LIE GROUPS

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ABSTRACT. These are notes on Lie groups taken during graduate school. These notes assume a working knowledge of smooth manifold theory. Typos may be present; please send any corrections to junaid.aftab1994@gmail.com.

CONTENTS

Part 1. Lie Groups

Lie groups play a central role in modern geometry, topology, and mathematical physics, offering a natural framework for studying continuous symmetries. We begin by introducing the basic definitions and properties of Lie groups, then move on to their infinitesimal counterparts, Lie algebras. The exponential map, which connects Lie algebras back to Lie groups, is then explored, followed by a discussion of the Baker–Campbell–Hausdorff (BCH) formula and related constructions. Finally, we examine the deep correspondence between Lie groups and Lie algebras, which underpins much of the theory developed in this area.

1. Definition & Examples

A (real) Lie group is a group endowed with the structure of a finite-dimensional smooth manifold. In fact, a (real) Lie group is a group object in the category of finite-dimensional smooth manifolds. Lie groups are the objects that describe continuous symmetries, which is why they are considered to be important.

1.1. Definitions & Examples.

All smooth manifolds considered will be finite-dimensional. Hence, we will simply use the phrase 'smooth manifold' from now on. Therefore, the discussion below applies specifically to finite-dimensional Lie groups.

Definition 1.1. A **(real) Lie group** is a smooth manifold, G, that is also a group such that multiplication map $m : G \times G \to G$ and inversion map $i : G \to G$, given by

$$m(g, h) = gh, \qquad i(g) = g^{-1}$$

are both smooth. A Lie group is abelian if the underlying group is an abelian group.

Remark 1.2. *A complex Lie group is a complex manifold that is also a group such that the multiplication and inversion maps are holomorphic. We shall be mostly working with smooth manifolds and (real) Lie groups. We shall omit the phrase 'real' when it is clear from context. If we consider complex manifolds or complex Lie groups, we shall use the phrase 'complex.'*

Example 1.3. The following are examples of Lie groups.

(1) Every 0-dimensional smooth manifold, which is a countable set of isolated points, is a 0-dimensional Lie group because the multiplication and inversion maps are locally constant and hence are smooth maps. For example, $\mathbb{Z}$ and $\mathbb{Z}_n$ for $n \geq 1$ are Lie groups.

(2) $\mathbb{R}^n$ and $\mathbb{C}^n$ are abelian Lie groups since addition and subtraction are smooth functions[1].

(3) $\mathbb{S}^1$ is a Lie group. Identifying $\mathbb{S}^1$ with complex numbers of norm one, we have that $\mathbb{S}^1$ inherits a group structure, given by

$$(x, y) \cdot (x', y') := (xx' - yy', \quad xy' + x'y),$$
$$(x, y)^{-1} = (x, -y).$$

Using the smooth manifold structure on $\mathbb{S}^1$, it is easy to now verify that $\mathbb{S}^1$ is a Lie group[2].

[1] Note that $\mathbb{C}^n$ is also a complex Lie group.

[2] We have $\mathbb{S}^1 \cong \mathsf{U}(1)$. So this claim also follows from results mentioned later in the section.

(4) Let $\mathsf{GL}(n,\mathbb{R})$ denote the general linear group of invertible $n \times n$ over $\mathbb{R}$. Consider the map

$$\det : \mathbb{R}^{n^2} \to \mathbb{R},$$
$$A \mapsto \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\, a_{1,\sigma(1)} \cdots a_{n,\sigma(n)},$$

Since det is a polynomial map, det is a smooth function. Note that $\mathsf{GL}(n,\mathbb{R}) = \det^{-1}(\mathbb{R}^\times)$, Hence, $\mathsf{GL}(n,\mathbb{R})$ is an open subset of $\mathbb{R}^{n^2}$, and hence is a smooth manifold of dimension n^2. Clearly, $\mathsf{GL}(n,\mathbb{R})$ is a group. Matrix multiplication is a smooth map (given by polynomials) and matrix inverse is a smooth map (by Cramer's rule). Hence $\mathsf{GL}(n,\mathbb{R})$ is a Lie group. Note that $\mathsf{GL}(n,\mathbb{R})$ is a non-abelian Lie group for $n \geq 2$.

(5) Similarly, $\mathsf{GL}(n,\mathbb{C})$ is a (real) Lie group of dimensions $2n^2$. It is non-abelian for $n \geq 2$.

(6) A direct product of Lie groups is a Lie group. This can be easily checked. In particular,

$$\mathbb{T}^n := \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$$

is an abelian Lie group.

(7) If G is Lie group and $H \subseteq G$ is an open subgroup then, H is a Lie group with the inherited group structure and smooth manifold structure. H is called a Lie subgroup of G. For example, note that $\mathsf{GL}^+(n,\mathbb{R})$, the open subgroup of $\mathsf{GL}(n,\mathbb{R})$ consists of invertible matrices with positive determinant, is a Lie group.

Remark 1.4. *More generally, if G is a Lie group and $H \subseteq G$ is a closed subgroup, then H is a Lie subgroup of G. This is Cartan's Closed Subgroup Theorem which is non-trivial to prove.*

We can play off the group and smooth manifold structure of a Lie group to define the notion of a 'smooth group homomorphism.'

Definition 1.5. Let G, H be Lie groups. A **Lie group homomorphism** is a smooth map $F : G \to H$ that is also a group homomorphism. A **Lie group isomorphism** is a Lie group homomorphism that is also a diffeomorphism.

It is easy to verify Lie groups form a subcategory of the category of smooth manifold. We denote this category as LieGrp.

Example 1.6. The following are examples of Lie group homomorphisms:

(1) The map $\exp : \mathbb{R} \to \mathbb{R}^\times$[3] given by $\exp(t) = e^t$ is smooth, and is a Lie group homomorphism because $e^{s+t} = e^s \cdot e^t$. The image of exp is the open Lie subgroup $\mathbb{R}^+$, and $\exp : \mathbb{R} \to \mathbb{R}^+$ is a Lie group isomorphism with inverse $\log : \mathbb{R}^+ \to \mathbb{R}$.

(2) Similarly, $\exp : \mathbb{C} \to \mathbb{C}^\times$[4] given by $\exp(z) = e^z$ is a (real) Lie group homomorphism. It is not a Lie group isomorphism because its kernel consists of the complex numbers of the form $2\pi i k$, where $k \in \mathbb{Z}$.

(3) Let G be a Lie group, and let $g \in G$. The inner automorphism of G is the map $C_g : G \to G$ given by $C_g(h) = ghg^{-1}$ (conjugation by g). Because multiplication and

[3] $\mathbb{R}^\times \cong \mathsf{GL}(1,\mathbb{R})$. Hence, $\mathbb{R}^\times$ is a Lie group

[4] $\mathbb{C}^\times \cong \mathsf{GL}(1,\mathbb{C})$. Hence $\mathbb{C}^\times$ is a Lie group.

inversion are smooth, C_g is smooth; inner automorphisms are group isomorphisms, so this is a Lie group isomorphism.

Remark 1.7. *The group and smooth manifold structure of a Lie group can be conveniently played off of each other. For instance, the multiplication map gives rise to two all-important families of diffeomorphisms of G: the left-translation and right-translation maps $L_g, R_g : G \to G$ for $g \in G$:*

$$L_g(h) = gh,$$
$$R_g(h) = hg.$$

It is easily seen that both maps are diffeomorphisms. For example, letting $\iota_g : G \to G \times G$ be the map $\iota_g(h) = (g, h)$ which is clearly smooth, note that $L_g = m \circ \iota_g$ is smooth as well. Since L_g is a bijection such that the inverse is $L_{g^{-1}}$, L_g is a diffeomorphism for all $g \in G$. Similarly, R_g is a diffeomorphism for all $g \in G$. Many of the important properties of Lie groups follow from the fact that we can systematically map any point to any other point by such a global diffeomorphism.

As an application of the comments made in Remark 1.7, we can show that every Lie group homomorphism is of constant rank:

Proposition 1.8. *Every Lie group homomorphism is of constant rank.*

Proof. Let G, H be Lie groups, and let $F : G \to H$ be a Lie group homomorphism. Let $g_0 \in G$, and denote the identity of G as e_G (and the identity of H as e_H). Since F is a homomorphism, we have, for all $g \in G$,

$$F(L_{g_0}(g)) = F(g_0 g) = F(g_0)F(g) = L_{F(g_0)}(F(g)).$$

That is: $F \circ L_{g_0} = L_{F(g_0)} \circ F$. Taking differentials of both sides at the identity, the chain rule then tells us

$$dF_{g_0} \circ d(L_{g_0})_{e_G} = d(L_{F(g_0)})_{e_H} \circ dF_{e_G},$$

Since L_{g_0} and $L_{F(g_0)}$ are diffeomorphisms, their differentials at any points are isomorphisms. It follows, therefore, that dF_{g_0} has the same rank as F_{e_G}. As this holds true for any g_0, we see that dF_{g_0} has constant rank. □

1.2. **Lie Group Actions.** Lie groups are group objects in the category of smooth manifolds. Therefore, we can define the notion of a smooth group action on a manifold, which will in turn allow us to study manifolds using tools from group theory.

Definition 1.9. Let G be a Lie group and let M be a smooth manifold. A **smooth left action** of G on M is a smooth map

$$\theta : G \times M \mapsto M$$
$$(g, p) \mapsto g \cdot p,$$

which satisfies the following two conditions:

$$g_1 \cdot (g_2 \cdot p) = (g_1 g_2) \cdot p,$$
$$e \cdot p = p.$$

for all $g_1, g_2 \in G$ and $p \in M$.

Remark 1.10. *We can also talk about smooth right actions which are defined similarly. All remarks made below above equally well to smooth right actions.*

Remark 1.11. *For each $g \in G$, we denote by θ_g the map $M \to M$ defined by $\theta_g(p) = g \cdot p$.*

Definition 1.12. Let G be a Lie group and let M be a smooth manifold and θ be a smooth group action.

(1) For $p \in M$, the **orbit of** P of p is the set
$$G \cdot p = \{g \cdot p : g \in G\}$$
(2) For $p \in M$, the **stabilizer of** p is the set
$$G_p = \{g \in G : g \cdot p = p\}$$
That is, it is the set of group elements that fix p. Note that G_p is a subgroup.
(3) An action is said to be **transitive** for each pair $p, q \in M$, there is some $g \in G$ with $g \cdot p = q$.
(4) An action is said to be **free** if all stabilizers are trivial: $G_p = \{e\}$ for all p. In other words, only the group unit fixes any element.

Example 1.13. Here are some examples of Lie group actions on manifolds.

(1) If G is any Lie group and M is any smooth manifold, the trivial action of G on M is defined by $g \cdot p = p$ for all $g \in G$, $p \in M$. It is smooth[5], each orbit is a single point and $G_p = G$ for each $p \in M$.
(2) If G is a connected Lie group, then any smooth action on a discrete smooth manifold, M, is the trivial action. Consider $G \cdot p$, the orbit of $p \in M$. $G \cdot p$ is connected, so it must be a singleton, as the only connected non-empty subsets of a discrete space are singletons. Hence, the action must be the trivial action.
(3) Let $G = \mathsf{GL}(n, \mathbb{R})$ and $M = \mathbb{R}^n$. G acts on M by matrix multiplication. It is clearly a smooth action. Note that $A \cdot 0 = 0$ for all $A \in G$, so the orbit of 0 is just $\{0\}$. For any two non-zero vectors x, y, there is some invertible matrix A with $Ax = y$. Hence, there is only one other orbit, $\mathbb{R}^n \setminus \{0\}$.

Group actions allow is to impart some nice properties of Lie groups to the manifolds they act on. This can be described through a property called equivariance.

Definition 1.14. Let M, N be smooth manifolds, and let $F : M \to N$ be a smooth map. Suppose that M, N both possess smooth (left) actions by some Lie group G. F is an **equivariant smooth map** under the actions of G if
$$F(g \cdot_M p) = g \cdot_N F(p), \quad \text{for all } g \in G, p \in M.$$
If the action of G on M is denoted by θ and the action of G on N is denoted by φ, this condition is often expressed as a commutative diagram:
$$\begin{tikzcd} M \arrow[r, "F"] \arrow[d, "\theta_g"] & N \arrow[d, "\varphi_g"] \\ M \arrow[r, "F"] & N \end{tikzcd}$$

We know that all Lie group homomorphisms have constant rank by Proposition 1.8. This property extends to the much wider class of equivariant maps under transitive actions.

[5] θ is just the projection map $G \times M \to M$

Proposition 1.15. ***(Equivariant Rank Theorem)*** *Let $F : M \to N$ be a smooth map between manifolds. Let G be a Lie group that acts smoothly on both M and N, and suppose the action on M is transitive. If F is equivariant with respect to these actions, then F has constant rank.*

Proof. Denote the action on M by θ and the action on N by φ. Let $p, q \in M$. By the transitivity assumption, there is some $g \in G$ with $\theta_g(p) = q$. The equivariance of F is the statement that

$$F \circ \theta_g = \varphi_g \circ F$$

We now apply the chain rule at the point p:

$$dF_q \circ (d\theta_g)_p = (d\varphi_g)_{F(p)} \circ dF_p$$

Since θ_g and φ_g are diffeomorphisms, the differentials $(d\theta_g)_p$ and $(d\varphi_g)_{F(p)}$ are linear isomorphisms, and it follows that dF_p and dF_q have the same rank.

$$\begin{tikzcd} \mathsf{T}_p M \arrow[r, "dF_p"] \arrow[d, "d(\theta_g)_p"] & \mathsf{T}_{F(p)} M \arrow[d, "d(\varphi_g)_{F(q)}"] \\ \mathsf{T}_q M \arrow[r, "dF_q"] & \mathsf{T}_{F(q)} N \end{tikzcd}$$

This completes the proof. $\square$

Example 1.16. (Quarternions) Let $\mathbb{R}\{1, i, j, k\}$ be the free $\mathbb{R}$-vector space on the set $\{1, i, j, k\}$. Let I be the ideal generated by the relations

$$i^2 = j^2 = k^2 = -1, \quad ij = k, \quad jk = i, \quad ki = j$$

$\mathbb{H}$ is defined as

$$\mathbb{H} := \mathbb{R}\{1, i, j, k\}/I$$

It is a simple but tedious matter to check that $\mathbb{H}$ is a division algebra. If $x = a+bi+cj+dk \in \mathbb{H}$, we define $\overline{x} = a - bi - cj - dk$. It can be checked that the map $x \mapsto x\overline{x} := |x|^2$ defines a norm on $\mathbb{H}$. It turns out to be much more convenient to work with a matrix representation of $\mathbb{H}$. Let I, i', j', and k' be the following matrices:

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad i' = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad j' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad k' = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

It is easily seen that the matrices satisfy the relations mentioned above. Hence, the map

$$1 \to I, \quad i \to i', \quad j \to j', \quad k \to k',$$

defines a matrix representation of $\mathbb{H}$. From now on we identify $\mathbb{H}$ with its matrix representation. A simple derivation shows that every matrix in $\mathbb{H}$ is of the form

$$A = \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix},$$

where $\alpha, \beta \in \mathbb{C}$. Hence the conjugate of a quaternions is now identified with the matrix conjugated transpose. Moreover, the norm of a quaternion is now defined as the determinant of the associated matrix[6]. A simple argument then shows that $A \in \mathsf{GL}_2(\mathbb{C})$ is identified

[6]THis association easily implies that $\overline{xy} = \overline{y}\,\overline{x}$ and $|xy| = |x||y|$.

with a non-zero quaternion if and only if $A^*A = \det(A)I_2$. Let $\mathbb{H}^\times$ denote the non-zero quaternions. We have a map

$$\begin{aligned}\Phi : \mathsf{GL}(2,\mathbb{C}) &\to M(2,\mathbb{C})\\ X &\mapsto \det(X)^{-1}X^*X\end{aligned}$$

Clearly, $\mathbb{H}^\times = \Phi^{-1}(I_2)$. As before, it can be checked that Φ is a smooth equivariant map under suitable right and left actions of $\mathsf{GL}(2,\mathbb{C})$. By Proposition 1.15, Φ is of constant rank. By the constant rank theorem, $\mathbb{H}^\times$ is an embedded submanifold of $\mathsf{GL}(2,\mathbb{C})$. This allows us to immediately conclude that $\mathbb{H}^\times$ is a Lie group. The unit quaternions, $\mathbb{H}_u$, consist of all $A \in \mathbb{H}$ with determinant one. $\mathbb{H}_u$ is also a Lie group[7]. Simply consider the map

$$\begin{aligned}\Phi : \mathsf{GL}(2,\mathbb{C}) &\to M(2,\mathbb{C})\\ X &\mapsto X^*X.\end{aligned}$$

and apply the argument as above. Note that we can identity $\mathbb{H}_u$ with $\mathbb{S}^3$. This also shows that $\mathbb{S}^3$ as a Lie group structure.

Remark 1.17. *Note that $\mathsf{GL}(n,\mathbb{H})$ is a (real) Lie group of dimensions $4n^2$. It is non-abelian for $n \geq 1$.*

1.3. **Matrix Lie Groups.** The most famous example of a Lie group is the general linear groups $\mathsf{GL}(n,\mathbb{K})$, where $\mathbb{K} = \mathbb{R},\mathbb{C},\mathbb{H}$. A closed subgroup of $\mathsf{GL}(n,\mathbb{K})$ is called a matrix Lie group. In this section, we discuss some important examples of the so-called classical Lie groups, which are well-known examples of matrix Lie groups.

Remark 1.18. *In the following examples, we will not explicitly verify that the given Lie groups are indeed groups, as this verification is straightforward.*

1.3.1. *Special Linear group.* As an application of the constant rank theorem, we can furnish further examples of Lie group by appealing to the constant rank theorem. Let

$$\mathsf{SL}(n,\mathbb{R}) = \det{}^{-1}\{1\}$$

$\det : \mathsf{GL}(n,\mathbb{R}) \to \mathbb{R}^\times$ is a smooth map. We show that det has constant rank 1. Let $X \in T_{I_n}\mathsf{GL}(n,\mathbb{R}) \cong \mathbb{R}^{n^2}$ and consider the curve $\gamma(t) = I_n + tX$ in $\mathbb{R}^{n^2}$[8]. We compute

$$\left.\frac{d}{dt}\right|_{t=0} \det(I+tX)$$

Note that to first order:

$$\begin{aligned}\det(I+tX) &= \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma)\cdot(I+tX)_{1,\sigma(1)}\cdot(I+tX)_{2,\sigma(2)}\cdot\dots\cdot(I+tX)_{n,\sigma(n)}\\ &= \prod_{i=1}^n(1+tX_{ii}) + O(t^2) = 1 + t\sum_{i=1}^n X_{ii} + O(t^2)\end{aligned}$$

Therefore,

$$\left.\frac{d}{dt}\right|_{t=0} \det(I+tX) = \sum_{i=1}^n X_{ii} = \operatorname{Tr} X$$

[7]It is indeed a group as it can be easily verified.

[8]For small enough t, $\gamma(t)$ is contained in $\mathsf{GL}(n,\mathbb{R})$ since $\mathsf{GL}(n,\mathbb{R})$ is an open subset of $\mathbb{R}^{n^2}$ so the map is well-defined for small enough t.

Clearly, the linear map $X \mapsto \operatorname{Tr} X$ is surjective. More generally, if $X \in \mathbb{R}^{n^2}$, we have:

$$d(\det)_{I_n}(X) = \operatorname{Tr}(X)$$

More generally, we can easily compute the differential of the det map at any $A \in \mathsf{GL}(n, \mathbb{R})$. Indeed, for $A \in$ consider the path $\gamma(t) = A + tX$ which is well-defined for small enough values of t. Then

$$\begin{aligned} d(\det)_A(X) &= \frac{d}{dt}\bigg|_{t=0} \det(A + tX) \\ &= \frac{d}{dt}\bigg|_{t=0} \det(A)\det\big(I + tA^{-1}X\big) \\ &= \det(A)\frac{d}{dt}\bigg|_{t=0} \det\big(I + tA^{-1}X\big) \\ &= \det(A)\operatorname{Tr}(A^{-1}X) \end{aligned}$$

Clearly, the linear map $X \mapsto \det(A)\operatorname{Tr}\big(A^{-1}X\big)$ is surjective. This shows that det has constant rank. By the constant rank theorem $\mathsf{SL}(n, \mathbb{R})$ is an embedded subamanifold such that

$$\dim \mathsf{SL}(n, \mathbb{R}) = n^2 - 1$$

Clearly, $\mathsf{SL}(n, \mathbb{R})$ is group. Hence, $\mathsf{SL}(n, \mathbb{R})$ is a Lie group.

Remark 1.19. *Similarly, $\mathsf{SL}(n, \mathbb{C})$ is (real) Lie group of dimension $2n^2 - 2$.*

Remark 1.20. *Since $\mathbb{H}$ is a non-commutative ring, multilinearity and alternating properties are incompatible in $\mathsf{GL}(n, \mathbb{H})$ for $n \geq 2$. Hence, there is no canonical way to define a determinant of a matrix in $\mathsf{GL}(n, \mathbb{H})$ for $n \geq 2$.*

1.3.2. *Orthogonal & Unitary Groups.* As an application of Proposition 1.15, we can furnish further examples of Lie group by appealing to the equivariant rank theorem.

Example 1.21. Let $\mathsf{O}(n, \mathbb{R})$ be the group of $n \times n$ real orthogonal matrices that preserve the Euclidean inner product:

$$\mathsf{O}(n, \mathbb{R}) = \{A \in \mathsf{GL}(n, \mathbb{R}) \mid A^{\mathrm{T}}A = I_n\}$$

Define

$$\begin{aligned} \Phi : \mathsf{GL}(n, \mathbb{R}) &\to \mathbb{R}^{n^2} \\ A &\mapsto A^T A \end{aligned}$$

Clearly, $\mathsf{O}(n, \mathbb{R}) = \Phi^{-1}(I_n)$. By defining suitable group actions on $\mathsf{GL}(n, \mathbb{R})$ and $\mathbb{R}^{n^2}$ and appealing to Proposition 1.15, we can show that Φ is of constant rank: Let $G = \mathsf{GL}(n, \mathbb{R})$ act on $\mathsf{GL}(n, \mathbb{R})$ by matrix multiplication. This action is clearly transitive. Define a right action of $\mathsf{GL}(n, \mathbb{R})$ on $\mathbb{R}^{n^2}$ by

$$X \cdot B = B^T X B$$

for $X \in \mathbb{R}^{n^2}$ $B \in \mathsf{GL}(n, \mathbb{R})$. It is easy to check that this is a smooth action, and $\cdot$ is equivariant because

$$\Phi(AB) = (AB)^T(AB) = B^T A^T AB = B^T\Phi(A)B = \Phi(A) \cdot B$$

Appealing to Proposition 1.15, $\mathsf{O}(n, \mathbb{R})$ is an embedded submanifold of $\mathsf{GL}(n, \mathbb{R})$. We compute its dimension by computing the rank of the differential of Φ at I_n. Fix any $A \in \mathbb{R}^{n^2}$.

For any small enough $\varepsilon > 0$, consider a curve $\gamma : (-\varepsilon, \varepsilon) \to \mathsf{O}(n, \mathbb{R})$ such that $\gamma(0) = I_n$ and $\gamma'(0) = A$. We have:

$$d\Phi_{I_n}(A) = (\Phi \circ \gamma)'(0) = \frac{d}{dt}\gamma(t)^T\gamma(t)\bigg|_{t=0} = \gamma'(0)^T\gamma(0) + \gamma(0)^T\gamma'(0) = A + A^T$$

Since $A + A^T$ is symmetric, the image of $d\Phi_{I_n}$ is contained in the vector space of n-by-n symmetric matrices. In fact, it is equal to this vector space. This is because for any

$$d\Phi_{I_n}(B/2) = \frac{B + B^T}{2} = B$$

for any n-by-n symmetric matrix, B. Therefore,

$$\dim \mathsf{O}(n, \mathbb{R}) = n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2}$$

Example 1.22. Consider the special orthogonal group,

$$\mathsf{SO}(n, \mathbb{R}) := \mathsf{O}(n, \mathbb{R}) \cap \mathsf{SL}(n, \mathbb{R})$$

It is easy to show that every matrix in $\mathsf{O}(n, \mathbb{R})$ has determinant ± 1. Hence, $\mathsf{SO}(n, \mathbb{R})$ is the subset of those matrices in $\mathsf{O}(n, \mathbb{R})$ having determinant 1. In fact it is an open subset of $\mathsf{O}(n, \mathbb{R})$ since the det map restricts to a map

$$\det : \mathsf{O}(n, \mathbb{R}) \to \{\pm 1\}$$

and $\mathsf{SO}(n, \mathbb{R}) = \det^{-1}(+1)$. Hence, $\mathsf{SO}(n, \mathbb{R})$ is a Lie group of dimension $n(n-1)/2$. Note that $\mathsf{O}(n, \mathbb{R})$ and $\mathsf{SO}(n, \mathbb{R})$ fit into a short exact sequence:

$$1 \to \mathsf{SO}(n, \mathbb{R}) \to \mathsf{O}(n, \mathbb{R}) \xrightarrow{\det} \{\pm 1\} \to 1$$

Example 1.23. Let $\mathsf{U}(n, \mathbb{C})$ be the group of n by n complex orthogonal matrices that preserve the Hermitian inner product:

$$\mathsf{U}(n, \mathbb{C}) = \{A \in \mathsf{GL}(n, \mathbb{C}) \mid A^*A = I_n\}$$

Define

$$\begin{aligned} \Phi : \mathsf{GL}(n, \mathbb{C}) &\to \mathbb{C}^{n^2} \\ A &\mapsto A^*A \end{aligned}$$

Clearly, $\mathsf{U}(n, \mathbb{C}) = \Phi^{-1}(I_n)$. Let $G = \mathsf{GL}(n, \mathbb{C})$ act on $\mathsf{GL}(n, \mathbb{C})$ by matrix multiplication. This action is clearly transitive. Define a right action of $\mathsf{GL}(n, \mathbb{C})$ on $\mathbb{C}^{n^2}$ by

$$X \cdot B = B^*XB$$

for $X \in \mathbb{C}^{n^2}$ $B \in \mathsf{GL}(n, \mathbb{C})$. It is easy to check that this is a smooth action, and $\cdot$ is equivariant because

$$\Phi(AB) = (AB)^*(AB) = B^*A^*AB = B^*\Phi(A)B = \Phi(A) \cdot B$$

Appealing to the Appealing to Proposition 1.15, $\mathsf{U}(n, \mathbb{C})$ is an embedded submanifold of $\mathsf{GL}(n, \mathbb{C})$. We compute its dimension by computing the rank of the differential of Φ at I_n. Fix any $A \in \mathbb{C}^{n^2}$. For any small enough $\varepsilon > 0$, consider a curve $\gamma : (-\varepsilon, \varepsilon) \to \mathsf{GL}(n, \mathbb{R})$ such that $\gamma(0) = I_n$ and $\gamma'(0) = A$. We have:

$$d\Phi_{I_n}(A) = (\Phi \circ \gamma)'(0) = \frac{d}{dt}\gamma(t)^*\gamma(t)\bigg|_{t=0} = \gamma'(0)^*\gamma(0) + \gamma(0)^T\gamma'(0) = A + A^*$$

Since $A + A^*$ is self-adjoint, the image of $d\Phi_{I_n}$ is contained in the vector space of n-by-n self-adjoint matrices. In fact, it is equal to this vector space. This is because for any

$$d\Phi_{I_n}(B/2) = \frac{B + B^*}{2} = B$$

for any n-by-n self-adjoint matrix, B. We have:

$$\dim \mathsf{U}(n, \mathbb{C}) = 2n^2 - n^2 = n^2$$

This is because the vector of all matrices of the form $A = A^*$ has dimension $n+4n(n-1)/2 = n^2$.

Example 1.24. Consider the special unitary group:

$$\mathsf{SU}(n, \mathbb{C}) := \mathsf{U}(n, \mathbb{C}) \cap \mathsf{SL}(n, \mathbb{C})$$

It is easy to show that every matrix in $\mathsf{U}(n, \mathbb{C})$ has determinant of absolute value 1. As above, the det map restricts to a map

$$\det : \mathsf{U}(n, \mathbb{C}) \to \mathbb{S}^1$$

and $\mathsf{SU}(n, \mathbb{C}) = \det^{-1}(1)$. Clearly, det is of full rank as before. The constant rank theorem then implies that

$$\dim \mathsf{SU}(n, \mathbb{C}) = n^2 - 1$$

Thus $\mathsf{SU}(n, \mathbb{C})$ is a (real) Lie group of dimension n^2. Note that $\mathsf{U}(n, \mathbb{C})$ and $\mathsf{SU}(n, \mathbb{C})$ fit into a short exact sequence:

$$1 \to \mathsf{SU}(n, \mathbb{C}) \to \mathsf{U}(n, \mathbb{C}) \xrightarrow{\det} \mathbb{S}^1 \to 1$$

Remark 1.25. *$\mathsf{U}(n, \mathbb{C})$ and $\mathsf{SU}(n, \mathbb{C})$ are not complex Lie groups! We verify this later.*

Remark 1.26. *Let $\mathsf{U}(n, \mathbb{H})$ be the group of n by n quarternionic orthogonal matrices that preserve the quarternionic inner product:*

$$\mathsf{U}(n, \mathbb{H}) = \{A \in \mathsf{GL}(n, \mathbb{H}) \mid A^H A = I_n\}$$

Here A^H is the quarterionic conjugate transpose. It can checked that $\mathsf{U}(n, \mathbb{H})$ is a (real) Lie group of dimension $n(2n+1)$. The argument is the same as in Example 1.23. Indeed, the analog of the differential of the map in Example 1.23 has image the set of all matrices of the form $A = A^H$. Therefore,

$$\dim \mathsf{U}(n, \mathbb{H}) = 4n^2 - n(2n-1) = n(2n+1)$$

This is because the vector of all matrices of the form $A = A^H$ has dimension $n + 4n(n-1)/2 = n(2n-1)$.

1.3.3. *Symplectic Groups.* Consider the skew-symmetric bilinear form ω on $\mathbb{R}^{2n}$ defined as follows:

$$\omega(x, y) = \sum_{j=1}^{n} (x_{n+j} y_j - x_j y_{n+j}) = x^T \begin{pmatrix} 0_n & -I_n \\ I_n & 0_n \end{pmatrix} y := x^T J y = \langle x, Jy \rangle_{\mathbb{R}^n}$$

The set of all $2n \times 2n$ real matrices A which preserve ω is the real symplectic group $\mathsf{Sp}(n, \mathbb{R})$

$$\mathsf{Sp}(n, \mathbb{R}) = \{A \in \mathsf{GL}(2n, \mathbb{R}) \mid \omega(Ax, Ay) = \omega(x, y)\}$$

It is easily shown that

$$\mathsf{Sp}(n, \mathbb{R}) = \{A \in \mathsf{GL}(2n, \mathbb{R}) \mid A^T J A = J\}$$

Define

$$\begin{aligned}\Phi : \mathsf{GL}(2n,\mathbb{R}) &\to \mathbb{R}^{(2n)^2}\\ A &\mapsto A^TJA\end{aligned}$$

Clearly, $\mathsf{Sp}(n,\mathbb{R}) = \Phi^{-1}(J)$. Let $G = \mathsf{GL}(2n,\mathbb{R})$ ct on $\mathsf{GL}(2n,\mathbb{R})$ by matrix multiplication. Moreover, let $\cdot$ denote the corresponding action in Example 1.21.$\cdot$ is equivariant because

$$\Phi(AB) = (AB)^TJ(AB) = B^TA^TJAB = B^T\Phi(A)B = \Phi(A)\cdot B$$

Appealing to Proposition 1.15, $\mathsf{Sp}(n,\mathbb{R})$ is an embedded submanifold of $\mathsf{GL}(n,\mathbb{R})$. Since $\mathsf{Sp}(n,\mathbb{R})$ is clearly a group, we have that $\mathsf{Sp}(n,\mathbb{R})$ is a Lie group. Similarly, we can define the complex symplectic group:

$$\begin{aligned}\mathsf{Sp}(n,\mathbb{C}) &= \{A \in \mathsf{GL}(2n,\mathbb{C}) \mid \omega(Ax,Ay) = \omega(x,y)\}\\ &= \{A \in \mathsf{GL}(2n,\mathbb{C}) \mid A^TJA = J\}\end{aligned}$$

As above, $\mathsf{Sp}(n,\mathbb{C})$ is a Lie group. We will derive the dimension of these Lie groups by computing the dimension of the associated Lie algebras in the next section.

1.3.4. *Indefinite Orthogonal Group.* Let $p,q \in \mathbb{N}$ such that $p+q = n$. Consider the indefinite bilinear form $\beta_{p,q}$ on $\mathbb{R}^n$ defined as follows:

$$\beta_{p,q}(x,y) = \sum_{j=1}^{p} x_jy_j - \sum_{j=1}^{q} x_{p+j}y_{p+j} = x^T\begin{pmatrix} I_p & 0\\ 0 & -I_q\end{pmatrix}y := x^Tg_{p,q}y = \langle x, g_{p,q}y\rangle_{\mathbb{R}^n}$$

The set of all $n\times n$ real matrices A which preserve $\beta_{p,q}$ is the indefinite orthogonal group $\mathsf{O}(p,q) \subseteq \mathsf{GL}(n,\mathbb{R})$

$$\mathsf{O}(p,q) = \{A \in \mathsf{GL}(n,\mathbb{R}) \mid \beta_{p,q}(Ax,Ay) = \beta_{p,q}(x,y)\}$$

It is easily shown that

$$\mathsf{O}(p,q) = \{A \in \mathsf{GL}(n,\mathbb{R}) \mid A^Tg_{p,q}A = g_{p,q}\}$$

Clearly, $\mathsf{O}(p,q)$ is a group. An argument as in Section 1.3.3 shows that $\mathsf{O}(p,q)$ is a Lie group. Of particular interest in physics is the Lorentz group $\mathsf{O}(3,1)$. It is easily verified that if $A \in \mathsf{O}(p,q)$, then $\det A = \pm 1$. Hence, we can also define

$$\mathsf{SO}(p,q) = \mathsf{O}(p,q) \cap \mathsf{SL}(n,\mathbb{R})$$

It is also a Lie group. We will derive the dimension of these Lie groups by computing the dimension of the associated Lie algebras in the next section.

1.4. **Topological Properties.** We discuss topological properties of the classical Lie groups.

1.4.1. *Compactness.* We determine which classical Lie groups discussed above are compact.

Proposition 1.27. *The following statements are true:*

1. $\mathsf{GL}(n,\mathbb{R})$ *and* $\mathsf{GL}(n,\mathbb{C})$ *are not compact for* $n \geq 1$.
2. $\mathsf{SL}(n,\mathbb{R})$ *and* $\mathsf{SL}(n,\mathbb{C})$ *are not compact for* $n \geq 2$.
3. $\mathsf{O}(n,\mathbb{R})$ *and* $\mathsf{U}(n,\mathbb{C})$ *are compact for* $n \geq 1$.
4. $\mathsf{SO}(n,\mathbb{R})$ *and* $\mathsf{SU}(n,\mathbb{C})$ *are compact for* $n \geq 1$.
5. $\mathsf{O}(p,q)$ *is not compact for all* $n \geq 1$ *such that* $p+q = n$ *and* $q \neq 0$.

Proof. The proof is given below:

(1) Clearly, $\mathsf{GL}(n,\mathbb{R})$ is not compact for $n \geq 1$. Otherwise, the image of $\mathsf{GL}(n,\mathbb{R})$ under the determinant map would be a compact set. However, $\det(\mathsf{GL}(n,\mathbb{R})) = \mathbb{R}^\times$ which is not compact. Similarly, $\mathsf{GL}(n,\mathbb{C})$ is not compact for $n \geq 1$.

(2) If $n = 1$, we have that $\mathsf{SL}(n,\mathbb{R}) \cong \{1\}$ which is compact. Similarly, $\mathsf{SL}(n,\mathbb{C}) \cong \mathbb{S}^1$ which is compact. Let $n \geq 2$. Consider the set,

$$A = \{A_m \in \mathsf{GL}(n,\mathbb{R}) \mid m \in \mathbb{R}^\times\}, \qquad A_m = \begin{pmatrix} m & 0 & 0 & \cdots & 0 \\ 0 & 1/m & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

We have[9] $\|A_m\|_\infty = m$ for $m \geq 1$ and $\|A_m\|_\infty \to \infty$ as $m \to \infty$. Hence, $\mathsf{SL}(n,\mathbb{R})$ is not a bounded set for $n \geq 2$. Therefore, $\mathsf{SL}(n,\mathbb{R})$ is not compact for $n \geq 2$. Similarly, $\mathsf{SL}(n,\mathbb{C})$ is not compact for $n \geq 2$.

(3) $\mathsf{O}(n,\mathbb{R})$ is clearly a closed subset. Moreover, if $A \in \mathsf{O}(n,\mathbb{R})$, then $|A_{jk}| \leq 1$ for each $j, k = 1, \cdots n$ since the columns of $A \in G$ are required to be unit vectors. Hence, $\|A\|_\infty \leq 1$ for each $A \in \mathsf{O}(n,\mathbb{R})$. Hence, $\mathsf{O}(n,\mathbb{R})$ is compact for $n \geq 1$. Similarly, $\mathsf{U}(n,\mathbb{C})$ is compact for $n \geq 1$.

(4) $\mathsf{SO}(n,\mathbb{R})$ is a closed subset of $\mathsf{O}(n,\mathbb{R})$ for $n \geq 1$. Hence, $\mathsf{SO}(n,\mathbb{R})$ is compact for $n \geq 1$. Similarly, $\mathsf{SU}(n,\mathbb{C})$ is compact for $n \geq 1$.

(5) WLOG, let $n = 2$ and $p, q = 1$. A similar argument applies in the general case. Note that

$$\begin{pmatrix} 1 & x \\ 0 & y \end{pmatrix} \in \mathsf{O}(1,1) \iff x^2 - y^2 = -1$$

The set of solutions of $x^2 - y^2 = -1$ is an unbounded set. This is sufficient to conclude that $\mathsf{O}(1,1)$ is not compact. An entirely similar argument shows that $\mathsf{O}(p,q)$ is not compact as long as $q \neq 0$.

This completes the proof. $\square$

Remark 1.28. *Let G be a Lie group. The previous proposition shows that G is not necessarily a compact group. However, G is always a locally compact group. This is a general fact about smooth manifolds.*

1.4.2. *Connectedness.* We determine which classical Lie groups discussed above are connectedness.

Remark 1.29. *Recall that a smooth manifold is connected if and only if it is path-connected. We shall make use of this characterization of connectedness below.*

Proposition 1.30. *The following statements are true:*

(1) $\mathsf{GL}(n,\mathbb{C})$ *is connected for $n \geq 1$. However,* $\mathsf{GL}(n,\mathbb{R})$ *is not connected for $n \geq 1$.*
(2) $\mathsf{SO}(n,\mathbb{R})$ *is connected for $n \geq 1$. However,* $\mathsf{O}(n,\mathbb{R})$ *is not connected for $n \geq 1$ and it has two connected components.*
(3) $\mathsf{GL}^{\pm}(n,\mathbb{R})$ *is connected for $n \geq 1$. Hence,* $\mathsf{GL}(n,\mathbb{R})$ *has two connected components.*
(4) $\mathsf{SL}(n,\mathbb{R})$ *and* $\mathsf{SL}(n,\mathbb{C})$ *are connected for $n \geq 1$.*
(5) $\mathsf{U}(n,\mathbb{C})$ *and* $\mathsf{SU}(n,\mathbb{C})$ *are connected for $n \geq 1$.*

[9]Here $\|\cdot\|_\infty$ is the infinity norm. Recall that all norms on finite-dimensional vector spaces are equivalent.

(6) $\mathsf{SO}(p,q)$ *is not connected for all* $n \geq 1$ *such that* $p+q=n$ *and* $q \neq 0$. *In fact,* $\mathsf{SO}(p,q)$ *has two connected components.*
(7) $\mathsf{O}(p,q)$ *is not connected for all* $n \geq 1$ *such that* $p+q=n$ *and* $q \neq 0$. *In fact,* $\mathsf{O}(p,q)$ *has four connected components.*

Proof. The proof is given below:

(1) $\mathsf{GL}(n,\mathbb{R})$ is not connected for $n \geq 1$. Otherwise, the image of $\mathsf{GL}(n,\mathbb{R})$ under the determinant map would be a connected set. However, $\det(\mathsf{GL}(n,\mathbb{R})) = \mathbb{R}^\times$ which is not connected. On the other hand, $\mathsf{GL}(n,\mathbb{C})$ is connected. To see this fact, recall that every matrix in $\mathbb{C}^{n^2}$ is similar to an upper triangular matrix. That is, we can express any $A \in M_n(\mathbb{C})$ in the form
$$A = CBC^{-1},$$
where
$$B = \begin{pmatrix} \lambda_1 & \cdots & * \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{pmatrix}.$$
If $A \in \mathsf{GL}(n,\mathbb{C})$ in particular, each λ_i must be nonzero. Let $B(t)$ be obtained by multiplying the part of B above the diagonal by $(1-t)$, for $0 \leq t \leq 1$, and let $A(t) = CB(t)C^{-1}$. Then $A(t)$ is a continuous path lying in $\mathsf{GL}(n,\mathbb{C})$ which starts at A and ends at CDC^{-1}, where D is the diagonal matrix with diagonal entries $\lambda_1,\ldots,\lambda_n$. We can now define paths $\lambda_j(t)$ connecting λ_j to 1 in $\mathbb{C}$ as t goes from 1 to 2, and we can define $A(t)$ on the interval $1 \leq t \leq 2$ by
$$A(t) = C \begin{pmatrix} \lambda_1(t) & 0 & \cdots & 0 \\ 0 & \lambda_2(t) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n(t) \end{pmatrix} C^{-1}.$$
Then $A(t)$, for $0 \leq t \leq 2$, is a continuous path in $\mathsf{GL}(n,\mathbb{C})$ connecting A to I. Hence, $\mathsf{GL}(n,\mathbb{C})$ is connected for $n \geq 1$.

(2) $\mathsf{O}(n,\mathbb{R})$ is not connected for $n \geq 1$ since $+I_n$ and $-I_n$ cannot be connected by a continuous path by the continuity of the determinant function. Moreover, we have
$$\mathsf{O}(n,\mathbb{R}) = \mathsf{O}^+(n,\mathbb{R}) \coprod \mathsf{O}^-(n,\mathbb{R}) := \mathsf{SO}(n,\mathbb{R}) \coprod \mathsf{SO}^-(n,\mathbb{R})$$
We show that $\mathsf{SO}(n,\mathbb{R})$ is connected. An entirely similar argument shows that $\mathsf{SO}^-(n,\mathbb{R})$. It follows that $\mathsf{O}(n,\mathbb{R})$ has two connected components. We show every $A \in \mathsf{SO}(n,\mathbb{R})$ can be connected to I_n. First, we argue that given any two unit vectors $v,w \in \mathbb{R}^n$, there is a path $\gamma(t) \in \mathsf{SO}(n,\mathbb{R})$ such that:
$$\gamma(0)v = v, \quad \gamma(1)v = w$$
That is, any two unit vectors in $\mathbb{R}^n$ can be *continuously rotated.* Choose a $u \in \mathbb{R}^n$ as follows:

(a) If v and w are linearly independent, apply the Gram-Schmidt algorithm and choose u such that $u \perp v$ and $u \in \operatorname{span}\{v,w\}$.
(b) If v and w are linearly dependent ($w = -v$), then take u to be any unit vector in $v^\perp$.

Let $V = \text{span}\{v, u\}$. One can then consider a one-parameter family of rotations, $R_\phi \in \mathsf{SO}(2, \mathbb{R})$ that act on V. Since $w \in V$, there is an angle ϕ_0 such that (in the above constructed basis):

$$w = \begin{bmatrix} R_{\phi_0} & 0 \\ 0 & I_{n-2} \end{bmatrix} v.$$

Define the path

$$\gamma(t) := \begin{bmatrix} R_{t\phi_0} & 0 \\ 0 & I_{n-2} \end{bmatrix}$$

The image of γ is clearly contained in $\mathsf{SO}(n, \mathbb{R})$ and is such that

$$\gamma(0) = R(0)v = v$$
$$\gamma(1) = R(1)v = w$$

Any $A \in \mathsf{SO}(n, \mathbb{R})$ is represented by an orthonormal basis $(a_1, \ldots, a_n)$ over vectors in $\mathbb{R}^n$. Apply the above procedure recursively: find a path $\gamma_1(t) \in \mathsf{SO}(n, \mathbb{R})$ such that

$$\gamma_1(t)a_1 = e_1$$

Then choose a path γ_2 taking $\gamma_1(1)a_2$ to e_2. Note that any such γ_2 leaves e_1 invariant. Indeed $e_1 \perp e_2, \gamma_1(1)a_2$ [10]. So, e_1 is in the complement of the subspace in which the rotation happens and is thus left invariant. Proceed recursively now and consider the paths $\gamma_1(t), \cdots, \gamma_n(t)$. Consider

$$\gamma = \gamma_n \circ \cdots \circ \gamma_1$$

Based on the above remarks, it is clear that

$$\gamma(0)a_i = a_i$$
$$\gamma(1)a_i = e_i$$

for $i = 1, \cdots, n$. Hence, $\mathsf{SO}(n, \mathbb{R})$ is path-connected and hence connected since $\mathsf{SO}(n, \mathbb{R})$ is a smooth manifold.

(3) It suffices to show that $\mathsf{GL}^+(n, \mathbb{R})$ is connected since $\mathsf{GL}^-(n, \mathbb{R})$ is diffeomorphic to $\mathsf{GL}^+(n, \mathbb{R})$. We use the singular value decomposition. Let

$$A = U\Sigma V$$

be the singular value decomposition of A. Here U and V are unitary matrices and Σ is a diagonal matrix consisting of the singular values of A which are all non-negative[11]. Since A has positive determinant, the singular values of A are all positive real numbers. Since $\det A > 0$, $\det U = \det V$. Therefore, both U and V are in the same component of $\mathsf{O}(n, \mathbb{R})$. WLOG, assume that both matrices are contained in $\mathsf{SO}(n, \mathbb{R})$. Since $\mathsf{SO}(n, \mathbb{R})$ is connected, there exist paths $\gamma_1(t)$ and $\gamma_2(t)$ in $\mathsf{SO}(n, \mathbb{R})$ such that

$$\gamma_1(0) = U \quad \gamma_1(1) = I_n$$
$$\gamma_1(0) = V \quad \gamma_1(1) = I_n$$

Consider the path

$$\gamma(t) = \gamma_1(t)\Sigma\gamma_2(t)$$

[10]Applying γ_1 to an orthonormal basis results in an orthonormal basis

[11]This is crucial in this proof.

Clearly, $\gamma(t)$ is in $\mathsf{SO}(n,\mathbb{R})$ such that

$$\gamma_1(0) = A \quad \gamma_1(1) = \Sigma$$

Since Γ has positive entries, there exists a smooth curve β such that $\beta(s) \in \mathsf{SO}(n,\mathbb{R})$ and that

$$\beta_1(0) = \Sigma \quad \beta(1) = I_n$$

Simply consider $\beta \circ \gamma$. This shows that $\mathsf{GL}^+(n,\mathbb{R})$ is connected. This clearly implies that $\mathsf{GL}(n,\mathbb{R})$ has two connected components.

(4) Consider the continuous surjective map

$$\Psi : \mathsf{GL}^+(n,\mathbb{R}) \mapsto \mathsf{SL}(n,\mathbb{R})$$
$$A \mapsto \frac{A}{\det(A)^{1/n}}$$

Since Ψ is surjective and $\mathsf{GL}^+(n,\mathbb{R})$ is connected for $n \geq 1$, $\mathsf{SL}(n,\mathbb{R})$ is connected for $n \geq 1$. Moreover, $\mathsf{SL}(n,\mathbb{C})$ is connected for $n \geq 1$. The proof is almost the same as for $\mathsf{GL}(n,\mathbb{C})$ in (a), except by choosing $\lambda_n(t)$, in the second part of the preceding proof, to be equal to $(\lambda_1(t)\cdots\lambda_{n-1}(t))^{-1}$, we can ensure that the path is contained in $\mathsf{SL}(n,\mathbb{C})$.

(5) Every $A \in \mathsf{U}(n,\mathbb{C})$ unitary matrix has an orthonormal basis of eigenvectors, with eigenvalues having absolute value 1. Thus, each $U \in \mathsf{U}(n,\mathbb{C})$ can be written as $U = U_1 D U_1^{-1}$, where $U_1 \in \mathsf{U}(n,\mathbb{C})$ and D is diagonal with diagonal entries $e^{i\theta_1}, \ldots, e^{i\theta_n}$. We may then define

$$U(t) = U_1 \begin{pmatrix} e^{i(1-t)\theta_1} & 0 & \cdots & 0 \\ 0 & e^{i(1-t)\theta_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{i(1-t)\theta_n} \end{pmatrix} U_1^{-1}, \quad 0 \leq t \leq 1.$$

It is easy to see that $U(t) \in \mathsf{U}(n,\mathbb{C})$ for all t, and $U(t)$ connects U to I_n. Hence, $\mathsf{U}(n,\mathbb{C})$ is connected for $n \geq 1$. Similarly, $\mathsf{SU}(n,\mathbb{C})$ is connected for $n \geq 1$.

(6)

(7)

This completes the proof. □

Remark 1.31. *The previous proposition shows that a Lie group is not necessarily connected. However, a Lie group is always locally path-connected. This is a general fact about smooth manifolds.*

We end this section with some properties of the connected component of a Lie group.

Proposition 1.32. *Let G be a Lie group. Let G_0 be the connected component of the identity.*

1. *G_0 is open.*
2. *G_0 is a normal subgroup of G.*
3. *G/G_0 is a discrete group.*
4. *If G is connected, then G is generated by every neighborhood of the the identity.*
5. *If G is connected, a discrete normal subgroup, Γ, of G must be contained in the center of G.*

Proof. The proof is given below:

(1) This is a general fact about topological manifolds.
(2) For all $g \in G_0$, we have that gG_0 is connected, open, and closed since G_0 has these properties and L_g is a diffeomorphism. Since $g \in gG_0$, we have that $gG_0 = G_0$. Similarly, G_0^{-1} is connected, open, and closed containing e, so that $G_0^{-1} = G_0$. It follows that G_0 is a subgroup of G. Similarly, for all $g \in G$, we have that gG_0g^{-1} is connected, open, and closed. Since $e \in gG_0g^{-1}$, we have that $gG_0g^{-1} = G$, i.e., G_0 is normal.
(3) Using the fact that L_g is a diffeomorphism for each $g \in G$, (2) implies that all connected components of G are cl-open. Since each connected component is cl-open, G/G_0 is discrete.
(4) Let U be an open neighborhood of the identity. For each $n \in \mathbb{N}$, we denote by U_n the set of elements of the form $u_1 \cdots u_n$, where each $u_i \in U$. Let $W := \bigcup_{n\in\mathbb{N}} U_n$. Each U_n is an open set[12]. Hence, W is an open set. We now see check that W is a closed set. Let $g \in \overline{W}$, the closure of W. Since gU^{-1} is an open neighborhood of g, it must intersect W. Thus, let $h \in W \cap gU^{-1}$. We have the following:
   - Since $h \in gU^{-1}$, then $h = gu^{-1}$ for some element $u \in U$.
   - Since $h \in W$, then $h \in U_n$ for some $n \in \mathbb{N}$, i.e., $h = u_1 \cdots u_n$ with each $u_i \in U$.

   We then have $g = u_1 \cdots u_n u$, i.e., $g \in U_{n+1} \subseteq W$. Hence, W is closed. Since G is connected, we must have $W = G$. This means that G is generated by U.
(5) Let $x \in \Gamma$. Consider the map

$$\begin{aligned} C'_x : G &\to G \\ g &\mapsto gxg^{-1} \end{aligned}$$

   Since Γ is a normal subgroup, we have that $C'_x(G) \subseteq \Gamma$. Since G is connected, $C'_x(G)$ is connected. Since Γ is discrete, $C'_x(G)$ is a singleton. Since $x \in C'_x(G)$, we have that $C'_x(G) = \{x\}$. This shows that x is in the center of G. Hence, Γ is contained in the center of G.

This completes the proof. □

1.5. **Low Dimensional Examples.** We discuss some low dimensional examples.

Example 1.33. ($\mathsf{Sp}(1, \mathbb{R})$) Note that

$$\begin{aligned} A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathsf{Sp}(1,\mathbb{R}) &\iff \begin{pmatrix} d & c \\ b & a \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\ &\iff \begin{pmatrix} 0 & -(ad-bc) \\ ad-bc & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \end{aligned}$$

Therefore, we have

$$\mathsf{Sp}(1,\mathbb{R}) = \mathsf{SL}(2,\mathbb{R}).$$

Example 1.34. Let $A \in \mathsf{SO}(2,\mathbb{R})$. Since the columns of A are orthonormal, it readily follows that every matrix in $\mathsf{SO}(2,\mathbb{R})$ is of the form:

$$A_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

[12]This hold by induction.

Define a map

$$F : \mathbb{S}^1 \cong \mathsf{U}(1, \mathbb{C}) \to \mathsf{SO}(2, \mathbb{R})$$
$$e^{i\theta} \mapsto \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

F is clearly bijective. It can be easily checked that F is also smooth. Indeed, we can use stereographic coordinates on $\mathbb{S}^1$ and think of F as a map into $\mathbb{R}^4$, since $\mathsf{SO}(2, \mathbb{R})$ is an embedded submanifold of $\mathbb{R}^4$. We can then restrict the codomain accordingly. Using the usual angle sum identities, we can check that F is a homomorphism. Hence, F is a bijective Lie group homomorphism. Hence, F is a Lie group isomorphism[13]. Hence,

$$\mathsf{SO}(2, \mathbb{R}) \cong \mathbb{S}^1$$

Example 1.35. Let's now discuss in $\mathsf{SU}(2, \mathbb{C})$. Let $A \in \mathsf{SU}(2, \mathbb{C})$ and write A as

$$A = \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}$$

Since $A^{-1} = A^*$ and $\det(A) = 1$, we have

$$\frac{1}{\det(A)} \begin{pmatrix} \delta & -\gamma \\ -\beta & \alpha \end{pmatrix} = \begin{pmatrix} \alpha^* & \beta^* \\ \gamma^* & \delta^* \end{pmatrix} \quad \Rightarrow \quad A = \begin{pmatrix} \alpha & -\beta^* \\ \beta & \alpha^* \end{pmatrix}$$

Since the columns of A are orthonormal, we must also have that $|\alpha|^2 + |\beta|^2 = 1$. Hence, any $A \in \mathsf{SU}(2, \mathbb{C})$ is of the form

$$A_{\alpha,\beta} = \begin{pmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix}, \qquad |\alpha|^2 + |\beta|^2 = 1$$

This argument and Example 1.16 readily show that

$$\mathsf{SU}(2, \mathbb{C}) \cong \mathbb{H}_u \cong \mathbb{S}^3.$$

Example 1.35 implies that every plane rotation A_θ by an angle θ is represented by multiplication by the complex number $e^{i\theta} \in \mathsf{U}(1, \mathbb{C}) \cong \mathbb{S}^1$ in the sense that for all $z, z' \in \mathbb{C}$,

$$z' = \rho_\theta(z) \iff z' = e^{i\theta} z.$$

In some sense, the quaternions generalize the complex numbers in such a way that rotations of $\mathbb{R}^3$ are represented by multiplication by quaternions of unit length. We will explore this link now.

Example 1.36. ($\mathsf{SO}(3, \mathbb{R})$ and $\mathsf{SU}(2, \mathbb{C})$) Consider $\mathbb{H}_u$. We can identify $\mathbb{R}^3 \subseteq \mathbb{H}_u$ with unit quarternions such that $a = 0$. Using our matrix representation, we can equivalently consider the matrices,

$$A_{x_1,x_2,x_3} = \begin{pmatrix} ix_1 & x_2 + ix_3 \\ -x_2 + ix_3 & -ix_1 \end{pmatrix}$$

In what follows, instead identify $(x_1, x_2, x_3) \in \mathbb{R}^3$ with

$$A_{x_1,x_2,x_3} = \begin{pmatrix} x_1 & x_2 + ix_3 \\ x_2 - ix_3 & -x_1 \end{pmatrix}$$

[13]Here we use the fact that a bijective Lie group homomorphism is a Lie group isomorphism.

We have simply factored i and replaced x_2 by $-x_3$. Such matrices clearly form a subspace. Call it V. Note that V can be identified with 2×2 complex matrices which are self-adjoint and have trace zero. If we identify V with $\mathbb{R}^3$, the inner product on $\mathbb{R}^3$ can be computed as

$$\langle (x_1, x_2, x_3), (x_1', x_2', x_3') \rangle = \frac{1}{2} \operatorname{Tr}\Big(A_{x_1,x_2,x_3} A_{x_1',x_2',x_3'} \Big).$$

This is a straightforward computation. For each $U \in \mathsf{SU}(2, \mathbb{C})$, define a linear map $\Phi_U : V \to V$ by

$$\Phi_U(X) = UXU^{-1}.$$

This is well-defined since $\operatorname{Tr}(\Phi_U(X)) = \operatorname{Tr}(X) = 0$ and

$$(UXU^{-1})^\dagger = (U^{-1})^\dagger X^\dagger U^\dagger = UXU^{-1},$$

showing that UXU^{-1} is again in V. Furthermore,

$$\frac{1}{2} \operatorname{Tr}\big((UX_1U^{-1})(UX_2U^{-1})\big) = \frac{1}{2} \operatorname{Tr}\big(UX_1X_2U^{-1}\big) = \frac{1}{2} \operatorname{Tr}(X_1X_2),$$

Thus, each Φ_U preserves the inner product on $V \cong \mathbb{R}^3$. It follows that the we have a map

$$\Phi : \mathsf{SU}(2, \mathbb{C}) \to \mathsf{SO}(3, \mathbb{R})$$

A priori, Φ is only a map into $\mathsf{O}(3, \mathbb{R})$. Since $\mathsf{SU}(2, \mathbb{C})$ is connected Φ must actually lie in $\mathsf{SO}(3, \mathbb{R})$ for all $U \in \mathsf{SU}(2, \mathbb{C})$. It is easy to see that Φ is a group homomorphism.

Here is an example computation. Suppose, for example, that U is the matrix

$$U = \begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix}.$$

We obtain

$$U \begin{pmatrix} x_1 & x_2 + ix_3 \\ x_2 - ix_3 & -x_1 \end{pmatrix} U^{-1} = \begin{pmatrix} x_1' & x_2' + ix_3' \\ x_2' - ix_3' & x_1' \end{pmatrix}$$

where $x_1' = x_1$ and

$$x_2' + ix_3' = e^{i\theta}(x_2 + ix_3) = (x_2 \cos\theta - x_3 \sin\theta) + i(x_2 \sin\theta + x_3 \cos\theta).$$

In this case, Φ_U is a rotation by angle θ in the (x_2, x_3)-plane.

Proposition 1.37. *The map* $\ker \Phi$ *is a* 2*-to-*1 *covering map.*

Proof. We just have to check that $\ker \Phi \cong \mathbb{Z}_2$ and that Φ is surjective. ☐

2. Lie Algebra of a Lie Group

2.1. **Linearizing a Lie Group.** A Lie group can be quite difficult to understand. Fortunately, since a Lie group is a smooth manifold, we can consider its linearized version by looking at its tangent space at the identity, T_eG, which should be thought of as a *linearized* version of G.

The multiplication and inversion maps are in general non-linear, smooth maps. However, we can take the differential of these maps which takes elements from $\mathsf{T}_e(G \times G)$ (or T_eG) to T_eG. Hence, the differential of the multiplication and inversion maps should be thought of as linear approximations to to both multiplication and inverse maps on a Lie group. We compute these differentials. The differential of m at e is

$$\begin{aligned} dm_{(e,e)} : \mathsf{T}_e G \oplus \mathsf{T}_e G &\to \mathsf{T}_e G \\ (X, Y) &\mapsto X + Y \end{aligned}$$

where we have identified $\mathsf{T}_e(G \times G) \cong \mathsf{T}_e G \oplus \mathsf{T}_e G$. Indeed, we have

$$dm_{(e,e)}(X,Y) = dm_{(e,e)}(X,0) + dm_{(e,e)}(0,Y) = dm^1_e(X) + dm^2_e(Y)$$

where $m^1 : G \to G$ is defined by $x \mapsto m(x,e)$ and $m^2 : G \to G$ defined by $y \mapsto m(e,y)$. Since $m^1 = m^2 = \mathrm{Id}_G$, so

$$dm_{(e,e)}(X,Y) = X + Y$$

The differential of i at e

$$\begin{aligned} di_e : \mathsf{T}_e G &\to \mathsf{T}_e G \\ X &\mapsto -X \end{aligned}$$

Consider the constant map $1_G : G \to G$. $d(1_G)_e$ is clearly the zero map. 1_G can be thought of being given by the following composition:

$$g \mapsto (g, i(g)) \mapsto m(g, i(g)) = e$$

Therefore, we have

$$0 = d(1_G)_e(X) = (X, di_e(X)) = X + di_e(X) \quad \Longrightarrow \quad di_e(X) = -X$$

This shows that 'near the identity', multiplication behaves as addition and inversion behaves as subtraction.

Remark 2.1. *It turns out that the smoothness of the inversion map in a Lie group follows form the smoothness of the multiplication map. Let $\Delta = \{(g, g^{-1}) \in G \times G\}$. Then Δ is an embedded submanifold of $G \times G$[14]. Consider the diagram below:*

$$\begin{tikzcd} & & & G \\ G \arrow[r, "d"] & \Delta \arrow[r, hook, "\iota"] & G \times G \arrow[ur, "\pi_1"] \arrow[dr, "\pi_2"'] & \\ & & & G \end{tikzcd}$$

Here d is the map $g \mapsto (g, g^{-1})$, ι is the canonical embedding Δ in $G \times G$ and π_1 and π_2 are projection maps. Clearly, ι, π_1 and π_2 are smooth maps. We claim d is smooth as well. Consider $\pi_1 \circ \iota : \Delta \to G \times G \to G$, which maps $(g, g^{-1}) \mapsto g$. This is clearly a homeomorphism, and by the inverse function theorem, a diffeomorphism as well. But d is its inverse, and is thus smooth. But then the inversion map is just,

$$\pi_2 \circ \iota \circ d, \qquad g \mapsto g^{-1},$$

which is the composition of smooth maps and is thus smooth.

[14]Consider the map $m : G \times G \to G$ given by multiplication. This is a smooth map by assumption and $\Delta = m^{-1}(e)$. Since m is a Lie group homomorphism and m has constant rank, it suffices to show that for $(e,e) \in \Delta$, $dm|_{(e,e)} : \mathsf{T}_{(e,e)}(G \times G) \to \mathsf{T}_e(G)$ is surjective. But this is actually clear from the remark above.

This change of perspective has resulted in some loss of information about the Lie group. Indeed, T_pM can be computed for each $p \in M$ where M is a smooth manifold. If $M = G$ is a Lie group, what is special about T_eG? Can T_eG be interpreted in a different manner allowing us to further glean into the structure of G. For instance, G will in general be a non-commutative group. Therefore, the multiplication and inversion maps will in general be non-commutative. Is it possible to endow T_eG an additional structure that captures this non-commutativity? We explore this detail next.

2.2. **Left-Invariant Vector Fields.**

Definition 2.2. Let G be a Lie group. A vector field X on G is said to be **left-invariant** if it is invariant under left translations. That is,

$$d(L_g)_{g_0} \cdot X_{g_0} = X_{gg_0},$$

for all $g, g_0 \in G$. We denote the set of left-invariant vector fields as $\mathfrak{X}^L(G)$.

If X and Y are left-invariant vector fields, then we have

$$(dL_g)_{g_0}(aX_{g_0}bY_{g_0}) = a(dL_g)_{g_0}(X_{g_0}) + b(dL_g)_{g_0}(Y_{g_0}) = X_{gg_0} + Y_{gg_0}$$

for all $g \in G$ and $a, b \in \mathbb{R}$, we see that $\mathfrak{X}^L(G)$ is a linear subspace of $\mathfrak{X}(G)$, the vector space of all vector fields on G. We now show that $\mathfrak{X}^L(G)$ is isomorphic to T_eG as vector spaces.

Proposition 2.3. *Let G be a Lie group and let $\mathfrak{X}^L(G)$ denote the vector space of left-invariant vector fields on G. Then $\mathsf{T}_eG \cong \mathfrak{X}^L(G)$ as vector spaces via the map*

$$\begin{aligned} \varepsilon : \mathfrak{X}^L(G) &\to \mathsf{T}_eG \\ X &\mapsto X_e \end{aligned}$$

Proof. Clearly, ε is linear over $\mathbb{R}$. Moreover, ε is injective. Indeed, if $\varepsilon(X) = X_e = 0$ for some $X \in \mathfrak{X}^L(G)$, then left-invariance of X implies that

$$X_g = d(L_g)_e(X_e) = d(L_g)_e(0) = 0$$

for every $g \in G$. So $X = 0$. Let $v \in \mathsf{T}_eG$ be arbitrary. We can define a (rough) vector field v^L on G by

$$v^L|_g = d(L_g)_e(v).$$

Clearly, if ε is surjective, then we must have that $\varepsilon(v^L) = v$. Thus it suffices to show that v^L is a smooth, left-invariant vector field. We show v^L is a left-invariant vector field.

$$\begin{aligned} d(L_h)_g(v^L|_g) &= d(L_h)_h \cdot d(L_g)_e(v) \\ &= d(L_h \circ L_g)_e(v) \\ &= d(L_{hg})_e(v) = v^L_{hg} \end{aligned}$$

Hence, v^L is a left-invariant vector field, $v^L \in \mathfrak{X}^L(G)$. The proof of smoothness of v^L is skipped. ∎

The proof of Proposition 2.3 once again relies on an astute application of the comment made in Remark 1.7. We can go a step further. We now can use the multiplication operator to see how integral curves transform under the action of the diffeomorphism generated by left multiplication.

Lemma 2.4. *Let G be a Lie group.*

(1) *Every left-invariant vector field on G is complete, i.e. its corresponding integral curves are defined for all $t \in \mathbb{R}$.*

(2) *If γ is an integral curve of some left-invariant vector field, then*
$$\gamma(t+s) = \gamma(t)\gamma(s)$$
for each $s, t \in \mathbb{R}$.

(3) *Conversely, if $\gamma : \mathbb{R} \to G$ is a smooth curve such that*
$$\gamma(t+s) = \gamma(t)\gamma(s)$$
for $s, t \in \mathbb{R}$, then γ is the integral curve of some left-invariant vector field.

Proof. The proof is given below:

(1) Let $X \in \mathfrak{X}^L(G)$. There exists a maximal integral curve $\gamma_e : (-\varepsilon, \varepsilon) \to G$ such that $0 \in (a,b)$, $\gamma_e(0) = e$ and $\gamma_e'(t) = X_{\gamma_e(t)}$ and $\varepsilon > 0$[15]. Since
$$\left.\frac{d}{dt}\right|_{t=0} L_g(\gamma(t)) = d(L_g)_e(X_{\gamma(0)}) = X_g,$$
we have that $\gamma_g := L_g \circ \gamma_e$ is an integral curve of X starting at $g \in G$ for each $g \in G$. Assume $\varepsilon < \infty$ and let $s = \gamma(\varepsilon/2)$. Define a curve $\phi : (-\varepsilon, 3\varepsilon/2) \to M$ by
$$\phi(t) = \begin{cases} \gamma_e(t), & \text{for } -\varepsilon < t < \varepsilon, \\ \gamma_g(t-\varepsilon/2), & \text{for } -\varepsilon/2 < t < 3\varepsilon/2. \end{cases}$$
These two definitions agree on the overlap. Hence, $\phi(t)$ is an integral curve starting at e. Since $3\varepsilon/2 > \varepsilon$, this is a contradiction. Hence X is complete.

(2) Let $s \in \mathbb{R}$. The map $t \mapsto \gamma(s+t)$ is an integral curve[16] with initial point $g = \gamma(s)$. However by (1), $t \mapsto L_g \circ \gamma(t)$ is also an integral curve with initial point $g = \gamma(s)$. By uniqueness,
$$\gamma(s+t) = L_g \circ \gamma(t) = \gamma(s)\gamma(t)$$

(3) Let $X_e = d\gamma(\partial_t|_0)$ and let X denote the corresponding left-invariant field. The assumption,
$$\gamma(t+s) = \gamma(t)\gamma(s)$$
for $s, t \in \mathbb{R}$ implies that
$$\gamma \circ L_t = L_{\gamma(t)} \circ \gamma$$
for each $t \in \mathbb{R}$. Therefore,
$$d\gamma \circ dL_s = dL_{\gamma(s)} \circ d\gamma$$
For any $t_0 \in \mathbb{R}$, we have
$$\begin{aligned} \gamma'(t_0) = d\gamma\left(\left.\frac{d}{dt}\right|_{t_0}\right) &= d\gamma\left(dL_{t_0}(d\gamma\left(\left.\frac{d}{dt}\right|_0\right))\right) \\ &= dL_{\gamma(t_0)}\left(d\gamma\left(\left.\frac{d}{dt}\right|_0\right)\right) = dL_{\gamma(t_0)}(X_e) = X_{\gamma(t_0)} \end{aligned}$$
so γ is an integral curve of X.

This completes the proof. □

[15]WLOG, we have assumed the domain of the maximal integral curve is symmetric.

[16]This follows from the translation lemma. This is covered in [1, Chapter 9].

Given $X \in \mathfrak{X}^L(G)$, the integral curve $\gamma : \mathbb{R} \to G$ determined by X such that $\gamma(t+s) = \gamma(t)\gamma(s)$ for each $t, s \in \mathbb{R}$ is called the one-parameter subgroup generated of G by X. Thus there are one-to-one correspondences

$$\{\text{one-parameter subgroups of } G\} \longleftrightarrow \mathfrak{X}^L(G) \longleftrightarrow \mathsf{T}_e G.$$

Remark 2.5. *Note that the correspondence set above is only a bijective correspondence. We only have a vector space isomorphism between $\mathfrak{X}^L(G)$ and $\mathsf{T}_e G$.*

2.3. **Lie Algebra of a Lie Group.** We are now in a position to develop a measure of the 'non-commutativity' of a Lie group. Let $X, Y \in \mathfrak{X}^L(G)$, and let ϕ^X, ϕ^Y be the corresponding one-parameter subgroups (and hence maximal integral curves) of X and Y respectively such that

$$\phi^X(0) = e = \phi^Y(0) \qquad (\phi^X)'(0) = X_e, \;\; (\phi^Y)'(0) = Y_e$$

for $e \in G$. A heuristic argument suggests that for $XY - YX$ might be able to measure the non-commutativity of multiplication and inversion maps. If X and Y are (left-invariant) smooth vector fields on G, then XY might not be a vector field. However, a remarkable fact is that the difference

$$[X, Y] := XY - YX$$

is a vector field. We verify this claim. Recalling that vector fields are in one-to-one correspondence with derivations of $C^\infty(M)$, it suffices to check that $[X, Y]$ is a derivation. Linearity is clear. We verify the Leibniz rule. If $f, g \in C^\infty(M)$, then we have,

$$\begin{aligned}
[X,Y](fg) &= X(Y(fg)) - Y(X(fg)) \\
&= X(fY(g) + Y(f)g) - Y(fX(g) + X(f)g) \\
&= f(XYg) + (Xf)(Yg) + (XYf)g + (Yf)(Xg) \\
&\quad - (Yf)(Xg) - f(YXg) - (YXf)g - (Xf)(Yg) \\
&= f(XYg - YXg) + (XYf - YXf)g \\
&= f([X,Y]g) + ([X,Y]f)g.
\end{aligned}$$

In fact, if X, Y are left-invariant vector fields, then $[X, Y]$ is a left-invariant vector field. This can be easily checked. It is useful to write $[X, Y]$ in terms of its components in some chart. Let (U, φ) be a co-ordinate chart on G. We can write

$$X = X^i \partial_i \quad \text{and} \quad Y = Y^j \partial_j$$

Note that

$$[\partial_i, \partial_j]f = \partial_i \partial_j f - \partial_j \partial_i f = \frac{\partial^2}{\partial x^i \partial x^j}\left(f \circ \varphi^{-1}\right) - \frac{\partial^2}{\partial x^j \partial x^i}\left(f \circ \varphi^{-1}\right) = 0.$$

Therefore, we have

$$[X, Y] = X^i \partial_i Y^j \partial_j - Y^j \partial_j X^i \partial_i = \sum_{i,j=1}^{n} \left(X^i \partial_i Y^j - Y^i \partial_i X^j\right) \partial_j.$$

In any case, we can define the notion of a Lie bracket which is a 'measure this non-commutativity' of multiplication in a Lie group.

Definition 2.6. Let G be a Lie group. The **Lie bracket** of G is map bilinear map given by

$$\begin{aligned}[\cdot,\cdot] : \mathfrak{X}^L(G) \times \mathfrak{X}^L(G) &\to \mathfrak{X}^L(G),\\ (X,Y) &\mapsto XY - YX.\end{aligned}$$

We can now make the above claim precise by showing that the Lie bracket measures the extent to which the derivatives in directions X and Y do not commute.

Proposition 2.7. *Let G be a Lie group. Let $X, Y \in \mathfrak{X}^L(G)$, and let ϕ^X be the flow of X. Then*

$$[X,Y]_e = \frac{d}{dt}\left(d(\phi^X_{-t})_{\phi^X(t)} Y_{\phi^X(t)} \right)\Big|_{t=0} = \lim_{t\to 0} \frac{d(\phi^X_{-t})_{\phi^X(t)}(Y_{\phi^X(t)}) - Y_e}{t}$$

Here ϕ^X_{-t} denotes the map $\phi^X_{-t} : G \to G$ generated by flowing along integral curve generated by $-X$ for t units of time.

Proof. Choose any chart ϕ for G about e. In this chart, we can write uniquely $X = X^j \partial_j$ and $Y = Y^k \partial_k$, where the coefficients X^j and Y^k are smooth functions on a neighborhood of e. Working in co-ordinates, we have,

$$\begin{aligned}\frac{d}{dt}\left(d(\phi^X_{-t})_{\phi^X(t)} Y_{\phi^X(t)} \right)^j\Big|_{t=0} &= (\partial_t \partial_k \phi^{X,j}_{-t})\Big|_{t=0} Y^k_e + (\partial_k \phi^{X,j}_{-t}) \partial_t Y^k_{\phi^X(t)}\Big|_{t=0}\\ &= (\partial_k \partial_t \phi^{X,j}_{-t})\Big|_{t=0} Y^k_e + \delta^j_k X^i_e \partial_i Y^k_e\\ &= -Y^k_e \partial_k X^j_e + X^i_e \partial_i Y^j_e\\ &= X^i_e \partial_i Y^j_e - Y^i_e \partial_i X^j_e\\ &= [X,Y]^j_e.\end{aligned}$$

This completes the proof. □

Definition 2.8. Let G be a Lie group. The **Lie algebra of** G of G, denoted as $\mathfrak{g}$, is $\mathsf{T}_e G$ endowed with the Lie bracket as defined in Definition 2.6.

Remark 2.9. *Note that* $\dim \mathfrak{g} = \dim G$.

Example 2.10. Let $G = \mathsf{GL}(n,\mathbb{K})$. Then G is an open subset of $\mathbb{K}^{n^2}$. Hence, the corresponding Lie algebra is $\mathfrak{gl}(n,\mathbb{K}) = M(n,\mathbb{K})$.

2.4. **Abstract Lie Algebras.** Let $X, Y, Z \in \mathfrak{X}^L(G)$ and $a, b \in \mathbb{R}$. We note that the Lie bracket satisfies the following properties:

1. $[X,Y] = -[Y,X]$
2. $[aX + bY, Z] = a[X,Z] + b[Y,Z]$
3. (Jacobi Identity) $[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0$

The first two properties are immediate. The Jacobi identity can be verified directly. First note that we have,

$$[[X,Y],Z]f = [X,Y]Zf - Z[X,Y]f = XYZf - YXZf - ZXYf + ZYXf$$

As a result, we have:

$$
\begin{aligned}
[[X,Y],Z]f+[[Y,Z],X]f+[[Z,X],Y]f &= XYZf-YXZf-ZXYf+ZYXf \\
&\quad +YZXf-ZYXf-XYZf+XZYf \\
&\quad +ZXYf-XZYf-YZXf+YXZf \\
&= 0.
\end{aligned}
$$

These observations motivate the following definition of a (finite-dimensional) abstract real Lie algebra.

Definition 2.11. A (finite-dimensional) **Lie algebra** is a $\mathbb{R}$-vector space, $\mathfrak{g}$, together with a map $[\cdot,\cdot]:\mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$ called the Lie bracket with the following properties:

1. $[X,Y]$ is $\mathbb{R}$-bilinear,
2. $[X,Y]=-[Y,X]$ for all $X,Y\in\mathfrak{g}$, and
3. (Jacobi Identity) $[X,[Y,Z]]+[Y,[Z,X]]+[Z,[X,Y]]=0$ for all $X,Y,Z\in\mathfrak{g}$.

A **Lie algebra homomorphism** is a linear map $T:\mathfrak{g}\to\mathfrak{h}$ that preserves the Lie bracket.

Remark 2.12. *If $\mathfrak{g}$ is a finite-dimensional Lie algebra and $T_1,\dots,T_n$ is a vector space basis for $\mathfrak{g}$, then we can write*

$$[T_a,T_b]=\sum_{c=1}^{n} f_{ab}^{c}T_c,$$

where the coefficients $f_{ab}^c\in\mathbb{R}$ are called the structure constants for the given basis $\{T_a\}$. Because of bilinearity, the structure constants determine all commutators between elements of V. The structure constants satisfy,

$$
\begin{aligned}
f_{ab}^{c} &= -f_{ba}^{c}, \\
f_{ab}^{d}f_{dc}^{e}+f_{bc}^{d}f_{da}^{e}+f_{ca}^{d}f_{db}^{e} &= 0.
\end{aligned}
$$

Here we have used the Einstein summation convention and sum over d. Conversely, every set of n^3 numbers $f_{ab}^c\in\mathbb{R}$ satisfying these two conditions defines a Lie algebra structure on $V=\mathbb{K}^n$.

Example 2.13. The following is a list of examples of Lie algebras.

1. Let $\mathfrak{g}=\mathsf{GL}(n,\mathbb{K})$ Then $\mathfrak{g}$ is a Lie algebra with bracket operation given by

$$[X,Y]=XY-YX$$

   The bilinearity and skew symmetry of the bracket are evident. To verify the Jacobi identity, note that each double bracket generates four terms, for a total of 12 terms. It can be verified that the product of X, Y, and Z in each of the six possible orderings occurs twice, once with a plus sign and once with a minus sign.
2. Let $\mathfrak{g}=\mathbb{R}^3$ and let $[\cdot,\cdot]:\mathbb{R}^3\times\mathbb{R}^3\to\mathbb{R}^3$ be given by

$$[x,y]=x\times y$$

   where $x\times y$ is the cross product (or vector product). Then $\mathfrak{g}$ is a Lie algebra. Once again, the bilinearity and skew symmetry of the bracket are evident. Jacobi identity can be verified using a tedious computation.

Let LieAlg denote the category of finite-dimensional real Lie algebras. We note that we have defined a functor Lie from the category of Lie algebras, associating to each Lie group its Lie algebra:

$$\mathsf{Lie} : \mathsf{LieGrp} \to \mathsf{LieAlg}$$

This is essentially because if $F : G \to H$ is a Lie group homomorphism and $X, Y \in \mathfrak{g}$, then

$$dF_e[X,Y]_{\mathfrak{g}} = [dF_e(X), dF_e(Y)]_{\mathfrak{h}}.$$

Proposition 2.14. *Let G, H be Lie groups. If $F : G \to H$ is a Lie group homomorphism and $X, Y \in \mathfrak{g}$, then*

$$dF_{e_G}[X,Y]_{\mathfrak{g}} = [dF_{e_G}(X), dF_{e_G}(Y)]_{\mathfrak{h}}.$$

Proof. dF_{e_G} is clearly linear. We show that dF_{e_G} preserves the Lie bracket. Let X, Y be left-invariant vector fields on G. For any $f \in C^\infty(H)$, we have

$$dF_{e_G}([X,Y]_{e_G})f = [X,Y]_{e_G}(f \circ F) = X_{e_G}(Y(f \circ F)) - Y_{e_G}(X(f \circ F))$$

Similarly, we have

$$\begin{aligned}[dF(X), dF(Y)]_{e_H} f &= dF_{e_G}(X_{e_G})(dF(Y)f) - dF_{e_G}(Y_{e_G})(dF(X)f) \\ &= X_{e_G}(dF(Y)f \circ F) - Y_{e_G}(dF(X)f \circ F).\end{aligned}$$

So it is enough to check that, as functions on G, $Y(f \circ F) = dF(Y)f \circ F$. In fact, for any $g \in G$, we have

$$Y(f \circ F)(a) = Y_g(f \circ F) = dL_g(Y_{e_G})(f \circ F) = Y_{e_G}(f \circ F \circ L_g) = Y_{e_G}(f \circ L_{F(g)} \circ F),$$

Similarly, we have

$$dF(Y)f \circ F(g) = dF(Y)(f)(F(g)) = dL_{F(g)} \circ dF_{e_G}(Y_{e_G})(f) = Y_{e_G}(f \circ L_{F(g)} \circ F).$$

This completes the proof. $\square$

Example 2.15. Let $G = \mathbb{R}^n$. For any $a \in \mathbb{R}^n$, the left translation L_a is just the usual translation map on $\mathbb{R}^n$. So dL_a is the identity map, as long as we identify $\mathsf{T}_a\mathbb{R}^n \cong \mathbb{R}^n$. It follows that any left-invariant vector field is, in fact, a constant vector field, i.e.,

$$X_{\mathbf{v}} = v^1 \frac{\partial}{\partial x_1} + \cdots + v^n \frac{\partial}{\partial x_n},$$

for $\mathbf{v} = (v^1, \cdots, v^n) \in \mathsf{T}_0\mathbb{R}^n$. Since ∂_i commutes with ∂_j for any pair (i,j), the Lie bracket of any two left-invariant vector fields vanishes. Hence, In other words, the Lie algebra of $G = \mathbb{R}^n$ is $\mathfrak{g} = \mathbb{R}^n$ with a vanishing Lie bracket.

Note that $\mathfrak{gl}(n, \mathbb{K})$ can be thought of as a Lie algebra in two different ways. First, it is a Lie algebra identified as the tangent space to $\mathsf{GL}(n, \mathbb{K})$, with the Lie bracket given by the Lie bracket on vector fields. Second, it can be identified as an abstract Lie algebra with the Lie bracket given by the commutator of matrices. A natural question arises: what is the relationship between these two Lie algebra structures? In fact, these two notions coincide in the sense that there is a Lie algebra isomorphism between these two Lie algebra structures on $\mathfrak{gl}(n, \mathbb{K})$. We prove the result below for $\mathbb{K} = \mathbb{R}$.

Proposition 2.16. *The natural map $\mathsf{T}_{I_n}\mathsf{GL}(n, \mathbb{R}) \to \mathfrak{gl}(n, \mathbb{R})$ is a Lie algebra isomorphism.*

Proof. The natural isomorphism takes the form

$$A^i_j \frac{\partial}{\partial X^i_j}\bigg|_{I_n} \mapsto A^i_j.$$

Any matrix $A = (A^i_j) \in \mathfrak{gl}(n, \mathbb{R})$ determines a left-invariant vector field $A^L \in \mathsf{T}_{I_n} \mathsf{GL}(n, \mathbb{R})$

$$A^L\big|_X = d(L_X)_{I_n}(A) = d(L_X)_{I_n} \left(A^i_j \frac{\partial}{\partial X^i_j}\bigg|_{I_n} \right) = X^i_j A^j_k \frac{\partial}{\partial X^i_k}\bigg|_X.$$

Given $A, B \in \mathfrak{gl}(n, \mathbb{R})$, the Lie bracket of the corresponding left-invariant vector fields is given by

$$\begin{aligned}
[A^L, B^L] &= \left[X^i_j A^j_k \frac{\partial}{\partial X^i_k}, X^p_q B^q_r \frac{\partial}{\partial X^p_r} \right] \\
&= X^i_j A^j_k \frac{\partial}{\partial X^i_k} \left(X^p_q B^q_r \right) \frac{\partial}{\partial X^p_r} - X^p_q B^q_r \frac{\partial}{\partial X^p_r} \left(X^i_j A^j_k \right) \frac{\partial}{\partial X^i_k} \\
&= X^i_j A^j_k B^k_r \frac{\partial}{\partial X^i_r} - X^p_q B^q_r A^r_k \frac{\partial}{\partial X^p_k} \\
&= \left(X^i_j A^j_k B^k_r - X^i_j B^j_k A^k_r \right) \frac{\partial}{\partial X^i_r}.
\end{aligned}$$

Evaluating this last expression when X is equal to the identity matrix, we get

$$[A^L, B^L]\big|_{I_n} = \left(A^i_k B^k_r - B^i_k A^k_r \right) \frac{\partial}{\partial X^i_r}\bigg|_{I_n}.$$

This is the vector corresponding to the matrix commutator bracket $[A, B]$. Since the left-invariant vector field $[A^L, B^L]$ is determined by its value at the identity, this implies that

$$[A^L, B^L] = [A, B]^L,$$

which implies that the natural map is a Lie algebra isomorphism. □

3. Exponential Map

If we have a Lie group G and a Lie algebra $\mathsf{T}_e G$, we aim to find a method to map elements of the algebra back onto the group. This mapping is important because the Lie algebra provides a linearized approximation of the Lie group near the identity element, and understanding how to map from the Lie algebra back to the Lie group is crucial for many applications in differential geometry, representation theory, and physics. Let's see how to do this in the case of matrix Lie groups. Let G be a matrix Lie group over $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ with associated Lie algebra, $\mathfrak{g}$. If $X \in \mathfrak{g}$, the corresponding one-parameter subgroup satisfies the initial value problem (IVP):

$$\gamma(0) = I_n, \quad \gamma'(0) = X.$$

The solution to this IVP is given by the matrix exponential:

$$\gamma(t) = e^{tX} := \sum_{n=0}^{\infty} \frac{t^n X^n}{n!} \quad t \in \mathbb{R}.$$

It can be checked that e^X converges for all $X \in M(n, \mathbb{K})$ and that e^X is a continuous function of X. Thus, the matrix exponential maps the element X of the Lie algebra to a one-parameter subgroup of the Lie group, making it a powerful method for understanding the structure of the Lie group and its algebra.

Example 3.1. We compute the exponential map in some basic cases, using the characterization of Lie algebras of certain classical Lie groups, as discussed later in Example 3.23.

(1) We will see later that the Lie algebra of $\mathbb{R}$ is $\mathbb{R}$. Hence, we have

$$\begin{aligned}\exp\colon \mathbb{R} &\to \mathsf{U}(1,\mathbb{C}) \cong \mathbb{R},\\ x &\mapsto e^x.\end{aligned}$$

(2) We will see later on that the Lie algebra of $\mathsf{U}(1,\mathbb{C})$ is isomorphic $i\mathbb{R}$. Hence, we have

$$\begin{aligned}\exp\colon i\mathbb{R} &\to \mathsf{U}(1,\mathbb{C}) \cong \mathbb{S}^1,\\ ix &\mapsto e^{ix}.\end{aligned}$$

(3) We will see later on that the Lie algebra of $\mathsf{SO}(2,\mathbb{R})$ is isomorphic $\mathbb{R}$ with generator

$$X = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

It is easy to see that

$$\begin{aligned}X^{2n} &= (-1)^n I_2,\\ X^{2n+1} &= (-1)^n X.\end{aligned}$$

Hence, we have

$$\begin{aligned}e^{tX} &= \sum_{n=0}^{\infty} \frac{t^n X^n}{n!}\\ &= \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n)!}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)!}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}\end{aligned}$$

Example 3.2. The matrix exponential can be computed effectively for certain special 2×2 matrices:

(1) We compute

$$\exp\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$$

First assume $a = d$. In this case, an inductive argument shows that

$$\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}^n = \begin{pmatrix} a^n & a^{-1+n} bn \\ 0 & a^n \end{pmatrix}$$

for $n \geq 1$. Hence,

$$\exp\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} = \begin{pmatrix} \sum_{n=0}^{\infty} \frac{a^n}{n!} & b\sum_{n=1}^{\infty} \frac{a^{n-1}}{(n-1)!} \\ 0 & \sum_{n=0}^{\infty} \frac{a^n}{n!} \end{pmatrix} = \begin{pmatrix} e^a & be^a \\ 0 & e^a \end{pmatrix}$$

If $a \neq d$, an inductive argument shows that

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}^n = \begin{pmatrix} a^n & \frac{b(a^n - d^n)}{a-d} \\ 0 & d^n \end{pmatrix}$$

for $n \geq 1$. Hence,

$$\exp\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = \begin{pmatrix} \sum_{n=0}^{\infty} \frac{a^n}{n!} & \frac{b}{a-d} \sum_{n=1}^{\infty} \frac{(a^n - d^n)}{n!} \\ 0 & \sum_{n=0}^{\infty} \frac{d^n}{n!} \end{pmatrix} = \begin{pmatrix} e^a & b\frac{e^a - e^d}{a-d} \\ 0 & e^d \end{pmatrix}$$

(2) Let $X \in \mathrm{M}(2, \mathbb{C})$ such that $\mathrm{trace}(X) = 0$. It is not too hard to check that such a X satisfies

$$X^2 = -\det(X) I.$$

We have

$$\begin{aligned} \exp(X) &= \sum_{n=0}^{\infty} \frac{X^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{X^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{X^{2n+1}}{(2n+1)!} \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{\det(X)^n I}{(2n)!} + \sum_{n=0}^{\infty} (-1)^n \frac{\det(X)^n X}{(2n+1)!} \\ &= \cos\left(\sqrt{\det(X)}\right) I + \frac{\sin\left(\sqrt{\det(X)}\right)}{\sqrt{\det(X)}} X. \end{aligned}$$

Here $\sqrt{\det X}$ is either of the two (possibly complex) square roots of $\det X$.

More generally, we can define an exponential map for an arbitrary Lie group. The exponential map provides a natural way of mapping $\mathsf{T}_e G$ onto G such that exp acts as a homomorphism when restricted to any line in $\mathsf{T}_e G$.

Definition 3.3. Let G be a Lie group with Lie algebra $\mathfrak{g} := \mathsf{T}_e G$. The **exponential map** of G is the map

$$\begin{aligned} \exp : \mathfrak{g} &\to G \\ X &\mapsto \gamma(1) \end{aligned}$$

where $\gamma : \mathbb{R} \to G$ is the integral curve associated with the left-invariant vector field, X.

Remark 3.4. *We choose $\gamma(1)$ because we want* exp *to be its own derivative, similar to e^X.*

Proposition 3.5. *Let G be a Lie group and let $\mathfrak{g}$ be its Lie algebra. Let* exp *denote the exponential map.*

(1) *For any $X \in \mathfrak{g}$, $\gamma(t) = \exp(tX)$ is the one-parameter subgroup of G generated by X.*
(2) exp *is a smooth map.*
(3) *For any $X \in \mathfrak{g}$ and $s, t \in \mathbb{R}$,*

$$\begin{aligned} \exp((s+t)X) &= \exp(sX)\exp(tX), \\ (\exp(X))^{-1} &= \exp(-X) \end{aligned}$$

(4) *For any $X \in \mathfrak{g}$ and $n \in \mathbb{Z}$, $(\exp(X))^n = \exp(nX)$.*
(5) *The differential $(d\exp)_0 : \mathfrak{g} \cong \mathsf{T}_0\mathfrak{g} \to \mathsf{T}_e G$ is the identity map.*
(6) *The exponential map restricts to a diffeomorphism from some neighborhood of 0 in $\mathfrak{g}$ to a neighborhood of e in G.*

(7) *If H is another Lie group with Lie algebra $\mathfrak{h}$ and $f : G \to H$ is a Lie group homomorphism, the following diagram commutes:*

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "f_*"] \arrow[d, "\exp"] & \mathfrak{h} \arrow[d, "\exp"'] \\ G \arrow[r, "f"] & H \end{tikzcd}$$

That is,

$$f(e^X) = e^{f_*(X)}$$

Here $f_* = d_0 f$ is the map induced by the Lie *functor. This shows that* exp *defines a natural transformation between the functors* Lie *and the identity functor.*

Proof. The proof is given below:

(1) Let $\gamma : \mathbb{R} \to G$ be the one-parameter subgroup generated by X. For any fixed $s \in \mathbb{R}$, it follows that $\gamma_s(t) = \gamma(st)$ is the integral curve of sX starting at e. Hence,

$$\exp(sX) = \gamma_s(1) = \gamma(s)$$

(2) Define a map $\varphi : \mathbb{R} \times (G \times \mathfrak{g}) \to G \times \mathfrak{g}$ by

$$\varphi(t, g, X) = (g \cdot \exp(tX), X),$$

Note that this is the flow of the left-invariant vector field $(X, 0)$ on $G \times \mathfrak{g}$. Thus, it is smooth as the flow of a smooth vector field. Now we can decompose exp as

$$\exp = \pi_1 \circ \varphi \circ i = \mathfrak{g} \xrightarrow{i} \mathbb{R} \times G \times \mathfrak{g} \xrightarrow{\varphi} G \times \mathfrak{g} \xrightarrow{\pi_1} G,$$

because

$$\pi_1(\varphi(i(X))) = \pi_1(\varphi(1, e, X)) = \pi_1(\exp(X), X) = \exp(X)$$

for every $X \in \mathfrak{g}$. We conclude that exp is smooth as a composition of smooth maps.

(3) This follows from (1) since γ is group homomorphism.

(4) This follows from (3) and induction.

(5) Let $X \in \mathfrak{g}$ and let $\gamma : \mathbb{R} \to \mathfrak{g}$ be the curve $\gamma(t) = tX$. Then $\gamma'(0) = X$, and (1) implies

$$(d\exp)_0(X) = (d\exp)_0(\gamma'(0)) = (\exp \circ \gamma)'(0) = \left.\frac{d}{dt}\right|_{t=0} \exp(tX) = X.$$

(6) This follows from (5) and the inverse function theorem.

(7) We will show that for all $t \in \mathbb{R}$,

$$\exp(tf_*(X)) = f(\exp(tX)).$$

By (1), the left-hand side is the one-parameter subgroup generated by $f_*(X)$. Thus, if $\gamma(t) = f(\exp(tX))$, it suffices to show that $\gamma : \mathbb{R} \to H$ is a Lie group homomorphism satisfying $\gamma'(0) = f_*(X)$. It is a Lie group homomorphism because it is the composition of the homomorphisms f and $t \mapsto \exp(tX)$. Note that we have:

$$\begin{aligned} \gamma'(0) &= \left.\frac{d}{dt}\right|_{t=0} f(\exp(tX)) \\ &= df_0\left(\left.\frac{d}{dt}\right|_{t=0} \exp(tX)\right) \\ &= df_0(X) = f_*(X). \end{aligned}$$

This completes the proof. $\square$

Remark 3.6. *Note that Proposition 3.5(3) implies that* $\exp(0) = e_G$.

The intuition behind Proposition 3.5(6) is that the exponential map can be used to reconstruct a Lie group from its Lie algebra, at least locally near the identity. We now make precise this intuition.

Proposition 3.7. *Let G be a Lie group with Lie algebra $\mathfrak{g}$. The exponential map generates G_0, the connected component of the identity. In particular, if G is connected, every $g \in G$ can be written as*

$$g = \exp(X_1) \cdots \exp(X_n)$$

for some $X_1, \cdots, X_n \in \mathfrak{g}$.

Proof. Proposition 3.5(6) implies that there exists open neighbourhods $0 \in V \subseteq \mathfrak{g}$ and $e_G \in U \subseteq G$ such that $U = \exp(V)$ is a diffeomorphism. For any $g \in G_0$, choose a continuous path $\gamma : [0,1] \to G$ with $\gamma(0) = e_G$ and $\gamma(1) = g$. We can find some $\delta > 0$ such that if $|s-t| < \delta$, then $\gamma(s)\gamma(t)^{-1} \in U$[17]. Divide $[0,1]$ into m pieces, where $1/m < \delta$. Then, for $j = 1, \ldots, m$, we see that $\gamma((j-1)/m)^{-1}\gamma(j/m)$ belongs to U, so that

$$\gamma((j-1)/m)^{-1}\gamma(j/m) = \exp(X_j)$$

for some elements $X_1, \ldots, X_m$ of $\mathfrak{g}$. Thus,

$$\begin{aligned} g &= \gamma(0)^{-1}\gamma(1) \\ &= \gamma(0)^{-1}\gamma(1/m)\gamma(1/m)^{-1}\gamma(2/m) \cdots \gamma((m-1)/m)^{-1}\gamma(1) \\ &= \exp(X_1) \cdots \exp(X_n) \end{aligned}$$

This completes the proof. $\square$

However, it is not true that one can globally recover a Lie group from a Lie algebra via the exponential map. This is because the exponential map need not be surjective.

Example 3.8. Let $G = \mathsf{SL}(2, \mathbb{R})$. Consider the matrix:

$$A = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$$

We claim that we cannot find $X \in \mathfrak{sl}(2, \mathbb{R})$ such that $A = e^X$. Assume to the contrary that there is a trace zero matrix X such that,

$$B = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} = e^X = e^{X/2}e^{X/2} = (e^{X/2})^2$$

However, we show that A doesn't have a square root in $\mathsf{GL}(2, \mathbf{R})$. Assume this is not the case. Then we have,

$$A = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^2 = \begin{pmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{pmatrix}, \qquad ad - bc \neq 0.$$

[17]This follows from a compactness argument.

Comparing coefficients, we have the system of equations,

$$\begin{aligned} a^2 + bc &= -1, \\ ab + bd &= 1, \\ ac + cd &= 0, \\ bc + d^2 &= -1. \end{aligned}$$

Note that we can't have $c = 0$ or else this would imply that we have $a^2 = -1$ by the first equation. Hence $c \neq 0$ implies that we have $d = -a$ by the third equation. But then the second equation implies we have,

$$1 = ab + bd = ab - ba = 0,$$

a contradiction. However, Proposition 3.7 implies that each $G = \mathsf{SL}(2, \mathbb{R})$ can be written as a product of finitely many expressions in $\exp(\mathfrak{g})$.

Remark 3.9. *Later on, we will see that a sufficient condition for the exponential map to be surjective is that G is a compact, connected Lie group.*

Remark 3.10. *If G is a matrix Lie group, we could also have that $e^X \in G$ but $X \notin \mathfrak{g}$. Indeed, let $G = \mathsf{SL}(n, \mathbb{C})$, with Lie algebra $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$. Then $X := 2\pi i I_n$ does not lie in $\mathfrak{sl}(n, \mathbb{C})$, but $e^X = I_n \in \mathsf{SL}(n, \mathbb{C})$.*

Example 3.11. The exponential map can be onto but not one-to-one. Let $G = \mathsf{U}(n, \mathbb{C})$ and $\mathfrak{g} = \mathfrak{u}(n)$. We first show that it is surjective. Let $U \in \mathsf{U}(n, \mathbb{C})$. By the spectral theorem for normal matrices, there exists a unitary matrix V such that $U = VD_UV^*$, where D is diagonal with

$$D_U = \operatorname{diag}(e^{i\theta_1}, \cdots, e^{i\theta_n})$$

We have $D_U = \exp(D_X)$ where

$$D_X = i \operatorname{diag}(\theta_1, \cdots, \theta_n)$$

Now, let $X = VD_XV^*$. We have

$$\exp(X) = \exp(VD_XV^*) = V \exp(D_X)V^* = VD_UV^* = U$$

Moreover, $X \in \mathfrak{u}(n)$ since

$$X^* = (VD_XV^*)^* = VD_X^*V^* = -VD_XV^* = -X$$

However, exp is not injective. This follows because $\exp(\operatorname{diag}(2\pi i m, \cdots, 2\pi i m)) = I_n$ for $m \in \mathbb{Z}$.

We conclude with an important characterization of the Lie bracket in terms of the differentials of the conjugation map. This identification follows from the formula in Proposition 2.7, now made precise by expressing the integral curves explicitly using the exponential map.

Proposition 3.12. *Let G be a Lie group with Lie algebra $\mathfrak{g}$. For $X, Y \in \mathfrak{g}$, we have*

$$[X, Y]_e = \frac{d}{dt} \left(dC_{\exp(tX)}(Y) \right) \Big|_{t=0}$$

Here $C_{(\cdot)}$ denotes the conjugation map.

Proof. For $g \in G$, note that left multiplication map by L_g takes integral curves of X to integral curves of X. Thus, the map $\gamma(t) = L_g(\exp tX)$ is the integral curve such that $\gamma(0) = g$ and $\gamma'(0) = X_g$. It follows that

$$R_{\exp tX}(g) = g \exp tX = L_g(\exp tX) = \gamma(t).$$

Recall from Proposition 2.7 that we have

$$[X, Y]_e = \frac{d}{dt}\left(d(\phi^X_{-t})_{\phi^X(t)}(Y_{\phi^X(t))})\right)\Big|_{t=0}.$$

We have $Y_{\phi^X(t)} = Y_{\exp(tX)}$ and $R_{\exp(-tX)}(\exp(tX))$ is the integral curve starting at $\exp(tX)$ and generated by $-X$ for time t. Hence, we have

$$\begin{aligned} d(\phi^X_{-t})_{\phi^X(t)}(Y_{\phi^X(t)}) &= dR_{\exp(-tX)}(Y_{\exp(tX)}) \\ &= dR_{\exp(-tX)}(dL_{\exp(tX)}(Y)) \\ &= d(R_{\exp(-tX)} \circ L_{\exp(tX)})(Y) = dC_{\exp(tX)}(Y) \end{aligned}$$

Therefore, we have

$$[X, Y]_e = \frac{d}{dt}\left(d(\phi^X_{-t})_{\phi^X(t)}Y_{\phi^X(t)}\right)\Big|_{t=0} = \frac{d}{dt}\left(dC_{\exp(tX)}(Y)\right)\Big|_{t=0}$$

This completes the proof. □

3.1. **Abelian Lie Groups.** We can use the exponential map, along with the fact that it can be used to construct a Lie group from its Lie algebra locally, to classify abelian Lie groups. We first prove a lemma which we state for a general Lie group.

Lemma 3.13. *Let G be a connected Lie group with Lie algebra $\mathfrak{g}$.*

(1) *If $X, Y \in \mathfrak{g}$, then*
$$\exp(tX)\exp(sY) = \exp(sY)\exp(tX) \iff [X, Y] = 0$$
for all $t, s \in \mathbb{R}$.

(2) *G is abelian if and only if $\mathfrak{g}$ is abelian. That is, $[X, Y] = 0$ for all $X, Y \in \mathfrak{g}$.*

(3) *We have*
$$\exp(X)\exp(Y) = \exp(X + Y)$$
for all $X, Y \in \mathfrak{g}$ if and only if G is abelian.

(4) *If G is abelian, then the exponential map is surjective that is a group homomorphism of abelian groups.*

Proof. The proof is given below:

(1) We first prove the forward implication. Since G is connected, the assumption implies that $C_g(h) = h$ for all $g, h \in G$. Hence, $[X, Y] = 0$ for all $X, Y \in \mathfrak{g}$ by Proposition 3.12 since the differential is of C_g is zero for all $g \in G$. The converse follows similarly.

(2) If G is abelian, the equation in (1) is true. Hence, $\mathfrak{g}$ is abelian. Conversely, if $\mathfrak{g}$ is abelian, then (1) implies that we have
$$\exp(tX)\exp(sY) = \exp(sY)\exp(tX)$$
for all $X, Y \in \mathfrak{g}$ and $t, s \in \mathbb{R}$. Since G is connected, As this implies that $C_g(h) = h$ for all $g, h \in G$. Hence, G is abelian.

(3) The forward implication is clear from (1) and (2). Conversely, assume that G is abelian. Consider the map:

$$\gamma(t) = (\exp(tX))(\exp(tY))$$

We have

$$\begin{aligned}\gamma(t+s) &= (\exp((t+s)X))(\exp((t+s)Y)) \\ &= (\exp(tX))(\exp(sX))(\exp(tY))(\exp(sY)) \\ &= (\exp(tX))(\exp(tY))(\exp(sX))(\exp(sY)) \\ &= \gamma(t)\gamma(s).\end{aligned}$$

Hence, γ is a 1-parameter subgroup. Note that $\gamma(0) = e$ and $\gamma'(0) = X + Y$ Hence,

$$\gamma(t) = (\exp(tX))(\exp(tY)) = \exp(t(X+Y))$$

Plug in $t = 1$ now.

(4) Consider the exponential map:

$$\exp : \mathfrak{g} \to G.$$

Since G is connected and abelian if $g \in G$

$$g = \exp(X_1)\cdots\exp(X_n) = \exp(X_1 + \cdots + X_n) \in \exp(\mathbb{R})$$

for $X_i \in \mathfrak{g}$. The last equality follows from (3). Thus, exp is surjective. Clearly, exp is a group homomorphism of abelian groups.

This completes the proof. □

Proposition 3.14. *Let G be a connected Lie group.*

(1) *If G is 1-dimensional, then G is isomorphic to $\mathbb{R}$ or $\mathbb{S}^1$.*
(2) *If $\dim G = n$ and G is abelian, then*

$$G \cong (\mathbb{S}^1)^s \times \mathbb{R}^{n-k}$$

for some $0 \leq s \leq n$.

Proof. The proof is given below:

(1) Since G is one-dimensional, $\mathfrak{g} \cong \mathbb{R}$ is an abelian Lie algebra. By Lemma 3.13, G is abelian. Since G is connected as well, the exponential map is a surjective group homomorphism. We can think of $\mathbb{R}$ as a smooth manifold/Lie group. Hence, exp is a surjective Lie group homomorphism. If $\ker\exp = \{0\}$, exp is a bijective Lie group homomorphism. Hence, it is a Lie group isomorphism[18]. Hence,

$$G \cong \mathbb{R}.$$

Otherwise, assume that $\ker\exp \neq \{0\}$. We claim that $\ker\exp = r\mathbb{Z}$ for some $r > 0$. Indeed, Let

$$r = \inf\{a \in A : a > 0\}$$

Since exp is injective on some neighborhood of 0, we have $r > 0$. Moreover, $r \in A$ since A is closed. Thus, $r\mathbb{Z} \subseteq A$ since A is a group. We now show that $A \subseteq r\mathbb{Z}$. Let $a \in A$ and suppose that $a \notin r\mathbb{Z}$. Then, there exists $k \in \mathbb{Z}$ such that $0 < a - kr < r$.

[18] A bijective Lie group homomorphism is a Lie group isomorphism.

But $a - kr \in A$ since $r \in A$, which contradicts the definition of r. Thus, $A = r\mathbb{Z}$. In this case $\mathbb{S}^1 \cong \mathbb{R}/r\mathbb{Z}$. The exp descends to a bijective group homomorphism:

$$\widetilde{\exp} : \mathbb{S}^1 \cong \frac{\mathbb{R}}{\ker \exp} \to G$$

This is a smooth map since $\mathbb{R} \to \mathbb{S}^1$ is a smooth submersion. Hence, $\widetilde{\exp}$ is a bijective Lie group homomorphism. Hence,

$$G \cong \mathbb{S}^1$$

(2) Consider the exponential map:

$$\exp : \mathfrak{g} \to G$$

Since $\dim G = n$ and G is abelian, $\mathfrak{g}$ is also abelian. Hence, $\mathfrak{g} \cong \mathbb{R}^n$ with the trivial Lie bracket. Since G is connected, the exponential map is surjective group homomorphism. Since exp is a local diffeomorphism, exp has discrete kernel. This follows because $\mathbb{R}^n$ is a Lie group and it is homogenous. Using the general fact that a discrete subgroup of $\mathbb{R}^n$ is isomorphic to $\mathbb{Z}^s$ for some $0 \leq s \leq n$. Hence, as in (1) we have

$$G \cong \frac{\mathbb{R}^n}{\mathbb{Z}^s} \cong (\mathbb{S}^1)^s \times \mathbb{R}^{n-s}.$$

This completes the proof. □

3.2. **Lie Subalgebras & Lie Subgroups.** We have defined Lie groups and Lie algebras. We would now like to define sub-objects for Lie groups and Lie algebras. In this section, we define subobjects of a Lie algebra.

Definition 3.15. Let $\mathfrak{g}$ be a Lie algebra. A **Lie subalgebra**, $\mathfrak{h}$, is a vector subspace of $\mathfrak{g}$ such that $[X, Y] \in \mathfrak{h}$ holds for all $X, Y \in \mathfrak{h}$.

Example 3.16. The simplest example is $\mathfrak{g} = \mathbb{R}^2$, endowed with the trivial Lie algebra $[\cdot, \cdot] \equiv 0$. Then any vector subspace of $\mathbb{R}^n$ is a Lie subalgebra of $\mathfrak{g}$.

Many standard linear algebra facts carry over to the setting of Lie algebras and Lie subalgebras.

Lemma 3.17. *Let $A : \mathfrak{g} \to \mathfrak{h}$ be a Lie algebra homomorphism. Then* $\ker A \subseteq \mathfrak{g}$ *and* $\operatorname{im} A \subseteq \mathfrak{h}$ *are Lie subalgebras.*

Proof. The kernel and image of A are linear subspaces for algebraic reasons, so it suffices to check that they are closed under the brackets on $\mathfrak{g}$ and $\mathfrak{h}$, respectively. For any $X, Y \in \ker A$ we have

$$A([X, Y]) = [A(X), A(Y)] = [0, 0] = 0$$

so $[X, Y] \in \ker A$ and the kernel is closed under brackets. Similarly, for any $u, v \in \mathfrak{g}$ the equation

$$[A(X), A(Y)] = A([X, Y])$$

implies that $[A(X), A(Y)] \in \operatorname{Im} A$. Hence the image is closed under brackets. □

Remark 3.18. *In fact,* $\ker A$ *is an ideal. That is,* $[X, Y] \in \ker A$ *if* $X \in \mathfrak{g}$ *and* $Y \in \ker A$. *This is easily seen to be true since*

$$A[X, Y] = [A(X), A(Y)] = [A(X), 0] = 0.$$

The matrix Lie groups discussed before are all examples of (closed) Lie subgroups, to be defined shortly. In fact, these are embedded submanifolds of the general linear group. Should we require a Lie subgroup to be an embedded manifold? The answer is no.

Let $\mathfrak{g} = \mathbb{R}^2$. We consider Lie subalgebras of $\mathfrak{g}$ that are 1-dimensional subspaces of $\mathbb{R}^2$. All such Lie subalgebras are of the form:

$$\mathfrak{h}_\alpha = \text{the line passing through the origin in } \mathbb{R}^2 \text{ whose slope equals } \alpha.$$

If $\alpha = \frac{p}{q}$, where p, q are co-prime integers, then

$$G_{p,q} = \left\{ \begin{pmatrix} e^{ipt} & 0 \\ 0 & e^{iqt} \end{pmatrix} : t \in \mathbb{R} \right\} \subseteq \mathsf{GL}(2, \mathbb{C})$$

can be easily seen to be a matrix Lie group of $\mathsf{GL}(2, \mathbb{C})$ such that the Lie algebra of $G_{p,q}$ is $\mathfrak{h}_\alpha$. All such $G_{p,q}$ are difeomorphic to $\mathbb{S}^1$ and are all embedded submanifolds in $\mathbb{S}^1 \times \mathbb{S}^1$. However, if α is irrational, then

$$G_\alpha = \left\{ \begin{pmatrix} e^{it} & 0 \\ 0 & e^{i\alpha t} \end{pmatrix} : t \in \mathbb{R} \right\} \subseteq \mathsf{GL}(2, \mathbb{C})$$

is a (non-matrix) Lie group with Lie algebra $\mathfrak{h}_\alpha$. Note that G_α is an immersed manifold of $\mathbb{S}^1 \times \mathbb{S}^1$ such that $\overline{G_\alpha} = \mathbb{S}^1 \times \mathbb{S}^1$.

Definition 3.19. Let G be a Lie group. A **Lie subgroup**, H, is a subgroup of G that is also an immersed submanifold such that $m_{H\times H}$ and $i_H : H \to H$ are smooth maps. That is, $\iota : H \hookrightarrow G$ is a Lie group homomorphism that is a smooth immersion.

Note that we identify and define $\mathsf{T}_e H$ as a subspace of $\mathsf{T}_e G$. Hence, if H is a Lie subgroup of G, then the Lie algebra of H is a Lie subalgebra of the Lie algebra of G.

Remark 3.20. *According to the discussion above, a Lie subgroup of a compact Lie group could be a non-compact Lie subgroup!*

Proposition 3.21. *Let G be a Lie group with Lie algebra $\mathfrak{g}$. Let H be a Lie subgroup of G with Lie algebra $\mathfrak{h}$.*

1. *The one-parameter subgroups of H are precisely those one-parameter subgroups of G whose initial velocities lie in $\mathsf{T}_e H$.*
2. *The exponential map of H is the restriction to $\mathfrak{h}$ of the exponential map of G.*
3. *We have then*

$$\mathfrak{h} = \{X \in \mathfrak{g} \mid \exp_G(tX) \in H \textit{ for all } t \in \mathbb{R}\}.$$

Proof. The proof is given below:

1. Let $\gamma : \mathbb{R} \to H$ be a one-parameter subgroup. Then the composite map
$$\mathbb{R} \xrightarrow{\varphi} H \hookrightarrow G$$
is a one-parameter subgroup of G, which clearly satisfies $\gamma'(0) \in \mathsf{T}_e H$. Conversely, suppose $\gamma : \mathbb{R} \to G$ is a one-parameter subgroup whose initial velocity lies in $\mathsf{T}_e H$. Let $\widetilde{\gamma} : \mathbb{R} \to H$ be the one-parameter subgroup of H with the same initial velocity. By composing with the inclusion map, we can also consider $\widetilde{\gamma}$ as a one-parameter subgroup of G. Since γ and $\widetilde{\gamma}$ are both one-parameter subgroups of G with the same initial velocity, they are equal.
2. This follows from (1).

(3) If $X \in \mathfrak{h}$, then (1) implies that $\exp_G(tX) \in H$ for all $t \in \mathbb{R}$. Now assume that $\exp_G(tX) \in H$ for all $t \in \mathbb{R}$. It can be shown that $\exp_G(tX)$ is a smooth map into H [1, Theorem 19.25]. Hence, $\exp_G(tX)$ is a one-parameter subgroup of H. By (1), $\exp_G(tX)$ is a one-parameter subgroup of G such that $X \in \mathsf{T}_e H = \mathfrak{h}$.

This completes the proof. □

We can now compute the Lie algebras of some classical matrix Lie groups for $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$. Note that Proposition 3.21 and the fact that the Lie algebra of $\mathsf{GL}(n, \mathbb{K})$ is $M(n, \mathbb{K})$ implies that elements in the Lie algebra of a matrix Lie group are contained in $M(n, \mathbb{K})$. We first prove an important lemma.

Lemma 3.22. *Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$ and let $X \in M(n, \mathbb{K})$. We have*

$$\det e^X = e^{\operatorname{Tr} X}$$

Proof. Consider the Lie group homomorphism: $\det : \mathsf{GL}(n, \mathbb{K}) \to \mathbb{K}^\times$ We know that $d(\det)_{I_n}(X) = \operatorname{Tr}(X)$. Proposition 3.5(7) then implies that

$$\det e^X = e^{\operatorname{Tr} X}.$$

This completes the proof. □

We can now compute the Lie algebras of some matrix Lie groups:

Example 3.23. Let $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$.

(1) Assume that $\mathbb{K} \neq \mathbb{H}$. Let $G = \mathsf{SL}(n, \mathbb{K})$. If $X \in \mathfrak{sl}(n, \mathbb{K})$, then consider $\gamma(t) = e^{tX}$. Since $\gamma(t) \in \mathsf{SL}(n, \mathbb{K})$, we must have that

$$1 = \det \gamma(t) = \det e^{tX} = e^{t \operatorname{Tr} X}.$$

Hence, we have

$$0 = \frac{d}{dt} e^{t \operatorname{Tr} X} \bigg|_{t=0} = \operatorname{Tr} X \frac{d}{dt} e^{t \operatorname{Tr} X} \bigg|_{t=0} = \operatorname{Tr} X.$$

Hence, the Lie algebra $\mathfrak{sl}(n, \mathbb{K})$ of $\mathsf{SL}(n, \mathbb{K})$ is given by

$$\mathfrak{sl}(n, \mathbb{K}) = \{X \in M(n, k) \mid \operatorname{Tr} X = 0\}.$$

(2) Let $G = \mathsf{O}(n, \mathbb{R})$. Consider $\gamma(t) = e^{tX} \in \mathsf{O}(n, \mathbb{R})$. Differentiating the equation $\gamma(t)^T \gamma(t) = I_n$ at the identity $t = 0$, we have

$$0 = \frac{d}{dt}\left(\gamma(t)^T \gamma(t)\right) = \frac{d}{dt}\left((e^{tX})^T e^{tX}\right) \bigg|_{t=0} = \left(X^T e^{tX} + (e^{tX})^T X\right) \bigg|_{t=0} = X^T + X$$

Hence, the Lie algebra $\mathfrak{o}(n, \mathbb{R})$ of $\mathsf{O}(n, \mathbb{R})$ is given by

$$\mathfrak{o}(n, \mathbb{R}) = \{X \in M(n, \mathbb{R}) : X^T = -X\}.$$

(3) Using (3), we can obtain the Lie algebra $\mathfrak{su}(n, \mathbb{R})$ of $G = \mathsf{SO}(n, \mathbb{R})$. We observe that $\gamma(t)$ as in (3) additionally satisfies the constraint:

$$1 = \det \gamma(t) = \det e^{tX} = e^{t \operatorname{Tr} X} \qquad \text{for all } t \in \mathbb{R}.$$

Therefore, we have

$$\mathfrak{so}(n, \mathbb{R}) = \{X \in \mathfrak{o}(n, \mathbb{R}) : \operatorname{Tr} X = 0\} = \mathfrak{o}(n, \mathbb{R})$$

The last equality follows since each matrix in $\mathfrak{o}(n, \mathbb{R})$ already has zero trace.

(4) Let $G = \mathsf{U}(n,\mathbb{C})$. Consider $\gamma(t) = e^{tX} \in \mathsf{U}(n,\mathbb{C})$. Differentiating the equation $\gamma(t)^*\gamma(t) = I_n$ at the identity $t = 0$, we have

$$0 = \frac{d}{dt}\left(\gamma(t)^*\gamma(t)\right) = \frac{d}{dt}\left((e^{tX})^* e^{tX}\right)\bigg|_{t=0} = \left(X^* e^{tX} + (e^{tX})^* X\right)\bigg|_{t=0} = X^* + X$$

Hence, the Lie algebra $\mathfrak{u}(n,\mathbb{C})$ of $\mathsf{U}(n,\mathbb{C})$ is given by

$$\mathfrak{u}(n,\mathbb{C}) = \{X \in M(n,\mathbb{C}) : X^* = -X\}.$$

It is easy to check that $\mathfrak{u}(n,\mathbb{C})$ is not a complex Lie algebra. Hence, $\mathsf{U}(n,\mathbb{C})$ is not a complex Lie group.

(5) Using (5), we can obtain the Lie algebra $\mathfrak{su}(n,\mathbb{C})$ of $G = \mathsf{SU}(n,\mathbb{C})$. We observe that $\gamma(t)$ as in (5) additionally satisfies the constraint:

$$1 = \det \gamma(t) = \det e^{tX} = e^{t\,\mathrm{Tr} X} \qquad \text{for all } t \in \mathbb{R}.$$

Therefore, we have

$$\mathfrak{su}(n,\mathbb{C}) = \{X \in \mathfrak{u}(n,\mathbb{C}) : \mathrm{Tr} X = 0\}.$$

It is easy to check that $\mathfrak{su}(n,\mathbb{C})$ is not a complex Lie algebra. Hence, $\mathsf{SU}(n,\mathbb{C})$ is not a complex Lie group.

(6) Let $G = \mathsf{U}(n,\mathbb{H})$. A similar argument as in (5) shows that the Lie algebra $\mathfrak{u}(n,\mathbb{H})$ of $\mathsf{U}(n,\mathbb{H})$ is given by

$$\mathfrak{u}(n,\mathbb{H}) = \{X \in M(n,\mathbb{H}) : X^H = -X\}.$$

(7) Let $G = \mathsf{Sp}(n,\mathbb{R})$. Consider $\gamma(t) = e^{tX} \in \mathsf{Sp}(n,\mathbb{R})$. Differentiating the equation $\gamma(t)^T J \gamma(t) = J$ at the identity $t = 0$, we have

$$0 = \frac{d}{dt}\left(\gamma(t)^T J\gamma(t)\right) = \frac{d}{dt}\left((e^{tX})^T J e^{tX}\right)\bigg|_{t=0} = \left(X^T e^{tX} J + (e^{tX})^T JX\right)\bigg|_{t=0} = X^T J + JX$$

Hence, the Lie algebra $\mathfrak{sp}(n,\mathbb{R})$ of $\mathsf{Sp}(n,\mathbb{R})$ is given by

$$\mathfrak{sp}(n,\mathbb{R}) = \{X \in M(2n,\mathbb{R}) : X^T J = -JX\}.$$

A simple counting argument shows that $\dim \mathfrak{sp}(n,\mathbb{R}) = n(2n+1)$.

(8) Let $G = \mathsf{Sp}(n,\mathbb{C})$. An argument as in (8) shows that the Lie algebra $\mathfrak{sp}(n,\mathbb{C})$ of $\mathsf{Sp}(n,\mathbb{C})$ is given by

$$\mathfrak{sp}(n,\mathbb{C}) = \{X \in M(2n,\mathbb{C}) : X^T J = -JX\}.$$

A simple counting argument shows that $\dim \mathfrak{sp}(n,\mathbb{C}) = 2n(2n+1)$.

(9) Let $G = \mathsf{O}(p,q)$. An argument as in (8) shows that the Lie algebra $\mathfrak{o}(p,q)$ of $\mathsf{O}(p,q)$ is given by

$$\mathfrak{o}(p,q) = \{X \in M(n,\mathbb{C}) : X^T g_{p,q} = -g_{p,q} X\}.$$

A simple counting argument shows that $\dim \mathfrak{o}(p,q) = n(n-1)/2$.

3.3. **Cartan's Theorem.** Note that all matrix Lie groups discussed thus far are embedded submanifolds of $\mathsf{GL}(n, \mathbb{K})$ which are closed. Cartan's theorem states the converse is true in general: any closed subgroup of a Lie group is a Lie subgroup that is also an embedded submanifold. Before proving Cartan's theorem, we prove a key lemma first. The result effectively states that group multiplication in G is reflected to first order in the vector space structure of its Lie algebra.

Lemma 3.24. *Let G be a Lie group and let $\mathfrak{g}$ be its Lie algebra and let $X, Y \in g$.*

(1) *There is a smooth function $Z : (-\delta, \delta) \to \mathfrak{g}$ for some $\varepsilon > 0$ such that*
$$\exp(tX)\exp(tY) = \exp\left(t(X+Y) + t^2 Z(t)\right)$$
for all $t \in (-\varepsilon, \varepsilon)$.

(2) *(**Lie-Trotter Product Formula**) We have*
$$\lim_{n\to\infty}\left(\exp\left(\frac{t}{n}X\right)\exp\left(\frac{t}{n}Y\right)\right)^n = \exp(t(X+Y))$$

Proof. The proof is given below:

(1) Let $0 \in U \subseteq \mathfrak{g}$ be neighbourhood such that that $\exp|_U : U \to \exp(U)$ is a diffeomorphism. If $X, Y \in \mathfrak{g}$, we can find an ε sufficiently small so that $\exp(tX)\exp(tY) \in U$ for all $|t| < \varepsilon$. Define $f : (-\varepsilon, \varepsilon) \to \mathfrak{g}$ by $f(t) = \exp^{-1}(\exp(tX)\exp(tY))$. The map f is smooth as it is the composition of
$$(-\varepsilon, \varepsilon) \xrightarrow{\exp_X \times \exp_Y} \exp(U) \times \exp(U) \xrightarrow{m} \exp(U) \xrightarrow{\exp^{-1}} U$$
where $\exp_X(t) = \exp(tX)$ and $\exp_Y(t) = \exp(tY)$ Taking the differential at zero yields
$$f'(0) = (d\exp)_0^{-1}\left(d(\exp_X)_0(\partial_t|_{t=0}) + d(\exp_X)_0\partial_t|_{t=0})\right) = X + Y.$$
Therefore, Taylor's theorem yields
$$f(t) = f(0) + tf'(0) + t^2 Z(t) = 0 + t(X+Y) + t^2 Z(t)$$
for some smooth function Z.

(2) For any $t \in \mathbb{R}$ and any sufficiently large $n \in \mathbb{Z}$, (1) implies that
$$\exp\left(\frac{t}{n}X\right)\exp\left(\frac{t}{n}Y\right) = \exp\left(\frac{t}{n}(X+Y) + \frac{t^2}{n^2}Z\left(\frac{t}{n}\right)\right)$$
Using properties of the exponential map, we have
$$\begin{aligned}\lim_{n\to\infty}\left(\exp\left(\frac{t}{n}X\right)\exp\left(\frac{t}{n}Y\right)\right)^n &= \lim_{n\to\infty}\exp\left(\frac{t}{n}(X+Y) + \frac{t^2}{n^2}Z\left(\frac{t}{n}\right)\right)^n \\ &= \lim_{n\to\infty}\exp\left(t(X+Y) + \frac{t^2}{n}Z\left(\frac{t}{n}\right)\right) \\ &= \exp\left(t(X+Y) + \lim_{n\to\infty}\frac{t^2}{n}Z\left(\frac{t}{n}\right)\right) \\ &= \exp\left(t(X+Y)\right)\end{aligned}$$

This completes the proof. ∎

We will discuss results similar to that of Lemma 3.24 later on.

Proposition 3.25. ***(Cartan's Closed Subgroup Theorem)*** *Let G be a Lie group and let H be a closed subgroup. Then H is a Lie subgroup of G that is an embedded submanifold of G.*

Proof. The proof has been commented out from the note for brevity. ☐

Example 3.26. Consider the group:

$$H = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \in M(3, \mathbb{R}) \;\middle|\; x, y, z \in \mathbb{R} \right\}$$

It is easily checked that H is a closed subgroup under matrix multiplication of $\mathsf{GL}(3, \mathbb{R})$. Hence, Proposition 3.25 implies that H is a Lie subgroup of $\mathsf{GL}(3, \mathbb{R})$. It is a non-abelian Lie group group called the Heisenberg group. We denote the Lie algebra of H as $\mathfrak{heis}$. We have

$$\begin{aligned} \mathfrak{heis} &= \{X \in M(3, \mathbb{R}) : e^{tX} \in H \text{ for all } t \in \mathbb{R}\} \\ &= \{X \in M(3, \mathbb{R}) : X \text{ is strictly upper triangular}\} \end{aligned}$$

This is because if X is strictly upper triangular, X^m will be strictly upper triangular for $m \in \mathbb{N}$. Thus, for any such X we will have $e^{tX} \in H$. Conversely, if $e^{tX} \in H$ for all real t, then all of the entries of e^{tX} on or below the diagonal are independent of t. Thus, X will be strictly upper triangular. Clearly, $\dim \mathfrak{heis} = 3$ as expected.

4. Baker-Campbell-Hausdorff Formula

The starting point for the Baker-Campbell-Hausdorff (BCH) formula can be considered to be the formula in Lemma 3.24, which loosely states that

$$\exp(tX)\exp(tY) = \exp\left(t(X+Y) + \text{higher-order terms}\right).$$

The BCH formula provides the solution for the nature of these higher-order terms. One of the main applications of the BCH formula is to prove Lie's Third Theorem, which will be covered in the next section. To simplify the expression of nested commutators appearing in the BCH formula, we introduce the adjoint operator.

Definition 4.1. For an element X in a Lie algebra $\mathfrak{g}$, define the map

$$\begin{aligned} \mathrm{ad}_X : \mathfrak{g} &\to \mathfrak{g}, \\ Y &\mapsto [X, Y]. \end{aligned}$$

That is, ad_X is the linear map given by taking the Lie bracket with X. This notation is particularly useful when writing series expansions involving iterated commutators, such as those that appear in the BCH formula. We have the following result.

Lemma 4.2. *Let G be a Lie group with Lie algebra $\mathfrak{g}$. The conjugation map $C_g : G \to G$ defined by $C_g(h) = ghg^{-1}$ is such that*

$$\mathrm{ad}_X = \left.\frac{d}{dt}\right|_{t=0} \mathrm{Ad}_{\exp(tX)}.$$

where $dC_g := \mathrm{Ad}_g$ and $X \in \mathfrak{g}$.

Proof. Let $X \in \mathfrak{g}$. By Proposition 3.12, we have

$$[X, Y]_e = \frac{d}{dt}\bigg|_{t=0} \left(dC_{\exp(tX)}(Y)\right) := \frac{d}{dt}\bigg|_{t=0} \left(\mathrm{Ad}_{\exp(tX)}(Y)\right)$$

for each $Y \in \mathfrak{g}$. The claim follows by invoking the definition of ad_X. $\square$

4.1. **Motivation.** Before we state and prove the BCH formula, we first examine the structure of the higher-order terms by analyzing the expression to first order in t. This preliminary discussion provides insight into the nature of the corrections beyond the linear approximation and offers a glimpse into the structure of the full BCH formula. This will be a consequence of Taylor's expansion formula.

Proposition 4.3. *Let G be a Lie group and let $f \in C^\infty(G)$. For $X_1, X_2 \in \mathfrak{g}$, we have*

$$f(\exp(tX_1)\exp(tX_2)) = f(e) + t(X_1 + X_2)f(e) + \frac{t^2}{2}\left(X_1^2 + X_2^2 + 2X_1X_2\right) f(e) + O(t^3).$$

Proof. For each $g \in G$, we have:

$$(Xf)(g) = (X_g f) = (dL_g(X_e))f = X_e(f \circ L_g) = \frac{d}{dt}\bigg|_{t=0} f(g\exp(tX)).$$

More generally, for any $t \in \mathbb{R}$, we compute:

$$\begin{aligned}(Xf)(g\exp(tX)) &= \frac{d}{ds}\bigg|_{s=0} f(g\exp(tX)\exp(sX)) \\ &= \frac{d}{ds}\bigg|_{s=0} f(g\exp((t+s)X)) = \frac{d}{dt} f(g\exp(tX)).\end{aligned}$$

Using this identity and induction, one obtains that for any $k \geq 0$,

$$X^k f(g\exp(tX)) = \frac{d^k}{dt^k} f(g\exp(tX)).$$

In particular,

$$X^k f(g) = \frac{d^k}{dt^k}\bigg|_{t=0} f(g\exp(tX)).$$

The formulae above can be generalized to the bivariate case:

$$\begin{aligned}(X_1X_2f)(g) &= \frac{d}{dt_1}\bigg|_{t_1=0} (X_2f)(g\exp(t_1X_1)) \\ &= \frac{d}{dt_1}\bigg|_{t_1=0} \frac{d}{dt_2}\bigg|_{t_2=0} f(g\exp(t_1X_1)\exp(t_2X_2)).\end{aligned}$$

More generally,

$$(X_{\alpha(1)} \cdots X_{\alpha(k)} f)(g) = \frac{\partial^k}{\partial t_1 \cdots \partial t_k}\bigg|_{t_1=\cdots=t_k=0} f\left(g\exp\left(t_1X_{\alpha(1)}\right) \cdots \exp\left(t_kX_{\alpha(k)}\right)\right).$$

Here each $X_{\alpha(i)}$ is either X_1 or X_2. The claim follows. $\square$

We can now use the computation above to derive the BCH formula up to first order.

Corollary 4.4. *Let G be a Lie group. For $X_1, X_2 \in \mathfrak{g}$ and $|t|$ sufficiently small, we have*

$$\exp(tX_1)\exp(tX_2) = \exp\left(t(X_1 + X_2) + \frac{t^2}{2}[X_1, X_2] + O(t^3)\right).$$

Proof. We apply Proposition 4.3 to the inverse of the exponential map near the identity element e, that is, the map f given by $f(\exp(tX)) = tX$ for t sufficiently small. We have $f(e) = 0$. For any $X \in \mathfrak{g}$, we have

$$(Xf)(e) = \frac{d}{dt}\bigg|_{t=0} f(\exp(tX)) = \frac{d}{dt}\bigg|_{t=0} (tX) = X,$$
$$(X^n f)(e) = \frac{d^n}{dt^n}\bigg|_{t=0} f(\exp(tX)) = \frac{d^n}{dt^n}\bigg|_{t=0} (tX) = 0 \quad \text{for } n \geq 2.$$

The claim follows upon making the observation that

$$X_1^2 + X_2^2 + 2X_1X_2 = (X_1 + X_2)^2 + [X_1, X_2],$$
$$\exp(tX_1)\exp(tX_2) = \exp\left(f\left(\exp(tX_1)\exp(tX_2)\right)\right).$$

This completes the proof. □

Hence, we see that beyond the sum $X_1 + X_2$, the BCH formula includes terms involving the Lie bracket $[X_1, X_2]$, reflecting the non-commutativity of the underlying Lie algebra structure.

4.2. **General BCH Formula.** We now derive the general BCH formula. For simplicity, we denote by log the inverse of exp near $0 \in \mathfrak{g}$.The trick is to consider the formula:

$$Z(t) = \log(\exp(X)\exp(tY)).$$

Clearly, $Z(0) = X$ and $Z(1)$ is the element satisfying

$$e^{Z(1)} = \exp(X)\exp(Y).$$

The idea is to find an analytic expression for the derivative $Z'(t)$, and then integrate this derivative from 0 to 1 to compute $Z(1)$. We will make use of the fact that that if $\gamma_1(t)$, $\gamma_2(t)$ are smooth curves in a Lie group, G, and $\gamma(t) = \gamma_1(t)\gamma_2(t)$. Then

$$\dot{\gamma}(t) = dL_{\gamma_1(t)}\left(\dot{\gamma}_2(t)\right) + dR_{\gamma_2(t)}\left(\dot{\gamma}_1(t)\right).$$

Notice the fact that $\gamma(t) = m(\gamma_1(t), \gamma_2(t))$, where m is the multiplication operation on G. The formula above follows from the differential of the multiplication map at an arbitrary point (a, b):

$$dm_{a,b}(X_a, Y_b) = (dL_a)_b(Y_a) + (dR_b)_a(X_a).$$

More generally, by using induction one can easily see that if $\gamma(t) = \gamma_1(t)\cdots\gamma_m(t)$ where each $\gamma_i(t)$ is a smooth curve, then

$$\dot{\gamma}(t) = \sum_{k=1}^{m} \left(dL_{\gamma_1(t)}\cdots dL_{\gamma_{k-1}(t)}\, dR_{\gamma_{k+1}(t)}\cdots dR_{\gamma_m(t)}\right)(\dot{\gamma}_k(t)).$$

We can now use these formulas to compute the derivative of the exponential map.

Lemma 4.5. *Let G be a Lie group. For all $X, Y \in \mathfrak{g}$, we have*

$$\Delta(X, Y) := \frac{d}{dt}e^{X+tY}\bigg|_{t=0} = (dL_{\exp X})_e \circ \phi(\mathrm{ad}_X)(Y),$$

where ϕ is the function

$$\phi(z) = \frac{1 - e^{-z}}{z} = \sum_{m=0}^{\infty} \frac{(-1)^m}{(m+1)!} z^m.$$

More generally, if $Z(t)$ is a smooth Lie-algebra valued function, then

$$\frac{d}{dt}e^{Z(t)} = \left(dL_{\exp(Z(t))}\right)_e \phi(\mathrm{ad}_{Z(t)})\left(\frac{dZ}{dt}\right),$$

Proof. For each $m \in \mathbb{N}$, we have

$$\begin{aligned}
\Delta(X,Y) &= \frac{d}{dt}\bigg|_{t=0} (\exp(X/m + tY/m))^m \\
&= \sum_{k=0}^{m-1} (dL_{\exp X/m})^{m-k-1}(dR_{\exp X/m})^k \, \Delta(X/m, Y/m) \\
&= \frac{1}{m}(dL_{\exp X})^{m-1} \sum_{k=0}^{m-1} (dL_{\exp X/m})^{-k}(dR_{\exp X/m})^k \, \Delta(X/m, Y).
\end{aligned}$$

Here we have used that $\Delta(X,Y)$ is $\mathbb{R}$-linear in Y. Since the differential of the conjugation map $C_g = L_g R_{g^{-1}}$ is the map Ad_g, we have

$$\begin{aligned}
(dL_{\exp X/m})^{-k}(dR_{\exp X/m})^k &= \left(dC_{\exp(-X/m)}\right)^k \\
&= \left(\mathrm{Ad}_{\exp(-X/m)}\right)^k \\
&= \exp\left(-\left(\frac{\mathrm{ad}_X}{m}\right)\right)^k.
\end{aligned}$$

Hence, we have

$$\Delta(X,Y) = (dL_{\exp X/m})^{m-1}\frac{1}{m}\sum_{k=0}^{m-1} \left(\exp(-\mathrm{ad}_{X/m})\right)^k \Delta(X/m, Y)$$

As $m \to \infty$, we have

$$\begin{aligned}
(dL_{\exp X/m})^{m-1} &\to dL_{\exp((m-1)X/m)} \to dL_{\exp X}, \\
\Delta(X/m, Y) &\to \Delta(0, Y) = Y.
\end{aligned}$$

Moreover, since $\mathrm{ad}_X \in \mathrm{End}(\mathfrak{g})$ is a linear operator (i.e., a matrix), we have the following:

$$\begin{aligned}
\frac{1}{m}\sum_{k=0}^{m-1} \exp\left(-\frac{\mathrm{ad}_X}{m}\right)^k &= \frac{1}{m}\sum_{k=0}^{m-1}\sum_{n=0}^{\infty}\frac{1}{n!}\left(-\frac{k\,\mathrm{ad}_X}{m}\right)^n \\
&= \sum_{n=0}^{\infty}\left[\frac{1}{m}\sum_{k=0}^{m-1}\left(\frac{k}{m}\right)^n\right]\frac{(-1)^n}{n!}(\mathrm{ad}_X)^n \\
&\to \sum_{n=0}^{\infty}\left[\int_0^1 x^n dx\right]\frac{(-1)^n}{n!}(\mathrm{ad}_X)^n \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)!}(\mathrm{ad}_X)^n = \phi(\mathrm{ad}_X)
\end{aligned}$$

This proves the formula for $\Delta(X,Y)$. The formula for the derivative of $\exp(Z(t))$ follows from the chain rule. $\square$

We are now in a position to prove the Baker–Campbell–Hausdorff (BCH) formula.

Proposition 4.6. *Let G be a Lie group. For $X, Y \in \mathfrak{g}$ such that $\|X\|, \|Y\|$ and t are sufficiently small (here $\|\cdot\|$ is any norm on $\mathfrak{g}$), we have*

$$\log\left(e^X e^Y\right) = X + \int_0^1 g(e^{\operatorname{ad}_X} e^{t \operatorname{ad}_Y})(Y)\, dt,$$

Proof. Consider the expression:

$$Z(t) = \log(\exp(X)\exp(tY)) \iff \exp(Z(t)) = \exp(X)\exp(tY)$$

We have

$$\begin{aligned} \frac{d}{dt}\exp(Z(t)) &= dL_{\exp X}\frac{d}{dt}(\exp tY) \\ &= dL_{\exp X}\, dL_{\exp tY}\, \phi(\operatorname{ad}_{tY})(Y) \\ &= dL_{\exp Z(t)}(Y) \end{aligned}$$

On the other hand, Lemma 4.5 implies that

$$\frac{d}{dt}\exp(Z(t)) = dL_{\exp Z(t)}\, \phi(\operatorname{ad}_{Z(t)})\frac{dZ}{dt}$$

Since $\|X\|, \|Y\|$ are sufficiently small, then $\|Z(t)\|$ will also be small, so that the operator $I - e^{-\operatorname{ad}_{Z(t)}}$ is close to zero, making

$$\frac{I - e^{-\operatorname{ad}_{Z(t)}}}{\operatorname{ad}_{Z(t)}}$$

close to the identity and therefore invertible. Hence,

$$\frac{dZ}{dt} = \phi(\operatorname{ad}_{Z(t)})^{-1}(Y) := \left(\frac{I - e^{-\operatorname{ad}_{Z(t)}}}{\operatorname{ad}_{Z(t)}}\right)^{-1}(Y)$$

Since $Z(t) = \exp(t)\exp(tY)$, we have the following relationships:

$$\begin{aligned} \operatorname{Ad}_{Z(t)} &= \operatorname{Ad}_X \operatorname{Ad}_{e^{tY}} \\ e^{\operatorname{ad}_{Z(t)}} &= e^{\operatorname{ad}_X} e^{t \operatorname{ad}_Y} \\ \operatorname{ad}_{Z(t)} &= \log\left(e^{\operatorname{ad}_X} e^{t\operatorname{ad}_Y}\right) \end{aligned}$$

Therefore, we have

$$\frac{dZ}{dt} = \left(I - \frac{(e^{\operatorname{ad}_X} e^{t\operatorname{ad}_Y})^{-1}}{\log\left(e^{\operatorname{ad}_X} e^{t\operatorname{ad}_Y}\right)}\right)^{-1}(Y) := g(e^{\operatorname{ad}_X} e^{\operatorname{ad}_{tY}})(Y),$$

where $g(\cdot)$ is the function $g(z) = (1 - z^{-1}/\log z)^{-1}$. Noting that $Z(0) = X$ gives

$$\log\left(e^X e^Y\right) = Z(1) = X + \int_0^1 g(e^{\operatorname{ad}_X} e^{t\operatorname{ad}_Y})(Y)\, dt$$

This completes the proof. □

One can expand the integrand as a power series in terms of iterated commutators; however, we will not perform that expansion here.

5. Lie Group-Lie Algebra Correspondence

Part 2. Representations of Compact Lie Groups

Representation theory is the study of group actions on vector spaces. It lies at the intersection of group theory and linear algebra and serves as a foundational tool in many areas of mathematics, including number theory, algebraic geometry, and harmonic analysis. It also has profound applications outside of mathematics, such as in quantum mechanics (e.g., the simple harmonic oscillator) and chemistry (e.g., the analysis of atomic spectra like that of hydrogen). We begin with the basic theory of group representations, then specializes to the case of compact Lie groups. We explore examples of finite group and compact Lie group representations and gradually build toward the structural theory, including fundamental results such as the Peter–Weyl Theorem.

Remark 5.1. *As before, all Lie groups are assumed to be finite-dimensional.*

6. General Theory

Studying Lie groups (and Lie algebras) through their representations provides a powerful way to understand their structure. Representations allow us to translate abstract group elements into linear transformations, facilitating easier computation and insight into the group's properties.

6.1. Definitions & Examples.

We discuss basic definitions and examples in this section. Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$.

Definition 6.1. Let G be a Lie group. A **Lie group representation** is a $\mathbb{K}$-vector space V with a Lie group homomorphism $\Pi : G \to \mathsf{GL}(V)$.

Remark 6.2. *If V is finite-dimensional, it is customary to write $\mathsf{GL}(n, \mathbb{K})$ instead of $\mathsf{GL}(V)$. In this case, we say that the representation is a linear representation. The choice of the base field is usually left implicit in the discussion. It is also customary to refer to either the vector space V or the homomorphism Π as a representation. However, note that it is the pair (Π, V) that constitutes the representation.*

Note that for each $g \in G$, we have

$$\Pi(g^{-1}) \circ \Pi(g) = \Pi(g) \circ \Pi(g^{-1}) = \Pi(e) = \mathrm{Id}_V$$

Hence, each $\Pi(g)$ is a linear isomorphism of V.

Definition 6.3. Let G be a Lie group and let (Π_1, V_1) and (Π_2, V_2) be two representations over $\mathbb{K}$. A morphism between the representations is a linear map $T : V_1 \to V_2$ such that

$$\Pi_2(g) \circ T = T \circ \Pi_1(g) \quad \text{for all } g \in G.$$

We call T an intertwining map.

$$\begin{tikzcd} V_1 \arrow[r, "T"] \arrow[d, "\Pi_1(g)"'] & V_2 \arrow[d, "\Pi_2(g)"] \\ V_1 \arrow[r, "T"] & V_2 \end{tikzcd}$$

We now observe that group representations naturally organize into a mathematical structure called a category. This viewpoint allows us to use the language and tools of category theory to study representations more systematically.

Proposition 6.4. *Let $\mathbb{K}$ be a field and G be a group. The class of representations of G over $\mathbb{K}$, written as $\mathsf{Rep}_{\mathbb{K}}(G)$, forms a category.*

Proof. Objects and morphisms $\mathsf{Rep}_{\mathbb{K}}(G)$ have been defined above. Let $\mathrm{Hom}_G(V_1, V_2)$ denote the set of all such morphisms between (V_1, Π_1) and (V_2, Π_2). The identity morphism in $\mathrm{Hom}_G(V, V)$ is simply the identity morphism of the underlying vector spaces. If $T \in \mathrm{Hom}_G(V_1, V_2)$ and $S \in \mathrm{Hom}_G(V_2, V_3)$, then $S \circ T \in \mathrm{Hom}_G(V_1, V_3)$ is defined such that the following diagram commutes

$$\begin{tikzcd}
V_1 \arrow[r, "T"] \arrow[rr, bend left, "S\circ T"] \arrow[d, "\Pi_1(g)"'] & V_2 \arrow[r, "S"] \arrow[d, "\Pi_2(g)"] & V_3 \arrow[d, "\Pi_3(g)"] \\
V_1 \arrow[r, "T"] \arrow[rr, bend right, "S\circ T"'] & V_2 \arrow[r, "S"] & V_3
\end{tikzcd}$$

for all $g \in G$. The identity and associativity axioms are easy to verify. $\square$

We have the notion of an isomorphism, which is a morphism with an inverse, in $\mathsf{Rep}_{\mathbb{K}}(G)$. Isomorphic representations are also called equivalent representations. Let's present a few basic examples:

Example 6.5. Let G be a Lie group with Lie algebra $\mathfrak{g}$. The following is a basic list of examples of representations of Lie groups and Lie algebras:

(1) Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$. The trivial representation of G is
$$\begin{aligned} \Pi : G &\to \mathrm{GL}(V) \\ g &\mapsto \mathrm{Id}_V \end{aligned}$$

(2) Let $\mathbb{K} = \mathbb{C}$. If $G \subseteq \mathsf{GL}(n, \mathbb{C})$ is a matrix Lie group, the standard representation of G is
$$\begin{aligned} \Pi : G &\to \mathrm{GL}(n, \mathbb{C}) \\ g &\mapsto g \end{aligned}$$

(3) Let $\mathbb{K} = \mathbb{R}$. The adjoint representation of G is
$$\begin{aligned} \mathrm{Ad} : G &\to \mathrm{GL}(\mathfrak{g}) \\ g &\mapsto \mathrm{Ad}_g \end{aligned}$$
Here Ad_g is the differential of the conjugation map, C_g.

(4) Let $\mathbb{K} = \mathbb{R}$. Assume G acts on a smooth manifold M, and let $F(M)$ be the space of complex-valued functions on M. The action induces a representation of G on $F(M)$ by
$$\begin{aligned} \Pi : G &\to \mathsf{GL}(F(M)) \\ (\Pi(g) f)(x) &= f(g^{-1}x) \end{aligned}$$

The Lie algebra representations are defined by the same way:

Definition 6.6. Let $\mathfrak{g}$ be a Lie algebra. A **Lie algebra representation** is a $\mathbb{K}$-vector space V with a Lie algebra homomorphism $\pi : G \to \mathfrak{gl}(V)$

We have the following important proposition:

Proposition 6.7. *Let G be a Lie group with Lie algebra $\mathfrak{g}$. Let $\Pi : G \to V$ be a (finite-dimensional) representation of G over $\mathbb{K}$.*

(1) *Then there is a unique representation π of $\mathfrak{g}$ acting on the same space such that*

$$\Pi(e^X) = e^{\pi(X)}$$

The representation Π can be computed as

$$\pi(X) = \left.\frac{d}{dt}\right|_{t=0} \Pi\left(e^{tX}\right).$$

(2) *If G is connected and simply connected, then any representation of $\mathfrak{g}$ can be uniquely lifted to a representation of G.*

Proof. The proof is given below:

(1) Consider the diagram:

$$\begin{tikzcd} G \arrow[r, "\Pi"] & \mathsf{GL}(V) \\ \mathfrak{g} \arrow[u, "\exp"] \arrow[r, "\pi"] & \mathfrak{gl}(V) \arrow[u, "\exp"'] \end{tikzcd}$$

Commutativity of the diagram shows that we have

$$e^{t\pi(X)} = \Pi(e^{tX})$$

for $t \in \mathbb{R}$ and $X \in \mathfrak{g}$. Hence, we have,

$$\pi(X) = \left.\frac{d}{dt}\right|_{t=0} e^{t\pi(X)} = \left.\frac{d}{dt}\right|_{t=0} \Pi\left(e^{tX}\right)$$

(2) Simply apply the Lie group-Lie algebra correspondence.

This completes the proof. □

Remark 6.8. *In the discussion that follows, the base field is usually left implicit. If not otherwise specified, we assume $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. When we specialize to either $\mathbb{R}$ or $\mathbb{C}$, this will be explicitly indicated.*

6.2. **Operations on Representations.** We can construct a broader class of examples of representations by extending the standard linear algebraic constructions. We discuss some constructions in this section. Let G be a Lie group with Lie algebra $\mathfrak{g}$.

6.2.1. *Subrepresentations and Quotient representations.* Let (V, Π) be a finite-dimensional representation of G. A subrepresentation of V is a G-invariant linear subspace $W \subseteq V$ together with the restriction of Π to $W, \Pi|_W$. Suppose W is a subrepresentation of V. Then since W is a linear subspace of V, one can form the quotient space V/W. It follows from the G-invariance of W that the G-action on V descends to a G-action on V/W by

$$\Pi(g) \cdot (v + W) = \Pi(g) \cdot v + W.$$

This gives a representation of G on the quotient V/W, and is called the quotient representation of V under W.

Remark 6.9. *The quotient space construction also works for possibly infinite-dimensional representations. In this case, one usually works with V a Banach or Hilbert space, and the subspace W is additionally assumed to be closed.*

Remark 6.10. *It is easy to check that if G is a connected Lie group, and V a representation of G, then $W \subseteq V$ is a subrepresentation of V if and only if it is a subrepresentation of $\mathfrak{g}$.*

6.2.2. *Direct Sums.* Let $\Pi_1 : G \to \mathrm{GL}(V_1)$ and $\Pi_2 : G \to \mathrm{GL}(V_2)$ be two finite-dimensional representations. The direct sum representation is defined as

$$\begin{aligned} \Pi_1 \oplus \Pi_2 : G &\to \mathsf{GL}(V_1 \oplus V_2) \\ g &\mapsto \Pi_1(g) \oplus \Pi_2(g) \end{aligned}$$

Remark 6.11. *The direct sum construction also works for possibly infinite-dimensional representations. In this case, one usually works with V as a Hilbert space, and the direct sum is then considered as a Hilbert space.*

6.2.3. *Tensor Products.* Let $\Pi_1 : G \to \mathsf{GL}(V_1)$ and $\Pi_2 : G \to \mathsf{GL}(V_2)$ be two finite-dimensional representations. The tensor product representation

$$\begin{aligned} \Pi_1 \otimes \Pi_2 : G &\to \mathsf{GL}(V_1 \otimes V_2) \\ g &\mapsto \Pi_1(g) \otimes \Pi_2(g) \end{aligned}$$

What Lie algebra representation does $\Pi_1 \otimes \Pi_2$ induce? If $X \in \mathfrak{g}$ and $v_1 \in V_1$ and $v_2 \in V_2$, note that we have

$$X \cdot (v \otimes w) = \left.\frac{d}{dt}\right|_{t=0} \left(e^{tX} v \otimes e^{tX} w\right) = (X \cdot v) \otimes w + v \otimes (X \cdot w).$$

Therefore, we can make the following construction. Let $\pi_1 : \mathfrak{g} \to \mathfrak{gl}(V_1)$ and $\pi_2 : \mathfrak{g} \to \mathfrak{gl}(V_2)$ be two Lie algebra. The tensor product representation of Lie algebras is defined

$$\begin{aligned} \pi_1 \otimes \pi_2 : \mathfrak{g} &\to \mathfrak{gl}(V_1 \otimes V_2) \\ X &\mapsto \pi_1(X) \otimes \mathrm{Id}_{V_2} + \mathrm{Id}_{V_1} \otimes \pi_2(X) \end{aligned}$$

Remark 6.12. *The tensor product construction also works for possibly infinite-dimensional representations. In this case, one usually works with V as a Hilbert space, and the tensor product is then considered as a Hilbert space.*

6.3. **Irreducible Representations.**

Definition 6.13. Let G be a Lie group and let $\Pi : G \to \mathsf{GL}(V)$ be a representation. Then Π is **irreducible** if it has no non-trivial G-invariant subspace. That is, there exists no subspace $0 \subsetneq W \subsetneq V$ such that $\Pi(g)W \subsetneq W$ for each $g \in G$.

Remark 6.14. *A representation which is not irreducible is called reducible.*

Example 6.15. The following is a list of irreducible representations:

1. Any 1-dimensional representation of a group is irreducible since there is no non-trivial proper subspace of a 1-dimensional vector space.
2. If $G = \{1\}$, the trivial group, then the only irreducible representation is a 1-dimensional representation. Indeed, any representation $G \to \mathsf{GL}(V)$ just maps 1 to Id_V. Since every subspace of V is G-invariant, the representation is irreducible if and only if $\dim V = 1$.
3. Let $\mathbb{K} = \mathbb{C}$. Consider the standard representation of $\mathsf{GL}(n, \mathbb{C})$. The fact that $\mathsf{GL}(n, \mathbb{C})$ acts transitively on $\mathbb{C}^n \setminus \{0\}$ implies that the standard representation of $\mathsf{GL}(n, \mathbb{C})$ is irreducible.

(4) Let $\mathbb{K} = \mathbb{C}$. Consider the standard representation of $\mathsf{SO}(n, \mathbb{R})$. The fact that $\mathsf{SO}(n, \mathbb{R})$ acts transitively on $\mathbb{S}^n$ is implies that the standard representation $\mathsf{SO}(n, \mathbb{R})$ is irreducible.
(5) Let $\mathbb{K} = \mathbb{C}$. Consider the standard representation of $\mathsf{SU}(n, \mathbb{C})$. An argument similar to (2) shows that the standard representation of $\mathsf{SU}(n, \mathbb{R})$ is irreducible.

Proposition 6.16. *Let G be a connected Lie group with (finite-dimensional) representation $\Pi : G \to \mathsf{GL}(V)$ and let $\pi : \mathfrak{g} \to \mathfrak{gl}(V)$ denote the induced representation. Π is irreducible if and only if π is irreducible.*

Proof. Suppose first that Π is irreducible. Let $W \subseteq V$ be an invariant subspace for π. Since G is connected, Proposition 3.7 states that any $g \in G$ can be written as

$$g = e^{X_1} \cdots e^{X_m}$$

for some $X_1, \dots, X_m \in \mathfrak{g}$. Since W is invariant under $\pi(X_j)$, it will also be invariant under

$$\exp(\pi(X_j)) = I + \pi(X_j) + \frac{\pi(X_j)^2}{2} + \cdots ,$$

and hence under

$$\Pi(g) = \Pi(e^{X_1} \cdots e^{X_m}) = \Pi(e^{X_1}) \cdots \Phi(e^{X_m}) = e^{\pi(X_1)} \cdots e^{\pi(X_m)}.$$

Since Π is irreducible and W is invariant under each $\Pi(g)$, W must be either $\{0\}$ or V. This shows that π is irreducible. Now suppose that π is irreducible. Let $W \subseteq V$ be an invariant subspace for Π. Then W is invariant under $\Phi(\exp(tX))$ for all $X \in \mathfrak{g}$ and, hence, under

$$\pi(X) = \frac{d}{dt}\Phi(\exp(tX))\bigg|_{t=0}.$$

Thus, since πi is irreducible, W is either $\{0\}$ or V, and we conclude that Π is irreducible. $\square$

Remark 6.17. *A similar argument shows that if (Π_1, V_1) and (Π_2, V_2) are two (finite-dimensional) Lie group representations, then Π_1 and Π_2 are isomorphic if and only if π_1 and π_2 are isomorphic, where π_1 and π_2 are the induced representations.*

The next proposition is an extremely useful tool in the theory of representations

Proposition 6.18. ***(Schur's Lemma).*** *Let G be (a compact Lie) group and let (Π_1, V_1) and Π, V_2 be finite-dimensional irreducible representations . Then:*

(1) *A morphism $T : V_1 \to V_2$ is either zero or an isomorphism.*
(2) *If $\mathbb{K} = \mathbb{C}$, every morphism $T : V_1 \to V_1$ has the form $f(v) = \lambda v$ for some $\lambda \in \mathbb{C}$.*
(3) *If $\mathbb{K} = \mathbb{C}$, we have*

$$\dim \mathrm{Hom}_G(V_1, V_2) = \begin{cases} 1 & \text{if } V_1 \cong V_2 \\ 0 & \text{if } V_1 \ncong V_2 \end{cases}$$

Proof. The proof is given below:

(1) Since V is irreducible, the kernel of T is either $\{0\}$ or V. In the latter case, T is zero, and in the former, T is injective. If T is injective, its image is a non-zero G-invariant subspace of W, and hence is all of W by assumption. We conclude that T is an isomorphism.

(2) Assume that T is non-trivial and let λ be any eigenvalue of T, and W the corresponding eigenspace. Thus,
$$W = \{v \in V_1 \mid T(v) = \lambda v\}$$
and one easily checks that W is G-invariant. Hence $W = V_1$. Hence, $T = \lambda \operatorname{Id}_{V_1}$.

(3) The follows directly from (1) and (2).

This completes the proof. □

The following fact is quite useful:

Proposition 6.19. *Let G be an abelian (Lie) group. An irreducible (finite-dimensional) representation $\Pi : G \to \mathsf{GL}(V)$ G over $\mathbb{C}$ is one-dimensional.*

Proof. Since G is abelian, $\Pi(g)$ is a morphism of $\Pi(g) : V \to V$ of representations for each $g \in G$. By Proposition 6.18(ii), every $\Pi(g)$ is multiplication by $\lambda(g) \in \mathbb{C}$. This implies that any subspace of V is G-invariant. The result follows, since if $\dim V > 1$, V would have a one-dimensional subspace, and since all subspaces are G-invariant, this would contradict the irreducibility of V. □

7. Haar Measure

We want to study the representations of compact Lie groups. It will be shown that the representation theory of compact Lie groups closely parallels that of finite groups. The key tool underpinning this similarity is the existence of a G-invariant measure on every compact Lie group, called the Haar measure, which serves as the foundation for many of the results in this theory. Recall that to integrate a function on a manifold, one typically begins with a fixed volume form, which requires the manifold to be orientable. Suppose G is a Lie group. Since any Lie group is orientable (as the tangent bundle is trivial), volume forms always exist on G. Naturally, we would like to select a volume form that behaves well under the group operations.

Definition 7.1. Let G be a Lie group. A volume form ω on a Lie group G is called **left-invariant** if $L_g^*\omega = \omega$ for all $g \in G$, where L_g denotes the left multiplication map.

Proposition 7.2. *Let G be a Lie group. G admits a left-invariant volume form that is unique up to a multiplicative constant.*

Proof. Let $\dim G = n$. Take any basis of T_e^*G and let ω a non-zero element $\omega_e \in \Lambda^n \mathsf{T}_e^* G$. Then define an n-form ω on G by letting $\omega_g = L_{g^{-1}}^*\omega_e$, where L_g denotes the left multiplication map. This is left-invariant since
$$(L_g^*\omega)_h = L_g^*\omega_{gh} = L_g^* L_{h^{-1}g^{-1}}^*\omega_e = (L_{h^{-1}g^{-1}} \circ L_g)^*\omega_e = L_{h^{-1}}^*\omega_e = \omega_h.$$
Moreover, suppose ω' is any left-invariant volume form on G. Since $\dim \Lambda^n \mathsf{T}_e^* G = 1$, there exists some non-zero constant C such that $\omega_e' = C\omega_e$. It follows from left-invariance that for any g,
$$\omega_g' = L_{g^{-1}}^*\omega_e' = C L_{g^{-1}}^*\omega_e = C\omega_g.$$
Thus, the left-invariant volume form is unique up to a multiplicative constant. □

Definition 7.3. Let G be a Lie group. A left-invariant measure is called a **left-invariant Haar measure**.

Corollary 7.4. *Let G be a connected Lie group.*

(1) *There exists a left-invariant Haar measure on any compact Lie group.*
(2) *If G is compact, the Haar measure can be taken to be normalized and is unique.*

Proof. The proof is given below:

(1) Suppose ω is a left-invariant volume form on G. Replacing ω by $-\omega$ if necessary, we may assume ω is positive with respect to the orientation of G. We have a linear map

$$\begin{aligned} I : C_c(G) &\to \mathbb{R} \\ f &\mapsto \int_G f(g)\,\omega(g) \end{aligned}$$

For any $h \in G$, note that we have

$$I(f) = \int_G f\omega = \int_G L_h^*(f\omega) = \int_G (L_h^* f)\omega = I(L_h^* f).$$

Let $\sigma(G)$ denote the Borel sigma algebra. We can define a map,

$$\begin{aligned} \mu : \sigma(G) &\to [0, \infty] \\ E &\mapsto I(\chi_E) \end{aligned}$$

This defines a left-invariant measure on G.

(2) If G is compact, we can replace ω by $1/\mathrm{Vol}(G)\omega$ where

$$\mathrm{Vol}(G) = \int_G \omega$$

This is well-defined since the volume of a compact Lie group is always finite. Since any two volume forms differ by a multiplicative constant, the claim follows.

This completes the proof. □

Remark 7.5. *We usually write $\omega = dg$. Note that the left invariance means*

$$d(hg) = dg,$$

or equivalently,

$$\int_G f(hg)\,dg = \int_G f(g)\,dg.$$

Remark 7.6. *Similarly one can define the right invariant volume forms and right Haar measures on a Lie group, and prove their existence and uniqueness (up to a constant)*

Remark 7.7. *In what follows we will not distinguish this measure and the corresponding positive volume form.*

In general a left Haar measure need not be a right Haar measure.

Example 7.8. (Sketch) Consider

$$G = \left\{ \begin{bmatrix} y & x \\ 0 & 1 \end{bmatrix} \,\middle|\, x, y \in \mathbb{R},\, y > 0 \right\},$$

One can check that G is a compact Lie group. One can also check that, up to a multiplicative constant $\omega_L = y^{-2}\,dx\,dy$ is the left Haar measure on G, and $\omega_R = y^{-1}\,dx\,dy$ is the right Haar measure on G.

Proposition 7.9. *Let G be a Lie group and ω a left Haar measure on G. Let $i : G \to G$ denote the inversion map.*

(1) *$i^*\omega$ is a right-invariant Haar measure on G.*
(2) *For any $g \in G$ and any left Haar measure ω, $R_g^*\omega$ is also left-invariant.*

Proof. The proof is given below:

(1) Using the relation $i \circ R_g = L_{g^{-1}} \circ i$, we get
$$R_g^*(i^*\omega) = (i \circ R_g)^*\omega = (L_{g^{-1}} \circ i)^*\omega = i^* L_{g^{-1}}^*\omega = i^*\omega.$$
(2) This follows from the fact that any left multiplication commutes with any right multiplication:
$$L_h^*(R_g^*\omega) = (R_g \circ L_h)^*\omega = (L_h \circ R_g)^*\omega = R_g^* L_h^*\omega = R_g^*\omega.$$

This completes the proof. □

It follows that there exists a positive constant, $\Delta(g)$, such that
$$\omega = \Delta(g) R_g^*\omega.$$
Note that the number $\Delta(g)$ is independent of the choice of a left Haar measure ω, since any two left Haar measures differ only by a constant.

Definition 7.10. The function $\Delta : G \to \mathbb{R}_+$ is called the *modular function* of G.

Proposition 7.11. *Let G be a Lie group and let ω be a left-invariant Haar measure.*

(1) *The modular function $\Delta : G \to \mathbb{R}_+$ is a Lie group homomorphism.*
(2) *$(i^*\omega)_g = \Delta(g)\omega_g$.*

Proof. The proof is given below:

(1) Obviously, Δ is continuous. Moreover, by definition,
$$\omega = \Delta(g_1 g_2) R_{g_1 g_2}^*\omega = \Delta(g_1 g_2)(R_{g_2} \circ R_{g_1})^*\omega = \Delta(g_1 g_2) R_{g_1}^* R_{g_2}^*\omega.$$
On the other hand, we have
$$R_{g_2}^*\omega = \Delta(g_1) R_{g_1}^* R_{g_2}^*\omega,$$
and thus
$$\omega = \Delta(g_2) R_{g_2}^*\omega = \Delta(g_2)\Delta(g_1) R_{g_1}^* R_{g_2}^*\omega.$$
It follows that $\Delta(g_1 g_2) = \Delta(g_1)\Delta(g_2)$.
(2) We first prove that $\Delta(g)\omega(g)$ is right-invariant:
$$R_h^*(\Delta(g)\omega_g) = \Delta(gh)(R_h^*\omega)_g = \Delta(g)\Delta(h)(R_h^*\omega)_g = \Delta(g)\omega_g.$$
It follows that there exists a positive constant C such that
$$\Delta(g)\omega_g = C(i^*\omega)_g.$$
It remains to show that $C = 1$. This follows from the fact
$$\omega_g = \Delta(g^{-1}) C (i^*\omega)_g = C i^*(\Delta\omega)_g = C^2 (i^*(i^*\omega))_g = C^2\omega_g.$$

This completes the proof. □

As a consequence, we see that for any $f \in C_c(G)$ and any left Haar measure,

$$\int_G f(g^{-1})dg = \int_G f(g)\Delta(g)dg.$$

We are interested in those Lie groups whose left Haar measure are also right-invariant.

Definition 7.12. Let G be a Lie group. G is called **unimodular** if $\Delta(g) \equiv 1$ for any $g \in G$.

Note that by definition, a Lie group is unimodular if and only if every left Haar measure is also a right Haar measure. So we can speak of "Haar measure" on unimodular Lie groups, without indicating left or right.

Proposition 7.13. *Let G be a compact Lie group.*

(1) *Then G is unimodular.*
(2) *The normalized Haar measure dg on a compact Lie group is left invariant, right invariant and invariant under inversion:*

$$\int_G f(hg)\, dg = \int_G f(gh)\, dg = \int_G f(g^{-1})\, dg = \int_G f(g)\, dg.$$

Corollary 7.14. *The proof is given below:*

(1) *Let G be a compact, the image $\Delta(G)$ of G is a compact subgroup of $\mathbb{R}_+$. However, the only compact subgroup of $\mathbb{R}_+$ is $\{1\}$.*
(2) *This is clear.*

8. Unitary Representations

The representation theory of compact Lie groups resembles that of finite groups in that finite-dimensional representations decompose into irreducibles, and characters form an orthonormal basis for class functions. A key ingredient is the Haar measure, which enables averaging similar to the finite case.

Proposition 8.1. *Let G be a compact Lie group. Any finite-dimensional representation (V, Π) of G admits a G-invariant inner product.*

Proof. Let $\langle , \rangle$ be any inner product on V. We define a new inner product by

$$\langle v, w\rangle_G := \int_G \langle g \cdot v, g \cdot w\rangle\, dg,$$

where dg is the Haar measure on G. It is straightforward to verify that $\langle \cdot, \cdot\rangle_G$ is an inner product on V and is invariant under the G-action:

$$\langle g \cdot v, g \cdot w\rangle_{\text{new}} = \langle v, w\rangle_{\text{new}}.$$

This follows from the left-invariance of the Haar meausure on G. □

We immediately get the following result:

Proposition 8.2. ***(Maschke's Theorem)*** *Let G be a compact Lie group and let (V, Π) be a finite-dimensional representation.*

(1) *If $(W, \Pi|_W)$ is a subrepresentation, then there is a complementary subrepresentation $(W', \Pi|_{W'})$ such that*

$$V = W \oplus W'.$$

(2) (V, Π) *is completely reducible.*

Proof. The proof is given below:

(1) Choose a G-invariant inner product on V and let W' be the orthogonal complement of W in V. It is easy to check that $(W', \Pi_{W'})$ is a subrepresentation. Clearly, V is a direct sum of W and W'.
(2) WLOG, assume that $V \neq \{0\}$. This follows by induction on $\dim V$. If $U \neq \{0\}$ is reducible, then $V = W' \oplus W'$ with $0 < \dim W < \dim V$. We can now proceed by induction.

This completes the proof. $\square$

Remark 8.3. *Proposition 8.2(1) fails in general if G is not a compact Lie group. Let $G = \mathbb{R}$. Consider the representation*

$$\rho : \mathbb{R} \to \mathsf{GL}(2, \mathbb{R})$$
$$t \mapsto \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

ρ leaves the x-axis fixed, so the x-axis is an invariant subspace. However, there is no complementary subspace since $\rho(t)$ is not diagonalizable for each $t \in \mathbb{R}^\times$.

Definition 8.4. Let G be a Lie group.

(1) Let $\mathbb{K} = \mathbb{C}$. A **unitary representation** is a vector space V with a Lie group homomorphism $\Pi : G \to \mathsf{U}(V)$.
(2) Let $\mathbb{K} = \mathbb{R}$. An **orthogonal representation** is a vector space V with a Lie group homomorphism $\Pi : G \to \mathsf{O}(V)$.

Proposition 8.2 implies that any finite-dimensional representation of a compact Lie group is equivalent to a unitary ($\mathbb{K} = \mathbb{C}$) or orthogonal ($\mathbb{K} = \mathbb{R}$) representation.

Remark 8.5. *A non-compact Lie group might have no non-trivial finite-dimensional unitary representation. Consider $G = \mathsf{SL}(2, \mathbb{R})$. Let $\Pi : \mathsf{SL}(2, \mathbb{R}) \to \mathsf{U}(n)$ be a representation of $\mathsf{SL}(2, \mathbb{R})$. For $m \in \mathbb{N}$, in $\mathsf{SL}(2, \mathbb{R})$ we have that*

$$\begin{pmatrix} m & 0 \\ 0 & m^{-1} \end{pmatrix} A_t \begin{pmatrix} m & 0 \\ 0 & m^{-1} \end{pmatrix}^{-1} = A_{m^2 t} = A_t^{m^2},$$

where A_t is the matrix as in Remark 8.3. Therefore, we have,

$$\Pi \begin{pmatrix} m & 0 \\ 0 & m^{-1} \end{pmatrix} \Pi(A_t) \Pi \begin{pmatrix} m & 0 \\ 0 & m^{-1} \end{pmatrix}^{-1} = \Pi(A_{m^2 t}) = \Pi(A_t)^{m^2}$$

If λ is an eigenvalue of $\Pi(A_t)$, the computation above implies that for every positive integer λ^{m^2} is an eigenvalue of $\Pi(A_t)$. Since the number of the eigenvalues of $\Pi(A_t)$ is finite, there exists $n \neq m$ such that $\lambda^{n^2} = \lambda^{m^2}$. Hence, the order of λ is finite and is a root of unity. Let N be a multiple of the order of the eigenvalues of $\Pi(A_t)$. The eigenvalues of $\Pi(A_t)^{N^2}$ are the eigenvalues of $\Pi(A_t)$. But the eigenvalues of $\Pi(A_t)^{N^2}$ are all 1. Hence, all eigenvalues of $\Pi(A_t)$ are 1. This implies that $\Pi(A_t) = I_n$ for each A_t. By the classification of normal subgroups of $\mathsf{SL}(2, \mathbb{R})$, the only normal subgroups of $\mathsf{SL}(2, \mathbb{R})$ are the trivial group, the whole group, and $\{\pm I_2\}$. Thus, the normal subgroup generated by A_t is $\mathsf{SL}(2, \mathbb{R})$. This readily implies that $\Pi(g) = I_n$. Hence, Π is trivial.

9. Character Theory

Let $\mathbb{K} = \mathbb{C}$ in this section. We now develop an analogue of character theory for finite groups in the context of compact Lie groups. We start off with the some observations. Let (V, Π) be a representation of a (compact Lie) group, G. If we choose a basis $e_1, \dots, e_n$ of V, we can identify V with $\mathbb{C}^n$, and represent any $g \in G$ by a matrix:

$$\Pi(g)v = \begin{pmatrix} \Pi_{11}(g) & \cdots & \Pi_{1n}(g) \\ \vdots & \ddots & \vdots \\ \Pi_{n1}(g) & \cdots & \Pi_{nn}(g) \end{pmatrix} \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix},$$

for $v = \sum_{i=1}^{n} v_i e_i$. If $e_1^*, \dots, e_n^*$ is the dual basis of V^*, we have

$$\Pi_{ij}(g) = e_i^*(\Pi(g)e_j).$$

This motivates the following definition:

Definition 9.1. Let G be a (compact Lie) group and let (V, Π) be a finite-dimensional representation of G. For any $v \in V$ and $\psi \in V^*$, the map

$$\begin{aligned} \varphi : G &\to \mathbb{C}, \\ g &\mapsto \psi(\Pi(g)v), \end{aligned}$$

is called a **matrix coefficient** of G.

Clearly any matrix coefficients of G is a continuous function on G. In fact, they form a subring of $C(G)$:

Lemma 9.2. *Let G be a (compact Lie) group and $\varphi_1, \varphi_2 \in C(G)$ be matrix coefficients of G. Then $\varphi_1 + \varphi_2$ and $\varphi_1 \cdot \varphi_2$ are also matrix coefficients for G.*

Proof. Let (Π_i, V_i) be (finite-dimensional) representations of G, $v_i \in V_i$, and $\psi_i \in V_i^*$ such that $\varphi_i(g) = \psi_i(\Pi_i(g)v_i)$. Then $(\Pi_1 \oplus \Pi_2, V_1 \oplus V_2)$ is a representation of G, $\psi_1 \oplus \psi_2 \in V_1^* \oplus V_2^* = (V_1 \oplus V_2)^*$, and

$$(\psi_1 \oplus \psi_2)((\Pi_1 \oplus \Pi_2)(g)(v_1, v_2)) = \varphi_1(g) + \varphi_2(g).$$

Similarly, we have a linear functional $\psi_1 \otimes \psi_2$ on $V_1 \otimes V_2$ satisfying

$$(\psi_1 \otimes \psi_2)(v_1 \otimes v_2) = \psi_1(v_1)\psi_2(v_2),$$

and thus

$$(\psi_1 \otimes \psi_2)((\Pi_1 \otimes \Pi_2)(g)(v_1 \otimes v_2)) = \varphi_1(g)\varphi_2(g).$$

This completes the proof. $\square$

Now assume that G is a compact Lie group. Recall that $L^2(G)$, the space of square-integrable functions with respect to the Haar measure, is the completion of the space of continuous functions on G with respect to the inner product

$$\langle f_1, f_2 \rangle_{L^2(G)} = \int_G \overline{f_1(g)} f_2(g)\, dg.$$

Proposition 9.3. ***(Schur Orthogonality)*** *Let G be a compact Lie group and let (V_i, Π_i) for $i = 1, 2$ be two non-isomorphic finite-dimensional irreducible representations of G. Fix G-invariant inner products on V_1 and V_2 respectively.*

1. *Every matrix coefficient of Π_1 is orthogonal in $L^2(G)$ to every matrix coefficient of Π_2.*

(2) *We have*

$$\int_G \langle \Pi(g)w_1, v_1\rangle_1 \, \overline{\langle \Pi(g)w_2, v_2\rangle_1} \, dg = \frac{1}{\dim V} \langle w_1, w_2\rangle_1 \overline{\langle v_1, v_2\rangle_1}$$

Proof. The proof is given below:

(1) Suppose

$$\varphi_i(g) = \langle \Pi_i(g)v_i, w_i\rangle_2, \quad i = 1, 2$$

are matrix coefficients for Π_i, where $v_i, w_i \in V_i$. Every matrix coefficient is of the form by the Riesz Representation Theorem. We show that

$$\begin{aligned}
\int_G \overline{\varphi_1(g)} \, \varphi_2(g) \, dg &= \int_G \overline{\langle \Pi_1(g)v_1, w_1\rangle_2} \, \langle \Pi_2(g)v_2, w_2\rangle_2 \, dg \\
&= \int_G \langle w_1, \Pi_1(g)v_1\rangle_2 \, \langle \Pi_2(g)v_2, w_2\rangle_2 \, dg \\
&= \int_G \langle \Pi_1(g^{-1})w_1, v_1\rangle_2 \, \langle \Pi_2(g)v_2, w_2\rangle_2 \, dg = 0
\end{aligned}$$

Fix a basis of V_1 such that $e_1 = v_1$. Define a linear map $f : V_1 \to V_2$ by $f(v_1) = v_2$ and $f(e_k) = 0$ for all $k \geq 2$. Consider the map

$$\begin{aligned}
F : V_1 &\to V_2 \\
v &\mapsto \int_G \Pi_2(g) \, f\big(\Pi_1(g^{-1})v\big) \, dg
\end{aligned}$$

It is clear that F is linear and G-invariant. By Schur's Lemma (Proposition 6.18), $F(v) = 0$ for any $v \in V_1$, and in particular,

$$\langle F(v), w_2\rangle = 0.$$

On the other hand, for any j, we have

$$\begin{aligned}
\Pi_2(g) f\big(\Pi_1(g^{-1})e_j\big) &= \Pi_2(g) f\left(\sum_k \Pi_1(g^{-1})_{kj} e_k\right) \\
&= \sum_k \Pi_1(g^{-1})_{kj} \Pi_2(g) f(e_k) \\
&= \Pi_1(g^{-1})_{1j} \Pi_2(g)(v_2),
\end{aligned}$$

where $\Pi_1(g^{-1})_{kj} = \langle \Pi_1(g^{-1})e_j, e_k\rangle$ are the matrix coefficients of Π_1 with respect to the basis $\{e_1, \dots, e_n\}$. It follows that

$$0 = \langle F(v_2), w_2\rangle = \int_G \langle \Pi_1(g^{-1})e_j, e_1\rangle_2 \langle \Pi_2(g)v_2, w_2\rangle_2 \, dg$$

for any j. By linearity and that $e_1 = v_1$, we have

$$\int_G \langle \Pi_1(g^{-1})w_1, v_1\rangle_2 \langle \Pi_2(g)v_2, w_2\rangle_2 \, dg = 0.$$

(2) Skipped.

This completes the proof. □

We now introduce characters of representations.

Definition 9.4. Let G be a compact Lie group and let (V, Π) be a finite-dimensional representation of. The **character** of Π is the function

$$\begin{aligned} \chi_\Pi : G &\to \mathbb{C} \\ g &\mapsto \operatorname{Tr}(\Pi(g)) \end{aligned}$$

Remark 9.5. *The character is a class function. Recall that a class function on a group G is a function $f : G \to \mathbb{C}$ that is invariant under conjugation; that is, for all $g, h \in G$,*

$$f(hgh^{-1}) = f(g).$$

This means that f is constant on conjugacy classes of G, depending only on the equivalence class of elements under conjugation. Moreover, one can easily check the character of representations ssatisfies the following properties:

1. $\chi_\Pi(e) = \dim V$.
2. $\chi_{\Pi_1 \oplus \Pi_2} = \chi_{\Pi_1} + \chi_{\Pi_2}$.
3. $\chi_{\Pi_1 \otimes \Pi_2} = \chi_{\Pi_1} \cdot \chi_{\Pi_2}$.

Note that the character of a representation is determined by specific matrix coefficients. In particular, the character

$$\chi_\Pi(g) = \sum_i \Pi_{i,i}(g)$$

is a linear combination of matrix coefficients. Therefore, Schur orthogonality (cf.Proposition 9.3) applies to characters as well.

Corollary 9.6. *Let G be a compact Lie group.*

1. *If (V, Π) is a finite-dimensional irreducible representation of G, then*

$$\langle \chi_\Pi, \chi_\Pi \rangle = \int_G |\chi_\Pi(g)|^2 \, dg = 1.$$

2. *If (V_1, Π_1) and (V_2, Π_2) are non-isomorphic finite-dimensional irreducible representations, then*

$$\langle \chi_{\Pi_1}, \chi_{\Pi_2} \rangle = \int_G \overline{\chi_{\Pi_1}(g)} \chi_{\Pi_2}(g) \, dg = 0.$$

Proof. This follows at once from Proposition 9.3. □

We can now show that the character of a representation completely determines the representation itself. This is a fundamental result in representation theory of compact Lie groups because characters encode essential information about representations in a highly efficient way. Since characters are class functions—meaning they are constant on conjugacy classes—they reduce the complexity of studying representations by focusing on conjugacy-invariant data rather than the full action of the group. The remarkable fact that two irreducible representations with the same character must be equivalent implies that the character serves as a powerful invariant, classifying irreducible representations up to isomorphism.

Proposition 9.7. *Let G be a compact Lie group. Two finite-dimensional representations (V_1, Π_1) and (V_2, Π_2) of G are isomorphic if and only if their characters coincide:*

$$\chi_{\Pi_1} = \chi_{\Pi_2}.$$

Proof. Since G is compact, any finite-dimensional representation of G is completely reducible (Proposition 8.2). Hence, we can decompose (V_1, Π_1) and (V_2, Π_2) into irreducible representations as

$$(V_1, \Pi_1) = \bigoplus_i m_i (W_i, \rho_i),$$
$$(V_2, \Pi_2) = \bigoplus_i n_i (W_i, \rho_i).$$

where (W_i, ρ_i) are pairwise non-isomorphic irreducible representations of G, and m_i, n_i are nonnegative integers. We have,

$$\chi_{\Pi_1} = \sum_i m_i \chi_{\rho_i},$$
$$\chi_{\Pi_2} = \sum_i n_i \chi_{\rho_i}.$$

Suppose $\chi_{\Pi_1} = \chi_{\Pi_2}$. Taking inner products with χ_{ρ_i} and using the orthogonality relations (Proposition 9.7), we get

$$m_i = \langle \chi_{\Pi_1}, \chi_{\rho_i} \rangle = \langle \chi_{\Pi_2}, \chi_{\rho_i} \rangle = n_i.$$

It follows that (V_1, Π_1) is isomorphic to (V_2, Π_2). The converse is clear. □

The argument in Proposition 9.7 implies that for any finite-dimensional representation, of a compact Lie group, there is a unique way to decompose it into irreducible ones:

$$(V, \Pi) = \bigoplus_{\rho \in \widehat{G}} \langle \chi_\Pi, \chi_\rho \rangle \, (W_\rho, \rho),$$

where $\widehat{G}$ is the set of equivalence classes of all irreducible representations of G.

Corollary 9.8. *Let G be a compact Lie group and (V, Π) be a finite-dimensional representation of G. Then (V, Π) is irreducible if and only if*

$$\langle \chi_\Pi, \chi_\Pi \rangle = \int_G |\chi_\Pi(g)|^2 \, dg = 1.$$

Proof. Let $(V, \Pi) = \bigoplus_i n_i (W_i, \rho_i)$ be the decomposition as above. Then

$$\int_G |\chi_\Pi(g)|^2 \, dg = \sum_i n_i^2.$$

The representation is irreducible if and only in $\sum_i n_i^2 = 1$. The claim follows. □

References

[1] John M. Lee. *Introduction to Smooth Manifolds*. Springer, 2012.