Algebraic Topology

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Preface

These notes on algebraic topology were compiled during my graduate studies while attending a year-long course on the subject. They reflect my developing conviction that the language of category theory should be introduced at the outset, and that spectral sequences merit early presentation, as they provide a unified framework for deriving many fundamental homological results in algebraic topology. The notes proceed systematically through core topics, including fundamental groups, homological algebra, homology, cohomology, higher homotopy groups, and the Serre spectral sequence. Some sections remain incomplete, and typographical errors may be present. Corrections and suggestions are welcome and can be sent to junaid.aftab1994@gmail.com.

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CHAPTER 1

Introduction

1.1. What is Algebraic Topology?

Topology may be regarded as a "deformable" form of geometry—often informally described as the study of "squishy" shapes. In contrast to classical geometry, which concerns itself with rigid structures determined by distances and angles, topology focuses on properties of spaces that remain invariant under continuous deformations such as stretching, bending, or compressing, but not tearing or gluing. This viewpoint gives rise to a mathematical theory of shape that abstracts away geometric notions.

The foundational ideas of algebraic topology were introduced in Poincaré's seminal work *Analysis Situs* [Poi10], which laid the groundwork for the modern discipline. At its essence, algebraic topology employs tools from abstract algebra to investigate topological spaces, chiefly by associating to them algebraic objects that encode essential topological features.

In contemporary terms, these conceptual foundations are elegantly articulated through the framework of category theory. Within this framework, one seeks to construct functors from a "topological category" (for instance, the category of topological spaces Top or the homotopy category hTop) to an "algebraic category" (such as the category of abelian groups Ab or the category of commutative rings CRing). This process assigns topological invariants to spaces—quantities that remain unchanged under continuous maps, homeomorphisms, and homotopies—thereby enabling their study via algebraic methods.

Among the most fundamental constructions in this paradigm are the fundamental group, homology and cohomology theories, as well as the higher homotopy groups.

1.1.1. Fundamental Group. The most basic topological invariant of a space is the fundamental group (also called the first homotopy group). For a pointed topological space (X, x_0), the fundamental group is denoted by $\pi_1(X, x_0)$ and encodes the structure of continuous maps $\gamma : [0,1] \to X$ satisfying $\gamma(0) = \gamma(1) = x_0$, called loops based at x_0, up to homotopy relative to their endpoints. Intuitively, the fundamental group measures the "1-dimensional loop structure" of a space. For example, we shall see that the circle $\mathbb{S}^1$ possesses nontrivial loops, whereas the sphere $\mathbb{S}^2$ does not. By assigning to each pointed topological space its fundamental group, one obtains a covariant functor

$$\pi_1 : \mathsf{Top}_* \longrightarrow \mathsf{Grp},$$

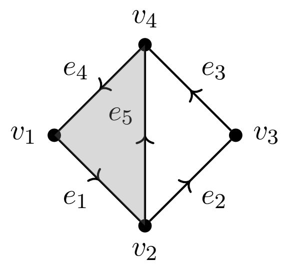
where Top_* denotes the category of pointed topological spaces and basepoint-preserving continuous maps. The study of the fundamental group illustrates a central principle of algebraic topology: replacing geometric or topological data with algebraic structures in order to facilitate classification and computation.

1.1.2. Homology. Fix $n \geq 0$. Homology gives rise to a family of covariant functors

$$H_n : \mathsf{Top} \longrightarrow \mathsf{Ab},$$

each of which assigns to a topological space X an abelian group $H_n(X)$ that, heuristically, encodes the "n-dimensional hole structure" of X. For example, consider the topological space X depicted in Figure 1.

The "1-dimensional hole structure" of X can be computed using a combinatorial homology theory known as simplicial homology. We now outline the argument. Intuitively, the boundary of an oriented



A simplicial complex illustrating edges and a 2-simplex

edge in the diagram can be viewed as the formal difference between its "target" and "source" vertices. For example, the boundary of e_1 is given by

$$\partial e_1 = v_2 - v_1.$$

A chain of paths is defined as a formal sum of oriented edges. For instance, we have the 1-chains

$$\begin{aligned} c_1 &= e_1 + e_5 + e_4, \\ c_2 &= e_2 + e_3 + e_5^{-1}, \\ c_3 &= e_1 + e_2 + e_3 + e_4, \end{aligned}$$

where e_5^{-1} denotes the edge e_5 traversed in the opposite orientation. In this terminology, a cycle in X is a 1-chain whose boundary vanishes. For example, each of c_1, c_2, and c_3 is a cycle. However, the loop c_1 can be contracted to a point: the path $e_4 + e_1$ may be continuously deformed to e_5 within the interior of c_1. Algebraically, this is detected by the fact that c_1 is the boundary of the 2-simplex with vertices v_1, v_2, v_4. By contrast, c_2 cannot be contracted to a point, since the 2-simplex with vertices v_2, v_3, v_4 is absent (the corresponding triangular region is hollow). Hence, we anticipate the presence of a single 1-dimensional hole in X. The first simplicial homology group will detect this hole.

Remark 1.1.1. *The above intuition can be made precise by the Hurewicz theorem which states that*

$$H_1^{\mathrm{Simp}}(X) \cong \frac{\pi_1(X)}{[\pi_1(X), \pi_1(X)]}$$

That is $H_1^{\mathrm{Simp}}(X)$ is isomorphic to the abelianization of $\pi_1(X)$, the fundamental group of X which quite literally is a measure of holes in a topological space.

More generally, we shall see that the n-th simplicial homology group measures the existence of 'n-dimensional holes' in a topological space, X.

Remark 1.1.2. *The statement above is only meant for intuition and should be taken with a grain of salt. In general, there is only a group homomorphism $\pi_n(X) \to H_n^{\mathrm{Simp}}(X)$ for $n \geq 2$ if X is path-connected.*

We shall encounter several homology theories, including simplicial homology, singular homology, and cellular homology. Although these theories arise from distinct constructions, under appropriate conditions they are naturally isomorphic, and hence yield the same topological invariants. Each theory possesses its own computational strengths and conceptual advantages.

1.1.3. Cohomology. Fix $n \geq 0$. Cohomology gives rise to a family of contravariant functors

$$H^n : \mathsf{Top} \longrightarrow \mathsf{Ab}.$$

Cohomology provides a dual perspective to homology. While both theories assign graded abelian groups to topological spaces, cohomology often captures more refined invariants and exhibits a richer algebraic structure. In particular, cohomology groups admit a natural graded ring structure via the cup product, yielding a contravariant functor

$$H^* : \mathsf{Top} \longrightarrow \mathsf{CRing},$$

where CRing denotes the category of commutative graded rings. This additional structure enables deeper connections with other areas of mathematics, including geometry, bundle theory, and differential topology. For example, one can construct invariants of topological spaces from geometric objects such as differential forms (de Rham cohomology) or vector bundles (topological K-theory). Such invariants are typically contravariant by nature and are naturally encoded in cohomological data.

1.1.4. Homotopy. Homotopy groups form a class of fundamental invariants in algebraic topology that classify continuous maps up to homotopy, thereby capturing intrinsic shape and deformation properties of topological spaces. The first homotopy group coincides with the fundamental group. For $n \geq 2$, the higher homotopy groups $\pi_n(X, x_0)$ generalize this notion to homotopy classes of based maps from the n-sphere $\mathbb{S}^n$ into X, thus detecting higher-dimensional analogues of holes and obstructions to contractibility. While these groups encode richer and more subtle topological information than homology or cohomology, they are typically far more difficult to compute.

1.2. Preliminaries

1.2.1. Notation. Here is a list of some standard notation used throughout the notes:

$\mathbb{R}^n$	Euclidean space
$\mathbb{D}^n = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$	n-dimensional disk
$\mathbb{S}^{n-1} = \{x \in \mathbb{D}^n \mid \|x\| = 1\}$	$(n-1)$-dimensional sphere
$\mathbb{B}^n = \mathbb{D}^n \setminus \mathbb{S}^{n-1}$	n-dimensional unit open ball
$I^n = \{x \in \mathbb{R}^n \mid 0 \leq x_i \leq 1\}$	n-dimensional unit cube
$\partial I^n = \{x \in I^n \mid x_i = 0 \text{ or } 1 \text{ for some } i\}$	boundary of I^n

1.2.2. Category Theory. Throughout these notes, we assume familiarity with the language of category theory[1], and it is freely used throughout the notes. References include [Rie17; Lei14; Mac13]. Here is a review of basic notions in category theory:

(1) A category C consists of
   (a) a collection of objects $\mathrm{Ob}(\mathsf{C})$,
   (b) for each pair of objects $X, Y \in \mathsf{C}$, a set of morphisms $\mathrm{Hom}_{\mathsf{C}}(X, Y)$,
   (c) identity morphisms Id_X for each X,
   (d) a composition law $\circ$ satisfying associativity and unitality.

(2) A functor $\mathsf{F} : \mathsf{C} \to \mathsf{D}$ assigns to each object $X \in \mathsf{C}$ an object $\mathsf{F}(X) \in \mathsf{D}$, and to each morphism $f : X \to Y$ a morphism $\mathsf{F}(f) : \mathsf{F}(X) \to \mathsf{F}(Y)$, such that:

$$\mathsf{F}(\mathrm{Id}_X) = \mathrm{Id}_{\mathsf{F}(X)}$$
$$\mathsf{F}(g \circ f) = \mathsf{F}(g) \circ \mathsf{F}(f)$$

[1] This is covered in detail in my other notes.

The class of morphisms between $X, Y \in \mathsf{C}$ is denoted by $\mathrm{Hom}_{\mathsf{C}}(\cdot,\cdot)$. We usually wite $\mathrm{Hom}_{\mathsf{C}}(\cdot,\cdot)$ as simply $\mathrm{Hom}(\cdot,\cdot)$.

(3) A natural transformation $\eta : \mathsf{F} \Rightarrow \mathsf{G}$ between functors $\mathsf{F}, \mathsf{G} : \mathsf{C} \to \mathsf{D}$ assigns to each object X in C a morphism $\eta_X : \mathsf{F}(X) \to \mathsf{G}(X)$ in D such that for every morphism $f : X \to Y$ in C, the following square commutes:

$$\begin{array}{ccc} \mathsf{F}(X) & \xrightarrow{F(f)} & \mathsf{F}(Y) \\ \Big\downarrow{\scriptstyle \eta_X} & & \Big\downarrow{\scriptstyle \eta_Y} \\ \mathsf{G}(X) & \xrightarrow{G(f)} & \mathsf{G}(Y) \end{array}$$

(4) A functor $\mathsf{F} : \mathsf{C} \to \mathsf{D}$ is an equivalence of categories if there exists a functor $G : \mathsf{D} \to \mathsf{C}$

$$\mathsf{F} \circ \mathsf{G} \cong \mathrm{Id}_{\mathsf{D}}$$
$$\mathsf{G} \circ \mathsf{F} \cong \mathrm{Id}_{\mathsf{C}}$$

Equivalently, $\mathsf{F} : \mathsf{C} \to \mathsf{D}$ is an equivalence of categories if and only if:

(a) F is fully faithful, i.e., for all objects $X, Y \in \mathsf{C}$, the map

$$\mathrm{Hom}_{\mathsf{C}}(X, Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathsf{D}}(\mathsf{F}(X), \mathsf{F}(Y))$$

is a bijection; and

(b) F is essentially surjective, i.e., for every object $D \in \mathsf{D}$, there exists an object $C \in \mathsf{C}$ such that $\mathsf{F}(C) \cong D$ in D.

(5) A diagram in a category C is a functor

$$D : \mathsf{J} \to \mathsf{C}$$

from an index category J. The limit (resp. colimit) of such a diagram is a universal cone (resp. cocone) over D. Examples of limits include products and pullbacks; examples of colimits include coproducts and pushouts.

(6) A functor $\mathsf{F} : \mathsf{C} \to \mathsf{D}$ is said to be left adjoint to $\mathsf{G} : \mathsf{D} \to \mathsf{C}$ if there is a natural isomorphism:

$$\mathrm{Hom}_{\mathsf{D}}(\mathsf{F}(X), Y) \cong \mathrm{Hom}_{\mathsf{C}}(X, \mathsf{G}(Y))$$

for all $X \in \mathsf{C}, Y \in \mathsf{D}$. In this case, $F \dashv G$ and G is called a right adjoint.

Remark 1.2.1. *The language of derived functors and resolutions is reviewed in the appendix (Chapter 15).*

We will frequently make use of various "algebraic categories." A few standard examples are listed below:

(1) Grp : The objects of Grp are groups and $\mathrm{Hom}(G, H)$ consists of group homomorphisms from G to H. Ab, the category of abelian groups, is defined similarly.

(2) GrGrp : The objects of GrGrp are graded groups, that is, groups $G = \bigoplus_{n\in\mathbb{Z}} G_n$ equipped with a direct sum decomposition indexed by integers. The morphisms $\mathrm{Hom}_{\mathsf{GrGrp}}(G, H)$ are group homomorphisms $f : G \to H$ that respect the grading; explicitly, $f(G_n) \subseteq H_n$ for each $n \in \mathbb{Z}$. GrAb, the category of graded abelian groups, is defined similarly.

(3) CRing: The objects of CRing are commutative rings, and $\mathrm{Hom}(R, S)$ consists of ring homomorphisms from R to S. $\mathsf{GrCring}$, the category of graded commutative rings, is defined similarly as above.

(4) Mod_R: The objects of Mod_R are modules over a fixed (commutative) ring R, and $\mathrm{Hom}_R(M, N)$ consists of R-linear homomorphisms from the R-module M to the R-module N. The category of graded R-modules, GrMod_R, is defined similarly as above.

(5) Grpd : The objects of Grpd are groupoids, which are categories in which each morphism is an isomorphism, and $\mathrm{Hom}(X, Y)$ consists of functors between groupoids from X to Y.

1.2.3. Topology. Basic notions in topology will be assumed throughout. We usually assume that a fixed base point $x_0 \in X$ has been chosen, in which case X is called a pointed topological space. A continuous function $f\colon (X, x_0) \to (Y, y_0)$ between pointed topological spaces is assumed to satisfy $f(x_0) = y_0$. Such functions are called pointed continuous maps. The following categories naturally arise in algebraic topology:

(1) Top: The objects are topological spaces and $\mathrm{Hom}(X, Y)$ is the set of continuous functions from X to Y.
(2) Top_*: The objects are pointed topological spaces, (X, x_0) and $\mathrm{Hom}((X, x_0), (Y, y_0))$ consists of continuous maps $f : (X, x_0) \to (Y, y_0)$ such that $f(x_0) = y_0$. Such maps are called pointed continuous maps.
(3) Top^2: The objects are all pairs (X, A) where X is a topological space and $A \subseteq X$ is a subspace and $\mathrm{Hom}((X, A), (Y, B))$ is simply the set of continuous map $f : X \to Y$ such that $f(A) \subseteq B$.
(4) Top^3: The objects are all triples (X, A, B) where X is a topological space and $B \subseteq A \subseteq X$ is a subspace. Then $\mathrm{Hom}((X, A, C), (Y, B, D))$ is simply the set of continuous maps $f : X \to Y$ such that $f(A) \subseteq B$ and $f(C) \subseteq D$.

Additional categories arising in algebraic topology will be introduced as needed later in the notes. Below, we recall some basic universal properties that will be invoked frequently throughout.

(1) Top has products and co-products. The product of X, Y is given by the Cartesian product $X \times Y$ of topological spaces with the product topology. The product is an example of a categorical pullback:

$$\begin{tikzcd}
A \arrow[drr, bend left, "f"] \arrow[dr, dashed] \arrow[ddr, bend right, "g"'] & & \\
& X \times Y \arrow[r, "\pi_X"'] \arrow[d, "\pi_Y"] & X \\
& Y &
\end{tikzcd}$$

The co-product of X, Y is given by the disjoint union $X \sqcup Y$ with the disjoint union topology. The disjoint union is an an example of a categorical pushout:

$$\begin{tikzcd}
& X \arrow[d, "\iota_X"'] \arrow[ddr, bend left, "g"] & \\
Y \arrow[r, "\iota_Y"] \arrow[drr, bend right, "f"'] & X \coprod Y \arrow[dr, dashed] & \\
& & A
\end{tikzcd}$$

(2) The category Top_* has products and co-products. The product of $(X, x_0), (Y, y_0)$ is given by the pointed Cartesian product $(X \times Y, (x_0, y_0))$. The pointed Cartesian product is an example

of a categorical pullback:

$$
\begin{tikzcd}
A \arrow[dr, dashed] \arrow[drr, bend left, "f"] \arrow[ddr, bend right, "g"'] & & \\
& X \times Y \arrow[r, "\pi_X"'] \arrow[d, "\pi_Y"] & X \arrow[d] \\
& Y \arrow[r] & \{*\}
\end{tikzcd}
$$

The coproduct of $(X, x_0), (Y, y_0)$ is given by the wedg sum

$$X \vee Y := (X \times Y)/\sim$$

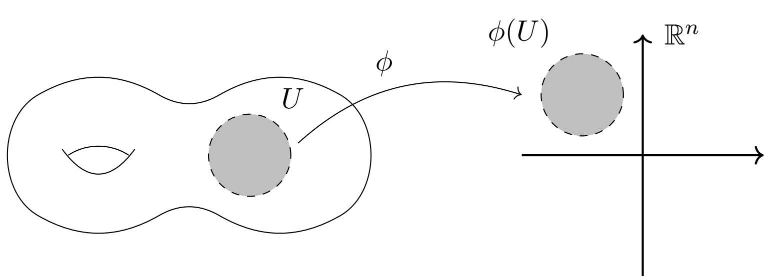
where the quotient identifies the basepoints x_0 and y_0 to a single point. The wedge sum is an instance of a categorical pushout:

$$
\begin{tikzcd}
\{*\} \arrow[r] \arrow[d] & X \arrow[d, "\iota_X"'] \arrow[ddr, bend left, "g"] & \\
Y \arrow[r, "\iota_Y"] \arrow[drr, bend right, "f"'] & X \vee Y \arrow[dr, dashed] & \\
& & A
\end{tikzcd}
$$

1.3. Topological Manifolds

One of the principal classes of spaces studied in algebraic topology via topological invariants is the class of topological manifolds. These are spaces that locally resemble Euclidean space. Familiar examples include plane curves such as circles and parabolas, as well as two-dimensional surfaces such as spheres and tori. A comprehensive treatment of topological manifolds can be found in [Lee10].

Definition 1.3.1. A topological space, X, is a topological n-manifold if X is a second-countable, Hausdorff space that is locally homemorphic to $\mathbb{R}^n$. That is, each point of X is contained in a coordinate chart, which is a pair (U, ϕ), where U is an open subset of X and $\phi : U \to \phi(U) \subseteq \mathbb{R}^n$ is a homeomorphism from U to an open subset $\phi(U)$ of $\mathbb{R}^n$.



Remark 1.3.2. *A collection of charts (U_α, ϕ_α) such that $\bigcup_\alpha U_\alpha = M$ is an atlas for X.*

Remark 1.3.3. *The number n is attached to a single chart and might apriori depend on the chart itself. This turns out to be not the case. This result is called the invariance of dimension and will be proved later.*

We discuss the implications of the conditions imposed in Definition 1.3.1. Since a topological manifold is locally Euclidean, it is easy to see that it inherits a number of properties of Euclidean space locally. For instance, we have the following:

Proposition 1.3.4. *Let X be a topological n-manifold. Then X is locally compact, locally path-connected and locally contractible.*

Proof. Every point of X has a neighborhood homeomorphic to the open unit ball in $\mathbb{R}^n$. Each open ball in $\mathbb{R}^n$ is locally compact, locally compact and locally path-connected, locally contractible. The claim follows. $\square$

The locally Euclidean condition does not impose any global topological restrictions. To address this, the second-countability and Hausdorff conditions are imposed. Intuitively, Hausdorff spaces possess "enough open sets," ensuring that familiar properties hold; for instance, in a Hausdorff space, finite subsets are closed, and limits of convergent sequences are unique. Moreover, this condition excludes certain pathological examples, such as the "line with two origins." On the other hand, second-countability restricts the space from having an excessively large collection of open sets necessary to cover it. The following is a prototypical global topological property of a topological n-manifold.

Proposition 1.3.5. *Let X be a topological n-manifold. X has a countable basis of precompact coordinate balls.*

Proof. First consider the special case in which X can be covered by a single chart. Suppose $\varphi : M \to U \subseteq \mathbb{R}^n$ is a global coordinate chart. Let

$$\mathcal{B} = \{B_r(x) : x \in \mathbb{Q},\ x \in \mathbb{Q}^n\ B_{r'}(x) \subseteq U \text{ for some } r' < r\}$$

Each $B_r(x) \in \mathcal{B}$ is pre-compact in U, and it is easy to check that $\mathcal{B}$ is a countable basis for the topology of U. Because φ is a homeomorphism, it follows that $\varphi^{-1}(\mathcal{B})$ is a countable basis for X, consisting of pre-precompact coordinate balls. More generally, each $p \in M$ is in the domain of a coordinate chart. Since X is second-countable, X is covered by countably many coordinate charts $\{(U_i, \varphi_i)\}_{i=1}^{\infty}$. By the argument in the preceding paragraph, each U_i has a countable basis of coordinate balls that are pre-compact in U_i. If $V \subseteq U_i$ is one of these balls, then the closure of V in U_i is compact, and because X is Hausdorff, it is closed in X. It follows that the closure of V in X is the same as its closure in U_i, so V is precompact in X as well. Clearly, the union of all these countable bases is a countable basis for X. $\square$

Example 1.3.6. The following is a list of examples of topological manifolds.

(1) $\mathbb{R}^n$ is a topological n-manifold. $\mathbb{R}^n$ is covered by a single chart $(\mathbb{R}^n, \mathrm{Id}_{\mathbb{R}^n})$, where $\mathrm{Id}_{\mathbb{R}^n} : \mathbb{R}^n \to \mathbb{R}^n$ is the identity map.

(2) **(Spheres)** The unit n-sphere, $\mathbb{S}^n$, is Hausdorff and second-countable because it is a topological subspace of $\mathbb{R}^{n+1}$. For each $1 \leq i \leq n+1$, consider the sets:

$$U_i^+ = \{(u^1, \cdots, u^{n+1}) \in \mathbb{R}^{n+1} \mid u^i > 0\}$$
$$U_i^- = \{(u^1, \cdots, u^{n+1}) \in \mathbb{R}^{n+1} \mid u^i < 0\}$$

Let $f : \mathbb{B}^n \to \mathbb{R}$ be the continuous function defined by

$$f(x) = \sqrt{1 - \|u\|^2}$$

For each $1 \leq i \leq n$, $U_i^{\pm} \cap \mathbb{S}^n$ is the graph of the function

$$u^i = \pm f(u^1, \cdots, \widehat{u^i}, \cdots, u^{n+1}),$$

where the hat indicates that u^i is omitted. Thus, each subset $U_i^{\pm} \cap \mathbb{S}^n$ is locally Euclidean of dimension n, and the maps $\phi_i \colon U_i^{\pm} \cap \mathbb{S}^n \to \mathbb{B}^n$ given by

$$\phi_i(u^1, \cdots, u^{n+1}) = (u^1, \cdots, \widehat{u^i}, \cdots, u^{n+1})$$

defines the desired homemorphism.

(3) **(Real Projective Space)** The real projective space, $\mathbb{RP}^n$, is defined as the quotient space, $\mathbb{RP}^n = (\mathbb{R}^{n+1} \setminus \{0\})/\sim$ with the equivalence relation

$$x \sim y \text{ in } \mathbb{R}^{n+1} \setminus \{0\} \iff x = \lambda y \text{ for some } \lambda \in \mathbb{R}^{\times}$$

It is made into a topological space by giving it the quotient topology via the map

$$\pi \colon \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{RP}^n,$$

where $[x] := \pi(x) = \text{span}\{x\}$. It can be easily checked that

$$\mathbb{RP}^n \cong \mathbb{S}^n/\sim$$

where $\sim$ is the equivalence relation on $\mathbb{S}^n$ such that $x \sim -x$ (i.e., antipodal points are identified). We check that $\mathbb{RP}^n$ is both second-countable and Hausdorff:

(a) Consider the quotient map: $q : \mathbb{S}^n \to \mathbb{S}^n/\sim$ Note that q is an open map. Indeed for any open subset $V \subseteq \mathbb{S}^n$, we have:

$$q^{-1}(q(V)) = V \cup -V,$$

Since $\mathbb{S}^n$ is second-countable, $\mathbb{RP}^n \cong \mathbb{S}^n/\sim$ is also second-countable as q is an open map.

(b) If $[x], [y] \in \mathbb{S}^n/\sim$, then one can choose $\varepsilon > 0$ small enough that

$$U = \mathbb{B}(x, \varepsilon) \cap \mathbb{S}^n$$
$$V = \mathbb{B}(y, \varepsilon) \cap \mathbb{S}^n$$

are open sets in $\mathbb{S}^n$ such that $\pm U, \pm V$ are pairwise disjoint. Since,

$$q^{-1}(q(U)) = U \cup -U$$
$$q^{-1}(q(V)) = V \cup -V$$

$q^{-1}(q(U))$ and $q^{-1}(q(V))$ are open disjoint subsets of $\mathbb{S}^n/\sim$ containing $[x]$ and $[y]$. Hence, $\mathbb{RP}^n \cong \mathbb{S}^n/\sim$ is Hausdorff.

For each $1 \leq i \leq n+1$, consider the sets:

$$\widetilde{U}_i = \{(u^1, \cdots, u^{n+1}) \in \mathbb{R}^{n+1} \mid u^i \neq 0\}$$

Let $U_i = \pi(\widetilde{U}_i)$. By properties of the quotient topology, U_i is an open subset of $\mathbb{R}^n$. Consider the map $\phi_i \colon U_i \to \mathbb{R}^n$ defined as:

$$\phi_i\left([u]\right) = \left(\frac{u^1}{u^i}, \cdots, \frac{u^{i-1}}{u^i}, 1, \frac{u^{i+1}}{u^i}, \cdots, \frac{u^{n+1}}{u^i}\right).$$

This map is well-defined because its value is unchanged by multiplying x by a nonzero constant. By properties of the quotient topology, ϕ_i is continuous. In fact, ϕ_i is a homeomorphism because it has a continuous inverse given by

$$\phi_i^{-1}(u^1, \cdots, u^n) = [u^1, \cdots, u^{i-1}, 1, u^i, \cdots, u^n];$$

This shows that $\mathbb{RP}^n$ is locally Euclidean of dimension n.

(4) (**Tori**) For a positive integer $n \geq 2$, the $(n-1)$-torus is the product space

$$\mathbb{S}^1 \times \cdots \times \mathbb{S}^1$$

It is clear that a product of topological manofolds is a topolopgical manifold. Hence, T is topological n-manifold since $\mathbb{S}^1$ is a 1-manifold.

Remark 1.3.7. *For $n \geq 2$, we usually abbreviate the n-torus as $\mathbb{T}^n$.*

Sets such as closed intervals in $\mathbb{R}$ and closed balls in $\mathbb{R}^n$ fail to be both topological manifolds since they 'have a boundary of sorts.' We make precise the notion of a topological manifold with boundary.

Definition 1.3.8. Let X be a topological space. X is a topological n-manifold with boundary if X is a second countable, Hausdorff space such that each point $x \in M$ is contained in a coordinate chart, (U, ϕ), such that:

(1) (**Interior Chart**) Either $\phi : U \to \phi(U) \subseteq \mathbb{R}^n$ is a homeomorphism from U to an open subset $\phi(U)$ of $\mathbb{R}^n$.
(2) (**Boundary Chart**) Or $\phi : U \to \phi(U) \subseteq \mathbb{H}^n$ is a homeomorphism from U to an open subset $\phi(U)$ of $\mathbb{H}^n$, the upper-half plane, such that $\phi(x) \cap \partial\mathbb{H}^n \neq \emptyset$.

A point $p \in M$ is called an interior point of X if it is in the domain of some interior chart or a boundary chart (U, ϕ) such that $\phi(U) \cap \partial\mathbb{H}^n = \emptyset$. It is a boundary point of X if it is in the domain of a boundary chart that sends p to $\partial\mathbb{H}^n$. The boundary of X (the set of all its boundary points) is denoted by ∂M; similarly, its interior, the set of all its interior points, is denoted by $\text{Int}(M)$.

Remark 1.3.9. *A point $p \in M$ might apriori simultaneously be a boundary point and an interior point, meaning that there is one interior chart whose domain contains p, and another boundary chart that sends p to $\partial\mathbb{H}^n$. This turns out not to be the case. This result is called the invariance of boundary and will be proved later.*

Example 1.3.10. (Sketch) The following is a list of basic examples of a topological manifold with boundary.

(1) $\overline{\mathbb{B}^n}$ is smooth n-manifold with boundary. One can prove this by definition. We skip details.
(2) If X is a n-dimesnional manifold with boundary, then ∂M is a $(n-1)$-dimensional manifold without boundary. We skip details.

1.4. CW Complexes

Another important class of spaces studied via topological invariants in algebraic topology is the class of CW complexes. These spaces are constructed combinatorially by successively attaching basic building blocks, namely disks of varying dimensions, via continuous attaching maps. For further details, see [Hat02; Lee10].

1.4.1. Definitions.

An arbitrary topological space X can be challenging to visualize and analyze. Therefore, we shall primarily focus on the subcategory of topological spaces that can be constructed inductively using open cells. This subcategory is known as the category of CW complexes. This approach enables the meaningful study of a wide range of topological spaces.

Definition 1.4.1. An open n-cell is a topological space that is homeomorphic to the open unit ball $\mathbb{B}^n$. A closed n-cell is a topological space homeomorphic to $\mathbb{D}^n$.

Remark 1.4.2. *We will only use the phrase n-cell when the context is clear.*

New topological spaces can be constructed from old topological spaces by attaching an n-cell. Let X be a topological space. Suppose there is a map $\phi : \mathbb{S}^{n-1} \to X$ a map. One can form a new topological space, $X \coprod_\phi \mathbb{D}^n$, from the disjoint union $X \coprod \mathbb{D}^n$ by identifying each $\phi(x) \in \mathbb{S}^{n-1}$ with $\phi(x) \in X$ for all $x \in \mathbb{S}^{n-1}$, and equipping the resulting set with the quotient topology. The map ϕ is called the characteristic map. We refer to the space $X \coprod_\phi \mathbb{D}^n$ as being obtained from X by 'attaching an n-cell', and call $\phi : \mathbb{S}^{n-1} \to X$ the attaching map. Using the the universal properties of the disjoint union and quotient topology, we have the following commutative diagram.

$$\begin{tikzcd}
\mathbb{S}^{n-1} \arrow[r, "\phi"] \arrow[d, hook, "\iota"] & X \arrow[d] \arrow[ddr, bend left, "f_1"] & \\
\mathbb{D}^n \arrow[r] \arrow[drr, bend right, "f_2"'] & X \coprod_\phi \mathbb{D}^n \arrow[dr, dashed, "f"] & \\
& & Y
\end{tikzcd}$$

This shows that $X \coprod_\phi \mathbb{D}^n$ is a pushout in Top. One can also attach more than one n-cell. Let $\{\mathbb{D}^n_\alpha\}_{\alpha \in A_n}$ be a collection of n-cells and let $\phi^n_\alpha : \mathbb{S}^{n-1}_\alpha \to X$ be a collection of continuous maps. One can form a new topological space, $X \coprod_{\phi^n_\alpha} \mathbb{D}^n_\alpha$, by attaching the aforementioned collection of n-cells using the rule prescribed above. Once again, we have a commutative diagram.

$$\begin{tikzcd}
\coprod_{\phi^n_\alpha} \mathbb{S}^{n-1}_\alpha \arrow[r, "f"] \arrow[d, hook, "\iota"] & X \arrow[d] \\
\coprod_{\phi^n_\alpha} \mathbb{D}^n_\alpha \arrow[r] & X \coprod_{\phi^n_\alpha} \mathbb{D}^n_\alpha
\end{tikzcd}$$

This shows that $X \coprod_{\phi^n_\alpha} \mathbb{D}^n_\alpha$ is a pushout in Top.

Definition 1.4.3. Let X be a topological space. A CW decomposition of X is a sequence of subspaces

$$X^0 \subseteq X^1 \subseteq X^2 \subseteq \cdots \subseteq X^n \subseteq \cdots,$$

of X such that the following three conditions are satisfied:

(1) The space X^0 is discrete.
(2) The space X^n is obtained from X^{n-1} by attaching a (possibly) infinite number of n-cells $\{\mathbb{D}^n_\alpha\}_{\alpha \in A_n}$ via attaching maps $\phi^n_\alpha : \mathbb{S}^{n-1}_\alpha \to X^{n-1}$.
(3) The topology of X is compatible with quotient topology on X that makes the

$$\coprod_{n \in \mathbb{N}} X^n \to X$$

continuous. In other words, $A \subseteq X$ is open if and only if $A \cap X^n$ is open for all $n \geq 0$.

Remark 1.4.4. *If X admits a CW decomposition, then it can be easily checked that X is a colimit of $\{X^n\}_{n \in \mathbb{N} \cup \{0\}}$. In particular, X is the colimit of the diagram*

$$X^0 \xrightarrow{j_0} X^1 \to \cdots \to X^n \xrightarrow{j_n} X^{n+1} \to \cdots$$

in Top. *Here j_n is the inclusion of X_n into X_{n+1}.*

We can now define the category of CW complexes, CW. The objects of the category are topological spaces that admit a CW structure (CW complexes), and morphisms between CW complexes are cellular continuous maps. That is, $f(X^n) \subseteq Y^n$ for each $n \geq 0$ where f is a continuous map. In other words, if X and Y are CW complexes and we have a commutative diagram

$$\begin{tikzcd}
X^0 \arrow[r] \arrow[d] & X^1 \arrow[r] \arrow[d] & \cdots \arrow[r] & X^n \arrow[r] \arrow[d] & X^{n+1} \arrow[r] \arrow[d] & \cdots \\
Y^0 \arrow[r] & Y^1 \arrow[r] & \cdots \arrow[r] & Y^n \arrow[r] & Y^{n+1} \arrow[r] & \cdots
\end{tikzcd}$$

then, on forming the colimits, we obtain an induced map $f : X \to Y$ which is a cellular map.

Remark 1.4.5. *CW_* is the category of pointed CW complexes defined analogously to Top_*. Similarly, CW^2 is the category of pairs of CW complexes defined analogously to Top^2.*

Each cell $\mathbb{D}^n_\alpha$ has its characteristic map ϕ^n_α, which is by definition the composition of continuous maps:

$$\begin{tikzcd}
\mathbb{D}^n_\alpha \arrow[r, hook] \arrow[rrr, bend left, "\phi^n_\alpha"] & X^{n-1} \coprod \mathbb{D}^n_\alpha \arrow[r, two heads] & X^{n-1} \coprod_\phi \mathbb{D}^n_\alpha \arrow[r, hook] & X
\end{tikzcd}$$

Proposition 1.4.6. *Let X be a topological space with a CW decomposition. $A \subseteq X$ is open if and only if $(\phi^n_\alpha)^{-1}(\mathbb{D}^n_\alpha)$ is continuous for each $\alpha \in A_n$ and $n \in \mathbb{N}$. In particular, X is a quotient space of $\coprod_{\alpha \in A_n} \mathbb{D}^n_\alpha$*

Proof. The forward implication is clear. Conversely, suppose $(\phi^n_\alpha)^{-1}(\mathbb{D}^n_\alpha)$ is open in $\mathbb{D}^n_\alpha$ for each for each $\alpha \in A_n$ and $n \in \mathbb{N}$. Suppose by induction on n that $A \cap X^{n-1}$ is open in X^{n-1}. Since $(\phi^n_\alpha)^{-1}(\mathbb{D}^n_\alpha)$ is open in $\mathbb{D}^n_\alpha$ for all $\alpha \in A_n$, then $A \cap X^n$ is open in X^n by the definition of the quotient topology on X^n. The last implication is clear by definition. $\square$

Definition 1.4.7. Let X be a topological space. X is a CW complex if X admits a CW decomposition satisfying the following two properties:

(1) The closure of each open cell is contained in a union of finitely many cells.
(2) The topology of X is coherent with $\{\{\overline{\mathbb{D}^n_\alpha}\}_{\alpha \in A_n} : n \in \mathbb{N}\}$[2].

Remark 1.4.8. *If $X \in \mathsf{Top}$ admits a CW decomposition, we say that X belongs to the subcategory CW.*

A CW complex is finite (or finite-dimensional) if there are only finitely many cells involved. Every finite CW decomposition is automatically a finite CW complex. In fact, every locally finite CW decomposition is automatically a CW complex as we show below.

Proposition 1.4.9. *Let X be a topological space endowed with a CW decomposition. If $\{\mathbb{D}^n_\alpha \mid \alpha \in A_n,\ n \in \mathbb{N}\}$ is a locally finite collection, then X is a CW complex.*

Proof. By assumption, every point $\mathbb{D}^n_\alpha$ has a neighborhood that intersects only finitely many cells. Since $\mathbb{D}^n_\alpha$ is compact, it is covered by finitely many such neighborhoods. This readily implies (1) in Definition 1.4.7. Suppose $A \subseteq X$ is a subset such that $A \cap \mathbb{D}^n_\alpha$ is closed for each $\alpha \in A_n$ and $n \in \mathbb{N}$. Given $x \in X \setminus A$, let W_x be a neighborhood of x that intersects the closures of only finitely many cells, say $\mathbb{D}^{n_1}_1, \ldots, \mathbb{D}^{n_k}_k$. Since $A \setminus \mathbb{D}^{n_j}_j$ is closed in $\mathbb{D}^{n_j}_j$ and thus in X, it follows that

$$W \setminus A = W \setminus (A \cap \mathbb{D}^{n_1}_1) \cup \ldots \cup \left(A \cap \mathbb{D}^{n_k}_k\right)$$

is a neighborhood of x contained in $X \setminus A$. Thus $X \setminus A$ is open, so A is closed. This readily implies (2) in Definition 1.4.7. $\square$

[2] That is, $A \subseteq X$ is open/closed if and only if $A \cap \overline{\mathbb{D}^n_\alpha}$ is open/closed for each $\alpha \in A_n$ and $n \in \mathbb{N}$.

1.4.2. Examples. In the examples that follows, we will not explicitly check that condition (3) in Definition 1.4.3 is satisfied. It should be straightforward to do verify these claims, though.

Example 1.4.10. Let $N = (0, \ldots, 0, 1)$ in $\mathbb{S}^n$. Consider the map $\sigma_N : \mathbb{S}^n \setminus \{N\} \to \mathbb{R}^n$ by [3]

$$\sigma_N(u^1, \ldots, u^{n+1}) = \left(\frac{u^1}{1-u^{n+1}}, \ldots, \frac{u^n}{1-u^{n+1}}\right)$$

Similarly, consider $\beta_N : \mathbb{R}^n \to \mathbb{S}^n \setminus \{N\}$

$$\beta_N(u^1, \ldots, u^n) = \left(\frac{2u^1}{|u|^2+1}, \cdots, \frac{2u^n}{|u|^2+1}, \frac{|u|^2-1}{|u|^2+1}\right).$$

It is easy to check that σ_N, β_N are inverses of each other. Hence, $\mathbb{R}^n \cong \mathbb{S}^n \setminus \{N\}$. The map σ_N is called the stereographic projection. $\mathbb{S}^n$ can now be given a CW structure with one 0-cell ($\mathbb{D}^0$) and one n-cell ($\mathbb{D}^n$). The attaching map for the n-cell is $\phi : \mathbb{S}^{n-1} = \partial\mathbb{D}^n \to \{*\}$.

Example 1.4.11. $\mathbb{S}^n$ can be given a different CW structure with two k-cells in each dimension for $0 \leq k \leq n$. Let $X^0 = \mathbb{S}^0 = \{\mathbb{D}^0_1, \mathbb{D}^0_2\}$. Then $X^1 = \mathbb{S}^1$ where the two 1-cells $\mathbb{D}^1_1, \mathbb{D}^1_2$ are attached to the 0-cells by homeomorphisms on their boundary. Similarly, two 2-cells can be attached to $X^1 = \mathbb{S}^1$ by homeomorphism on their boundary, giving $X^2 = \mathbb{S}^2$. We can now proceed inductively to construct the described CW complex structure on $\mathbb{S}^n$.

Example 1.4.12. There are natural inclusions

$$\mathbb{S}^0 \subseteq \mathbb{S}^1 \subseteq \cdots \subseteq \mathbb{S}^n \subseteq \cdots \subseteq$$

We can then define $\mathbb{S}^\infty = \varinjlim_{n\in\mathbb{N}} \mathbb{S}^n$. If $\mathbb{S}^n$ is given a CW structure as in Example 1.4.11 for each $n \geq 0$, then $\mathbb{S}^\infty$ is a CW complex as well. Note that $\mathbb{S}^\infty$ is a colimit of the $\mathbb{S}^n$'s for $n \geq 0$.

Example 1.4.13. Consider $\mathbb{RP}^n$ as the quotient of $\mathbb{S}^n$ with anti-podal points identified. An easy observation shows that $\mathbb{RP}^n$ is a quotient of $\mathbb{D}^n$ by the relation $x \sim -x$ on the boundary $\mathbb{S}^{n-1}$[4]. Thus, $\mathbb{RP}^n$ can be obtained from $\mathbb{RP}^{n-1}$ by attaching a one cell.

$$\begin{tikzcd}
\mathbb{S}^{n-1} \arrow[r, hook] \arrow[d, two heads] & \mathbb{D}^n \arrow[d, two heads] \\
\mathbb{RP}^{n-1} \arrow[r, hook] & \mathbb{RP}^n
\end{tikzcd}$$

Thus $\mathbb{RP}^n$ can be built as a CW complex with a single cell in each dimension $\leq n$.

Example 1.4.14. There are natural inclusions

$$\mathbb{RP}^0 \subseteq \mathbb{RP}^1 \subseteq \cdots \subseteq \mathbb{RP}^n \subseteq \cdots \subseteq$$

We can then define $\mathbb{RP}^\infty = \varinjlim_{n\in\mathbb{N}} \mathbb{RP}^n$. Note that $\mathbb{RP}^\infty$ is a colimit of the $\mathbb{RP}^n$'s for $n \geq 0$. We can define $\mathbb{CP}^\infty$ similarly to $\mathbb{RP}^\infty$.

---

[3] Let $x = (x^1, \ldots, x^{n+1}) \in \mathbb{S}^n \setminus \{N\}$. The line through N and x is parameterized by

$$u^1 = x^1 t, \ldots, u^n = x^n t, u^{n+1} = (x^{n+1} - 1)t + 1$$

The intersection of this line with $u^{n+1} = 0$ occurs when $t = \frac{1}{1-x^{n+1}}$. Hence, the intersection point is $(\sigma_N(x), 0)$, as desired. Therefore, $\sigma_N(x)$ is the intersection of the line through N and x with the $\mathbb{R}^n$ plane.

[4] It is easy to check that these identifications are consistent with out discussion of the real projective plane, which is $\mathbb{RP}^2$.

Example 1.4.15. The complex projective space, $\mathbb{CP}^n$, is defined as the quotient space $\mathbb{CP}^n = (\mathbb{C}^{n+1} \setminus \{0\})/\sim$ with the equivalence relation $x \sim y$ in $\mathbb{C}^{n+1} \setminus \{0\}$ if and only if $x = \lambda y$ for some $\lambda \neq 0$. Note that there is a map

$$\mathbb{D}^{2n} \to \mathbb{CP}^n$$
$$(z_0, \dots, z_{n-1}) \mapsto [z_0, \dots, z_{n-1}, \sqrt{1 - \|z\|}]$$

The boundary of $\mathbb{D}^{2n}$ (where $\sqrt{1 - \|z\|} = 0$) is sent to $\mathbb{CP}^{n-1}$. In this way, $\mathbb{CP}^n$ is obtained from $\mathbb{CP}^{n-1}$ by attaching one $2n$-cell. So $\mathbb{CP}^n$ has a CW structure with one cell in each even dimension $0, 2, \dots, 2n$.

Example 1.4.16. There are natural inclusions

$$\mathbb{CP}^0 \subseteq \mathbb{CP}^1 \subseteq \cdots \subseteq \mathbb{CP}^n \subseteq \cdots \subseteq$$

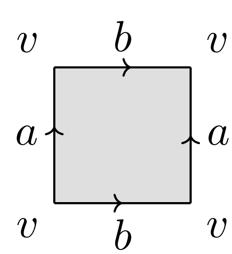
We can then define $\mathbb{CP}^\infty = \varinjlim_{n \in \mathbb{N}} \mathbb{CP}^n$ as before.

Let's discuss some 2-dimensional examples. It is well-known that compact, connected 2-dimensional manifolds are classified into the following types:

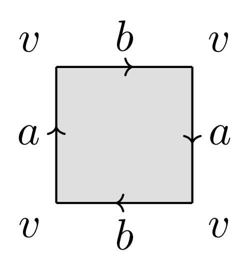
(1) $\mathbb{S}^2$,
(2) A connected sum of g-tori $\mathbb{T}$ (or a g-hold torus) for $g \geq 2$,
(3) A connected sum of g-projective spaces $\mathbb{RP}^2$, for $g \geq 2$.

Example 1.4.17. Consider $X = \mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ (the 1-torus) or $\mathbb{RP}^2$ (the real projective plane). Both spaces can be constructed as quotients of a rectangle by identifying edges according to specific rules. For the torus, opposite edges are identified in the same orientation, whereas for $\mathbb{RP}^2$, one pair of opposite edges is identified in the usual manner, and the other pair is identified with reversed orientation. These identification diagrams provide a convenient means of visualizing the topology of each space. Each space admits a natural CW complex structure with the following cells:

(1) a single 0-cell representing the vertex of the rectangle,
(2) two 1-cells corresponding to the edges of the rectangle,
(3) a single 2-cell which is attached via a continuous map from the boundary circle $\mathbb{S}^1$ into the 1-skeleton.



$X = \mathbb{T}$

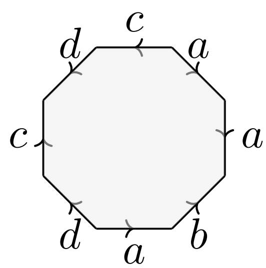


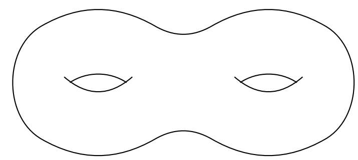
$X = \mathbb{RP}^2$

Example 1.4.18. For $g \geq 1$, a model for a connected sum of g copies of the torus $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ is denoted by M_g, and is known as an orientable surface of genus g. The surface M_g can be constructed by taking a polygon with $4g$ sides and identifying its edges in pairs according to the word $a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1}$, which encodes the edge identifications that yield a closed orientable surface. Each pair a_i, a_i^{-1} and b_i, b_i^{-1}

contributes a 'handle,' so M_g can be visualized as a torus with g holes, or a g-holed doughnut. This construction endows M_g with a natural CW complex structure consisting of:

(1) a single 0-cell where all loops based on the edges are attached;
(2) $2g$ 1-cells corresponding to the edges of the polygon;
(3) a single 2-cell attached along the loop described by the edge word above.



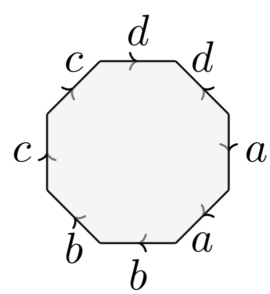


Example 1.4.19. For $g \geq 2$, a model for the connected sum of g copies of the real projective plane $\mathbb{RP}^2$ is denoted by N_g, and is known as a non-orientable surface of genus g. The surface N_g can be constructed from a polygon with g sides by identifying the edges according to the word

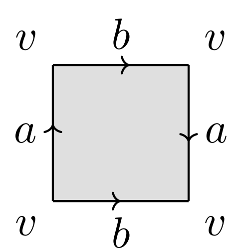
$$a_1 a_1 \cdots a_g a_g,$$

where each pair $a_i a_i$ represents an edge identification. This construction yields a closed surface that is non-orientable and has genus g. The surface N_g admits a CW complex structure consisting of:

(1) a single 0-cell to which all loops are attached;
(2) g 1-cells corresponding to the edges of the polygon;
(3) a single 2-cell attached via a loop following the word $a_1 a_1 \cdots a_g a_g$.



Remark 1.4.20. *N_2 is usually called a Klein bottle. Another model for the Klein bottle is given by the CW structure shown below:*



It can be checked that both models are homeomorphic.

1.4.3. Properties. A sub-complex of X is a subspace $Y \subseteq X$ that is a union of open cells of X, such that if Y contains a cell, it also contains its closure. It follows immediately that the union and the intersection of any collection of sub-complexes are themselves sub-complexes. Examples of a sub-complexes would be be the subspaces X^n for $n \geq 0$ in the definition of a CW complex.

Proposition 1.4.21. *Suppose X is a CW complex and Y is a sub-complex of X. Then Y is closed in X, and with the subspace topology and the cell decomposition that it inherits from X, it is a CW complex.*

Proof. Let $\mathbb{B}^n \subseteq Y$ denote such an open n-cell in Y. Since $\overline{\mathbb{B}^n} \subseteq Y$, the finitely many cells of X that have nontrivial intersections with $\mathbb{B}^n$ must also be cells of Y. So condition (1) in Definition 1.4.7 is automatically satisfied by Y. In addition, any characteristic map $\phi : \overline{\mathbb{B}^n} \to X$ for $\mathbb{D}^n$ in X also serves as a characteristic map for $\overline{\mathbb{B}^n}$ in Y. Suppose $A \subseteq Y$ is a subset such that $A \cap \mathbb{D}^n$ is closed in $\mathbb{D}^n$ for every n-cell $\mathbb{D}^n$ contained in Y. Let $\mathbb{D}^n$ be a n-cell of X that is not contained in Y. We know that $\overline{\mathbb{D}^n} \setminus \mathbb{B}^n$ is contained in the union of finitely many open cells of X; some of these, say $\mathbb{B}_1^{n_1}, \ldots, \mathbb{B}_k^{n_k}$, might be contained in Y. Then $\overline{\mathbb{B}_1^{n_1}} \cup \ldots \cup \overline{\mathbb{B}_k^{n_k}} \subseteq Y$, and

$$A \cap \mathbb{D}^n = A \cap (\overline{\mathbb{B}_1^{n_1}} \cup \ldots \cup \overline{\mathbb{B}_k^{n_k}}) \cap \mathbb{D}^n = ((A \cap \overline{\mathbb{B}_1^{n_1}}) \cup \cdots \cup (A \cap \overline{\mathbb{B}_k^{n_k}})) \cap \mathbb{D}^n$$

which is closed in $\mathbb{D}^n$. It follows that A is closed in X and therefore in Y. This implies (2) in Definition 1.4.7. Hence Y is a CW complex. Taking $A = Y$ shows that Y is closed. $\square$

Proposition 1.4.22. *The following is a list of some categorical/topological properties of CW complexes.*

(1) *If A is a sub-complex of X, then the inclusion $\iota : A \hookrightarrow X$ is a cellular map.*
(2) *If A is a sub-complex of X, then X/A is a CW complex such that the quotient map $X \to X/A$ is a cellular map.*
(3) *If X and Y are finite CW complexes, then $X \times Y$ is a CW complex.*
(4) *The closure of each cell in a CW complex is contained in a finite sub-complex.*
(5) *A subset of a CW complex is compact if and only if it is closed and contained in a finite sub-complex. In particular, a CW complex is compact if and only if it is a finite complex.*
(6) *A CW complex is locally compact if and only if it is locally finite.*
(7) *A CW complex is locally path-connected.*

Proof. The proof of some of the properties is given below:

(1) This is clear given the definition of a sub-complex.
(2) The cells of the quotient space X/A consist of the cells of X that lie in the complement $X \setminus A$, together with a single new 0-cell corresponding to the image of the inclusion

$$A \hookrightarrow X \to X/A$$

This is well-defined because since $A \subseteq X$ is a subcomplex, every cell of X is either contained in A or in $X \setminus A$. Let $\phi_\alpha^n : \mathbb{S}_\alpha^{n-1} \to X^{n-1}$ be the attaching map of an n-cell in $X \setminus A$. The corresponding n-cell in the quotient X/A is attached via the composite map

$$\mathbb{S}_\alpha^{n-1} \xrightarrow{\phi_\alpha^n} X^{n-1} \to X^{n-1}/A^{n-1},$$

where $A^{n-1} \subseteq X^{n-1}$ is the $(n-1)$-skeleton of A, and this inclusion holds because A is a subcomplex of X. Moreover, the image of the cellular filtration satisfies $q(X^n) \subseteq X^n/A^n = (X/A)^n$, so the quotient X/A inherits a CW-complex structure and is therefore cellular.
(3) Skipped.
(4) Let $\mathbb{D}^n$ be an n-cell of a CW complex. We prove the claim by induction on n. If $n = 0$, then $\overline{\mathbb{D}^0} = \mathbb{D}^0$ is itself a finite subcomplex. Assume the claim is true for every cell of dimension less than n. By (1) in Definition 1.4.7, $\overline{\mathbb{D}^n} \setminus \mathbb{D}^n$ is contained in the union of finitely many cells of lower

dimension, each of which is contained in a finite subcomplex by the inductive hypothesis. The claim now follows by taking a union of these these finite subcomplexes together with $\mathbb{D}^n$.

(5) Every finite subcomplex $Y \subseteq X$ is compact because it is the union of finitely many closed cells. Thus, if $K \subseteq X$ is closed and contained in a finite subcomplex, it is also compact. Conversely, suppose $K \subseteq X$ is compact. If K intersects infinitely many cells, by choosing one point of K in each such cell, we obtain an infinite discrete subset of K, which is impossible. Therefore, K is contained in the union of finitely many cells, and thus in a finite subcomplex by (1).

(6) This follows from (4).

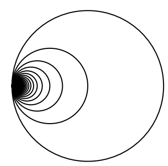
(7) Consider the spaces $X^n \subseteq X$[5]. We induct on $n \in \mathbb{N}$. X^0 is obviously locally path-connected. If X^{n-1} is locally path-connected then X^n is also locally path-connected since it is the the quotient of the disjoint union of X^{n-1} and a bunch of n-cells which are locally path-connected. Therefore, $\coprod_{n\in\mathbb{N}} X_n$ is locally path-connected. Since

$$\coprod_{n\in\mathbb{N}} X_n \to X$$

is a quotient map, X is locally-path connected.

This completes the proof. □

Remark 1.4.23. *Every topological space is not a CW complex. Consider the Hawaiian earring, X:*



The easiest way to see the Hawaiian earring has no CW decomposition is using information about the first homology group. If X were a CW-complex, then it would have to be a finite CW-complex by Proposition 1.4.22(6) since it is compact. Since every finite CW-complex has finitely generated homology, it suffices to show that the homology of X is not finitely generated. Observe that for any $n \in \mathbb{N}$, X has a retract which is a wedge of n circles - namely, the union of n of the circles that make up X (the retraction just maps all the other circles to the origin). The first homology group of a wedge of n circles is $\mathbb{Z}^n$, which cannot be generated by fewer than n elements. It follows that $H_1(X)$ cannot be generated by fewer than n elements for any $n \in \mathbb{N}$, and thus cannot be finitely generated. We have

$$\mathsf{CW} \subsetneq \mathsf{Top}$$

as inclusion of categories.

[5]We will use the following facts from general topology. A disjoint union of locally path-connected spaces is locally path-connected. Moreover, a quotient of a locally path-connected space is locally path-connected.

Part 1

First Homotopy Group

CHAPTER 2

Fundamental Group

The fundamental group is a central concept in algebraic topology, encapsulating essential information about the shape and structure of a topological space via the study of loops and their homotopy classes. It serves as a powerful algebraic invariant that aids in distinguishing between topologically distinct spaces and forms the foundation for many profound results in topology. For further details, see [Hat02; Lee10; May99].

2.1. Paths & Homotopy

2.1.1. Paths and π_0. We first consider paths in a topological space and the notion of path-connected components. The set of path-connected components of a space X is captured by the zeroth homotopy group, denoted $\pi_0(X)$.

Definition 2.1.1. Let $X \in \mathsf{Top}$. A path in X from x to y is a continuous map $f : [0, a] \to X$ such that $f(0) = x$ and $f(a) = y$ for some $a \geq 0$.

Proposition 2.1.2. *Let $X \in \mathsf{Top}$. Paths in X form a category, called the path category, Paths_X.*

Proof. The objects of this category are points of X and a morphism between two points, x, y, is simply a path. Composition of paths is defined as: if $f_1, : [0, a_1]$ and $f_2, : [0, a_2]$ such that $f_1(a_1) = f_2(0)$ are two paths, then the product path is defined as follows:

$$\begin{aligned} f_2 \cdot f_1 : [0, a_1 + a_2] &\to X \\ t &\mapsto \begin{cases} f_1(t) & \text{if } t \in [0, a_1] \\ f_2(t - a_1) & \text{if } t \in [a_1, a_1 + a_2] \end{cases} \end{aligned}$$

For each $x \in X$, the identity path Id_x is simply the path $\mathrm{Id}_x : [0, 0] \to X$ such that $\mathrm{Id}_x(t) = x$ for each $t \in [0, 0]$. Associativity and the identity axiom can be easily checked. $\square$

Being connected by paths is an equivalence relation on X: each $x \in X$ is connected to x via the identity path. if x is connected to y by a path $f : [0, a] \to X$ such that $f(0) = x$ and $f(a) = y$, then y is connected to x via the reverse path:

$$\begin{aligned} f_r : [0, a] &\to X \\ t &\mapsto f(a - t) \end{aligned}$$

If x is connected to y by a path f and y is connected to z via a path g, then x is connected to z via the path $f_2 \cdot f_1$.

Definition 2.1.3. Let $X \in \mathsf{Top}$. An equivalence relation on X under the equivalence relation of being connected by a path is a path-component.

We denote by $\pi_0(X)$ the set of path components, and by $\pi_0(x)$ the path component of the point x. π_0 then defines a functor

$$\pi_0 : \mathsf{Top} \to \mathsf{Sets}$$
$$X \mapsto \pi_0(X)$$

Indeed, a map $f : X \to Y$ induces $\pi_0(f) : \pi_0(X) \to \pi_0(Y)$ given by $\pi_0(x) \mapsto \pi_0(f(x))$ for each $x \in X$. π_0 assigns an invariant to a topological space in the sense that if X and Y are homeomorphic topological space via a map $f : X \to Y$, then

$$\pi_0(X) \cong \pi_0(Y)$$

as sets. This can be easily checked. See Proposition 2.1.12 for a more general argument. Hence, the cardinality of $\pi_0(X)$ can be used to distinguish some simple topological spaces.

Example 2.1.4. $\mathbb{R}$ is not homeomorphic to $\mathbb{R}^n$ for $n > 1$. Suppose $f : \mathbb{R} \to \mathbb{R}^n$ is a homeomorphism. Then $\mathbb{R} \cong \mathbb{R}^n$. WLOG let $f(0) = 0$. Hence, $\mathbb{R} \setminus 0 \cong \mathbb{R}^n \setminus 0$. $\mathbb{R} \setminus 0$ has two path-components and $\mathbb{R}^n \setminus 0$ has a single path-component, a contradiction.

Example 2.1.5. Let

$$X = \{(x, y) \in \mathbb{R}^2 : x = 0 \text{ or } y = 0\}$$

be the union of the x and y axes in $\mathbb{R}^2$. X is not homeomorphic to $\mathbb{R}$ since $X \setminus \{(0, 0)\}$ has four path-components and $\mathbb{R} \setminus \{0\}$ has two path-components.

Example 2.1.6. Assume that $\mathbb{R} \cong X \times Y$. Then $X \times Y$ and hence X, Y are path-connected. Assume $|X|, |Y| \geq 2$. Let $(x_0, y_0) \in X \times Y$. Then $X \times Y \setminus (x_0, y_0)$ is path-connected [1]. However, $\mathbb{R} \setminus \{*\}$ is not path-connected. Hence, either $|X| = 1$ or $|Y| = 1$.

2.1.2. Homotopy. Topology can at best be thought of as 'squishy geometry.' Perhaps it is possible to continuously deform a path while still retaining its underlying topological properties. More generally, perhaps two functions $f, g : X \to Y$ can be 'deformed into each other' This leads to the notion of homotopy.

Definition 2.1.7. Let $X, Y \in \mathsf{Top}$ and let $f, g : X \to Y$ be continuous maps. A homotopy from f to g is a continuous map

$$H : X \times [0, 1] \to Y$$

such that $f(x) = H(x, 0)$ and $g(x) = H(x, 1)$ for $x \in X$. In this case, we write $f \sim g$. H is said to be relative to $A \subseteq X$ if the restriction $H|_A$ is constant on A. In this case, we write $f \sim_A g$.

Example 2.1.8. A homotopy between paths $f_i : [0, a_i] \to X$ from x to y is a continuous map

$$H : I \times I \to X$$

such that

$$h(s, 0) = f_1(s)$$
$$h(s, 1) = f_2(s)$$
$$h(0, t) = x$$
$$h(1, t) = y$$

[1] Let $(a, b), (c, d) \in X \times Y$. If $a = c \neq x_0$ or $b = d \neq y_0$, then exists a path between (a, b) & (c, d) in either $\{a\} \times Y$ or $X \times \{b\}$ resp. avoiding (x_0, y_0). If $a = c = x_0$, then $b, d \neq y_0$. Choose a point $x \neq x_0 \in X$. Consider the path $(a, b) \to (x, b) \to (x, d) \to (c, d)$ which avoids (x_0, y_0). A similar argument works if $b = d = y_0$. If $a \neq c$ and $b \neq d$, consider two paths: $(a, b) \to (c, b) \to (c, d)$ and $(a, b) \to (a, d) \to (c, d)$. (x_0, y_0) cannot be on both paths. This covers all cases.

for all $s, t \in I$. In other words, we have a homotopy relative to the set $\{x, y\}$.

Proposition 2.1.9. *The homotopy operation satisfies the following properties:*

(1) $\sim$ *is an equivalence relation.*
(2) $\sim$ *is compatible with composition of maps.*
(3) *If* $f : X \to Y$ *is a continuous function, then* $f \circ \mathrm{Id}_X \sim \mathrm{Id}_Y \circ f$

Proof. The proof is as follows:

(1) Any map $f : X \to Y$ is homotopic to itself via the constant homotopy
$$\begin{aligned} H(x,t) : X \times [0,1] &\to Y \\ (x,t) &\mapsto f(x) \end{aligned}$$
Hence, $f \sim f$. Given $H : f \sim g$, the inverse homotopy
$$\begin{aligned} H(x,t) : X \times [0,1] &\to Y \\ (x,t) &\mapsto H(x, 1-t) \end{aligned}$$
shows $g \sim f$. Let $K : f \sim g$ and $L : g \sim h$ be given. The product homotopy $K * L$ is defined by
$$(K * L)(x,t) = \begin{cases} K(x, 2t) & 0 \le t \le \frac{1}{2} \\ L(x, 2t-1) & \frac{1}{2} \le t \le 1 \end{cases}$$
and shows $f \sim h$.

(2) Consider continuous functions
$$\begin{aligned} f_i &: X \to Y \\ g_i &: Y \to Z \end{aligned}$$
for $i = 1, 2$. Assume $f_1 \sim f_2$ via a homotopy F and $g_1 \sim g_2$ via a homotopy G. Define a homotopy:
$$\begin{aligned} G \circ F : X \times I &\to Z \\ (x,t) &\mapsto G(F(x,t), t) \end{aligned}$$
This shows that $f_2 \circ f_1 \sim g_2 \circ g_1$.

(3) This is clear.

This completes the proof. $\square$

We now consider a new category hTop, whose objects coincide with those of Top, but whose morphisms are given by homotopy classes of continuous maps. Proposition 2.1.9 establishes that hTop is well-defined. For $X, Y \in \mathsf{hTop}$, the set of homotopy classes of continuous maps from X to Y is denoted by $[X, Y]$.

Remark 2.1.10. *We can also define* hTop_* *corresponding to* Top_*. *For instance, if* $(X, x_0), (Y, y_0) \in \mathsf{hTop}_*$, *then a pointed homotopy in* hTop_* *is a continuous function such that*
$$H : (X, x_0) \times I \to (Y, y_0)$$
such that $H|_{(X,x_0)\times\{t\}}$ *for each* $t \in I$ *is a pointed map. The set of pointed homotopy classes from* X *to* Y *is denoted as* $[X, Y]_*$. *Note that* $[X, Y]_*$ *is itself a pointed set with the basepoint given by the homotopy class of the constant map* $X \to y_0$.

Remark 2.1.11. *If $A = \{\bullet\}$, then Definition 2.1.7 is a statement about the homotopy of maps considered as morphisms in Top_*. We can also define a notion of homotopy for morphisms in Top^2. If there are two maps $f, g : (X, A) \to (Y, B)$ in Top^2, a homotopy of pairs from f to g is a homotopy $H : f \simeq g$ that, in addition, satisfies $H(a, t) \in B$ for all $t \in [0, 1]$ and $a \in A$. This defines a new category, hTop^2.*

We now show that π_0 is a topological invariant in hTop.

Proposition 2.1.12. *Let $X, Y \in \mathsf{Top}$. If X and Y are homotopy equivalent, then*

$$\pi_0(X) \cong \pi_0(Y)$$

as sets.

Proof. Let $f : X \to Y$ and $g : Y \to X$ be the homotopy equivalent maps. We have a function

$$f_* : \pi_0(X) \to \pi_0(Y),$$

that sends the path component $[x]$ in X to the path component $[f(x)]$ in Y. Clearly, this is well-defined. We similarly have a function

$$g_* : \pi_0(Y) \to \pi_0(X),$$

that sends the path component $[y]$ in Y to the path component $[g(y)]$ in X. Moreover, homotopic maps give the same function, since $I \times I$ is path-connected. Since $g \circ f \cong \mathrm{Id}_X$ and $f \circ g \cong \mathrm{Id}_Y$, we must have that $\pi_0(X) \cong \pi_0(Y)$. □

We discuss a few basic but useful results:

Proposition 2.1.13. *The following statements are true:*

(1) *Let $A, X, Y \in \mathsf{Top}$. If $f_0 \sim f_1 : A \to X$ and $g_0 \sim g_1 : A \to Y$, then*

$$(f_0, g_0) \sim (f_1, g_1) : A \to X \times Y.$$

That is,

$$[A, X] \times [A, Y] \cong [A, X \times Y]$$

(2) *Let $X, Y, B \in \mathsf{Top}_*$. If $f_0 \sim f_1 : X \to B$ and $g_0 \sim g_1 : Y \to B$, then*

$$\{f_0, g_0\} \sim \{f_1, g_1\} : X \vee Y \to B.$$

That is,

$$[X, B]_* \times [Y, B]_* \cong [X \vee Y, B]_*.$$

Proof. For (1), let H_t be the homotopy between f_0 and f_1 and G_t the homotopy between g_0 and g_1. Then $(H_t, G_t) : A \to X \times Y$ is a homotopy between (f_0, g_0) and (f_1, g_1). The proof of (2) is similar, and hence omitted. □

An important example of a homotopy arises from the question of whether a topological space can be continuously deformed into a "smaller" subspace. This motivates the notion of a *deformation retraction*, which is a particular type of homotopy exhibiting such a deformation.

Definition 2.1.14. Let $X \in \mathsf{Top}$. A deformation retraction of X onto a subspace A is a homotopy

$$H : X \times [0, 1] \to X$$

such that $H(\cdot, 0) = \mathrm{Id}_X$, $H(\cdot, 1) = A$, and $H(\cdot, t)|_A = \mathrm{Id}_A$ for all $t \in [0, 1]$. X is said to be contractible if deformation retracts to a point $A = \{*\}$.

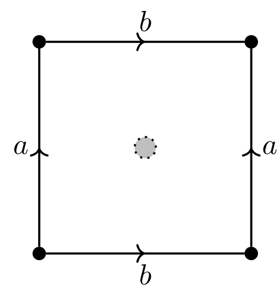
Example 2.1.15. The following are examples of some deformation retractions:

(1) $\mathbb{R}^n$ is contracible. More genrally, any star-shaped region is contractible. Indeed, if X is star-shaped with respect to some point $a \in X$, then $H(x,t) = (1-t)x + ta$ defines a homotopy between the constant map and the identity map. Hence star-shaped sets are contractible.

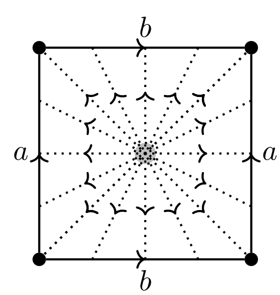
(2) $\mathbb{R}^n \setminus 0$ deformation retracts to $\mathbb{S}^{n-1}$. Simply consider the straight-line homotopy:
$$H(x,t) = (1-t)x + \frac{tx}{\|x\|}.$$

(3) $\mathbb{S}^\infty$ is contractible. Let H_1 be given by
$$\begin{aligned} H_1 : \mathbb{R}^\infty \times I &\to \mathbb{R}^\infty, \\ (x,t) &\mapsto (1-t)(x_1, x_2, x_3, \ldots) + t(0, x_1, x_2, \ldots). \end{aligned}$$
Note that $H_1(-,1)$ is the right shift map. For any $x \in \mathbb{S}^\infty$, the vector $H_1(x,-)$ is not a multiple of x, so the line segment between them does not pass through the origin. Thus, we can define a homotopy from the identity on $\mathbb{S}^\infty$ by setting $H_1/\|H_1\|$. The idea is now to contract the image of $H_1(-,1)$, which is a codimension-1 sphere, to a point not on it-say, $(1,0,0,0,0,\ldots)$. Let
$$\begin{aligned} H_2 : \mathbb{R}^\infty \times I &\to \mathbb{R}^\infty \\ (x,t) &= (1-t)(0, x_1, x_2, \ldots) + t(1,0,0,\ldots). \end{aligned}$$
Clearly, $H_2 = H_2/\|H_2\|$ is a homotopy from the map $H_1(-,1)$ to the constant map at $(1,0,0,\ldots)$ on $\mathbb{S}^\infty$. The composition is the desired homotopy that shows that $\mathbb{S}^\infty$ is contractible.

(4) Let Y be the topological space obtained by identifying opposite sides $[-1,1] \times [-1,1]$. Let $X = Y \setminus \{(0,0)\}$. See the diagram below:



Consider the homotopy $H : X \times I \to X$ defined by the formula
$$H((x,y),t) = \begin{cases} (t\frac{x}{|y|} + (1-t)x, t\frac{y}{|y|} + (1-t)y) & |y| > |x| \\ (t\frac{x}{|x|} + (1-t)x, t\frac{y}{|x|} + (1-t)y) & |x| > |y| \end{cases}$$
If $(x,y) \in X$ and $|y| > |x|$, the homotopy H linearly slides (x,y) onto a point such that the x-coordinate of $H((x,y),1)$ is $\operatorname{sgn}(x)$. See the diagram below:



The image of $H(\cdot, 1)$ is clearly the identified edges of X, which is, geometrically, a figure eight: a graph consisting of two circles intersecting in a point.

(5) Consider the Mobius strip, M, obtained as the quotient space of the square $[0,1] \times [0,1]$ by identifying
$$(x, 0) \sim (1-x, 1) \quad \text{for all } 0 \leq x \leq 1.$$
The line $\{(x, \frac{1}{2}) : x \in [0,1]\} \subseteq \mathbb{S}^1$ is $\mathbb{S}^1$ as a subspace of M. Then the map
$$\begin{aligned} H : M \times [0,1] &\to M \\ ((x,y),t) &\mapsto \left(x, (1-t)y + \frac{t}{2}\right) \end{aligned}$$
gives a well-defined strong deformation retract of M to $\mathbb{S}^1$ (as can be checked).

The following example is quite important:

Example 2.1.16. $\mathbb{D}^n \times I$ deformation retracts onto $\mathbb{D}^n \times \{0\} \cup \mathbb{S}^{n-1} \times I$. Define
$$r(x,t) = \begin{cases} (\frac{2x}{2-t}, 0) & \|x\| \leq \frac{2-t}{2} \\ (\frac{x}{\|x\|}, 2 - \frac{2-t}{\|x\|}) & \|x\| \geq \frac{2-t}{2} \end{cases}$$
It is easy to check that this is a well-defined continuous map. For $t = 0$ we get $\frac{2-t}{2} = 1$ and thus $r(x, 0) = (x, 0)$ for all $x \in \mathbb{D}^n$. For $x \in \mathbb{S}^{n-1}$ we have $r(x,t) = (x,t)$. Thus r is a retraction.

Remark 2.1.17. *There is a geometric interpretation of r in Example 2.1.16. For each (x,t) consider the line $L_{x,t}$ through $(0,2)$ and (x,t). This line intersects $\mathbb{D}^n \times \{0\} \cup \mathbb{S}^{n-1} \times I$ in a single point $r(x,t)$.*

Example 2.1.18. Let $X = \{\bullet_1, \bullet_2\}$ be a two-point topological space. If X is given the discrete topology, then X is not contractible. Indeed, contractible spaces are path-connected and X is not path connected with the discrete topology. If X is given the Sierpinski topology, $\{\emptyset, X, \{\bullet_1\}\}$, then X is contractible. Define
$$H : X \times [0,1] \to X$$
so that $H(x,0) = x$ for all $x \in X$, and $H(x,t) = \bullet_1$ for all $x \in X$ and $t \in (0,1]$. It is easy to see that H is continuous and hence defines a homotopy.

Definition 2.1.19. A map $f : X \to Y$ defines a homotopy equivalence if there exists $g : Y \to X$ such that $g \circ f$ and $f \circ g$ are both homotopic to the identity. X and Y are homotopy equivalent if there exists a homotopy equivalence. In this case, we write $X \sim Y$.

Example 2.1.20. For any $X \in \mathsf{Top}$, $X \times I$ is homotopy equivalent to X. Consider the maps
$$\begin{aligned} p_1 : X \times I &\to X, & i_0 : X &\to X \times I, \\ (x,t) &\mapsto x. & x &\mapsto (x, 0). \end{aligned}$$
Note that $\pi_1 \circ i_0 = \mathrm{Id}_X$. Moreover, $i_0 \circ \pi_1 \sim \mathrm{Id}_{X \times I}$ via the homotopy:
$$\begin{aligned} H : (X \times I) \times I &\to X \times I, \\ ((x,t),s) &\mapsto (x, (1-s)t). \end{aligned}$$

Remark 2.1.21. *Example 2.1.20 can generalized to prove that if X is a topological space and Y is a contractible topological space, then the projection*
$$p_1 : X \times Y \to X$$
is a homotopy equivalence.

Proposition 2.1.22. *Let X, Y be topological space. The following are some properties of the homotopy and homotopy equivalence concept.*

(1) *X is contractible if and only if every map $f : X \to Y$, for arbitrary Y, is homotopic to a constant map. Similarly, X is contractible if and only if every map $f : Y \to X$ is homotopic to a constant map.*
(2) *Let $f : X \to Y$ be a continuous map. Suppose there exist $g, h : Y \to X$, possibly different, such that $f \circ g \simeq \mathrm{Id}_Y$ and $h \circ f \simeq \mathrm{Id}_X$. Then f is a homotopy equivalence.*

Proof. The proof is given below:

(1) Suppose X is contractible. Let $H : X \times I \to X$ be a homotopy such that $H(\cdot, 0) = \mathrm{Id}_X$ and $H(\cdot, 1)$ is the constant map with value x_0.
 (a) If $f : X \to Y$ is a continuous map for any topological space Y, then $f \circ G : X \times I \to Y$ is a homotopy from f to the constant map with value $f(x_0)$. Thus, f is homotopic to a constant map. Conversely, letting $Y = X$ and $f = \mathrm{Id}_X$ shows that X is contractible.
 (b) If $f : Y \to X$ is a continuous map for any topological space Y, then the map
$$H : Y \times I \to X \qquad H(y, t) \mapsto H(f(y), t)$$
is a homotopy from f to the constant map with value x_0. Thus, f is homotopic to a constant map. Conversely, letting $Y = X$ and $f = \mathrm{Id}_X$ shows that X is contractible.
(2) If $h \circ f \sim \mathrm{Id}_X$ and $f \circ g \sim \mathrm{Id}_Y$, then
$$g \sim \mathrm{Id}_X \circ g \sim (h \circ f) \circ g \sim h \circ (f \circ g) \sim h \circ \mathrm{Id}_Y \sim h$$
Thus, $g \circ f \sim h \circ f \sim \mathrm{Id}_X$, and since $f \circ g \sim \mathrm{Id}_Y$, g is a homotopy equivalent to f.

This completes the proof. $\square$

2.2. Fundamental Group

2.2.1. π_1. Recall the definition of homotopy from the previous section. In this section, we focus on homotopy of paths relative to the boundary of $I = [0, 1]$, denoted as ∂I. We have the following observations:

Lemma 2.2.1. *The product of paths (read left to right) has the following properties:*

(1) *Let $\alpha : I \to I$ be continuous and $\alpha(0) = 0$, $\alpha(1) = 1$. Then $f \sim f \circ \alpha$.*
(2) $f \cdot (g \cdot h) \sim (f \cdot g) \cdot h$[2]
(3) *$f \sim f'$ and $g \sim g'$ implies $f \cdot g \sim f' \cdot g'$.*
(4) *If c_x denotes the constant path at $x \in X$, then $c_{f(0)} \cdot f \sim f \sim f \cdot c_{f(1)}$*
(5) *$f \cdot f_r \sim c_{f(1)}$ and $f_r \cdot f \sim c_{f(0)}$ where f_r is the reverse of f*

Proof. The proof is given below:

(1) α defines a reparamterization of the identity map from I to I. Let $H : I \times I \to I$ denote the straight-line homotopy from Id_I to α. Then $f \circ H$ is a path homotopy from f to $f \circ \alpha$
(2) We provide a proof in words. We need to show that
$$(f \cdot g) \cdot h \sim f \cdot (g \cdot h)$$
for any three paths in X such that the left-hand side is well-defined. The first path follows f and then g at quadruple speed for $s \in [0, \frac{1}{2}]$, and then follows h at double speed for $s \in [\frac{1}{2}, 1]$, while the second follows f at double speed and then g and h at quadruple speed. The two paths are therefore reparametrizations of each other and thus homotopic.

[2]This assumes that at least one side of the equation is well-defined.

(3) We show that $c_{f(0)} \cdot f \sim f$. The other homotopy follows similarly. Define $H : I \times I \to X$ as

$$H(s,t) = \begin{cases} f(0), & t \geq 2s, \\ f\left(\frac{2s-t}{2-t)}\right), & t \leq 2s. \end{cases}$$

Geometrically, this maps the portion of the square on the left of the line $t = 2s$ to the point $f(0)$, and it maps the portion on the right along the path f at increasing speeds as t goes from 0 to 1. This map is continuous by the gluing lemma, and we have that $H(s,0) = f(s)$ and $H(s,1) = c_{f(0)} * f(s)$. The claim follows.

(4) We just show that $f \cdot f_r \simeq c_{f(1)}$. Define a homotopy by the following recipe: at any time t, the path H_t follows f as far as $f(t)$ at double speed while the parameter s is in the interval $[0, t/2]$; then for $s \in [t/2, 1-t/2]$, it stays at $f(t)$; then it retraces f at double speed back to p. Formally,

$$H(s,t) = \begin{cases} f(2s), & 0 \leq s \leq t/2, \\ f(t), & t/2 \leq s \leq 1-t/2, \\ f(2-2s), & 1-t/2 \leq s \leq 1. \end{cases}$$

It is easy to check that H is a homotopy from $c_{f(1)}$ to $f \cdot f_r$.

This completes the proof. □

For $X \in \mathsf{Top}$, Lemma 2.2.1 implies that one can consider a category $\Pi(X)$ whose objects are points of X and morphisms are homotopy classes of paths between points of X relative to ∂I. $\Pi(X)$ is called the fundamental groupoid of X because each element in $\mathrm{Hom}_{\Pi(X)}(\cdot,\cdot)$ has an inverse path. In particlar, $\mathrm{Hom}_{\Pi(X)}(X, x_0)$ is a group for each $x \in X$.

Definition 2.2.2. Let $X \in \mathsf{Top}$ and $x_0 \in X$. The fundamental group of X at x is

$$\pi_1(X, x_0) = \mathrm{Aut}_{\Pi(X)}(x_0)$$

X is simply connected (or 1-connected) if it is path connected and its fundamental group is trivial.

Remark 2.2.3. *A loop based at $x_0 \in X$ is a map $f : I \to X$ such that $f(0) = f(1) = x$. Since $I/\partial I \cong \mathbb{S}^1$ where the homemorphism is given by the exponential function, $\varepsilon(t) = \exp(2\pi i t)$, f descends to a continuous map from $\mathbb{S}^1$ to X.*

$$\begin{tikzcd} I \arrow[d, "\varepsilon"] \arrow[dr, "f"] & \\ \mathbb{S}^1 \arrow[r, dashed, "\tilde{f}"] & X \end{tikzcd}$$

Therefore, we have

$$\pi_1(X, x_0) \cong [(\mathbb{S}^1, *), (X, x_0)]$$

where $[(\mathbb{S}^1, *), (X, x_0)]$ denotes the set of homotopy classes of maps from $(\mathbb{S}^1, *)$ to (X, x_0) such that $* \mapsto x_0$.

We next state an important lemma:

Lemma 2.2.4. ***(Square Lemma)*** *Let $F : W \to I \times X$ be a continuous map, and let f, g, h, and k be the paths in X defined by:*

$$f(s) = F(s,0) \quad g(s) = F(1,s) \quad h(s) = F(0,s) \quad k(s) = F(s,1)$$

Then $f \cdot g \sim h \cdot k$.

Proof. (Sketch) Consider an appropriate straight-line homotopy from the corners of the square $I \times I$. □

Proposition 2.2.5. *Let $(X, x_0) \in \mathsf{Top}_*$. The following are some properties of the fundamental group of X at x_0.*

(1) *For each $x_0' \in X$, such that $x_0' \in \pi_0(x_0)$, we have*

$$\pi_1(X, x_0) \cong \pi_1(X, x_0')$$

More generally, if X_0 is a path component of X that contains x_0, and $i : X_0 \hookrightarrow X$ is the inclusion map, then

$$i_* : \pi_1(X_0, x_0) \to \pi_1(X, x_0)$$

is an isomorphism.

(2) *$\Pi(\cdot)$ is a functor from Top to Grpd, the category of groupoids.*

(3) *π_1 is a functor from Top_* to Grp, the category of groups.*

(4) *If (X, x_0) and (Y, y_0) are pointed topological spaces, then*

$$\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \cong \pi_1(Y, y_0)$$

That is, π_1 preserves products.

(5) *If $f, g : X \to Y$ are homotopic with a homotopy $H : X \times I \to Y$ and h is the path $h(t) = H(x, \cdot)$, then the following diagram commutes:*

$$\begin{tikzcd}
 & \pi_1(X, f(x_0)) \arrow[dd, "\Phi_g"] \\
\pi_1(X, x_0) \arrow[ur, "f_*"] \arrow[dr, "g_*"'] & \\
 & \pi_1(X, g(x_0))
\end{tikzcd}$$

(6) *If $f : X \to Y$ is a homotopy equivalence, then the induced homomorphism*

$$f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$$

is an isomorphism.

Proof. The proof is given below:

(1) Let α be a path from x to x'. Consider the map

$$\begin{aligned} \Phi_\alpha : \pi(X, x_0) &\to \pi(X, x_0') \\ \beta &\mapsto \alpha \cdot \beta \cdot \alpha_r \end{aligned}$$

Note that Φ_α is a homomorphim:

$$\begin{aligned} \Phi_\alpha[\beta_2 \cdot \beta_1] &= [\alpha \cdot \beta_2 \cdot \beta_1 \cdot \alpha_r] \\ &= [\alpha \cdot \beta_2 \cdot \alpha_r \cdot \alpha \cdot \beta_1 \cdot \alpha_r] \\ &= [\alpha \cdot \beta_2 \cdot \alpha_r] \cdot [\alpha \cdot \beta_1 \cdot \alpha_r] \\ &= \Phi_\alpha[\beta_2] \cdot \Phi_\alpha[\beta_1] \end{aligned}$$

It is clear that Φ_α is bijective with inverse

$$\begin{aligned} \Phi_{\alpha_r} : \pi(X, x_0') &\to \pi(X, x_0) \\ \beta &\mapsto \alpha_r \cdot \beta \cdot \alpha \end{aligned}$$

More generally, any loop in X based at x must in fact be a loop in X_0, so it is necessary only to check that two homotopic loops in X are homotopic in X_0. But this is immediate since if

$$F : I \times I \to X$$

is a homotopy whose image contains x_0, its image must lie entirely in X_0, because $I \times I$ is path-connected.

(2) A continuous map $f : X \to Y$ induces a homomorphism

$$f_* : \Pi(X) \to \Pi(Y)$$

defined by $f_*([\alpha]) = [f \circ \alpha]$. We have

$$\begin{aligned} f_*([\beta] \cdot [\alpha]) &= f_*([\beta \cdot \alpha]) \\ &= [f \circ (\beta \cdot \alpha)] \\ &= [(f \circ \beta) \cdot (f \circ \alpha)] \\ &= f_*[\beta] \circ f_*[\alpha] \end{aligned}$$

The rest of the axioms can be checked in a straightforward way.

(3) This is similar to (2).

(4) Consider the map

$$\begin{aligned} \Phi : \pi_1(X \times Y, (x_0, y_0)) &\to \pi_1(X, x_0) \times \pi_1(Y, y_0) \\ [\alpha] &\mapsto ([\alpha_X], [\alpha_Y]) \end{aligned}$$

It is clear that Φ is well-defined since $f \sim g$ implies that

$$\alpha_X = \pi \circ \alpha \sim \pi \circ \alpha = \alpha_X$$

Similarly, $\alpha_Y \sim \alpha_Y$. The universal property of the product topology shows that Φ is a surjective. Moreover, if $[\alpha_X] = [c_x]$ and $[\alpha_Y] = [c_y]$ we can choose we choose homotopies H_x and H_y. Then the map $H : I \times I \to X \times Y$ given by

$$H(s,t) = (H_x(s,t), H_y(s,t))$$

is a homotopy from f to the constant loop $c_{(x_0,y_0)}$. Checking that Φ is a homomorphism is easy.

$$\begin{tikzcd} & I \arrow[d, "f"] \arrow[ddl, bend right, "f_X"'] \arrow[ddr, bend left, "f_Y"] & \\ & X \times Y \arrow[dl, "\pi_X"] \arrow[dr, "\pi_Y"'] & \\ X & & Y \end{tikzcd}$$

(5) Let α be any loop in X based at x. What we need to show is

$$\begin{aligned} g_*[\alpha] = \Phi_g \circ f_*[\alpha] & \\ \iff \quad & g \circ \alpha \sim h \cdot (f \circ \alpha) \cdot h_r \\ \iff \quad & (f \circ \alpha) \cdot h \sim h \cdot (g \circ \alpha) \end{aligned}$$

This readily follows from the square lemma applied to the map $F : I \times I \to Y$ defined by $F(s,t) = H(\alpha(s), t)$.

(6) Let $g : Y \to X$ be a homotopy inverse for g, so that $f \circ g \simeq 1_Y$ and $g \circ f \simeq 1_X$. Consider the maps:

$$\pi_1(X, x_0) \xrightarrow{f_*} \pi_1(Y, f(x_0)) \xrightarrow{g_*} \pi_1(X, g(f(x_0))) \xrightarrow{f_*} \pi_1(Y, f(g(f(x_0))))$$

The composition of the first two maps is an isomorphism by (4). In particular, f_* is injective. The same reasoning with the second and third maps shows that g_* is injective. Thus the first two of the three maps are injections and their composition is an isomorphism, so g_* must be injective and surjective.

This completes the proof. □

Remark 2.2.6. *Note that the homomorphism Φ_α in Proposition 2.2.5(a) depends only on the homotopy class of α. Indeed, assume that $\alpha \cong \alpha'$ where α and α' are continuous path joining x and x'. Then:*

$$\Phi_\alpha[\beta] = [\alpha \cdot \beta \cdot \alpha_r] = [\alpha] \cdot [\beta] \cdot [\alpha_r] = [\alpha'] \cdot [\beta] \cdot [\alpha'_r] = \Phi_{\alpha'}[\beta].$$

Hence we have $\Phi_\alpha = \Phi_{\alpha'}$.

Remark 2.2.7. *Proposition 2.2.5 implies that if X is a path-connected space, then*

$$\pi_1(X, x_0) \cong \pi_1(X, x'_0)$$

for each $x, x' \in X$. We shall mostly be concerned with path-connected spaces. Therefore, we shall not write the basepoint from now on.

How does one compute fundamental groups? This might be a difficult problem. But we can at the very least state some trivial calculations:

Proposition 2.2.8. *The following are calculations of some fundamental groups:*

(1) *If $X = \{\bullet\}$ is a one-point space, then $\pi_1(X) = \{1\}$, is the trivial group.*
(2) *If X is contractible, then $\pi_1(X, x_0) = \{1\}$ is the trivial group.*

Proof. The proof is given below:

(1) A one-point space has only the constant loop. Hence, its fundamental group is trivial.
(2) This follows from the Proposition 2.2.5(5) and (1) above.

This completes the proof. □

Remark 2.2.9. *A topological space, X, is simply connected if its fundamental group is the trivial group. One can easily check that X is simply connected if and only if there is a unique homotopy class of paths connecting any two points in X. For the forward direction, let $x, y \in X$, and $f, g : I \to X$ are paths from x to y. Then we have the following sequence of homotopies:*

$$f \sim f * c_y \sim f * \bar{g} * g \sim c_x * g \sim g,$$

where we use the fact that $\bar{g} * g$ and $f * \bar{g}$ are loops at y and x, respectively, and hence are homotopic to the respective constant paths. For the reverse direction, take $x = y$. By hypothesis, any loop γ at $x \in X$ is in the homotopy class of the constant loop c_x.

As we shall see by way of examples, fundamental groups are rarely abelian. However, there is an important class of groups for which the fundamental groups are abelian. Before identifying this class, we identify when a fundamental group is abelian.

Lemma 2.2.10. *Let X be a path-connected topological space X. For any $x \in X$, $\pi_1(X, x_0)$ is abelian if and only if all basepoint-change homomorphisms Φ_α depend only on the endpoints of the path α.*

PROOF. Assume $\pi_1(X, x_0)$ is abelian and consider two paths α, α' with same endpoints x and x'. Since $\pi_1(X, x_0)$ is abelian, $\pi_1(X, x_0')$ is also abelian since $\pi_1(X, x_0) \cong \pi_1(X, x_0')$. We have:

$$\begin{aligned}
\Phi_\alpha[\beta] &= [\alpha] \cdot [\beta] \cdot [\alpha_r] \\
&= [\alpha] \cdot [\beta] \cdot [c_x] \cdot [\alpha_r] \\
&= [\alpha] \cdot [\beta] \cdot [\alpha_r' \cdot \alpha'] \cdot [\alpha_r] \\
&= ([\alpha] \cdot [\beta] \cdot [\alpha_r']) \cdot ([\alpha' \cdot \alpha_r]) \\
&= ([\alpha' \cdot \alpha_r]) \cdot ([\alpha] \cdot [\beta] \cdot [\alpha_r']) \\
\Phi_\alpha[\beta] &= [\alpha'] \cdot [\beta] \cdot [\alpha_r'] = \Phi_{\alpha'}(\beta).
\end{aligned}$$

Hence $\Phi_\alpha = \Phi_{\alpha'}$. Conversely, assume all basepoint-change homomorphisms Φ_α depend depend only on the endpoints of the path α. Consider $x' = x$ and loops c_x (constant loop) and $[\beta] \in \pi_1(X, x_0)$. Then $\Phi_\beta = \Phi_{c_x}$. We can easily see that this is implies

$$[\beta] \cdot [\beta'] = [\beta'] \cdot [\beta]$$

for each $[\beta'] \in \pi_1(X, x_0)$. Hence $\pi_1(X, x_0)$ is abelian. □

Example 2.2.11. We argue that the fundamental group of a topological group, G, is abelian. Let e_G be the identity element chosen as the base point. Let $[\alpha], [\beta] \in \pi_1(G, e_G)$. Define a map

$$\begin{aligned}
F : I \times I &\to G \\
(t, s) &\mapsto \alpha(t) \cdot \beta(s)
\end{aligned}$$

In $I \times I$, let

$$(0,0) \xrightarrow{\epsilon_1} (1,0), \quad (1,0) \xrightarrow{\epsilon_2} (1,1), \quad (0,0) \xrightarrow{\epsilon_3} (0,1), \quad (1,0) \xrightarrow{\epsilon_4} (1,1), \quad (0,0) \xrightarrow{\epsilon_5} (1,1)$$

be the straight line paths. Applying F ε_5 yields a path

$$\alpha * \beta(t) = \alpha(t) \cdot \beta(t)$$

Since $I \times I$ is convex, we have $\epsilon_2 \cdot \epsilon_1 \simeq \epsilon_5 \simeq \epsilon_4 \cdot \epsilon_3$ since all three are paths from $(0,0) \to (1,1)$. Applying F to this gives

$$\beta \cdot \alpha \sim \alpha * \beta \sim \alpha \cdot \beta$$

Hence $[\beta \cdot \alpha] = [\alpha \cdot \beta]$. Hence, $\pi_1(G, e_G)$ is abelian.

Remark 2.2.12. *We remark that t if X be a path-connected space, for each point $x_0 \in X$, the inclusion*

$$\mathsf{B}\pi_1(X, x_0) \hookrightarrow \Pi(X)$$

is an equivalence of categories. Here $\mathsf{B}\pi_1(X, x_0)$ is the 1-object category defined by $\pi_1(X, x_0)$.

2.3. Seifert-Van Kampen Theorems

The Seifert Van Kampen (SVK) theorems gives a method for computing the fundamental groups of spaces that can be decomposed into simpler spaces whose fundamental groups are already known.

Proposition 2.3.1. ***(SVK for Groupoids)*** *Let $X \in \mathsf{Top}$, and let $\{U_i\}_{i \in I}$ be an open cover of X such that that the intersection of finitely many open sets again belongs to the open cover. Then*

$$\Pi(X) = \varinjlim_{i \in I} \Pi(U_i)$$

in the category of groupoids.

Remark 2.3.2. *Note that Proposition 2.3.1 states that* Π *preserves colimits.*

Proof. We verify the universal property of colimits in the category of groupoids. Let G be some groupoid and let $\Gamma_i : \Pi(U_i) \to G$ be groupoid morphisms. We show that there exists a unique groupoid morphism $\Phi : \Pi(X) \to G$ such that the diagram

$$\begin{tikzcd}
\Pi(\bigcap_{j\in J\subseteq I} U_j) \arrow[r, hook] \arrow[d, hook] & \Pi(U_{i_{|J|}}) \arrow[d, hook] \arrow[ddr, bend left, "\Gamma_{i_{|J|}}"] & \\
\Pi(U_{i_1}) \arrow[r, hook] \arrow[drr, bend right, "\Gamma_{i_1}"'] & \Pi(X) \arrow[dr, dashed, "\Phi"] & \\
& & G
\end{tikzcd}$$

for each subset $J \subseteq I$. Consider the following observations:

(1) An object of $\Pi(X)$ is a point $x \in X$ and so lies in one of U_i. If $x \in U_i$, we are forced to set $\Gamma(x) = \Gamma_i(x)$. If x is contained in the intersection of finitely many U_i's, these definitions agree by the commutative square above[3].

(2) A morphism in $\Pi(X)$ is a homotopy class of a path α in X. If α is solely contained in some U_i, we would be forced to set $\Phi(\alpha) = \Gamma_i(\alpha)$. Since the open cover is closed under finite intersections, this specification is independent of the choice of U_i if α *lies entirely in more than one* U_i. What if a path intersects $\cap_{j\in J\subseteq I} U_j$ for some subset $J \subseteq I$ such that $|J| \geq 2$? If $\alpha : I \to X$, then the Lebesgue covering lemma implies that there is a decomposition

$$0 = t_0 < t_1 < \cdots < t_{m-1} < t_m = 1$$

such that $\alpha([t_i, t_{i+1}])$ is contained in solely on of U_i. In this case, we are forced to set

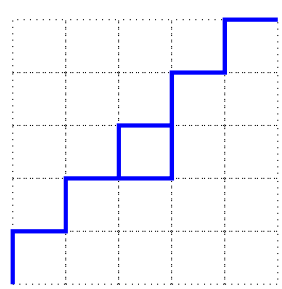
$$\Phi(\alpha) = F_1(\alpha_1) \circ \cdots \circ F_m(\alpha_m)$$

where each F_k is one of the Γ_i's as necessary.

The observations above pin down the map Φ. However, in order for Φ to be well-defined, we must show that it is independent of the choice of a path, α, in a homotopy class of paths. Let $H : I \times I \to X$ be a homotopy of paths from x to y. By the Lebesgue covering lemma, there exists $n \in \mathbb{N}$ such that H sends each sub-square

$$\left[\frac{i}{n}, \frac{i+1}{n}\right] \times \left[\frac{j}{n}, \frac{j+1}{n}\right]$$

into one of U_i. Consider edge-paths in the subdivided square $I \times I$ which differ by a sub-square, as indicated in the following figure.



[3]We implicitly use here the fact that the open cover is closed under finite intersections.

We apply H and obtain two paths in X. They yield the same result since they differ by a homotopy on some subinterval which stays inside one of the sets U_i. Changes of this type allow us to pass inductively from H on the lower to H on the upper boundary path from $(0,0)$ to $(1,1)$. Hence, Φ is well-defined. It is easy to check that Φ is indeed a functor between $\Pi(X)$ and G. By construction, the diagram commutes. $\square$

Proposition 2.3.1 contains a lot of redundant information since we only want to know how to compute $\pi_1(X, x_0)$ for some $x \in X$.

Corollary 2.3.3. ***(SVK for Groups)*** *Let $(X, x_0) \in \mathsf{Top}$ be path-connected. Let $\{U_i\}_{i\in I}$ be an open cover of X by path-connected open subsets such that $\{U_i\}_{i\in I}$ is closed under taking finite intersections and such that $x_0 \in U_i$ for each $i \in I$, then*

$$\pi_1(X, x_0) = \varinjlim_{i\in I} \pi(U_i, x_0)$$

Proof. We will only prove the case where the open cover is finite. The proof for the general case can be found in See [May99, Section 2.7]. We need to verify the universal property of colimits in the category of groups. Let G be some group and let $\Gamma_i : \pi(U_i, x_0) \to G$ be group homomorphisms. We show that there exists a unique group homomorphism $\Phi : \pi_1(X, x_0) \to G$ such that the diagram

$$\begin{tikzcd}
\pi_1(\bigcap_{j\in J\subseteq I} U_j, x_0) \arrow[r, hook] \arrow[d, hook] & \pi_1(U_{i_{|J|}}, x_0) \arrow[d, hook] \arrow[ddr, bend left, "\Gamma_{i_{|J|}}"] & \\
\pi_1(U_{i_1}, x_0) \arrow[r, hook] \arrow[drr, bend right, "\Gamma_{i_1}"'] & \pi_1(X, x_0) \arrow[dr, dashed, "\Phi"] & \\
& & G
\end{tikzcd}$$

for each subset $J \subseteq I$. Recall that the inclusion of categories $\mathsf{I} : \pi_1(X, x_0) \to \Pi(X)$ is actually an equivalence of categories. An inverse equivalence $\mathsf{F} : \Pi(X) \to \pi_1(X, x_0)$ is determined by a choice of path classes $x \to y$ for $y \in X$. We choose c_x when $y = x$ and so ensure that $\mathsf{F} \circ \mathsf{I} = \mathrm{Id}_{\pi_1(X,x_0)}$. Because the cover is finite and closed under finite intersections, we can choose our paths inductively so that the path $x \to y$ lies entirely in *every* U_i for *all* U_i such that $y \in U_i$[4]. This ensures that the chosen paths determine compatible inverse equivalences $\mathsf{F}_{U_i} : \Pi(U_i) \to \pi_1(U_i, x_0)$ to the inclusions $\mathsf{I}_{U_i} : \pi_1(U_i, x_0) \to \Pi(U_i)$. Thus, the functors

$$\Pi(U_i) \xrightarrow{\mathsf{F}_{U_i}} \pi_1(U_i, x_0) \xrightarrow{\Gamma_{U_i}} G$$

specify diagram of groupoids. By Corollary 2.3.3, there is a unique map of groupoids

$$\widetilde{\Gamma} : \Pi(X) \to G$$

that restricts to $\Gamma_{U_i} \circ \mathsf{F}_{U_i}$ on $\Pi(U_i)$ for each U_i. The composite

$$\Gamma : \pi_1(X, x_0) \xrightarrow{\mathsf{I}} \Pi(X) \xrightarrow{\widetilde{\Gamma}} G$$

It restricts to Γ_i on $\pi_1(U_i, x_0)$ by a diagram chase argument and the fact that $\mathsf{F}_{U_i} \circ \mathsf{I}_{U_i} = \mathrm{Id}_{\pi_1(U_i,x_0)}$. Indeed, we have that the following diagram commutes:

[4] We assume here that the open cover is finite

$$
\begin{tikzcd}
\pi_1(X, x_0) \arrow[d, "\mathsf{I}"'] & & \\
\Pi(X) \arrow[d, "\widetilde{\Gamma}"'] & \Pi(U_i) \arrow[l, hook'] & \pi_1(U_i, x_0) \arrow[l, "\mathsf{I}_{U_i}"'] \arrow[ull, bend right, "\Gamma_i"'] \arrow[dll, bend left, "\widetilde{\Gamma}_i \circ \mathsf{I}_{U_i}"] \\
G & &
\end{tikzcd}
$$

It is unique because $\widetilde{\Gamma}$ is unique. Indeed, if we are given $\Gamma' : \pi_1(X, x_0) \to G$ that restricts to Γ_i on each $\pi_1(U, x_0)$, then $\Gamma' \circ \mathsf{F} : \Pi(X) \to G$ restricts to $\Gamma_i \circ \mathsf{F}_{U_i}$ on each $\Pi(U_i)$. Therefore $\widetilde{\Gamma} = \Gamma' \circ \mathsf{F}$ and thus $\widetilde{\Gamma} \circ \mathsf{I} = \Gamma'$. This completes the proof. $\square$

Note that the Seifert-Van Kampen theorem does not apply when the open sets in the cover we consider do not have path-connected intersections. An important example is $\mathbb{S}^1$. If we cover it by two open semicircles, their intersection would be two disjoint open intervals which is not path-connected. This leads to the idea of constructing a "fundamental group with multiple basepoints" and its corresponding Seifert-Van Kampen theorem. We discuss this approach briefly.

Definition 2.3.4. Let $X \in \mathsf{Top}$ be path-connected. For a set $A \subseteq X$, let $\Pi(X, A)$ denote the full subcategory of $\Pi(X)$ on the objects in A.

As before, our strategy for proving the Seifert-Van Kampen theorem for multiple basepoints will be to deduce it from the version for the full fundamental groupoid.

Proposition 2.3.5. ***(SVK for Groupoids - Multiple Base-points)*** *Let $X \in \mathsf{Top}$ be path-connected. Let $\{U_i\}_{i \in I}$ be an open cover of X of open subsets such that $\{U_i\}_{i \in I}$ is closed under taking finite intersections. Let $A \subseteq X$ (not necessarily a singleton) such that A contains one point from each path-component of U_i. Then*

$$\Pi(X, A) = \varinjlim_{i \in I} \Pi(U_i, A)$$

Remark 2.3.6. *We will need to invoke the notion of a retract of a diagram in a category, C. Recall that an object $X \in \mathsf{C}$ in a category is called a retract of an object $Y \in \mathsf{C}$ if there are morphisms*

$$i : X \to Y, \qquad r : Y \to X$$

such that $r \circ i = Id_X$. In this case, r is called a retraction of Y onto X. A commutative diagram, D_1, in C is a retract of another commutative diagram, D_2, in C if each 'corner' of D_1 is a rectract of the corresponding corner of D_2 such that all of the inclusions and retractions are compatible with one another in the sense that the diagram obtained by 'pasting together' D_1 and D_2 via the inclusions and retractions commutes. We will use below the categorical fact that the retract of a colimit diagram in category is a colimit diagram.

Proof. (Sketch) Consider the diagram determined by $\Pi(U_i)$'s and also consider the diagram determined by $\Pi(U_i, A)$'s. Denote the diagrams D_1 and D_2 respectively. We claim that D_2 is a retraction of D_1. The inclusions at each 'corner' are the just inclusions

$$\Pi(U_i, A) \hookrightarrow \Pi(U_i)$$

The retractions are built as follows. To retract $\Pi(U_i)$ onto $\Pi(U_i, A)$, pick for every point $x \in U_i$, a path α_x from x to some point $y \in A$ but do this in such a way that if $x \in A$, then α_x is the identity morphism at x[5]. We define the retraction by sending each $x \in U_i$ to the other endpoint of α_x, and each morphism

[5]We can always pick these paths because the hypothesis includes that A has at least one point in each component of U_i.

$\beta : x \to y$ to the morphism $\alpha_y \circ \beta \circ \alpha_x^{-1}$[6]. The claim follows by noting that D_1 is a colimit diagram. See [Bro06, Proposition 6.7.2] and [Die08, Theorem 2.6.2] for some relevant partial details. □

2.4. Computations

We calculate the fundamental group of some topological spaces. Our main working tool will be the SVK theorems. Let's first discuss a general example.

2.4.1. Fundamental Group of Circle. We cannot use Corollary 2.3.3 to compute the the fundamental group of $\mathbb{S}^1$. This is because if we cover $\mathbb{S}^1$ by two open semi-circles, their intersection would be two disjoint open intervals which is not path-connected. Instead, we use Proposition 2.3.5. Let

$$\begin{aligned} U_1 &= \mathbb{S}^1 \setminus \{(0,1)\} \\ U_2 &= \mathbb{S}^1 \setminus \{(0,-1)\} \\ A &= \{(-1,0),(1,0)\} \end{aligned}$$

Then U_1, U_2 are simply connected (they are both homeomorphic to $\mathbb{R}$) while $U_1 \cap U_2$ is a homeomorphic to a disjoint union of two copies of $\mathbb{R}$. What is $\Pi(U, A)$? There are clearly morphisms

$$\begin{aligned} (1,0) &\to (-1,0) \\ (-1,0) &\to (1,0) \end{aligned}$$

and they are inverses of each other since any path $(1,0) \to (-1,0) \to (1,0)$ can be shrunk to $(1,0)$ alone. So $\Pi(U, A)$ is simply a category with two objects and a single isomorphism between them. Similar remarks apply to $\Pi(V, A)$. Similarly, $\Pi(U \cap V, A)$ is a category with two distinct objects and no morphisms between the distinct objects. In other words, it is a two object discrete category. What is $\Pi(X, A)$? It is a groupoid with two objects and *two* isomorphisms between. One isomorphism comes from $\Pi(U, A)$ and the other from $\Pi(V, A)$. Denote the isomorphisms as i_U and i_V. Beyond that it is free as possible. So, for example, all the composites $(i_V^{-1} \circ i_U)^n$ are distinct (because there is no reason for them not to be). We get that

$$\pi_1(\mathbb{S}^1, (1,0)) \cong \{i_V^{-1} \circ i_U)^n \mid n \in \mathbb{Z}\} \cong \mathbb{Z}$$

Remark 2.4.1. *Usually, x_0 is chosen to be the point $(1,0)$ if we consider $\mathbb{S}^1 \subseteq \mathbb{C}$. We denote the basepoint as $*$. We can also use the theory of covering spaces (which are special instances of fiber bundles) that*

$$\pi_1(\mathbb{S}^1, *) \cong \mathbb{Z}$$

We now derive a number of consequences of this result:

Proposition 2.4.2. *The following statements are true:*

(1) *We have $\pi_1(\mathbb{R}^2 \setminus \{0\}, x_0) \cong \mathbb{Z}$*
(2) *We have*

$$\pi_1\Big(\underbrace{\mathbb{S}^1 \times \cdots \times \mathbb{S}^1}_{n-\text{times}}, (*_1, \cdots, *_n)\Big) \cong \underbrace{\pi_1(\mathbb{S}^1, *_1) \times \cdots \times \pi_1(\mathbb{S}^1, *_n)}_{n-\text{times}} \cong \mathbb{Z}^n$$

(3) *$\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^n$ for $n \geq 3$*[7].

[6]To ensure that the cube formed by the two van Kampen squares and the four retractions commutes, simply always pick the same α_x for x in all of the groupoids it appears in.

[7]Clearly, $\mathbb{R}^2$ is not homemorphic to $\mathbb{R}^0$. We have already checked above that $\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^1$. Clearly, $\mathbb{R}^2$ is homeomorphic to $\mathbb{R}^2$.

(4) ***(Brouwer's Fixed Point Theorem)*** *If $f : \mathbb{D}^2 \to \mathbb{D}^2$ is a continuous function, then f has a fixed point. That is, there is a $x \in \mathbb{D}^2$ such that $f(x) = x$.*
(5) ***(Fundamental Theorem of Algebra)*** *Any non-constant polynomial $p \in \mathbb{C}[x]$ has a root.*
(6) *There are no retractions $r : X \to A$ in the following cases:*
  (a) *$X = \mathbb{R}^3$ with A any subspace homeomorphic to $\mathbb{S}^1$.*
  (b) *$X = \mathbb{S}^1 \times \mathbb{D}^2$ with A its boundary torus $\mathbb{S}^1 \times \mathbb{S}^1$.*
  (c) *$X = \mathbb{D}^2 \vee \mathbb{D}^2$ with A its boundary $\mathbb{S}^1 \vee \mathbb{S}^1$.*

Proof. The proof is given below:
(1) This follows since $\mathbb{R}^2 \setminus \{0\}$ deformation retracts to $\mathbb{S}^1$.
(2) This is a straightforward consequence of Proposition 2.2.5 and that $\pi_1(\mathbb{S}^1, *) \cong \mathbb{Z}$.
(3) Suppose $f : \mathbb{R}^2 \to \mathbb{R}^n$ is a homeomorphism. Without loss of generality, let $f(0) = 0$. Hence, $\mathbb{R}^2 \setminus \{0\} \cong \mathbb{R}^n \setminus \{0\}$. $\mathbb{R}^2 \setminus 0$ deformation retracts to $\mathbb{S}^1$ and $\mathbb{R}^n \setminus \{0\}$ deformation retracts to $\mathbb{S}^{n-1}$. Therefore,
$$\mathbb{Z} \cong \pi_1(\mathbb{S}^1, *) \cong \pi_1(\mathbb{R}^2 \setminus 0) \cong \pi_1((\mathbb{R}^n \setminus 0) \cong \pi_1(\mathbb{S}^{n-1}, *) \cong \{1\}.$$
a contradiction. Here we use the fact that $\pi_1(\mathbb{S}^{n-1}, *) \cong \{1\} = 0$ for $n \geq 3$. See [Lee10, Lemma 7.19 & Theorem 7.20] for a proof of this fact. See also below.
(4) Assume that $f(x) \neq x$ for all $x \in \mathbb{D}^2$. There is then a deformation retraction $r : \mathbb{D}^2 \to \mathbb{S}^1$ that carries a point $x \in \mathbb{D}^2$ to the intersection of the ray from $f(x)$ to x with the boundary circle $\mathbb{S}^1$. Hence, we have the following diagram:
$$\begin{tikzcd} \mathbb{S}^1 \arrow[r, hook] \arrow[rr, bend left, "\mathrm{Id}_{\mathbb{S}^1}"] & \mathbb{D}^2 \arrow[r, "r"'] & \mathbb{D}^2 \end{tikzcd}$$
Applying π_1, we have the following diagram:
$$\begin{tikzcd} \mathbb{Z} \cong \pi_1(\mathbb{S}^1, *) \arrow[r, hook] \arrow[rr, bend left, "\mathrm{Id}_{\mathbb{Z}}"] & \{\bullet\} \arrow[r] & \pi_1(\mathbb{S}^1, *) \cong \mathbb{Z} \end{tikzcd}$$
This is clearly a contradiction.
(5) We may assume that the polynomial $p(z)$ is of the form
$$p(z) = z^n + a_1 z^{n-1} + \cdots + a_n.$$
Suppose that $p(z)$ has no roots. Fix a $R > 0$ such that
$$R > \max\left\{1, \sum_{i=1}^{n} |a_i|\right\}$$
Then for $|z| = R$ we have
$$\begin{aligned} |z^n| &> (|a_1| + \cdots + |a_n|)|z^{n-1}| \\ &> |a_1 z^{n-1}| + \cdots + |a_n| \\ &\geq |a_1 z^{n-1} + \cdots + a_n|. \end{aligned}$$
From the inequality $|z^n| > |a_1 z^{n-1} + \cdots + a_n|$, it follows that the polynomial
$$p_t(z) = z^n + t(a_1 z^{n-1} + \cdots + a_n)$$

has no roots on the circle $|z| = R$ when $0 \leq t \leq 1$. Note that p_t defines a homotopy between the polynomials z^n and $p(z)$. Consider the formula

$$f_t(s) = \frac{p(tRe^{2\pi is})/p(tR)}{|p(tRe^{2\pi is})/p(tR)|}$$

defined on $[0,1] \times [0,1]$. For each fixed t, Then each $f_t(s)$ defines a loop in the unit circle $\mathbb{S}^1 \subseteq \mathbb{C}$ based at 1. Note that

$$f_0(s) = 1, \quad f_1(s) = \frac{p(Re^{2\pi is})/p(R)}{|p(Re^{2\pi is})/p(R)|}$$

Write $p(z) = z^n + q(z)$. Consider

$$H_t(s) = \frac{[re^{2\pi is})^n + tq(re^{2\pi is})/(r^n + tq(r))}{|[re^{2\pi is})^n + tq(re^{2\pi is})/(r^n + tq(r))|}$$

This defines a homotopy between f_1 and $\omega_n = e^{2\pi ins}$. Since f_0 is homotopic to the constant map and f_0 is homotopic to f_1, we have that ω_n is homotopic to the constant map. Hence, $n = 0$. This is a contradiction.

(6) We use the fact that if $r : X \to A$ is retraction, then the induced map on fundamental groups is injective.
   (a) This follows because there is no injection of the form $0 \to \mathbb{Z}$.
   (b) This follows because there is no injection of the form $\mathbb{Z} \times \mathbb{Z}$ to $\mathbb{Z}$. Let $h : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ be any group homomorphism. Suppose $h(1,0) = a$ and $h(0,1) = b$. It follows that $h(-b,a) = (0,0)$, and hence $\ker(h) \neq (0,0)$.
   (c) This follows because there is no injection of the form $0 \to \mathbb{Z} * \mathbb{Z}$.

This completes the proof. $\square$

Remark 2.4.3. *Here are two cute applications of Brouwer's fixed point theorem:*

(1) *A 3×3 real invertible matrix with non-negative entries has a real positive eigenvalue. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear map corresponding to a matrix A. Define*

$$B = \mathbb{S}^2 \cap \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1, x_2, x_3 \geq 0\right\} \cong \mathbb{D}^2.$$

If $x \in B$, then all coordinates of $Tx = Ax$ are non-negative and not all zero since A is non-singular and not all coordinates of $x \in B$ can be zero. Therefore, the normalized vector $Tx/\|Tx\|$ lies in B. Now, consider the continuous map $f : B \to B$ defined by

$$f(x) = \frac{Tx}{\|Tx\|}.$$

By Brouwer's Fixed Point Theorem, there exists a point $x_0 \in B$ such that $f(x_0) = x_0$, which implies $Tx_0 = \|Tx_0\|x_0$. Setting $\lambda = \|Tx_0\|$, we conclude that λ is an eigenvalue of A, with $\lambda \in \mathbb{R}$ and $\lambda > 0$.

(2) *A 3×3 real matrix with positive entries has a positive real eigenvalue. This follows as in (1).*

2.4.2. Fundamental Group of Spheres.

Let $X = \mathbb{S}^n$ for $n \geq 2$. Let

$$U = \mathbb{S} \setminus \{N\}, \quad V = \mathbb{S} \setminus \{S\}$$

Clearly, $U, V \cong \mathbb{R}^{n-1}$ via the stereographic projection. Hence, U and V are two open simply connected subsets of $\mathbb{S}^n$ with path connected intersection. Hence, we have that $\mathbb{S}^n$ is simply connected for $n \geq 2$, since the colimit of two trivial groups is the trivial the group. Let's consider a basic application:

Example 2.4.4. Consider $X_m = \mathbb{R}^n - \{m \text{ points}\}$ for $n \geq 3$. We compute the fundamental group of X_n by induction on n. The case $m = 1$ is easy since

$$\mathbb{R}^n - \{\text{a point}\} \cong \mathbb{S}^{n-1} \times \mathbb{R}^+$$

which is simply connected:

$$\pi_1(\mathbb{R}^n - \{\text{a point}\}) \cong \pi_1(\mathbb{S}^{n-1}, *) \times \pi_1(\mathbb{R}^+, 1) \cong \{\bullet\}$$

Now assume that $m > 1$. Divide the points in into two sets of smaller size, A and B. A and B can be separated by the hyperplane H, and that N^+ and N^- are two open neighborhoods of the half-spaces that result. For an arbitrary base-point $x_0 \in H$, Van Kampen theorem applies, giving a surjection

$$\pi_1(N^+ \setminus A, x_0) * \pi_1(N^- \setminus B, x_0) \twoheadrightarrow \pi_1(\mathbb{R}^n - \{m \text{ points}\}, x_0)$$

By the induction hypothesis[8], both $N^+ \setminus A$ and $N^- \setminus B$ are simply connected. Hence,

$$\pi_1(\mathbb{R}^n - \{m,\ \text{points}\}, x_0) \cong 0.$$

Example 2.4.5. Let V be a finite-dimensional real vector space and $W \subsetneq V$ a (proper) linear subspace. We compute the fundamental group $\pi_1(V \setminus W)$. Since every finite-dimensional real vector space is linearly homeomorphic to some $\mathbb{R}^n$, we can assume WLOG that $V = \mathbb{R}^n$ and $W = \mathbb{R}^m$ for $m < n$. Projecting first onto $(\mathbb{R}^m)^\perp$ and then unit sphere shows that $\mathbb{R}^n \setminus \mathbb{R}^m$ is homotopically equivalent to $\mathbb{S}^{n-m-1}$, Therefore, we have:

$$\pi_1(\mathbb{R}^n \setminus \mathbb{R}^m) = \begin{cases} \mathbb{Z}, & \text{if } m = n-2, \\ 0, & \text{otherwise.} \end{cases}$$

2.4.3. Fundamental Group of Wedge Sums. In order to use Van Kampen's theorem to compute the fundamental group of the wedge sum, we need to put a mild restriction on the type of base points we consider. A point p in a topological space X is said to be a non-degenerate base point if p has a neighborhood that admits a strong deformation retraction onto p.

Lemma 2.4.6. *Suppose $x_i \in X_i$ is a non-degenerate base point for $i = 1, \ldots, n$. Then $\bigvee_{i=1}^n x_i$ is a non-degenerate base point in $X_1 \vee \cdots \vee X_n$.*

Proof. For each i, choose a neighborhood W_i of x_i that admits a deformation retraction $r_i : W_i \to \{x_i\}$, and let $H_i : W_i \times I \to W_i$ be the associated homotopy. Define a map

$$H : \coprod_{i=1}^n W_i \times I \to \coprod_{i=1}^n W_i$$

by letting $H = H_i$ on $W_i \times I$. Let W be the image of $\coprod_{i=1}^n W_i$ under the quotient map

$$q : \coprod_{i=1}^n X_i \to \bigvee_{i=1}^n X_i$$

Since $\coprod_{i=1}^n W_i$ is a saturated open set, W is an open set of $X_1 \vee \cdots \vee X_n$ that is a neighbourhood of $\bigvee_{i=1}^n x_i$. We have that

$$q \times \mathrm{Id}_I : \coprod_{i=1}^n W_i \times I \to W \times I$$

is a quotient map. Since $q \circ H$ respects the identifications made by $q \times \mathrm{Id}_I$, it descends to the quotient and yields a deformation retraction of W onto $\vee_{i=1}^n x_i$. ☐

[8] Here we use the observation that a open half space in $\mathbb{R}^n$ is homeomorphic to $\mathbb{R}^n$.

Proposition 2.4.7. *Let $X_1, \dots, X_n$ be spaces with non-degenerate base points $x_i \in X_i$. The map*

$$\Phi : \pi_1(X_1, x_1) * \cdots * \pi_1(X_n, x_n) \to \pi_1\left(\bigvee_{i=1}^n X_i, \bigvee_{i=1}^n x_i\right)$$

induced by $\iota_i : \pi_1(X_i, x_i) \to \pi_1(\bigvee_{i=1}^n X_i, \bigvee_{i=1}^n x_i)$ is an isomorphism.

Proof. It suffices to consider the case $n = 2$. The general case follows by induction. Choose neighborhoods W_i in which x_i is a deformation retract, and let

$$U = q(X_1 \coprod W_2), \qquad V = q(W_1 \coprod X_2)$$

where $q : X_1 \coprod X_2 \to X_1 \vee X_2$ is the quotient map. Since $X_1 \coprod W_2$ and $W_1 \coprod X_2$ are saturated open sets in $X_1 \coprod X_2$, the restriction of q to each of them is a quotient map onto its image, and U and V are open in the wedge sum. The three maps

$$\begin{aligned} \{*\} &\hookrightarrow U \cap V, \\ X_1 &\hookrightarrow U, \\ X_2 &\hookrightarrow V \end{aligned}$$

are all homotopy equivalences. Because $U \cap V$ is contractible, we have: $U \hookrightarrow X_1 \vee X_2$ and $V \hookrightarrow X_1 \vee X_2$ induce an isomorphism

$$\pi_1(U) * \pi_1(V) \cong \pi_1(X_1 \vee X_2).$$

Moreover, the injections $\phi_1 : X_1 \hookrightarrow U$ and $\phi_2 : X_2 \hookrightarrow V$, which are homotopy equivalences, induce isomorphisms

$$\begin{aligned} \pi_1(X_1, x_1) &\cong \pi_1(U) \\ \pi_1(X_2, x_2) &\cong \pi_1(V) \end{aligned}$$

Hence,

$$\pi_1(X_1, x_1) * \pi_1(X_2, x_2) \cong \pi_1(X_1 \vee X_2, x_1 \vee x_2).$$

The general case follows by induction. □

Example 2.4.8. The following is a list of computations based on the information about the fundamental group of wedge sums:

(1) Consider$X = \bigvee_{i=1}^n \mathbb{S}^1$. We have

$$\pi_1(\mathbb{S}^1 \vee \cdots \vee \mathbb{S}^1, \bigvee_{i=1}^n *_i) \cong \pi_1(\mathbb{S}^1, *_1) * \cdots * \pi_1(\mathbb{S}^1, *_n) \cong \underbrace{\mathbb{Z} * \cdots * \mathbb{Z}}_{n \text{ times}}$$

(2) Let X be the union of n lines through the origin in $\mathbb{R}^3$. Then $\mathbb{R}^3 - X$ deformation retracts to $\mathbb{S}^2$ minus $2n$ points, which is homeomorphic to $\mathbb{R}^2$ minus $2n - 1$ points. This in turn admits a deformation retraction to a wedge of $2n - 1$ circles, so

$$\pi_1(\mathbb{R}^3 - X, x_0) \cong \underbrace{\mathbb{Z} * \cdots * \mathbb{Z}}_{2n-1 \text{ times}}$$

2.4.4. Fundamental Group of Graphs.

Definition 2.4.9. A graph is a CW complex of dimension 0 or 1. The 0-cells of a graph are called its vertices, and the 1-cells are called its edges.

It follows from the definition of a CW complex that for each edge e, the set $\overline{e} \setminus e$ consists of one or two vertices. If a vertex v is contained in $\overline{e}$, we say that v and e are incident. A subgraph is a subcomplex of a graph. Thus, if a subgraph contains an edge, it also contains the vertex or vertices incident with it. Here is some more important terminology:

- An edge path in a graph is a finite sequence $(v_0, e_1, v_1, \ldots, v_{k-1}, e_k, v_k)$ that starts and ends with vertices and alternates between vertices and edges, such that for each i, $\{v_{i-1}, v_i\}$ is the set of vertices incident with the edge e_i.
- An edge path is said to be closed if $v_0 = v_k$, and simple if no edge or vertex appears more than once, except that v_0 might be equal to v_k.
- A cycle is a nontrivial simple closed edge path.
- A tree is a connected graph that contains no cycles.

Lemma 2.4.10. *Let G be a finite graph.*

(1) *If G is a tree, then G is contractible and hence simply connected. In fact, if v_0 is a vertex of G, then v_0 is a deformation retract of G.*
(2) *If G is a connected graph, then G contains a maximal tree - called a spanning tree.*

Proof. The proof is given below:

(1) We induct on the number of edges, n. If $n = 1$, it G is homeomorphic to an interval I. The claim is clearly true in this case. Assume the claim is true for $n \in \mathbb{N}$ and consider the case $n + 1 \in \mathbb{N}$. Since G is simple, every edge of G is incident with exactly two vertices. If every vertex in G is incident with at least two edges, then, we can construct sequences $(v_j)_{j\in\mathbb{Z}}$ of vertices and $(e_j)_{j\in\mathbb{Z}}$ of edges such that for each j, v_{j-1} and v_j are the two vertices incident with e_j, and e_j, e_{j+1} are two different edges incident with v_j. Because T is finite, there must be some integers n and $n + k > n$ such that $v_n = v_{n+k}$. If n and k are chosen so that k is the minimum positive integer with this property, this means that $(v_n, e_{n+1}, \ldots, e_{n+k}, v_{n+k})$ is a cycle, contradicting the assumption that G is a tree. Hence, we can choose $v_1 \in G$ such that v_1 is incident to only one edge. Let v_1' denote the other vertex. Then e deformation retracts onto the vertex v_1'. The result is then a tree with n edges, which deformation retracts onto v_0.
(2) Since the empty subgraph is a tree, an application of Zorn's lemma shows that G contains a maximal subtree - a subgraph that is a tree and is not properly contained in any larger tree in G.

This completes the proof. $\square$

Remark 2.4.11. *Lemma 2.4.10 can be extended to the case of infinite graphs.*

Remark 2.4.12. *A spanning tree $T \subseteq G$ contains every vertex of G. Indeed, suppose that there is a vertex $v \in G$ that is not contained in T. Because G is connected, there is an edge path from a vertex $v_0 \in T$ to v, say $(v_0, e_1, \ldots, e_k, v_k = v)$. Let v_i be the last vertex in the edge path that is contained in T. Then the edge e_{i+1} is not contained in T, because if it were, v_{i+1} would also be in T since T is a subgraph. The subgraph $T' = T \cup \{e_{i+1}\}$ properly contains T, so it is not a tree, and therefore contains a cycle. This cycle must include e_{i+1} or v_{i+1}, because otherwise it would be a cycle in T. However, since e_{i+1} is the only edge of T' that is incident with v_{i+1}, and v_{i+1} is the only vertex of T' incident with e_{i+1}, there can be no such cycle.*

Proposition 2.4.13. *Let G be a finite graph and let $T \subseteq G$ be a spanning tree. Let $n_{G\setminus T}$ denote the number of edges in $G \setminus T$. If $v_0 \in G$, then*

$$\pi_1(G, v_0) \cong \underbrace{\mathbb{Z} * \cdots * \mathbb{Z}}_{n_{G\setminus T} times}$$

Proof. The proof is by induction on the number of edges in $G \setminus T$. Let $n = 1$. Clearly, $G/T \cong \mathbb{S}^1$. Consider the map

$$q : G \to G/T \cong \mathbb{S}^1.$$

We show q is a homotopy equivalence. We define a map

$$q' : G/T \cong \mathbb{S}^1 \to G.$$

Let e be the edge not contained in T. Pick paths α_1 and α_2 in T from v_0 to v_1 and v_2, respectively. Consider the loop $\alpha_1 \circ e \circ \alpha_2^{-1}$. It is easily checked that $q \circ q'$ and $q' \circ q$ are homotopic to the identity maps. If $n > 1$ and assume that the claim is true for $n \in \mathbb{N}$. we can use Van Kampen's theorem to prove the case for $n + 1 \in \mathbb{N}$. Let $e_1, \cdots, e_{n+1}$ be edges in $G \setminus T$. For each $i = 1, \ldots, n + 1$, choose a point $x_i \in e_i$ Let

$$U = G \setminus \{x_1, \ldots, x_n\}$$
$$V = G \setminus \{x_{n+1}\}$$

Both U and V are open in G. Just as before, it is easy to construct deformation retractions to show that

$$U \cap V \simeq T, \quad U \simeq T \cup e_{n+1} \quad V \simeq G \setminus e_{n+1}.$$

By the inductive hypothesis, we have $\pi_1(V, v_0) \cong \mathbb{Z}$ and

$$\pi_1(V, v_0) = \underbrace{\mathbb{Z} * \cdots * \mathbb{Z}}_{n \text{ times}}$$

The claim follows by Van Kampen's theorem noting that Since $U \cap V \cong T$ is simply connected. □

2.4.5. Fundamental Group of CW Complexes. Let X be a connected CW complex. If $X = X^0$, then X is a point and the fundamental group of X is the trivial. If $X = X^1$, then X is a graph and we have already covered that case.

Proposition 2.4.14. *Let X be a path-connected CW complex such that $X = X^2$. Let $x_0 \in X$ and let $\varphi_\alpha : \mathbb{S}^1 \to X$ be the attaching maps of the 2-cells $\mathbb{D}_\alpha$ and let $\gamma_\alpha : I \to X$ be a path from x_0 to $\varphi_\alpha(1)$. Then*

$$\pi_1(X, x_0) \cong \pi_1(X^1, x_0)/N,$$

where N is the normal subgroup generated by the path $\{\gamma_\alpha \circ \varphi_\alpha \circ \overline{\gamma}_\alpha\}$.

Proof. The proof is given below:

(1) Let A be a subcomplex generated by the union of the 2-cells, $\mathbb{D}^2_\alpha$ and toegether with the φ_α's. Then A is a contractible subcomplex. Hence,

$$\pi_1(A, x_0) \cong \{1\}$$

(2) Choose points $x_\alpha \in \mathbb{D}^2_\alpha$ and define the subset $B = X^2 - \bigcup_\alpha \{x_\alpha\}$. Then B retracts to X^1. Hence,

$$\pi_1(B, x_0) \cong \pi_1(X^1, x_0)$$

(3) We have $X^2 = A \cup B$ and $A \cap B$ consists of precisely those edge-cycles starting at x_0 that make up loops homotopic to the boundaries of 2-cells, or in other words, the images of $\mathbb{S}^1_\alpha$ under the attaching maps. Therefore, each element of $\pi_1(A \cap B, x_0)$ represents a an of $\{\gamma_\alpha \circ \varphi_\alpha \circ \overline{\gamma}_\alpha\}$.

(4) By Van-Kampen's theorem,

$$\pi_1(X, x_0) \cong \pi_1(X^2, x_0) \cong \frac{\pi_1(A, x_0) * \pi_1(B, x_0)}{N} \cong \pi_1(X^1, x_0)/N$$

This completes the proof. □

In fact, we now show that the fundamental group of a CW complex only depends on its 2-skeleton with basepoint x_0.

Corollary 2.4.15. *Let X be a path-connected CW complex. If $x_0 \in X$, then*

$$\pi_1(X, x_0) \cong \pi_1(X^2, x_0).$$

Proof. This follows simply because $A \cap B$ as in Proposition 2.4.14 will comprised on boundaries of n-cells for $n \geq 3$. These are all contractible. Hence, an application of Van-Kampen's theorem yields the desired result. □

Example 2.4.16. We can use the discussion in the previous section to compute the fundamental groups of topological spaces introduced Section 1.4.2.

(1) Let $X = \mathbb{S}^1 \times \mathbb{S}^1$. We have already computed the fundamental group of X, but we can also compute it using the discussion above. We have,

$$\pi_1(X, x_0) \cong \frac{\mathbb{Z} * \mathbb{Z}}{\langle aba^{-1}b^{-1} \rangle} \cong \langle a, b \mid aba^{-1}b^{-1} \rangle \cong \mathbb{Z} \times \mathbb{Z}$$

(2) Let $X = \mathbb{RP}^2$. We have,

$$\pi_1(X, x_0) = \frac{\mathbb{Z}}{\langle a^2 \rangle} \cong \langle a \mid a^2 \rangle \cong \mathbb{Z}_2$$

In general, if $X = \mathbb{RP}^n$, we have

$$\pi_1(\mathbb{RP}^n, x_0) \cong \mathbb{Z}_2$$

This follows at once from the computation above and that 2-skeleton of $\mathbb{RP}^n$ is just $\mathbb{RP}^2$.

(3) If $X = \mathbb{CP}^n$ then X is simply-connected. This is because the 1-skeleton of X consists of a single 0-cell.

(4) Let X be the g-holed surface in Example 1.4.18. We have

$$\pi_1(X, x_0) = \langle a_1, b_1, \ldots, a_g, b_g \mid [a_1, b_1] \cdots [a_g, b_g] = 1 \rangle$$

This follows pretty much by the definition of X and the fact that the 1-skeleton is a wedge sum of $2g$ circles.

(5) Let $X = K$ (Klein bottle). We have,

$$\pi_1(X, x_0) = \frac{\mathbb{Z} * \mathbb{Z}}{\langle abab^{-1} \rangle} \cong \langle a, b \mid abab^{-1} \rangle$$

Let A be the subgroup generated by a and B be the subgroup generated by b. We have $A, B \cong \mathbb{Z}$. Then since $bab^{-1} = a^{-1}$, we have that B is a normal subgroup. Clearly, A and B generate $\pi_1(X, v)$ and since every element has a unique representation in the form $b^n a^m$, we have that and $A \cap B = \{e\}$. Hence,

$$\pi_1(X, x_0) \cong \mathbb{Z} \rtimes \mathbb{Z}$$

Remark 2.4.17. *The fundamental group π_1 is not a complete topological invariant. Let $X = \mathbb{RP}^n$ and $Y = \mathbb{RP}^m$ for integers $m, n \geq 2$ with $m \neq n$. Then $\pi_1(X) \cong \pi_1(Y) \cong \mathbb{Z}/2\mathbb{Z}$. However, one can compute their homology groups to confirm that X and Y are not homotopy equivalent (Example 6.3.8). Thus, π_1 does not fully distinguish topological or homotopy types.*

CHAPTER 3

Covering Spaces

Covering spaces form a central concept in algebraic topology, closely linked to the study of the fundamental group. Intuitively, a covering space allows us to 'lift' paths and homotopies in a controlled way to a space that often has simpler or more symmetric structure. By understanding covering spaces, we gain deeper insight into the topological and algebraic structure of spaces, particularly through techniques such as path lifting and covering maps. References include [Hat02; Lee10; May99].

3.1. Definitions & Examples

Covering spaces offer a powerful framework in topology by enabling the study of complex spaces through simpler, well-behaved ones. One of their most compelling features is the ability to lift paths and homotopies from the base space to the covering space. This lifting property allows us to analyze the behavior of loops and paths in the base space by observing their images in the covering space, where the geometry and topology are often easier to handle. Importantly, this process reveals rich information about the fundamental group of the base space.

3.1.1. Definitions.

Definition 3.1.1. Let X be a topological space. A covering space of X is a topological space $\bar{X}$ together with a continuous surjective map $p : \bar{X} \to X$ called a covering map such that for every point $x \in X$, there exists an open neighborhood $U_x \subseteq X$ and a discrete topological spaces, D_x, such that

$$p^{-1}(U_x) = \coprod_{\alpha \in D_x} U_\alpha$$

where V_d is an open set of $\bar{X}$ homeomrophic to U_x.

Remark 3.1.2. *Covering spaces are special examples of fiber bundles where the fiber is a discrete topological space. In a covering space $p : \bar{X} \to X$, the local triviality condition resembles that of fiber bundles: for each $x \in X$, there exists an open neighborhood $U_x \subseteq X$ such that $p^{-1}(U_x) \cong U_x \times F$, where F is a discrete set (the fiber).*

Remark 3.1.3. *If X is a connected topological space, the cardinality of D_x in Definition 3.1.1 is constant. That is, $|D_x| = |D_y|$ for each $x, y \in X$. Indeed, let $x \in X$ and let U_x be an open set as in Definition 3.1.1. Then for each $y \in U_x$ and $\alpha \in D_x$, the intersection $p^{-1}(\{y\}) \cap U_\alpha$ contains exactly one point, since the restriction of p to U_α is a homeomorphism onto U_x. Now fix $a \in X$, and define the set*

$$A := \{x \in X \mid |p^{-1}(\{x\})| = |p^{-1}(\{a\})|\}.$$

By the above argument, the cardinality of the fiber is locally constant so both A and its complement A^c are open sets in X. Since X is connected, it follows that $X \setminus A = \emptyset$, and thus $A = X$. If the cardinality is n, we say that p is a n-sheeted covering map.

Example 3.1.4. The following is a list of examples of covering spaces:

(1) Any homeomorphism is a covering map.

(2) If X is any topological space and D is a discrete topological spaces, then the projection $p : X \times D \to X$ is a covering map.

(3) The map $p : \mathbb{R} \to \mathbb{S}^1$ defined by

$$p(t) = (\cos 2\pi t, \sin 2\pi t).$$

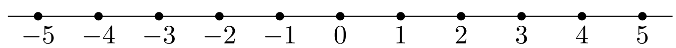
Indeed, let $x = (x_1, x_2) \in \mathbb{S}^1$ be a point on the unit circle such that $x_1 > 0$. Consider the open set

$$U := \{(x_1, x_2) \in \mathbb{S}^1 \mid x_1 > 0\},$$

which is an open neighborhood of x in $\mathbb{S}^1$. The pre-image of U under p is the disjoint union

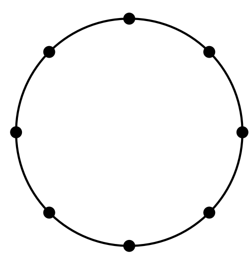
$$p^{-1}(U) = \bigsqcup_{n\in\mathbb{Z}} \left(n - \tfrac{1}{4},\, n + \tfrac{1}{4}\right),$$

where each interval $(n - 1/4,\ n + 1/4)$ is mapped homeomorphically onto U by p. Hence, p is an ∞-sheeted covering map. The fiber over the point $1 \in \mathbb{S}^1$ is given by $\mathbb{Z}$.



The fiber over the point $1 \in \mathbb{S}^1$ is given by $\mathbb{Z}$.

(4) The map $p : \mathbb{S}^1 \to \mathbb{S}^1$ defined by $p(z) = z^n$ for $n \in \mathbb{N}$ is an n-sheeted covering map. The fiber of 1 are the n-th roots of unity.



The fiber of 1 are the 8-th roots of unity.

Let's now prove some general properties bout covering spaces:

Proposition 3.1.5. *The following are properties of covering spaces.*

(1) *A covering map is an open map.*
(2) *A covering map is a local homeomorphism.*
(3) *The restriction of a covering map is a convering map.*
(4) *A finite product of covering maps is a covering map.*

Proof. The proof is given below:

(1) Let $p : \bar{X} \to X$ be a covering map. Let U be an open set in $\bar{X}$, and fix a point $p(x) \in p(U)$. Since p is a covering map, let $U_{p(x)} \subseteq X$ be as in Definition 3.1.1. Let U_α be the slice of $p^{-1}(U_{p(x)})$ containing x. Then p maps U_α homeomorphically onto $U_{p(x)}$. So $p(U_\alpha \cap U) \subseteq p(U)$ is open in X. Hence, p is an open map.
(2) This is clear.

(3) Let $p : \bar{X} \to X$ be a covering map. Let $X_0 \subseteq X$ and consider the restricted map $p|_{p^{-1}(X_0)} : p^{-1}(X_0) \to X_0$. The map is clearly continuous. Since $p : \bar{X} \to X$ is a covering space, for each $x \in X$ there exists an open set U_x such that $p^{-1}(U_x)$ satisfies Definition 3.1.1. The map $p|_{p^{-1}(X_0)}$ satisfies Definition 3.1.1 if we choose the open set to be $U_x \cap X_0$.

(4) $p_i : \bar{X}_i \to X_i$ be covering maps for $i = 1, 2$. Choose $(x_1, x_2) \in X_1 \times X_2$. Then there is a neighborhood U_{x_i} of x_i in X_i such that $p_i^{-1}(U_{x_i})$ satisfies Definition 3.1.1. Then $U_{x_1} \times U_{x_2}$ is an open set of $X_1 \times X_2$ such that $(p_1 \times p_2)^{-1}(U_{x_1} \times U_{x_2})$ satisfies Definition 3.1.1.

This completes the proof. □

How are two covering spaces of a topological space related? Can we define a map between two covering spaces to identify them up to isomorphism? This question leads us to the definition of homomorphisms between covering spaces.

Definition 3.1.6. Let $(\bar{X}_1, p_1)$ and $(\bar{X}_2, p_2)$ be covering spaces of a topological space X. A morphism of $(\bar{X}_1, p_1)$ into $(\bar{X}_2, p_2)$ is a continuous map $\phi : \bar{X}_1 \to \bar{X}_2$ such that the following diagram commutes:

$$\begin{tikzcd} \bar{X}_1 \arrow[rr, "\phi"] \arrow[dr, "p_1"'] & & \bar{X}_2 \arrow[dl, "p_2"] \\ & X & \end{tikzcd}$$

A homomorphism ϕ of $(\bar{X}_1, p_1)$ into $(\bar{X}_2, p_2)$ is an isomorphism if there exists a homomorphism ψ of $(\bar{X}_2, p_2)$ into $(\bar{X}_1, p_1)$ such that $\psi \circ \phi$ and $\phi \circ \psi$ are the identity maps on $\bar{X}_1$ and $\bar{X}_2$.

The discussion above defines a category, $\mathsf{Cov}_{/X}$, the category of covering spaces over X. Its objects are covering maps, and its morphisms are morphisms of covering maps.

3.1.2. Examples. Group actions on topological spaces provide a rich source of covering maps, which have significant applications in various areas of mathematics, particularly in geometric topology and geometric group theory. For instance, in geometric topology the quotient spaces formed by group actions often inherit interesting topological properties, which can be studied via covering space theory. These spaces provide insights into the structure of manifolds and their fundamental groups.

Proposition 3.1.7. *Let G be a topological group acting on a topological space X. Assume that for each $x \in X$, there exists an open set $U_x \subseteq X$ containing x such that for each $g \in G$ with $g \neq e$, we have $gU_x \cap U_x = \emptyset$. The quotient map $p : X \to X/G$ is a covering map.*

Remark 3.1.8. *A group action satisfying the condition in Proposition 3.1.7 is called a covering space action. We use this terminology from now.*

Proof. (Proposition 3.1.7) Since $q^{-1}(q(U_x)) = \bigsqcup_{g \in G} gU_x$, the set $q(U_x) = GU_x$ is open in X/G and satisfies Definition 3.1.1. Hence, $p : X \to X/G$ is a covering map. □

Example 3.1.9. The following is a list of examples of covering maps generated by group actions:

(1) Let $\mathbb{Z}$ act on $\mathbb{R}$ by translations: for each $n \in \mathbb{Z}$, define the action $n \cdot x = x + n$. This action satisfies the assumptions in Proposition 3.1.7. Hence, the map $\mathbb{R} \to \mathbb{R}/\mathbb{Z} \cong \mathbb{S}^1$ is a covering map. This reproves Example 3.1.4(1).

(2) Let $\mathbb{Z}_2$ act on the n-sphere $\mathbb{S}^n$ by the antipodal map:

$$g \cdot x = -x, \quad \text{for all } x \in \mathbb{S}^n.$$

where $g \neq e \in \mathbb{Z}_2$. This action satisfies Proposition 3.1.7. Hence, the projection map $p : \mathbb{S}^n \to \mathbb{S}^n/\mathbb{Z}_2 \cong \mathbb{RP}^n$ is a covering space. In fact, it is a two-sheeted covering map since each point in $\mathbb{RP}^n$ corresponds to a pair of antipodal points on $\mathbb{S}^n$.

(3) We can generalize (2). Let $\mathbb{Z}_p$ act on the odd-dimensional sphere $\mathbb{S}^{2n-1} \subseteq \mathbb{C}^n$ by

$$\zeta \cdot (z_1, z_2, \ldots, z_n) = (\zeta^{q_1} z_1, \zeta^{q_2} z_2, \ldots, \zeta^{q_n} z_n),$$

where $\zeta \in \mathbb{Z}_p$ is a primitive p-th root of unity (i.e., $\zeta = e^{2\pi i/p}$) and $q_1, \ldots, q_n \in \mathbb{Z}$ are integers coprime to p. This action satisfies Proposition 3.1.7, so the quotient

$$\mathbb{S}^{2n-1} \to \mathbb{S}^{2n-1}/\mathbb{Z}_p$$

is a covering map. $L(p; q_1, \ldots, q_n) := \mathbb{S}^{2n-1}/\mathbb{Z}_p$ is called a lens space. The projection map is a p-sheeted covering map.

Remark 3.1.10. *Lens spaces generalize real projective spaces $\mathbb{RP}^{2n-1} \cong L(2; 1, \ldots, 1)$. and play an important role in low-dimensional topology and the study of 3-manifolds.*

3.2. Lifting Properties

Studying the lifting properties of maps in covering spaces is fundamental because it allows us to transfer complex topological problems to simpler, often more manageable spaces. Lifting of paths helps in understanding how loops and homotopies in the base space relate to the structure of the covering space, providing key insights into the fundamental group.

Proposition 3.2.1. *Let $p : \bar{X} \to X$ be a covering map. Any path $f : I \to X$ with initial point x_0 can be uniquely lifted to a path $\bar{f} : I \to \bar{X}$ with an initial point in $p^{-1}(x_0)$ such that $p \circ \bar{f} = f$.*

Proof. We first prove existence. First assume that $f(I) \subseteq U_{x_0}$, where U_{x_0} satisfies Definition 3.1.1. For any $\bar{x}_0 \in p^{-1}(x_0)$, let $\bar{U}$ be an open set containing $\bar{x}_0$ that is mapped homemorphically to U_{x_0}. the path component of $p^{-1}(U)$ which contains $\bar{x}_0$. Clearly, the path

$$\bar{f} = p^{-1}|_{U_{x_0}} \circ f : I \to \bar{X}$$

is a path such that such that $p \circ \bar{f} = f$. Now assume that the image of f is not contained in U_{x_0} or in a single open set. In this case, let $\{U_i\}_i$ be an open cover of X by open sets satisfying Definition 3.1.1. Then, $\{f^{-1}(U_i)\}_i$ is an open covering of I. Let λ be the Lebesgue number of the covering. Now, choose $n \in \mathbb{N}$ such that $1/n < \lambda$. Divide the interval I into the closed sub-intervals of length $1/n$. Since the diameter of these intervals is less than λ, f maps each of these intervals inside some U_i. We can now apply the argument above. We now show uniqueness by proving that given any two maps $\bar{f}_0, \bar{f}_1 : I \to \bar{X}$ with same initial point such that $p \circ \bar{f}_0 = p \circ \bar{f}_1$, the set

$$A = \{t \in I \mid \bar{f}_0(t) = \bar{f}_1(t)\}$$

is either empty or all of I. It suffices to show that A is cl-open. First we will see that it is a closed set. Let t be in the closure of A and let $x = p \circ \bar{f}_0(t) = p \circ \bar{f}_1(t)$. Assume $\bar{f}_0(t) \neq \bar{f}_1(t)$. We will see that this leads to a contradiction. Let $x \in U_x$ be an open satisfying Definition 3.1.1, and let $\bar{U}_0$ and $\bar{U}_1$ be open sets of $\bar{X}$ containing $\bar{f}_0(t)$ and $\bar{f}_1(t)$ respectively that are mapped homeomorphically to x. Since $\bar{f}_0$ and $\bar{f}_1$ are both continuous, we can find a neighborhood $t \in W \subseteq I$ such that $\bar{f}_0(W) \subseteq \bar{U}_0$ and $\bar{f}_1(W) \subseteq \bar{U}_1$. But $\bar{U}_0 \cap \bar{U}_1 = \emptyset$. This is a contradiction to the fact that every neighborhood of t must intersect the set A. This shows that A is closed. Analogously, we can argue that every point in A is an interior point and therefore the set is open. Since $\bar{f}_0$ and $\bar{f}_1$ agree on at least one point in I, i.e., $\bar{f}_0(0) = \bar{f}_1(0)$, they have to be equal. □

Proposition 3.2.2. *Let $p : \bar{X} \to X$ be a covering map. Any homotopy $H : Y \times I \to X$ can be uniquely lifted to $\bar{X}$ if $H_0 : Y \to X$ can be lifted to $\bar{X}$ provide that $\bar{H}_0$ has been specified.*

PROOF. A more general version of Proposition 3.2.2 will be proved in Proposition 11.2.4. For an alternative approach, see [Lee10], which demonstrates how Proposition 3.2.2 can be applied by generalizing the argument in Proposition 3.2.1. □

Let us now use Proposition 3.2.1 and Proposition 3.2.2 to recompute the fundamental group of $\mathbb{S}^1$.

Example 3.2.3. (Homotopy Classification of Loops in $\mathbb{S}^1$) Consider the covering space $p : \mathbb{R} \to \mathbb{S}^1$. We compute the fundamental group of $\mathbb{S}^1$ in the following steps:

(1) If $f : I \to \mathbb{S}^1$, then any two lifts $\bar{f}_1, \bar{f}_2(0)$ such that $\bar{f}_1(0) = \bar{f}_2$ differ by an integer. Indeed,
$$p(\bar{f}_1) = p(\bar{f}_2)) \Rightarrow \bar{f}_1(t) - \bar{f}_2(t) \in \mathbb{Z}$$
for each $t \in I$. Since $\bar{f}_1 - \bar{f}_2$ is a continuous function from the connected space I into the discrete space $\mathbb{Z}$, it must be constant.

(2) Let $f_0, f_1 : I \to \mathbb{S}^1$ be two paths in $\mathbb{S}^1$ with same initial and terminal points. If $\bar{f}_0(0) = \bar{f}_1(0)$, then
$$f_0 \sim f_1 \iff \bar{f}_0(1) = \bar{f}_1(1)$$
The forward direction is clear since $\mathbb{R}$ is simply-connected (Remark 2.2.9). For the reverse direction, suppose that $f_0 \sim f_1$. Let $H : I \times I \to \mathbb{S}^1$ be a between f_0 and f_1. Then Proposition 3.2.2 implies that H lifts to a homotopy
$$\bar{H} : I \times I \to \mathbb{R}$$
such that $\bar{H}(\cdot, 0) = \bar{f}_0$. Now $\bar{H}_1(\cdot, 1) : I \to \mathbb{R}$ is a path of that is a lift of f_1 starting at $\bar{f}_1(0)$. By uniqueness of lifts, it must be equal to $\bar{f}_1$. Thus, $\bar{f}_0 \sim \bar{f}_1$ and this implies that $\bar{f}_0(1) = \bar{f}_1(1)$.

(3) (**Winding Number**) Suppose $f : I \to \mathbb{S}^1$ is a loop based at a point $x_0 \in \mathbb{S}^1$. If $\bar{f} : I \to \mathbb{R}$ is any lift of f, then $\bar{f}(1)$ and $\bar{f}(0)$ are both points in the fiber $p^{-1}(x_0)$, so they differ by an integer. Since any other lift differs from $\bar{f}$ by an additive constant, the difference
$$\bar{f}(1) - \bar{f}(0)$$
is an integer that depends only on f, and not on the choice of lift. This integer is denoted by $N(f)$, and is called the winding number of f. (1) and (2) at once imply that two loops in $\mathbb{S}^1$ based at the same point are path-homotopic if and only if they have the same winding number.

(4) (**Fundamental Group of $\mathbb{S}^1$**) We can now show that $\pi_1(\mathbb{S}^1, 1) \cong \mathbb{Z}$ generated by $[\omega]$ where $\omega : I \to \mathbb{S}^1$ such that $\omega(t) = e^{2\pi i t}$. Define the maps
$$J : \mathbb{Z} \to \pi_1(\mathbb{S}^1, 1), \qquad K : \pi_1(\mathbb{S}^1, 1) \to \mathbb{Z},$$
$$n \mapsto [\omega^n] \qquad [f] \mapsto N(f)$$
It is clear that J, K are well-defined and that J, K are homomorphisms. We show that J, K are two-sided inverses. To prove that $K \circ J = \mathrm{Id}_{\mathbb{Z}}$, let $n \in \mathbb{Z}$ be arbitrary. Note that
$$K(J(n)) = K([\omega^n]) = K([\alpha_n]) = N(\alpha_n) = n,$$
where $\alpha_n : I \to \mathbb{S}^1$ is the map $\alpha_n(t) = e^{2\pi i n t}$. To prove that $J \circ K = \mathrm{Id}_{\pi_1(\mathbb{S}^1, 1)}$, suppose f is any element of $\pi_1(\mathbb{S}^1, 1)$, and let n be the winding number of f. Then f and α_n are path-homotopic because they are loops based at 1 with the same winding number. Therefore,
$$J(K([f])) = J(n) = [\omega]^n = [\alpha_n].$$

Let us now use Proposition 3.2.1 and Proposition 3.2.2 to determine how the fundamental groups of the based space and covering space in a covering map relate to each other.

Proposition 3.2.4. *Let $p : (\bar{X}, \bar{x}_0) \to (X, x_0)$ be a covering map such that $x_0 = p(\bar{x}_0)$.*

(1) *The induced homomorphism*

$$p_* : \pi_1(\bar{X}, \bar{x}_0) \to \pi_1(X, x_0)$$

is injective. Hence, $\pi_1(\bar{X}, \bar{x}_0)$ *can be identified with a subgroup of* $\pi_1(X, x_0)$*.*

(2) *If* $\bar{X}$ *is path-connected, the subgroups* $p_*(\pi_1(\bar{X}, \bar{x}))$ *for* $\bar{x} \in p^{-1}(x_0)$ *are exactly the conjugacy class of subgroups of* $p_*(\pi_1(\bar{X}, \bar{x}_0))$*.*

(3) *If* $\bar{X}$ *is path-connected, then*

$$\text{Number of sheets of } p = |\pi_1(X, x_0) : \pi_1(\bar{X}, \bar{x}_0)|$$

(4) *If* $\bar{X}$ *is path-connected and simply connected, then*

$$\pi_1(X, x_0) \cong p^{-1}(x_0)$$

as sets.

Proof. The proof is given below:

(1) Let $[\alpha]$ and $[\beta]$ be two homotopy classes of paths in $\bar{X}$ and suppose that $p_*[\alpha] = p_*[\beta]$. If $f_\alpha \in [\alpha]$ and $f_\beta \in [\beta]$, then $p \circ g_\alpha \sim p \circ g_\beta$. It follows from Proposition 3.2.2 that $g_\alpha \sim g_\beta$ in $\bar{X}$. So, $[\alpha] = [\beta]$. Thus the map is injective.

(2) First suppose that $\bar{x}_0, \bar{x}_1 \in p^{-1}(x_0)$. Let γ be a path from $\bar{x}_0$ to $\bar{x}_1$. This defines an isomorphism (Proposition 2.2.5):

$$\begin{aligned} \phi : \pi_1(\bar{X}, \bar{x}_0) &\to \pi_1(\bar{X}, \bar{x}_1) \\ [\alpha] &\mapsto [\gamma \circ \alpha \circ \gamma^{-1}] \end{aligned}$$

We thus have the following commutative diagram:

$$\begin{tikzcd} \pi_1(\bar{X}, \bar{x}_0) \arrow[r, "p_*"] \arrow[d, "\phi"'] & \pi_1(X, x_0) \arrow[d, "\psi"] \\ \pi_1(\bar{X}, \bar{x}_1) \arrow[r, "p_*"] & \pi_1(X, x_0) \end{tikzcd}$$

Here, ψ is defined such that

$$\psi([\beta]) = [(p_*(\gamma))^{-1} \circ \beta \circ (p_*(\gamma))]$$

We conclude that the images of $\pi_1(\bar{X}, \bar{x}_0)$ and $\pi_1(\bar{X}, \bar{x}_1)$ are conjugate via $[p_*(\gamma)]$. Conversely, any subgroup in the conjugacy class of $p_*(\pi_1(\bar{X}, \bar{x}_0))$ is of the form

$$[\alpha^{-1}]\, p_*(\pi_1(\bar{X}, \bar{x}_0))\, [\alpha]$$

for some $[\alpha] \in \pi_1(X, x_0)$. Let $f \in [\alpha]$. By Proposition 3.2.1 $g : I \to \bar{X}$ is a unique lift of f initial point $\bar{x}_0$. Let $\bar{x}_1$ be the terminal point of the lifted path. Then

$$p_*(\pi_1(\bar{X}, \bar{x}_1)) = [\alpha^{-1}]\, p_*(\pi_1(\bar{X}, \bar{x}_0))\, [\alpha]$$

(3) Let $H = p_*(\pi_1(\bar{X}, \bar{x}_0)$. Define a map

$$\begin{aligned} \phi : \frac{\pi_1(X, x_0)}{H} &\to p^{-1}(x_0) \\ [f] + H &\mapsto \bar{f}(1) \end{aligned}$$

Here $\bar{f}$ is a lift of the path f. We claim that ϕ is well-defined. Given $[f] \in \pi_1(X, x_0)$ and $[h] \in H$, let $\bar{h}$ be a loop in $\bar{X}$ based at $\bar{x}_0$. Thus, $(\bar{h} \cdot \bar{f})(1) = \bar{f}(1)$. This shows that ϕ is well-defined. We claim that ϕ is a bijection. Since $\bar{X}$ is path-connected, for any $\bar{x} \in p^{-1}(x_0)$, there exists a path $\bar{g}$ from $\bar{x}_0$ to $\bar{x}$, and it must project to a loop g in X based at x_0. Thus, ϕ is surjective. Now suppose

$$\phi([f]+H) = \phi([f']+H)$$

Then $\bar{f}(1) = \bar{f}'(1)$, and so the path $f\cdot(f')^{-1}$ lifts to a loop in $\bar{X}$ based at $\bar{x}_0$, i.e., $[f][f']^{-1} \in H$. This shows that ϕ is connected.

(4) This follows from (3).

This completes the proof. $\square$

Example 3.2.5. Consider the covering $p : \mathbb{S}^n \to \mathbb{RP}^n$. For $n \geq 2$, $\mathbb{S}^n$ is path-connected and simply-connected. Hence, Proposition 3.2.4 implies that

$$\pi_1(\mathbb{RP}^n, x_0) \to p^{-1}(x_0)$$

as sets. Since $|p^{-1}(x_0)| = 2$ and there is a unique group of order 2, it follows that

$$\pi_1(\mathbb{RP}^n, x_0) \cong \mathbb{Z}/2\mathbb{Z}$$

for $n \geq 2$. For $n = 1$, we have that $\mathbb{RP}^1 \cong \mathbb{S}^1$. Hence, $\pi_1(\mathbb{RP}^1, x_0) \cong \mathbb{Z}$.

Previously, we considered a path in the unit interval I within X and lifted it to a corresponding path in the covering space $\bar{X}$. We now extend this concept by studying the lifting of paths in X from an arbitrary connected space Y. Let $p : (\bar{X}, \bar{x}_0) \to (X, x_0)$ be a covering space. Let (Y, y_0) be a topological space and let $f : (Y, y_0) \to (X, x_0)$ be a pointed continuous map. We seek to determine conditions under which there exists a map $\phi : (Y, y_0) \to (X, \bar{x}_0)$ such that the following diagram commutes:

$$\begin{tikzcd} & (\bar{X}, \bar{x}_0) \arrow[d, "p"] \\ (Y, y_0) \arrow[ur, "\bar{\phi}"] \arrow[r, "\phi"] & (X, \bar{x}_0) \end{tikzcd}$$

If ϕ exists, we say that ϕ can be lifted to $\bar{X}$. We refer to ϕ as a lifting of ϕ. Note that if ϕ exists, then the following commutative diagram of group homomorphisms holds:

$$\begin{tikzcd} & \pi_1(\bar{X}, \bar{x}_0) \arrow[dd, "p_*"] \\ \pi_1(Y, y_0) \arrow[ur, "\bar{\phi}_*"] \arrow[dr, "\phi_*"] & \\ & \pi_1(X, x_0) \end{tikzcd}$$

Since p_* is injective, for the diagram to commute it is necessary that

$$\phi_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(\bar{X}, \bar{x}_0))$$

This condition is also sufficient.

Proposition 3.2.6. *Let $p : (\bar{X}, \bar{x}_0) \to (X, x_0)$ be a covering map. Let (Y, y_0) be a connected and locally path-connected space. Given a pointed continuous map $\phi : (Y, y_0) \to (X, x_0)$, there exists a lifting ϕ : $(Y, y_0) \to (\bar{X}, \bar{x}_0)$ if and only if*

$$\phi_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(\bar{X}, \bar{x}_0))$$

Proof. Skipped. $\square$

3.2.1. Covering Space Automorphisms. An automorphism of a covering map is an isomorphism from a covering space to itself. An automorphism of a covering map interchanges points in the fiber point in the base space. The set of all automorphisms of forms a group under composition and can be interpreted as the symmetries of the covering space.

Remark 3.2.7. *Automorphisms of a covering map are also called deck transformations.*

Using Proposition 3.2.1 and Proposition 3.2.6, we first establish various properties of the morphisms of a covering map.

Corollary 3.2.8. *Let $(\bar{X}_1, p_1)$ and $(\bar{X}_2, p_2)$ be covering spaces of a topological space (X, x_0) such that $p_1(\bar{x}_1) = p_2(\bar{x}_2) = x_0$*

1. *Let ϕ_0 and ϕ_1 be homomorphisms of $(\bar{X}_1, p_1)$ into $(\bar{X}_2, p_2)$. If there exists a point $x \in \bar{X}_1$ such that $\phi_0(x) = \phi_1(x)$, then $\phi_0 = \phi_1$.*
2. *There exists a morphism ϕ of $(\bar{X}_1, p_1)$ into $(\bar{X}_2, p_2)$ such that $\phi(\bar{x}_1) = \bar{x}_2$ if and only if*
$$p_{1*}(\pi_1(\bar{X}_1, \bar{x}_1)) \subseteq p_{2*}(\pi_1(\bar{X}_2, \bar{x}_2))$$
3. *The morphism in (2) is an isomorphism if and only if*
$$p_{1*}(\pi_1(\bar{X}_1, \bar{x}_1)) = p_{2*}(\pi_1(\bar{X}_2, \bar{x}_2))$$
4. *$(\bar{X}_1, p_1)$ and $(\bar{X}_2, p_2)$ are isomorphic if and only if the subgroups $p_{1*}(\pi_1(\bar{X}_1, \bar{x}_1))$ and $p_{2*}(\pi_1(\bar{X}_2, \bar{x}_2))$ of $\pi_1(X, x_0)$ belong to the same conjugacy class.*

Proof. The proof is given below:

1. This follows from Proposition 3.2.1.
2. This is a special case of Proposition 3.2.6.
3. This follows from (2).
4. This follows (3) and Proposition 3.2.4(3).

This completes the proof. □

We can now specialize to the case of an automorphism of a covering map.

Corollary 3.2.9. *Let $p : (\bar{X}, \bar{x}_0) \to (X, x_0)$ be a covering map.*

1. *If ϕ is an automorphism $(\bar{X}, \bar{x}_0)$ and ϕ is not the identity map then ϕ has no fixed points.*
2. *Let $\bar{x}_1, \bar{x}_2 \in p^{-1}(x_0)$. There exists an automorphism $\phi \in \mathrm{Aut}(\bar{X}, p)$ such that $\phi(\bar{x}_1) = \bar{x}_2$ if and only if*
$$p_*(\pi_1(\bar{X}, \bar{x}_1)) = p_*(\pi_1(\bar{X}, \bar{x}_2))$$

Proof. The proof is given below:

1. This follows from Corollary 3.2.8(1).
2. This follows from Corollary 3.2.8(3).

This completes the proof. □

3.3. Action of Fundamental Group on Fibers

We have seen in Proposition 3.2.4 if $p : (\bar{X}, \bar{x}_0) \to (X, x_0)$ is a covering map, then if $\bar{X}$ is path-connected and simply-connected, then

$$\pi_1(X, x_0) \cong p^{-1}(x_0)$$

as sets. We provide another perspective on this bijection of sets by observing that $\pi_1(X, x_0)$ acts naturally on $p^{-1}(x_0)$. For any point $\bar{x} \in p^{-1}(x_0)$ and any $[\alpha] \in \pi_1(X, x_0)$, define $\bar{x} \cdot [\alpha] \in p^{-1}(x)$ as follows:

let $\bar{\alpha}$ be the lift of α to $\bar{X}$ starting at $\bar{x}$, so that $p_*(\bar{\alpha}) = \alpha$. Then define $\bar{x} \cdot [\alpha]$ to be the terminal point of the path class $\bar{\alpha}$.

Remark 3.3.1. *It can be verified that the action defined above is well-defined.*

It follows from the definition that:

(1) $\bar{x} \cdot [c_{x_0}] = \bar{x}$
(2) $(\bar{x} \cdot \alpha) \cdot \beta = \bar{x} \cdot (\alpha\beta)$

Therefore, this defines a right group action of $\pi_1(X, x_0)$ on the set $p^{-1}(x_0)$.

Proposition 3.3.2. *Let $p : (\bar{X}, \bar{x}_0) \to (X, x_0)$ be a covering map. If $\bar{X}$ is path-connected, the action of $\pi_1(X, x_0)$ on $p^{-1}(x_0)$ is transitive. As a right $\pi_1(X, x_0)$-space, we have*

$$p^{-1}(x_0) \cong \frac{\pi_1(X, x)}{p_*(\pi_1(\bar{X}, \bar{x}_0))}$$

Proof. Let $\bar{x}, \bar{y} \in p^{-1}(x_0)$. Since $\bar{X}$ is path-connected, there exists a path $\bar{\alpha}$ in $\bar{X}$ with the initial point $\bar{x}$ and terminal point $\bar{y}$. Let $[\alpha] = [p_*(\bar{\alpha})]$. We have $\bar{x} \cdot [\alpha] = \bar{y}$. This shows that the action is transitive. Note that the isotropy subgroup of any $\bar{x}_0$ is the set.

$$\{[\alpha] \in \pi_1(X, x) \mid \bar{x}_0 \cdot [\alpha] = \bar{x}_0\} \cong p_*(\pi_1(X, \bar{x}_0))$$

The desired isomorphism of $\pi_1(X, x_0)$-sets follows by the orbit-stabilizer theorem. □

In fact, the automorphism group of the covering space, denoted as $\mathrm{Aut}(\bar{X}, p)$, acts on the fiber $p^{-1}(x_0)$ as a right $\pi_1(X, x)$-space. This action is compatible with the group action of $\pi_1(X, x_0)$ on the fiber. Indeed, let $\phi \in \mathrm{Aut}(\bar{X}, p)$, any point $\bar{x} \in p^{-1}(x_0)$, and any $[\alpha] \in \pi_1(X, x_0)$. Lift α to a path $\bar{\alpha}$ in $\bar{X}$ with initial point $\bar{x}$, such that $p_*(\bar{\alpha}) = \alpha$. Note that $\bar{x} \cdot [\alpha]$ is the terminal point of $\bar{\alpha}$. Now consider the path $\phi \circ \bar{\alpha}$ in $\bar{X}$. Its initial point is $\phi(\bar{x})$ and the terminal point is $\phi(\bar{x} \cdot [\alpha])$. Observe that:

$$p(\phi \circ \bar{\alpha}) = (p \circ \phi)(\bar{\alpha}) = p(\bar{\alpha}) = \alpha.$$

This implies that $\phi \circ \bar{\alpha}$ is also a lifting of α. By definition, $(\phi(\bar{x})) \cdot [\alpha]$ is the terminal point of $\phi \circ \bar{\alpha}$. Therefore, we have

$$\phi(\bar{x} \cdot [\alpha]) = \phi(\bar{x}) \cdot [\alpha]$$

We now state an important result relating automorphisms of covering spaces to automorphisms of the fiber.

Proposition 3.3.3. *Let $p : (\bar{X}, \bar{x}_0) \to (X, x_0)$ be a covering map.*

(1) $\mathrm{Aut}(\bar{X}, p)$ *is naturally isomorphic to the group of automorphisms of the set $p^{-1}(x_0)$ considered as a right $\pi_1(X, x)$-set.*
(2) *The automorphism group* $\mathrm{Aut}(\bar{X}, p)$ *is isomorphic to the quotient group*

$$\frac{N(p_*(\pi_1(\bar{X}, \bar{x}_0))}{p_*(\pi_1(\bar{X}, \bar{x}_0))},$$

where $N(p_*(\pi_1(\bar{X}, \bar{x}_0))$ denotes the normalizer of $\pi_1(\bar{X}, \bar{x}_0))$ in $\pi_1(X, x_0)$.

Proof. The proof is given below:

(1) Note that if ϕ is an automorphism of $\bar{X}$, then $\phi|_{p^{-1}(x)}$ is an automorphism of the fiber $p^{-1}(x_0)$. We will prove that the map

$$\phi \mapsto \phi|_{p^{-1}(x_0)}$$

is bijective. Suppose $\phi|_{p^{-1}(x_0)} = \psi|_{p^{-1}(x_0)}$. This implies that $(\phi \circ \psi^{-1})|_{p^{-1}(x)} = \mathrm{Id}_{p^{-1}(x_0)}$. Since automorphisms of covering spaces have no fixed points unless they are the identity (Corollary 3.2.8)(1)), it follows that $\phi \circ \psi^{-1} = \mathrm{Id}_{(\bar{X},p)}$, and thus $\phi = \psi$. This shows the map is injective. If ϕ is an automorphism of the fiber $p^{-1}(x_0)$ such that $\phi(\bar{x}_0) = \bar{x}_1$, where $\bar{x}_1 \in p^{-1}(x_0)$. Then

$$p_*(\pi_1(\bar{X}, \bar{x}_0)) = p_*(\pi_1(\bar{X}, \bar{x}_1))$$

By Corollary 3.2.9(2), there exists an automorphism $\psi \in \mathrm{Aut}(X, \bar{p})$ such that $\psi(\bar{x}_0) = \bar{x}_1$. This shows the map is surjective.

(2) This follows from (1) and the group-theoretic fact that if Z is a transtive G-set and H is the isotropy subgroup of some $z \in Z$. Then the automorphism group $\mathrm{Aut}(Z)$ is isomorphic to the quotient group

$$\mathrm{Aut}(Z) \cong \frac{N(H)}{H}$$

This completes the proof. □

We now state two important corollaries of the previous result:

Corollary 3.3.4. *Let $p : (\bar{X}, \bar{x}_0) \to (X, x_0)$ be a covering map.*

(1) *If $p_*(\pi_1(\bar{X}, \bar{x}_0))$ is a normal subgroup of $\pi_1(X, x_0)$, then*

$$\mathrm{Aut}(\bar{X}, p) \cong \frac{\pi_1(X, x_0)}{p_*(\pi_1(\bar{X}, \bar{x}_0))}$$

(2) *If $\bar{X}$ is simply-connected then*

$$\mathrm{Aut}(\bar{X}, p) \cong \pi_1(X, x_0)$$

Proof. (1) follows at once from Proposition 3.3.3. (2) also follows from (1) since $N(\{c_{x_0}\} = \pi_1(X, x_0)$. □

Corollary 3.3.4 provides key insights into the structure of the automorphism group of a covering space.

(1) The first part shows that when the image of the induced map on the fundamental group $p_*(\pi_1(\bar{X}, \bar{x}_0))$ is a normal subgroup of $\pi_1(X, x_0)$, the automorphism group of the covering space is isomorphic to the quotient of the fundamental group of the base space by this normal subgroup. Moreover, for any $\bar{x} \in p^{-1}(x_0)$, we have

$$p_*(\pi_1(X, \bar{x}_0)) \cong p_*(\pi_1(X, \bar{x}))$$

since there is only one conjugacy class of $p_*(\pi_1(X, \bar{x}))$. Covering spaces that satisfy this property are called *regular/normal* covering spaces.

(2) The second part of the corollary, which applies when $\bar{X}$ is simply-connected, shows that the automorphism group of such a covering space is isomorphic to the fundamental group of the base space. Covering spaces that are simply connected are called *universal* covering spaces

3.4. Classification of Covering Spaces

If $p : (\bar{X}, \bar{x}_0) \to (X, x_0)$ is a covering map, we have proven that a covering space $(\bar{X}, p)$ is determined up to isomorphism by the conjugacy class of the subgroup $p_*(\pi_1(X, \bar{x}_0))$ of $\pi_1(X, x_0)$ (Corollary 3.2.9). Now, we address the inverse question:

Suppose (X, x_0) is a topological space and we are given a conjugacy class of subgroups of $\pi_1(X, x)$. Does there exist a topological space $(\bar{X}, \bar{x}_0)$ and a covering map $p : (\bar{X}, \bar{x}_0) \to (X, x_0)$ such that $p_*(\pi_1(\bar{X}, \bar{x}))$ belongs to the given conjugacy class?

We will see that the properties of regular and universal covering spaces are closely related to this question.

Proposition 3.4.1. *Let (X, x_0) be a topological space that is connected, locally path-connected, and semi-locally simply connected. Then, given any conjugacy class of subgroups of $\pi_1(X, x_0)$, there exists a topological space $(\bar{X}, \bar{x}_0)$ and a covering map $p : (\bar{X}, \bar{x}_0) \to (X, x_0)$ such that $p_*(\pi_1(\bar{X}, \bar{x}))$ belongs to the given conjugacy class.*

Remark 3.4.2. *A topological space (X, x_0) is semi-locally simply connected if every $x \in X$ has a neighborhood U_x such that the homomorphism*

$$\pi_1(U_x, x) \to \pi_1(X, x)$$

is trivial. That is, every loop in U_x can be contracted to x within X. Note that U need not be simply connected since every loop in U may not be contractible within U. For this reason, a space can be semi-locally simply connected without being locally simply connected. The definition of the latter is obvious. It turns out that (X, x_0) has a universal cover if and only if (X, x_0) is connected, locally path-connected, and semi-locally simply connected. See [Lee10; Hat02] for details. Universal covering spaces are called universal because they satisfy the following property: let $q : (\bar{Y}, \bar{y}_0) \to (X, x_0)$ be a covering map such that $(\bar{Y}, \bar{y}_0)$ is a universal covering space. Then for any other covering space $p : (\bar{X}, \bar{x}_0) \to (X, x_0)$, there exists a unique covering map

$$\phi : (Y, y_0) \to (\bar{X}, \bar{x}_0)$$

such that the following diagram commutes:

$$\begin{tikzcd}
(Y, y_0) \arrow[rr, "\phi"] \arrow[dr, "q"'] & & (\bar{X}, \bar{x}_0) \arrow[dl, "p"] \\
& (X, x_0) &
\end{tikzcd}$$

This follow at once from Corollary 3.2.8(2).

Proof. (Proposition 3.4.1) The assumptions on (X, x_0) imply that there exists a universal covering space, (Y, y_0), for (X, x_0). Let $q : (Y, y_0) \to (X, x_0)$ denote the corresponding (universal) covering map. We know the following facts:

(1) $\pi_1(X, x_0)$ acts freely and transitively on the set $q^{-1}(x_0)$.
(2) $\mathrm{Aut}(Y, q) \cong \pi_1(X, x_0)$.

Choose a subgroup $G \subseteq \pi_1(X, x_0)$ that lies in the given conjugacy class. Consider the following subgroup:

$$H := \{\phi : \mathrm{Aut}(Y, q) \mid \text{there exists } \alpha_\phi \in G \text{ such that } \phi(y) = y \cdot [\alpha] \in p^{-1}(x_0)\}$$

Note that $G \cong H$ under the correspondence $\phi \mapsto \alpha_\phi$. Since H is a subgroup of $\mathrm{Aut}(Y, q)$, it is a satisfies the hypothesis of Proposition 3.1.7. Hence, the quotient map $r : Y \to Y/G := \bar{X}$ is a covering map. The universal property of universal covering spaces (Remark 3.4.2) implies that we have a commutative diagram:

$$\begin{tikzcd}
(Y, y_0) \arrow[dd, "q"] \arrow[dr, "r"] & \\
& (\bar{X}, \bar{x}_0) \arrow[dl, dashed, "p"'] \\
(Y, x_0) &
\end{tikzcd}$$

Here $p : \bar{X} \to X$ is a map induced by q and $\bar{x}_0 = r(y_0) \in p^{-1}(x_0)$. It is not hard to verify that $p : \bar{X} \to X$ is a covering map. Thus, the group $\pi_1(X, x_0)$ acts transitively on the right of the set $p^{-1}(x_0)$. Since Y is simply-connected, we have $\text{Aut}(Y, r) \cong \pi_1(\bar{X}, \bar{x}_0)$. We claim that $\text{Aut}(Y, r) = H$. Clearly, $H \subseteq \text{Aut}(Y, r)$. Suppose $y_1, y_2 \in Y$, and let $\phi \in G$ be such that $\phi(y_1) = y_2$. Since ϕ is a covering transformation, we can choose an automorphism $\psi \in \text{Aut}(Y, p)$ such that $\psi(y_1) = y_2$. It follows that $\phi = \psi$. Hence, $\text{Aut}(Y, p) \subseteq G$, and therefore $G = \text{Aut}(Y, p)$[1]. Hence, we have

$$\text{Aut}(Y, r) \cong H \cong G.$$

So p_* maps $\pi_1(\bar{X}, \bar{x}_0)$ onto G. This completes the proof. □

Remark 3.4.3. *Proposition 3.4.1 proves the additional fact that if a group acts on a simply-connected space X such that the group action satisfies the hypotheses in Proposition 3.1.7, then the quotient map $p : X \to X/G$ is a regular covering map, and*

$$\text{Aut}(X/G, p) \cong \pi_1(X/G, x_0) \cong G$$

Example 3.4.4. Consider the Lens space $L(p; q_1, \ldots, q_n) \cong \mathbb{S}^{2n-1}/\mathbb{Z}_p$. Since $\mathbb{S}^{2n-1}$ is simply-connected and the action of $\mathbb{Z}_p$ on $\mathbb{S}^{2n-1}$ satisfies Proposition 3.1.7, Remark 3.4.3 implies that

$$\pi_1(\mathbb{S}^{2n-1}/\mathbb{Z}_p, x_0) \cong \mathbb{Z}_p$$

We have shown that there is a one-to-one correspondence:

$$\{\text{Conjugacy classes of } \pi_1(X, x_0)\} \longleftrightarrow \{\text{Covering maps } p : (\bar{X}, \bar{x}_0) \to (X, x_0)\}$$

provided that (X, x_0) is connected, locally path-connected, and semi-locally simply connected. Since the universal covering space is connected, we in fact have a one-to-one correspondence for connected covering maps. Let us now consider some examples to illustrate this correspondence.

Example 3.4.5. (**Coverings of $\mathbb{S}^1$**) Sice $\mathbb{R}$ is simply connected, the covering map

$$p : \mathbb{R} \to \mathbb{S}^1$$

is a universal covering map. We know that $\pi_1(\mathbb{S}^1, x_0) \cong \mathbb{Z}$. Every connected covering space of $\mathbb{S}^1$ corresponds to a subgroup of $\mathbb{Z}$. Every non-trivial subgroup of $\mathbb{Z}$ is of the form $n\mathbb{Z}$ for some integer $n \geq 1$. Note that the index of $n\mathbb{Z}$ is n. For each $n \geq 1$, Proposition 3.4.1 implies that there exists a unique (up to isomorphism) connected n-fold covering space of $\mathbb{S}^1$, which can be described as the quotient $\mathbb{R}/n\mathbb{Z} \cong \mathbb{S}^1$. The associated covering map is the n-th power map on $\mathbb{S}^1$.

$$\begin{aligned} p : \mathbb{S}^1 &\to \mathbb{S}^1 \\ z &\mapsto z^n \end{aligned}$$

[1]This shows that $r : Y \to Y/G$ is a regular covering map.

Part 2

Homology

CHAPTER 4

Homological Algebra

Homological algebra provides the foundational framework and tools essential for systematically studying and computing algebraic invariants arising in topology, geometry, and algebra. Central to this framework are exact sequences, (co)chain complexes, and spectral sequences, which play a fundamental role in algebraic topology. The theory is developed within the category Ab, with all definitions and results extending naturally to any abelian category. Standard references for this material include [Wei94; Rot09].

4.1. Motivation via Simplicial Homology

To keep the discussion grounded and to motivate the forthcoming material, we present a detailed treatment of simplicial homology in this section. This serves to illustrate the application of homological algebra techniques to topological contexts. Simplicial homology is particularly advantageous due to its computational tractability, as it applies to topological spaces that admit triangulations. In particular, we will define simplicial homology in terms of Δ-complexes, which will serve as the fundamental building blocks for our triangulations.

Definition 4.1.1. Let $[v_0, v_1, \ldots, v_n]$ be an ordered tuple in $\mathbb{R}^m$.

(1) $[v_0, v_1, \ldots, v_n] \subseteq \mathbb{R}^m$ is said to be affinely independent if the set

$$\{v_1 - v_0, v_2 - v_0, \ldots, v_n - v_0\}$$

is linearly independent[1].

(2) Given an affinely independent ordered tuple $[v_0, v_1, \ldots, v_n] \subseteq \mathbb{R}^m$, the n-simplex generated by $[v_0, v_1, \ldots, v_n]$ is the convex span in $\mathbb{R}^m$ of the $n+1$ points $v_0, \ldots, v_n$:

$$\operatorname{conv}[v_0, v_1, \ldots, v_n] = \left\{ x = \sum_{i=0}^{n} t_i v_i \in \mathbb{R}^m \mid t_i \geq 0, \sum_i t_i = 1 \right\},$$

We call the points v_i the vertices of the n-simplex $[v_0, v_1, \ldots, v_n]$.

(3) Given an n-simplex $\operatorname{conv}[v_0, v_1, \ldots, v_n]$, the face opposite to v_i is the $(n-1)$-simplex:

$$\operatorname{conv}[v_0, \ldots, \widehat{v}_i, \ldots, v_n] := \{x \in \operatorname{conv}[v_0, v_1, \ldots, v_n] \mid t_i = 0\}.$$

The boundary of an n-simplex is the union of its faces.

Geometrically, an n-simplex can be understood as the smallest convex subset containing the vertices $\{v_0, \ldots, v_n\}$, where these points do not lie within any hyperplane of dimension less than n. As an example, consider the standard n-simplex:

Definition 4.1.2. The standard simplex, $\Delta^n \subseteq \mathbb{R}^{n+1}$, is

$$\Delta^n = \left\{ (t_0, \ldots, t_n) \in \mathbb{R}^{n+1} \mid t_i \geq 0, \sum_i t_i = 1 \right\}.$$

[1] Thus necessarily $n \leq m$

Remark 4.1.3. *The standard simplex allows one to induce coordinates on all n-simplices by sending $e_i \mapsto v_i$ inducing a map of simplicies:*

$$(t_0, \dots, t_n) \mapsto \sum_{i=0}^{n} t_i v_i$$

$(t_0, \dots, t_n)$ *are are called barycentric coordinates.*

Definition 4.1.4. A Δ-complex structure on a topological space X is a collection of maps $\{\sigma_j^n : \Delta^n \to X\}_{n \geq 0}^{j \in J_n}$ such that:

(1) The restriction $\sigma_j^n|_{\mathrm{Int}(\Delta^n)}$ is injective, and each point of X is in the image of exactly one such $\sigma_j^n|_{\mathrm{Int}(\Delta^n)}$.
(2) Restriction of each σ_j^n to a face of Δ^n is one of the maps $\sigma_k^{n-1} : \Delta^{n-1} \to X$.
(3) A set $A \subseteq X$ is open if and only if $(\sigma_j^n)^{-1}(A)$ is open in X for each σ_j^n.

Remark 4.1.5. *In what follows, we shall identify a $\sigma_j^n : \Delta^n \to X$ with a n-simplex $[v_0, \cdots, v_n]$.*

Our goal is to define the simplicial homology groups of a Δ-complex structure on a topological space X. Let $\Delta_n(X)$ denote the free abelian group generated by the open n-simplices of X. Elements of $\Delta_n(X)$ are called n-chains. These can be expressed as finite formal sums

$$\sum_{j \in J_n} n_j \sigma_j^n, \quad n_j \in \mathbb{Z}.$$

Definition 4.1.6. Let X be a topological space with a Δ-complex structure. The boundary operator

$$\partial_n^\Delta : \Delta_n(X) \to \Delta_{n-1}(X)$$

is defined on each basis element of $\Delta_n(X)$ as:

$$\partial_n^\Delta [v_0, \dots, v_n] = \sum_{i=0}^{n} (-1)^i [v_0, \dots, \widehat{v_i}, \dots, v_n]$$

We say that

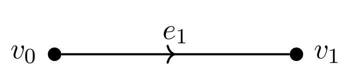
$$\partial_n^\Delta \Big(\sum_{j \in J_n} n_j \sigma_j^n \Big) \in \Delta_{n-1}(X)$$

is the boundary of $\sum_{j \in J_n} n_i \sigma_j^n$ in X.

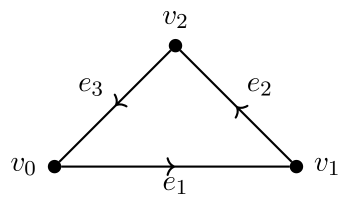
Remark 4.1.7. *Note that the boundary of an n-simplex in X is a $\mathbb{Z}$-linear combination (with coefficients ± 1) of $n-1$-simplices. This provides one motivation as to why we consider $\mathbb{Z}$-linear combinations of n-simplices. Moreover, heuristically the signs are inserted to take orientations into account, so that all the faces of a simplex are coherently oriented. See Example 4.1.8.*

Example 4.1.8. The following are examples of boundaries some standard simplexes.

(1) Consider $X = \Delta^1$. Then $\partial_1^\Delta [v_0, v_1] = [v_1] - [v_0]$



(2) Consider $X = \Delta^2$. Then $\partial_2^\Delta[v_0, v_1, v_2] = [v_1, v_2] - [v_0, v_2] + [v_0, v_1]$



Lemma 4.1.9. *Let X be a topological space with a Δ-complex structure. The map,*

$$\partial_{n-1}^\Delta \circ \partial_n^\Delta : \Delta_n(X) \xrightarrow{\partial_n^\Delta} \Delta_{n-1}(X) \xrightarrow{\partial_{n-1}^\Delta} \Delta_{n-2}(X)$$

is zero for each $n \geq 0$.

Proof. Note that:

$$\sum_{0\leq j<i\leq n} (-1)^i(-1)^j[v_0, \cdots, \widehat{v_j}, \cdots, \widehat{v_i}, \cdots, v_n] + \sum_{0\leq i<j\leq n} (-1)^i(-1)^{j-1}[v_0, \cdots, \widehat{v_i}, \cdots, \widehat{v_j}, \cdots, v_n] = 0$$

The latter two summations cancel since after switching i and j in the second sum, it becomes the negative of the first. ☐

Remark 4.1.10. *Note that $\Delta^1 \in \ker \partial_1^\Delta$ if and only if $v_0 = v_1$. In this case, Δ^1 can be thought of as a circle or a 1-loop. Indeed, this observation motivates the the observation that n-loops in X correspond to elements of $\ker \partial_n^\Delta$ for each $n \geq 1$. Moreover, the condition $\partial_n^\Delta \circ \partial_{n+1}^\Delta = 0$ is the observation that the boundary of a $\mathbb{Z}$-linear combination of $(n+1)$-simplicies is a n-loop.*

Let $C_n^\Delta(X) = \Delta_n(X)$ for each $n \geq 0$. Purely algebraically, we have a sequence of homomorphisms of abelian groups:

$$\cdots \xrightarrow{\partial_{n+1}^\Delta} C_n^\Delta(X) \xrightarrow{\partial_n^\Delta} C_{n-1}^\Delta(X) \xrightarrow{\partial_{n-1}^\Delta} C_{n-2}^\Delta(X) \xrightarrow{\partial_{n-2}^\Delta} \cdots$$

The boundary map $\partial_n^\Delta : C_n^\Delta(X) \longrightarrow C_{n-1}^\Delta(X)$ is such that

$$\partial_{n-1}^\Delta \circ \partial_n^\Delta = 0$$

That is,

$$\mathrm{im}(\partial_{n+1}^\Delta) \subseteq \ker(\partial_n^\Delta)$$

Elements of $\ker(\partial_n^\Delta)$ are called n-cycles (or n-loops) and elements of $\mathrm{im}(\partial_{n+1}^\Delta)$ are called n-boundaries.

Definition 4.1.11. Let X be a topological space with a Δ-complex structure. The n-th simplicial homology group of X with $\mathbb{Z}$-coefficients of the associated chain complex $(C_n^\Delta(X), \partial_n^\Delta)_{n\in\mathbb{N}}$ is

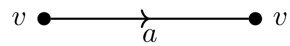
$$H_n^\Delta(X;\mathbb{Z}) = \frac{\ker(\partial_n^\Delta)}{\mathrm{im}(\partial_{n+1}^\Delta)}$$

$H_n^\Delta(X;\mathbb{Z})$ is called the n-th simplicial homology group of X.

Remark 4.1.12. *In what follows, we will not explicitly verify Definition 4.1.4(3). For instance, we will not explicitly verify that the Δ-complex structure on the circle, $\mathbb{S}^1$, in Example 4.1.13(1) is compatible with the topology on $\mathbb{S}^1$. Similarly, Example 4.1.13(2)-(8) we will not explicitly verify that the Δ-complex structure is compatible with the underlying quotient topology. It should be straightforward to do verify these claims, though.*

Example 4.1.13. We compute simplicial homology groups of various topological spaces below.

(1) **(Circle)** Consider $X = \mathbb{S}^1$ with a Δ-complex structure with a single 1-simplex and a single 0-simplex.



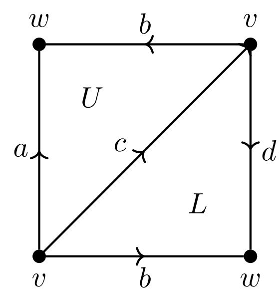
We have a chain complex of the following form:

$$\cdots \longrightarrow 0 \xrightarrow{\partial_2^\Delta} \mathbb{Z} \xrightarrow{\partial_1^\Delta} \mathbb{Z} \xrightarrow{\partial_0^\Delta} 0.$$

Here ∂_1^Δ is the zero map. Therefore, we have:

$$H_n^\Delta(\mathbb{S}^1; \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } n = 0, 1 \\ 0 & \text{otherwise} \end{cases}$$

(2) (**Mobius Band**) Consider $X = M$, the Mobius band. A Δ-complex structure on M is pictured below.



We have a complex of the following form:

$$\cdots 0 \xrightarrow{\partial_3^\Delta} \mathbb{Z}^{\oplus 2} \xrightarrow{\partial_2^\Delta} \mathbb{Z}^{\oplus 4} \xrightarrow{\partial_1^\Delta} \mathbb{Z}^{\oplus 2} \xrightarrow{\partial_0^\Delta} 0.$$

We have

$$\begin{aligned} &\partial_1^\Delta a = \partial_1^\Delta b = \partial_1^\Delta d = w - v \\ &\partial_1^\Delta c = 0 \end{aligned}$$

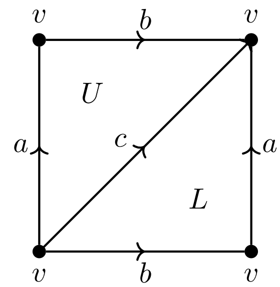
Hence $\operatorname{Im} \partial_1^\Delta \cong \mathbb{Z}$, implying that $H_0^\Delta(X) \cong \mathbb{Z}^{\oplus 2}/\mathbb{Z} \cong \mathbb{Z}$. Also

$$\begin{aligned} \partial_2^\Delta U &= a - b - c \\ \partial_2^\Delta L &= b - d - c \end{aligned}$$

This implies ∂_2^Δ is injective. Hence $H_2^\Delta(X) \cong 0$. A basis for $\ker \partial_1^\Delta$ is $\{x = a - d, y = b - d, z = c\}$. Hence $\ker \partial_1^\Delta \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$. A basis for $\operatorname{Im} \partial_2^\Delta$ is $\{x - y - z, y - z\}$. An equivalent basis is $\{x, y - z\}$. Hence $H_1^\Delta(X) \cong \mathbb{Z}$.

$$H_n^\Delta(M; \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{for } n = 0, 1 \\ 0 & \text{for } n \geq 2 \end{cases}$$

(3) (**Torus**) Consider the $X = \mathbb{T}$, the torus, with the Δ-complex structure is pictured below having one vertex, three edges a, b, and c, and two 2-simplices U and L.



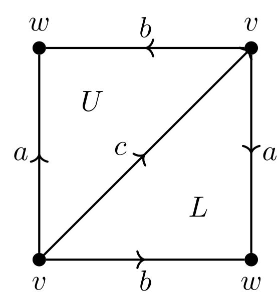
We have a complex of the following form:

$$\cdots 0 \xrightarrow{\partial_3^\Delta} \mathbb{Z}^{\oplus 2} \xrightarrow{\partial_2^\Delta} \mathbb{Z}^{\oplus 3} \xrightarrow{\partial_1^\Delta} \mathbb{Z} \xrightarrow{\partial_0^\Delta} 0.$$

As in the previous example, $\partial_1^\Delta = 0$. Also $\partial_2^\Delta U = a + b - c = \partial_2^\Delta L$. Since $\partial_1^\Delta = 0$, $H_0^\Delta(T) \cong \mathbb{Z}$. Since $\{a, b, a+b-c\}$ is a valid basis for $\mathbb{Z}^{\oplus 3}$, it follows that $H_1^\Delta(T) \cong \mathbb{Z}^2$ with basis the homology classes $[a]$ and $[b]$. Since there are no 3-simplices, $H_2^\Delta(T)$ is equal to $\ker \partial_2^\Delta$, which is infinite cyclic generated by $U - L$. Thus,

$$H_n^\Delta(T;\mathbb{Z}) \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & \text{for } n = 1 \\ \mathbb{Z} & \text{for } n = 0, 2 \\ 0 & \text{for } n \geq 3 \end{cases}$$

(4) (**Real Projective Plane**) Consider $X = \mathbb{RP}^2$. The delta complex structure is pictured below.



We have a complex of the following form:

$$\cdots 0 \xrightarrow{\partial_3^\Delta} \mathbb{Z}^{\oplus 2} \xrightarrow{\partial_2^\Delta} \mathbb{Z}^{\oplus 3} \xrightarrow{\partial_1^\Delta} \mathbb{Z}^{\oplus 2} \xrightarrow{\partial_0^\Delta} 0.$$

We have

$$\partial_1^\Delta b = \partial_1^\Delta a = w - v \qquad \partial^\Delta c = 0.$$

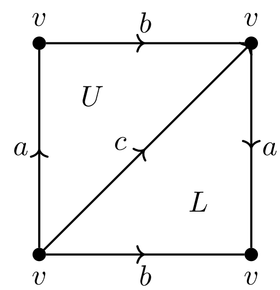
Hence $\operatorname{Im} \partial_1^\Delta \cong \mathbb{Z}$, implying that $H_0^\Delta(X) = \mathbb{Z}^{\oplus 2}/\mathbb{Z} \cong \mathbb{Z}$. Also

$$\partial_2^\Delta U = a - b - c \qquad \partial_2^\Delta L = b - a - c$$

This implies ∂_2^Δ is injective. Hence $H_2^\Delta(X) \cong 0$. A basis for $\ker \partial_1^\Delta$ is $\{x = a - b, y = c\}$. Hence $\ker \partial_1^\Delta \cong \mathbb{Z} \oplus \mathbb{Z}$. A basis for $\operatorname{Im} \partial_2^\Delta$ is $\{x - y, -x - y\}$. An equivalent basis is $\{x - y, 2y\}$. Hence $H_1^\Delta(X) \cong \mathbb{Z}_2$.

$$H_n^{\Delta}(\mathbb{RP}^2;\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{for } n = 0 \\ \mathbb{Z}_2 & \text{for } n = 1 \\ 0 & \text{for } n \geq 2 \end{cases}$$

(5) **(Klein Bottle)** Consider $X = K$, the Klein bottle, with the Δ-complex structure is pictured below having one vertex, three edges a, b and c, and two 2-simplices U and L.:



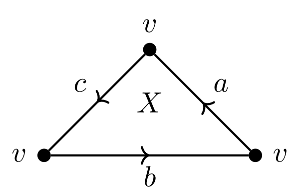
We have a complex of the following form:

$$\cdots 0 \xrightarrow{\partial_3^{\Delta}} \mathbb{Z}^{\oplus 2} \xrightarrow{\partial_2^{\Delta}} \mathbb{Z}^{\oplus 3} \xrightarrow{\partial_1^{\Delta}} \mathbb{Z} \xrightarrow{\partial_0^{\Delta}} 0.$$

Clearly, $\partial_1^{\Delta} = 0$. $\partial_2^{\Delta} U = a + b - c$ and $\partial_2^{\Delta} L = a - b + c$. Since $\partial_1^{\Delta} = 0$, $H_0^{\Delta}(K) \cong \mathbb{Z}$. We have $\mathrm{Im}(\partial_2^{\Delta}) = \mathrm{span}\{2a, a + b - c\}$. Since $\{a, a + b - c, c\}$ is a valid basis for $\mathbb{Z}^{\oplus 3}$, it follows that $H_1^{\Delta}(K) \cong \mathbb{Z} \oplus \mathbb{Z}_2$. Since there are no 3-simplices, $H_2^{\Delta}(K)$ is equal to $\ker \partial_2^{\Delta}$, which is easily seen to be trivial. Thus,

$$H_n^{\Delta}(K;\mathbb{Z}) \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z}_2 & \text{for } n = 1 \\ \mathbb{Z} & \text{for } n = 0 \\ 0 & \text{for } n \geq 2 \end{cases}$$

(6) **(Triangular Parachute)** Let X be a triangular parachute obtained from Δ^2 by identifying its three vertices to a single point.



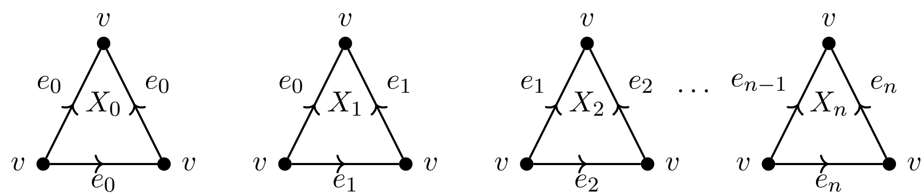
We have 1 face, 3 edges, and 1 vertex so that $\Delta^2(X), \Delta^0(X) \cong \mathbb{Z}$, $\Delta^1(X) \cong \mathbb{Z}^3$. Note that

$$\begin{aligned} &\partial_2^{\Delta}(X) = b + a - c \\ &\partial_1^{\Delta}(a) = \partial_1^{\Delta}(b) = \partial_1^{\Delta}(c) = \partial_0^{\Delta}(v) = 0 \end{aligned}$$

Hence $\ker \partial_2^{\Delta} = 0$, $\ker \partial_1^{\Delta} = \mathbb{Z}^3$, $\ker \partial_0^{\Delta} = \mathbb{Z}$. On the other hand, $\mathrm{Im}\, \partial_2^{\Delta} = \mathbb{Z}$ as the subgroup $\langle b + a - c \rangle$ is free on one generator. Hence we have,

$$H_n^{\Delta}(X;\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{for } n = 0 \\ \mathbb{Z}^{\oplus 2} & \text{for } n = 1 \\ 0 & \text{for } n \geq 2 \end{cases}$$

(7) Let X be the topological space obtained obtained from $n+1$ 2-simplices $\Delta_0^2, \ldots, \Delta_n^2$ by identifying all three edges of Δ_0^2 to a single edge, and for $i > 0$ identifying the edges $[v_0, v_1]$ and $[v_1, v_2]$ of Δ_i^2 to a single edge and the edge $[v_0, v_2]$ to the edge $[v_0, v_1]$ of Δ_{i-1}^2.



We have 1 vertex, $n+1$ edges, and $n+1$ faces so that $\Delta_0(X) \cong \mathbb{Z}$, $\Delta_1(X), \Delta_2(X) \cong \mathbb{Z}^{n+1}$. We have a complex of the following form:

$$\cdots 0 \xrightarrow{\partial_3^\Delta} \mathbb{Z}^{\oplus n+1} \xrightarrow{\partial_2^\Delta} \mathbb{Z}^{\oplus n+1} \xrightarrow{\partial_1^\Delta} \mathbb{Z}^{\oplus 1} \xrightarrow{\partial_0^\Delta} 0.$$

Clearly, $\partial_0^\Delta = 0$ and Im $\partial_1^\Delta = 0$. Hence $H_0^\Delta(X) \cong \mathbb{Z}$. Let's compute Im ∂_2. Note that:

$$\partial_2 X_i = \begin{cases} e_0 & \text{if } i = 0 \\ 2e_i - e_{i-1} & \text{if } i > 1 \end{cases}$$

It is clear that a basis for Im $\partial_2 = \{e_0\} \cup \{2e_i - e_{i-1} : 1 \leq i \leq n\}$. Note that in $H_1^\Delta(X) = \ker \partial_1 / \text{Im } \partial_2$, we set $e_0 = 0$ and $2e_i - e_{i-1} = 0$ so that $e_0 = 0, 2e_i = e_{i-1}$. This implies that

$$2e_1 = e_0 = 0 \quad 2^2 e_2 = e_0 = 0 \quad \cdots \quad 2^k e_k = e_0 = 0$$

so that Therefore:

$$H_1^\Delta(X) \cong \mathbb{Z}^{n+1} / (\mathbb{Z} \times 2\mathbb{Z} \times \cdots \times 2^n \mathbb{Z}) \cong \mathbb{Z}_2 \times \mathbb{Z}_4 \times \cdots \times \mathbb{Z}_{2^n}$$

It is easy to see that $\ker \partial_2^\Delta = 0$. Hence $H_2^\Delta(X) = 0$. Therefore, we have:

$$H_n^\Delta(X; \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{for } n = 0 \\ \mathbb{Z}_2 \times \mathbb{Z}_4 \times \cdots \times \mathbb{Z}_{2^n} & \text{for } n = 1 \\ 0 & \text{for } n \geq 2 \end{cases}$$

(8) Let X_n be obtained from an n-simplex by identifying all faces of the same dimension. Since there is only one k-simplex for each $k \leq n$, we see that $\Delta^k(X_n) \cong \mathbb{Z}$ for $k \leq n$. Choose a generator σ_k for each of these. Note that the restriction of σ_k to a $(k-1)$-dimensional face will just be σ_{k-1}. Thus,

$$\partial_k^\Delta \sigma_k = \sum_{i=0}^{k} (-1)^i \sigma_{k-1} = \begin{cases} 0 & \text{if } k = 0, \\ 0 & \text{if } k \leq n, \text{ and } k \text{ is odd}, \\ \sigma_{k-1} & \text{if } k \leq n, \text{ and } k \text{ is even}, \\ 0 & \text{if } k > n. \end{cases}$$

Therefore:

$$\ker(\partial_k^\Delta) = \begin{cases} \mathbb{Z} & \text{if } k = 0, \\ \mathbb{Z} & \text{if } k \leq n, \text{ and } k \text{ is odd}, \\ 0 & \text{if } k \leq n, \text{ and } k \text{ is even}, \\ 0 & \text{if } k > n. \end{cases} \qquad \text{Im}(\partial_k) = \begin{cases} 0 & \text{if } k = 0, \\ 0 & \text{if } k \leq n, \text{ and } k \text{ is odd}, \\ \mathbb{Z} & \text{if } k \leq n, \text{ and } k \text{ is even}, \\ 0 & \text{if } k > n. \end{cases}$$

Hence:

$$H_k^{\Delta}(X_n;\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } k=0, \\ \mathbb{Z} & \text{if } k=n, \text{ and } n \text{ is odd}, \\ 0 & \text{else.} \end{cases}$$

4.2. (Co)-Chain Complexes & (Co)homology

We now turn to the formal study of homological algebra. In particular, we will discuss (co)chain complexes and their associated (co)homology theories. These algebraic structures provide a systematic framework for investigating topological and algebraic invariants. Our aim is to develop the foundational tools essential for the study of algebraic topology.

4.2.1. Exact Sequences. Before defining (co)chain complexes and their associated (co)homology, it is necessary to first discuss exact sequences. Exact sequences are a central tool in homological algebra, encoding the manner in which one algebraic object maps into another. They play a crucial role in detecting kernels and images of homomorphisms, which are fundamental in the definition and computation of homology and cohomology groups.

Definition 4.2.1. A sequence

$$A \xrightarrow{f} B \xrightarrow{g} C$$

of two homomorphisms of abelian groups is said to be exact at B if $\operatorname{im} f = \ker g$. More generally, a sequence

$$\ldots \to A_{n+1} \xrightarrow{f_{n+1}} A_n \xrightarrow{f_n} A_{n-1} \to \ldots$$

is said to be exact if it is exact at A_n for each $n \in \mathbb{Z}$. Such a sequence is called a long exact sequence of abelian groups.

The following is an important special case:

Definition 4.2.2. A short exact sequence of abelian groups is an exact sequence of the form

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0.$$

that is exact in each degree.

Example 4.2.3. Using the notion of exactness, we can rephrase familiar definitions from basic algebra. Suppose $f : A \to B$ is a homomorphism of abelian groups.

(1) f is injective if and only if $0 \to A \xrightarrow{f} B$ is exact. Indeed, the sequence is exact at A if and only if $\ker f = 0$ if and only if f is injective.
(2) f is surjective if and only if $A \xrightarrow{f} B \to 0$ is exact. Indeed, the sequence is exact at B if and only if $\operatorname{im} f = B$ if and only if f is surjective.
(3) f is an isomorphism if and only if $0 \to A \xrightarrow{f} B \to 0$ is exact. This follows from the two statements above.

Remark 4.2.4. *Functors in Ab_R (or in any abelian category) can preserve algebraic structure in different ways. A functor F is said to be left exact if it sends exact sequences of the form*

$$0 \to A \to B \to C$$

to exact sequences

$$0 \to \mathsf{F}(A) \to \mathsf{F}(B) \to \mathsf{F}(C).$$

Similarly, a functor is said to be right exact if it sends exact sequences of the form

$$A \to B \to C \to 0$$

to exact sequences

$$\mathsf{F}(A) \to \mathsf{F}(B) \to \mathsf{F}(C) \to 0.$$

A functor is called exact if it is both left exact and right exact. The notion of exact functors will be used later and is discussed in greater detail in the appendix (Chapter 15).

4.2.2. Co-(chain) Complexes. We now define the notions of (co)chain complexes and the (co)homology of these complexes. (Co)chain complexes provide the algebraic framework for computing (co)homology, encoding sequences of abelian groups connected by boundary maps. First, recall that in the category GrX, where $\mathsf{X} = \mathsf{Grp}, \mathsf{Ab}, \mathsf{Mod}_R$, a morphism $f : C \to D$ is said to be of degree k if for each $n \in \mathbb{Z}$, the component map satisfies $f : C_n \to D_{n+k}$.

Definition 4.2.5. A chain complex of abelian groups is a graded abelian group along with a homomorphism ∂ of degree -1 such that $\partial \circ \partial = 0$. In other words, a chain complex is a sequence of abelian groups and homomorphisms

$$\cdots \xrightarrow{\partial_{n+2}} C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots$$

for $n \in \mathbb{Z}$ which satisfies $\partial_n \circ \partial_{n+1} = 0,$ for each $n \in \mathbb{Z}$. That is,

$$\operatorname{im} \partial_{n+1} \subseteq \ker \partial_n \quad \iff \quad \partial_n \circ \partial_{n+1} = 0$$

We refer to the entire complex as $(C_\bullet, \partial_\bullet)$ or sometimes just $C_\bullet$. The maps ∂_n are called the boundary operators of the chain complex. Elements of $\ker \partial_n$ are called n-chains and elements of $\operatorname{im} \partial_{n+1}$ are called n-boundaries.

Example 4.2.6. Let X be a topological space. The chain complex

$$(C_n^\Delta(X), \partial_n^\Delta)_{n\in\mathbb{N}}$$

encountered in Section 4.1 is a chain complex of abelian groups. We call this the simplicial chain complex. Note that in this example the abelian groups are all zero for negative subscripts; this, however, is not part of the definition in general.

Remark 4.2.7. *There is a dual notion called a cochain complex of abelian groups, which is a graded abelian group along with a homomorphism ∂ of degree $+1$ such that $\partial \circ \partial = 0$. In other words, a co-chain complex is a sequence of abelian groups and homomorphisms*

$$\cdots \xrightarrow{\partial^{n-1}} C^n \xrightarrow{\partial^n} C^{n+1} \xrightarrow{\partial^{n+1}} C^{n+2} \xrightarrow{\partial^{n+2}} \cdots$$

for $n \in \mathbb{Z}$, satisfying the condition

$$\partial^{n+1} \circ \partial^n = 0, \quad \textit{for each } n \in \mathbb{Z}.$$

Equivalently,

$$\operatorname{im} \partial^n \subseteq \ker \partial^{n+1}.$$

We denote the co-chain complex by $(C^\bullet, \partial^\bullet)$ or simply $C^\bullet$, and the maps ∂^n are called the coboundary operators*. Elements of* $\ker \partial_n$ *are called n co-chains and elements of* $\operatorname{im} \partial_{n+1}$ *are called n-co-boundaries.*

The distinction between chain and co-chain complexes is purely formal. We will invoke either notion as needed throughout the text, depending on context and convenience. We now define the notion of a chain map between chain complexes.

Definition 4.2.8. Let $(C_\bullet, \partial_\bullet)$ and $(C'_\bullet, \partial'_\bullet)$ be chain complexes of abelian groups. A chain map between $(C_\bullet, \partial_\bullet)$ and $(C'_\bullet, \partial'_\bullet)$ is a sequence of abelian groups homomorphisms $f_n : C_n \to C'_n$ for $n \in \mathbb{Z}$ such that the diagram commutes:

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, "f_{n+1}"] & C_n \arrow[r, "\partial_n"] \arrow[d, "f_n"] & C_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \cdots \\
\cdots \arrow[r] & C'_{n+1} \arrow[r, "\partial'_{n+1}"] & C'_n \arrow[r, "\partial'_n"] & C'_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

Remark 4.2.9. *The definition of a co-chain map between co-chain complexes is similar.*

Proposition 4.2.10. *Chain complexes of abelian groups form a category, denoted as* $\mathsf{Chain}_{\mathsf{Ab}}$. *Similarly, co-hain complexes of abelian groups form a category, denoted as* $\mathsf{CoChain}_{\mathsf{Ab}}$.

Proof. It suffices to prove the first claim. Objects in $\mathsf{Chain}_{\mathsf{Ab}}$ are chain complexes of abelian groups and a morphism between chain complexes of abelian groups is a chain map. If $(C_\bullet, \partial_\bullet)$, $(C'_\bullet, \partial'_\bullet)$ and $(C''_\bullet, \partial''_\bullet)$ are chain complexes such that $f_\bullet : C_\bullet \to C'_\bullet$ and $g_\bullet : C'_\bullet \to C''_\bullet$ are two chain maps. Then

$$(g \circ f)_\bullet : C_\bullet \to C''_\bullet$$

is the chain map given by $(g \circ f)_n = g_n \circ f_n$. This is indeed a valid chain map as the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, "f_{n+1}"] & C_n \arrow[r, "\partial_n"] \arrow[d, "f_n"] & C_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \cdots \\
\cdots \arrow[r] & C'_{n+1} \arrow[r, "\partial'_{n+1}"] \arrow[d, "g_{n+1}"] & C'_n \arrow[r, "\partial'_n"] \arrow[d, "g_n"] & C'_{n-1} \arrow[r] \arrow[d, "g_{n-1}"] & \cdots \\
\cdots \arrow[r] & C''_{n+1} \arrow[r, "\partial''_{n+1}"] & C''_n \arrow[r, "\partial''_{n+1}"] & C''_{n-1} \arrow[r, "\partial''_{n+1}"] & \cdots
\end{tikzcd}$$

commutes essentially by construction as can be easily checked. This defines the composition of two chain maps. Moreover, the identity chain map $\mathrm{Id} : C_\bullet \to C_\bullet$ is the chain map given by $\mathrm{Id}_n = \mathrm{Id}_{C_n}$ where Id_{C_n} is the identity homomorphism from C_n to C_n.

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, "\mathrm{Id}_{n+1}"] & C_n \arrow[r, "\partial_n"] \arrow[d, "\mathrm{Id}_n"] & C_{n-1} \arrow[r] \arrow[d, "\mathrm{Id}_{n-1}"] & \cdots \\
\cdots \arrow[r] & C_{n+1} \arrow[r, "\partial'_{n+1}"] & C_n \arrow[r, "\partial'_n"] & C_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

All that is required is check that composition of chain maps satisfies the associativity property and the composition of a chain map with the identity chain map yields the original chain map. All these are routine checks. $\square$

Remark 4.2.11. *We can also define the category of short exact sequence of chain complexes,* $\mathsf{Chain}^{\mathsf{Exact}}_{\mathsf{Ab}}$. *Objects in* $\mathsf{Chain}^{\mathsf{Exact}}_{\mathsf{Ab}}$ *are short exact sequences of chain complexes. A morphism between short exact sequence of* *chain complexes is a diagram*

$$\begin{tikzcd} 0_\bullet \arrow[r] & A_\bullet \arrow[r, "i_\bullet"] \arrow[d, "f_\bullet"] & B_\bullet \arrow[r, "j_\bullet"] \arrow[d, "g_\bullet"] & C_\bullet \arrow[r] \arrow[d, "h_\bullet"] & 0_\bullet \\ 0_\bullet \arrow[r] & A'_\bullet \arrow[r, "i'_\bullet"] & B'_\bullet \arrow[r, "j'_\bullet"] & C'_\bullet \arrow[r] & 0_\bullet \end{tikzcd}$$

such that $f_\bullet, g_\bullet, h_\bullet$ are chain maps. We will not go through the pain of writing the diagram out explicitly. The category $\mathsf{CoChain}_{\mathsf{Ab}}^{\mathsf{Exact}}$ *is defined similarly.*

4.2.3. Co(homology). Given a chain complex $(C_\bullet, \partial_\bullet)$, where each C_n is a abelian group and $\partial_n : C_n \to C_{n-1}$ is a boundary map, the defining condition of a chain complex is that the composition of consecutive boundary maps is zero; that is, $\partial_n \circ \partial_{n+1} = 0$ for all $n \in \mathbb{Z}$. This condition ensures that

$$\operatorname{im} \partial_{n+1} \subseteq \ker \partial_n.$$

This containment motivates the introduction of homology which serves to measure the failure of this inclusion to be an equality.

Definition 4.2.12. Let $(C_\bullet, \partial_\bullet)$ be a chain complex. The n-th homology group is defined as

$$H_n(C_\bullet) := \frac{\ker \partial_n}{\operatorname{im} \partial_{n+1}}.$$

Remark 4.2.13. *There is a dual notion of cohomology of co-chain complexes. Given a co-chain complex $(C^\bullet, d^\bullet)$, where each C^n is an abelian group and $\partial^n : C^n \to C^{n+1}$ is a co-boundary map satisfying $\partial^{n+1} \circ \partial^n = 0$, we define cohomology to measure the failure of exactness:*

$$H^n(C^\bullet) := \frac{\ker \partial^n}{\operatorname{im} \partial^{n-1}}.$$

Once again, the distinction between homology and cohomology is purely formal. We will invoke either notion as needed throughout the text, depending on context and convenience.

Example 4.2.14. Let's compute the homology of some simple chain complexes:

(1) Consider the chain complex

$$C_\bullet : \cdots \to 0 \to \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\partial_2} \mathbb{Z} \xrightarrow{\partial_1} 0 \to \cdots ,$$

where the chain groups are given by

$$\begin{aligned} C_1 &= \mathbb{Z}, \\ C_2 &= \mathbb{Z} \oplus \mathbb{Z}, \\ C_n &= 0 \quad \text{for } n \neq 1, 2. \end{aligned}$$

The homomorphism ∂_2 is defined by $\partial_2(x, y) = 3x + 3y$. Note that we have the following

$$\ker \partial_n \cong \begin{cases} \mathbb{Z}, & \text{if } n = 1 \text{ or } n = 2, \\ 0, & \text{if } n \neq 1, 2. \end{cases}$$

Similarly, we have

$$\operatorname{im} \partial_n \cong \begin{cases} 3\mathbb{Z}, & \text{if } n = 2, \\ 0, & \text{if } n \neq 2. \end{cases}$$

Therefore, the homology of the chain complex is given as:

$$H_n(C_\bullet) \cong \begin{cases} \mathbb{Z}_3, & \text{if } n = 1, \\ \mathbb{Z}, & \text{if } n = 2, \\ 0, & \text{if } n \neq 1, 2. \end{cases}$$

(2) Consider the chain complex:

$$\cdots \xrightarrow{\partial} \mathbb{Z}/8\mathbb{Z} \xrightarrow{\partial} \mathbb{Z}/8\mathbb{Z} \xrightarrow{\partial} \mathbb{Z}/8\mathbb{Z} \xrightarrow{\partial} \mathbb{Z}/8\mathbb{Z} \to 0 \to 0 \to \cdots$$

where $C_n = \mathbb{Z}/8\mathbb{Z}$ for $n \leq 0$ and $C^n = 0$ for $n > 0$ and the map ∂ is given by $x \bmod 8 \mapsto 4x \bmod 8$. It is easy to see that

$$\ker\,\partial = \{\overline{0}, \overline{2}, \overline{4}, \overline{6}\} \cong \mathbb{Z}/4\mathbb{Z}$$
$$\text{im}\,\partial = \{\overline{0}, \overline{4}\} \cong \mathbb{Z}/2\mathbb{Z}$$

for $n < 0$. Hence,

$$H_n(C_\bullet) \cong \frac{\mathbb{Z}/4\mathbb{Z}}{\mathbb{Z}/2\mathbb{Z}} \cong \mathbb{Z}/2\mathbb{Z}$$

Trivially, $H_n(C_\bullet) \cong 0$ for $n > 0$ and $H_0(C_\bullet) \cong \mathbb{Z}/4\mathbb{Z}$.

We now discuss an important observation that (co)homology defines a functor from the category of (co)-chain complexes to the category of abelian groups. This is formalized in the following proposition.

Proposition 4.2.15. *For each $n \in \mathbb{Z}$, there is a functor*

$$H_n : \mathsf{Chain}_{\mathsf{Ab}} \to \mathsf{Ab}$$

that associates to a chain complex over abelian groups its n-th homology group Similarly, for each $n \in \mathbb{Z}$ there is a functor

$$H^n : \mathsf{CoChain}_{\mathsf{Ab}} \to \mathsf{Ab}$$

that associates to a co-chain complex over abelian groups is n-th cohomology group.

Proof. It suffices to prove the first claim. Consider a chain map between chain complexes given by the following diagram:

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & C_{n+1} & \xrightarrow{\partial_{n+1}} & C_n & \xrightarrow{\partial_n} & C_{n-1} & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle f_{n+1}} & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} & & \\ \cdots & \longrightarrow & C'_{n+1} & \xrightarrow{\partial'_{n+1}} & C'_n & \xrightarrow{\partial'_n} & C'_{n-1} & \longrightarrow & \cdots \end{array}$$

The relation $f_n\partial_{n+1} = \partial'_{n+1} f_{n+1}$ implies that f_n takes n-cycles to takes n-cycles for each $n \in \mathbb{N}$. This is because if $\partial_n c = 0$, then

$$\partial'_n(f_n(c)) = f_{n-1}(\partial_n c) = 0$$

Also, f_n takes n boundaries to n-boundaries since

$$f_n(\partial_{n+1}c) = \partial'_{n+1}(f_{n+1}c)$$

Hence f_n descends to a homomorphism

$$H_n(f) : H_n(C_\bullet) \to H_n(C'_\bullet)$$

It remains to check that $H_n(g \circ f) = H_n(g) \circ H_n(f)$ and that $H_n(\text{Id}_X) = \text{Id}_{H_n(X)}$. Both of these are immediate from the definitions. $\square$

4.2.4. (Co)-chain homotopy. We conclude this section by introducing the notion of a chain homotopy between chain complexes. Chain homotopy allows us to compare chain maps up to a 'deformation,' playing a crucial role in establishing when two chain complexes have the same homological properties. As expected, there exists an analogous notion of a co-chain homotopy between co-chain complexes. We will not repeat the definitions in this case.

Definition 4.2.16. Suppose $(C_\bullet, \partial)$ and $(C'_\bullet, \partial')$ are two chain complexes with chain maps $f_\bullet, g_\bullet$. A chain homotopy between $f_\bullet, g_\bullet$ is a series of maps $T_n : C_n$ and C_{n+1} such that

$$\begin{aligned} f_n - g_n &= \partial'_{n+1} T_n + T_{n-1}\partial_n && n \geq 1 \\ T_0 \circ \partial_1 &= f_0 - g_0 && n = 0 \end{aligned}$$

That is, the following diagram commutes

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, "f_{n+1}"', "g_{n+1}"] \arrow[dl, "T_{n+1}"'] & C_n \arrow[r, "\partial_n"] \arrow[d, "f_n"', "g_n"] \arrow[dl, "T_n"] & C_{n-1} \arrow[r] \arrow[d, "f_{n-1}"', "g_{n-1}"] \arrow[dl, "T_{n-1}"'] & \cdots \arrow[dl, "T_{n-2}"] \\
\cdots \arrow[r] & C'_{n+1} \arrow[r, "\partial'_{n+1}"'] & C'_n \arrow[r, "\partial'_n"'] & C'_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

Remark 4.2.17. *It can be verified that (co)chain homotopy is an equivalence relation. Consequently, Definition 4.2.16 defines a new category,* $\mathsf{hChain}_{\mathsf{Ab}}$*, whose objects are chain complexes over abelian groups and whose morphisms are chain maps modulo the chain homotopy equivalence relation. The category* $\mathsf{hCoChain}_{\mathsf{Ab}}$ *is defined analogously, with cochain complexes and cochain maps modulo cochain homotopy.*

Remark 4.2.18. *We will provide a geometric intuition behind the definition of a chain homotopy in Section 5.2.*

Proposition 4.2.19. *Let* $(C, \partial_\bullet), (C', \partial'_\bullet)$ *be chain complexes and let* $f_\bullet, g_\bullet$ *be chain maps between the chain complexes. If there is a chain homotopy* $f_\bullet$ *and* $g_\bullet$*, then the induced maps in homology are equal, i.e., we have:*

$$H_n(f) = H_n(g) : H_n(C_\bullet) \to H_n(C'_\bullet)$$

Proof. Let $(T_n)_{n\geq 1}$ be the sequence of maps defining a chain homotopy. Let $[c] \in H_n(C)$. If $n = 0$, we have

$$H_0(f)([c]) = [f_0(c)] = [g_0(c) + \partial_1 T_0(c)] = [g_0(c)] = H_0(g)([c])$$

For $n \geq 1$, we have:

$$\begin{aligned} H_n(f)([c]) &= [f_n(c)] \\ &= [g_n(c) + \partial'_{n+1} T_n(c) + T_{n-1}\partial_n(c)] \\ &= [g_n(c)] \\ &= H_n(g)([c]) \end{aligned}$$

The third equality uses that c is a n-cycle and that a homology class is not changed if we add a n-boundary. The claim follows. $\square$

Proposition 4.2.19 shows that for each $n \geq 0$, the homology functor defined in Proposition 4.2.15 factors through the homotopy category $\mathsf{hChain}_{\mathsf{Ab}}$.

$$\begin{tikzcd}
\mathsf{Chain}_{\mathsf{Ab}} \arrow[r, "\gamma"] \arrow[dr, "H_n"'] & \mathsf{hChain}_{\mathsf{Ab}} \arrow[d, dashed, "\overline{H}_n"] \\
& \mathsf{Ab}
\end{tikzcd}$$

Of course, a similar statement holds for the cohomology functors and the category $\mathsf{hCoChain}_{\mathsf{Ab}}$.

4.3. Motivation for Spectral Sequences

In algebra, one often seeks to compute a graded object M^*, which may be, for instance, any of the following:

(1) a graded R-module for a ring R,
(2) a graded $\mathbb{K}$-vector space for a field $\mathbb{K}$,
(3) a graded $\mathbb{K}$-algebra for a field $\mathbb{K}$.

The computation of M^* is frequently nontrivial. Significant progress can often be made through an *approximation argument*, especially when M^* is equipped with additional structure that facilitates such methods. A common scenario arises when M^* is endowed with a filtration by a (possibly unbounded) descending sequence of subobjects:

$$\cdots \supseteq M_n \supseteq M_{n+1} \supseteq \cdots , \tag{1}$$

satisfying $\bigcup_{n=0}^{\infty} M_n = M^*$ and $\bigcap_{n=0}^{\infty} M_n = 0$. Alternatively, one may consider filtrations given by possibly unbounded increasing sequences of subobjects. In general, these sequences need not be bounded. We illustrate this with the following example:

Example 4.3.1. Let M^* be a possibly infinite-dimensional $\mathbb{K}$-vector space. For instance, consider $M^* = \mathbb{K}^\infty$, the countably infinite-dimensional $\mathbb{K}$-vector space with basis $\{e_0, e_1, \dots\}$. Define

$$M_n := \operatorname{span}\{e_p \mid p \geq n\}$$

Then $\{M_n\}_{n\in\mathbb{N}}$ defines a filtration as described in Equation (1).

In fact, Example 4.3.1 possesses additional structure, in the sense that $\mathbb{K}^\infty$ can be recovered from its filtration as follows. The filtration of $\mathbb{K}^\infty$ gives rise to a new graded $\mathbb{K}$-vector space known as the associated graded $\mathbb{K}$-vector space defined by M_n/M_{n+1}. One can recover $\mathbb{K}^\infty$ up to isomorphism from its associated $\mathbb{K}$-vector space by taking direct sums:

$$\mathbb{K}^\infty \cong \bigoplus_{n=0}^{\infty} M_n/M_{n+1} \cong \bigoplus_{n=0}^{\infty} \frac{\operatorname{span}\{e_p \mid p \geq n\}}{\operatorname{span}\{e_p \mid p \geq n+1\}} \cong \bigoplus_{n=0}^{\infty} \operatorname{span}\{e_n\}$$

In general, it may not be possible to fully compute an arbitrary graded object M^* via such filtrations. For example, if M^* is an arbitrary graded R-module, extension problems may arise that obstruct the reconstruction of M^* solely from its associated graded R-module. Nevertheless, the associated graded module of a filtration on M^* serves as a natural first approximation to M^*, which can potentially be refined through a limiting process. This principle underlies the theory of spectral sequences:

A spectral sequence is an algorithm for computing a graded object by taking successive approximations.

Spectral sequences arise as a natural computational and conceptual framework in the study of filtered complexes within homological algebra and algebraic topology. They prove especially useful in contexts where direct computation of homology is intractable, yet a filtration endows the problem with a more accessible structure. To illustrate this informally, we now consider an example that will naturally motivate the formal definition of a spectral sequence.

Example 4.3.2. (**Filtered Co-Chain Complexes**) Let $C^\bullet$ be a co-chain complex of abelian groups equipped with a decreasing filtration[2] by sub-complexes:

$$\cdots \supseteq F_pC^\bullet \supseteq F_{p+1}C^\bullet \supseteq \cdots$$

[2]We could also consider an increasing filtration; however, to remain consistent with the filtration introduced earlier in Equation (1), we choose to work with a decreasing filtration. Later on, we will work with both decreasing and increasing filtrations.

$$\begin{tikzcd}
E^0_{-1,1} \arrow[d] & E^0_{0,1} \arrow[d] & E^0_{1,1} \arrow[d] \\
E^0_{-1,0} \arrow[d] & E^0_{0,0} \arrow[d] & E^0_{1,0} \arrow[d] \\
E^0_{-1,-1} & E^0_{0,-1} & E^0_{1,-1}
\end{tikzcd}$$

(A) Zeroth page

$$\begin{tikzcd}
E^0_{-1,1} \arrow[r] & E^0_{0,1} \arrow[r] & E^0_{1,1} \\
E^0_{-1,0} \arrow[r] & E^0_{0,0} \arrow[r] & E^0_{1,0} \\
E^0_{-1,-1} \arrow[r] & E^0_{0,-1} \arrow[r] & E^0_{1,-1}
\end{tikzcd}$$

(B) First page

$$\begin{tikzcd}
{} & E^0_{-1,1} & E^0_{0,1} & E^0_{1,1} & {} \\
{} & E^0_{-1,0} & E^0_{0,0} \arrow[ull] & E^0_{1,0} \arrow[ull] & {} \arrow[ull] \\
{} & E^0_{-1,-1} & E^0_{0,-1} \arrow[ull] & E^0_{1,-1} \arrow[ull] & {} \arrow[ull]
\end{tikzcd}$$

(C) Second page

Snapshots of first three pages of a homological spectral sequence

Such a filtration provides a decomposition of the complex into progressively more refined components. The central question is how the cohomology of $C^\bullet$ can be reconstructed from the data of the filtration. A natural first step is to consider the associated graded complex:

$$\mathrm{G}_p\, C^\bullet := F_p C^\bullet / F_{p+1} C^\bullet. \tag{2}$$

For each $p \in \mathbb{Z}$, one may compute the cohomology of $\mathrm{G}_p\, C^\bullet$, yielding an approximation to the cohomology of the co-chain complex. However, this information may be insufficient to fully determine the cohomology of $C^\bullet$. To overcome this limitation, one introduces a spectral sequence: for $r \in \mathbb{N}$, a sequence of pages $\{E_r^{p,q}\}_{(p,q)\in\mathbb{Z}^2}$ such that $E_r^{p,q}$ is a bi-graded group. The first page is derived from the cohomology of the associated graded complex in Equation (2). Subsequent pages refine this approximation as each subsequent page is defined as the cohomology of the preceding page.

Remark 4.3.3. *Under appropriate convergence conditions, the spectral sequence is expected to stabilize at a terminal page which captures the associated graded components of the homology of the co-chain complex. We will see how the exact definition of convergence of a spectral sequence naturally arises in explicit constructions.*

4.4. Definition of a Spectral Sequence

Based on the discussion in Section 4.3, we introduce the definition of a spectral sequence in this section and comment on some details surrounding the definition.

Definition 4.4.1. Let $r_0 \in \mathbb{N}$. A homological spectral sequence, E, of abelian groups consists of the following data:

(1) A collection of abelian groups, $E^r_{p,q}$, with integers $p, q \geq 0$ and $r \geq r_0$,
(2) A collection of differentials

$$d^r_{p,q} : E^r_{p,q} \to E^r_{p-r,q+r-1}$$

such that $d^r_{p-r,q+r-1} \circ d^r_{p,q} = 0$ and $E^{r+1}_{p,q}$ is the homology at $E^r_{p,q}$, i.e.

$$E^{r+1}_{p,q} \cong \frac{\ker d^r_{p,q}}{\operatorname{im} d^r_{p+r,q-r+1}}$$

The collection $E^r = \{(E^r_{p,q}, d^r_{p,q}) : p, q \in \mathbb{Z}\}$ for a fixed r is called the r-th page.

One way to look at a homological spectral sequence is to imagine an infinite book, where each page is a Cartesian plane with the integral lattice points (p, q) consisting of objects in the category of abelian groups and differentials between the objects forming a chain complex. The homology groups of these chain complexes are precisely the groups which appear on the next page. The customary picture is shown in Figure 1.

The intuition behind the definition of the differential maps can be understood as follows. Since the differentials $d^r_{p,q}$ are designed to compute homology, we anticipate that the total degree of the map decreases by one. Consequently, if the domain is $E^r_{p,q}$, it is natural for the codomain of $d^r_{p,q}$ to be $E^r_{p-r,\,q+r-1}$. The particular choice of shifts by r will be justified later, when we explicitly construct spectral sequences. We also recall the analogous definition in the cohomological setting:

Definition 4.4.2. Let $r_0 \in \mathbb{N}$. A cohomological spectral sequence, E, of abelian groups consists of the following data:

(1) A collection of abelian groups, $E^{p,q}_r$, with integers $p, q \geq 0$ and $r \geq r_0$,
(2) A collection of differentials

$$d^{p,q}_r : E^{p,q}_r \to E^{p+r,q-r+1}_r$$

such that $d^{p+r,q-r+1}_r \circ d^{p,q}_r = 0$ and $E^{p,q}_{r+1}$ is the homology at $E^{p,q}_r$, i.e.

$$E^{p,q}_{r+1} \cong \frac{\ker d^{p,q}_r}{\operatorname{im} d^{p-r,q+r-1}_r}$$

The collection $E_r = \{(E^{p,q}_r, d^{p,q}_r) : p, q \geq 0\}$ for a fixed r is called the r-th page.

$$\begin{tikzcd}
E_0^{-1,1} & E_0^{0,1} & E_0^{1,1} \\
E_0^{-1,0} \arrow[u] & E_0^{0,0} \arrow[u] & E_0^{1,0} \arrow[u] \\
E_0^{-1,-1} \arrow[u] & E_0^{0,-1} \arrow[u] & E_0^{1,-1} \arrow[u]
\end{tikzcd}$$

(A) Zeroth page

$$\begin{tikzcd}
E_1^{-1,1} \arrow[r] & E_1^{0,1} \arrow[r] & E_1^{1,1} \\
E_1^{-1,0} \arrow[r] & E_1^{0,0} \arrow[r] & E_1^{1,0} \\
E_1^{-1,-1} \arrow[r] & E_1^{0,-1} \arrow[r] & E_1^{1,-1}
\end{tikzcd}$$

(B) First page

$$\begin{tikzcd}
E_2^{-1,1} \arrow[drr] & E_2^{0,1} & E_2^{1,1} \\
E_2^{-1,0} \arrow[drr] & E_2^{0,0} & E_2^{1,0} \\
E_2^{-1,-1} & E_2^{0,-1} & E_2^{1,-1}
\end{tikzcd}$$

(C) Second page

Snapshots of first three pages of a cohomological spectral sequence

Remark 4.4.3. *Spectral sequences refine the process of calculating homology in the sense that the computation of homology at the r-th page not only yields the homology at the $(r+1)$-st page, but also determines the differential maps between the homology on the $(r+1)$-st page. Hence, a spectral sequence encodes a significant amount of additional information. However, the differentials can be difficult to compute explicitly. In practice, educated guesswork and ad-hoc techniques are often required to determine the differential maps.*

Since homology is defined as a sub-quotient (i.e., the quotient of an abelian group), we expect the modules appearing on the $(r+1)$-th page to be, in some sense, "smaller" or more refined than those on the r-th page. Fixing a position $(p, q) \in \mathbb{Z}^2$, we consider the sequence of abelian groups $E^r_{p,q}$ as $r \to \infty$. This motivates the definition of the limiting page of the spectral sequence, referred to as the E^∞ page, as well as the notion of convergence of spectral sequences. The definition will be provided in a later section. The best way to understand how this definition arises is by examining a concrete construction where we can explicitly determine the ingredients that determine the definition of convergence of a spectral sequence. For the time being, we consider a formal construction in which issues of convergence do not arise.

Example 4.4.4. (**First Quadrant Spectral Sequence**) First-quadrant homological spectral sequences are significantly more tractable than general homological spectral sequences, both computationally and conceptually. A homological spectral sequence is a first quadrant homological spectral sequence if $E^r_{p,q} = 0$ for $p < 0$ or $q < 0$. Fix $(p, q) \in (\mathbb{N} \cup \{0\})^2$. In a first quadrant homological spectral sequence, for r (as a function of (p, q)) large enough, the differential with co-domain $E^r_{p,q}$ has domain 0 and the differential with domain $E^r_{p,q}$ has co-domain 0. Therefore, we get

$$E^{r+1}_{p,q} \cong \frac{\ker d^r_{p,q}}{\operatorname{im} d^r_{p+r,q-r+1}} \cong \frac{E^r_{p,q}}{0} \cong E^r_{p,q}.$$

The stable value $E^r_{p,q} = E^k_{p,q}$ for $k \geq r(p,q)$ is named $E^\infty_{p,q}$. In this case, we can determine the entries on the E^∞ page in a finite number of steps, and there are no issues of convergence. A first-quadrant cohomological spectral sequence is defined similarly.

The distinction between homological and cohomological indexing is purely a matter of convention. We will use both notations as appropriate and convenient in the discussions that follow.

4.5. Spectral Sequence of a Filtered Complex

Remark 4.5.1. *In this section, we work with cohomological spectral sequences. The constructions are largely formal and have analogous counterparts for homological spectral sequences, which we will freely use later on.*

Definition 4.5.2. Let $C^\bullet = \{C^n, \partial^n\}_{n\in\mathbb{Z}}$ be a co-chain complex of abelian groups. A decreasing filtration of $C^\bullet$ is a sequence

$$\cdots \supseteq F_p C^\bullet \supseteq F_{p+1} C^\bullet \supseteq \cdots$$

such that each $F_p C^\bullet$ is a sub-complex of $C^\bullet$ and the differential ∂^n restricts to a map

$$F_p C^n \to F_p C^{n+1}$$

for all $n \in \mathbb{Z}$ that is compatible with the filtration.

Remark 4.5.3. *Increasing filtrations for chain complexes are defined similarly.*

Remark 4.5.4. *We write the j-th entry of $F_i C^\bullet$ as C^j_i for $i, j \in \mathbb{Z}$. Note that we have $\partial^j_i(C^j_i) \subseteq C^{j+1}_i$ for each $i, j \in \mathbb{Z}$. We can visualize the filtered co-chain complex as follows:*

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \cup & & \cup & & \cup & & \\
\cdots & \longrightarrow & C^{-1}_{-1} & \xrightarrow{\partial^{-1}_{-1}} & C^0_{-1} & \xrightarrow{\partial^0_{-1}} & C^1_{-1} & \longrightarrow & \cdots \\
 & & \cup & & \cup & & \cup & & \\
\cdots & \longrightarrow & C^{-1}_0 & \xrightarrow{\partial^{-1}_0} & C^0_0 & \xrightarrow{\partial^0_0} & C^1_0 & \longrightarrow & \cdots \\
 & & \cup & & \cup & & \cup & & \\
\cdots & \longrightarrow & C^{-1}_1 & \xrightarrow{\partial^{-1}_1} & C^0_1 & \xrightarrow{\partial^0_1} & C^1_1 & \longrightarrow & \cdots \\
 & & \cup & & \cup & & \cup & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

We say that the filtration is exhaustive and separated if for each $j \in \mathbb{Z}$, we have

$$\bigcap_{i\in\mathbb{Z}} C^j_i = 0 \quad \textit{and} \quad \bigcup_{i\in\mathbb{Z}} C^j_i = C^j.$$

The first condition is the exhaustive condition, and the second condition is the separated condition. In other words, the filtration must eventually become arbitrarily small and arbitrarily large at each degree.

From the data of a filtration on a co-chain complex, one constructs a spectral sequence that approximates the cohomology of $C^\bullet$ through successive approximations. Let's first discussion the motivation behind the construction. We let the E_0 page of the spectral sequence be the associated graded co-chain complex. That is[3],

$$E_0^{p,q} \cong G_p C^{p+q} \cong F_p C^{p+q}/F_{p+1}C^{p+q}$$

with induced differential

$$d_0^{p,q} : \frac{F_p C^{p+q}}{F_{p+1}C^{p+q}} \cong E_0^{p,q} \to E_0^{p,q+1} \cong \frac{F_p C^{p+q+1}}{F_{p+1}C^{p+q+1}}$$

induced by the map $\partial_p^{p+q} : F_p C^{p+q} \to F_p C^{p+q+1}$. The map is well-defined because $\partial_p^{p+q}(F_{p+1}C^{p+q}) \subseteq F_{p+1}C^{p+q+1}$. It is clear that these maps compose to zero. We then let the E_1 page denote the cohomology of the associated graded co-chain complex. That is,

$$\begin{aligned} E_1^{p,q} &\cong H^{p+q}(G_p C^\bullet) \\ &= \frac{\ker\left(d_0^{p,q} : E_0^{p,q} \to E_0^{p,q+1}\right)}{\operatorname{im}\left(d_0^{p,q-1} : E_0^{p,q-1} \to E_0^{p,q}\right)} = \frac{\ker\left(d_0^{p,q} : G_p C^{p+q} \to G_p C^{p+q+1}\right)}{\operatorname{im}\left(d_0^{p,q-1} : G_p C^{p+q-1} \to G_p C^{p+q}\right)}. \end{aligned}$$

We think of $E_1^{p,q}$ as a 'first-order approximation' to $H^{p+q}(C^\bullet)$. The question now is how to construct the differential $d_1^{p,q}$? Let's construct $d_1^{p,q}$. Note that a cohomology class $[\alpha] \in E_1^{p,q}$ represents a chain $c \in F_p C^{p+q}$ with differential $\partial_p^{p+q} c \in F_{p+1}C^{p+q+1}$. With this in mind, we define

$$\begin{aligned} d_1^{p,q} : E_1^{p,q} &\to E_1^{p+1,q} \\ [\alpha] &\mapsto [\partial_p^{p+q} c]. \end{aligned}$$

One easily sees that $d_1^{p+1,q} \circ d_1^{p,q} = 0$. So we are justified in defining

$$E_2^{p,q} := \frac{\ker(d_1^{p,q} : E_1^{p,q} \to E_1^{p+1,q})}{\operatorname{im}(d_1^{p-1,q} : E_1^{p-1,q} \to E_1^{p,q})}.$$

We can continue to construct higher-order approximations. Note that a cohomology class $[\alpha] \in E_2^{p,q}$ can be represented by some $[x] \in E_1^{p,q}$ with differential $d_1^{p,q}[x] = 0 \in E_1^{p+1,q}$. Since $d_1^{p,q}[x] = [\partial_p^{p+q} c]$, where $c \in F_p C^{p+q}$ is any chain representing c, we can choose $\partial_p^{p+q} c$ to be the zero element in $\ker(d_0^{p+1,q})$, meaning that $\partial_p^{p+q} c \in F_{p+2}C^{p+q+1}$. This suggests that we can define a map

$$d_2^{p,q} : E_2^{p,q} \to E_2^{p+2,q-1}.$$

Based on what we've seen so far, it seems that elements of an rth-order approximation $E_r^{p,q}$ should ultimately be represented by co-cycles $x \in F_p C^{p+q}$ such that $dx \in F_{p+r}C^{p+q+1}$. This turns out to be exactly the case. For each $n \in \mathbb{Z}$, we have a filtration

$$\cdots \supseteq F_{p-1}C^n \supseteq F_p C^n \supseteq F_{p+1}C^n \supseteq \cdots$$

of the object C_n. We think of elements of C_n further down the filtration as being "closer to zero." The idea of a cohomological spectral sequence of a filtered co-chain complex is to asymptotically approximate the cohomology of $C^\bullet$ by refining co-cycles and co-boundaries through their r-approximations.

(1) Specifically, an r-almost co-cycle is a co-chain whose differential vanishes modulo terms that are r steps lower in the filtration.

[3]Although the choice of C^{p+q} instead of C^q may initially appear unusual, for fixed p the index $p+q$ is merely a shift of q by a constant, and thus poses no problem. The necessity of this choice will become apparent when the spectral sequence is constructed in detail.

(2) An r-almost co-boundary in filtration degree p is a co-cycle that is the differential of a co-chain which may be up to r steps higher in filtration degree.

We now state and prove the desired result.

Proposition 4.5.5. *Every decreasing filtration of a co-chain complex $C^\bullet$ determines a cohomological spectral sequence.*

Remark 4.5.6. *We will see in the proof that the zeroth page of the spectral sequence is the associated graded co-chain complex*

$$G_pC^\bullet = F_pC^\bullet/F_{p+1}C^\bullet,$$

and that the first page is the cohomology of this co-chain complex. Hence, the construction is consistent with the remarks made in Section 4.3.

Proof. Choose the E_0 page of the spectral sequence such that

$$E_0^{p,q} = F_pC^{p+q}/F_{p+1}C^{p+q} := G_pC^{p+q}$$

For $r \geq 0$, we define r-almost (p,q)-co-cycles and r-almost (p,q)-co-boundaries as the following abelian groups:

(1) The abelian group of r-almost (p,q)-co-cycles is defined as

$$\begin{aligned} Z_r^{p,q} &= \left\{c \in F_pC^{p+q} \mid \partial_p^{p+q}(c) \in F_{p+r}C^{p+q+1}\right\}/F_{p+1}C^{p+q} \\ &= \frac{F_pC^{p+q} \cap (\partial_p^{p+q})^{-1}(F_{p+r}C^{p+q+1}) + F_{p+1}C^{p+q}}{F_{p+1}C^{p+q}} := \frac{K_r^{p,q} + F_{p+1}C^{p+q}}{F_{p+1}C^{p+q}} \end{aligned}$$

In other words, $Z_r^{p,q}$ consists of co-chains in F_pC^{p+q} whose co-boundaries lie in $F_{p+r}C^{p+q+1}$ modulo $F_{p+1}C^{p+q}$.

(2) The abelian group of r-almost (p,q)-co-boundaries is defined as

$$\begin{aligned} B_r^{p,q} &= \partial_{p-r+1}^{p+q-1}(F_{p-r+1}C^{p+q-1}) \cap F_pC^{p+q} \\ &= \frac{\partial_{p-r+1}^{p+q-1}(F_{p-r+1}C^{p+q-1}) \cap F_pC^{p+q} + F_{p+1}C^{p+q}}{F_{p+1}C^{p+q}} \\ &= \frac{\partial_{p-r+1}^{p+q-1}(K_{r-1}^{p-r+1,1+r-2}) + F_{p+1}C^{p+q}}{F_{p+1}C^{p+q}} := \frac{I_r^{p,q} + F_{p+1}C^{p+q}}{F_{p+1}C^{p+q}} \end{aligned}$$

In other words, $B_r^{p,q}$ consists of co-chains in F_pC^{p+q} that are in the image of $F_{p-r+1}C^{p+q-1}$ modulo $F_{p+1}C^{p+q}$.

Note that the reason we quotient out by $F^{p+1}C^{p+q}$ in the definitions of $Z_r^{p,q}$ and $B_r^{p,q}$ is that we want to localize our attention to the p-th graded piece of the filtered complex, and avoid interference from deeper levels of the filtration. This allows us to consider approximate co-cycles and approximate co-boundaries in the associated graded co-chain complex. Since the differentials in the co-chain complex compose to zero, note that we have,

$$B_r^{p,q} \subseteq Z_r^{p,q}$$

We can therefore define the r-almost (p,q)-cohomology by

$$E_r^{p,q} = \frac{Z_r^{p,q}}{B_r^{p,q}} \cong \frac{K_r^{p,q} + F_{p+1}C^{p+q}}{I_r^{p,q} + F_{p+1}C^{p+q}}$$

Note that we have a canonical surjective homomorphism:

$$\eta_r^{p,q} : K_r^{p,q} \longrightarrow K_r^{p,q} + F_{p+1}C^{p+q} \longrightarrow \frac{K_r^{p,q} + F_{p+1}C^{p+q}}{I_r^{p,q} + F_{p+1}C^{p+q}} \cong E_r^{p,q}.$$

mapping $x \in K_r^{p,q}$ to $[x + 0]$. Note that the kernel can be identified with $I_r^{p,q} \subseteq K_r^{p,q}$. Moreover, note that ∂_p^{p+q} restricts to a map from $K_r^{p,q}$ to $K_r^{p+r,q-r+1}$. Since $\partial_p^{p+q}(I_r^{p,q}) = 0$, we have a commutative diagram:

$$\begin{tikzcd} K_r^{p,q} \arrow[r, "\partial_p^{p+q}"] \arrow[d, "\eta_r^{p,q}"'] & K_r^{p+r,q-r+1} \arrow[d, "\eta_r^{p+r,q-r+q}"] \\ E_r^{p,q} \arrow[r, dashed, "d_r^{p,q}"'] & E_r^{p+r,q-r+1} \end{tikzcd}$$

It is clear by construction that $d_r^{p+r,q-r+1} \circ d_r^{p,q} = 0$. We now show that

$$E_{r+1}^{p,q} \cong \frac{\ker\left(d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}\right)}{\operatorname{im}\left(d_r^{p-r,q+r-1} : E_r^{p-r,q+r-1} \to E_r^{p,q}\right)}.$$

A quick computation shows that

$$\ker(d_r^{p,q}) \cong \frac{K_{r+1}^{p,q} + F^{p+1}C^{p+q}}{I_r^{p,q} + F^{p+1}C^{p+q}},$$
$$\operatorname{im}(d_r^{p-r,q+r-1}) \cong \frac{I_{r+1}^{p,q} + F^{p+1}C^{p+q}}{I_r^{p,q} + F^{p+1}C^{p+q}}.$$

Therefore, we have

$$\frac{\ker(d_r^{p,q})}{\operatorname{im}(d_r^{p-r,q+r-1})} \cong \frac{(K_{r+1}^{p,q} + F^{p+1}C^{p+q})/(I_r^{p,q} + F^{p+1}C^{p+q})}{(I_{r+1}^{p,q} + F^{p+1}C^{p+q}/(I_r^{p,q} + F^{p+1}C^{p+q})} \cong \frac{K_{r+1}^{p,q} + F_{p+1}C^{p+q}}{I_{r+1}^{p,q} + F_{p+1}C^{p+q}} \cong E_{r+1}^{p,q}$$

It is clear from the definitions that

$$Z_1^{p,q} = \ker\left(G_pC^{p+q} \to G_pC^{p+q+1}\right),$$
$$B_1^{p,q} = \operatorname{im}\left(G_pC^{p+q-1} \to G_pC^{p+q}\right).$$

Hence, $E_1^{p,q} = H^{p+q}(G_pC^\bullet)$. This completes the proof. □

Remark 4.5.7. *A homological spectral sequence associated to an increasing filtration of a chain complex is constructed similarly.*

4.5.1. Bounded Convergence. We discuss the notion of convergence of spectral sequences informally in this section, focusing on the special case of bounded filtrations, where convergence issues do not essentially arise. The idea behind the construction of the spectral sequence is that as r becomes large, the approximate co-cycles and co-boundaries of degree r approach the actual co-cycles and co-boundaries. Therefore, we expect $E_r^{p,q}$ to approach something related to the cohomology of the co-chain complex. There are subtle issues of convergence involved, but we can attempt to identify the 'limiting page' in the special case where the filtration is bounded. For a bounded, exhaustive and separated filtration, for each $l \in \mathbb{Z}$ there exist $m(l) > n(l) \in \mathbb{Z}$ such that

$$F_nC^l = C^l,$$
$$F_mC^l = 0.$$

Fix any $p, q \in \mathbb{Z}$, and choose any $r > \max\{m(p+q+1)-p, p-n(p+q+1)+1, 0\}$. Then,

$$F_{p+r}C^{p+q+1} \subseteq F_m C^{p+q+1} = 0,$$
$$F_{p-r+1}C^{p+q-1} \supseteq F_n C^{p+q-1} = C^{p+q-1}.$$

Therefore, we have

$$Z_r^{p,q} = \frac{F_pC^{p+q} \cap \ker \partial^{p+q} + F_{p+1}C^{p+q}}{F_{p+1}C^{p+q}}, \quad B_r^{p,q} = \frac{F_pC^{p+q} \cap \operatorname{im} \partial^{p+q-1} + F_{p+1}C^{p+q}}{F_{p+1}C^{p+q}}.$$

With these descriptions stated, we obviously have a surjective map

$$F_pH^{p+q}(C^\bullet) \twoheadrightarrow E_r^{p,q}.$$

The kernel of this map will be the cohomology classes $\alpha \in F_pH^{p+q}(C^\bullet)$ represented by a cycle $x \in F_{p+1}C^{p+q}$. That is, the kernel is exactly $F_{p+1}C^{p+q}$. Hence, for $r > \max\{m(p+q+1)-p, p-n(p+q+1)+1, 0\}$ we have the isomorphism

$$G_pH^{p+q}(C^\bullet) \cong E_r^{p,q}.$$

Hence, we see that if the filtration is bounded, then for sufficiently large r, the r-almost (p,q) cohomology coincides with the associated graded cohomology abelian groups. Hence, for the case of a bounded filtration, we say that the "limiting page" of a cohomological spectral sequence, denoted $E_\infty^{p,q}$, satisfies

$$E_\infty^{p,q} \cong G_pH^{p+q}(C^\bullet).$$

We write $E_r^{p,q} \Rightarrow G_p(H^{p+q}) = E_\infty^{p,q}$ and say that the spectral sequence converges weakly. The general case is dealt with by definition through the notion of a convergence of spectral sequences.

Remark 4.5.8. *A similar remark regarding convergence applies to homological spectral sequences associated to an increasing bounded filtration of a chain complex.*

4.5.2. Spectral Sequence of a Double Complex. We now discuss the construction of a cohomological spectral sequence arising from a double complex. Our focus will be on first-quadrant cohomological double complexes, where convergence issues do not arise. The construction of a homological spectral sequence arising from a first-quadrant homological double complex is similar.

Definition 4.5.9. A first quadrant cohomological double complex, $C^{\bullet,\bullet}$, of abelian groups consists of a collection of abelian groups $\{C^{p,q}\}_{(p,q)\in\mathbb{N}^2}$ arranged in a bi-graded grid, together with two differentials:

$$d_{p,q}^{\mathrm{H}} : C^{p,q} \to C^{p+1,q}, \quad d_{p,q}^{\mathrm{V}} : C^{p,q} \to C^{p,q+1}$$

such that the following conditions hold:

$$d_{p+1,q}^H \circ d_{p,q}^H = 0,$$
$$d_{p,q+1}^V \circ d_{p,q}^V = 0,$$
$$d_{p,q+1}^H \circ d_{p,q}^V + d_{p+1,q}^V \circ d_{p,q}^H = 0,$$

for all $p, q \in \mathbb{N}$.

A first quadrant cohomological double complex can be visualized as a grid of abelian groups arranged in the first quadrant, with horizontal and vertical differentials.

$$\begin{tikzcd}
\vdots & \vdots & \vdots & \vdots & \\
C^{0,2} \arrow[u, "d^V_{0,2}"] \arrow[r, "d^H_{0,2}"] & C^{1,2} \arrow[u, "d^V_{1,2}"] \arrow[r, "d^H_{1,2}"] & C^{2,2} \arrow[u, "d^V_{2,2}"] \arrow[r, "d^H_{2,2}"] & C^{3,2} \arrow[u, "d^V_{3,2}"] \arrow[r, "d^H_{3,2}"] & \cdots \\
C^{0,1} \arrow[u, "d^V_{0,1}"] \arrow[r, "d^H_{0,1}"] & C^{1,1} \arrow[u, "d^V_{1,1}"] \arrow[r, "d^H_{1,1}"] & C^{2,1} \arrow[u, "d^V_{2,1}"] \arrow[r, "d^H_{2,1}"] & C^{3,1} \arrow[u, "d^V_{3,1}"] \arrow[r, "d^H_{3,1}"] & \cdots \\
C^{0,0} \arrow[u, "d^V_{0,0}"] \arrow[r, "d^H_{0,0}"] & C^{1,0} \arrow[u, "d^V_{1,0}"] \arrow[r, "d^H_{1,0}"] & C^{2,0} \arrow[u, "d^V_{2,0}"] \arrow[r, "d^H_{2,0}"] & C^{3,0} \arrow[u, "d^V_{3,0}"] \arrow[r, "d^H_{3,0}"] & \cdots
\end{tikzcd}$$

The total differential $d = d^V + d^H$ on the associated total complex $\mathrm{Tot}^\bullet(C^{\bullet,\bullet})$, defined by $\mathrm{Tot}^n(C^\bullet) = \bigoplus_{p+q=n} C^{p,q}$ satisfies $d \circ d = 0$, making $\mathrm{Tot}(C^\bullet)$ a co-chain complex. Each element in $C^{p,q} \subseteq \mathrm{Tot}^n(C^\bullet)$ is mapped, via both the horizontal and vertical differentials of the double complex to the corresponding summands in $(\mathrm{Tot}\, C)^{p+q+1}$.

Remark 4.5.10. *A* first quadrant homological double complex $C_{\bullet,\bullet}$ *of abelian groups can be defined similarly. It consists of a collection of abelian groups* $\{C_{p,q}\}_{(p,q)\in\mathbb{N}^2}$ *arranged in a bi-graded grid, together with two differentials:*

$$d_{\mathrm{H}}^{p,q} : C_{p,q} \to C_{p-1,q}, \quad d_{\mathrm{V}}^{p,q} : C_{p,q} \to C_{p,q-1}$$

satisfying the following conditions for all $p, q \in \mathbb{N}$*:*

$$\begin{aligned}
&d_{\mathrm{H}}^{p-1,q} \circ d_{\mathrm{H}}^{p,q} = 0, \\
&d_{\mathrm{V}}^{p,q-1} \circ d_{\mathrm{V}}^{p,q} = 0, \\
&d_{\mathrm{H}}^{p,q-1} \circ d^{\mathrm{V}}_{p,q} + d_{\mathrm{V}}^{p-1,q} \circ d_{\mathrm{H}}^{p,q} = 0.
\end{aligned}$$

A first quadrant homological double complex can be visualized as a grid of abelian groups arranged in the first quadrant, with horizontal and vertical differentials.

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
C_{0,2} \arrow[d, "d_V^{0,2}"'] & C_{1,2} \arrow[l, "d_H^{1,2}"'] \arrow[d, "d_V^{1,2}"'] & C_{2,2} \arrow[l, "d_H^{2,2}"'] \arrow[d, "d_V^{2,2}"'] & C_{3,2} \arrow[l, "d_H^{3,2}"'] \arrow[d, "d_V^{3,2}"'] & \cdots \arrow[l] \\
C_{0,1} \arrow[d, "d_V^{0,1}"'] & C_{1,1} \arrow[l, "d_H^{1,1}"'] \arrow[d, "d_V^{1,1}"'] & C_{2,1} \arrow[l, "d_H^{2,1}"'] \arrow[d, "d_V^{2,1}"'] & C_{3,1} \arrow[l, "d_H^{3,1}"'] \arrow[d, "d_V^{3,1}"'] & \cdots \arrow[l] \\
C_{0,0} & C_{1,0} \arrow[l, "d_H^{1,0}"'] & C_{2,0} \arrow[l, "d_H^{2,0}"'] & C_{3,0} \arrow[l, "d_H^{3,0}"'] & \cdots \arrow[l]
\end{tikzcd}$$

Given a cohomological double complex, we construct a cohomological spectral sequence by filtering our double complex in two different ways. We first consider the following filtration:

$$(C_{\mathrm{I}}^{i,j})_p = \begin{cases} 0 & \text{if } i < p, \\ C^{i,j} & \text{if } i \geq p. \end{cases}$$

Note that we have a decreasing filtration by columns. The total complexes of these truncations of $C^{\bullet,\bullet}$ give rise to a decreasing, exhaustive, seperated and bounded filtration on the total complex of $C^{\bullet,\bullet}$.

$$F_p \operatorname{Tot}_n^I(C^{\bullet,\bullet}) = \bigoplus_{i \geq p} C^{i,n-i}$$

Using Proposition 4.5.5 and Section 4.5.1, we have the following result:

Proposition 4.5.11. *Consider a first quadrant cohomological double complex, $C^{\bullet,\bullet}$, of abelian groups. There exists a cohomological spectral sequence $E_r^{p,q}$ for $r \geq 0$ such that:*

(1) *The zeroth page is given by the original double complex:*

$$E_0^{p,q} = \frac{F_p \operatorname{Tot}_{p+q}^I(C^{\bullet,\bullet})}{F_{p+1} \operatorname{Tot}_{p+q}^I(C^{\bullet,\bullet})} = C^{p,q}$$

and the differentials $d_0^{p,q} : E_0^{p,q} \to E_0^{p,q+1}$ are the vertical differentials d^V of the double complex

(2) *The first page is given by the cohomology computed from the zeroth page and the differentials $d_1^{p,q} : E_1^{p,q} \to E_1^{p+1,q}$ are naturally induced by the horizontal differentials d^H.*

Moreover, for each $(p,q) \in \mathbb{N}^2$ there exists a $R(p,q)$ such that for $r > R(p,q)$ we have

$$E_r^{p,q} = E_\infty^{p,q} = G_p H^{p+q}(\operatorname{Tot} C^{\bullet,\bullet}).$$

We could easily have used the vertical truncations of the double complex.

$$(C_{\mathrm{I}}^{i,j})_p = \begin{cases} 0 & \text{if } j < p, \\ C^{i,j} & \text{if } j \geq p. \end{cases}$$

Note that we have a decreasing filtration by rows. The total complexes of these truncations of $C^{\bullet,\bullet}$ give rise to a decreasing, exhaustive, seperated and bounded filtration on the total complex of $C^{\bullet,\bullet}$.

$$F_p \operatorname{Tot}_n^{II}(C^{\bullet,\bullet}) = \bigoplus_{j \geq p} C^{n-j,j}$$

Using Proposition 4.5.5 and Section 4.5.1, we have the following result:

Proposition 4.5.12. *Consider a first quadrant cohomological double complex, $C^{\bullet,\bullet}$, of abelian groups. There exists a cohomological spectral sequence $E_r^{p,q}$ for $r \geq 0$ such that:*

(1) *The zeroth page is given by the 'transposed' original double complex:*

$$E_0^{p,q} = \frac{F_p \operatorname{Tot}_{p+q}^{II}(C^{\bullet,\bullet})}{F_{p+1} \operatorname{Tot}_{p+q}^{II}(C^{\bullet,\bullet})} = C^{q,p}$$

and the differentials $d_0^{p,q} : E_0^{p,q} \to E_0^{p,q+1}$ are the induced by the horizontal differentials d^H of the double complex

(2) *The first page is given by the cohomology computed from the zeroth page and the differentials $d_1^{p,q} : E_1^{p,q} \to E_1^{p+1,q}$ are naturally induced by the vertical differentials d^V of the double complex.*

Moreover, for each $(p,q) \in \mathbb{N}^2$ there exists a $R(p,q)$ such that for $r > R(p,q)$ we have

$$E_r^{p,q} = E_\infty^{p,q} = G_p H^{p+q}(\operatorname{Tot} C^{\bullet,\bullet}).$$

Remark 4.5.13. *Of course, we could have derived a homological spectral sequence associated to first-quadrant homological double complexes. We obtain two types of spectral sequences, which we described in words:*

(1) *The first spectral sequence is obtained by filtering columns. The zeroth page is the double complex, and the differentials are the vertical (downward facing) maps from the double complex. The first page is the homology of the first page and the maps are the horizontal (rightward facing) maps induced from the double complex.*
(2) *The first spectral sequence is obtained by filtering rows. The zeroth page is the 'transposed' double complex, and the differentials induced by the horizontal differentials from the double complex. The first page is the homology of the first page and the differentials are the vertical maps induced from the double complex.*

We will freely use the analogous results below.

4.6. Applications

Why all the fuss about homological algebra, and in particular spectral sequences? Their significance stems from the ability to systematically decompose complex computations into more tractable, stepwise analyses. Let us now apply the machinery we have developed.

4.6.1. Diagram Chasing Lemmas. We first establish several diagram-chasing lemmas that play a foundational role in homological algebra.

Proposition 4.6.1. ***(Five Lemma)*** *Consider the following diagram of R-modules:*

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\alpha"'] & B \arrow[r] \arrow[d, "\beta"'] & C \arrow[r] \arrow[d, "\gamma"'] & D \arrow[r, "g"] \arrow[d, "\delta"'] & E \arrow[d, "\epsilon"'] \\
A' \arrow[r, "f'"] & B' \arrow[r] & C' \arrow[r] & D' \arrow[r, "g'"] & E'
\end{tikzcd}$$

be a commutative diagram with exact rows of R-modules. We have the following:

(1) *If α is a surjective homomorphism and β, δ are injective homomorphisms, then γ is an injective homomorphism.*
(2) *If ϵ is an injective homomorphism and β, δ are surjective homomorphisms, then γ is a surjective homomorphism.*

Remark 4.6.2. *We use the homological spectral sequence associated with a first-quadrant homological double complex in the argument below.*

Proof. We only prove (1), noting that the proof of (2) proceeds analogously. To construct the desired double complex, we begin with the given diagram, reflect it appropriately, and adjoin the necessary kernels and cokernels on the left and right. By assigning zero objects to all remaining entries, we obtain a first quadrant homological double complex.

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] \\
0 \arrow[d] & 0 \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] \\
\operatorname{coker} g' \arrow[d] & E \arrow[l] \arrow[d] & D \arrow[l] \arrow[d] & C \arrow[l] \arrow[d] & B \arrow[l] \arrow[d] & A \arrow[l] \arrow[d] & \ker f \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] \\
\operatorname{coker} g'' & E' \arrow[l] & D' \arrow[l] & C' \arrow[l] & B' \arrow[l] & A' \arrow[l] & \ker f' \arrow[l] & 0 \arrow[l]
\end{tikzcd}$$

If we consider the homological spectral sequence arising from filtering the double complex by rows, the E^1-page is computed by taking homology in the horizontal direction. Since the double complex is exact along rows, it follows that all entries on the E^1-page vanish. Consequently, the spectral sequence converges weakly to zero. Similarly, the homological spectral sequence obtained by filtering the double complex by columns also converges weakly to zero. In this case, the E^1 page is obtained by taking the the homology of the double complex in the horizontal direction:

$$* \longleftarrow \ker\varepsilon \longleftarrow \ker\delta \longleftarrow \ker\gamma \longleftarrow \ker\beta \longleftarrow \ker\alpha \longleftarrow *$$

$$* \longleftarrow \operatorname{coker}\varepsilon \longleftarrow \operatorname{coker}\delta \longleftarrow \operatorname{coker}\gamma \longleftarrow \operatorname{coker}\beta \longleftarrow \operatorname{coker}\alpha \longleftarrow *$$

By assumption $\ker\delta = \ker\beta = \operatorname{coker}\alpha = 0$. By taking homology once more, we arrive at the E^2 page:

$$\begin{tikzcd}
* & * & * & \ker\gamma & * & * & * \\
* & * & * \arrow[ull] & * \arrow[ull] & * \arrow[ull] & 0 \arrow[ull] & * \arrow[ull]
\end{tikzcd}$$

As the spectral sequence converges weakly to 0, we know that on the E^∞ page, no entry can remain. But this means that $\ker\gamma$ must vanish, since otherwise it could never disappear on the subsequent pages. This proves the claim. □

We now prove the Snake Lemma using homological spectral sequences. The Snake Lemma is a fundamental result in homological algebra, playing a crucial role in the construction of long exact sequences that arise naturally from short exact sequences of chain complexes.

Proposition 4.6.3. ***(Snake Lemma)*** *Consider the following diagram of R-modules:*

$$\begin{tikzcd}
 & A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[r, "g"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' &
\end{tikzcd}$$

Then there is an exact sequence of R-modules:

$$\ker\alpha \longrightarrow \ker\beta \longrightarrow \ker\gamma \overset{\delta}{\dashrightarrow} \operatorname{coker}\alpha \longrightarrow \operatorname{coker}\beta \longrightarrow \operatorname{coker}\gamma$$

The map δ is called the connecting homomorphism.

Proof. The proof proceeds analogously to that of Proposition 4.6.1. As in that case, we construct a first quadrant homological double complex by adjoining kernels and cokernels to the given diagram and assigning zero objects to the remaining entries.

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] \\
0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] \\
0 \arrow[d] & 0 \arrow[l] \arrow[d] & C \arrow[l] \arrow[d] & B \arrow[l] \arrow[d] & A \arrow[l] \arrow[d] & \ker f \arrow[l] \arrow[d] & 0 \arrow[l] \arrow[d] \\
0 & \operatorname{coker} g' \arrow[l] & C' \arrow[l] & B' \arrow[l] & A' \arrow[l] & 0 \arrow[l] & 0 \arrow[l]
\end{tikzcd}$$

If we consider the homological spectral sequence obtained by filtering the double complex by rows, the E^1-page is computed by taking homology in the horizontal direction. Since the double complex is exact along rows, all horizontal homology R-modules should vanish, and thus the E^1-page consists entirely of zero objects. Consequently, the spectral sequence converges weakly to zero. Similarly, the homological spectral sequence obtained by filtering by columns also converges weakly to 0. For this spectral sequence, the E^1 page is obtained by taking the the homology of the double complex in the horizontal direction:

$$0 \longleftarrow \operatorname{coker}\bar{g} \longleftarrow \ker\gamma \xleftarrow[\bar{g}]{} \ker\beta \xleftarrow[\bar{f}]{} \ker\alpha$$

$$\operatorname{coker}\gamma \xleftarrow[\bar{g}']{} \operatorname{coker}\beta \xleftarrow[\bar{f}']{} \operatorname{coker}\alpha \longleftarrow \ker\bar{f}' \longleftarrow 0$$

The maps shown above are induced by the morphisms in the original commutative diagram. We show that we have exactness at $\ker\beta$ and $\operatorname{coker}\beta$.

(1) Note that we have $\ker\bar{g} = \ker g \cap \ker\beta$. By exactness of the original diagram, we have $\ker g = \operatorname{im} f$. Hence, we have

$$\begin{aligned}\ker g \cap \ker\beta &= \operatorname{im} f \cap \ker\beta \\ &= f(f^{-1}(\ker\beta)) \\ &= f(\ker(\beta\circ f)) \\ &= f(\ker(f'\circ\alpha)) \\ &= f(\ker\alpha)\end{aligned}$$

The last equality follows since f' is injective. Hence, we have $\ker\bar{g} = \operatorname{im}\bar{f}$.

(2) By exactness of the original diagram, we have $\ker g' = \operatorname{im} f'$. Note that we have

$$\begin{aligned}\frac{(g')^{-1}(\operatorname{im}\gamma)}{\operatorname{im}\beta} &= \frac{(g')^{-1}(\operatorname{im}(\gamma\circ g))}{\operatorname{im}\beta} \\ &= \frac{(g')^{-1}(\operatorname{im}(g'\circ\beta))}{\operatorname{im}\beta} \\ &= \frac{\operatorname{im}\beta + \ker g'}{\operatorname{im}\beta} \\ &= \frac{\operatorname{im}\beta + \operatorname{im} f'}{\operatorname{im}\beta}\end{aligned}$$

The first equality follows since g is surjective, and the third equality follows from exactness at B'. Hence, $\ker\bar{g}' = \operatorname{im} f'$.

We take homology once more to examine the E^2 page.

$$\begin{tikzcd}
0 & \operatorname{coker}\bar{g} & 0 & 0 & 0 \\
0 & 0 & 0 \arrow[ull] & \ker\bar{f}' \arrow[ull] & 0 \arrow[ull]
\end{tikzcd}$$

Since all entries must vanish on the E^∞ page, the one remaining map must necessarily be an isomorphism. By inverting this isomorphism, we obtain a connecting homomorphism:

$$\begin{tikzcd}
\ker\alpha \arrow[r, "\bar{f}"] & \ker\beta \arrow[r, "\bar{g}"] & \ker\gamma \arrow[r, dashed, "\delta"] \arrow[d, "\pi"] & \operatorname{coker}\alpha \arrow[r, "\bar{f}'"] & \operatorname{coker}\beta \arrow[r, "\bar{g}'"] & \operatorname{coker}\gamma \\
 & & \operatorname{coker}\bar{g} \arrow[r, "\cong"] \arrow[d] & \ker\bar{f}' \arrow[u] & & \\
 & & 0 & 0 \arrow[u] & &
\end{tikzcd}$$

Let's show exactness at $\ker\gamma$. Using the commutative square, we have that $\ker\gamma = \ker\pi$. But we also have that $\ker\pi = \operatorname{im}\bar{g}$. Hence, $\ker\gamma = \operatorname{im}\bar{g}$. A similar argument shows that the sequence is exact at $\operatorname{coker}\alpha$. □

We now provide a proof of the Braid Lemma using a diagram chasing argument. While spectral sequences are powerful tools in homological algebra, I am not aware of a standard proof of the Braid Lemma that relies on them. As we shall see, the Braid Lemma plays a crucial role in constructing the long exact sequence associated with triples of topological spaces.

Proposition 4.6.4. ***(Braid Lemma)*** *Suppose three long exact sequences and a chain complex we have a commutative diagram. Then the chain complex is also a long exact sequence*

$$\begin{tikzcd}
\cdots \arrow[rr] \arrow[dr] & & A \arrow[rr, "g_1"] \arrow[dr, "f_1"] & & D \arrow[rr, "h_3"] \arrow[dr, "g_2"] & & G \arrow[rr, "j_4"] \arrow[dr, "h_4"] & & J \arrow[dr] & \\
\cdots & O \arrow[ur, "g_0"] \arrow[dr, "j_0"] & & C \arrow[ur, "h_2"] \arrow[dr, "f_2"] & & F \arrow[ur, "j_3"] \arrow[dr, "g_3"] & & I \arrow[ur, "f_5"] \arrow[dr, "h_5"] & & \cdots \\
\cdots \arrow[rr] \arrow[ur] & & B \arrow[rr, "j_1"] \arrow[ur, "h_1"] & & E \arrow[rr, "f_3"] \arrow[ur, "j_2"] & & H \arrow[rr, "g_4"] \arrow[ur, "f_4"] & & K \arrow[ur] &
\end{tikzcd}$$

Proof. WLOG, assume that the f maps describe the chain complex. By symmetry, it suffices to show exactness at C, E and H:

(1) $\ker(f_2) \subseteq \operatorname{im}(f_1)$: Let $x \in \ker(f_2)$. Then $0 = f_2(x) = j_2 f_2(x) = g_2 h_2(x)$ by commutativity. It follows that $h_2(x) \in \ker(g_2) = \operatorname{im}(g_1)$. So $\exists x_1 \in A$ such that $g_1(x_1) = h_2(x)$. By commutativity, $g_1(x_1) = h_2 f_1(x_1)$. So we have that $0 = g_1(x_1) - h_2(x) = h_2(f_1(x_1) - x)$. Let $x_2 := f_1(x_1) - x \in \ker(h_2) = \operatorname{im}(h_1)$. Then $\exists x_3 \in B$ such that $h_1(x_3) = x_2$. Note that
$$j_1(x_3) = f_2 h_1(x_3) = f_2(x_2) = f_2(f_1(x_1) - x) = 0,$$
where the last equality follows from $f_2 \circ f_1 = 0$ and $f_2(x) = 0$. We therefore have that $x_3 \in \ker(j_1) = \operatorname{im}(j_0)$. So there exists $x_4 \in O$ such that $j_0(x_4) = x_3$. Consider$g_0(x_4)$. It satisfies
$$f_1 g_0(x_4) = h_1 j_0(x_4) = h_1(x_3) = x_2 = f_1(x_1) - x$$
Therefore, we have $x = f_1(x_1 - g_0(x_4))$. This shows that $x \in \operatorname{im}(f_1)$.

$$\begin{tikzcd}
& g_0(x_4), x_1 \arrow[rr, mapsto, "g_1"] \arrow[dr, "f_1"] & & h_2(x) \arrow[dr, mapsto, "g_2"] & \\
x_4 \arrow[ur, "g_0"] \arrow[dr, "j_0"] & & x_2, x \arrow[ur, mapsto, "h_2"] \arrow[dr, mapsto, "f_2"] & & 0 \\
& x_3 \arrow[rr, "j_1"] \arrow[ur, mapsto, "h_1"] & & 0 \arrow[ur, mapsto, "j_2"] &
\end{tikzcd}$$

(2) $\ker(f_3) \subseteq \operatorname{im}(f_2)$: Let $x \in E$ be such that $f_3(x) = 0$. By commutativity, $g_3 j_2(x) = 0$, so $j_2(x) \in \ker(g_3) = \operatorname{im}(g_2)$. Then $\exists x_1 \in D$ such that $g_2(x_1) = j_2(x)$. It satisfies $h_3(x_1) = j_3 g_2(x_1) = j_3 j_2(x) = 0$, as (j_i) is a chain complex. So $x_1 \in \ker(h_3) = \operatorname{im}(h_2)$. Therefore, there exists $x_2 \in C$ such that $h_2(x_2) = x_1$. This element is such that $j_2 f_2(x_2) = g_2 h_2(x_2) = g_2(x_1) = j_2(x)$. We therefore have $j_2(f_2(x_2) - x) = 0$. Let $x_3 := f_2(x_2) - x$. Then $x_3 \in \ker(j_2) = \operatorname{im}(j_1)$. Let $x_4 \in B$ be such that $j_1(x_4) = x_3$. x_4 is such that $f_2 h_1(x_4) = j_1(x_4) = x_3 = f_2(x_2) - x$. Finally, we see that $x = f_2(x_2 - h_1(x_4))$, so $x \in \operatorname{im}(f_2)$ as required.

$$\begin{tikzcd}
& & x_1 \arrow[rr, mapsto, "h_3"] \arrow[dr, mapsto, "g_2"] & & 0 \\
& h_1(x_4), x_2 \arrow[ur, mapsto, "h_2"] \arrow[dr, mapsto, "f_2"] & & 0, j_2(x) \arrow[ur, mapsto, "j_3"] \arrow[dr, mapsto, "g_3"] & \\
x_4 \arrow[ur, mapsto] \arrow[rr, mapsto, "j_1"] & & x_3, x \arrow[ur, mapsto, "j_2"] \arrow[rr, mapsto, "f_3"] & & 0
\end{tikzcd}$$

(3) $\ker(f_4) \subseteq \operatorname{im}(f_3)$: Let $x \in H$ be such that $f_4(x) = 0$. Then $0 = h_5 f_4(x) = g_4(x)$. So $x \in \ker(g_4) = \operatorname{im}(g_3)$. Let $x_1 \in F$ be such that $g_3(x_1) = x$. Then $h_4 j_3(x_1) = f_4 g_3(x_1) = f_4(x) = 0$. So $j_3(x_1) \in \ker(h_4) = \operatorname{im}(h_3)$. Let $x_2 \in D$ be such that $h_3(x_2) = j_3(x_1)$. Then $j_3(x_1) = j_2 g_2(x_2)$, such that $x_3 := g_2(x_2) - x_1 \in \ker(j_3) = \operatorname{im}(j_2)$. Let $x_4 \in E$ be such that $j_2(x_4) = x_3$. Then $f_3(x_4) = g_3 j_2(x_4) = g_3(x_3) = g_3(g_2(x_2) - x_1) = -g_3(x_1) = -x$. Therefore, $x = f_3(-x_4)$, and $x \in \operatorname{im}(f_3)$ as required.

$$\begin{tikzcd}
x_2 \arrow[rr, mapsto, "h_3"] \arrow[dr, mapsto, "g_2"] & & j_3(x_1) \arrow[dr, mapsto, "h_4"] & & \\
& x_3, x_1 \arrow[ur, mapsto, "j_3"] \arrow[dr, mapsto, "g_3"] & & 0 \arrow[dr, mapsto, "h_5"] & \\
x_4 \arrow[ur, mapsto, "j_2"] \arrow[rr, mapsto, "f_3"] & & -x, x \arrow[ur, mapsto, "f_4"] \arrow[rr, mapsto, "g_4"] & & 0
\end{tikzcd}$$

This completes the proof. □

4.6.2. Short Exact Sequence via Spectral Sequences. Spectral sequences often encode a wealth of homological information, organizing it across multiple pages of approximations. Even when the data appears sparse, the structure of a spectral sequence can be leveraged to extract compact and meaningful results. We show how they can also be employed to derive short exact sequences that are both non-trivial and extremely useful in computations.

Proposition 4.6.5. *(Two Column Sequence) Let $\{E_r^{p,q}\}_{r \geq 1}$ be a cohomological spectral sequence associated to a decreasing, exhaustive and separated filtration. Assume that $E_2^{p,q} = 0$ unless $p = 0, 1$. For each $n \in \mathbb{Z}$, we have a short exact sequence:*

$$0 \longrightarrow E_2^{0,n} \longrightarrow M_n \longrightarrow E_2^{-1,n+1} \longrightarrow 0.$$

Proof. The E_2 pages looks like the following:

$$\begin{tikzcd}
0 \arrow[drr] & 0 \arrow[drr] & E_2^{0,1} \arrow[drr] & E_2^{1,1} \arrow[drr] & 0 & 0 \\
0 \arrow[drr] & 0 \arrow[drr] & E_2^{0,0} \arrow[drr] & E_2^{1,0} \arrow[drr] & 0 & 0 \\
0 & 0 & E_2^{0,-1} & E_2^{1,-1} & 0 & 0
\end{tikzcd}$$

Hence, we see that $E_2^{p,q} = E_\infty^{p,q}$. Assume that the spectral sequence converges weakly to $\{M_n\}_{n \in \mathbb{Z}}$. Hence, we have

$$E_2^{p,q} = E_\infty^{p,q} \cong \frac{F_p M_{p+q}}{F_{p+1} M_{p+q}}$$

If $p \neq 0, 1$, we get

$$0 = E_2^{p,q} = \frac{F_p M_{p+q}}{F_{p+1} M_{p+q}},$$

which tells us $F_p H_{p+q} = F_{p+1} H_{p+q}$ for all $q \in \mathbb{Z}$ such that $p \neq 0, 1$. Therefore the filtration looks like

$$\cdots = F_{-2} M_n = F_{-1} M_n \supseteq F_0 M_n \supseteq F_1 M_n = F_2 M_n = \cdots$$

Since the filtration is assumed to be exhaustive and separated, we have that

$$\begin{aligned} F_{-1} M_n &= F_{-2} M_n = \cdots = M_n, \\ F_1 M_n &= F_2 M_n = \cdots = 0. \end{aligned}$$

For $p = 0$, we notice that

$$E_2^{0,n} = E_\infty^{0,n} \cong \frac{F_0 M_n}{F_1 M_n} = F_0 M_n.$$

For $p = 1$, we get

$$E_2^{-1,n+1} = E_\infty^{1,n-1} \cong \frac{F_{-1} M_n}{F_0 H_n} = \frac{M_n}{F_0 M_n}.$$

For each $n \in \mathbb{Z}$, the short exact sequence

$$0 \longrightarrow F_0 M_n \longrightarrow M_n \longrightarrow \frac{M_n}{F_0 M_n} \longrightarrow 0$$

turns into the short exact sequence

$$0 \longrightarrow E_2^{0,n} \longrightarrow M_n \longrightarrow E_2^{-1,n+1} \longrightarrow 0.$$

$\square$

Remark 4.6.6. *If we had a homological spectral sequence, the analogous statement would be that for each $n \in \mathbb{Z}$ there is a short exact sequence:*

$$0 \longrightarrow E^2_{0,n} \longrightarrow M_n \longrightarrow E^2_{1,n-1} \longrightarrow 0$$

4.6.3. Long Exact Sequence in Homology.

We now employ an argument analogous to that of Proposition 4.6.3 to demonstrate that any short exact sequence of chain complexes gives rise to a long exact sequence in homology. As this reasoning is entirely algebraic in nature, we carry out the proof in the general algebraic setting.

Proposition 4.6.7. ***(Long Exact Sequence in Homology)*** *Consider a short exact sequence in $\mathsf{Chain}_{\mathsf{Mod}_R}$:*

$$0_\bullet \to A_\bullet \xrightarrow{i_\bullet} B_\bullet \xrightarrow{j_\bullet} C_\bullet \to 0_\bullet$$

For each $n \geq 1$, there exist connecting morphisms $\delta_n : H_n(C_\bullet) \to H_{n-1}(A_\bullet)$ such that there is a long exact sequence in homology:

$$\begin{tikzcd} \cdots \arrow[r] & H_{n+1}(B_\bullet) \arrow[r] & H_{n+1}(C_\bullet) \arrow[dll, "\delta_{n+1}"] \\ H_n(A_\bullet) \arrow[r] & H_n(B_\bullet) \arrow[r] & H_n(C_\bullet) \arrow[dll, "\delta_n"] \\ H_{n-1}(A_\bullet) \arrow[r] & H_{n-1}(B_\bullet) \arrow[r] & \cdots \end{tikzcd}$$

In fact, the above construction defines a functor from $\mathsf{Chain}^{\mathsf{Exact}}_{\mathsf{Mod}_R}$ to $\mathsf{Chain}^{\mathsf{Long}}_{\mathsf{Mod}_R}$, the category of long exact sequences of abelian groups.

Proof. (Sketch) Note that a short exact sequence of chain complexes naturally gives rise to a first quadrant homological double complex: the chain complexes are arranged in rows, with horizontal maps given by the differentials within each complex and vertical maps given by the maps from the short exact sequence at each degree. The resulting double complex lies in the first quadrant because the indices of the R-modules in each chain complex are drawn from the natural numbers. The exactness at each degree ensures the resulting diagram satisfies the conditions for forming a double complex. The short exact sequence of chain complexes can be drawn more explicitly as:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots & A_{n-1} \arrow[l] \arrow[d, "i_{n+1}"] & A_n \arrow[l, "\partial_n"'] \arrow[d, "i_n"] & A_{n+1} \arrow[l, "\partial_{n+1}"'] \arrow[d, "i_{n-1}"] & \cdots \arrow[l] \\
\cdots & B_{n-1} \arrow[l] \arrow[d, "i_{n+1}"] & B_n \arrow[l, "\partial'_n"'] \arrow[d, "i_n"] & B_{n+1} \arrow[l, "\partial'_{n+1}"'] \arrow[d, "i_{n-1}"] & \cdots \arrow[l] \\
\cdots & C_{n-1} \arrow[l] \arrow[d, "i_{n+1}"] & C_n \arrow[l, "\partial''_n"'] \arrow[d, "i_n"] & C_{n+1} \arrow[l, "\partial''_{n+1}"'] \arrow[d, "i_{n-1}"] & \cdots \arrow[l] \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

If we consider the homological spectral sequence obtained by filtering the double complex by rows, the E^1-page is computed by taking homology in the horizontal direction. Since the double complex is exact along rows, all horizontal homology R-modules should vanish, and thus the E^1-page consists entirely of zero objects. Consequently, the spectral sequence converges weakly to zero. Similarly, the homological spectral sequence obtained by filtering by columns also converges weakly to 0. For this spectral sequence, the E^1 page is obtained by taking the the homology of the double complex in the horizontal direction:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots & H_{n-1}(A) \arrow[l] \arrow[d] & H_n(A) \arrow[l, "\bar{\partial}_n"'] \arrow[d] & H_{n+1}(A) \arrow[l, "\bar{\partial}_{n+1}"'] \arrow[d] & \cdots \arrow[l] \\
\cdots & H_{n-1}(B) \arrow[l] \arrow[d] & H_n(B) \arrow[l, "\bar{\partial}'_n"'] \arrow[d] & H_{n+1}(B) \arrow[l, "\bar{\partial}'_{n+1}"'] \arrow[d] & \cdots \arrow[l] \\
\cdots & H_{n-1}(C) \arrow[l] \arrow[d] & H_n(C) \arrow[l, "\bar{\partial}''_n"'] \arrow[d] & H_{n+1}(C) \arrow[l, "\bar{\partial}''_{n+1}"'] \arrow[d] & \cdots \arrow[l] \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

Let us focus on the sub-diagram involving only the indices $n-1$ and n. Upon rotating the sub-diagram, the resulting sub-diagram is as follows.

$$\begin{tikzcd}
0 \arrow[r] & H_n(A) \arrow[r] \arrow[d, "\partial_n"] & H_n(B) \arrow[r] \arrow[d, "\partial'_n"] & H_n(C) \arrow[r] \arrow[d, "\partial''_n"] & 0 \\
0 \arrow[r] & H_{n-1}(A) \arrow[r] & H_{n-1}(B) \arrow[r] & H_{n-1}(C) \arrow[r] & 0
\end{tikzcd}$$

By applying Proposition 4.6.3, we obtain the following long exact sequence:

$$H_n(A) \to H_n(B) \to H_n(C) \xrightarrow{\delta_n} H_{n-1}(A) \to H_{n-1}(B) \to H_{n-1}(C)$$

By assembling these short exact sequences via the connecting homomorphisms, we obtain the desired long exact sequence in homology. In fact, we can explicitly describe the connecting morphism δ_n in this case, and the construction given below can be shown to be compatible with the abstract existence of the connecting homomorphism in Proposition 4.6.3. The connecting morphisms δ_n are constructed as follows: let $c \in C_n$ be a cycle representative for $[\alpha] \in H_n(C)$. Then, since j_n is surjective, there exists $b \in B_n$ such that $c = j_n(b)$. Therefore, we have that $\partial'_n(b) \in B_{n-1}$. By the commutativity of the diagram, we know that

$$j_{n-1}(\partial''_n(b)) = \partial''_n(j_{n-1}(b)) = \partial''_n(c) = 0,$$

since c is a cycle. Therefore, $\partial'_n(b) \in \ker j_{n-1} = \operatorname{im} i_{n-1}$. So, there exists a (unique, since i_{n-1} is injective) $a \in A_{n-1}$ with $\partial'_n(b) = i_{n-1}(a)$. We show that a is a cycle. Note that

$$i_{n-2}(\partial_{n-1}(a)) = \partial'_{n-1}(i_{n-1}(a)) = \partial'_{n-1}(\partial'_n(b)) = 0$$

Since i_{n-2} is injective, this implies that $\partial_{n-1}(a) = 0$. Finally, we define $\delta_n([\alpha]) = [a] \in H_{n-1}(A)$. We have to show that this assignment is independent of all choices.

(1) Suppose we choose $b' \in B_n$ such that $j_n(b') = c$. Then, $b' - b \in \ker j_n = \operatorname{im} i_n$. So, there exists $a' \in A_n$ such that $b' - b = i_n(a')$. Therefore,

$$\begin{aligned}\partial'_n(b') &= \partial'_n(b) + \partial'_n(i_n(a')) \\ &= \partial(b) + i_{n-1}(\partial_n(a')) \\ &= i_{n-1}(a) + i_{n-1}(\partial_n(a')) \\ &= i_{n-1}(a + \partial_n(a')).\end{aligned}$$

So we see that changing b to b' amounts to changing a by a homologous cycle $a + \partial_n(a')$.

(2) If instead of c we use $c + \partial''_{n+1}(c')$ for some $c' \in C_{n+1}$. But then, $c' = j_{n+1}(b')$ for some $b' \in B_{n+1}$. So,

$$\begin{aligned}c + \partial''_{n+1}(c') &= c + \partial''_{n+1}(j_{n+1}(b')) \\ &= c + j_n(\partial'_{n+1}(b')) \\ &= j_n(b + \partial'_{n+1}(b'))\end{aligned}$$

Then b will be replaced by $b + \partial'_{n+1}(b')$, which leaves $\partial'_n(b)$ unchanged, hence a unchanged.

We now show that the above construction defines a functor from $\mathsf{Chain}^{\mathsf{Exact}}_{\mathsf{Mod}_R}$ to $\mathsf{Mod}^{\mathsf{Long}}_R$. Consider the following diagram in $\mathsf{Chain}^{\mathsf{Exact}}_{\mathsf{Mod}_R}$:

$$\begin{tikzcd} 0_\bullet \arrow[r] & A_\bullet \arrow[r, "i_\bullet"] \arrow[d, "f_\bullet"] & B_\bullet \arrow[r, "j_\bullet"] \arrow[d, "g_\bullet"] & C_\bullet \arrow[r] \arrow[d, "h_\bullet"] & 0_\bullet \\ 0_\bullet \arrow[r] & A'_\bullet \arrow[r, "i'_\bullet"] & B'_\bullet \arrow[r, "j'_\bullet"] & C'_\bullet \arrow[r] & 0_\bullet \end{tikzcd}$$

We show that induces the following commutative diagram.

$$\begin{tikzcd} \cdots \arrow[r] & H_n(A) \arrow[r, "H_n(i)"] \arrow[d, "H_n(f_n)"] & H_n(B) \arrow[r, "H_n(j)"] \arrow[d, "H_n(g_n)"] & H_n(C) \arrow[r, "\delta_n"] \arrow[d, "H_n(h_n)"] & H_{n-1}(A) \arrow[r, "H_{n-1}(i_{n-1})"] \arrow[d, "H_{n-1}(f_{n-1})"] & H_{n-1}(B) \arrow[r] \arrow[d, "H_{n-1}(g_{n-1})"] & \cdots \\ \cdots \arrow[r] & H_n(A') \arrow[r, "H_n(i'_n)"] & H_n(B') \arrow[r, "H_n(j'_n)"] & H_n(C') \arrow[r, "\delta'_n"] & H_{n-1}(A') \arrow[r, "H_n(i'_{n-1})"] & H_{n-1}(B') \arrow[r] & \cdots \end{tikzcd}$$

The commutativity of the first two squares and the last square is obvious since n-th homology is a functor. It suffices to check that the the diagram

$$\begin{tikzcd} H_n(C) \arrow[r, "\delta_n"] \arrow[d, "H_n(h_n)"'] & H_{n-1}(A) \arrow[d, "H_{n-1}(f_{n-1})"] \\ H_n(C') \arrow[r, "\delta'_n"] & H_{n-1}(A') \end{tikzcd}$$

is commutative. Recall that the map $\delta_n : H_n(C) \to H_{n-1}(A)$ was defined by $\delta_n[c] = [a]$ where $c = j_n(b)$ and $i_{n-1}(a) = \partial'_n b$. Consider $h_n(c) \in C'_n$. Note that

$$h_n(c) = h_n(j_n(b)) = j'_n(g_n(b))$$
$$i'_{n-1}(f_{n-1}(a)) = g_{n-1}(i_{n-1}(a)) = g_{n-1}(\partial'_n(b)) = d'_n(g_n(b)).$$

Here d'_n is the map from B'_n to B'_{n-1}. Hence,

$$[\delta'_n h_n(c))] = [f_{n-1}(a)] = [f_{n-1}\delta_n(c)]$$

This shows that the construction defines a functor from the $\mathsf{Chain}^{\mathsf{Exact}}_{\mathsf{Mod}_R}$ to $\mathsf{Mod}^{\mathsf{Long}}_R$. $\square$

4.7. Convergence of a Spectral Sequence

We have seen in Section 4.5 that a bounded descending filtration naturally induces a *convergent* cohomological spectral sequence, in the sense that the entries $E_r^{p,q}$ stabilize for sufficiently large r, as a function of (p, q), allowing us to define the limiting page of a spectral sequence. We now turn to the question of convergence for a general spectral sequence. We first need to define the notion of the limiting page of a general spectral sequence. If $\{E_r^{p,q}, d_r^{p,q}\}_{r\in\mathbb{N}}$ is a a cohomological spectral sequence, we have a tower of R-submodules

$$B_0^{p,q} \subseteq B_1^{p,q} \subseteq B_2^{p,q} \subseteq \cdots \subseteq \cdots \subseteq Z_2^{p,q} \subseteq Z_1^{p,q} \subseteq Z_0^{p,q} \tag{3}$$

Here $Z_r^{p,q}$, $B_r^{p,q}$ are defined as in Section 4.5. Define

$$Z_\infty^{p,q} = \bigcap_{r=1}^{\infty} Z_r^{p,q}, \quad B_\infty^{p,q} = \bigcup_{r=1}^{\infty} B_r^{p,q}$$

Note that $B_r^{p,q} \subseteq Z_\infty^{p,q}$ for each $(p, q) \in \mathbb{Z}^2$. Clearly, the construction generalizes to the case where we have a cohomological spectral sequence starting on the r_0-th page, for some $r_0 \in \mathbb{N}$. This allows us to define a potential candidate for the limit of a cohomological spectral sequence.

Definition 4.7.1. Let $r_0 \in \mathbb{N}$, and let $\{E_r^{p,q}, d_r^{p,q}\}_{r\geq r_0}$ be a cohomological spectral sequence of R-modules. The E_∞ page of $\{E_r^{p,q}, d_r^{p,q}\}_{r\geq r_0}$ is defined such that

$$E_{p,q}^{\infty} = \frac{Z_{p,q}^{\infty}}{B_{p,q}^{\infty}}$$

In the specific instances examined in Section 4.5, the spectral sequence was shown to converge to the associated graded cohomology of a co-chain complex. This observation motivates the following definition.

Definition 4.7.2. Let $r_0 \in \mathbb{N}$. Let $\{M_n\}_{n\in\mathbb{Z}}$ be a family of R-modules, and let $\{E_r^{p,q}, d_r^{p,q}\}_{r\geq r_0}$ be a cohomological spectral sequence of R-modules.

(1) We say that $\{E_r^{p,q}, d_r^{p,q}\}_{r\geq r_0}$ converges weakly to $\{M_n\}_{n\in\mathbb{Z}}$ if there exists a decreasing exhaustive filtration

$$\cdots \supseteq F_{p-1}M_n \supseteq F_pM_n \supseteq F_{p+1}M_n \supseteq \cdots$$

for each $n \in \mathbb{Z}$ and furthermore, there exist isomorphisms

$$E_\infty^{p,q} \cong G_p(M_{p+q}) := \frac{F_p M_{p+q}}{F_{p+1} M_{p+q}}.$$

We write

$$E_r^{p,q} \Rightarrow G_p(M_{p+q})$$

(2) We say that $\{E_r^{p,q}, d_r^{p,q}\}_{r \geq r_0}$ approaches $\{M_n\}_{n\in\mathbb{Z}}$ if the filtration in (1) is exhaustive and separated.

(3) We say that $\{E_r^{p,q}, d_r^{p,q}\}_{r \geq r_0}$ converges strongly to $\{M_n\}_{n\in\mathbb{Z}}$ if it approaches $\{M_n\}_{n\in\mathbb{Z}}$ and

$$M_n = \varprojlim_{p\in\mathbb{Z}} (M_n / F_p M_n).$$

CHAPTER 5

Singular Homology

Singular homology is a powerful tool in algebraic topology that assigns a sequence of abelian groups to a topological space by studying continuous maps from standard simplices into the space. It enjoys wide applicability due to its functorial properties and satisfies the Eilenberg–Steenrod axioms, making it a foundational invariant in the subject. References include [Hat02; Lee10; May99].

5.1. Definitions

Singular homology is difficult to compute, but singular homology has nice theoretical properties which allows us to prove a host of properties about a homology theory. It can be checked that simplicial homology and singular homology is coincide as we will do later on. Hence, simplicial homology provides a computational tool to compute homology, and singular homology provides a theoretical tool to study homology theoretically.

Definition 5.1.1. Let X be a topological space. A singular n-simplex is a continuous map $\sigma : \Delta^n \to X$. The set of all such continuous maps is denoted as $\mathrm{Hom}(\Delta^n, X)$.

Example 5.1.2. Since Δ^0 is a point, a 0-simplex in X is simply a point in X. Since Δ^1 is a closed interval, a 1-simplex is a path in X. Since Δ^1 is a solid triangle, 2-simplex is the image of a solid triangle.

Remark 5.1.3. *The phrase 'singular' is used here to express the idea that σ need not be an embedding or a homeomorphism but can have 'singularities' where its image does not look at all like a simplex. All that is required is that σ be continuous.*

We can express our constructions categorically. Consider the category Δ, whose objects are the finite ordered sets

$$[n] = \{0 < 1 < \cdots < n\}$$

for each integer $n \geq 0$, and whose morphisms from $[n]$ to $[m]$ are given by strictly increasing functions. The assignment of the n-simplex as a topological space defines a functor

$$\Delta_\bullet : \Delta \longrightarrow \mathsf{Top}$$

that sends $[n]$ to Δ^n. A morphism $\alpha : [n] \to [m]$ in Δ is sent to the continuous injection

$$\begin{aligned} \Delta^n &\to \Delta^m \\ (t_0, \ldots, t_n) &\mapsto (s_0, \ldots, s_m), \end{aligned}$$

where $s_{\alpha(i)} = t_i$ for each $i = 0, \ldots, n$, and the remaining coordinates are set to zero. For example, the inclusion $[n-1] \to [n]$ which skips the i-th entry induces the face inclusion

$$\begin{aligned} d_i^n : \Delta^{n-1} &\to \Delta^n, \\ (t_0, \ldots, t_{n-1}) &\mapsto (t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_{n-1}), \end{aligned}$$

Note that we obtain the sets $\mathrm{Hom}(\Delta^n, X)$ by considering continuous maps $\Delta^n \to X$, and we have a collection of functions

$$\overline{d}_i^n : \mathrm{Hom}(\Delta^n, X) \to \mathrm{Hom}(\Delta^{n-1}, X),$$
$$f \mapsto f \circ d_i^n.$$

The construction of the sets $\mathrm{Hom}(\Delta^n, X)$ and the maps between them is given by the composition of functors

$$\mathrm{Sing}_\bullet(X) := \mathrm{Hom}(\Delta_\bullet(-), X) : \Delta \longrightarrow \mathsf{Top} \longrightarrow \mathsf{Sets}^{\mathrm{Op}}.$$

Note that the second functor is a contravariant functor. Such functors are known as simplicial sets. In words, a semi-simplicial set is a contravariant functor

$$X_\bullet : \Delta \to \mathsf{Set}^{\mathrm{Op}}$$

that consists of a collection of sets $\{X_n\}_{n\geq 0}$, where each X_n is called the set of n-simplices. Each strictly order-preserving injection $[n] \to [m]$ in Δ induces a function $f_{n,m} : X_m \to X_n$.

Remark 5.1.4. *Since every morphism in Δ can be written as a composition of elementary injections of the form $d_i^n : [n-1] \to [n]$, it suffices to specify the effect of each such d_i^n. The functor $X_\bullet$ assigns to each such injection a function $\overline{d}_i^n : X_n \to X_{n-1}$.*

The data of a semi-simplicial set can be encoded in a category.

Definition 5.1.5. The category ssSet consists of semi-simplicial sets is the functor category $\mathsf{Func}(\Delta, \mathsf{Sets}^{\mathrm{Op}})$, the category of contravariant functors from Δ to Sets.

Explicitly, a semi-simplicial is a contravariant functor $X_\bullet : \Delta \to \mathsf{Set}$, and a morphism $f_\bullet : X_\bullet \to Y_\bullet$ of semi-simplicial sets is a collection of functions $f_n : X_n \to Y_n$ for each $n \geq 0$, such that for all $n \geq 1$ and $0 \leq i \leq n$, the following compatibility condition holds:

$$\overline{d}_i^n \circ f_n = f_{n-1} \circ \overline{d}_i^{n-1}.$$

The constructions can be summarized by noting that there exists a singular semi-simplicial set functor

$$\mathrm{Sing}_\bullet : \mathsf{Top} \longrightarrow \mathsf{ssSet},$$
$$X \mapsto \mathrm{Sing}_\bullet(X).$$

$\mathrm{Sing}_\bullet$ is defined as a composition of functors:

$$\mathsf{Top} \to \mathsf{Func}(\mathsf{Top}, \mathsf{Sets}^{\mathrm{Op}}) \to \mathsf{Func}(\Delta, \mathsf{Sets}^{\mathrm{Op}})$$
$$X \mapsto \mathrm{Hom}(-, X) \mapsto \Delta_\bullet^* \circ \mathrm{Hom}(-, X)$$

Here $\Delta_\bullet^* \circ \mathrm{Hom}(-, X)$ is the functor that maps n to $\mathrm{Hom}(\Delta^n, X)$. Singular homology studies a topological space by probing it through $\mathbb{Z}$-linear combination of singular simplices.

Definition 5.1.6. Let X be a topological space. The group of n-chains, $C_n(X)$, is the free abelian group with basis the set of singular n-simplices in X:

$$C_n(X) := \mathbb{Z}[\mathrm{Hom}(\Delta^n, X)] \left\{ \sum_{i=0}^{n} n_i \sigma_i : n_i \in \mathbb{Z}, \sigma_i : \Delta^n \to X \text{ continuous} \right\}$$

where each formal sum $\sum_{i=0}^n n_i \sigma_i$ is finite, i.e., all but finitely many n_i are zero. The n-th boundary map

$$\partial_n : C_n(X) \to C_{n-1}(X)$$

is defined on the basis of $C_n(X)$ by the formula

$$\partial_n(\sigma) = \sum_{i=0}^{n} (-1)^i \sigma \circ d_i^n$$

Remark 5.1.7. *What is the meaning of the expression $\sigma \circ d_i^n$? Note that the image $d_i(\Delta^{n-1}) \subseteq \Delta^n$ can be identified with the the i-th face of Δ^n. Hence, if $\sigma : \Delta^n \to X$ is a singular n-simplex in X, then the composition $\sigma \circ d_i^n$ restricts σ to its i-th face. For the most part, however, we shall use the notation $\sigma|_{[v_0,\ldots,\widehat{v_i},\ldots,v_n]}$ to refer to the map $\sigma \circ d_i^n$.*

Lemma 5.1.8. *Let X be a topological space. The composition*

$$\partial_{n-1} \circ \partial_n : C_n(X) \xrightarrow{\partial_n} C_{n-1}(X) \xrightarrow{\partial_{n-1}} C_{n-2}(X)$$

is the zero map.

Proof. The crucial observation about the maps d_i's we need is that for every $n \geq 2$ and every $0 \leq j < i \leq n$, we have:

$$d_i^n \circ d_j^{n-1} = d_j^n \circ d_{i-1}^{n-1} : \Delta^{n-2} \to \Delta^n$$

Indeed, it is easy to verify that both maps are given by

$$(t_0, t_1, \ldots, t_{n-2}) \mapsto (t_0, \ldots, t_{j-1}, 0, t_j, \ldots, t_{i-1}, 0, t_i, \ldots, t_{n-2})$$

Note that

$$\begin{aligned}
\partial_{n-1} \circ \partial_n(\sigma) &= \sum_{j=0}^{n-1} \sum_{i=0}^{n} (-1)^{i+j} \sigma \circ d_i^n \circ d_j^{n-1} \\
&= \sum_{j=0}^{n-1} \sum_{i=0}^{j} (-1)^{i+j} d_i^n \circ d_j^{n-1} + \sum_{j=0}^{n-1} \sum_{i=j+1}^{n} (-1)^{i+j} d_i^n \circ d_j^{n-1} \\
&= \sum_{j=0}^{n-1} \sum_{i=0}^{j} (-1)^{i+j} d_i^n \circ d_j^{n-1} + \sum_{j=0}^{n-1} \sum_{i=j+1}^{n} (-1)^{i+j} d_j^n \circ d_{i-1}^{n-1} \\
&= \sum_{j=0}^{n-1} \sum_{i=0}^{j} (-1)^{i+j} d_i^n \circ d_j^{n-1} + \sum_{j=0}^{n-1} \sum_{i=j}^{n-1} (-1)^{i+j+1} d_j^n \circ d_i^{n-1} \\
&= \sum_{j=0}^{n-1} \sum_{i=0}^{j} (-1)^{i+j} d_i^n \circ d_j^{n-1} - \sum_{j=0}^{n-1} \sum_{i=j}^{n-1} (-1)^{i+j} d_j^n \circ d_i^{n-1}
\end{aligned}$$

The second last equality follows by a shift of the inner summation index in the second nested sum. If we now interchange the roles of i and j in the second sum, the two nested sums cancel. ☐

Remark 5.1.9. *In what follows, we shall write the boundary operator as*

$$\partial_n(\sigma) = \sum_{i=0}^{n} (-1)^i \sigma|_{[v_0,\ldots,\widehat{v_i},\ldots,v_n]}$$

Note that the following computation:

$$\sum_{j<i} (-1)^i (-1)^j \sigma \mid_{[v_0,\cdots,\widehat{v_j},\cdots,\widehat{v_i},\cdots,v_n]} + \sum_{j>i} (-1)^i (-1)^{j-1} \sigma \mid_{[v_0,\cdots,\widehat{v_i},\cdots,\widehat{v_j},\cdots,v_n]} = 0.$$

The latter two summations cancel since after switching i and j in the second sum, it becomes the negative of the first.

The discussion above can be summarized as extracting a chain complex $C_\bullet(X)$ from the semi-simplicial set $\text{Sing}_\bullet(X)$. Therefore, the procedure described above can be viewed as being defined as a functor:

$$\mathsf{Top} \to \mathsf{ssSet} \to \mathsf{Chain}_{\mathsf{Ab}}.$$

We will discuss the second functor in more detail in Proposition 5.1.11. Purely algebraically, we have a sequence of homomorphisms of abelian groups:

$$\cdots \xrightarrow{\partial_{n+1}} C_n(X) \xrightarrow{\partial_n} C_{n-1}(X) \xrightarrow{\partial_{n-1}} C_{n-2}(X) \xrightarrow{\partial_{n-2}} \cdots$$

The boundary map $\partial_n : C_n(X) \longrightarrow C(X)_{n-1}$ is such that

$$\partial_n \circ \partial_{n+1} = 0$$

That is:

$$\text{im}(\partial_{n+1}) \subseteq \text{ker}(\partial_n)$$

A sequence $(C_n(X), \partial_n)_{n\in\mathbb{N}}$ satisfying these properties is is called a singular chain complex. Elements of $\text{ker}(\partial_n)$ are called (singular) n-cycles and elements of $\text{im}(\partial_{n+1})$ are called (singular) n-boundaries.

Definition 5.1.10. Let X be a topological space. The n-th homology of the chain complex $(C_n(X), \partial_n)_{n\in\mathbb{N}}$ is

$$H_n(X;\mathbb{Z}) = \frac{\text{ker}(\partial_n)}{\text{im}(\partial_{n+1})}$$

$H_n(X)$ is called the n-th singular homology group of X with $\mathbb{Z}$ coefficients.

We now verify that tht singular homology defines a functor from Top to Ab. Thus, singular homology yields an invariant that can distinguish spaces. More importantly, it provides a systematic and general way to study topological spaces using algebraic methods. Unlike simplicial homology, which require specific decompositions, singular homology applies to all topological spaces, making it a powerful and flexible theoretical tool in algebraic topology.

Proposition 5.1.11. *For each $n \geq 0$,*

$$H_n : \mathsf{Top} \to \mathsf{Ab}$$

is a covariant functor for each $n \geq 0$.

Proof. H_n can be described as the composite functor:

$$\begin{tikzcd} \mathsf{Top} \arrow[r, "\text{Sing}_\bullet"] \arrow[rrr, bend left, "H_n"] & \mathsf{ssSet} \arrow[r, "\mathbb{Z}(-)"] & \mathsf{Chain}_{\mathsf{Ab}} \arrow[r] & \mathsf{Ab} \end{tikzcd}$$

We already know that $\text{Sing}_\bullet$ is a covariant functor. The last functor is the homology functor which computes the n-th homology of a chain complex of abelian groups. We already know from Proposition 4.2.15 that this functor is covariant. Hence, it suffices to prove that the middle functor is also covariant. Our construction of the chain complex $C_\bullet(X)$ from $\text{Sing}_\bullet(X)$ can be understood as arising from this middle functor. This functor extends naturally to any semi-simplicial set. Given a semi-simplicial set $X_\bullet$, we define

$$\mathbb{Z}(X_\bullet)_n := \mathbb{Z}[X_n],$$

the free abelian group generated by the set X_n. We equip this graded group with a differential given in terms of the face maps by the formula

$$d : \mathbb{Z}[X_n] \longrightarrow \mathbb{Z}[X_{n-1}]$$

defined on a generator $x \in X_n$ by

$$d(x) = \sum_{i=0}^{n} (-1)^i f^i_{n-1,n}.$$

Here, $f^i_{n-1,n}$ denotes the map $X_n \to X_{n-1}$ induced by the standard injection $[n-1] \to [n]$. For a morphism $f_\bullet : X_\bullet \to Y_\bullet$ of semi-simplicial sets, the induced chain map

$$\mathbb{Z}(f_\bullet)_n : \mathbb{Z}[X_n] \to \mathbb{Z}[Y_n]$$

is defined on generators by sending $x \in X_n$ to $f_n(x) \in Y_n$, extended linearly. It can be verified that this is a chain map and that $\mathbb{Z}$ preserves identities and composition [1]. This proves that the middle map

$$\mathbb{Z}(-) : \mathsf{ssSet} \longrightarrow \mathsf{Chain}_{\mathsf{Ab}},$$
$$X_\bullet \mapsto \mathbb{Z}(X_\bullet)$$

is a functor. This proves the claim. □

Remark 5.1.12. *We can collect all the functors in Proposition 5.1.11 to define a single homology functor*

$$H_\bullet : \mathsf{Top} \to \mathsf{GrAb},$$

where GrAb *is the category of graded abelian groups. This functor takes a topological space X and maps it to the graded abelian group*

$$H_\bullet(X) = \bigoplus_{n \geq 0} H_n(X).$$

Calculation with singular homology is difficult because each C_n is generally a free abelian group on uncountably many generators! Eventually, however, we will show that simplicial homology and singular homology are isomorphic.

Remark 5.1.13. *We will also introduce cellular homology which is isomorphic to singular homology and is amenable to computation.*

Here is a trivial computation:

Example 5.1.14. **(Singular Homology of a Point)** If X is a single point, then there is exactly one map $\sigma_n : \Delta^n \to X$, and it is continuous, so $C_n(X) = \mathbb{Z}$ for all n. Moreover,

$$\partial_n(\sigma_n) = \sum_{i=0}^{n-1} (-1)^i \sigma_{n-1} = \begin{cases} 0 & \text{for } n \text{ odd} \\ \sigma_{n-1} & \text{for } n \text{ even} \end{cases}$$

We end up with:

$$\cdots \xrightarrow{\cong} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0$$

Thus, we can quotient out to get the homology:

$$H_n(X; \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{for } n = 0 \\ 0 & \text{for } n \geq 1 \end{cases}$$

[1]This construction is the same construction we have used to defined chain maps $C_\bullet(X) \to C_\bullet(Y)$ if there is a morphism $X \to Y$.

On the other hand, singular homology is much nicer theoretically, because we don't have to worry about choosing a Δ-complex structure, so it provides a convenient tool to prove various properties about a homology theory. For instance:

Proposition 5.1.15. *Let X be a topological space.*

(1) *Let $(X_\alpha)_{\alpha\in A}$ are the path-connected components of X. Then,*
$$H_n(X;\mathbb{Z}) \cong \bigoplus_{\alpha\in A} H_n(X_\alpha;\mathbb{Z})$$
(2) *(0-**th Singular Homology Groups**) If X is non-empty and path-connected, then $H_0(X) \cong \mathbb{Z}$. Hence, for any space X, $H_0(X;\mathbb{Z})$ is a direct sum of $\mathbb{Z}$'s, one for each path-component of X.*

Proof. The proof is given below:

(1) Since Δ^n is path-connected, and an n-simplex $\sigma : \Delta^n \to X$ is a continuous map, we have that $\operatorname{im}(\sigma) \subseteq X_\alpha$ for some α. Therefore, we get a decomposition:
$$C_n(X) \cong \bigoplus_{\alpha} C_n(X_\alpha).$$
The boundary maps preserve this decomposition, i.e., $\partial_n(C_n(X_\alpha)) \subseteq C_{n-1}(X_\alpha)$. Hence $\ker(\partial_n)$ and $\operatorname{im}(\partial_{n+1})$ split similarly as direct sums, and the result follows.

(2) By definition, $H_0(X;\mathbb{Z}) = C_0(X)/\operatorname{im}\partial_1$. Define a homomorphism
$$\begin{aligned}\varepsilon : C_0(X) &\to \mathbb{Z}\\ \sum_{i=0}^{n} n_i\sigma_i &\mapsto \sum_{i=0}^{n} n_i\end{aligned}$$
This is obviously surjective if X is non-empty. We claim that $\ker\varepsilon = \operatorname{im}\partial_1$ if X is path-connected. Observe first that $\operatorname{im}\partial_1 \subseteq \ker\varepsilon$ since for a singular 1-simplex $\sigma : \Delta^1 \to X$, we have
$$\varepsilon \circ \partial_1(\sigma) = \varepsilon(\sigma\big|_{[v_1]} - \sigma\big|_{[v_0]}) = 1 - 1 = 0$$
For the reverse inclusion, suppose $\varepsilon\left(\sum_{i=0}^{n} n_i\sigma_i\right) = 0$. Hence, $\sum_{i=0}^{n} n_i = 0$. The σ_i's are singular 0-simplices which are simply points of X. Choose a path $\tau_i : I \to X$ from a basepoint, x_0, to $\sigma_i(v_0)$, and let σ_0 be the singular 0-simplex with image x_0. We can view τ_i as a singular 1-simplex, a map $\tau_i : [v_0, v_1] \to X$, and then we have $\partial_1(\tau_i) = \sigma_i - \sigma_0$. Hence,
$$\partial_1\left(\sum_{i=0}^{n} n_i\tau_i\right) = \sum_{i=0}^{n} n_i\sigma_i - \sum_{i=0}^{n} n_i\sigma_0 = \sum_{i=0}^{n} n_i\sigma_i,$$
since $\sum_{i=0}^{n} n_i = 0$. Thus, $\sum_{i=0}^{n} n_i\sigma_i$ is a boundary. Hence, $\ker\varepsilon \subseteq \operatorname{im}\partial_1$. Hence, ε induces an isomorphism $H_0(X;\mathbb{Z}) \cong \mathbb{Z}$.

This completes the proof. □

It is often very convenient to have a slightly modified version of homology for which a point has trivial homology groups in all dimensions, including zero. This is done by defining the reduced homology groups, $\widetilde{H}_n(X)$. Let $\{*\} \in X$ be a one-point space. There is a unique map $X \to \{*\}$. Moreover, a map $\{*\} \to X$ assigns a basepoint, x_0, to X. Since the composition
$$\{*\} \xrightarrow{x_0} X \to \{*\}$$

is the identity, for each $n \geq 0$ the induced maps on homology satisfy

$$H_n(\{*\}) \xrightarrow{H_n(x_0)} H_n(X) \longrightarrow H_n(\{*\})$$

also compose to the identity. Hence, $H_n(\{*\})$ is an abelian subgroup of $H_n(X)$ for each $x_0 \in X$. For each $n \geq 0$, we define the reduced homology to be

$$\widetilde{H}_n(X) = \frac{H_n(X)}{H_n(\{*\})}$$

By Example 5.1.14, we have

$$\widetilde{H}_n(X) = \begin{cases} H_n(x)/\mathbb{Z} & n = 0 \\ H_n(X) & n > 0 \end{cases}$$

Note that we have $H_0(X) \cong \widetilde{H}_0(X) \oplus \mathbb{Z}$ because $H_0(X)$ is a free $\mathbb{Z}$-module.

5.2. Eilenberg–Steenrod Axioms

We have met two homology theories: simplicial homology and singular homology. Later on, we will discuss cellular homology. In fact, there are many other homology theories in mathematics that satisfy remarkably similar properties. Eilenberg and Steenrod unified these different homology theories by formulating a set of axioms that all such theories satisfy.

Definition 5.2.1. (Eilenberg–Steenrod Axioms) A homology theory with $\mathbb{Z}$ coefficients consists of

(1) A family of functors $H_n : \mathsf{Top}^2 \to \mathsf{Ab}$ for $n \geq 0$, and
(2) A family of natural transformations

$$\delta_n : H_n \to H_{n-1} \circ p$$

where p is the functor sending (X, A) to $(A, \varnothing)$ and $f : (X, A) \to (Y, B)$ to $f|_A : (A, \varnothing) \to (B, \varnothing)$.

such that the following axioms are satisfied:

(a) (Homotopy invariance) If $f, g : (X, A) \to (Y, B)$ are homotopic maps, then the induced maps

$$H_n(f), H_n(g) : H_n(X, A; \mathbb{Z}) \to H_n(Y, B; \mathbb{Z})$$

are such that $H_n(f) = H_n(g)$ for $n \geq 0$[2].

(b) (Long exact sequence) The inclusions

$$(A, \emptyset) \hookrightarrow (X, \emptyset) \hookrightarrow (X, A)$$

give rise to a long exact sequence

$$\cdots \longrightarrow H_{n+1}(X; \mathbb{Z}) \longrightarrow H_{n+1}(X, A; \mathbb{Z}) \xrightarrow{\delta_{n+1}} H_n(A; \mathbb{Z}) \longrightarrow H_n(X; \mathbb{Z}) \longrightarrow \cdots$$

(c) (Excision) If $Z \subseteq A \subseteq X$ are topological spaces such that $\overline{Z} \subseteq \text{Int}(A)$, the inclusion of pairs $(X \setminus Z, A \setminus Z) \subseteq (X, A)$ induces isomorphisms

$$H_n(X \setminus Z, A \setminus Z; \mathbb{Z}) \to H_n(X, A; \mathbb{Z})$$

for all $n \geq 0$.

[2]In other words, H_n may be regarded as a functor from hTop to Ab.

(d) (Additivity) If $X = \coprod_\alpha X_\alpha$ is the disjoint union of a family of topological spaces X_α, then
$$H_n(X;\mathbb{Z}) = \bigoplus_\alpha H_n(X_\alpha;\mathbb{Z})$$
for each $n \in \mathbb{N}$.

If a homology theory satisfies the following additional dimension axiom, which states that for any one-point space $X = \{*\}$,
$$H_n(X;\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } n = 0, \\ 0 & \text{otherwise,} \end{cases}$$
then the homology theory is called an ordinary homology theory with $\mathbb{Z}$-coefficients, and the corresponding Eilenberg–Steenrod axioms are referred to as the ordinary Eilenberg–Steenrod axioms for homology.

Remark 5.2.2. *Not all homology theories satisfy the dimension axiom. Such theories are known as extraordinary/generalized homology theories. Although they still satisfy the other Eilenberg–Steenrod axioms, they fail the dimension axiom. Examples of extraordinary/generalized homology theories include: Examples of extraordinary homology theories include:*

(1) *Unoriented bordism homology,*
(2) *Oriented bordism homology,*
(3) *Homology theories represented by spectra in stable homotopy theory that do not satisfy the dimension axiom.*

5.2.1. Relative Homology. We introduce the notion of relative homology functors to make sense of the family of functors in Definition 5.2.1:
$$H_n : \mathsf{Top}^2 \to \mathsf{Ab}$$
Given $(X, A) \in \mathsf{Top}^2$, we have $C_n(A) \subseteq C_n(X)$ such that ∂_n restricts to a map from $C_n(A)$ to $C_{n-1}(A)$. Therefore, we can consider a chain complex $(C_\bullet(A), \partial_\bullet|_A)$ which is a sub-complex [3] of the chain complex $(C_\bullet, \partial_\bullet)$ The chain complex $(C_\bullet(A), \partial_\bullet|_A)$ is usually drawn as:
$$\cdots \longrightarrow C_2(A) \xrightarrow{\partial_2|_A} C_1(A) \xrightarrow{\partial_1|_A} C_0(A)$$
Note that $C_n(A)$ is an abelian subgroup of $C_n(X)$. Hence, we e can consider quotient group
$$C_n(X, A) = \frac{C_n(X)}{C_n(A)}$$
Since the boundary map
$$\partial_n : C_n(X) \to C_{n-1}(X)$$
takes $C_n(A)$ to $C_{n-1}(A)$, it induces a quotient boundary map
$$\overline{\partial}_n : C_n(X, A) \to C_{n-1}(X, A)$$
Since $\partial_{n+1} \circ \partial_n = 0$ on $C_n(X)$, we have that $\overline{\partial}_{n+1} \circ \overline{\partial}_n = 0$ on $C_n(X, A)$. Therefore, we get a chain complex $(C_\bullet(X, A), \overline{\partial}_\bullet)$ The chain complex is usually drawn as:
$$\cdots \longrightarrow C_2(X, A) \xrightarrow{\overline{\partial}_2} C_1(X, A) \xrightarrow{\overline{\partial}_1} C_0(X, A)$$
The above discussion implies that the construction of relative singular chain complexes defines a functor from Top^2 to $\mathsf{Chain}_{\mathsf{Ab}}$.

[3]Given a chain complex $(C_\bullet, \partial_\bullet)$, a subcomplex of $(C_\bullet, \partial_\bullet)$ is given by a family of subgroups $C_n' \subseteq C_n$ such that the boundary operator $\partial_n' : C_n \to C_{n-1}$ restricts to a homomorphism $C_n' \to C_{n-1}'$ for all n.

Definition 5.2.3. Let $(X, A) \in \mathsf{Top}^2$. The n-th relative homology group with $\mathbb{Z}$ coefficients, $H_n(X, A)$, is the n-th homology group of the chain complex $(C_\bullet(X, A), \overline{\partial}_\bullet)$. That is:

$$H_n(X, A; \mathbb{Z}) = \frac{\text{Ker } \overline{\partial}_n}{\text{Im } \overline{\partial}_{n+1}}$$

Remark 5.2.4. *It is clear that the n-th relative homology group with $\mathbb{Z}$ coefficients defines a functor from Top^2 to Ab.*

Remark 5.2.5. *Since the homology of the empty set is trivial for all $n \geq 0$, we have:*

$$H_n(X, \emptyset; \mathbb{Z}) = H_n(X; \mathbb{Z})$$

for each $n \geq 0$. Similarly, we have

$$H_n(X, X; \mathbb{Z}) = 0$$

for each $n \geq 0$.

By considering the definition of the relative boundary map we see that:

(1) Elements of $H_n(X, A; \mathbb{Z})$ are represented by relative n-cycles: n-chains $\alpha \in C_n(X)$ such that $\partial_n(\alpha) \in C_{n-1}(A)$.
(2) A relative n-cycle, α, is trivial in $H_n(X, A; \mathbb{Z})$ iff it is a relative n-boundary: $\alpha = \partial_{n+1}(\beta) + \gamma$ for some $\beta \in C_{n+1}(X)$ and $\gamma \in C_n(A)$.

5.2.2. Long Exact Sequence in Singular Homology. We now prove that singular homology satisfies the long exact sequence axiom. The importance of the long exact sequence axiom is that is allows us to compute homology groups of various spaces in using an 'inductive' and/or 'bottom-up' approach, as we shall see in various examples later on. We have a short exact sequence of chain complexes:

$$0_\bullet \to (C_\bullet(A), \partial_\bullet|_A) \xrightarrow{i_\bullet} (C_\bullet, \partial_\bullet) \xrightarrow{j_\bullet} (C_\bullet(X, A), \overline{\partial}_\bullet) \to 0_\bullet$$

By Proposition 4.6.7, we have the following long exact sequence is homology associated to the pair of spaces (X, A):

$$\cdots \to H_{n+1}(X; \mathbb{Z}) \to H_{n+1}(X, A; \mathbb{Z}) \xrightarrow{\delta_{n+1}} H_n(A) \to H_n(X; \mathbb{Z}) \to \cdots$$

By Proposition 4.6.7, the boundary map $\delta_n : H_n(X, A; \mathbb{Z}) \to H_{n-1}(A; \mathbb{Z})$ has a very simple description: if a class $[\alpha] \in H_n(X, A; \mathbb{Z})$ is represented by a relative cycle α, then $\delta_n[\alpha]$ is the class of the cycle $\delta_n \alpha$ in $H_{n-1}(A; \mathbb{Z})$.

Remark 5.2.6. *An easy generalization of the long exact sequence of a pair (X, A) is the long exact sequence of a triple $(X, A, B) \in \mathsf{Top}^3$. Indeed, we have $(X, A), (X, B), (A, B) \in \mathsf{Top}^2$. The three long exact sequences assemble in the following diagram:*

$$\begin{tikzcd}
H_{n+2}(X;\mathbb{Z}) \arrow[rr, dashed, "g_1"] \arrow[dr, "f_1"] & & H_{n+2}(X,A;\mathbb{Z}) \arrow[rr, Rightarrow] \arrow[dr, dashed, "g_2"] & & H_{n+1}(A,B;\mathbb{Z}) \arrow[rr, dashed, "j_4"] \arrow[dr, Rightarrow] & & H_n(B;\mathbb{Z}) \\
& H_{n+2}(X,B;\mathbb{Z}) \arrow[ur, Rightarrow] \arrow[dr, "f_2"] & & H_{n+1}(A;\mathbb{Z}) \arrow[ur, dashed, "j_3"] \arrow[dr, dashed, "g_3"] & & H_{n+1}(X,B;\mathbb{Z}) \arrow[ur, "f_5"] \arrow[dr, Rightarrow] & \\
H_{n+2}(A,B;\mathbb{Z}) \arrow[rr, dashed, "j_1"] \arrow[ur, Rightarrow] & & H_{n+1}(B;\mathbb{Z}) \arrow[rr, "f_3"] \arrow[ur, dashed, "j_2"] & & H_{n+1}(X;\mathbb{Z}) \arrow[rr, dashed, "g_4"] \arrow[ur, "f_4"] & & H_{n+1}(X,A;\mathbb{Z})
\end{tikzcd}$$

The braid lemma (Proposition 4.6.4) implies that the chain complex labeled with $\Rightarrow$ arrows is a chain complex. This is the desired long exact sequence in homology generated by (X, A, B).

5.3. Homotopy Invariance of Singular Homology

We establish that singular homology groups satisfy the homotopy invariance axiom. To this end, we begin with the non-relative case. Recall from Proposition 5.1.11 that for each $n \geq 0$, the singular homology group H_n defines a functor from the category Top of topological spaces to the category Ab of abelian groups. The homotopy invariance axioms amounts to the claim that H_n factors as follows:

$$\begin{tikzcd}
\mathsf{Top} \arrow[r, "\gamma"] \arrow[dr, "H_n"'] & \mathsf{hTop} \arrow[d, dashed, "\overline{H}_n"] \\
& \mathsf{Ab}
\end{tikzcd}$$

Here, $\gamma : \mathsf{Top} \to \mathsf{hTop}$ is the canonical projection functor that identifies homotopic maps. We we will make use of the notion of a chain homotopy between chain complexes.

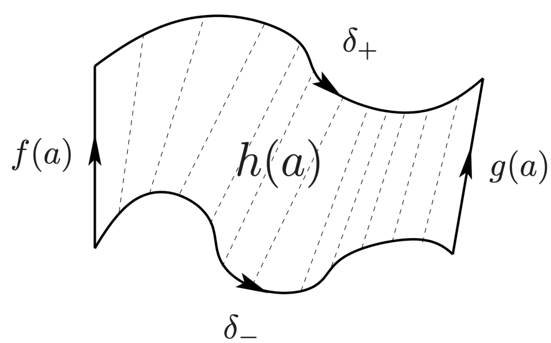
Remark 5.3.1. *What is the geometric interpretation of a chain homotopy? Let $h\colon X \times I \to Y$ be a homotopy between two continuous maps $f, g\colon X \to Y$. Consider a 1-chain $a \in C_1(X)$. Then $f_*(a)$ and $g_*(a)$ are 1-chains in Y. The homotopy h interpolates between f and g, and thus maps the endpoints of $f_*(a)$ to those of $g_*(a)$. To understand this, consider the boundary of the image of a under h, as depicted in Figure 1. Traversing the boundary of $h_*(a)$ counterclockwise starting at the bottom right, we observe:*

$$\partial_2 h_*(a) = g_*(a) - \delta_+ - f_*(a) + \delta_-,$$

where δ_+ and δ_- correspond to the images under h of the endpoints of a. The difference $\delta_+ - \delta_-$ precisely equals $h_*(\partial_1 a)$. Therefore, we obtain:

$$\partial_2 h_*(a) = g_*(a) - f_*(a) - h_*(\partial_1 a).$$

This expression illustrates that a chain homotopy mirrors the notion of a homotopy between maps at the level of chain complexes.



The image is taken from [Alu21].

For each $n \geq 0$, recall that the functor H_n is given as the composite functor:

$$\begin{tikzcd}
\mathsf{Top} \arrow[r, "\mathrm{Sing}_\bullet"] \arrow[rrr, bend left, "H_n"] \arrow[rr, bend right, "S_\bullet"'] & \mathsf{ssSet} \arrow[r, "\mathbb{Z}(-)"] & \mathsf{Chain}_{\mathsf{Ab}} \arrow[r] & \mathsf{Ab}
\end{tikzcd}$$

It suffices to show that the functor $S_\bullet$ is homotopy invariant, since we already know from Proposition 4.2.19 that the functor $\mathsf{Chain}_{\mathsf{Ab}} \to \mathsf{Ab}$ descends to a well-defined functor

$$\mathsf{hChain}_{\mathsf{Ab}} \to \mathsf{Ab}.$$

In other words, it suffices to show that if $f, g\colon X \to Y$ are homotopic continuous maps, then the induced chain maps $f_*, g_*\colon C_\bullet(X) \to C_\bullet(Y)$ are chain homotopic; that is, there exists a chain homotopy between them. A homotopy $H\colon X \times I \to Y$ between maps $f, g\colon X \to Y$ determines the data given in the following diagram:

$$\begin{tikzcd}
X \times \{0\} \arrow[dr, "i_0"'] \arrow[drr, bend left, "f"] & & \\
& X \times I \arrow[r, "H"'] & Y \\
X \times \{1\} \arrow[ur, "i_1"'] \arrow[urr, bend right, "g"'] & &
\end{tikzcd}$$

Since $S_\bullet\colon \mathsf{Top} \to \mathsf{Chain}_{\mathsf{Ab}}$ is a functor, applying it to the diagram yields the following equalities at the level of chain complexes:

$$S_\bullet(f) = S_\bullet(H) \circ S_\bullet(i_0),$$
$$S_\bullet(g) = S_\bullet(H) \circ S_\bullet(i_1).$$

Since composing a chain homotopy with a chain map[4] yields a chain homotopy it suffices to prove that the maps

$$S_\bullet(i_0), S_\bullet(i_1)\colon C_\bullet(X) \to C_\bullet(X \times I)$$

are chain homotopic since $S_\bullet(H)$ is a chain map. Thus, we need to construct a chain homotopy whose components

$$C_n(X) \to C_{n+1}(X \times I)$$

are defined for each $n \geq 0$. Given a generator $\sigma\colon \Delta^n \to X$ in $S_n(X)$, it is natural to consider the map

$$\sigma \times \mathrm{Id}\colon \Delta^n \times I \to X \times I.$$

Although $\Delta^n \times I$ is not itself an $(n+1)$-simplex, we show that it can be expressed as a union of $(n+1)$-simplices. This decomposition allows us to define the desired map by assembling it piecewise on these simplices. Let's provide an intuition foe the this construction. In $\Delta^n \times I$, let

$$\Delta^n \times 0 = [v_0, \ldots, v_n]$$
$$\Delta^n \times 1 = [w_0, \ldots, w_n]$$

where v_i and w_i have the same image under the projection $\Delta^n \times I \to \Delta^n$. We can pass from $[v_0, \ldots, v_n]$ to $[w_0, \ldots, w_n]$ by interpolating a sequence of n simplices, each obtained from the preceding one by moving one vertex v_i up to w_i, starting with v_n and working backwards to v_0. For instance,

$$[v_0, \ldots, v_i, w_{i+1}, \ldots, w_n]$$

moves up to

$$[v_0, \ldots, v_{i-1}, w_i, \ldots, w_n]$$

The region between these two n simplices is exactly the $(n+1)$ simplex

$$[v_0, \ldots, v_i, w_i, \ldots, w_n]$$

Lemma 5.3.2. *$\Delta^n \times I$ is the union of $n+1$ copies of Δ^{n+1}.*

[4]This is clear.

Proof. For $i = -1, 0, \dots, n-1$, let $g_i : \Delta^n \to I$ denote the map

$$g_i(s_0, s_1, \dots, s_n) = \sum_{i<j} s_j.$$

Let $G_i \subseteq \Delta^n \times I$ denote the graph of g_i. Then G_i is homeomorphic to Δ^n via the projection $\Delta^n \times I \to \Delta^n$ surjective the first factor. Let us now label the vertices at the bottom (i.e., $\Delta^n \times \{0\}$) of $\Delta^n \times I$ by $v_0, v_1, \dots, v_n$ and those at the top (i.e., $\Delta^n \times \{1\}$) by $w_0, w_1, \dots, w_n$. Then G_i is the n-simplex

$$G_i = [v_0, \dots, v_i, w_{i+1}, \dots, w_n].$$

Since G_i lies below G_{i-1} as $g_i \leq g_{i-1}$, it follows that the region between G_i and G_{i-1} is the $(n+1)$-simplex $[v_0, \dots, v_i, w_i, \dots, w_n]$; this is indeed an $(n+1)$-simplex as w_i is not in G_i and hence not in the n-simplex $[v_0, \dots, v_i, w_i, \dots, w_n]$. Since

$$0 = g_n \leq g_{n-1} \leq \dots \leq g_0 \leq g_{-1} = 1,$$

we see that $\Delta^n \times I$ is the union of the regions between the G_i, and hence the union of $n+1$ different $(n+1)$-simplices $[v_0, \dots, v_i, w_i, \dots, w_n]$, each intersecting the next in an n-simplex face. □

We can now prove the desired result.

Proposition 5.3.3. ***(Homotopy Invariance)*** *For each $n \geq 0$, the functor H_n in Proposition 5.1.11 descends to a functor $\overline{H}_n : \mathsf{hTop} \to \mathsf{Ab}$.*

$$\begin{tikzcd} \mathsf{Top} \arrow[r, "\gamma"] \arrow[dr, "H_n"'] & \mathsf{hTop} \arrow[d, dashed, "\overline{H}_n"] \\ & \mathsf{Ab} \end{tikzcd}$$

In other words, if X and Y are topological spaces and $f, g : X \to Y$ are homotopic maps, then

$$H_n(f) = H_n(g) : H_n(X; \mathbb{Z}) \to H_n(Y; \mathbb{Z})$$

for each $n \geq 0$.

Proof. As noted above, it suffices to prove that

$$S_\bullet(i_0), S_\bullet(i_1) \colon C_\bullet(X) \to C_\bullet(X \times I)$$

are chain homotopic. Given a $\sigma : \Delta^n \to X$ in $C_n(X)$, we can consider the map:

$$\sigma \times \mathrm{Id} : \Delta^n \times I \to X \times I \to Y$$

We can define prism operators $P_n : C_n(X) \to C_{n+1}(X \times I)$ by the following formula:

$$P_n(\sigma) = \sum_{i=0}^{n+1} (-1)^i (\sigma \times \mathrm{Id})|_{[v_0, \cdots, v_i, w_i, \cdots, w_n]}.$$

Note that $P_n(\sigma) \in C_{n+1}(X \times I)$ since Lemma 5.3.2 shows that $[v_0, \dots, v_i, w_i, \dots, w_n]$ is an $(n+1)$-simplex contained in $\Delta^n \times I$. The prism operator is our proposed chain homotopy. A simple computation shows that we have

$$(i_1)_n - (i_0)_n = \partial'_{n+1} P_n + P_{n-1} \partial_n$$

Indeed:

$$\begin{aligned} \partial'_{n+1} P_n(\sigma) &= \sum_{j \leq i} (-1)^i (-1)^j (\sigma \times \mathrm{Id})|_{[v_0, \dots, \widehat{v_j}, \dots, v_i, w_i, \dots, w_n]} \\ &\quad + \sum_{j \geq i} (-1)^i (-1)^{j+1} (\sigma \times \mathrm{Id})|_{[v_0, \dots, v_i, w_i, \dots, \widehat{w_j}, \dots, w_n]} \end{aligned}$$

The terms with $i = j$ in the two sums cancel except for

$$\begin{aligned} F \circ (\sigma \times \mathrm{Id})\big|_{[\widehat{v_0}, w_0, \dots, w_n]} &= \sigma|_{[w_0, \dots, w_n]} = (i_1)_n(\sigma), \\ -F \circ (\sigma \times \mathrm{Id})\big|_{[v_0, \dots, v_n, \widehat{w_n}]} &= -\sigma|_{[v_0, \dots, v_n]} = -(i_0)_n(\sigma) \end{aligned}$$

The terms with $i \neq j$ are exactly $-P_{n-1}\partial_n(\sigma)$. Hence, the maps $\{P_n\}_{n \geq 0}$ define the desired chain homotopy. ☐

Corollary 5.3.4. *If $f : X \to Y$ is a homotopy equivalence, then $H_n(X; \mathbb{Z}) \cong H_n(Y; \mathbb{Z})$ for each $n \geq 0$. In particular, if X is contractible, then $H_n(X; \mathbb{Z}) = 0$ for each $n > 0$.*

Proof. Let $g\colon Y \to X$ be a homotopy inverse of f. Then $g \circ f \sim \mathrm{Id}_X$ and $f \circ g \sim \mathrm{Id}_Y$. By Proposition 5.3.3, it follows that

$$\begin{aligned} H_n(f) \circ H_n(g) &= \mathrm{Id}_{H_n(X;\mathbb{Z})}, \\ H_n(g) \circ H_n(f) &= \mathrm{Id}_{H_n(Y;\mathbb{Z})}. \end{aligned}$$

Hence, we conclude that

$$H_n(X; \mathbb{Z}) \cong H_n(Y; \mathbb{Z}).$$

for each $n \geq 0$. The second statement follows immediately from the first statement together along with the fact that $H_n(\{*\}; \mathbb{Z}) = 0$ for all $n \geq 1$. ☐

Corollary 5.3.5. *For each $n \geq 0$, the functor $H_n : \mathsf{Top}^2 \to \mathsf{Ab}$ descends to a functor $\overline{H}_n : \mathsf{hTop}^2 \to \mathsf{Ab}$.*

$$\begin{tikzcd} \mathsf{Top}^2 \arrow[r, "\gamma"] \arrow[dr, "H_n"'] & \mathsf{hTop}^2 \arrow[d, dashed, "\overline{H}_n"] \\ & \mathsf{Ab} \end{tikzcd}$$

In other words, if $(X, A), (Y, B) \in \mathsf{Top}^2$, and $f, g : (X, A) \to (Y, B)$ are homotopic maps, then

$$H_n(f) = H_n(g) : H_n(X, A; \mathbb{Z}) \to H_n(Y, B; \mathbb{Z})$$

for each $n \geq 0$.

Proof. It suffices to prove that

$$S_\bullet(i_0), S_\bullet(i_1)\colon C_\bullet((X, A)) \to C_\bullet((X \times I, (A \times I)$$

are chain homotopic. Consider the chain homotopy constructed in Proposition 5.3.3. The prism operator $P_n : C_n(X) \to C_{n+1}(X \times I)$ sends $C_n(A)$ into $C_{n+1}(A \times I)$. Consequently, it induces a chain homotopy between the quotient chain maps

$$\frac{C_\bullet(X \times \{0\})}{C_\bullet(A \times \{0\})} \to \frac{C_\bullet(X \times I)}{C_\bullet(A \times I)} \quad \text{and} \quad \frac{S_\bullet(X \times \{1\})}{S_\bullet(A \times \{1\})} \to \frac{S_\bullet(X \times I)}{S_\bullet(A \times I)}.$$

The claim follows. ☐

5.4. Acyclic Models

There are several theorems in algebraic topology whose proofs require significant computation. For example, consider the argument used to prove Proposition 5.3.3. We used categorical arguments to predict and provide that for each $X \in \mathsf{Top}$, there exists a chain homotopy

$$C_\bullet(X) \to C_\bullet(X \times I).$$

If $X = \Delta^n$ and $\Delta^n \to \Delta^n$ is the identity map, we can consider Δ^n as an element of $C_n(\Delta^n)$. The argument in the proof of Proposition 5.3.3 relied on writing $\Delta^n \times I$ as a union of Δ^{n+1}'s. This construction, in turn, can be thought of as producing a map

$$C_n(\Delta^n) \to C_{n+1}(\Delta^{n+1} \times I).$$

The proof then bootstrapped this construction to apply to any $X \in \mathsf{Top}$. The method of acyclic models is an abstraction of this strategy. It provides a powerful framework to prove various theorems in algebraic topology by breaking complex problems down into manageable steps:

(1) Start with a simple "model"—an object or class of objects.
(2) Show that on these models the construction is straightforward or canonical.
(3) Extend the construction uniquely and naturally to more complicated objects by inductively "building up" from the "model" cases.

The method of acyclic models goes a step further by also addressing when certain constructions are natural in the categorical sense. Note that we did not prove that the chain map

$$C_\bullet(X) \to C_\bullet(X \times I)$$

is natural in the categorical sense. That is, if $X \to Y$ is a continuous map, then there is a commutative diagram:

$$\begin{tikzcd} C_\bullet(X) \arrow[r] \arrow[d] & C_\bullet(X \times I) \arrow[d] \\ C_\bullet(Y) \arrow[r] & C_\bullet(Y \times I) \end{tikzcd}$$

In fact, this statement is more general than what we have done to prove Proposition 5.3.3, in the sense that we constructed a chain homotopy

$$C_\bullet(\Delta^n) \to C_\bullet(\Delta^n \times I)$$

and then extended it to a chain homotopy

$$C_\bullet(X) \to C_\bullet(X \times I).$$

for any $X \in \mathsf{Top}$. The method of acyclic models allows us to prove naturality questions related to functors within a categorical framework. Hence, the method of acyclic models can be applied to various "bottom-up" constructions in algebraic topology that we expect to be natural in a categorical sense.

Definition 5.4.1. Let C be a category, and let M be a collection of (model) objects in C. Let $\mathsf{F} : \mathsf{C} \to \mathsf{Chain}^{\geq 0}_{\mathsf{Mod}_R}$, the category of non-negatively graded chain complexes of R-modules.

(1) F is acyclic on M if, for each $M \in \mathsf{M}$, the complex $\mathsf{F}(M)$ is acyclic in positive degrees. That is, $\mathsf{F}(M)$ has zero homology in all positive degrees.
(2) F is free on M if there exist objects $M_\alpha \in \mathsf{M}$ (possibly with repetitions and not necessarily including all objects in M) and elements $m_\alpha \in \mathsf{F}(M_\alpha)$ such that, for any object $X \in \mathsf{C}$, the set

$$\{\mathsf{F}(f)(m_\alpha) \mid \text{for all } \alpha \text{ and } f : M_\alpha \to X\}$$

forms a basis of $\mathsf{F}(X)$.

Remark 5.4.2. *Note that we do not require* F *to be acyclic in degree zero in Definition 5.4.1.*

Let us try to motivate Definition 5.4.1. Let $\mathsf{C} = \mathsf{Top}$ and $\mathsf{D} = \mathsf{Chain}^{\geq 0}_{\mathsf{Mod}_R}$. We should think of the collection $\mathsf{M} = \{\Delta^n\}_{n\geq 0}$ as a collection of "model" objects in $\mathsf{C} = \mathsf{Top}$. We have two functors from Top to $\mathsf{Chain}^{\geq 0}_{\mathsf{Mod}_R}$.

(1) $X \mapsto C_\bullet(X)$
(2) $X \mapsto C_\bullet(X \times I)$

We know that the first functor satisfies the acyclic condition on M. The freeness condition means that, often, one only needs to define constructions on the objects M_α in order to define them everywhere. Note that the functor $X \to C_\bullet(X)$ satisfies the freeness condition since $\mathrm{Hom}(\Delta^n, X)$ forms a basis for $C_n(X)$ for each $X \in \mathsf{Top}$ and $n \geq 0$. We can state the main result:

Proposition 5.4.3. *Let C be a category with a collection, M, of (model) objects. Let $\mathsf{F} : \mathsf{C} \to \mathsf{Chain}^{\geq 0}_{\mathsf{Mod}_R}$ be functors such that F is free on M and G is acylic on M.*

(1) *Given a natural transformation $H_0(\mathsf{F}) \Rightarrow H_0(\mathsf{G})$, there exists a natural transformation $\mathsf{F} \Rightarrow \mathsf{G}$ inducing the given natural transformation.*
(2) *Given two natural transformations $\mathsf{F} \Rightarrow \mathsf{G}$ inducing the same natural transformation $H_0(\mathsf{F}) \Rightarrow H_0(\mathsf{G})$, there exists a natural chain homotopy between $\mathsf{F}(X)$ and $\mathsf{G}(X)$ for all $X \in \mathsf{C}$.*

Proof. The proof is given below:

(1) For $X \in \mathsf{C}$, we need to define a natural transformation $\tau : \mathsf{F} \Longrightarrow \mathsf{G}$ such that $\tau(X)$ is a chain map ($\partial_{\mathsf{G}(X)} \circ \tau(X) = \tau(X) \circ \partial_{\mathsf{F}(X)}$) and for each $g \in \mathrm{Hom}(X, Y)$, the following diagram commutes in each degree q:

$$\begin{tikzcd} \mathsf{F}_q(X) \arrow[r, "\tau_q(X)"] \arrow[d, "\mathsf{F}_q(g)"'] & \mathsf{G}_q(X) \arrow[d, "\mathsf{G}_q(g)"] \\ \mathsf{F}_q(Y) \arrow[r, "\tau_q(Y)"'] & \mathsf{G}_q(Y) \end{tikzcd}$$

Since F is free, for each $q \geq 0$ there exist objects $\{M^q_{\alpha_j}\}_{j \in J^q}$ in M and elements $m^q_{\alpha_j} \in \mathsf{F}(M^q_{\alpha_j})$ such that for any $X \in \mathsf{C}$ and

$$\{\mathsf{F}_q(f)(m^q_{\alpha_j}) \mid \text{for all } \alpha \text{ and } f : M^q_{\alpha_j} \to X\}$$

forms a basis of $\mathsf{F}_q(X)$. Let $x^q_{\alpha_j, f} := \mathsf{F}_q(f)(m^q_{\alpha_j})$. The naturality condition reads that for each $f \in \mathrm{Hom}(M^q_\alpha, X)$, we must have

$$\tau_q(X)(x^q_{\alpha_j, f}) = \mathsf{G}_q(f)(\tau_q(M^q_{\alpha_j})(m^q_{\alpha_j})) \tag{4}$$

By freeness of F, it suffices to ensure that the above equality holds. We define $\tau_q(X)$ inductively. First let $q = 0$. Define $\tau_0(M^0_{\alpha_j})(m^0_{\alpha_j})$ to be any element in $\mathsf{G}_0(M^0_{\alpha_j})$ such that its homology class satisfies

$$[\tau_0(M^0_{\alpha_j})(m^0_{\alpha_j})] = \phi(\overline{m^0_{\alpha_j,}}) \in H_0(\mathsf{G}(M_{\alpha_j})),$$

where $\overline{m^0_{\alpha_j}}$ denotes the homology class of $m^0_{\alpha_j}$ in $\mathsf{F}(M^0_{\alpha_j, f})$. Here ϕ is the map from $H_0(\mathsf{F})$ to $H_0(\mathsf{G})$. Hence, we just need to pick a representative in the homology class of $\phi(\overline{m^0_{\alpha_j}})$. τ is then defined such that Equation (4) holds. Hence, we get a map

$$\tau(X) : \mathsf{F}_0(X) \to \mathsf{G}_0(X)$$

such that it induces ϕ on H_0. This is because the following diagram is commutative by construction:

$$\begin{tikzcd}
H_0(\mathsf{F}(M^0_{\alpha_j})) \arrow[r, "\phi(M^q_{\alpha_j})"] & H_0(\mathsf{G}(M^0_{\alpha_j})) \\
\mathsf{F}_0(M^0_{\alpha_j}) \arrow[r, "\tau_0(M^0_{\alpha_j})"] \arrow[u, two heads] \arrow[d, "\mathsf{F}_0(f)"'] & \mathsf{G}_0(M^0_{\alpha_j}) \arrow[u, two heads] \arrow[d, "\mathsf{G}_0(f)"] \\
\mathsf{F}_0(X) \arrow[r, "\tau_0(X)"'] & \mathsf{G}_0(X)
\end{tikzcd}$$

Now assume that τ_i has been defined for $i < q$. Since τ_q must be a chain map, it suffices to define τ_q so that

$$\partial_{\mathsf{G}(M^q_{\alpha_j})}\, \tau_q(M^q_{\alpha_j})(m^q_{\alpha_j}) = \tau_{q-1}(M^{q-1}_{\alpha_j})(\partial_{\mathsf{F}(M^q_{\alpha_j})} m^q_{\alpha_j}),$$

This is sufficient because naturality ensures that the appropriate square involving boundary maps commutes for any object. Note that the right hand side is a cycle. Since G is acylic on models, we can choose $\tau_q(M^q_{\alpha_j})(m^q_{\alpha_j})$ to satisfy the compatibility condition with the boundary map. Proceeding inductively defines a natural transformation

(2) Skipped.

This completes the proof. □

Remark 5.4.4. *The closest analogy to Proposition 5.4.3 is the comparison theorem (Proposition 15.4.14) for projective resolutions, which states that if we have a projective resolution $P_\bullet \to M$ and a resolution $Q_\bullet \to N$, then any morphism $M \to N$ induces a chain map $P_\bullet \to Q_\bullet$ that is unique up to chain homotopy.*

Remark 5.4.5. *There is a more general version of the acyclic models theorem where the target category is replaced by an abelian category. See [Bar02] for more details.*

We conclude this section by showing how Definition 5.4.1 can be applied to prove Proposition 5.3.3. The only remaining requirement is to verify that the functor $X \mapsto C_\bullet(X \times I)$ is acyclic. In other words, we must show that $H_n(\Delta^k \times I) = 0$ for all k and all $n > 0$. This will follow from the observation that $H_n(X) = 0$ for $n > 0$ if X is star-shaped.

Remark 5.4.6. *A star-shaped domain, X, is contractible. However, note that we cannot use Corollary 5.3.4 a priori, since we would like to use Definition 5.4.1 to give an alternative proof of Proposition 5.3.3, which implies Corollary 5.3.4!*

If X is star-shaped, then there is a chain map

$$\epsilon : C_\bullet(X) \longrightarrow \mathbb{Z},$$

determined uniquely by sending each generator of $C_0(X)$ to 1. Here, we think of $\mathbb{Z}$ as a chain complex that is nonzero in degree 0 and zero everywhere else. Define a chain map $\eta : \mathbb{Z} \to S_\bullet(X)$ as follow. Let $x_0 \in X$ be the point with respect to which X is star-shaped. Then η is defined in degree 0 by sending $1 \in \mathbb{Z}$ to the 0-simplex corresponding to x_0, and zero in all other degrees.

$$\begin{tikzcd}
\cdots \arrow[r] & C_2(X) \arrow[r, "\partial_2"] \arrow[d, bend left, "\epsilon_2"] & C_1(X) \arrow[r, "\partial_1"] \arrow[d, bend left, "\epsilon_1"] & C_0(X) \arrow[d, bend left, "\epsilon_0"] \\
\cdots \arrow[r] & 0 \arrow[r] \arrow[u, bend left, "\eta_2"] & 0 \arrow[r] \arrow[u, bend left, "\eta_1"] & \mathbb{Z} \arrow[u, bend left, "\eta_0"]
\end{tikzcd}$$

Clearly, $\epsilon \circ \eta = \mathrm{Id}_{\mathbb{Z}}$. We show that η and ϵ are inverse up to chain homotopy by showing that $\eta \circ \epsilon$ is chain homotopic to $\mathrm{Id}_{C_\bullet}$. Let $\sigma \in C_n(X)$. Define $H_n(\sigma) : \Delta^{n+1} \longrightarrow X$ by

$$H_n(\sigma)(t_0, \dots, t_{n+1}) := \begin{cases} t_0 x_0 + (1-t_0)\,\sigma\left(\dfrac{t_1, \dots, t_{n+1}}{1-t_0}\right) & \text{if } t_0 \neq 1, \\ x_0 & \text{if } t_0 = 1, \end{cases}$$

A simple computation shows that

$$\partial_{n+1}\big(H_n(\sigma)\big) = \sigma + \sum_{i=1}^{n+1} (-1)^i\, h_{n-1}\big(\partial_{i-1}(\sigma)\big)$$

As $\eta \circ \epsilon$ vanishes in positive degrees, this can be rewritten as

$$\partial_{n+1} \circ H_n + h_{n-1} \circ \partial_n = \mathrm{Id}_{C_\bullet(X)} - \eta \circ \epsilon$$

for $n > 0$. For $n = 0$, note that we have

$$\partial_1(H_0(\sigma)) = \sigma - x_0 = \mathrm{Id}_{C_\bullet(X)}(\sigma) - \eta \circ \epsilon(\sigma).$$

In any case, we have the formula

$$\partial_{n+1} \circ H_n + h_{n-1} \circ \partial_n = \mathrm{Id}_{C_\bullet(X)} - \eta \circ \epsilon$$

This shows that $\eta \circ \epsilon$ is chain homotopic to $\mathrm{Id}_{C_\bullet(X)}$.

$$\begin{tikzcd}
\cdots \arrow[r] & C_2(X) \arrow[r, "\partial_2"] \arrow[d, "\eta_2 \circ \epsilon_2"] \arrow[dl, "h_2"'] & C_1(X) \arrow[r, "\partial_1"] \arrow[d, "\eta_1 \circ \epsilon_1"] \arrow[dl, "h_1"'] & C_0(X) \arrow[d, "\eta_0 \circ \epsilon_0"] \arrow[dl, "H_0"'] \\
\cdots \arrow[r] & C_2(X) \arrow[r, "\partial_2"] & C_1(X) \arrow[r, "\partial_1"] & C_0(X)
\end{tikzcd}$$

We can now provide an alternative argument proving Proposition 5.3.3. For $X \in \mathsf{Top}$, let $i_{0,1} : X \to X \times I$ denote the two inclusion maps. We have two maps $C_\bullet(X) \to C_\bullet(X \times I)$ induced by i_0 and i_1, and these are natural transformations between the functors

$$X \mapsto C_\bullet(X)$$
$$X \mapsto C_\bullet(X \times I)$$

They induce the same map on zeroth homology because the points $(x, 0)$ and $(x, 1)$ are connected by a path in $X \times I$, and thus the zero-singular simplices $(x, 0)$ and $(x, 1)$ differ by the boundary of a 1-simplex. We can now invoke Proposition 5.4.3(2) to conclude that singular homology is homotopy invariant.

Remark 5.4.7. *There si a version of Proposition 5.4.3 which states that if* C *is a category with models* M*, and let* $\mathsf{F}, \mathsf{G} : \mathsf{C} \to \mathsf{Chain}^{\mathrm{Aug}}_{\mathsf{Mod}_R}$ *be functors to the category of augmented chain complexes. If* F *and* G *are free and acyclic on* M*. Then* F *and* G *are naturally chain equivalent as functors of augmented chain complexes. We will freely invoke this variation of the result below as necessary.*

5.5. Excision in Singular Homology

We now establish that singular homology satisfies the excision axiom. Excision is a statement that homology is "localizable." Intuitively, it asserts that if $A \subseteq X$, and if the n-chains in question lie "sufficiently deep" inside A, then removing a suitable subspace of A does not alter the relative homology groups $H_n(X, A; \mathbb{Z})$. We now state the formal version of the excision property that we aim to prove:

Proposition 5.5.1. *Suppose $Z \subseteq A \subseteq X$ are topological spaces such that $\overline{Z} \subseteq \operatorname{Int}(A)$. Then there is an inclusion of the pair $(X \setminus Z, A \setminus Z) \subseteq (X, A)$, and the induced map*

$$H_n(X \setminus Z, A \setminus Z; \mathbb{Z}) \to H_n(X, A; \mathbb{Z})$$

is an isomorphism for all $n \geq 0$. Equivalently, for subspaces $A, B \subseteq X$ whose interiors cover X, the inclusion $(B, A \cap B) \hookrightarrow (X, A)$ induces isomorphisms

$$H_n(B, A \cap B; \mathbb{Z}) \cong H_n(X, A; \mathbb{Z})$$

Remark 5.5.2. *To see that the two statements of the Excision Theorem are equivalent, just take $B = X \setminus Z$ (or $Z = X \setminus B$). Then $A \cap B = A \setminus Z$, and the condition $\overline{Z} \subset \operatorname{int}(A)$ is equivalent to $X = \operatorname{int}(A) \cup \operatorname{int}(B)$.*

To build some intuition, suppose that $X = \operatorname{int}(A) \cup \operatorname{int}(B)$. In this setting, we might expect that the relative homology group $H_n(X, A)$ remains unaffected when we excise A, i.e., remove it from consideration. This expectation is valid when all singular chains lie entirely within either A or B. However, complications arise when a chain does not lie completely within either A or B. To address this, we employ the technique of barycentric subdivision, which systematically replaces "large" simplices with "smaller" ones that are sufficiently localized to lie within A or B. This technique is also called the locality principle.

Proposition 5.5.3. ***(Locality Principle)*** *Let X be a topological space, and let $\mathcal{U} = \{U_j\}$ be a collection of subspaces of X whose interiors form an open cover of X. Let $C^{\mathcal{U}}_\bullet(X) \subseteq C_\bullet(X)$ denote the chain sub-complex of simplices whose image is contained entirely within one of the open sets of $\mathcal{U}$. The functors $X \mapsto C_\bullet(X)$ and $X \mapsto C^{\mathcal{U}}_\bullet(X)$ are naturally chain homotopic.*

Proof. (Sketch) The proof invokes Proposition 5.4.3. We sketch the details of the argument that allow us to invoke Proposition 5.4.3.

(1) Let C denote the category of spaces with open coverings, which has objects $(X, \mathcal{U})$, where $\mathcal{U}$ is an open cover of X, and whose morphisms respect these coverings in the natural way. Let the model spaces be $\{I^n\}_{n \geq 0}$, where I^n is the n-cube.
(2) One can show that both functors acyclic and free. The argument involves Lebesgue number/compactness argument via Barycentric division.
(3) It can be verified that $H_0(X^{\mathcal{U}}) \cong H_0(X)$ are isomorphic. Here $H_0(X^{\mathcal{U}})$ is the homology with respect to the chain complex $C^{\mathcal{U}}_\bullet(X)$. This essentially follows because $\mathcal{U}$ is an open cover of X.

Therefore, these functors are naturally chain homotopic. □

Proof. (Proposition 5.5.1) Assume that $X = A \cup B$. WLOG, assume that A and B are open sets. We have

$$C^{\mathcal{U}}_n(X) = C_n(A) + C_n(B),$$
$$C_n(A \cap B) = C_n(A) \cap C_n(B).$$

Therefore, we have

$$\frac{C_n(B)}{C_n(A \cap B)} = \frac{C_n(B)}{C_n(A) \cap C_n(B)} \cong \frac{C_n(A) + C_n(B)}{C_n(A)} \cong \frac{C^{\mathcal{U}}_n(X)}{C_n(A)}$$

Proposition 5.5.3 implies that the inclusion

$$C_n^{\mathcal{U}}(X)/C_n(A) \hookrightarrow C_n(X)/C_n(A)$$

induces an isomorphism on homology. Since

$$C_n^{\mathcal{U}}(X)/C_n(A) = \frac{C_n(B)}{C_n(A \cap B)},$$

we have that

$$H_n(B, A \cap B; \mathbb{Z}) \cong H_n(X, A; \mathbb{Z})$$

for each $n \geq 0$. This completes the proof. □

The case of an open cover $\mathcal{U} = \{A, B\}$ consisting of two subsets gives rise to an alternative formulation of the excision principle, known as the Mayer–Vietoris sequence. This is a long exact sequence in homology that provides a powerful computational tool, especially in inductive arguments. Specifically, if a homological property is known to hold for A, B, and their intersection $A \cap B$, the Mayer–Vietoris sequence can be used to deduce that the same property holds for the union $A \cup B$.[5]

Proposition 5.5.4. ***(Mayer-Vietoris Sequence)*** *Let $A, B \subseteq X$ be open sets such that $X = A \cup B$. Let*

$$\begin{aligned} i_A &: A \hookrightarrow X, \\ i_B &: B \hookrightarrow X, \\ j_A &: A \cap B \to A, \\ j_B &: A \cap B \to B \end{aligned}$$

denote inclusions maps. Then there is a long exact sequence

$$\cdots \to H_n(A \cap B; \mathbb{Z}) \xrightarrow{\alpha_n} H_n(A; \mathbb{Z}) \oplus H_n(B; \mathbb{Z}) \xrightarrow{\beta_n} H_n(X; \mathbb{Z}) \xrightarrow{\partial_n} H_{n-1}(A \cap B; \mathbb{Z}) \to \cdots$$

where $\alpha_n = (H_n(i_A), H_n(i_B))$ and $\beta_n = H_n(j_A) - H_n(j_B)$.

Proof. Let $\mathcal{U} = \{A, B\}$. Consider the long exact sequence associated to the short exact sequence of chain complexes:

$$0 \to C_\bullet(A \cap B) \xrightarrow{\alpha} C_\bullet(A) \oplus C_\bullet(B) \xrightarrow{\beta} C_\bullet^{\mathcal{U}}(X) \to 0,$$

The associated long exact sequence in homology yields the desired result by Proposition 5.5.3. □

Remark 5.5.5. *By using augmented chain complexes, we also obtain a corresponding Mayer-Vietoris sequence for the reduced homology groups.*

Example 5.5.6. **(Homology of Spheres)** We can use of the Mayer–Vietoris sequence to carry out our first non-trivial computation: computing homology groups of a sphere in arbitrary dimension. Let $X = \mathbb{S}^n$, $A = \mathbb{S}^n \setminus \{S\}$, and $B = \mathbb{S}^n \setminus \{N\}$, where S and N are the south pole and north pole, respectively. Then

$$A \simeq \mathbb{R}^n \quad B \simeq \mathbb{R}^n \quad A \cap B \simeq \mathbb{S}^{n-1}$$

From the Mayer-Vietoris sequence for reduced homology groups, we get $\widetilde{H}_k(\mathbb{S}^n) \simeq \widetilde{H}_{k-1}(\mathbb{S}^{n-1})$ for all k. By induction, we find that the reduced homology groups of spheres are given as:

$$\widetilde{H}_k(\mathbb{S}^n; \mathbb{Z}) \simeq \begin{cases} \mathbb{Z}, & k = n \\ 0, & k \neq n. \end{cases}$$

[5]The Mayer–Vietoris sequence can also be interpreted as an abelian analogue of the Seifert–van Kampen theorem.

The computation above readily implies the following:

$$H_k(\mathbb{S}^n;\mathbb{Z}) \cong \begin{cases} \mathbb{Z}\oplus\mathbb{Z}, & \text{if } k=0, n=0 \\ \mathbb{Z} & \text{if } k=0, n>1 \\ \mathbb{Z} & \text{if } k=n>0 \\ 0 & \text{otherwise} \end{cases}$$

We end this section by proving the suspension theorem. This result provides a fundamental link between the homology of a space and the homology of its suspension, allowing us to relate different dimensions of homology groups. It also plays a key role in the study of stable phenomena in algebraic topology. Suspension and other categorical constructions will be discussed in more detail later on when we study homotopy theory.

Proposition 5.5.7. ***(Suspension Theorem)*** *Let X be a topological space and let SX be its suspension:*

$$SX = \frac{X\times I}{(X\times\{0\}, X\times\{1\})}$$

We have

$$\widetilde{H}_n(X;\mathbb{Z}) \cong \widetilde{H}_{n+1}(SX;\mathbb{Z})$$

for $n \geq -1$.

Proof. For $n=-1$, $\widetilde{H}_{-1}(X;\mathbb{Z})$ is the trivial group. Since SX is path-connected, $\widetilde{H}_0(SX;\mathbb{Z})$ is also the trivial group. Let $n\geq 0$. Let P, Q denote the collapsed spaces $X\times\{0\}$ and $X\times\{1\}$ respectively. Let $A = SX - \{P\}$ and let $B = SX - \{Q\}$. A and B are homeomorphic to the cone space

$$CX = (X\times I)/(X\times\{0\})$$

By the Mayer-Vietoris sequence for reduced homology, since $A\cap B = X\times(0,1)$, we obtain the exact sequence

$$\cdots\to\widetilde{H}_{n+1}(A;\mathbb{Z})\oplus\widetilde{H}_{n+1}(B;\mathbb{Z})\to\widetilde{H}_{n+1}(SX;\mathbb{Z})\to\widetilde{H}_n(A\cap B;\mathbb{Z})\to\widetilde{H}_n(A;\mathbb{Z})\oplus\widetilde{H}_n(B;\mathbb{Z})\to\cdots$$

for all n. Note that CX is contractible[6]. Moreover, $X\times(0,1)$ deformation retracts down to X. Hence, the sequence simplifies to:

$$\cdots\to 0\to\widetilde{H}_{n+1}(SX;\mathbb{Z})\to\widetilde{H}_n(X;\mathbb{Z})\to 0\to\cdots$$

This proves the claim. □

5.6. Equivalence of Simplicial & Singular Homologies

Let X be a topological space that admits a Δ-complex structure. We say that a subspace $A\subseteq X$ admits a Δ-sub-complex structure on X if A is a union of simplicies of X. Relative simplicial homology group can be defined in the same way as relative (singular) homology groups. That is, the n-th relative simplicial homology group, $H_n^{\Delta}(X,A;\mathbb{Z})$, is the n-th homology group of the chain complex:

$$\cdots\longrightarrow\frac{\Delta_2(X)}{\Delta_2(A)}\xrightarrow{\overline{\partial}_2^{\Delta}}\frac{\Delta_1(X)}{\Delta_1(A)}\xrightarrow{\overline{\partial}_1^{\Delta}}\frac{\Delta_0(X;\mathbb{Z})}{\Delta_0(A)}$$

That is:

$$H_n^{\Delta}(X,A;\mathbb{Z}) = \frac{\operatorname{Ker}\overline{\partial}_n^{\Delta}}{\operatorname{Im}\overline{\partial}_{n+1}^{\Delta}}$$

[6]Indeed, the homotopy $H_t(x,s) = (x,(1-t)s)$ continuously shrinks CX down to its vertex point.

As before, this yields a long exact sequence of simplicial homology groups for the pair $(X, A; \mathbb{Z})$ by the same algebraic argument as for singular homology. We now show that the simplicial homology groups of X corresponding to any Δ-complex structure on X coincides with its singular homology groups of X.

Proposition 5.6.1. *Let X be a topological space that admits a Δ-complex structure and let A be a Δ-subcomplex of X. The inclusion map*

$$\Delta_n(X, A) \hookrightarrow C_n(X, A)$$

induces an isomorphism

$$H_n^{\Delta}(X, A; \mathbb{Z}) \cong H_n(X, A; \mathbb{Z})$$

for each $n \geq 0$. Taking $A = \emptyset$, we obtain the equivalence of absolute singular and simplicial homology.

Remark 5.6.2. *A spectral sequence argument can be used to prove the equivalence of simplicial and singular homologies. However, we choose to present a different argument here. Later on, we will prove that cellular and singular homologies are equivalent via a spectral sequence argument.*

Proof. Our strategy will be to proceed by induction on the skeleta X_k^{Δ}, consisting of all simplices of dimension k or less. The proof proceeds in several steps:

(1) First suppose that X is finite dimensional. That is, $X_m^{\Delta} = \emptyset$ for $m \geq n$ for some $n \in \mathbb{N}$. Assume that $A = \emptyset$. Consider the following diagram:

$$\begin{tikzcd}
H_{n+1}^{\Delta}(X_k^{\Delta}, X_{k-1}^{\Delta}\ Z) \arrow[r] \arrow[d, "f_1"] & H_n^{\Delta}(X_{k-1}^{\Delta}\ Z) \arrow[r] \arrow[d, "f_2"] & H_n^{\Delta}(X_k^{\Delta}\ Z) \arrow[r] \arrow[d, "f_3"] & H_n^{\Delta}(X_k^{\Delta}, X_{k-1}^{\Delta}\ Z) \arrow[r] \arrow[d, "f_4"] & H_{n-1}^{\Delta}(X_{k-1}^{\Delta}\ Z) \arrow[d, "f_5"] \\
H_{n+1}(X_k^{\Delta}, X_{k-1}^{\Delta}\ Z) \arrow[r] & H_n(X_{k-1}^{\Delta}\ Z) \arrow[r] & H_n(X_k^{\Delta}\ Z) \arrow[r] & H_n(X_k^{\Delta}, X_{k-1}^{\Delta}\ Z) \arrow[r] & H_{n-1}(X_{k-1}^{\Delta}\ Z)
\end{tikzcd}$$

Note that $\Delta_k(X_k^{\Delta}, X_{k-1}^{\Delta})$ is a free abelian group generated by k-simplices and $\Delta_n(X_k^{\Delta}, X_{k-1}^{\Delta}) = \emptyset$ for $n \neq k$. Therefore, we have:

$$\Delta_k(X_k^{\Delta}, X_{k-1}^{\Delta}) = \begin{cases} \text{free abelian group generated by } k\text{-simplices} & \text{if } n = k \\ \emptyset & \text{if } n \neq k \end{cases}$$

A simple calculation shows that:

$$H_n^{\Delta}(X_k^{\Delta}, X_{k-1}^{\Delta}\ Z) = \begin{cases} \mathbb{Z}^{\#k-\text{simplicies}} & \text{if } n = k \\ 0 & \text{if } n \neq k \end{cases}$$

It is easy to check that $(X_k^{\Delta}, X_{k-1}^{\Delta})$ is a good pair and

$$X_k^{\Delta}/X_{k-1}^{\Delta} = \bigvee_{i=1}^{\#k-\text{simplicies}} \mathbb{S}^k$$

Therefore, Corollary 6.1.9 implies

$$H_n(X_k^{\Delta}, X_{k-1}^{\Delta}\ Z) = \begin{cases} \mathbb{Z}^{\#k-\text{simplicies}} & \text{if } n = k \\ 0 & \text{if } n \neq k \end{cases}$$

Therefore, both f_1 and f_4 are isomorphisms. An induction argument shows that f_2 and f_5 are isomorphisms. The five lemma (Proposition 4.6.1) then implies that f_3 is an isomorphism.

(2) Suppose that X is possibly infinite-dimensional. Assume that $A = \emptyset$. Note that a compact set $C \subseteq X$ can meet only finitely many open simplices of X. If not, C would contain an infinite sequence of points x_i, each lying in a different open simplex. Then the sets

$$U_i = X - \bigcup_{i \neq j} \{x_j\}$$

which are open since their pre-images under the characteristic maps of all the simplices are clearly open, form an open cover of C with no finite sub-cover. This can be applied to show the map

$$H_n^\Delta(X; \mathbb{Z}) \to H_n(X; \mathbb{Z})$$

is bijective. For surjectivity, let $[c] \in H_n^\Delta(X; \mathbb{Z})$. Choose a representative n-cycle, α, of $[c]$. Now α is a linear combination of finitely many singular simplices with compact images, meeting only finitely many open simplices of X. Hence, α in X_k^Δ for some k. We have shown that

$$H_n(X_k^\Delta\ Z) \cong H_n^\Delta(X_k^\Delta\ Z)$$

So there exists a n-cycle $v \in \Delta_n(X_k^\Delta)$ such that $[v]$ gets mapped to $[c]$. This proves surjectivity. Injectivity is similar so we omit details.

(3) Now consider the general case where $A \neq \emptyset$. Consider the following diagram:

$$\begin{tikzcd}
H_{n+1}^\Delta(X, A; \mathbb{Z}) \arrow[r] \arrow[d, "f_1"] & H_n^\Delta(A; \mathbb{Z}) \arrow[r] \arrow[d, "f_2"] & H_n^\Delta(X; \mathbb{Z}) \arrow[r] \arrow[d, "f_3"] & H_n^\Delta(X, A; \mathbb{Z}) \arrow[r] \arrow[d, "f_4"] & H_{n-1}^\Delta(A; \mathbb{Z}) \arrow[d, "f_5"] \\
H_{n+1}(X, A; \mathbb{Z}) \arrow[r] & H_n(A; \mathbb{Z}) \arrow[r] & H_n(X; \mathbb{Z}) \arrow[r] & H_n(X, A; \mathbb{Z}) \arrow[r] & H_{n-1}(A; \mathbb{Z})
\end{tikzcd}$$

By (2), f_2, f_3, f_5 are isomorphisms. The claim now follows by induction and the five-lemma. This completes the proof. □

Example 5.6.3. Let $X = \mathbb{S}^1 \times \mathbb{S}^1$. Note that X is homemorphic to the torus, T, considered in Section 4.1. Hence, Proposition 5.6.1 implies that

$$H_n(X; \mathbb{Z}) = H_n^\Delta(T; \mathbb{Z}) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & \text{for } n = 1 \\ \mathbb{Z} & \text{for } n = 0, 2 \\ 0 & \text{for } n \geq 3 \end{cases}$$

CHAPTER 6

Computations & Applications

Building upon the foundations of singular homology, practical computations and significant applications arise through tools such as cellular homology, which provides an effective computational framework for CW complexes. The Euler characteristic emerges as a fundamental invariant linking topological and algebraic properties. Key theoretical results—including the Universal Coefficient Theorem (UCT), relating homology groups with different coefficients, and the Künneth theorem, describing the homology of product spaces—further enhance computational methods. Together, these concepts illustrate the broad applicability and strength of homology theory in both calculation and topological understanding. References include [Hat02; Lee10; May99].

6.1. Interpretation of Relative Homology

We first examine relative homology in greater depth in order to develop a clearer understanding of its structure and significance. This deeper insight will provide us tools to better interpret relative homology groups and their applications.

Lemma 6.1.1. *Let $A \subseteq X$ be topological spaces. Consider an exact sequence of abelian groups:*

$$A \to B \to C \to D \to E$$

1. *$C = 0$ if and only if the map $A \to B$ is surjective and $D \to E$ is injective.*
2. *For a pair of spaces $(X, A) \in \mathsf{Top}^2$, the inclusion $A \hookrightarrow X$ induces isomorphisms on all homology groups if and only if $H_n(X, A; \mathbb{Z}) = 0$ for all $n \geq 0$.*

Proof. The proof is as follows:

1. Let $\alpha, \beta, \gamma, \delta$ be the corresponding maps. By exactness,
$$\operatorname{im}(\alpha) = \ker(\beta), \quad \operatorname{im}(\beta) = \ker(\gamma), \quad \operatorname{im}(\gamma) = \ker(\delta).$$
Note that α is surjective iff $\ker(\beta) = B$ iff $\operatorname{im}(\beta) = 0$, and δ is injective iff $\operatorname{im}(\gamma) = 0$ iff $\ker(\gamma) = C$. Putting both together, α is surjective and δ is injective iff $C = 0$, since $\operatorname{im}(\beta) = \ker(\gamma)$.
2. Consider the following part of the the long exact sequence in homology:
$$\cdots \to H_{n+1}(X;\mathbb{Z}) \to H_{n+1}(X, A; \mathbb{Z}) \to H_n(A;\mathbb{Z}) \to H_n(X;\mathbb{Z}) \to \cdots$$
The maps $H_n(A;\mathbb{Z}) \to H_n(X;\mathbb{Z})$ are isomorphisms for all $n \geq 0$ if and only if they are both injective and surjective for all $n \geq 0$. By re-indexing, this is true if and only if the leftmost map in our five-term exact sequence is surjective and the rightmost map is injective for all $n \geq 0$. But (1), this is true if and only if the middle group vanishes for all $n \geq 0$.

This completes the proof. □

As per Lemma 6.1.1, we can think of $H_n(X, A; \mathbb{Z})$ as measuring the failure of the induced morphism $H_n(A;\mathbb{Z}) \to H_n(X;\mathbb{Z})$ to be an isomorphism for each $n \geq 0$. Based on Lemma 6.1.1, we can characterize relative homology groups for $n = 0, 1$.

Proposition 6.1.2. *Let $A \subseteq X$ be topological spaces.*

(1) $H_0(X, A; \mathbb{Z}) = 0$ *if and only if A meets each path-component of X. In other words,*
$$H_0(X, A; \mathbb{Z}) = \mathbb{Z}[\textit{path-components of } X \textit{ not intersecting } A]$$
(2) $H_1(X, A; \mathbb{Z}) = 0$ *if and only if $H_1(A; \mathbb{Z}) \to H_1(X; \mathbb{Z})$ is surjective and each path-component of X contains at most one path-component of A.*
(3) *Let (X, x_0) be a pointed topological space. Then*
$$H_n(X, x_0; \mathbb{Z}) \cong H_n(X; \mathbb{Z}) \cong \widetilde{H}_n(X; \mathbb{Z})$$
for each $n \geq 1$.

Proof. The proof is given below:

(1) We first prove the special case that if X is a non-empty *path-connected* space and $A \subseteq X$, then $H_0(X, A; \mathbb{Z}) = 0$ if and only if A is not-empty. Consider the end of the long exact sequence for the pair $(X, A; \mathbb{Z})$:
$$H_0(A; \mathbb{Z}) \to \mathbb{Z} \to H_0(X, A; \mathbb{Z}) \to 0$$
If A is empty, the sequence is,
$$0 \to \mathbb{Z} \to H_0(X, A; \mathbb{Z}) \to 0$$
Hence, $\mathbb{Z} \cong H_0(X, A; \mathbb{Z})$ and $H_0(X, A; \mathbb{Z})$ must be non-zero. If A is non-empty, pick a point $a \in A$ and consider the homology class $[a] \in H_0(A; \mathbb{Z})$. The image of $[a]$ under
$$H_0(A; \mathbb{Z}) \to \mathbb{Z}$$
is the homology class of a point, which generates the co-domain. So $H_0(A; \mathbb{Z}) \to \mathbb{Z}$ is surjective. Hence
$$H_0(A; \mathbb{Z}) \to H_0(X, A; \mathbb{Z})$$
is surjective as well implying that and $H_0(X, A; \mathbb{Z}) = 0$. More generally, suppose X has multiple connected components. Assume that A meets each path component of X. If X_i is a component of X, then $H_0(A \cap X_i; \mathbb{Z}) \to H_0(X_i; \mathbb{Z})$ is surjective. But then
$$H_0(A; \mathbb{Z}) = \bigoplus_i H_0(A \cap X_i; \mathbb{Z}) \to \bigoplus_i H_0(X_i; \mathbb{Z}) = H_0(X; \mathbb{Z})$$
is surjective. Therefore, So $H_0(X, A; \mathbb{Z}) = 0$. Conversely, if A does not meet a component of X, say X_j, then $H_0(X_j, A; \mathbb{Z}) \neq 0$. But then $H_0(X_j, A; \mathbb{Z}) \neq 0$ is a direct summand of $H_0(X, A; \mathbb{Z})$. Hence $H_0(X, A; \mathbb{Z})$ must be non-zero.

(2) If $H_1(X, A; \mathbb{Z}) = 0$, then $H_1(A; \mathbb{Z}) \to H_1(X; \mathbb{Z})$ is surjective and $H_0(A; \mathbb{Z}) \to H_0(X; \mathbb{Z})$ is injective by Lemma 6.1.1. This last statement can't be true if some path component X_i of X contains multiple components of A because then $H_0(A \cap X_i) \cong \mathbb{Z}^n$ for some $n \geq 2$ while $H_0(X_i) = \mathbb{Z}$. So then
$$H_0(A \cap X_i) \to H_0(X_i)$$
can't be one-to-one, and the same follows for
$$H_0(A; \mathbb{Z}) \to H_0(X; \mathbb{Z})$$
If $H_1(A; \mathbb{Z}) \to H_1(X; \mathbb{Z})$ is surjective, then the kernel of the map $H_1(X; \mathbb{Z}) \to H_1(X, A; \mathbb{Z})$ is $H_1(X; \mathbb{Z})$. So the map $H_1(X; \mathbb{Z}) \to H_1(X, A; \mathbb{Z})$ is the 0 map. Similarly, if each component of X contains at most one component of A, then $H_0(A; \mathbb{Z}) \to H_0(X; \mathbb{Z})$ is injective. So its kernel is 0, so the image of $H_1(X, A; \mathbb{Z}) \to H_0(A; \mathbb{Z})$ is 0. But then by exactness, $0 = H_1(X, A; \mathbb{Z})$.

(3) If $n \geq 2$, then $H_n(X;\mathbb{Z}) = 0$ and $H_{n-1}(x;\mathbb{Z}) = 0$, and thus we immediately see $H_n(X, x_0) \cong H_n(X;\mathbb{Z})$ by inspecting the long exact sequence in relative homology. For $n = 1$, consider the following part of the long exact sequence in relative homology:

$$0 \to H_1(X;\mathbb{Z}) \to H_1(X, x_0;\mathbb{Z}) \to \mathbb{Z} \to H_0(X;\mathbb{Z}) \to H_0(X, x_0;\mathbb{Z}) \to 0$$

Proposition 6.1.2(1) readily implies that

$$0 \to H_1(X;\mathbb{Z}) \to H_1(X, x;\mathbb{Z})$$

is surjective if and only if $\mathbb{Z} \to H_0(X;\mathbb{Z})$ is injective if and only if it is not the zero map. The last equivalence follows from the observation that $H_0(X;\mathbb{Z})$ is a free abelian group. If it were the zero map, the map $H_0(X;\mathbb{Z}) \to H_0(X, x_0;\mathbb{Z})$ will be injective. However, this is not the case since the point $x_0 \in X$ defines a generator $\langle x_0 \rangle$ of $H_0(X;\mathbb{Z})$ that is is in the kernel of the map $H_0(X;\mathbb{Z}) \to H_0(X, x_0)$. Therefore, the claim is true for $n = 1$ as well.

This completes the proof. □

We now explore the relationship between reduced and relative homology.

Definition 6.1.3. Let (X, A) be in Top^2. If $A \subseteq X$ is a closed subspace such that there exists a neighborhood V of X such that A is a strong deformation retract of V.

Remark 6.1.4. *Colloquially, we say that a pair of topological spaces (X, A) satisfying Definition 6.1.3 is a 'good' of topological spaces.*

The next proposition provides an alternative interpretation of relative homology in many cases of interest. Intuitively, it asserts that for *good* pairs of topological spaces (X, A), the relative homology group of (X, A) is isomorphic to the reduced homology of the quotient space X/A.

Proposition 6.1.5. *Let $(X, A) \in \mathsf{Top}^2$ such that there exists $B \subseteq X$ such that the following conditions are satisfied:*

(1) $A \subseteq B \subseteq X$,
(2) $\overline{A} \subseteq \operatorname{Int}(B)$,
(3) *A is a strong deformation retract of B.*

For each $n \geq 0$, we have

$$H_n(X, A;\mathbb{Z}) \cong H_n(X/A, *;\mathbb{Z}) \cong \widetilde{H}_n(X/A;\mathbb{Z})$$

Remark 6.1.6. *A pair (X, A) satisfying the hypothesis in Proposition 6.1.5 is called a 'good' pair. The associated triple (A, B, X) is called an excisive triple. Note that these conditions are satisfied is A is a closed subspace and there exists an open neighbourhood of A that retracts surjective A.*

Proof. Consider the following diagram:

$$\begin{tikzcd}
H_n(X, A;\mathbb{Z}) \arrow[r] \arrow[d] & H_n(X, B;\mathbb{Z}) \arrow[d] & H_n(X \setminus A, B \setminus A;\mathbb{Z}) \arrow[l] \arrow[d] \\
H_n(X/A, *;\mathbb{Z}) \arrow[r] & H_n(X/A, B/A;\mathbb{Z}) & H_n(X/A \setminus *, B/A \setminus *;\mathbb{Z}) \arrow[l]
\end{tikzcd}$$

We make the following observations:

(1) The upper-left horizontal map is an isomorphism by virtue of the long exact sequence of the triple (X, B, A) (see Remark 5.2.6) since the groups $H_n(B, A)$ vanish for all $n \geq 0$. This vanishing follows from the fact that a deformation retraction of B surjective A induces a homotopy equivalence of pairs $(B, A) \simeq (A, A) \simeq (*, *)$. Hence, $H_n(B, A) \cong H_n(*, *) = 0$.

(2) The deformation retraction of B surjective A descends to a deformation retraction of the quotient B/A surjective the point A/A[1]. Therefore, the same reasoning as above applies to show that the lower-left horizontal map is also an isomorphism.
(3) The remaining two horizontal maps are isomorphisms by the excision property.

Since the right-hand vertical map is an isomorphism, it follows by the five lemma (Proposition 4.6.1) that the left-hand vertical map must also be an isomorphism. This completes the proof. □

Corollary 6.1.7. *If $(X, A; \mathbb{Z})$ is a* good *pair, then there is an exact sequence:*

$$\cdots \to \widetilde{H}_n(A;\mathbb{Z}) \to \widetilde{H}_n(X;\mathbb{Z}) \to \widetilde{H}_n(X/A;\mathbb{Z}) \to \widetilde{H}_{n-1}(A;\mathbb{Z}) \to \widetilde{H}_{n-1}(X;\mathbb{Z}) \to \cdots$$

Proof. This is clear. □

Example 6.1.8. (**Homology of Spheres**) We first recompute the homology groups of spheres using the long exact sequence in relative homology. The reduced homology groups of spheres are given as:

$$\widetilde{H}_k(\mathbb{S}^n;\mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & \text{if } k = n \\ 0, & \text{if } k \neq n \end{cases}$$

Since $(\mathbb{D}^n, \mathbb{S}^{n-1})$ is a *good* pair and $\mathbb{D}^n/\mathbb{S}^{n-1} \cong \mathbb{S}^n$, the long exact sequence in relative reduced homology yields:

$$\cdots \to \widetilde{H}_k(\mathbb{S}^{n-1};\mathbb{Z}) \to \widetilde{H}_k(\mathbb{D}^n;\mathbb{Z}) \to \widetilde{H}_k(\mathbb{S}^n;\mathbb{Z}) \to \widetilde{H}_{k-1}(\mathbb{S}^{n-1};\mathbb{Z}) \to \widetilde{H}_{k-1}(\mathbb{D}^n;\mathbb{Z}) \to \cdots$$

Since $\mathbb{D}^n$ is contractible, $\widetilde{H}_k(\mathbb{D}^n;\mathbb{Z}) = 0$ for $k \geq 0$. Therefore,

$$\widetilde{H}_k(\mathbb{S}^{n-1};\mathbb{Z}) \to 0 \to \widetilde{H}_k(\mathbb{S}^n;\mathbb{Z}) \to \widetilde{H}_{k-1}(\mathbb{S}^{n-1};\mathbb{Z}) \to 0 \to \cdots$$

Hence, we have:

$$\widetilde{H}_k(\mathbb{S}^n;\mathbb{Z}) \cong \widetilde{H}_{k-1}(\mathbb{S}^n;\mathbb{Z})$$

The result now follows via induction and the observation that

$$\widetilde{H}_0(\mathbb{S}^0;\mathbb{Z}) \cong \mathbb{Z} \qquad \widetilde{H}_k(\mathbb{S}^0;\mathbb{Z}) \cong 0 \quad k > 0$$

The computation above readily implies the following:

$$H_k(\mathbb{S}^n;\mathbb{Z}) \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z}, & \text{if } k = 0, n = 0 \\ \mathbb{Z} & \text{if } k = 0, n > 1 \\ \mathbb{Z} & \text{if } k = n > 0 \\ 0 & \text{otherwise} \end{cases}$$

Corollary 6.1.9. *Let $(X_\alpha, x_\alpha)_{\alpha \in I}$ be a collection of* good *pairs in Top_*. Let $X = \bigvee_{\alpha \in I} X_\alpha$ with the basepoint $x = (x_\alpha)_{\alpha \in I}$ in Top_*. Then*

$$\widetilde{H}_n(X;\mathbb{Z}) \cong \bigoplus_{\alpha \in I} \widetilde{H}_n(X_\alpha;\mathbb{Z})$$

for $n \geq 1$.

[1]This uses the fact that taking the product with I commutes with taking quotients, since I is a compact Hausdorff space.

PROOF. Since $(X_\alpha, x_\alpha)_{\alpha\in I}$ be a collection of *good* pairs, (X,x) is also a *good* pair. We have:

$$\begin{aligned}\widetilde{H}_n(X;\mathbb{Z}) &= \widetilde{H}_n\left(\coprod_{\alpha\in I} X_\alpha \middle/ \coprod_{\alpha\in I}\{x_\alpha\};\mathbb{Z}\right)\\ &\cong H_n\left(\coprod_{\alpha\in I} X_\alpha, \coprod_{\alpha\in I}\{x_\alpha\};\mathbb{Z}\right)\\ &\cong \bigoplus_{\alpha\in I} H_n(X_\alpha, x_\alpha;\mathbb{Z})\\ &\cong \bigoplus_{\alpha\in I} \widetilde{H}_n(X_\alpha;\mathbb{Z}).\end{aligned}$$

The first and third equivalences follow by Proposition 6.1.2. The second equivalence follows by observing that the additivity axiom holds in Top^2 as can be checked. □

Remark 6.1.10. *Let $X = \mathbb{S}^1\times\mathbb{S}^1$ and $Y = \mathbb{S}^1\vee\mathbb{S}^1\vee\mathbb{S}^2$. Using Corollary 6.1.9 we have that:*

$$H_n(Y) = \begin{cases}\mathbb{Z}\oplus\mathbb{Z} & \textit{for } n=1\\ \mathbb{Z} & \textit{for } n=0,2\\ 0 & \textit{for } n\geq 3\end{cases}$$

We see that

$$H_n(X;\mathbb{Z}) = H_n(Y)$$

for $n\geq 0$. It can be checked that the covering spaces of X and Y have different homology groups. Hence, X and Y are not homotopy equivalent. Therefore, singular homology is not a complete topological invariant, as it may fail to distinguish between topological spaces that are not homotopy equivalent.

6.2. Local Homology

Using the homology groups of spheres, we can distinguish spheres from Euclidean spaces in any dimension. This distinction allows us to prove the invariance of dimension (Remark 1.3.3) and the invariance of boundary (Remark 1.3.9) for topological manifolds. It also provides the foundation for introducing the concept of local homology groups.

Corollary 6.2.1. *Let $m\neq n$. The following statements are true:*

(1) *$\mathbb{R}^m$ and $\mathbb{R}^n$ are not homeomorphic or even homotopy equivalent.*
(2) *If $U\subseteq\mathbb{R}^m$ and $V\subseteq\mathbb{R}^n$ are non-empty homeomorphic open sets, then $m=n$.*

PROOF. Note that $\mathbb{S}^m$ is not homotopy equivalence to $\mathbb{S}^n$. This follows from Example 6.1.8 since the homology groups are not isomorphic for $m\neq n$.

(1) If m or n is zero, this is clear. So let $m,n>0$. Assume we have a homeomorphism $f:\mathbb{R}^m\to\mathbb{R}^n$. WLOG assume that $f(0)=0$. This restricts to a homeomorphism $\mathbb{R}^m\setminus\{0\}\to\mathbb{R}^n\setminus\{f(0)\}$. But these spaces are homotopy equivalent to spheres of different dimension, yielding a contradiction.
(2) For all $x\in U$ and for all $k\in\mathbb{Z}$, we have

$$H_k(U,U\setminus\{x\})\cong H_k(\mathbb{R}^m,\mathbb{R}^m\setminus\{x\})$$

by the Excision Theorem. Combining this with the long exact sequence for the reduced homology of $(\mathbb{R}^m,\mathbb{R}^m\setminus\{x\})$ and the fact that $\mathbb{R}^m\setminus\{x\}$ is homotopy equivalent to $\mathbb{S}^{m-1}$, we obtain for all $x\in U$ and all $k\in\mathbb{Z}$:

$$H_k(U, U \setminus \{x\}) \cong H_k(\mathbb{R}^m, \mathbb{R}^m \setminus \{x\}) \cong \widetilde{H}_{k-1}(\mathbb{R}^m \setminus \{x\}) \cong \begin{cases} \mathbb{Z}, & k = m \\ 0, & k \neq m. \end{cases}$$

Similarly,

$$H_k(V, V \setminus \{x\}) \cong H_k(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\}) \cong \widetilde{H}_{k-1}(\mathbb{R}^n \setminus \{x\}) \cong \begin{cases} \mathbb{Z}, & k = n \\ 0, & k \neq n. \end{cases}$$

If U, V are homeomorphic via $f : U \to V$, then

$$H_k(U, U \setminus \{x\}) \cong H_k(V, V \setminus \{f(x)\})$$

The claim follows by comparing homology groups.

This completes the proof. □

Remark 6.2.2. *If X is a topological space, $x \in X$, and $U \subseteq X$ is an open neighborhood of x, then for all $n \in \mathbb{Z}$, the excision theorem yields that*

$$H_n(X, X \setminus \{x\}) \cong H_n(U, U \setminus \{x\}).$$

In particular, for all $n \in \mathbb{Z}$, the group $H_n(X, X \setminus \{x\})$ depends only on the topology of a neighborhood of x. Therefore, these homology groups are called the local homology groups of X at x.

Using the discussion about the homology groups of spheres and Corollary 6.2.1, we can prove the invariance of dimension (Remark 1.3.3) and the invariance of boundary (Remark 1.3.9) results about topological manifolds.

Proof. Let X be a topological n-manifold.

(1) Working in local coordinate charts, this follows from Corollary 6.2.1(2).
(2) This is skipped.

This completes the proof. □

6.3. Cellular Homology

We define the cellular homology of a CW complex X in terms of a given cell structure, then we show that it coincides with the singular homology, so it is in fact independent on the cell structure. Cellular homology is very useful for computations. Before discussing cellular homology, we compute the relative homology groups of a topological space, X, that can be given the structure of a CW complex.

Lemma 6.3.1. *Let X be a topological space that can be endowed with the structure of a CW complex. Then:*

(1) *The relative romology $H_k(X^n, X^{n-1}; \mathbb{Z})$ is given by:*

$$H_k(X^n, X^{n-1}; \mathbb{Z}) = \begin{cases} 0, & \text{if } k \neq n \\ \mathbb{Z}^{\#n\text{-cells}}, & \text{if } k = n. \end{cases}$$

for $k \geq 1$.

(2) *$H_k(X^n; \mathbb{Z}) = 0$ if $k > n \geq 1$. In particular, if X is finite dimensional, then $H_k(X; \mathbb{Z}) = 0$ if $k > \dim(X)$.*
(3) *The inclusion $i : X^n \hookrightarrow X$ induces an isomorphism $H_k(X^n; \mathbb{Z}) \cong H_k(X)$ if $k < n$.*

Proof. The proof is given below:

(1) Since (X^n, X^{n-1}) is a *good* pair, we have:

$$\begin{aligned} H_k(X^n, X^{n-1}; \mathbb{Z}) &\cong \widetilde{H}_k(X^n/X^{n-1}; \mathbb{Z}) \\ &= H_k(X^n/X^{n-1}; \mathbb{Z}) \\ &\cong \bigvee_{i=1}^{\#n\text{-cells}} \mathbb{S}^n \cong \begin{cases} 0, & \text{if } k \neq n, \\ \mathbb{Z}^{\#n\text{-cells}}, & \text{if } k = n. \end{cases} \end{aligned}$$

(2) Since (X^n, X^{n-1}) is a *good* pair for each $n \geq 1$, we can consider the following portion of the long exact sequence:

$$H_{k+1}(X^n, X^{n-1}; \mathbb{Z}) \to H_k(X^{n-1}; \mathbb{Z}) \to H_k(X^n; \mathbb{Z}) \to H_k(X^n, X^{n-1}; \mathbb{Z})$$

If $k+1 \neq n$ and $k \neq n$, we have from (1) we have that $H_{k+1}(X^n, X^{n-1}; \mathbb{Z}) = 0$ and $H_k(X^n, X^{n-1}) = 0$. Thus

$$H_k(X^{n-1}; \mathbb{Z}) \cong H_k(X^n; \mathbb{Z})$$

Hence, if $k > n$ (so in particular, $n \neq k+1$ and $n \neq k$), we get by iteration that

$$H_k(X^n; \mathbb{Z}) \cong H_k(X^{n-1}; \mathbb{Z}) \cong \cdots \cong H_k(X^0; \mathbb{Z})$$

Note that X^0 is just a collection of points, so $H_k(X^0; \mathbb{Z}) = 0$. Thus, when $k > n \geq 1$, we have $H_k(X^n; \mathbb{Z}) = 0$ as desired.

(3) We only prove the statement for finite-dimensional CW complexes. Let $k < n$, and consider the following portion of the long exact sequence:

$$\cdots \to H_{k+1}(X^{n+1}, X^n; \mathbb{Z}) \to H_k(X^n; \mathbb{Z}) \to H_k(X^{n+1}; \mathbb{Z}) \to \cdots$$

Since $k < n$, we have $k+1 \neq n+1$ and $k \neq n+1$, so by part (1), we get that $H_{k+1}(X^{n+1}, X^n; \mathbb{Z}) = 0$ and $H_k(X^{n+1}, X^n; \mathbb{Z}) = 0$. Thus, $H_k(X^n) \cong H_k(X^{n+1}; \mathbb{Z})$. By repeated iteration, we obtain:

$$H_k(X^n; \mathbb{Z}) \cong H_k(X^{n+1}; \mathbb{Z}) \cong \cdots \cong H_k(X^{n+l}; \mathbb{Z}) = H_k(X; \mathbb{Z}),$$

where l is such that $X^{n+l} = X$ since we assumed X is finite dimensional. See [Hat02] for the case when X is infinite-dimensional.

This completes the proof. □

In what follows we define the cellular homology of a CW complex, X, in terms of a given cell structure, then we show that it coincides with the singular homology.

Definition 6.3.2. The cellular homology $H^{\mathrm{CW}}(X)$ of a CW complex X is the homology of the cellular chain complex $(C_\bullet(X), d_*)$ indexed by the cells of X, i.e.,

$$C_n(X) := H_n(X^n, X^{n-1}; \mathbb{Z}) = \mathbb{Z}^{\#n\text{-cells}},$$

and with differentials $d_n : C_n(X) \to C_{n-1}(X)$ defined by the following diagram: d_n etc. are defined in the obvious way to make the diagram commute. It is easy to check that $d_{n+1} \circ d_n = 0$ since the composition of these two maps induces two successive maps in one of the diagonal exact sequences.

$$
\begin{tikzcd}
 & & & & H_n(X^{n+1},X^n;\mathbb{Z})=0 & \\
 & 0=H_n(X^{n-1};\mathbb{Z}) \arrow[dr] & & H_n(X^{n+1};\mathbb{Z})\cong H_n(X;\mathbb{Z}) \arrow[ur] & & \\
 & & H_n(X^n;\mathbb{Z}) \arrow[ur, "i_n"] \arrow[dr, "j_n"] & & & \\
 & H_{n+1}(X^{n+1},X^n;\mathbb{Z}) \arrow[ur, "\partial_{n+1}"] \arrow[rr, "d_{n+1}"] & & H_n(X^n,X^{n-1};\mathbb{Z}) \arrow[rr, "d_n"] \arrow[dr, "\partial_n"'] & & H_{n-1}(X^{n-1},X^{n-2};\mathbb{Z}) \\
 & & & & H_{n-1}(X^{n-1};\mathbb{Z}) \arrow[ur, "j_{n-1}"] & \\
 & & & H_{n-1}(X^{n-2};\mathbb{Z})=0 \arrow[ur] & &
\end{tikzcd}
$$

Proposition 6.3.3. *Let X be a topological space that admits a CW-complex structure. We have:*

$$H_n^{\mathrm{CW}}(X)\cong H_n(X;\mathbb{Z})$$

for all $n\geq 0$, where $H_n(X;\mathbb{Z})$ is the singular homology of X.

Proof. We present an argument based on spectral sequences. Let $C_\bullet(X)$ denote its singular chain complex. We filter $C_\bullet(X)$ by setting

$$F^pC_n(X)=\{\sigma\in C_n(X)\mid \sigma|_{C_n(X^p)}=0\}=\ker(C_n(X^p)\to C_n(X)),$$

where the map $C_n(X^p)\to C_n(X)$ is the natural restriction map. This defines an increasing filtration. By the analog of Proposition 4.5.5, we get a homological spectral sequence such that

$$E^0_{p,q}=\mathrm{G}^p\,C_{p+q}(X)\Rightarrow H_{p+q}(X).$$

We claim that $E^0_{p,q}\cong C_{p+q}(X^{p+1},X^p)$. Note that we have a homomorphism of short exact sequences:

$$
\begin{tikzcd}
0 \arrow[r] & F^{p-1}C_{p+q}(X) \arrow[r] \arrow[d] & C_{p+q}(X) \arrow[r] \arrow[d, "\mathrm{Id}_{C_{p+q}(X)}"] & C_{p+q}(X^{p+1}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F^pC_{p+q}(X) \arrow[r] & C_{p+q}(X) \arrow[r] & C_{p+q}(X^p) \arrow[r] & 0
\end{tikzcd}
$$

Since the middle map is an isomorphism, the snake lemma (Proposition 4.6.3) tells us that the left map is injective, and that its cokernel is isomorphic to $\ker(C_{p+q}(X^{p+1})\to C_{p+q}(X^p))$. Hence, we have

$$E^0_{p,q}\cong\frac{F^pC_{p+q}(X)}{F^{p-1}C_{p+q}(X)}\cong\ker(C_{p+q}(X^{p+1})\to C_{p+q}(X^p))=C_{p+q}(X^{p+1},X^p)$$

Hence, the E^1 page is defined such that:

$$E^1_{p,q}=H_{p+q}(X^{p+1},X^p)$$

Recall that $H_{p+q}(X^{p+1},X^p)=0$ if $q\neq 1$, so the only nontrivial differentials on the E_1 page are

$$d^1_{p,1}:H_{p+q}(X^{p+1},X^p)\to H_{p+2}(X^{p+2},X^{p+1}).$$

One easily checks that these agree with the differentials defining cellular homology, so the E_2 page is given by

$$E_2^{p,q}=\begin{cases}H^{\mathrm{CW}}_{p+1}(X) & \text{if } q=1,\\ 0 & \text{otherwise.}\end{cases}$$

Since there are no further non-trivial differentials in the spectral sequence, we have $E^2_{p,q} = E^\infty_{p,q}$. Moreover, because each diagonal $p+q=n$ contains at most one nonzero term, it follows that the associated graded pieces stabilize, and we obtain an isomorphism

$$H_p(X) \cong E^\infty_{p-1,1} \cong H^{\mathrm{CW}}_p(X).$$

Therefore, the singular and cellular cohomology groups of X are isomorphic. □

Let's make some observations which are immediate:

(1) If X has no n-cells, then $H_n(X;\mathbb{Z}) = 0$. Indeed, in this case we have $C_n = H_n(X^n, X^{n-1};\mathbb{Z}) = 0$. Therefore, $H^{\mathrm{CW}}_n(X;\mathbb{Z}) = 0$.
(2) If X is connected and has a single 0-cell, then $d_1 : C_1 \to C_0$ is the zero map. Indeed, since X contains only a single 0-cell, $C_0 = \mathbb{Z}$. Also, since X is connected, $H_0(X) = \mathbb{Z}$. So, by the above theorem, $\mathbb{Z} = H_0(X;\mathbb{Z}) = \ker d_0/\operatorname{Im} d_1 = \mathbb{Z}/\operatorname{Im} d_1$. This implies that $\operatorname{Im} d_1 = 0$, so d_1 is the zero map as desired.

If X has no cells in adjacent dimensions, then $d_n = 0$ for all n, and $H_n(X;\mathbb{Z}) \cong \mathbb{Z}^{\#n\text{-cells}}$ for all n. Indeed, in this case, all maps d_n vanish. So for any n, $H^{\mathrm{CW}}_n(X) \cong C_n \cong \mathbb{Z}^{\#n\text{-cells}}$. Let's look at two examples:

Example 6.3.4. When $n > 1$, $\mathbb{S}^n \times \mathbb{S}^n$ has one 0-cell, two n-cells, and one $2n$-cell. Since $n > 1$, these cells are not in adjacent dimensions. Hence:

$$H_k(\mathbb{S}^n \times \mathbb{S}^n;\mathbb{Z}) = \begin{cases} \mathbb{Z}, & \text{if } k = 0, 2n \\ \mathbb{Z}^2, & \text{if } k = n \\ 0, & \text{otherwise.} \end{cases}$$

Example 6.3.5. Recall that $\mathbb{CP}^n$ has one cell in each even dimension $0, 2, 4, \dots, 2n$. So $\mathbb{CP}^n$ has no two cells in adjacent dimensions. Hence:

$$H_k(\mathbb{CP}^n;\mathbb{Z}) = \begin{cases} \mathbb{Z}, & \text{if } k = 0, 2, 4, \dots, 2n \\ 0, & \text{otherwise.} \end{cases}$$

We next discuss how to compute, in general, the maps

$$d_n : C_n(X) = \mathbb{Z}^{\#n\text{-cells}} \to C_{n-1}(X) = \mathbb{Z}^{\#(n-1)\text{-cells}}$$

of the cellular chain complex. Let us consider the n-cells $\{e^\alpha_n\}_\alpha$ as the basis for $C_n(X)$ and the $(n-1)$-cells $\{e^\beta_{n-1}\}_\beta$ as the basis for $C_{n-1}(X)$. In particular, we can write:

$$d_n(e^\alpha_n) = \sum_\beta d_{\alpha,\beta} \cdot e^\beta_{n-1} \qquad d_{\alpha,\beta} \in \mathbb{Z},$$

Proposition 6.3.6. ***(Cellular Boundary Formula)*** *The coefficient $d_{\alpha,\beta}$ is equal to the degree of the map $\Delta_{\alpha,\beta} : \mathbb{S}^{n-1}_\alpha \to \mathbb{S}^{n-1}_\beta$ defined by the composition:*

$$\mathbb{S}^{n-1}_\alpha = \partial\mathbb{D}^n_\alpha \xrightarrow{\varphi^n_\alpha} X^{n-1} = X^{n-2} \cup_\gamma \mathbb{D}^{n-1}_\gamma \xrightarrow{\text{collapse}} X^{n-1}/(X^{n-2} \cup_{\gamma\neq\beta} \mathbb{D}^{n-1}_\gamma) = \mathbb{S}^{n-1}_\beta,$$

where φ^n_α is the attaching map of $\mathbb{D}^n_\alpha$, and the collapsing map sends $X^{n-2} \cup_{\gamma\neq\beta} \mathbb{D}^{n-1}_\gamma$ to a point.

Proof. We will proceed with the proof by chasing the following diagram:

$$\begin{tikzcd}
H_n(\mathbb{D}^n_\alpha, \mathbb{S}^{n-1}_\alpha;\mathbb{Z}) \arrow[r, "\cong"] \arrow[d, "(\Phi^n_\alpha)_*"'] & \widetilde{H}_n(\mathbb{S}^{n-1}_\alpha;\mathbb{Z}) \arrow[r, "(\Delta_{\alpha,\beta})_*"] \arrow[d, "(\phi^n_\alpha)_*"'] & \widetilde{H}_n(\mathbb{S}^{n-1}_\beta;\mathbb{Z}) \\
H_n(X^n, X^{n-1};\mathbb{Z}) \arrow[r, "\partial_n"] \arrow[dr, "d_n"'] & \widetilde{H}_{n-1}(X^{n-1};\mathbb{Z}) \arrow[r, "q_*"] \arrow[d, "j_{n-1}"] & \widetilde{H}_{n-1}(X^{n-1}/X^{n-2};\mathbb{Z}) \arrow[u, "q_{\beta,*}"'] \arrow[d, "\cong"'] \\
 & H_{n-1}(X^{n-1}, X^{n-2};\mathbb{Z}) \arrow[r, "\cong"'] & H_{n-1}(X^{n-1}/X^{n-2}, X^{n-2}/X^{n-2};\mathbb{Z})
\end{tikzcd}$$

The maps are as follows:

(1) Φ^n_α is the characteristic map of the cell e^n_α, and ϕ^n_α is its attaching map.

(2) The map

$$q_* : \widetilde{H}_{n-1}(X^{n-1};\mathbb{Z}) \to \widetilde{H}_{n-1}(X^{n-1}/X^{n-2};\mathbb{Z}) = \bigoplus_\beta \widetilde{H}_{n-1}(\mathbb{D}^{n-1}_\beta/\partial\mathbb{D}^{n-1}_\beta;\mathbb{Z})$$

is induced by the quotient map $q : X^{n-1} \to X^{n-1}/X^{n-2}$.

(3) $q_\beta : X^{n-1}/X^{n-2} \to \mathbb{S}^{n-1}_\beta$ collapses the complement of the cell e^{n-1}_β to a point, the resulting quotient sphere being identified with $\mathbb{S}^{n-1}_\beta = \mathbb{D}^{n-1}_\beta/\partial\mathbb{D}^{n-1}_\beta$ via the characteristic map Φ^{n-1}_β.

(4) $\Delta_{\alpha\beta} : \mathbb{S}^{n-1}_\alpha \to \mathbb{S}^{n-1}_\beta$ is the composition $q_\beta \circ q \circ \phi^n_\alpha$, i.e., the attaching map of e^n_α followed by the quotient map $X^{n-1} \to \mathbb{S}^{n-1}_\beta$ collapsing the complement of $\mathbb{D}^{n-1}_\beta$ in X^{n-1} to a point.

The top left-hand square commutes by naturality of the long-exact sequence in reduced homology. The top right-hand square commutes by the definition of $\Delta_{\alpha,\beta}$. The bottom left-hand triangle commutes by definition of d_n . The bottom right-hand square commutes due to the relationship between reduced and relative homology. The map $(\Phi^n_\alpha)_*$ takes the generator $[\mathbb{D}^n_\alpha] \in H_n(\mathbb{D}^n_\alpha, \mathbb{S}^{n-1}_\alpha)$ to a generator of the $\mathbb{Z}$-summand of $H_n(X^n, X^{n-1})$ corresponding to $\mathbb{D}^n_\alpha$, i.e.,

$$(\Phi^n_\alpha)_*([\mathbb{D}^n_\alpha]) = \mathbb{D}^n_\alpha$$

Since the top left square and the bottom left triangle both commute, this gives that

$$\sum_\beta d_{\alpha,\beta}\mathbb{D}^{n-1}_\beta = d_n(\mathbb{D}^n_\alpha) = d_n \circ (\Phi^\alpha_n)_*([\mathbb{D}^n_\alpha]) = j_{n-1} \circ (\phi^n_\alpha)_*([\mathbb{D}^n_\alpha]).$$

Here we have implicitly identified $H_n(\mathbb{D}^n_\alpha, \mathbb{S}^{n-1}_\alpha)$ with $H_n(\mathbb{S}^{n-1}_\alpha)$. Looking to the bottom right square, recall that since X is a CW complex, (X^n, X^{n-1}) is a *good* pair. This gives the isomorphism

$$\begin{aligned}
H_{n-1}(X^{n-1}, X^{n-2};\mathbb{Z}) &\cong \widetilde{H}_{n-1}(X^{n-1}/X^{n-2};\mathbb{Z}) \\
&\cong H_{n-1}(X^{n-1}/X^{n-2}, X^{n-2}/X^{n-2};\mathbb{Z}).
\end{aligned}$$

Notice that the map q_β, collapsing all the $n-1$ cells of X to the $n-1$ cell $\mathbb{S}^{n-1}_\beta$, induces the map $q_{\beta,*}$, which projects linear combinations of $\{\mathbb{D}^{n-1}_{\beta'}\}$ onto its summand of $\mathbb{D}^{n-1}_\beta$. Therefore, the value of $d_n(\mathbb{D}^n_i)$ is going to be the sum of the projections $q_{\beta',*}$ on the $n-1$ dimensional cells e^{n-1}_β. In other words:

$$\sum_\beta d_{\alpha,\beta}\mathbb{D}^\beta_\beta = d_n(\mathbb{D}^\alpha_n) = \sum_\beta q_{\beta*} \circ q_* \circ (\phi^\alpha_n)_* \circ [\mathbb{D}^\alpha_n].$$

As noted before, we have defined $(\Delta_{\alpha\beta})_* = q_{\beta*} \circ q_* \circ (\phi^\alpha_n)_*$. The result now follows. $\square$

Example 6.3.7. Let $X = \mathbb{S}^2$. We $\mathbb{S}^2$ with $\mathbb{D}^2/\sim$ such that

$$(x, y) \sim (x', y') = x',\ y = |y'|$$

This induces a cell decomposition into one 2-cell, the image of the interior, one 1-cell, the image of $\mathbb{S}^1 \setminus \{(0,1), (0,-1)\}$, and two 0-cells, the images of $(0,1)$ and $(0,1)$ which are N and S. Let $A = \{N, S\}$. Since A is a sub-complex, X/A inherits a CW complex structure with one one 2-cell, one 1-cell and one 0-cell. We have

$$0 \to \mathbb{Z} \xrightarrow{d_2} \mathbb{Z} \xrightarrow{d_1} \mathbb{Z} \to 0$$

Since X/A is connected as has a single 0-cell, $d_1 \equiv 0$. The attaching map of the two-cell in either case

$X = \mathbb{S}^2$ $\quad\quad$ $X = \mathbb{S}^2/\{N, S\}$

can be identified with the map:

$$\phi_{1,2}(e^{\phi i}) = \begin{cases} e^{i\phi} & 0 \leq \phi \leq \pi \\ e^{-i\phi} & \pi \leq \phi \leq 2\pi \end{cases}$$

The map has degree 0. Hence, $d_2 \equiv 0$. As a result, we have

$$H_n(\mathbb{S}^2/\{N, S\}; \mathbb{Z}) = \begin{cases} \mathbb{Z}, & \text{if } n = 0, 1, 2 \\ 0, & \text{otherwise.} \end{cases}$$

Example 6.3.8. Recall that $\mathbb{RP}^n$ has a CW structure with one k-cell $\mathbb{D}^k$ in each dimension $0 \leq k \leq n$. The attaching map for $\mathbb{D}^k$ is the standard 2-fold covering map $\phi : \mathbb{S}^{k-1} \to \mathbb{RP}^{k-1}$ identifying a point and its antipodal point in $\mathbb{S}^{k-1}$. To compute the boundary map d_k, we compute the degree of the composition

$$f : \mathbb{S}^{k-1} \to \mathbb{RP}^{k-1} \to \frac{\mathbb{RP}^{k-1}}{\mathbb{RP}^{k-2}} = \mathbb{S}^{k-1}$$

We consider a neighborhood V of y and the two neighborhoods U_1 and U_2 given to exist by the local homeomorphism property of f. One of the homeomorphisms is the identity map and the other homemorphisms is the anti-podal map. Then by the local degree formula implies

$$d_k = 1 + (-1)^k$$

It follows that

$$d_k = \begin{cases} 0 & \text{if } k \text{ is odd,} \\ 2 & \text{if } k \text{ is even,} \end{cases}$$

and therefore we obtain that

$$H_k(\mathbb{RP}^n; \mathbb{Z}) = \begin{cases} \mathbb{Z}_2 & \text{if } k \text{ is odd, } 0 < k < n, \\ \mathbb{Z} & \text{if } k = 0, n \text{ is odd,} \\ 0 & \text{otherwise.} \end{cases}$$

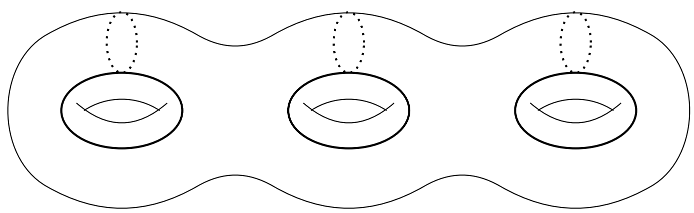
Example 6.3.9. Let M_g be the closed oriented surface of genus g, with its usual CW structure: one 0-cell, $2g$ 1-cells $\{a_1, b_1, \dots, a_g, b_g\}$, and one 2-cell attached by the product of commutators $[a_1, b_1] \cdot \dots \cdot [a_g, b_g]$. The associated cellular chain complex of M_g is:

$$0 \xrightarrow{d_3} \mathbb{Z} \xrightarrow{d_2} \mathbb{Z}^{2g} \xrightarrow{d_1} \mathbb{Z} \xrightarrow{d_0} 0$$

Since M_g is connected and has only one 0-cell, we get that $d_1 = 0$. We claim that d_2 is also the zero map. As the attaching map sends the generator to $a_1 b_1 a_1^{-1} b_1^{-1} \dots a_g b_g a_g^{-1} b_g^{-1}$, when we collapse all 1-cells (except a_i) to a point, the word defining the attaching map $a_1 b_1 a_1^{-1} b_1^{-1} \dots agbga_g^{-1} b_g^{-1}$ reduces to $a_i a_i^{-1}$. Hence, the coefficient$d_{ea_i} = 1 - 1 = 0$. Altogether, $d_2(e) = 0$. So the homology groups of M_g are given by

$$H_n(M_g; \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } n = 0, 2, \\ \mathbb{Z}^{2g} & \text{for } n = 1, \\ 0 & \text{otherwise.} \end{cases}$$

For $g = 3$, refer to Figure 3 to visualize the $2g = 6$ generators of $H_1(M_3)$:



M_3 with three pairs of loops indicated generating $H_1(M_3)$.

6.4. Equivalence of Homology Theories

We have encountered various homology theories, including singular, simplicial, and cellular homology, and have seen that they all coincide in specific cases. For instance, if a topological space admits a Δ-complex structure, the singular and simplicial homologies coincide. Similarly, if a topological space admits a CW-complex structure, the singular and cellular homologies coincide. We now demonstrate that this is a specific instance of a more general principle: homology theories are uniquely determined on well-behaved topological spaces, particularly within the category of CW pairs.

Proposition 6.4.1. *Let h_* be a homology theory in the sense of Definition 5.2.1 with $\mathbb{Z}$ coefficients defined as a collection of functors*

$$h_n : \mathsf{CW}^2 \to \mathsf{Ab}$$

If $h_n(*; \mathbb{Z}) \cong 0$ for $n \neq 0$, then there exists a natural isomorphism

$$h_n(X, A) \cong H_n(X, A; G)$$

for all CW-pairs (X, A) and for all $n \geq 1$, where $G := h_0(*; \mathbb{Z}) \in \mathsf{Ab}$.

Proof. Since (X, A) is a good pair, we have an isomorphism

$$h_n(X, A; \mathbb{Z}) \cong \tilde{h}_n(X/A; \mathbb{Z})$$

for all $n \geq 0$. This is a formal consequence of Eilenberg-Steenrod axioms that we have verified for singular homology. Hence, we only need to check the absolute case. Just as for singular homology, we have

$$h_n^{\mathrm{CW}}(X; \mathbb{Z}) \cong h_n(X; \mathbb{Z})$$

The hypothesis that $h_n(*;\mathbb{Z}) = 0$ for $n \neq 0$ is used here. The long exact sequences of h_* homology groups for the pairs (X_n, X_{n-1}) give rise to a cellular chain complex.

$$\cdots \to h_n^{\mathrm{CW}}(X_n, X_{n-1};\mathbb{Z}) \xrightarrow{d_n} h_{n-1}^{\mathrm{CW}}(X_{n-1}, X_{n-2};\mathbb{Z}) \to \cdots$$

We also have

$$\cdots \to H_n^{\mathrm{CW}}(X_n, X_{n-1};G) \xrightarrow{\partial_n} H_{n-1}^{\mathrm{CW}}(X_{n-1}, X_{n-2};G) \to \cdots$$

The individual groups are isomorphic, since

$$h_n^{\mathrm{CW}}(X_n, X_{n-1};\mathbb{Z}) \cong G^{\#n\text{-cells}} \cong H_n^{\mathrm{CW}}(X_n, X_{n-1};G).$$

Thus, it remains to show that $d_n = \partial_n$ for $n \geq 1$. For $n = 1$, we can pass from X to S^2X since suspension is a natural isomorphism in any homology theory. S^2X has no 1-cells, so immediately $d_1 = 0 = \partial_1$. Now let $n > 1$. The calculation of cellular boundary maps d_n for $n > 1$ in terms of degrees of certain maps between spheres works equally well for h_*, where degree now means degree with respect to the h_* theory. But a map

$$f : \mathbb{S}^n \to \mathbb{S}^n$$

of degree m in the usual sense is simply multiplication by m on $H_n(\mathbb{S}^n;G) \cong G \cong h_n(\mathbb{S}^n;G)$. The claim follows. □

6.5. Euler Characteristic

The Euler characteristic was first introduced by Leonhard Euler in the 18th century as a formula for convex polyhedra:

$$\chi = V - E + F,$$

where V, E, and F represent the numbers of vertices, edges, and faces, respectively. Euler discovered that for any convex polyhedron, this quantity is always equal to 2. This remarkable result was a precursor to the development of modern topology. We can *categorify* Euler's formula by expressing it purely in terms of the ranks of certain homology groups. This has the advantage that the Euler characteristic can be computed for a large class of topological spaces. Moreover, it allows us to demonstrate that the Euler characteristic is indeed a topological invariant, independent of any particular decomposition of a polyhedron into vertices, edges, and faces, justifying that χ is always equal to 2 for a convex polyhedron, regardless of its decomposition into vertices, edges, and faces, is a key step in showing that the Euler characteristic is a topological invariant.

Definition 6.5.1. Let X be a n-dimensional CW complex. The Euler characteristic of X is defined as:

$$\chi(X) = \sum_{i=0}^{n} (-1)^i \cdot \#n - \text{cells} := \sum_{i=0}^{n} (-1)^i \cdot \mathrm{rank}(C_i^{\mathrm{CW}})$$

Here, C_i^{CW} denotes the i-th abelian group in the chain complex that defines the cellular homology of X. A priori, the Euler characteristic appears to depend on the chosen cell structure of X. However, we now show that the Euler characteristic is actually independent of this choice. To establish this, it suffices to show that the Euler characteristic depends only on the cellular homology of the space X. Indeed, since cellular homology is isomorphic to singular homology, and singular homology is independent of the cell structure, the Euler characteristic is therefore a topological invariant.

Proposition 6.5.2. *Let X be a n-dimensional CW complex. The Euler characteristic can be computed as:*

$$\chi(X) = \sum_{i=0}^{n} (-1)^i \cdot \mathrm{rank}(H_i^{\mathrm{CW}}(X;\mathbb{Z}))$$

In particular, $\chi(X)$ is independent of the chosen cell structure on X.

PROOF. Let $B_i = \operatorname{im}(d_{i+1})$, $Z_i = \ker(d_i)$, and $H_i^{\mathrm{CW}} = Z_i/B_i$. From the definitions of the cellular chain complex, we have two short exact sequences for each degree i:

$$0 \longrightarrow Z_i \longrightarrow C_i \longrightarrow B_{i-1} \longrightarrow 0,$$
$$0 \longrightarrow B_i \longrightarrow Z_i \longrightarrow H_i^{\mathrm{CW}} \longrightarrow 0,$$

The additivity of rank yields that

$$\operatorname{rank}(C_i) = \operatorname{rank}(Z_i) + \operatorname{rank}(B_{i-1})$$
$$\operatorname{rank}(Z_i) = \operatorname{rank}(B_i) + \operatorname{rank}(H_i^{\mathrm{CW}})$$

We have

$$\begin{aligned}\sum_{i=0}^{n}(-1)^i a_i &= \sum_{i=0}^{n}(-1)^i \operatorname{rank}(C_i)\\ &= \sum_{i=0}^{n}(-1)^i(\operatorname{rank}(Z_i) + \operatorname{rank}(B_{i-1}))\\ &= \sum_{i=0}^{n}(-1)^i(\operatorname{rank}(B_i) + \operatorname{rank}(H_i^{\mathrm{CW}}) + \operatorname{rank}(B_{i-1})).\end{aligned}$$

The terms $\operatorname{rank}(B_i) + \operatorname{rank}(B_{i-1})$ cancel out due to the alternating sum, leaving

$$\chi(X) = \sum_{i=0}^{n}(-1)^i \operatorname{rank}(H_i^{\mathrm{CW}})$$

Since cellular homology is naturally isomorphic to singular homology, and singular homology is invariant under homotopy equivalence, the result follows immediately. □

We conclude this section by discussing some basic properties and examples of the Euler characteristic.

Proposition 6.5.3. *Let X, Y be finite-dimensional CW complexes and let*

$$\chi(X) = \sum_{i=0}^{n}(-1)^i a_i, \qquad \chi(Y) = \sum_{j=0}^{m}(-1)^j b_j$$

Here a_i is the number of i-cells in X. Similarly, b_j is the number of j-cells in B The Euler characteristic enjoys some nice properties:

(1) $\chi(X \times Y) = \chi(X) \times \chi(Y)$
(2) *If $X = A \cup B$ such that A, B are sub-complexes of X. Then*

$$\chi(X) = \chi(A) + \chi(B) - \chi(A \cap B)$$

(3) *If $p : \widetilde{X} \to X$ is an n-sheeted covering space, then*

$$\chi(\widetilde{X}) = n\chi(X)$$

PROOF. The proof is given below:

(1) For any index k, k-cells in $X \times Y$ are created by considering products of r-cells and $k-r$ cells from X and Y respectively where $0 \leq r \leq k$. Hence the number of k-cells is

$$\sum_{r=0}^{k} a_r b_{k-r}$$

Therefore,

$$\begin{aligned}\chi(X) \times \chi(Y) &= \left(\sum_{i=0}^{n}(-1)^i a_i\right) \times \left(\sum_{j=0}^{m}(-1)^j b_j\right) \\ &= \sum_{k=0}^{m+n}(-1)^k \sum_{r=0}^{k} a_r b_{k-r} = \chi(X \times Y).\end{aligned}$$

(2) Let a_i^A denote the number of i-cells in A. Similarly, let a_i^B be the number of i-cells in B. Similarly, let $a_i^{A\cap B}$ be the number of i-cells in $A \cap B$. We have

$$a_i = a_i^A + a_i^B - a_i^{A\cap B}$$

for $i = 1, \cdots, n$. Therefore, we have,

$$\begin{aligned}\chi(X) &= \sum_{i=0}^{n}(-1)^i a_i \\ &= \sum_{i=0}^{n}(-1)^i a_i^A + \sum_{i=0}^{n}(-1)^i a_i^B - \sum_{i=0}^{n}(-1)^i a_i^{A\cap B} \\ &= \chi(A) + \chi(B) - \chi(A \cap B)\end{aligned}$$

(3) Recall that if $\mathbb{D}_\alpha^k$ is a k-cell in X, then $\widetilde{X}$ has n k-cells. Therefore, it is clear that

$$\chi(\widetilde{X}) = n\chi(X)$$

This completes the proof. $\square$

Example 6.5.4. Let M_g be the oriented surface of genus g, and let N_g be the oriented surface of genus g. We have

$$\begin{aligned}\chi(M_g) &= 2 - 2g \\ \chi(N_g) &= 2 - g\end{aligned}$$

Thus all the M_g, N_g are distinguished from each other by their Euler characteristics. There are only the relations

$$\chi(M_g) = \chi(N_{2g})$$

6.6. Tor Functor

We now discuss the Tor (derived) functor which will play an important role in the discussion of homology with coefficients. Further details on derived functors and related topics can be found in Chapter 15.

Remark 6.6.1. *We work with commutative rings below. Hence, we don't make any distinction between the categories of left R-modules and right R-modules. We use the generic phrase 'R-module' to refer to a left/right R-module.*

Recall that the tensor product, $\otimes_R$, defines a functor from the category of R-modules to itself such that if N is a R-module, then

$$- \otimes_R M(N) = N \otimes_R M$$

Moreover, if $f : N \to N'$ is a R-module morphism, then

$$- \otimes_R M(f) : N \otimes_R M \xrightarrow{f \otimes_R \mathrm{Id}_M} N' \otimes_R M$$

It can be checked that $- \otimes_R M$ is a right exact functor. However, $- \otimes_R M$ is not a left exact functor in general.

Example 6.6.2. The functor $- \otimes_R M$ need not be left exact functor. Let $R = \mathbb{Z}$. Consider the sequence:

$$0 \to \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z}$$

Here $\cdot n$ is the multiplication by n map. Let $M = \mathbb{Z}/n\mathbb{Z}$ we obtain a map:

$$\mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \xrightarrow{\cdot n \otimes_{\mathbb{Z}} \mathrm{Id}_{\mathbb{Z}/n\mathbb{Z}}} \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/n\mathbb{Z}$$

However, this is the zero map since we have

$$\cdot n \otimes_{\mathbb{Z}} \mathrm{Id}_{\mathbb{Z}/n\mathbb{Z}}(1 \otimes_{\mathbb{Z}} \overline{m}) = n \otimes_{\mathbb{Z}} \overline{m} = 1 \otimes_{\mathbb{Z}} \overline{nm} = 0.$$

The zero map is not injective.

Remark 6.6.3. *A R-module M is called flat if $- \otimes_R M$ is a left exact functor. If M is a projective R-module, then $- \otimes_R M$ is a left exact functor. This follows because a projective R-module is a direct summand of a free R-module, a free R-module is a flat module and that a R-module is flat if and only if each summad is a flat R-module.*

Since the $- \otimes_R M$ functor is a right exact functor which in general is not a left exact functor, we can consider its left derived functor.

Definition 6.6.4. Let R be a ring and let M be a R-module. The i-th Tor functor is the i-th left derived functor of $- \otimes_R M$. It is denoted as

$$\mathrm{Tor}_i^R(-, M)$$

By definition, $\mathrm{Tor}_i^R(-, M)$ is computed as follows. If N is a R-module, take any projective resolution

$$\cdots \to P^1 \to P^0 \to N \to 0,$$

and form the chain complex:

$$\cdots \to P^2 \otimes_R M \to P^1 \otimes_R M \to P^0 \otimes_R M$$

Then $\mathrm{Tor}_i^R(N, M)$ is the homology of this complex at position i.

$$\mathrm{Tor}_i^R(N, M) = H_i((P^i \otimes_R M)_\bullet)$$

Remark 6.6.5. *General results about derived functors (Chapter 15) show that the homology is independent of the choice of the projective resolution.*

If R is a commutative ring and M is a R-module, we can define another functor $M \otimes_R -$. The definition is similar to that of the functor defined above. It can also be checked that $M \otimes_R -$ is right exact functor that is, in general, not left exact. Hence, we can attempt to construct a left-derived functor associated to $M \otimes_R -$ as above. We label that derived functor $\mathrm{Tor}_i^R(M, -)$. We have the following result:

Proposition 6.6.6. ***(Balanacing Tor)*** *Let R be a ring, M, N be R-modules. Denote by $\operatorname{Tor}_*^R(N, -)$ the left-derived functors of the tensor product functor $N \otimes_R -$, and by $\operatorname{Tor}_*^R(-, M)$ the left-derived functors of $- \otimes_R M$. We have that*

$$\operatorname{Tor}_*^R(N, M) \cong \operatorname{Tor}_*^R(M, N)$$

Proof. We provide an argument based on homological spectral sequences of double complexes. We choose projective resolutions

$$\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow N \longrightarrow 0$$
$$\cdots \longrightarrow Q_2 \longrightarrow Q_1 \longrightarrow Q_0 \longrightarrow M \longrightarrow 0$$

of N and M, respectively. We define a first quadrant homological double complex $C_{\bullet,\bullet}$ by $C_{p,q} = P_p \otimes Q_q$, where the maps are the induced ones coming from the maps in the projective resolutions. The double complex can be visualized as follows:

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
P_0 \otimes Q_2 \arrow[d] & P_1 \otimes Q_2 \arrow[l] \arrow[d] & P_2 \otimes Q_2 \arrow[l] \arrow[d] & \cdots \arrow[l] \\
P_0 \otimes Q_1 \arrow[d] & P_1 \otimes Q_1 \arrow[l] \arrow[d] & P_2 \otimes Q_1 \arrow[l] \arrow[d] & \cdots \arrow[l] \\
P_0 \otimes Q_0 & P_1 \otimes Q_0 \arrow[l] & P_2 \otimes Q_0 \arrow[l] & \cdots \arrow[l]
\end{tikzcd}$$

Since projective modules are flat modules, the rows and columns of this double complex are indeed complexes, and the squares in the double complex are commutative. We first filter this double complex by columns. Note that the homology in the vertical direction determines the E^1 page such that

$$E^1_{p,q} = \operatorname{Tor}_q^R(P_p, M).$$

Since P_p are projective and hence flat modules, we have $\operatorname{Tor}_q^R(P_p, M) = 0$ for $q > 0$, and $\operatorname{Tor}_0^R(P_p, M) = P_p \otimes M$. Thus, the E^1 page is:

$$\begin{tikzcd}
\vdots & \vdots & \vdots & \\
0 & 0 \arrow[l] & 0 \arrow[l] & \cdots \arrow[l] \\
P_0 \otimes M & P_1 \otimes M \arrow[l] & P_2 \otimes M \arrow[l] & \cdots \arrow[l]
\end{tikzcd}$$

Taking homology of the E^1 page yields the E^2 page:

$$\begin{tikzcd}
0 & 0 & 0 & 0 & 0 & 0 \\
\operatorname{Tor}_0^R(N, M) & \operatorname{Tor}_1^R(N, M) & \operatorname{Tor}_2^R(N, M) \arrow[ull] & \operatorname{Tor}_3^R(N, M) \arrow[ull] & \operatorname{Tor}_4^R(N, M) \arrow[ull] & \operatorname{Tor}_5^R(N, M) \arrow[ull]
\end{tikzcd}$$

In a similar manner, we can filter the double complex by rows, and we obtain a spectral sequence whose E^2-page looks like:

$$\begin{tikzcd}
0 & 0 & 0 & 0 & 0 & 0 \\
\operatorname{Tor}_0^R(M,N) & \operatorname{Tor}_1^R(M,N) & \operatorname{Tor}_2^R(M,N) \arrow[ull] & \operatorname{Tor}_3^R(M,N) \arrow[ull] & \operatorname{Tor}_4^R(M,N) \arrow[ull] & \operatorname{Tor}_5^R(M,N) \arrow[ull]
\end{tikzcd}$$

For both spectral sequences, we have $E_2^{p,q} = E_\infty^{p,q}$. Since both spectral sequences converge to the associated graded object, we can conclude that

$$\operatorname{Tor}_*^R(N,M) \cong \operatorname{Tor}_*^R(M,N).$$

This completes the proof. □

Remark 6.6.7. *In light of Proposition 6.6.6, we can identify the two Tor functors. This allows us to compute projective resolutions of either N or M to compute $\operatorname{Tor}_i^R(N,M)$ for each $i \geq 0$.*

Proposition 6.6.8. *Let R be a commutative ring and let M be a R-module. The Tor functor satisfies the following properties:*

(1) *$\operatorname{Tor}_0^R(N,M) \cong N \otimes_R M$ for any R-modules M, N.*
(2) *If N is a projective R-module, then $\operatorname{Tor}_i^R(N,M) = 0$ for all $i \geq 1$*
(3) *Any $f : N_1 \to N_2$ R-module homomorphism induces a morphism*

$$f_*^i : \operatorname{Tor}_i^R(N_1,M) \longrightarrow \operatorname{Tor}_i^R(N_2,M)$$

for each $i \geq 0$.

(4) *Any short exact sequence $0 \to N_1 \xrightarrow{\phi} N_2 \xrightarrow{\psi} N_3 \to 0$ of R-modules induces a long exact sequence:*

$$\cdots \to \operatorname{Tor}_1^R(N_1,M) \to \operatorname{Tor}_1^R(N_2,M) \to \operatorname{Tor}_1^R(N_3,M) \to N_1 \otimes_R M \to N_2 \otimes_R M \to N_3 \otimes_R M \to 0$$

Proof. (1) and (2) follow from general properties of derived functors (Corollary 15.4.15). For (3), let $P_1^\bullet$ be a projective resolution of N_1 and $P_2^\bullet$ be a projective resolution of N_2. General properties about projective resolutions imply that that f lifts to a chain map $\varphi^\bullet : P_1^\bullet \longrightarrow P_2^\bullet$. Then, $\varphi^\bullet$ induces a morphism of chain complexes $P_1^\bullet \otimes_R M \longrightarrow P_2^\bullet \otimes_R M$ which, in turn, induces a morphism:

$$f_*^i : \operatorname{Tor}_i^R(N_1,M) \longrightarrow \operatorname{Tor}_i^R(N_2,M)$$

for each $i \geq 0$. For (4), let $P^\bullet$ be a projective resolution of M. Then there is an induced short exact sequence of chain complexes:

$$0 \to N_1 \otimes_R P^\bullet \to N_2 \otimes_R P^\bullet \to N_3 \otimes_R P^\bullet \to 0$$

because each module P^i is projective. Applying the long exact sequence in homology produces the required long exact sequence. □

We now specialize to the category of $\mathbb{Z}$-modules. In what follows, we fix G to be an abelian group. We have the following result:

Lemma 6.6.9. *For any abelian group A, we have $\operatorname{Tor}_i^{\mathbb{Z}}(A,G) = 0 \quad$ if $i > 1$.*

Proof. Recall that any abelian group, A, admits a two-step free resolution.

$$0 \to F_1 \to F_0 \to A \to 0$$

Thus, $\operatorname{Tor}_i^{\mathbb{Z}}(A,G) = 0$ if $i > 1$. □

Remark 6.6.10. *Only* $\mathrm{Tor}_1^{\mathbb{Z}}(-,G)$ *encodes any interesting information. In what follows, we adopt the notation:* $\mathrm{Tor}(-,G) := \mathrm{Tor}_1^{\mathbb{Z}}(-,G)$.

Proposition 6.6.11. *If* $R = \mathbb{Z}$*, the Tor functor satisfies the following properties:*

(1) $\mathrm{Tor}\left(\bigoplus_i A_i, G\right) \cong \bigoplus_i \mathrm{Tor}(A_i, G)$.
(2) *If* A *is a free abelian group, then* $\mathrm{Tor}(A,G) = 0$.
(3) $\mathrm{Tor}(\mathbb{Z}/n\mathbb{Z}, G) \cong \ker(G \xrightarrow{n} G)$.
(4) *For a short exact sequence:* $0 \to B \to C \to D \to 0$ *of abelian groups, there is a natural exact sequence:*

$$0 \to \mathrm{Tor}(B,G) \to \mathrm{Tor}(C,G) \to \mathrm{Tor}(D,G) \to B \otimes_R G \to C \otimes_R G \to D \otimes_R G \to 0.$$

Proof. The proof is given below:

(1) This follows from the identity,

$$\left(\bigoplus_i A_i\right) \otimes_{\mathbb{Z}} G = \bigoplus_i (A_i \otimes_{\mathbb{Z}} G)$$

and noting that taking direct sums of projective resolutions of A_i forms a projective resolution for $\bigoplus_i A^i$, and that homology commutes with direct sums.
(2) If A is free, then $0 \to A \to A \to 0$ is a projective resolution of A, so $\mathrm{Tor}(A,G) = 0$.
(3) The exact sequence $0 \to \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$ is a projective resolution of $\mathbb{Z}/n\mathbb{Z}$. Tensoring with G and dropping the right-most term yields the complex:

$$G \cong \mathbb{Z} \otimes_R G \xrightarrow{\cdot n \otimes_R 1_G} G \cong \mathbb{Z} \otimes_R G \to 0,$$

Thus, $\mathrm{Tor}(\mathbb{Z}/n\mathbb{Z}, G) = \ker(G \xrightarrow{n} G)$.
(4) This follows from Proposition 6.6.8(4).

This completes the proof. □

6.7. Universal Coefficient Theorem

We extend our discussion of homology by introducing homology with coefficients in an abelian group ,G. While homology with integer coefficients captures much of the essential topological information of a space, considering more general coefficient groups allows for greater flexibility and reveals additional structure. In particular, it can clarify torsion phenomena and simplify computations in certain contexts. We then turn to the Universal Coefficient Theorem (UCT) for homology, which provides a precise relationship between homology with integer coefficients and homology with coefficients in an arbitrary abelian group. This result plays a central role in understanding how the choice of coefficients affects the computation of homology groups of a topological space.

Definition 6.7.1. Let G be an abelian group and X a topological space. The homology of X with G-coefficients, denoted $H_n(X;G)$ for $n \in \mathbb{N}$, is the homology of the chain complex:

$$C_\bullet(X;G) = C_\bullet(X) \otimes_{\mathbb{Z}} G$$

consisting of finite formal sums $\sum_i \eta_i \cdot \sigma_i$ with $\eta_i \in G$, and with boundary maps given by

$$\partial_n^G := \partial_n \otimes_{\mathbb{Z}} \mathrm{Id}_G\,.$$

Remark 6.7.2. *Since* ∂_n *satisfies* $\partial_n \circ \partial_{n+1} = 0$*, it follows that* $\partial_n^G \circ \partial_{n+1}^G = 0$*. Hence,* $(C_\bullet(X); G, \partial_\bullet^G)$ *is indeed a chain complex.*

We can construct versions of the usual modified homology groups (relative, reduced, etc.) in the most natural way.

(1) **(Relative homology with G-coefficients)** Consider the augmented chain complex:
$$C_1(X;G) \to C_0(X;G) \to G \to 0$$
where $\epsilon(\sum_i \eta_i \sigma_i) = \sum_i \eta_i \in G$. Reduced homology with G-coefficients is defined as the homology of the augmented chain complex.

(2) **(Relative chain Complex with G-coefficients)** Define relative chains with G-coefficients by:
$$C_n(X,A;G) := C_n(X;G)/C_n(A;G),$$
Consider the chain complex:
$$C_1(X,A;G) \to C_0(X,A;G) \to 0$$
The relative homology with G-coefficients is defined as the homology of the augmeneted chain complex.

(3) **(Cellular homology with G-coefficients)** We can build cellular homology with G-coefficients by defining
$$C_n^G(X) = H_n(X_n, X_{i-1}; G) \cong G^{(\text{number of } i\text{-cells})}$$
The cellular boundary maps are given by:
$$d_n^G\,(e_n^\alpha) = \sum_\beta d_{\alpha\beta} e_{i-1}^\beta,$$
where $d_{\alpha\beta}$ is as before the degree of a map $\Delta_{\alpha\beta} : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$. As it is the case for integers, we get an isomorphism:
$$H_n^{\mathrm{CW}}(X;G) \cong H_n(X;G)$$

Example 6.7.3. Let's look at some examples:

(1) By considering the chain complex with coefficients in an abelian group G, we deduce that the homology groups of a single-point space satisfy
$$H_n(\{*\};G) = \begin{cases} G & \text{if } n = 0, \\ 0 & \text{if } n \neq 0. \end{cases}$$

(2) (Sketch) By induction and using the long exact sequence of the pair $(\mathbb{D}^n, \mathbb{S}^{n-1})$, the homology of the sphere with coefficients in an abelian group G is given by
$$H_i(\mathbb{S}^n;G) = \begin{cases} G & \text{if } i = 0 \text{ or } i = n, \\ 0 & \text{otherwise}. \end{cases}$$

Remark 6.7.4. *The computations in Example 6.7.3 will follow more easily from our discussion of the universal coefficient theorem below.*

We now turn to the proof of the Universal Coefficient Theorem (UCT), which establishes a fundamental relationship between homology groups computed with different coefficient groups. Since the choice of coefficients can significantly affect the structure of homology groups, it is crucial to understand how these variations are interconnected. The UCT provides an explicit algebraic framework that allows one to express homology with arbitrary coefficients in terms of homology with integer coefficients and certain algebraic constructions involving the chosen coefficient group. The UCT not only facilitates practical computations of homology groups in diverse contexts but also offers deep theoretical insights into how algebraic and topological information is encoded within homological invariants. Since the result holds in a general homological algebra framework, we state it here in its full generality.

Proposition 6.7.5. ***(Universal Coefficient Theorem)*** *Consider a chain complex $(C_\bullet, \partial_\bullet)$ of free abelian groups. For each $n \in \mathbb{N}$, the homology groups $H_n(C_\bullet; \mathbb{Z})$ and $H_n(C_\bullet; G)$ are determined by the short exact sequence:*

$$0 \to H_n(C_\bullet; \mathbb{Z}) \otimes_{\mathbb{Z}} G \to H_n(C_\bullet; G) \xrightarrow{h} \mathrm{Tor}(H_{n-1}(C_\bullet; \mathbb{Z}), G) \to 0$$

Proof. Choose a projective resolution for G:

$$\cdots \to P_2 \to P_1 \to P_0 \to G \to 0.$$

We define a first quadrant homological double complex $C_{\bullet,\bullet}$ by $C_{p,q} = P_p \otimes C_q$, where the maps are the induced ones coming from the maps in the projective resolutions. The double complex can be visualized as follows:

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
P_0 \otimes C_2 \arrow[d] & P_1 \otimes C_2 \arrow[l] \arrow[d] & P_2 \otimes C_2 \arrow[l] \arrow[d] & \cdots \arrow[l] \\
P_0 \otimes C_1 \arrow[d] & P_1 \otimes C_1 \arrow[l] \arrow[d] & P_2 \otimes C_1 \arrow[l] \arrow[d] & \cdots \arrow[l] \\
P_0 \otimes C_0 & P_1 \otimes C_0 \arrow[l] & P_2 \otimes C_0 \arrow[l] & \cdots \arrow[l]
\end{tikzcd}$$

We first filter this double complex by columns. Taking the homology in the vertical direction, we obtain the E^1 page:

$$\begin{tikzcd}
\vdots & \vdots & \vdots & \\
P_0 \otimes H_2(C) & P_1 \otimes H_2(C) \arrow[l] & P_2 \otimes H_2(C) \arrow[l] & \cdots \arrow[l] \\
P_0 \otimes H_1(C) & P_1 \otimes H_1(C) \arrow[l] & P_2 \otimes H_1(C) \arrow[l] & \cdots \arrow[l] \\
P_0 \otimes H_0(C) & P_1 \otimes H_0(C) \arrow[l] & P_2 \otimes H_0(C) \arrow[l] & \cdots \arrow[l]
\end{tikzcd}$$

This follows because $P_p \otimes -$ is an exact functor. The rows here correspond to the complexes used to calculate $\mathrm{Tor}_*^{\mathbb{Z}}(G, -)$, so the (p,q)-th entry on the E^2 page is $\mathrm{Tor}_q^{\mathbb{Z}}(G, H_p(C; \mathbb{Z}))$. Let's examine this more closely. Applying $G \otimes -$ to the short exact sequence of chain complexes

$$0 \to \operatorname{im} d_\bullet \to \ker d_\bullet \to H_\bullet(C_\bullet) \to 0$$

and deriving gives a long exact sequence:

$$\cdots \to \mathrm{Tor}_2^{\mathbb{Z}}(G, H_n(C)) \to \mathrm{Tor}_1^{\mathbb{Z}}(G, \operatorname{im} d_{n-1}) \to \mathrm{Tor}_1^{\mathbb{Z}}(G, \ker d_n) \to \mathrm{Tor}_1^{\mathbb{Z}}(G, H_n(C)) \to \mathrm{Tor}_0^{\mathbb{Z}}(G, \operatorname{im} d_{n-1}) \to \cdots$$

But since $\operatorname{im} d_{n-1}$ and $\ker d_n$ are subgroups of the free abelian group C_n, they are themselves free. Therefore, the higher Tor groups vanish. By the long exact sequence, it follows that $\mathrm{Tor}_q^{\mathbb{Z}}(G, H_p(C; \mathbb{Z})) = 0$

for all $q \geq 2$. Hence, the E^2 page looks as follows:

$$\begin{tikzcd}
\operatorname{Tor}_0^{\mathbb{Z}}(G, H_2(C_\bullet;\mathbb{Z})) & \operatorname{Tor}_1^{\mathbb{Z}}(G, H_2(C_\bullet;\mathbb{Z})) & 0 & 0 \\
\operatorname{Tor}_0^{\mathbb{Z}}(G, H_1(C_\bullet;\mathbb{Z})) & \operatorname{Tor}_1^{\mathbb{Z}}(G, H_1(C_\bullet;\mathbb{Z})) & 0 \arrow[ull] & 0 \arrow[ull] \\
\operatorname{Tor}_0^{\mathbb{Z}}(G, H_0(C_\bullet;\mathbb{Z})) & \operatorname{Tor}_1^{\mathbb{Z}}(G, H_0(C_\bullet;\mathbb{Z})) & 0 \arrow[ull] & 0 \arrow[ull]
\end{tikzcd}$$

By Remark 4.6.6, we have

$$0 \to E^2_{0,n} \to M_n \to E^2_{1,n-1} \to 0$$

Note that $E^2_{0,n} = \operatorname{Tor}_0^{\mathbb{Z}}(G, H_n(C_\bullet;\mathbb{Z})) = G \otimes H_n(C_\bullet;\mathbb{Z})$ and $E^2_{1,n-1} = \operatorname{Tor}_1^{\mathbb{Z}}(G, H_{n-1}(C_\bullet;\mathbb{Z}))$. Hence, the exact sequence becomes

$$0 \to G \otimes H_n(C_\bullet;\mathbb{Z}) \to M_n \to \operatorname{Tor}_1^{\mathbb{Z}}(G, H_{n-1}(C_\bullet;\mathbb{Z})) \to 0$$

To identify M_n, we now filter the double complex by rows. This amounts to considering a spectral sequence whose E^0 page is the transposed double complex:

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
P_2 \otimes C_0 & P_2 \otimes C_1 \arrow[l] \arrow[d] & P_2 \otimes C_2 \arrow[l] \arrow[d] & \cdots \arrow[l] \\
P_1 \otimes C_0 & P_1 \otimes C_1 \arrow[l] \arrow[d] & P_1 \otimes C_2 \arrow[l] \arrow[d] & \cdots \arrow[l] \\
P_0 \otimes C_0 & P_0 \otimes C_1 \arrow[l] & P_0 \otimes C_2 \arrow[l] & \cdots \arrow[l]
\arrow[from=2-1, to=3-1] \arrow[from=3-1, to=4-1]
\end{tikzcd}$$

Here the vertical maps are induced by the horizontal maps of the double complex. Hence, taking the homology in the vertical direction of the transposed double complex is equivalent to taking the homology of the double complex in the horizontal direction. Thus, the (p,q)-th entry of the E^1 page is $\operatorname{Tor}_p(G, C_q)$. For $p \geq 1$, this vanishes because each C_q is a free abelian group. Moreover, we have $\operatorname{Tor}_0(G, C_q) = G \otimes C_q$. Hence, the E^1 page is given by:

$$\begin{tikzcd}
\vdots & \vdots & \vdots & \\
0 & 0 \arrow[l] & 0 \arrow[l] & \cdots \arrow[l] \\
G \otimes C_0 & G \otimes C_1 \arrow[l] & G \otimes C_2 \arrow[l] & \cdots \arrow[l]
\end{tikzcd}$$

Here the horizontal differentials are induced by the vertical differentials of the double complex. Taking homology, on the E^2 page, everything except the bottom row is zero. In the bottom row, we have:

$$H_0(G \otimes C_\bullet;\mathbb{Z}) \quad H_1(G \otimes C_\bullet;\mathbb{Z}) \quad H_2(G \otimes C_\bullet;\mathbb{Z}) \quad \cdots$$

This shows that $M_n = H_n(G \otimes C_\bullet;\mathbb{Z})$. Therefore, we obtain the short exact sequence:

$$0 \longrightarrow G \otimes H_n(C_\bullet;\mathbb{Z}) \longrightarrow H_n(G \otimes C_\bullet;\mathbb{Z}) \longrightarrow \operatorname{Tor}_1^{\mathbb{Z}}(G, H_{n-1}(C_\bullet;\mathbb{Z})) \longrightarrow 0$$

This completes the proof. □

Remark 6.7.6. *It can be checked that the sequence in Proposition 6.7.5 splits.*

Remark 6.7.7. *There is also a universal coefficient theorem for homology where $\mathbb{Z}$ is replaced by a PID, R and G is a R-module. In this case, we have*

$$0 \to H_n(C_\bullet) \otimes_R G \to H_n(C_\bullet; G) \xrightarrow{h} \mathrm{Tor}_1^R(H_{n-1}(C_\bullet), G) \to 0$$

This comes from first establishing that Tor_i^R vanishes for $i \geq 2$ for when R is a PID, and then going through a proof for universal coefficient theorem essentially as above.

Example 6.7.8. Suppose $X = K$ is the Klein bottle, and $G = \mathbb{Z}/4$. Recall that $H_1(K;\mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}/2$, and $H_2(K;\mathbb{Z}) = 0$, so:

$$\begin{aligned} H_2(K;\mathbb{Z}/4) &= (H_2(K;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}/4) \oplus \mathrm{Tor}(H_1(K), \mathbb{Z}/4) \\ &= \mathrm{Tor}(\mathbb{Z}, \mathbb{Z}/4) \oplus \mathrm{Tor}(\mathbb{Z}/2, \mathbb{Z}/4) \\ &= 0 \oplus \mathbb{Z}/2 \\ &= \mathbb{Z}/2. \end{aligned}$$

Example 6.7.9. Let $X = \mathbb{RP}^n$ and $G = \mathbb{Z}/2\mathbb{Z}$. Recall that we have

$$H_k(\mathbb{RP}^n;\mathbb{Z}) = \begin{cases} \mathbb{Z}/2\mathbb{Z} & \text{if } k \text{ is odd, } 0 < k < n, \\ \mathbb{Z} & \text{if } k = 0, n \text{ is odd,} \\ 0 & \text{otherwise.} \end{cases}$$

We compute $H_k(\mathbb{RP}^n;\mathbb{Z}/2\mathbb{Z})$. We consider multiple cases. For $k = 0$, we have:

$$H_0(\mathbb{RP}^n;\mathbb{Z}/2\mathbb{Z}) \cong H_0(\mathbb{RP}^n;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} = \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z}.$$

For $k = 1$, we have:

$$\begin{aligned} H_1(\mathbb{RP}^n;\mathbb{Z}/2\mathbb{Z}) &\cong H_1(\mathbb{RP}^n;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \oplus \mathrm{Tor}(H_0(\mathbb{RP}^n), \mathbb{Z}/2\mathbb{Z}) \\ &= (\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}) \oplus \mathrm{Tor}(\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \\ &= \mathbb{Z}/2\mathbb{Z} \oplus 0 = \mathbb{Z}/2\mathbb{Z}. \end{aligned}$$

For $1 < k < n$, such that k is an odd integer, we have

$$\begin{aligned} H_k(\mathbb{RP}^n;\mathbb{Z}/2\mathbb{Z}) &\cong (H_k(\mathbb{RP}^n;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}) \oplus \mathrm{Tor}(H_{k-1}(\mathbb{RP}^n;\mathbb{Z}), \mathbb{Z}/2\mathbb{Z}) \\ &= (\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}) \oplus \mathrm{Tor}(0, \mathbb{Z}/2\mathbb{Z}) \\ &= \mathbb{Z}/2\mathbb{Z} \end{aligned}$$

For $1 < k < n$, such that k is an even integer, we have

$$\begin{aligned} H_k(\mathbb{RP}^n;\mathbb{Z}/2\mathbb{Z}) &\cong (H_k(\mathbb{RP}^n;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}) \oplus \mathrm{Tor}(H_{k-1}(\mathbb{RP}^n;\mathbb{Z}), \mathbb{Z}/2\mathbb{Z}) \\ &= (0 \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}) \oplus \mathrm{Tor}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \\ &= \mathbb{Z}/2\mathbb{Z} \end{aligned}$$

For $k = n$ even, we have

$$H_k(\mathbb{RP}^n;\mathbb{Z}/2\mathbb{Z}) = (0 \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}) \oplus \mathrm{Tor}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$$

If $k = n$ is odd, we have

$$H_k(\mathbb{RP}^n;\mathbb{Z}/2\mathbb{Z}) = (\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}) \oplus \mathrm{Tor}(0, \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$$

All in all, we have

$$H_k(\mathbb{RP}^n;\mathbb{Z}/2\mathbb{Z}) = \begin{cases} \mathbb{Z}/2\mathbb{Z} & \text{if } k = 0,\cdots,n \\ 0 & \text{otherwise.} \end{cases}$$

6.8. Künneth Formula

The Künneth Formula is a fundamental result in algebraic topology that relates the homology of a product space $X \times Y$ to the homology of the individual spaces X and Y. It provides a practical way to compute $H_n(X \times Y; G)$, where G is an abelian group, by expressing it in terms of the homology groups $H_\bullet(X;G)$ and $H_\bullet(Y;G)$. Suppose X and Y are finite CW complexes. Their product $X \times Y$ inherits a CW structure where the n-cells are built from products of p-cells in X and q-cells in Y with $p+q=n$. This suggests that the chain complex of the product, $C_\bullet(X \times Y)$, should be related to the tensor product of the chain complexes $C_\bullet(X) \otimes C_\bullet(Y)$. We use the method of acyclic models to construct a natural chain map

$$C_\bullet(X) \otimes C_\bullet(Y) \to C_\bullet(X \times Y)$$

The proof uses acyclic models.

Proposition 6.8.1. *Let $X, Y \in \mathsf{Top}$. There is a natural (in both X and Y) chain equivalence*

$$C_\bullet(X \times Y) \simeq C_\bullet(X) \otimes C_\bullet(Y),$$

where $C_\bullet(X) \otimes C_\bullet(Y)$ denotes the tensor product of chain complexes defined in degree n by

$$(C_\bullet(X) \otimes C_\bullet(Y))_n = \bigoplus_{p+q=n} C_p(X) \otimes C_q(Y),$$

Proof. Let $\mathsf{C} = \mathsf{Top}^2$. Consider the functors

$$(X,Y) \longmapsto C_\bullet(X \times Y),$$
$$(X,Y) \longmapsto C_\bullet(X) \otimes C_\bullet(Y).$$

Call these functors F and G. Let $\mathsf{M} = \{(\Delta^n, \Delta^m)\}_{n,m\geq 0}$. It is clear that F is acylic since $\Delta^n \times \Delta^m$ is contractible for each $n, m \geq 0$. We show that G is free. Indeed, G_n is free on

$$\coprod_{p+q=n} \mathrm{Hom}(\Delta^p, X) \times \mathrm{Hom}(\Delta^q, Y) \cong \mathrm{Hom}((\Delta^p, \Delta^q), (X,Y))$$

Hence, G is free on M. The result follows by Proposition 5.4.3. □

Remark 6.8.2. *We have used abstract arguments to establish the existence of such a map. In fact, it is possible to construct a chain map*

$$C_\bullet(X \times Y) \to C_\bullet(X) \otimes C_\bullet(Y),$$

by an inductive argument. By Proposition 5.4.3(2), any other natural chain map will be chain homotopic to the one constructed by an inductive argument.

We have thus reduced the problem to computing the homology of the tensor product of chain complexes. This can be done using the algebraic Künneth formula.

Proposition 6.8.3. *Let G be an abelian group, and let $C_\bullet$, $D_\bullet$ be chain complexes of abelian groups. Assume that each C_n is a free abelian group. There is a short exact sequence*

$$0 \to \bigoplus_{p+q=n} H_p(C) \otimes H_q(D) \to H_n(C_\bullet \otimes D_\bullet) \to \bigoplus_{p+q=n-1} \mathrm{Tor}(H_p(C), H_q(D)) \to 0,$$

The short exact sequence splits.

Proof. The proof is similar to a spectral sequence argument used to prove Proposition 6.7.5. $\square$

Example 6.8.4. Let $X = \mathbb{T}^k$. Since the homology groups of $\mathbb{S}^1$ are free, all Tor terms will drop out. For $k \geq 2$, we have

$$\begin{aligned} H_n(\mathbb{T}^k; \mathbb{Z}) &\cong \bigoplus_{i_1+\cdots+i_k=n} H_{i_1}(\mathbb{S}^1; \mathbb{Z}) \otimes \cdots \otimes H_{i_k}(\mathbb{S}^1; \mathbb{Z}) \\ &= \bigoplus_{\substack{i_1+\cdots+i_k=n \\ i_j \in \{0,1\}}} H_{i_1}(\mathbb{S}^1; \mathbb{Z}) \otimes \cdots \otimes H_{i_k}(\mathbb{S}^1; \mathbb{Z}) \cong \mathbb{Z}^{\binom{k}{n}} \end{aligned}$$

Part 3

Cohomology

CHAPTER 7

Singular Cohomology

Singular cohomology is a topological invariant of topological spaces analogous to singular homology, which is a contravariant functor rather than a covariant functor. That is, a continuous map $f : X \to Y$ between topological spaces induces a homomorphism

$$f^* : H^*(Y) \to H^*(X).$$

of abelian groups. We will show that this functor actually factors through $\mathsf{hTop}^{\mathsf{Op}}$. Hence, singular cohomology is a homotopy invariant. There are several reasons for caring about the fact singular cohomology is a contravariatnt functor:

(1) Many geometric objects pull back along continuous maps $f : X \to Y$. Examples include differential forms, covering spaces, or vector bundles. Topological invariants of such objects that are compatible with pullback must take values in an abelian group which is contravariantly assigned to X.
(2) For every topological space there is a diagonal map $\Delta : X \to X \times X$. Cohomology will turn this into a map
$$\Delta^* : H^*(X \times X) \to H^*(X),$$
from which we can extract a product on $H^*(X)$. This additional algebraic structure turns cohomology groups into a ring structure, which makes cohomology a ring-valued invariant, unlike singular homology.
(3) Categorically, cohomology is a better invariant than homology is that cohomology is a representable functor.

References include [Hat02; Lee10; May99].

7.1. Definition

Formally, we can develop he theory of singular homology similarly to that of singular cohomology. Let G be an abelian group and let X be a topological space with a singular chain complex $(C_\bullet, \partial_\bullet)$ of abelian groups:

$$\cdots \xrightarrow{\partial_{n+1}} C_n(X) \xrightarrow{\partial_n} C_{n-1}(X) \xrightarrow{\partial_{n-1}} C_{n-2}(X) \xrightarrow{\partial_{n-2}} \cdots .$$

Consider $C_n^*(X) = \operatorname{Hom}(C_n(X), G)$, the group of singular n co-chains of X with G-coefficients. This defines the dual chain complex:

$$\xleftarrow{\partial_{n+1}^*} C_n^*(X) \xleftarrow{\partial_n^*} C_{n-1}^*(X) \xleftarrow{\partial_{n-1}^*} C_{n-2}^*(X) \xleftarrow{\partial_{n-2}^*} \cdots .$$

Remark 7.1.1. *We write $(C^\bullet, \partial^\bullet)$ for the above diagram which is called a singular co-chain complex. We often abbreviate $(C^\bullet, \partial^\bullet)$ as $C^\bullet$. We write C^n for $C_n^* = \operatorname{Hom}(C_n, G)$. Moreover, we shall also write the boundary map ∂_{n+1}^* as δ^n for the boundary map.*

The boundary maps are $\partial_n^* : C_{n-1}^* \to C_n^*$ defined as:

$$(\partial_n^* \psi)(\alpha) = (\psi \circ \partial_n)(\alpha)$$

for $\psi \in C_{n-1}^*, \ \alpha \in C_n$. Note that the boundary map are such that $\partial_{n+1}^* \circ \partial_n^* = 0$ for $n \in \mathbb{Z}$. Indeed,

$$(\partial_{n+1}^* \circ \partial_n^*)(\psi) = \psi(\partial_{n+1} \circ \partial_n) = 0$$

for $\psi \in C_{n-1}^*$. Hence, we can now make the following definition:

Definition 7.1.2. Let G be an abelian group, and let $(C_\bullet, \partial_\bullet)$ be a chain complex of free abelian groups. The n-th cohomology group of $(C_\bullet, \partial_\bullet)$ with G-coefficients is defined as

$$H^n((C_\bullet, \partial_\bullet); G) := H_n((C^\bullet, \partial^\bullet); G)$$

Elements of $\ker \partial_{n+1}^*$ are called n-cocycles, and elements of $\mathrm{Im}\ \partial_n^*$ are called n-coboundaries. We shall write $Z^n(X)$ for $\ker \partial_{n+1}^* = \ker \delta^n$ and $B^n(X)$ for $\mathrm{Im}\ \partial_n^* = \ker \delta^{n-1}$.

Remark 7.1.3. *Recall that chain complexes of abelian groups for a category,* $\mathsf{Chain}_{\mathsf{Ab}}$. *The dual category,* $\mathsf{Chain}_{\mathsf{Ab}}^{\mathsf{Op}}$, *is called the category of co-chain complexes of abelian groups. Singular co-chain complexes are elements of* $\mathsf{Chain}_{\mathsf{Ab}}^{\mathsf{Op}}$. *It can be checked that both* $\mathsf{Chain}_{\mathsf{Ab}}$ *and* $\mathsf{Chain}_{\mathsf{Ab}}^{\mathsf{Op}}$ *are abelian categories. Thus, all results that hold for* $\mathsf{Chain}_{\mathsf{Ab}}$, *or singular chain complexes in particular continue to hold in* $\mathsf{Chain}_{\mathsf{Ab}}^{\mathsf{Op}}$, *or singular co-chain co-chain complexes in particular. For instance, we have various diagram-chasing lemmas such as the five lemma, the nine lemma, and the snake lemma. We shall not repeat these details in these notes. In any case, the proofs are similar to those discussed in the context of homology.*

Proposition 7.1.4. *Let G be an abelian group. For each $n \geq 0$,*

$$H^n : \mathsf{Top} \to \mathsf{Ab}$$

is a contravariant functor. Moreover, functor H^n descends to a contravariant functor $\overline{H}^n : \mathsf{hTop} \to \mathsf{Ab}$.

$$\begin{tikzcd}
\mathsf{Top} \arrow[r, "\gamma"] \arrow[dr, "H_n"'] & \mathsf{hTop} \arrow[d, dashed, "\overline{H}_n"] \\
& \mathsf{Ab}
\end{tikzcd}$$

In other words, if X and Y are topological spaces and $f, g : X \to Y$ are homotopic maps, then

$$H^n(f) = H^n(g) : H^n(X; G) \to H^n(Y; G)$$

for each $n \geq 0$.

Proof. Fix $n \geq 0$. The construction of H^n can be defined as the following composite functor:

$$\begin{tikzcd}
\mathsf{Top} \arrow[r, "\mathrm{Sing}_\bullet"] \arrow[rr, bend right, "S_\bullet"'] \arrow[rrr, bend left, "H_n"] & \mathsf{ssSet} \arrow[r, "\mathbb{Z}(-)"] & \mathsf{Chain}_{\mathsf{Ab}} \arrow[r, "\mathrm{Hom}(;G)"] & \mathsf{CoChain}_{\mathsf{Ab}} \arrow[r] & \mathsf{Ab}
\end{tikzcd}$$

Since $\mathrm{Hom}(-; G)$ is a contravariant functor and all the other functors involved are covariant functors, it follows that H^n is a contravariant functor. Moreover, note that dualizing a chain homotopy gives a co-chain homotopy. Thus, H^n factors through $\mathsf{hoTop}^{\mathsf{Op}}$. $\square$

Similar to singular homology, singular cohomology can be challenging to compute. One must develop computational tools to calculate singular cohomology, such as long exact sequences, excision, and other techniques. However, at the very least, we can explicitly compute H^0 by invoking the construction of H_0.

Example 7.1.5. $H_0(X;G)$ consists of those functions $f : C_0(X) \to G$ such that $f \circ \partial_1 = 0$. This means that $f \circ \partial_1(\sigma) = 0$ for every singular 1-simplex. Hence, f is constant on path components. We conclude that

$$H^0(X;G) = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^{|\pi_0(X)|}, G).$$

7.2. Ext Functor

We now discuss the Ext (derived) functor, which arises as a derived functor of $\mathrm{Hom}(-, G)$, and plays a crucial role in the formulation of the universal coefficient theorem for singular cohomology.

Remark 7.2.1. *We work with commutative rings below. Hence, we don't make any distinction between the categories of left R-modules and right R-modules. We use the generic phrase 'R-module' to refer to a left/right R-module.*

In the category of R-modules, recall that the $\mathrm{Hom}(X, -)$ functor defines a covariant functor from the category of R-modules to itself. If M is an R-module, then

$$\mathrm{Hom}(X, -)(M) = \mathrm{Hom}(X, M).$$

Moreover, if $f : M \to M'$ is a morphism of R-modules, then the functor acts on morphisms by

$$\mathrm{Hom}(X, -)(f) : \mathrm{Hom}(X, M) \longrightarrow \mathrm{Hom}(X, M')$$

defined. It can be checked that $\mathrm{Hom}(X, -)$ is a left exact functor. However, $\mathrm{Hom}(X, -)$ is not a right exact functor in general.

Example 7.2.2. The functor $\mathrm{Hom}(X, -)$ is not a right exact functor in general. Let $R = \mathbb{Z}$. Consider the short exact sequence of abelian groups:

$$0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0.$$

Apply the functor $\mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}, -)$ to this sequence. We obtain:

$$0 \longrightarrow \mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) \xrightarrow{(\cdot 2)_*} \mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) \longrightarrow \mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})$$

The resulting sequence is:

$$0 \to 0 \to 0 \to \mathbb{Z}/2\mathbb{Z},$$

which is not an exact since $\mathbb{Z}/2\mathbb{Z} \to 0$ is not a surjective function.

Definition 7.2.3. Let R be a ring and let X be a R-module. The i-th Ext functor is the i-th left derived functor of $\mathrm{Hom}(X, -) := h_X$. It is denoted as

$$\mathrm{Ext}_I^i(X, -)$$

Remark 7.2.4. *The subscript I denotes that we have taken an injective resolution.*

By definition, $\mathrm{Ext}_I^i(X, -)$ is computed as follows: for an R-module Y take any injective resolution

$$0 \to Y \to I_0 \to I_1 \to \cdots$$

and form the co-chain complex:

$$\mathrm{Hom}(X, I_0) \to \mathrm{Hom}(X, I_1) \to \cdots .$$

For each integer i, $\mathrm{Ext}^i(X, Y)$ is the homology of this co-chain complex at position i:

$$\mathrm{Ext}_I^i(X, Y) = H_i(\mathrm{Hom}(X, I_i)^\bullet)$$

Similarly, we can consider the contravariant Hom functor and consider its right derived functor. Since it is a contravariant functor, we take projective resolutions now.

Definition 7.2.5. Let R be a ring and let Y be a R-module. The i-th Ext functor is the i-th left derived functor of $\mathrm{Hom}(-, Y) := h^Y$. It is denoted as

$$\mathrm{Ext}^i_P(-, Y)$$

Remark 7.2.6. *The subscript P denotes that we have taken a projective resolution.*

By definition, $\mathrm{Ext}^i_P(-, Y)$ is computed as follows: for an R-module X take any projective resolution

$$\cdots \to P^1 \to P^0 \to X \to 0,$$

and form the co-chain complex:

$$\mathrm{Hom}(P^0, Y) \to \mathrm{Hom}(P^1, Y) \to \cdots .$$

Then $\mathrm{Ext}^i_P(X, Y)$ is the homology of this co-chain complex at position i:

$$\mathrm{Ext}^i_P(X, Y) = H_i(\mathrm{Hom}(P^i, Y)^\bullet)$$

The left exact $\mathrm{Hom}(-, -)$ functor can be thought of as a bifunctor which is covariant in the second variable and contravariant in the first variable. The discussion above seemingly provides us with with two different strategies to compute the Ext functor. Fortunately, it turns out that we can use either strategy as formalized by the following proposition.

Proposition 7.2.7. ***(Balancing Ext)*** *Let X, Y be R-modules. Then*

$$\mathrm{Ext}^i_P(X, Y) \cong \mathrm{Ext}^i_I(X, Y)$$

for each $i \geq 0$.

Proof. A spectral sequence argument analogous to that used in Proposition 6.6.6 can be employed to establish this result. □

Therefore, one can work with either strategy mentioned above. Therefore, we can now unambiguously write $\mathrm{Ext}^i(X, Y)$.

Proposition 7.2.8. *The Ext functor satisfies the following properties:*

(1) $\mathrm{Ext}^0(X, Y) \cong \mathrm{Hom}(X, Y)$ *for all R-modules X, Y..*
(2) *If X is a projective R-module, then* $\mathrm{Ext}^i(X, Y) = 0$ *for all $i \geq 1$*
(3) *If Y is an injective R-module, then* $\mathrm{Ext}^i(X, Y) = 0$ *for all $i \geq 1$*
(4) *Any $f : X_1 \to X_2$ induces a morphism*

$$f^{*,i} : \mathrm{Ext}^i(X_2, Y) \longrightarrow \mathrm{Ext}^i(X_1, Y)$$

for each $i \geq 0$.

(5) *Any $g : Y_1 \to Y_2$ induces a morphism*

$$g^i_* : \mathrm{Ext}^i(X, Y_1) \longrightarrow \mathrm{Ext}^i(X, Y_2)$$

for each $i \geq 0$.

(6) *Any short exact sequence $0 \to Y_1 \xrightarrow{\phi} Y_2 \xrightarrow{\psi} Y_3 \to 0$ induces a long exact sequence:*

$$0 \to \mathrm{Ext}^0(X, Y_1) \to \mathrm{Ext}^0(X, Y_2) \to \mathrm{Ext}^0(X, Y_3) \to \mathrm{Ext}^1(X, Y_1) \to \mathrm{Ext}^2(X, Y_2) \to \cdots$$

(7) *Any short exact sequence $0 \to X_1 \xrightarrow{\phi} X_2 \xrightarrow{\psi} X_3 \to 0$ induces a long exact sequence:*

$$0 \to \mathrm{Ext}^0(X_3, Y) \to \mathrm{Ext}^0(X_2, Y) \to \mathrm{Ext}^0(X_1, Y) \to \mathrm{Ext}^1(X_3, Y) \to \mathrm{Ext}^2(X_2, Y) \to \cdots$$

Proof. (1), (2) and (3) all follow from general properties of derived functors (Corollary 15.4.15). For (4) Let $P_1^\bullet$ be a projective resolution of X_1 and $P_2^\bullet$ be a projective resolution of X_2. General properties about resolutions implies that f lifts to a chain map $\varphi^\bullet : P_1^\bullet \longrightarrow P_2^\bullet$. Then, $\varphi^\bullet$ induces a morphism of chain complexes $\mathrm{Hom}(P_2^\bullet, Y) \longrightarrow \mathrm{Hom}(P_1^\bullet, Y)$ which, in turn, induces a morphism:

$$f^{*,i} : \mathrm{Ext}^i(X_2, Y) \longrightarrow \mathrm{Ext}^i(X_1, Y)$$

for each $i \geq 0$. For (5), let $P^\bullet$ be a projective resolution of X. Then, there is a morphism of chain complexes $\beta^\bullet : \mathrm{Hom}(P^\bullet, Y_1) \longrightarrow \mathrm{Hom}(P^\bullet, Y_2)$ induced by g, which, in turn, induces a morphism:

$$g_*^i : \mathrm{Ext}^i(X, Y_1) \longrightarrow \mathrm{Ext}^i(X, Y_2)$$

for each $i \geq 0$. For (6), let $P^\bullet$ be a projective resolution of X. Then there is an induced short exact sequence of chain complexes:

$$0 \to \mathrm{Hom}(P^\bullet, Y_1) \to \mathrm{Hom}(P^\bullet, Y_2) \to \mathrm{Hom}(P^\bullet, Y_3) \to 0$$

because each module P^i is projective. Indeed, at each degree i, P^i this sequence is

$$0 \to \mathrm{Hom}(P^i, Y_1) \to \mathrm{Hom}(P^i, Y_2) \to \mathrm{Hom}(P^i, Y_3) \to 0$$

obtained by applying the functor $\mathrm{Hom}(P^i, -)$, which is exact as P^i is projective. It is then easily checked that this gives a short exact sequence of chain complexes. Thus, applying the long exact sequence in homology produces the required long exact sequence. For (7), Let $P^\bullet$ be a projective resolution of X_1 and let $Q^\bullet$ be a projective resolution of X_3. By the horseshoe lemma (Lemma 15.4.16), there exists a projective resolution $R^\bullet$ of X_2 and a short exact sequence of chain complexes

$$0 \to P^\bullet \to R^\bullet \to Q^\bullet \to 0,$$

Since Q^i is projective, applying $\mathrm{Hom}(-, Y)$ yields

$$0 \to \mathrm{Hom}(Q^i, Y) \to \mathrm{Hom}(R^i, Y) \to \mathrm{Hom}(P^i, Y) \to 0$$

for each $i \geq 0$. It follows that there is a s.e.s. of cochain complexes

$$0 \to \mathrm{Hom}(Q^\bullet, Y) \to \mathrm{Hom}(R^\bullet, Y) \to \mathrm{Hom}(P^\bullet, Y) \to 0.$$

The associated long exact sequence in cohomology is the required long exact sequence. □

The above proposition show that the Ext groups 'measure' and 'repair' the non-exactness of the functors $\mathrm{Hom}(-, Y)$ and $\mathrm{Hom}(X, -)$. Let us now specialize to $R = \mathbb{Z}$. In what follows, let G be a fixed abelian group.

Lemma 7.2.9. *For any abelian group A, we have that*

$$\mathrm{Ext}^n(A, G) = 0 \text{ if } n > 1,$$

Proof. Any abelian group, A, admits a two-step free resolution.

$$0 \to F_1 \to F_0 \to A \to 0$$

Thus, $\mathrm{Ext}^n(A, G) = 0$ if $n > 1$. □

Remark 7.2.10. *Only* $\mathrm{Ext}^1(A, G)$ *encodes interesting information for abelian groups. We write* $\mathrm{Ext}(A, G) := \mathrm{Ext}^1(A, G)$.

Proposition 7.2.11. *The Ext functor satisfies the following properties:*

(1) $\mathrm{Ext}\left(\bigoplus_i A_i, G\right) \cong \prod_i \mathrm{Ext}(A_i, G)$.
(2) *If A is free, then* $\mathrm{Ext}(A, G) = 0$.
(3) $\mathrm{Ext}(\mathbb{Z}/n\mathbb{Z}, G) = G/nG$.

(4) *If H is a finitely generated abelian group, then:*

$$\operatorname{Ext}(H, G) = \operatorname{Ext}(\operatorname{Torsion}(H), G) = \mathit{Torsion}(H) \otimes_{\mathbb{Z}} G$$

(5) *For a short exact sequence: $0 \to A \to A' \to A'' \to 0$ of abelian groups, there is a natural exact sequence:*

$$0 \to \operatorname{Hom}(A'', G) \to \operatorname{Hom}(A', G) \to \operatorname{Hom}(A, G) \to \operatorname{Ext}(A'', G) \to \operatorname{Ext}(A', G) \to \operatorname{Ext}(A, G) \to 0$$

Proof. The proof is given below:

(1) This follows from the identity,

$$\operatorname{Hom}\left(\bigoplus_i A_i, G\right) = \prod_i \operatorname{Hom}(A_i, G),$$

and noting that taking direct sums of projective resolutions of A_i forms a projective resolution for $\bigoplus_i A^i$, and that homology commutes with arbitrary direct products.

(2) If A is free, then

$$0 \to A \to A \to 0$$

is a projective resolution of A, so $\operatorname{Ext}(A, G) = 0$.

(3) Consider the projective resolution of $\mathbb{Z}/n\mathbb{Z}$ given by

$$0 \to \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$$

dualize it and use the fact that $\operatorname{Hom}(\mathbb{Z}, G) \cong G$ to conclude that $\operatorname{Ext}(\mathbb{Z}/n\mathbb{Z}, G) = G/nG$.

(4) This follows at once from the previous statement.

(5) This follows from Proposition 7.2.8.

This completes the proof. □

Remark 7.2.12. *The discussion above implies has dealt with the case of $\mathbb{Z}$-modules (abelian groups). The general case can be more involved. For instance, consider $\mathbb{Z}_2$ as a $\mathbb{Z}_4$-module. Let $\mathbb{Z}_4 \xrightarrow{q} \mathbb{Z}_2$ denote the quotient map. Let $\mathbb{Z}_4 \xrightarrow{\times 2} \mathbb{Z}_4$ denote multiplication by 2. $\mathbb{Z}_2$ has the following free resolution over $\mathbb{Z}_4$:*

$$\cdots \xrightarrow{\times 2} \mathbb{Z}_4 \xrightarrow{\times 2} \mathbb{Z}_4 \xrightarrow{\times 2} \mathbb{Z}_4 \xrightarrow{\times 2} \mathbb{Z}_4 \xrightarrow{q} \mathbb{Z}_2 \to 0.$$

Since $\operatorname{Hom}_{\mathbb{Z}_4}(\mathbb{Z}_4, \mathbb{Z}_2) \cong \mathbb{Z}_2$ (by mapping the generator of $\mathbb{Z}_4$ to either 0 or 1), the dual of $\times 2 : \mathbb{Z}_4 \to \mathbb{Z}_4$ is simply the zero map. Hence, we have the dual sequence

$$0 \to \mathbb{Z}_2 \to \mathbb{Z}_2 \xrightarrow{0} \mathbb{Z}_2 \xrightarrow{0} \mathbb{Z}_2 \xrightarrow{0} \mathbb{Z}_2 \to \cdots$$

Consider the truncated sequence

$$\mathbb{Z}_2 \xrightarrow{0} \mathbb{Z}_2 \xrightarrow{0} \mathbb{Z}_2 \xrightarrow{0} \mathbb{Z}_2 \to \cdots$$

The homology of this complex is $\mathbb{Z}_2$ for every degree. Hence,

$$\operatorname{Ext}^n_{\mathbb{Z}_4}(\mathbb{Z}_2, \mathbb{Z}_2) \cong \mathbb{Z}_2$$

is nonzero for all $n \in \mathbb{N}$. This is stark contrast Remark 7.2.10.

Remark 7.2.13. *The name Ext comes from the phrase extension. We say X is an extension of A by B if*

$$0 \to B \to X \to A \to 0$$

is exact. Given A and B, there is always the trivial extension $X = A \oplus B$, corresponding to the isomorphism class of the split exact sequence. It can be shown that isomorphism classes of extensions of A by B are in 1-1 correspondence with elements of $\operatorname{Ext}^1(A, B)$, with the trivial extension corresponding to 0.

7.3. Universal Coefficient Theorem

Recall the construction of singular cohomology in Section 7.1. Since everything is determined in terms of $(C_\bullet, \partial_\bullet)$, can we compute cohomology groups using information about homology groups? The answer is a qualified yes. This is the universal coefficient theorem (UCT) for cohomology, which we now discuss. We first motivate the statement of UCT. As a first guess, we might think that

$$H^n(C_\bullet; G) := H_n(C^\bullet; G) \cong \mathrm{Hom}(H_n(C_\bullet), G)$$

This turns out to be almost true. We indeed have a natural map:

$$\varphi : H^n(C_\bullet, G) \longrightarrow \mathrm{Hom}(H_n(C_\bullet), G).$$

Denote $Z_n = \ker \partial_n \subseteq C_n$ and $B_n = \operatorname{Im} \partial_{n+1} \subseteq C_n$. We have $B_n \subseteq Z_n$. A class in $H^n(C^\bullet; G)$ is represented by a homomorphism $\phi : C_n \to G$ such that $\partial^*_{n+1}\phi = 0$. That is, $\phi\partial_{n+1} = 0$, or in words, ϕ vanishes on B_n. The restriction $\phi_0 = \phi|Z_n$ then induces a quotient homomorphism

$$\bar{\phi}_0 : Z_n/B_n \to G,$$

an element of $\mathrm{Hom}(H_n(C_\bullet), G)$. If ϕ is in $\operatorname{Im} \partial^*_n$, say $\phi = \psi\partial_n$ for some $\psi \in C^*_{n-1}$, then ϕ is zero on Z_n since $\partial_n \circ \partial_{n+1} = 0$. So $\phi_0 = 0$ and hence also $\bar{\phi}_0 = 0$.

$$\begin{tikzcd}
\cdots \arrow[r, "\partial_{n+2}"] & C_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[dr, dashed, "\phi\circ\partial_{n+1}"'] & C_n \arrow[r, "\partial_n"] \arrow[d, "\phi"] & C_{n-1} \arrow[r, "\partial_{n-1}"] \arrow[dl, dashed, "\psi"] & \cdots \\
& & G & &
\end{tikzcd}$$

Thus, there is a well-defined quotient map

$$h : H^n(C_\bullet, G) \to \mathrm{Hom}(H_n(C), G)$$

sending the cohomology class of $[\phi]$ to $\bar{\phi}_0$. Obviously h is a homomorphism.

Proposition 7.3.1. ***(Universal Coefficient Theorem)*** *If a chain complex $(C_\bullet, \partial_\bullet)$ of free abelian groups has homology groups $H_n(C_\bullet)$, then the cohomology groups $H^n(C_\bullet; G)$ of the cochain complex $\mathrm{Hom}(C_n, G)$ are determined by the short exact sequence:*

$$0 \longrightarrow \mathrm{Ext}(H_{n-1}(C_\bullet), G) \longrightarrow H^n(C_\bullet; G) \xrightarrow{h} \mathrm{Hom}(H_n(C_\bullet), G) \longrightarrow 0$$

Proof. The proof is based on a spectral sequence argument and is analogous to that of Proposition 6.7.5. Moreover, it can be verified that the sequence in Proposition 7.3.1 splits. ☐

Corollary 7.3.2. *Let $(C_\bullet, \partial_\bullet)$ be a chain complex so that its $\mathbb{Z}$-homology groups are finitely generated. Let $T_n = \mathit{Torsion}(H_n)$. We have*

$$0 \to T_{n-1} \to H^n(C_\bullet; \mathbb{Z}) \to H_n/T_n \to 0$$

This sequence splits[1]*, so:*

$$H^n(C^\bullet; \mathbb{Z}) \cong T_{n-1} \oplus H_n/T_n.$$

Proof. Clear. ☐

Example 7.3.3. Let us now derive some immediate consequences of Proposition 7.3.1.

(1) If $n = 0$, we have

$$H^0(X; G) = \mathrm{Hom}_{\mathbb{Z}}(H_0(X), G) \cong \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^{|\pi_0(X)|}, G)$$

This is consistent with Example 7.1.5.

[1]Since H_n/T_n is free and hence projective.

(2) If $n = 1$, the Ext-term vanishes since $H_0(X)$ is free, so we get:

$$H^1(X;G) = \mathrm{Hom}_{\mathbb{Z}}(H_1(X), G)$$

Remark 7.3.4. *There is also a universal coefficient theorem for cohomology where $\mathbb{Z}$ is replaced by a PID, R and G is a R-module. In this case, we have*

$$0 \to \mathrm{Ext}^1_R(H_{n-1}(X;R), G) \to H^n(X;G) \xrightarrow{h} \mathrm{Hom}_R(H_n(X;R), G) \to 0.$$

This comes from first establishing that Ext^n_R vanishes for $n \geq 2$ for when R is a PID, and then going through a proof for universal coefficient theorem as above.

7.4. Eilenberg-Steenrod Axioms

We have defined singular cohomology. There are many other cohomology theories: sheaf cohomology, Čech cohomology, etc. All these cohomology theories satisfy the Eilenberg-Steenrod axioms. The purpose of this section is to state these axioms and prove that singular cohomology satisfies these axioms.

Definition 7.4.1. (Eilenberg-Steenrod Axioms) Let G be an abelian group. A (unreduced) cohomology theory consists of

(1) A family of functors $H^n : \mathsf{Top}^2 \to \mathsf{Ab}$ for $n \geq 0$, and
(2) A family of natural transformations $\gamma^n : H^n \to H^{n+1} \circ p$, where p is the functor sending (X, A) to $(A, \varnothing)$ and $f : (X, A) \to (Y, B)$ to $f|_B : (A, \varnothing) \to (B, \varnothing)$.

such that the following axioms are satisfied:

(a) (Homotopy invariance) If $f, g : (X, A) \to (Y, B)$ are homotopic maps, then the induced maps
$$H^n(f), H^n(g) : H^n(X, A) \to H^n(Y, B)$$
are such that $H^n(f) = H^n(g)$ for $n \geq 0$. In other words, H^n may be regarded as a functor from hTop to Ab.

(b) (Long exact sequence) For every pair (X, A), the inclusions
$$(A, \emptyset) \overset{i}{\hookrightarrow} (X, \emptyset) \overset{j}{\hookrightarrow} (X, A)$$
give rise to a long exact sequence
$$\cdots \to H^n(X, A) \xrightarrow{j_n^*} H^n(X) \xrightarrow{i_n^*} H^n(A) \xrightarrow{\delta^n} H^{n+1}(X, A) \to \cdots$$

(c) (Excision) If $Z \subseteq A \subseteq X$ are topological spaces such that $\overline{Z} \subseteq \mathrm{Int}(A)$, the inclusion of pairs $(X \setminus Z, A \setminus Z) \subseteq (X, A)$ induces isomorphisms
$$H^n(X \setminus Z, A \setminus Z) \to H^n(X, A)$$
for all $n \geq 0$.

(d) (Multiplicativity) If $X = \coprod_\alpha X_\alpha$ and $A = \coprod_\alpha A_\alpha$ is the disjoint union of a family of topological spaces X_α, then
$$H^n(X, A) = \prod_\alpha H^n(X_\alpha, A_\alpha)$$
for each $n \geq 0$.

If a cohomology theory satisfies the following additional dimension axiom, which states that for any one-point space $X = \{*\}$,

$$H_n(X;G) = \begin{cases} G & \text{if } n = 0, \\ 0 & \text{otherwise,} \end{cases}$$

then the cohomology theory is called an ordinary cohomology theory with G-coefficients, and the corresponding Eilenberg–Steenrod axioms are referred to as the ordinary Eilenberg–Steenrod axioms for cohomology.

Remark 7.4.2. *Not all cohomology theories satisfy the dimension axiom. Such theories are known as extraordinary/generalized cohomology theories. Although they still satisfy the other Eilenberg–Steenrod axioms, they fail the dimension axiom. Examples of extraordinary/generalized cohomology theories include:*

(1) *Topological K-theory,*
(2) *Complex cobordism,*
(3) *Cobordism theories associated with structured manifolds, e.g., spin cobordism,*
(4) *Morava K-theories,*
(5) *Cohomology theories represented by spectra in stable homotopy theory that do not satisfy the dimension axiom.*

We now verify that singular cohomology satisfies the Eilenberg–Steenrod axioms. This is largely a formal exercise. This follows because singular homology satisfies the corresponding Eilenberg–Steenrod axioms for homology, and singular cohomology and singular homology are related via the Universal Coefficient Theorem (Proposition 7.3.1). An alternative line of argument is that we can dualize all the proofs showing that singular homology satisfies the Eilenberg–Steenrod axioms for homology and thereby conclude that singular cohomology satisfies the Eilenberg–Steenrod axioms for cohomology.

7.4.1. Relative cohomology groups.

We first construct the relative cohomology group that shall allow us to construct the appropriate functors from Top to Ab. We apply the $\mathrm{Hom}(-, G)$ functor to the the relative singular chain complex to get

$$C^n(X, A; G) := \mathrm{Hom}(C_n(X, A), G).$$

The group $C^n(X, A; G)$ can be identified with functions from the set of n-simplices in X to G that vanish on simplices in A, so we have a natural inclusion

$$C^n(X, A; G) \hookrightarrow C^n(X; G)$$

The relative coboundary maps

$$\overline{\delta}^n : C^n(X, A; G) \to C^{n+1}(X, A; G)$$

are obtained by restricting δ^n. We have a co-chain complex $(C^\bullet(X, A), \overline{\delta}^\bullet)$.

Definition 7.4.3. Let $A \subseteq X$ be a subspace of a topological space X. The n-th relative cohomology group, $H^n(X, A)$, is the n-th homology group of the chain complex $(C^\bullet(X, A), \overline{\delta}^\bullet)$. That is:

$$H^n(X, A) = \frac{\mathrm{Ker}\, \overline{\delta}^n}{\mathrm{Im}\, \overline{\delta}^{n+1}}$$

Similar to Proposition 7.1.4, it is easily checked that each H^n is a functor from Top^2 to Ab. This effectively checkes the first two conditons in the definition of the Eilenberg-Steenrod axioms.

Remark 7.4.4. *Since the cohomology of the empty set is trivial for all $n \geq 0$, we have:*

$$H^n(X, \emptyset) = H^n(X), \quad \forall n \geq 0.$$

Remark 7.4.5. *There is also a Universal Coefficient Theorem for relative cohomology. The argument is identical to that in Proposition 7.3.1.*

We now prove that singular cohomology satisfies the long exact sequence axiom. The importance of the long exact sequence axiom is that is allows us to compute cohomology groups of various spaces in using an 'inductive' and/or 'bottom-up' approach. Applying by the $\mathrm{Hom}(-, G)$ functor to the short exact sequence,

$$0 \to C_n(A) \xrightarrow{i_n} C_n(X) \xrightarrow{j_n} C_n(X, A) \to 0,$$

we get another short exact sequence[2]

$$0 \leftarrow C^n(A; G) \xleftarrow{i^*} C^n(X; G) \xleftarrow{j^*} C^n(X, A; G) \leftarrow 0.$$

Since i_n and j_n commute with the boundary maps, it follows that i_n^* and j_n^* commute with co-boundary maps. So we obtain a short exact sequence of cochain complexes:

$$0 \leftarrow C^\bullet(A; G) \xleftarrow{i^*} C^\bullet(X; G) \xleftarrow{j^*} C^\bullet(X, A; G) \leftarrow 0.$$

By taking the associated long exact sequence of homology groups, we get the long exact sequence for the cohomology groups of the pair (X, A):

$$\cdots \longrightarrow H^n(X, A; G) \xrightarrow{j_n^*} H^n(X; G) \xrightarrow{i_n^*} H^n(A; G) \xrightarrow{\gamma^n} H^{n+1}(X, A; G) \xrightarrow{j_{n+1}^*} H^{n+1}(X; G) \xrightarrow{i_{n+1}^*} \cdots$$

This shows that the long exact sequence axiom is satisfied.

7.4.2. Homotopy Invariance. We now show that singular cohomology satisfies the homotopy invariance property. In fact, we have already formally established this in Proposition 7.1.4. Nonetheless, we explicitly detail the proof below using a dualization argument.

Proposition 7.4.6. ***(Homotopy Invariance)*** *Let X, Y be topological spaces, and let G be an abelian group. If $f \simeq g : X \to Y$ are homotopic maps, then*

$$H^n(f) = H^n(g) : H^n(Y, G) \to H^n(X, G).$$

Proof. Recall from the proof of the similar statement for homology that a chain homotopy between $C_\bullet(X, A; G)$ and $C_\bullet(Y, B; G)$ is given by a prism operator

$$T_n : C_n(X, A; G) \to C_{n+1}(Y, B; G)$$

satisfying

$$f_n - g_n = T_{n-1} \circ \partial_n + \partial'_{n+1} \circ T_n$$

with f_n and g_n being the induced maps on singular chain complexes. The claim about cohomology follows by applying the $\mathrm{Hom}(-.G)$ functor to the prism operator to get

$$T^n : C^{n+1}(Y, B; G) \to C^n(X, A; G)$$

which satisfies

$$f^n - g^n = \partial_n^* \circ T^{n-1} + T^n \circ \partial_{n+1}^{*'}.$$

Hence, we have a chain homotopy between $C^\bullet(X, A; G)$ and $C^\bullet(Y, B; G)$. It is now a standard fact that a chain homotopy induces the same maps on homology groups. Hence, $H^n(f) = H^n(g)$ for each $n \geq 0$. □

Corollary 7.4.7. *If X is contractible, then $H^n(X) = 0$ for all $n \geq 1$.*

Proof. This is immediate from the homotopy invariance of singular cohomology and that $H^n(\{*\}) = 0$ for $n \geq 1$. □

[2] $\mathrm{Hom}(-, G)$ is only a left exact functor in general. But it can be checked in this case that the resulting sequence is both left exact and right exact.

7.4.3. Excision. We now prove that singular cohomology satisfies the excision axiom. As in the case of singular homology, the importance of the excision axiom is that if $A \subseteq X$ and n-cochains are "sufficiently inside" A, we can remove A without affecting the relative cohomology groups $H^n(X, A)$. Here is the formal statement, which parallels the corresponding statement for singular homology.

Proposition 7.4.8. *(**Excision**) Let X be a topological space such that $Z \subseteq A \subseteq X$, with $\overline{Z} \subseteq int(A)$. Then the inclusion of pairs $i : (X \setminus Z, A \setminus Z) \hookrightarrow (X, A)$ induces isomorphisms:*

$$i^n : H^n(X, A; G) \to H^n(X \setminus Z, A \setminus Z; G)$$

for all $n \geq$. Equivalently, if A and B are subsets of X with $X = \operatorname{int}(A) \cup \operatorname{int}(B)$, then the inclusion map $(B, A \cap B) \hookrightarrow (X, A)$ induces isomorphisms in cohomology.

Proof. Proposition 7.3.1 and Proposition 5.5.1 imply that the left and right maps in the diagram below are isomorphisms.

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Ext}(H_{n-1}(X, A), G) \arrow[r] & H^n(X, A; G) \arrow[r] & \operatorname{Hom}(H_n(X, A), G) \arrow[r] & 0 \\
0 \arrow[r] & \operatorname{Ext}(H_{n-1}(X \setminus Z, A \setminus Z), G) \arrow[r] \arrow[u] & H^n(X \setminus Z, A \setminus Z; G) \arrow[r] \arrow[u] & \operatorname{Hom}(H_n(X \setminus Z, A \setminus Z), G) \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

The five-lemma (Proposition 4.6.1) then implies that the middle map is also an isomorphism. This completes the proof. $\square$

7.4.4. Dimension Axiom. We now verify that singular cohomology satisfies the dimension axiom. This follows formally as a consequence of Proposition 7.3.1. Let $X = \{*\}$ be a single point space. By Proposition 7.3.1, we have:

$$H^n(X; G) = \operatorname{Hom}(H_n(X), G) \oplus \operatorname{Ext}(H_{n-1}(X), G).$$

Since $H_n(X)$ satisfies the dimension axiom for singular homology and $\operatorname{Ext}(H_{n-1}(X), G) = 0$ for all $n \geq 0$ since each $H_n(X)$ is a free abelian group, we have the following consequence:

$$H^n(X; G) = \begin{cases} G, & \text{if } n = 0 \\ 0, & \text{otherwise} \end{cases}.$$

7.4.5. Multiplicativity Axiom. We now prove the multiplicativity axiom. This is easily seen to hold using a UCT in relative cohomology (Remark 7.4.5). Let $X = \coprod_\alpha X_\alpha$ and $A = \coprod_\alpha A_\alpha$. We have:

$$\begin{aligned}
H^n(X, A; G) &= \operatorname{Ext}(H_{n-1}(X, A); G) \oplus \operatorname{Hom}(H_n(X, A); G) \\
&= \operatorname{Ext}(H_{n-1}(\coprod_\alpha X_\alpha, \coprod_\alpha A_\alpha); G) \oplus \operatorname{Hom}(H_n(\coprod_\alpha X_\alpha, \coprod_\alpha A_\alpha); G) \\
&= \operatorname{Ext}(\bigoplus_\alpha H_{n-1}(X_\alpha, A_\alpha); G) \oplus \operatorname{Hom}(\bigoplus_\alpha H_n(X_\alpha, A_\alpha); G) \\
&= \prod_\alpha \operatorname{Ext}(H_{n-1}(X_\alpha, A_\alpha); G) \oplus \prod_\alpha \operatorname{Hom}(H_n(X_\alpha, A_\alpha); G) \\
&= \prod_\alpha \operatorname{Ext}(H_{n-1}(X_\alpha, A_\alpha); G) \oplus \operatorname{Hom}(H_n(X_\alpha, A_\alpha); G) = \prod_\alpha H^n(X_\alpha, A_\alpha; G)
\end{aligned}$$

Hence, we see that singular cohomology satisfies the Eilenberg-Steenrod axioms.

Remark 7.4.9. *We have previously seen that the Mayer–Vietoris sequence for singular homology is a formal consequence of the Eilenberg–Steenrod axioms for homology. Similarly, the Mayer–Vietoris sequence for singular cohomology is a formal consequence of the Eilenberg–Steenrod axioms for cohomology. Mayer–Vietoris sequence for singular cohomology states that if X be a topological space, and A and B are open subsets of X such that $X = \text{int}(A) \cup \text{int}(B)$, then there is a long exact sequence of cohomology groups:*

$$\cdots \to H^n(X;G) \xrightarrow{\psi} H^n(A;G) \oplus H^n(B;G) \xrightarrow{\phi} H^n(A \cap B;G) \to \cdots$$

We also note in passing that have a Mayer-Vietoris sequence in relative cohomology groups.

Remark 7.4.10. *We can also define reduced cohomology. Consider the augmented singular chain complex for X:*

$$\cdots \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

where $\varepsilon(\sum_i n_i x_i) = \sum_i n_i$. After applying the $\text{Hom}(-,G)$ functor, we get the augmented co-chain complex:

$$\xleftarrow{\partial_3^*} C_2^* \xleftarrow{\partial_2^*} C_1^* \xleftarrow{\partial_1^*} C_0^* \xleftarrow{\varepsilon^*} \mathbb{Z} \leftarrow 0$$

Note that since $\epsilon \circ \partial_1 = 0$, we get by by applying the $\text{Hom}(-,G)$ functor that $\partial_1^* \circ \epsilon^* = 0$. The cohomology of this augmented cochain complex is the reduced cohomology *of X with G-coefficients, denoted by $\widetilde{H}^n(X;G)$. It follows by definition that*

$$\widetilde{H}^n(X;G) = H^n(X;G) \quad n > 0$$

and by the universal coefficient theorem (applied to the augmented chain complex), we get

$$\widetilde{H}^0(X;G) = \text{Hom}(\widetilde{H}^0(X), G).$$

Remark 7.4.11. *If (X, A) is a good pair, then the long exact sequence in reduced cohomology holds true. This is because the analogous result is a formal consequence of the Eilenberg-Steenrod axioms.*

$$\cdots \to H^n(X, A;G) \to \widetilde{H}^n(X;G) \to \widetilde{H}^n(A;G) \to H^{n+1}(X, A;G) \to \cdots$$

In particular, if $A = \{*\}$ is a point in X, we get that

$$\widetilde{H}^n(X;G) \cong H^n(X, x_0;G)$$

for $n \geq 1$. Moreover, we have

$$H^n(X, A;G) \cong H^n(X/A;G)$$

for all $n \in \mathbb{N}$. The proof is the same as in the homology case since it is a formal consequence of the Eilenberg-Steenrod axioms and the hypothesis on the space. Also, if each X_α is path-connected, we have

$$\widetilde{H}^n(\bigvee_\alpha X_\alpha) = \prod_\alpha \widetilde{H}^n(X_\alpha)$$

for $n \geq 0$. Once again, the proof is similar to the proof in the case of singular homology. We also have a Mayer-Vietoris sequence in reduced cohomology.

Remark 7.4.12. *We can define simplicial cohomology and cellular cohomology in exactly the same way as expected. As expexted, simplicial cohomology and cellular cohomology are isomorphic to singular cohomology. The proofs are identical in the homology case.*

7.5. First Computations

The purpose of this section is to compute cohomology groups of some topological spaces. We begin by looking at some specific examples.

Example 7.5.1. (Contractible Spaces) Let X be a contractible topological space. We have:

$$H^n(X;G) = \begin{cases} G, & \text{if } n = 0 \\ 0, & \text{otherwise} \end{cases}.$$

This follows immediately by the homotopy invariance of cohomology groups since X homotopy equivalent to a point.

Example 7.5.2. (Spheres) Let $X = \mathbb{S}^n$. Then we have

$$H_k(\mathbb{S}^n\ \mathbb{Z}) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z}, & \text{if } k = n = 0 \\ \mathbb{Z}, & \text{if } k = n > 0,\ k = 0, n > 0 \\ 0, & \text{otherwise} \end{cases}$$

Since, $\mathbb{Z}$ and $\mathbb{Z} \oplus \mathbb{Z}$ are free-abelian groups, the Ext term in the UCT for cohomology vanishes for each k. Hence,

$$H^k(\mathbb{S}^n, G) \cong \operatorname{Hom}(H_k(\mathbb{S}^n, G), \mathbb{Z}) \cong \begin{cases} G \oplus G, & \text{if } k = n = 0 \\ G, & \text{if } k = n > 0,\ k = 0, n > 0 \\ 0, & \text{otherwise} \end{cases}.$$

for each $k \geq 0$.

Remark 7.5.3. *We can also compute the cohomology groups of $\mathbb{S}^n$ by using the above Mayer-Vietoris sequence. Cover $\mathbb{S}^n$ by two open sets $A = \mathbb{S}^n \setminus \{N\}$ and $B = \mathbb{S}^n \setminus \{S\}$, where N and S are the North and South poles of S^n. Then we have*

$$A \cap B \simeq \mathbb{S}^{n-1} \qquad A \simeq B \simeq \mathbb{R}^n$$

Thus, by the Mayer-Vietoris sequence for reduced cohomology, homotopy invariance, and induction, we get:

$$\widetilde{H}^k(\mathbb{S}^n; G) \cong \widetilde{H}_{k-n}(\mathbb{S}^0; G) \cong \begin{cases} G, & \textit{if } k = n \\ 0, & \textit{otherwise} \end{cases}$$

for each $k \geq 0$.

Example 7.5.4. (Mobius Band) Let M denote the Mobius band. Since M is homotopic to $\mathbb{S}^1$. we have,

$$H^k(M, G) \cong \begin{cases} G, & \text{if } k = 0, 1 \\ 0, & \text{otherwise} \end{cases}.$$

for each $k \geq 0$.

Example 7.5.5. (Torus) Let $X = \mathbb{S}^1 \times \mathbb{S}^1$. Recall that we have,

$$H_n(\mathbb{S}^1 \times \mathbb{S}^1, \mathbb{Z}) \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & \text{for } n = 1 \\ \mathbb{Z} & \text{for } n = 0, 2 \\ 0 & \text{for } n \geq 3 \end{cases}$$

Since, $\mathbb{Z}$ and $\mathbb{Z} \oplus \mathbb{Z}$ are free-abelian groups, the Ext term in the UCT for cohomology vanishes for each k. Hence,

$$H^n(\mathbb{S}^1 \times \mathbb{S}^1, G) \cong \mathrm{Hom}_{\mathbb{Z}}(H_n(\mathbb{S}^1 \times \mathbb{S}^1, \mathbb{Z}), G) \cong \begin{cases} G \oplus G & \text{for } n = 1 \\ G & \text{for } n = 0, 2 \\ 0 & \text{for } n \geq 3 \end{cases}$$

Example 7.5.6. **(Klein Bottle)** Let $X = K$ be the Klein bottle. Recall that we have,

$$H_n(K\ \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{for } n = 0 \\ \mathbb{Z} \oplus \mathbb{Z}_2 & \text{for } n = 1 \\ 0 & \text{for } n \geq 2 \end{cases}$$

Note that we have,

$$\begin{aligned} &\mathrm{Ext}(H_0(K, \mathbb{Z}), G) = 0, \\ &\mathrm{Ext}(H_1(K, \mathbb{Z}), G) \cong \mathrm{Ext}(\mathbb{Z}_2, G) \cong G/2G \end{aligned}$$

Therefore, we have

$$\begin{aligned} H^n(K, G) &\cong \mathrm{Hom}_{\mathbb{Z}}\left(H_n(K, \mathbb{Z}), G\right) \oplus \mathrm{Ext}\left(H_{n-1}(K, \mathbb{Z}), G\right) \\ &\cong \begin{cases} G, & \text{for } n = 0, \\ G \oplus G/2G, & \text{for } n = 1, \\ G/2G, & \text{for } n = 2, \\ 0, & \text{for } n \geq 3. \end{cases} \end{aligned}$$

The case $G = \mathbb{Z}_2$ is important. Then,

$$H^n(K\ \mathbb{Z}_2) \cong \begin{cases} \mathbb{Z}_2 & \text{for } n = 0, 2, \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{for } n = 1 \\ 0 & \text{for } n \geq 3 \end{cases}$$

Example 7.5.7. **(Real Projective Space)** Let $X = \mathbb{RP}^n$. Recall that we have,

$$H_k(\mathbb{RP}^n, \mathbb{Z}) = \begin{cases} \mathbb{Z}_2 & \text{if } k \text{ is odd, } 0 < k < n, \\ \mathbb{Z} & \text{if } k = 0, n \text{ is odd,} \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_2, G) \cong G_2 = \{g \in G \mid 2g = 0\}$ and $\mathrm{Ext}(\mathbb{Z}_2, G) \cong G/2G$. If n is odd, we have:

$$H^k(\mathbb{RP}^n, G) \cong \begin{cases} G & \text{if } k = 0, n, \\ G/2G & \text{if } k \text{ is even, } 0 < k < n, \\ G_2 & \text{if } k \text{ is odd, } 0 < k < n, \\ 0 & \text{otherwise.} \end{cases}$$

If n is even, we have:

$$H^k(\mathbb{RP}^n, G) \cong \begin{cases} G & \text{if } k = 0, \\ G/2G & \text{if } k \text{ is even, } 0 < k \leq n, \\ G_2 & \text{if } k \text{ is odd, } 0 < k < n, \\ 0 & \text{otherwise.} \end{cases}$$

Remark 7.5.8. *The case $G = \mathbb{Z}_2$ in Example 7.5.7 is important. We have*

$$H^k(\mathbb{RP}^n\ \mathbb{Z}_2) \cong \begin{cases} \mathbb{Z}_2 & k = 0, \cdots, n, \\ 0 & \textit{otherwise}. \end{cases}$$

Example 7.5.9. (**Complex Projective Space**) Let $X = \mathbb{CP}^n$. Recall that we have,

$$H_k(\mathbb{CP}^n\ \mathbb{Z}) = \begin{cases} \mathbb{Z}, & \text{if } k = 0, 2, 4, \ldots, 2n \\ 0, & \text{otherwise.} \end{cases}$$

Since $\mathbb{Z}$ is a free abelian group, all the Ext terms in the UCT for cohomology vanish. Hence,

$$H^k(\mathbb{CP}^n, G) \cong \mathrm{Hom}_{\mathbb{Z}}(H_k(\mathbb{CP}^n, \mathbb{Z}), G) \cong \begin{cases} G, & \text{if } k = 0, 2, 4, \ldots, 2n \\ 0, & \text{otherwise.} \end{cases}$$

CHAPTER 8

de-Rham Cohomology

The transition from singular cohomology to de Rham cohomology focuses on a cohomology theory intrinsically adapted to smooth manifolds. De Rham cohomology provides a geometric realization of singular cohomology with real coefficients, establishing an isomorphism between these theories in the smooth setting. Defined via differential forms and their exterior derivatives, de Rham cohomology captures topological invariants through differential-geometric structures. The kth de Rham cohomology group $H^k_{\mathrm{dR}}(M)$ measures the extent to which closed k-forms fail to be exact on a smooth manifold M, quantifying the obstruction to extending local primitives to global ones. This reflects a fundamental principle across mathematics: cohomology theories serve to measure the failure of local data to assemble into global structures.

8.1. Definition & Propeties

Let M be a smooth manifold. Since

$$d^2 = d \circ d : \Omega^{k-1}(M) \xrightarrow{d} \Omega^k(M) \xrightarrow{d} \Omega^{k+1}(M)$$

is the zero operator for every $k \geq 1$, we have

$$\operatorname{im}\left(d : \Omega^{k-1}(M) \to \Omega^k(M)\right) \subseteq \ker\left(d : \Omega^k(M) \to \Omega^{k+1}(M)\right).$$

Thus, im d is a subspace of ker d for all $k \geq 1$.

Remark 8.1.1. *Let M be a smooth n-dimensional manifold. For convenience, we set $\Omega^k(M) = \{0\}$ for all $k < 0$ and $k > n$. Moreover, we set*

$$d = 0 : \Omega^k(M) \to \Omega^{k+1}(M)$$

for all $k < 0$ and $k \geq n$. Then the inclusion above holds for all $k \in \mathbb{Z}$.

Definition 8.1.2. Let M be a smooth manifold. The quotient vector space

$$H^k_{\mathrm{dR}}(M) = \frac{\ker(d : \Omega^k(M) \to \Omega^{k+1}(M))}{\operatorname{im}(d : \Omega^{k-1}(M) \to \Omega^k(M))} = \frac{\{\omega \in \Omega^k(M) : d\omega = 0\}}{\{d\omega : \omega \in \Omega^{k-1}(M)\}}$$

is the k-th de Rham cohomology group of M.

Let M be a smooth manifold. A form $\omega \in \Omega^k(M)$ is closed if $d\omega = 0$ and exact if there exists a $(k-1)$-form $\tau \in \Omega^{k-1}(M)$ for which $d\tau = \omega$. Since $d \circ d = 0$, every exact form is closed. hus,

$$H^k_{\mathrm{dR}}(M) = \frac{\{\text{closed } k\text{-forms in } M\}}{\{\text{exact } k\text{-forms in } M\}}$$

This suggests that Definition 8.1.2 measures the failure of closed forms to be exact forms. Indeed, every closed form need not be exact:

Example 8.1.3. Consider the 1-form on $\mathbb{R}^2 \setminus \{0\}$ defined by:

$$\omega = \frac{x\,dy - y\,dx}{x^2 + y^2}$$

Then,

$$\begin{aligned} d\omega &= \frac{(dx \wedge dy - dy \wedge dx)(x^2+y^2) - (2x\,dx + 2y\,dy)(x\,dy - y\,dx)}{(x^2+y^2)^2} \\ &= \frac{2(x^2+y^2)\,dx \wedge dy - (2x^2\,dx \wedge dy - 2y^2\,dy \wedge dx)}{(x^2+y^2)^2} = 0 \end{aligned}$$

So, ω is closed. But writing ω in polar coordinates and integrating around a circle centered at 0 in $\mathbb{R}^2 \setminus \{0\}$ gives

$$\int_{\mathbb{S}^1} \omega = 2\pi.$$

If $\omega = d\eta$ were exact, Stokes' theorem would imply

$$0 = \int_{\emptyset} \eta = \int_{\partial \mathbb{S}^1} \eta = \int_{\mathbb{S}^1} d\eta = \int_{\mathbb{S}^1} \omega = 2\pi.$$

Hence, ω is not exact.

Remark 8.1.4. *Elements $H^k(M)$ are equivalence classes of k-forms. Given $\omega \in \ker(d : \Omega^k(M) \to \Omega^{k+1}(M))$, we denote the equivalence class by*

$$[\omega] = \{\omega + d\tau \in \Omega^k(M) : \tau \in \Omega^{k-1}(M)\}.$$

Therefore, $H^k_{dR}(M)$ is a vector space that classifies the closed k-forms in M up to exact forms.

8.1.1. Properties of de Rham cohomology. We now discuss several algebraic properties of de Rham cohomology, which are analogous to the properties of singular cohomology for general topological spaces. We first show that the de Rham cohomology defines a contravariant functor from the category of smooth manifolds, Man, to the category of abelian groups, Ab.

Proposition 8.1.5. *Let M, N be smooth manifolds and let $F : M \to N$ be a smooth map. For each $k \in \mathbb{Z}$, let $F^* : \Omega^k(N) \to \Omega^k(M)$ be the pullback map.*

(1) *For each $k \in \mathbb{Z}$, F^* descends to a linear map $F^\# : H^k_{dR}(N) \to H^k_{dR}(M)$ between the de Rham cohomology groups given by $F^\#[\omega] = [F^*\omega]$.*
(2) ***(Functoriality)*** *For each $k \in \mathbb{Z}$, $H^k_{dR} : \mathsf{Man} \to \mathsf{Ab}$ is a contravariant functor.*

Proof. We shall use the fact that the exterior derivative commutes with pullbacks. The proof is given below:

(1) Let ω is a closed form. Then

$$d(F^*\omega) = F^*(d\omega) = 0$$

Hence, $F^*\omega$ is also closed a form. This shows that $F^*\omega$ restricts to a map

$$F^* : \{\text{closed } k-\text{forms on } N\} \to \{\text{closed } k-\text{forms on } M\}$$

Now let $\omega = d\eta$ be an exact form. Then

$$F^*\omega = F^*(d\eta) = d(F^*\eta),$$

Hence, $F^*\omega$ is also an exact form. This shows that F^* descends to a well-defined map map

$$F^\# : H^k_{\mathrm{dR}}(N) \to H^k_{\mathrm{dR}}(M)$$

given by $F^\#[\omega] = [F^*\omega]$.

(2) This follows from (1).

This completes the proof. □

Proposition 8.1.6. ***(de Rham Cohomolgy of Disjoint Unions)*** *Let $\{M_j\}_{j\in J}$ be a countable collection of smooth n-dimensional manifolds. Let $M = \bigsqcup_{j\in J} M_j$. For each $k \in \mathbb{Z}$, the inclusion maps $i_j : M_j \hookrightarrow M$ induce an isomorphism*

$$H^k_{dR}(M) \cong \prod_{j\in J} H^k_{dR}(M_j)$$

Proof. The pullback maps $i_j^* : H^k(M) \to H^k(M_j)$ induce an isomorphism from

$$i : H^k(M) \to \prod_{j\in J} H^k(M_j), \qquad i(\omega) \mapsto \big(i_j^*(\omega)\big)_{j\in J} = \big(\omega|_{M_j}\big)_{j\in J}$$

This map is injective because any smooth k-form whose restriction to each M_j is zero must itself be zero, and it is surjective because giving an arbitrary smooth k-form on each M_j defines one on M. □

We now discuss the homotopy invariance of de Rham cohomology, allowing us to show that de Rham cohomology is a topological invariant. If M and N are smooth manifolds, and $F, G : M \to N$ are smooth maps, we shall show homotopy invariance by constructing a co-chain homotopy between $F^\#$ and $G^\#$ which are given by linear maps

$$h_k : \Omega^k(N) \to \Omega^{k-1}(M)$$

for each $k \in \mathbb{Z}$ such that

$$F^\#(\omega) - G^\#(\omega) = d(h_k\omega) - h_{k+1}(d\omega)$$

for each $\omega \in \Omega^k(N)$ and $k \in \mathbb{Z}$.

$$\begin{tikzcd}
\cdots \arrow[r, "d"] & \Omega^{k-1}(N) \arrow[r, "d"] \arrow[d, "F^\# - G^\#"'] & \Omega^k(N) \arrow[r, "d"] \arrow[d, "F^\# - G^\#"'] \arrow[dl, "h_k"'] & \Omega^{k+1}(N) \arrow[r, "d"] \arrow[d, "F^\# - G^\#"'] \arrow[dl, "h_{k+1}"'] & \cdots \\
\cdots \arrow[r, "d"] & \Omega^{k-1}(M) \arrow[r, "d"] & \Omega^k(M) \arrow[r, "d"] & \Omega^{k+1}(M) \arrow[r, "d"] & \cdots
\end{tikzcd}$$

The key to our proof of homotopy invariance is to construct a homotopy operator first in the following special case. Let M be a smooth manifold, and for each $t \in I$, let

$$i_t : M \to M \times I$$

be the map $i_t(x) = (x, t)$. We first construct a co-chain homotopy between $i_0^\#$ and $i_1^\#$.

Lemma 8.1.7. *Let M be a smooth n-dimensional manifold. There exists a co-chain homotopy between the two maps $i_0^\#$ and $i_1^\#$.*

Proof. For each $k \in \mathbb{Z}$, we need to define a linear map

$$h_k : \Omega^k(M \times I) \to \Omega^{k-1}(M)$$

such that

$$h_{k+1}(d\omega) + d(h_k\omega) = i_1^\#(\omega) - i_0^\#(\omega) \tag{*}$$

for each $\omega \in \Omega^k(M \times I)$. Let S be the vector field on $M \times \mathbb{R}$ given by $S(p,s) = (0, \frac{\partial}{\partial s}\big|_s)$. Given $\omega \in \Omega^k(M \times I)$, define $h_k(\omega) \in \Omega^{k-1}(M)$ by

$$h_k(\omega) = \int_0^1 i_t^\#(S \lrcorner \omega)\, dt.$$

We shall verify the formula in $(*)$ in local coordinates. For $p \in M$, let $U = (x^1, \cdots, x^n)$ denote a co-ordinate chart containing then. Then $U \times \mathbb{R} = (x^1, \cdots, x^n, s)$ is a co-cordinate chart containing (p,s) for each $s \in \mathbb{R}$. In coordinates:

$$\omega = \sum_I \omega_I^1(x,s)\, ds \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k} + \sum_J \omega_I^2(x,s)\, dx^{j_1} \wedge \cdots \wedge dx^{j_k}$$

where I, J range over all increasing k-multi-indices over $\{1, \cdots, n\}$. We have,

$$S \lrcorner \omega = \sum_I \omega_I^1(x,s)\, dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

$$i_t^\#(S \lrcorner \omega) = i_t^\# \left(\sum_I \omega_I^1(x,s)\, dx^{i_1} \wedge \cdots \wedge dx^{i_k} \right) = \sum_I \omega_I^1(x,s)\, dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

We have,

$$\begin{aligned} d(h_k\omega) &= d\int_0^1 i_t^\#(S \lrcorner \omega)\, dt \\ &= d\int_0^1 \left(\sum_I \omega_I^1(x,t)\, dx^{i_1} \wedge \cdots \wedge dx^{i_k} \right) dt = \sum_I \int_0^1 \left(\frac{\partial \omega_I^1(x,t)}{\partial x^j}\, dx^j \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k} \right) dt. \end{aligned}$$

We now compute $h_{k+1}(d\omega)$. Here d is the exterior derivative on $M \times I$. First note that,

$$d\omega = \sum_I \frac{\partial \omega_I^1(x,s)}{\partial x^j}\, dx^j \wedge dt \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k} + \sum_J \frac{\partial \omega_I^2(x,s)}{\partial x^l}\, dx^l \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_k} + \sum_J \frac{\partial \omega_I^2(x,s)}{\partial s}\, ds \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_k}$$

We now find $h_{k+1}(d\omega)$, which is given by the expression:

$$h_{k+1}(d\omega) = \int_0^1 i_t^\#(S \lrcorner d\omega) dt$$

We have,

$$S \lrcorner d\omega = \sum_J \frac{\partial \omega_I^2(x,s)}{\partial s}\, ds \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_k} - \sum_I \frac{\partial \omega_I^1(x,s)}{\partial x^j}\, dt \wedge dx^j \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

Therefore, we have,

$$\begin{aligned} h_{k+1}(d\omega) &= \int_0^1 i_t^\#(S \lrcorner d\omega) dt \\ &= \int_0^1 \left(\sum_J \frac{\partial \omega_I^2(x,t)}{\partial s}\, dx^{j_1} \wedge \cdots \wedge dx^{j_k} - \sum_I \frac{\partial \omega_I^1(x,t)}{\partial x^j}\, dx^j \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k} \right) dt \end{aligned}$$

We have,

$$d(h_k\omega) + h_{k+1}(d\omega) = \int_0^1 \left(\sum_J \frac{\partial \omega_I^2(x,t)}{\partial s}\, dx^{j_1} \wedge \cdots \wedge dx^{j_k} \right) dt$$

Noting that,

$$i_t^\#(\omega) = \sum_J \omega_I^2(x,t)\, dx^{j_1} \wedge \cdots \wedge dx^{j_k},$$

we have,

$$\frac{di_t^\#(\omega)}{dt} = \sum_J \frac{\partial \omega_I^2(x,t)}{\partial t}\, dx^{j_1} \wedge \cdots \wedge dx^{j_k}$$

As a result, we have,

$$d(h_k\omega) + h_{k+1}(d\omega) = \int_0^1 \frac{di_t^\#(\omega)}{dt}\, dt = i_1^\#(\omega) - i_0^\#(\omega)$$

Hence, $(*)$ holds in every co-ordinate chart. This proves the claim. ☐

Proposition 8.1.8. *Let M and N be smooth manifolds. If $F, G : M \to N$ are smoothly homotopic smooth maps, then the induced cohomology maps $F^*, G^* : H^k_{dR}(N) \to H^k_{dR}(M)$ are equal for each $k \in \mathbb{Z}$.*

Proof. There exists a homotopy $H : M \times I \to N$ from F to G such that $F = H \circ i_0, G = H \circ i_1$ We have,

$$F^\# = (H \circ i_0)^\# = i_0^\# \circ H^*,$$
$$G^\# = (H \circ i_1)^\# = i_1^\# \circ H^*.$$

By Lemma 8.1.7, we know the maps $i_0^\#$ and $i_1^\#$ are equal from $H^k_{\mathrm{dR}}(M \times I)$ to $H^k_{\mathrm{dR}}(M)$ for each $k \in \mathbb{Z}$. Therefore,

$$F^\# = (H \circ i_0)^\# = i_0^\# \circ H^* = i_1^\# \circ H^* = G^\#$$

This proves the claim. ☐

Corollary 8.1.9. ***(Smooth Homotopy Invariance)*** *Let M and N be smoothly homotopy equivlaent smooth manifolds. Then*

$$H^k_{dR}(M) \cong H^k_{dR}(N)$$

for each $k \in \mathbb{Z}$.

Proof. Let $F : M \to N$ and $G : N \to M$ be smooth maps such that

$$G \circ F \simeq \mathrm{Id}_M$$
$$F \circ G \simeq \mathrm{Id}_N$$

We have,

$$(G \circ F)^\# = F^\# \circ G^\# = \mathrm{Id}_M^\# = \mathrm{Id}_{H^k(M)},$$
$$(F \circ G)^\# = G^\# \circ F^\# = \mathrm{Id}_N^\# = \mathrm{Id}_{H^k(N)}.$$

Since $\mathrm{Id}_M^\#$ is a surjective map, then $F^\#$ is surjective. Moreover, since $\mathrm{Id}_N^\#$ is an injective map, then $F^\#$ is an injective map. Hence, $F^\#$ is a linear map bijection, and hence an isomorphism. Hence, we have

$$H^k_{\mathrm{dR}}(M) \cong H^k_{\mathrm{dR}}(N)$$

for each $k \in \mathbb{Z}$. ☐

It is clear that if $M = \{*\}$, then $H^k_{\mathrm{dR}}(M) = 0$ for all $k > 0$. We will verify this explicitly later on. If M is a star-like manifold, then by smooth homotopy invariance, $H^k_{\mathrm{dR}}(M) = 0$ for all $k > 0$ since M is contractible. This is immediately implies that the famous Poincaré lemma which states that if U is an open star-shaped subset of $\mathbb{R}^n$, then every closed form on U is exact. A consequence of the Poincaré lemma is that every closed form on a smooth manifold, M, is locally exact. This suggests that the the obstruction of solving the equation

$$d\eta = \omega,$$

is connected to a global problem. This hints that the de Rham cohomology group is not affected by the differential structure that is of local nature. This is made precise below:

Corollary 8.1.10. ***(Topological Invariance of de Rham Cohomology)*** *If M and N are homotopy equivalent,*

$$H^k_{dR}(M) \cong H^k_{dR}(N)$$

for each $k \in \mathbb{Z}$.

Proof. By Whitney's approximation theorem, every topological homotopy equivalence can be approximate is homotopic to a smooth homotopy equivalence. The result then follows from Corollary 8.1.9. □

8.1.2. Mayer–Vietoris Sequence. Suppose M is a smooth manifold, and let U and V be open subsets of M such that $U \cup V = M$. The main goal of using the Mayer-Vietoris Sequence is to compute $H^k_{\mathrm{dR}}(M)$ in terms of $H^k_{\mathrm{dR}}(U)$, $H^k_{\mathrm{dR}}(V)$, and $H^k_{\mathrm{dR}}(U \cap V)$ where $\{U, V\}$ is an open cover of M. We have the following inclusions:

$$\begin{tikzcd}
 & U \arrow[dr, hook, "k"] & \\
U \cap V \arrow[ur, hook, "i"] \arrow[dr, hook, "j"'] & & M \\
 & V \arrow[ur, hook, "l"'] &
\end{tikzcd}$$

For each $k \in \mathbb{Z}$, these inclusion induce pullback maps on differential forms

$$\begin{tikzcd}
 & \Omega^k(U) \arrow[dl, "i^*"'] & \\
\Omega^k(U \cap V) & & \Omega^k(M) \arrow[ul, "k^*"'] \arrow[dl, "l^*"] \\
 & \Omega^k(V) \arrow[ul, "j^*"] &
\end{tikzcd}$$

Note that these pullbacks are in fact just restrictions. If we take some $\omega \in \Omega^k(M)$ and apply the map $k^* \oplus \ell^*$, we get

$$k^* \oplus \ell^*(\omega) : \Omega^k(M) \to \Omega^k(U) \oplus \Omega^k(V), \qquad k^* \oplus \ell^*(\omega) = (k^*\omega, \ell^*\omega) = (\omega|_U, \omega|_V)$$

Furthermore, if we take $(\omega, \eta) \in \Omega^p(U) \oplus \Omega^p(V)$ and apply the map $i^* - j^*$, we have

$$i^* - j^* : \Omega^k(U) \oplus \Omega^k(V) \to \Omega^k(U \cap V) \qquad (i^* - j^*)(\omega, \eta) = \omega\big|_{U \cap V} - \eta\big|_{U \cap V}$$

In other words, we have the following diagram

$$\begin{tikzcd}
 & \Omega^k(U) \arrow[dl, "i^*"'] & \\
\Omega^k(U\cap V) & \Omega^k(U)\oplus\Omega^k(V) \arrow[l, "i^*-j^*"'] & \Omega^k(M) \arrow[l, "k^*\oplus l^*"'] \arrow[ul, "k^*"'] \arrow[dl, "l^*"] \\
 & \Omega^k(V) \arrow[ul, "j^*"] &
\end{tikzcd}$$

Proposition 8.1.11. ***(Mayer–Vietoris Sequence)*** *Let M be a smooth manifold, and let U, V be open subsets of M such that $M = U \cup V$. For each $k \in \mathbb{Z}$, there is a linear map $\delta^k : H^k_{dR}(U\cap V) \to H^{k+1}_{dR}(M)$ such that the following sequence, called the Mayer-Vietoris sequence for the open cover $\{U, V\}$, is exact:*

$$\cdots \xrightarrow{\delta^{k-1}} H^k_{dR}(M) \xrightarrow{k^\# \oplus l^\#} H^k_{dR}(U) \oplus H^k_{dR}(V) \xrightarrow{i^\# - j^\#} H^k_{dR}(U\cap V) \xrightarrow{\delta^k} H^{k+1}_{dR}(M) \xrightarrow{k^\# \oplus l^\#} \cdots$$

Proof. Consider the following sequence:

$$0 \to \Omega^k(M) \xrightarrow{k^*\oplus l^*} \Omega^k(U)\oplus\Omega^k(V) \xrightarrow{i^*-j^*} \Omega^k(U\cap U) \to 0$$

We show that this sequence is a short exact sequence. We first show that $k^*\oplus l^*$ is injective. Suppose that $\sigma \in \Omega^p(M)$ satisfies

$$(k^*\oplus l^*)(\sigma) = (\sigma|_U,\, \sigma|_V) = (0,0)$$

This means that the restrictions of σ to U and V are both zero. Since $\{U, V\}$ is an open cover of M, this implies that σ is zero. We now show exactness at $\Omega^k(U)\oplus\Omega^k(V)$. Note that

$$(i^* - j^*)\circ(k^*\oplus l^*)(\sigma) = (i\circ j)(\sigma|_U, \sigma|_V) = \sigma|_{U\cap V} - \sigma|_{U\cap V} = 0,$$

This shows that $\mathrm{Im}(k^*\oplus l^*) \subseteq \ker(i^* - j^*)$. Conversely, suppose we are given $(\alpha,\alpha') \in \Omega^k(U)\oplus\Omega^k(V)$ such that $(i^*\circ j^*)(\alpha,\alpha') = 0$. This means that $\alpha|_{U\cap V} = \alpha'|_{U\cap V}$. So there is $\sigma \in \Omega^k(M)$ defined by

$$\sigma = \begin{cases} \alpha \text{ on } U, \\ \alpha' \text{ on } V. \end{cases}$$

Clearly, $(\alpha,\alpha') = (k\oplus l)(\sigma)$. So $\ker(i^* - j^*) \subseteq \mathrm{im}(k^*\oplus l^*)$. We now show that $i^* - j^*$ is surjective. Let $\omega \in \Omega^k(U\cap U)$. Let $\{\varphi,\psi\}$ be a smooth partition of unity subordinate to the open cover $\{U, V\}$ of M. Define $\alpha \in \Omega^k(U)$ and $\alpha' \in \Omega^k(V)$ by

$$\alpha = \begin{cases} \psi\omega \text{ on } U\cap V, \\ 0 \text{ on } U \setminus \operatorname{supp}\psi \end{cases} \qquad \alpha' = \begin{cases} -\varphi\omega \text{ on } U\cap V, \\ 0 \text{ on } U \setminus \operatorname{supp}\varphi \end{cases}$$

We have

$$(i^* - j^*)(\alpha,\alpha') = \alpha|_{U\cap V} - \alpha'|_{U\cap V} = \psi\omega - (-\varphi\omega) = (\psi - \varphi)\omega = \omega.$$

Hence, the sequence is indeed a short exact sequence. Because pullback maps commute with the exterior derivative, above short exact sequence induces the following this will show that we have the following short exact sequence:

$$0 \to H^k_{\mathrm{dR}}(M) \xrightarrow{k^\#\oplus l^\#} H^k_{\mathrm{dR}}(U)\oplus H^k_{\mathrm{dR}}(V) \xrightarrow{i^\# - j^\#} H^k_{\mathrm{dR}}(U\cap U) \to 0$$

Since this is true for each $k \in \mathbb{Z}$ we get a short exact sequence of co-chain complexes involing the de-Rham cohomology groups. The Mayer–Vietoris theorem then a formal consequence of the snake lemma. $\square$

Remark 8.1.12. *The snake lemma defines the connecting morphism*

$$H^k_{dR}(U \cap V) \xrightarrow{\delta^k} H^{k+1}_{dR}(M)$$

A characterization of the connecting homomorphism is given in the proof of the snake lemma (Proposition 4.6.3). Recalling it and adapting it to our case, we have that $\delta^k[\omega] = [\sigma]$, provided there exists $(\alpha, \alpha) \in \Omega^k(U) \oplus \Omega^k(V)$ such that

$$i^*\alpha - j^*\alpha' = \omega, \quad (k^*\sigma, l^*\sigma) = (d\alpha, d\alpha').$$

α, α' can be defined as in Proposition 8.1.11 to satisfy the first equation. Given such forms (α, α'), the fact that ω is closed implies that $d\alpha = d\alpha'$ on $U \cap V$. Thus, there is a smooth $(k+1)$-form σ on M that is equal to $d\alpha$ on U and $d\alpha'$ on V, and it satisfies the second equation.

It is quite easy to characterize the de Rham cohomology in degree zero and degree one. We first discuss the degree zero case.

Proposition 8.1.13. *Let M is a connected smooth manifold. Then $H^0_{dR}(M)$ is equal to the space of constant functions. Therefore,*

$$H^0_{dR}(M) \cong \mathbb{R}$$

Proof. Note that

$$H^0_{\text{dR}}(M) \cong \{\text{closed 0 forms on } M\} \cong \{f \in C^\infty(M) \mid df = 0\}$$

Since M is connected, $df = 0$ if and only if f is constant real-valued function. Therefore,

$$H^0_{\text{dR}}(M) \cong \mathbb{R}$$

This completes the proof. □

Corollary 8.1.14. *Let M be a smooth manifold. Then*

$$H^0_{dR}(M) \cong \mathbb{R}^{|J|},$$

where $|J|$ is the number of connected components of M.

Proof. We have,

$$M = \coprod_{j \in J} M_j,$$

where each M_j is a connected component of M and J is at most countably infinite. By Proposition 8.1.6 and Proposition 8.1.13, we have,

$$H^0_{\text{dR}}(M) \cong \prod_{j \in J} H^0_{\text{dR}}(M_j \cong) \prod_{j \in J} \mathbb{R} \cong \mathbb{R}^{|J|}$$

This completes the proof. □

Another case in which we can say quite a lot about de Rham cohomology is in degree one. Let $\text{Hom}(\pi_1(M, p), \mathbb{R})$ denote the set of group homomorphisms from $\pi_1(M, p)$ to the additive group $\mathbb{R}$. We define a linear map $\Phi \colon H^1_{\text{dR}}(M) \to \text{Hom}(\pi_1(M, p), \mathbb{R})$ as follows: given a cohomology class $[\omega] \in H^1_{\text{dR}}(M)$, define $\Phi([\omega]) \colon \pi_1(M, p) \to \mathbb{R}$ by

$$\Phi([\omega])([\gamma]) = \int_\gamma \omega,$$

where $[\gamma]$ is any path homotopy class in $\pi_1(M, p)$, and γ is any piecewise smooth curve representing the same path class.

Proposition 8.1.15. *Suppose M is a connected smooth manifold. For each $q \in M$, the linear map $\Phi \colon H^1_{\mathrm{dR}}(M) \to \mathrm{Hom}(\pi_1(M,p),\mathbb{R})$ is well defined and injective.*

Proof. (Sketch) Given $[\gamma] \in \pi_1(M,p)$, it follows from the Whitney approximation theorem that there is some smooth closed curve segment $\widetilde{\gamma}$ in the same path class as γ. We use without proof the fact that

$$\int_{\widetilde{\gamma}} \omega = \int_{\widetilde{\widetilde{\gamma}}} \omega$$

for every closed forms, ω and every other smooth closed curve $\widetilde{\widetilde{\gamma}}$ in the same path class as γ. If $\widetilde{\omega}$ is another smooth 1-form in the same cohomology class as ω, then $\widetilde{\omega} - \omega = df$ for some smooth function f, which implies

$$\int_{\widetilde{\gamma}} \widetilde{\omega} - \int_{\widetilde{\gamma}} \omega = \int_{\widetilde{\gamma}} df = f(q) - f(q) = 0.$$

Thus, Φ is well defined. It follows from properties of the line integral that $\Phi([\omega])$ is a group homomorphism from $\pi_1(M,p)$ to $\mathbb{R}$, and that Φ itself is a linear map. Suppose $\Phi([\omega])$ is the zero homomorphism. This means that $\int_{\widetilde{\gamma}} \omega = 0$ for every piecewise smooth closed curve $\widetilde{\gamma}$ with basepoint q. If γ is a piecewise smooth closed curve starting at some other point $q_0 \in M$, we can choose a piecewise smooth curve α from q to q_0, so that the path product $\alpha \cdot \gamma \cdot \overline{\alpha}$ is a closed curve based at q. It then follows that

$$0 = \int_{\alpha\cdot\gamma\cdot\overline{\alpha}} \omega = \int_{\alpha} \omega + \int_{\gamma} \omega + \int_{\overline{\alpha}} \omega = \int_{\alpha} \omega + \int_{\gamma} \omega - \int_{\alpha} \omega = \int_{\gamma} \omega.$$

Thus, ω is conservative and therefore exact. □

Corollary 8.1.16. *If M is a connected smooth manifold with finite fundamental group, then $H^1_{\mathrm{dR}}(M) = 0$.*

Proof. There are no nontrivial homomorphisms from a finite group to $\mathbb{R}$. The claim follows from Proposition 8.1.15. □

Remark 8.1.17. *If M is a connected smooth manifold whose fundamental group is a torsion group, then $H^1_{\mathrm{dR}}(M) = 0$. This is because $\mathbb{R}$ has no torsion elements. Hence, $\mathrm{Hom}(\pi_1(M,p),\mathbb{R}) = 0$ in this case.*

8.2. Examples & Applications

We discuss some example computations of de Rham cohomology.

Example 8.2.1. (**0-Dimensions**) Let M be a 0-dimensional smooth manifold. We have,

$$M \cong \coprod_{i\in I} \{*\}$$

where $|I|$ is the cardinality[1] of M. Then

$$H^k_{\mathrm{dR}}(M) = \begin{cases} \mathbb{R}^{|I|}, & \text{if } k = 0 \\ 0, & \text{otherwise} \end{cases}.$$

where $|I|$ is the cardinality of M. This follows at once from Proposition 8.1.6 and Proposition 8.1.13.

Example 8.2.2. (**Contractible Manifolds**) Let M be contractible manifold. Then,

$$H^k_{\mathrm{dR}}(M) = \begin{cases} \mathbb{R}^{|J|}, & \text{if } k = 0 \\ 0, & \text{otherwise} \end{cases}.$$

[1] $|I|$ is at most countably infinite.

where $|J|$ is the number of connected components of M. This follows immediately from Example 8.2.1 and Corollary 8.1.14.

Remark 8.2.3. *If M is a star-like manifold, then by homotopy invariance, $H^k_{\mathrm{dR}}(M) = 0$ for all $k > 0$ since M is contractible. This is immediately implies that the famous Poincaré lemma which states that if U is an open star-shaped subset of $\mathbb{R}^n$, then every closed form on U is exact. A consequencee of the Poincaré lemma is that every closed form on a smooth manifold, M, is locally exact.*

Example 8.2.4. **(Circle)** Let's compute the de-Rham cohomology of $\mathbb{S}^1$. Clearly, $H^0_{\mathrm{dR}}(\mathbb{S}^1) = \mathbb{R}$ since $\mathbb{S}^1$ is connected. Write $\mathbb{S}^1 = U \cup V$, where U, V represent the 'nothern hemisphere' and 'southern heisphere'. U, V are contractible and $U \cap V \cong \{\pm 1\}$. The Mayer-Vietoris theorem implies

$$0 \to \mathbb{R} \to \mathbb{R} \oplus \mathbb{R} \to \mathbb{R} \oplus \mathbb{R} \to H^1_{\mathrm{dR}}(\mathbb{S}^1) \to 0 \to 0 \to \cdots$$

This clearly implies that $H^k_{\mathrm{dR}}(\mathbb{S}^1) = 0$ for $k > 2$. We can immediately conclude via exactness that $H^1_{\mathrm{dR}}(\mathbb{S}^1) = \mathbb{R}$. Hence,

$$H^k_{\mathrm{dR}}(\mathbb{S}^1) = \begin{cases} \mathbb{R}, & \text{if } k = 0, 1 \\ 0, & \text{otherwise} \end{cases}.$$

We can compute the generator for $H^1_{\mathrm{dR}}(\mathbb{S}^1)$. The generator of is the angular 1-form $d\theta$. Notice that $d\theta$ is not globally defined on the circle since it is a multiple-valued function. Therefore, $d\theta$ is not zero in cohomology and generates $H^1_{\mathrm{dR}}(\mathbb{S}^1)$.

Example 8.2.5. **(Spheres)** Let's compute the de Rham cohomology of $\mathbb{S}^n$ for $n \geq 1$. We proceed by induction on k to show that

$$H^k_{\mathrm{dR}}(\mathbb{S}^n) = \begin{cases} \mathbb{R}, & \text{if } k = 0, n \\ 0, & \text{otherwise} \end{cases}.$$

We have verified the claim for $k = 1$ in Example 8.2.4. Now assume the claim is true for $n - 1$. Let $U = \mathbb{S}^n \setminus \{N\}$ and $V = \mathbb{S}^n \setminus \{S\}$. We have

$$U \cap V \simeq \mathbb{S}^{n-1} \quad U \simeq V \simeq \mathbb{R}^n$$

The Mayer-Vietoris sequence implies

$$\cdots \to 0 \to H^{k-1}_{\mathrm{dR}}(\mathbb{S}^{n-1}) \to H^k_{\mathrm{dR}}(\mathbb{S}^n) \to 0 \to \cdots$$

This implies that $H^{k-1}_{\mathrm{dR}}(\mathbb{S}^{n-1}) \cong H^k_{\mathrm{dR}}(\mathbb{S}^n)$. The claim now follows via induction and Example 8.2.4.

Example 8.2.6. **(Punctured Euclidean Space)** Let $p \in \mathbb{R}^n$ for $n \geq 2$. WLOG, we can assume that $p = 0$. We have

$$H^k_{\mathrm{dR}}(\mathbb{R}^n \setminus \{p\}) = \begin{cases} \mathbb{R}, & \text{if } k = 0, n-1 \\ 0, & \text{otherwise} \end{cases}.$$

Indeed, the inclusion $\mathbb{S}^{n-1} \to \mathbb{R}^n$ is a homotopy equivalence. The claim now follows from Example 8.2.5.

We can now discuss some elementary applications of de-Rham cohomology. We can now prove the topological invariance of the dimension of smooth manifolds.

Proposition 8.2.7. *If $m \neq n$, then $\mathbb{R}^n$ is not homeomorphic to $\mathbb{R}^m$. In particular, if M be a topological n-manifold then its dimension is uniquely determined.*

Proof. Assume that $\mathbb{R}^m \cong \mathbb{R}^n$. If $f : \mathbb{R}^n \to \mathbb{R}^m$ is a homeomorphism, then $f : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^m \setminus \{f(0)\}$ is a homeomorphism. So,

$$H^k_{\mathrm{dR}}(\mathbb{R}^n \setminus \{0\}) = H^k_{\mathrm{dR}}(\mathbb{R}^m \setminus \{f(0)\}),$$

for each $k \in \mathbb{Z}$. But $\mathbb{R}^n \setminus \{0\} \cong \mathbb{S}^{n-1}$ and $\mathbb{R}^m \setminus \{f(0)\} \cong \mathbb{S}^{m-1}$. So,

$$H^k_{\mathrm{dR}}(\mathbb{S}^{m-1}) = H^k_{\mathrm{dR}}(\mathbb{S}^{n-1})$$

for each $k \in \mathbb{Z}$. This is a contradiction by Example 8.2.5. The claim for a topological manifold follows by working in co-ordinate charts. $\square$

We can also show that the rank of the de-Rham cohomology groups is finite for *most manifolds*. We first need a definition:

Definition 8.2.8. Let M be a smooth n-manifold and $\{U_\alpha\}_{\alpha\in\Lambda}$ an open cover of M. We say $\{U_\alpha\}_{\alpha\in\Lambda}$ is a good cover if for any finite subset $I = \{\alpha_1, \dots, \alpha_k\} \subseteq \Lambda$ of indices, the intersection

$$U_I := U_{\alpha_1} \cap U_{\alpha_2} \cap \cdots \cap U_{\alpha_k}$$

is either empty or diffeomorphic to $\mathbb{R}^n$.

Remark 8.2.9. *By using the theory of geodesically convex neighborhoods in Riemannian geometry, one can show that any open cover of any smooth manifold M admits a refinement which is a good cover. In particular, if M is compact, then M admits a good cover which contains only finitely many open sets. See [BT13].*

Proposition 8.2.10. *Let M be a smooth n-manifold. If M admits a finite good cover,* $\dim H^k_{\mathrm{dR}}(M) < \infty$ *for each $k \in \mathbb{Z}$.*

Proof. We proceed by induction on the number of sets in a finite good cover of M. If M admits a good cover that contains only one open set, then that open set has to be M itself. In this case, M is diffeomorphic to $\mathbb{R}^n$, and the conclusion follows. Now suppose the theorem holds for any manifold that admits a good cover containing $k-1$ open sets. Let M be a manifold with a good cover $\{U_1, \dots, U_k\}$. We denote

$$U = U_1 \cup \cdots \cup U_{k-1} \quad \text{and} \quad V = U_k.$$

Then $U \cap V$ admits a finite good cover $\{U_1 \cap U_k, \dots, U_{k-1} \cap U_k\}$. By the induction hypothesis, all the de Rham cohomology groups of U, V, and $U \cap V$ are finite-dimensional. Now consider the Mayer-Vietoris sequence:

$$\cdots \to H^{k-1}_{\mathrm{dR}}(U \cap V) \xrightarrow{\delta^{k-1}} H^k_{\mathrm{dR}}(M) \xrightarrow{\alpha^k} H^k_{\mathrm{dR}}(U) \oplus H^k_{\mathrm{dR}}(V) \longrightarrow \cdots .$$

The conclusion follows since

$$\begin{aligned} \dim \mathrm{im}(\alpha_k) &\leq \dim H^k_{\mathrm{dR}}(U) \oplus H^k_{\mathrm{dR}}(V) < \infty, \\ \dim \ker(\alpha_k) = \dim \mathrm{im}(\delta_{k-1}) &\leq \dim H^{k-1}_{\mathrm{dR}}(U \cap V) < \infty. \end{aligned}$$

This completes the proof. $\square$

Corollary 8.2.11. *If M is a compact manifold (or M is homotopy equivalent to a compact manifold), then* $\dim H^k_{\mathrm{dR}}(M) < \infty$ *for all $k \in \mathbb{Z}$.*

Proof. This follows from Proposition 8.2.10. $\square$

8.3. Compactly Supported de-Rham Cohomology

Let M be an orientable smooth manifold. Integration is a pairing between compactly supported forms and oriented manifolds. This observation motivates that $H^n_{\text{dR}}(M)$ is important for studying orientations on M. Unfortunately, if M is non-compact, the integration of a n-form is not nicely defined unless the differential form is compactly supported. This observation motivates the study of de-Rham cohomology with compact support.

Definition 8.3.1. Let M be a smooth n-manifold and let $\omega \in \Omega^k(M)$. The **support** of ω is

$$\text{supp}(\omega) = \{p \in M \mid \omega_p \neq 0\}.$$

ω is compactly supported if $\text{supp}(\omega)$ is a compact set.

We set,

$$\Omega^k_c(M) = \{\omega \in \Omega^k(M) \mid \omega \text{ is compactly supported}\},$$

be the set of all compactly supported smooth k-forms. Clearly, the following facts are true:

(1) if ω_1, ω_2 are compactly supported k-forms, so is $c_1\omega_1 + c_2\omega_2$;
(2) if ω is compactly supported, then $d\omega$ is also compactly supported.

So $\Omega^k_c(M)$ are real vector spaces for each $k \in \mathbb{Z}$, and the exterior derivative makes these vector spaces a co-chain complex:

$$0 \to \Omega^0_c(M) \xrightarrow{d} \Omega^1_c(M) \xrightarrow{d} \Omega^2_c(M) \xrightarrow{d} \Omega^3_c(M) \xrightarrow{d} \cdots .$$

Definition 8.3.2. Let M be a smooth manifold. The quotient vector space

$$H^k_{\text{dR,c}}(M) = \frac{\ker(d : \Omega^k_c(M) \to \Omega^{k+1}_c(M))}{\text{im}(d : \Omega^{k-1}_c(M) \to \Omega^k_c(M))} = \frac{\{\omega \in \Omega^k_c(M) : d\omega = 0\}}{\{d\omega : \omega \in \Omega^{k-1}_c(M)\}}$$

is the k-th de Rham cohomology group with compact support of M.

Example 8.3.3. Let M be a smooth manifold. For $k = 0$, by definition

$$H^k_{\text{dR,c}}(M) = \{f \in C^\infty(M) \mid df = 0 \text{ and } \text{supp}(f) \text{ is compact}\}.$$

But $df = 0$ if and only if f is locally constant, i.e., f is constant on each connected component. Moreover, a locally constant compactly supported function has to be zero on any non-compact connected component. So we conclude

$$H^0_{\text{dR,c}}(M) \cong \mathbb{R}^{m_c},$$

where m_c is the number of *compact* connected components of M. In particular,

$$H^0_{\text{dR,c}}(\text{pt}) = \mathbb{R}, \quad \text{and} \quad H^0_{\text{dR,c}}(\mathbb{R}^n) = 0$$

for all $n \geq 1$.

Remark 8.3.4. *Since $\mathbb{R}^n$ is homotopy equivalent to $\{pt\}$, we conclude that $H^0_{\text{dR,c}}(M)$ is* no longer *a homotopy invariant.*

We now discuss the analog of the Mayer-Vietoris sequence for the compactly supported case. If $F : M \to N$ is a smooth map between smooth manifolds, note that by definition,

$$\text{supp}(F^*\omega) \subseteq F^{-1}(\text{supp}(\omega)).$$

So if $\omega \in \Omega^k_c(N)$, in general we may have $F^*\omega \notin \Omega^k_c(M)$. Hence, we cannot expect to pull back compactly-supported cohomology classes on N to compactly-supported cohomology classes on M!

Remark 8.3.5. *If $F : M \to N$ is proper map, then the pull-back $F^*\omega$ of a compactly supported differential form $\omega \in \Omega_c^k(N)$ is still compactly supported. In this case, we have an induced map:*

$$F^* : H^k_{\mathrm{dR,c}}(N) \to H^k_{\mathrm{dR,c}}(M)$$

In this case, one can prove that if $F_0, F_1 : M \to N$ are proper smooth maps that are properly homotopic, then the induced maps are equal:

$$F_1^* = F_2^* : H^k_{\mathrm{dR,c}}(N) \to H^k_{\mathrm{dR,c}}(M).$$

Note that any homeomorphism is proper. So, in particular, the compactly supported de Rham cohomology groups are still topological invariants up to homeomorphisms. That is, if $M \cong N$ as smooth manifolds, then

$$H_c^k(M) = H_c^k(N)$$

for each $k \in \mathbb{Z}$. We have aready seen that compactly supported de Rham cohomology groups is not a topological invariant up to homotopy equivalence.

So how do we prove an analog of the Mayer-Vietoris sequence for the compactly supported case. Note that we can now instead pushforward compactly supported differential forms and hence cohomology classes. If $U \subseteq M$ is an open set, the inclusion $i : U \hookrightarrow M$ induces a map

$$i_* : \Omega_c^n(U) \to \Omega_c^n(M)$$

that sends a compactly supported differential form on U to the same differential form extended by zero outside of U.

Lemma 8.3.6. *For each $k \in \mathbb{Z}$, the map i_* commutes with the exterior derivative.*

Proof. For $\omega \in \Omega_c^n(U)$, we have $d\omega \in \Omega_c^{n+1}(U)$. Thus, applying $(i_* \circ d)$ to ω results in $d\omega$ extended by zero outside of U. If we first apply i_*, we obtain

$$i_*(\omega) = \begin{cases} 0, & \text{on } M \setminus U, \\ \omega, & \text{on } U. \end{cases}$$

Taking the exterior derivative, we get

$$d(i_*\omega) = \begin{cases} 0, & \text{on } M \setminus U, \\ d\omega, & \text{on } U. \end{cases}$$

Thus, i_* commutes with d. That is, $i_* \circ d = d \circ i_*$. □

Lemma 8.3.6 allows us to establish the following version of the Mayer-Vietoris sequence for the compactly supported case.

Proposition 8.3.7. *Let M be a smooth n-manifold and let $U, V \subseteq M$ be open sets such that $U \cup V = M$. Then there exists linear maps $\delta_k^c : H_c^k(M) \to H_c^{k+1}(U \cap V)$ so that the following sequence is exact:*

$$\cdots \to H_c^k(U \cap V) \to H_c^k(U) \oplus H_c^k(V) \to H_c^k(M) \xrightarrow{\delta_k^c} H_c^{k+1}(U \cap V) \to \cdots$$

Proof. The proof is so much like the original Mayer-Vietoris proof, and it involves a diagram chase. We omit details. □

Example 8.3.8. We compute $H^k_{\mathrm{dR,c}}(\mathbb{R}^n)$ for $k < n$. We have seen $H^0_{\mathrm{dR,c}}(\mathbb{R}^n) = 0$. Now we show that

$$H^k_{\mathrm{dR,c}}(\mathbb{R}^n) = 0$$

for $1 \leq k < n$. We identify $\mathbb{R}^n$ with then open set $\mathbb{S}^n - \{N\}$. Then we get an inclusion map

$$\iota : \mathbb{R}^n \to \mathbb{S}^n,$$

(1) Let $k=1$. Let $\omega \in \Omega^1_c(\mathbb{R}^n)$ such that $d\omega = 0$. Since d commutes with i as seen above, we have that $\iota_*\omega \in \Omega^1_c(\mathbb{S}^n)$ such that $d(\iota_*\omega)=0$. Since[2]

$$H^1_{\text{dR,c}}(\mathbb{S}^n) = H^1_{\text{dR}}(\mathbb{S}^n) = 0$$

there exists $\eta \in \Omega^0(\mathbb{S}^n) = C^\infty_c(\mathbb{S}^n)$ such that $\iota_*\omega = d\eta$. Noting that $\iota_*\omega$ is supported in $\mathbb{S}^n - U$ for open set U containing N, we have $d\eta = \iota^*\omega = 0$ on U. This implies that $\eta|_U \equiv c$ for some constant $c \in \mathbb{R}$. It follows that if we take $\tilde{\eta} = \eta - c$, then $\tilde{\eta} \in \Omega^0_c(\mathbb{S}^n - \{N\}) = \Omega^0_c(\mathbb{R}^n)$ and $d\tilde{\eta} = \omega$.

(2) Let $k>1$. Let $\omega \in \Omega^c_k(\mathbb{R}^n)$ such that $d\omega = 0$. As above, $\iota_*\omega \in \Omega^c_k(\mathbb{R}^n)$ such that $d(\iota_*\omega)=0$, and $\iota_*\omega$ is supported in $\mathbb{S}^n - U$ for open set U containing N. Since[3]

$$H^k_{\text{dR,c}}(\mathbb{S}^n) = H^k_{\text{dR}}(\mathbb{S}^n) = 0$$

there exists $\eta \in \Omega^{k-1}(\mathbb{S}^n)$ such that $\iota_*\omega = d\eta$. By shrinking the neighborhood U of N, we can assume that U is contractible. Then the fact that $d\eta = \iota_*\omega = 0$ in U implies that there exists a $\mu \in \Omega^{k-2}_c(U)$ such that $\eta|_U = d\mu$. Now pick a bump function ψ on $\mathbb{S}^n$ which compactly supported in U that equals 1 on N. Then

$$\tilde{\eta} = \eta - d(\psi\mu) \in \Omega^{k-1}_c(\mathbb{S}^n)$$

and $\tilde{\eta} = 0$ near N. By construction, $d\tilde{\eta} = d\eta = \omega$.

8.3.1. Top Degree Cohomology. We now set up the machinery to argue that the degree k de Rham cohomology with compact support is related to the orientation of smooth manifolds. First, an example:

Example 8.3.9. Let's compute $H^1_{\text{dR,c}}(\mathbb{R})$. Consider the integration map

$$\int_{\mathbb{R}} : \Omega^1_c(\mathbb{R}) \to \mathbb{R}, \quad \omega \mapsto \int_{\mathbb{R}} \omega.$$

This map is clearly linear and surjective. Moreover, if $\omega = df$ is a compactly supported exact form, then

$$\int_{-\infty}^{\infty} df\, dx = \int_{-R}^{R} \frac{df}{dx}\, dx = f(R) - f(-R),$$

for each $R>0$. Since $f \in C^\infty_c(\mathbb{R})$, $f(R) = f(-R) = 0$ for R large enough. So it induces a surjective linear map

$$\int_{\mathbb{R}} : H^1_{\text{dR,c}}(\mathbb{R}) \to \mathbb{R}.$$

Moreover, if $\int_{\mathbb{R}} f(t)\, dt = 0$, where $f \in C^\infty_c(\mathbb{R})$, then consider the function

$$g(t) = \int_{-\infty}^{t} f(\tau)\, d\tau$$

Clearly, g is smooth. If we choose $T>0$ and $R<0$ large enough, we get

$$F(T) = \int_{-\infty}^{T} f(t)\, dt = \int_{-\infty}^{\infty} f(t)\, dt = 0.$$
$$F(R) = \int_{-\infty}^{R} f(t)\, dt = \int_{-\infty}^{R} 0\, dt = 0.$$

[2]Note that $k=1<n$

[3]Once again, note that $k<1<n$

Hence, $g \in C_c^\infty(\mathbb{R})$. Since $dg = f$, we have $[f(t)dt]$ in $H^1_{\text{dR,c}}(\mathbb{R})$. Thus, $\int_{\mathbb{R}}$ is an isomorphism between $H_c^1(\mathbb{R})$ and $\mathbb{R}$, i.e.,

$$H_c^1(\mathbb{R}) \cong \mathbb{R}.$$

The same method as in Example 8.3.9 works generally. Let M be a connected, oriented n-manifold, and let $\omega \in \Omega_c^n(M)$ be a compactly supported n-form. Then ω is closed, and we have defined the integral $\int_M \omega$. So we get a map

$$\int_M : \Omega_c^n(M) \to \mathbb{R}, \quad \omega \mapsto \int_M \omega.$$

Suppose $\omega = d\eta$ for some $\eta \in \Omega_c^{n-1}(M)$. We can take a compact set $K \subseteq M$ such that $\operatorname{supp}(\eta) \subseteq K$. By Stokes' theorem,

$$\int_M \omega = \int_M d\eta = \int_K d\eta = \int_{\partial K} \eta = 0.$$

Thus, $\int_M$ induces a linear map

$$\int_M : H^n_{\text{dR,c}}(M) \to \mathbb{R}, \quad [\omega] \mapsto \int_M \omega$$

Proposition 8.3.10. *Let M be an oriented smooth n-manifold. Then the map $\int_M : H^n_{\text{dR,c}}(M) \to \mathbb{R}$ is surjective.*

Proof. Fix a n-form (a volume form) ω on M. For any $c \in \mathbb{R}$, one can find a smooth function f that is compactly supported in a coordinate chart U, such that $\int_U f\omega = c$. □

We can prove the following corollary based on Proposition 8.3.10:

Corollary 8.3.11. *The following statements are true:*

(1) *If $\omega \in \Omega^n(\mathbb{S}^n)$ and $\int_{\mathbb{S}^n} \omega = 0$, then ω is exact.*

(2) *We have*

$$H^k_{\text{dR,c}}(\mathbb{R}^n) = \begin{cases} \mathbb{R}, & k = n, \\ 0, & k \neq n. \end{cases}$$

(3) *Let M be a smootn n-manifold. if M admits a finite good cover, then* $\dim H^k_{\text{dR,c}}(M) < \infty$ *for all $k \in \mathbb{Z}$.*

Proof. The proof is given below:

(1) Note that

$$H^n_{\text{dR}}(\mathbb{S}^n) = H^n_{\text{dR,c}}(\mathbb{S}^n) \cong \mathbb{R}$$

Hence, the map in Proposition 8.3.10 is in fact a linear isomorphism. In other words, if $\int_{\mathbb{S}^n} \omega = 0$, then $[\omega] = 0$, i.e., ω is exact.

(2) Example 8.3.9 proves the case $n = 1$ and Example 8.3.8 takes care of the case $1 \leq k < n$ for $n \geq 2$. We discuss the case $k = n \geq 2$. It suffices to show that the surjective linear map

$$\int_{\mathbb{R}^n} : H^n_{\text{dR,c}}(\mathbb{R}^n) \to \mathbb{R}, \quad [\omega] \mapsto \int_{\mathbb{R}^n} \omega$$

is in fact an isomorphism. We show that the map is injective. Assume that $\int_{\mathbb{R}^n} \omega = 0$ for some $\omega \in \Omega_c^n(\mathbb{R}^n)$. Automatically, we have $d\omega = 0$. As before, consider the inclusion map $\iota : \mathbb{R}^n \to \mathbb{S}^n$. Then $\iota_*\omega \in \Omega^n(S^n)$. Since

$$\int_{\mathbb{S}^n} \iota_*\omega = \int_{\mathbb{R}^n} \omega = 0,$$

by (1), we see $\iota_*\omega = d\eta$ for some $\eta \in \Omega^{n-1}(S^n)$. The rest of the proof is similar to that of Example 8.3.8(2).

(3) We can use Mayer-Vertoris sequence compactly supported de Rham cohomology and induction and the number of open sets in a good cover. The same as the proof for the ordinary de Rham cohomology.

This completes the proof. □

We now reach the punchline for this section. We argue that Proposition 8.3.10 is, in fact, a linear isomorphism if the underlying smooth manifold is connected and orientable.

Proposition 8.3.12. *Let M be a smooth connected orientable n-manifold. The map in Proposition 8.3.10 is an isomorphism. In particular,*

$$H^n_{\text{dR,c}}(M) \cong \mathbb{R}$$

PROOF. In Proposition 8.3.10, we have already checked that the map is a surjective linear isomorphism. We check that it is injective. Let $\omega \in \Omega^c_c(M)$ such that $\int_M \omega = 0$. Since M is connected and $\operatorname{supp}(\omega)$ is compact, we can take a connected compact set $\operatorname{supp}(\omega) \subseteq K_\omega$. If we can cover K_ω by a good cover which contains only one chart, then Corollary 8.3.11(2) implies that $\omega = d\mu$ for some $\mu \in \Omega^{n-1}_c(M)$. We can now proceed by induction. Suppose the claim is true if K_ω can be covered by $k-1$ 'good charts,' and suppose $\omega \in \Omega^n_c(M)$ satisfies the property that K_ω admits a good cover $\{U_1, \ldots, U_k\}$. There exists one U_i, say U_k for simplicity, such that both $U = U_1 \cup \cdots \cup U_{k-1}$ and $V = U_k$ are connected[4]. Pick a partition of unity $\{\rho_U, \rho_V\}$ of $U \cup V$ subordinate to the cover $\{U, V\}$, and let $\omega|_U = \rho_U\omega, \omega|_V = \rho_V\omega$. Since K_ω is connected, $U \cap V \neq \emptyset$. We pick an n-form ω_0 compactly supported in $U \cap V$ so that

$$\int_M \omega_0 = \int_M \omega|_U.$$

Then $\omega|_U - \omega_0$ is compactly supported in U, which is connected and admits a good cover of $k-1$ good charts, and

$$\int_M (\omega|_U - \omega_0) = 0.$$

So by the induction hypothesis,

$$\omega_U - \omega_0 = d\eta|_U$$

for some $\eta_U \in \Omega^{n-1}_c(M)$. Similarly,

$$\int_M (\omega|_V + \omega_0) = -\int_M \omega|_U + \int_M \omega_0 = 0$$

implies

$$\omega_V + \omega_0 = d\eta|_V$$

for some $\eta|_V \in \Omega^{n-1}_c(M)$. It follows that

$$\omega = \omega_U + \omega_V = d(\eta_U + \eta_V),$$

where $\eta|_U + \eta|_V \in \Omega^{n-1}_c(M)$. This completes the proof. □

8.4. de-Rham's Theorem

[4] This needs proof.

CHAPTER 9

Products & Duality

Cohomology acquires a rich algebraic structure through the cup product, which endows cohomology groups with a graded ring structure fundamental to distinguishing and classifying topological spaces. The interplay between homology and cohomology is further illuminated by Poincaré duality, a powerful theorem revealing a deep symmetry on orientable smooth manifolds that connects geometric properties with algebraic invariants. These concepts form the backbone of much of modern algebraic topology, providing essential tools for understanding manifold structures, intersection theory, and the algebraic encoding of geometric information. References include [Hat02; Lee10; May99].

9.1. Cup Product

Let's revert back to singular cohomology. However, we will keep on referring to de Rham cohomology for some down-to-earth motivation. We have worked with coefficients G, where G is some abelian group. Cohomology groups with G-coefficients can be 'summed up' to yield a direct sum decomposition:

$$H^*(X;G) = \bigoplus_{n\geq 0} H^n(X;G)$$

We now show that if we take $G = R$ to be a commutative ring R, then the singular cohomology with coefficients in R also forms a ring under the cup product operation. This suggests that cohomology is a stronger topological invariant than homology. First, let's define the algebraic object over which we define the ring structure.

Definition 9.1.1. Let X be a topological space, and let R be a commutative ring. The total cohomology of X with coefficients in R is given by

$$H^\bullet(X;R) := \bigoplus_{n\geq 0} H^n(X;R).$$

Our aim is to make $H^\bullet(X;R)$ into a graded ring when R is a commutative ring. We shall do this by first making

$$C^\bullet(X;R) := \bigoplus_{n\geq 0} C^n(X;R)$$

into a graded ring, and then showing that the ring structure descends to cohomology. This will be done by introducing a cup product structure on $C^\bullet(X;R)$.

Example 9.1.2. We first discuss the special case of de Rham cohomology. The advantage here is that we can directly work at the de Rham cohomology groups. Let M be a smooth manifold, and let $\omega \in \Omega^k(M)$, $\eta \in \Omega^l(M)$ be closed forms. If $[\omega] = [\omega']$ and $[\eta] = [\eta']$, we have

$$\omega = \omega' + d\alpha, \qquad \eta = \eta' + d\beta$$

for $\alpha \in \Omega^{k-1}(M)$ and $\beta \in \Omega^{l-1}(M)$. Note that we have

$$\begin{aligned}\omega \wedge \eta &= (\omega' + d\alpha) \wedge (\eta' + d\beta) \\ &= \omega' \wedge \eta' + \omega' \wedge d\beta + d\alpha \wedge \eta' + d(\alpha \wedge \beta) \\ &= \omega' \wedge \eta' - d\beta \wedge \omega' + d\alpha \wedge \eta' + d(\alpha \wedge \beta) \\ &= \omega' \wedge \eta' - d(\beta \wedge \omega') + d(\alpha \wedge \eta') + d(\alpha \wedge \beta) \\ &= \omega' \wedge \eta' - d(\beta \wedge \omega' + \alpha \wedge \eta' + \alpha \wedge \beta).\end{aligned}$$

Hence, $[\omega \wedge \eta] = [\omega' \wedge \eta']$. This shows that the wedge product

$$\wedge : \Omega^k(M) \times \Omega^l(M) \to \Omega^{k+l}(M)$$

descends to a well-defined bilinear map

$$\begin{aligned}\smile\colon\ H^k_{\text{dR}}(M) \times H^l_{\text{dR}}(M) &\to H^{k+l}_{\text{dR}}(M), \\ [\omega] \smile [\eta] &\mapsto [\omega \wedge \eta].\end{aligned}$$

This is called the cup product in de-Rham cohomology.

Let's now move back to the singular cohomology case and define the cup product. We first define it at the level of $C^\bullet(X; R)$.

Definition 9.1.3. Let $\phi \in C^k(X; R)$ and $\psi \in C^l(X; R)$. The cup product $\phi \smile \psi \in C^{k+l}(X; R)$ is defined by:

$$(\phi \smile \psi)(\sigma : \Delta^{k+l} \to X) = \phi(\sigma|_{[v_0,\ldots,v_k]}) \cdot \psi(\sigma|_{[v_k,\ldots,v_{k+l}]})$$

where '$\cdot$' denotes the multiplication in the ring R.

Remark 9.1.4. *Technically, these restricted maps in Definition 9.1.3 have the wrong domains; they aren't the standard k, l-simplices. But we just pre-compose with the 'obvious' maps from the standard simplices. We shall not do this below.*

The cup product extends by linearity to define a function $C^k(X; R) \times C^l(X; R) \to C^{k+l}(X; R)$ by

$$\left(\sum_i \phi_i\right) \smile \left(\sum_j \psi_j\right) := \sum_{i,j} \phi_i \smile \psi_j.$$

Let us first check this gives us a ring structure.

Lemma 9.1.5. *Let X be a topological space and let R be a commutative ring. Then $C^\bullet(X; R)$ is a graded ring under the cup product. If R has an identity then $C^\bullet(X; R)$ also has an identity.*

Proof. Suppose $\phi \in C^k(X; R)$ and $\psi, \gamma \in C^l(X; R)$. We claim that $\phi \smile (\psi + \gamma) = \phi \smile \psi + \phi \smile \gamma$. For this, take $\sigma : \Delta^{k+l} \to X$. Then

$$\begin{aligned}(\phi \smile (\psi + \gamma))(\sigma) &= \phi(\sigma_{[v_0,\ldots,v_k]}) \cdot (\psi + \gamma)(\sigma_{[v_k,\ldots,v_{k+l}]}) \\ &= \phi(\sigma_{[v_0,\ldots,v_k]}) \cdot \psi(\sigma_{[v_k,\ldots,v_{k+l}]}) + \phi(\sigma_{[v_0,\ldots,v_k]})) \cdot \gamma(\sigma_{[v_k,\ldots,v_{k+l}]}) \\ &= \phi \smile \psi(\sigma_{[v_k,\ldots,v_{k+l}]}) + \phi \smile \gamma(\sigma_{[v_k,\ldots,v_{k+l}]}).\end{aligned}$$

A similar computation shows that $(\phi + \psi) \smile \gamma = \phi \smile \gamma + \psi \smile \gamma$. Associativity follows by a similar computation. Let 1_R denote the identity in R. Define a cochain $\nu \in C^0(X; R)$ by $\nu(x) = 1_R \quad \forall x \in X$ and extend by linearity. It is clear that

$$\nu \smile \phi = \phi = \phi \smile \nu$$

for any $\phi \in C^n(X;R)$ and any $n \geq 0$. Thus, $C^\bullet(X;R)$ is indeed a graded ring. $\square$

Unfortunately, the ring structure on $C^\bullet(X;R)$ is not very useful, as it is too "large" and almost impossible to compute. However, as we will now see, the total cohomology $H^\bullet(X;R)$ also inherits a ring structure, and this structure is much nicer. We need the following result:

Lemma 9.1.6. *Let $\phi \in C^k(X;R)$ and $\psi \in C^l(X;R)$*

$$\delta^{k+l}(\phi \smile \psi) = \delta^k\phi \smile \psi + (-1)^k \phi \smile \delta^l \psi$$

Proof. For $\sigma : \Delta^{k+l+1} \to X$, we have

$$(\delta^k\phi \smile \psi)(\sigma) = \sum_{i=0}^{k} (-1)^i \phi(\sigma_{[v_0,\ldots,\widehat{v_i},\ldots,v_{k+1}]}) \cdot \psi(\sigma_{[v_{k+1},\ldots,v_{k+l+1}]}),$$

$$(-1)^k(\phi \smile \delta\psi)(\sigma) = \sum_{i=k}^{k+l+1} (-1)^i \phi(\sigma_{[v_0,\ldots,v_k]}) \cdot \psi(\sigma_{[v_k,\ldots,\widehat{v_i},\ldots,v_{k+l+1}]}).$$

When we add these two expressions, the last term of the first sum cancels with the first term of the second sum, and the remaining terms are exactly $\delta^{k+l}(\phi \smile \psi)(\sigma) = (\phi \smile \psi)(\partial_{k+l+1}\sigma)$ since

$$\partial_{k+l+1}\sigma = \sum_{i=0}^{k+l+1} (-1)^i \sigma_{[v_0,\ldots,\widehat{v_i},\ldots,v_{k+l+1}]}.$$

This completes the proof. $\square$

Corollary 9.1.7. *The following statements are true:*

(1) *If $\phi \in C^k(X;R)$ and $\psi \in C^l(X;R)$ are cocycles, then $\delta^{k+l}(\phi \smile \psi) = 0$.*
(2) *If $\phi \in C^k(X;R)$ and $\psi \in C^l(X;R)$ are such that one of ϕ or ψ is a cocycle and the other a coboundary, then $\phi \smile \psi$ is a coboundary.*

Proof. The proof is given below:

(1) Since $\delta^k\phi = 0$ and $\delta^l\psi = 0$, we have that that
$$\delta^{k+l}(\phi \smile \psi) = \delta^k\phi \smile \psi + (-1)^k \phi \smile \delta^l\psi = 0$$
(2) Say $\delta^k\phi = 0$ and $\psi = \delta^{l-1}\eta$. Then
$$\delta^{k+l-1}(\phi \smile \eta) = (-1)^k \phi \smile \delta^{l-1}\eta = (-1)^k \phi \smile \psi$$
The other case is similar.

This completes the proof $\square$

It follows that we get an induced cup product on cohomology:

$$\begin{aligned} \smile : H^k(X;R) \times H^l(X;R) &\to H^{k+l}(X;R) \\ [\phi] \times [\psi] &\mapsto [\phi \smile \psi] \end{aligned}$$

Well-definedness follows from Corollary 9.1.7. Indeed, if $[\phi] = [\phi']$ and $[\psi] = [\psi']$, then

$$\phi = \phi' + \alpha, \quad \psi = \psi' + \beta$$

where α, β are co-chains. We have

$$\begin{aligned} \phi \smile \psi &= (\phi' + \alpha) \smile (\psi' + \beta) \\ &= \phi' \smile \psi' + (\phi' \smile \beta + \alpha \smile \psi' + \alpha \smile \beta) \end{aligned}$$

Corollary 9.1.7 implies that the term in paranthesis is a coboundary. Hence,

$$[\phi \smile \psi] = [\phi' \smile \psi']$$

The operation is distributive and associative since it is so on the co-chain level. If R has an identity element, then there is an identity element for the cup product, namely the class $[1] \in H^0(X;R)$ defined by the 0-cocycle taking the value 1_R on each singular 0-simplex. Considering the cup product as an operation on the direct sum of all cohomology groups, we get a (graded) ring structure on $H^\bullet(X;R)$.

Definition 9.1.8. Let X be a topological space and let R be a commutative ring. The cohomology ring of X is the graded ring

$$H^\bullet(X;R) := \left(\bigoplus_{n\geq 0} H^n(X;R), \smile \right)$$

with respect to the cup product operation. If R has an identity, then so does $H^\bullet(X;R)$.

Remark 9.1.9. *We can also define the relative cup product. The cup product on cochains*

$$C^k(X;R) \times C^l(X;R) \to C^{k+l}(X;R)$$

restricts to cup products

$$\begin{aligned} C^k(X,A;R) \times C^l(X;R) &\to C^{k+l}(X,A;R), \\ C^k(X,A;R) \times C^l(X,A;R) &\to C^{k+l}(X,A;R), \\ C^k(X;R) \times C^l(X,A;R) &\to C^{k+l}(X,A;R). \end{aligned}$$

since $C^i(X,A;R)$ can be regarded as the set of cochains vanishing on chains in A, and if φ or ψ vanishes on chains in A, then so does $\varphi \smile \psi$. So there exist relative cup products:

$$\begin{aligned} H^k(X,A;R) \times H^l(X;R) &\to H^{k+l}(X,A;R), \\ H^k(X,A;R) \times H^l(X,A;R) &\to H^{k+l}(X,A;R), \\ H^k(X;R) \times H^l(X,A;R) &\to H^{k+l}(X,A;R). \end{aligned}$$

In particular, if A is a point, we get a cup product on the reduced cohomology $\widetilde{H}^*(X;R)$. More generally, we can define

$$H^k(X,A;R) \times H^\ell(X,B;R) \to H^{k+\ell}(X,A\cup B;R)$$

when A and B are open subsets of X or sub-complexes of the CW complex X.

Normally, no one computes cohomology rings using the definition of the cup product, as this can be quite tedious for the most part. However, we compute a couple of basic examples:

Example 9.1.10. (Spheres) Let $X = \mathbb{S}^n$ for $n \geq 1$ and $R = \mathbb{Z}$. We have

$$H^k(\mathbb{S}^n;\mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & \text{if } k = 0, n \\ 0, & \text{otherwise} \end{cases}.$$

The generating element in $H^0(\mathbb{S}^n;\mathbb{Z})$ is the identity element. We label the generators of $H^0(\mathbb{S}^n;\mathbb{Z}))$ and $H^n(\mathbb{S}^n;\mathbb{Z})$ as 1 and x respectively. We have the following relations

$$1 \smile 1 = 1, \quad 1 \smile x = x, \quad x \smile 1 = x, \quad x \smile x = 0$$

The last relation is true since $H^{2n}(\mathbb{S}^n;\mathbb{Z}) = 0$. Hence, we have

$$H^*(\mathbb{S}^n;\mathbb{Z}) \cong \frac{\mathbb{Z}[x]}{\langle x^2 \rangle} = \mathbb{Z}[x]/(x^2) \cong \Lambda_{\mathbb{Z}}[x]$$

Here $\Lambda_{\mathbb{Z}}[x]$ is the exterior algebra on two generator over $\mathbb{Z}$.

Remark 9.1.11. *We can define a cup product for simplicial cohomology by the same formula as for singular cohomology. It can be checked that the isomorphism between simplicial and singular cohomology respects cup products. Hence, we can compute cup products using simplicial cohomology.*

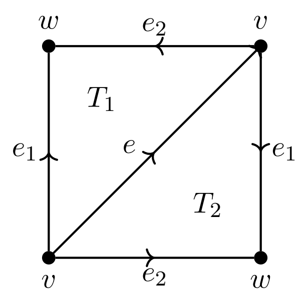
Example 9.1.12. **(Real Projective Plane)** Let $X = \mathbb{RP}^2$ and $R = \mathbb{Z}_2$. We have

$$H^k(\mathbb{RP}^2;\mathbb{Z}_2) \cong \begin{cases} \mathbb{Z}_2 & k = 0,1,2, \\ 0 & \text{otherwise.} \end{cases}$$

Let α be the generator of $H^1(\mathbb{RP}^2;\mathbb{Z}_2)$. Consider

$$\alpha^2 := \alpha \smile \alpha \in H^2(\mathbb{RP}^2;\mathbb{Z}_2).$$

We claim that $\alpha^2 \neq 0$, so α^2 is in fact the generator of $H^2(\mathbb{RP}^2;\mathbb{Z}_2) \cong \mathbb{Z}_2$. Consider the cell structure on $\mathbb{RP}^2$ shown in the figure below. The 2-cell T_1 is attached by the word $e_1e_2^{-1}e^{-1}$, and the 2-cell T_2 is attached by the word $e_2e_1^{-1}e^{-1}$.



Since α is a generator of $H^1(\mathbb{RP}^2;\mathbb{Z}/2\mathbb{Z}) \cong \mathrm{Hom}_{\mathbb{Z}}(H_1(\mathbb{RP}^2;\mathbb{Z}),\mathbb{Z}/2\mathbb{Z})$, it is represented by a cocycle

$$\varphi : C_1(\mathbb{RP}^2) \to \mathbb{Z}/2\mathbb{Z}$$

with $\varphi(e) = 1$, where e represents the generator of $H_1(\mathbb{RP}^2;\mathbb{Z}) \cong \mathbb{Z}_2$. The co-cycle condition for φ translates into the identities:

$$0 = (\delta\varphi)(T_1) = \varphi(\partial T_1) = \varphi(e_1) - \varphi(e_2) - \varphi(e),$$
$$0 = (\delta\varphi)(T_2) = \varphi(\partial T_2) = \varphi(e_2) - \varphi(e_1) - \varphi(e_1).$$

As $\varphi(e) = 1$, we may WLOG take $\varphi(e_1) = 1$ and $\varphi(e_2) = 0$. Note that α^2 is represented by $\varphi \smile \varphi$, and we have:

$$(\varphi \smile \varphi)(T_1) = \varphi(e_1) \cdot \varphi(e) = 1.$$

Similarly,

$$(\varphi \smile \varphi)(T_2) = \varphi(e_2) \cdot \varphi(e) = 0.$$

Since the generator of $C_2(\mathbb{RP}^2)$ is $T_1 + T_2$, and we have

$$(\varphi \smile \varphi)(T_1 + T_2) = (\varphi \smile \varphi)(T_1) + (\varphi \smile \varphi)(T_2) = 1 + 0 = 1,$$

it follows that $\alpha^2 = [\varphi \smile \varphi]$ is the generator of $H^2(\mathbb{RP}^2;\mathbb{Z}/2\mathbb{Z})$. Let I denote the ideal generated by the relations. Hence, we have

$$H^*(\mathbb{RP}^n;\mathbb{Z}_2) = \frac{\mathbb{Z}_2[x]}{I} \cong \mathbb{Z}_2[x]/(x^3)$$

Let's prove some important facts about the cup product.

Proposition 9.1.13. *Let X, Y be topological spaces and let let $f : X \to Y$ be a continuous map. For each $n \in \mathbb{Z}$, the induced maps*

$$f_n^* = H^n(Y;R) \to H^n(X;R)$$

are ring homomorphisms. That is,

$$f_n^*(\alpha \smile \beta) = f^*(\alpha) \smile f^*(\beta)$$

for each $\alpha, \beta \in H^k(Y;R)$.

Proof. It suffices to show the following co-chain formula:

$$f^\#(\varphi \smile \psi) = f^\#(\varphi) \smile f^\#(\psi).$$

For $\varphi \in C^k(Y;\mathbb{R})$ and $\psi \in C^l(Y;\mathbb{R})$, we have:

$$\begin{aligned}(f^\#\varphi \smile f^\#\psi)(\sigma : \Delta^{k+l} \to X) &= (f^\#\varphi)(\sigma|_{[v_0,\ldots,v_k]}) \cdot (f^\#\psi)(\sigma|_{[v_k,\ldots,v_{k+l}]}) \\ &= \varphi((f^\#\sigma)|_{[v_0,\ldots,v_k]}) \cdot \psi((f^\#\sigma)|_{[v_k,\ldots,v_{k+l}]}) \\ &= (\varphi \smile \psi)(f^\#\sigma) \\ &= (f^\#(\varphi \smile \psi))(\sigma).\end{aligned}$$

This completes the proof. □

Corollary 9.1.14. *If $f : X \to Y$ is a continuous map, then there is a ring homomorphism*

$$f^* : H^*(Y;R) \to H^*(X;R).$$

Proof. We have

$$H^*(Y;R) = \bigoplus_{n \geq 0} H^n(Y;R), \quad H^*(X;R) = \bigoplus_{n\geq 0} H^n(X;R)$$

If we define f^* such that $f^*|_{H^n(Y;R)} = f_n$, the claim follows via Proposition 9.1.13. □

Remark 9.1.15. *The discussion above implies that the operation of taking the cohomology ring is a (contravariant) functor from* Top *to* CRing.

Example 9.1.16. The isomorphisms

$$H^*\left(\coprod_\alpha X_\alpha; R\right) \cong \prod_\alpha H^*(X_\alpha;R)$$

whose coordinates are induced by the inclusions $i_\alpha : X_\alpha \hookrightarrow \coprod_\alpha X_\alpha$, is a ring isomorphism with respect to the coordinatewise multiplication in a ring product, since each coordinate function i_α^* is a ring homomorphism. Similarly, the group isomorphism

$$H^*(\bigvee_\alpha X_\alpha; R) \cong \prod_\alpha H^*(X_\alpha;R)$$

is a ring isomorphism.

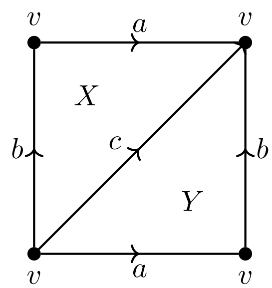
We now show that the cup product is graded anti-commutative.

Proposition 9.1.17. *Let X be a topological space and let R be a commutative ring. Let $\alpha \in H^k(X;R)$ and $\beta \in H^l(X;R)$. We have*

$$\alpha \smile \beta = (-1)^{kl} \beta \smile \alpha$$

Proof. See [Hat02]. □

Example 9.1.18. Let $X = \mathbb{S}^1 \times \mathbb{S}^1 = T^2$. We can use Proposition 9.1.17 to compute $H^*(T^2; R)$. Consider the following simplicial complex structure on T^2:



The generator $1 \in H^0(T^2; \mathbb{Z})$ is the unit. By examining the dimensions of the other generators, the only non-identity generators which could multiply together and give something non-zero are the generators of $H^1(T^2; \mathbb{Z})$. Let $\alpha, \beta \in H^1(T^2; \mathbb{Z})$ be generators of $H^1(T^2; \mathbb{Z})$. We compute

$$\alpha \smile \alpha, \quad \alpha \smile \beta, \quad \beta \smile \alpha, \quad \beta \smile \beta.$$

By Proposition 9.1.17, we must have $\alpha \smile \alpha = \beta \smile \beta = 0$. But let's verify it explicitly. α is represented by a cocycle

$$\varphi_\alpha : C_1(T^2) \to \mathbb{Z}$$

with $\varphi_\alpha(a) = 1, \varphi_\alpha(b) = 0$. Here a, b are generators of $H_1(T^2; \mathbb{Z})$. The co-cycle condition for φ translates into the identities:

$$0 = (\delta\varphi_\alpha)(X) = \varphi_\alpha(X) = \varphi_\alpha(a) - \varphi_\alpha(c) + \varphi_\alpha(b),$$

$$0 = (\delta\varphi_\alpha)(Y) = \varphi_\alpha(Y) = \varphi_\alpha(b) - \varphi_\alpha(c) + \varphi_\alpha(a).$$

As $\varphi_\alpha(a) = 1, \varphi_\alpha(b) = 0$, we must have $\varphi_\alpha(c) = 1$. Note that α^2 is represented by $\varphi \smile \varphi$, and we have:

$$(\varphi_\alpha \smile \varphi_\alpha)(X) = \varphi_\alpha(b) \cdot \varphi_\alpha(a) = 1.$$

$$(\varphi_\alpha \smile \varphi_\alpha)(Y) = \varphi_\alpha(a) \cdot \varphi_\alpha(b) = 1.$$

Hence, $\varphi_\alpha \smile \varphi_\alpha = 0$. This shows that $\alpha \smile \alpha = 0$. If we choose β to be represented by a cocycle

$$\varphi_\beta : C_1(T^2) \to \mathbb{Z}$$

with $\varphi_\beta(b) = 1, \varphi_\beta(a) = 0$, we similarly have $\beta \smile \beta = 0$. We now compute $\alpha \smile \beta$. Note that $\alpha \smile \beta$ is represented by $\varphi_\alpha \smile \varphi_\beta$. We have

$$(\varphi_\alpha \smile \varphi_\beta)(X) = \varphi_\alpha(b) \cdot \varphi_\alpha(a) = 0.$$

$$(\varphi_\alpha \smile \varphi_\beta)(Y) = \varphi_\alpha(a) \cdot \varphi_\alpha(b) = 1.$$

Since the generator of $C_2(T^2)$ is $X + Y$, and $(\varphi_\alpha \smile \varphi_\beta)(X + Y) = 1$, it follows that $\alpha \smile \beta$ is the generator of $H^2(T^2; \mathbb{Z})$. By Proposition 9.1.17, we have $\beta \smile \alpha = -\alpha \smile \beta$. Hence, we have

$$H^*(T^2; \mathbb{Z}) \cong \frac{\mathbb{Z}[x, y]}{\langle x^2, y^2, xy + yx \rangle} \cong \Lambda_{\mathbb{Z}}[x, y]$$

Here $\Lambda_{\mathbb{Z}}[x, y]$ is the exterior algebra on two generator over $\mathbb{Z}$.

9.2. Poincaré Duality for Smooth Manifolds

We discuss Poincaré duality for smooth, oriented, n-manifolds in this section. We can prove this special case by leveraging de Rham cohomology. Using Stokes' theorem, Poincaré duality for smooth, oriented n-manifolds asserts that there is a non-degenerate pairing between de Rham cohomology groups:

$$H^k_{\mathrm{dR}}(M) \times H^{n-k}_{\mathrm{dR,c}}(M) \to \mathbb{R}, \quad ([\alpha], [\beta]) \mapsto \int_M \alpha \wedge \beta.$$

It is easily checked that the pairing defined above is well-defined. The pairing above can be equivalently defined as a linear map from $H^k_{\mathrm{dR}}(M)$ to $(H^{n-k}_{\mathrm{dR,c}}(M))^*$. We show that this linear map is an isomorphism.

Proposition 9.2.1. *Let M be a smooth, oriented, n-manifold that admits a good finite cover. Then*

$$H^k_{dR}(M) \cong (H^{n-k}_{dR,c}(M))^*$$

for each $0 \leq k \leq n$.

9.3. Poincaré Duality

Poincaré duality is a fundamental result in algebraic topology that relates the homology and cohomology groups of an orientable closed manifold. It states that for an n-dimensional orientable manifold M, there exists an isomorphism

$$H_k(M;\mathbb{Z}) \cong H^{n-k}_c(M;\mathbb{Z})$$

This duality provides deep insights into the topology of manifolds, constraining their possible homology groups and aiding in the computation of topological invariants. It also plays a crucial role in intersection theory. Before defining Poincaré duality, we need to define the notation of a fundamental class. In order to define a fundamental class, we need to define the notation of an orientation.

Part 4

Homotopy Theory

CHAPTER 10

Categorical Nuances

The category hTop is the appropriate framework for studying homotopy theory. However, not all concepts from the category Top carry over directly to hTop. For instance, we have the following pushout diagram in Top:

$$\begin{tikzcd} \mathbb{S}^{n-1} \arrow[r] \arrow[d, hook] & \{*\} \arrow[d] \\ \mathbb{D}^n \arrow[r] & \mathbb{S}^n \end{tikzcd}$$

On the other hand, we also have the pushout diagram in Top:

$$\begin{tikzcd} \mathbb{S}^{n-1} \arrow[r] \arrow[d] & \{*\} \arrow[d] \\ \{*\} \arrow[r] & \{*\} \end{tikzcd}$$

Therefore, even though $\mathbb{D}^n$ is homotopy equivalent to $\{*\}$, the two pushouts are not homotopy equivalent. Therefore, contrary to expectation, the pushout diagrams in hTop are not the same. This example suggests that further analysis and applications of the homotopy notion require a certain amount of formal (categorical) considerations. In this section, we discuss some basic constructions of a categorical nature. More advanced constructions such as homotopy pullbacks and homotopy pushouts will be discussed as necessary later on.

10.1. Cones & Suspensions

In this section, we discuss the the categorical constructions of cones and suspensions.

10.1.1. Cone & Suspension. Let $I = [0,1] \subseteq \mathbb{R}$. The space $X \times I$ is called a cylinder over X, and the subspaces $X \times \{0\}$, $X \times \{1\}$ are the bottom and top "bases". Now we will construct new spaces out of the cylinder $X \times I$.

Definition 10.1.1. Let X be a topological space. The cone of X is the quotient space:

$$CX = X \times I / (X \times \{0\})$$

Remark 10.1.2. *CX has a natural basepoint given by the collaposed space $X \times \{0\}$. Hence, we have a functor*

$$C : \mathsf{Top} \to \mathsf{Top}_*$$

Indeed, if $f : X \to Y$ is a continuous map, we have a continuous map $f \times id_I : X \times I \to Y \times Y$ and if we define $C(f)$ to be the map

$$\begin{aligned} C(f) : CX &\to CY, \\ [x,t] &\mapsto [f(x), t], \end{aligned}$$

We have $Cf \circ q_X = q_Y \circ (f \times \mathrm{id}_I)$ where q_X, q_Y are quotient maps defining CX and CY.

$$\begin{array}{ccc} X \times I & \xrightarrow{f \times id_I} & Y \times I \\ \Big\downarrow q_X & & q_Y \Big\downarrow \\ CX & \xrightarrow{C(f)} & CY \end{array}$$

The universal property of quotient topology implies that Cf is continuous.

The cone of a topological space is always a contractible space.

Proposition 10.1.3. *Let $X \in \mathsf{Top}$. Then CX is contractible.*

PROOF. A homotopy between the identity on CX and the map to the basepoint is given by:

$$F : CX \times I \to CX,$$
$$([x,t], s) \mapsto [x, (1-s)t]$$

This completes the proof. □

The motivation for introducing the cone of a topological space is given by the following proposition:

Proposition 10.1.4. *Let $X, Y \in \mathsf{Top}$. A map $f : X \to Y$ is nullhomotopic if and only if it extends to a map $\overline{f} : CX \to Y$.*

PROOF. Consider a continuous map $H : X \times [0,1] \to Y$ with $H(\cdot, 0) = f(\cdot)$. Note that $H(x,1)$ is constant for all $x \in X$ if and only if $X \times \{1\}$ is contained in a fiber of H, which in turn, by the universal property of quotient spaces, says that H factors uniquely through the canonical quotient map $X \times [0,1] \to CX$. This proves the claim. □

Remark 10.1.5. *Proposition 10.1.4 implies that a continuous map $f : \mathbb{S}^n \to X$ is null-homotopic if and only if f extends to a continuous map $\overline{f} : \mathbb{D}^{n+1} \to X$. This is because $C\mathbb{S}^n \cong \mathbb{D}^{n+1}$*

We now define the suspension of a topological space.

Definition 10.1.6. Let $X \in \mathsf{Top}$. The suspension of X is the quotient space:

$$SX = X \times I/(X \times \{0\}, X \times \{1\})$$

Remark 10.1.7. *S defines a a functor $S : \mathsf{Top} \to \mathsf{Top}$ This follows by a similar reasoning that cone is a functor.*

Example 10.1.8. The suspension of $\mathbb{S}^0 = \{x, x_1\}$ consists of two lines (one over each point in $\mathbb{S}^0$) joined at 0 and 1, giving $\mathbb{S}^1$. In fact,

$$\mathbb{S}^{n+1} \cong S\mathbb{S}^n$$

in general. To see this, WLOG, replace $I = [0,1]$ by $I = [-1,1]$. Define

$$f : \mathbb{S}^n \times [-1,1] \to \mathbb{S}^{n+1}$$

by

$$f((x, \ldots, x_n), t) = (x \cdot \sqrt{1-t^2}, \ldots, x_n \cdot \sqrt{1-t^2}, t)$$

It is clear that f is continuous and surjective. Moreover, f agrees on the fibers of $S\mathbb{S}^n$. Hence, f descends to a continuous bijection $\widetilde{f}$ from $S\mathbb{S}^n$ to $\mathbb{S}^{n+1}$. Since $S\mathbb{S}^n$ is compact and $\mathbb{S}^{n+1}$ is Hausdorff, $\widetilde{f}$ is a homemorphism.

$$\begin{tikzcd}
\mathbb{S}^n \times [-1,1] \arrow[d, "q"] \arrow[dr, "f"] & \\
S\mathbb{S}^n \arrow[r, dotted, "\widetilde{f}"'] & \mathbb{S}^{n+1}
\end{tikzcd}$$

10.2. Compact Open Topology, Path & Loop Spaces

10.2.1. Compact Open Topology. If $(X, x_0) \in \mathsf{Top}_*$, note that $\pi_1(X, x_0)$ is, in particular, a space of continuous functions. Hence, we would like to discuss what appropriate topology to put on the function space of continuous maps between topological spaces.

Definition 10.2.1. Let $X, Y \in \mathsf{Top}$ and let $\mathrm{C}(X, Y)$ denote the set of of continuous maps $X \to Y$. $\mathrm{C}(X, Y)$ carries a natural topology, called the compact-open topology, generated by a subbasis formed by the sets of the form

$$B(K, U) = \{f : X \to Y \mid f(K) \subseteq U\}$$

where $K \subseteq X$ is a compact set and $U \subseteq Y$ is an open set.

Remark 10.2.2. *The topological space given by this compact-open topology will be denoted by* $\mathrm{Maps}(X, Y)$*.*

Remark 10.2.3. *For a map $f : X \to Y$, one can form a typical basis open neighborhood by choosing compact subsets $K_1, \ldots, K_n \subseteq X$ and small open sets $U_i \subseteq X$ with $f(K_i) \subseteq U_i$ to get a neighborhood O_f of f,*

$$O_f = B(K_1, U_1) \cap \cdots \cap B(K_n, U_n).$$

The collection of all such sets forms a basis for the compact-open topology.

What is the motivation behind the definition of the compact-open topology? If X is compact Hausdorff and Y a metric space, then one can consider the supremum norm on $(C(X, Y), \|\cdot\|_\infty)$. It can be checked that in this case $\mathrm{Maps}(X, Y) = (C(X, Y), \|\cdot\|_\infty)$. We prove a slightly more genral claim:

Proposition 10.2.4. *Let $X, Y \in \mathsf{Top}$ such that (Y, d) is a metric space. The compact-open topology and the topology of uniform convergence on compact sets coincide on $C(X, Y)$.*

Proof. We first prove that the topology of compact convergence is finer than the compact-open topology. Let $B(K, U)$ be a subbasis element for the compact-open topology, and let $f \in B(K, U)$. Because f is continuous, $f(K)$ is a compact subset of U. Therefore, we can choose $\varepsilon > 0$ so that the ε-neighborhood of $f(K)$ is contained in U. Then, as desired,

$$E_K(f, \varepsilon) \subseteq B(C, U).$$

Here

$$E_K(f, \varepsilon) = \{g \in C(X, Y) \mid \|f - g\|_{\infty, K} < \varepsilon\}$$

is a basis element of the topology of compact convergence. We now prove that the compact-open topology is finer than the topology of compact convergence. Let $f \in C(X, Y)$ and consider $E_K(f, \varepsilon)$ for some $\varepsilon > 0$. Every $x \in X$ has a neighborhood V_x such that $f(\overline{V_x})$ lies in an open set $U_{f(x)}$ of Y having diameter less than ε. For example, choose V_x so that $f(V_x)$ lies in the $\varepsilon/4$-neighborhood of $f(x)$. Then $f(\overline{V_x})$ lies in the $\varepsilon/3$-neighborhood of $f(x)$, which has diameter at most $2\varepsilon/3$. Cover K by finitely many such sets V_x, say for $x = x_1, \ldots, x_n$. Let $K_x = \overline{V_x} \cap K$. Then K_x is compact, and the basis element

$$B(K_{x_1}, U_{x_1}) \cap \cdots \cap B(K_{x_n}, U_{x_n})$$

contains f and lies in $E_K(f, \varepsilon)$, as desired. $\square$

Remark 10.2.5. *If Y is not a metric space, we need to redefine the notion of proximity between maps. Suppose $f, g \in \mathrm{Maps}(X, Y)$ are two continuous maps. Let $K \subseteq X$ be a compact subset and $U \subseteq Y$ be an open subset such that $f(K) \subseteq U$. Assume that Y is Hausdorff, which ensures that closed sets in Y behave well under continuous maps. Since $f(K)$ is compact and Y is Hausdorff, $f(K)$ is closed in Y, and intuitively, small perturbations of $f(K)$ should remain within U. Thus, to define the topology on $\mathrm{Maps}(X, Y)$, we say that a neighborhood of f is the set of maps $g \in \mathrm{Maps}(X, Y)$ such that $g(K) \subseteq U$. This formalizes the notion that g is 'close' to f if it maps the compact set K into the same open set U that contains $f(K)$.*

Proposition 10.2.6. ***(Exponential Law)*** *Let $X, Y, Z \in \mathsf{Top}$. If X is Hausdorff and Y is locally compact, then*

$$\varphi : \mathrm{Maps}(X \times Y, Z) \to \mathrm{Maps}(X, \mathrm{Maps}(Y, Z)), \quad \varphi(g)(x)(y) = g(x, y)$$

is a continuous bijection.

Proof. We first show that φ is well-defined. Suppose g is continuous, and choose an arbitrary sub-basis open set $B(K, U)$ in $\mathrm{Maps}(Y, Z)$. Choose $x \in \varphi(g)^{-1}(B(K, U))$, so $g(\{x\} \times K) \subseteq U$. Since K is compact and g is continuous, there are open sets $V \ni x$ and $W \supseteq K$ such that $g(V \times W) \subseteq U$. Then V is a neighborhood of x with $\varphi(g)(V) \subseteq B(K, U)$, showing that $\varphi(g)^{-1}(B(K, U))$ is open. Hence, φ is well-defined. φ is obviously an injection. We now show that φ is continuous and surjective.

We first show that φ is continuous. Let $g : X \times Y \to Z$ be a continuous map. Let $K_1 \subseteq X$ and $K_2 \subseteq Y$ be compact subsets, $U \subseteq Z$ be an open set. Let

$$B(K_1, B(K_2, U)) = \{g : X \to \mathrm{Maps}(Y, Z) \mid g(K_1)(K_2) \subseteq U\}$$

be an open neighborhood of $\varphi(g)$. Then $[K_1 \times K_2, U]$ is an open neighborhood of g in $\mathrm{Maps}(X \times Y, Z)$[1] such that

$$\varphi(B(K_1 \times K_2, U)) \subseteq B(K_1, B(K_2, U))$$

This shows that φ is continuous. We now show that φ is surjective. Let $f : X \to \mathrm{Maps}(Y, Z)$ be a continuous map. Let $(x, y) \in X \times Y$ and W be an open neighborhood of $\varphi^{-1}(f)(x, y)$ in Z. We find neighborhoods $x \in U \subseteq X$ and $y \in V \subseteq Y$ such that $\varphi^{-1}(f)(U \times V) \subseteq W$. Since $f(x) : Y \to Z$ is continuous, $f(x)^{-1}(W)$ is an open neighborhood of y in Y. Since Y is locally compact, there exists a compact set $K \subseteq f(x)^{-1}(W)$ such that $y \in \mathrm{Int}(K) \subseteq K$. Then $[K, W]$ is an open neighborhood of $f(y)$, and since $f : X \to \mathrm{Maps}(Y, Z)$ is continuous at x, there exists an open neighborhood $x \in U \subseteq X$ such that $f(U) \subseteq [K, W]$. This implies that $F(U \times K) \subseteq W$. We can take $V = \mathrm{Int}(K)$, and thus we have $F(U \times V) \subseteq W$. $\square$

Remark 10.2.7. *If X is locally compact Hausdorff, the continuous bijection in Proposition 10.2.6 is in fact a homeomorphism.*

Corollary 10.2.8. *Let $X, Y \in \mathsf{Top}$ such that X is locally compact and Y is Hausdorff. The evaluation map*

$$\begin{aligned} \mathrm{Ev}_{X,Y} : X \times \mathrm{Maps}(X, Y) &\to Y, \\ (x, f) &\mapsto f(x). \end{aligned}$$

is continuous.

Proof. We take for granted the statement that Y is Hausdorff implies that $\mathrm{Maps}(X, Y)$ is Hausdorff. Proposition 10.2.6 implies there is a continuous bijection:

$$\mathrm{Maps}(\mathrm{Maps}(X, Y) \times X, Y) \cong \mathrm{Maps}(\mathrm{Maps}(X, Y), \mathrm{Maps}(X, Y))$$

The inverse image of $\mathrm{Id}_{\mathrm{Maps}(X,Y)}$ is $\mathrm{Ev}_{X,Y}$. $\square$

[1] We need that X is Hausdorff here. See Lemma XII.5.1 (a) of Dugundji's *Topology*.

Remark 10.2.9. *Here is an important observation. If $X, Y \in \mathsf{Top}$, then a homotopy between two maps $f, g : X \to Y$ as an element of* $\mathrm{Maps}(X \times I, Y)$. *Based on Proposition 10.2.6, it is possible to reinterpret a homotopy between two maps $f, g : X \to Y$ as an element of* $\mathrm{Maps}(X, \mathrm{Maps}(I, Y))$ *or* $\mathrm{Maps}(I, \mathrm{Maps}(X, Y))$. *The latter says that a homotopy is a path in* $\mathrm{Maps}(X, Y)$.

Proposition 10.2.10. *If $X, Y \in \mathsf{Top}$ are locally compact Hausdorff spaces, then the function*

$$\Phi_{X,Y,Z} : \mathrm{Maps}(X, Y) \times \mathrm{Maps}(Y, Z) \to \mathrm{Maps}(X, Z)$$

given by composition is continuous.

Proof. By Proposition 10.2.6 we have the bijection

$$\mathrm{Maps}(\mathrm{Maps}(X, Y) \times \mathrm{Maps}(Y, Z), \mathrm{Maps}(X, Z)) \cong \mathrm{Maps}(\mathrm{Maps}(X, Y) \times \mathrm{Maps}(Y, Z), \times X, Z)$$

Hence, $\Phi_{X,Y,Z}$ is continuous if and only if the image of $\Phi_{X,Y,Z}$, denoted $\Phi'_{X,Y,Z}$, under the exponential law is continuous. Let $(f, g) \in \mathrm{Maps}(X, Y) \times \mathrm{Maps}(Y, Z)$ and $x \in X$. We have

$$\Phi'_{X,Y,Z}((f, g), x) = (T(f, g))(x) = f(g(x)).$$

We can decompose $\Phi'_{X,Y,Z}$ as the following composition:

$$\mathrm{Maps}(Y, Z) \times \mathrm{Maps}(X, Y) \times X \xrightarrow{(f,g,x)\mapsto(f,g(x))} \mathrm{Maps}(Y, Z) \times Y \xrightarrow{(g,y)\mapsto g(y)} Z.$$

The first map is just $\mathrm{Id}_{\mathrm{Maps}(Y,Z)} \times \mathrm{Ev}_{X,Y}$ and the second map is $\mathrm{Ev}_{Y,Z}$. Both these maps are continuous by Corollary 10.2.8. The claim follows. □

10.2.2. Path & Loop Spaces. We can consider special instances of the function space discussed above to define loop spaces. For instance, if $X \in \mathsf{Top}$, the space $\Lambda(X) = \mathrm{Maps}(\mathbb{S}^1, X) \in \mathsf{Top}$ is the free loop space of X. Similarly, the space $P(X) = \mathrm{Maps}(I, X) \in \mathsf{Top}$ is the free path space of X.

Remark 10.2.11. *If $X = I = [0, 1]$, Y is locally compact and Z is a topological space, then Proposition 10.2.6 reads*

$$\begin{aligned} \mathrm{Maps}(Y, \mathrm{Maps}(I, Z)) &\cong \mathrm{Maps}(I \times Y, Z) \\ &\cong \mathrm{Maps}(I, \mathrm{Maps}(Y, Z)) \end{aligned}$$

This is called the cylinder-free path adjunction. This is because $I \times Y$ is a cylinder on Y and $(\mathrm{Maps}(I, Z)))$ *is the path space on Z. Note that* $\mathrm{Maps}(I \times Y, Z) = [Y, Z]$.

We now make the following definition:

Definition 10.2.12. Let $(X, x_0) \in \mathsf{Top}_*$.

(1) The path space $P(X, x_0) \in \mathsf{Top}_*$ of (X, x_0) is the pointed space given by

$$P(X, x_0) = \{\gamma \in P(X) \mid \gamma(0) = x_0\}.$$

with the constant path c_x at x as the base point.

(2) The loop space $\Omega(X, x_0) \in \mathsf{Top}_*$ of (X, x_0) is the pointed space

$$\Omega(X, x_0) = \{\gamma \in P(X) \mid \gamma(0) = x_0 = \gamma(1)\}$$

with the constant loop c_x at x as the base point.

Remark 10.2.13. *Note that $\Omega(X, x_0)$ consists of pointed loops $(\mathbb{S}^1, *) \to (X, x_0)$. Moreover, note that $P(X, x_0)$ can be thought of as a pullback:*

$$\begin{tikzcd}
P(X, x_0) \arrow[r] \arrow[d] & X^I \arrow[d, "\mathrm{Ev}_0"] \\
\{x_0\} \arrow[r, hook] & X
\end{tikzcd}$$

Proposition 10.2.14. *Let $X \in \mathsf{Top}$. The path space, $P(X, x_0)$, is contractible.*

Proof. A homotopy between the identity on $P(X, x_0)$ and the map to the basepoint (the constant path) is given by:

$$\begin{aligned} F : P(X, x_0) \times I &\to P(X, x_0), \\ (\gamma, s) &\mapsto (t \mapsto \gamma((1-s)t)) . \end{aligned}$$

This completes the proof. ☐

Remark 10.2.15. *We discuss some applications of function space* $\mathrm{Maps}(X, Y)$ *to establish some basic facts:*

(1) *A point in X can be identified with a map $x : * \to X$ sending the unique point $*$ to x. Hence, we have a bijective correspondence*

$$X \cong \mathrm{Maps}(*, X).$$

(2) *In the case of a pointed space, we have a bijective correspondence*

$$\begin{aligned} (X, x_0) &\cong \mathrm{Maps}\left((\{*, *'\}, *'), (X, x_0)\right) \\ &\cong \mathrm{Maps}\left((\mathbb{S}^0, 1), (X, x_0)\right). \end{aligned}$$

(3) *Note that we have*

$$[X, Y] \cong \pi_0(\mathrm{Maps}(X, Y))$$

(4) *Let X, Y be locally compact Hausdorff spaces. Consider the continuous function*

$$T : \mathrm{Maps}(X, Y) \times \mathrm{Maps}(Y, Z) \to \mathrm{Maps}(X, Z)$$

Hence, T induces a map

$$\begin{aligned} [X, Y] \times [Y, Z] &= \pi_0(\mathrm{Maps}(X, Y)) \times \pi_0(\mathrm{Maps}(Y, Z)) \\ &= \pi_0(\mathrm{Maps}(X, Y) \times \mathrm{Maps}(Y, Z)) \\ &\to \pi_0(\mathrm{Maps}(X, Z)) \\ &= [X, Z] \end{aligned}$$

In particular, a continuous function $f : X \to Y$ induces a map

$$f^{\#} : [Y, Z] \to [X, Z]$$

and a continuous function $g : Y \to Z$ induces a map

$$g_{\#} : [X, Y] \to [X, Z].$$

(5) *Let $f : X \to Y$ be a homotopy equivalence with homotopy inverse and $g : Y \to X$. Using (3), we have two induced maps*

$$f^{\#} : [Y, Z] \to [X, Z] \quad f_{\#} : [Z, X] \to [Z, Y]$$

The maps $g^{\#}$ and $g_{\#}$ are inverses of f^* and $f_{\#}$ respectively. Hence, we have a bijection of sets

$$[Y, Z] \cong [X, Z], \qquad [Z, X] \cong [Z, Y]$$

10.3. Smash Products

We introduce the notion of a smash product that *forces us to take basepoints seriously*. The need for the smash product arises based on the need to consider the pointed analog of Maps$(\cdot,\cdot)$.

Definition 10.3.1. Let $(X,x_0),(Y,y_0)\in \mathsf{Top}_*$. The pointed space Maps$((X,x_0),(Y,y_0))$ is defined to be subspace of Maps(x_0,y_0) consisting of pointed maps, along with the natural basepoint given by constant map $X\to y_0$.

Remark 10.3.2. *We have* $[(X,x_0),(Y,y_0)]=\pi_0(\text{Maps}((X,x_0),(Y,y_0)))$.

We now define the smash product:

Definition 10.3.3. Let $(X,x_0),(Y,y_0)\in \mathsf{Top}_*$. The smash product is defined as the quotient space

$$(X,x_0)\wedge(Y,y_0)=(X\times Y,(x_0,y_0))/(X\vee Y)$$

Remark 10.3.4. *The wedge sum $X\vee Y$ of two pointed spaces is naturally a pointed subspace of $(X,x_0)\times(Y,y_0)$. For pointed spaces (X,x_0) and (Y,y_0), the pointed product $(X\times Y,(x_0,y_0))$ comes naturally with an inclusion map of (X,x_0) given by*

$$\begin{aligned}(X,x_0)&\to(X\times Y,(x_0,y_0)),\\ x&\mapsto(x,y_0).\end{aligned}$$

There is a similar map $(Y,y_0)\to(X\times Y,(x_0,y_0))$. Since $X\vee Y$ is a pushout in Top_*, we obtain a pointed map $X\vee Y\to(X\times Y,(x_0,y_0))$ which yields the desired inclusion.

Remark 10.3.5. *It can be checked that the smash product defines a functor*

$$\wedge:\mathsf{Top}_*\to\mathsf{Top}_*$$

The motivation behind the definition of a smash product is to extend Proposition 10.2.6 to Top_*. Let $(X,x_0),(Y,y_0),(Z,z_0)\in\mathsf{Top}_*$. Since y_0 and z_0 are basepoints in Y and Z, respectively, then Maps$((Y,y_0),(X,x_0))$ has a basepoint given by the constant function $X\to z_0$. We want a map $f:(X,x_0)\to\text{Maps}((Y,y_0),(Z,z_0))$ to preserve basepoints, meaning that it must satisfy

$$f(x_0)(y)=z_0\quad\text{for all } y\in Y.$$

Additionally, for any $x\in X$, the map $f(x):Y\to Z$ must also preserve basepoints, i.e.,

$$f(x)(y_0)=z_0\quad\text{for all } x\in X.$$

Therefore, if Proposition 10.2.6 is to be extend to Top_*, then a a map $f:X\times Y\to Z$ f must be constant on

$$(\{x_0\}\times Y)\cup(X\times\{y_0\}),$$

sending it to z_0. This is exactly how we have defined the smash product, which yields the following result:

Proposition 10.3.6. *Let $(X,x_0),(Y,y_0),(Z,z_0)\in\mathsf{Top}_*$. If Y is locally compact and X is Hausdorff, then the smash product satsifes the pointed version of the exponential law. That is we have a continuous bijection:*

$$\text{Maps}_*((X,x_0)\wedge(Y,y_0),(Z,z))\cong\text{Maps}_*((X,x_0),\text{Maps}((Y,y_0),(Z,z)))$$

Proof. Clear. Invoke the discussion above, and note that Proposition 10.2.6 descends to yield the desired result. □

Remark 10.3.7. *If X is locally compact Hausdorff, the continuous bijection in Proposition 10.3.6 is in fact a homeomorphism.*

Remark 10.3.8. *Let M, N be locally compact Hausdorff spaces. Then their one-point compactifications M_∞, N_∞ are compact Hausdorff spaces, and each is equipped with a canonical basepoint. We continue to write (M_∞, ∞_M) as M_∞. The product $M \times N$ is locally compact Hausdorff and we have the basic relation*

$$(M \times N)_\infty \cong M_\infty \wedge N_\infty.$$

Indeed, there is canonical continuous map

$$u : M_\infty \times N_\infty \to (M \times N)_\infty$$

which maps $M \times N \subseteq M_\infty \times N_\infty$ via the identity onto $M \times N \subseteq (M \times Y)_\infty$ and maps $M_\infty \times \{\infty_N\} \cup \{\infty_M\} \times N_\infty$ to $\{\infty_{M\times N}\}$. Therefore it induces a continuous bijection

$$u' : M_\infty \wedge N_\infty \to (M \times Y)_\infty$$

on the quotient space $M_\infty \wedge N_\infty$ of $M_\infty \times N_\infty$. This space is comapct, therefore u' is a homeomorphism.

Example 10.3.9. Each $(\mathbb{S}^n, *)$ is a pointed topological space. We have

$$(\mathbb{S}^n, *) = (\mathbb{S}^1, *) \wedge (\mathbb{S}^1, *)$$

Note that $(\mathbb{S}^1, *) \times (\mathbb{S}^1, *)$ is a torus. Visualizing the torus as quotient of a square with endpoints identified appropriately, $\mathbb{S}^1 \vee \mathbb{S}^1$ corresponds to the to the boundary of the square. The smash product identifies all these boundary points to a single point, yielding $(\mathbb{S}^2, *)$. More generally, we have

$$(\mathbb{S}^n, *) = (\mathbb{S}^1, *) \wedge \cdots \wedge (\mathbb{S}^1, *)$$

Indeed,

$$\begin{aligned}(\mathbb{S}^{m+n}, *) &\cong (\mathbb{R}^{m+n})_\infty \\ &= (\mathbb{R}^m \times \mathbb{R}^n)_\infty \\ &\cong (\mathbb{R}^m)_\infty \wedge (\mathbb{R}^n)_\infty \cong (\mathbb{S}^m, *) \wedge (\mathbb{S}^n, *).\end{aligned}$$

Example 10.3.10. Let $(X, x_0) \in \mathsf{Top}_*$. We can define the reduced cone of (X, x_0) as

$$\widetilde{C}(X, x_0) = (X, x_0) \times (I, 0)/((X, x_0) \times \{x_0\} \cup \{*\} \times (I, 0)).$$

Essentially by definition,

$$\widetilde{C}(X, x_0) \cong (X, x_0) \wedge (I, 0)$$

We can also define the notion of a reduced suspension.

Definition 10.3.11. Let $(X, x_0) \in \mathsf{Top}_*$. The reduced suspension $\Sigma(X, x_0) \in \mathsf{Top}_*$ is the pointed space

$$\Sigma(X, x_0) = ((X, x_0) \times (\mathbb{S}^1, *))/\left(\{x_0\} \times (\mathbb{S}^1, *) \cup (X, x_0) \times \{*\}\right),$$

where the base point is given by the collapsed subspace.

Remark 10.3.12. *Using the quotient map $I \to I/\partial I \cong \mathbb{S}^1$, an alternative description of the reduced suspension $\Sigma(X, x_0)$ is given by*

$$\Sigma(X, x_0) = ((X, x_0) \times (I, 0))/\left(\{x_0\} \times I \cup X \times \{0, 1\}\right),$$

Example 10.3.13. Let $(X, x_0) \in \mathsf{Top}_*$. We have

$$\Sigma(X, x_0) = (X, x_0) \wedge (\mathbb{S}^1, *)$$

Indeed, consider the quotient map $f : (I, 0) \to (\mathbb{S}^1, *)$ given by $f(t) = e^{2\pi i t}$, and the diagram:

$$\begin{tikzcd}
(X, x_0) \times (I, 0) \arrow[r, "1\times f"] \arrow[d, "p"'] & (X, x_0) \times (\mathbb{S}^1, *) \arrow[d, "q"] \\
\Sigma(X, x_0) & (X, x_0) \wedge (\mathbb{S}^1, *)
\end{tikzcd}$$

It can be checked that We show that $1 \times f$ is a quotient map. The characteristic property of the quotient topology now implies that

$$\Sigma(X, x_0) \cong (X, x_0) \wedge (\mathbb{S}^1, *)$$

Remark 10.3.14. *Along with Example 10.3.9, the previous examples readily implies that we have*

$$\Sigma(\mathbb{S}^n, *) = (\mathbb{S}^{n+1}, *)$$

Corollary 10.3.15. *Let $(X, x_0), (Y, y_0) \in \mathsf{Top}_*$ such that X is Hausdorff. Then there is a continuous bijective correspondence*

$$\mathrm{Maps}_*(\Sigma(X, x_0), (Y, y_0)) \cong \mathrm{Maps}_*((X, x_0), \Omega(Y, y_0))$$

Passing to π_0, we have

$$[\Sigma(X, x_0), (Y, y_0)] \cong [(X, x_0), \Omega(Y, y_0)].$$

Proof. This follows from Proposition 10.3.6 and that Remark 10.3.14. $\square$

10.4. Compactly Generated Spaces

The bijection in Proposition 10.2.6 relies on the fact that Y is locally compact. A number of topological spaces in homotopy theory are non-locally finite CW complexes. Fundamental examples include $\mathbb{RP}^\infty$ and $\mathbb{CP}^\infty$. We now look at a category of topological spaces where we expect homotopy theoretic propositions to be true without additional assumptions.

10.4.1. Compactly Generated Spaces.

Informally, a compactly generated space is a topological space whose topology is determined by all continuous maps from arbitrary compact spaces.

Definition 10.4.1. Let $X \in \mathsf{Top}$. A subset $A \subseteq X$ is called k-closed in X if, for any compact Hausdorff space K and continuous map $f : K \to X$, the preimage $f^{-1}(A) \subseteq K$ is closed in K.

The collection of k-closed subsets of X forms a topology, which contains the original topology of X. Let kX denote the topological space whose underlying set is that of X, but equipped with the topology of k-closed subsets of X. Because the k-topology contains the original topology on X, the identity function $\mathrm{Id} : kX \to X$ is continuous.

Definition 10.4.2. Let $X \in \mathsf{Top}$. X is compactly generated (CG) if $\mathrm{Id} : kX \to X$ is a homeomorphism.

Let **CG** denote the full subcategory of **Top** consisting of compactly generated spaces. Let's discuss categorical properties:

Proposition 10.4.3. *Let $X \in \mathsf{Top}$.*

1. *The k-ification is a functor.*
2. *For any space X, the map $k^2X \to kX$ is a homeomorphism. Hence, $k^2X \cong kX$.*
3. *The k-ification functor is right adjoint to the forgetful functor. That is,*

$$\mathrm{Hom}_{\mathsf{CG}}(X, kY) = \mathrm{Hom}_{\mathsf{Top}}(X, Y)$$

   for all $X \in \mathsf{CG}$ and $Y \in \mathsf{Top}$.
4. *The k-ification functor commutes with limits. Hence, limits exist in CG.*

(5) *Disjoint unions of compactly generated spaces are compactly generated. Quotients of compactly generated spaces by equivalence relations are compactly generated.*
(6) *Colimits exist in* CG *and can simply be computed in* Top.

PROOF. The proof is given below:

(1) Suppose $f : X \to Y$ is any continuous map and $A \subseteq Y$ is compactly closed. For any map $u : K \to X$, the set $u^{-1}(f^{-1}(A))$ is closed in K. Thus, $f^{-1}(A)$ is compactly closed in X. This means that $f : kX \to kY$ is continuous.
(2) Given a compact Hausdorff space X and a (set) map $f : K \to X$, the map f is continuous if and only if $f : X \to kX$ is continuous. So the compactly closed sets of X are the same as the compactly closed sets of kX. In other words, $kx \cong k^2X$.
(3) It suffices to show that $f : X \to Y$ is continuous if and only if $\bar{f} : X \to kY$ is continuous. Since the k-ification topology is finer, we assume that f is continuous and show that $\bar{f}$ is continuous. But $k(f) : kX \to kY$ is continuous and $kX \cong X$.
(4) This follows from (3) and categorical arguments. Indeed, we have:

$$\begin{aligned}\mathrm{Hom}_{\mathsf{CG}}(X, k(\varprojlim_i Y_i)) &\cong \mathrm{Hom}_{\mathsf{Top}}(X, \varprojlim_i Y_i)\\ &\cong \varprojlim_i \mathrm{Hom}_{\mathsf{Top}}(X, Y_i)\\ &= \varprojlim_i \mathrm{Hom}_{\mathsf{CG}}(X, kY_i) = \mathrm{Hom}_{\mathsf{CG}}(X, \varprojlim_i kY_i)\end{aligned}$$

for all $X \in \mathsf{CG}$ and $Y \in \mathsf{Top}$. Hence,

$$k(\varprojlim_i Y_i) = \varprojlim_i kY_i$$

(5) Let $X = \coprod_i X_i$ such that each X_i is compactly generated. Let $A \subseteq X$ be k-closed. Then A has the form $\coprod_i A_i$, where $A_i = A \cap X_i$, and it is sufficient to check that A_i is closed in X_i. As X_i is CG, it is enough to check that A_i is k-closed in X_i. Consider a map $f : K \to X_i$. Then the composite $i \circ f : K \to X_i \hookrightarrow X)$ is continuous and

$$f^{-1}(A_i) = (i \circ f')^{-1}(A),$$

which is closed because A is k-closed in X. Now let X be compactly generated and let $q : X \to Y$ a quotient map. Since X is compactly generated, q induces a continuous map $\tilde{q} : X \to kY$ as shown below:

$$\begin{tikzcd} X \arrow[r, "\tilde{q}"] \arrow[rr, bend left, "q"] & kY \arrow[r, "\mathrm{Id}"] & Y \end{tikzcd}$$

Let $A \subseteq Y$ be a k-open subset of Y. Hence, $\mathrm{Id}^{-1}(A) \subseteq kY$ is open in kY. Then the preimage

$$q^{-1}(A) = (\mathrm{Id} \circ \tilde{q})^{-1}(A) = \tilde{q}^{-1}(\mathrm{Id}^{-1}(A)) \subseteq X$$

is open in X since $\tilde{q} : X \to kY$ is continuous. Therefore, $A \subseteq Y$ is open in Y since q is a quotient map.
(6) Colimits in Top can be constructed by taking disjoint unions and quotients. The colimit of compactly generated spaces in the category Top is a compactly generated space. Thus, it is also the colimit in CG.

This completes the proof. □

Remark 10.4.4. *In Proposition 10.4.3(3) we have shown that $f : X \to Y$ is continuous if and only if $f : X \to kY$ is continuous for all $X \in \mathsf{CG}$ and $Y \in \mathsf{Top}$. This can be summarized such that following diagram commutes:*

$$\begin{tikzcd} kY \arrow[r, "Id"] & Y \\ X \arrow[u, dotted, "\bar{f}"] \arrow[ur, "f"'] & \end{tikzcd}$$

Note that $\bar{f}$ has the same underlying function as f. This exhibits $kY \to Y$ as the 'closest approximation' of Y by a CG space.

Proposition 10.4.5. *Every locally compact Hausdorff space and CW-complex is CG.*

Proof. Let X be a locally compact space assume $f^{-1}(A) \subseteq K$ is closed for every compact Hausdorff space, K. We show A is closed by showing that A^c is open. Let $x \in A^c$. By local compactness, there exists a compact neighbourhood of x, say K_x. Let U_x be an open neighbourhood of x such that $x \in U_x \subseteq K_x$. Because $K_x \cap A$ is closed by hypothesis (consider the inclusion map $i_x : K_x \to X$), we have that $(K_x \cap A)^c$ is open. Therefore,

$$(K_x \cap A)^c \cap U_x = U_x^c := V_x$$

is an open neighbourhood of x not intersecting A. We have

$$A^c = \bigcup_{x \in A^c} V_x,$$

and therefore A^c is open. A CW complex is a colimit constructed by consideting closed disks. Since closed disks are in CG and CG is closed under taking colimits, every CW complex is in CG. $\square$

Corollary 10.4.6. *Let $X \in \mathsf{Top}$. Then $X \in \mathsf{CG}$ if and only if X is a quotient space of a locally compact space.*

Proof. Let $X \in \mathsf{CG}$. The converse follows from Proposition 10.4.3. Consider the following collection:

$$\mathcal{K} = \{f_K(K) \mid f_K : K \to X \text{ is continuous and } K \text{ compact Hausdorff}\}$$

Let $Y = \bigoplus_{f_K(K) \in \mathcal{K}} f_K(K)$ where each $f_K(K) \in \mathcal{K}$ has the subspace topology inherited from X. Then Y is a locally compact space. Let

$$q : Y \to X$$

be the map maps each $f_K(K)$ onto the corresponding compact subset $f_K(K) \subseteq X$ by the identity map. We claim that the quotient topology, τ_q, generated by this mapping coincides with the original topology, τ, on X. Clearly, $\tau \subseteq \tau_q$ since q is a continuous map. Let $U \in \tau_q$. Let $g : L \to X$ be continuous such that L is compact. Since $U \in \tau_q$, we have that $q^{-1}(U)$ is open in Y. Since $g(L)$ is open in Y, it follows that $q^{-1}(U) \cap g(L)$ is open in Y. But $q^{-1}(U) \cap g(L) = g^{-1}(U)$. Thus, $g^{-1}(U)$ is open in L and consequently, $U \in \tau$. $\square$

Remark 10.4.7. *Limits in CG need not coincide with limits in Top. Let $X, Y \in \mathsf{CG}$ such that $X = \mathbb{R} \setminus \{1, 1/2, 1/3, \dots\}$ with the subspace topology, and let $Y = \mathbb{R}/\mathbb{Z}$ with the quotient topology. $X \in \mathsf{CG}$ since X is a CW complex[2] and $Y \in \mathsf{CG}$ by Corollary 10.4.6. In fact, Y is also a CW complex since Y is an infinite bouqet of circles. However, $X \times Y$ is not compactly generated. Let*

$$A = \bigcup_{i=1}^{\infty} A_i = \bigcup_{i,j=1}^{\infty} \left\{ \left(\frac{1}{i} + \frac{a_i}{j}, i + \frac{0.5}{j} \right) \in X \times Y : j \in \mathbb{N} \right\}, \quad a_i = \left(\frac{1}{i} - \frac{1}{i+1} \right) 10^{-i}.$$

[2] Right?

The closure of A closure contains $(0,0)$. Hence, A is not closed. But for any compact subset $K \subseteq X \times Y$, the set $A \cap K$ has only finitely many points. This is because for fixed $i \in \mathbb{N}$, there are only finitely many $j \in \mathbb{N}$, and also there can be only finitely many i. Hence A k-closed. This shows that $X \times Y$ is not in CG*. See [Eng89] for details.*

Remark 10.4.8. *We have*

$$\mathsf{CW} \subsetneq \mathsf{CG} \subsetneq \mathsf{Top}$$

as inclusion of categories. The inclusions are in general strict. Indeed, the Hawaiian earring is in CG *since it is compact and hence locally compact. However, we have already seen that it admits no CW decomposition. For the inclusion* $\mathsf{CG} \subsetneq \mathsf{Top}$*, consider the example in Remark 10.4.7.*

We can now discuss the mapping spaces with the notion of a compactly generated space in place. We need to modify the definition given in the previous section a bit since we deal with compact Hausdorff spaces in this section.

Definition 10.4.9. Let $C_0(X,Y)$ be the set of continuous functions from X to Y with the compact-open topology that is generated by a subbasis formed by the sets of the form

$$B(u,K,U) = \{f : K \to Y \mid f(u(K)) \subseteq U,\ u : K \to X \text{ is cts. s.t } K \text{ is cpt. Hausdorff}\}$$

We define $C(X,Y) = kC_0(X,Y)$.

Remark 10.4.10. *If $X, Y \in$ CG, then $X \times Y$ might not be in* CG*. See Remark 10.4.7. In this case, we can consider $k(X \times Y)$. Below, we write $k(X \times Y)$ as $X \times_k Y$.*

Remark 10.4.11. *If $X \in$ CG and $Y \in$ Top is locally compact, it turns our that $X \times Y \in$ CG. Since $X \in$ CG, we have $X = Z/\sim$ such that Z is locally compact by Corollary 10.4.6. In other words, we have a quotient map $q : Z \to X$. Consider the map*

$$q \times \mathrm{Id}_Y : Z \times Y \to X \times Y$$

It is a standard fact that $q \times \mathrm{Id}_Y$ is a quotient map since Y is assumed to be locally compact. It is clear that

$$X \times Y = \frac{Z}{\sim} \times Y = \frac{Z \times Y}{\sim'},$$

where $(z,y) \sim' (z',y')$ if and only if $z \sim z'$. In other words, we have "$\sim' = \sim \times \mathrm{Id}$". Here we have implicitly used the fact that the bijection of sets

$$X \times Y \cong \frac{Z}{\sim} \times Y \cong \frac{Z \times Y}{\sim'}$$

is in fact a homeomorphism in Top *essentially because the product topology (left hand side) and the quotient topology (right hand side) are the same. Since $Z \times Y$ is locally compact, the claim follows from Corollary 10.4.6.*

Proposition 10.4.12. *Let $X, Y, Z \in$ CG.*

(1) *For $X, Y \in$ CG, $C(X, \cdot)$ is a covariant functor from* CG *to* Sets*. Similarly, $C(\cdot, Y)$ is a contravariant functor from* CG *to* Sets*.*
(2) *The evaluation map*

$$\mathrm{Ev}_{X,Y} : X \times C(X,Y) \to Y$$

and the injection map

$$i_{X,Y} : Y \to C(X \times_k C(X,Y))$$

are continuous.

(3) ***(Exponential Law)*** *The map*

$$\varphi : C(X \times_k Y, Z) \to C(X, C(Y, Z)),$$

as discussed in Proposition 10.2.6 is a homeomorphism.

(4) *The composition map*

$$\Phi_{X,Y,Z} : C(X, Y) \times_k C(Y, Z) \to C(X, Z)$$

is continuous.

Proof. The proof is given below:

(1) We prove the the covariant case. It suffices to check that $C_0(X, \cdot)$ is a covariant functor. We have to check that if $g : Y \to Z$ is a continuous map, then $g_* = C_0(X, Y) \to C_0(X, Z)$ is continuous. But we have

$$(g_*)^{-1} B(u, K, U) = B(u, K, g^{-1}(U))$$

The claim follows.

(2) It suffices to show that $Y \to C_0(X, X \times_k Y)$ or equivalently that $i^{-1}B(u, K, U)$ is open in Y. As $Y \in \mathsf{CG}$, it is equivalent to check that $v^{-1}i^{-1}B(u, K, U)$ is open in L for every test map $v : L \to Y$, where L is a compact Hausdorff space. Note that $u \times v : K \times_k L \to X \times_k Y$ is a test map, so $(u \times v)^{-1}(U)$ is open in $K \times_k L$. By the Tube Lemma, the set

$$\{b \in L : K \times \{b\} \subseteq (u \times v)^{-1}(U)\}$$

is open in L. It is easy to check that this set is the same as $v^{-1}\text{inj}^{-1}B(u, K, U)$, which completes the proof.

Consider an open set $U \subseteq Y$, and a map $u : K \to X \times_k C(X, Y)$. We show that $V = u^{-1}\text{Ev}^{-1}(U)$ is open in K. Let $v : K \to X$ and $w : K \to C(X, Y)$ be the two components of u, so

$$V = \{a \in K : w(a)(v(a)) \in U\}.$$

Suppose that $a \in V$. As $w(a) \circ v : K \to Y$ is continuous, we can choose a compact neighbourhood L of a in K such that $w(a)(v(L)) \subseteq U$. This means that $w(a) \in B(v, L, U) \subseteq C(X, Y)$. As $w : K \to C(X, Y)$ is continuous, the set $N = w^{-1}(B(v, L, U))$ is a neighbourhood of a in K. If $b \in N \cap L$, then $w(b)(v(b)) \in w(b)(v(L)) \subseteq U$, so $b \in V$. Thus, the neighbourhood $N \cap L$ of a is contained in V. This shows that V is open, as required.

(3) We first show that it is a bijection at the level of sets. If $f : X \to C(Y, Z)$ is continuous, then its image

$$X \times_k Y \xrightarrow{f \times \text{Id}} C(Y, Z) \times_k Y \xrightarrow{\text{Ev}_{Y,Z}} Z$$

is continuous. On the other hand, if $g : X \times_{\text{CG}} Y \to Z$ is continuous, then its image

$$X \xrightarrow{\text{inj}_{X,Y}} C(Y, X \times_k Y) \xrightarrow{\text{Ev}_{X,Y}} Y$$

is continuous. This shows that the exponential map is bijection. Moreover, if $W \in \mathbf{CG}$ we have bijections:

$$\begin{aligned} C(W, C(X, C(Y, Z))) &\cong C(W \times_k X, C(Y, Z)) \\ &\cong C(W \times_k X \times_k Y, Z) \\ &\cong C(W, C(X \times_k Y, Z)). \end{aligned}$$

This means that $C(X, C(Y, Z))$ and $C(X \times_k Y, Z)$ represent the same contravariant functor and the claim now follows by Yoneda's Lemma.

(4) The proof is similar to Proposition 10.2.10.

This completes the proof. $\square$

Thus, we have obtained a category CG that contains all locally compact Hausdorff spaces, CW-complexes, admits all limits and colimits, and is Cartesian closed.

10.4.2. Weakly Hausdorff Spaces. The category CG still contains some bad topological spaces, like the Sierpinski space. These do not satisfy the Hausdorff condition and we would like to exclude them by imposing a Hausdorff like condition.

Definition 10.4.13. Let $X \in \mathsf{Top}$. Then X is weakly Hausdorff (WH) if for every compact Hausdorff space K and every continuous map $u : K \to X$, the image $u(K) \subseteq X$ is closed in X.

Example 10.4.14. If X is a Hausdorff space, then X is weakly-Hausdorff since $u(K)$ is compact and thus closed in X. Every CW-complex is Hausdorff, hence in particular weakly Hausdorff.

Proposition 10.4.15. *Let X be a weakly Hausdorff topological space.*

(1) *Any finer topology on X is still weakly Hausdorff. In particular, kX is weakly Hausdorff.*
(2) *Any subspace of X is weakly Hausdorff.*

Proof. The proof is given below:

(1) Let x be the set X equipped with a topology containing the original topology, i.e., the identity function $\mathrm{Id} : x \to X$ is continuous. For any compact Hausdorff space K and continuous map $u : K \to x$, the composite $\mathrm{Id} \circ u : K \to X$ is continuous, and so its image $(\mathrm{Id} \circ u)(K) \subseteq X$ is closed in X. Thus, $u(K) = \mathrm{Id}^{-1}((\mathrm{Id} \circ u)(K))$ is closed in x.
(2) Let $i : A \hookrightarrow X$ be the inclusion of a subspace in X. For any compact Hausdorff space K and continuous map $u : K \to A$, the composite $i \circ u : K \to X$ is continuous, and so its image $(i \circ u)(K) \subseteq X$ is closed in X, and thus in A as well.

This completes the proof. $\square$

Let **CGWH** denote the full subcategory of **CG** consisting of compactly generated weakly Hausdorff spaces. We have

$$\mathsf{CW} \subsetneq \mathsf{CGWH} \subsetneq \mathsf{CG} \subsetneq \mathsf{Top}$$

as inclusion of categories. The inclusion $\mathsf{CGWH} \subsetneq \mathsf{CG}$ is strict since the Sierpinski space is in CG but not in CGWH. Similarly, the inclusion $\mathsf{CW} \subsetneq \mathsf{CGWH}$ is strict. Simply consider the Hawaiian earring.

Proposition 10.4.16. *Let $X \in \mathsf{CG}$. Then $X \in \mathsf{CGWH}$ if and only if the diagonal subspace $\Delta_X = \{(x, x) \mid x \in X\}$ is k-closed in $X \times_k X$.*

Proof. Suppose that X is weakly Hausdorff. First, observe that every one-point set $\{x\} \subseteq X$ is certainly a continuous image of a compact Hausdorff space and thus is closed in X, so X is T_1. Next, consider a test map $u = (v, w) : K \to X \times_k X$. It will be enough to show that the set $u^{-1}(\Delta_X) = \{a \in K : v(a) = w(a)\}$ is closed in K. Suppose that $a \notin u^{-1}(\Delta_X)$, so $v(a) \neq w(a)$. Then the set

$$U = \{b : v(b) \neq w(a)\}$$

is an open neighbourhood of a (because $\{w(a)\}$ is closed in X). Now K is compact Hausdorff and therefore regular, so there is an open neighbourhood V of a in K such that $\overline{V} \subseteq U$, or equivalently $w(a) \notin v(\overline{V})$. This means that a lies in the set

$$W = w^{-1}(v(\overline{V})^c).$$

The weak Hausdorff condition implies that $v(\overline{V})$ is closed in X, and thus W is open in K. We claim that $(V \cap W) \cap u^{-1}(\Delta_X) = \emptyset$. Indeed, if $b \in V \cap W$, then $v(b) \in v(\overline{V})$ but $w(b) \in v(\overline{V})^c$ by the definition of W, so $v(b) \neq w(b)$, which implies $u(b) = (v(b), w(b)) \notin \Delta_X$. This shows that $u^{-1}(\Delta_X)$ is closed in K, as required.

Conversely, suppose that Δ_X is k-closed in $X \times_k X$. Let $u : K \to X$ be a test map. Given any other test map $v : L \to X$, we define

$$M = \{(a, b) \in K \times L : u(a) = v(b)\} \subseteq K \times L.$$

This can also be described as $(u \times v)^{-1}(\Delta_X)$, so it is closed in $K \times L$ and thus compact. It follows that the projection $\pi_L(M)$ is compact and hence closed in L. However, it is easy to see that $\pi_L(M) = v^{-1}(u(K))$. This shows that $u(K)$ is k-closed in X, and hence closed. This means that X is weakly Hausdorff. $\square$

Remark 10.4.17. *Proposition 10.4.16 is an important characterization of weakly Hausdorff spaces. In* Top, *the criteria in Proposition 10.4.16 is exactly the characterization of a Hausdorff space. That is* $X \in \mathsf{Top}$ *is Hausdorff if and only if* $\Delta_X = \{(x, x) \mid x \in X\}$ *is closed in* $X \times X$. *Here* $X \times X$ *is the product in* Top.

We now have two additional functors. The first is the forgetful functor from CGWH to CG. Another is weak-Hausdorffification, dented as h, from CG to CGWH. We need to construct this functor h. We fist need some additional facts about wtaking quotients in CG.

Lemma 10.4.18. *Let* $X, Y \in \mathsf{CG}$ *and let* $\sim$ *be an equivalence relation on* X.

(1) *We have*
$$(X \times_k Y)/(\sim \times \mathrm{Id}) \cong (X/\sim) \times_k Y$$
(2) *Let q be the map*
$$q : X \times_k X \to (X/\sim) \times_k (X/\sim)$$
The set $q^{-1}(\Delta_{X/\sim}) \subseteq X \times_k X$ *is closed if and only if* $X/\sim\, \in \mathsf{CGWH}$.

Proof. The proof is given below:

(1) The standard map
$$f : X \times_k Y \to (X/\sim) \times_k Y$$
respects the relation $\sim \times \mathrm{Id}$ and thus factors as
$$\overline{f} : (X \times_k Y)/(\sim \times \mathrm{Id}) \to (X/\sim) \times_k Y$$
Let g_1 be the projection map.
$$g_1 : (X/\sim) \times_k Y \to (X \times_k Y)/(\sim \times \mathrm{Id})$$
Using the exponential law, we get a map
$$g_2 : X \to C\left(Y, (X \times_k Y)/(\sim \times \mathrm{Id})\right)$$
This respects $\sim$ on the level of sets, and so factors to give a map
$$\overline{g}_2 : X/\sim \to C\left(Y, (X \times_k Y)/(\sim \times \mathrm{Id})\right)$$
Using the exponential law again, we get a map
$$\overline{g} : X/\sim \times_k Y \to (X \times_k Y)/(\sim \times \mathrm{Id})$$
$\overline{f}$ and $\overline{g}$ are clearly inverses.

(2) By applying (1) twice, we have

$$(X \times_k X)/(\sim \times \sim) \cong (X/\sim) \times_k (X/\sim)$$

Thus, $\Delta_{X/\sim}$ is closed if and only if $q^{-1}(\Delta_{X/\sim})$ is closed if and only if $X/\sim$ is in CGWH.

This completes the proof. $\square$

Proposition 10.4.19. *There exists a functor $h : \mathsf{CG} \to \mathsf{CGWH}$ that is a left adjoint to the forgetful functor $\mathsf{CGWH} \to \mathsf{CG}$. That is,*

$$\mathrm{Hom}_{\mathsf{CG}}(h(X), Y) = \mathrm{Hom}_{\mathsf{CGWH}}(X, Y)$$

for all $X \in \mathsf{CG}$ and $Y \in \mathsf{CGWH}$.

Proof. We first construct h. For $X \in \mathsf{CG}$, consider the smallest equivalence relation on $X \times_k X$ that is closed. We can take the intersection of all closed equivalence relations. Then $hX := X/\sim \in \mathsf{CGWH}$ and there is a natural projection map $p\colon X \to X/\sim$. We now show that h is left-adjoint to the forgetful functor. It suffices to show that every $f : X \to Y$ factors through $X \to hX$. Since $Y \in \mathsf{CGWH}$ $\Delta_Y \subseteq Y \times_k Y$ is closed and hence $f^{-1}(\Delta_Y)$ is closed in $X \times_k Y$. This is an equivalence relation that contains $\sim$ since it is closed. Thus $X \to Y$ respects $\sim$ and factors through $X \to hX$. Moreover, h is a functor since for $f : X \to Y$ such that $X, Y \in \mathsf{CG}$, we have $X \to Y \to hY$ which in turn gives $hX \to hY$. $\square$

Remark 10.4.20. *The functors discussed are summarized in the diagram below:*

$$\begin{tikzcd}
\mathsf{Top} \arrow[r, bend left, "\mathrm{k}"] \arrow[r, bend right, "\text{Forgetful}"'] & \mathsf{CG} \arrow[r, bend left, "\mathrm{h}"] & \mathsf{CGWH} \arrow[l, bend left, "\text{Forgetful}"]
\end{tikzcd}$$

Corollary 10.4.21. *The following properties hold:*

(1) *Limits exist in CGWH and can simply be computed in CG. In fact, small colimits in CGWH can be computed in CG.*
(2) *h commutes with colimits. In particular, colimits in exist in CGWH and are obtained by applying h to the colimit in CG. In particular, the category CGWH is admits small colimits exist because CG admits admits small colimits.*
(3) *For $X \in \mathsf{CG}$ and $Y \in \mathsf{CGWH}$, $C(X, Y) \in \mathsf{CGWH}$. Hence, CGWH is Cartesian closed.*

Proof. (Sketch) The proof is given below:

(1) (Sketch) This is because an arbitrary product in CG of CGWH spaces is still WH, and so is an equalizer in CG of two maps.
(2) This follows since h is left adjoint to the forgetful functor.
(3) Define

$$\mathrm{Ev}_x : C(X, Y) \cong \{x\} \times C(X, Y) \hookrightarrow X \times C(X, Y) \xrightarrow{\mathrm{Ev}} Y$$

We have

$$\Delta_{C(X,Y)} = \bigcap_{x \in X} (\mathrm{Ev}_x \times \mathrm{Ev}_x)^{-1}(\Delta_Y)$$

This is closed. Hence, $C(X, Y) \in \mathsf{CGWH}$,

This completes the proof. $\square$

Thus, we have obtained a category CGWH that contains all locally compact Hausdorff spaces, CW-complexes, admits all limits and colimits, and is Cartesian closed.

Remark 10.4.22. *All results about* $\mathrm{Maps}(X, Y)$ *that hold under the hypothesis of locally compact and Hausdorff hold without any additional assumptions in* $\mathrm{C}(X, Y)$.

CHAPTER 11

Fibrations

Fibrations play a central role in algebraic topology by providing a framework to study spaces via continuous maps that locally resemble product spaces. They generalize the notion of fiber bundles and enable powerful tools for understanding the relationship between total spaces, base spaces, and fibers. The concept of a fibration allows the use of long exact sequences in homotopy and cohomology, as well as spectral sequences, to analyze complex topological structures. Establishing a suitable categorical setting ensures the technical foundations are robust and well-behaved, facilitating the development of the theory. From this point onward, we adopt the following conventions:

(1) All spaces and maps are assumed to lie in the category **CGWH** of compactly generated weak Hausdorff spaces.
(2) For simplicity, we denote **CGWH** by **Top**.
(3) The fibered product $X \times_k Y$ will be written simply as $X \times Y$.

These conventions enable the development of the theory of fibrations without additional technical restrictions. References include [Hat02; May99].

11.1. Fibrations

Fibrations play a fundamental role in homotopy theory. In a sense, fibrations can be thought of as 'homotopically nice projections,' a notion made precise below. We will introduce two types of fibrations - the Hurewicz fibrations and Serre fibrations - which are both obtained by imposing certain homotopy lifting properties. Prominent examples of fibrations are fiber covering spaces and fiber bundles, which are introduced in the next section. These fibrations provide powerful tools for understanding the relationships between the base and total spaces, and they allow us to analyze the homotopy type of complex spaces by studying simpler ones.

11.1.1. Definition & Examples.

Definition 11.1.1. Let $X, E \in \mathsf{Top}$. A continuous surjective map $p : E \to X$ satisfies the homotopy lifting for $A \in \mathsf{Top}$ if for any homotopy $H : A \times I \to X$ and map $f : A \times \{0\} \to E$, there exists a homotopy $\widetilde{H} : A \times I \to E$ such that the following diagram commutes:

$$\begin{tikzcd} A \times \{0\} \arrow[r, "H_0"] \arrow[d, "i_0"'] & E \arrow[d, "p"] \\ A \times I \arrow[r, "H"] \arrow[ur, dotted, "\widetilde{H}"] & X \end{tikzcd}$$

(1) We say a continuous surjective map $p : E \to X$ is a Hurewicz fibration if it satisfies the homotopy lifting property for any $A \in \mathsf{Top}$.
(2) We say a continuous surjective map $p : E \to X$ is a Serre fibration if it satisfies the homotopy lifting property for any $I^n \in \mathsf{Top}$ for each $n \geq 0$.

Remark 11.1.2. *Clearly, a Hurewicz fibration is a Serre fibration. It is clear that a fibration is a generalization of the notion of a covering space since covering spaces satisfy the homotopy lifting property.*

Remark 11.1.3. *A continuous surjective map $p : E \to X$ satisfies the homotopy lifting property for $A \in \mathsf{Top}$ if and only if the following diagram commutes:*

$$\begin{tikzcd} A \arrow[drr, bend left, "H"] \arrow[dr, dotted, "\widetilde{H}"] \arrow[ddr, bend right, "f"'] & & \\ & E^I \arrow[r, "p_*"] \arrow[d, "\mathrm{Ev}_0"] & X^I \arrow[d, "\mathrm{Ev}_0"'] \\ & E \arrow[r, "p"] & X \end{tikzcd}$$

Here Ev_0 *is the evaluation at* 0 *map and* X^I *denotes* $\mathrm{Maps}(I, X)$.

Remark 11.1.4. *If $p : E \to X$ is a fibration and $x \in X$, then $F_x := p^{-1}(x) \subseteq E$ is called the fiber of p over x. We write*

$$F_x \to E \to X$$

Example 11.1.5. Let's look at some examples of fibrations:

(1) For any $X \in \mathsf{Top}$, the unique map $X \to *$ is a Hurewicz fibration. This is clear.

(2) Any projection $p : X \times Y \to X$ is a Hurewicz fibration. For $A \in \mathsf{Top}$, let $H : A \times I \to X$ be a homotopy such that H_0 lifts to a map $A \to X \times Y$. We can define $\widetilde{H}$ by

$$\begin{aligned} \widetilde{H} : A \times I &\to X \times Y, \\ (a, t) &\mapsto (H(a, t), H(a, 0)). \end{aligned}$$

It is clear that $\widetilde{H}$ satisfies the definition.

(3) A homeomorphism $f : X \to Y$ is a Hurewicz fibration since we can simply define $\widetilde{H} = f^{-1} \circ H$.

(4) Consider the evaluation map

$$\begin{aligned} \mathrm{Ev}_{0,1} : \mathrm{Maps}(I, X) &\to X \times X \\ \gamma &\mapsto (\gamma(0), \gamma(1)) \end{aligned}$$

We show that $\mathrm{Ev}_{0,1}$ is a Hurewicz fibration. Consider the diagram:

$$\begin{tikzcd} A \cong A \times \{0\} \arrow[r, "H_0"] \arrow[d, "i_0"] & \mathrm{Maps}(I, X) \arrow[d, "\mathrm{Ev}_{0,1}"] \\ A \times I \arrow[r, "H"] \arrow[ur, dotted, "\widetilde{H}"] & X \times X \end{tikzcd}$$

Equivalently, we are given a continuous map

$$\varphi : (A \times \{0\} \times I) \cup (A \times I \times \{0, 1\}) \to X$$

which we wish to extend to $A \times I \times I$. But

$$(\{0\} \times I) \cup (I \times \{0, 1\}) := J_1 \subseteq I^2$$

is a retract of I^2. The argument is similar to Example 2.1.16. Hence so is $A \times J_1$ of $A \times I \times I$. Therefore, we can simply pre-compose φ with the retraction

$$r : A \times I \times I \to A \times J_1$$

to find the required extension.

Proposition II.1.6. *The following statements are true:*

(1) *The composition of Hurewicz fibrations is a Hurewicz fibration.*
(2) *The product of Hurewicz fibrations is a Hurewicz fibration.*
(3) *The pullback of a Hurewicz fibration is a Hurewicz fibration.*
(4) ***(Universal Test Space)*** *Let $p : E \to X$ be a continuous surjective map and let N_p be the following pullback:*

$$\begin{tikzcd} N_p \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & X^I \arrow[d, "\mathrm{Ev}_0"] \\ E \arrow[r, "p"] & X \end{tikzcd}$$

Here

$$N_p := E \times_X X^I = \{(e, \gamma) \in E \times X^I \mid p(e) = \gamma(0)\}.$$

If p satisfies the homotopy lifting property for $N_p \in \mathsf{Top}$, then $p : E \to X$ is a Hurewicz fibration.

PROOF. The proof is given below:

(1) Let $p_1 : E_1 \to X$ and $p_2 : E_2 \to E_1$ be fibrations and consider the following diagram:

$$\begin{tikzcd} A \arrow[rrrd, bend left, "H"] \arrow[rd, dotted, "\widetilde{\widetilde{H}}"'] \arrow[rrd, dotted, "\widetilde{H}"] \arrow[rdd, bend right, "f"'] & & & \\ & E_2^I \arrow[r, "(p_2)_*"'] \arrow[d, "\mathrm{Ev}_0"] & E_1^I \arrow[r, "(p_1)_*"'] \arrow[d, "\mathrm{Ev}_0"] & X^I \arrow[d, "\mathrm{Ev}_0"'] \\ & E_2 \arrow[r, "p_2"] & E_1 \arrow[r, "p_1"] & X \end{tikzcd}$$

Since p_1 is a Hurewicz fibration, $\widetilde{H}$ exists to make the right-hand side of the diagram commute.

Since p_2 is a Hurewicz fibration, $\widetilde{\widetilde{H}}$ exists to make the left-hand side of the diagram commute. The claim follows.

(2) This is clear. The same argument as in covering space theory applies here.
(3) Let $p : E \to X$ be a Hurewicz fibration and consider a pullback square:

$$\begin{tikzcd} E' \arrow[r] \arrow[d, "q"] & E \arrow[d, "p"'] \\ X' \arrow[r] & X \end{tikzcd}$$

Consider the diagram below:

$$\begin{tikzcd} A \times \{0\} \arrow[r, "f"] \arrow[d, "i_0"'] & E' \arrow[r] \arrow[d, dashed, "q"] & E \arrow[d, "p"'] \\ A \times I \arrow[r, "H"] \arrow[ru, dotted, "\widetilde{H}"] \arrow[rru, dotted] & X' \arrow[r] & X \end{tikzcd}$$

The unmarked dotted arrow from $A \times I$ to E can be completed since p is a Hurewicz fibration. The fact that the square is a pullback square then implies the existence of H.

(4) Consider the following diagram:

$$\begin{tikzcd}
A \arrow[dr, dotted, "q"] \arrow[rrrdd, bend left, "H"] \arrow[dddrr, bend right, "f"'] & & & \\
& N_p \arrow[dr, dotted, "s"] \arrow[drr, bend left, "\pi_2"] \arrow[ddr, bend right, "\pi_1"] & & \\
& & E^I \arrow[r, "p_*"'] \arrow[d, "\mathrm{Ev}_0"] & X^I \arrow[d, "\mathrm{Ev}_0"'] \\
& & E \arrow[r, "p"] & X
\end{tikzcd}$$

Suppose p satisfies the homotopy lifting property for N_p. Then the map s exists as in the diagram. By the universal property of pullbacks, there exists a map $q : Y \to N_p$ such that the diagram commutes. Then $s \circ q :\to E^I$ solves the problem.

This completes the proof. $\square$

Remark 11.1.7. *We give N_p the subspace topology with respect to the compact open topology. We say that N_p is a universal test space $p : E \to X$.*

11.1.2. Mapping Path Space. N_p is an instance of the construction of a mapping path space which we now describe.

Definition 11.1.8. Let $f : X \to Y$ be a continuous map. The mapping math space is the topological space

$$N_f = X \times_Y Y^I = \{(x, \gamma) \in X \times Y^I \mid f(x) = \gamma(0)\}.$$

We give N_f the subspace topology with respect to the compact open topology.

Note that N_f is defined as a pullback:

$$\begin{tikzcd}
N_f \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & Y^I \arrow[d, "\mathrm{Ev}_0"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

We can now use the mapping path space construction in Proposition 11.1.6(4) to argue that any continuous map $f : X \to Y$ can be decomposed as a composition of a homotopy equivalence and a Hurewicz fibration.

Proposition 11.1.9. *Let $f : X \to Y$ be a continuous map. Then f can be decomposed as*

$$\begin{tikzcd}
X \arrow[r, "i"'] \arrow[rr, bend left, "f"] & N_f \arrow[r, "p"'] & Y
\end{tikzcd}$$

where i is a homotopy equivalence and p is a Hurewicz fibration.

Proof. We have $X \subseteq N_f$ via mapping $x \mapsto (x, c_{f(x)})$, where $c_{f(x)}$ is the constant path based at the image of x under f. Call this map i as in the diagram above. Define

$$\begin{aligned} p \colon N_f &\to Y \\ (x, \gamma) &\mapsto \gamma(1) \end{aligned}$$

Clearly, $f = p \circ i$. We first show that i is a homotopy equivalence. Let $\pi_1 : N_f \to X$ be the projection onto X. Then $\pi_1 \circ i = \mathrm{Id}_X$ and we have a homotopy

$$H : N_f \times I \to N_f$$
$$((x, \gamma), t) \mapsto (x, s \mapsto \gamma((1-t)s))$$

from $i \circ \pi_1$ to Id_{N_f}. We now check that p is a Hurewicz fibration. Consider the following diagram:

$$\begin{tikzcd} A \times \{0\} \arrow[r, "H_0"] \arrow[d, "i_0"] & N_f \arrow[d, "p"] \\ A \times I \arrow[r, "H"] & Y \end{tikzcd}$$

First note that we have the following commutative diagram:

$$\begin{tikzcd} A \cong A \times \{0\} \arrow[r, "H_0"] \arrow[d, "i_0"] \arrow[drr, "\mathrm{Id}"] & N_f \arrow[r, "\pi_1"] & X \\ A \times I \arrow[rr, "\pi_A"] & & A \cong A \times \{0\} \arrow[u, "\pi_1 \circ H_0"'] \end{tikzcd}$$

If we write $H_0(a) = (I(a), J(a))$, then $\pi_A \circ (\pi \circ H_0)(a, t) = I(a)$. Hence, we identify $\pi_A \circ (\pi_1 \circ H_0)$ with I. Moreover, using Example II.1.5(4), we have the following commutative diagram:

$$\begin{tikzcd} A \times \{0\} \arrow[r, "H_0"] \arrow[d, "i_0"] & N_f \arrow[r, "\pi_2"] & Y^I \arrow[d, "\mathrm{Ev}_{0,1}"] \\ A \times I \arrow[urr, "K"] \arrow[rr, "(f \circ I, H)"] & & Y \times Y \end{tikzcd}$$

Hence, we can define $\widetilde{H}(a, t) = (I(a, t), K(a, t))$. The image of $\widetilde{H}$ is in N_f. This is because

$$f(I(a)) = \mathrm{Ev}_0(K(a, t))$$

Moreover, the intended diagram commutes since

$$(p \circ \tilde{H})(a, t) = K(a, 1) = \mathrm{Ev}_1 K(a, t) = H(a, t),$$
$$\tilde{H} \circ i_0(a) = \tilde{H}(a, 0) = H_0(a).$$

This completes the proof. $\square$

Motivated by Proposition II.1.9 we can make the following definition of the homotopy fiber of any arbitrary continuous map $f : X \to Y$.

Definition II.1.10. Let $f : X \to Y$ be a continuous map. Let $p : N_f \to Y$ denote the map as in Proposition II.1.9. The homotopy fiber of f over $y_0 \in Y$ is

$$\mathrm{hFiber}_f(y_0) := p^{-1}(f) = \{(x, \gamma) \mid \gamma(0) = f(x), \gamma(1) = y_0\}$$

Remark II.1.11. *For each $y_0 \in Y$, note that there is a canonical map from the fiber of f over x to the homotopy fiber of f over x:*

$$f^{-1}(y_0) \to \mathrm{hFiber}_f(y_0)$$
$$x \mapsto (x, c_{f(x)})$$

Thus, the fiber sits in the homotopy fiber while the homotopy fiber can be thought of as a 'relaxed' version of the fiber: a point of the homotopy fiber is a pair (x,γ) consisting of $x \in X$ together with a path γ in Y 'witnessing' that x 'lies in the fiber up to homotopy.'

Remark II.I.12. *Let $f : X \to Y$ be a fibration. We check that the canonical map*

$$f^{-1}(y_0) \to \mathrm{hFiber}_f(y_0)$$

is a homotopy equivalence in this case. Define a homotopy

$$\begin{aligned} H : N_f \times I &\to Y \\ ((x,\gamma),t) &\mapsto \gamma(t) \end{aligned}$$

Note that $H_0(x,\gamma) = \gamma(0) = f(x)$, and H_0 lifts through f by $\bar{H}_0 : N_f \to X$, $\bar{H}_0(x,\gamma) = x$. That is, $f \circ \bar{H}_0 = H_0$. Because $X \to Y$ is a fibration, there is a full lift

$$\bar{H} : N_f \times I \to X$$

of H through f. In other words, $\bar{H}_t$ satisfies the following equation:

$$f(\bar{H}_t(x,\gamma)) = \gamma(t)$$

Now restrict everything to the fibers. Let

$$\begin{aligned} h_t : \mathrm{hFiber}_f(y_0) &\to \mathrm{hFiber}_f(y_0) \\ (x,\gamma) &\mapsto \left(\bar{H}_t(x,\gamma), \gamma_{|[t,1]}\right) \end{aligned}$$

Then h_0 is the identity, whereas $h_1(x,\gamma) = (\bar{H}_1(x,\gamma), c_{y_0})$ is in the image of $i : f^{-1}(y_0) \to \mathrm{hFiber}_f(y_0)$. Now that h_t is a homotopy between $i \circ h_1$ and the identity, while the restriction of h_t is a homotopy between $h_1 \circ i$ and the identity. This verifies the assertion.

II.I.3. Fiber Homotopy Equivalence. It is important to study fibrations over a given base space $X \in \mathsf{Top}$, working in the category of spaces over X which we denote as Top_X. An object in Top_X is a continuous map $p : E \to X$. Moreover, a morphism in Top_X is a commutative diagram

$$\begin{tikzcd} E_1 \arrow[rr, "f"] \arrow[dr, "p_1"'] & & E_2 \arrow[dl, "p_2"] \\ & X & \end{tikzcd}$$

A homotopy in Top_X is commutative diagram

$$\begin{tikzcd} E_1 \times I \arrow[rr, "H"] \arrow[dr, "p_1"'] & & E_2 \arrow[dl, "p_2"] \\ & X & \end{tikzcd}$$

such that for all $t \in I$, we have the following commutative diagram:

$$\begin{tikzcd} E_1 \times \{t\} \arrow[rr, "H|_{E_1 \times \{t\}}"] \arrow[dr, "p_1|_{E_1 \times \{t\}}"'] & & E_2 \arrow[dl, "p_2"] \\ & X & \end{tikzcd}$$

Definition 11.1.13. Let $X \in \mathsf{Top}$ and $E_1, E_2 \in \mathsf{Top}_X$. An object $f : E_1 \to E_2$ in Top_X is homotopy equivalent if there exists another object $g : E_2 \to E_1$ in Top_X such that

$$g \circ f \sim \mathrm{Id}_{E_1}, \qquad f \circ g \sim \mathrm{Id}_{E_2},$$

in Top_X. The maps f and g are called fibre homotopy equivalences.

The following result will be useful later on:

Proposition 11.1.14. *Let $X \in \mathsf{Top}$. Let $p_1 : E_1 \to X$ and $p_2 : E_2 \to X$ be fibrations in Top_X. Let $f : E_1 \to E_2$ be a map such that $p_2 \circ f = p_1$. Suppose that f is a homotopy equivalence in Top. Then f is a fiber homotopy equivalence in Top_X.*

Proof. The proof is skipped. $\square$

11.1.4. Characterization of Fibrations.

We end with a criterion that allows us to recognize Hurewicz fibrations. The criterion will also allow us to deduce that covering spaces and fiber bundles over nice spaces are Hurewicz fibrations.

Definition 11.1.15. Let $\mathcal{U}$ be an open cover of $X \in \mathsf{Top}$. We say that $\mathcal{U}$ is numerable if there are maps $\lambda_U : X \to I$ for each $U \in \mathcal{U}$ such that $\lambda_U^{-1}((0,1]) = U$.

Proposition 11.1.16. *Let $p : E \to X$ be a continuous surjective map and let $\mathcal{U}$ be a locally finite numerable open cover of X. Then p is a Hurewicz fibration if and only if $p|_U : p^{-1}(U) \to U$ is a Hurewicz fibration for every $U \in \mathcal{U}$.*

Proof. The proof can be found in [May99]. We will see in Proposition 11.2.4 that fiber bundles are Serre fibrations. This suffices for most purposes. $\square$

11.2. Fibre & Principal Bundles

In this section, we discuss fiber bundles providing the key definitions required to introduce some important examples of interest. Our primary interest in fiber bundles arises from the fact that important examples of Serre fibrations are given by fiber bundles.

11.2.1. Definitions.

Definition 11.2.1. Let $E, X, F \in \mathsf{Top}$. A continuous surjective map $p : E \to X$ is a F-fibre bundle if it satisfies the following conditions:

(1) There is an open cover $\{U_\alpha\}_\alpha$
(2) There are homeoomorphisms $\varphi_\alpha : p^{-1}(U_\alpha) \to U_\alpha \times F$ such that the following diagram commutes:

$$\begin{tikzcd} p^{-1}(U_\alpha) \arrow[r, "\varphi_\alpha"] \arrow[d, "p"'] & U_\alpha \times F \arrow[dl, "\mathrm{pr}_1"] \\ U_\alpha & \end{tikzcd}$$

Remark 11.2.2. *If $p : E \to X$ is a continuous surjective map, we will henceforth use the term fibre bundle to refer to a F-fiber bundle when the fiber F is clear from context.*

Example 11.2.3. Here is a basic list of examples of fibre bundles:

(1) A trivial fibre bundle is of the form $F \times X$ with fibre X. It is clear that this is a fibre bundle since $p : F \times X \to X$ is a continuous surjective map and the following diagram commutes:

$$\begin{tikzcd} X \times F \arrow[r, "\mathrm{Id}"] \arrow[d, "p"'] & X \times F \arrow[dl, "\mathrm{pr}_1"] \\ X & \end{tikzcd}$$

(2) Let $p : E \to X$ be a covering space with a discrete fibres, F. Then $p : E \to X$ is a fibre bundle with fibre F.
(3) Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$. Let $p : E \to X$ be a rank n $\mathbb{K}$-vector bundle. Then $p : E \to X$ is a fibre bundle with fibre $\mathbb{K}^n$.

Why are we interested in fiber bundles in homotopy theory? We demonstrate that a fiber bundle is a Serre fibration.

Proposition 11.2.4. *Let $p : E \to X$ be a fibre bundle. Then $p : E \to X$ is a Serre fibration.*

Proof. Consider the following diagram:

$$\begin{tikzcd} I^n \times \{0\} \arrow[r, "H_0"] \arrow[d, "i_0"] & E \arrow[d, "p"'] \\ I^n \times I \arrow[r, "H"] & X \end{tikzcd}$$

Since $\pi : E \to B$ is a fiber bundle, there exists an open covering by subsets U_α such that $p^{-1}(U_\alpha) \cong U_\alpha \times F$ (over U_α). We can cover I^{n+1} by the open subsets $H^{-1}(U_\alpha)$. Since I^{n+1} is compact, the Lebesgue number lemma implies there exists a $k \in \mathbb{N}$ such that, for any sequence $(j_1, \ldots, j_n)$ of numbers $0 \leq j_1, \ldots, j_n \leq k-1$, the small cube

$$\left[\frac{j_1}{k}, \frac{j_1+1}{k}\right] \times \cdots \times \left[\frac{j_n}{k}, \frac{j_n+1}{k}\right]$$

is mapped by H into an open set $U_\alpha \subseteq X$. We construct the lift H incrementally, one cube at a time. Thus, we may assume that no further subdivision of I^{n+1} is necessary and that H maps I^{n+1} entirely into some U_α. Moreover, we are given H_0 defined on $I^n \times \{0\}$ that can be extended onto $\partial I^n \times I$. Consequently, we can also assume that p is the trivial fiber bundle $U_\alpha \times F$. Thus, we can construct such a lift as

$$\widetilde{H} : I^{n+1} \to U_\alpha \times F; \quad (x_1, \ldots, x_{n+1}) \mapsto \big(H(x_1, \ldots, x_{n+1}), f(x_1, \ldots, x_n)\big),$$

where f is the composition $I^{n+1} \to I^n \times \{0\} \cup \partial I^n \times I \to I^n \times \{0\} \to U_\alpha \times F \to F$. Here the first map is a deformation retraction. $\square$

Remark 11.2.5. *Proposition 11.1.16 implies that fibre bundles with paracompact base spaces are Hurewicz fibrations. Recall that a paracompact space is a topological space in which every open cover has an open refinement that is locally finite. Moreover, every open cover on a paracompact space can be shown to be numerable by working with the existence of bump functions guaranteed to exist by Urysohn's lemma.*

Let us consider some examples. We construct examples of fiber bundles (and hence Serre fibrations) via group actions. In fact, the examples we construct will be principal bundles, which are specific instances of fiber bundles. To proceed, we first define the notion of a principal bundle.

Definition 11.2.6. Let $p : E \to X$ be a fibre bundle with a topological group, G, as its fibre. Then $p : E \to X$ is a principal G-bundle if the following hold:

(1) There is a continuous, free group action $E \times G \to E$,
(2) For each $x \in X$, the action of G preserves the fibre E_x and the orbit map $G \to E_x$ is a homeomorphism,
(3) The locally trivalizing cover $\{U_\alpha, \varphi_\alpha\}_\alpha$ is such that each φ is G-equivariant. That is,

$$\varphi_\alpha(e \cdot g) = \varphi_\alpha(e) \cdot g$$

The group G is called the structure group of the principal G-bundle.

The examples of principal G-bundles we construct will be derived from the category of smooth manifolds. Consequently, the remainder of this section is adapted for the category of smooth manifolds. We will use the following important result:

Proposition 11.2.7. *Let G be a Lie group and M be a smooth manifold. A smooth, free, properly discontnuous action of G on M induces a smooth manifold structure on M/G such that the map $M \to M/G$ is principle G-bundle.*

PROOF. The proof is skipped. □

Example 11.2.8. (**Hopf Fibrations**) We discuss the all important example of Hopf fibrations over $k = \mathbb{R}, \mathbb{C}, \mathbb{H}$.

(1) Let $\mathbb{S}^n \subseteq \mathbb{R}^{n+1}$. Let $\mathbb{Z}_2$ acts on $\mathbb{S}^n$ via

$$\mathbb{S}^n \times \mathbb{Z}_2 \to \mathbb{S}^{2n+1}, \quad (w, \pm 1) \mapsto \pm w.$$

The action is free. Since $\mathbb{Z}_2$ is compact, the action is proper as well. Proposition 11.2.7 implies that the $\mathbb{S}^n/\mathbb{Z}_2$ is principal $\mathbb{Z}_2$-bundle. In fact, we have

$$\mathbb{Z}_2 \to \mathbb{S}^n \to \mathbb{S}^n/\mathbb{Z}_2 \cong \mathbb{RP}^n$$

is a principal $\mathbb{Z}_2$-bundle called the real Hopf bundle. By letting $n \to \infty$, we get:

$$\mathbb{Z}_2 \to \mathbb{S}^\infty \to \mathbb{RP}^\infty$$

(2) Let $\mathbb{S}^{2n+1} \subseteq \mathbb{C}^{n+1}$ be a sphere of odd dimension. Let $\mathbb{S}^1 \cong \mathrm{U}(1) \subseteq \mathbb{C}$ acts on $\mathbb{S}^{2n+1}$ via

$$\mathbb{S}^{2n+1} \times \mathbb{S}^1 \to \mathbb{S}^{2n+1}, \quad (w, z) \mapsto wz.$$

The action is free. Since $\mathbb{S}^1$ is compact, the action is proper as well. Proposition 11.2.7 implies that the $\mathbb{S}^{2n+1}/\mathrm{U}(1)$ is principal $\mathbb{S}^1$-bundle. Hence,

$$\mathbb{S}^1 \to \mathbb{S}^{2n+1} \to \mathbb{S}^{2n+1}/\mathrm{U}(1) \cong \mathbb{CP}^n$$

is a principal $\mathbb{S}^1$-bundle called the complex Hopf bundle. By letting $n \to \infty$, we get:

$$\mathbb{S}^1 \to \mathbb{S}^\infty \to \mathbb{CP}^\infty$$

(3) Let $\mathbb{S}^3 \subseteq \mathbb{H}$. and $\mathbb{S}^{4n+3} \subseteq \mathbb{H}^{n+1}$. An argument as in (2) shows that

$$\mathbb{S}^3 \to \mathbb{S}^{4n+3} \to \mathbb{S}^{4n+3}/\mathbb{S}^3 \cong \mathbb{HP}^n$$

is a principal $\mathbb{S}^3$-bundle called the quarternionic Hopf bundle. By letting $n \to \infty$, we get:

$$\mathbb{S}^3 \to \mathbb{S}^\infty \to \mathbb{HP}^\infty$$

Remark 11.2.9. *For $n = 1$, the Hopf fibrations reduce to:*

$$\mathbb{S}^0 \to \mathbb{S}^1 \to \mathbb{S}^1,$$
$$\mathbb{S}^1 \to \mathbb{S}^3 \to \mathbb{S}^2,$$
$$\mathbb{S}^3 \to \mathbb{S}^7 \to \mathbb{S}^4.$$

There is also an octonionic fibration:

$$\mathbb{S}^7 \to \mathbb{S}^{15} \to \mathbb{S}^8,$$

but there are no higher octonionic versions of the Hopf fibrations.

Let G be a Lie group and $H \subseteq G$ is a closed Lie subgroup. The natural of H on G by right multiplication is smooth, free and proper. Hence, Proposition 11.2.7 implies that G/H is a H-principal bundle.

Example 11.2.10. (Homoegenous Spaces) The following is a list of examples of homogenous space which are principal bundles.

(1) Consider $\mathrm{O}(n-1)$ a closed subgroup acting naturally on $\mathrm{O}(n)$. Note that

$$\mathrm{O}(n)/\,\mathrm{O}(n-1) \cong \mathbb{S}^{n-1}$$

This follows from the standard transitive action of $\mathrm{O}(n)$ on $\mathbb{S}^{n-1}$, the orbit-stabilizer theorem and the characteristic property of smooth submersions. Hence,

$$\mathrm{O}(n-1) \to \mathrm{O}(n) \to \mathbb{S}^{n-1}$$

is a principal $\mathrm{O}(n-1)$-bundle.

(2) Consider $\mathrm{SO}(n-1)$ a closed subgroup acting naturally on $\mathrm{SO}(n)$. Note that

$$\mathrm{SO}(n)/\,\mathrm{SO}(n-1) \cong \mathbb{S}^{n-1}$$

This follows from the standard transitive acton of $\mathrm{SO}(n)$ on $\mathbb{S}^{n-1}$, the orbit-stabilizer theorem and the characteristic property of smooth submersions. Hence,

$$\mathrm{SO}(n-1) \to \mathrm{SO}(n) \to \mathbb{S}^{n-1}$$

is a principal $\mathrm{SO}(n-1)$-bundle.

(3) Consider $\mathrm{U}(n-1)$ a closed subgroup acting naturally on $\mathrm{U}(n)$. Note that

$$\mathrm{U}(n)/\,\mathrm{U}(n-1) \cong \mathbb{S}^{2n-1}$$

This follows from the standard transitive acton of $\mathrm{U}(n)$ on $\mathbb{S}^{2n-1}$, the orbit-stabilizer theorem and the characteristic property of smooth submersions. Hence,

$$\mathrm{U}(n-1) \to \mathrm{U}(n) \to \mathbb{S}^{2n-1}$$

is a principal $\mathrm{U}(n-1)$-bundle.

We can generalize the above examples by introducing the notion of Stiefel manifolds.

Definition 11.2.11. Let $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$. A k-frame in $\mathbb{K}^n$ is an ordered orthonormal set of vectors $\{v_1, \ldots, v_k\} \subseteq \mathbb{K}^n$[1]. The set of all k-frames, $V_k(\mathbb{K}^n)$, is called the Stiefel manifold.

Remark 11.2.12. *It can be verified that $V_k(\mathbb{K}^n)$ is a compact smooth manifold. Note that we have the following identifications:*

(1) $V_1(\mathbb{R}^n) \cong \mathbb{S}^{n-1}$
(2) $V_1(\mathbb{C}^n) \cong \mathbb{S}^{2n-1}$
(3) $V_n(\mathbb{R}^n) \cong \mathrm{O}(n)$
(4) $V_n(\mathbb{C}^n) \cong \mathrm{U}(n)$

[1]Here we take the standard inner product.

Example 11.2.13. Consider $\mathrm{O}(n-k)$ a closed subgroup acting naturally on $\mathrm{O}(n)$. Note that

$$\mathrm{O}(n)/\,\mathrm{O}(n-k) \cong V_k(\mathbb{R}^n)$$

The group $\mathrm{O}(n)$ acts on the set $V_k(\mathbb{R}^n)$ via

$$A \cdot (v_1, \dots, v_k) = (Av_1, \dots, Av_k).$$

Since the vectors $v_1, \dots, v_k$ can be completed to form an orthonormal basis of $\mathbb{R}^n$, and $\mathrm{O}(n)$ acts transitively on orthonormal bases, it follows that the action of $\mathrm{O}(n)$ on $V_k(\mathbb{R}^n)$ is also transitive. The isotropy group of the point

$$p = (e_1, \dots, e_k) \in V_k(\mathbb{R}^n)$$

is given by

$$\mathrm{O}(n)_p = \left\{ \begin{pmatrix} E_k & 0 \\ 0 & A \end{pmatrix} \middle| \, A \in \mathrm{O}(n-k) \right\} \cong \mathrm{O}(n-k).$$

The characteristic property of smooth submersions now implies that

$$V_k(\mathbb{R}^n) = O(n)/O(n-k).$$

as smooth manifolds. The discussion of homogenous spaces implies that

$$\mathrm{O}(n-k) \to \mathrm{O}(n) \to V_k(\mathbb{R}^n)$$

is a principal $\mathrm{O}(n-k)$-bundle.

Remark 11.2.14. *Similarly, it can be shown that*

$$V_k(\mathbb{C}^n) = \mathrm{U}(n)/\,\mathrm{U}(n-k),$$
$$V_k(\mathbb{H}^n) = \mathrm{Sp}(n)/\,\mathrm{Sp}(n-k).$$

Hence, we have additional examples:

$$\mathrm{U}(n-k) \to \mathrm{U}(n) \to V_k(\mathbb{C}^n),$$
$$\mathrm{Sp}(n-k) \to \mathrm{Sp}(n) \to V_k(\mathbb{H}^n).$$

We can also define the notion of a Grassmannian that can be used to generated additional principal G-bundles.

Example 11.2.15. Let $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$. The set of all k-dimensional subspaces of $\mathbb{K}^n$, $G_k(\mathbb{K}^n)$, is called the Grassmannian.

There is a natural surjection

$$p : V_k(\mathbb{K}^n) \longrightarrow G_k(\mathbb{K}^n)$$
$$\{v_1, \dots, v_n\} \mapsto \mathrm{span}\{v_1, \dots, v_n\}.$$

The fact that p is onto follows from the Gram-Schmidt procedure. Thus, $G_k(\mathbb{K}^n)$ is a topological space endowed with the quotient topology via p.

Example 11.2.16. (Sketch) Note that $\mathrm{O}(k)$ acts on $V_k(\mathbb{R}^n)$ smoothly, freely and properly discontinuously. We have that

$$V_k(\mathbb{R}^n)/\,\mathrm{O}(k) \cong G_k(\mathbb{R}^n)$$

Hence,

$$\mathrm{O}(k) \to V_k(\mathbb{R}^n) \to G_k(\mathbb{R}^n)$$

is a principal $\mathrm{O}(k)$-bundle. If we let $n \to \infty$, we get:

$$\mathrm{O}(k) \to V_k(\mathbb{R}^\infty) \to G_k(\mathbb{R}^\infty)$$

Here $G_k(\mathbb{R}^\infty)$ is the infinite Grassmannian.

Remark 11.2.17. *Similarly, it can be shown that we have the following examples:*

$$\mathsf{U}(k) \to V_k(\mathbb{C}^n) \to G_k(\mathbb{C}^n)$$

If we let $n \to \infty$, we get:

$$\mathsf{U}(k) \to V_k(\mathbb{C}^\infty) \to G_k(\mathbb{C}^\infty)$$

11.3. Based Fibrations

Fibrations discussed above are called unbased fibrations. We can now define pointed fibrations.

Definition 11.3.1. Let $(E, e_0), (X, x_0) \in \mathsf{Top}_*$. A pointed in $p : (E, e_0) \to (X, x_0)$ is a based fibration if all the relevant maps in Definition 11.1.1 are pointed maps.

Remark 11.3.2. $F = p^{-1}(x_0)$ *is called the pointed fiber of p over x. We have a sequence*

$$F \xrightarrow{i} E \xrightarrow{p} X$$

Remark 11.3.3. *Let $(X, x_0), (Y, y_0) \in \mathsf{Top}_*$. If $f : (X, x_0) \to (Y, y_0)$ is a pointed map, we redefine N_f to be*

$$N_f = X \times_f Y^I : \{(x, \gamma) \in X \times Y^I \mid f(x) = \gamma(1)\}.$$

The proof in Proposition 11.1.9 goes through and f can be decomposed as:

$$\begin{tikzcd} (X, x_0) \arrow[r, "i"'] \arrow[rr, bend left, "f"] & N_f \arrow[r, "p"'] & (Y, y_0) \end{tikzcd}$$

Here i is a homotopy equivalence as defined in Proposition 11.1.9 and p is a (Hurewicz) fibration such that

$$p \colon N_f \to Y$$
$$(x, \gamma) \mapsto \gamma(0)$$

Example 11.3.4. Let $(X, x_0) \in \mathsf{Top}_*$ and let $f : \{x_0\} \hookrightarrow X$ denote the inclusion of a singleton. In this case, $N_f \cong P(X, x_0)$. Here N_f is as redefined in Remark 11.3.3. We have

$$\begin{tikzcd} \{x_0\} \arrow[r] \arrow[rr, bend left, "f"] & P(X, x_0) \arrow[r, "p"'] & (X, x_0) \end{tikzcd}$$

Here p is a fibration called the path space fibration. Note that we have $p^{-1}(x_0) = \Omega(X, x_0)$. Hence, we have the following sequence

$$\Omega(X, x_0) \to P(X, x_0) \to (X, x_0).$$

Remark 11.3.5. *We shall only use the phrase fibration when working with a based fibration.*

We would to generate a long exact sequence of homotopy groups associated to a continuous map. Here is the strategy. Let $f : X \to Y$ be a pointed continuous map of pointed topological spaces. Using Remark 11.3.3, we can decomopose f as:

$$\begin{tikzcd} X \arrow[r, "i"'] \arrow[rr, bend left, "f"] & N_f \arrow[r, "p'"'] & Y \end{tikzcd}$$

Consider the homotopy fiber:

$$\mathrm{hFiber}_f(y_0) := (p')^{-1}(y) = \{(x,\gamma) \in X \times P(Y,y_0) \mid \gamma(1) = f(x), \gamma(0) = y_0\}$$

For brevity, we write $\mathrm{hFiber}_f(y_0)$ and hFib_f. Note that hFib_f is a pullback:

$$\begin{tikzcd}
\mathrm{hFib}_f \arrow[r] \arrow[d] & P(Y,y_0) \arrow[d, "\mathrm{Ev}_1"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

Therefore, Proposition 11.1.6 implies that the projection $p : \mathrm{hFib}_f \to X$ is a fibration. We get a sequence

$$\mathrm{hFib}_f \xrightarrow{p} X \xrightarrow{f} Y$$

We would like to iterate this construction. Since $x \in X_0$ is the basepoint of X, note that the fiber of the fibration p is

$$p^{-1}(x_0) = \{x_0\} \times \{\gamma \in P(Y,y_0) \mid \gamma(1) = f(x_0) = y_0, \gamma(0) = y_0\} \cong \Omega(Y,y_0)$$

Hence, the usual fiber of p over x_0 can be identified with $\Omega(Y,y_0)$. As before, we have an inclusion of $p^{-1}(x_0)$ into the homotopy fiber hFib_p. Since p is a fibration, this inclusion is a homotopy equivalence by Remark 11.1.12. Hence, we have a sequence

$$\begin{tikzcd}
\Omega(Y,y_0) \arrow[r, "i"] \arrow[d, "\cong"] & \mathrm{hFib}_f \arrow[r, "p"] \arrow[d, equal] & X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[d, equal] \\
\mathrm{hFib}_p \arrow[r, "\mathrm{proj}"] & \mathrm{hFib}_f \arrow[r, "p"] & X \arrow[r, "f"] & Y
\end{tikzcd}$$

Here i is the inclusion mapping $\gamma \to (x_0,\gamma)$. The left most square commutes by construction. Hence, the diagram above commutes in Top_*. How shall we extend the sequence? The answer is given by the following result:

Lemma 11.3.6. *Let $(E,e_0),(X,x_0) \in \mathsf{Top}_*$ and let $f : (E,e_0) \to (X,x_0)$ be a pointed map. Let $F = p^{-1}(x_0)$ and*

$$F \xrightarrow{g} E \xrightarrow{f} X$$

be the associated fiber sequence. The homotopy fibre hFiber_g of g is homotopy equivalent to $\Omega(X,x_0)$.

Proof. Using Remark 11.3.3, decompose f as:

$$\begin{tikzcd}
(E,e_0) \arrow[rr, "i"] \arrow[dr, "f"'] & & N_f \arrow[dl, "p"] \\
& (X,x_0) &
\end{tikzcd}$$

Since i is a homotopy equivalence and f,p are fibrations, Proposition 11.1.14 implies that i is in fact a fiber homotopy equivalence. It follows that

$$i|_F : F \to \mathrm{hFiber}_f$$

is a homotopy equivalence. From the discussion above hFiber_f is defined as a pullback and the projection $\mathrm{hFiber}_f \to E$ is a fibration. We can furthermore decompose g as:

$$F \to N_g \to E$$

We know that N_g is also defined as a pullback, the map $F \to N_g$ is a homotopy equivalence and $N_g \to E$ is a fibration. The fiber of the fibration $N_g \to E$ is hFiber$_g$ and the fiber of the fibration hFiber$_f \to E$ is $\Omega(X, x_0)$. All in all, we have the following diagram:

$$\begin{tikzcd}
\mathrm{hFiber}_g \arrow[r, hook] \arrow[dd, dotted] & N_g \arrow[d, "\cong"] \arrow[dr, "\text{fib.}"] & \\
 & F \arrow[r, "g"] \arrow[d, "\cong"] & E \\
\Omega(X, x_0) \arrow[r, hook] & \mathrm{hFiber}_f \arrow[ur, "\text{fib.}"'] &
\end{tikzcd}$$

Since the map from $N_g \to$ hFiber$_f$, Proposition 11.1.14 implies that the map is in fact a fiber homotopy equivalence over E. In particular, the map restricts to a homotopy equivalnce between hFiber$_g$ and $\Omega(X, x_0)$. □

We can now use Lemma 11.3.6 to continue to construction of the sequence.

$$\begin{tikzcd}
\Omega(X, x_0) \arrow[r, "\Omega f"] \arrow[d, "j' \circ \mathrm{inv}"] & \Omega(Y, y_0) \arrow[r, "i"] \arrow[d, "j"] & \mathrm{hFib}_f \arrow[r, "p"] \arrow[d, equal] & X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[d, equal] \\
\mathrm{hFib}_i \arrow[r, "\mathrm{proj}'"] & \mathrm{hFib}_p \arrow[r, "\mathrm{proj}"] & \mathrm{hFib}_f \arrow[r, "p"] & X \arrow[r, "f"] & Y
\end{tikzcd}$$

Here j is the homotopy equivalence discussed above and j' that exists by Lemma 11.3.6. Moreover, inv is the map

$$\begin{aligned} \mathrm{inv} : \Omega(X, x_0) &\to \Omega(X, x_0), \\ \gamma &\mapsto \gamma^{-1}. \end{aligned}$$

Since hFib$_p \cong \Omega(Y, y_0)$, we identify proj$'$ to be simply the projection onto $\Omega(Y, y_0)$. We claim that the diagram above commutes in in hTop_*. The first and second squares are clearly commutative. The third square commutes as discussed above. It suffices to consider the left most square. Let

$$k = \mathrm{proj}' \circ (j' \circ \mathrm{inv})$$

We claim that $k \sim j \circ \Omega f$. Note that we have

$$\begin{aligned} k[\gamma] &= (c_{y_0}, [\gamma^{-1}]) \\ j \circ \Omega f[\gamma] &= ([f \circ \gamma], c_{x_0}) \end{aligned}$$

The desired homotopy is given by

$$H([\gamma], t) = (f(\gamma|_{[t,1]}]), [\gamma^{-1}|_{[0,t]}])$$

Iterating the above construction, we get the following sequence in hTop_*:

$$\begin{tikzcd}
\cdots \arrow[r] & \Omega(\mathrm{hFib}_f) \arrow[r, "\Omega p_1"] \arrow[d, "\cong"] & \Omega(X, x_0) \arrow[r, "\Omega f"] \arrow[d, "\cong"] & \Omega(Y, y_0) \arrow[r, "i"] \arrow[d, "\cong"] & \mathrm{hFib}_f \arrow[r, "p"] \arrow[d, equal] & X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[d, equal] \\
\cdots \arrow[r] & \mathrm{hFib}_{\Omega f} \arrow[r] & \mathrm{hFib}_i \arrow[r] & \mathrm{hFib}_p \arrow[r] & \mathrm{hFib}_f \arrow[r] & X \arrow[r] & Y
\end{tikzcd}$$

For each pair of adjacent maps, the first is the inclusion of the homotopy fibre of the next, up to homotopy equivalence. What now? For a fixed Y, w can take the homotopy classes of maps $[Y, -]_*$, where $\cdot$ is a space in the sequence above. We need the following lemma and a definition.

Definition 11.3.7. A sequence of functions of pointed sets

$$(A, a) \xrightarrow{f} (B, b) \xrightarrow{g} (C, c)$$

is exact if $f(A) = g^{-1}(c)$.

Lemma 11.3.8. *Let $(E, e_0), (X, x_0), (Z, z_0) \in \mathsf{Top}_*$. Let $p : (E, e_0) \to (X, x_0)$ be a fibration and let $F = p^{-1}(x_0)$. The sequence*

$$F \xrightarrow{i} E \xrightarrow{p} X$$

induces an exact sequence of sets[2]*:*

$$[Z, F]_* \xrightarrow{i_\#} [Z, E]_* \xrightarrow{p_\#} [Z, X]_*$$

Proof. Let $[g] \in [Z, F]_*$. Then

$$\begin{aligned} p_\# \circ i_\#([g]) : Z &\to X \\ y &\mapsto x_0 \end{aligned}$$

and so

$$i_\#([Z, F]_*) \subseteq p_\#^{-1}([c_{x_0}])$$

where c_{x_0} is the constant map $E \to x_0$. Now, let $[f] \in p_\#^{-1}([c_{x_0}])$. So $f : Z \to E$ is such that

$$p_\#([f]) = [p \circ f] = [c_{x_0}]$$

That is $p \circ f$ is homotopic to c_{x_0}. Let $G : Z \times I \to X$ be the corresponding homotopy. Now define $H : Z \times I \to E$ via the homotopy lifting property as in the following commutative diagram.

$$\begin{tikzcd} Z \times \{0\} \arrow[r, "f"] \arrow[d, "i_0"'] & E \arrow[d, "p"] \\ Z \times I \arrow[r, "G"] \arrow[ur, dotted, "H"] & X \end{tikzcd}$$

Then

$$p \circ H(z, 1) = G(z, 1) = c_{x_0}$$

Hence $H(Z, 1) \subseteq F$. So $z \mapsto H(z, 1)$ can be restricted to a map $f' : Z \to F$. But $H(z, 0) = f(z)$, so we have

$$f \cong i \circ f'$$

That is, $[f] = i_\#([f'])$ and so $[f] \in i_\#([Y, F])$. This completes the proof. □

Let's now use Lemma 11.3.8 to get the following result:

Proposition 11.3.9. ***(Exact Puppe Sequence)*** *Let $(X, x_0), (Y, y_0) \in \mathsf{Top}_*$ and let $f : (X, x_0) \to (Y, y_0)$ be a pointed continuous map. The sequence*

$$\cdots \longrightarrow \Omega(\mathrm{hFib}_f) \xrightarrow{\Omega p_1} \Omega(X, x_0) \xrightarrow{\Omega f} \Omega(Y, y_0) \xrightarrow{i} \mathrm{hFib}_f \xrightarrow{p} X \xrightarrow{f} Y$$

is exact.

---

[2]A sequence of functions of pointed sets $(A, a) \xrightarrow{f} (B, b) \xrightarrow{g} (C, c)$is exact if $f(A) = g^{-1}(c)$.

Proof. Let $(Z, z_0) \in \mathsf{Top}_*$. First consider

$$\mathrm{hFib} \xrightarrow{p} X \xrightarrow{f} Y$$

Instead consider the sequence

$$\mathrm{hFib}_f \xrightarrow{i} N_f \xrightarrow{p'} Y$$

Here the map p' is an honest fibration and hFib_f is the fibre of p'. For $Y \in \mathsf{Top}_*$, we can apply Lemma 11.3.6 to get an exact sequence of sets:

$$[Z, \mathrm{hFib}_f]_* \to [Z, N_f]_* \to [Z, Y]_*$$

However, note that $[Z, N_f]_* \cong [Z, X]_*$ since $P(Y, y_0)$ is contractible. Hence, we find that the sequence

$$\mathrm{hFib} \xrightarrow{p} X \xrightarrow{f} Y$$

is exact. Moreover,

$$\Omega^k(\mathrm{hFib}) \xrightarrow{p} \Omega^k(X, x_0) \xrightarrow{f} \Omega^k(Y, y_0)$$

is exact for each $k \geq 1$. This is because the sequence

$$[Z, \Omega^k(\mathrm{hFib})]_* \to [Z, \Omega^k(X, x_0)] \to [Z, \Omega^k(Y, y_0)]$$

can be written as

$$[\Sigma^k Z, \mathrm{hFib}]_* \to [\Sigma^k Z, (X, x_0)]_* \to [\Sigma^k Z, (Y, y_0)]_*$$

which is know to be exact. Hence, the given long exact sequence is an exact sequence. $\square$

CHAPTER 12

Cofibrations

CHAPTER 13

Higher Homotopy Groups

While the fundamental group captures information about loops in a space, higher homotopy groups generalize this idea to spheres of higher dimensions. These groups provide deeper insight into the global topological structure of a space and are crucial in distinguishing spaces that share the same fundamental group but differ in higher-dimensional features. As before, we adopt the following conventions from now on:

(1) We will assume that we work in the category **CGWH**.
(2) Abusing notation, we will write **CGWH** as **Top**.
(3) We will write $X \times_k Y$ simply as $X \times Y$.

References include [Hat02; May99].

13.1. Definitions

In this section, we generalize the definition of the first homotopy group.

Definition 13.1.1. Let $(X, x_0) \in \mathsf{Top}_*$ be a path-connected pointed topological space. The n-th homotopy group of (X, x_0), denoted as $\pi_n(X, x_0)$, is defined as

$$\pi_n(X, x_0) = [(\mathbb{S}^n, *), (X, x_0)] := [\mathbb{S}^n, X]_*$$

X is or n-connected if $\pi_k(X, x_0)$ for $1 \leq k \leq n$. We say that X is weakly contractible or ∞-connected if $\pi_k(X, x_0) = 0$ for all $k \in \mathbb{N}$.

Remark 13.1.2. *Note that*

$$(I^n/\partial I^n, \partial I^n/\partial I^n) \cong (\mathbb{S}^n, *)$$

Hence, we have the following commutative diagram:

$$\begin{tikzcd}
(I^n, \partial I^n) \arrow[r, "f"] \arrow[d, "q"'] & (X, x_0) \\
(I^n/\partial I^n, \partial I^n/\partial I^n) \arrow[ur, "g"] &
\end{tikzcd}$$

Equivalently, $\pi_n(X, x_0)$ consists of homotopy classes of maps $f : I^n \to X$ for which ∂I^n is mapped onto x_0. This is because the properties of the quotient topology imply that every f in the diagram above uniquely factors through a g in the same diagram.

Remark 13.1.3. *If $n = 0$, then $\pi_0(X, x_0)$ is the set of connected components of X. Indeed, we have $I^0 = \{*\}$ and $\partial I^0 = \emptyset$. Hence,*

$$\pi_0(X, x_0) = \{[y] \mid y \in X\}$$

Moreover, $[y] \sim [y']$ if and only if there is a path between y and y'. Hence, $\pi_0(X, x_0)$ consists of homotopy classes of maps from a point into the space X.

Proposition 13.1.4. *Let $(X, x_0) \in \mathsf{Top}_*$. Then there is a bijection of pointed sets:*

$$\pi_{n-1}(\Omega(X, x_0), c_{x_0}) \cong \pi_n(X, x_0)$$

for each $n \geq 1$. Here c_{x_0} is the constant loop at x_0.

Proof. We have:

$$\begin{aligned}
\pi_{n-1}(\Omega(X, x_0), c_{x_0}) &= \pi_0(\text{Maps}((\mathbb{S}^{n-1}, *), \Omega(X, x_0))) \\
&\cong \pi_0(\text{Maps}(\Sigma(\mathbb{S}^{n-1}, *), (X, x_0))) \\
&\cong \pi_0(\text{Maps}((\mathbb{S}^{n}, *), (X, x_0))) \\
&= \pi_n(X, x_0).
\end{aligned}$$

for each $n \geq 1$. □

Remark 13.1.5. *We can also make the following computation:*

$$\begin{aligned}
\pi_n(X, x_0) &\cong \pi_0(\text{Maps}((\mathbb{S}^{n}, *), (X, x_0))) \\
&\cong \pi_0(\text{Maps}((\mathbb{S}^{1}, *) \wedge (\mathbb{S}^{n-1}, *), (X, x_0))) \\
&\cong \pi_0(\text{Maps}((\mathbb{S}^{1}, *), \Omega'^{n-1}(X, x_0))) \\
&:= \pi_1(\Omega'^{n-1}(X, x_0)).
\end{aligned}$$

for each $n \geq 1$. Here we denote

$$\Omega'^{n-1}(X, x_0) := [(\mathbb{S}^{n-1}, *), (X, x_0)]_*$$

Note that we have

$$\begin{aligned}
(\Omega^{n-1}(X, x_0), c_{x_0}) &\cong [(\mathbb{S}^{0}, *), \Omega^{n-1}(X, x_0)] \\
&\cong [\Sigma^{n-1}(\mathbb{S}^{0}, *), (X, x_0)] \\
&\cong [(\mathbb{S}^{n-1}, *), (X, x_0)] \\
&:= \Omega'^{n-1}(X, x_0).
\end{aligned}$$

This identification follows since $\Omega^{n-1}(X, x_0)$ is a pointed topological space with the constant loop as the base point. Hence, we have

$$\pi_n(X, x_0) = \pi_1(\Omega^{n-1}(X, x_0), c_{x_0})$$

for $n \geq 1$.

Proposition 13.1.6. *Let $(X, x_0) \in \mathsf{Top}_*$. The set $\pi_n(X, x_0)$ forms a group for $n \geq 2$.*

Proof. This follows immediately from the fact that $\pi_n(X, x_0) = \pi_1(\Omega^{n-1}(X, x_0), c_{x_0})$ and $\pi_1(\cdot)$ is a group. We can also give a more direct argument. Consider the following map:

$$(f+g)(s_1, s_2, \ldots, s_n) = \begin{cases} f(2s_1, s_2, \ldots, s_n) & \text{if } 0 \leq s_1 \leq \frac{1}{2} \\ g(2s_1 - 1, s_2, \ldots, s_n) & \text{if } \frac{1}{2} \leq s_1 \leq 1 \end{cases}$$

Note that since only the first coordinate is involved in this operation, the same argument used to prove that $\pi_1(X, x_0)$ is a group is valid here as well. In particular, the identity element is the constant map taking all of I^n to x_0 and the inverse element is given by

$$-f(s_1, s_2, \ldots, s_n) = f(1 - s_1, s_2, \ldots, s_n).$$

This completes the proof. □

The additive notation for the group operation is used because $\pi_n(X, x_0)$ is abelian for $n \geq 2$.

Lemma 13.1.7. ***(Eckmann–Hilton Argument)*** *Let X be a set equipped with two binary operations, $\circ$ and $\otimes$, such that:*

- *$\circ$ and $\otimes$ are both unital. That is, there are identity elements $1_\circ$ and $1_\otimes$ such that*
$$1_\circ \circ a = a = a \circ 1_\circ$$
$$1_\otimes \otimes a = a = a \otimes 1_\otimes$$
for all $a \in X$.
- *We have,*
$$(a \otimes b) \circ (c \otimes d) = (a \circ c) \otimes (b \circ d),$$
for all $a, b, c, d \in X$.

Then $\circ$ and $\otimes$ are the same and in fact commutative and associative.

Proof. Observe that the units of the two operations coincide:
$$1_\circ = 1_\circ \circ 1_\circ = (1_\otimes \otimes 1_\circ) \circ (1_\circ \otimes 1_\otimes) = (1_\otimes \circ 1_\circ) \otimes (1_\circ \circ 1_\otimes) = 1_\otimes \otimes 1_\otimes = 1_\otimes.$$
We denote the common identity as 1. Now, let $a, b \in X$. Then
$$a \circ b = (1 \otimes a) \circ (b \otimes 1) = (1 \circ b) \otimes (a \circ 1) = b \otimes a = (b \circ 1) \otimes (1 \circ a) = (b \otimes 1) \circ (1 \otimes a) = b \circ a.$$
This establishes that the two operations coincide and are commutative. For associativity,
$$(a \otimes b) \otimes c = (a \otimes b) \otimes (1 \otimes c) = (a \otimes 1) \otimes (b \otimes c) = a \otimes (b \otimes c).$$
This completes the proof. ∎

Proposition 13.1.8. *Let $(X, x_0) \in \mathsf{Top}_*$. If $n \geq 2$, $\pi_n(X, x_0)$ is an abelian group.*

Proof. We use Lemma 13.1.7. To use Lemma 13.1.7, we define an alternative binary operation. Let $[f], [g] \in \pi_n(X, x_0)$. Then define $[f] \times [g]$ to be the homotopy class of the map $f \times g$ defined by
$$(f \times g)(t_1, \ldots, t_n) = \begin{cases} f(t_1, 2t_2, t_3, \ldots, t_n) & \text{if } t_2 \in [0, 1/2], \\ g(t_1, 2t_2 - 1, t_3, \ldots, t_n) & \text{if } t_2 \in [1/2, 1]. \end{cases}$$
It is clear that $\times$ is a well-defined operation on $\pi_n(X, x_0)$. Moreover, $\times$ is a unital operation with the identity element given by the constant map taking all of I^n to x_0. To make use of Lemma 13.1.7, it remains to prove that for any $[f], [g], [h], [k] \in \pi_n(X, x_0)$,
$$([f] \times [g]) + ([h] \times [k]) = ([f] + [h]) \times ([g] + [k]).$$
The left-hand side is defined to be the homotopy class of
$$(([f] \times [g]) + ([h] \times [k]))(t_1, \ldots, t_n) = \begin{cases} f(2t_1, 2t_2, t_3, \ldots, t_n) & \text{if } t_1 \leq \frac{1}{2},\ t_2 \leq \frac{1}{2}, \\ g(2t_1, 2t_2 - 1, t_3, \ldots, t_n) & \text{if } t_1 \leq \frac{1}{2},\ t_2 \geq \frac{1}{2}, \\ h(2t_1 - 1, 2t_2, t_3, \ldots, t_n) & \text{if } t_1 \geq \frac{1}{2},\ t_2 \leq \frac{1}{2}, \\ k(2t_1 - 1, 2t_2 - 1, t_3, \ldots, t_n) & \text{if } t_1 \geq \frac{1}{2},\ t_2 \geq \frac{1}{2}. \end{cases}$$
The right-hand side is the homotopy class of
$$(([f] + [h]) \times ([g] + [k]))(t_1, \ldots, t_n) = \begin{cases} f(2t_1, 2t_2, t_3, \ldots, t_n) & \text{if } t_1 \leq \frac{1}{2},\ t_2 \leq \frac{1}{2}, \\ h(2t_1 - 1, 2t_2, t_3, \ldots, t_n) & \text{if } t_1 \geq \frac{1}{2},\ t_2 \leq \frac{1}{2}, \\ g(2t_1, 2t_2 - 1, t_3, \ldots, t_n) & \text{if } t_1 \leq \frac{1}{2},\ t_2 \geq \frac{1}{2}, \\ k(2t_1 - 1, 2t_2 - 1, t_3, \ldots, t_n) & \text{if } t_1 \geq \frac{1}{2},\ t_2 \geq \frac{1}{2}. \end{cases}$$

Both these maps are exactly the same map. By Lemma 13.1.7, $+$ is commutative, so $\pi_n(X, x_0)$ is abelian for $n \geq 2$. □

Remark 13.1.9. *The proof of Proposition 13.1.8 makes it clear why $\pi_1(X, x_0)$ need not be abelian. We simply do not have "enough space" in $[0, 1]$ to carry out the same argument.*

We end this section is devoted to discussing various properties of higher homotopy groups that are analogous to those of the fundamental group.

Proposition 13.1.10. *Let $(X, x_0) \in \mathsf{Top}_*$. The following are some properties of $\pi_n(X, x_0)$ for $n \geq 1$.*

(1) *For each $x_0' \in X$, such that $x_0' \in \pi_0(x)$, we have*
$$\pi_n(X, x_0) \cong \pi_n(X, x')$$
(2) *For $n \geq 2$, π_n is a functor from Top_* to Ab, the category abelian of groups.*
(3) *For each $n \geq 1$, π_n preserves products. If $(Y, y) \in \mathsf{Top}_*$ then*
$$\pi_n(X \times Y, (x_0, y_0)) \cong \pi_n(X, x_0) \cong \pi_n(Y, y_0)$$
That is, π_n preserves products for $n \geq 1$.
(4) *If $f : (X, x_0) \to (Y, y_0)$ is a pointed homotopy equivalence, then the induced homomorphism $f_* : \pi_n(X, x_0) \to \pi_n(Y, y_0)$ is an isomorphism.*
(5) *If $(\widetilde{X}, \tilde{x}_0) \in \mathsf{Top}$ and $p : \widetilde{X} \to X$ is a covering map, then $p_* : \pi_n(\widetilde{X}, \tilde{x}_0) \to \pi_n(X, p(\tilde{x}_0))$ is an isomorphism for all $n \geq 2$.*

Proof. The proof is given below:

(1) This follows because
$$\pi_n(X, x_0) \cong \pi_1(\Omega^{n-1}(X, x_0), c_{x_0}) \cong \pi_1(\Omega^{n-1}(X, x_0'), c_{x_0'}) \cong \pi_n(X, x_0')$$
Here we have used the fact that $\Omega^{n-1}(X, x_0)$ and $\Omega^{n-1}(X, x_0')$ are homeomorphic topological spaces.
(2) Let $\phi : (X, x_0) \to (Y, y_0)$ be a continuous map. If $f \sim g$, then $\varphi \circ f \sim \varphi \circ g$ as before. Hence, the induced map $\phi_* : \pi_n(X, x_0) \to \pi_n(Y, y)$ is well-defined. Moreover, from the definition of the group operation on π_n, it is clear that we have
$$\varphi \circ (f + g) = (\varphi \circ f) + (\varphi \circ g)$$
Thus, $\varphi_*([f + g]) = \varphi_*([f]) + \varphi_*([g])$. Hence, φ_* is a group homomorphism.
(3) The proof in the case of π_1 goes through as before.
(4) Let $g : (Y, y_0) \to (X, x_0)$ be an inverse pointed homotopy equivalence so that we have:
$$g \circ f \sim \mathrm{Id}_X \text{ rel } x_0 \quad \text{and} \quad f \circ g \sim \mathrm{Id}_Y \text{ rel } y_0.$$
Homotopy invariance gives[1] $g_* \circ f_* = (g \circ f)_* = \mathrm{Id}_{\pi_n(X, x_0)}$, and similarly $f_* \circ g_* = \mathrm{Id}_{\pi_n(Y, y_0)}$.
(5) First, we show that p_* is surjective. Let $x_0 = p(\tilde{x}_0)$ and consider $f : (\mathbb{S}^n, *) \to (X, x_0)$. Since $n \geq 2$, we have $\pi_1(\mathbb{S}^n, *) = 0$, so
$$f_*(\pi_1(\mathbb{S}^n, *)) = \{0\} \subseteq p_*(\pi_1(\tilde{X}, \tilde{x})).$$
By the path lifting criterion, f admits a lift to $(X, \tilde{x}_0)$. That is there exists $\tilde{f} : (\mathbb{S}^n, *) \to (\tilde{X}, \tilde{x}_0)$ such that $p \circ \tilde{f} = f$. Then $[f] = [p \circ f] = p_*([f])$. Next, we show that p_* is injective. Suppose $[\tilde{f}] \in \ker p_*$. So $[p \circ \tilde{f}] = 0$. Let $p \circ \tilde{f} = f$. Then $f \sim c_{x_0}$ via some homotopy
$$H_t : (\mathbb{S}^n, *) \to (X, x_0)$$

[1]It is easy to see that homotopic maps induce identical homotopic maps and hence identical maps on homotopy groups.

with $\varphi_1 = f$ and $\varphi_0 = c_{x_0}$. By the homotopy lifting criterion, there is a unique $\tilde{H}_t : (\mathbb{S}^n, *) \to (\tilde{X}, \tilde{x}_0)$ with $p \circ \tilde{H}_t = H_t$. Then we have $p \circ \tilde{H}_1 = H_1 = f$ and $p \circ \tilde{H}_0 = H_0 = c_{x_0}$, so by the uniqueness of lifts, we must have $\tilde{H}_1 = \tilde{f}$ and $\tilde{H}_0 = c_{\tilde{x}_0}$. Then H_t is a homotopy between $\tilde{f}$ and $c_{\tilde{x}}$. So $[\tilde{f}] = 0$. Thus, p_* is injective.

This completes the proof. □

Remark 13.1.11. *Proposition 13.1.10(5) can be interpreted as mentioning that covering spaces cannot be used to compute higher homotopy groups!*

How does one compute higher homotopy groups? In general, this is a difficult problem. But we can at the very least state some trivial calculations for definition and basic properties:

Proposition 13.1.12. *The following are calculations of some higher homotopy groups:*

(1) *If $X = \{\bullet\}$ is a one-point space, then $\pi_n(X) = \{0\}$, is the trivial abelian group for $n \geq 2$.*
(2) *If (X, x_0) is contractible, then $\pi_n(X, x_0) = \{0\}$ is the trivial abelian group for $n \geq 2$.*
(3) *We have*
$$\pi_n(\mathbb{S}^1) = \begin{cases} \mathbb{Z}, & \text{if } n = 0, 1, \\ 0, & \text{if } n \geq 2 \end{cases}$$
(4) *We have*
$$\pi_n\Big(\underbrace{\mathbb{S}^1 \times \cdots \times \mathbb{S}^1}_{k\text{-times}}, (*_1, \cdots, *_k)\Big) = \{0\}$$
for $n \geq 2$.
(5) *We have $\pi_n(\mathbb{RP}^k, *) \cong \pi_n(\mathbb{S}^k, *)$ for $n \geq 2$.*
(6) *We have $\pi_n(\mathbb{RP}^\infty, *) = 0$ for $n \geq 2$.*

Proof. The proof is given below:

(1) A one-point space has only the constant loop. Hence, each higher homotopy group is trivial.
(2) This follows from the Proposition 13.1.10(5) and (1) above.
(3) Consider $\mathbb{S}^1$ with its universal covering map $p : \mathbb{R} \to \mathbb{S}^1$. If $n \geq 2$, we have
$$\pi_n(\mathbb{S}^1, *) = \pi_n(\mathbb{R}, 0) = 0$$
We already know the result for $n = 0, 1$.
(4) This follows from (3) and Proposition 13.1.10(3). We can also apply a covering space argument as in (3).
(5) This is because $\mathbb{S}^n \to \mathbb{RP}^n$ is a covering map.
(6) This is because $\mathbb{S}^\infty \to \mathbb{RP}^\infty$ is a covering map and $\mathbb{S}^\infty$ is contractible.

This completes the proof. □

13.2. Cellular Approximation

Homotopy theory of CW complexes is more tractable. For instance, a key results includes the cellular approximation, which allows for approximating maps by cellular ones. We prove the cellular approximation theorem in this section. Since CW complexes are built inductively, the following strategy will not come as a surprise. Given a map $f : X \to Y$ of CW complexes, we will try to deform f cell by cell into a cellular map. As an important building block for the proof of the theorem, there is the following case of a single cell.

Lemma 13.2.1. *Let $X, Y \in \mathsf{Top}$ such that we have a pushout diagram*

$$\begin{tikzcd} \mathbb{S}^{n-1} \arrow[r] \arrow[d] & X \arrow[d] \\ \mathbb{D}^n \arrow[r, "\chi"] & Y \end{tikzcd}$$

Any map $f : (\mathbb{D}^m, \mathbb{S}^{m-1}) \to (Y, X)$ with $m < n$ is homotopic relative to $\mathbb{S}^{m-1}$ to a map g satisfying $g(\mathbb{D}^m) \subseteq X$.

Let us describe the strategy of the proof. The attaching map $\chi : \mathbb{D}^n \to Y$ restricts to a homeomorphism $\chi_{\mathrm{Int}(\mathbb{D}^n)}$. Hence, we identify $\mathrm{Int}(\mathbb{D}^n) \subseteq Y$. We show that we can construct a homotopy relative to $\mathbb{S}^{m-1}$ such that $f \simeq h$ and $0 \notin h(\mathbb{D}^m)$. Here 0 is the origin of $\mathrm{Int}(\mathbb{D}^n)$. To see that this is enough, consider $Y \setminus \{0\}$. The inclusion $i : X \to Y \setminus \{0\}$ is the inclusion of a strong deformation retraction $r : Y \setminus \{0\} \to X$ induced by collapsing $\mathrm{Int}(\mathbb{D}^n) \setminus \{0\}$ onto $\mathbb{S}^{n-1}$. Hence, we have

$$\mathrm{Id}_{Y-\{0\}} \simeq i \circ r$$

relative to X which induces the desired relative homotopy

$$h = \mathrm{Id}_{Y-\{0\}} \circ h \simeq i \circ r \circ h = g$$

relative to $\mathbb{S}^{m-1}$. Putting these two homotopies together, we conclude that $f \simeq g$ relative to $\mathbb{S}^{m-1}$.

Proof. We induct on n. Let $n = 1, m = 0$. In this case, $\mathbb{S}^{m-1} = \emptyset$ and $\mathbb{D}^m = \{*\}$. A map

$$f : (\{*\}, \emptyset) \to (Y, X)$$

is essentially the same as a point $y \in Y$. For some $x \in X$, there is a path $\omega : I \to Y$ with $\omega(0) = y$ and $\omega(1) = x \in X$. This path defines the desired homotopy. Assume the claim has been prove for $n-1$. We list the following consequences of our inductive assumption:

(1) Any map $\mathbb{S}^k \to \mathbb{S}^{n-1}$ for $k < n-1$ is homotopic to a constant map. Indeed, apply the inductive hypothesis to for $n-1$ to the standard map $(\mathbb{D}^k, \mathbb{S}^{k-1}) \to (\mathbb{D}^{n-1} \coprod_{\mathbb{S}^{n-2}} *, *)$.
(2) Any map $\mathbb{S}^k \to \mathbb{S}^{n-1} \times (a, b)$ for $k < n-1$ is homotopic to a constant map. This follows from (1) and that (a, b) is contractible.
(3) Any map $\mathbb{S}^k \to \mathbb{S}^{n-1} \times (a, b)$ for $k < n-1$ can be extended $\mathbb{D}^k$. This follows from (2) and Proposition 10.1.4.

We now construct a homotopy $f \simeq h : \mathbb{D}^m \to Y$ relative to $\mathbb{S}^{m-1}$ such that $0 \notin h(\mathbb{D}^m)$.Consider the subsets

$$U_0 = \{x \in \mathbb{D}^n \mid \|x\| < 2/3\},$$
$$V_0 = \{x \in \mathbb{D}^n \mid \|x\| > 1/3\}.$$

and define two subsets of Y by setting $U = \chi(U_0)$ and $V = X \coprod_{\mathbb{S}^{n-1}} \chi(V_0)$. Note that

$$U \cap V \cong \mathbb{S}^{n-1} \times (1/3, 2/3).$$

We construct a homotopy $f \simeq h$ such that the image of h entirely lies in V, and hence avoids the point $0 \in Y$. WLOG replace $(\mathbb{D}^m, \mathbb{S}^{m-1})$ by $(I^m, \partial I^m)$. Note that $\{U, V\}$ is an open cover for Y. Pulling back the open cover of Y along f induces an open cover $f^{-1}(U), f^{-1}(V)$ of of I^m. The Lebesgue number lemma implies there exists a $N > 0$ such that the image of each m-cube

$$I^m_{k_1,\ldots,k_m} = \left[\frac{k_1}{N}, \frac{k_1+1}{N}\right] \times \cdots \times \left[\frac{k_m}{N}, \frac{k_m+1}{N}\right], \quad 0 \le k_i < N$$

under f is contained in either U or V. We construct homotopies to modify f only on those $I^m_{k_1,\dots,k_m}$ which are not entirely mapped to V. We define a filtration on I^m,

$$\partial I^m \subseteq Z^{(-1)} \subseteq Z^{(0)} \subseteq \cdots \subseteq Z^{(m)} = I^m.$$

Let J^{-1} be the index set for all l-dimensional sub-cubes, $0 \leq l \leq m$, of all cubes $I^m_{k_1,\dots,k_m}$ which are already completely mapped to V by f. Let us denote the l-dimensional sub-cube corresponding to such an index $\varphi \in J^{-1}$ by I^l_φ. We then set

$$Z^{(-1)} = \bigcup_{\varphi \in J^{-1}} I^l_\varphi,$$

and it follows from our assumption on f that $\partial I^m \subseteq Z^{(-1)}$. We now take care of the remaining sub-cubes, and this will be done by induction over the dimension of these sub-cubes. For each $0 \leq k \leq m$, let J_k be the index set for all k-dimensional sub-cubes, I^k_φ for $\varphi \in J_k$, of the cubes $I^m_{k_1,\dots,k_m}$ which satisfy $f(I^k_\varphi) \nsubseteq V$. Set

$$Z^{(k)} = Z^{(k-1)} \coprod \bigcup_{\varphi \in J_k} I^k_\varphi.$$

This defines a filtration for X. We now want to inductively construct maps $h_k : Z^{(k)} \to Y$, $k \geq -1$, such that:

- The map h_{-1} is obtained from f by restriction.
- The map h_k sends the cubes I^k_φ to $U \cap V$ for all $\varphi \in J_k$ and $k \geq 0$.
- The map h_k extends h_{k-1}, i.e., we have $h_k|_{Z^{(k-1)}} = h_{k-1}$ for all $k \geq 0$.

For h_0, note that $Z^{(0)}$ is obtained from $Z^{(-1)}$ by possibly adding some vertices which are mapped to U. For each such vertex, choose a path to a point in $U \cap V$. This defines h_0. Inductively, assume that h_{k-1} has already been constructed. For each $\varphi \in J_k$, we have that $h_{k-1}(\partial I^\varphi_k) \subseteq U \cap V$. Since $U \cap V \cong \mathbb{S}^{n-1} \times (1/3, 2/3)$, the induction hypothesis and (3) above implies that we can find extensions as indicated in the following diagram:

$$\begin{tikzcd}
\partial I^k_\varphi \arrow[r, "h_{k-1}"] \arrow[d] & U \cap V \\
I^k_\varphi \arrow[ur, dotted, "h_{k,\varphi}"'] &
\end{tikzcd}$$

These maps $h_{k,\varphi}$ and h_{k-1} can be assembled together in order to define a map $h_k : Z^{(k)} \to Y$ with the desired properties. If we set $h = h_m : I^m = X^{(m)} \to Y$, then we have $h(I^m) \subseteq V$. Hence, it suffices to show that $f \cong h$ relative to ∂I^m. We show that in fact we construct such a homotopy relative to $Z^{(-1)}$. Both maps f and h coincide on $Z^{(-1)}$. Moreover, the restrictions of both maps to $I^m - Z^{(-1)}$ can be considered as maps taking values in U. But, U is homeomorphic to an open n-disc, hence convex, so that the two restrictions are homotopic via linear homotopies. This homotopy, together with the constant homotopy on $Z^{(-1)}$, can be assembled together to give the desired homotopy $f \sim h$ relative to $Z^{(-1)}$. $\square$

We now state and prove the cellular approximation theorem.

Proposition 13.2.2. *Let (X, A) be a finite-dimensional CW pair, let Y be a CW complex. If $f : X \to Y$ is a continuous map such that $f|_A : A \to Y$ is a cellular map, then f is homotopic to a cellular map $g : X \to Y$ relative to A. In particular, any map of CW complexes is homotopic to a cellular map.*

Proof. We have a filtration

$$A = X^{(-1)} \subseteq X^{(0)} \subseteq X^{(1)} \subseteq \dots X^{(m)} = X,$$

Similarly, we also have a filtration for Y. Let $g_{-1} = f$. We construct maps $g_n : X \to Y$ and homotopies $g_{n-1} \simeq g_n$ such that

(1) The map g_n sends the relative n-cells to $Y^{(n)}$.
(2) The homotopy $g_{n-1} \simeq g_n$ is relative to $X^{(n-1)}$.

We proceed by induction. Let J_n denoting the set of relative n-cells. We have a pushout diagram:

$$\begin{tikzcd} \coprod_{\sigma \in J_n} \partial \mathbb{D}^n_\sigma \arrow[r] \arrow[d] & X^{(n-1)} \arrow[d] \\ \coprod_{\sigma \in J_n} \mathbb{D}^n_\sigma \arrow[r] & X^{(n)}. \end{tikzcd}$$

For each such cell $\mathbb{D}^n_\sigma$ such that $g_{n-1}(\mathbb{D}^n_\sigma)$ is not contained in $Y^{(n-1)}$, there is a finite relative subcomplex Y' with

$$Y^{(n)} \subseteq Y' \subseteq Y$$

such that $g_{n-1}(\mathbb{D}^n_\sigma) \subseteq Y'$. Choose a cell of maximal dimension in Y' which has a nontrivial intersection with $g_{n-1}(\mathbb{D}^n_\sigma)$. By Lemma 13.2.1, this cell can be avoided up to relative homotopy. Repeating this finitely many times and gluing the relative homotopies together, we obtain a homotopy $H_{n,\sigma} : g_{n-1} \simeq g_{n,\sigma} : \mathbb{D}^n_\sigma \to Y$ relative to $\partial \mathbb{D}^n_\sigma$ such that $g_{n,\sigma}(\mathbb{D}^n_\sigma) \subseteq Y^{(n)}$. Recalling that $X^{(n)} \times [0,1]$ carries the quotient topology with respect to the map

$$\left(X^{(n-1)} \sqcup \left(\coprod_{\sigma \in J_n} \mathbb{D}^n_\sigma \right) \right) \times [0,1] \longrightarrow X^{(n)} \times [0,1].$$

We can glue the homotopies $H_{n,\sigma}$, the constant homotopies on $g_{n-1} : \mathbb{D}^n_\sigma \to Y$ for all n-cells with $g_{n-1}(e^n_\sigma) \subseteq Y^{(n)}$, and the constant homotopy on $g_{n-1}|_{X^{(n-1)}}$ together in order to obtain a homotopy

$$\widetilde{H}_n : X^{(n)} \times [0,1] \to Y.$$

relative to $X^{(n)}$. Since $X^{(n)} \to X$ is a cofibation, we obtain a homotopy

$$H_n : X \times [0,1] \to Y.$$

which admits a solution since the inclusion $X^{(n)} \to X$ is a cofibration. We set $g_n = H_n(-,1)$. Since $X^{(m)} = X$, it suffices to compose the finitely many homotopies H_k, $0 \leq k \leq n$, to obtain a homotopy $H : f \simeq g = g_n$ relative to A such that $g : X \to Y$ is a cellular map. $\square$

Remark 13.2.3. *Proposition 13.2.2 can be extended to the case where X is infinite-dimensional.*

13.3. Relative Homotopy Groups

We define relative homotopy groups. We also state and prove the long exact sequence in homotopy groups, which is a crucial tool for computations.

Definition 13.3.1. Let $(X, A, x_0) \in \mathsf{Top}^*_2$ such that A contains the basepoint x_0. For $n \geq 1$, the n-th relative homotopy group, denoted as $\pi_n(X, A, x_0)$, is defined as

$$\pi_n(X, A, x_0) = [(\mathbb{D}^n, \mathbb{S}^{n-1}, *), (X, A, x_0)]$$

We say (X, A, x_0) is n-connected if $\pi_k(X, A, x_0)$ for $0 \leq k \leq n$.

One can think about relative homotopy groups differently. Consider the relative path space

$$P(X, A, x_0) = \{\gamma \in P(X, x_0) \mid \gamma(1) \in A\}.$$

Note that $P(X, A, x_0)$ is a based topological sapce with basepoint the constant path at $I \to x_0$. Consider $\pi_{n-1}(P(X, A, x_0), c_{x_0})$ An element of $\pi_{n-1}(P(X, A, x_0), c_{x_0})$ is a map $(\mathbb{S}^{n-1}, *) \to P(X, A, c_{x_0})$ up to homotopy that sends $*$ to c_{x_0}. Equivalently, it is a map $(\mathbb{S}^{n-1}, *) \times I \to X$ up to homotopy that:

(1) Maps $(\mathbb{S}^{n-1}, *) \times \{0\}$ to x_0, as every path starts at x_0.
(2) Maps $\{*\} \times I$ to c_{x_0}
(3) Maps $(\mathbb{S}^{n-1}, *) \times \{1\}$ into A as paths end in A.

This is nothing but the data of a map

$$(\mathbb{D}^n, \mathbb{S}^{n-1}, *) \to (X, A, x_0)$$

defined up to homotopy. Hence,

$$\pi_n(X, A, x_0) = \pi_{n-1}(P(X, A, x_0), c_{x_0})$$

This shows that $\pi_n(X, A, x_0)$ is a group for $n \geq 2$ and an abelian group for $n \geq 3$. We have a long exact sequence in relative homotopy groups:

Proposition 13.3.2. *Let $(X, A, x_0) \in \mathsf{Top}_*^2$ such that A is a closed subspace. Then there is an exact sequence of homotopy groups:*

$$\cdots \to \pi_n(A, x_0) \to \pi_n(X, x_0) \to \pi_n(X, A, x_0) \to \pi_{n-1}(A, x_0) \to \cdots$$

Proof. Consider the inclusion map $i : A \to X$. The homotopy fiber of i is $P(X, A, x_0)$. Note that

$$\begin{aligned}[(\mathbb{S}^0, *), \Omega^n(P(X, A, x_0), c_{x_0})] &\cong [\Sigma^n(\mathbb{S}^0, *), (P(X, A, x_0), c_{x_0})] \\ &\cong [(\mathbb{S}^n, *), (P(X, A, x_0), c_{x_0})] \\ &= \pi_n(P(X, A, x_0), c_{x_0}) = \pi_{n+1}(X, A, x_0).d\end{aligned}$$

The claim now follows by letting $(Z, z_0) = (\mathbb{S}^0, *)$ and applying $[(\mathbb{S}^0, *), -]$ to the exact sequence in Proposition 11.3.9. ☐

Remark 13.3.3. *Using Proposition 13.3.2 and some algebraic manipulations, one can show if $(B \subset A \subset X) \in \mathsf{Top}_*^3$ such that $B \subseteq A \subseteq A$ and B contains the base point x_0, then there is a long exact sequence:*

$$\cdots \to \pi_n(A, B, x_0) \to \pi_n(X, B, x_0) \to \pi_n(X, A, x_0) \to \pi_{n-1}(A, B, x_0) \to \cdots$$

13.4. Freudenthal's Suspension Theorem

The purpose of this section is to prove Freudenthal's suspension theorem. We first state a notion of the excision theorem for homotopy groups. Recall that a remarkable fact about homology groups is that the relative homology groups satisfy the excision property. However, this is not the case for relative homotopy groups. However, there is a version of excision that holds for CW complexes that have rather strong connectedness properties.

Proposition 13.4.1. *Let X be a CW complex such that $X = A \cup B$ and $A \cap B$ is non-empty and connected. If $(A, A \cap B)$ is k-connected, $(B, A \cap B)$ is l-connected, and $i : (A, A \cap B) \to (X, B)$ is the inclusion map, then for any $x_0 \in A \cap B$, the induced map*

$$i_* : \pi_n(A, A \cap B, x_0) \to \pi_n(X, B, x_0)$$

is an isomorphism when $n < k + l$ and is a surjection when $n = k + l$.

Proof. The proof is lengthy and technical. See [May99; Hat02]. ☐

We now state and prove Freudenthal's Suspension Theorem:

Proposition 13.4.2. *Let (X, x_0) be an $(k-1)$-connected pointed CW complex. For any map $f : \mathbb{S}^k \to (X, x_0)$, consider its suspension,*

$$\Sigma f : \Sigma\mathbb{S}^n = \mathbb{S}^{n+1} \to \Sigma(X, x_0).$$

The assignment

$$\pi_n((X, x_0)) \to \pi_{n+1}(\Sigma(X, x_0))$$
$$[f] \mapsto [\Sigma f]$$

is a homomorphism which is an isomorphism for $n < 2k-1$ and a surjection for $n = 2k-1$.

Proof. We can think of ΣX as two copies of CX, which we call C^+X and C^-X, identified along their bases. Define j to be the composition of the following three maps drawn in the diagram below, each of which is induced by the obvious inclusion maps.

$$\pi_n(X, x_0) \xrightarrow{\cong} \pi_{n+1}(C^+X, X, x_0) \to \pi_{n+1}(\Sigma X, C^-X, x_0) \xrightarrow{\cong} \pi_{n+1}(\Sigma X, x_0)$$

For any n, the leftmost and rightmost maps are isomorphisms because of the long exact sequences of the CW pairs (C^+X, X) and $(\Sigma X, C^-X)$, respectively, since $\pi_n(C^\pm X)$ is always trivial. Also, when X is $(k-1)$-connected, the CW pair $(C^\pm X, X)$ is k-connected by the long exact sequence of the pair $(C^\pm X, X)$. This allows us to apply Proposition 13.4.1, so the middle map in the diagram above is an isomorphism when $n+1 < 2k$ and a surjection when $n+1 = 2k$. $\square$

Remark 13.4.3. *Let X be a k-connected CW complex. For any $n \in \mathbb{N}$, consider the sequence of maps:*

$$\cdots \to \pi_n(X, x_0) \to \pi_{n+1}(\Sigma X, x_0) \to \pi_{n+2}(\Sigma^2 X, x_0) \to \ldots$$

Since X is an k-connected CW complex, Proposition 13.4.2 implies that us that $\pi_n(X, x_0) \cong \pi_{n+1}(\Sigma X, x_0)$ if $n < 2k+1$. We make the following observations:

(1) *In particular,*

$$\pi_n(\Sigma X, x_0) \cong \pi_{n-1}(X, x_0) \cong 0$$

if $0 < n \leq k+1$. If X is 0-connected (path-connected) then ΣX is also 0-connected (path-connected). Hence, we have that

$$X \text{ is } k\text{-connected} \Longrightarrow \Sigma X \text{ is } (k+1)\text{-connected}$$

More generally, we have that $\Sigma^n X$ is $(k+n)$-connected for any $n \in \mathbb{N}$.

(2) *Let $N(k,n) = n-1-2k$, and observe that when $i > N(k,n)$, we have $n+i < 2(k+i)+1$. Thus, the groups $\pi_{n+i}(\Sigma^i X)$ are isomorphic for all $i > N(k,n)$. Let $N = N(n,k)$, and define $\pi_{n+N}(\Sigma^N X, x_0)$ as the n-th stable homotopy group of X.*

More generally, the n-th stable homotopy group of any $X \in \mathsf{Top}_*$,

$$\pi_n^s(X) := \varinjlim_{i\in\mathbb{N}} \pi_{n+i}(\Sigma^i X, x_0).$$

Observe that since ΣX is always 0-connected, we do not actually need X to be 0-connected, so every space X has n-th stable homotopy groups for all $n \in \mathbb{N}$. Moreover, Proposition 13.4.2 proves this colimit is realized after finitely many elements along the sequence. This is the start of the subject of stable homotopy theory.

13.5. Some Computations

The purpose of this section is to compute the homotopy groups. Our main computational tool will be the long exact sequence associated with a fibration, as proved below. We begin by considering a basic example.

Example 13.5.1. **(Spheres)** Let $k \geq 2$. We compute $\pi_n(\mathbb{S}^k, *)$.

(1) Let $1 \leq n < k$. Let $\mathbb{S}^n \to \mathbb{S}^k$ be a continuous map. WLOG, we can assume that f is a cellular map by Proposition 13.2.2. If $\mathbb{S}^k$ is given the standard cellular structure with a 0-cell and a k-cell, then the n-th skeleton of $\mathbb{S}^k$ for $n < k$ is simply the 0-cell. Therefore, $f : \mathbb{S}^n \to \mathbb{S}^k$ is homotopic to the constant map. Hence, $\pi_n(\mathbb{S}^k, *) = 0$ for $1 \leq n < k$.

(2) Let $n = k$. Since $\mathbb{S}^n$ is $(n-1)$-connected by (1), Proposition 13.4.2 implies that $\pi_j(\mathbb{S}^n, *) \to \pi_{j+1}(\mathbb{S}^{n+1}, *)$ is an isomorphism for $j < 2n - 1$. Therefore,

$$\pi_n(\mathbb{S}^n, *) \to \pi_{n+1}(\mathbb{S}^{n+1}, *)$$

for $n \geq 2$. Moreover, Proposition 13.4.2 implies that $\mathbb{Z} \cong \pi_1(\mathbb{S}^1, *) \to \pi_2(\mathbb{S}^2, *)$ is surjective. We will show that in Example 13.5.6 that in fact this map is an isomorphism since $\pi_2(\mathbb{S}^2, *) \cong \mathbb{Z}$. Therefore, we have

$$\mathbb{Z} \cong \pi_2(\mathbb{S}^2, *) \cong \pi_3(\mathbb{S}^3, *) \cong \pi_4(\mathbb{S}^4, *) \cong \cdot$$

That is, $\pi_n(\mathbb{S}^n, *) \cong \mathbb{Z}$.

Hence, for $k \geq 2$ we have

$$\pi_n(\mathbb{S}^k) = \begin{cases} 0, & \text{if } 1 \leq n < k, \\ \mathbb{Z}, & \text{if } n = k. \end{cases}$$

Remark 13.5.2. *Example 13.5.1 implies that it remains to compute $\pi_n(\mathbb{S}^k)$ for $n > k$. This turns out to be a very difficult problem.*

Remark 13.5.3. *We can now also argue that homotopy groups are not a perfect topological invariant. Consider*

$$X = \mathbb{S}^2 \times \mathbb{RP}^3, \quad Y = \mathbb{RP}^2 \times \mathbb{S}^3.$$

By Example 13.5.1, we have:

$$\pi_1(X) \cong \pi_1(\mathbb{S}^2) \times \pi_1(\mathbb{RP}^3) \cong 0 \times \mathbb{Z}_2 \cong \mathbb{Z}_2$$
$$\pi_1(Y) \cong \pi_1(\mathbb{RP}^2) \times \pi_1(\mathbb{S}^3) \cong \mathbb{Z}_2 \times 0 \cong \mathbb{Z}_2$$

The universal cover of both X and Y is homeomorphic to $\mathbb{S}^5$. Hence,

$$\pi_n(X) \cong \pi_n(Y)$$

for $n \geq 2$. Hence, X and Y have same homotopy groups. However, X and Y are not homotopy equivalent. Indeed,

$$H_5(X) = \mathbb{Z}, \quad H_5(Y) = 0$$

since X is compact and orientable, and Y is compact and non-orientable.

Example 13.5.4. Since $\mathbb{S}^k$ is $(k-1)$-connected, Proposition 13.4.2 implies we have

$$\pi_{n+k}(\mathbb{S}^k) \cong \pi_{n+k+1}(\mathbb{S}^{k+1})$$

whenever $n + k < 2k - 1$. It follows that the groups $\pi_{n+k}(\mathbb{S}^k)$ stabilize when $k > n + 1$.

We now establish the long exact sequence associated with a fibration. Notably, we can leverage the properties of fibrations along with the exact Puppe sequence (Proposition 11.3.9) to derive a long exact sequence of homotopy groups.

Proposition 13.5.5. *Let $(E, e_0), (X, x_0) \in \mathsf{Top}_*$ and let $p : (E, e_0) \to (X, x_0)$ be a based fibration. Let $F = p^{-1}(x_0)$. Then we have an exact sequence of homotopy groups:*

$$\cdots \to \pi_n(F, e_0) \to \pi_n(E, e_0) \to \pi_n(X, x_0) \to \pi_{n-1}(F, e_0) \to \pi_{n-1}(E, e_0) \to \pi_{n-1}(X, e_0) \to \cdots$$

Proof. Since p is a fibration, we have that the homotopy fiber of p is homotopy equivalent to F. Observe that

$$\begin{aligned}
[(\mathbb{S}^0, *), \Omega^n(X, x_0)]_* &\cong [\Sigma^n(\mathbb{S}^0, *), (X, x_0)]_* \\
&\cong [(\mathbb{S}^n, *), (X, x_0)]_* \\
&= \pi_n(X, x_0)
\end{aligned}$$

The claim now follows by letting $(Z, z_0) = (\mathbb{S}^0, *)$ and applying $[(\mathbb{S}^0, *), -]$ to the exact sequence in Proposition 11.3.9. □

We can perform further computations by exploiting the long exact sequence of homotopy groups associated to fibrations.

Example 13.5.6. (**Hopf Fibration**) Consider the Hopf Fibration:

$$\mathbb{S}^1 \to \mathbb{S}^3 \to \mathbb{S}^2$$

Barring π_0, the long exact sequence reads:

$$\cdots \to 0 \to \pi_3(\mathbb{S}^3) \to \pi_3(\mathbb{S}^2) \to 0 \to 0 \to \pi_2(\mathbb{S}^2) \to \mathbb{Z} \to 0 \to 0$$

Hence,

$$\begin{aligned}
&\pi_2(\mathbb{S}^2, *) \cong \mathbb{Z} \\
&\pi_n(\mathbb{S}^3, *) \cong \pi_n(\mathbb{S}^2, *), \quad n \geq 2
\end{aligned}$$

In particular,

$$\pi_3(\mathbb{S}^2, *) \cong \mathbb{Z}$$

Remark 13.5.7. *The computation in Example 13.5.6 is independent of the computation in Example 13.5.1. This shows that the claim $\pi_n(\mathbb{S}^n, *) \cong \mathbb{Z}$ made in Example 13.5.1 is correct.*

Example 13.5.8. Consider the fibration

$$\mathbb{S}^1 \to \mathbb{S}^\infty \to \mathbb{CP}^\infty$$

Barring π_0, the long exact sequence reads:

$$\cdots 0 \to \pi_3(\mathbb{CP}^\infty) \to 0 \to \pi_2(\mathbb{CP}^\infty) \to \mathbb{Z} \to 0 \to \pi_1(\mathbb{CP}^\infty) \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}$$

Hence,

$$\pi_n(\mathbb{CP}^\infty) \cong \begin{cases} \mathbb{Z} & k = 0, 2 \\ 0 & \text{otherwise} \end{cases}$$

Example 13.5.9. (Special Orhthogonal Group) Consider the fibration:

$$\mathrm{SO}(k-1) \to \mathrm{SO}(k) \to \mathbb{S}^{k-1}$$

We have a long exact sequence of homotopy groups:

$$\cdots \to \pi_n(\mathrm{SO}(k-1)) \to \pi_n(\mathrm{SO}(k)) \to \pi_n(\mathbb{S}^{k-1}) \to \pi_{n-1}(\mathrm{SO}(k-1)) \to \pi_{n-1}(\mathrm{SO}(k)) \to \pi_{n-1}(\mathbb{S}^{k-1}) \to \cdots$$

For $n \leq k-3$, we have

$$\cdots \to \pi_n(\mathrm{SO}(k-1)) \to \pi_n(\mathrm{SO}(k)) \to 0 \to \cdots$$

This implies that

$$\pi_n(\mathrm{SO}(k)) \cong \pi_n(\mathrm{SO}(k-1))$$

for $k \geq n+3$. This isomorphism doesn't hold for $k = n+2$ ($n = k-2$). Indeed, if $k = 3$ and $n = 1$, we have

$$\mathbb{Z}_2 \cong \pi_1(\mathbb{RP}^3) \cong \pi_1(\mathrm{SO}(3)) \not\cong \pi_1(\mathrm{SO}(2)) \cong \pi_1(\mathbb{S}^1) \cong \mathbb{Z}$$

In any case if $k \geq n+3$, we have

$$\pi_n(\mathrm{SO}(k)) \cong \pi_n(\mathrm{SO}(k-1)) \cong \pi_n(\mathrm{SO}(n+2))$$

In particular, we have

$$\pi_0(\mathrm{SO}(k)) \cong \mathbb{Z}, \qquad \pi_1(\mathrm{SO}(k)) \cong \begin{cases} 0 & k=1 \\ \mathbb{Z} & k=2 \\ \mathbb{Z}_2 & k=3 \\ \mathbb{Z}_2 & k \geq 4 \end{cases}$$

Example 13.5.10. (Unitary Group) Consider the fibration:

$$\mathrm{U}(k-1) \to \mathrm{U}(k) \to \mathbb{S}^{2k-1}$$

We have a long exact sequence of homotopy groups:

$$\cdots \to \pi_n(\mathrm{U}(k-1)) \to \pi_n(\mathrm{U}(k)) \to \pi_n(\mathbb{S}^{2k-1}) \to \pi_{n-1}(\mathrm{U}(k-1)) \to \pi_{n-1}(\mathrm{U}(k)) \to \pi_{n-1}(\mathbb{S}^{2k-1}) \to \cdots$$

For $n \leq 2k-3$, we have

$$\cdots \to \pi_n(\mathrm{U}(k-1)) \to \pi_n(\mathrm{U}(k)) \to 0 \to \cdots$$

This implies that

$$\pi_n(\mathrm{U}(k)) \cong \pi_n(\mathrm{U}(k-1))$$

for $n \leq 2k-3$. This isomorphism doesn't hold for $n = 2k-1$. Indeed, if $k = 2$ and $n = 3$, we have

$$\begin{aligned} \mathbb{Z} &\cong \pi_3(\mathbb{S}^3) \\ &\cong \pi_3(\mathrm{SU}(2)) \\ &\cong \pi_3(\mathrm{SU}(2)) \times \pi_3(\mathbb{S}^1) \\ &\cong \pi_3(\mathrm{SU}(2) \times \mathbb{S}^1) \\ &\cong \pi_3(\mathrm{U}(2)) \not\cong \pi_3(\mathrm{U}(1)) \\ &\cong \pi_3(\mathbb{S}^1) \cong 0. \end{aligned}$$

In any case if $k \geq \lceil (n+3)/2 \rceil$, we have

$$\pi_n(\mathrm{U}(k)) \cong \pi_n(\mathrm{U}(k-1)) \cong \pi_n(\mathrm{U}(\lceil (n+3)/2 \rceil - 1))$$

In particular, we have

$$\pi_0(\mathrm{U}(k)) \cong \mathbb{Z}, \qquad \pi_1(\mathrm{U}(k)) \cong \begin{cases} \mathbb{Z} & k=1 \\ \mathbb{Z} & k \geq 2 \end{cases} \qquad \pi_2(\mathrm{U}(k)) \cong \begin{cases} 0 & k=1 \\ 0 & k=2 \\ 0 & k \geq 3 \end{cases} \qquad \pi_3(\mathrm{U}(k)) \cong \begin{cases} 0 & k=1 \\ \mathbb{Z} & k=2 \\ \mathbb{Z} & k \geq 3 \end{cases}$$

13.6. Classification of Principal G-bundles

We can use homotopy theory to provide a homotopy-theoretic classification of principal G-bundles. More precisely, we show that the functor taking a topological space to the set of principal G-bundles on it is representable in the homotopy category. Recall that a principal G-bundle is defined as follows:

Definition 13.6.1. Let $E, X \in \mathsf{Top}$ such that $p : E \to X$ be a fibre bundle with a topological group, G, as its fibre. Then $p : E \to X$ is a principal G-bundle if the following hold:

(1) There is a continuous, free group action $E \times G \to E$,
(2) For each $x \in X$, the action of G preserves the fibre E_x and the orbit map $G \to E_x$ is a homeomorphism,
(3) The locally trivalizing cover $\{U_\alpha, \varphi_\alpha\}_\alpha$ is such that each φ is G-equivariant. That is,

$$\varphi_\alpha(e \cdot g) = \varphi_\alpha(e) \cdot g$$

The group G is called the structure group of the principal G-bundle.

We now define the notion of morphisms of principal G-bundles.

Definition 13.6.2. Let $p_i : E_i \to X_i$ be two principal G-bundles. A morphism of principal G-bundles is given by a pair of smooth functions $f : E_1 \to E_2$ and $g : X_1 \to X_2$ such f is a G-equivariant map and the diagram

$$\begin{tikzcd} E_1 \arrow[r, "f"] \arrow[d, "1"'] & E_2 \arrow[d, "p_2"] \\ X_1 \arrow[r, "g"'] & X_2 \end{tikzcd}$$

commutes. A morphism of principal G-bundles is an isomorphism of principal G-bundles if f, g are diffeomorphisms.

Remark 13.6.3. *If $p_i : E_i \to X_i$ be two principal G-bundles for $i = 1, 2$, any G-equivariant map $f : E_1 \to E_2$ defines a morphism of principal G-bundles. This is because*

$$f(e_1 \cdot g) = g(e_1) \cdot g$$

implies that f maps fibers of E_1 to fibers of E_2. Hence, defining

$$\begin{aligned} g : X_1 &\to X_2 \\ x_1 &\mapsto p_2(f(e_1)) \end{aligned}$$

is well-defined for any choice of $e_1 \in p_1^{-1}(x_1)$ and uniquely determines the base map $g : X_1 \to X_2$.

Remark 13.6.4. *An important special case we will consider is when $X_1 = X_2 = X$. In this case, $g = \mathrm{Id}_X$.*

Before stating the classification theorem, we need to introduce some constructions of principal G-bundles, with the first important one being the pullback construction.

Example 13.6.5. The following are examples of constructions of principal G-bundles.

(1) Let $X_1, X_2 \in \mathsf{Top}$ and let $p_i : E_i \to X_i$ be a principal G-bundles for $i = 1, 2$. Then
$$p_1 \times p_2 : E_1 \times E_2 \to X_1 \times X_2$$
is a principal G-bundle. Indeed, $p_1 \times p_2$ is a fibre bundle. This is in my other notes. The group action on $E_1 \times E_2$ is given by the diagonal action:
$$(e_1, e_2) \cdot g = (e_1 \cdot g, e_2 \cdot g)$$
It is clear that Definition 13.6.1 is satisfied.

(2) Let $X \in \mathsf{Top}$ and let $p : E \to X$ be a principal G-bundle. If $X' \subseteq X$ is a subspace of X, then
$$p|_{X'} : p^{-1}(X') \to X'$$
is a principal G-bundle. Indeed, $p|_{X'}$ is a fibre bundle. This is in my other notes. Moreover, Definition 13.6.1 is easily satisfied.

(3) **(Pullback)** Let $X, Y \in \mathsf{Top}$ and let $p : E \to Y$ be a principal G-bundle. Consider a continuous map $f : X \to Y$. We construct a principal G-bundle $f^*p : f^*E \to X$ that fits into the following commutative diagram:
$$\begin{tikzcd} f^*E \arrow[r, "\pi_2"] \arrow[d, "f^*p"'] & E \arrow[d, "p"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$
Consider
$$f^*E := \{(x, e) \in X \times E \mid f(x) = p(e)\} := X \times_Y E$$
We endow f^*E with the subspace topology of the product topology. Let f^*p and π_2 be projections onto first and second factors respectively. Consider the product principal G-bundle.
$$\mathrm{Id}_X \times p : X \times E \to X \times Y$$
Consider the graph of f:
$$\Gamma_f = \{(x, y) \in X \times Y \mid y = f(x)\} \subseteq X \times Y$$
Note that we have
$$(x, e) \in (\mathrm{Id}_X \times p)^{-1}(\Gamma_f) \iff f(x) = p(e).$$
Hence, the inverse image of Γ_f is f^*E. This shows that f^*E is a principal G-bundle. Uniqueness follows from categorical nonsense.

We now prove the important fact the pullbacks of principal G-bundles along homotopic maps are isomorphic.

Proposition 13.6.6. *Let $X, Y \in \mathsf{Top}$ be paracompact Hausdorff topological spaces. Let $p : E \to Y$ be a principal G-bundle. If $h_0, h_1 : X \to Y$ are homotopic maps, then $h_0^*(E) \cong h_1^*(E)$.*

Remark 13.6.7. *We will invoke the following facts on principal G-bundles in the proof of Proposition 13.6.6.*

(1) *Every morphism of principal G-bundles is an isomorphism.*
(2) *There is a bijection between morphisms of G-bundles $E_1 \to E_2$ over a common base and global sections of the associated bundle $E_1 \times_G E_2$[2] over X with fiber E_2.*

These constructions and facts are covered in my other notes.

[2]Here E_2 is endowed with the left action $g \cdot e_2 := e_2 \cdot g^{-1}$

Proof. Consider a homotopy
$$H : X \times I \to Y$$
such that $H_0 = h_0$ and $H_1 = h_1$. Pulling back $p : E \to Y$ along H, we get a principal G-bundle $H^*E \to X \times I$ such that
$$H^*E|_{X\times\{0\}} = h_0^*(E)$$
$$H^*E|_{X\times\{1\}} = h_1^*(E)$$
Hence, it suffices to sow that for any principal G-bundle $q : F \to X \times I$, the restrictions $q|_{X\times\{0\}}$ and $q|_{X\times\{1\}}$ are isomorphic. Denote the restrictions as
$$q_0 : F_0 \to X \times \{0\} \cong X$$
$$q_1 : F_1 \to X \times \{1\} \cong X$$
It suffices to prove that $F \cong F_0 \times I$ as prinicpal G-bundles over $X \times I$, since then restriction to $X \times \{1\}$ gives the isomorphism
$$F|_{X\times\{1\}} \equiv F_1 \cong (F_0 \times I)|_{X\times\{1\}} \equiv F_0.$$
It suffices to find a global section of $F \times_G (F_0 \times I) \to X \times I$. Now, $F \times_G (F_0 \times I)$ has a section over $X \times \{0\}$, since
$$F|_{X\times\{0\}} \cong F_0 \cong F_0 \times I|_{X\times\{0\}}$$
Since X is paracompact Hausdorff, $X \times I$ is paracompact Hausdorff. Hence,
$$F \times_G (F_0 \times I) \to X \times I$$
is a fibration by Remark 11.2.5. The claim now follows from the homotopy lifting property of fibrations.
$$\begin{tikzcd} X \times \{0\} \arrow[r] \arrow[d] & F \times_G (F_0 \times I) \arrow[d] \\ X \times I \arrow[r, "\mathrm{Id}"] \arrow[ur, dashed] & X \times I \end{tikzcd}$$
This completes the proof. $\square$

Let $\mathscr{P}(X, G)$ denote the set of isomorphism classes of principal G-bundles over X, and let $\mathscr{P}(G)$ denote the isomorphism classes of all principal G-bundles. The assignment
$$\mathsf{Top} \to \mathscr{P}(G)$$
$$X \mapsto \mathscr{P}(X, G)$$
is a contravariant (set-valued) functor. Indeed, this follows from the pullback construction. Proposition 13.6.6 states that the functor actually descends to the homotopy category:
$$\mathsf{Top} \to \mathscr{P}(G)$$
The homotopy theoretic classification of principal G-bundles argues that this functor is representable. We restrict ourselves to the category CW complexes. The representability below can be generalized to other categories. We need the following facts about CW complexes.

Lemma 13.6.8. *Let (X, Y) be a CW pair and let $p : E \to X$ be a fiber bundle with fiber F. Assume that $p_k(F) = 0$ for each k such that $X \setminus Y$ has cells of dimension $k + 1$.*

(1) *Every map $f : Y \to F$ extends to a map $\bar{f} : X \to F$.*
(2) *Every section $s \in \Gamma(Y, E)$ can be extended to a global section $\bar{s} \in \Gamma(X, E)$. In particular (taking $Y = \emptyset$) $p : E \to X$ admits global sections if F is k-connected where $k = \dim(X)$.*

Remark 13.6.9. *Let $p : E \to X$ be fiber bundle with fiber F. We will invoke the following facts in the proof of Lemma 13.6.8.*

(1) *There is a bijection between local sections defined over a locally trivial cover U_α and smooth maps $U_\alpha \to F$*

This is proved in my other notes.

Proof. The proof is given below:

(1) We have a relative CW-complex structure:
$$Y \subseteq Z^{(-1)} \subseteq Z^{(0)} \subseteq \cdots \subseteq Z^{(m)} = X.$$
We induct on k. Assume that f has been extended to $Z^{(k)}$. The base case $k = -1$ follows by assumption. For each $k+1$-cell $\mathbb{D}^{k+1} \subseteq X$ with attaching map $\varphi : \mathbb{S}^k \to Z^{(k)}$, the composition
$$f \circ \varphi : \mathbb{S}^k \to F$$
is nullhomotopic by assumption. Hence $f \circ \varphi$ can be extended to $\mathbb{D}^{k+1}$ and hence to $Z^{(k)} \cup_\varphi \mathbb{D}^{k+1}$. Extending f in this way for each $k+1$-cell completes the induction.

(2) If $E = X \times F$, the claim follows from (1) and Remark 13.6.9. Generally, we proceed as above by induction on k. Assume a section has been extended to $Z^{(k)}$, so $s \in \Gamma(Z^{(k)}, E)$. Given a $k+1$ cell $\mathbb{D}_{k+1}$ of X, we can subdivide $\mathbb{D}_{k+1} \cong I^{k+1}$ into sufficiently small cubes and reduce to the case $\mathbb{D}_{k+1} \subseteq U_\alpha$, where U_α is a locally trivial open set. The claim follows as above.

This completes the proof. □

We now state and prove the main result regarding classification of principal G-bundles.

Proposition 13.6.10. *Let X be a CW-complex. For a topological group, G, $p_G : EG \to BG$ be a principal G-bundle such that EG is weakly contractible. There is a bijective correspondence*
$$\Phi : [X, BG] \to \mathscr{P}(X, G)$$
$$[f] \mapsto [f^* p_G]$$
BG is called the classifying space for principal G-bundles.

Remark 13.6.11. *We will invoke the following facts on principal G-bundles in the proof of Proposition 13.6.10.*

(1) *There is a bijection between morphisms of G-bundles $E_1 \to X_1$ and $E_2 \to X_2$ and global sections of the associated bundle $E_1 \times_G E_2$[3] over X_1 with fiber E_2.*

This is covered in my other notes.

Proof. Φ is well-defined by Proposition 13.6.6. We first show Φ is surjective. Suppose $p : E \to X$ is a principal G-bundle. We need to find $f : B \to BG$ and a principal G-bunle morphism $\hat{f} : E \to EG$ such that the following diagram commutes:

$$\begin{tikzcd} E \cong f^*(EG) \arrow[r, "\bar{f}"] \arrow[d, "p"] & EG \arrow[d, "p_G"] \\ X \arrow[r, "f"'] & BG \end{tikzcd}$$

This is equivalent to the existence of a global section of the associated bundle $E \times_G EG \to X$ with fiber EG. Since EG is weakly contractible, such a section exists by Lemma 13.6.8(2). We now show Φ is

[3]Here E_2 is endowed with the left action $g \cdot e_2 := e_2 \cdot g^{-1}$

injective. Suppose that $f_0, f_1 : B \to BG$ are two maps such that $E_0 := f_0^*(EG) \cong f_1^*(EG) := E_1$. Let $p_0 : E_0 \to X$ and $p_1 : E_1 \to X$. We show that $f_0 \sim f_1$. We have a commutative diagram:

$$\begin{tikzcd} E_0 \times \{0,1\} \arrow[r, "(\bar{f}_0, \bar{f}_1)"] \arrow[d, "p_0 \times \mathrm{Id}"] & EG \arrow[d, "p_G"] \\ X \times \{0,1\} \arrow[r, "(f_0, f_0)"] & BG \end{tikzcd}$$

We extend it to a commutative diagram:

$$\begin{tikzcd} E_0 \times I \arrow[r, "\bar{H}"] \arrow[d, "p_0 \times \mathrm{Id}"] & EG \arrow[d, "p_G"] \\ X \times I \arrow[r, "H"] & BG \end{tikzcd}$$

This yields the desired homotopy $H : X \times I$ between f_0 and g_1. This is equivalent to finding a section of the associated bundle bundle $(E_0 \times I) \times_G EG \to X \times I$ with fiber EG. We have already have a section of the associated bundle $(E_0 \times \{0,1\}) \times_G EG \to X \times \{0,1\}$. Under the obvious inclusion

$$(E_0 \times \{0,1\}) \times_G EG \subseteq (E_0 \times I) \times_G EG,$$

this section can be regarded as a section of $(E_0 \times I) \times_G EG \to X \times I$ over $X \times \{0,1\}$. Since EG is weakly contractible, the section can be extended via Lemma 13.6.8(2). $\square$

The question remains: how does one construct the universal bundle $EG \to BG$? We will not present a general construction ; rather, we will explicitly find such universal bundles for specific examples. However, we can prove that such a universal bundle is defined uniquely up to homotopy.

Proposition 13.6.12. *Let G be a topological group. Then a universal principal G-bundle $p_G : EG \to BG$ such that EG is weakly contractible exists. Moreover, the construction is functorial in the sense that a continuous group homomorphism $\mu : G \to H$ induces a bundle map*

$$\begin{tikzcd} EG \arrow[r, "E\mu"] \arrow[d, "p_G"] & EH \arrow[d, "p_H"] \\ BG \arrow[r, "B\mu"] & BH \end{tikzcd}$$

Furthermore, the classifying space BG is unique up to homotopy.

Proof. (Sketch) There is a general construction due to Milnor of BG associated to any locally compact topological group G. We don't discuss it here. We first show that BG is unique up to homotopy. Assume we are given two universal principal G-bundles

$$p_G : EG \to BG$$
$$p_{G'} : EG' \to BG'$$

By regarding each as a universal principal G-bundle for the other principal G-bundle, we obtain the following commutative diagram:

$$\begin{tikzcd} EG \arrow[r, "\bar{f}"] \arrow[d, "p_G"] & EG' \arrow[r, "\bar{f}"] \arrow[d, "p_{G'}"] & EG \arrow[d, "p_G"] \\ BG \arrow[r, "g"] & BG' \arrow[r, "f"] & BG \end{tikzcd}$$

By Proposition 13.6.10, $f \circ g = \mathrm{Id}_{BG}$ and $g \circ f = \mathrm{Id}_{BG'}$. This shows uniqueness up to homotopy. Functoriality is clear. ☐

How does one construct the classifying space BG? Note that if $EG \to BG$ is a principal G-bundle, then G acts freely on EG such that $BG \cong EG/G$. Hence, it suffices to find a weakly contractible space EG on which G acts freely.

Example 13.6.13. The following is a list of examples of some classifying spaces:

(1) Let $G = \mathbb{Z}$. We can take $EG = \mathbb{R}$ since $\mathbb{Z}$ acts freely on $\mathbb{R}$ by translations. Hence, we have

$$\begin{aligned} B\mathbb{Z} &\cong E\mathbb{Z}/\mathbb{Z} \\ &\cong \mathbb{R}/\mathbb{Z} \cong \mathbb{S}^1 \end{aligned}$$

(2) Let $G = \mathbb{Z}^n$. We can take $EG = \mathbb{R}^n$ since $\mathbb{Z}^n$ acts freely on $\mathbb{R}^n$ by translations. Hence, we have

$$\begin{aligned} B\mathbb{Z}^n &\cong E\mathbb{Z}^n/\mathbb{Z}^n \\ &\cong \mathbb{R}^n/\mathbb{Z}^n \cong \underbrace{\mathbb{S}^1 \times \cdots \times \mathbb{S}^1}_{n\text{-times}} \end{aligned}$$

(3) Let $G = \mathbb{Z}_2$. We can take $EG = \mathbb{S}^\infty$ since $\mathbb{Z}_2$ acts freely on $\mathbb{S}^\infty$ and $\mathbb{S}^\infty$ is contractible. Hence, we have

$$\begin{aligned} B\mathbb{Z}_2 &\cong E\mathbb{Z}_2/\mathbb{Z}_2 \\ &\cong \mathbb{S}^\infty/\mathbb{Z}_2 \cong \mathbb{RP}^\infty \end{aligned}$$

(4) Let $G = \mathbb{S}^1$. We can take $EG = \mathbb{S}^\infty$ since $\mathbb{S}^1$ acts freely on $\mathbb{S}^\infty$ and $\mathbb{S}^\infty$ is contractible. Hence, we have

$$\begin{aligned} B\mathbb{S}^1 &\cong E\mathbb{S}^1/\mathbb{S}^1 \\ &\cong \mathbb{S}^\infty/\mathbb{S}^1 \cong \mathbb{CP}^\infty \end{aligned}$$

(5) Let $G = \mathrm{O}(k)$. It can be checked that $V_k(\mathbb{R}^\infty)$ is contractible. Hence, we can take $EG = V_k(\mathbb{R}^\infty)$ since $\mathrm{O}(k)$ acts freely on $V_k(\mathbb{R}^\infty)$. Hence, we have

$$\begin{aligned} B\,\mathrm{O}(k) &\cong E\,\mathrm{O}(k)/\,\mathrm{O}(k) \\ &\cong V_k(\mathbb{R}^\infty)/\,\mathrm{O}(k) \cong G_k(\mathbb{R}^\infty) \end{aligned}$$

(6) Let $G = \mathrm{U}(k)$. It can be checked that $V_k(\mathbb{C}^\infty)$ is contractible. Hence, we can take $EG = V_k(\mathbb{C}^\infty)$ since $\mathrm{U}(k)$ acts freely on $V_k(\mathbb{C}^\infty)$. Hence, we have

$$\begin{aligned} B\,\mathrm{U}(k) &\cong E\,\mathrm{U}(k)/\,\mathrm{U}(k) \\ &\cong V_k(\mathbb{C}^\infty)/\,\mathrm{U}(k) \\ &\cong G_k(\mathbb{C}^\infty) \end{aligned}$$

Remark 13.6.14. *The following observation is quite useful. Since $EG \to BG$ is a principal G-bundle, the long exact sequence in homotopy associated to a fibration reads:*

$$\cdots \to \pi_{n+1}(BG) \to \pi_n(G) \to \pi_n(EG) \to \pi_n(BG) \to \pi_{n-1}(G) \to \dots$$

As EG is weakly contractible, $\pi_n(EG) = 0$ for $n > 0$. Hence, we see that

$$\pi_{n+1}(BG) \cong \pi_n(G)$$

for $n \geq 1$.

13.7. Eilenberg-Maclane Spaces

We can use homotopy theory to show that the singular cohomology functor is representable in the homotopy category. If G is an abelian group, assume there exists a topological space Z_n such that

$$H^n(X,G) = [X, Z_n]$$

for all topological spaces and all $X \in \mathsf{Top}$. If $X = \mathbb{S}^k$, note that

$$\pi_k(Z_n) = [\mathbb{S}^k, Z_n] = H^n(\mathbb{S}^k, G) = \begin{cases} G & \text{if } n = 0, k \\ 0 & \text{otherwise} \end{cases}$$

Hence, we see that $\pi_k(Z_n)$ is non-trivial for exactly one value of $k \in \mathbb{N}$. This motivates the following definition.

Definition 13.7.1. Let $X \in \mathsf{Top}$, $G \in \mathsf{Grp}$ If X has only one non-trivial homotopy group such that

$$\pi_n(X) \cong G$$

for some $n \in \mathbb{N}$, then X is called an Eilenberg-MacLane space.

A generic Eilenberg-Maclane space is denoted as $K(G, n)$. The question remains: how does one construct an Eilenberg-Maclane space $K(G, n)$. We will not present a general existence and uniqueness argument; rather, we will explicitly find $K(G, n)$ for specific examples. We first discuss a link between classifying spaces and Eilenberg-Maclane spaces for discrete groups:

Proposition 13.7.2. *Let G be a discrete abelian group. Then $BG \cong K(G, 1)$.*

Proof. Since G is discrete, we have

$$\pi_n(G) = \begin{cases} G & \text{if } n = 0 \\ 0 & \text{otherwise} \end{cases}$$

By Remark 13.6.14, we have $\pi_n(BG) = 0$ for $n \geq 2$. Since $EG \to BG$ is a universal covering map with discrete fibers G, covering space theory implies that $\pi_1(BG) \cong G$. This proves the claim. □

Example 13.7.3. The following is a list of Eilenberg-Maclane space:

(1) $\mathbb{S}^1$ is a model of $K(\mathbb{Z}, 1)$. This follows from Proposition 13.7.2 and that $B\mathbb{Z} \cong \mathbb{S}^1$.
(2) $\mathbb{RP}^\infty$ is a model of $K(\mathbb{Z}_2, 1)$. This follows from Proposition 13.7.2 and that $B\mathbb{Z}_2 \cong \mathbb{RP}^\infty$.
(3) More generally, the cyclic group $\mathbb{Z}/m$ acts on $\mathbb{S}^\infty$ when thought of as a direct limit of spheres in complex vector spaces $\mathbb{C}^n$, where the action is by multiplication of each coordinate by $e^{2\pi i/m}$. We have a principal $\mathbb{Z}/m$-bundle:

$$\mathbb{Z}/m \to \mathbb{S}^\infty \to \mathbb{S}^\infty/\mathbb{Z}/m$$

The quotient $S^\infty/\mathbb{Z}/m$ is called the infinite-dimensional lens space, which is a $K(\mathbb{Z}/m, 1)$.
(4) $\mathbb{CP}^\infty$ is a model of $K(\mathbb{Z}, 2)$. This follows from Example 13.5.8.
(5) The wedge sum of n-circles is a model space for $K(F_n, 1)$, where F_n is the free group on n generators. Clearly, we have

$$\pi_1\left(\bigvee_{i=1}^n \mathbb{S}^1\right) \cong F_n$$

Moreover, the higher homotopy groups of a wedge sum of n-circles vanish since its universal covering space, which is the Cayley graph on n generators, is contractible. By Proposition 13.7.2, we also have

$$BF_n \cong \bigvee_{i=1}^{n} \mathbb{S}^1$$

The following is a basic list of properties of $K(G,1)$.

Proposition 13.7.4. *Let G be a group.*

(1) *If G' is another group, we have*

$$K(G,n) \times K(G',n) \cong K(G \times G', n)$$

(2) *$K(G,1)$ exists for any finitely-generated abelian group.*
(3) *If $X \cong K(G,n)$, then $\Omega X \cong K(G,n-1)$.*

Proof. The proof is given below:

(1) This is clear π_n is a functor that preserves products.
(2) We have constructed a $K(\mathbb{Z},1)$ and a $K(\mathbb{Z}/m,1)$ for each $m \geq 2$. Thus, we can construct a $K(G,1)$ for any finitely generated G by (1).
(3) This follows because $\pi_n(\Omega X) \cong \pi_{n+1}(X)$

This completes the proof. □

Remark 13.7.5. *If G is a finitely-generated abelian group, then if G has torsion, then the $K(G,1)$ contains an infinite-dimensional lens space in the product. Since a $K(G,1)$ is unique up to homotopy equivalence (assumed without proof), a finite-dimensional $K(G,1)$ cannot exist if G is finitely generated and has torsion.*

We now use homotopy theory to show that the singular cohomology functor is representable in the homotopy category in terms of Eilenberg-Maclane spaces. We will invoke the definition of reduced cohomology. We first prove the following lemma:

Lemma 13.7.6. *Let h^* be an unreduced cohomology theory with $\mathbb{Z}$ coefficients defined as a collection of functors*

$$h^n : \mathsf{CW}^2 \to \mathsf{Ab}$$

If $h^n(*;\mathbb{Z}) \cong 0$ for $n \neq 0$, then there exists a natural isomorphism

$$h^n(X,A) \cong H^n(X,A;G)$$

for all CW-pairs (X,A) and for all $n \geq 1$, where $G := h^0(*;\mathbb{Z}) \in \mathsf{Ab}$.

Proof. (Sketch) The proof is similar to the proof for homology theories defined on CW^2. (Proposition 6.4.1). See [Hat02] for the difference that needs to be accounted for. □

Remark 13.7.7. *There is also a version of Lemma 13.7.6 for reduced cohomology.*

We now prove the desired result:

Proposition 13.7.8. *Let H^* be an unreduced singular cohomology theory with $\mathbb{Z}$ coefficients defined as a collection of functors*

$$H^n : \mathsf{CW}^2_* \to \mathsf{Ab}$$

There exists a natural isomorphism

$$T_n : H^n(X;G) \to [X, K(G,n)]_*$$

for all $X \in \mathsf{CW}_*$ and any abelian group G for all $n \geq 1$.

Proof. Using Proposition 13.7.4 we have that $\Omega K(G,n) \cong K(G,n-1)$. Define the functors

$$\begin{aligned} L^n : \mathsf{CW}_* &\to \mathsf{Ab} \\ X &\mapsto [X, K(G,n)]_* \end{aligned}$$

We claim that these functors define a reduced cohomology theory on CW_*.

(1) **(Homotopy invariance)** A map $f : X \to Y$ induces a map
$$f^* : [Y, K(G,n)]_* \to [X, K(G,n)]_*$$
which depends only on the basepoint-preserving homotopy class. It can be checked that f^* is indeed a homomorphism by replacing $K(G,n)$ with $\Omega K(G,n+1)$.

(2) **(Wedge sum axiom)** Let $i_\alpha : X_\alpha \hookrightarrow \bigvee_{\alpha\in A} X_\alpha$ be the inclusion. We want to show that the map
$$\prod_{\alpha\in A} i_\alpha^* : \Big[\bigvee_{\alpha\in A} X_\alpha, K(G,n)\Big]_* \to \prod_{\alpha\in A} [X_\alpha, K(G,n)]_*$$
is an isomorphism for all n. This follows from **??**.

(3) (**Suspension Axiom**) We have
$$\begin{aligned} L^{n+1}(\Sigma X) &= [\Sigma X, K(G,n+1)]_* \\ &= [X, \Omega K(G,n+1)]_* \\ &= [X, K(G,n)]_* \\ &= L^n(X) \end{aligned}$$
for all n. Hence, the suspension axiom holds.

(4) (**Long Exact Sequence**) This follows from the coexact Puppe sequence (which is not included in the notes for now).

Hence, we have an reduced cohomology theory. The reduced version of Lemma 13.7.6 shows that there exists natural isomorphism
$$T_n : H^n(X;G) \to [X, K(G,n)]_*$$
This completes the proof. □

Remark 13.7.9. *In the proof Proposition 13.7.8 we used the fact that the family of spaces $\{K(G,n)\}_{n\geq 0}$ for a fixed $G \in \mathsf{Ab}$ is such that*
$$K(G,n) \cong \Omega K(G,n+1)$$
We say that $\{K(G,n)\}_{n\geq 0}$ is an Ω-spectrum. This suggests that Ω-spectrum can be used to defined cohomology theories. This is the start of the study of spectra in stable homotopy theory.

CHAPTER 14

Serre Spectral Sequence

The Serre spectral sequence is a fundamental computational tool in algebraic topology, particularly in the study of fibrations. It provides a systematic method for relating the homology or cohomology of a total space with those of its base and fiber, facilitating calculations that would otherwise be intractable. By organizing complex algebraic information into successive approximations, the spectral sequence reveals deep connections between the topology of fiber bundles and their constituent spaces, playing a central role in the analysis of fibered spaces and their invariants. References include [Hat04; McC01].

14.1. Construction

The Serre spectral sequence is a powerful computational tool in algebraic topology that arises in the study of the homology and cohomology of fibrations. It allows one to relate the (co)homology of the total space of a fibration to that of its base and fiber, often turning otherwise intractable computations into manageable ones. We present the Serre spectral sequence and illustrate its use through examples and applications. We will treat the general theory of spectral sequences largely as a black box, relying on established results without reproving them here.[1]

Remark 14.1.1. *There is a version of the Serre spectral sequence for both homology and cohomology. In these notes, we focus on the cohomological version, as it is the one most commonly used in practice. The homological version is very similar in structure and can be invoked when needed. For further details, the reader is referred to [Hat04].*

[1]Some of these general results are developed in more detail in my other notes.

CHAPTER 15

Appendix

This appendix provides a concise overview of key categorical concepts fundamental to homological algebra. It discusses the Hom functor and tensor product functor, highlighting their roles as bifunctors and their adjoint relationships. The treatment extends to derived functors, explaining their construction and significance in capturing higher-dimensional algebraic information. These categorical tools underpin much of the algebraic topology developed in this document. Throughout, let A be a locally small abelian category to ensure that the Hom functors are set-valued.

15.1. Hom Functors

We briefly review the Hom functors.

Definition 15.1.1. Let $A \in \mathsf{A}$. The Hom functor $\mathrm{Hom}(A, -) : \mathsf{A} \to \mathsf{Ab}$, is defined by

$$\mathrm{Hom}(A, -)(B) = \mathrm{Hom}(A, B),$$

for all $B \in \mathsf{A}$.

Let's verify that $\mathrm{Hom}(A, -)$ is indeed a functor.

Lemma 15.1.2. *For $A \in \mathsf{A}$, $\mathrm{Hom}(A, -)$ is a covariant functor.*

Proof. If $f : B \to B'$ is a morphism A, then $\mathrm{Hom}(A, -)(f) : \mathrm{Hom}(A, B) \to \mathrm{Hom}(A, B')$ is given by $h \mapsto f \circ h$. Note that the composite $f \circ h$ makes sense:

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[rr, bend right, "f\circ h"'] & B \arrow[r, "f"] & B' \end{tikzcd}$$

We call $\mathrm{Hom}(A, -)(f)$ the induced map, and we denote it by f_*. If f is the identity map $1_B : B \to B$, then

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[rr, bend right, "h"'] & B \arrow[r, "1_B"] & B \end{tikzcd}$$

Hence so that $(1_B)_* = 1_{\mathrm{Hom}(A,B)}$. Suppose now that $g : B' \to B''$. We have the following diagram:

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[rr, bend right, "f\circ h"'] \arrow[rrr, bend left, "(g\circ f)\circ h"] & B \arrow[r, "f"] & B' \arrow[r, "g"] \arrow[r, bend right, "g\circ(f\circ h)"'] & B'' \end{tikzcd}$$

Clearly, $g \circ (f \circ h) = (g \circ f) \circ h$ Therefore, we have $(g \circ f)_* = g_* \circ f_*$. $\square$

We now discuss the contravariant Hom functor.

Definition 15.1.3. Let $B \in \mathsf{A}$. The contravariant Hom functor $\mathrm{Hom}(A, -) : \mathsf{A} \to \mathsf{Ab}$, is defined by

$$\mathrm{Hom}(-, A)(B) = \mathrm{Hom}(A, B),$$

for all $A \in \mathsf{A}$.

Remark 15.1.4. *It can be verified, similarly to Lemma 15.1.2, that the contravariant Hom functor is indeed a well-defined contravariant functor.*

We now show that the Hom functors are are also left and right exact depending on the choice of the Hom functor.

Proposition 15.1.5. *Let $A \in \mathsf{A}$.*

(1) *The functor $\mathrm{Hom}(A, -) : \mathsf{A} \to \mathsf{Ab}$ is a left exact functor.*
(2) *The functor $\mathrm{Hom}(-, A) : \mathsf{A} \to \mathsf{Ab}$ is a left exact functor.*

Proof. The proof is as follows:

(1) Let

$$0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$$

be an exact sequence in A. Applying $\mathrm{Hom}(A, -)$, which we denote as h_A in the rest of the proof, we obtain homomorphisms

$$0 \to \mathrm{Hom}_{\mathsf{A}}(A, X) \xrightarrow{h_A(f)} \mathrm{Hom}_{\mathsf{A}}(A, Y) \xrightarrow{h_A(g)} \mathrm{Hom}_{\mathsf{A}}(A, Z)$$

of abelian groups. We claim that this sequence is exact. If $h_A(f)(\alpha) = 0$, then $f \circ \alpha = 0$, but f is a monomorphism, so $\alpha = 0$.

$$\begin{tikzcd}
& A \arrow[d, "\alpha"] \arrow[dr, "0"] & \\
0 \arrow[r] & X \arrow[r, "f"] & Y
\end{tikzcd}$$

Since h_A is a functor, we have $h_A(g) \circ h_A(f) = 0$. If $\beta \in \ker h_A(g)$, then $g \circ \beta = 0$. The universal property of the kernel implies that β factors through a morphism $X \to \ker g$.

$$\begin{tikzcd}
& & A \arrow[d, "\beta"] \arrow[dr, "0"] & & \\
0 \arrow[r] & X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r] & 0
\end{tikzcd}$$

But we have canonical isomorphisms

$$X \xrightarrow{\sim} \mathrm{coim}\, f \xrightarrow{\sim} \mathrm{im}\, f \xrightarrow{\sim} \ker g$$

the first as f is a monomorphism, the second by the first isomorphism theorem in a small abelian category and the third because the sequence is exact at Y. The composite of the composite of these with the canonical morphism $\ker g \to B$ is g.

$$\begin{tikzcd}
& A \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\
X \arrow[rr, "f"] & & Y
\end{tikzcd}$$

Therefore, we obtain a morphism $\alpha : X \to A$ satisfying $f \circ \alpha = \beta$.

(2) The statement in (2) is the dual of the statement in (1).

This completes the proof. $\square$

In fact, as the next lemma shows, exactness of a sequence can be checked by studying all possible Hom functors. More precisely:

Proposition 15.1.6. *Let* A *be a small abelian category. A sequence*

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

is exact if every sequence

$$\mathrm{Hom}(A, X) \xrightarrow{h_A(f)} \mathrm{Hom}(A, Y) \xrightarrow{h_A(g)} \mathrm{Hom}_{\mathsf{A}}(A, Z)$$

is exact for each $A \in \mathsf{A}$.

Proof. For $A = X$, we get

$$g \circ f = h_X(g) \circ h_X(f)(\mathrm{id}_A) = 0,$$

so we have a monomorphism $s : \mathrm{im}\, f \to \ker g$.

$$\begin{tikzcd} X \arrow[d, "\mathrm{Id}_X"'] & & \\ X \arrow[r, "f"] \arrow[rr, bend right, "0"'] & Y \arrow[r, "g"] & Z \end{tikzcd}$$

For $A = \ker g$ and $\iota : \ker g \hookrightarrow Y$, we have $h_X(g)(\iota) = g \circ \iota = 0$, so there exists $\alpha : \ker g \to X$ with $f \circ \alpha = \iota$.

$$\begin{tikzcd} & \ker g \arrow[dl, "\alpha"'] \arrow[d, hook, "\iota"] & \\ X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \end{tikzcd}$$

Then ι factors as a morphism $t : \ker g \to \mathrm{im}\, f$ which is the inverse to s. □

Corollary 15.1.7. *Let* A *be a small abelian category. A sequence*

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

is exact if every sequence

$$\mathrm{Hom}(Z, A) \xrightarrow{h_A(g)} \mathrm{Hom}(Y, A) \xrightarrow{h_A(f)} \mathrm{Hom}_{\mathsf{A}}(X, Z)$$

is exact for each $A \in \mathsf{A}$.

Proof. The statement is dual to the statement in Proposition 15.1.6. □

Example 15.1.8. The functor $\mathrm{Hom}(A, -)$ need not be right exact. To see this, let $\mathsf{A} = \mathsf{Ab}$ be the category of abelian groups and let $A = \mathbb{Z}/2\mathbb{Z}$. Consider the short exact sequence:

$$0 \to \mathbb{Z} \xrightarrow{\mathsf{Dot2}} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/2\mathbb{Z} \to 0$$

Applying $\mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}, -)$ and noting that

$$\mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) = 0$$
$$\mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z},$$

we obtain the sequence:

$$0 \to 0 \to 0 \to \mathbb{Z}/2\mathbb{Z} \to 0$$

This sequence is not right exact since $0 \to \mathbb{Z}/2\mathbb{Z}$ is not a surjective map.

15.2. Tensor Product Functor

Let's now introduce the tensor product functor. We assume the construction of the tensor product functor is known. Note that the tensor product functor is only defined in $\mathbf{Mod}_R$, the category of left R-modules. In what follows, we assume that R is a commutative ring, so we do not need to distinguish between left and right R-modules. Using a clever argument exploiting the adjunction between the Hom and tensor product functors, we can show the following:

Proposition 15.2.1. *Let R be a ring and let $\mathbf{Mod}_R$ be the category of left R-modules. Let M be a right R-module. The functor $M \otimes_R -$ is a right exact functor.*

Proof. Let $0 \to A \to B \to C \to 0$ be an exact sequence in $\mathbf{Mod}_R$. We show that

$$M \otimes_R A \to M \otimes_R B \to M \otimes_R C \to 0$$

is an exact sequence. Proposition 15.1.5 and Corollary 15.1.7 imply that

$$M \otimes_R A \to M \otimes_R B \to M \otimes_R C \to 0$$

is an exact sequence if and only if

$$0 \to \operatorname{Hom}(M \otimes_R C, X) \to \operatorname{Hom}(M \otimes_R B, X) \to \operatorname{Hom}(M \otimes_R A, X)$$

is an exact sequence for each left R-module X. We have

$$\operatorname{Hom}(M \otimes_R N, X) = \operatorname{Hom}(N, \operatorname{Hom}(M, X)),$$

for all R-modules N. Hence, the sequence above can be written as

$$0 \to \operatorname{Hom}(C, \operatorname{Hom}(M, X)) \to \operatorname{Hom}(B, \operatorname{Hom}(M, X)) \to \operatorname{Hom}(A, \operatorname{Hom}(M, X))$$

which is indeed exact by Proposition 15.1.5. □

Example 15.2.2. The functor $M \otimes_R -$ need not be left exact functor. To see this, take $R = \mathbb{Z}$. Consider the sequence:

$$0 \to \mathbb{Z} \hookrightarrow \mathbb{Q}$$

Letting $M = \mathbb{Z}$ and noting that,

$$\begin{aligned} \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q} &\cong 0 \\ \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z} &\cong \mathbb{Z}, \end{aligned}$$

we obtain the sequence:

$$0 \to \mathbb{Z} \to 0$$

which is not left exact since the map $\mathbb{Z} \to 0$ is not a surjective map.

15.3. Projective & Injective Objects

We now introduce special objects that can rectify the failure of the exactness of the Hom and tensor product functors.

15.3.1. Projective Objects.

We first define the notion of projective objects.

Definition 15.3.1. An object $P \in \mathsf{A}$ is called projective if the functor $\mathrm{Hom}(P, -)$ is an exact functor.

Remark 15.3.2. *An object P is projective if and only if for every morphism $Y \to Z \to 0$ and $P \to Z$, there exists a morphism $P \to Y$ such that the diagram*

$$\begin{tikzcd}
 & P \arrow[dl, dashed] \arrow[d] & \\
Y \arrow[r] & Z \arrow[r] & 0
\end{tikzcd}$$

commutes.

Example 15.3.3. The following are examples of projective objects:

(1) The zero object in a small abelian category is projective.
(2) In $\mathbf{Mod}_R$, the object R is projective: indeed, the functor

$$\begin{aligned}\mathrm{Hom}(R, -) : \mathrm{Mod}\, R &\to \mathsf{Ab} \\ M &\mapsto M\end{aligned}$$

is just the forgetful functor, and hence clearly is exact.

Proposition 15.3.4. *An object P in A is projective if and only if every exact sequence*

$$0 \to X \xrightarrow{f} Y \xrightarrow{p} P \to 0$$

in A splits.

PROOF. Skipped. □

Proposition 15.3.5. *A direct summand of a projective object is a projective object. Moreover, an arbitrary direct sum of projective objects is a projective object*

PROOF. Let $P_1, P_2 \in \mathsf{A}$ such that $P_1 \oplus P_2$ such that $P_1 \oplus P_2$ is a projective object. Consider an epimorphism $f : Y \to Z \to 0$ and a morphism $\beta : P_1 \to Z$. Along with the zero morphism from P_2 to Z, the universal property of the co-product implies that there is a unique morphism $\gamma : P_1 \oplus P_2 \to Z$. Since $P_1 \oplus P_2$ is a projective object, there is a morphism $\gamma' : P_1 \oplus P_2 \to Y$ such that the diagram

$$\begin{tikzcd}
 & P \arrow[d, "\iota_{P_1}"] \arrow[ddl, dashed] \arrow[dd, bend left, "\beta"] & \\
 & P \oplus Q \arrow[d, dashed, "\gamma"'] \arrow[dl, dashed, "\gamma'"] & Q \arrow[l, "\iota_{P_2}"'] \arrow[dl, "0"] \\
Y \arrow[r, "f"] & Z \arrow[r] & 0
\end{tikzcd}$$

commutes. The required morphism is then $\gamma' \circ \iota_{P_1}$. A similar argument as above shows that a direct sum of projective objects is a projective object. □

Projective objects in $\mathbf{Mod}_R$ can be easily characterized in terms of free R-modules, which we now define:

Definition 15.3.6. A left R-module, F, is a **free module** if it is isomorphic to an arbitrary direct sum of copies of R as a left R-module. That is,

$$F \cong \bigoplus_{i \in I} R := R^I$$

Remark 15.3.7. *Any free R-module F has a basis B in bijection with its indexing set, and therefore a map $F \to A$ for some left R-module A is prescribed uniquely by its (arbitrary) values on B.*

$$\mathrm{Hom}_{\mathbf{Mod}_R}(F, A) = \mathrm{Hom}_{\mathbf{Sets}}(B, A)$$

Proposition 15.3.8. *A free R-module, F, is a projective module.*

Proof. Consider $\beta : F \to Z$ and a surjective R-module homomorphism[1] $f : Y \to Z \to 0$. Let B be a basis for F.

$$\begin{tikzcd} & F \arrow[dl, dashed, "g"'] \arrow[d, "\beta"] & \\ Y \arrow[r, "f"] & Z \arrow[r] & 0 \end{tikzcd}$$

For each $b \in B$, the element $\beta(b) \in Z$ has the form $f(b) = p(a_b)$ for some $a_b \in A$, because f is surjective. By the Axiom of Choice, there is a function $u : B \to Y$ with $u(b) = a_b$ for all $b \in B$. By the remark above, we have an R-homomorphism $g : F \to Y$ with $g(b) = a_b$ for all $b \in B$. Clearly, g is the required morphism. $\square$

Proposition 15.3.9. *The following statements are equivalent:*

(1) *P is projective in $\mathbf{Mod}_R$.*
(2) *There is a module Q such that $P \oplus Q \cong R^I$ for some set I. The module R^I is a called a free module.*

Proof. Assume that P is a projective object and let I be the set of generators of P and let R^I denote a free module on the set of generators of P. Consider the natural map $\pi : R^I \to P$. It clearly is a surjective, and, since P is projective, it splits. Therefore,

$$P \oplus \ker \pi \cong R^I$$

The converse follows since a free module is projective and a direct summand of projective module is a projective module by Proposition 15.3.5. $\square$

Remark 15.3.10. *Every projective module need not be free. For example, consider*

$$R = \mathbb{Z}/6\mathbb{Z} = \mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$$

$\mathbb{Z}/3\mathbb{Z}$ is a projective $\mathbb{Z}/6\mathbb{Z}$-module since $\mathbb{Z}/6\mathbb{Z}$ is a projective $\mathbb{Z}/6\mathbb{Z}$-module. However, $\mathbb{Z}/3\mathbb{Z}$ is not a free $\mathbb{Z}/6\mathbb{Z}$-module: a (finitely generated) free $\mathbb{Z}/6\mathbb{Z}$-module F is a direct sum of, say, n copies of $\mathbb{Z}/6\mathbb{Z}$, and so F has 6^n elements. Therefore, $\mathbb{Z}/3\mathbb{Z}$ is not a free $\mathbb{Z}/6\mathbb{Z}$ since it has only three elements.

Example 15.3.11. Let $\mathsf{A} = \mathsf{Ab}$. The functor $\mathrm{Hom}(\mathbb{Z}, -)$ is an exact functor. This is because $\mathbb{Z}$ is a free object in Ab.

Example 15.3.12. Let $\mathsf{A} = \mathsf{Ab}$. The functor $\mathrm{Hom}(\mathbb{Q}, -)$ is not an exact functor. This is because $\mathbb{Q}$ is not a projective object in Ab since $\mathbb{Q}$ cannot be a summand of a free $\mathbb{Z}$-module because a free $\mathbb{Z}$-module is not divisible but $\mathbb{Q}$ is a divisible group.

[1] Epimorphisms and surjective R-module homomorphisms coincide in the category of R-modules.

15.3.2. Injective Objects.

We now define the notion of injective objects.

Definition 15.3.13. Let A be a small abelian category. An object $I \in \mathsf{A}$ is called injective if the functor $\operatorname{Hom}(-, I)$ is an exact functor.

Remark 15.3.14. *Injective objects in A are just projective objects in A^{op}.*

Remark 15.3.15. *An object I is injective if and only if for every morphisms $0 \to X \to Y$ and $X \to Y$, there exists a unique morphism $Y \to I$ such that the diagram*

$$\begin{tikzcd} & I & \\ 0 \arrow[r] & X \arrow[u] \arrow[r] & Y \arrow[ul, dashed] \end{tikzcd}$$

commutes.

Proposition 15.3.16. *Let A be a small abelian category. An object I in A is injective if and only if every exact sequence*

$$0 \to I \xrightarrow{i} Y \xrightarrow{f} Z \to 0$$

in A splits.

Proof. Skipped. □

Proposition 15.3.17. *A direct summand of an injective object is an injective object. Moreover, an arbitrary product of injective objects is an injective object.*

Proof. Let $I_1, \oplus I_2 \in \mathsf{A}$ such that $I_1 \oplus I_2$ is an injective object. Consider a monomorphism $f : 0 \to X \to Y$ and a morphism $\gamma : X \to I_1$. Note that $\iota_1 \circ \gamma$ is a morphism from X to $E_1 \oplus E_2$, where ι_1 is the canonical inlcusion map. Since $E_1 \oplus E_2$ is injective, there is a morphism $\gamma' : Y \to E_1 \oplus E_2$ such that the diagram

$$\begin{tikzcd} & E_1 \arrow[r, bend left, "\iota_1"] & E_1 \oplus E_2 \arrow[l, bend left, "\pi_1"] \\ 0 \arrow[r] & X \arrow[u, "\gamma"] \arrow[r, "f"] & Y \arrow[u, dashed, "\gamma'"'] \end{tikzcd}$$

commutes. Then $\pi_1 \circ \gamma'$ is the required morphism, where π_1 is the canonical projection map. A similar argument as above shows that a product of injective objects is an injective object. □

We now characterize injective objects in $\mathbf{Mod}_R$.

Proposition 15.3.18. ***(Baer's criterion)*** *An R-module, I, is injective if and only if for every left ideal $J \subseteq R$ and every R-module homomorphism $g : J \to I$, there exists $g' : R \to I$ such that the following diagram*

$$\begin{tikzcd} 0 \arrow[r] & J \arrow[r] \arrow[d, "g"] & R \arrow[dl, dashed, "g'"] \\ & I & \end{tikzcd}$$

commutes.

Proof. The forward implication is clear. For the reverse implication, consider the diagram:

$$\begin{tikzcd} 0 \arrow[r] & J \arrow[r] \arrow[d, "g"] & R \\ & I & \end{tikzcd}$$

Consider the set of all intermediate extensions:

$$S = \{(C, h) \mid J \subseteq C \subseteq R \text{ submodule, } h \in \mathrm{Hom}(C, I) \text{ and } h|_J = g\}$$

Set $(C, h) \leq (C', h')$ if and only if $C \subseteq C'$ and $h'|_C = h$. Note that $S \neq \emptyset$ because we can choose $C = J$. Suppose $\{(C_x, h_x)\}_{x \in I}$ is a chain for an index set I such that for any $x, y \in I$, $(C_x, h_x) \leq (C_y, h_y)$. Let

$$C = \bigcup_{x \in I} C_x$$

and define $h : C \to I$ by setting $h(a) = h_x(a)$ if $a \in C_x$ for some $x \in I$. This is well-defined by assumption, and $h|_{C_x} = h_x$ for any $x \in I$. Hence $(C_x, h_x) \leq (C, h)$ for any $x \in I$, showing that (C, h) is an upper bound. By Zorn's lemma, the chain has a maximal element, (C, h). If $C = R$, we are done. Otherwise, let $b \in R \setminus C$. Consider the sequence:

$$0 \to J \xrightarrow{f_1} R \oplus C \xrightarrow{f_2} Rb \oplus C \to 0 \qquad f_2(r, c) = rb + c \quad f_1(r) = (r, -rb)$$

where $J = \{a \in R \mid ab \in C\}$. Let $g : J \to I$, $g(a) = h(ab)$ and hence there exists a g' such that the diagram

$$\begin{tikzcd} 0 \arrow[r] & J \arrow[r] \arrow[d, "g"] & R \arrow[dl, dashed, "g'"] \\ & I & \end{tikzcd}$$

commutes. Consider a morphism:

$$\begin{aligned} \hat{h} : Rb \oplus C &\to I \\ rb + c &\mapsto h(c) + rg'(1) \end{aligned}$$

We show that $\hat{h}$ is well-defined. If $rb + c = r'b + c'$, then $(r - r')b = c' - c \in C$. It follows that $(r - r') \in J$. Therefore, $h((r - r')b)$ and $g(r - r')$ are defined. Moreover,

$$h(c' - c) = h((r - r')b) = g(r - r') = g'(r - r') = (r - r')g'(1).$$

Thus,

$$h(c') - h(c) = rg'(1) - r'g'(1),$$

which implies that

$$h(c') + r'g'(1) = h(c) + rg'(1)$$

Clearly, $\hat{h}(c) = h(c)$ so $\hat{h}'$ extends h. With $\hat{C} = Rb + c$, we have that $(C, h) \leq (\hat{C}, \hat{h})$, so $(C, h) = (\hat{C}, \hat{h})$. Hence $b \in C$, a contradiction. This completes the proof. ☐

Example 15.3.19. The following are examples of injective objects as can be easily deduced from Proposition 15.3.18.

(1) $\mathbb{Z}/n\mathbb{Z}$ is an injective $\mathbb{Z}/n\mathbb{Z}$-module for any $n \geq 1$.
(2) $\mathbb{Z}/3\mathbb{Z}$ is an injective $\mathbb{Z}/6\mathbb{Z}$-module, but not an injective $\mathbb{Z}/9\mathbb{Z}$-module.
(3) $\mathbb{Q}$ is an injective $\mathbb{Z}$-module. A homomorphism $f \colon n\mathbb{Z} \to \mathbb{Q}$ extends to a homomorphism $g \colon \mathbb{Z} \to \mathbb{Q}$. Just take $y \in \mathbb{Q}$ such that $ny = f(n)$ and define $g(z) = zy$.

Corollary 15.3.20. *Let $A \in \mathsf{Ab}$. A is an injective $\mathbb{Z}$-module if and only if A is a divisible group.*

Proof. Assume that A is an injective $\mathbb{Z}$-module. Let $a \in A$ and $n \in \mathbb{Z}$. Consider the group homomorphism

$$\begin{aligned} f\colon n\mathbb{Z} &\to \mathbb{Z} \\ n &\mapsto a \end{aligned}$$

By assumption, f extends to a group homomorphism

$$\hat{f} : \mathbb{Z} \to I$$

such that $\hat{f}(nk) = f(nk)$ for each $k \in \mathbb{Z}$. Note that we have

$$a = f(n) = \hat{f}(n\mathsf{D}ot1) = n\hat{f}(1)$$

Hence, A is a divisible group. Conversely, assume that A is a divisible group. We show that the criterion in Proposition 15.3.18 is satisfied. Let $J \subseteq \mathbb{Z}$ be an abelian subgroup and let $g : J \to A$ be a group homomorphism. Let $\{((K, g')\}$ be the set of pairs (K, g') such that $J \subseteq K \subseteq \mathbb{Z}$ and $g' : K \to \mathbb{Z}$ is a homomorphism with $g'|_U = g$. The set is non-empty since as it contains (J, J), and it is partially ordered by

$$(K_1, g_1') \leq (K_2, g_2') \quad \Leftrightarrow \quad K_1 \subseteq K_2 \text{ and } g_2'|_{K_1} = g_1'.$$

It is clear that any ascending chain has an upper bound. By Zorn's Lemma, the set contains a maximal element (K, g'). We claim that $K = \mathbb{Z}$. Suppose not. Let $k \in \mathbb{Z} \setminus K$. If

$$\langle k\rangle \mathsf{A}pK = \{0\},$$

the sum $K + \langle k\rangle$ is in fact a direct sum, and we can extend g' to $K + \langle k\rangle$ by choosing an arbitrary image of k in $\mathbb{Z}$ and extending linearly. This is a contradiction. Hence, assume that

$$nk \in \langle k\rangle \mathsf{A}pK$$

for some $n \neq 0$. Choose n_0 such that n_0 is minimal. Since $n_0 \in K$, and g' is defined on K, $g'(nk)$ is well-defined. Since A is divisible, there exists $a \in A$ such that

$$na = g'(nk).$$

It is now easy to see that we can extend g' to $K + \langle k\rangle$ by defining $g'(k) = a$. This is also a contradiction. □

Example 15.3.21. Let $\mathsf{A} = \mathsf{Ab}$ and let k be a field of characteristic zero. The functor $\mathrm{Hom}(-, k)$ is an exact functor. This is because k is a divisible group since for any $g \in k$ and $n \in \mathbb{Z}$, there exists an $h \in k$ such that $hn = g$. since $\mathbb{Q} \subseteq k$.

15.4. Resolutions & Derived Functors

An arbitrary R-module, M, might be quite complicated to study; however, one can always find a set of (possibly infinite) generator for M[2]. In other words, one can always find a surjective morphism $F^0 \to M \to 0$, where F^0 is a free R-module. Since M is not a free R-module, the morphism

$$F^0 \to M \to 0$$

is in general not injective; indeed, the any non-trivial relationship between generators of M will force the kernel to be non-zero. However, we can repeat the construction as above: if we take a generating set for the kernel of the morphism $F^0 \to M \to 0$, one can always find a morphism $F^1 \to F^0$, which is surjective

[2]A fact we used in a proof in the previous section.

onto the kernel of the morphism $F^0 \to M \to 0$, and where F^1 is a free R-module on the generating set of the kernel of the morphism $F^0 \to M \to 0$. We have the following sequence:

$$F^1 \to F^0 \to M \to 0$$

We can repeat the above process unless it terminates, which only happens when there is no non-trivial relationship among elements generating the free module at the left end of the sequence

$$F^i \to \cdots \to F^1 \to F^0 \to M \to 0$$

This motivates the idea of taking a resolution of an object in a category by special types of objects (free R-modules in the case considered above) in order to study the structure of the original object in the category.

15.4.1. Projective & Injective Resolutions. For a object $X \in \mathsf{A}$, we will first discuss taking a resolution of X in A by projective objects in A. An arbitrary category may not have projective objects, though.

Example 15.4.1. Let $\mathsf{A} = \mathsf{Ab}_{\mathbf{Fin}}$ be the category of finite abelian groups. A has no projective objects except for the trivial abelian group. Indeed, the exact sequence

$$0 \to \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/2n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$$

is non-split, since $\mathbb{Z}/2n\mathbb{Z}$ is not isomorphic to $\mathbb{Z}/n\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. Hence, $\mathbb{Z}/n\mathbb{Z}$ is not projective. But every other non-zero finite abelian group has a direct summand $\mathbb{Z}/n$ and the direct summand of a projective object is a projective object.

This motivates the following definition:

Definition 15.4.2. A has enough projectives if for every $X \in \mathsf{A}$ there exists an epimorphism $f : P \to X \to 0$ where P is a projective object.

Example 15.4.3. Clearly, the category of R-modules has enough projective objects. Indeed, free modules are projective objects and free module exist in abdundance in the category of R-modules.

Definition 15.4.4. A projective resolution of $X \in \mathsf{A}$ is a nonnegative complex $P^\bullet$ together with a morphism $\epsilon : P^0 \to M$ such that

$$\cdots \to P^3 \to P^2 \to P^1 \to P^0 \xrightarrow{\varepsilon} M \to 0$$

is exact and the P^i's are projective objects.

Example 15.4.5. In Ab, the abelian group $\mathbb{Z}/n\mathbb{Z}$ has a projective resolution

$$0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0.$$

Proposition 15.4.6. *If A has enough projectives, then every object has a projective resolution.*

Proof. Take any $X \in \mathsf{A}$. There is an epimorphism $P^0 \to X \to 0$ from a projective object. P^0. Taking the kernel $K^0 \to P^0$, we have a projective P^1 with an epimorphism $P^1 \to K^0 \to 0$. We take its kernel

$K^1 \to P^1$ and again get a projective $P^2 \to K^1 \to$. This way, we get that the diagram:

$$
\begin{tikzcd}
& 0 \arrow[dr] & & 0 & & & & \\
& & K^1 \arrow[ur] \arrow[dr] & & & & & \\
\cdots \arrow[r] & P^2 \arrow[ur] \arrow[rr] & & P^1 \arrow[rr] \arrow[dr] & & P^0 \arrow[r] & X \arrow[r] & 0 \\
& & & & K^0 \arrow[ur] \arrow[dr] & & & \\
& & & 0 \arrow[ur] & & 0 & &
\end{tikzcd}
$$

Continuing, this gives a projective resolution of X. □

We can similarly define injective resolutions.

Definition 15.4.7. A has enough injectives if for every $X \in \mathsf{A}$ there exists a monomorphism $f : 0 \to X \to I$ where I is an injective object.

Proposition 15.4.8. *The category of R-modules has enough injective objects.*

Proof. See [Rot09, Theorem 3.38]. □

Definition 15.4.9. An injective resolution of $X \in \mathsf{A}$ is a non-negative complex $I_\bullet$ together with a morphism $\epsilon : 0 \to X \to I_0$ such that

$$0 \to X \xrightarrow{\varepsilon} I_0 \to I_1 \to \cdots$$

is exact and the I_i's are injective objects.

Proposition 15.4.10. *If A has enough injectives, then every object has a injective resolution.*

Proof. The statement is the dual of the statement in Proposition 15.4.6, so it is clearly true. □

Example 15.4.11. In Ab, an injective resolution of $\mathbb{Z}$ is

$$0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$$

and an injective resolution of $\mathbb{Z}/n\mathbb{Z}$ is

$$0 \to \mathbb{Z}/n\mathbb{Z} \to \mathbb{Q}/\mathbb{Z} \to \mathbb{Q}/\mathbb{Z} \to 0$$

15.4.2. Derived functors. Derived functors provide us with a tool to quantitatively measure the failure of a functor to be an exact functor. The philosophy behind derived functors is the following: if $\mathsf{F} : \mathsf{A} \to \mathsf{D}$ is a left exact functor between two abelian categories, then any short exact sequence in A,

$$0 \to A \to B \to C \to 0,$$

gets transformed to a left exact sequence in D:

$$0 \to \mathsf{F}(A) \to \mathsf{F}(B) \to \mathsf{F}(C),$$

A right derived functor is a a sequence of functors $R^i\mathsf{F} : \mathsf{A} \to \mathsf{D}$ for all $i \geq 0$ and a functorial isomorphism $R^0\mathsf{F} \cong \mathsf{F}$ such that that for any short exact sequence,

$$0 \to A \to B \to C \to 0,$$

in A there is a long exact sequence,

$$0 \to R^0\mathsf{F}(A) \to R^0\mathsf{F}(B) \to R^0\mathsf{F}(C) \to R^1\mathsf{F}(A) \to R^1\mathsf{F}(B) \to R^1\mathsf{F}(C) \to R^2\mathsf{F}(A) \to \cdots,$$

for all $i \geq 0$. We expect that $R^1\mathsf{F}(A)$ to quantitatively measure the failure of F to be a right exact functor since $R^1\mathsf{F}(A) = 0$ if and only if the sequence,

$$0 \to \mathsf{F}(A) \to \mathsf{F}(B) \to \mathsf{F}(C),$$

is a right exact sequence.

On the other hand, if $\mathsf{F} : \mathsf{A} \to \mathsf{D}$ is a right exact sequence, a left derived functor is a sequence of functors $L^i\mathsf{F} : \mathsf{A} \to \mathsf{D}$ along with a functorial isomorphism $L^0\mathsf{F} \cong \mathsf{F}$ yieldsing a long exact sequence

$$\cdots \to L^1\mathsf{F}(A) \to L^1\mathsf{F}(B) \to L^1\mathsf{F}(C) \to L^0\mathsf{F}(A) \to L^0\mathsf{F}(B) \to L^0\mathsf{F}(C) \to 0.$$

The theory of left and right derived functors is quite similar. Therefore, in what follows we shall only focus on left derived functors of covariant functors. The theory of left derived functors of contravariant functors is similar to the theory of left derived functors of covariant functors, which we now describe. Left derived functors are constructed by means of projective resolutions.

Definition 15.4.12. Let A be a locally small abelian category with enough projectives, D be an abelian category, and $\mathsf{F} : \mathsf{A} \to \mathsf{D}$ be a right exact functor. Given $X \in \mathsf{A}$, choose a projective resolution of X:

$$\cdots \to P^3 \to P^2 \to P^1 \to P^0 \xrightarrow{\varepsilon} X \to 0.$$

Apply F to the above complex to obtain (the truncated) complex:

$$\cdots \to \mathsf{F}(P^3) \to \mathsf{F}(P^2) \to \mathsf{F}(P^1) \to \mathsf{F}(P^0).$$

The i-th left derived functor of F is defined as:

$$L^i(\mathsf{F}(X)) = H_i(\mathsf{F}(P^\bullet)).$$

Here $H_i(\mathsf{F}(P^\bullet))$ is the i-th homology (defined similarly to cohomology) of $P^\bullet$,

Remark 15.4.13. *If* F *is a right exact* contravariant *functor, then the left derived functor is defined by taking an* injective resolution*.*

The above definition naturally begs the question: is the definition of a left-derived functor well-defined? If this is the case, the definition of a left-derived functor should be independent of the projective resolution chosen. We show that this is indeed the case.

Proposition 15.4.14. ***(Comparison Theorem)*** *Let* A *be a locally small abelian category with enough projectives and let* $f : X \to Y$ *be a morphism in* A*. Let*

$$\cdots \xrightarrow{d_2^X} P^1 \xrightarrow{d_1^X} P^0 \xrightarrow{d_0^X} X \longrightarrow 0$$

and

$$\cdots \xrightarrow{d_2^Y} Q^1 \xrightarrow{d_1^Y} Q^0 \xrightarrow{d_0^Y} Y \longrightarrow 0$$

be projective resolutions for X *and* Y*. Then there is a sequence of homomorphisms* $f^i : P^i \to Q^i$ *such that the following diagram commutes:*

$$\begin{tikzcd} \cdots \arrow[r, "d_2^X"] & P^1 \arrow[r, "d_1^X"] \arrow[d, "f^1"] & P^0 \arrow[r, "d_0^X"] \arrow[d, "f^0"] & X \arrow[r] \arrow[d, "f"] & 0 \\ \cdots \arrow[r, "d_2^Y"] & Q^1 \arrow[r, "d_1^Y"] & Q^0 \arrow[r, "d_0^Y"] & Y \arrow[r] & 0 \end{tikzcd}$$

Furthermore, any two such extensions of f *are chain homotopic.*

Proof. (Existence) We proceed by induction on i. For the base case, note that since P^0 is projective, the morphism $f \circ d_0^X$ lifts to a unique morphism f^0 such that the right most square in the diagram above commutes. Assume that $f^i : P^i \to Q^i$ has been constructed. Denote by $\ker d_i^X$ and $\ker d_i^Y$ denote the kernels of d_i^X and d_i^Y, respectively. Since d_{i+1}^X factors through $\mathrm{im} d_{i+1}^X$ which is isomorphicm to $\ker d_i^X$, we can think of d_{i+1}^X as mapping into $\ker d_i^X$. Moreover, f^i factors into $\ker d_i^Y$ since $f^{i-1}d_i^X = d_i^Y f^i$. Thus, consider the diagram:

$$\begin{tikzcd} P^{i+1} \arrow[r, "d_{i+1}^X"] \arrow[d, dashed, "f^{i+1}"] & \ker d_i^X \arrow[r] \arrow[d, "f^i"] & 0 \\ Q^{i+1} \arrow[r, "d_{i+1}^Y"] & \ker d_i^Y \arrow[r] & 0 \end{tikzcd}$$

The composition $f^i d_{i+1}^X$ gives a map from P^{i+1} to $\ker d_i^Y$, onto which d_{i+1}^Y surjects. Thus, the map f^{i+1} is furnished by the defining property of the projective object P^{i+1}, completing the induction.

(Uniqueness) To show that two extensions $\{f^i\}$ and $\{g^i\}$ are chain homotopic, we consider the difference $h^i := f^i - g^i$ and construct a chain homotopy $s^i : P^i \to Q^{i+1}$ such that

$$h^i = d_{i+1}^Y s^i + s^{i-1} d_i^X$$

We proceed by induction. Observe that $h^{-1} = f - f \equiv 0$, so that h^0 maps P^0 $\ker d_0^Y$ by the universal property of kernels, and therefore lifts to a map $s^0 : P^0 \to Q^1$ as in the following diagram:

$$\begin{tikzcd} & P^0 \arrow[r, "d_0^X"] \arrow[dl, dashed, "s^0"'] \arrow[d, "h^0"] & A \arrow[d, "h^{-1}"] \arrow[dl, dashed, "0"'] \\ Q^1 \arrow[r, "d_1^Y"'] & \ker d_0^Y \arrow[r] & 0 \end{tikzcd}$$

This gives the base case for the induction. Suppose that $s^i : P^i \to Q^{i+1}$ has been constructed such that $h^i = d_{i+1}^Y s^i + s^{i-1} d_i^X$. It follows that the map $h^{i+1} - s^i d_{i+1}^X$ maps P^{i+1} into $\ker d_{i+1}^Y$ since

$$d_{i+1}^Y(h^{i+1} - s^i d_{i+1}^X) = h^i d_{i+1}^X - (h^i - s^{i-1} d_i^X) d_{i+1}^X = h^i d_{i+1}^X - h^i d_{i+1}^X = 0.$$

$$\begin{tikzcd} & P^{i+1} \arrow[dl, dashed, "s^{i+1}"'] \arrow[d, "h^{i+1} - s^i d_{i+1}^X"] & \\ Q^{i+2} \arrow[r, "d_{i+2}^Y"'] & \ker d_{i+1}^Y \arrow[r] & 0 \end{tikzcd}$$

Thus we have the diagram above and projectivity furnishes the map s^{i+1} such that $d_{i+2}^Y s^{i+1} = h^{i+1} - s^i d_{i+1}^X$. □

As a consequence of the comparison theorem, if $P^\bullet$ is a projective resolution for X and $Q^\bullet$ is a projective resolution for Y such that there is a morphism $f : X \to Y$, we get a well-defined map

$$H_i(F(P^\bullet)) \longrightarrow H_i(F(Q^\bullet)),$$

which is an isomorphism by the chain homotopy conclusion in the comparison theorem. Similarly, we have:

Corollary 15.4.15. *Let* A, D *be abelian categories. The following are corollaries of Proposition 15.4.14:*

(1) *Suppose that $P^\bullet$ and $Q^\bullet$ are projective resolutions of $X \in \mathsf{A}$. Then there is a canonical isomorphism between $H_i(\mathsf{F}(P^\bullet))$ and $H_i(\mathsf{F}(Q^\bullet))$ for each $i \geq 0$.*
(2) *let $\mathsf{F} : \mathsf{A} \to \mathsf{D}$ be a right exact functor. For any $X \in \mathsf{A}$, $L^0\mathsf{F}(X) \cong \mathsf{F}(X)$.*
(3) *Let $\mathsf{F} : \mathsf{A} \to \mathsf{D}$ be a functor. If P is a projective object in A, then $L^i\mathsf{F}(P) = 0$ for $i \geq 1$.*

Proof. The proof proceeds as follows:

(1) If $P^\bullet$ and $Q^\bullet$ are two projective resolutions of $X \in \mathsf{A}$, then $\mathrm{Id}_X : X \to X$ gives rise to unique maps (up to homotopy) by the Proposition 15.4.14 such that the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & P^1 \arrow[r] \arrow[d, bend left, "f^1"] & P^0 \arrow[r] \arrow[d, bend left, "f^0"] & X \arrow[d, bend left, "\mathrm{Id}_X"] \\
\cdots \arrow[r] & Q^1 \arrow[r] \arrow[u, bend left, "g^1"] & Q^0 \arrow[r] \arrow[u, bend left, "g^1"] & X \arrow[u, bend left, "\mathrm{Id}_X"]
\end{tikzcd}$$

commutes. Hence, there are two chain homotopies

$$s : H_\bullet(\mathsf{F}(P^\bullet)) \to H_\bullet(\mathsf{F}(Q^\bullet))$$
$$q : H_\bullet(\mathsf{F}(Q^\bullet)) \to H_\bullet(\mathsf{F}(P^\bullet))$$

such that both sq and qs compose to the identity (by uniqueness up to homotopy). Hence the derived functor is well-defined: for two choices of projective resolutions of objects, the construction yields isomorphic derived functors.

(2) Choose a projective resolution of X:

$$\cdots \to P^3 \to P^2 \to P^1 \to P^0 \xrightarrow{\varepsilon} X \to 0.$$

Since F is right exact, the sequence

$$\mathsf{F}(P^1) \xrightarrow{\varphi} \mathsf{F}(P^0) \xrightarrow{\psi} \mathsf{F}(X) \to 0$$

is exact. Hence, ψ is an epimorphism and ψ is the cokernel of φ. By the first isomorphism theorem,

$$\mathsf{F}(X) \cong \frac{\mathsf{F}(P^0)}{\ker \psi} \cong \frac{\mathsf{F}(P^0)}{\mathrm{im}\varphi} \cong \mathrm{coker}\varphi$$

Hence, we have

$$L^0\mathsf{F}(X) = H_0(\mathsf{F}(P^\bullet)) \cong \mathrm{coker}\varphi \cong \mathsf{F}(X).$$

(3) Consider the projective resolution:

$$\cdots \to 0 \cdots \to 0 \to P \xrightarrow{\mathrm{Id}_P} P \to 0$$

Hence, we consider the homology of the complex:

$$\cdots \to 0 \cdots \to 0 \to \mathsf{F}(P)$$

and it is clear that

$$L^i\mathsf{F}(P) \cong H_i(\mathsf{F}(P^\bullet)) = 0$$

for $i \geq 1$.

□

We now prove the horseshoe lemma (Lemma 15.4.16). The horsehoe lemma allows us to construct a short exact sequence of projective resolutions given a short exact sequence of objects in an abelian category. In the statement and the proof of the horseshoe lemma, for ease of notation we use subscripts instead of superscipts to label indices of all projective objects.

Lemma 15.4.16. ***(Horseshoe Lemma)*** *Let* A *be an abelian category and let*

$$0 \to A \to A' \to A'' \to 0$$

be a short exact sequence in A*. Assume that there are projective resolutions* $P^\bullet$ *and* $(P'')^\bullet$ *of* A *and* A'' *respectively. Then there is projective resolution* $(P')^\bullet$ *of* A' *such that the following diagram commutes.*

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & A & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & P'_1 & \longrightarrow & P'_0 & \longrightarrow & A' & \longrightarrow & 0 \\
 & & \downarrow j & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & P''_1 & \longrightarrow & P''_0 & \longrightarrow & A'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Proof. Composition gives a map $P_0 \to A'$, and a map $P_0^{''} \to A'$ is furnished by projectivity. Using the universal property of the co-product, these combine to give a map $P_0 \oplus P_0^{''} \to A'$, and we set $P'_0 := P_0 \oplus P_0^{''}$. The sequence $P_0 \to P_0 \oplus P''_0 \to P_0^{''}$ is obviously split exact, we will show that the morphism $P_0 \oplus P''_0 \to A'$ is an epimorphism. This follows by applying the snake lemma to the two right most exact columns, yielding a morphism $\ker \varepsilon''_0 \to 0 \to \operatorname{coker} \varepsilon'_0$.

$$\begin{array}{ccccccc}
0 & & 0 & & 0 & & \\
\downarrow & & \downarrow & & \downarrow & & \\
\ker \epsilon_0 & \longrightarrow & P_0 & \xrightarrow{\varepsilon_0} & A & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\ker \epsilon'_0 & \longrightarrow & P_0 \oplus P''_0 & \xrightarrow{\varepsilon'_0} & A' & \longrightarrow & \operatorname{coker}\varepsilon'_0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\ker \epsilon''_0 & \longrightarrow & P''_0 & \xrightarrow{\varepsilon''_0} & A'' & \longrightarrow & 0\text{”} \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
\end{array}$$

By the snake lemma (Proposition 4.6.3), the left most column is exact and the connecting morphism yields a sequence which has a subsequence of the form

$$\cdots \to 0 \to \operatorname{coker} \varepsilon'_0 \to 0 \to \cdots$$

Hence, $\operatorname{coker} \varepsilon'_0 = 0$ and the morphism $P_0 \oplus P''_0 \to A'$ is an epimorphism. We then apply the same procedure to the diagram with kernels to construct $P_1 \oplus P''_1 \to \ker\varepsilon'_0$, where the product is projective and the map is an epimorphism onto the kernel.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
 & P_1 \arrow[r] \arrow[d] \arrow[dr, dashed] & \ker \epsilon_0 \arrow[r] \arrow[d] & P_0 \arrow[r, "\varepsilon_0"] \arrow[d] & A \arrow[r] \arrow[d] & 0 \\
 & P_1 \oplus P_1'' \arrow[r] \arrow[d] & \ker \epsilon_0' \arrow[r] \arrow[d] & P_0 \oplus P_0'' \arrow[r, "\varepsilon_0'"] \arrow[d] & A' \arrow[r] \arrow[d] & 0 \\
 & P_1'' \arrow[r] \arrow[d] \arrow[ur, dashed] & \ker \epsilon_0'' \arrow[r] \arrow[d] & P_0'' \arrow[r, "\varepsilon_0''"] \arrow[d] & A'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 & 0 &
\end{tikzcd}$$

We continue this way iteratively and construct $P_n = P_n \oplus P_n''$ at the n-th step with the desired properties.

□

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