus g. We have

$$\chi_{N_g} = 2 - g$$

Thus all the M_g, N_g are distinguished from each other by their Euler characteristics. There are only the relations

$$\chi(M_g) = \chi(N_{2g})$$

14. Homology with Coefficients

In this section, we discuss homology with coefficients and the universal coefficient theorem in homology. Let G be an abelian group and X a topological space. We define the homology of X with G-coefficients, denoted $H_n(X;G)$ for $n \in \mathbb{N}$, as the homology of the chain complex:

$$C_\bullet(X;G) = C_\bullet(X) \otimes G$$

consisting of finite formal sums $\sum_i \eta_i \cdot \sigma_i$ with $\eta_i \in G$, and with boundary maps given by

$$\partial_n^G := \partial_n \otimes \mathrm{Id}_G.$$

Since ∂_n satisfies $\partial_n \circ \partial_{n+1} = 0$, it follows that $\partial_n^G \circ \partial_{n+1}^G = 0$. Hence,

$$(C_\bullet(X); G, \partial_\bullet^G)$$

is indeed a chain complex. We can construct versions of the usual modified homology groups (relative, reduced, etc.) in the most natural way.

(1) **(Relative homology with G-coefficients)** Consider the augmented chain complex:

$$C_1(X;G) \longrightarrow C_0(X;G) \longrightarrow G \longrightarrow 0$$

where $\epsilon(\sum_i \eta_i \sigma_i) = \sum_i \eta_i \in G$. Reduced homology with G-coefficients is defined as the homology of the augmeneted chain complex.

(2) **(Relative chain Complex with G-coefficients)** Define relative chains with G-coefficients by:

$$C_n(X,A;G) := C_n(X;G)/C_n(A;G),$$

Consider the chain complex:

$$C_1(X,A;G) \longrightarrow C_0(X,A;G) \longrightarrow 0$$

The relative homology with G-coefficients is defined as the homology of the augmeneted chain complex.

(3) **(Cellular homology with G-coefficients)** We can build cellular homology with G-coefficients by defining

$$C_n^G(X) = H_n(X_n, X_{i-1}; G) \cong G^{(\text{number of } i\text{-cells})}$$

The cellular boundary maps are given by:

$$d_n^G(e_n^\alpha) = \sum_\beta d_{\alpha\beta} e_{i-1}^\beta,$$

where $d_{\alpha\beta}$ is as before the degree of a map $\Delta_{\alpha\beta} : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$. As it is the case for integers, we get an isomorphism:

$$H_n^{CW}(X;G) \cong H_n(X;G)$$

Example 14.1. By studying the chain complex with G-coefficients, it follows that:

$$H_n(\{*\};G) = \begin{cases} G & \text{if } n = 0, \\ 0 & \text{if } n \neq 0. \end{cases}$$

Example 14.2. (Sketch) The homology of a sphere as before by induction and using the long exact sequence of the pair $(\mathbb{D}^n, \mathbb{S}^n)$ to be:

$$H_n(\mathbb{S}^n; G) = \begin{cases} G & \text{if } i = 0, n, \\ 0 & \text{otherwise.} \end{cases}$$

14.1. **Tor Functor.** We discuss the Tor functor which will play an important role in the discussion of homology with coefficients. We first briefly review the tensor product construction.

Remark 14.3. *We work with commutative rings below. Hence, we don't make any distinction between the categories of left R-modules and right R-modules. We use the generic phrase 'R-module' to refer to a left/right R-module.*

Definition 14.4. Let R be a commutative ring and let M, N be two R-modules. The tensor product of M and N, denoted $M \otimes N$, is the R-module

$$M \otimes_R N := M \times N / \sim$$

where $\sim$ is enforces the relations

$$\begin{aligned} (m + m') \otimes_R n &= m \otimes_R n + m' \otimes_R n, \\ m \otimes_R (n + n') &= m \otimes_R n + m \otimes_R n' \\ m \otimes_R rn &= mr \otimes_R n \end{aligned}$$

for $r \in R, m \in M$ and $n \in N$.

Remark 14.5. *It is easily checked that $M \otimes_R N$ is a R-module. For example, the zero element of $M \otimes_R N$ is given by*

$$0 \otimes_R n = m \otimes_R 0 = 0 \otimes_R 0 := 0_{M \otimes N}$$

since, for example,

$$0 \otimes_R n = (0 + 0) \otimes_R n = 0 \otimes n + 0 \otimes n$$

which implies $0 \otimes n = 0_{M \otimes N}$. Similarly, the inverse of an element $m \otimes_R n$ is

$$-(m \otimes_R n) = (-m) \otimes_R n = m \otimes_R (-n)$$

Remark 14.6. *The tensor product satisfies the following universal property, which asserts that if $\varphi : M \times N \to P$ is any R-bilinear map, then there exists a unique R-linear map $\overline{\varphi} : M \otimes_R N \to P$ such that $\varphi = \overline{\varphi} \circ i$, where $i : M \times N \to M \otimes_R N$ is the natural map given by $(m, n) \mapsto m \otimes_R n$.*

We prove some properties of the tensor product:

Proposition 14.7. *The tensor product satisfies the following properties:*

1. $M \otimes_R N \cong N \otimes_R M$ *via the isomorphism* $m \otimes_R n \mapsto n \otimes_R m$.
2. $(\bigoplus_i M_i) \otimes_R N \cong \bigoplus_i (M_i \otimes_R N)$ *via the isomorphism* $(m_i)_i \otimes_R n \mapsto (m_i \otimes_R n)_i$.
3. $M \otimes_R (N \otimes_R P) \cong (M \otimes_R N) \otimes_R P$ *via the isomorphism* $m \otimes_R (n \otimes_R p) \mapsto (m \otimes_R n) \otimes_R p$.
4. $R \otimes_R M \cong M$ *via the isomorphism* $r \otimes_R m \mapsto rm$.

Proof. The proof is given below:

(1) The map $\varphi : M \times N \to N \otimes_R M$ defined by $(m, n) \mapsto n \otimes_R m$ is clearly R-bilinear and therefore induces a R-linear map

$$\overline{\varphi} : M \otimes_R N \to N \otimes_R M$$
$$m \otimes_R n \mapsto n \otimes_R m$$

Similarly, the map $\psi : N \times M \to M \otimes_R N$ defined by $(n, m) \mapsto m \otimes_R n$, is R-bilinear and therefore induces a R-linear map

$$\overline{\psi} : N \otimes_R M \to M \otimes_R N$$
$$n \otimes_R m \mapsto m \otimes_R n$$

Clearly, $\overline{\varphi}$ and $\overline{\psi}$ are inverses. The claim follows.

(2) Same as (1).

(3) Same as (1).

(4) The map $\varphi : \mathbb{Z} \times M \to M$ defined by $(n, a) \mapsto na$ is a R-bilinear map and therefore induces a R-bilinear map

$$\overline{\varphi} : \mathbb{Z} \otimes_R M \to M$$
$$n \otimes a \mapsto na.$$

Suppose $\overline{\varphi}(n \otimes m) = 0$. Then $nm = 0$ and

$$n \otimes m = 1 \otimes (nm) = 1 \otimes 0 = 0_{\mathbb{Z} \otimes M}.$$

Thus, $\overline{\varphi}$ is injective. Moreover, if $m \in M$, then

$$\overline{\varphi}(1 \otimes m) = m$$

and $\overline{\varphi}$ is surjective as well.

This completes the proof. □

The tensor product, $\otimes_R$, defines a functor from the category of R-modules to itself such that if N is a R-module, then

$$- \otimes_R M(N) = N \otimes_R M$$

Moreover, if $f : N \to N'$ is a R-module morphism, then

$$- \otimes_R M(f) : N \otimes_R M \xrightarrow{f \otimes_R \mathrm{Id}_M} N' \otimes_R M$$

Using a clever argument exploiting the adjunction between the Hom and tensor product functors (which we take for granted in these notes), we can show the following:

Proposition 14.8. *Let R be a commutative ring and let M be a R-module. The functor $- \otimes_R M$ is a right exact functor.*

Remark 14.9. *We freely invoke results about the Hom functor discussed in the cohomology notes in the proof of Proposition 14.8 below.*

Proof. (Sketch) Let

$$0 \to A \to B \to C \to 0$$

be an exact sequence in the category of R-modules. We show that

$$A \otimes_R M \to B \otimes_R M \to C \otimes_R M \to 0$$

is an exact sequence. Results about the Hom functor imply that

$$A \otimes_R M \to B \otimes_R M \to C \otimes_R M \to 0$$

is an exact sequence if and only if

$$0 \to \mathrm{Hom}(C \otimes_R M, X) \to \mathrm{Hom}(B \otimes_R M, X) \to \mathrm{Hom}(A \otimes_R M, X)$$

is an exact sequence for all R-module X. We have

$$\mathrm{Hom}(N \otimes_R M, X) = \mathrm{Hom}(N, \mathrm{Hom}(M, X)),$$

for all left R-modules N. Hence, the sequence above can be written as

$$0 \to \mathrm{Hom}(C, \mathrm{Hom}(M, X)) \to \mathrm{Hom}(B, \mathrm{Hom}(M, X)) \to \mathrm{Hom}(A, \mathrm{Hom}(M, X))$$

which is indeed exact by the fact that the $\mathrm{Hom}(-, \mathrm{Hom}(M, X))$ functor is left exact. $\square$

Example 14.10. The functor $- \otimes_R M$ need not be left exact functor. To see this, take $R = \mathbb{Z}$. Consider the sequence:

$$0 \to \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z}$$

Here $\cdot n$ is the multiplication by n map. Let $M = \mathbb{Z}/n\mathbb{Z}$ we obtain a map:

$$\mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \xrightarrow{\cdot n \otimes_{\mathbb{Z}} \mathrm{Id}_{\mathbb{Z}/n\mathbb{Z}}} \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/n\mathbb{Z}$$

However, this is the zero map since we have

$$\cdot n \otimes_{\mathbb{Z}} \mathrm{Id}_{\mathbb{Z}/n\mathbb{Z}}(1 \otimes_{\mathbb{Z}} \overline{m}) = n \otimes_{\mathbb{Z}} \overline{m} = 1 \otimes_{\mathbb{Z}} \overline{nm} = 0.$$

The zero map is not injective.

Remark 14.11. *A R-module M is called flat if $- \otimes_R M$ is a left exact functor. If M is a projective R-module, then $- \otimes_R M$ is a left exact functor. This follows because a projective R-module is a direct summand of a free R-module, a free R-module is a flat module and that a R-module is flat if and only if each summad is a flat R-module. Details skipped.*

Since the $- \otimes_R M$ functor is a right exact functor which in general is not a left exact functor, we can consider its left derived functor.

Definition 14.12. Let R be a ring and let M be a R-module. The i-th Tor functor is the i-th left derived functor of $- \otimes_R M$. It is denoted as

$$\mathrm{Tor}_i^R(-, M)$$

By definition, $\mathrm{Tor}_i^R(-, M)$ is computed as follows. If N is a R-module, take any projective resolution

$$\cdots \to P^1 \to P^0 \to N \to 0,$$

and form the chain complex:

$$\cdots \to P^2 \otimes_R M \to P^1 \otimes_R M \to P^0 \otimes_R M$$

Then $\mathrm{Tor}_i^R(N, M)$ is the homology of this complex at position i.

$$\mathrm{Tor}_i^R(N, M) = H_i((P^i \otimes_R M)_\bullet)$$

Remark 14.13. *General results about derived functors show that the homology is independent of the choice of the projective resolution.*

If R is a commutative ring and M is a R-module, we can define another functor $M \otimes_R -$. The definition is similar to that of the functor defined above. It can also be checked that $M \otimes_R -$ is right exact functor that is, in general, not left exact. Hence, we can attempt to construct a left-derived functor associated to $M \otimes_R -$ as above. We label that derived functor $\overline{\mathrm{Tor}}_i^R(M, -)$. We have the following result:

Proposition 14.14. *(**Balanacing Tor**) Let R be a commutative ring and let M be a R-module. Then*

$$\operatorname{Tor}_i^R(-, M) \cong \overline{\operatorname{Tor}}_i^R(M, -)$$

That is, for every R-module N, we have

$$\operatorname{Tor}_i^R(N, M) \cong \overline{\operatorname{Tor}}_i^R(M, N)$$

Proof. See [Wei94] for a proof. $\square$

Remark 14.15. *In light of Proposition 14.14, we can identify Tor_i^R with $\overline{\operatorname{Tor}}_i^R$ for each $i \geq 0$. This allows us to compute projective resolutions of either N or M to compute $\operatorname{Tor}_i^R(N, M)$ for each $i \geq 0$.*

Proposition 14.16. *Let R be a commutative ring and let M be a R-module. The Tor functor satisfies the following properties:*

(1) *$\operatorname{Tor}_0^R(N, M) \cong N \otimes_R M$ for any R-modules M, N.*
(2) *If N is a projective R-module, then $\operatorname{Tor}_i^R(N, M) = 0$ for all $i \geq 1$*
(3) *Any $f : N_1 \to N_2$ R-module homomorphism induces a morphism*

$$f_*^i : \operatorname{Tor}_i^R(N_1, M) \longrightarrow \operatorname{Tor}_i^R(N_2, M)$$

for each $i \geq 0$.

(4) *Any short exact sequence $0 \to N_1 \xrightarrow{\phi} N_2 \xrightarrow{\psi} N_3 \to 0$ of R-modules induces a long exact sequence:*

$$\cdots \to \operatorname{Tor}_1^R(N_1, M) \to \operatorname{Tor}_1^R(N_2, M) \to \operatorname{Tor}_1^R(N_3, M) \to N_1 \otimes_R M \to N_2 \otimes_R M \to N_3 \otimes_R M \to 0$$

Proof. (1) and (2) follow from general properties of derived functors. For (3), let $P_1^\bullet$ be a projective resolution of N_1 and $P_2^\bullet$ be a projective resolution of N_2. General properties about projective resolutions imply that that f lifts to a chain map $\varphi^\bullet : P_1^\bullet \longrightarrow P_2^\bullet$. Then, $\varphi^\bullet$ induces a morphism of chain complexes $P_1^\bullet \otimes_R M \longrightarrow P_2^\bullet \otimes_R M$ which, in turn, induces a morphism:

$$f_*^i : \operatorname{Tor}_i^R(N_1, M) \longrightarrow \operatorname{Tor}_i^R(N_2, M)$$

for each $i \geq 0$. For (4), let $P^\bullet$ be a projective resolution of M. Then there is an induced short exact sequence of chain complexes:

$$0 \to N_1 \otimes_R P^\bullet \to N_2 \otimes_R P^\bullet \to N_3 \otimes_R P^\bullet \to 0$$

because each module P^i is projective. Applying the long exact sequence in homology produces the required long exact sequence. $\square$

The above proposition show that the Ext groups 'measure' and 'repair' the non-exactness of the tensor product functor. We now specialize to the category of $\mathbb{Z}$-modules. In what follows, we fix G to be an abelian group. We have the following result:

Lemma 14.17. *For any abelian group A, we have*

$$\operatorname{Tor}_i^{\mathbb{Z}}(A, G) = 0 \quad \textit{if } i > 1.$$

Proof. Recall that any abelian group, A, admits a two-step free resolution.

$$0 \to F_1 \to F_0 \to A \to 0$$

Thus, $\operatorname{Tor}_i^{\mathbb{Z}}(A, G) = 0$ if $n > 1$. $\square$

Remark 14.18. *Only* $\mathrm{Tor}_1(-,G)$ *encodes any interesting information. In what follows, we adopt the notation:*

$$\mathrm{Tor}(-,G) := \mathrm{Tor}_1^{\mathbb{Z}}(-,G).$$

Proposition 14.19. *If* $R=\mathbb{Z}$*, the Tor functor satisfies the following properties:*

1. $\mathrm{Tor}\left(\bigoplus_i A_i, G\right) \cong \bigoplus_i \mathrm{Tor}(A_i,G)$.
2. *If* A *is a free abelian group, then* $\mathrm{Tor}(A,G)=0$.
3. $\mathrm{Tor}(\mathbb{Z}/n\mathbb{Z},G) \cong \mathit{ker}(G \xrightarrow{n} G)$.
4. *For a short exact sequence:* $0 \to B \to C \to D \to 0$ *of abelian groups, there is a natural exact sequence:*

$$0 \to \mathrm{Tor}(B,G) \to \mathrm{Tor}(C,G) \to \mathrm{Tor}(D,G) \to B \otimes G \to C \otimes G \to D \otimes G \to 0.$$

Proof. The proof is given below:

1. This follows from the identity,

$$\left(\bigoplus_i A_i\right) \otimes_{\mathbb{Z}} G = \bigoplus_i (A_i \otimes_{\mathbb{Z}} G)$$

   and noting that taking direct sums of projective resolutions of A_i forms a projective resolution for $\bigoplus_i A^i$, and that homology commutes with direct sums.
2. If A is free, then

$$0 \to A \to A \to 0$$

   is a projective resolution of A, so $\mathrm{Tor}(A,G)=0$.
3. The exact sequence

$$0 \to \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$$

   is a projective resolution of $\mathbb{Z}/n\mathbb{Z}$. Tensoring with G and dropping the right-most term yields the complex:

$$G \cong \mathbb{Z} \otimes G \xrightarrow{\cdot n \otimes 1_G} G \cong \mathbb{Z} \otimes G \to 0,$$

   Thus, $\mathrm{Tor}(\mathbb{Z}/n\mathbb{Z},G) = \ker(G \xrightarrow{n} G)$.
4. This follows from Proposition 14.16(4).

This completes the proof. ☐

14.2. **Universal Coefficient Theorem.**

We now prove an important theorem that relates how homology with different coefficients are related.

Proposition 14.20. ***(Universal Coefficient Theorem)*** *If a chain complex* $(C_\bullet, \partial_\bullet)$ *of free abelian groups has homology groups* $H_n(C_\bullet)$*, then the homology groups* $H_n(C_\bullet;G)$ *are determined by the short exact sequence:*

$$0 \longrightarrow H_n(C_\bullet) \otimes_{\mathbb{Z}} G \longrightarrow H_n(C_\bullet;G) \xrightarrow{h} \mathrm{Tor}(H_{n-1}(C_\bullet),G) \longrightarrow 0$$

Proof. We have

$$B_n = \operatorname{im} \partial_{n+1} \subseteq Z_n = \ker \partial_n \subseteq C_n$$

Since $\partial_n|_{Z_n} = 0$ and $\partial_{n-1}|_{B_{n-1}} = 0$, we have the following short exact sequence of chain complexes:

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & Z_{n+1} \arrow[r, hook, "i_{n+1}"] \arrow[d, "\partial_{n+1}=0"] & C_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, "\partial_{n+1}"] & B_n \arrow[r] \arrow[d, "\partial_n=0"] & 0 \\
0 \arrow[r] & Z_n \arrow[r, hook, "i_n"] \arrow[d, "\partial_n=0"] & C_n \arrow[r, "\partial_n"] \arrow[d, "\partial_n"] & B_{n-1} \arrow[r] \arrow[d, "\partial_{n-1}=0"] & 0 \\
0 \arrow[r] & Z_{n-1} \arrow[r, hook, "i_{n-1}"] \arrow[d] & C_{n-1} \arrow[r, "\partial_{n-1}"] \arrow[d] & B_{n-2} \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

Since C_n is a free abelian group, Z_n, B_n are also free abelian groups. Hence, $\otimes_{\mathbb{Z}}$ is an exact functor if applied to each row. Hence, we have a short exact sequence of chain complexes:

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & Z_{n+1} \otimes_{\mathbb{Z}} G \arrow[r, hook, "i_{n+1} \otimes_{\mathbb{Z}} \mathrm{id}_G"] \arrow[d, "0"] & C_{n+1} \otimes_{\mathbb{Z}} G \arrow[r, "\partial_{n+1} \otimes_{\mathbb{Z}} \mathrm{id}_G"] \arrow[d, "\partial_{n+1} \otimes_{\mathbb{Z}} \mathrm{id}_G"] & B_n \otimes_{\mathbb{Z}} G \arrow[r] \arrow[d, "0"] & 0 \\
0 \arrow[r] & Z_n \otimes_{\mathbb{Z}} G \arrow[r, hook, "i_n \otimes_{\mathbb{Z}} \mathrm{id}_G"] \arrow[d, "0"] & C_n \otimes_{\mathbb{Z}} G \arrow[r, "\partial_n \otimes_{\mathbb{Z}} \mathrm{id}_G"] \arrow[d, "\partial_n \otimes_{\mathbb{Z}} \mathrm{id}_G"] & B_{n-1} \otimes_{\mathbb{Z}} G \arrow[r] \arrow[d, "0"] & 0 \\
0 \arrow[r] & Z_{n-1} \otimes_{\mathbb{Z}} G \arrow[r, hook, "i_{n-1} \otimes_{\mathbb{Z}} \mathrm{id}_G"] \arrow[d] & C_{n-1} \otimes_{\mathbb{Z}} G \arrow[r, "\partial_{n-1} \otimes_{\mathbb{Z}} \mathrm{id}_G"] \arrow[d] & B_{n-2} \otimes_{\mathbb{Z}} G \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

By the snake lemma (Proposition 5.19) we have a connecting homomorphism

$$\Delta_n : B_n \otimes_{\mathbb{Z}} G \to Z_n \otimes_{\mathbb{Z}} G$$

Following the construction of the connecting homomorphism in the proof of the snake lemma, we see that $\Delta_n = j_n \otimes \mathrm{Id}_M$ where $j_n : B_n \to Z_n$ is the inclusion $B_n \subseteq Z_n$. Additionally, we get an exact sequence

$$\cdots \to H_n(B \otimes_{\mathbb{Z}} G) \xrightarrow{(j_n \otimes_{\mathbb{Z}} \mathrm{id}_G)_*} H_n(Z \otimes_{\mathbb{Z}} G) \xrightarrow{(i_n \otimes_{\mathbb{Z}} \mathrm{id}_G)_*} H_n(C \otimes_{\mathbb{Z}} G) \xrightarrow{(\partial_n \otimes_{\mathbb{Z}} \mathrm{id}_G)_*} H_{n-1}(B \otimes_{\mathbb{Z}} G) \to \cdots$$

Since the chain complexes $Z \otimes_{\mathbb{Z}} G$ and $B \otimes_{\mathbb{Z}} G$ have null boundary maps, we deduce that the sequence

$$\cdots \to B_n \otimes_{\mathbb{Z}} G \xrightarrow{j_n \otimes_{\mathbb{Z}} \mathrm{id}_G} Z_n \otimes_{\mathbb{Z}} G \xrightarrow{i_n \otimes_{\mathbb{Z}} \mathrm{id}_G} H_n(C \otimes_{\mathbb{Z}} G) \xrightarrow{(\partial_n \otimes_{\mathbb{Z}} \mathrm{id}_G)_*} B_{n-1} \otimes_{\mathbb{Z}} G \to \cdots$$

We claim that this sequence is natural. Suppose that we have another chain complex $(C'_\bullet, \partial'_\bullet)$ and a chain map $f : C_\bullet \to C'_\bullet$. We want to see that the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & B_n(C)\otimes_{\mathbb{Z}} M \arrow[r, "j_n\otimes_{\mathbb{Z}}\mathrm{id}_G"] \arrow[d, "f_n\otimes_{\mathbb{Z}}\mathrm{Id}_M"] & Z_n(C)\otimes_{\mathbb{Z}} M \arrow[r, "i_n\otimes_{\mathbb{Z}}\mathrm{id}_G"] \arrow[d, "f_n\otimes_{\mathbb{Z}}\mathrm{Id}_M"] & H_n(C\otimes_{\mathbb{Z}} M) \arrow[r, "(\partial_n\otimes_{\mathbb{Z}}\mathrm{id}_G)_*"] \arrow[d, "(f_n\otimes_{\mathbb{Z}}\mathrm{Id}_M)_*"] & B_{n-1}(C)\otimes_{\mathbb{Z}} M \arrow[r] \arrow[d, "f_n\otimes_{\mathbb{Z}}\mathrm{Id}_M"] & \cdots \\
\cdots \arrow[r] & B_n(C')\otimes_{\mathbb{Z}} M \arrow[r, "j'_n\otimes_{\mathbb{Z}}\mathrm{id}_G"] & Z_n(C')\otimes_{\mathbb{Z}} M \arrow[r, "i'_n\otimes_{\mathbb{Z}}\mathrm{id}_G"] & H_n(C'\otimes_{\mathbb{Z}} M) \arrow[r, "(\partial'_n\otimes_{\mathbb{Z}}\mathrm{id}_G)_*"] & B_{n-1}(C')\otimes_{\mathbb{Z}} M \arrow[r] & \cdots
\end{tikzcd}$$

commutes. It suffices to show that the three squares commute. For the left-most square, given an n-boundary b_n, its image $f_n(b_n)$ is also an n-boundary, hence $f_n\circ j_n(b_n) = f_n(b_n) = j'_n \circ f_n(b_n)$. Therefore, we deduce that

$$(f_n \otimes \mathrm{id}_G)\circ(j_n\otimes \mathrm{id}_G) = (j'_n\otimes \mathrm{id}_G)\circ(f_n\otimes \mathrm{id}_G).$$

The middle square commutes for similar reasons. Finally, the right most square commutes since

$$(f_n\otimes_{\mathbb{Z}} \mathrm{id}_G)\circ(\partial_n\otimes_{\mathbb{Z}} \mathrm{id}_G) = (\partial'_n\otimes_{\mathbb{Z}} \mathrm{id}_G)\circ(f_n\otimes_{\mathbb{Z}} \mathrm{id}_G).$$

From this discussion, we deduce the natural short exact sequences:

$$0 \to \mathrm{Coker}(j_n\otimes_{\mathbb{Z}} \mathrm{id}_G) \xrightarrow{i_n\otimes \mathrm{id}} H_n(C\otimes_{\mathbb{Z}} G) \xrightarrow{(\partial_n\otimes_{\mathbb{Z}}\mathrm{id}_G)_*} \mathrm{Ker}(j_{n-1}\otimes_{\mathbb{Z}} \mathrm{id}_G) \to 0.$$

Consider the projections $\pi_n : Z_n \to H_n(C)$, and the short exact sequence

$$0 \to B_n \xrightarrow{j_n} Z_n \xrightarrow{\pi_n} H_n(C) \to 0.$$

Note that these sequences are free resolutions of the homology modules $H_n(C)$. Hence, by properties of Tor, we have

$$0 \to \mathrm{Tor}(H_n(C), G) \to B_n\otimes_{\mathbb{Z}} G \xrightarrow{\iota_n\otimes \mathrm{id}_G} C_n\otimes_{\mathbb{Z}} G \xrightarrow{\pi_n\otimes\mathrm{id}} H_n(C)\otimes_{\mathbb{Z}} G \to 0.$$

We can now identify $\mathrm{Ker}(j_{n-1}\otimes_{\mathbb{Z}}\mathrm{id}_G)$ as $\mathrm{Tor}(H_{n-1}(C), G)$ and $\mathrm{Coker}(j_n\otimes_{\mathbb{Z}}\mathrm{id}_G)$ as $H_n(C)\otimes_{\mathbb{Z}} G$. The claim follows. ☐

Remark 14.21. *The sequence in Proposition 14.20 splits. This is because in the beginning of the proof, we considered the the exact sequences*

$$0 \to Z_n \xrightarrow{i_n} C_n \xrightarrow{\partial_n} B_{n-1} \to 0,$$

This sequence splits since B_{n-1} is a free abelian group. Hence, there must exist group homomorphisms $l_n : C_n \to Z_n$ such that $l_n \circ i_n = id$: $Z_n \to Z_n$ for all $n \in \mathbb{Z}$. Hence the compositions $(\pi_n \circ l_n)_{n\in\mathbb{N}}$ induce maps in homology

$$((\pi_n\circ l_n)\otimes id_G)_* : H_n(C\otimes_{\mathbb{Z}} G) \to H_n(C)\otimes_{\mathbb{Z}} G,$$

which are such that

$$((\pi_n\circ l_n)\otimes id_G)_* \circ ((i_n)_*\otimes id_G) = (id_G\otimes id_G)_*.$$

As a consequence, the sequence splits and we obtain

$$H_n(C_\bullet\otimes_{\mathbb{Z}} G) \cong (H_n(C_\bullet)\otimes_{\mathbb{Z}} G)\oplus \mathrm{Tor}(H_{n-1}(C), G).$$

Remark 14.22. *There is also a universal coefficient theorem for homology where $\mathbb{Z}$ is replaced by a PID, R and G is a R-module. In this case, we have*

$$0 \longrightarrow H_n(C_\bullet)\otimes_R G \longrightarrow H_n(C_\bullet; G) \xrightarrow{\ h\ } \mathrm{Tor}_1^R(H_{n-1}(C_\bullet), G) \longrightarrow 0$$

This comes from first establishing that Tor_i^R vanishes for $i \geq 2$ for when R is a PID, and then going through a proof for universal coefficient theorem essentially as above.

Example 14.23. Suppose $X = K$ is the Klein bottle, and $G = \mathbb{Z}/4$. Recall that $H_1(K;\mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}/2$, and $H_2(K;\mathbb{Z}) = 0$, so:

$$\begin{aligned} H_2(K;\mathbb{Z}/4) &= (H_2(K;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}/4) \oplus \mathrm{Tor}(H_1(K), \mathbb{Z}/4) \\ &= \mathrm{Tor}(\mathbb{Z}, \mathbb{Z}/4) \oplus \mathrm{Tor}(\mathbb{Z}/2, \mathbb{Z}/4) \\ &= 0 \oplus \mathbb{Z}/2 \\ &= \mathbb{Z}/2. \end{aligned}$$

Example 14.24. Let $X = \mathbb{RP}^n$ and $G = Z/2\mathbb{Z}$. Recall that we have

$$H_k(\mathbb{RP}^n;\mathbb{Z}) = \begin{cases} \mathbb{Z}/2\mathbb{Z} & \text{if } k \text{ is odd, } 0 < k < n, \\ \mathbb{Z} & \text{if } k = 0, n \text{ is odd,} \\ 0 & \text{otherwise.} \end{cases}$$

We compute $H_k(\mathbb{RP}^n;\mathbb{Z}/2\mathbb{Z})$. We consider multiple cases. For $k = 0$, we have:

$$H_0(\mathbb{RP}^n;\mathbb{Z}/2\mathbb{Z}) \cong H_0(\mathbb{RP}^n;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} = \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z}.$$

For $k = 1$, we have:

$$\begin{aligned} H_1(\mathbb{RP}^n;\mathbb{Z}/2\mathbb{Z}) &\cong H_1(\mathbb{RP}^n;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \oplus \mathrm{Tor}(H_0(\mathbb{RP}^n), \mathbb{Z}/2\mathbb{Z}) \\ &= (\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}) \oplus \mathrm{Tor}(\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \\ &= \mathbb{Z}/2\mathbb{Z} \oplus 0 = \mathbb{Z}/2\mathbb{Z}. \end{aligned}$$

For $1 < k < n$, such that k is an odd integer, we have

$$\begin{aligned} H_k(\mathbb{RP}^n;\mathbb{Z}/2\mathbb{Z}) &\cong (H_k(\mathbb{RP}^n;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}) \oplus \mathrm{Tor}(H_{k-1}(\mathbb{RP}^n;\mathbb{Z}), \mathbb{Z}/2\mathbb{Z}) \\ &= (\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}) \oplus \mathrm{Tor}(0, \mathbb{Z}/2\mathbb{Z}) \\ &= \mathbb{Z}/2\mathbb{Z} \end{aligned}$$

For $1 < k < n$, such that k is an even integer, we have

$$\begin{aligned} H_k(\mathbb{RP}^n;\mathbb{Z}/2\mathbb{Z}) &\cong (H_k(\mathbb{RP}^n;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}) \oplus \mathrm{Tor}(H_{k-1}(\mathbb{RP}^n;\mathbb{Z}), \mathbb{Z}/2\mathbb{Z}) \\ &= (0 \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}) \oplus \mathrm{Tor}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \\ &= \mathbb{Z}/2\mathbb{Z} \end{aligned}$$

For $k = n$ even, we have

$$H_k(\mathbb{RP}^n;\mathbb{Z}/2\mathbb{Z}) = (0 \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}) \oplus \mathrm{Tor}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$$

If $k = n$ is odd, we have

$$H_k(\mathbb{RP}^n;\mathbb{Z}/2\mathbb{Z}) = (\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}) \oplus \mathrm{Tor}(0, \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$$

All in all, we have

$$H_k(\mathbb{RP}^n;\mathbb{Z}/2\mathbb{Z}) = \begin{cases} \mathbb{Z}/2\mathbb{Z} & \text{if } k = 0, \cdots, n \\ 0 & \text{otherwise.} \end{cases}$$

15. Equivalence of Homology Theories

We have encountered various homology theories, including singular, simplicial, and cellular homology, and have seen that they all coincide in specific cases. For instance, if a topological space admits a Δ-complex structure, the singular and simplicial homologies coincide. Similarly, if a topological space admits a CW-complex structure, the singular and cellular homologies coincide. We now demonstrate that this is a specific instance of a more general principle: homology theories are uniquely determined on well-behaved topological spaces, particularly within the category of CW pairs.

Proposition 15.1. *Let h_* be a homology theory in the sense of Definition 7.1 with $\mathbb{Z}$ coefficients defined as a collection of functors*

$$h_n : \mathbf{CW}^2 \to \mathbf{Ab}$$

If $h_n(*; \mathbb{Z}) \cong 0$ for $n \neq 0$, then there exists a natural isomorphism

$$h_n(X, A) \cong H_n(X, A; G)$$

for all CW-pairs (X, A) and for all $n \geq 1$, where $G := h_0(*; \mathbb{Z}) \in \mathbf{Ab}$.

Proof. Since (X, A) is a good pair, we have an isomorphism

$$h_n(X, A; \mathbb{Z}) \cong \tilde{h}_n(X/A; \mathbb{Z})$$

for all $n \geq 0$. This is a formal consequence of Eilenberg-Steenrod axioms that we have verified for singular homology. Hence, we only need to check the absolute case. Just as for singular homology, we have

$$h_n^{\mathrm{CW}}(X; \mathbb{Z}) \cong h_n(X; \mathbb{Z})$$

The hypothesis that $h_n(*; \mathbb{Z}) = 0$ for $n \neq 0$ is used here. The long exact sequences of h_* homology groups for the pairs (X_n, X_{n-1}) give rise to a cellular chain complex.

$$\cdots \to h_n^{\mathrm{CW}}(X_n, X_{n-1}; \mathbb{Z}) \xrightarrow{d_n} h_{n-1}^{\mathrm{CW}}(X_{n-1}, X_{n-2}; \mathbb{Z}) \to \cdots$$

We also have

$$\cdots \to H_n^{\mathrm{CW}}(X_n, X_{n-1}; G) \xrightarrow{\partial_n} H_{n-1}^{\mathrm{CW}}(X_{n-1}, X_{n-2}; G) \to \cdots$$

The individual groups are isomorphic, since

$$h_n^{\mathrm{CW}}(X_n, X_{n-1}; \mathbb{Z}) \cong G^{\#n\text{-cells}} \cong H_n^{\mathrm{CW}}(X_n, X_{n-1}; G).$$

Thus, it remains to show that $d_n = \partial_n$ for $n \geq 1$. For $n = 1$, we can pass from X to S^2X since suspension is a natural isomorphism in any homology theory. S^2X has no 1-cells, so immediately

$$d_1 = 0 = \partial_1$$

Now let $n > 1$. The calculation of cellular boundary maps d_n for $n > 1$ in terms of degrees of certain maps between spheres works equally well for h_*, where degree now means degree with respect to the h_* theory. But a map

$$f : \mathbb{S}^n \to \mathbb{S}^n$$

of degree m in the usual sense is simply multiplication by m on $H_n(\mathbb{S}^n; G) \cong G \cong h_n(\mathbb{S}^n; G)$. The claim follows. $\square$

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