SCHEME THEORY

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Abstract. These notes represent the efforts of a non-specialist in algebraic geometry to scheme theory. The notes cover basic sheaf theory, affine schemes, schemes and properties of schemes. These notes were written at various stages during graduate school as part of an attempt to develop an understanding of modern algebraic geometry. The notes are highly incomplete and remain a work in progress. There may be errors or typographical mistakes; corrections and suggestions are most welcome. Please send them to junaid.aftab1994@gmail.com.

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1. Why Algebraic Geometry & Schemes?

Classical algebraic geometry is the study of affine algebraic sets, $X \subseteq \mathbb{C}^n$, given by the common zero set of a bunch of polynomials,

$$X = \{f_1(x) = \ldots = f_k(x) = 0\},$$

for some $f_1, \ldots, f_k \in \mathbb{C}[x_1, \ldots, x_n]$.

Remark 1.1. *Classical algebraic geometry also studies projective algebraic sets.*

Classical algebraic geometry is captured by the slogan:

"algebra = geometry"

The slogan "algebra = geometry" is captured in the algebra-geometry correspondence. This correspondence forms a fundamental bridge between geometric objects and algebraic structures. This correspondence allows us to translate geometric problems into algebraic ones and vice versa. This duality is central to many powerful methods and results in affine algebraic geometry, enabling a deep interplay between geometry and algebra. When $\mathbb{K}$ is algebraically closed, this leads to the classical algebra-geometry correspondence:

$$\{\text{Affine algebraic subsets of } X \subseteq \mathbb{K}^n\} \longleftrightarrow \{\text{Radical Ideals of } \mathbb{K}[x_1, \cdots, x_n]\}$$

Scheme theory is the language of modern algebraic geometry. While the slogan "algebra = geometry" is already embodied in the classical algebra-geometry correspondence, why go further? One motivation lies in the following meta-principle:

A scheme is to a variety as an abstract
manifold is to an embedded submanifold of $\mathbb{R}^n$.

Recall the Whitney embedding theorem, which states that any smooth finite-dimensional manifold can be embedded in $\mathbb{R}^n$ for some $n \in \mathbb{N}$. Yet, smooth manifold theory is not about studying objects distinct from submanifolds of $\mathbb{R}^n$, but about understanding them in a way that emphasizes intrinsic properties, free from artifacts of any specific embedding. Similarly, scheme theory seeks to study classical affine and projective varieties intrinsically—beyond their realization as subsets of affine or projective space. This perspective not only clarifies foundational aspects but also provides the natural framework for advanced topics in algebraic geometry.

(1) **Intersection Theory**: Consider a basic example from intersection theory: the intersection of the line $y = 0$ and the parabola $y = x^2$. Classically, their intersection is the single point $(0, 0)$, but this misses an important feature—namely, tangency. From a scheme-theoretic perspective, the intersection is given by

$$\operatorname{Spec} \mathbb{R}[x, y]/(y, y - x^2) \cong \operatorname{Spec} \mathbb{R}[x]/(x^2),$$

which reflects the fact that the curves are tangent at the origin. Thus, the scheme-theoretic approach captures geometric information—like tangency—that the classical viewpoint overlooks.

(2) **Moduli Spaces**:

Thus, scheme theory becomes indispensable in modern algebraic geometry, enabling us to work with geometric objects in a more general and powerful setting.

Part 1. Spectrum of a Ring

2. Affine Algebraic Sets

The goal of affine algebraic geometry is to study the solution sets of polynomial equations in several variables over a fixed ground field. We introduce the main objects of study and outline the relationship between algebra and geometry.

Remark 2.1. *We denote affine n-space over a field $\mathbb{K}$ by*

$$\mathbb{A}^n_{\mathbb{K}} := \{(a_1, \ldots, a_n) \mid a_i \in \mathbb{K} \textit{ for } i = 1, \ldots, n\},$$

which is just $\mathbb{K}^n$ viewed geometrically. We will abbreviate $\mathbb{A}^n_{\mathbb{K}}$ as $\mathbb{A}^n$.

Let $\mathbb{K}[x_1, \ldots, x_n]$ be the polynomial ring in n variables over $\mathbb{K}$. We begin by defining an affine algebraic set.

Definition 2.2. For a subset $S \subseteq \mathbb{K}[x_1, \ldots, x_n]$ of polynomials, the affine zero locus of S,

$$\mathbb{V}(S) := \{x \in \mathbb{A}^n : f(x) = 0 \text{ for all } f \in S\} \subseteq \mathbb{A}^n,$$

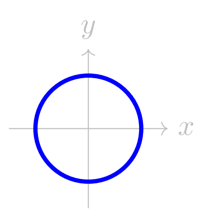
is an **affine algebraic set**.

Note that if $\mathfrak{a}$ is the ideal generated by S, then $\mathbb{V}(S) = \mathbb{V}(\mathfrak{a})$. Moreover, $\mathbb{V}(\mathfrak{a}) = \mathbb{V}(\sqrt{\mathfrak{a}})$ where $\sqrt{\mathfrak{a}}$ is the radical ideal of $\mathfrak{a}$ (**??**).

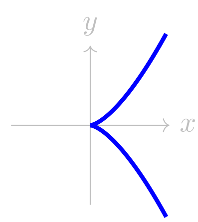
Remark 2.3. *If $S = \{f_1, \ldots, f_r\}$ is a finite set, we will write $\mathbb{V}(S) = \mathbb{V}(f_1, \ldots, f_r)$. Since $\mathbb{K}[x_1, \cdots, x_n]$ is a Noetherian ring, Hilbert's basis theorem (**??**) implies that every $\mathbb{V}(S)$ is of the form $\mathbb{V}(f_1, \ldots, f_k)$ for some $f_1, \cdots, f_r \in \mathbb{K}[x_1, \cdots, x_n]$.*

Example 2.4. The following is a list of affine algebraic sets:

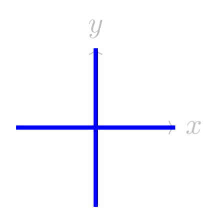
(1) Any point in $\mathbb{A}^n$ with $a = (a_1, \ldots, a_n) \in \mathbb{A}^n$ is an affine algebraic set.
(2) Linear subspaces of $\mathbb{A}^n$ are algebraic sets.
(3) Let $\mathbb{K} = \mathbb{R}$. Some affine algebraic sets in $\mathbb{R}^2$ are shown below:



$\mathbb{V}(x^2 + y^2 - 1)$



$\mathbb{V}(y^2 - x^3 - 1)$



$\mathbb{V}(xy)$

Our goal is to study geometric properties of affine algebraic sets through their defining polynomials from an algebraic perspective. However, it is not sufficient to consider only the initially given polynomials, since they are not unique. For example,

$$\mathbb{V}(x^2 + y^2 - 1) = \mathbb{V}((x^2 + y^2 - 1)^2),$$

even though the defining expressions differ. This motivates the need to consider all polynomials that vanish on a given affine algebraic set—that is, its vanishing ideal.

Definition 2.5. Let $X \subseteq \mathbb{A}^n$ be any subset. The **ideal of** X is the set:

$$\mathbb{I}(X) := \{f \in \mathbb{K}[x_1, \ldots, x_n] : f(x) = 0 \text{ for all } x \in X\}.$$

Remark 2.6. $\mathbb{I}(X)$ *is indeed an ideal, as can be easily verified. In fact, it is a radical ideal.*

Example 2.7. Let $a = (a_1, \ldots, a_n) \in \mathbb{A}^n$ be a point. We claim that

$$\mathbb{I}(a) = (x_1 - a_1, \ldots, x_n - a_n).$$

(1) If $f \in \mathbb{I}(a)$, then $f(a) = 0$. This means that replacing each x_i by a_i in f gives zero, i.e., that f is zero modulo $(x_1 - a_1, \ldots, x_n - a_n)$. Hence $f \in (x_1 - a_1, \ldots, x_n - a_n)$.
(2) If $f \in (x_1 - a_1, \ldots, x_n - a_n)$, then $f = \sum_{i=1}^{n}(x_i - a_i)f_i$ for some $f_1, \ldots, f_n \in \mathbb{K}[x_1, \ldots, x_n]$, and so certainly $f(a) = 0$, i.e., $f \in \mathbb{I}(a)$.

Note that we now have two distinct operations, $\mathbb{V}(\cdot)$ and $\mathbb{I}(\cdot)$. Moreover, these operations allow us to move and forth between subsets of $\mathbb{A}^n$ and subsets of $\mathbb{K}[x_1, \cdots, x_n]$.

$$\{\text{Subsets of } \mathbb{A}^n\} \longleftrightarrow \{\text{Subsets of } \mathbb{K}[x_1, \cdots, x_n]\}$$
$$X \mapsto \mathbb{I}(X)$$
$$\mathbb{V}(S) \leftarrow\!\shortmid S$$

Actually, $\mathbb{I}(X)$ is a radical ideal of of $\mathbb{K}[x_1, \cdots, x_n]$ and $\mathbb{I}(S)$ is an affine algebraic subset of $\mathbb{A}^n$. Hence, we have the following maps:

$$\{\text{Affine algebraic subsets of } \mathbb{A}^n\} \longleftrightarrow \{\text{Radical ideals of } \mathbb{K}[x_1, \cdots, x_n]\}$$
$$X \mapsto \mathbb{I}(X)$$
$$\mathbb{I}(S) \leftarrow\!\shortmid S$$

This begs the question: are the operations $\mathbb{V}(\cdot)$ and $\mathbb{I}(\cdot)$ and inverses of each other? An investigation of this question is important since since it is in some sense the central question in affine algebraic geometry: is there a bijective correspondence between geometric objects (affine algebraic sets) and algebraic objects (radical ideals)?

Conjecture 2.8. *Let $\mathbb{K}$ be an algebraically closed field. Consider the operations $\mathbb{V}(\cdot)$ and $\mathbb{I}(\cdot)$ define above.*

$$\{\textit{Affine algebraic subsets of } \mathbb{A}^n\} \longleftrightarrow \{\textit{Radical ideals of } \mathbb{K}[x_1, \cdots, x_n]\}$$
$$X \mapsto \mathbb{I}(X)$$
$$\mathbb{I}(S) \leftarrow\!\shortmid S$$

Then $\mathbb{V}(\cdot)$ and $\mathbb{I}(\cdot)$ yield an inclusion-reversing bijective correspondence between affine algebraic sets of $\mathbb{A}^n$ and radical ideals $\mathbb{K}[x_1, \cdots, x_n]$

We will prove Conjecture 2.8 in Proposition 3.6. For now, we assume the validity of the bijective correspondence between affine algebraic sets and their vanishing ideals, and explore some of its consequences. Throughout, we assume $\mathbb{K}$ is an algebraically closed field. If $X \subseteq \mathbb{A}^n$ is a fixed affine algebraic set, we are often interested in identifying polynomials in $\mathbb{K}[x_1, \ldots, x_n]$ that take the same values at every point of X. This leads to the following definition:

Definition 2.9. Let $X \subseteq \mathbb{A}^n$ be an affine algebraic subset. The **coordinate ring** of X is the quotient ring:

$$A(X) = \frac{\mathbb{K}[x_1, \cdots, x_n]}{\mathbb{I}(X)}$$

Remark 2.10. *In $A(X)$, we identify two polynomials $f, g \in \mathbb{K}[x_1, \dots, x_n]$ if and only if $f - g$ vanishes on X; that is, $f(x) = g(x)$ for all $x \in X$. Thus, an element $f \in A(X)$ can be viewed as a function $X \to \mathbb{K}$, given by evaluating a polynomial at points of X, where functions differing by a polynomial vanishing on X are considered equal.*

Given a fixed affine algebraic set X, one may focus on studying affine algebraic subsets of X. This motivates the following definition:

Definition 2.11. Let $X \subseteq \mathbb{A}^n$ be an affine algebraic subset and $A(X)$ be the associated co-ordinate ring. For any $S \subseteq A(X)$, the **X-affine algebraic subsets** is the zero locus

$$\mathbb{V}_X(S) = \{x \in X : f(x) = 0 \text{ for all } f \in S\} \subseteq X$$

For subset $Y \subseteq X$,

$$\mathbb{I}_X(Y) = \{f \in A(X) : f(x) = 0 \text{ for all } x \in Y\} \trianglelefteq A(X),$$

is the ideal of all polynomials on X that vanish on Y.

The assumed bijective correspondence can now be refined as follows:

Conjecture 2.12. *Let $\mathbb{K}$ be an algebraically closed field. Let $X \subseteq \mathbb{A}^n$ be an affine algebraic subset. There is an inclusion reversing bijective correspondence:*

$$\begin{aligned} \{\textit{Affine algebraic subsets of } X \subseteq \mathbb{A}^n\} &\longleftrightarrow \{\textit{Radical ideals of } A(X)\} \\ Y &\mapsto \mathbb{I}_X(Y) \\ \mathbb{V}_X(S) &\leftarrow\!\shortmid S \end{aligned}$$

The bijective correspondences described above imply that every algebraic operation on (radical) ideals admits a geometric interpretation.

We now illustrate this principle with several examples.

Example 2.13. We can give a geometric interpretation of various operations of ideals:

(1) Clearly, $\mathbb{V}_X(0) = X$ and $\mathbb{V}_X(A(X)) = \emptyset$.

(2) For any two ideals I, J be ideals in $A(X)$, note that $I + J$ is the ideal generated by $I \cup J$. We have:

$$\begin{aligned} \mathbb{V}_X(I \cup J) &= \mathbb{V}_X(I + J) \\ &= \{x \in X : f(x) = 0 \text{ for all } f \in I \cup J\} \\ &= \{x \in X : f(x) = 0 \text{ for all } f \in I\} \cap \{x \in X : f(x) = 0 \text{ for all } f \in J\} \\ &= \mathbb{V}_X(I) \cap \mathbb{V}_X(J). \end{aligned}$$

Hence, the ideal $I \cup J$ corresponds to the union of the algebraic sets $\mathbb{V}_X(I)$ and $\mathbb{V}_X(J)$. In particular, if $I + J = A(X)$, then I and J are coprime ideals, and

$$\mathbb{V}_X(I) \cap \mathbb{V}_X(J) = \emptyset.$$

(3) For any two X-affine algebraic subsets $Y, Z \subseteq X$:

$$\begin{aligned} \mathbb{I}_X(Y \cup Z) &= \{f \in A(X) : f(x) = 0 \text{ for all } x \in Y \cup Z\} \\ &= \{f \in A(X) : f(x) = 0 \text{ for all } x \in Y\} \cap \{f \in A(X) : f(x) = 0 \text{ for all } x \in Z\} \\ &= \mathbb{I}_X(Y) \cap \mathbb{I}_X(Z). \end{aligned}$$

Hence, the ideal $\mathbb{I}_X(Y) \cap \mathbb{I}_X(Z)$ corresponds to the union of the affine algebraic subsets $Y, Z \subseteq X$.

Algebraic Subsets	**Ideals of** $\mathbb{K}[x_1, \dots, x_n]$
$\mathbb{A}^n$	(0)
$\emptyset$	$\mathbb{K}[x_1, \dots, x_n]$
X	$A(X)$
$Y \cap Z$	$I + J$
$Y \cup Z$	$I \cap J$
$Y \subseteq Z$	$I : J$
$Y \cap Z = \emptyset$	$I + J = A(X)$

Algebra–geometry correspondence. Here $\mathbb{V}_X(I) = Y$ and $\mathbb{V}_X(J) = Z$.

(4) For any two X-affine algebraic subsets $Y, Z \subseteq X$:

$$\begin{aligned}\mathbb{I}_X(Y \setminus Z) &= \{f \in A(X) : f(x) = 0 \text{ for all } x \in Y \setminus Z\} \\ &= \{f \in A(X) : f(x)g(x) = 0 \text{ for all } x \in Y \text{ and } g \in \mathbb{I}_X(Z)\} \\ &= \{f \in A(X) : f \cdot \mathbb{I}_X(Z) \subseteq \mathbb{I}_X(Y)\} \\ &= \mathbb{I}_X(Y) : \mathbb{I}_X(Z)\end{aligned}$$

So taking the set-theoretic difference $Y \setminus Z$ corresponds to quotient ideals.

Given $X \subseteq \mathbb{A}^n$ and $Y \subseteq \mathbb{A}^m$, we can also consider functions between affine algebraic subsets.

Definition 2.14. Let $X \subseteq \mathbb{A}^n$ and $Y \subseteq \mathbb{A}^m$ be affine algebraic sets. A **polynomial morphism** from X to Y is a set-theoretic map

$$f : X \to Y$$

such that there exist polynomials $f_1, \dots, f_m \in \mathbb{K}[x_1, \dots, x_n]$ satisfying

$$f(x) = (f_1(x), \dots, f_m(x)) \in Y$$

for all $x \in X$.

Given the algebra-geometry correspondence discussed above, a natural question arises: what is the algebraic counterpart at the level of coordinate rings of a polynomial morphism between affine algebraic sets? The answer is provided by the following definition:

Definition 2.15. Let $X \subseteq \mathbb{A}^n$ and $Y \subseteq \mathbb{A}^m$ be affine algebraic sets and $f : X \to Y$ be a polynomial morphism. Then f induces a ring homomorphism

$$\begin{aligned}\phi : A(Y) &\to A(X) \\ g &\mapsto g \circ f = g(f_1, \dots, f_m)\end{aligned}$$

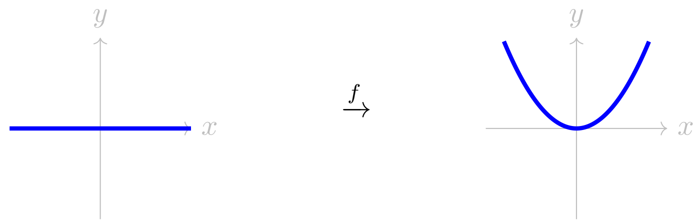
given by composing a polynomial function on Y with f to obtain a polynomial function on X.

Remark 2.16. *It is easy to check that the ϕ defined above is a $\mathbb{K}$-algebra homomorphism.*

Example 2.17. Let $\mathbb{K} = \mathbb{R}$, $X = \mathbb{A}^1$ (with coordinate x) and $Y = \mathbb{A}^2$ (with coordinates y_1 and y_2). Then $A(X) = \mathbb{R}[x]$ and $A(Y) = \mathbb{R}[y_1, y_2]$. Consider a polynomial morphism of affine algebraic sets,

$$\begin{aligned} f : X &\to Y \\ x &\mapsto (x, x^2) \end{aligned}$$

The image is obviously the standard parabola $Z = \mathbb{V}(y_2 - y_1^2)$.



The associated ring homomorphism $A(Y) = \mathbb{R}[y_1, y_2] \to \mathbb{R}[x] = A(X)$ is given by composing a polynomial function defined on Z with f, i.e., by plugging in x and x^2 for y_1 and y_2, respectively:

$$\begin{aligned} \mathbb{R}[y_1, y_2] &\to \mathbb{R}[x] \\ g &\mapsto g(x, x^2). \end{aligned}$$

Example 2.18. Let $f : X \to Y$ be a polynomial morphism of affine algebraic sets, and let $\phi : A(Y) \to A(X)$, $g \mapsto g \circ f$ be the associated map between the coordinate rings.

(1) For any X-affine algebraic subset Z, we have

$$\begin{aligned} \mathbb{I}(f(Z)) &= \{g \in A(Y) : g(f(x)) = 0 \text{ for all } x \in Z\} \\ &= \{g \in A(Y) : \phi(g) \in \mathbb{I}(Z)\} \\ &= \phi^{-1}(\mathbb{I}(Z)) \end{aligned}$$

Hence, taking images of X-affine algebraic subsets corresponds to the contraction of ideals.

(2) For any Y-affine algebraic subset Z, the zero locus of the extension $I(Z)$ by ϕ is

$$\begin{aligned} \mathbb{V}(\phi(\mathbb{I}(Z))) &= \{x \in X : g(f(x)) = 0 \text{ for all } g \in \mathbb{I}(Z)\} \\ &= f^{-1}(\{y \in Y : g(y) = 0 \text{ for all } g \in \mathbb{I}(Z)\}) \\ &= f^{-1}(\mathbb{V}(\mathbb{I}(Z))) \\ &= f^{-1}(Z) \end{aligned}$$

Hence, taking inverse images of Y-affine algebraic subsets corresponds to the extension of ideals.

Remark 2.19. *One can keep on asking similar questions:*

(1) *What X-affine algebraic sets correspond to maximal ideals in of $A(X)$?*
(2) *What X-affine algebraic sets correspond to prime ideals on of $A(X)$?*

We will answer this question in Proposition 3.19 by arguing that maximal ideals in $A(X)$ correspond to points in X and prime ideals in $A(X)$ correspond to irreducible[1] X-affine algebraic sets.

[1] We will need to make sense of irreducible algebraic subsets as well.

3. Algebra-Geometry Correspondence

We now set out to prove the algebra-geometry correspondence for affine algebraic sets. To do so, we first need to define a topology on the set of affine algebraic sets. This can be done via the following result:

Proposition 3.1. *The following properties are true for affine algebraic sets in $\mathbb{A}^n$:*

(1) *The empty set and the whole space are affine algebraic sets.*
(2) *The intersection of any family of affine algebraic sets is an affine algebraic set.*
(3) *The union of two affine algebraic sets is an affine algebraic set.*

Proof. The proof proceeds in the following steps:

(1) The empty set is $\emptyset = \mathbb{V}(1)$, and the whole space is $\mathbb{A}^n = \mathbb{V}(0)$.
(2) If $Y_\alpha = \mathbb{V}(T_\alpha)$ is any family of algebraic sets, then

$$\bigcap_\alpha Y_\alpha = \mathbb{V}\left(\bigcup_\alpha Y_\alpha\right),$$

so $\bigcap_\alpha Y_\alpha$ is also an affine algebraic set.
(3) Let $Y_1 = \mathbb{V}(T_1)$ and $Y_2 = \mathbb{V}(T_2)$, where $T_1, T_2 \subseteq \mathbb{K}[x_1, \dots, x_n]$. Then

$$Y_1 \cup Y_2 = \mathbb{V}(T_1 T_2),$$

where $T_1 T_2$ denotes the set of all finite sums of products fg with $f \in T_1$ and $g \in T_2$.
   (a) If $x \in Y_1 \cup Y_2$, then x is a zero of every polynomial in $T_1 T_2$ since $x \in Y_1$ implies $f(x) = 0$ for all $f \in T_1$, and similarly for Y_2.
   (b) Conversely, suppose $x \in \mathbb{V}(T_1 T_2)$ but $x \notin Y_1$. Then there exists $f \in T_1$ such that $f(x) \neq 0$. For any $g \in T_2$, since $(fg)(x) = f(x)g(x) = 0$ and $f(x) \neq 0$, it follows that $g(x) = 0$. Hence, $x \in Y_2$.

This completes the proof. □

Proposition 3.1 implies that the collection of affine algebraic sets is closed under arbitrary intersections and finite unions. This observation motivates the following definition of a topology:

Definition 3.2. The **(classical) Zariski topology** on $\mathbb{A}^n$ is defined by taking open subsets to be the complements of affine algebraic sets. This is a topology by Proposition 3.1.

What are the basis open sets in the (classical) Zariski topology? Let $U \subseteq \mathbb{A}^n$ be open in the Zariski topology. By definition, its complement U^c is an affine algebraic set, so

$$U^c = \mathbb{V}(I)$$

for some ideal $I \subseteq \mathbb{K}[x_1, \dots, x_n]$. Since

$$U^c = \bigcap_{f \in I} \mathbb{V}(f),$$

it follows that

$$U = \bigcup_{f \in I} \mathbb{V}(f)^c := \bigcup_{f \in I} D(f),$$

where

$$D(f) = \{x \in \mathbb{A}^n \mid f(x) \neq 0\}.$$

Hence, the collection

$$\{D(f) \mid f \in \mathbb{K}[x_1, \dots, x_n]\}$$

forms a basis for the Zariski topology. Sets of this form are called *distinguished open sets*.

Example 3.3. Let $\mathbb{K}$ be an algebraically closed field. Consider the Zariski topology on $\mathbb{A}^1$. Since every ideal in $\mathbb{K}[x]$ is principal, every algebraic set is the zero locus of a single polynomial. Given that $\mathbb{K}$ is algebraically closed, every nonzero polynomial $f(x) \in \mathbb{K}[x]$ factors as

$$f(x) = c(x - a_1) \cdots (x - a_n),$$

for some $c, a_1, \dots, a_n \in \mathbb{K}$. Thus,

$$\mathbb{V}(f) = \{a_1, \dots, a_n\}.$$

Consequently, the (closed) algebraic sets in $\mathbb{A}^1$ are exactly the finite subsets (including the empty set) and the entire space (corresponding to the zero polynomial). In particular, this implies that the Zariski topology on $\mathbb{A}^1$ is not Hausdorff.

Example 3.4. Using properties of the classical Zariski topology, one can show that the zero loci of transcendental functions are not necessarily algebraic sets. Consider the set

$$X = \{(x, y) \in \mathbb{R}^2 \mid y - \cos x = 0\}$$

Assume that X is an affine algebraic set. Proposition 3.1 implies that $W = X \cap \{(x, 0) \mid x \in \mathbb{R}\}$ is an affine algebraic set since $\{(x, 0) \mid x \in \mathbb{R}\}$ is the zero set of $g(x, y) = y$. But W is an infinite subset of $\mathbb{R}$, and the only non-trivial affine algebraic subsets of $\mathbb{R}$ are finite subsets (Example 3.3). Its Zariski closure is

$$\bar{X} = \mathbb{V}(\mathbb{I}(X)) = \mathbb{V}(0) = \mathbb{R}^2$$

Proposition 3.5. *(*Hartshorne I.1.4) *If we identify $\mathbb{A}^2$ with $\mathbb{A}^1 \times \mathbb{A}^1$ in the natural way, the Zariski topology on $\mathbb{A}^2$ is not the product topology of the Zariski topologies on the two copies of $\mathbb{A}^1$.*

PROOF. Consider the affine algebraic set $\mathbb{V}(y - x) \subseteq \mathbb{A}^2$. Clearly, $\mathbb{V}(y - x)$ is closed in the Zariski topology. However, $\mathbb{V}(y-x)$ is not closed in the product topology on $\mathbb{A}^2 = \mathbb{A}^1 \times \mathbb{A}^1$ equipped with the Zariski topology on each factor. Indeed, if $\mathbb{V}(y - x)$ were closed in the product topology, then its complement would be open. Any point in the complement would then be contained in a basis open set of the product topology. Since open sets in the Zariski topology on $\mathbb{A}^1$ are complements of finite sets, every basis open set in $\mathbb{A}^2$ is a product of cofinite sets and therefore must intersect $\mathbb{V}(y - x)$. This contradicts the assumption that the complement is open, proving that $\mathbb{V}(y - x)$ is not closed in the product topology. $\square$

We now use Hilbert's Nullstellensatz (**??**), we are now able to prove the algebra-geometry correspondence (Conjecture 2.8).

Proposition 3.6. *Let $\mathbb{K}$ be an algebraically closed field. There is an inclusion-reversing bijection between radical ideals in $\mathbb{K}[x_1, \cdots, x_n]$and algebraic sets in $\mathbb{A}^n$. More specifically,*

(1) If $T_1 \subseteq T_2$ are subsets of $\mathbb{K}[x_1, \cdots, x_n]$, then $\mathbb{V}(T_2) \subseteq \mathbb{V}(T_1)$.
(2) If $Y_1 \subseteq Y_2$ are subsets of $\mathbb{A}^n$, then $\mathbb{I}(Y_2) \subseteq \mathbb{I}(V_1)$.
(3) For any ideal $\mathfrak{a}$ in $\mathbb{K}[x_1, \cdots, x_n]$, $\mathbb{I}(\mathbb{V}(a)) = \sqrt{\mathfrak{a}}$, the radical of a.
(4) For any subset $Y \subseteq \mathbb{A}^n$, $\mathbb{V}(\mathbb{I}(Y)) = \overline{Y}$, the closure of Y.

Remark 3.7. *In what follows, let $R = \mathbb{K}[x_1, \cdots, x_n]$.*

Proof. (1), (2) are clear. The $\supseteq$ inclusion (3) is clear. For $\subseteq$ inclusion, assume that $f \notin \sqrt{\mathfrak{a}}$. We first argue that:

$$\sqrt{\mathfrak{a}} = \bigcap_{\mathfrak{a} \subseteq \mathfrak{m} \ \mathfrak{m} \text{ maximal}} \mathfrak{m}$$

The $\subseteq$ inclusion is clear. For the opposite inclusion $\supseteq$, let $f \in R$ with $f \notin \sqrt{\mathfrak{a}}$; we have to find a maximal ideal $\mathfrak{m} \supseteq \mathfrak{a}$ with $f \notin \mathfrak{m}$. Consider the multiplicatively closed set $S = \{f^n : n \in \mathbb{N}\}$. Since $f \notin \sqrt{\mathfrak{a}}$, $\mathfrak{a} \cap S = \emptyset$. Hence, $\mathfrak{a}$ can be thought of as a prime ideal in the $S^{-1}R$. A standard Zorn's lemma argument then shows that there is a prime ideal $\mathfrak{p}$ with $\mathfrak{p} \supseteq \mathfrak{a}$ and $\mathfrak{p} \cap S = \emptyset$ such that $S^{-1}\mathfrak{p} := \mathfrak{p}_f$ is maximal. It only remains to show that $\mathfrak{p}$ is maximal in R. Consider the ring extension

$$k \to R/\mathfrak{p} \hookrightarrow (R/\mathfrak{p})_f = R_f/\mathfrak{p}_f$$

Note that the second map is, in fact, an inclusion since $R/\mathfrak{p}$ is an integral domain. Moreover, $R_f/\mathfrak{p}_f$ is a field since $\mathfrak{p}_f$ is maximal and finitely generated as a k-algebra. So $k \subseteq R_f/\mathfrak{p}_f$ is a finite field extension, and hence integral. But then $R/\mathfrak{p} \subset R_f/\mathfrak{p}_f$ is integral as well, which means that $R/\mathfrak{p}$ is a field since $R_f/\mathfrak{p}_f$ is. Hence, $\mathfrak{p}$ is maximal. Now there is then a maximal ideal $\mathfrak{m}$ with $\mathfrak{m} \supseteq \mathfrak{a}$ and $f \notin \mathfrak{m}$. By Hilbert's Nullstellensatz (Proposition 5.19), $\mathfrak{m}$ has to be of the form

$$\mathbb{I}(a) = (x_1 - a_1, \cdots, x_n - a_n)$$

for some point $a \in \mathbb{A}^n$. Now $\mathbb{I}(a) \supseteq \mathfrak{a}$ implies $\mathfrak{a} \in \mathbb{V}(I)$, and $f \notin I(\mathfrak{a})$ means $f(a) \neq 0$. Hence, $f \notin \mathbb{I}(\mathbb{V}(I))$. In particular, for any radical ideal $\mathfrak{a}$, we have $\mathbb{I}(\mathbb{V}(\mathfrak{a})) = \mathfrak{a}$.

To prove (4), we note that $Y \subseteq \mathbb{V}(\mathbb{I}(Y))$, which is a closed set in the Zariski topology on $\mathbb{A}^n$, so clearly $\overline{Y} \subseteq \mathbb{V}(I(Y))$. On the other hand, let W be any closed set containing Y. Then $W = \mathbb{V}(\mathfrak{b})$ for some ideal $\mathfrak{b}$. So $\mathbb{V}(\mathfrak{b}) \supseteq Y$, and by (3), $\mathbb{I}(\mathbb{V}(\mathfrak{b})) \subseteq \mathbb{I}(Y)$. But certainly $\mathfrak{b} \subseteq \mathbb{I}(\mathbb{V}(\mathfrak{b}))$, so we have $W = \mathbb{V}(\mathfrak{b}) \supseteq \mathbb{V}(\mathbb{I}(Y))$. Thus, $\mathbb{V}(\mathbb{I}(Y)) = \overline{Y}$. In particular, if Y is an algebraic subset of $\mathbb{A}^n$, then $\mathbb{V}(\mathbb{I}(Y)) = Y$. □

We immediately have the following corollary:

Corollary 3.8. *(Conjecture 2.8) Let $\mathbb{K}$ be an algebraically closed field. There is an inclusion-reversing bijective correspondence:*

$$\{\textit{Closed affine Algebraic Sets of } \mathbb{A}^n\} \longleftrightarrow \{\textit{Radical Ideals of } \mathbb{K}[x_1, \cdots, x_n]\}$$
$$X \longrightarrow \mathbb{I}(X)$$
$$\mathbb{I}(\mathfrak{a}) \longleftarrow \mathfrak{a}$$

We also have the following corollory:

Corollary 3.9. *(Conjecture 2.12) Let $\mathbb{K}$ be an algebraically closed field. Let $X \subseteq \mathbb{A}^n$ be an affine algebraic subset. There is an inclusion reversing bijective correspondence:*

$$\{\textit{Closed affine algebraic subsets of } X \subseteq \mathbb{A}^n\} \longleftrightarrow \{\textit{Radical ideals of } A(X)\}$$
$$Y \mapsto \mathbb{I}_X(Y)$$
$$\mathbb{V}_X(S) \leftarrow\!\shortmid S$$

Remark 3.10. *Corollary 3.9 follows from Corollary 3.8. We omit details.*

Example 3.11. If we had not assumed $\mathbb{K}$ to be algebraically closed, then Corollary 3.8 would break down in the simple example with the the prime (and hence radical) ideal

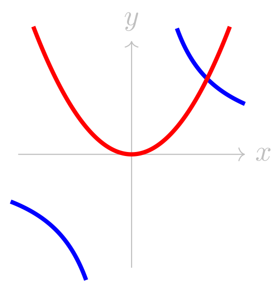
$$\mathfrak{a} = (x^2 + 1) \subseteq \mathbb{R}[x]$$

The ideal has an empty zero locus in $\mathbb{A}^1$ (over $\mathbb{R}$ of course), so we would obtain $\mathbb{I}(\mathbb{V}(\mathfrak{a})) = \mathbb{I}(\emptyset) = \mathbb{R}[x] \neq \sqrt{\mathfrak{a}} = \mathfrak{a}$.

Remark 3.12. *In light of Corollary 3.8 and Example 3.11, we almost exclusively assume that $\mathbb{K}$ is an algebraically closed field. In particular, we will work over $\mathbb{K} = \mathbb{C}$.*

We have now established a correspondence between affine algebraic sets of $\mathbb{A}^n$ and radical ideals of $\mathbb{K}[x_1, \cdots, x_n]$. This correspondence allows us us translate from thinking about affine algebraic sets as from either a geometric or an algebraic vantage point. Before moving on, let's look at an example:

Example 3.13. (*Hartshorne I.1.1*) Let Y_1 be the plane curve $y = x^2$ and let Y_2 be the plane curve $xy = 1$. The real locus of the plane curves is drawn below[2]. We claim that



$A(Y_1) = \mathbb{C}[x, y]/(y - x^2)$ is isomorphic to polynomial ring in one variable over $\mathbb{C}$. Consider the following morphism:

$$\begin{aligned} \phi : \mathbb{C}[x, y] &\longrightarrow \mathbb{C}[t] \\ x &\mapsto t \\ y &\mapsto t^2 \end{aligned}$$

The map is clearly a $\mathbb{C}$-algebra morphism. Note that the polynomial $y - x^2$ is contained in the kernel of ϕ. Therefore, ϕ descends to the map:

$$\overline{\phi} : \mathbb{C}[x, y]/(y - x^2) \longrightarrow \mathbb{C}[t]$$

It is easily seen that the map:

$$\begin{aligned} \phi : \mathbb{C}[t] &\longrightarrow \mathbb{C}[x, y] \\ t &\mapsto \bar{x} \end{aligned}$$

is a $\mathbb{C}$-algebra morphism that is the inverse of ϕ. Therefore,

$$\mathbb{C}[x, y]/(y - x^2) \cong \mathbb{C}[t].$$

On the other hand, $A(Y_2) = \mathbb{C}[x, y]/(y - 1/x)$ is not isomorphic to polynomial ring in one variable over $\mathbb{C}$. In the coordinate ring, x has a unit, namely $y = 1/x$. However, the indeterminate of a polynomial ring in one variable over $\mathbb{C}$ is never a unit.

Using the algebra-geometry correspondence, we can characterize irreducible affine algebraic sets.

Definition 3.14. Let X be a topological space. A non-empty subset $Y \subseteq X$ is **irreducible** if it cannot be expressed as the union $Y = Y_1 \cup Y_2$ of two proper closed subsets of Y.

[2] We shall only be able to plot the real locus of affine algebraic sets.

Example 3.15. Let $\mathbb{K}$ be an algebraically closed field. The following is a basic list of examples of irreducible and reducible affine algebraic sets:

(1) If $\mathbb{K}$ is infinite, $\mathbb{A}^1$ is irreducible in the Zariski topology, because its only proper closed subsets are finite, yet $\mathbb{A}^1$ is infinite[3].

(2) The affine algebraic $\mathbb{V}(xyz) \subseteq \mathbb{A}^3$ is not irreducible, as it can be written as a union of three coordinate planes:
$$\mathbb{V}(xyz) = \mathbb{V}(x) \cup \mathbb{V}(y) \cup \mathbb{V}(z).$$

Proposition 3.16. *(*Hartshorne I.1.6) *Any nonempty open subset of an irreducible topological space is dense and irreducible. If Y is a subset of a topological space X, which is irreducible in its induced topology, then $\overline{Y}$ is also irreducible.*

Proof. The proof has been commented out as it follows from standard results in point-set topology. □

With the notion of irreducible sets in a topological space in mind, we can now define the concept of an affine variety which are irreducible algebraic sets.

Definition 3.17. Let $\mathbb{K}$ be an algebraically closed field. An **affine variety** is an irreducible affine algebraic set of $\mathbb{A}^n$

Remark 3.18. *An affine variety cannot be written as a union of two non-empty affine algebraic sets.*

Proposition 3.19. *Let $\mathbb{K}$ be an algebraically closed field, and let $Y \subseteq \mathbb{A}^n$ be an affine algebraic set. Y is as affine variety if and only if $\mathbb{I}(Y)$ is a prime ideal.*

Proof. Assume Y is irreducible. If $fg \in \mathbb{I}(Y)$, then using that $\mathbb{V}(fg) = \mathbb{V}(f) \cup \mathbb{V}(g)$, we have
$$Y = Y \cap \mathbb{V}(fg) = (Y \cap \mathbb{V}(f)) \cup (Y \cap \mathbb{V}(g)),$$
both being closed subsets of Y. Since Y is irreducible, we have either $Y = Y \cap \mathbb{V}(f)$, in which case $Y \subseteq \mathbb{V}(f)$, or $Y \subseteq \mathbb{V}(g)$. Hence either $f \in \mathbb{I}(Y)$ or $g \in \mathbb{I}(Y)$. Conversely, assume that $\mathbb{I}(Y)$ is a prime ideal and $Y = Y_1 \cup Y_2$. If $Y = Y_1$, we are done. Hence, assume that $Y \neq Y_1 := \mathbb{V}(\mathfrak{a}_1)$. Then there is a $f_1 \in \mathfrak{a}_1$ and $y \in Y$ such that $f_1(y) \neq 0$. Writing $Y_2 := \mathbb{V}(\mathfrak{a}_2)$, we have that for every $f_2 \in \mathfrak{a}_2$, $f_1 f_2$ vanishes on Y and hence $f_1 f_2 \in \mathbb{I}(Y)$. Since $f_1 \notin \mathbb{I}(Y)$ and $\mathbb{I}(Y)$ is a prime ideal, we have that $f_2 \in \mathbb{I}(Y)$. This implies that $\mathfrak{a}_2 \subseteq \mathbb{I}(Y)$. But this implies that
$$Y \subseteq \mathbb{V}(\mathbb{I}(Y)) \subseteq \mathbb{V}(\mathfrak{a}_2) = Y_2.$$
Hence, $Y = Y_2$ and we are done. □

We have shown the following bijection:

$\{$Closed, irreducible affine Algebraic Sets of $\mathbb{A}^n\} \longleftrightarrow \{$Prime Ideals of $\mathbb{K}[x_1, \cdots, x_n]\}$

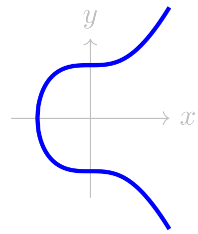
If $X \subseteq \mathbb{A}^n$ is an X-affine algebra set, we also have a bijection:

$\{$Closed, irreducible affine Algebraic Subsets of $X \subseteq \mathbb{A}^n\} \longleftrightarrow \{$Prime Ideals of $A(X)\}$

Example 3.20. Let $\mathbb{K}$ be an algebraically closed field. The following is a basic list of affine varieties:

(1) $\mathbb{A}^n$ is irreducible since it corresponds to (0) which is a prime ideal (0) as $\mathbb{K}$ is a field.

[3]Having $\mathbb{K}$ be infinite is crucial here.



$\mathbb{V}(y^2 - x^3 - 1)$

(2) Let $\mathbb{K} = \mathbb{C}$ and $X = \mathbb{V}(y^2 - x^3 - 1)$. We claim that X is irreducible by showing that the ideal $\mathbb{I}(y^2 - x^3 - 1)$ is prime in $\mathbb{C}[x, y]$. We consider the quotient ring:

$$R = \mathbb{C}[x, y]/(y^2 - x^3 - 1) \cong \mathbb{C}[x][y]/(y^2 - x^3 - 1)$$

Suppose $y^2 - x^3 - 1$ is reducible in $\mathbb{C}[x][y]$. Then it must factor as:

$$y^2 - x^3 - 1 = (y - f(x))(y - g(x))$$

for some $f(x), g(x) \in \mathbb{C}[x]$. This is because $y^2 = x^3 + 1$ in R. Expanding:

$$(y - f(x))(y - g(x)) = y^2 - (f(x) + g(x))y + f(x)g(x)$$

Comparing coefficients with $y^2 - x^3 - 1$, we get:

$$f(x) + g(x) = 0$$
$$f(x)g(x) = -x^3 - 1$$

From the first equation, $g(x) = -f(x)$. Substituting into the second:

$$f(x)(-f(x)) = -x^3 - 1 \implies -f(x)^2 = -x^3 - 1 \implies f(x)^2 = x^3 + 1$$

This implies that $x^3 + 1$ must be a perfect square in $\mathbb{C}[x]$. However:

$$x^3 + 1 = (x + 1)(x^2 - x + 1)$$

which is not a square in $\mathbb{C}[x]$. Hence, $\mathbb{I}(y^2 - x^3 - 1)$ is a prime ideal in $\mathbb{C}[x, y]$.

Remark 3.21. *We have an additional bijection correspondences:*

$$\{\textit{Points in } \mathbb{A}^n\} \longleftrightarrow \{\textit{Maximal Ideals of } \mathbb{K}[x_1, \cdots, x_n]\}$$
$$\{\textit{Points of } X \subseteq \mathbb{A}^n\} \longleftrightarrow \{\textit{Maximal Ideals of } A(X)\}$$

Here $X \subseteq \mathbb{A}^n$ is an affine algebra set. This can be easily proven given what we know now.

4. Morphisms of Affine Algebraic Sets

We first study the case of morphisms from an affine algebraic set to the affine algebraic set $\mathbb{A}^1 = \mathbb{K}$. Classical affine algebraic geometry studies the zero loci of polynomial functions, which correspond to morphisms from $\mathbb{A}^n$ to $\mathbb{A}^1$, or more generally, from an affine algebraic set X to $\mathbb{A}^1$, that is, elements of the coordinate ring $A(X)$. However, a broader class of functions is also important in this setting: rational functions. These only make sense in a restricted context, since for an affine algebraic set X, the coordinate ring $A(X)$ is not, in general, an integral domain. As shown in Proposition 3.19, this is the case if and only if X is an affine variety. Hence, we can make the following definition.

Definition 4.1. Let X be an affine variety, and let $A(X)$ denote its coordinate ring. The function field of X, denoted $K(X)$, is defined to be the field of fractions of $A(X)$:

$$K(X) := \mathrm{Frac}(A(X)).$$

Elements of $K(X)$ are called rational functions on X. A rational function $f \in K(X)$ is said to be regular at a point $x \in X$ if there exists $g, h \in A(X)$ such that $f = g/h$ and $h(x) \neq 0$.

Remark 4.2. *One can check that Definition 4.1 is independent of the choice of representative fraction.*

For $x \in X$, the set of rational functions that are regular at x forms a subring of $K(X)$, called the local ring of X at x:

$$\mathcal{O}_{X,x} := \{f \in K(X) \mid f \text{ is regular at } x\}$$

The ring $\mathcal{O}_{X,x}$ is indeed a local ring, with maximal ideal

$$\mathfrak{m}_x := \{f \in \mathcal{O}_{X,x} \mid f(x) = 0\}.$$

In fact, if $x = (a_1, \dots, a_n) \in X$, and we write $\mathfrak{m}_x = (x_1 - a_1, \dots, x_n - a_n)$ for the maximal ideal in $A(X)$, then

$$\mathcal{O}_{X,x} = A(X)_{\mathfrak{m}_x}.$$

To see this, note that since $A(X)$ is an integral domain, the localization $A(X)_{\mathfrak{m}_x}$ is naturally a subring of $K(X)$. An element $f \in K(X)$ lies in $A(X)_{\mathfrak{m}_x}$ if and only if it can be written as a/b with $b \notin \mathfrak{m}_x$, which is equivalent to f being regular at x. If $U \subseteq X$ is an open set, we define

$$\mathcal{O}_X(U) := \{f \in K(X) \mid f \text{ is regular at every point in } U\} = \bigcap_{x \in U} \mathcal{O}_{X,x}.$$

Note that each $\mathcal{O}_X(U)$ is a sub-$\mathbb{K}$-algebra of $K(X)$. Moreover, if $V \subseteq U$, then any $f \in K(X)$ that is regular on U is also regular on V, so there is an inclusion

$$\mathcal{O}_X(U) \subseteq \mathcal{O}_X(V).$$

Remark 4.3. *We will later see that $\mathcal{O}_X$ is a sheaf.*

Proposition 4.4. *Let X be an affine variety.*

(1) *If a rational function $g \in K(X)$ is regular at every point of X, then g is a polynomial function. In other words,*

$$\mathcal{O}_X(X) = A(X).$$

(2) *If $f \in A(X)$ and $D(f) := \{x \in X \mid f(x) \neq 0\}$, then*

$$\mathcal{O}_X(D(f)) = A(X)_f.$$

Proof. (1) follows from (2) by taking $f = 1$. Clearly $A(X)_f \subseteq \mathcal{O}_X(D(f))$. Conversely, given $g \in K(X)$, define the ideal

$$I_g := \{b \in A(X) \mid bg \in A(X)\}.$$

This ideal has the property that g is regular at a point $x \in X$ if and only if $x \notin \mathbb{V}(I_g)$. Note that $x \notin \mathbb{V}(I_g)$ if and only if there exists some $b \in I_g$ with $b(x) \neq 0$, which is equivalent to g being of the form $g = a/b$ with $b(x) \neq 0$. Therefore, if g is regular on all of $D(f)$, it follows that $\mathbb{V}(I_g) \subseteq \mathbb{V}(f)$. By Proposition 5.19, we conclude that $f^n \in I_g$ for some $n > 0$. But then $f^n g \in A(X)$, which shows that $g \in A(X)_f$. $\square$

Remark 4.5. *Proposition 4.4 is a sort of 'local-to-global principle': being regular is a local condition, which has to be verified near every point, but the conclusion is that a rational function which is regular at every point can be represented globally by a polynomial function.*

Example 4.6. Let $U = \mathbb{A}^1 \setminus \{0\} = \mathbb{V}(x)$. Then $K(\mathbb{A}^1) = \mathbb{K}(x)$ and $\mathcal{O}_{\mathbb{A}^1}(U) = \mathbb{K}[x, x^{-1}]$.

We now turn to the notion of morphisms between affine varieties. We define morphisms between affine varieties in a way that captures the algebraic structure encoded by regular functions. To do this, we adopt a functorial perspective: a morphism of varieties will be defined in terms of how it pulls back regular functions. The utility of this functorial definition will become apparent later on when we derive an appropriate equivalence of categories.

Definition 4.7. Let X, Y be affine varieties. A morphism between X and Y is a continuous map $f\colon X \to Y$ such that the pullback of any regular function is again regular. That is, for every open set $V \subseteq Y$ and every $g \in \mathcal{O}_Y(V)$, the composition $g \circ f$ lies in $\mathcal{O}_X(f^{-1}(V))$.

Remark 4.8. *We write the pullback of f as f^*.*

Given n regular functions $f_1, \ldots, f_n \in A(X)$, we can define a morphism

$$\begin{aligned} f\colon X &\to \mathbb{A}^n \\ x &\mapsto (f_1(x), \ldots, f_n(x)). \end{aligned}$$

Let's check that f is continuous. Let $W = \mathbb{V}(g_1, \ldots, g_r) \subseteq \mathbb{A}^n$ be a closed set., then

$$f^{-1}(W) = \{x \in X \mid g_i(f_1(x), \ldots, f_n(x)) = 0 \text{ for all } i = 1, \ldots, r\} = \mathbb{V}(f^*(g_1), \ldots, f^*(g_r)),$$

which is closed in X. Let $g \in \mathbb{K}(y_1, \ldots, y_n)$ be a rational function on $\mathbb{A}^n$, and assume g is regular on an open set $V \subseteq \mathbb{A}^n$. Let $x \in f^{-1}(V)$ and set $y = f(x)$. Locally around y, we may write $g = a/b$ where $a, b \in \mathbb{K}[y_1, \ldots, y_n]$. Then in a neighborhood of $x \in f^{-1}(V)$, we have

$$f^*(g)(x) = \frac{a(f_1(x), \ldots, f_n(x))}{b(f_1(x), \ldots, f_n(x))}.$$

After expanding, this can be written as a quotient of polynomials where the denominator does not vanish at x. Hence $f^*(g) \in \mathcal{O}_X(f^{-1}(V))$, and f is a morphism. In fact, we can now argue that all morphisms with target affine variety $\mathbb{A}^n$ are of this form:

Proposition 4.9. *Let X be an affine variety. Then every morphism $f\colon X \to \mathbb{A}^n$ is of the form*

$$\begin{aligned} f\colon X &\to \mathbb{A}^n \\ x &\mapsto (f_1(x), \ldots, f_n(x)), \end{aligned}$$

where $f_i \in A(X)$ for $i = 1, \ldots, n$.

Proof. Let $f\colon X \to \mathbb{A}^n$ be a morphism. The coordinate functions $y_1, \ldots, y_n$ are regular functions on $\mathbb{A}^n$, so their pullbacks $f^*(y_i)$ are regular on X. The morphism defined by these functions,

$$\begin{aligned} X &\to \mathbb{A}^n \\ x &\mapsto \big(f^*(y_1)(x), \ldots, f^*(y_n)(x)\big). \end{aligned}$$

coincides with f, since both are determined by the same set of regular functions. $\square$

Grothendieck emphasized studying affine algebraic sets via their coordinate rings rather than the sets themselves. This is also the philosophy underlying the algebra-geometry correspondence: the geometry of X reflects the algebra of its coordinate ring

$$A(X) = \mathbb{K}[x_1, \dots, x_n]/\mathbb{I}(X),$$

and vice versa. This can be formalized by arguing that we have an appropriate equivalence of categories between affine algebraic sets and finitely generated reduced $\mathbb{K}$-algebras. These categories are defined as follows:

(1) The category AffVar of affine varieties and morphisms of affine algebraic varieties.
(2) The category $\mathsf{fgAlg}_{\mathbb{K}}^{\mathsf{Dom}}$ of finitely-generated $\mathbb{K}$-algebras that are integral domains and morphisms of $\mathbb{K}$-algebras.

Proposition 4.10. *Let $\mathbb{K}$ be an algebraically closed field. The categories AffVar and $\mathsf{fgAlg}_{\mathbb{K}}^{\mathsf{Dom}}$ are equivalent.*

Proof. Consider the functor that assigns to each affine variety $X \subseteq \mathbb{A}^n$ its coordinate ring $A(X) = \mathbb{K}[x_1, \dots, x_n]/\mathbb{I}(X)$. Clearly, $A(X)$ is finitely generated as a $\mathbb{K}$-algebra since $\mathbb{K}[x_1, \dots, x_n]$ is finitely generated. Moreover, $A(X)$ is reduced because $\mathbb{I}(X)$ is a radical ideal. The functor is essentially surjective. Indeed, if $A \in \mathsf{fgAlg}_{\mathbb{K}}^{\mathsf{Dom}}$ we can choose a presentation

$$A = \mathbb{K}[x_1, \dots, x_n]/\mathfrak{p}$$

for some ideal $\mathfrak{p}$. Since A is an integral domain, $\mathfrak{p}$ is a prime ideal. Then $X = \mathbb{V}(\mathfrak{p}) \subseteq \mathbb{A}^n$ is an affine variety with $A(X) = A$. We claim that

$$\begin{aligned} \mathrm{Hom}_{\mathsf{AffVar}}(X, Y) &\longrightarrow \mathrm{Hom}_{\mathsf{fgAlg}_{\mathbb{K}}^{\mathsf{Dom}}}(A(Y), A(X)) \\ f &\mapsto f^* \end{aligned}$$

is a bijection. We first show that the map is surjective. Let $\phi\colon A(Y) \to A(X)$ be a morphism of $\mathbb{K}$-algebras. Suppose $Y \subseteq \mathbb{A}^n$, and let $y_1, \dots, y_n$ be the coordinate functions on $\mathbb{A}^n$. Define $f_i := \phi(y_i)$ for $i = 1, \dots, n$. Then the functions $f_1, \dots, f_n \in A(X)$ define a morphism

$$\begin{aligned} f\colon X &\longrightarrow \mathbb{A}^n \\ x &\mapsto \big(f_1(x), \dots, f_n(x)\big) \end{aligned}$$

This induces a map $\mathbb{K}[x_1, \cdots, x_n] \to A(X)$ via the pullback (Definition 2.15). More explicitly for $h \in \mathbb{K}[x_1, \cdots, x_n]$, we have

$$f^*(h)(x) = h(f(x)) = h(f_1(x), \dots, f_n(x)) = \phi\big(h(y_1, \dots, y_n)\big)(x)$$

The last equality holds because both sides agree on the generators $y_1, \dots, y_n$, taking values $f_1, \dots, f_n$, respectively. This shows that

$$f^*(h) = \phi(h) = 0 \quad \text{for every } h \in \mathbb{I}(Y),$$

since h is zero in $A(Y)$. Therefore, the image of f is contained in $Y = \mathbb{V}(\mathbb{I}(Y))$. This shows that f^* factors through $\mathbb{A}(Y) = K[x_1, \cdots, x_n]/\mathbb{I}(Y)$. Hence, the map is surjective. The map is also injective since $f^* = \phi$. $\square$

5. Spectrum of a Ring

Recall the classical algebra-geometry correspondence (Proposition 3.6). There are some limitations of this correspondence:

(1) The bijection isn't natural. since morphisms of affine (projective) algebraic sets assumes an embedding of affine (projective) algebraic sets in some underlying affine (projective) space.
(2) The classical algebra-geometry correspondence for affine (projective) algebraic sets holds for algebraically closed fields. How does one study the analog of affine (projective) algebraic sets over non-algebraically closed fields?

We address these limitations by providing an intrinsic characterization of relevant concepts for affine algebraic sets. The classical algebra-geometry correspondence states that there we have bijections:

$$\{\text{Points of } \mathbb{A}^n\} \longleftrightarrow \{\text{Maximal ideals of } \mathbb{K}[x_1, \cdots, x_n]\}$$
$$\{\text{Closed irreducible affine algebraic sets } \mathbb{A}^n\} \longleftrightarrow \{\text{Prime ideals of } \mathbb{K}[x_1, \cdots, x_n]\}$$

We can now attempt to generalize to the case of an arbitrary commutative ring, R. Previously, we have kept track of the data consisting of all irreducible algebraic subsets of $\mathbb{A}^n$, which correspond to prime ideals of $R = \mathbb{K}[x_1, \dots, x_n]$. This leads us to the following definition:

Definition 5.1. Let R be a commutative ring. The **spectrum of a ring** R, denoted as $\operatorname{Spec} R$, is the set of all prime ideals of R.

Remark 5.2. *All rings, unless otherwise specified, will be commutative with an identity. From now on, we shall use the phrase 'let R be a ring.' We will use the phrases the spectrum of a ring and affine schemes interchangeably to refer to the set* $\operatorname{Spec} R$. *We will justify the use of the phrase affine schemes in Section 14 when we formally define schemes.*

Remark 5.3. *The construction* $R \mapsto \operatorname{Spec} R$ *is functorial in* R *in a contravariant sense. That is, given a ring homomorphism* $f : R \to S$, *there is an induced map:*

$$F : \operatorname{Spec} S \to \operatorname{Spec} R,$$

defined by sending a prime ideal $\mathfrak{p} \subseteq S$ *to its preimage* $f^{-1}(\mathfrak{p}) \subseteq R$, *which is easily verified to be a prime ideal. This establishes a contravariant functor*

$$\operatorname{Spec} : \mathsf{CRing} \to \mathsf{Sets},$$

from the category of commutative rings with identity to the category of sets.

Example 5.4. The following is a list of basic examples of the spectrum of a ring:

(1) The spectrum of a ring is empty if and only if the ring is the zero ring.
(2) If $R = \mathbb{K}$ is a field, then the spectrum of $\mathbb{K}$ is a single point, (0). The corresponds to the notion of a point in affine algebraic geometry.
(3) Let $R = \mathbb{C}[x_1, \dots, x_n]$. Classical affine algebraic geometry can be reformulated as the study of the scheme $\operatorname{Spec} \mathbb{C}[x_1, \dots, x_n]$. More generally, for any ring R, we define

$$\mathbb{A}^n_R := \operatorname{Spec} R[x_1, \dots, x_n],$$

which is called affine n-space over R. When the base ring R is clear from context, we will simply write $\mathbb{A}^n$.

(4) Let $R = \mathbb{Z}$. We have

$$\operatorname{Spec}\mathbb{Z} = \{(0)\} \cup \{(p) \mid p \text{ prime}\}$$

Schemes over $\mathbb{Z}$, $\mathbb{Q}$, or more generally over number fields, play a central role in the application of scheme theory to number theory.

Remark 5.5. *Example 5.4(b) raises an important concern. There are many non-isomorphic fields; however, the spectrum of all fields is the same singleton set. We will see later that the structure sheaf will allow us to distinguish between non-isomorphic fields.*

We now endow the spectrum of a ring with a topology. In affine algebraic geometry, the Zariski topology is defined as the coarsest topology in which all affine algebraic sets are closed. The definition of the Zariski topology on a a spectrum of a ring (or an affine scheme) is motivated by its classical counterpart in affine algebraic geometry.

Definition 5.6. Let R be a ring. The **Zariski topology** on $\operatorname{Spec} R$ is given by declaring the closed sets to be of the form,

$$V(S) = \{\mathfrak{p} \in \operatorname{Spec} R \mid S \subseteq \mathfrak{p}\},$$

for all subsets S of R.

Remark 5.7. *Note that $V(S) = V(\langle S \rangle)$.*

Remark 5.8. *If $\mathbb{K}$ is an algebraically closed field and $R = \mathbb{K}[x_1, \cdots, x_n]$ the classical algebra-geometry correspondence implies that $V(S)$ can be identified with the set of all irreducible affine algebraic sets in $\mathbb{A}^n$ that are contained in the affine algebraic set $\mathbb{V}(S)$.*

Remark 5.9. *Here is another way to think of $V(S)$. For $\mathfrak{p} \in \operatorname{Spec} R$, define the residue field $\kappa(\mathfrak{p})$ to be the field of fractions of the integral domain $R/\mathfrak{p}$. For $f \in R$, note that we have*

$$V(f) = \{\mathfrak{p} \in \operatorname{Spec} R \mid f(\mathfrak{p}) = 0 \text{ in } \kappa(\mathfrak{p})\}.$$

Here $f(\mathfrak{p}) := \overline{(f)}$ is the image of (f) in $R/\mathfrak{p}$. In other words, we can think of $f \in R$ as a function on $\operatorname{Spec} R$, and $f(\mathfrak{p})$ is the element $\overline{(f)}$ in $R/\mathfrak{p}$. Consider $R = \mathbb{C}[z]$. For $f \in \mathbb{C}[z]$, note $(\overline{f}) = 0$ in $R/(z-a)$ if and only if $f(a) = 0$. More precisely, the evaluation map

$$\begin{aligned} \operatorname{Ev}_a : \mathbb{C}[z] &\to \mathbb{C} \\ f &\mapsto f(a) \end{aligned}$$

induces an isomorphism $\mathbb{C}[z]/(z-a) \cong \mathbb{C}$. For instance, we have

$$V(z^2 + z + 1) = \{(z - e^{i\pi/3}), (z - e^{i2\pi/3})\}.$$

Let us verify that the Zariski topology on $\operatorname{Spec}(R)$ is indeed a topology. The argument is similar to that of Proposition 3.1, but in this setting we must work purely algebraically to check that the collection of closed sets satisfies the axioms of a topology.

Lemma 5.10. *Let R be a ring. The following statements are true:*

(1) $V(0) = \operatorname{Spec}(R), V(1) = \emptyset$.
(2) If $\mathfrak{a}$ and $\mathfrak{b}$ are two ideals of A, then $V(\mathfrak{a}\mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b})$.
(3) If $\{\mathfrak{a}_i\}$ is any set of ideals of A, then $V(\sum_i \mathfrak{a}) = \bigcap_i V(\mathfrak{a}_i)$.
(4) If $\mathfrak{a}$ and $\mathfrak{b}$ are two ideals, $V(\mathfrak{a}) \subseteq V(\mathfrak{b})$ if and only if $\sqrt{\mathfrak{b}} \subseteq \sqrt{\mathfrak{a}}$.
(5) $V(\mathfrak{a}) = V(\sqrt{\mathfrak{a}})$

Proof. The proof is given below:

1. This is clear.
2. If $\mathfrak{a} \subseteq \mathfrak{p}$ or $\mathfrak{b} \subseteq \mathfrak{p}$, then $\mathfrak{ab} \subseteq \mathfrak{p}$. Conversely, if $\mathfrak{ab} \subseteq \mathfrak{p}$, and if $\mathfrak{b}$ is not contained in $\mathfrak{p}$, for example, then there is a $b \in \mathfrak{b}$ such that $b \notin \mathfrak{p}$. Now, for any $a \in \mathfrak{a}$, $ab \in \mathfrak{p}$, so we must have $a \in p$ since $\mathfrak{p}$ is a prime ideal. Thus, $\mathfrak{a} \subseteq \mathfrak{p}$.
3. $\mathfrak{p}$ contains $\sum_i \mathfrak{a}_i$ if and only if $\mathfrak{p}$ contains each $\mathfrak{a}_i$, simply because $\sum_i \mathfrak{a}_i$ is the smallest ideal containing all of the ideals $\mathfrak{a}_i$.
4. Recall that the radical of $\mathfrak{a}$ is the intersection can be defined as:
$$\sqrt{a} = \bigcap_{\mathfrak{a} \subseteq \mathfrak{p}} \mathfrak{p}.$$
This shows that
$$\sqrt{\mathfrak{a}} \subseteq \sqrt{\mathfrak{b}} \iff V(\mathfrak{a}) \supseteq V(\mathfrak{b})$$
5. This follows from the definition of $\sqrt{\mathfrak{a}}$.

This completes the proof. $\square$

Lemma 5.10 shows that finite unions and arbitrary intersections of sets of the form $V(\mathfrak{a})$ are again of that form. Hence, these sets define the closed sets of a topology on $\operatorname{Spec}(R)$.

Example 5.11. Consider $\operatorname{Spec}\mathbb{Z}$. Let us describe the closed subsets. These are of the form $V(I)$ where $I \subseteq \mathbb{Z}$ is an ideal, so $I = (n)$ for some $n \in \mathbb{Z}$.

1. If $n = 0$, the closed subset is all of $\operatorname{Spec}(\mathbb{Z})$.
2. If $n \neq 0$, then n has finitely many prime divisors. Hence
$$V(n) = \{(p_1), \ldots, (p_{k_n}) \mid p_i \text{ is prime and } p_i \mid n\}$$
For example, $V(33)$ is $\{(3), (11)\}$.

We have seen that Spec induces a contravariant functor from $\mathsf{CRings} \to \mathsf{Sets}$ (Remark 5.3). Next, we check that the morphisms induced on Spec's from a ring-homomorphism are in fact continuous maps of topological spaces.

Lemma 5.12. Spec *induces a contravariant functor*
$$\operatorname{Spec} : \mathsf{CRing} \to \mathsf{Top},$$
from the category of CRing *commutative rings to the category* Top *of topological spaces.*

Proof. Let $f : R \to S$ be a ring homomorphism. We claim that the induced map
$$F : \operatorname{Spec} S \to \operatorname{Spec} R,$$
$$\mathfrak{p} \mapsto f^{-1}(\mathfrak{p}).$$
is continuous in the Zariski topology. Consider a closed subset $V(I) \subseteq \operatorname{Spec} R$, where $I \subseteq R$ is an ideal. Then the preimage under F is given by
$$F^{-1}(V(I)) = \{\mathfrak{p} \in \operatorname{Spec} S \mid f^{-1}(\mathfrak{p}) \supseteq I\}.$$
This is precisely the set of prime ideals $\mathfrak{p} \subseteq S$ such that $\mathfrak{p} \supseteq f(I)$, which is the definition of the closed subset $V(f(I)) \subseteq \operatorname{Spec} S$[4]. Thus,
$$F^{-1}(V(I)) = V(f(I)) = V(\langle f(I) \rangle),$$

[4] For $\mathfrak{p} \in \operatorname{Spec} S$, we have $F(\mathfrak{p}) \in V(I)$ if and only if $f^{-1}(\mathfrak{p}) \supseteq I$, which holds if and only if $\mathfrak{p} \supseteq f(I)$, i.e., $\mathfrak{p} \in V(f(I))$.

showing that F is continuous. □

Note that we can also characterize the Zariski topology in terms of open sets. Indeed, for any $f \in R$, define

$$U_f := \{\mathfrak{p} \in \operatorname{Spec} R \mid f \notin \mathfrak{p}\}.$$

Then U_f is the subset of $\operatorname{Spec} R$ consisting of prime ideals that do not contain f. This is precisely the complement of the closed set $V(f)$, so U_f is open in the Zariski topology.

Lemma 5.13. *Let R be a ring. The sets U_f form a basis for the Zariski topology on* $\operatorname{Spec} R$*.*

Proof. Suppose $U \subseteq \operatorname{Spec} R$ is open. We have $U = V(I)^c$ for some ideal I. Note that we have

$$V(I) = \bigcap_{f \in I} V(f) \Longrightarrow U = \bigcup_{f \in I} V^c(f) = \bigcup_{f \in I} U_f$$

This completes the proof. □

Example 5.14. Let $R = \operatorname{Spec} \mathbb{C}[z]$. For $f \in \mathbb{C}[z]$, the open set U_f consists of all prime ideals $(x - z)$ such that $f(z) \neq 0$. Therefore,

$$V(f) = U_f^c \cong \{z \in \mathbb{C} \mid f(z) = 0\}$$

If f is not the zero polynomial, then $V(f)$ is a finite subset of $\mathbb{C}$. Otherwise, $V(0) = \mathbb{C}$.

Proposition 5.15. *Let R be a a ring. The collection of open sets U_f have the following properties:*

(1) $U_1 = \operatorname{Spec} R$ and $U_0 = \emptyset$
(2) $U_{fg} = U_f \cap U_g$.
(3) $U_f = \emptyset$ iff f is nilpotent.
(4) $U_f = \operatorname{Spec} R$ iff f is a unit.
(5) More generally, $\bigcup_{i \in I} U_{f_i} = \operatorname{Spec} R$ if and only if the ideal generated by $\{f_i\}_{i \in I}$ is R.
(6) $U_f \subseteq U_g$ iff $f \in \sqrt{\mathfrak{g}}$.

Proof. The proof is given below:

(1) This follows because because prime ideals are not allowed to contain the unit element and because every prime ideal contains 0.
(2) This follows because fg lies in a prime ideal $\mathfrak{p}$ if and only if one of f, g does.
(3) This follows because $U_f = \emptyset$ iff every prime ideal contains f iff f is in the intersection of all prime ideals, i.e., nilradical.
(4) This follows because $U_f = \operatorname{Spec} R$ iff no prime ideal contains f iff f is a unit.
(5) This follows because $\bigcup_{i \in I} U_{f_i} = \operatorname{Spec} R$. if and only if the ideal generated by $\{f_i\}_{i \in I}$ is R. Note that $\bigcup_{i \in I} U_{f_i} = \operatorname{Spec} R$ iff that all prime ideals cannot contain all f_i's iff for every prime ideal, $\mathfrak{p}$, there exists a $i \in I$ such that $f_i \notin \mathfrak{p}$ iff no prime ideal contains the ideal generated by $\{f_i\}_{i \in I}$ iff the ideal generated by $\{f_i\}_{i \in I}$ is R.
(6) This follows directly from Lemma 5.10.

This completes the proof. □

Remark 5.16. *For basic open sets, we will use the notation $U_f, V_f, D(f)$ interchangeably*

Let us define an operation that is, in a certain sense, an "inverse" to the process of taking closed subsets in $\operatorname{Spec} R$. This construction is analogous to the definition of $\mathbb{I}$ in classical affine algebraic geometry, where one assigns to a subset of affine space the ideal of all polynomials vanishing on it. This perspective will allow us to formulate and prove an analogue of the algebra-geometry correspondence in the more general setting of spectrum of a ring (or an affine scheme).

Definition 5.17. Let R be a ring. Given a subset $S \subseteq \operatorname{Spec} R$, define

$$I(S) = \{f \in R \mid f \in \mathfrak{p} \text{ for all } \mathfrak{p} \in S\} = \bigcap_{\mathfrak{p} \in S} \mathfrak{p} \subseteq R$$

Example 5.18. Let $R = \operatorname{Spec} \mathbb{C}[z_1, \cdots, z_n]$. Then

$$I(S) = \bigcap_{(z-a) \in S} \{f \in \mathbb{C}[z_1, \cdots, z_n] \mid f(a) = 0\}$$

Identifying $S \subseteq \operatorname{Spec} \mathbb{C}[z_1, \cdots, z_n]$ with a subset of $\mathbb{C}^n$, $I(S)$ is the set of polynomials in $\mathbb{C}[z_1, \cdots, z_n]$ that vanish at all points of S. Hence, the operator $I(\cdot)$ generalizes the operator $\mathbb{I}(\cdot)$ discussed before.

Proposition 5.19. *The $I(\cdot)$ operation has the following properties:*

(1) $I(S)$ is an an ideal of R.
(2) $I(\cdot)$ is inclusion-reversing: if $S_1 \subseteq S_2$, then $I(S_2) \subseteq I(S_1)$.
(3) $I(S_1 \cup S_2) = I(S_1) \cap I(S_2)$
(4) $V(I(S)) = \overline{S}$.
(5) $I(V(J)) = \sqrt{J}$ for an ideal J in R.
(6) (**Nullstellensatz**) We have the following bijection:

$$\begin{aligned} \{\text{Closed sets in Spec} R\} &\longleftrightarrow \{\text{Radical ideals in } R\} \\ S &\mapsto I(S) \\ V(\mathfrak{a}) &\leftarrow\!\shortmid \mathfrak{a} \end{aligned}$$

Proof. (1), (2) and (3) are clear. For (4), note that $S \subseteq V(I(S))$, because if $\mathfrak{p} \in S$, then clearly

$$I(S) = \bigcap_{\mathfrak{p} \in S} \mathfrak{p} \subseteq \mathfrak{p},$$

and so $\mathfrak{p} \in V(I(S))$. Conversely, if $V(\mathfrak{a})$ is any closed subset containing S, then this means that for any $\mathfrak{b} \in S$, we have $\mathfrak{b} \in V(\mathfrak{a})$ and hence $\mathfrak{a} \subseteq \mathfrak{b}$. Taking the intersection over all $\mathfrak{b}$ in S, we see that $I(S) \supseteq \mathfrak{a}$. Since I reverses inclusions we get that $V(I(S)) \subseteq V(\mathfrak{a})$. This shows that $V(I(S))$ is the smallest closed set which contains S. (5) follows because

$$I\left(V(\mathfrak{a})\right) := \bigcap_{\mathfrak{p} \in V(\mathfrak{a})} \mathfrak{p} = \bigcap_{\mathfrak{a} \subseteq \mathfrak{p}} \mathfrak{p} = \sqrt{\mathfrak{a}}$$

(6) follows from (4) and (5). □

Proposition 5.19 is the analog of the classical algebra-geometry correspondence (Proposition 3.6). While the classical correspondence connects radical ideals in a polynomial ring with the geometry of their zero sets in affine space, Proposition 5.19 provides a similar bridge in a more general setting.

6. Properties of Zariski Topology

We now discuss various topological properties of the Zariski topology on the spectrum of a ring. These properties include closed points, compactness, characterization of irreducible closed subsets. Understanding these features is essential for grasping the geometric intuition behind the algebraic structure of the spectrum.

6.1. Closed Points.

Note that Proposition 5.19(5) characterizes closed sets in the Zariski topology. In particular, we can characterize closed *points* in the Zariski topology as well.

Proposition 6.1. *Let R be a ring and $\mathfrak{p}$ be a prime ideal. We have $\overline{\{\mathfrak{p}\}} = V(\mathfrak{p})$. In particular, closed points of Spec R correspond to the maximal ideals of R.*

Proof. Observe that $V(E)$ is a closed set that contains the point $\mathfrak{p}$ if and only if $E \subseteq \mathfrak{p}$. Hence, we have

$$\overline{\{\mathfrak{p}\}} = \bigcap_{E \subseteq \mathfrak{p}} V(E) = V\left(\bigcup_{E \subseteq \mathfrak{p}} E\right) = V(\mathfrak{p}).$$

Hence, $\{\mathfrak{p}\}$ is closed if and only if

$$\{\mathfrak{p}\} = \overline{\{\mathfrak{p}\}} = V(\mathfrak{p})$$

if and only if there doesn't exist any prime ideal $\mathfrak{q}$ of $\operatorname{Spec} R$ properly containing $\mathfrak{p}$ if and only if $\mathfrak{p}$ is a maximal ideal. $\square$

We have a bijection:

$$\{\text{Closed Points in Spec } R\} \longleftrightarrow \{\text{Maximal Ideals of } R\}$$

Proposition 6.1 illustrates that the Zariski topology on $\operatorname{Spec} R$ behaves quite differently from the Euclidean topology on manifolds. In particular, $\operatorname{Spec} R$ is not Hausdorff if R contains a prime ideal that is not a maximal ideal. Consider the following example:

Example 6.2. Let R be an integral domain. Then (0) is a prime ideal that is not a maximal ideal. Hence, (0) is not a closed set since $\overline{(0)} = R$. In particular, $\operatorname{Spec} R$ is not Hausdorff.

Example 6.2 motivates the concept of a generic point.

Definition 6.3. Let R be a ring, A point $\mathfrak{p} \in \operatorname{Spec} R$ is called a generic point if $\overline{\{\mathfrak{p}\}} = \operatorname{Spec} R$.

We conclude this section with a proof that $\operatorname{Spec} R$ is a T_0 space. Recall that a topological space, X, is T_0 if $x, y \in X$ are distinct points, then either there exists a neighborhood of x that does not contain y, or there exists a neighborhood of y that does not contain x.

Proposition 6.4. *Let R be a ring.* $\operatorname{Spec} R$ *is a T_0 space.*

Proof. Let $\mathfrak{p}, \mathfrak{q} \in \operatorname{Spec} R$ be distinct points. Then we have $\mathfrak{p} \neq \mathfrak{q}$ as ideals. WLOG assume that $\mathfrak{q} \not\subseteq \mathfrak{p}$. Then there exists an element $\alpha \in \mathfrak{q}$ such that $\alpha \notin \mathfrak{p}$. The distinguished open set U_α is then an open neighborhood of $\mathfrak{p}$ that does not contain $\mathfrak{q}$. $\square$

6.2. **Irreducible Closed Sets.** We have characterized the closed subsets of the spectrum of a ring, R. We now turn to characterizing the irreducible closed subsets of $\operatorname{Spec} R$. In analogy with the classical algebra-geometry correspondence, these irreducible closed subsets correspond precisely to the prime ideals of R.

Proposition 6.5. *Let R be a ring. A closed subset $Z \subseteq \operatorname{Spec} R$ is irreducible if and only if Z is of the form $Z = V(\mathfrak{p})$ for some prime ideal $\mathfrak{p} \in \operatorname{Spec} R$.*

Proof. First assume that $Z = V(\mathfrak{p})$ for some prime ideal $\mathfrak{p}$. By Proposition 5.19(5), we have

$$Z = V(\mathfrak{p}) = V(I(\mathfrak{p})) = \overline{\{\mathfrak{p}\}}$$

If Z is reducible, then

$$\mathfrak{p} \in \overline{\{\mathfrak{p}\}} \subseteq V(\mathfrak{a}) \cup V(\mathfrak{b})$$

Hence, $\mathfrak{p} \in V(\mathfrak{a})$ or $\mathfrak{p} \in V(\mathfrak{b})$ and since $V(\mathfrak{a}), V(\mathfrak{b})$ are closed sets we have $Z = \overline{\{\mathfrak{p}\}} \in V(\mathfrak{a})$ or $V(\mathfrak{b})$, contradicting that Z is reducible. Hence, Z is irreducible. Conversely, assume that Z is a closed irreducible set. Let

$$Z = V(\mathfrak{a}) = V(\sqrt{\mathfrak{a}})$$

for some ideal $\mathfrak{a} \in \operatorname{Spec} R$. It suffices to show that $\sqrt{\mathfrak{a}}$ is a prime ideal. Assume this is not the case. Then there exist $b, c \in R$ such that $bc \in \sqrt{\mathfrak{a}}$ but $b, c \notin \sqrt{\mathfrak{a}}$. If $\mathfrak{b} = (b)$ and $\mathfrak{c} = (c)$, then $\mathfrak{bc} \in \sqrt{\mathfrak{a}}$ but $\sqrt{\mathfrak{b}}, \sqrt{\mathfrak{c}} \notin \sqrt{\mathfrak{a}}$. We claim that $V(\sqrt{\mathfrak{a}}) \subseteq V(\mathfrak{b}) \cup V(\mathfrak{c})$. Note that

$$V(\sqrt{\mathfrak{a}}) \subseteq V(\mathfrak{b}) \cup V(\mathfrak{c}) \iff \sqrt{\mathfrak{a}} = I(V(\sqrt{\mathfrak{a}})) \supseteq I(V(\mathfrak{b}) \cup V(\mathfrak{c})) = I(V(\mathfrak{b})) \cap I(V(\mathfrak{c})) = \sqrt{\mathfrak{b}} \cap \sqrt{\mathfrak{c}}$$

Since $\mathfrak{b}$ and $\mathfrak{c}$ are principal ideals, we have

$$\sqrt{\mathfrak{a}} \supseteq \sqrt{\mathfrak{b}} \cap \sqrt{\mathfrak{c}} \iff \sqrt{\mathfrak{a}} \supseteq \mathfrak{b} \cap \mathfrak{c}$$

The latter condition is clearly true. Hence, Z is reducible, a contradiction. □

We have a bijection:

$$\{\text{Closed irreducible Subsets of Spec } R\} \longleftrightarrow \{\text{Prime Ideals of } R\}$$

Corollary 6.6. *Let R be a ring. $\operatorname{Spec} R$ is irreducible if and only if the nilradical of R is a prime ideal.*

Proof. This follows from Proposition 6.5 and the fact that $\operatorname{Spec} R = V(0)$ and $\sqrt{0}$ is the nilradical of R. □

Example 6.7. Let R be an integral domain and let $ab \in \mathscr{N}(R)$. By definition of $\mathscr{N}(R)$, we have $a^n b^n = (ab)^n = 0$ for some $n \in \mathbb{N}$. Since R is an integral domain, $a^n = 0$ or $b^n = 0$ if and only if a or b is contained in $\mathscr{N}(R)$ if and only if $\mathscr{N}(R)$ is a prime ideal. Hence, $\operatorname{Spec} R$ is irreducible.

An irreducible component of a topological space is a maximal irreducible subset. Recall that any topological space can be written as a union of its irreducible components. We can also characterize irrecucible components of the spectrum of a ring.

Proposition 6.8. *Let R be a ring. The irreducible components of $\operatorname{Spec} R$ are the closed sets $V(\mathfrak{p})$, where $\mathfrak{p}$ is a minimal prime ideal of R.*

Proof. A maximal irreducible subspace of $\operatorname{Spec} A$ must be closed, that is of the form $V(\mathfrak{a})$ for some ideal a of A. Since it is irreducible, we can assume the ideal to be prime, so it is $V(\mathfrak{p})$. Now $\mathfrak{p}$ is minimal because $V(\mathfrak{p})$ is maximal. □

6.3. **Compactness & Noetherian-ness.** We first show that the spectrum of a ring is always a compact topological space. This is a very convenient property to have. However, a general scheme need not be compact, as we shall see later.

Proposition 6.9. *Let R be a ring. Then* $\operatorname{Spec} R$ *is compact. More generally, U_f is compact for every $f \in R$.*

Proof. Assume that

$$U_1 = \operatorname{Spec} R = \bigcup_{i \in I} U_{h_i}$$

is a union of basic open sets. This is true if and only if

$$V(1) = \bigcap_{i \in I} V(h_i) = V\left(\sum_{i \in I} h_i\right).$$

By Lemma 5.10, this implies $1 \in \sqrt{\sum_i (h_i)}$, or $1 \in \sum_i (h_i)$ for some n. This means that 1 can be expressed as a finite sum $1 = \sum_i b_i h_i$, $b_i \in R$. Hence a finite subset of the h_i's will do. □

Example 6.10. $\operatorname{Spec} R$ can have non-empty non-compact open sets. For instance, take $R = \mathbb{K}[x_1, x_2, \cdots]$. Then

$$\bigcup_{n \geq 1} D(x_n) \subseteq \operatorname{Spec}(\mathbb{K}[x_1, x_2, \dots])$$

is open but not quasi-compact. A similar argument shows that U_f is compact for every $f \in R$.

We now discuss an additional topological property of the spectrum of a ring. Recall that Noetherian rings form a special class of rings satisfying certain finiteness conditions. It is natural to define an analogous notion of Noetherian-ness at the level of topological spaces, yielding topological spaces that satisfy certain finiteness properties.

Definition 6.11. A topological space X is called **Noetherian** if it satisfies the descending chain condition for closed subsets. Any sequence

$$Z_1 \supseteq Z_2 \supseteq \cdots \supseteq Z_n \supseteq \cdots$$

of closed subsets eventually stabilizes. That is, there is a $r \in \mathbb{N}$ such that

$$Z_r = Z_{r+1} = \cdots .$$

Example 6.12. $\mathbb{A}^n$ is a Noetherian topological space. Indeed, if

$$Y_1 \supseteq Y_2 \supseteq \dots$$

is a descending chain of closed subsets, then

$$\mathbb{I}(Y_1) \subseteq \mathbb{I}(Y_2) \subseteq \dots$$

is an ascending chain of ideals in $R = \mathbb{K}[x_1, \dots, x_n]$. Since R is a Noetherian ring, this chain of ideals is eventually stationary. But for each i, $Y_i = \mathbb{V}(\mathbb{I}(Y_i))$, so the chain Y_i is also stationary.

Noetherian topological spaces possess desirable properties, making them more tractable due to the finiteness conditions imposed by the Noetherian property. We illustrate this with the following example result:

Proposition 6.13. *Suppose X is a Noetherian topological space. Then every nonempty closed subset Z can be expressed uniquely as a finite union $Z = Z_1 \cup \cdots \cup Z_n$ of irreducible closed subsets, none contained in any other.*

Proof. Consider the collection, Σ, of closed subsets of X that cannot be expressed as a finite union of irreducible closed subsets. We will show that it is empty. Assume that $\Sigma \neq \emptyset$. Since X is Noetherian, there is a minimal element in Σ. Call it Y. By construction, Y is not irreducible. So we can write $Y = Y' \cup Y''$ where Y' and Y'' are both proper closed subsets of Y. Both of these by hypothesis can be written as the union of a finite number of irreducible subsets, and hence so can Y, yielding a contradiction. Thus each closed subset can be written as a finite union of irreducible closed subsets. We can assume that none of these irreducible closed subsets contain any others, by discarding some of them. We now show uniqueness. Suppose

$$Z = Z_1 \cup Z_2 \cup \cdots \cup Z_r = Z_1' \cup Z_2' \cup \cdots \cup Z_s'$$

are two such representations. Then $Z_1' \subset Z_1 \cup Z_2 \cup \cdots \cup Z_r$, so $Z_1' = (Z_1 \cap Z_1') \cup \cdots \cup (Z_r \cap Z_1')$. Now Z_1' is irreducible, so one of these is Z_1' itself, say (without loss of generality) $Z_1 \cap Z_1'$. Thus $Z_1' \subseteq Z_1$. Similarly, $Z_1 \subseteq Z_j'$ for some j; but because $Z_1' \subseteq Z_1 \subseteq Z_j'$, and Z_1' is contained in no other Z_i', we must have $j = 1$, and $Z_1' = Z_1$. Thus each element of the list of Z's is in the list of Z''s, and vice versa, so they must be the same list. □

Example 6.14 (Hartshorne I.1.3)**.** Let $\mathbb{K}$ be a field, and let Y be the algebraic set in $\mathbb{A}^3$ defined by the two polynomials $x^2 - yz$ and $xz - x$. We have the following equalities of ideals in $\mathbb{K}[x, y, z]$:

$$\begin{aligned}(x^2 - yz,\ xz - x) &= (x^2 - yz,\ x) \cap (x^2 - yz,\ z - 1)\\ &= (x,\ yz) \cap (x^2 - y,\ z - 1)\\ &= (x,\ y) \cap (x,\ z) \cap (x^2 - y,\ z - 1).\end{aligned}$$

Therefore, Y is the union of three irreducible components: two of them are lines, and the third is a plane curve.

Proposition 6.15. *(*Hartshorne I.1.7) *The following statements are true:*

(1) *The following conditions are equivalent for a topological space X:*
   (i) *X is Noetherian.*
   (ii) *Every nonempty family of closed subsets has a minimal element.*
   (iii) *X satisfies the ascending chain condition for open subsets.*
   (iv) *Every nonempty family of open subsets has a maximal element.*
(2) *A Noetherian topological space is compact.*
(3) *Any subset of a Noetherian topological space is Noetherian in its induced topology.*
(4) *A Noetherian space which is also Hausdorff must be a finite set with the discrete topology.*

Proof. The proof has been commented out as it follows from standard results in point-set topology. □

We conclude by showing that if R is a Noetherian ring, then $\operatorname{Spec} R$ is a Noetherian topological space.

Proposition 6.16. *Let R be a Noetherian ring. Then* $\operatorname{Spec} R$ *is a Noetherian topological space.*

Proof. Suppose

$$V(I_1) \supseteq V(I_2) \supseteq \dots$$

is a descending sequence of closed subsets of $\operatorname{Spec} R$. Using Lemma 5.10, we have that,

$$\sqrt{I_1} \subseteq \sqrt{I_2} \subseteq \dots,$$

is an ascending sequence of ideals in R. Since R is a Noetherian ring, this sequence stabilizes. That is, there exist a r such that $\sqrt{I_r} = \sqrt{I_{r+1}} = \cdots$. Since $V(I) = V(\sqrt{I})$ for any ideal I in R, we have that the descending sequence of closed subsets also stabilizers. Hence $\operatorname{Spec}(R)$ is Noetherian. □

Example 6.17. Let R be a Notherian ring. Then $R[x_1]$ is a Noetherian ring. This is the celebrated the Hilbert basis theorem (**??**). Therefore, $R[x_1, \cdots, x_n]$ is also a Noetherian ring. Hence, $\operatorname{Spec} R[x_1, \cdots, x_n]$ is a Noetherian topological space. In particular, $\operatorname{Spec} \mathbb{K}[x_1, \cdots, x_n]$ is a Noetherian topological space for any field $\mathbb{K}$.

Remark 6.18. *If* $\operatorname{Spec} R$ *is a Noetherian topological space, R need not be Noetherian. One example is*

$$R = \mathbb{K}[x_1, x_2, x_3, \dots]/(x_1, x_2^2, x_3^3, \dots)$$

Then $\operatorname{Spec} R$ *has one point, so it is a Noetherian topological space. But T is not a Noetherian ring.*

6.4. **Conectedness.** We can now characterize when the spectrum of a ring is disconnected. To do so, we invoke properties of the structure sheaf that is naturally endowed on the spectrum of a ring (see Proposition 11.8). The structure sheaf itself will be introduced in detail later, when we discuss sheaves in Part 2.

Proposition 6.19. *Let R be a ring.* $\operatorname{Spec} R$ *is disconnected if and only if R is a direct product of rings. More generally, the following are equivalent:*

(a) $\operatorname{Spec} R$ *is disconnected.*
(b) *There exist nonzero elements $e_1, e_2 \in R$ such that $e_1 e_2 = 0$, $e_1^2 = e_1$, $e_2^2 = e_2$, and $e_1 + e_2 = 1$ (these elements are called orthogonal idempotents).*
(c) *R is isomorphic to a direct product $R_1 \times R_2$ of two nonzero rings.*

Proof. The proof is given below:

(1) (a) implies (c): Assume $\operatorname{Spec} R$ is disconnected. Then $\operatorname{Spec} R$ can be written as a union of two disjoint clopen subsets. Let these subsets be $V(\mathfrak{a})$ and $V(\mathfrak{b})$ for some ideals $\mathfrak{a}, \mathfrak{b} \subset R$. Noting that $V(\mathfrak{a}) = \operatorname{Spec}(R/\mathfrak{a})$ and $V(\mathfrak{b}) = \operatorname{Spec}(R/\mathfrak{b})$, we have:

$$\operatorname{Spec} R = V(\mathfrak{a}) \sqcup V(\mathfrak{b}) = \operatorname{Spec}(R/\mathfrak{a}) \sqcup \operatorname{Spec}(R/\mathfrak{b}) \cong \operatorname{Spec}(R/(\mathfrak{a} \times \mathfrak{b})). \tag{*}$$

Since we can recover the ring by taking global sections of the structure sheaf[5], we conclude that

$$R \cong R/\mathfrak{a} \times R/\mathfrak{b}.$$

(2) (c) implies (b): Simply take $e_1 = (1, 0)$ and $e_2 = (0, 1)$. It is straightforward to verify that the desired properties hold.

[5] Structure sheaf will be discussed in the next section.

(3) (b) implies (a): Since $e_1e_2 = 0$, for every prime ideal $\mathfrak{p} \subset R$, we must have either $e_1 \in \mathfrak{p}$ or $e_2 \in \mathfrak{p}$. Hence, the closed subsets $V(e_1)$ and $V(e_2)$ cover $\operatorname{Spec} R$. Suppose a prime ideal $\mathfrak{p}$ lies in both $V(e_1)$ and $V(e_2)$. Then $e_1, e_2 \in \mathfrak{p}$, and so $1 = e_1 + e_2 \in \mathfrak{p}$, implying $\mathfrak{p} = R$, which contradicts the definition of a prime ideal. Therefore, $V(e_1) \cap V(e_2) = \varnothing$, and the cover $\operatorname{Spec} R = V(e_1) \cup V(e_2)$ is by two disjoint closed subsets. It follows that $\operatorname{Spec} R$ is disconnected.

This completes the proof. $\square$

Remark 6.20. *The last equality in Equation (*) follows from the fact that the prime ideals of the product ring $R_1 \times R_2$ are precisely those of the form $\mathfrak{p}_1 \times R_2$ and $R_1 \times \mathfrak{p}_2$, where $\mathfrak{p}_i$ is a prime ideal of R_i for $i = 1, 2$.*

7. Examples

The purpose of this section is to present a collection of examples of spectra of various rings. These examples serve to illustrate the diversity of behaviors that can arise and will be revisited throughout the text as more theory is developed. We begin by examining examples of spectra of rings that recover classical affine algebraic geometry as special cases.

Example 7.1. Let $\mathbb{K}$ be an algebraically closed field.

(1) Let's describe $\operatorname{Spec} \mathbb{K}[x]$, which is called the affine line over $\mathbb{K}$. Since $\mathbb{K}$ is algebraically closed and $\mathbb{K}[x]$ is a PID, we have

$$\operatorname{Spec} \mathbb{K}[x] = \{(x-a) \mid a \in \mathbb{K}\} \cup \{0\}$$

We can identify each $a \in \mathbb{C}$ with the prime ideal $(x - a)$. Thus, the non-zero prime ideals of $\operatorname{Spec} \mathbb{K}[x]$ correspond bijectively to the points of $\mathbb{K}$.

(2) Consider the following spectrum of a ring:

$$\operatorname{Spec} \mathbb{K}[x, y].$$

The ring $\mathbb{K}[x, y]$ is not a principal ideal domain, since the ideal (x, y) is not principal. Nevertheless, one can show that every prime ideal of $\mathbb{K}[x, y]$ is of one of the following forms:

$$(0), \quad (x-a, y-b), \quad (f),$$

where $a, b \in \mathbb{K}$, and $f \in \mathbb{K}[x, y]$ is an irreducible polynomial.

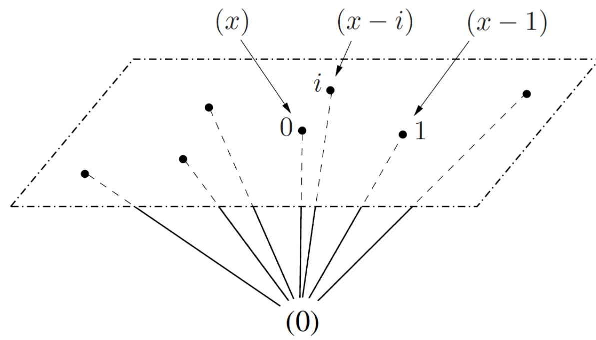
Remark 7.2. *Note that the zero ideal* (0) *in* $\mathbb{K}[x]$ *is contained in every non-zero prime (in this case, maximal) ideal.* (0) *is called the generic point of* $\mathbb{K}[x]$*. Generic point of* $\operatorname{Spec} R$ *will formally defined later.*

We can also examine the spectra of fields that are not algebraically closed. This provides one of the first indications that the language of schemes is well-suited for studying algebraic geometry over arbitrary fields, not just algebraically closed ones.

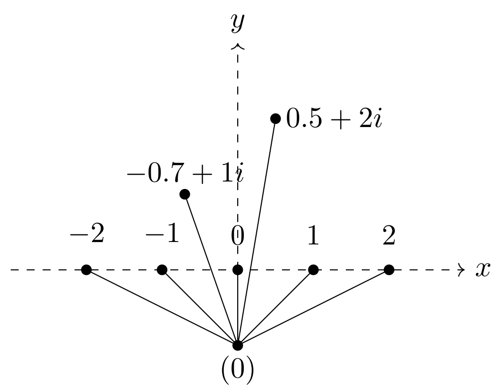
Example 7.3. Let $R = \operatorname{Spec} \mathbb{R}[x]$. We have

$$\operatorname{Spec}(\mathbb{R}[x]) = \{(0)\} \cup \{(x-a) \mid a \in \mathbb{R}\} \cup \{p(x)) \mid p(x) \text{ is an irreducible quadratic}\}$$

Note that (0) and $(x - a)$ for $a \in \mathbb{R}$ are all the maximal ideals of $\mathbb{R}[x]$. Maximal ideals of $\mathbb{R}[x]$ correspond to $\mathbb{R}$. Each $(p(x))$ can be identified with a complex number with positive imaginary part that is the root of $p(x)$.



A picture of $\operatorname{Spec} \mathbb{C}[z]$. This picture is taken from [Vak17].



A picture of $\operatorname{Spec} \mathbb{R}[x]$

One might ask: why do we identify an irreducible quadratic $f(x) \in \mathbb{R}[x]$ with its complex root having positive imaginary part, rather than the conjugate root with negative imaginary part? This choice is a matter of convention, not mathematical necessity. The two complex conjugate roots correspond to the same prime ideal $(f(x))$ in $\operatorname{Spec} \mathbb{R}[x]$, since $f(x)$ is irreducible over $\mathbb{R}$. Therefore, selecting the root with positive imaginary part is simply a canonical way to represent the conjugate pair.

Remark 7.4. *Let $\mathbb{K}$ be a perfect field. Consider $\operatorname{Spec} \mathbb{K}[x]$. Since $\mathbb{K}$ is a field, $\mathbb{K}[x]$ is a PID. Therefore,*

$$\operatorname{Spec} \mathbb{K}[x] = \{f(x)) \mid p(x) \text{ is an irreducible polynomial}\}$$

Since $\mathbb{K}$ is perfect, f has distinct roots in $\overline{\mathbb{K}}$[6] *and these roots $r_1, \cdots, r_n$ form an orbit of $\overline{\mathbb{K}}$ under the action of the Galois group, G, of the field extension $\overline{\mathbb{K}}/\mathbb{K}$. Indeed, if any non-trivial subset of $S \subsetneq \{r_1, \cdots, r_n\}$ is G-invariant, then*

$$\prod_{i \in S} (x - r_i)$$

[6] Because it is co-prime to its derivative.

would be an element of $\mathbb{K}[x]$ *dividing* f[7]*, contradicting that* f *is irreducible. Conversely, given any* G*-invariant finite subset* $S \subsetneq \overline{\mathbb{K}}$ *which has no non-trivial* G*-invariant subsets, the polynomial*

$$\prod_{s \in S}(x - s)$$

is in $\mathbb{K}$ *and irreducible by the same logic. So we have a bijection between* $\overline{\mathbb{K}}/G$ *and the non-zero prime ideals of* $\mathbb{K}[x]$*.*

Let us now discuss the spectrum of quotient of a ring. If $I \subseteq R$ is an ideal, recall that there is a bijection:

$$\{\text{Prime ideals of } R/I\} \longleftrightarrow \{\text{Prime ideals of } R \,\text{containing}\, I\}$$

Thus we can picture $\operatorname{Spec} R/I$ as a subset of $\operatorname{Spec} R$. In fact, $\operatorname{Spec} R/I \cong V(I)$.

Example 7.5. Consider the following examples:

(1) Consider $\mathbb{K}[x]/(x^2)$. An elementary argument shows that

$$\operatorname{Spec} \mathbb{K}[x]/(x^2) = \{(x)\}$$

(2) Consider $\mathbb{K}[x]/(x(x-1))$. Using the Chinese Remainder Theorem:

$$\mathbb{K}[x]/(x(x-1)) \cong \mathbb{K}[x]/(x) \times \mathbb{K}[x]/(x-1) \cong \mathbb{K} \times \mathbb{K}$$

Therefore, the spectrum of the ring is:

$$\operatorname{Spec} \mathbb{K}[x]/(x(x-1)) = \{(0) \times \mathbb{K},\ \mathbb{K} \times (0)\}$$

Remark 7.6. *The classical algebraic geometry correspondence implies that if* $\mathbb{K}$ *is an algebraically closed field , then points in* $\operatorname{Spec}(\mathbb{K}/I)$ *correspond to the points on* $\mathbb{V}(I)$*, as well as to the irreducible algebraic subsets of* $\mathbb{V}(I)$*. For example, points of* $\operatorname{Spec} \mathbb{C}[x,y]/(xy)$ *correspond to the points on the coordinate axes and the irreducible algebraic subsets of their union. One should use this intuition in the general case for a arbitrary commutative ring* R*.*

Let's now discuss the spectrum of a localized ring. Consider $\operatorname{Spec} S^{-1}R$, where S is a multiplicatively closed subset of R containing 1. Recall that there is a bijection between $\operatorname{Spec} S^{-1}R$ and the set of prime ideals $\mathfrak{p} \subseteq R$ such that $\mathfrak{p} \cap S = \emptyset$.

$$\{\text{Prime ideals of } S^{-1}R\} \longleftrightarrow \{\text{Prime ideals of } R \,\text{that don't intersect with}\, S\}$$

Example 7.7. Let R be a ring. Consider the following examples:

(1) Let $S_f = \{1, f, f^2, \ldots\}$. The prime ideals of $R_f := S_f^{-1}R$ correspond to the prime ideals of R that do not contain f. Hence,

$$\operatorname{Spec} R_f = \{\text{Prime ideals of } R \text{ that don't contain } f\}$$

(2) Let $S_{\mathfrak{p}} = \mathfrak{p}^c$, where $\mathfrak{p}$ is a prime ideal. The prime ideals of $R_{\mathfrak{p}} := S_{\mathfrak{p}}^{-1}R$ are precisely the prime ideals of R that are contained in $\mathfrak{p}$. Hence,

$$\operatorname{Spec} R_{\mathfrak{p}} = \{\text{Prime ideals of } R \text{ contained in } \mathfrak{p}\}$$

[7] This follows from Galois theory.

Remark 7.8. *If $R = \mathbb{C}[x_1, \ldots, x_n]$, the classical algebra-geometry correspondence implies that we can picture* $\operatorname{Spec} S_f^{-1}\mathbb{C}[x_1, \ldots, x_n]$ *as the set of all points in $\mathbb{C}^n$ that do not lie on the zero set of f, along with irreducible affine algebraic sets not contained in the zero set of f.*

Local rings play a fundamental role in algebraic geometry, as they capture the behavior of schemes at a single point. The spectrum of a local ring provides a simple but illustrative example of a scheme with a distinguished closed point (corresponding to the maximal ideal) and a *generi*c point. These examples are crucial for understanding local properties of schemes.

Example 7.9. Consider the localization of the polynomial ring $\mathbb{K}[x]$ at the prime ideal (x), denoted by $\mathbb{K}[x]_{(x)}$. Its spectrum, $\operatorname{Spec}(\mathbb{K}[x]_{(x)})$, consists of exactly two prime ideals:

$$\operatorname{Spec}(\mathbb{K}[x]_{(x)}) = \{(0), (x)\}$$

This provides an example of a scheme with a unique closed point (x) and a *generic point* (0).

Part 2. Basic Sheaf Theory

A *space*, such as a topological space or a smooth manifold, can often be studied through the algebra of functions defined on it. However, a generic space may admit few *globally defined* functions—for instance, consider a non-normal topological space or bounded holomorphic functions on $\mathbb{C}^n$. A more precise perspective is that the structure of a *space* can be understood by studying *locally defined* functions. This richer perspective is formalized using a mathematical object called a pre-sheaf. Pre-sheaves and sheaves are objects studied in sheaf theory. We will use sheaves to endow the spectrum of a ring with a collection of functions that generalize the polynomial and rational functions studied in classical affine algebraic geometry. More generally, sheaves can be used to define objects like vector bundles by specifying their spaces of sections over any open set and describing how those sections restrict to one another and glue together. Hence, sheaves are an important tool in algebraic geometry for keeping track of locally defined geometric data. By associating data—such as functions, sections, etc.—to open subsets of a space and ensuring compatibility across overlaps, sheaves enable a cohesive transition from local to global phenomena.

8. Definitions

Before we define sheaves, we first want to introduce the notion of pre-sheaves, which is simpler and yet very helpful in understanding sheaves. Philosophically, pre-sheaves provide a powerful framework for systematically organizing and locally defined data. More precisely, given a topological space, X, the idea of a pre-sheaf is to associate each open set in X with an object in a category, C in such a way that we can establish a map from a bigger open set to a smaller open set inside it. More formally, we have the following definition:

Definition 8.1. Let X be a topological space and let C be a category. Let $\mathsf{Open}(X)$ denote the category of open sets on X A C**-valued pre-sheaf** on X is a contravariant functor:

$$\mathscr{F} : \mathsf{Open}(X) \to \mathsf{C}$$

Remark 8.2. *If $\mathscr{F}$ is a pre-sheaf on X, we refer to $\mathscr{F}(U)$ as the sections of the pre-sheaf $\mathscr{F}$ over the open set U. We sometimes use the notation $\Gamma(U, \mathscr{F})$ to denote $\mathscr{F}(U)$. If $U \subseteq V$, we write $\rho_{V,U}$, $res_{V,U}$ or $|_U$ for the morphism between $\mathscr{F}(V)$ and $\mathscr{F}(U)$.*

Given a pre-sheaf on X, a natural question to ask is the extent to which its sections over an open set $U \subseteq X$ are determined by their restrictions to the open subsets of U. A sheaf is roughly speaking a pre-sheaf where the aforementioned question can be answered affirmatively.

Definition 8.3. Let X be a topological space and let C be a category admitting all limits[8]. A C**-valued sheaf** on X is a pre-sheaf if the following diagram is an equalizer for every open cover $\mathscr{U} = \{U_i\}_{i \in I}$ of any open set U:

$$\mathscr{F}(U) \longrightarrow \prod_i \mathscr{F}(U_i) \rightrightarrows \prod_{i,j} \mathscr{F}(U_i \cap U_j)$$

The first map in Definition 8.3 is the product of the restriction maps

$$\mathrm{res}_{U,U_i} : \mathscr{F}(U) \to \mathscr{F}(U_i),$$

[8]It would be sufficient to require that C admits all products and equalizers. However, this assumption implies that C admits all limits.

and the pair of arrows are the products of the two sets of restrictions:

$$\begin{aligned}\mathrm{res}_{U_i,U_i\cap U_j} &: \mathscr{F}(U_i) \to \mathscr{F}(U_i \cap U_j),\\ \mathrm{res}_{U_j,U_i\cap U_j} &: \mathscr{F}(U_j) \to \mathscr{F}(U_i \cap U_j).\end{aligned}$$

If $\mathsf{C} = \mathsf{Sets}, \mathsf{Ab}, R\text{-}\mathsf{Mod}$, the condition in the definition of a sheaf simplify to the conditions:

1. (**Identity Axiom**) If $\{U_i\}_{i\in I}$ is an open cover of U, and $f_1, f_2 \in \mathscr{F}(U)$, and $f_1|_{U_i} = f_2|_{U_i}$ for all i, then $f_1 = f_2$.
2. (**Gluing Axiom)** Suppose $\{U_i\}_{i\in I}$ is an open cover of U. Suppose for each i we have $f_i \in \mathscr{F}(U_i)$ such that $f_i = f_j$ in $\mathscr{F}(U_i \cap U_j)$. Then there is a unique $f \in \mathscr{F}(U)$ such that $f|_{U_i} = f_i$.

We can package the above argument in the form an equalizer diagram to reconstruct Definition 8.3. For each open cover $U = \bigcup_{i\in I} U_i$ of an open set $U \subseteq X$, there is a sequence

$$0 \longrightarrow \mathscr{F}(U) \xrightarrow{\alpha} \prod_{i\in I} \mathscr{F}(U_i) \xrightarrow{\beta} \prod_{i,j\in I} \mathscr{F}(U_i \cap U_j),$$

where the maps α and β are defined by the assignments

$$\begin{aligned}\alpha(s) &= (s|_{U_i})_{i\in I}\,,\\ \beta\left((s_i)_{i\in I}\right) &= \left(s_i|_{U_i\cap U_j} - s_j|_{U_i\cap U_j}\right)_{i,j\in I}.\end{aligned}$$

Then $\mathscr{F}$ is a sheaf if and only if these sequences are exact. Indeed, exactness at $\mathscr{F}(U)$ means that α is injective, i.e., that $s|_{U_i} = 0$ for all $i \in I$ implies $s = 0$. Exactness in the middle means that $\ker\beta = \operatorname{Im}\alpha$; that is, elements $(s_i) \in \prod \mathscr{F}(U_i)$ satisfying

$$s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j} \quad \text{for all } i, j \in I$$

come from a global section $s \in \mathscr{F}(U)$ such that $s|_{U_i} = s_i$.

Remark 8.4. *We'll mostly be working with* $\mathsf{Ab}, R\text{-}\mathsf{Mod}$*,* CRing.

In some cases, it is assumed *a priori* that $\mathcal{F}(\emptyset)$ is a terminal object in C. For instance, if $\mathsf{C} = \mathsf{Ab}$, it is often assumed *a priori* that $\mathcal{F}(\emptyset) = 0$, where 0 denotes the trivial abelian group. However, we show that this is actually a consequence of the definition of a sheaf.

Lemma 8.5. *Let X be a topological space and let C be a category with all limits and a terminal object, T. If $\mathscr{F} : \mathsf{Open}(X) \to \mathsf{C}$ is a sheaf, then $\mathscr{F}(\emptyset) \cong T$.*

PROOF. Let $U = \emptyset$. Since the product over objects in C indexed over $\emptyset$ is a terminal object, the equalizer condition becomes

$$\begin{tikzcd} \mathscr{F}(\emptyset) \arrow[r, "g"] & T \arrow[r, shift left, "\mathrm{Id}_T"] \arrow[r, shift right, "\mathrm{Id}_T"'] & T \end{tikzcd}$$

The morphism $g : \mathscr{F}(\emptyset) \to T$ is unique since T is a terminal object. Let $X \in \mathsf{C}$ and let $f : X \to T$ be a unique morphism from X to T. The equalizer condition states that there exists a unique morphism $f' : X \to \mathscr{F}(\emptyset)$ such that the following diagram commutes:

$$\begin{tikzcd} \mathscr{F}(\emptyset) \arrow[r, "g"] & T \arrow[r, shift left, "\mathrm{Id}_T"] \arrow[r, shift right, "\mathrm{Id}_T"'] & T \\ & X \arrow[ul, dashed, "f'"] \arrow[u, "f"] \arrow[ur, "f"'] & \end{tikzcd}$$

On the other hand, *any* morphism $f' : X \to \mathscr{F}(\emptyset)$ makes the left-most triangle commute. Therefore $\mathscr{F}(\emptyset)$ is an object such for any X there exists a unique morphism $X \to F(\emptyset)$. In other words, $F(\emptyset) \cong T$. $\square$

Category theory teaches us to always define morphisms between mathematical objects. We now define morphisms of pre-sheaves, and similarly for sheaves. In other words, we will describe the category of pre-sheaves and the category of sheaves

Definition 8.6. Let X be a topological space and $\mathscr{F}$ and $\mathscr{G}$ be C-valued pre-sheaves. A **morphism of pre-sheaves**, $\varphi : \mathscr{F} \to \mathscr{G}$, is a natural transformation. That is, for each open set $U \subseteq X$ there exists a morphism from $\mathscr{F}(U) \to \mathscr{G}(U)$ such that whenever $U \subseteq V$, the following diagram

$$\begin{tikzcd} \mathscr{F}(V) \arrow[r, "\varphi(V)"] \arrow[d, "\mathrm{res}_{V,U}"'] & \mathscr{G}(V) \arrow[d, "\mathrm{res}_{V,U}"] \\ \mathscr{F}(U) \arrow[r, "\varphi(U)"'] & \mathscr{G}(U) \end{tikzcd}$$

commutes.

Remark 8.7. *We denote by $\varphi(V) : \mathscr{F}(V) \to \mathscr{G}(V)$ the morphism on sections over an open set $V \subseteq X$, and we write its restriction to an open subset $U \subseteq V$ as*

$$\varphi(V)|_U : \mathscr{F}(V)|_U \to \mathscr{G}(V)|_U.$$

If C is a concrete category, and $s \in \mathscr{F}(V)$, the commutativity of the restriction diagram (written in componentwise notation) is expressed by the equation:

$$\varphi(s)|_U = \varphi_i(s|_U),$$

where $\varphi_i : \mathscr{F}|_{U_i} \to \mathscr{G}|_{U_i}$ is the given morphism on a piece of an open cover $\{U_i\}_{i\in I}$, and $U \subseteq U_i \cap V$.

Remark 8.8. *If $\mathscr{F}$ and $\mathscr{G}$ are sheaves of abelian groups, R-modules etc. then we require all the maps $\varphi(U)$ to be homomorphisms in the appropriate category.*

Definition 8.6 makes the collection all pre-sheaves on X into a category, which we denote as $\mathsf{PreShv}(X, \mathsf{C})$. The category of sheaves on X, which we denote as $\mathsf{Shv}(X, \mathsf{C})$, is then a full subcategory of the category of pre-sheaves on X satisfying the identity and gluing axioms.

Example 8.9. Note the following basic examples:

(1) Consider the simplest case where $X = \{*\}$ is a one-point space. Then the category of presheaves $\mathsf{PreShv}(X, \mathsf{C})$ is equivalent to C itself.
(2) There is a natural forgetful functor

$$\mathsf{Shv}(X, \mathsf{C}) \to \mathsf{PreShv}(X, \mathsf{C}),$$

reflecting that every sheaf is in particular a presheaf.

9. Examples

Sheaves offer a unified language to study and solve problems that involve patching local solutions into a global context. This makes them essential tools in diverse fields, including algebraic geometry, topology, and complex analysis. Let's look at some examples of sheaves.

Example 9.1. Let $\mathsf{C} = \mathsf{Sets}$ and let $X = \{*\}$ be a one-point topological space. Lemma 8.5 implies that a C-valued sheaf on X is defined by $\mathscr{F}(*) = S$ and $\mathscr{F}(\emptyset) = \{\mathrm{pt}\}$, where $S \in \mathsf{Sets}$.

Sheaves of functions form an important example of sheaves. For instance, the assignment that sends each open set $U \subseteq X$ to the ring of functions on U defines a sheaf. These examples are central in areas such as differential geometry and complex analysis.

Example 9.2. The following is a list of some examples of sheaves of functions:

(1) If X is a topological space, the pre-sheaf of continuous functions, $\mathscr{C}$, defined by $U \mapsto \mathscr{C}(U)$, where $\mathscr{C}(U)$ is the abelian group of continuous functions on U (with usual restrictions), is a sheaf.
(2) If X is a topological space, the pre-sheaf of nowhere vanishing continuous functions, $\mathscr{C}^\times$, defined by $U \mapsto \mathscr{C}^\times(U)$, where $\mathscr{C}^\times(U)$ is the abelian group of no-where vanishing continuous functions on U (with usual restrictions), is a sheaf.
(3) If $X = \mathbb{C}^n$, the pre-sheaf of holomorphic functions, $\mathscr{O}$, defined by $U \mapsto \mathscr{H}(U)$, where $\mathscr{H}(U)$ is the abelian group of holomorphic functions on U (with usual restrictions), is a sheaf.
(4) If $X = \mathbb{C}^n$, the pre-sheaf of nowhere vanishing holomorphic functions, $\mathscr{O}^\times$, defined by $U \mapsto \mathscr{H}^\times(U)$, where $\mathscr{H}^\times(U)$ is the abelian group of non-where vanishing holomorphic functions on U (with usual restrictions), is a sheaf.

Note that $\mathscr{F}(\emptyset) = 0$[9] is forced by Lemma 8.5 in all examples above.

Remark 9.3. *We can easily generalize Example 9.2 by considering the sheaf of functions restricted to open subsets of the appropriate space. Moreover, all the examples discussed above are, in fact, examples of R-module valued sheaves with $R = \mathbb{R}, \mathbb{C}$ as appropriate.*

Arguably the most important example of a sheaf of functions in algebraic geometry is the sheaf of regular functions on an affine variety. This is the sheaf of functions that can be written as rational functions—that is, quotients of polynomials—that are regular on their domain of definition.

Example 9.4. (Sheaf of Regular Functions) Let $\mathbb{K}$ be an algebraically closed field, and let $X \subseteq \mathbb{A}^n$ be an affine variety. For an open set $U \subseteq X$, let $\mathscr{R}_X(U)$ be the ring of all rational functions which are regular on U:

$$\mathscr{R}_X(U) = \{f \in K(X) \,|\, f \text{ is regular on } U\}.$$

Note that if $V \subseteq U$, then $\mathscr{R}_X(U) \subseteq \mathscr{R}_X(V)$, so this defines a pre-sheaf by letting the restriction maps be the inclusion maps. The sheaf axioms are also satisfied.

(1) The identity axiom holds because if $f \in K(X)$ restricts to zero in some $\mathscr{R}_X(U)$, then this simply means that $f = 0$ in $K(X)$.
(2) The gluing axiom holds because if $\{f_i \in \mathscr{R}_X(U_i)\}$ is a collection of rational functions that agree on the overlaps $U_i \cap U_j$ of an open covering, then they are all equal to the same element $f \in K(X)$. This rational function f must in turn be regular on all of $U = \bigcup_i U_i$ because if $p \in U$, then p lies in some U_j, and hence $f = f_j$ can be written as a/b with $b(p) \neq 0$.

It follows that $\mathscr{R}_X$ defines a CRing-valued sheaf, called the sheaf of regular functions on X.

[9] Here 0 is the trivial abelian group.

The next examples concern constant pre-sheaves and sheaves, which, despite their simplicity, play a foundational role in the development of sheaf cohomology. They also provide key intuition for understanding how local data can fail to glue globally.

Example 9.5. Let X be a topological space and let $\mathsf{C} = \mathsf{Ab}$. Let $A \in \mathsf{Ab}$ with the discrete topology. The following are two examples of Ab-valued pre-sheaves:

(1) For any non-empty open set $U \in \mathsf{Open}(X)$, let $\overline{A}(U) = A$. Clearly, $\overline{A}$ is a pre-sheaf with restriction maps the identity. This is called the **constant pre-sheaf**.
(2) For any non-emptyset open set $U \in \mathsf{Open}(X)$, let $\underline{A}(U)$ be the abelian group of all continuous maps of U into A. Then with the usual restriction maps (as in the previous example), we obtain a sheaf. Note that each function in $\underline{A}(U)$ is locally constant for each open set of X. This is called the **constant sheaf**.

Once again, $\mathscr{F}(\emptyset) = 0$ is forced by Lemma 8.5.

Remark 9.6. *Let $\underline{A}$ be the constant sheaf. Note that for every connected open set U, $\underline{A}(U) \cong A$ since the image of a continuous map from a connected set to a discrete space is constant. This justifies the terminology.*

Example 9.7. All examples discussed in Example 9.2 are examples of sheaves of R-modules with $R = \mathbb{R}, \mathbb{C}$ as appropriate.

Example 9.8. Let X be a topological space and let 0 denote the trivial abelian group. Fix any abelian group, A, and $x \in X$. Consider the assignment

$$i_x^A(U) = \begin{cases} A & \text{if } x \in U, \\ 0 & \text{if } x \notin U. \end{cases}$$

This is can be made into a pre-sheaf if for open sets $U, V \subseteq X$ such that $V \subseteq U$, the map $i_x^A(U) \to i_x^A(V)$ is defined such that:

(1) If $x \notin U$, then the map is simply the identity morphism $0 \to 0$
(2) If $x \in V$, then the map is simply the identity morphism $A \to A$
(3) If $x \in U \setminus V$, then the map is simply the unique morphism $A \to 0$.

It is easy to verify that this defines a pre-sheaf. Let $\{U_i\}_{i \in I}$ for an open cover for an open set $U \subseteq X$. The identity and gluing axioms are essentially satisfied since U contains x if and only if some U_i contains x. This is called the **skyscraper sheaf**.

Next, we consider the sheaf of sections, which is fundamental in relating sheaf theory to geometry by associating sheaves to vector bundles and other fibered structures.

Example 9.9. (Sheaf of Sections) Let X, Y be topological space and let $\pi : Y \to X$ be a continuous map. Recall that a section of π is a continuous map $\sigma : X \to Y$ such that $\pi \circ \sigma = \mathrm{Id}_X$. For an open non-empty set $U \subseteq X$, define $\mathscr{E}(U)$ to be the set of sections of π on U. That is,

$$\mathscr{E}(U) = \{\sigma : U \to Y \mid \sigma \text{ is continuous and } \pi \circ \sigma = \mathrm{Id}_U\}.$$

The empty set is sent to the singleton set and the restriction maps are are given by restriction of functions. This is called the pre-sheaf of sections of π. In fact, pre-sheaf of sections of π is a sheaf of sets. Indeed, since sections are indeed continuous function, it is clear that the identity axiom is satisfied. Similarly, the gluing axiom is also satisfied if we note that if $\{U_i\}_{i \in I}$ is an open cover of of U and $\sigma_i \in \mathscr{E}(U_i)$ such that $\sigma_i = \sigma_j \in \mathscr{E}(U_i \cap U_j)$, then the function $\sigma : U \to Y$ such that $\sigma|_{U_i} = \sigma_i$ is indeed a section.

Remark 9.10. *If Y is a topological group, the sheaf of sections is a sheaf of groups.*

Remark 9.11. *If $Y = X \times \mathbb{R}$, and π is projecting onto the first factor, then sections of π are just continuous maps $X \to \mathbb{R}$. In other words, the sheaf of sections generalizes the sheaf of real-valued continuous functions.*

Finally, we examine presheaves that fail to satisfy the sheaf axioms and thus are not sheaves. These examples illustrate the necessity of the gluing and locality conditions in the definition of a sheaf, and understanding such presheaves is crucial for constructing sheafifications and for deeper insights in sheaf cohomology.

Remark 9.12. *A pre-sheaf may not be a sheaf. Here are two examples:*

(1) *(Identity axiom fails) Let $X = \{*_1, *_2\}$ with the discrete topology. Let $\mathscr{F}$ be a pre-sheaf of abelian groups defined as follows:*

$$\mathscr{F}(\{*_1, *_2\}) = \mathbb{Z}, \quad \mathscr{F}(\{*_1\}) = \mathbb{Z}_2, \quad \mathscr{F}(\{*_2\}) = \mathbb{Z}_2, \quad \mathscr{F}(\emptyset) = 0,$$

with the obvious homomorphisms $\mathbb{Z} \to 0$ and $\mathbb{Z} \to \mathbb{Z}_2$ However, this is not a sheaf. Indeed, $X = \{*_1\} \cup \{*_2\}$. Then $2, 4 \in \mathbb{Z}$ such that these elements restrict to the 0 element in $\mathbb{Z}_2$. However, $2 \neq 4$.

(2) *(Gluing axiom fails) Let $X = \mathbb{R}$, and let $\mathscr{F}(U)$ be the abelian group of bounded functions on non-empty open sets U. Then $\mathscr{F}$ defines a pre-sheaf but not a sheaf. Indeed, let $X = \bigcup_{i \in \mathbb{Z}}^{\infty} (i, i+1]$ and let $f_i \equiv i$ on $(i, i+1]$. Since $V_i \cap V_j = \emptyset$ for $i \neq j$, trivially we have that $f_i = f_j$ on each $V_i \cap V_j = \emptyset$ for $i \neq j$. However, there is not $f \in \mathscr{F}(\mathbb{R})$ such that $f|_{(i,i+1]} = f_i$; otherwise, f must be an unbounded function.*

The constant pre-sheaf is usually not a sheaf. Let X be a topological space with two open sets, U_1, U_2, such that $U_1 \cap U_2 = \emptyset$. Let $A \in \mathsf{Ab}$ be a non-trivial abelian group and let $\overline{A}$ be the corresponding constant pre-sheaf on X. Specifically,

$$\overline{A}(U) = \begin{cases} A, & U \neq \emptyset, \\ 0, & U = \emptyset. \end{cases}$$

Let's show that the gluing axiom fails. Let $a_1 \neq a_2 \in A$ such that $a_i \in \overline{A}(U_i)$ for $i = 1, 2$. Let $U = U_1 \cup U_2$. The overlap condition is trivially satisfied. However, it is clear we cannot find any $a \in \overline{A}(U)$ such that

$$a|_{U_1} = a_1 \quad \text{and} \quad a|_{U_2} = a_2,$$

Hence, this constant pre-sheaf is not a sheaf. Conceptually, if we view elements of $\overline{A}(U)$ as constant functions $U \to A$, this example illustrates that we are attempting to take constant functions on U_1 and U_2 with different values and glue them to obtain a constant function on $U_1 \cup U_2$, which is impossible.

10. Stalks

Stalks are fundamental tools in sheaf theory, providing a way to study the behavior of a sheaf at a single point. This construction allows us to isolate and analyze local data while still capturing the global structure of the sheaf. Stalks play a crucial role in understanding the local-to-global correspondence in mathematics, as they bridge the gap between local properties (encoded in sections over open sets) and global phenomena. Let's first motivate the definition of a stalk with the help of an example.

Example 10.1. Let $X = \mathbb{C}^n$ and let $\mathscr{O}$ denotes the sheaf of holomorphic functions on X. For each $x \in X$ and open set U containing x, we define an equivalence relation on $\mathscr{O}(U)$

$$f \sim g \iff \text{there exists an open set } W \subseteq U \text{ containing } p \text{ such that } f|_W = g|_W$$

The equivalence class of a function $f \in \mathscr{O}(U)$ is called the germ of f at x and is denoted by $[f]_x$. The stalk of $\mathscr{O}$ at x, denoted $\mathscr{O}_x$, is the vector space of all germs of holomorphic functions at x. Addition and scalar multiplication of germs are defined by performing these operations on any representatives that are defined on the same open set. For example, addition is defined as:

$$[f]_x + [g]_x = [f + g]_x$$

Let's check that addition is well-defined. Assume that $[f]_x = [f']_x$ and $[g]_x = [g']_x$. Then there exist open sets $V, W \subseteq U$ such that $f|_V = f'|_V$ and $g|_W = g'|_W$. It is clear that on $V \cap W \subseteq U$, we have

$$f + g|_{V\cap W} = f' + g'|_{V\cap W}$$

This shows addition is well-defined. Similarly, it can be checked that scalar multiplication is well-defined.

Remark 10.2. *$\mathscr{O}_x$ is actually a ring. This can be checked easily. In fact, $\mathscr{O}_x$ is a local ring. Let $\mathfrak{m}_x \subseteq \mathscr{F}_x$ denotes germs vanishing at x. This certainly forms an ideal. In fact, the ideal is maximal since $\mathscr{F}_x/\mathfrak{m}_x \cong \mathbb{C}$. This is the unique maximal ideal since any germ not contained in $\mathfrak{m}_x$ is invertible.*

The construction given above can be applied to continuous or smooth functions on an appropriate space. We can now give the general definition of a stalk of a pre-sheaf, abstracting away from the previous example.

Definition 10.3. Let X be a topological space and let C be a category admitting filtered colimits. Let $\mathscr{F}$ be a C-valued pre-sheaf on X. The **stalk** of a pre-sheaf, $\mathscr{F}$, at a point $x \in X$, denoted by $\mathscr{F}_x$, is

$$\mathscr{F}_x = \varinjlim_{x\in U} \mathscr{F}(U)$$

Remark 10.4. *The stalk of a sheaf is the stalk of the underlying pre-sheaf.*

If $\mathsf{C} = R\text{-}\mathsf{Mod}$, then colimits exist in C and the characterization of colimit of a directed system allows us to unpack the definition of the stalk of a pre-sheaf. For example, let $\mathscr{F}$ be a R-Mod-valued pre-sheaf on a topological space X. For each $x \in X$, the collection of R-modules $\mathscr{F}(U)$, where U ranges over all open sets containing x, together with the restriction maps, forms a direct system with the relation $U \leq V$ if $U \supseteq V$. The intersection of two open sets containing p serves as a common upper bound. Definition 10.3 defines the stalk of $\mathscr{F}$ at x as the direct limit of this system.

Remark 10.5. *Let's recall the direct limit of a directed system of R-modules. Recall that a directed set $(I, \leq)$ is a non-empty set I with a binary relation, $\leq$, that is reflexive and transitive, and where every pair of elements has a common upper bound. A direct system of R-modules consists of a family $\{M_\alpha\}_{\alpha\in I}$ of R-modules indexed by a directed set I, along with R-module homomorphisms $f_{\alpha\beta} : M_\alpha \to M_\beta$ for $\alpha \leq \beta$, satisfying*

$$f_{\alpha\alpha} = \mathrm{Id}_{M_\alpha},\ \alpha \in I$$
$$f_{\beta\gamma} \circ f_{\alpha\beta} = f_{\alpha\gamma},\ \alpha \leq \beta \leq \gamma$$

The direct limit (or colimit in this case) is defined by defining an equivalence relation on $\coprod_{\alpha \in I} M_\alpha$ such that

$$m_\alpha \sim m_\beta \iff \text{there exists some } \gamma \in I \text{ such tht } \alpha, \beta \leq \gamma \text{ and } f_{\alpha\gamma}(m_\alpha) = f_{\beta\gamma}(m_\beta) \in M_\gamma$$

The direct limit of the direct system is denoted as

$$\varinjlim_{\alpha \in I} M_\alpha = \left(\coprod_{\alpha \in I} M_\alpha \right) / \sim$$

It is easy to check that $\varinjlim_{\alpha \in I} M_\alpha$ is R-module. For example, addition is defined by

$$[m_\alpha] + [m_\beta] = [f_{\alpha\gamma}(m_\alpha) + f_{\beta\gamma}(m_\beta)],$$

where γ is some upper bound for α and β. This can be checked to be well-defined because all maps $f_{\alpha\beta}$ are homomorphisms. Other operations are defined in a similar manner.

Example 10.6. Let X, Y be topological space and let $\pi : Y \to X$ be a local homeomorphism. Let $\mathscr{E}$ be the pre-sheaf of sections. We show that for each $x \in X$, we have

$$\mathscr{E}_x \simeq E_x = \pi^{-1}(x)$$

For each $x \in X$ such that $x \in U$, define

$$\begin{aligned} \eta_U : \mathscr{E}(U) &\to E_x, \\ s &\mapsto s(x) \end{aligned}$$

We use the maps η_U to induce a map on the colimit:

$$\begin{aligned} \eta : \mathscr{E}_x = \varinjlim_{x \in U} \mathscr{E}(U) &\to E_x, \\ [s] &\mapsto s(x) \end{aligned}$$

We check that η is well-defined. Let $[s_1], [s_2] \in \mathscr{F}_x$ such that $[s_1] = [s_2]$ If s_1, s_2 are defined such that $s_1 : U_1 \to E$ and $s_2 : U_2 \to E$, then there is a neighborhood of $x \in W \subseteq U_1 \cap U_2$ such that $s_1|_W = s_2|_W$. In particular, $s_1(x) = s_2(x)$. Hence η is well-defined. We claim that η is a bijection. First, we show that η is surjective. Because π is a local homeomorphism, given $e \in E_x = \pi^{-1}(x)$, we can find open neighborhoods O_e of e and U_x of $x = \pi(e)$ such that

$$\pi|_{O_e} : O_e \to U_x$$

is a homeomorphism. Then

$$(\pi|_{O_e})^{-1} : U_x \to O_e$$

is a section of π, and

$$\eta((\pi|_{O_e})^{-1}) = (\pi|_{O_e})^{-1}(x) = (\pi|_{O_e})^{-1}(\pi(e)) = e.$$

Hence η is surjective. Now we prove that η is injective. Suppose $\eta[s_1] = \eta[s_2]$. Then $s_1(x) = s_2(x)$. By using properties of local homeomorphisms, we can check that there is an open neighborhood of x on which s_1, s_2 agree. That is, $[s_1] = [s_2]$. Thus, η is injective.

Example 10.7. Let $\mathscr{R}$ be the sheaf of regular functions on an affine variety. Then the stalk $\mathscr{R}_x$ is the local ring $\mathcal{O}_{X,x}$.

Category theory teaches us to focus on the properties of morphisms between objects rather than the objects themselves. Consequently, we infer the concept of the stalk of a morphism of sheaves from the following result:

Lemma 10.8. *Let X be a topological space. Let C be a category admitting filtered colimits and let $\mathscr{F}, \mathscr{G}$ be C-valued pre-sheaves on X. There is a functor*

$$\mathscr{S}_x : \mathsf{PreShv}(X, \mathsf{C}) \to \mathsf{C}$$

called the stalkification at x functor for each $x \in X$. In particular, if there is a morphism $\varphi : \mathscr{F} \to \mathscr{G}$, there is an induced morphism on stalks $\varphi_x : \mathscr{F}_x \to \mathscr{G}_x$ for each $x \in X$.

Proof. (Sketch) The functor is defined by mapping $\mathscr{F}$ to its stalk $\mathscr{F}_x$ for each $x \in X$. If $\varphi : \mathscr{F} \to \mathscr{G}$ is a morphism of sheaves, then the morphism $\varphi_x : \mathscr{F}_x \to \mathscr{G}_x$ is induced by the universal property of colimits. It is straightforward to verify that this construction defines a functor. □

Remark 10.9. *If $\mathsf{C} = R$-Mod, then the morphism $\varphi_x : \mathscr{F}_x \to \mathscr{G}_x$ of stalks can be described concretely. It is the morphism $\varphi_x : \mathscr{F}_x \to \mathscr{G}_x$ such that $\varphi_x([f]_x) = [\varphi(f)]_x$. Let's check that this is well-defined. Suppose $[f]_x = [f']_x$ such that $f \in \mathscr{F}(U)$ and $f' \in \mathscr{F}(U')$. Then there exists an open set $W \subseteq U \cap U'$ containing x such that $f|_W = f'|_W$. We have*

$$\mathscr{F}(f)|_W = \mathscr{F}(f|_W) = \mathscr{F}(f'|_W) = \mathscr{F}(f')|_W$$

Hence $[\mathscr{F}(f)]_x = [\mathscr{F}(f')]_x$. It is easy to check that $\mathscr{F}_x$ is a morphism in C.

Sheaves are an important tool for keeping track of locally defined data. Therefore, we expect that many properties of sheaves can be checked at the level of stalks. We discuss some properties of sheaves that can be determined by looking at the corresponding stalks. Here is a sample proposition when $\mathsf{C} = R-\mathsf{Mod}$.

Proposition 10.10. *Let X be a topological space and $\mathscr{F}$ be a sheaf of R-modules on X.*

(1) *(**Sections are determined by stalks**) For $U \in \mathsf{Open}(X)$, the natural map*

$$h : \mathscr{F}(U) \to \prod_{x \in U} \mathscr{F}_x$$

is a monomorphism. Equivalently, the natural map

$$h : \mathscr{F}(U) \to \prod_{x \in U} \mathscr{F}_x$$

is injective. That is, if $f, g \in \mathscr{F}(U)$ then $f = g$ if and only if $f_x = g_x$ for all $x \in U$.

(2) *(**Morphisms are determined by stalks**) Let $\varphi, \psi : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves such that $\varphi_x = \psi_x$ for each $x \in X$. Then $\varphi = \psi$.*

Proof. The proof is given below:

(1) The forward direction is trivial. Conversely, assume that $f_x = g_x$ for each $x \in U$. For a fixed $x_0 \in U$, $f_{x_0} = g_{x_0}$ if and only if there exists a neighborhood $x_0 \in U_{x_0} \subseteq U$ such that

$$f|_{U_{x_0}} = g|_{U_{x_0}}.$$

If we take all such neighborhoods U_x for all $x \in U$, we get an open cover for $U = \cup_{x \in U} U_x$, and by the definition of sheaves,

$$\mathscr{F}(U) \to \prod_{x \in U} \mathscr{F}(U_x)$$

is injective. Hence, $f = g$.

(2) Consider the following diagram:

$$\begin{tikzcd} \mathscr{F}(U) \arrow[r] \arrow[d] & \mathscr{G}(U) \arrow[d] \\ \prod_{x\in U}\mathscr{F}_x \arrow[r] & \prod_{x\in U}\mathscr{G}_x \end{tikzcd}$$

The top map is either $\varphi(U)$ or $\psi(U)$ and the bottom map is the corresponding induced map on stalks at each p. Since the diagram commutes by assumption, the following diagram commutes:

$$\begin{tikzcd} \mathscr{F}(U) \arrow[r, bend left, "\varphi(U)"] \arrow[r, bend right, "\psi(U)"'] & \mathscr{G}(U) \arrow[r] & \prod_{x\in x}\mathscr{G}_x \end{tikzcd}$$

Since the second map is a monomorphism by (1), we have $\varphi(U) = \psi(U)$ for each open set U. Hence, $\varphi = \psi$.

This completes the proof. $\square$

Remark 10.11. *Proposition 10.10 is false for general pre-sheaves. Let $X = \{*_1, *_2\}$ with the discrete topology. Let $\mathscr{F}$ be a pre-sheaf of abelian groups defined as follows:*

$$\mathscr{F}(\{*_1, *_2\}) = \mathbb{Z}, \quad \mathscr{F}(\{*_1\}) = \mathbb{Z}_2, \quad \mathscr{F}(\{*_2\}) = \mathbb{Z}_2, \quad \mathscr{F}(\emptyset) = 0,$$

with the obvious homomorphisms $\mathbb{Z} \to 0$ and $\mathbb{Z} \to \mathbb{Z}_2$. Note that $\mathscr{F}_{*_1}, \mathscr{F}_{*_2} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

(1) *Let $U = X$. Note that the map*

$$\mathbb{Z} = \mathscr{F}(U) \to \prod_{x\in U}\mathscr{F}_x \cong \mathbb{Z}_2^4$$

is clearly not injective. Hence, a section is not necessarily determined by stalks for a general pre-sheaf.

(2) *Let φ be a morphism of $\mathscr{F}$. Let φ be such that $\varphi(X) : \mathbb{Z} \to \mathbb{Z}$ is the identity map. The consistency conditions for φ to be a sheaf morphism implies that the maps $\mathbb{Z}_2 \to \mathbb{Z}_2$ are identity maps. However, if $\varphi(X)$ is changed to $\varphi'(X) : \mathbb{Z} \to \mathbb{Z}$ which is multiplication by n map where n is the odd, consistency conditions for φ' to be a sheaf morphism implies that the maps $\mathbb{Z}_2, \to \mathbb{Z}_2$ are identity maps. In either case, the induced maps on stalks are identity. This shows that morphisms are not necessarily determined by stalks for a general pre-sheaf.*

11. Structure Sheaf

We now look at an example of a sheaf, called the *structure sheaf*, that is important in algebraic geometry. We have endowed the spectrum of a ring with the structure of a topological space. For a commutative ring R, to view $\operatorname{Spec} R$ as a *geometric space*, we equip it with a sheaf analogous to the sheaf of regular functions on an affine variety (Example 9.4). The structure sheaf on $\operatorname{Spec} R$ is constructed by considering the "regular fnunctions" associated with the ring R.

11.1. **Sheaves Defined on a Basis.** We first need to look at the concept of sheaves defined on a basis. Let $\mathsf{C} = \mathsf{CRing}$. We can also take $\mathsf{C} = \mathsf{Ab}, R\text{-}\mathsf{Mod}$. Sheaves are defined with $\mathsf{Open}(X)$ as the domain category for a topological space X. Given a basis $\mathscr{B}$ for X, we can attempt to track the sheaf data at the level of open sets in $\mathscr{B}$.

Definition 11.1. Let X be a topological space with basis $\mathscr{B}$. A CRing-valued **$\mathscr{B}$-pre-sheaf** is a contravariant functor:

$$\mathscr{F} : \mathsf{Open}^{\mathscr{B}}(X) \to \mathsf{CRing}$$

Here $\mathsf{Open}^{\mathscr{B}}(X)$ is the category of open sets of X in $\mathscr{B}$. A CRing-valued sheaf $\mathscr{B}$ pre-sheaf is a **CRing-sheaf** if the following diagram is an equalizer for every open cover $\mathscr{U} = \{U_i\}_{i \in I} \subseteq \mathscr{B}$ of any open set $U \in \mathscr{B}$:

$$\mathscr{F}(U) \longrightarrow \prod_i \mathscr{F}(U_i) \rightrightarrows \prod_{V \subseteq U_i \cap U_j V \in \mathscr{B}} \mathscr{F}(V)$$

Lemma 11.2. *Let X be a topological space with basis $\mathscr{B}$, and let $\mathscr{F}$ be a CRing-valued $\mathscr{B}$-sheaf. We have*

$$\mathscr{F}(U) \cong \varprojlim_{\substack{V \subseteq U \\ V \in \mathscr{B}}} \mathscr{F}(V).$$

Proof. An element of $\varprojlim \mathscr{F}(V)$ defines a section on each base open set V, and these sections are compatible with restriction. Thus, by the gluing axiom this collection corresponds to a unique section over U, since the base open sets $V \in \mathscr{B}$ clearly form a cover of U. This gives a unique map

$$\varprojlim_{\substack{V \subseteq U \\ V \in \mathscr{B}}} \mathscr{F}(V) \longrightarrow \mathscr{F}(U)$$

such that for every base open set $W \subseteq U$, the following diagram commutes:

$$\begin{tikzcd}
\varprojlim_{\substack{V \subseteq U \\ V \in \mathscr{B}}} \mathscr{F}(V) \arrow[r] \arrow[dr] & \mathscr{F}(U) \arrow[d] \\
& \mathscr{F}(W)
\end{tikzcd}$$

Since any object with morphisms to each $\mathscr{F}(V)$ factors uniquely through the limit, this implies that $\mathscr{F}(U)$ satisfies the same universal property. This concludes the proof. $\square$

The primary motivation for introducing $\mathscr{B}$-sheaves is encapsulated in the following proposition, which states that, as anticipated, any $\mathscr{B}$-presheaf can be extended to a presheaf on X by *approximating* open sets in X through open sets in the basis $\mathscr{B}$.

Proposition 11.3. *Let X be a topological space with basis $\mathscr{B}$. Every CRing-valued $\mathscr{B}$-sheaf $\mathscr{F}$ extends to a CRing-valued sheaf $\overline{\mathscr{F}}$ on X, which is unique up to isomorphism.*

The idea of proof of Proposition 11.3 is to define $\overline{\mathscr{F}}(U)$ as the set of all possible gluings of sections of $\mathscr{F}$ over open sets in $\mathscr{B}$ that cover U. More precisely, a section $s \in \overline{\mathscr{F}}(U)$ is given by a collection of sections

$$s_i \in \mathscr{F}(V_i)$$

for some open cover $\{V_i\}_{i \in I}$ of U with $V_i \in \mathscr{B}$, such that

$$s_i|_W = s_j|_W$$

for any $W \in \mathscr{B}$ with $W \subseteq V_i \cap V_j$. The only drawback is that each section depends on the choice of a covering $\{V_i\}$. To define the group $\overline{\mathscr{F}}(U)$ in a manner that is independent of any particular cover, we consider the *largest cover* of U, consisting of all open sets $V \in \mathscr{B}$ such that $V \subseteq U$.

PROOF. Let $U \subseteq X$ be any open subset, and let $\mathscr{B}_U \subseteq \mathscr{B}$ denote the collection of open sets in $\mathscr{B}$ contained in U. Using Lemma 11.2, define

$$\bar{\mathscr{F}}(U) := \varprojlim_{V \in \mathscr{B}_U}$$

An element of $\mathscr{F}(U)$ is therefore given by a compatible family of sections $s_V \in \mathscr{F}(V)$. Note that we have

$$\overline{\mathscr{F}}(U) \ := \left\{ (s_V) \in \prod_{V \in \mathscr{B}_U} \mathscr{F}(V) : s_V|_W = s_W, \text{ for all } W \subseteq V \text{ with } W, V \in \mathscr{B}_U \right\}.$$

Note that if $U_1 \subseteq U$, then $\mathscr{B}_{U_1} \subseteq \mathscr{B}_U$, and the projection maps induce restriction maps

$$\overline{\mathscr{F}}(U) \to \overline{\mathscr{F}}(U_1).$$

This makes $\mathscr{F}$ into a pre-sheaf. The sheaf axioms are easily verified. Moreover, if U is an open set in $\mathscr{B}$, there is a canonical isomorphism

$$\mathscr{F}(U) \xrightarrow{\sim} \overline{\mathscr{F}}(U),$$

sending a section $t \in \mathscr{F}(U)$ to the collection $(s_V)_{V \in \mathscr{B}_U}$ where $s_V = t|_V$. The inverse is given by the projection onto the "U-th component". □

11.2. **Structure Sheaf.** To motivate the definition of the structure sheaf, let us recall the case when $X \subseteq \mathbb{A}^n$ is an affine variety, and $A(X)$ denotes its coordinate ring. For a distinguished open subset $D(f) \subseteq X$, we have

$$\mathcal{O}_X(D(f)) = A(X)_f,$$

where $A(X)_f$ is the localization of $A(X)$ at the element f (Proposition 4.4). Moreover, if $D(g) \subseteq D(f)$, the corresponding restriction map

$$\mathcal{O}_X(D(f)) \to \mathcal{O}_X(D(g))$$

is given by the canonical localization map $A(X)_f \to A(X)_g$. In fact, the sheaf $\mathcal{O}_X$ is completely determined by the values $\mathcal{O}_X(D(f)) = A_f$ for each $f \in A$. Indeed, we have

$$\mathcal{O}_X(U) = \bigcap_{D(f) \subseteq U} \mathcal{O}_X(D(f)).$$

Motivated by this discussion, we can define a pre-sheaf on a basis of open subsets of the spectrum of an arbitrary ring R.

Definition 11.4. Let R be a ring, and let $\mathscr{B}$ denote the basis of distinguished open subsets of $\operatorname{Spec} R$. The $\mathscr{B}$**-structured sheaf** is contravariant functor

$$\mathscr{S} : \mathsf{Open}^{\mathscr{B}}(\operatorname{Spec} R) \longrightarrow \mathsf{CRing}$$
$$U_f \longmapsto R_f$$

We now show that $\mathscr{S}$ is a $\mathscr{B}$-sheaf. The result is proved using an algebraic lemma, which is also stated below.

Proposition 11.5. *Let R be a ring, and let $\mathscr{B}$ denote the basis of distinguished open subsets of* $\operatorname{Spec} R$.

(1) *Let $g_1, \ldots, g_r \in R$ be elements generating the unit ideal. For any R-module M, the following sequence is exact:*

$$0 \to M \xrightarrow{\alpha} \bigoplus_{i=1}^{r} M_{g_i} \xrightarrow{\beta} \bigoplus_{i,j=1}^{r} M_{g_i g_j},$$

where the maps α and β are defined by

$$\alpha(s) = (s/1, \ldots, s/1),$$
$$\beta(s_1, \ldots, s_r)_{i,j} = s_i/1 - s_j/1.$$

(2) *$\mathscr{S}$ is a* CRing*-valued $\mathscr{B}$-sheaf.*

Proof. The proof is given below:

(1) Skipped.
(2) Let U_f is a distinguished open subset of $\operatorname{Spec} R$. First assume that $\{U_{f_i}\}_{i=1}^r$ is a finite open cover of U_f. Applying (1) to the ring R_f and the module $M = R_f$, we obtain an exact sequence:

$$0 \longrightarrow R_f \xrightarrow{\alpha} \bigoplus_{i=1}^{r} R_{f_i} \xrightarrow{\beta} \bigoplus_{i,j=1}^{r} R_{f_i f_j}.$$

The sheaf axioms are verified by the exactness of the sequence constructed above. Now assume that $\{U_{f_i}\}_{i\in I}$ is a general open cover of U_f. Since U_f is compact, there exists a finite subset $J \subseteq I$ such that $\{U_{f_j}\}_{j\in J}$ forms a finite subcover of U_f. We check the two sheaf axioms:

(a) If $s \in R_f$ maps to zero in R_{f_i} for every $i \in I$, then in particular, it maps to zero in R_{f_i} for each $i \in J$. The argument above for the finite cover case implies that $s = 0$ in R_f.
(b) Let $s_i \in R_{f_i}$ be compatible elements for $i \in I$. That is

$$s_i/1 = s_j/1$$

$R_{f_i f_j}$ for all $i, j \in I$. The argument above provides a unique element $s \in R_f$ such that $s_i = s/1 \in R_{f_i}$ for all $i \in J$. We show that this element s also induces the sections s_i for all $i \in I$. Fix an index $\alpha \in I$. Consider the finite covering

$$\{U_{f_i}\}_{i\in J\cup\{\alpha\}}$$

of U_f. The argument above implies there exists an element $s' \in R_f$ such that $s'/1 = s_i$ in R_{f_i} for all $i \in J$ and $s'/1 = s_\alpha$ in R_{f_α}. Since both s and s' restrict to the same elements in R_{f_i} for all $i \in J$, uniqueness implies that $s = s'$ in R_f. Hence, $s/1 = s_\alpha$ in R_{f_α} as well.

Hence, $\mathscr{S}$ is a $\mathscr{B}$-sheaf.

This completes the proof. □

The discussion above shows that the assignment $U_f \mapsto R_f$ define a sheaf on the basis of distinguished open sets of $\operatorname{Spec} R$. We now extend this to a sheaf on the entire space, giving the precise definition of the structure sheaf.

Definition 11.6. Let R be a ring. The **structure sheaf**, $\mathscr{O}_{\operatorname{Spec} R}$, is the unique sheaf extending the $\mathscr{B}$-sheaf, $\mathscr{S}$. In particular, $\mathscr{O}_{\operatorname{Spec} R}$ is defined as:

$$\mathscr{O}_{\operatorname{Spec}(R)}(U) = \left\{ (s_i) \in \prod_{i \in I} R_{f_i} \;\middle|\; s_i = s_j \text{ in } R_{f_i f_j} \text{ for all } i, j \in I \right\},$$

Remark 11.7. *Proposition 11.3 implies that Definition 11.6 is well-defined.*

Proposition 11.8. *Let R be a ring.*

(1) $\mathscr{O}_{\operatorname{Spec} R}(\operatorname{Spec} R) \cong R$.
(2) *For any $\mathfrak{p} \in \operatorname{Spec} R$, we have*

$$\mathscr{O}_{\operatorname{Spec} A, \mathfrak{p}} \cong R_{\mathfrak{p}}.$$

Proof. We defined $\mathscr{O}_{\operatorname{Spec} R}$ so that $\mathscr{O}_{\operatorname{Spec} R}(U_f) \cong R_f$ for every $f \in R$. Taking $f = 1$, we obtain

$$\mathscr{O}_{\operatorname{Spec} R}(\operatorname{Spec} R) \cong R$$

For (2), since the distinguished open sets form a basis for the topology, the stalk of $\mathscr{O}_{\operatorname{Spec} R}$ at a point $\mathfrak{p} \in \operatorname{Spec} R$ can be computed as the direct limit

$$\mathscr{O}_{\operatorname{Spec} R, \mathfrak{p}} \cong \varinjlim_{\mathfrak{p} \in U_f} \mathscr{O}_{\operatorname{Spec} R}(U_f) \cong \varinjlim_{f \notin \mathfrak{p}} R_f.$$

We claim that the natural map $\varinjlim_{f \notin \mathfrak{p}} R_f \to R_{\mathfrak{p}}$ induced by the maps $R_f \to R_{\mathfrak{p}}$ for $f \notin \mathfrak{p}$ is an isomorphism.

(a) Any element $a/s \in R_{\mathfrak{p}}$, with $s \in \mathfrak{p}^c$, lies in the image of the canonical map $R_s \to R_{\mathfrak{p}}$. Hence, the map is surjective.
(b) Suppose an element $a/f^n \in R_f$ maps to zero in $R_{\mathfrak{p}}$. This means there exists $s \in \mathfrak{p}^c$ such that $as = 0$ in R. Then $a/f^n = 0$ in R_g, where $g = sf$. Hence, the element vanishes in the direct limit. Hence, the map is injective.

Hence, we have

$$\mathscr{O}_{\operatorname{Spec} R, \mathfrak{p}} \cong \varinjlim_{f \notin \mathfrak{p}} R_f \cong R_{\mathfrak{p}}$$

This completes the proof. □

Example 11.9. The structure sheaf carries essential algebraic information beyond the underlying topological space. Let $\mathbb{K}$ be an algebraically closed field. Then $\operatorname{Spec} \mathbb{K}$ consists of a single point, (0) (Example 5.4). However, we have

$$\mathscr{O}_{\operatorname{Spec}(\mathbb{K})}(\mathbb{K}) \cong \mathbb{K}$$

Hence, the structure sheaf distinguishes non-isomorphic algebraically closed fields whose spectra are homeomorphic as single points.

12. Pushforward

We can make a plethora of functorial constructions with sheaves. For instance, given a continuous map between topological spaces, we can ask how the sheaves on X and Y are related. In this section, we discuss one such construction: the pushforward of a sheaf. This notion will play a crucial role in the definition of (locally) ringed spaces in Section 14, which combines the notion of the spectrum of a ring with its associated structure sheaf.

Definition 12.1. Let $X, Y \in \mathsf{Top}$ and $f : X \to Y$ be a continuous map. If $\mathscr{F}$ is a sheaf on X the **pushforward sheaf**, $f_*\mathscr{F}$, on Y defined by

$$(f_*\mathscr{F})(U) := \mathscr{F}(f^{-1}(U)),$$

for every open set $U \subseteq Y$ and the restriction maps

$$(f_*\mathscr{F})(U) \to (f_*\mathscr{F})(V)$$

are defined to be those induced by the restriction maps of $\mathscr{F}$, for open sets $V \subseteq U \subseteq Y$.

Let's check that the direct image pre-sheaf is in fact a pre-sheaf. We begin with a simple observation. Since f is continuous, the preimage of any open set in Y is still an open set in X, and the operation of taking preimages preserves inclusions. Consequently, f naturally induces a functor f^{-1} from $\mathsf{Open}(Y)$ to $\mathsf{Open}(X)$. Moreover, by reversing the arrows in both categories, f^{-1} retains its functoriality. Now, let $\mathscr{F}$ be a pre-sheaf of R-modules over X, and consider the following diagram:

$$f_*(\mathscr{G}) : \mathsf{Open}^{\mathsf{op}}(Y) \xrightarrow{f^{-1}} \mathsf{Open}^{\mathsf{op}}(X) \xrightarrow{A} R\text{-}\mathsf{Mod}.$$

This gives the desired contravariant functor. Hence, the direct image pre-sheaf if indeed a pre-sheaf. If $\mathscr{F}$ is a sheaf, it is clear that the direct image pre-seaf is in fact a sheaf.

Example 12.2. Let $i : \{x\} \hookrightarrow X$ be the inclusion of a closed point x into a topological space X, and let A be an abelian group, regarded as a constant sheaf on $\{x\}$. The pushforward sheaf is the skyscraper sheaf (Example 9.8):

$$i_*(A)(U) = \begin{cases} A & \text{if } x \in U, \\ 0 & \text{otherwise.} \end{cases}$$

We can now define a functor:

$$f_* : \mathsf{PreShv}(X, R\text{-}\mathsf{Mod}) \to \mathsf{PreShv}(Y, R\text{-}\mathsf{Mod})$$

Let's verify that f_* is indeed a functor. If $\varphi : \mathscr{F} \to \mathscr{F}'$ is a morphism of pre-sheaves on X, we have the following commutative diagrams for open sets $U \subseteq V \subseteq Y$:

$$\begin{tikzcd}
f_*\mathscr{F}(U) = \mathscr{F}(f^{-1}(U)) \arrow[d, "\varphi_{f^{-1}(U)}"'] & \mathscr{F}(f^{-1}(V)) = f_*\mathscr{F}(V) \arrow[l] \arrow[d, "\varphi_{f^{-1}(V)}"] \\
f_*\mathscr{F}'(U) = \mathscr{F}'(f^{-1}(U)) & \mathscr{F}'(f^{-1}(V)) = f_*\mathscr{F}'(V) \arrow[l]
\end{tikzcd}$$

Thus, $\varphi_{f^{-1}(-)}$ defines a pre-sheaf morphism $\varphi_{f^{-1}(-)} : f_*\mathscr{F} \to f_*\mathscr{F}'$. Hence, f_* is indeed a functor called the direct image functor. f_* also restricts to a functor

$$f_*|_{\mathsf{Shv}} : \mathsf{Shv}(X, R\text{-}\mathsf{Mod}) \to \mathsf{Shv}(Y, R\text{-}\mathsf{Mod})$$

This illustrates the functorial nature of sheaf-theoretic constructions—something we will explore further in later sections.

13. Gluing Sheaves

A central theme in modern geometry and topology is the principle of constructing global objects from compatible local data. In the theory of sheaves, this idea is made precise through the process of gluing. The ability to glue sheaves is fundamental not only to the construction of sheaves themselves but also to many deeper results, such as the formulation of sheaf cohomology and the development of schemes in algebraic geometry. We discuss how to glue morphisms of sheaves and sheaves.

Remark 13.1. *We work with sheaves taking values in a concrete category, such as* Ab *(the category of abelian groups) since sheaves defining schemes discussed later on take values in concrete categories. Arguments below can be adapted to apply to* C*-valued sheaves, where* C *is a locally small category, by using the Yoneda embedding to reduce to the case of set-valued sheaves.*

Gluing morphisms of sheaves is the easiest of case.

Proposition 13.2. *Let* X *be a topological space and* $\{U_i\}_{i\in I}$ *is an open cover of* X*. Let* $\mathscr{F}$ *and* $\mathscr{G}$ *be sheaves on* X *taking values in a concrete category such as* Ab*. Suppose we are given, for each* $i \in I$*, a morphism of sheaves* $\varphi_i : \mathscr{F}|_{U_i} \to \mathscr{G}|_{U_i}$ *such that for all* $i, j \in I$*, the restrictions agree on overlaps:*

$$\varphi_i|_{U_i\cap U_j} = \varphi_j|_{U_i\cap U_j}.$$

Then there exists a unique morphism of sheaves $\varphi : \mathscr{F} \to \mathscr{G}$ *satisfying*

$$\varphi|_{U_i} = \varphi_i$$

for all $i \in I$*.*

Proof. Let $V \subseteq X$ be an open set, and let $s \in \mathscr{F}(V)$. We define $\varphi(s) \in \mathscr{G}(V)$. The open sets $V_i := U_i \cap V$ form an open cover of V. For each $i \in I$, consider the section

$$\varphi_i(s|_{V_i}) \in \mathscr{G}(V_i).$$

We have

$$\varphi_i(s)|_{V_i\cap V_j} = \varphi_i(s|_{V_i\cap V_j}) = \varphi_j(s|_{V_i\cap V_j}) = \varphi_j(s)|_{V_i\cap V_j},$$

which shows that the sections $\varphi_i(s|_{V_i})$ agree on the overlaps $V_i \cap V_j$. By the gluing axiom for the $\mathscr{G}$, there exists a unique section $\varphi(s) \in \mathscr{G}(V)$ such that $\varphi(s)|_{V_i} = \varphi_i(s|_{V_i})$ for all $i \in I$. We define $\varphi(s)$ to be this glued section. By construction, $\varphi|_{U_i} = \varphi_i$ for all $i \in I$. For uniqueness, suppose φ and ψ are two morphisms of sheaves such that $\varphi|_{U_i} = \psi|_{U_i}$ for all $i \in I$. Let $V \subseteq X$ be an open set, and let $s \in \mathscr{F}(V)$. Then for each $i \in I$, we have

$$\varphi(s)|_{V_i} = \varphi_i(s|_{V_i}) = \psi_i(s|_{V_i}) = \psi(s)|_{V_i}.$$

Thus, $\varphi(s)$ and $\psi(s)$ agree on the open cover $\{V_i\}$ of V. By the identity axioms for the sheaf $\mathscr{G}$, it follows that $\varphi(s) = \psi(s)$. Hence, $\varphi = \psi$ as morphisms of sheaves. □

We now discuss how to glue sheaves. Suppose we are given a sheaf $\mathscr{F}_i$ on each open set U_i of an open cover $\{U_i\}_{i\in I}$ of a topological space X. The goal is to construct a global sheaf $\mathscr{F}$ on X such that $\mathscr{F}|_{U_i} \cong \mathscr{F}_i$ for each $i \in I$. A necessary condition for such a sheaf $\mathscr{F}$ to exist is that the local sheaves $\mathscr{F}_i$ must be isomorphic on the overlaps $U_i \cap U_j$. Moreover, by providing a collection of isomorphisms on the intersections together with compatibility on triple overlaps (i.e., satisfying the cocycle condition), this gluing data becomes not only necessary but also sufficient to construct such a global sheaf.

Proposition 13.3 (Hartshorne II.1.22). *Let X be a topological space with open cover $\{U_i\}_{i\in I}$. Suppose we are given:*

1. *for each $i \in I$, a sheaf $\mathscr{F}_i$ on U_i taking values in a concrete category such as* Ab,
2. *for each pair $i, j \in I$, an isomorphism of sheaves*

$$\tau_{ji} : \mathscr{F}_i|_{U_{ij}} \xrightarrow{\sim} \mathscr{F}_j|_{U_{ij}},$$

where $U_{ij} := U_i \cap U_j$,

satisfying the following cocycle conditions:

(a) $\tau_{ii} = \mathrm{Id}_{\mathscr{F}_i}$ *for all $i \in I$,*
(b) $\tau_{ji} = \tau_{ij}^{-1}$ *for all $i, j \in I$,*
(c) $\tau_{ki} = \tau_{kj} \circ \tau_{ji}$ *over $U_{ijk} := U_i \cap U_j \cap U_k$ for all $i, j, k \in I$.*

Then there exists a sheaf $\mathscr{F}$ on X taking values in a concrete category such as Ab, *together with isomorphisms*

$$\nu_i : \mathscr{F}|_{U_i} \xrightarrow{\sim} \mathscr{F}_i,$$

such that for all $i, j \in I$, the following compatibility condition holds over U_{ij}:

$$\nu_j = \tau_{ji} \circ \nu_i|_{U_{ij}}.$$

Moreover, the sheaf $\mathscr{F}$, together with the isomorphisms $\{\nu_i\}$, is unique up to unique isomorphism.

Proof. (Sketch) We first define $\mathscr{F}$ as a presheaf. A section of $\mathscr{F}$ over an open set $V \subseteq X$ is given by a collection of compatible sections $s_i \in \mathscr{F}_i(V_i)$, where $V_i := U_i \cap V$, satisfying the condition that for all $i, j \in I$, the identifications

$$\tau_{ji}(s_i|_{V_{ij}}) = s_j|_{V_{ij}} \quad \text{in } \mathscr{F}_j(V_{ij})$$

hold, where $V_{ij} := U_{ij} \cap V$. We define the set of sections of $\mathscr{F}$ over V as:

$$\mathscr{F}(V) := \left\{ (s_i)_{i\in I} \in \prod_{i\in I} \mathscr{F}_i(V_i) \;\middle|\; \tau_{ji}(s_i|_{V_{ij}}) = s_j|_{V_{ij}} \text{ for all } i, j \in I \right\}.$$

The restriction maps are defined componentwise: if $W \subseteq V$, then the restriction map is

$$\begin{aligned} \mathscr{F}(V) &\to \mathscr{F}(W) \\ (s_i)_{i\in I} &\mapsto (s_i|_{W_i})_{i\in I}. \end{aligned}$$

This is well-defined because the transition isomorphisms τ_{ji} are compatible with restrictions; that is,

$$\tau_{ji}(s_i|_{W_{ij}}) = s_j|_{W_{ij}} \quad \text{whenever} \quad \tau_{ji}(s_i|_{V_{ij}}) = s_j|_{V_{ij}}.$$

We next check the two sheaf axioms.

1. (Identity Axiom) Let $s = (s_i) \in \mathscr{F}(V)$ be a section, and suppose that $s|_{V_\alpha} = 0$ for every open set V_α in an open cover $\{V_\alpha\}_{\alpha\in\Lambda}$ of V. Then, for each i and α, we have

$$s_i|_{U_i \cap V_\alpha} = 0 \quad \text{in } \mathscr{F}_i(U_i \cap V_\alpha).$$

Since the sets $\{U_i \cap V_\alpha\}_{\alpha\in\Lambda}$ form an open cover of $U_i \cap V$, and each $\mathscr{F}_i$ is a sheaf on U_i, it follows that

$$s_i = 0 \quad \text{in } \mathscr{F}_i(U_i \cap V).$$

As this holds for every i, we conclude that

$$s = 0 \quad \text{in } \mathscr{F}(V).$$

(2) (Gluing Axiom) Let $\{s_\alpha\}$, with $s_\alpha \in \mathscr{F}(V_\alpha)$, be a compatible family of sections over an open cover $\{V_\alpha\}_{\alpha\in\Lambda}$ of V. Compatibility means that for all α, β,

$$s_\alpha|_{V_{\alpha\beta}} = s_\beta|_{V_{\alpha\beta}},$$

where $V_{\alpha\beta} := V_\alpha \cap V_\beta$. Fixing $i \in I$, this induces a compatible family of sections

$$s_{\alpha,i} := s_\alpha|_{U_i \cap V_\alpha} \in \mathscr{F}_i(U_i \cap V_\alpha).$$

Since each $\mathscr{F}_i$ is a sheaf on U_i, the sections $\{s_{\alpha,i}\}_\alpha$ glue uniquely to a section

$$s_i \in \mathscr{F}_i(U_i \cap V).$$

By construction, the compatibility condition on the transition maps holds:

$$\tau_{ij}(s_j)|_{U_{ij}\cap V} = s_i|_{U_{ij}\cap V},$$

since this equality holds on each $V_\alpha \cap U_{ij}$ and the s_α are compatible. Hence, the tuple $s = (s_i)$ defines an element of $\mathscr{F}(V)$, which by construction restricts to s_α on each V_α.

The proof that $\mathscr{F}|_{V_i} \cong \mathscr{F}_i$ is omitted here. This identification relies on condition (iii) of the gluing data. This completes the proof. $\square$

Part 3. Schemes

14. Locally Ringed Spaces

Geometrically, a spectrum of a ring with its structure sheaf encodes both the topological space of prime ideals of R and the ring of functions defined locally on this space. This naturally leads to a more general framework: a category of geometric spaces equipped with a sheaf of rings. To capture this structure abstractly, we introduce the category of locally ringed spaces, which formalizes the essential features of affine schemes and provides the appropriate categorical setting in which schemes naturally reside. We first define a ringed space.

Definition 14.1. A **ringed space** is a pair $(X, \mathscr{F}_X)$, where X is a topological space and $\mathscr{F}_X$ is a CRing-valued sheaf on X. A morphism of ringed spaces from $(X, \mathscr{F}_X)$ to $(Y, \mathscr{G}_Y)$ is a pair $(f, f^{\#})$, where $f : X \to Y$ is a continuous map and

$$f^{\#} : \mathscr{G}_Y \to f_* \mathscr{F}_X$$

is a morphism of sheaves of rings on Y.

Remark 14.2. *The definition of morphisms of ringed spaces given in* **??** *is inspired by the classical setting of affine varieties, where morphisms are continuous maps compatible with the pullback of regular functions. In particular, if X, Y are morphisms of affine varieties defined over an algebraically closed field, a morphism of affine varieties*

$$f : X \to Y$$

is precisely a continuous map such that the pullback defines a ring homomorphism

$$\mathscr{R}_Y(U) \to \mathscr{R}_X(f^{-1}(U)),$$

where $\mathscr{R}_{X,Y}$ are the sheaf of regular functions on X, Y respectively and $U \subseteq Y$ is an open set (Definition 4.7). In other words, we have a morphism of sheaves $\mathscr{R}_Y \to f_* \mathscr{R}_X$.

In a ringed space $(X, \mathscr{F}_X)$, it is natural to ask how to evaluate a section at a point. Given an open set $U \subseteq X$, a section $f \in \mathscr{F}_X(U)$, and a point $x \in U$, the germ $f_x \in \mathscr{F}_{X,x}$ captures the local behavior of f near x, but does not in itself define evaluation unless the stalks carry additional structure. In typical examples—such as the structure sheaf on $\operatorname{Spec} R$ or the sheaf of smooth functions on a manifold—the stalks are local rings. Each stalk $\mathscr{F}_{X,x}$ has a unique maximal ideal $\mathfrak{m}_x$ consisting of germs vanishing at x, and evaluation corresponds to the image of f_x in the residue field $\mathscr{F}_{X,x}/\mathfrak{m}_x$. This motivates the notion of a locally ringed space, where the stalks are required to be local rings, ensuring that evaluation at points is well-defined.

Definition 14.3. A ringed space $(X, \mathscr{F}_X)$ is a **locally ringed space** if for each point $x \in X$, the stalk $\mathscr{F}_{X,x}$ is a local ring with unique maximal ideal $\mathfrak{m}_x$. A morphism of locally ringed spaces $(f, f^{\#}) : (X, \mathscr{F}_X) \to (Y, \mathscr{F}_Y)$ is a morphism of ringed spaces such that the induced map

$$f^{\#}_x : \mathscr{F}_{Y,y} \to \mathscr{F}_{X,x}$$

maps $\mathfrak{m}_y$ into $\mathfrak{m}_x$ for every $x \in X$, $y \in Y$ such that $f(x) = y$.

Remark 14.4. *We now clarify the definition of a morphism between locally ringed spaces. Let $f : X \to Y$ be a morphism of topological spaces and suppose $x \in X$ with $f(x) = y$.*

The morphism of sheaves $f^\# : \mathscr{O}_Y \to f_*\mathscr{O}_X$ induces, for every open subset $V \subseteq Y$, a ring homomorphism

$$f^\#(V) : \mathscr{O}_Y(V) \to \mathscr{O}_X(f^{-1}(V)).$$

As V varies over all open neighborhoods of y, the preimages $f^{-1}(V)$ form a cofinal system of open neighborhoods of x. Passing to colimits, we obtain an induced map on stalks:

$$f^\#_x : \mathscr{O}_{Y,y} = \varinjlim_{y \in V} \mathscr{O}_Y(V) \longrightarrow \varinjlim_{x \in f^{-1}(V)} \mathscr{O}_X(f^{-1}(V)) = \mathscr{O}_{X,x}.$$

Hence, any morphism of ringed spaces $(f, f^\#)$ gives rise to a local map of stalks at each point $x \in X$. In the context of locally ringed spaces, we require that this induced map

$$f^\#_x : \mathscr{O}_{Y,y} \to \mathscr{O}_{X,x}$$

be a local homomorphism of local rings[10]*.*

Example 14.5. Let R be a ring. $(\operatorname{Spec} R, \mathscr{O}_{\operatorname{Spec} R})$ is a locally ringed space.

Remark 14.6. *It can be checked that locally ringed spaces assemble into a category, denoted* $\mathsf{LocRing}$*. For example, one can verify that the composition of morphisms of locally ringed spaces is a well-defined notion.*

15. Affine Schemes

Having introduced the spectrum of a ring and constructed the structure sheaf on this space, we are now ready to give the formal definition of an affine scheme.

Definition 15.1. Let R be a ring. An **affine scheme** is a pair $(\operatorname{Spec} R, \mathscr{O}_{\operatorname{Spec} R})$ consisting of the topological space $\operatorname{Spec} R$ together with the structure sheaf, $\mathscr{O}_{\operatorname{Spec} R}$.

Example 15.2 (Hartshorne II.2.11)**.** We describe the affine scheme $\operatorname{Spec} \mathbb{F}_p[x]$. Since $\mathbb{F}_p[x]$ is a PID, the set of prime ideals is in 1-1 correspondence with irreducible monic polynomials in $\mathbb{F}_p[x]$. Therefore,

$$\operatorname{Spec} \mathbb{F}_p[x] = \{(0)\} \cup \{(f) \mid f \text{ is an irreducible monic polynomial in } \mathbb{F}_p[x]\},$$

When $f = 0$, then $\mathscr{O}_{\operatorname{Spec} \mathbb{F}_p[x], 0} = \operatorname{Spec} \mathbb{F}_p[x]_{(0)} \cong \operatorname{Spec} \mathbb{F}_p(x)$ and the maximal idea $\mathfrak{m}_0$ is the zero ideal since $\operatorname{Spec} \mathbb{F}_p(x)$ is a field. When f is a non-zero irreducible monic polynomial of degree n, note that by definition,

$$\operatorname{Spec} \mathbb{F}_p[x]_{(f)} = \{g/h : g, h \in \operatorname{Spec} \mathbb{F}_p[x] \ f \nmid h\}$$

Recall that $\operatorname{Spec} \mathbb{F}_p[x]_{(f)}$ is a local ring since (f) is a prime ideal and that the unique maximal ideal of $\operatorname{Spec} \mathbb{F}_p[x]_{(f)}$ is given by:

$$\mathfrak{m}_{(f)} = \{a/b : a, b \in \operatorname{Spec} \mathbb{F}_p[x] \ f \nmid b \ f \mid a\}$$

Here $\mathfrak{m}_{(f)}$ is the ideal $\mathfrak{m} = (f)$ localized at (f).

Let AffSch be the full subcategory of $\mathsf{LocRing}$ consisting of locally ringed spaces isomorphic to the spectrum of some ring. We now come to the all-important result: the category AffSch is equivalent to CRing opposite category of commutative rings, establishing a deep duality between algebra and geometry.

Proposition 15.3. AffSch *is equivalent to the category* $\mathsf{Rings}^{\mathrm{op}}$*.*

[10] That is, if $(A, \mathfrak{m}_A)$ and $(B, \mathfrak{m}_B)$ are local rings, a ring homomorphism $\varphi : R \to S$ is called local if $\varphi^{-1}(\mathfrak{m}_B) = \mathfrak{m}_A$.

Remark 15.4. *We write* $\mathsf{AffSch} \simeq \mathsf{CRing}^{\mathrm{op}}$*.*

PROOF. Consider the functor:

$$\Gamma : \mathsf{AffSch} \longrightarrow \mathsf{Rings}^{\mathrm{op}}$$
$$(\operatorname{Spec} R, \mathscr{O}_{\operatorname{Spec} R}) \mapsto \mathscr{O}_{\operatorname{Spec} R}(\operatorname{Spec} R) = R$$

We show that Γ is fully faithful and essentially surjective. The latter follows since Proposition 11.8 implies that $\mathscr{O}_{\operatorname{Spec} R}(\operatorname{Spec} R) \cong R$. We now show that Γ is fully faithful by showing that for any $R, S \in \mathsf{CRing}$, the map

$$\tau_{R,S} : \mathrm{Hom}_{\mathsf{Rings}}(R, S) \longrightarrow \mathrm{Hom}_{\mathsf{AffSch}}(\operatorname{Spec} S, \operatorname{Spec} R)$$

defined by Lemma 5.12 is bijective. Let $\varphi : R \to S$ be a ring homomorphism, and let $f : \operatorname{Spec} S \to \operatorname{Spec} R$ induced map as in Lemma 5.12. We must show that f induces a morphism on the structure sheaves in order for it to define a morphism of locally ringed spaces. It suffices to define the morphism

$$f^{\#} : \mathscr{O}_{\operatorname{Spec} R} \to f_*(\mathscr{O}_{\operatorname{Spec} S})$$

on distinguished open sets. First note that

$$\mathscr{O}_{\operatorname{Spec} R}(U_h) \cong R_h,$$
$$\mathscr{O}_{\operatorname{Spec} R}(f^{-1}(U_h)) \cong S_{\varphi(h)}.$$

There is a ring homomorphism $R \to S \to S_{\varphi(g)}$. Since the image of g is invertible in $S_{\varphi(g)}$, the universal property of localization gives an induced ring homomorphism $R_g \to S_{\varphi(g)}$. This defines $f^{\#}$ and shows that $\tau_{R,S}$ is well-defined. The candidate for the inverse map is given by

$$\gamma_{R,S} : \mathrm{Hom}_{\mathsf{AffSch}}(\operatorname{Spec} S, \operatorname{Spec} R) \longrightarrow \mathrm{Hom}_{\mathsf{Rings}}(R, S)$$

defined by taking global sections. It is clear that $\gamma_{R,S} \circ \tau_{R,S}$ is the identity map. We now show that $\tau_{R,S} \circ \gamma_{R,S}$ is the identity map. Suppose given a morphism of locally ringed spaces

$$(f, f^{\#}) : \operatorname{Spec} S \to \operatorname{Spec} R$$

Taking global sections, $f^{\#}$ induces a homomorphism of rings $\varphi : R \to S$. Given $\mathfrak{p} \in \operatorname{Spec} S$, we obtain a morphism of local rings on stalks, which is compatible with φ and localization, yielding the following commutative diagram:

$$\begin{tikzcd} R \arrow[r, "\varphi"] \arrow[d] & S \arrow[d] \\ R_{f(\mathfrak{p})} \arrow[r] & S_{\mathfrak{p}} \end{tikzcd}$$

Since $f^{\#}$ is a local homomorphism, it follows that $\varphi^{-1}(\mathfrak{p}) = f(\mathfrak{p})$ [11]. This shows that the underlying map f coincides with the canonical map

$$\operatorname{Spec} R \xrightarrow{f} \operatorname{Spec} S$$

[11] Suppose $r \notin f(\mathfrak{p})$. Then the image of r in $R_{f(\mathfrak{p})}$ is a unit, so $f_p^{\#}(r)$ is a unit in $S_{\mathfrak{p}}$. Hence, $\varphi(r) \notin \mathfrak{p}$, i.e., $r \notin \varphi^{-1}(\mathfrak{p})$. Conversely, assume $r \notin \varphi^{-1}(\mathfrak{p})$, so that $\varphi(r) \notin \mathfrak{p}$, and thus the image of $\varphi(r)$ is a unit in $S_{\mathfrak{p}}$. Since $f_p^{\#} : R_{f(\mathfrak{p})} \to B_{\mathfrak{p}}$ is a local homomorphism, the preimage of a unit is a unit. Therefore, the image of r in $R_{f(\mathfrak{p})}$ is a unit, which implies $r \notin f(p)$. Thus, we conclude that $f(\mathfrak{p}) = \varphi^{-1}(\mathfrak{p})$.

induced by φ. It then follows that $f^{\#}$ is also the structure sheaf morphism induced by φ, so that the morphism $(f, f^{\#})$ of locally ringed spaces is indeed induced by the ring homomorphism φ. This shows $\tau_{R,S} \circ \gamma_{R,S}$ is the identity map $\square$

Example 15.5 (Hartshorne II.2.5)**.** $\mathbb{Z}$ is the initial object in CRing because any commutative ring R, there exists a unique ring homomorphism

$$\begin{aligned} \varphi : \mathbb{Z} &\to R \\ 1 &\mapsto 1_R \end{aligned}$$

Therefore, Proposition 15.3 implies that $\operatorname{Spec} \mathbb{Z}$ is the final object in AffSch. It follows that in AffSch, for any affine scheme $\operatorname{Spec} R$, there exists a unique morphism

$$\operatorname{Spec} R \to \operatorname{Spec} \mathbb{Z}.$$

We have established the category of affine schemes and demonstrated the fundamental result:

The category of affine schemes is equivalent
to the opposite category of commutative rings.

However, this does not capture the full scope of schemes. To illustrate the limitation, consider the analogy with complex manifolds, which are geometric spaces locally modeled on $(\mathbb{C}^n, \mathscr{H})$, where $\mathscr{H}$ denotes the sheaf of holomorphic functions. In this analogy, the category of affine schemes corresponds to the local models $(\mathbb{C}^n, \mathscr{H})$. Therefore, a *general scheme* is a topological space equipped with a sheaf of rings that is locally isomorphic to an affine scheme. Schemes are discussed in the next section.

16. General Schemes

A scheme is a locally ringed space that is locally isomorphic to an affine scheme. This construction mirrors the classical notion of a smooth manifold: whereas a manifold is a topological space locally modeled on open subsets of $\mathbb{R}^n$, equipped with a sheaf of smooth functions, a scheme is built by gluing together spectra of rings, each with a corresponding sheaf of regular functions. This local-to-global approach allows schemes to capture both geometric and arithmetic information in a unified framework.

Definition 16.1. A **scheme** is a locally ringed space $(X, \mathscr{F}_X)$ such that every $x \in X$ has an open neighborhood $x \in U_x$ such that $(U_x, \mathscr{F}_X|_{U_x})$ is an affine scheme.

Example 16.2. Let R be a ring. Then $(\operatorname{Spec} R, \mathscr{O}_{\operatorname{Spec} R})$ is a (affine) scheme.

A morphism of schemes is defined as a morphism of the underlying locally ringed spaces. That is, it consists of a continuous map between the underlying topological spaces together with a morphism of structure sheaves that respects the local ring structure at each point. This definition places the category of schemes, denoted by Sch, as a subcategory of the category of locally ringed spaces, $\mathsf{LocRing}$.

Remark 16.3. *The category of affine schemes, denoted* AffSch*, is a full subcategory of* Sch*.*

16.1. Basic Topological Properties.

We establish some basic topological properties of schemes. The topological space underlying a scheme is not generally expected to be Hausdorff. Indeed, as we have seen, even an affine scheme is not Hausdorff. However, an affine scheme is a T_0-space, and since any scheme can be covered by affine open subsets, we expect that a general scheme is also a T_0-space.

Proposition 16.4. *Let X be a scheme. Then X is a T_0-space*

Proof. e can cover X by affine open subsets. If x and y are contained in a common affine open subset, then the T_0-property follows from the fact that any affine scheme is a T_0-space (Proposition 6.4). Otherwise, there exists an affine open neighborhood of one point, say x, which does not contain the other point y. □

We now turn to a basic property of generic points in a scheme.

Proposition 16.5. *[Hartshorne II.2.9] If X is a scheme, then every non-empty irreducible closed subset $Z \subseteq X$ has a unique generic point, ξ.*

Proof. The proof is given below:

(1) Assume first that X is an affine scheme. Since $Z \subset X$ is a closed and irreducible subset, there exists a prime ideal $\mathfrak{p} \subset A$ such that $Z = V(\mathfrak{p})$. In particular, $\mathfrak{p} \in Z$, and by Proposition 6.1, we have
$$\overline{\{\mathfrak{p}\}} = V(\mathfrak{p}) = Z.$$
(2) Now assume that X is a general scheme. Let $x \in Z \neq \emptyset$, and let $U \subset X$ be a non-empty affine open subset containing x. Consider the set $V := U \cap Z$. Then V is a non-empty closed and irreducible subset of U. By (1), there exists a point $\xi \in U$ such that
$$\overline{\{\xi\}} = V \subseteq Z,$$
where the closure is taken in U. Consider $\overline{\{\xi\}}$ in X. Observe that we can write Z as the union of two closed subsets:
$$Z = \overline{\{\xi\}} \cup (U^c \cap Z) .$$
Since Z is irreducible and $Z \neq U^c \cap Z$, it follows that $Z = \overline{\{\xi\}}$.
(3) We now show uniqueness. Assume that there exist two distinct points $\xi_{1,2}$ such that $\overline{\{\xi_{1,2}\}} = Z$. Since X is a T_0-space (Proposition 16.4), there exists an open subset $U \subseteq X$ such that $\xi_1 \in U$ and $\xi_2 \notin U$. However, this contradicts the assumption that $\xi_1 \in \overline{\{\xi_2\}}$, which implies that every open neighborhood of ξ_1 must also contain ξ_2. Thus, $\xi_1 = \xi_2$.

This completes the proof. □

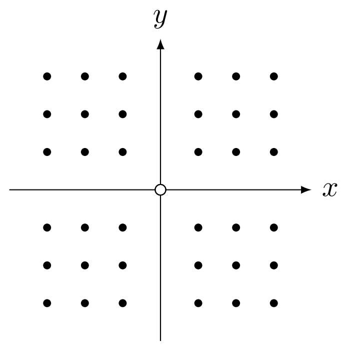
16.2. **Open Subschemes.** Recall that an open subset of a smooth manifold naturally inherits the structure of a manifold. We ask whether an analogous statement holds for schemes.

Proposition 16.6. *[Hartshorne II.2.1 & II.2.2] Let $(X, \mathscr{F}_X)$ be a scheme.*

(1) *If $(X, \mathscr{F}_X) \cong (\operatorname{Spec} R, \mathscr{O}_{\operatorname{Spec} R})$ is an affine scheme, then the locally ringed space $(U_f, \mathscr{O}_X|_{U_f})$ is isomorphic to $(\operatorname{Spec} R_f, \mathscr{O}_{\operatorname{Spec} R_f})$.*
(2) *Let $U \subseteq X$ be any open subset. Then $(U, \mathscr{F}_X|_U)$ is a scheme.*

Proof. The proof is given below:

(1) Recall that we have the following bijective correspondence:
$$\{\text{Prime ideals of } R_f\} \longleftrightarrow \{\text{Prime ideals of } R \text{ that don't contain } f\}$$



The scheme $\operatorname{Spec}\mathbb{K}[x,y]\setminus\{(0,0)\}$

Hence, $\operatorname{Spec} R_f \cong U_f$ as sets. This correspondence extends to the level of topological spaces. Indeed, $\{\operatorname{Spec} R_g\}_{g\in R}$ is a basis for the topology on $\operatorname{Spec} R$. On the other hand, the sets

$$\{\operatorname{Spec} R_f \cap \operatorname{Spec} R_g\}_{g\in R}$$

form a basis for the topology on $\operatorname{Spec} R_f$, and this is also a basis for the subspace topology on U_f. To conclude that $\operatorname{Spec} R_f \cong U_f$ as locally ringed spaces, it remains to verify that the structure sheaves agree under this identification. This follows from the construction of the structure sheaf: the stalks at a prime ideal $\mathfrak{p}$ in U_f are isomorphic to $R_{\mathfrak{p}}$, and since $f \notin \mathfrak{p}$, the element f is invertible in $R_{\mathfrak{p}}$. Thus, localization at f does not affect the local behavior at such prime ideals.

(2) For each $x \in U$ let $x \in V_x$ be an open set such that

$$(V_x, \mathscr{F}_X|_{V_x}) \cong (\operatorname{Spec} R_x, \mathscr{O}_{\operatorname{Spec} R_x}).$$

Because $x \in U \cap V_x$ is an open set in $(\operatorname{Spec} R_x, \mathscr{O}_{\operatorname{Spec} R_x})$, there exists a distinguished open set $x \in U_f \subseteq U \cap V_x$. By (1),

$$(U_f, \mathscr{O}_X|_{U_f}) \cong (\operatorname{Spec} R_f, \mathscr{O}_{\operatorname{Spec} R_f}).$$

Thus $(U, \mathscr{F}_X|_U)$ is a scheme.

This completes the proof. □

Remark 16.7. *Proposition 16.6 implies that the underlying topological space of every scheme has a base of open affine schemes*

Remark 16.8. *We refer to $(U, \mathscr{O}_X|_U)$ as an open sub-scheme of X.*

Armed with the notion of an open subscheme, we are now ready to examine an example of a scheme that is not affine. This example is geometrically well-motivated and illustrates that the passage from affine schemes to general schemes is both natural and necessary.

Example 16.9. Let $\mathbb{K}$ be a field. Consider the affine scheme $(\operatorname{Spec}\mathbb{K}[x,y], \mathscr{O}_{\operatorname{Spec}\mathbb{K}[x,y]})$. Let

$$X = \operatorname{Spec}\mathbb{K}[x,y] - \{(x,y)\}$$

Note that $X = U_x \cup U_y$. Hence, X is an open set of our affine scheme. Therefore, $(X, \mathscr{O}_{\operatorname{Spec}\mathbb{K}[x,y]}|_X)$ is an open sub-scheme. We show that this sub-scheme is not an affine scheme. We compute $\mathscr{O}_{\operatorname{Spec}\mathbb{K}[x,y]}|_X$. We find rational functions defined on U_x and U_y that agree on the intersection $U_x \cap U_y = U_{xy}$. Clearly, rational functions that have only powers of x in the denominator and also only powers of y in the denominator must, in fact, be polynomials. Thus, we conclude[12]:

$$\mathscr{O}_{\operatorname{Spec}\mathbb{K}[x,y]}|_X \cong \mathbb{K}[x,y].$$

If X were affine, then

$$\operatorname{Spec}\mathbb{K}[x,y] - \{(x,y)\} \cong X \cong \mathscr{O}_X(X) \cong \operatorname{Spec}\mathbb{K}[x,y]$$

However, X is not homeomorphic to $\operatorname{Spec}\mathbb{K}[x,y]$. Every proper ideal in an affine scheme has a nonempty vanishing locus, yet the ideal (x,y) has empty vanishing locus in X.

16.3. **Gluing Over a Base.** We will encounter a variety of constructions that demonstrate how to glue morphisms of schemes, as well as how to glue schemes themselves. These are essential techniques that enable us to build schemes and morphisms from local data, following a bottom-up approach. Rather than presenting all such constructions at once, we will introduce them as needed. In what follows, we begin by discussing how to glue morphisms of schemes when given morphisms defined on an open cover of the underlying topological space of a scheme. This result will be applied in the next section.

Proposition 16.10. *Let X and Y be schemes, and let $\mathscr{B}$ be a basis for the topology on X. Suppose we are given a family of morphisms*

$$\{f_U : U \to Y\}_{U\in\mathscr{B}},$$

such that for all $V, U \in \mathscr{B}$ with $V \subseteq U$, the restriction satisfies

$$f_U|_V = f_V.$$

Then there exists a unique morphism of schemes $\varphi : X \to Y$ such that for every $U \in \mathscr{B}$, the restriction of φ to U agrees with φ_U, i.e.,

$$f|_U = f_U.$$

Proof. Define a map $f : X \to Y$ by setting $f(x) := f_U(x)$, where $U \in \mathscr{B}$ is any open neighborhood of x. The assumption $f_U|_V = f_V$ for $V \subseteq U$ ensures this is well-defined. Since each f_U is continuous and $\mathscr{B}$ is a basis, it follows that f is continuous. We now define a morphism of sheaves

$$f^\# : \mathscr{F}_Y \to f_*\mathscr{F}_X.$$

Let $W \subseteq Y$ be an open set. For any basic open set $U \in \mathscr{B}$ such that $U \subseteq f^{-1}(W)$, we have a morphism

$$f_U^\#(W) : \mathscr{F}_Y(W) \longrightarrow \mathscr{F}_X|_U\big(f_U^{-1}(W)\big) = \mathscr{F}_X\big(f^{-1}(W)\cap U\big) = \mathscr{F}_X(U).$$

If $V \subseteq U$, we have the following commutative diagram:

$$\begin{tikzcd} \mathscr{F}_Y(W) \arrow[r, "f_U^\#(W)"] \arrow[dr, "f_V^\#(W)"'] & \mathscr{F}_X(U) \arrow[d, "\mathrm{res}_{U,V}"] \\ & \mathscr{F}_X(V) \end{tikzcd}$$

which is induced on sheaves by the relation

$$f_V = f_U|_V = f_U \circ i_V,$$

[12]In other words, the removal of the origin does not introduce any new global regular functions.

where $i_V : V \hookrightarrow U$ is the inclusion. For any element $s \in \mathscr{F}_Y(W)$, the family of sections

$$\left\{ f_U^{\#}(W)(s) \in \mathscr{F}_X(U) \right\}_{U \in \mathscr{B}, U \subseteq f^{-1}(W)}$$

defines a unique element of $\mathscr{F}_X(f^{-1}(W))$. It is clear that this definition is in fact a homomorphism. □

Corollary 16.11. *Let X and Y be schemes, and let $\mathscr{B}$ be a base for the topology on X. Suppose we have a family of open subsets $\{U_W\}_{W \in \mathscr{B}}$ covering Y such that*

$$U_V \subseteq U_W \quad \text{whenever } V \subseteq W \text{ in } \mathscr{B}.$$

Assume there is a family of morphisms of schemes

$$f_W : W \to U_W, \quad \text{for each } W \in \mathscr{B},$$

satisfying the following compatibility conditions if $V \in \mathscr{B}$ is contained in $W L$

(1) $f_W^{-1}(U_V) = V$,
(2) $f_W|^{U_V} = f_V$.

Then there exists a unique morphism of schemes $f : X \to Y$ such that

$$f|_W = f_W \quad \text{for all } W \in \mathscr{B}.$$

Proof. For any $W \in \mathscr{B}$, let $i_W : U_W \hookrightarrow Y$ denote the inclusion morphism. The family of morphisms

$$g_W := i_W \circ f_W : W \to Y$$

satisfies the hypotheses of Proposition 16.10. □

17. Reduced, Integral and Noetherian Schemes

Although the notion of a scheme is extremely general and flexible, this generality comes at a cost: various pathologies may arise in the absence of additional structure. To obtain schemes that more closely resemble classical geometric objects, or that exhibit desirable behavior under common constructions, it is often useful to restrict attention to schemes satisfying certain structural conditions. We introduce several important classes of schemes defined by properties of their rings of sections. In particular, we consider reduced, integral, and Noetherian schemes.

17.1. **Reduced Schemes.** Consider an element of a ring R that vanishes at every point of $\operatorname{Spec} R$; that is, an element contained in every prime ideal of R. One might initially expect such an element to be zero; however, this need not be the case. The condition of lying in all prime ideals is precisely equivalent to belonging to the nilradical of the ring, which may be nonzero[13]. In other words, thinking of elements of R as functions on $\operatorname{Spec} R$, the zero function need not be the only function that vanishes everywhere on $\operatorname{Spec} R$. This observation naturally motivates the following definition.

Definition 17.1. Let $(X, \mathscr{F})$ be a scheme. We say that X is a **reduced scheme** if for every open subset $U \subseteq X$, the ring $\mathscr{F}_X(U)$ has no non-zero nilpotent elements; that is, $\mathscr{F}_X(U)$ is a reduced ring.

Proposition 17.2. *(*Hartshorne II.2.3) *Let $(X, \mathscr{F}_X)$ be a scheme. $(X, \mathscr{F}_X)$ is reduced if and only if the stalk $\mathscr{F}_{X,x}$ is a reduced ring for all $x \in X$.*

[13]Geometrically, this means that a function vanishes at every point of the spectrum of a ring if and only if some power of it is zero.

Proof. Assume that $(X, \mathscr{F}_X)$ is a reduced scheme. For each point $x \in X$, the stalk is given by

$$\mathscr{F}_{X,x} = \varinjlim_{x \in U} \mathscr{F}_X(U),$$

where the limit is taken over all open neighborhoods U of x. Since each $\mathscr{F}_X(U)$ is a reduced ring by assumption, and the colimit of reduced rings is again reduced, it follows that $\mathscr{F}_{X,x}$ is a reduced ring. Conversely, assume that $\mathscr{F}_{X,x}$ is a reduced ring for all $x \in X$.

(1) Suppose that X is an affine scheme, $(X, \mathscr{F}_X) \cong (\operatorname{Spec} R, \mathscr{O}_{\operatorname{Spec} R})$. Then $\mathscr{O}_{\operatorname{Spec} R, \mathfrak{p}} \cong R_{\mathfrak{p}}$ is a reduced ring for all $\mathfrak{p} \in \operatorname{Spec} R$. This implies that R is a reduced ring[14].

(2) For a general scheme $(X, \mathscr{F}_X)$, we can cover X by an open affine cover $\{U_\alpha\}_\alpha$, where each U_α is an open subset isomorphic to $\operatorname{Spec} R_\alpha$ for some ring R_α. By (1), each U_α is a reduced affine scheme. Let $U \subseteq X$ be any open subset. Since the collection $\{U_\alpha\}$ covers X, the family $\{U_\alpha \cap U\}_\alpha$ forms an open cover of U. Suppose $r \in \mathscr{F}_X(U)$ is nilpotent. Then for each α, the restriction of r to $\mathscr{F}_X(U_\alpha \cap U)$ is also nilpotent. Since $\mathscr{F}_X(U_\alpha \cap U)$ is a subring of $\mathscr{F}_X(U_\alpha)$, and each $\mathscr{F}_X(U_\alpha)$ is reduced, it follows that the restriction of r to $\mathscr{F}_X(U_\alpha \cap U)$ must be zero. By the sheaf property, this implies that $r = 0$ in $\mathscr{F}_X(U)$. Hence, $(X, \mathscr{F}_X)$ is a reduced scheme.

This completes the proof. □

Corollary 17.3. *A ring R is a reduced ring if and only if Spec R is reduced.*

Proof. This follows from Proposition 17.2. □

Example 17.4. Let $\mathbb{K}$ be a field. The following is a basic list of examples of reduced and non-reduced schemes:

(1) Since $\mathbb{K}$ is a reduced ring, $\operatorname{Spec} \mathbb{K}[x_1, \ldots, x_n]$ is a reduced scheme.

(2) $\operatorname{Spec} \mathbb{K}[x]/(x^2)$ is a non-reduced scheme. This is because $\mathbb{K}[x]/(x^2)$ is not a reduced ring: the element $\overline{x}$ is nonzero, but satisfies $\overline{x}^2 = 0$ in $\mathbb{K}[x]/(x^2)$.

What if a scheme is not reduced? To any scheme X, one can associate a *reduced scheme* X_{red}, which has the same underlying topological space as X, but is equipped with a morphism of schemes

$$X_{\text{red}} \to X.$$

We call X_{red} the *reduced scheme* associated with X. For example, if $X = \operatorname{Spec} R$ is affine, then

$$X_{\text{red}} := \operatorname{Spec}(R/\mathscr{N}(R)),$$

where $\mathscr{N}(R)$ denotes the nilradical of R. The natural projection homomorphism

$$R \to R/\mathscr{N}(R)$$

induces the corresponding morphism of schemes $X_{\text{red}} \to X$. More generally, we have the following result:

Proposition 17.5. (Hartshorne II.2.3) *Let $(X, \mathscr{F}_X)$ be a scheme. Let $\mathscr{F}_{X,\text{red}}$ be the sheaf associated to the presheaf $U \mapsto \mathscr{F}_X(U)_{\text{red}}$. Then $(X, \mathscr{F}_{X,\text{red}})$ is a scheme, and there is a*

[14] This follows from a commutative algebra fact that R is reduced if and only if $R_{\mathfrak{p}}$ is reduced for all $\mathfrak{p} \in \operatorname{Spec} R$.

morphism of schemes $r : (X, \mathscr{F}_{X,\mathrm{red}}) \to (X, \mathscr{F})$*, which is a homeomorphism on the underlying topological spaces. Moreover,* $(X, \mathscr{F}_{X,\mathrm{red}})$ *satisfies the following universal property: for any morphism* $f : Y \to X$ *of schemes with* Y *reduced, there exists a unique morphism*

$$\theta : Y \to X_{\mathrm{red}}$$

such that the following diagram commutes:

$$\begin{tikzcd} Y \arrow[r, dashed, "\theta"] \arrow[dr, "f"'] & X_{\mathrm{red}} \arrow[d, "r"] \\ & X \end{tikzcd}$$

Remark 17.6. *The scheme* $(X, \mathscr{F}_{X,\mathrm{red}})$ *constructed in Proposition 17.5 is the reduced scheme associated to* X*.*

Proof. Let $\mathscr{B}$ denote the base for the topology of X consisting of all open affine schemes. Let $\mathscr{F}_{X,\mathrm{red}}$ be the sheaf associated to the $\mathscr{B}$-sheaf

$$U_\alpha = \operatorname{Spec} R_\alpha \mapsto R_\alpha/\mathscr{N}(R_\alpha),$$

Then $(X, \mathscr{F}_{X,\mathrm{red}})$ is a scheme. Indeed, it suffices to observe that $\operatorname{Spec}(R_\alpha/\mathscr{N}(R_\alpha))$ is naturally homeomorphic to $\operatorname{Spec}(R_\alpha)$ for any ring R_α. Moreover, the family of morphisms defined on each open affine subset of X by the projection

$$R_\alpha \to R_\alpha/\mathscr{N}(R_\alpha)$$

satisfies Corollary 16.11, and therefore gives rise to a morphism of schemes

$$r : (X_{\mathrm{red}}, \mathscr{F}_{X,\mathrm{red}}) \to (X, \mathscr{F}_X),$$

as required. Given any morphism $f : Y \to X$, and for each open affine subset $U \subseteq X$, we have an induced homomorphism

$$f_U^{\#} : \mathscr{F}_X(U) \to \mathscr{F}_Y(f^{-1}(U)),$$

whose kernel contains the nilradical of $\mathscr{F}_X(U)$ since Y is reduced. Hence, there exists a unique morphism

$$\theta_U^{\#} : \mathscr{F}_{X,\mathrm{red}}(U) \to \mathscr{F}_Y(f^{-1}(U))$$

such that the following diagram commutes:

$$\begin{tikzcd} \mathscr{F}_X(U) \arrow[d, "r_U^{\#}"] \arrow[dr, "f_U^{\#}"] & \\ \mathscr{F}_{X,\mathrm{red}}(U) \arrow[r, "\theta_U^{\#}"] & \mathscr{F}_Y(f^{-1}(U)) \end{tikzcd}$$

These maps define a unique morphism of sheaves $\theta^{\#} : \mathscr{F}_{X,\mathrm{red}} \to f_*\mathscr{F}_Y$ and hence a unique morphism of schemes $\theta : Y \to X_{\mathrm{red}}$ as required. □

17.2. **Integral Schemes.** We have seen that reduced schemes eliminate the pathological behavior caused by nilpotent elements. However, another natural condition we may wish to impose is that the scheme be "irreducible" in a global sense. An integral scheme is one that is both reduced and irreducible. This condition ensures that the structure sheaf behaves like the function field of an affine variety.

Definition 17.7. Let $(X, \mathscr{F})$ be a scheme. We say that X is an **integral scheme** if for every open subset $U \subseteq X$, the ring $\mathscr{F}_X(U)$ is an integral domain.

Proposition 17.8. *Let $(X, \mathscr{F})$ be a scheme. Then $(X, \mathscr{F})$ is an integral scheme if and only if it is both reduced and irreducible.*

Proof. An integral scheme is necessarily reduced, since integral domains contain no nonzero nilpotent elements. Moreover, if X is not irreducible, then there exist disjoint nonempty open subsets $U_1, U_2 \subseteq X$ such that $X = U_1 \cup U_2$. In this case, by the sheaf property, we have

$$\mathscr{F}_X(X) = \mathscr{F}_X(U_1 \cup U_2) = \mathscr{F}_X(U_1) \times \mathscr{F}_X(U_2),$$

which is not an integral domain. Hence, an integral scheme must be both reduced and irreducible. Conversely, assume that X is both irreducible and reduced. Let $U \subseteq X$ be an affine open subset, so that $U \cong \operatorname{Spec} R$ for some ring R. Since X is irreducible, $U = \operatorname{Spec} R$ is irreducible as a topological space (**??**). This implies that the nilradical of R is a prime ideal (Corollary 6.6). On the other hand, since X is reduced the nilradical of R is trivial; that is, it is the zero ideal. Combining these two facts, we see that the zero ideal is prime, which means R is an integral domain. Therefore, $\mathscr{F}_X(U)$ is an integral domain, where U is an open set corresponding to an affine scheme. Since the property of being an integral domain is local and we have verified it for an arbitrary affine open subset $U \subseteq X$, it follows that X is an integral scheme. The argument is similar to the analogous argument given in Proposition 17.2. □

Corollary 17.9. *Let R be a ring. Then $\operatorname{Spec} R$ is an integral if and only if R is an integral domain.*

Proof. If R is an integral domain, then $\operatorname{Spec} R$ is irreducible by Corollary 6.6, and reduced by Corollary 17.3. Conversely, if $\operatorname{Spec} R$ is both irreducible and reduced, then by the definition of an integral scheme and the identification $\mathscr{O}_{\operatorname{Spec} R}(\operatorname{Spec} R) \cong R$, it follows that R is an integral domain. □

Proposition 16.5 implies that an integral scheme, X, has a unique generic point ξ, which is characterized by the property that $X = \overline{\{\xi\}}$. We now that the local ring $\mathscr{F}_{X,\xi}$ of the generic point ξ of an integral scheme X is a field. Indeed, $\mathscr{F}_{X,\xi}$ is defined as the direct limit

$$\mathscr{F}_{X,\xi} = \varinjlim_{U \subseteq X} \mathscr{F}_X(U) = \varinjlim_{\substack{U \subseteq X \\ U \text{ affine}}} \mathscr{F}_X(U).$$

If $f_\xi \in \mathscr{F}_{X,\xi}$, then f_ξ is the equivalence class of a pair (U, f), where U is an open affine subset and $f \in \mathscr{F}_X(U)$. Since $\mathscr{F}_X(U)$ is an integral domain, f defines a *non-empty* distinguished open subset $U_f \subseteq U$. Now,

$$\mathcal{O}_X(U_f) = \mathscr{F}_X(U)_f,$$

and hence the pair $\left(U_f, f^{-1}\right)$ represents the element $f_\xi^{-1} \in \mathcal{O}_{X,\xi}$.

Proposition 17.10. *et X be an integral scheme with generic point ξ.*

(1) *(Hartshorne II.3.6) Let $U = \operatorname{Spec} R$ be any open affine subset of X containing ξ. Then the restriction homomorphism*

$$\mathscr{F}_X(U) \to \mathscr{F}_{X,\xi}$$

induces an isomorphism

$$\operatorname{Frac}(R) \cong \mathscr{F}_{X,\xi}.$$

(2) *By identifying $\mathscr{F}_X(U)$ and $\mathscr{F}_{X,x}$ as subrings of $\mathscr{F}_{X,\xi}$, we have*

$$\mathscr{F}_X(U) = \bigcap_{x \in U} \mathscr{F}_{X,x}.$$

Proof. The proof is given below:

1. The point ξ is also the generic point of U, and $\mathscr{F}_{X,\xi} = \mathscr{F}_{U,\xi}$. Observe that ξ corresponds to the zero ideal.
2. WLOG we may assume that U is affine, say $U = \operatorname{Spec} R$. Let $\gamma \in \operatorname{Frac} R$ be contained in all the localizations $R_{\mathfrak{p}}$ for every $\mathfrak{p} \in \operatorname{Spec} R$. Let

$$I = \{a \in R \mid a\gamma \in R\}$$

Then, recalling the definition of localization, for every $\mathfrak{p}$, there exists some $a \in I \setminus \mathfrak{p}$. This implies that I is not contained in any prime ideal, and therefore $I = R$. In particular, $1 \in I$, so $\gamma \in R$.

This completes the proof. □

Definition 17.11. Let $(X, \mathscr{F})$ be an integral scheme with generic point ξ. The field $\mathscr{F}_{X,\xi}$ by $K(X)$. We call $K(X)$ the **field of rational functions** (or the function field of X) and an element of $K(X)$ is called a **rational function** on X.

Remark 17.12. *We say that $f \in K(X)$ is* regular at $x \in X$ *if $f \in \mathscr{O}_{X,x}$. Proposition 17.10 affirms that a rational function which is regular at every point of an open subset $U \subseteq X$ is contained in $\mathscr{F}_X(U)$.*

Example 17.13. If $\mathbb{K}$ is algebraically closed, then $\operatorname{Spec} \mathbb{K}[x_1, \cdots, x_n]$ is an integral affine scheme. Its generic point corresponds to the zero ideal, and its field of rational functions is $K(x_1, \cdots, x_n)$. A rational function is thus given by a quotient of two polynomials, and it is regular on the whole space if and only if it is given by a single polynomial.

17.3. **Noetherian Schemes.** In many situations, it is useful to impose finiteness conditions on schemes to ensure manageable behavior both algebraically and topologically. One such condition comes from the notion of Noetherian rings. A Noetherian scheme is a scheme that is locally built from Noetherian rings and satisfies a finiteness condition on its topology. These schemes form a broad and important class that includes most examples of interest in algebraic geometry.

Definition 17.14. Let X be a scheme.

1. X is called a **locally Noetherian scheme** if it admits an open affine cover $\{\operatorname{Spec} R_i\}_{i \in I}$ such that each R_i is a Noetherian ring.
2. X is called a **Noetherian scheme** if it is locally Noetherian and quasicompact; equivalently, if it admits a finite open affine cover $\{\operatorname{Spec} R_i\}_{i=1}^n$ with each R_i a Noetherian ring.

In the definition of a locally Noetherian scheme, we do not require every open affine subset of X to be the spectrum of a Noetherian ring. Thus, while it is immediate from the definition that the spectrum of a Noetherian ring is a Noetherian scheme, the converse is less obvious. Establishing this converse amounts to proving that Noetherian-ness is a *local property* of schemes.

Proposition 17.15. *A scheme X is locally Noetherian if and only if for* every *open affine subset $U = \operatorname{Spec}(R)$, R is a Noetherian ring. In particular, an affine scheme $X = \operatorname{Spec}(R)$ is a Noetherian scheme if and only if the ring R is a Noetherian ring.*

Proof. Skipped. □

Part 4. **References**

References

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