HOMOTOPY THEORY

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ABSTRACT. These are assorted notes on homotopy theory in the context of algebraic topology. I took these notes during graduate school. There may be typos; please send corrections to junaid.aftab1994@gmail.com.

CONTENTS

1. CW Complexes

1.1. **Definitions.** Another primary class of spaces we aim to understand using topological invariants in algebraic topology are topological manifolds. An arbitrary topological space, X, can be difficult to visualize and analyze. We shall focus mostly focus on the subcategory of topological spaces that can be constructed inductively using open cells. This will be category of CW-complexes. This approach will allow us to meaningfully study a lot of topological spaces.

Definition 1.1. An open n-cell is a topological space that is homeomorphic to the open unit ball $\mathbb{B}^n$. A closed n-cell is a topological space homeomorphic to $\mathbb{D}^n$.

Remark 1.2. *We will only use the phrase n-cell when the context is clear.*

New topological spaces can be constructed from old topological spaces by attaching an n-cell. Let X be a topological space. Suppose there is a map $\varphi : \mathbb{S}^{n-1} \to X$ a map. One can form a new topological space, $X \coprod_\varphi \mathbb{D}^n$, from the disjoint union $X \coprod \mathbb{D}^n$ by identifying each $\varphi(x) \in \mathbb{S}^{n-1}$ with $\varphi(x) \in X$ for all $x \in \mathbb{S}^{n-1}$, and equipping the resulting set with the quotient topology. The map φ is called the characteristic map. We refer to the space $X \coprod_\varphi \mathbb{D}^n$ as being obtained from X by 'attaching an n-cell', and call $\varphi : \mathbb{S}^{n-1} \to X$ the attaching map. Using the the universal properties of the disjoint union and quotient topology, we have the following commutative diagram.

$$\begin{tikzcd}
\mathbb{S}^{n-1} \arrow[r, "\varphi"] \arrow[d, hook, "\iota"] & X \arrow[d] \arrow[ddr, bend left, "f_1"] & \\
\mathbb{D}^n \arrow[r] \arrow[drr, bend right, "f_2"'] & X \coprod_\varphi \mathbb{D}^n \arrow[dr, dotted, "f"] & \\
& & Y
\end{tikzcd}$$

Remark 1.3. *In fact, this shows that $X \coprod_\varphi \mathbb{D}^n$ is a pushout in* **Top**.

One can also attach more than one n-cell. Let $\{\mathbb{D}^n_i\}_{i\in I_n}$ be a collection of n-cells and let $\varphi^n_i : \mathbb{S}^{n-1}_i \to X$ be a collection of continuous maps. One can form a new topological space, $X \coprod_{\substack{i\in I_n \\ \varphi_i}} \mathbb{D}^n_i$, by attaching the aforementioned collection of n-cells using the rule prescribed above. Once again, we have a commutative diagram:

$$\begin{tikzcd}
\coprod_{\substack{i\in I_n \\ \varphi^n_i}} \mathbb{S}^{n-1}_i \arrow[r, "f"] \arrow[d, hook, "\iota"] & X \arrow[d] \\
\coprod_{\substack{i\in I_n \\ \varphi^n_i}} \mathbb{D}^n_i \arrow[r] & X \coprod_{\substack{i\in I_n \\ \varphi^n_i}} \mathbb{D}^n_i
\end{tikzcd}$$

Remark 1.4. *This shows that* $X \coprod_{\substack{i\in I_n \\ \varphi_i^n}} \mathbb{D}_i^n$ *is a pushout in* **Top**.

Definition 1.5. Let X be a topological space. A CW decomposition of X is a sequence of subspaces

$$X^0 \subseteq X^1 \subseteq X^2 \subseteq \cdots \subseteq X^n \subseteq \cdots \quad n \in \mathbb{N},$$

of X such that the following three conditions are satisfied:

(1) The space X^0 is discrete.
(2) The space X^n is obtained from X^{n-1} by attaching a (possibly) infinite number of n-cells $\{\mathbb{D}_i^n\}_{i\in I_n}$ via attaching maps $\varphi_i : \mathbb{S}_i^{n-1} \to X^{n-1}$.
(3) The topology of X is compatible with quotient topology on X that makes the

$$\coprod_{n\in\mathbb{N}} X^n \to X$$

continuous. In other words, $A \subseteq X$ is open if and only if $A \cap X^n$ is open for all $n \geq 0$.

Remark 1.6. *If X admits a CW decomposition, then it can be easily checked that X is a colimit of $\{X^n\}_{n\in\mathbb{N}\cup\{0\}}$. In particular, X is the colimit of the diagram*

$$X^0 \xrightarrow{j_0} X^1 \to \cdots \to X^n \xrightarrow{j_n} X^{n+1} \to \cdots$$

in **Top**. *Here j_n is the inclusion of X_n into X_{n+1}.*

We can now define the following categories:

(1) **CW** is the category of whose objects are topological spaces that admit a CW structure and morphisms between CW complexes are cellular continuous maps. That is, $f(X^n) \subseteq Y^n$ for each $n \geq 0$ where f is a continuous map. In other words, if X and Y are CW complexes and we have a commutative diagram

$$\begin{tikzcd}
X^0 \arrow[r] \arrow[d] & X^1 \arrow[r] \arrow[d] & \cdots \arrow[r] & X^n \arrow[r] \arrow[d] & X^{n+1} \arrow[r] \arrow[d] & \cdots \\
Y^0 \arrow[r] & Y^1 \arrow[r] & \cdots \arrow[r] & Y^n \arrow[r] & Y^{n+1} \arrow[r] & \cdots
\end{tikzcd}$$

then, on forming the colimits, we obtain an induced map $f : X \to Y$ which is a cellular map.
(2) $\mathbf{CW}_*$ is the category of pointed CW complexes defined analogously to $\mathbf{Top}_*$.
(3) $\mathbf{CW}^2$ is the category of pairs of CW complexes defined analogously to $\mathbf{Top}^2$.

Remark 1.7. *The categories* $\mathbf{CW}_*$ *and* $\mathbf{CW}^2$ *are defined similarly.*

Each cell $\mathbb{D}_i^n$ has its characteristic map Φ_i^n, which is by definition the composition of continuous maps:

$$\begin{tikzcd}
\mathbb{D}_i^n \arrow[r, hook] \arrow[rrr, bend left, "\Phi_i^n"] & X^{n-1} \coprod \mathbb{D}_i^n \arrow[r, two heads] & X^{n-1} \coprod_\varphi \mathbb{D}_i^n \arrow[r, hook] & X
\end{tikzcd}$$

Proposition 1.8. *Let X be a topological space with a CW decomposition. $A \subseteq X$ is open if and only if $(\Phi_i^n)^{-1}(\mathbb{D}_i^n)$ is continuous for each $i \in I_n$ and $n \in \mathbb{N}$. In particular, X is a quotient space of $\coprod_{\substack{n\in\mathbb{N}\\ i\in I_n}} \mathbb{D}_i^n$*

Proof. The forward implication is clear. Conversely, suppose $(\Phi_i^n)^{-1}(\mathbb{D}_i^n)$ is open in $\mathbb{D}_i^n$ for each for each $i \in I_n$ and $n \in \mathbb{N}$. Suppose by induction on n that $A \cap X^{n-1}$ is open in X^{n-1}. Since $(\Phi_i^n)^{-1}(\mathbb{D}_i^n)$ is open in $\mathbb{D}_i^n$ for all $i \in I_n$, then $A \cap X^n$ is open in X^n by the definition of the quotient topology on X^n. The last implication is clear by definition. □

Definition 1.9. Let X be a topological space. X is a CW complex if X admits a CW decomposition satisfying the following two properties:

(1) The closure of each open cell is contained in a union of finitely many cells.
(2) The topology of X is coherent with $\{\{\mathbb{D}_i^n\}_{i\in I_n} : n \in \mathbb{N}\}$[1].

A CW complex is finite (or finite-dimensional) if there are only finitely many cells involved. Every finite CW decomposition is automatically a finite CW complex. In fact, every locally finite CW decomposition is automatically a CW complex as we show below.

Proposition 1.10. *Let X be a topological space endowed with a CW decomposition. If $\{\{\mathbb{D}_i^n\}_{i\in I_n} : n \in \mathbb{N}\}$ is a locally finite collection, then X is a CW complex.*

Proof. By assumption, every point $\mathbb{D}_i^n$ has a neighborhood that intersects only finitely many cells. Since $\mathbb{D}_i^n$ is compact, it is covered by finitely many such neighborhoods. This readily implies (1) in Definition 1.9. Suppose $A \subseteq X$ is a subset such that $A \cap \mathbb{D}_i^n$ is closed for each $i \in I_n$ and $n \in \mathbb{N}$. Given $x \in X \setminus A$, let W_x be a neighborhood of x that intersects the closures of only finitely many cells, say $\mathbb{D}_1^{n_1}, \ldots, \mathbb{D}_k^{n_k}$. Since $A \setminus \mathbb{D}_j^{n_j}$ is closed in $\mathbb{D}_j^{n_j}$ and thus in X, it follows that

$$W \setminus A = W \setminus (A \cap \mathbb{D}_1^{n_1}) \cup \ldots \cup \left(A \cap \mathbb{D}_k^{n_k}\right)$$

is a neighborhood of x contained in $X \setminus A$. Thus $X \setminus A$ is open, so A is closed. This readily implies (2) in Definition 1.9. □

1.2. **Examples.** In the examples that follows, we will not explicitly check that condition (3) in Definition 1.5 is satisfied. It should be straightforward to do verify these claims, though.

Example 1.11. Let $N = (0, \ldots, 0, 1)$ in $\mathbb{S}^n$. Consider the map $\sigma_N : \mathbb{S}^n \setminus \{N\} \to \mathbb{R}^n$ by [2]

$$\sigma_N(u^1, \ldots, u^{n+1}) = \left(\frac{u^1}{1-u^{n+1}}, \ldots, \frac{u^n}{1-u^{n+1}}\right)$$

Similarly, consider $\beta_N : \mathbb{R}^n \to \mathbb{S}^n \setminus \{N\}$

$$\beta_N(u^1, \ldots, u^n) = \left(\frac{2u^1}{|u|^2+1}, \cdots, \frac{2u^n}{|u|^2+1}, \frac{|u|^2-1}{|u|^2+1}\right).$$

[1]That is, $A \subseteq X$ is open/closed if and only if $A \cap \overline{\mathbb{D}_i^n}$ is open/closed for each $i \in I_n$ and $n \in \mathbb{N}$.

[2]Let $x = (x^1, \ldots, x^{n+1}) \in \mathbb{S}^n \setminus \{N\}$. The line through N and x is parameterized by

$$u^1 = x^1 t, \ldots, u^n = x^n t, u^{n+1} = (x^{n+1} - 1)t + 1$$

The intersection of this line with $u^{n+1} = 0$ occurs when $t = \frac{1}{1-x^{n+1}}$. Hence, the intersection point is $(\sigma_N(x), 0)$, as desired. Therefore, $\sigma_N(x)$ is the intersection of the line through N and x with the $\mathbb{R}^n$ plane.

It is easy to check that σ_N, β_N are inverses of each other. Hence, $\mathbb{R}^n \cong \mathbb{S}^n \setminus \{N\}$. The map σ_N is called the stereographic projection. $\mathbb{S}^n$ can now be given a CW structure with one 0-cell ($\mathbb{D}^0$) and one n-cell ($\mathbb{D}^n$). The attaching map for the n-cell is $\varphi : \mathbb{S}^{n-1} = \partial\mathbb{D}^n \to \{*\}$.

Example 1.12. $\mathbb{S}^n$ can be given a different CW structure with two k-cells in each dimension for $0 \leq k \leq n$. Let $X^0 = \mathbb{S}^0 = \{\mathbb{D}^0_1, \mathbb{D}^0_2\}$. Then $X^1 = \mathbb{S}^1$ where the two 1-cells $\mathbb{D}^1_1, \mathbb{D}^1_2$ are attached to the 0-cells by homeomorphisms on their boundary. Similarly, two 2-cells can be attached to $X^1 = \mathbb{S}^1$ by homeomorphism on their boundary, giving $X^2 = \mathbb{S}^2$. Proceed inductively.

Example 1.13. There are natural inclusions

$$\mathbb{S}^0 \subseteq \mathbb{S}^1 \subseteq \cdots \subseteq \mathbb{S}^n \subseteq \cdots \subseteq$$

We can then define $\mathbb{S}^\infty = \varinjlim_{n\in\mathbb{N}} \mathbb{S}^n$. If $\mathbb{S}^n$ is given a CW structure as in Example 1.12 for each $n \geq 0$, then $\mathbb{S}^\infty$ is a CW complex as well. Note that $\mathbb{S}^\infty$ is a colimit of the $\mathbb{S}^n$'s for $n \geq 0$.

Example 1.14. Consider $\mathbb{RP}^n$ as the quotient of $\mathbb{S}^n$ with anti-podal points identified. An easy observation shows that $\mathbb{RP}^n$ is a quotient of $\mathbb{D}^n$ by the relation $x \sim -x$ on the boundary $\mathbb{S}^{n-1}$[3]. Thus, $\mathbb{RP}^n$ can be obtained from $\mathbb{RP}^{n-1}$ by attaching a one cell.

$$\begin{tikzcd}
\mathbb{S}^{n-1} \arrow[r, hook] \arrow[d, two heads] & \mathbb{D}^n \arrow[d, two heads] \\
\mathbb{RP}^{n-1} \arrow[r, hook] & \mathbb{RP}^n
\end{tikzcd}$$

Thus $\mathbb{RP}^n$ can be built as a CW complex with a single cell in each dimension $\leq n$.

Example 1.15. There are natural inclusions

$$\mathbb{RP}^0 \subseteq \mathbb{RP}^1 \subseteq \cdots \subseteq \mathbb{RP}^n \subseteq \cdots \subseteq$$

We can then define $\mathbb{RP}^\infty = \varinjlim_{n\in\mathbb{N}} \mathbb{RP}^n$. Note that $\mathbb{RP}^\infty$ is a colimit of the $\mathbb{RP}^n$'s for $n \geq 0$. We can define $\mathbb{CP}^\infty$ similarly to $\mathbb{RP}^\infty$.

Example 1.16. The complex projective space, $\mathbb{CP}^n$, is defined as the quotient space $\mathbb{CP}^n = (\mathbb{C}^{n+1} \setminus \{0\})/\sim$ with the equivalence relation $x \sim y$ in $\mathbb{C}^{n+1} \setminus \{0\}$ if and only if $x = \lambda y$ for some $\lambda \neq 0$. Note that there is a map

$$\mathbb{D}^{2n} \to \mathbb{CP}^n$$

$$(z_0, \ldots, z_{n-1}) \mapsto [z_0, \ldots, z_{n-1}, \sqrt{1 - \|z\|}]$$

The boundary of $\mathbb{D}^{2n}$ (where $\sqrt{1 - \|z\|} = 0$) is sent to $\mathbb{CP}^{n-1}$. In this way, $\mathbb{CP}^n$ is obtained from $\mathbb{CP}^{n-1}$ by attaching one $2n$-cell. So $\mathbb{CP}^n$ has a CW structure with one cell in each even dimension $0, 2, \ldots, 2n$.

Example 1.17. There are natural inclusions

$$\mathbb{CP}^0 \subseteq \mathbb{CP}^1 \subseteq \cdots \subseteq \mathbb{CP}^n \subseteq \cdots \subseteq$$

We can then define $\mathbb{CP}^\infty = \varinjlim_{n\in\mathbb{N}} \mathbb{CP}^n$ as before.

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[3] It is easy to check that these identifications are consistent with out discussion of the real projective plane, which is $\mathbb{RP}^2$.

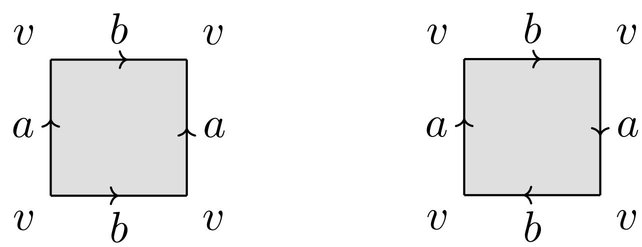
Let's discuss some 2-dimensional examples. It is well-known that compact, connected 2-dimensional manifolds are classified into the following types:

(1) $\mathbb{S}^2$,
(2) A connected sum of g-tori $\mathbb{T}$ (or a g-hold torus) for $g \geq 2$,
(3) A connected sum of g-projective spaces $\mathbb{RP}^2$, for $g \geq 2$.

We have already discussed a CW-structure on $\mathbb{S}^2$. We discuss examples of the other 2-manifolds below:

Example 1.18. Consider $X = \mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ (the 1-torus) or $\mathbb{RP}^2$ (the real projective plane). Both spaces can be constructed as quotients of a rectangle by identifying edges according to specific rules: for the torus, opposite edges are identified in the same direction, while for $\mathbb{RP}^2$, one pair of opposite edges are identified normally and the other pair with reversed orientation. These identification diagrams offer a convenient way to visualize the topology of each space. Each space admits a natural CW complex structure with the following cells:

(1) a single 0-cell representing the vertex of the rectangle,
(2) two 1-cells corresponding to the edges of the rectangle,
(3) a single 2-cell which is attached via a continuous map from the boundary circle $\mathbb{S}^1$ into the 1-skeleton.



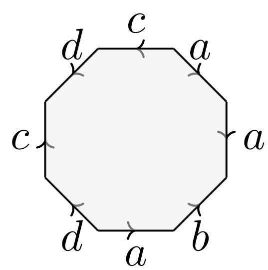
$X = \mathbb{T}$ $\qquad$ $X = \mathbb{RP}^2$

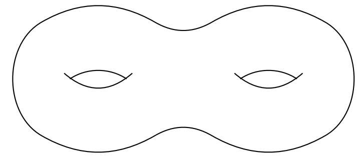
Example 1.19. For $g \geq 1$, a model for a connected sum of g copies of the torus $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ is denoted by M_g, and is known as an orientable surface of genus g. The surface M_g can be constructed by taking a polygon with $4g$ sides and identifying its edges in pairs according to the word

$$a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1},$$

which encodes the edge identifications that yield a closed orientable surface. Each pair a_i, a_i^{-1} and b_i, b_i^{-1} contributes a 'handle,' so M_g can be visualized as a torus with g holes, or a g-holed doughnut. This construction endows M_g with a natural CW complex structure consisting of:

(1) a single 0-cell where all loops based on the edges are attached;
(2) $2g$ 1-cells corresponding to the edges of the polygon;
(3) a single 2-cell attached along the loop described by the edge word above.



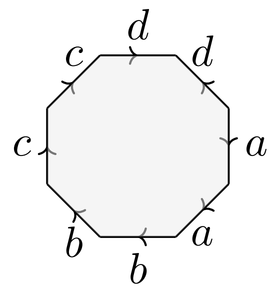


Example 1.20. For $g \geq 2$, a model for the connected sum of g copies of the real projective plane $\mathbb{RP}^2$ is denoted by N_g, and is known as a non-orientable surface of genus g. The surface N_g can be constructed from a polygon with g sides by identifying the edges according to the word

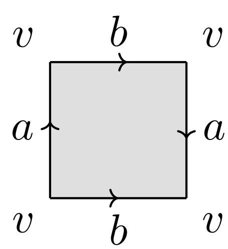
$$a_1 a_1 \cdots a_g a_g,$$

where each pair $a_i a_i$ represents an edge identification. This construction yields a closed surface that is non-orientable and has genus g. The surface N_g admits a CW complex structure consisting of:

(1) a single 0-cell to which all loops are attached;
(2) g 1-cells corresponding to the edges of the polygon;
(3) a single 2-cell attached via a loop following the word $a_1 a_1 \cdots a_g a_g$.



Remark 1.21. *N_2 is usually called a Klein bottle. Another model for the Klein bottle is given by the CW structure shown below:*



It can be checked that both models are homeomorphic.

1.3. **Properties.** A sub-complex of X is a subspace $Y \subseteq X$ that is a union of open cells of X, such that if Y contains a cell, it also contains its closure. It follows immediately that the union and the intersection of any collection of sub-complexes are themselves sub-complexes. Examples of a sub-complexes would be be the subspaces X^n for $n \geq 0$ in the definition of a CW complex.

Proposition 1.22. *Suppose X is a CW complex and Y is a sub-complex of X. Then Y is closed in X, and with the subspace topology and the cell decomposition that it inherits from X, it is a CW complex.*

Proof. Let $\mathbb{B}^n \subseteq Y$ denote such an open n-cell in Y. Since $\overline{\mathbb{B}^n} \subseteq Y$, the finitely many cells of X that have nontrivial intersections with $\mathbb{B}^n$ must also be cells of Y. So condition (1) in Definition 1.9 is automatically satisfied by Y. In addition, any characteristic map $\varphi : \overline{\mathbb{B}^n} \to X$ for $\mathbb{D}^n$ in X also serves as a characteristic map for $\overline{\mathbb{B}^n}$ in Y. Suppose $A \subseteq Y$ is a subset such that $A \cap \mathbb{D}^n$ is closed in $\mathbb{D}^n$ for every n-cell $\mathbb{D}^n$ contained in Y. Let $\mathbb{D}^n$ be a n-cell of X that is not contained in Y. We know that $\mathbb{D}^n \setminus \mathbb{B}^n$ is contained in the union of finitely many open cells of X; some of these, say $\mathbb{B}_1^{n_1}, \ldots, \mathbb{B}_k^{n_k}$, might be contained in Y. Then $\overline{\mathbb{B}_1^{n_1}} \cup \ldots \cup \overline{\mathbb{B}_k^{n_k}} \subseteq Y$, and

$$A \cap \mathbb{D}^n = A \cap (\overline{\mathbb{B}_1^{n_1}} \cup \ldots \cup \overline{\mathbb{B}_k^{n_k}}) \cap \mathbb{D}^n = ((A \cap \overline{\mathbb{B}_1^{n_1}}) \cup \cdots \cup (A \cap \overline{\mathbb{B}_k^{n_k}})) \cap \mathbb{D}^n$$

which is closed in $\mathbb{D}^n$. It follows that A is closed in X and therefore in Y. This implies (2) in Definition 1.9. Hence Y is a CW complex. Taking $A = Y$ shows that Y is closed. □

Proposition 1.23. *The following is a list of some categorical/topological properties of CW complexes.*

(1) *If A is a subcomplex of X, then the inclusion $\iota : A \hookrightarrow X$ is a cellular map.*
(2) *If A is a subcomplex of X, then X/A is a CW complex such that the quotient map $X \to X/A$ is a cellular map.*
(3) *If X and Y are finite CW complexes, then $X \times Y$ is a CW complex.*
(4) *The closure of each cell in a CW complex is contained in a finite subcomplex.*
(5) *A subset of a CW complex is compact if and only if it is closed and contained in a finite subcomplex.*
(6) *A CW complex is compact if and only if it is a finite complex.*
(7) *A CW complex is locally compact if and only if it is locally finite.*
(8) *A CW complex is locally path-connected.*
(9) *A CW complex is a T_1, normal space. Hence, a CW complex is a Hausdorff space.*

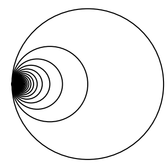
Proof. (**Sketch**) The proof of some of the properties is given below:

(1) The proof is skipped, but it is clear given the definition of a sub-complex.
(2) The proof is skipped.
(3) The proof is skipped.
(4) Let $\mathbb{D}^n$ be an n-cell of a CW complex. We prove the claim by induction on n. If $n = 0$, then $\overline{\mathbb{D}^0} = \mathbb{D}^0$ is itself a finite subcomplex. Assume the claim is true for every cell of dimension less than n. By (1) in Definition 1.9, $\overline{\mathbb{D}^n} \setminus \mathbb{D}^n$ is contained in the union of finitely many cells of lower dimension, each of which is contained in a finite subcomplex by the inductive hypothesis. The claim now follows by taking a union of these these finite subcomplexes together with $\mathbb{D}^n$.
(5) Every finite subcomplex $Y \subseteq X$ is compact because it is the union of finitely many closed cells. Thus, if $K \subseteq X$ is closed and contained in a finite subcomplex, it is also compact. Conversely, suppose $K \subseteq X$ is compact. If K intersects infinitely many cells, by choosing one point of K in each such cell, we obtain an infinite discrete subset of K, which is impossible. Therefore, K is contained in the union of finitely many cells, and thus in a finite subcomplex by (1).
(6) This follows from (5).

(7) This essentially follows from (5).
(8) Consider the spaces $X^n \subseteq X$[4]. We induct on $n \in \mathbb{N}$. X^0 is obviously locally path-connected. If X^{n-1} is locally path-connected then X^n is also locally path-connected since it is the the quotient of the disjoint union of X^{n-1} and a bunch of n-cells which are locally path-connected. Therefore, $\coprod_{n\in\mathbb{N}} X_n$ is locally path-connected. Since
$$\coprod_{n\in\mathbb{N}} X_n \to X$$
is a quotient map, X is locally-path connected.
(9) See [Hat02] for a proof.

This completes the proof. □

Remark 1.24. *Every topological space is not a CW complex. Consider the Hawaiian earring, X:*



The easiest way to see the Hawaiian earring has no CW decomposition is using information about the first homology group. If X were a CW-complex, then it would have to be a finite CW-complex by Proposition 1.23(6) since it is compact. Since every finite CW-complex has finitely generated homology, it suffices to show that the homology of X is not finitely generated. Observe that for any $n \in \mathbb{N}$, X has a retract which is a wedge of n circles - namely, the union of n of the circles that make up X (the retraction just maps all the other circles to the origin). The first homology group of a wedge of n circles is $\mathbb{Z}^n$, which cannot be generated by fewer than n elements. It follows that $H_1(X)$ cannot be generated by fewer than n elements for any $n \in \mathbb{N}$, and thus cannot be finitely generated. We have

$$\mathbf{CW} \subsetneq \mathbf{Top}$$

as inclusion of categories.

[4]We will use the following facts from general topology. A disjoint union of locally path-connected spaces is locally path-connected. Moreover, a quotient of a locally path-connected space is locally path-connected.

Part 1. First Homotopy Group

2. Paths & Homotopy

2.1. Paths and π_0.

Definition 2.1. Let $X \in \mathbf{Top}$. A path in X from x to y is a continuous map $f : [0, a] \to X$ such that $f(0) = x$ and $f(a) = y$ for some $a \geq 0$.

Proposition 2.2. *Let $X \in \mathbf{Top}$. Paths in X form a category, called the path category, $\mathbf{Paths}_X$.*

Proof. The objects of this category are points of X and a morphism between two points, x, y, is simply a path. Composition of paths is defined as: if $f_1, : [0, a_1]$ and $f_2, : [0, a_2]$ such that $f_1(a_1) = f_2(0)$ are two paths, then the product path is defined as follows:

$$f_2 \cdot f_1 : [0, a_1 + a_2] \to X$$
$$t \mapsto \begin{cases} f_1(t) & \text{if } t \in [0, a_1] \\ f_2(t - a_1) & \text{if } t \in [a_1, a_1 + a_2] \end{cases}$$

For each $x \in X$, the identity path Id_x is simply the path $\mathrm{Id}_x : [0, 0] \to X$ such that $\mathrm{Id}_x(t) = x$ for each $t \in [0, 0]$. Associativity and the identity axiom can be easily checked. $\square$

Being connected by paths is an equivalence relation on X: each $x \in X$ is connected to x via the identity path. if x is connected to y by a path $f : [0, a] \to X$ such that $f(0) = x$ and $f(a) = y$, then y is connected to x via the reverse path:

$$f_r : [0, a] \to X$$
$$t \mapsto f(a - t)$$

If x is connected to y by a path f and y is connected to z via a path g, then x is connected to z via the path $f_2 \cdot f_1$.

Definition 2.3. Let $X \in \mathbf{Top}$. An equivalence relation on X under the equivalence relation of being connected by a path is a path-component.

We denote by $\pi_0(X)$ the set of path components, and by $\pi_0(x)$ the path component of the point x. π_0 then defines a functor

$$\pi_0 : \mathbf{Top} \to \mathbf{Sets}$$
$$X \mapsto \pi_0(X)$$

Indeed, a map $f : X \to Y$ induces $\pi_0(f) : \pi_0(X) \to \pi_0(Y)$ given by $\pi_0(x) \mapsto \pi_0(f(x))$ for each $x \in X$. π_0 assigns an invariant to a topological space in the sense that if X and Y are homeomorphic topological space via a map $f : X \to Y$, then

$$\pi_0(X) \cong \pi_0(Y)$$

as sets. This can be easily checked. See Proposition 2.12 for a more general argument. Hence, the cardinality of $\pi_0(X)$ can be used to distinguish some simple topological spaces.

Example 2.4. $\mathbb{R}$ is not homeomorphic to $\mathbb{R}^n$ for $n > 1$. Suppose $f : \mathbb{R} \to \mathbb{R}^n$ is a homeomorphism. Then $\mathbb{R} \cong \mathbb{R}^n$. WLOG let $f(0) = 0$. Hence, $\mathbb{R} \setminus 0 \cong \mathbb{R}^n \setminus 0$. $\mathbb{R} \setminus 0$ has two path-components and $\mathbb{R}^n \setminus 0$ has a single path-component, a contradiction.

Example 2.5. Let

$$X = \{(x, y) \in \mathbb{R}^2 : x = 0 \text{ or } y = 0\}$$

be the union of the x and y axes in $\mathbb{R}^2$. X is not homeomorphic to $\mathbb{R}$ since $X \setminus \{(0,0)\}$ has four path-components and $\mathbb{R} \setminus \{0\}$ has two path-components.

Example 2.6. Assume that $\mathbb{R} \cong X \times Y$. Then $X \times Y$ and hence X, Y are path-connected. Assume $|X|, |Y| \geq 2$. Let $(x_0, y_0) \in X \times Y$. Then $X \times Y \setminus (x_0, y_0)$ is path-connected [5]. However, $\mathbb{R} \setminus \{*\}$ is not path-connected. Hence, either $|X| = 1$ or $|Y| = 1$.

2.2. **Homotopy.** Topology can at best be thought of as 'squishy geometry.' Perhaps it is possible to continuously deform a path while still retaining its underlying topological properties. More generally, perhaps two functions $f, g : X \to Y$ can be 'deformed into each other' This leads to the notion of homotopy.

Definition 2.7. Let $X, Y \in$ **Top** and let $f, g : X \to Y$ be continuous maps. A homotopy from f to g is a continuous map

$$H : X \times [0,1] \to Y$$

such that $f(x) = H(x, 0)$ and $g(x) = H(x, 1)$ for $x \in X$. In this case, we write $f \sim g$. H is said to be relative to $A \subseteq X$ if the restriction $H|_A$ is constant on A. In this case, we write $f \sim_A g$.

Example 2.8. A homotopy between paths $f_i : [0, a_i] \to X$ from x to y is a continuous map

$$H : I \times I \to X$$

such that

$$\begin{aligned} h(s, 0) &= f_1(s) \\ h(s, 1) &= f_2(s) \\ h(0, t) &= x \\ h(1, t) &= y \end{aligned}$$

for all $s, t \in I$. In other words, we have a homotopy relative to the set $\{x, y\}$.

Proposition 2.9. *The homotopy operation satisfies the following properties:*

(1) $\sim$ *is an equivalence relation.*
(2) $\sim$ *is compatible with composition of maps.*
(3) *If $f : X \to Y$ is a continuous function, then* $f \circ \mathrm{Id}_X \sim \mathrm{Id}_Y \circ f$

Proof. The proof is as follows:

(1) Any map $f : X \to Y$ is homotopic to itself via the constant homotopy

$$\begin{aligned} H(x,t) : X \times [0,1] &\to Y \\ (x,t) &\mapsto f(x) \end{aligned}$$

[5]Let $(a, b), (c, d) \in X \times Y$. If $a = c \neq x_0$ or $b = d \neq y_0$, then exists a path between (a, b) & (c, d) in either $\{a\} \times Y$ or $X \times \{b\}$ resp. avoiding (x_0, y_0). If $a = c = x_0$, then $b, d \neq y_0$. Choose a point $x \neq x_0 \in X$. Consider the path $(a, b) \to (x, b) \to (x, d) \to (c, d)$ which avoids (x_0, y_0). A similar argument works if $b = d = y_0$. If $a \neq c$ and $b \neq d$, consider two paths: $(a, b) \to (c, b) \to (c, d)$ and $(a, b) \to (a, d) \to (c, d)$. (x_0, y_0) cannot be on both paths. This covers all cases.

Hence, $f \sim f$. Given $H : f \sim g$, the inverse homotopy

$$\begin{aligned} H(x,t) : X \times [0,1] &\to Y \\ (x,t) &\mapsto H(x, 1-t) \end{aligned}$$

shows $g \sim f$. Let $K : f \sim g$ and $L : g \sim h$ be given. The product homotopy $K * L$ is defined by

$$(K * L)(x,t) = \begin{cases} K(x, 2t) & 0 \leq t \leq \frac{1}{2} \\ L(x, 2t-1) & \frac{1}{2} \leq t \leq 1 \end{cases}$$

and shows $f \sim h$.

(2) Consider continuous functions

$$\begin{aligned} f_i &: X \to Y \\ g_i &: Y \to Z \end{aligned}$$

for $i = 1, 2$. Assume $f_1 \sim f_2$ via a homotopy F and $g_1 \sim g_2$ via a homotopy G. Define a homotopy:

$$\begin{aligned} G \circ F : X \times I &\to Z \\ (x,t) &\mapsto G(F(x,t), t) \end{aligned}$$

This shows that $f_2 \circ f_1 \sim g_2 \circ g_1$.

(3) This is clear.

This completes the proof. □

We now have a new category **hTop**: objects in **hTop** are the same as objects as in **Top** and morphisms are homotopy classes of continuous maps. Proposition 2.9 shows that **hTop** is well-defined. If $X, Y \in$ **hTop**, the set of homotopy classes of continuous maps between X and Y is denoted by $[X, Y]$.

Remark 2.10. *We can also define* $\mathbf{hTop}_*$ *corresponding to* $\mathbf{Top}_*$. *For instance, if* $(X, x_0), (Y, y_0) \in \mathbf{hTop}_*$, *then a pointed homotopy in* $\mathbf{hTop}_*$ *is a continuous function such that*

$$H : (X, x_0) \times I \to (Y, y_0)$$

such that $H|_{(X,x_0)\times\{t\}}$ *for each* $t \in I$ *is a pointed map. The set of pointed homotopy classes from* X *to* Y *is denoted as* $[X, Y]_*$. *Note that* $[X, Y]_*$ *is itself a pointed set with the basepoint given by the homotopy class of the constant map* $X \to y_0$.

Remark 2.11. *If* $A = \{\bullet\}$, *then Definition 2.7 is a statement about the homotopy of maps considered as morphisms in* $\mathbf{Top}_*$. *We can also define a notion of homotopy for morphisms in* $\mathbf{Top}^2$. *If there are two maps* $f, g : (X, A) \to (Y, B)$ *in* $\mathbf{Top^2}$, *a homotopy of pairs from* f *to* g *is a homotopy* $H : f \simeq g$ *that, in addition, satisfies* $H(a, t) \in B$ *for all* $t \in [0, 1]$ *and* $a \in A$. *This defines a new category,* $\mathbf{hTop}^2$.

We now show that π_0 is a topological invariant in **hTop**.

Proposition 2.12. *Let* $X, Y \in$ **Top**. *If* X *and* Y *are homotopy equivalent, then*

$$\pi_0(X) \cong \pi_0(Y)$$

as sets.

Proof. Let $f : X \to Y$ and $g : Y \to X$ be the homotopy equivalent maps. We have a function
$$f_* : \pi_0(X) \to \pi_0(Y),$$
that sends the path component $[x]$ in X to the path component $[f(x)]$ in Y. Clearly, this is well-defined. We similarly have a function
$$g_* : \pi_0(Y) \to \pi_0(X),$$
that sends the path component $[y]$ in Y to the path component $[g(y)]$ in X. Moreover, homotopic maps give the same function, since $I \times I$ is path-connected. Since $g \circ f \cong \mathrm{Id}_X$ and $f \circ g \cong \mathrm{Id}_Y$, we must have that $\pi_0(X) \cong \pi_0(Y)$. □

We discuss a few basic but useful results:

Proposition 2.13. *The following statements are true:*

(1) *Let $A, X, Y \in$ **Top**. If $f_0 \sim f_1 : A \to X$ and $g_0 \sim g_1 : A \to Y$, then*
$$(f_0, g_0) \sim (f_1, g_1) : A \to X \times Y.$$
(2) *Let $X, Y, B \in$ **Top**$_*$. If $f_0 \sim f_1 : X \to B$ and $g_0 \sim g_1 : Y \to B$, then*
$$\{f_0, g_0\} \sim \{f_1, g_1\} : X \vee Y \to B.$$

Proof. For (1), let H_t be the homotopy between f_0 and f_1 and G_t the homotopy between g_0 and g_1. Then $(H_t, G_t) : A \to X \times Y$ is a homotopy between (f_0, g_0) and (f_1, g_1). The proof of (2) is similar. □

Remark 2.14. *Proposition 2.13 implies that we have bijections*
$$[A, X] \times [A, Y] \cong [A, X \times Y]$$
$$[X, B]_* \times [Y, B]_* \cong [X \vee Y, B]_*$$

An important instance of a homotopy arises when we consider the following question: perhaps it is possible to deform a topological space into a 'smaller' space continuously. This leads to the notion of a deformation retraction which is a specific instance of a homotopy.

Definition 2.15. Let $X \in$ **Top**. A deformation retraction of X onto a subspace A is a homotopy
$$H : X \times [0, 1] \to X$$
such that $H(\cdot, 0) = \mathrm{Id}_X$, $H(\cdot, 1) = A$, and $H(\cdot, t)|_A = \mathrm{Id}_A$ for all $t \in [0, 1]$. X is said to be contractible if deformation retracts to a point $A = \{*\}$.

Example 2.16. The following are examples of some deformation retractions:

(1) $\mathbb{R}^n$ is contracible. More genrally, any star-shaped region is contractible. Indeed, if X is star-shaped with respect to some point $a \in X$, then
$$H(x, t) = (1 - t)x + ta$$
defines a homotopy between the constant map and the identity map. Hence star-shaped sets are contractible.

(2) $\mathbb{R}^n \setminus 0$ deformation retracts to $\mathbb{S}^{n-1}$. Simply consider the straight-line homotopy:
$$H(x, t) = (1 - t)x + \frac{tx}{\|x\|}.$$

(3) $\mathbb{S}^\infty$ is contractible. Let H_1 be given by

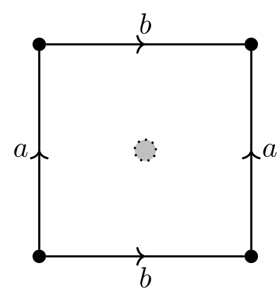
$$H_1 : \mathbb{R}^\infty \times I \to \mathbb{R}^\infty,$$
$$(x,t) \mapsto (1-t)(x_1, x_2, x_3, \ldots) + t(0, x_1, x_2, \ldots).$$

Note that $H_1(-,1)$ is the right shift map. For any $x \in \mathbb{S}^\infty$, the vector $H_1(x,-)$ is not a multiple of x, so the line segment between them does not pass through the origin. Thus, we can define a homotopy from the identity on $\mathbb{S}^\infty$ by setting $H_1/\|H_1\|$. The idea is now to contract the image of $H_1(-,1)$, which is a codimension-1 sphere, to a point not on it-say, $(1,0,0,0,0,\ldots)$. Let

$$H_2 : \mathbb{R}^\infty \times I \to \mathbb{R}^\infty$$
$$(x,t) = (1-t)(0, x_1, x_2, \ldots) + t(1, 0, 0, \ldots).$$

Clearly, $H_2 = H_2/\|H_2\|$ is a homotopy from the map $H_1(-,1)$ to the constant map at $(1,0,0,\ldots)$ on $\mathbb{S}^\infty$. The composition is the desired homotopy that shows that $\mathbb{S}^\infty$ is contractible.

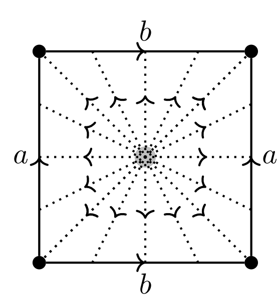
(4) Let Y be the topological space obtained by identifying opposite sides $[-1,1]\times[-1,1]$. Let $X = Y \setminus \{(0,0)\}$. See the diagram below:



Consider the homotopy $H : X \times I \to X$ defined by the formula

$$H((x,y),t) = \begin{cases} (t\frac{x}{|y|} + (1-t)x, t\frac{y}{|y|} + (1-t)y) & |y| > |x| \\ (t\frac{x}{|x|} + (1-t)x, t\frac{y}{|x|} + (1-t)y) & |x| > |y| \end{cases}$$

If $(x,y) \in X$ and $|y| > |x|$, the homotopy H linearly slides (x,y) onto a point such that the x-coordinate of $H((x,y),1)$ is $\mathrm{sgn}(x)$. See the diagram below:



The image of $H(\cdot,1)$ is clearly the identified edges of X, which is, geometrically, a figure eight: a graph consisting of two circles intersecting in a point.

(5) Consider the Mobius strip, M, obtained as the quotient space of the square $[0,1] \times [0,1]$ by identifying

$$(x,0) \sim (1-x,1) \quad \text{for all } 0 \leq x \leq 1.$$

The line $\{(x, \frac{1}{2}) : x \in [0,1]\} \subseteq \mathbb{S}^1$ is $\mathbb{S}^1$ as a subspace of M. Then the map

$$H : M \times [0,1] \to M$$
$$((x,y),t) \mapsto \left(x, (1-t)y + \frac{t}{2}\right)$$

gives a well-defined strong deformation retract of M to $\mathbb{S}^1$ (as can be checked).

The following example is quite important:

Example 2.17. $\mathbb{D}^n \times I$ deformation retracts onto $\mathbb{D}^n \times \{0\} \cup \mathbb{S}^{n-1} \times I$. Define

$$r(x,t) = \begin{cases} (\frac{2x}{2-t}, 0) & \|x\| \leq \frac{2-t}{2} \\ (\frac{x}{\|x\|}, 2 - \frac{2-t}{\|x\|}) & \|x\| \geq \frac{2-t}{2} \end{cases}$$

It is easy to check that this is a well-defined continuous map. For $t = 0$ we get $\frac{2-t}{2} = 1$ and thus $r(x,0) = (x,0)$ for all $x \in \mathbb{D}^n$. For $x \in \mathbb{S}^{n-1}$ we have $r(x,t) = (x,t)$. Thus r is a retraction.

Remark 2.18. *There is a geometric interpretation of r in Example 2.17. For each (x,t) consider the line $L_{x,t}$ through $(0,2)$ and (x,t). This line intersects $\mathbb{D}^n \times \{0\} \cup \mathbb{S}^{n-1} \times I$ in a single point $r(x,t)$.*

Example 2.19. Let $X = \{\bullet_1, \bullet_2\}$ be a two-point topological space. If X is given the discrete topology, then X is not contractible. Indeed, contractible spaces are path-connected and X is not path connected with the discrete topology. If X is given the Sierpinski topology, $\{\emptyset, X, \{\bullet_1\}\}$, then X is contractible. Define

$$H : X \times [0,1] \to X$$

so that $H(x,0) = x$ for all $x \in X$, and $H(x,t) = \bullet_1$ for all $x \in X$ and $t \in (0,1]$. It is easy to see that H is continuous and hence defines a homotopy.

Definition 2.20. A map $f : X \to Y$ defines a homotopy equivalence if there exists $g : Y \to X$ such that $g \circ f$ and $f \circ g$ are both homotopic to the identity. X and Y are homotopy equivalent if there exists a homotopy equivalence. In this case, we write $X \sim Y$.

Example 2.21. For any $X \in \mathbf{Top}$, $X \times I$ is homotopy equivalent to X. Consider the maps

$$\pi_1 : X \times I \to X, \qquad i_0 : X \to X \times I,$$
$$(x,t) \mapsto x. \qquad x \mapsto (x,0).$$

Note that $\pi_1 \circ i_0 = \mathrm{Id}_X$. Moreover, $i_0 \circ \pi_1 \sim \mathrm{Id}_{X \times I}$ via the homotopy:

$$H : (X \times I) \times I \to X \times I,$$
$$((x,t),s) \mapsto (x, (1-s)t).$$

Remark 2.22. *Example 2.21 can generalized to prove that if X is a topological space and Y is a contractible topological space, then the projection*

$$\pi_1 : X \times Y \to X$$

is a homotopy equivalence.

Proposition 2.23. *Let X, Y be topological space. The following are some properties of the homotopy and homotopy equivalence concept.*

(1) *X is contractible if and only if every map $f : X \to Y$, for arbitrary Y, is homotopic to a constant map. Similarly, X is contractible if and only if every map $f : Y \to X$ is homotopic to a constant map.*
(2) *Let $f : X \to Y$ be a continuous map. Suppose there exist $g, h : Y \to X$, possibly different, such that $f \circ g \simeq \mathrm{Id}_Y$ and $h \circ f \simeq \mathrm{Id}_X$. Then f is a homotopy equivalence.*

Proof. The proof is given below:

(1) Suppose X is contractible. Let $H : X \times I \to X$ be a homotopy such that $H(\cdot, 0) = \mathrm{Id}_X$ and $H(\cdot, 1)$ is the constant map with value x_0.
 (a) If $f : X \to Y$ is a continuous map for any topological space Y, then $f \circ G : X \times I \to Y$ is a homotopy from f to the constant map with value $f(x_0)$. Thus, f is homotopic to a constant map. Conversely, letting $Y = X$ and $f = \mathrm{Id}_X$ shows that X is contractible.
 (b) If $f : Y \to X$ is a continuous map for any topological space Y, then the map

$$H : Y \times I \to X \qquad H(y, t) \mapsto H(f(y), t)$$

 is a homotopy from f to the constant map with value x_0. Thus, f is homotopic to a constant map. Conversely, letting $Y = X$ and $f = \mathrm{Id}_X$ shows that X is contractible.
(2) If $h \circ f \sim \mathrm{Id}_X$ and $f \circ g \sim \mathrm{Id}_Y$, then

$$g \sim \mathrm{Id}_X \circ g \sim (h \circ f) \circ g \sim h \circ (f \circ g) \sim h \circ \mathrm{Id}_Y \sim h$$

 Thus, $g \circ f \sim h \circ f \sim \mathrm{Id}_X$, and since $f \circ g \sim \mathrm{Id}_Y$, g is a homotopy equivalent to f.

This completes the proof. □

3. Fundamental Group

3.1. π_1**.** Let $X \in$ **Top**. Recall the definition of homotopy from the previous section. In this section, we focus on homotopy of paths relative to the boundary of $I = [0, 1]$, denoted as ∂I. We have the following observations:

Lemma 3.1. *The product of paths (read left to right) has the following properties:*

(1) *Let $\alpha : I \to I$ be continuous and $\alpha(0) = 0$, $\alpha(1) = 1$. Then $f \sim f \circ \alpha$.*
(2) $f \cdot (g \cdot h) \sim (f \cdot g) \cdot h$[6]
(3) *$f \sim f'$ and $g \sim g'$ implies $f \cdot g \sim f' \cdot g'$.*
(4) *If c_x denotes the constant path at $x \in X$, then $c_{f(0)} \cdot f \sim f \sim f \cdot c_{f(1)}$*
(5) *$f \cdot f_r \sim c_{f(1)}$ and $f_r \cdot f \sim c_{f(0)}$ where f_r is the reverse of f*

Proof. The proof is given below:

(1) α defines a reparamterization of the identity map from I to I. Let $H : I \times I \to I$ denote the straight-line homotopy from Id_I to α. Then $f \circ H$ is a path homotopy from f to $f \circ \alpha$
(2) We provide a proof in words. We need to show that

$$(f \cdot g) \cdot h \sim f \cdot (g \cdot h)$$

[6]This assumes that at least one side of the equation is well-defined.

for any three paths in X such that the left-hand side is well-defined. The first path follows f and then g at quadruple speed for $s \in [0, \frac{1}{2}]$, and then follows h at double speed for $s \in [\frac{1}{2}, 1]$, while the second follows f at double speed and then g and h at quadruple speed. The two paths are therefore reparametrizations of each other and thus homotopic.

(3) We show that $c_{f(0)} \cdot f \sim f$. The other homotopy follows similarly. Define $H : I \times I \to X$ as

$$H(s,t) = \begin{cases} f(0), & t \geq 2s, \\ f\left(\frac{2s-t}{2-t)}\right), & t \leq 2s. \end{cases}$$

Geometrically, this maps the portion of the square on the left of the line $t = 2s$ to the point $f(0)$, and it maps the portion on the right along the path f at increasing speeds as t goes from 0 to 1. This map is continuous by the gluing lemma, and we have that $H(s, 0) = f(s)$ and $H(s, 1) = c_{f(0)} * f(s)$. The claim follows.

(4) We just show that $f \cdot f_r \simeq c_{f(1)}$. Define a homotopy by the following recipe: at any time t, the path H_t follows f as far as $f(t)$ at double speed while the parameter s is in the interval $[0, t/2]$; then for $s \in [t/2, 1 - t/2]$, it stays at $f(t)$; then it retraces f at double speed back to p. Formally,

$$H(s,t) = \begin{cases} f(2s), & 0 \leq s \leq t/2, \\ f(t), & t/2 \leq s \leq 1 - t/2, \\ f(2-2s), & 1 - t/2 \leq s \leq 1. \end{cases}$$

It is easy to check that H is a homotopy from $c_{f(1)}$ to $f \cdot f_r$.

This completes the proof. □

For $X \in \mathbf{Top}$, Lemma 3.1 implies that one can consider a category $\Pi(X)$ whose objects are points of X and morphisms are homotopy classes of paths between points of X relative to ∂I. $\Pi(X)$ is called the fundamental groupoid of X because each element in $\mathrm{Hom}_{\Pi(X)}(\cdot, \cdot)$ has an inverse path. In particlar, $\mathrm{Hom}_{\Pi(X)}(X, x_0)$ is a group for each $x \in X$.

Definition 3.2. Let $X \in \mathbf{Top}$ and $x_0 \in X$. The fundamental group of X at x is

$$\pi_1(X, x_0) = \mathrm{Aut}_{\Pi(X)}(x_0)$$

X is simply connected (or 1-connected) if it is path connected and its fundamental group is trivial.

Remark 3.3. *A loop based at $x_0 \in X$ is a map $f : I \to X$ such that $f(0) = f(1) = x$. Since $I/\partial I \cong \mathbb{S}^1$ where the homemorphism is given by the exponential function, $\varepsilon(t) = \exp(2\pi i t)$, f descends to a continuous map from $\mathbb{S}^1$ to X.*

$$\begin{tikzcd} I \arrow[d, "\varepsilon"'] \arrow[dr, "f"] & \\ \mathbb{S}^1 \arrow[r, dashed, "\tilde{f}"] & X \end{tikzcd}$$

Therefore, we have

$$\pi_1(X, x_0) \cong [(\mathbb{S}^1, *), (X, x_0)]$$

where $[(\mathbb{S}^1, *), (X, x_0)]$ denotes the set of homotopy classes of maps from $(\mathbb{S}^1, *)$ to (X, x_0) such that $* \mapsto x_0$.

We next state an important lemma:

Lemma 3.4. *(**Square Lemma**) Let $F : W \to I \times X$ be a continuous map, and let f, g, h, and k be the paths in X defined by:*

$$f(s) = F(s, 0) \quad g(s) = F(1, s) \quad h(s) = F(0, s) \quad k(s) = F(s, 1)$$

Then $f \cdot g \sim h \cdot k$.

Proof. (**Sketch**) Consider an appropriate straight-line homotopy from the corners of the square $I \times I$. □

Proposition 3.5. *Let $(X, x_0) \in$ **Top**$_*$. The following are some properties of the fundamental group of X at x_0.*

(1) *For each $x_0' \in X$, such that $x_0' \in \pi_0(x_0)$, we have*

$$\pi_1(X, x_0) \cong \pi_1(X, x_0')$$

More generally, if X_0 is a path component of X that contains x_0, and $i : X_0 \to X$ is the inclusion map, then

$$i_* : \pi_1(X_0, x_0) \to \pi_1(X, x_0)$$

is an isomorphism.

(2) *$\Pi(\cdot)$ is a functor from **Top** to **Grpd**, the category of groupoids.*

(3) *π_1 is a functor from **Top**$_*$ to **Grp**, the category of groups.*

(4) *if (X, x_0) and (Y, y_0) are pointed topological spaces, then*

$$\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \cong \pi_1(Y, y_0)$$

That is, π_1 preserves products.

(5) *If $f, g : X \to Y$ are homotopic with a homotopy $H : X \times I \to Y$ and h is the path $h(t) = H(x, \cdot)$, then the following diagram commutes:*

$$\begin{tikzcd} & \pi_1(X, f(x_0)) \arrow[dd, "\Phi_g"] \\ \pi_1(X, x_0) \arrow[ur, "f_*"] \arrow[dr, "g_*"'] & \\ & \pi_1(X, g(x_0)) \end{tikzcd}$$

(6) *If $f : X \to Y$ is a homotopy equivalence, then the induced homomorphism*

$$f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$$

is an isomorphism.

Proof. The proof is given below:

(1) Let α be a path from x to x'. Consider the map

$$\Phi_\alpha : \pi(X, x_0) \to \pi(X, x_0') \qquad \beta \mapsto \alpha \cdot \beta \cdot \alpha_r$$

Note that Φ_α is a homomorphim:

$$\begin{aligned}\Phi_\alpha[\beta_2 \cdot \beta_1] &= [\alpha \cdot \beta_2 \cdot \beta_1 \cdot \alpha_r] \\ &= [\alpha \cdot \beta_2 \cdot \alpha_r \cdot \alpha \cdot \beta_1 \cdot \alpha_r] \\ &= [\alpha \cdot \beta_2 \cdot \alpha_r] \cdot [\alpha \cdot \beta_1 \cdot \alpha_r] \\ &= \Phi_\alpha[\beta_2] \cdot \Phi_\alpha[\beta_1]\end{aligned}$$

It is clear that Φ_α is bijective with inverse

$$\Phi_{\alpha_r} : \pi(X, x_0') \to \pi(X, x_0) \qquad \beta \mapsto \alpha_r \cdot \beta \cdot \alpha$$

More generally, any loop in X based at x must in fact be a loop in X_0, so it is necessary only to check that two homotopic loops in X are homotopic in X_0. But this is immediate since if

$$F : I \times I \to X$$

is a homotopy whose image contains x_0, its image must lie entirely in X_0, because $I \times I$ is path-connected.

(2) A continuous map $f : X \to Y$ induces a homomorphism

$$f_* : \Pi(X) \to \Pi(Y)$$

defined by $f_*([\alpha]) = [f \circ \alpha]$. We have

$$\begin{aligned}f_*([\beta] \cdot [\alpha]) &= f_*([\beta \cdot \alpha]) \\ &= [f \circ (\beta \cdot \alpha)] \\ &= [(f \circ \beta) \cdot (f \circ \alpha)] \\ &= f_*[\beta] \circ f_*[\alpha]\end{aligned}$$

The rest of the axioms can be checked in a straightforward way.

(3) This is similar to (2).

(4) Consider the map

$$\begin{aligned}\Phi : \pi_1(X \times Y, (x_0, y_0)) &\to \pi_1(X, x_0) \times \pi_1(Y, y_0) \\ [\alpha] &\mapsto ([\alpha_X], [\alpha_Y])\end{aligned}$$

It is clear that Φ is well-defined since $f \sim g$ implies that

$$\alpha_X = \pi \circ \alpha \sim \pi \circ \alpha = \alpha_X$$

Similarly, $\alpha_Y \sim \alpha_Y$. The universal property of the product topology shows that Φ is a surjective. Moreover, if $[\alpha_X] = [c_x]$ and $[\alpha_Y] = [c_y]$ we can choose we choose homotopies H_x and H_y. Then the map $H : I \times I \to X \times Y$ given by

$$H(s, t) = (H_x(s, t), H_y(s, t))$$

is a homotopy from f to the constant loop $c_{(x_0, y_0)}$. Checking that Φ is a homomorphism is easy.

$$\begin{tikzcd} & I \arrow[d, "f"] \arrow[ddl, bend right, "f_X"'] \arrow[ddr, bend left, "f_Y"] & \\ & X \times Y \arrow[dl, "\pi_X"] \arrow[dr, "\pi_Y"'] & \\ X & & Y \end{tikzcd}$$

(5) Let α be any loop in X based at x. What we need to show is

$$\begin{aligned} g_*[\alpha] = \Phi_g \circ f_*[\alpha] & \\ \iff \quad & g \circ \alpha \sim h \cdot (f \circ \alpha) \cdot h_r \\ \iff \quad & (f \circ \alpha) \cdot h \sim h \cdot (g \circ \alpha) \end{aligned}$$

This readily follows from the square lemma applied to the map $F : I \times I \to Y$ defined by $F(s,t) = H(\alpha(s), t)$.

(6) Let $g : Y \to X$ be a homotopy inverse for g, so that $f \circ g \simeq 1_Y$ and $g \circ f \simeq 1_X$. Consider the maps:

$$\pi_1(X, x_0) \xrightarrow{f_*} \pi_1(Y, f(x_0)) \xrightarrow{g_*} \pi_1(X, g(f(x_0))) \xrightarrow{f_*} \pi_1(Y, f(g(f(x_0))))$$

The composition of the first two maps is an isomorphism by (4). In particular, f_* is injective. The same reasoning with the second and third maps shows that g_* is injective. Thus the first two of the three maps are injections and their composition is an isomorphism, so g_* must be injective and surjective.

This completes the proof. $\square$

Remark 3.6. *Note that the homomorphism Φ_α in Proposition 3.5(a) depends only on the homotopy class of α. Indeed, assume that $\alpha \cong \alpha'$ where α and α' are continuous path joining x and x'. Then:*

$$\Phi_\alpha[\beta] = [\alpha \cdot \beta \cdot \alpha_r] = [\alpha] \cdot [\beta] \cdot [\alpha_r] = [\alpha'] \cdot [\beta] \cdot [\alpha'_r] = \Phi_{\alpha'}[\beta].$$

Hence we have $\Phi_\alpha = \Phi_{\alpha'}$.

Remark 3.7. *Proposition 3.5 implies that if X is a path-connected space, then*

$$\pi_1(X, x_0) \cong \pi_1(X, x_0')$$

for each $x, x' \in X$. We shall mostly be concerned with path-connected spaces. Therefore, we shall not write the basepoint from now on.

How does one compute fundamental groups? This might be a difficult problem. But we can at the very least state some trivial calculations:

Proposition 3.8. *The following are calculations of some fundamental groups:*

(1) *If $X = \{\bullet\}$ is a one-point space, then $\pi_1(X) = \{1\}$, is the trivial group.*
(2) *If X is contractible, then $\pi_1(X, x_0) = \{1\}$ is the trivial group.*

Proof. The proof is given below:

(1) A one-point space has only the constant loop. Hence, its fundamental group is trivial.
(2) This follows from the Proposition 3.5(5) and (1) above.

This completes the proof. $\square$

Remark 3.9. *A topological space, X, is simply connected if its fundamental group is the trivial group. One can easily check that X is simply connected if and only if there is a unique homotopy class of paths connecting any two points in X. For the forward direction, let $x, y \in X$, and $f, g : I \to X$ are paths from x to y. Then we have the following sequence of homotopies:*

$$f \sim f * c_y \sim f * \bar{g} * g \sim c_x * g \sim g,$$

where we use the fact that $\bar{g} * g$ and $f * \bar{g}$ are loops at y and x, respectively, and hence are homotopic to the respective constant paths. For the reverse direction, take $x = y$. By hypothesis, any loop γ at $x \in X$ is in the homotopy class of the constant loop c_x.

As we shall see by way of examples, fundamental groups are rarely abelian. However, there is an important class of groups for which the fundamental groups are abelian. Before identifying this class, we identify when a fundamental group is abelian.

Lemma 3.10. *Let X be a path-connected topological space X. For any $x \in X$, $\pi_1(X, x_0)$ is abelian if and only if all basepoint-change homomorphisms Φ_α depend only on the endpoints of the path α.*

Proof. Assume $\pi_1(X, x_0)$ is abelian and consider two paths α, α' with same endpoints x and x'. Since $\pi_1(X, x_0)$ is abelian, $\pi_1(X, x_0')$ is also abelian since $\pi_1(X, x_0) \cong \pi_1(X, x_0')$. We have:

$$\begin{aligned}
\Phi_\alpha[\beta] &= [\alpha] \cdot [\beta] \cdot [\alpha_r] \\
&= [\alpha] \cdot [\beta] \cdot [c_x] \cdot [\alpha_r] \\
&= [\alpha] \cdot [\beta] \cdot [\alpha'_r \cdot \alpha'] \cdot [\alpha_r] \\
&= ([\alpha] \cdot [\beta] \cdot [\alpha'_r]) \cdot ([\alpha' \cdot \alpha_r]) \\
&= ([\alpha' \cdot \alpha_r]) \cdot ([\alpha] \cdot [\beta] \cdot [\alpha'_r]) \\
\Phi_\alpha[\beta] = [\alpha'] \cdot [\beta] \cdot [\alpha'_r] &= \Phi_{\alpha'}(\beta).
\end{aligned}$$

Hence $\Phi_\alpha = \Phi_{\alpha'}$. Conversely, assume all basepoint-change homomorphisms Φ_α depend depend only on the endpoints of the path α. Consider $x' = x$ and loops c_x (constant loop) and $[\beta] \in \pi_1(X, x_0)$. Then $\Phi_\beta = \Phi_{c_x}$. We can easily see that this is implies

$$[\beta] \cdot [\beta'] = [\beta'] \cdot [\beta]$$

for each $[\beta'] \in \pi_1(X, x_0)$. Hence $\pi_1(X, x_0)$ is abelian. □

Example 3.11. We argue that the fundamental group of a topological group, G, is abelian. Let e_G be the identity element chosen as the base point. Let $[\alpha], [\beta] \in \pi_1(G, e_G)$. Define a map

$$\begin{aligned}
F : I \times I &\to G \\
(t, s) &\mapsto \alpha(t) \cdot \beta(s)
\end{aligned}$$

In $I \times I$, let

$$(0,0) \xrightarrow{\epsilon_1} (1,0), \quad (1,0) \xrightarrow{\epsilon_2} (1,1), \quad (0,0) \xrightarrow{\epsilon_3} (0,1), \quad (1,0) \xrightarrow{\epsilon_4} (1,1), \quad (0,0) \xrightarrow{\epsilon_5} (1,1)$$

be the straight line paths. Applying F ε_5 yields a path

$$\alpha * \beta(t) = \alpha(t) \cdot \beta(t)$$

Since $I \times I$ is convex, we have

$$\epsilon_2 \cdot \epsilon_1 \simeq \epsilon_5 \simeq \epsilon_4 \cdot \epsilon_3,$$

since all three are paths from $(0,0) \to (1,1)$. Applying F to this gives

$$\beta \cdot \alpha \sim \alpha * \beta \sim \alpha \cdot \beta$$

Hence $[\beta \cdot \alpha] = [\alpha \cdot \beta]$. Hence, $\pi_1(G, e_G)$ is abelian.

3.2. **Categorical Remarks.** We end this section with some categorical remarks that will be useful in the next section. Recall that there is a useful notion of a skeleton of a category $\mathscr{C}$. This is a full subcategory with one object from each isomorphism class of objects of $\mathscr{C}$. We denote the skeleton as $\mathrm{Sk}\,\mathscr{C}$. The inclusion functor

$$\mathscr{J} : \mathrm{Sk}\,\mathscr{C} \hookrightarrow \mathscr{C}$$

is an equivalence of categories. Indeed, an inverse functor

$$\mathscr{C} : \mathscr{C} \to \mathrm{Sk}\,\mathscr{C}$$

is obtained by letting $\mathscr{F}(X)$ be the unique object in $\mathrm{Sk}\,\mathscr{C}$ that is isomorphic to X. We also choose an isomorphism $\alpha_X \in \mathrm{Hom}(X, \mathscr{F}(X))$. We choose $\alpha_X = \mathrm{Id}_X$ to be the identity morphism if $X \in \mathrm{Sk}\,\mathscr{C}$. If $f \in \mathrm{Hom}(x_0, y_0)$, we define

$$\mathscr{F}(f) = \alpha_Y \circ f \circ \alpha_X^{-1}$$

We have $\mathscr{F} \circ \mathscr{J} = \mathrm{Id}_{\mathrm{Sk}\,\mathscr{C}}$. Moreover, the α_X's specify a natural isomorphism

$$\alpha : \mathrm{Id}_X \to \mathscr{J} \circ \mathscr{F}$$

A category $\mathscr{C}$ is said to be connected if any two of its objects can be connected by a sequence of morphisms. For example, a sequence

$$A \leftarrow B \rightarrow C$$

connects A to C, although there need be no morphism $A \to C$. However, a groupoid $\mathscr{C}$ is connected if and only if any two of its objects are isomorphic. Hence if $\mathscr{C}$ is a groupoid, the group of endomorphisms of any object in $\mathscr{C}$ is then a skeleton of $\mathscr{C}$. Hence, we have:

Corollary 3.12. *Let X be a path-connected space. For each point $x_0 \in X$, the inclusion $\pi_1(X, x_0) \hookrightarrow \Pi(X)$ is an equivalence of categories.*

Proof. $\pi_1(X, x_0)$ is a category with a single object x, and it is a skeleton of $\Pi(X)$. $\square$

4. Seifert-Van Kampen Theorems

The Seifert Van Kampen (SVK) theorems gives a method for computing the fundamental groups of spaces that can be decomposed into simpler spaces whose fundamental groups are already known.

Proposition 4.1. *(**SVK for Groupoids**) Let $X \in$ **Top**, and let $\{U_i\}_{i \in I}$ be an open cover of X such that that the intersection of finitely many open sets again belongs to the open cover. Then*

$$\Pi(X) = \varinjlim_{i \in I} \Pi(U_i)$$

in the category of groupoids.

Remark 4.2. *Note that Proposition 4.1 states that Π preserves colimits.*

Proof. We verify the universal property of colimits in the category of groupoids. Let G be some groupoid and let $\Gamma_i : \Pi(U_i) \to G$ be groupoid morphisms. We show that there exists a unique groupoid morphism $\Phi : \Pi(X) \to G$ such that the diagram

$$\begin{tikzcd}
\Pi(\bigcap_{j\in J\subseteq I} U_j) \arrow[r, hook] \arrow[d, hook] & \Pi(U_{i_{|J|}}) \arrow[d, hook] \arrow[ddr, bend left, "\Gamma_{i_{|J|}}"] & \\
\Pi(U_{i_1}) \arrow[r, hook] \arrow[drr, bend right, "\Gamma_{i_1}"'] & \Pi(X) \arrow[dr, dashed, "\Phi"] & \\
& & G
\end{tikzcd}$$

for each subset $J \subseteq I$. Consider the following observations:

(1) An object of $\Pi(X)$ is a point $x \in X$ and so lies in one of U_i. If $x \in U_i$, we are forced to set $\Gamma(x) = \Gamma_i(x)$. If x is contained in the intersection of finitely many U_i's, these definitions agree by the commutative square above[7].

(2) A morphism in $\Pi(X)$ is a homotopy class of a path α in X. If α is solely contained in some U_i, we would be forced to set $\Phi(\alpha) = \Gamma_i(\alpha)$. Since the open cover is closed under finite intersections, this specification is independent of the choice of U_i if α *lies entirely in more than one* U_i. What if a path intersects $\cap_{j\in J\subseteq I} U_j$ for some subset $J \subseteq I$ such that $|J| \geq 2$? If $\alpha : I \to X$, then the Lebesgue covering lemma implies that there is a decomposition

$$0 = t_0 < t_1 < \cdots < t_{m-1} < t_m = 1$$

such that $\alpha([t_i, t_{i+1}])$ is contained in solely on of U_i. In this case, we are forced to set

$$\Phi(\alpha) = F_1(\alpha_1) \circ \cdots \circ F_m(\alpha_m)$$

where each F_k is one of the Γ_i's as necessary.

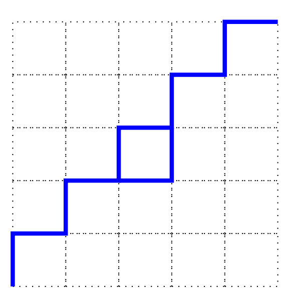
The observations above pin down the map Φ. However, in order for Φ to be well-defined, we must show that it is independent of the choice of a path, α, in a homotopy class of paths. Let

$$H : I \times I \to X$$

be a homotopy of paths from x to y. By the Lebesgue covering lemma, there exists $n \in \mathbb{N}$ such that H sends each sub-square

$$\left[\frac{i}{n}, \frac{i+1}{n}\right] \times \left[\frac{j}{n}, \frac{j+1}{n}\right]$$

into one of U_i. Consider edge-paths in the subdivided square $I \times I$ which differ by a sub-square, as indicated in the following figure.



[7]We implicitly use here the fact that the open cover is closed under finite intersections.

We apply H and obtain two paths in X. They yield the same result since they differ by a homotopy on some subinterval which stays inside one of the sets U_i. Changes of this type allow us to pass inductively from H on the lower to H on the upper boundary path from $(0,0)$ to $(1,1)$. Hence, Φ is well-defined. It is easy to check that Φ is indeed a functor between $\Pi(X)$ and G. By construction, the diagram commutes. $\square$

Proposition 4.1 contains a lot of redundant information since we only want to know how to compute $\pi_1(X, x_0)$ for some $x \in X$.

Corollary 4.3. *(**SVK for Groups**) Let $(X, x_0) \in$ **Top** be path-connected. Let $\{U_i\}_{i\in I}$ be an open cover of X by path-connected open subsets such that $\{U_i\}_{i\in I}$ is closed under taking finite intersections and such that $x_0 \in U_i$ for each $i \in I$, then*

$$\pi_1(X, x_0) = \varinjlim_{i\in I} \pi(U_i, x_0)$$

Proof. (**Sketch**) We will only prove the case where the open cover is finite. The proof for the general case can be found in See [May99, Section 2.7]. We need to verify the universal property of colimits in the category of groups. Let G be some group and let $\Gamma_i : \pi(U_i, x_0) \to G$ be group homomorphisms. We show that there exists a unique group homomorphism $\Phi : \pi_1(X, x_0) \to G$ such that the diagram

$$\begin{tikzcd}
\pi_1(\bigcap_{j\in J\subseteq I} U_j, x_0) \arrow[r, hook] \arrow[d, hook] & \pi_1(U_{i_{|J|}}, x_0) \arrow[d, hook] \arrow[ddr, bend left, "\Gamma_{i_{|J|}}"] & \\
\pi_1(U_{i_1}, x_0) \arrow[r, hook] \arrow[drr, bend right, "\Gamma_{i_1}"'] & \pi_1(X, x_0) \arrow[dr, dashed, "\Phi"] & \\
& & G
\end{tikzcd}$$

for each subset $J \subseteq I$. Recall that the inclusion of categories $\mathscr{J} : \pi_1(X, x_0) \to \Pi(X)$ is actually an equivalence of categories. An inverse equivalence $\mathscr{F} : \Pi(X) \to \pi_1(X, x_0)$ is determined by a choice of path classes $x \to y$ for $y \in X$. We choose c_x when $y = x$ and so ensure that $\mathscr{F} \circ \mathscr{J} = \mathrm{Id}_{\pi_1(X,x_0)}$. Because the cover is finite and closed under finite intersections, we can choose our paths inductively so that the path $x \to y$ lies entirely in *every* U_i for *all* U_i such that $y \in U_i$[8]. This ensures that the chosen paths determine compatible inverse equivalences $\mathscr{F}_{U_i} : \Pi(U_i) \to \pi_1(U_i, x_0)$ to the inclusions $\mathscr{J}_{U_i} : \pi_1(U_i, x_0) \to \Pi(U_i)$. Thus, the functors

$$\Pi(U_i) \xrightarrow{\mathscr{F}_{U_i}} \pi_1(U_i, x_0) \xrightarrow{\Gamma_{U_i}} G$$

specify diagram of groupoids. By Corollary 4.3, there is a unique map of groupoids

$$\widetilde{\Gamma} : \Pi(X) \to G$$

that restricts to $\Gamma_{U_i} \circ \mathscr{F}_{U_i}$ on $\Pi(U_i)$ for each U_i. The composite

$$\Gamma : \pi_1(X, x_0) \xrightarrow{\mathscr{J}} \Pi(X) \xrightarrow{\widetilde{\Gamma}} G$$

It restricts to Γ_i on $\pi_1(U_i, x_0)$ by a diagram chase argument and the fact that $\mathscr{F}_{U_i} \circ \mathscr{J}_{U_i} = \mathrm{Id}_{\pi_1(U_i,x_0)}$. Indeed, we have that the following diagram commutes:

[8] We assume here that the open cover is finite

$$\begin{tikzcd}
\pi_1(X,x_0) \arrow[d, "\mathscr{J}"'] & & \\
\Pi(X) \arrow[d, "\widetilde{\Gamma}"'] & \Pi(U_i) \arrow[l, hook'] & \pi_1(U_i,x_0) \arrow[l, "\mathscr{J}_{U_i}"'] \arrow[llu, bend right, "\Gamma_i"'] \arrow[lld, bend left, "\widetilde{\Gamma}_i \circ \mathscr{J}_{U_i}"] \\
G & &
\end{tikzcd}$$

It is unique because $\widetilde{\Gamma}$ is unique. Indeed, if we are given $\Gamma' : \pi_1(X, x_0) \to G$ that restricts to Γ_i on each $\pi_1(U, x_0)$, then $\Gamma' \circ \mathscr{F} : \Pi(X) \to G$ restricts to $\Gamma_i \circ \mathscr{F}_{U_i}$ on each $\Pi(U_i)$. Therefore $\widetilde{\Gamma} = \Gamma' \circ \mathscr{F}$ and thus $\widetilde{\Gamma} \circ \mathscr{J} = \Gamma'$. This completes the proof. $\square$

Note that the Seifert-Van Kampen theorem does not apply when the open sets in the cover we consider do not have path-connected intersections. An important example is $\mathbb{S}^1$. If we cover it by two open semi-circles, their intersection would be two disjoint open intervals which is not path-connected. This leads to the idea of constructing a "fundamental group with multiple basepoints" and its corresponding Seifert-Van Kampen theorem. We discuss this approach briefly.

Definition 4.4. Let $X \in \mathbf{Top}$ be path-connected. For a set $A \subseteq X$, let $\Pi(X, A)$ denote the full subcategory of $\Pi(X)$ on the objects in A.

As before, our strategy for proving the Seifert-Van Kampen theorem for multiple base-points will be to deduce it from the version for the full fundamental groupoid.

Proposition 4.5. *(**SVK for Groupoids - Multiple Base-points**) Let $X \in \mathbf{Top}$ be path-connected. Let $\{U_i\}_{i \in I}$ be an open cover of X of open subsets such that $\{U_i\}_{i \in I}$ is closed under taking finite intersections. Let $A \subseteq X$ (not necessarily a singleton) such that A contains one point from each path-component of U_i. Then*

$$\Pi(X, A) = \varinjlim_{i \in I} \Pi(U_i, A)$$

Remark 4.6. *We will need to invoke the notion of a retract of a diagram in a category, $\mathscr{C}$. Recall that an object $X \in \mathscr{C}$ in a category is called a retract of an object $Y \in \mathscr{C}$ if there are morphisms*

$$i : X \to Y, \qquad r : Y \to X$$

such that $r \circ i = Id_X$. In this case, r is called a retraction of Y onto X. A commutative diagram, $\mathscr{D}_1$, in $\mathscr{C}$ is a retract of another commutative diagram, $\mathscr{D}_2$, in $\mathscr{C}$ if each 'corner' of $\mathscr{D}_1$ is a rectract of the corresponding corner of $\mathscr{D}_2$ such that all of the inclusions and retractions are compatible with one another in the sense that the diagram obtained by 'pasting together' $\mathscr{D}_1$ and $\mathscr{D}_2$ via the inclusions and retractions commutes. We will use below the categorical fact that the retract of a colimit diagram in category is a colimit diagram.

Proof. Consider the diagram determined by $\Pi(U_i)$'s and also consider the diagram determined by $\Pi(U_i, A)$'s. Denote the diagrams $\mathscr{D}_1$ and $\mathscr{D}_2$ respectively. We claim that $\mathscr{D}_2$ is a retraction of $\mathscr{D}_1$. The inclusions at each 'corner' are the just inclusions

$$\Pi(U_i, A) \hookrightarrow \Pi(U_i)$$

The retractions are built as follows. To retract $\Pi(U_i)$ onto $\Pi(U_i, A)$, pick for every point $x \in U_i$, a path α_x from x to some point $y \in A$ but do this in such a way that if $x \in A$, then α_x is the identity morphism at x[9]. We define the retraction by sending each $x \in U_i$ to the other endpoint of α_x, and each morphism $\beta : x \to y$ to the morphism $\alpha_y \circ \beta \circ \alpha_x^{-1}$[10]. The claim follows by noting that $\mathscr{D}_1$ is a colimit diagram. See [Bro06, Proposition 6.7.2] and [Die08, Theorem 2.6.2] for some relevant partial details. ☐

5. Computations

We calculate the fundamental group of some topological spaces. Our main working tool will be the SVK theorems. Let's first discuss a general example.

5.1. **Fundamental Group of Circle.** We cannot use Corollary 4.3 to compute the the fundamental group of $\mathbb{S}^1$. This is because if we cover $\mathbb{S}^1$ by two open semi-circles, their intersection would be two disjoint open intervals which is not path-connected. Instead, we use Proposition 4.5. Let

$$U_1 = \mathbb{S}^1 \setminus \{(0,1)\}$$
$$U_2 = \mathbb{S}^1 \setminus \{(0,-1)\}$$
$$A = \{(-1,0),(1,0)\}$$

Then U_1, U_2 are simply connected (they are both homeomorphic to $\mathbb{R}$) while $U_1 \cap U_2$ is a homeomorphic to a disjoint union of two copies of $\mathbb{R}$. What is $\Pi(U, A)$? There are clearly morphisms

$$(1,0) \to (-1,0)$$
$$(-1,0) \to (1,0)$$

and they are inverses of each other since any path $(1,0) \to (-1,0) \to (1,0)$ can be shrunk to $(1,0)$ alone. So $\Pi(U, A)$ is simply a category with two objects and a single isomorphism between them. Similar remarks apply to $\Pi(V, A)$. Similarly, $\Pi(U \cap V, A)$ is a category with two distinct objects and no morphisms between the distinct objects. In other words, it is a two object discrete category. What is $\Pi(X, A)$? It is a groupoid with two objects and *two* isomorphisms between. One isomorphism comes from $\Pi(U, A)$ and the other from $\Pi(V, A)$. Denote the isomorphisms as i_U and i_V. Beyond that it is free as possible. So, for example, all the composites $(i_V^{-1} \circ i_U)^n$ are distinct (because there is no reason for them not to be). We get that

$$\pi_1(\mathbb{S}^1, (1,0)) \cong \{(i_V^{-1} \circ i_U)^n \mid n \in \mathbb{Z}\} \cong \mathbb{Z}$$

Remark 5.1. *Usually, x_0 is chosen to be the point $(1,0)$ if we consider $\mathbb{S}^1 \subseteq \mathbb{C}$. We denote the basepoint as $*$. We can also use the theory of covering spaces (which are special instances of fiber bundles) that*

$$\pi_1(\mathbb{S}^1, *) \cong \mathbb{Z}$$

We now derive a number of consequences of this result:

[9]We can always pick these paths because the hypothesis includes that A has at least one point in each component of U_i.

[10]To ensure that the cube formed by the two van Kampen squares and the four retractions commutes, simply always pick the same α_x for x in all of the groupoids it appears in.

Proposition 5.2. *The following statements are true:*

(1) *We have* $\pi_1(\mathbb{R}^2 \setminus \{0\}, x_0) \cong \mathbb{Z}$
(2) *We have*

$$\pi_1\Big(\underbrace{\mathbb{S}^1 \times \cdots \times \mathbb{S}^1}_{n-\text{times}}, (*_1, \cdots, *_n)\Big) \cong \underbrace{\pi_1(\mathbb{S}^1, *_1) \times \cdots \times \pi_1(\mathbb{S}^1, *_n)}_{n-\text{times}} \cong \mathbb{Z}^n$$

(3) $\mathbb{R}^2$ *is not homeomorphic to* $\mathbb{R}^n$ *for* $n \geq 3$[11].
(4) ***(Brouwer's Fixed Point Theorem)*** *If* $f : \mathbb{D}^2 \to \mathbb{D}^2$ *is a continuous function, then* f *has a fixed point. That is, there is a* $x \in \mathbb{D}^2$ *such that* $f(x) = x$.
(5) *(**Fundamental Theorem of Algebra**) Any non-constant polynomial* $p \in \mathbb{C}[x]$ *has a root.*
(6) *There are no retractions* $r : X \to A$ *in the following cases:*
   (a) $X = \mathbb{R}^3$, *with* A *any subspace homeomorphic to* $\mathbb{S}^1$.
   (b) $X = \mathbb{S}^1 \times \mathbb{D}^2$, *with* A *its boundary torus* $\mathbb{S}^1 \times \mathbb{S}^1$.
   (c) X *is the Möbius band and* A *its boundary circle.*

Proof. The proof is given below:

(1) This follows since $\mathbb{R}^2 \setminus \{0\}$ deformation retracts to $\mathbb{S}^1$.
(2) This is a straightforward consequence of Proposition 3.5 and that $\pi_1(\mathbb{S}^1, *) \cong \mathbb{Z}$.
(3) Suppose $f : \mathbb{R}^2 \to \mathbb{R}^n$ is a homeomorphism. Without loss of generality, let $f(0) = 0$. Hence, $\mathbb{R}^2 \setminus \{0\} \cong \mathbb{R}^n \setminus \{0\}$. $\mathbb{R}^2 \setminus 0$ deformation retracts to $\mathbb{S}^1$ and $\mathbb{R}^n \setminus \{0\}$ deformation retracts to $\mathbb{S}^{n-1}$. Therefore,

$$\mathbb{Z} \cong \pi_1(\mathbb{S}^1, *) \cong \pi_1(\mathbb{R}^2 \setminus 0) \cong \pi_1((\mathbb{R}^n \setminus 0) \cong \pi_1(\mathbb{S}^{n-1}, *) \cong \{1\}.$$

a contradiction. Here we use the fact that $\pi_1(\mathbb{S}^{n-1}, *) \cong \{1\} = 0$ for $n \geq 3$. See [Lee10, Lemma 7.19 & Theorem 7.20] for a proof of this fact. See also below.
(4) Assume that $f(x) \neq x$ for all $x \in \mathbb{D}^2$. There is then a deformation retraction $r : \mathbb{D}^2 \to \mathbb{S}^1$ that carries a point $x \in \mathbb{D}^2$ to the intersection of the ray from $f(x)$ to x with the boundary circle $\mathbb{S}^1$. Hence, we have the following diagram:

$$\begin{tikzcd} \mathbb{S}^1 \arrow[r, hook] \arrow[rr, bend left, "\mathrm{Id}_{\mathbb{S}^1}"] & \mathbb{D}^2 \arrow[r, "r"'] & \mathbb{D}^2 \end{tikzcd}$$

Applying π_1, we have the following diagram:

$$\begin{tikzcd} \mathbb{Z} \cong \pi_1(\mathbb{S}^1, *) \arrow[r, hook] \arrow[rr, bend left, "\mathrm{Id}_{\mathbb{Z}}"] & \{\bullet\} \arrow[r] & \pi_1(\mathbb{S}^1, *) \cong \mathbb{Z} \end{tikzcd}$$

This is clearly a contradiction.
(5) We may assume that the polynomial $p(z)$ is of the form

$$p(z) = z^n + a_1 z^{n-1} + \cdots + a_n.$$

Suppose that $p(z)$ has no roots. Fix a $R > 0$ such that

$$R > \max\left\{1, \sum_{i=1}^{n} |a_i|\right\}$$

[11] Clearly, $\mathbb{R}^2$ is not homemorphic to $\mathbb{R}^0$. We have already checked above that $\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^1$. Clearly, $\mathbb{R}^2$ is homeomorphic to $\mathbb{R}^2$.

Then for $|z| = R$ we have

$$\begin{aligned}|z^n| &> (|a_1| + \cdots + |a_n|)|z^{n-1}| \\ &> |a_1 z^{n-1}| + \cdots + |a_n| \\ &\geq |a_1 z^{n-1} + \cdots + a_n|.\end{aligned}$$

From the inequality $|z^n| > |a_1 z^{n-1} + \cdots + a_n|$, it follows that the polynomial

$$p_t(z) = z^n + t(a_1 z^{n-1} + \cdots + a_n)$$

has no roots on the circle $|z| = R$ when $0 \leq t \leq 1$. Note that p_t defines a homotopy between the polynomials z^n and $p(z)$. Consider the formula

$$f_t(s) = \frac{p(tRe^{2\pi is})/p(tR)}{|p(tRe^{2\pi is})/p(tR)|}$$

defined on $[0,1] \times [0,1]$. For each fixed t, Then each $f_t(s)$ defines a loop in the unit circle $\mathbb{S}^1 \subseteq \mathbb{C}$ based at 1. Note that

$$f_0(s) = 1, \quad f_1(s) = \frac{p(Re^{2\pi is})/p(R)}{|p(Re^{2\pi is})/p(R)|}$$

Write $p(z) = z^n + q(z)$. Consider

$$H_t(s) = \frac{[re^{2\pi is})^n + tq(re^{2\pi is})/(r^n + tq(r))}{|[re^{2\pi is})^n + tq(re^{2\pi is})/(r^n + tq(r))|}$$

This defines a homotopy between f_1 and $\omega_n = e^{2\pi ins}$. Since f_0 is homotopic to the constant map and f_0 is homotopic to f_1, we have that ω_n is homotopic to the constant map. Hence, $n = 0$. This is a contradiction.

(6) We use the fact that if $r : X \to A$ is retraction, then the induced map on fundamental groups is injective.
   (a) This follows because there is no injection from $0 \to \mathbb{Z}$.
   (b) This follows because there is no injection from $\mathbb{Z} \times \mathbb{Z}$ to $\mathbb{Z}$. Let $h : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ be any group homomorphism. Suppose $h(1,0) = a$ and $h(0,1) = b$. It follows that $h(-b, a) = (0,0)$, and hence $\ker(h) \neq (0,0)$.
   (c) Skipped.

This completes the proof. □

Remark 5.3. *Here are two cute applications of Brouwer's fixed point theorem:*

(1) *A 3×3 real invertible matrix with non-negative entries has a real positive eigenvalue. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear map corresponding to a matrix A. Define*

$$B = \mathbb{S}^2 \cap \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1, x_2, x_3 \geq 0\right\} \cong \mathbb{D}^2.$$

If $x \in B$, then all coordinates of $Tx = Ax$ are non-negative and not all zero since A is non-singular and not all coordinates of $x \in B$ can be zero. Therefore, the normalized vector $Tx/\|Tx\|$ lies in B. Now, consider the continuous map $f : B \to B$ defined by

$$f(x) = \frac{Tx}{\|Tx\|}.$$

By Brouwer's Fixed Point Theorem, there exists a point $x_0 \in B$ such that $f(x_0) = x_0$, which implies $Tx_0 = \|Tx_0\|x_0$. Setting $\lambda = \|Tx_0\|$, we conclude that λ is an eigenvalue of A, with $\lambda \in \mathbb{R}$ and $\lambda > 0$.

(2) *A* 3×3 *real matrix with positive entries has a positive real eigenvalue. This follows as in (1).*

5.2. **Fundamental Group of Spheres.** Let $X = \mathbb{S}^n$ for $n \geq 2$. Let

$$U = \mathbb{S} \setminus \{N\}, \quad V = \mathbb{S} \setminus \{S\}$$

Clearly, $U, V \cong \mathbb{R}^{n-1}$ via the stereographic projection. Hence, U and V are two open simply connected subsets of $\mathbb{S}^n$ with path connected intersection. Hence, we have that $\mathbb{S}^n$ is simply connected for $n \geq 2$, since the colimit of two trivial groups is the trivial the group. Let's consider a basic application:

Example 5.4. Consider $X_m = \mathbb{R}^n - \{m \text{ points}\}$ for $n \geq 3$. We compute the fundamental group of X_n by induction on n. The case $m = 1$ is easy since

$$\mathbb{R}^n - \{\text{a point}\} \cong \mathbb{S}^{n-1} \times \mathbb{R}^+$$

which is simply connected:

$$\pi_1(\mathbb{R}^n - \{\text{a point}\}) \cong \pi_1(\mathbb{S}^{n-1}, *) \times \pi_1(\mathbb{R}^+, 1) \cong \{\bullet\}$$

Now assume that $m > 1$. Divide the points in into two sets of smaller size, A and B. A and B can be separated by the hyperplane H, and that N^+ and N^- are two open neighborhoods of the half-spaces that result. For an arbitrary base-point $x_0 \in H$, Van Kampen theorem applies, giving a surjection

$$\pi_1(N^+ \setminus A, x_0) * \pi_1(N^- \setminus B, x_0) \twoheadrightarrow \pi_1(\mathbb{R}^n - \{m \text{ points}\}, x_0)$$

By the induction hypothesis[12], both $N^+ \setminus A$ and $N^- \setminus B$ are simply connected. Hence,

$$\pi_1(\mathbb{R}^n - \{m, \text{ points }\}, x_0) \cong 0.$$

Example 5.5. Let V be a finite-dimensional real vector space and $W \subset V$ a (proper) linear subspace. We compute the fundamental group $\pi_1(V \setminus W)$. Since every finite-dimensional real vector space is linearly homeomorphic to some $\mathbb{R}^n$, we can assume WLOG that $V = \mathbb{R}^n$ and $W = \mathbb{R}^m$ for $m < n$. Projecting first onto $(R^m)^\perp$ and then unit sphere shows that $\mathbb{R}^n \setminus \mathbb{R}^m$ is homotopically equivalent to $\mathbb{S}^{n-m-1}$, Therefore, we have:

$$\pi_1(\mathbb{R}^n \setminus \mathbb{R}^m) = \begin{cases} \mathbb{Z}, & \text{if } m = n-2, \\ 0, & \text{otherwise.} \end{cases}$$

5.3. **Fundamental Group of Wedge Sums.** In order to use Van Kampen's theorem to compute the fundamental group of the wedge sum, we need to put a mild restriction on the type of base points we consider. A point p in a topological space X is said to be a non-degenerate base point if p has a neighborhood that admits a strong deformation retraction onto p.

Lemma 5.6. *Suppose* $x_i \in X_i$ *is a non-degenerate base point for* $i = 1, \ldots, n$. *Then* $\bigvee_{i=1}^n x_i$ *is a non-degenerate base point in* $X_1 \vee \cdots \vee X_n$.

Proof. For each i, choose a neighborhood W_i of x_i that admits a deformation retraction $r_i : W_i \to \{x_i\}$, and let $H_i : W_i \times I \to W_i$ be the associated homotopy. Define a map

$$H : \coprod_{i=1}^n W_i \times I \to \coprod_{i=1}^n W_i$$

[12] Here we use the observation that a open half space in $\mathbb{R}^n$ is homeomorphic to $\mathbb{R}^n$.

by letting $H = H_i$ on $W_i \times I$. Let W be the image of $\coprod_{i=1}^n W_i$ under the quotient map

$$q : \coprod_{i=1}^n X_i \to \bigvee_{i=1}^n X_i$$

Since $\coprod_{i=1}^n W_i$ is a saturated open set, W is an open set of $X_1 \vee \cdots \vee X_n$ that is a neighbourhood of $\bigvee_{i=1}^n x_i$. We have that

$$q \times \mathrm{Id}_I : \coprod_{i=1}^n W_i \times I \to W \times I$$

is a quotient map. Since $q \circ H$ respects the identifications made by $q \times \mathrm{Id}_I$, it descends to the quotient and yields a deformation retraction of W onto $\vee_{i=1}^n x_i$. □

Proposition 5.7. *Let $X_1, \ldots, X_n$ be spaces with non-degenerate base points $x_i \in X_i$. The map*

$$\Phi : \pi_1(X_1, x_1) * \cdots * \pi_1(X_n, x_n) \to \pi_1\left(\bigvee_{i=1}^n X_i, \bigvee_{i=1}^n x_i\right)$$

induced by $\iota_i : \pi_1(X_i, x_i) \to \pi_1(\bigvee_{i=1}^n X_i, \bigvee_{i=1}^n x_i)$ is an isomorphism.

Proof. It suffices to consider the case $n = 2$. The general case follows by induction. Choose neighborhoods W_i in which x_i is a deformation retract, and let

$$U = q(X_1 \coprod W_2), \qquad V = q(W_1 \coprod X_2)$$

where $q : X_1 \coprod X_2 \to X_1 \vee X_2$ is the quotient map. Since $X_1 \coprod W_2$ and $W_1 \coprod X_2$ are saturated open sets in $X_1 \coprod X_2$, the restriction of q to each of them is a quotient map onto its image, and U and V are open in the wedge sum. The three maps

$$\begin{aligned} \{*\} &\hookrightarrow U \cap V, \\ X_1 &\hookrightarrow U, \\ X_2 &\hookrightarrow V \end{aligned}$$

are all homotopy equivalences. Because $U \cap V$ is contractible, we have: $U \hookrightarrow X_1 \vee X_2$ and $V \hookrightarrow X_1 \vee X_2$ induce an isomorphism

$$\pi_1(U) * \pi_1(V) \cong \pi_1(X_1 \vee X_2).$$

Moreover, the injections $\phi_1 : X_1 \hookrightarrow U$ and $\phi_2 : X_2 \hookrightarrow V$, which are homotopy equivalences, induce isomorphisms

$$\begin{aligned} \pi_1(X_1, x_1) &\cong \pi_1(U) \\ \pi_1(X_2, x_2) &\cong \pi_1(V) \end{aligned}$$

Hence,

$$\pi_1(X_1, x_1) * \pi_1(X_2, x_2) \cong \pi_1(X_1 \vee X_2, x_1 \vee x_2).$$

The general case follows by induction. □

Example 5.8. The following is a list of computations based on the information about the fundamental group of wedge sums:

(1) Consider$X = \bigvee_{i=1}^n \mathbb{S}^1$. We have

$$\pi_1(\mathbb{S}^1 \vee \cdots \vee \mathbb{S}^1, \bigvee_{i=1}^n *_i) \cong \pi_1(\mathbb{S}^1, *_1) * \cdots * \pi_1(\mathbb{S}^1, *_n) \cong \underbrace{\mathbb{Z} * \cdots * \mathbb{Z}}_{n \text{ times}}$$

(2) Let X be the union of n lines through the origin in $\mathbb{R}^3$. Then $\mathbb{R}^3 - X$ deformation retracts to $\mathbb{S}^2$ minus $2n$ points, which is homeomorphic to $\mathbb{R}^2$ minus $2n-1$ points. This in turn admits a deformation retraction to a wedge of $2n-1$ circles, so

$$\pi_1(\mathbb{R}^3 - X, x_0) \cong \underbrace{\mathbb{Z} * \cdots * \mathbb{Z}}_{2n-1 \text{ times}}$$

5.4. **Fundamental Group of Graphs.**

Definition 5.9. A graph is a CW complex of dimension 0 or 1. The 0-cells of a graph are called its vertices, and the 1-cells are called its edges.

It follows from the definition of a CW complex that for each edge e, the set $\overline{e} \setminus e$ consists of one or two vertices. If a vertex v is contained in $\overline{e}$, we say that v and e are incident. A subgraph is a subcomplex of a graph. Thus, if a subgraph contains an edge, it also contains the vertex or vertices incident with it. Here is some more important terminology:

- An edge path in a graph is a finite sequence $(v_0, e_1, v_1, \ldots, v_{k-1}, e_k, v_k)$ that starts and ends with vertices and alternates between vertices and edges, such that for each i, $\{v_{i-1}, v_i\}$ is the set of vertices incident with the edge e_i.
- An edge path is said to be closed if $v_0 = v_k$, and simple if no edge or vertex appears more than once, except that v_0 might be equal to v_k.
- A cycle is a nontrivial simple closed edge path.
- A tree is a connected graph that contains no cycles.

Lemma 5.10. *Let G be a finite graph.*

(1) *If G is a tree, then G is contractible and hence simply connected. In fact, if v_0 is a vertex of G, then v_0 is a deformation retract of G.*
(2) *If G is a connected graph, then G contains a maximal tree - called a spanning tree.*

Proof. The proof is given below:

(1) We induct on the number of edges, n. If $n = 1$, it G is homeomorphic to an interval I. The claim is clearly true in this case. Assume the claim is true for $n \in \mathbb{N}$ and consider the case $n + 1 \in \mathbb{N}$. Since G is simple, every edge of G is incident with exactly two vertices. If every vertex in G is incident with at least two edges, then, we can construct sequences $(v_j)_{j\in\mathbb{Z}}$ of vertices and $(e_j)_{j\in\mathbb{Z}}$ of edges such that for each j, v_{j-1} and v_j are the two vertices incident with e_j, and e_j, e_{j+1} are two different edges incident with v_j. Because T is finite, there must be some integers n and $n+k > n$ such that $v_n = v_{n+k}$. If n and k are chosen so that k is the minimum positive integer with this property, this means that $(v_n, e_{n+1}, \ldots, e_{n+k}, v_{n+k})$ is a cycle, contradicting the assumption that G is a tree. Hence, we can choose $v_1 \in G$ such that v_1 is incident to only one edge. Let v_1' denote the other vertex. Then e deformation retracts onto the vertex v_1'. The result is then a tree with n edges, which deformation retracts onto v_0.
(2) Since the empty subgraph is a tree, an application of Zorn's lemma shows that G contains a maximal subtree - a subgraph that is a tree and is not properly contained in any larger tree in G.

This completes the proof. $\square$

Remark 5.11. *Lemma 5.10 can be extended to the case of infinite graphs.*

Remark 5.12. *A spanning tree $T \subseteq G$ contains every vertex of G. Indeed, suppose that there is a vertex $v \in G$ that is not contained in T. Because G is connected, there is an edge path from a vertex $v_0 \in T$ to v, say $(v_0, e_1, \ldots, e_k, v_k = v)$. Let v_i be the last vertex in the edge path that is contained in T. Then the edge e_{i+1} is not contained in T, because if it were, v_{i+1} would also be in T since T is a subgraph. The subgraph $T' = T \cup \{e_{i+1}\}$ properly contains T, so it is not a tree, and therefore contains a cycle. This cycle must include e_{i+1} or v_{i+1}, because otherwise it would be a cycle in T. However, since e_{i+1} is the only edge of T' that is incident with v_{i+1}, and v_{i+1} is the only vertex of T' incident with e_{i+1}, there can be no such cycle.*

Proposition 5.13. *Let G be a finite graph and let $T \subseteq G$ be a spanning tree. Let $n_{G \setminus T}$ denote the number of edges in $G \setminus T$. If $v_0 \in G$, then*

$$\pi_1(G, v_0) \cong \underbrace{\mathbb{Z} * \cdots * \mathbb{Z}}_{n_{G\setminus T} \, times}$$

Proof. The proof is by induction on the number of edges in $G \setminus T$. Let $n = 1$. Clearly, $G/T \cong \mathbb{S}^1$. Consider the map

$$q : G \to G/T \cong \mathbb{S}^1.$$

We show q is a homotopy equivalence. We define a map

$$q' : G/T \cong \mathbb{S}^1 \to G.$$

Let e be the edge not contained in T. Pick paths α_1 and α_2 in T from v_0 to v_1 and v_2, respectively. Consider the loop $\alpha_1 \circ e \circ \alpha_2^{-1}$. It is easily checked that $q \circ q'$ and $q' \circ q$ are homotopic to the identity maps. If $n > 1$ and assume that the claim is true for $n \in \mathbb{N}$. we can use Van Kampen's theorem to prove the case for $n + 1 \in \mathbb{N}$. Let $e_1, \cdots, e_{n+1}$ be edges in $G \setminus T$. For each $i = 1, \ldots, n + 1$, choose a point $x_i \in e_i$ Let

$$U = G \setminus \{x_1, \ldots, x_n\}$$
$$V = G \setminus \{x_{n+1}\}$$

Both U and V are open in G. Just as before, it is easy to construct deformation retractions to show that

$$U \cap V \simeq T, \quad U \simeq T \cup e_{n+1} \quad V \simeq G \setminus e_{n+1}.$$

By the inductive hypothesis, we have $\pi_1(V, v_0) \cong \mathbb{Z}$ and

$$\pi_1(V, v_0) = \underbrace{\mathbb{Z} * \cdots * \mathbb{Z}}_{n \text{ times}}$$

The claim follows by Van Kampen's theorem noting that Since $U \cap V \cong T$ is simply connected. □

5.5. **Fundamental Group of CW Complexes.** Let X be a connected CW complex. If $X = X^0$, then X is a point and the fundamental group of X is the trivial. If $X = X^1$, then X is a graph and we have already covered that case.

Proposition 5.14. *Let X be a path-connected CW complex such that $X = X^2$. Let $x_0 \in X$ and let $\varphi_\alpha : \mathbb{S}^1 \to X$ be the attaching maps of the 2-cells $\mathbb{D}_\alpha$ and let $\gamma_\alpha : I \to X$ be a path from x_0 to $\varphi_\alpha(1)$. Then*

$$\pi_1(X, x_0) \cong \pi_1(X^1, x_0)/N,$$

where N is the normal subgroup generated by the path $\{\gamma_\alpha \circ \varphi_\alpha \circ \overline{\gamma}_\alpha\}$.

Proof. The proof is given below:

(1) Let A be a subcomplex generated by the union of the 2-cells, $\mathbb{D}^2_\alpha$ and toegether with the φ_α's. Then A is a contractible subcomplex. Hence,
$$\pi_1(A, x_0) \cong \{1\}$$
(2) Choose points $x_\alpha \in \mathbb{D}^2_\alpha$ and define the subset $B = X^2 - \bigcup_\alpha \{x_\alpha\}$. Then B retracts to X^1. Hence,
$$\pi_1(B, x_0) \cong \pi_1(X^1, x_0)$$
(3) We have $X^2 = A \cup B$ and $A \cap B$ consists of precisely those edge-cycles starting at x_0 that make up loops homotopic to the boundaries of 2-cells, or in other words, the images of $\mathbb{S}^1_\alpha$ under the attaching maps. Therefore, each element of $\pi_1(A \cap B, x_0)$ represents a an of $\{\gamma_\alpha \circ \varphi_\alpha \circ \overline{\gamma}_\alpha\}$.
(4) By Van-Kampen's theorem,
$$\pi_1(X, x_0) \cong \pi_1(X^2, x_0) \cong \frac{\pi_1(A, x_0) * \pi_1(B, x_0)}{N} \cong \pi_1(X^1, x_0)/N$$

This completes the proof. □

In fact, we now show that the fundamental group of a CW complex only depends on its 2-skeleton with basepoint x_0.

Corollary 5.15. *Let X be a path-connected CW complex. If $x_0 \in X$, then*
$$\pi_1(X, x_0) \cong \pi_1(X^2, x_0).$$

Proof. This follows simply because $A \cap B$ as in Proposition 5.14 will comprised on boundaries of n-cells for $n \geq 3$. These are all contractible. Hence, an application of Van-Kampen's theorem yields the desired result. □

Example 5.16. We can use the discussion in the previous section to compute the fundamental groups of topological spaces introduced Section 1.2.

(1) Let $X = \mathbb{S}^1 \times \mathbb{S}^1$. We have already computed the fundamental group of X, but we can also compute it using the discussion above. We have,
$$\pi_1(X, x_0) \cong \frac{\mathbb{Z} * \mathbb{Z}}{\langle aba^{-1}b^{-1} \rangle} \cong \langle a, b \mid aba^{-1}b^{-1} \rangle \cong \mathbb{Z} \times \mathbb{Z}$$
(2) Let $X = \mathbb{RP}^2$. We have,
$$\pi_1(X, x_0) = \frac{\mathbb{Z}}{\langle a^2 \rangle} \cong \langle a \mid a^2 \rangle \cong \mathbb{Z}_2$$
In general, if $X = \mathbb{RP}^n$, we have
$$\pi_1(\mathbb{RP}^n, x_0) \cong \mathbb{Z}_2$$
This follows at once from the computation above and that 2-skeleton of $\mathbb{RP}^n$ is just $\mathbb{RP}^2$.
(3) If $X = \mathbb{CP}^n$ then X is simply-connected. This is because the 1-skeleton of X consists of a single 0-cell.
(4) Let X be the g-holed surface in Example 1.19. We have
$$\pi_1(X, x_0) = \langle a_1, b_1, \ldots, a_g, b_g \mid [a_1, b_1] \cdots [a_g, b_g] = 1 \rangle$$
This follows pretty much by the definition of X and the fact that the 1-skeleton is a wedge sum of $2g$ circles.

(5) Let $X = K$ (Klein bottle). We have,

$$\pi_1(X, x_0) = \frac{\mathbb{Z} * \mathbb{Z}}{\langle abab^{-1} \rangle} \cong \langle a, b \mid abab^{-1} \rangle$$

Let A be the subgroup generated by a and B be the subgroup generated by b. We have $A, B \cong \mathbb{Z}$. Then since $bab^{-1} = a^{-1}$, we have that B is a normal subgroup. Clearly, A and B generate $\pi_1(X, v)$ and since every element has a unique representation in the form $b^n a^m$, we have that and $A \cap B = \{e\}$. Hence,

$$\pi_1(X, x_0) \cong \mathbb{Z} \rtimes \mathbb{Z}$$

5.6. **Fundamental Group of Knot Complements.**

Part 2. Covering Spaces

6. Definitions & Examples

Covering spaces offer a powerful framework in topology by enabling the study of complex spaces through simpler, well-behaved ones. One of their most compelling features is the ability to lift paths and homotopies from the base space to the covering space. This lifting property allows us to analyze the behavior of loops and paths in the base space by observing their images in the covering space, where the geometry and topology are often easier to handle. Importantly, this process reveals rich information about the fundamental group of the base space.

6.1. Definitions.

Definition 6.1. Let X be a topological space. A covering space of X is a topological space $\tilde{X}$ together with a continuous surjective map $p : \tilde{X} \to X$ called a covering map such that for every point $x \in X$, there exists an open neighborhood $U_x \subseteq X$ and a discrete topological spaces, D_x, such that

$$p^{-1}(U_x) = \coprod_{\alpha \in D_x} U_\alpha$$

where V_d is an open set of $\tilde{X}$ homeomrophic to U_x.

Remark 6.2. *Covering spaces are special examples of fiber bundles where the fiber is a discrete topological space. In a covering space $p : \tilde{X} \to X$, the local triviality condition resembles that of fiber bundles: for each $x \in X$, there exists an open neighborhood $U_x \subseteq X$ such that $p^{-1}(U_x) \cong U_x \times F$, where F is a discrete set (the fiber).*

Remark 6.3. *If X is a connected topological space, the cardinality of D_x in Definition 6.1 is constant. That is, $|D_x| = |D_y|$ for each $x, y \in X$. Indeed, let $x \in X$ and let U_x be an open set as in Definition 6.1. Then for each $y \in U_x$ and $\alpha \in D_x$, the intersection $p^{-1}(\{y\}) \cap U_\alpha$ contains exactly one point, since the restriction of p to U_α is a homeomorphism onto U_x. Now fix $a \in X$, and define the set*

$$A := \{x \in X \mid |p^{-1}(\{x\})| = |p^{-1}(\{a\})|\}.$$

By the above argument, the cardinality of the fiber is locally constant so both A and its complement A^c are open sets in X. Since X is connected, it follows that $X \setminus A = \emptyset$, and thus $A = X$. If the cardinality is n, we say that p is a n-sheeted covering map.

Example 6.4. The following is a list of examples of covering spaces:

(1) Any homeomorphism is a covering map.
(2) If X is any topological space and D is a discrete topological spaces, then the projection $p : X \times D \to X$ is a covering map.
(3) The map $p : \mathbb{R} \to \mathbb{S}^1$ defined by

$$p(t) = (\cos 2\pi t, \sin 2\pi t).$$

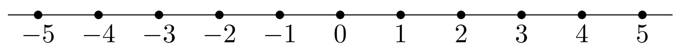
Indeed, let $x = (x_1, x_2) \in \mathbb{S}^1$ be a point on the unit circle such that $x_1 > 0$. Consider the open set

$$U := \{(x_1, x_2) \in \mathbb{S}^1 \mid x_1 > 0\},$$

which is an open neighborhood of x in $\mathbb{S}^1$. The pre-image of U under p is the disjoint union

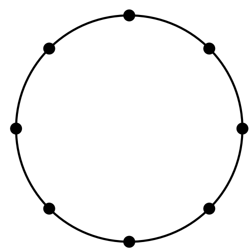
$$p^{-1}(U) = \bigsqcup_{n\in\mathbb{Z}} \left(n - \tfrac{1}{4},\, n + \tfrac{1}{4}\right),$$

where each interval $(n - 1/4,\, n + 1/4)$ is mapped homeomorphically onto U by p. Hence, p is an ∞-sheeted covering map. The fiber over the point $1 \in \mathbb{S}^1$ is given by $\mathbb{Z}$.



The fiber over the point $1 \in \mathbb{S}^1$ is given by $\mathbb{Z}$.

(4) The map $p : \mathbb{S}^1 \to \mathbb{S}^1$ defined by $p(z) = z^n$ for $n \in \mathbb{N}$ is an n-sheeted covering map. The fiber of 1 are the n-th roots of unity.



The fiber of 1 are the 8-th roots of unity.

Let's now prove some general properties bout covering spaces:

Proposition 6.5. *The following are properties of covering spaces.*

(1) *A covering map is an open map.*
(2) *A covering map is a local homeomorphism.*
(3) *The restriction of a covering map is a convering map.*
(4) *A finite product of covering maps is a covering map.*

Proof. The proof is given below:

(1) Let $p : \bar{X} \to X$ be a covering map. Let U be an open set in $\bar{X}$, and fix a point $p(x) \in p(U)$. Since p is a covering map, let $U_{p(x)} \subseteq X$ be as in Definition 6.1. Let U_α be the slice of $p^{-1}(U_{p(x)})$ containing x. Then p maps U_α homeomorphically onto $U_{p(x)}$. So $p(U_\alpha \cap U) \subseteq p(U)$ is open in X. Hence, p is an open map.
(2) This is clear.
(3) Let $p : \bar{X} \to X$ be a covering map. Let $X_0 \subseteq X$ and consider the restricted map $p|_{p^{-1}(X_0)} : p^{-1}(X_0) \to X_0$. The map is clearly continuous. Since $p : \bar{X} \to X$ is a covering space, for each $x \in X$ there exists an open set U_x such that $p^{-1}(U_x)$ satisfies Definition 6.1. The map $p|_{p^{-1}(X_0)}$ satisfies Definition 6.1 if we choose the open set to be $U_x \cap X_0$.
(4) $p_i : \bar{X}_i \to X_i$ be covering maps for $i = 1, 2$. Choose $(x_1, x_2) \in X_1 \times X_2$. Then there is a neighborhood U_{x_i} of x_i in X_i such that $p_i^{-1}(U_{x_i})$ satisfies Definition 6.1. Then $U_{x_1} \times U_{x_2}$ is an open set of $X_1 \times X_2$ such that $(p_1 \times p_2)^{-1}(U_{x_1} \times U_{x_2})$ satisfies Definition 6.1.

This completes the proof. □

How are two covering spaces of a topological space related? Can we define a map between two covering spaces to identify them up to isomorphism? This question leads us to the definition of homomorphisms between covering spaces.

Definition 6.6. Let $(\bar{X}_1, p_1)$ and $(\bar{X}_2, p_2)$ be covering spaces of a topological space X. A morphism of $(\bar{X}_1, p_1)$ into $(\bar{X}_2, p_2)$ is a continuous map $\phi : \bar{X}_1 \to \bar{X}_2$ such that the following diagram commutes:

$$\begin{tikzcd} \bar{X}_1 \arrow[rr, "\phi"] \arrow[dr, "p_1"'] & & \bar{X}_2 \arrow[dl, "p_2"] \\ & X & \end{tikzcd}$$

A homomorphism ϕ of $(\bar{X}_1, p_1)$ into $(\bar{X}_2, p_2)$ is an isomorphism if there exists a homomorphism ψ of $(\bar{X}_2, p_2)$ into $(\bar{X}_1, p_1)$ such that $\psi \circ \phi$ and $\phi \circ \psi$ are the identity maps on $\bar{X}_1$ and $\bar{X}_2$.

6.2. **Examples.** Group actions on topological spaces provide a rich source of covering maps, which have significant applications in various areas of mathematics, particularly in geometric topology and geometric group theory. For instance, in geometric topology the quotient spaces formed by group actions often inherit interesting topological properties, which can be studied via covering space theory. These spaces provide insights into the structure of manifolds and their fundamental groups.

Proposition 6.7. *Let G be a topological group acting on a topological space X. Assume that for each $x \in X$, there exists an open set $U_x \subseteq X$ containing x such that for each $g \in G$ with $g \neq e$, we have $gU_x \cap U_x = \emptyset$. The quotient map $p : X \to X/G$ is a covering map.*

Remark 6.8. *A group action satisfying the condition in Proposition 6.7 is called a covering space action. We use this terminology from now.*

Proof. (Proposition 6.7) Since $q^{-1}(q(U_x)) = \bigsqcup_{g \in G} gU_x$, the set $q(U_x) = GU_x$ is open in X/G and satisfies Definition 6.1. Hence, $p : X \to X/G$ is a covering map. □

Example 6.9. The following is a list of examples of covering maps generated by group actions:

(1) Let $\mathbb{Z}$ act on $\mathbb{R}$ by translations: for each $n \in \mathbb{Z}$, define the action $n \cdot x = x + n$. This action satisfies the assumptions in Proposition 6.7. Hence, the map $\mathbb{R} \to \mathbb{R}/\mathbb{Z} \cong \mathbb{S}^1$ is a covering map. This reproves Example 6.4(1).

(2) Let $\mathbb{Z}_2$ act on the n-sphere $\mathbb{S}^n$ by the antipodal map:

$$g \cdot x = -x, \quad \text{for all } x \in \mathbb{S}^n.$$

where $g \neq e \in \mathbb{Z}_2$. This action satisfies Proposition 6.7. Hence, the projection map $p : \mathbb{S}^n \to \mathbb{S}^n/\mathbb{Z}_2 \cong \mathbb{RP}^n$ is a covering space. In fact, it is a two-sheeted covering map since each point in $\mathbb{RP}^n$ corresponds to a pair of antipodal points on $\mathbb{S}^n$.

(3) We can generalize (2). Let $\mathbb{Z}_p$ act on the odd-dimensional sphere $\mathbb{S}^{2n-1} \subseteq \mathbb{C}^n$ by

$$\zeta \cdot (z_1, z_2, \ldots, z_n) = (\zeta^{q_1} z_1, \zeta^{q_2} z_2, \ldots, \zeta^{q_n} z_n),$$

where $\zeta \in \mathbb{Z}_p$ is a primitive p-th root of unity (i.e., $\zeta = e^{2\pi i/p}$) and $q_1, \ldots, q_n \in \mathbb{Z}$ are integers coprime to p. This action satisfies Proposition 6.7, so the quotient

$$\mathbb{S}^{2n-1} \to \mathbb{S}^{2n-1}/\mathbb{Z}_p$$

is a covering map. $L(p; q_1, \dots, q_n) := \mathbb{S}^{2n-1}/\mathbb{Z}_p$ is called a lens space. The projection map is a p-sheeted covering map.

Remark 6.10. *Lens spaces generalize real projective spaces $\mathbb{RP}^{2n-1} \cong L(2; 1, \dots, 1)$. and play an important role in low-dimensional topology and the study of 3-manifolds.*

7. Lifting Properties

Studying the lifting properties of maps in covering spaces is fundamental because it allows us to transfer complex topological problems to simpler, often more manageable spaces. Lifting of paths helps in understanding how loops and homotopies in the base space relate to the structure of the covering space, providing key insights into the fundamental group.

Proposition 7.1. *Let $p : \bar{X} \to X$ be a covering map. Any path $f : I \to X$ with initial point x_0 can be uniquely lifted to a path $\bar{f} : I \to \bar{X}$ with an initial point in $p^{-1}(x_0)$ such that $p \circ \bar{f} = f$.*

Proof. We first prove existence. First assume that $f(I) \subseteq U_{x_0}$, where U_{x_0} satisfies Definition 6.1. For any $\bar{x}_0 \in p^{-1}(x_0)$, let $\bar{U}$ be an open set containing $\bar{x}_0$ that is mapped homemorphically to U_{x_0}. the path component of $p^{-1}(U)$ which contains $\tilde{x}_0$. Clearly, the path $\bar{f} = p^{-1}|_{U_{x_0}} \circ f : I \to \bar{X}$ is a path such that such that $p \circ \bar{f} = f$. Now assume that the image of f is not contained in U_{x_0} or in a single. In this case, let $\{U_i\}_i$ be an open cover of X by open sets satisfying Definition 6.1. Then, $\{f^{-1}(U_i)\}_i$ is an open covering of I. Let λ be the Lebesgue number of the covering. Now, choose $n \in \mathbb{N}$ such that $1/n < \lambda$. Divide the interval I into the closed sub-intervals of length $1/n$. Since the diameter of these intervals is less than λ, f maps each of these intervals inside some U_i. We can now apply the argument above. We now show uniqueness by proving that given any two maps $\bar{f}_0, \bar{f}_1 : I \to \bar{X}$ with same initial point such that $p \circ \bar{f}_0 = p \circ \bar{f}_1$, the set

$$A = \{t \in I \mid \bar{f}_0(t) = \bar{f}_1(t)\}$$

is either empty or all of I. It suffices to show that A is cl-open. First we will see that it is a closed set. Let t be in the closure of A and let $x = p \circ \bar{f}_0(t) = p \circ \bar{f}_1(t)$. Assume $\bar{f}_0(t) \neq \bar{f}_1(t)$. We will see that this leads to a contradiction. Let $x \in U_x$ be an open satisfying Definition 6.1, and let $\bar{U}_0$ and $\bar{U}_1$ be open sets of $\bar{X}$ containing $\bar{f}_0(t)$ and $\bar{f}_1(t)$ respectively that are mapped homeomorphically to x. Since $\bar{f}_0$ and $\bar{f}_1$ are both continuous, we can find a neighborhood $t \in W \subseteq I$ such that $\bar{f}_0(W) \subseteq \bar{U}_0$ and $\bar{f}_1(W) \subseteq \bar{U}_1$. But $\bar{U}_0 \cap \bar{U}_1 = \emptyset$. This is a contradiction to the fact that every neighborhood of t must intersect the set A. This shows that A is closed. Analogously, we can argue that every point in A is an interior point and therefore the set is open. Since $\bar{f}_0$ and $\bar{f}_1$ agree on at least one point in I, i.e., $\bar{f}_0(0) = \bar{f}_1(0)$, they have to be equal. □

Proposition 7.2. *Let $p : \bar{X} \to X$ be a covering map. Any homotopy $H : Y \times I \to X$ can be uniquely lifted to $\bar{X}$ if $H_0 : Y \to X$ can be lifted to $\bar{X}$ provide that $\bar{H}_0$ has been specified.*

Proof. A more general version of Proposition 7.2 will be proved in Proposition 15.4. For an alternative approach, see [Lee10], which demonstrates how Proposition 7.2 can be applied by generalizing the argument in Proposition 7.1. □

Let us now use Proposition 7.1 and Proposition 7.2 to recompute the fundamental group of $\mathbb{S}^1$.

Example 7.3. (**Homotopy Classification of Loops in $\mathbb{S}^1$**) Consider the covering space $p : \mathbb{R} \to \mathbb{S}^1$. We compute the fundamental group of $\mathbb{S}^1$ in the following steps:

(1) If $f : I \to \mathbb{S}^1$, then any two lifts $\bar{f}_1, \bar{f}_2(0)$ such that $\bar{f}_1(0) = \bar{f}_2$ differ by an integer. Indeed, the fact that $p(\bar{f}_1) = p(\bar{f}_2))$ implies that $\bar{f}_1(t) - \bar{f}_2(t) \in \mathbb{Z}$ for each $t \in I$. Since $\bar{f}_1 - \bar{f}_2$ is a continuous function from the connected space I into the discrete space $\mathbb{Z}$, it must be constant.

(2) Let $f_0, f_1 : I \to \mathbb{S}^1$ be two paths in $\mathbb{S}^1$ with same initial and terminal points. If $\bar{f}_0(0) = \bar{f}_1(0)$, then $f_0 \sim f_1$ if and only if $\bar{f}_0(1) = \bar{f}_1(1)$. The forward direction is clear since $\mathbb{R}$ is simply-connected (Remark 3.9). For the reverse direction, suppose that $f_0 \sim f_1$. Let $H : I \times I \to \mathbb{S}^1$ be a between f_0 and f_1. Then Proposition 7.2 implies that H lifts to a homotopy

$$\bar{H} : I \times I \to \mathbb{R}$$

such that $\bar{H}(\cdot, 0) = \bar{f}_0$. Now $\bar{H}_1(\cdot, 1) : I \to \mathbb{R}$ is a path of that is a lift of f_1 starting at $\bar{f}_1(0)$. By uniqueness of lifts, it must be equal to $\bar{f}_1$. Thus, $\bar{f}_0 \sim \bar{f}_1$ and this implies that $\bar{f}_0(1) = \bar{f}_1(1)$.

(3) (**Winding Number**) Suppose $f : I \to \mathbb{S}^1$ is a loop based at a point $x_0 \in \mathbb{S}^1$. If $\tilde{f} : I \to \mathbb{R}$ is any lift of f, then $\tilde{f}(1)$ and $\tilde{f}(0)$ are both points in the fiber $p^{-1}(x_0)$, so they differ by an integer. Since any other lift differs from $\tilde{f}$ by an additive constant, the difference

$$\tilde{f}(1) - \tilde{f}(0)$$

is an integer that depends only on f, and not on the choice of lift. This integer is denoted by $N(f)$, and is called the winding number of f. (1) and (2) at once imply that two loops in $\mathbb{S}^1$ based at the same point are path-homotopic if and only if they have the same winding number.

(4) (**Fundamental Group of $\mathbb{S}^1$**) We can now show that $\pi_1(\mathbb{S}^1, 1) \cong \mathbb{Z}$ generated by $[\omega]$ where $\omega : I \to \mathbb{S}^1$ such that $\omega(t) = e^{2\pi i t}$. Define the maps

$$J : \mathbb{Z} \to \pi_1(\mathbb{S}^1, 1), \qquad K : \pi_1(\mathbb{S}^1, 1) \to \mathbb{Z},$$
$$n \mapsto [\omega^n] \qquad [f] \mapsto N(f)$$

It is clear that J, K are well-defined and that J, K are homomorphisms. We show that J, K are two-sided inverses. To prove that $K \circ J = \mathrm{Id}_{\mathbb{Z}}$, let $n \in \mathbb{Z}$ be arbitrary. Note that

$$K(J(n)) = K([\omega^n]) = K([\alpha_n]) = N(\alpha_n) = n,$$

where $\alpha_n : I \to \mathbb{S}^1$ is the map $\alpha_n(t) = e^{2\pi i n t}$. To prove that $J \circ K = \mathrm{Id}_{\pi_1(\mathbb{S}^1,1)}$, suppose f is any element of $\pi_1(\mathbb{S}^1, 1)$, and let n be the winding number of f. Then f and α_n are path-homotopic because they are loops based at 1 with the same winding number. Therefore,

$$J(K([f])) = J(n) = [\omega]^n = [\alpha_n].$$

Let us now use Proposition 7.1 and Proposition 7.2 to determine how the fundamental groups of the based space and covering space in a covering map relate to each other.

Proposition 7.4. *Let $p : (\bar{X}, \bar{x}_0) \to (X, x_0)$ be a covering map such that $x_0 = p(\tilde{x}_0)$.*

(1) *The induced homomorphism $p_* : \pi_1(\bar{X}, \bar{x}_0) \to \pi_1(X, x_0)$ is injective. Hence, $\pi_1(\bar{X}, \bar{x}_0)$ can be identified with a subgroup of $\pi_1(X, x_0)$.*

(2) *If $\bar{X}$ is path-connected, the subgroups $p_*(\pi_1(\tilde{X},\tilde{x}))$ for $\tilde{x} \in p^{-1}(x_0)$ are exactly the conjugacy class of subgroups of $p_*(\pi_1(\bar{X},\bar{x}_0))$.*

(3) *If $\bar{X}$ is path-connected, the the number of sheets of p equals the index of $p_* : \pi_1(\bar{X},\bar{x}_0)$ in $\pi_1(X,x_0)$.*

(4) *If $\bar{X}$ is path-connected and simply connected, then*

$$\pi_1(X,x_0) \cong p^{-1}(x_0)$$

as sets.

Proof. The proof is given below:

(1) Let $[\alpha]$ and $[\beta]$ be two homotopy classes of paths in $\bar{X}$ and suppose that $p_*[\alpha] = p_*[\beta]$. If $f_\alpha \in [\alpha]$ and $f_\beta \in [\beta]$, then $p \circ g_\alpha \sim p \circ g_\beta$. It follows from Proposition 7.2 that $g_\alpha \sim g_\beta$ in $\bar{X}$. So, $[\alpha] = [\beta]$. Thus the map is injective.

(2) First suppose that $\bar{x}_0, \bar{x}_1 \in p^{-1}(x_0)$. Let γ be a path from $\bar{x}_0$ to $\bar{x}_1$. This defines an isomorphism (Proposition 3.5):

$$\begin{aligned} \phi : \pi_1(\bar{X},\bar{x}_0) &\to \pi_1(\bar{X},\bar{x}_1) \\ [\alpha] &\mapsto [\gamma \circ \alpha \circ \gamma^{-1}] \end{aligned}$$

We thus have the following commutative diagram:

$$\begin{tikzcd} \pi_1(\tilde{X},\tilde{x}_0) \arrow[r, "p_*"] \arrow[d, "\phi"'] & \pi_1(X,x_0) \arrow[d, "\psi"] \\ \pi_1(\tilde{X},\tilde{x}_1) \arrow[r, "p_*"] & \pi_1(X,x_0) \end{tikzcd}$$

Here, ψ is defined such that

$$\psi([\beta]) = [(p_*(\gamma))^{-1} \circ \beta \circ (p_*(\gamma))]$$

We conclude that the images of $\pi_1(\tilde{X},\tilde{x}_0)$ and $\pi_1(\tilde{X},\tilde{x}_1)$ are conjugate via $[p_*(\gamma)]$. Conversely, any subgroup in the conjugacy class of $p_*(\pi_1(\tilde{X},\tilde{x}_0))$ is of the form

$$[\alpha^{-1}]\, p_*(\pi_1(\tilde{X},\tilde{x}_0))\, [\alpha]$$

for some $[\alpha] \in \pi_1(X,x_0)$. Let $f \in [\alpha]$. By Proposition 7.1 $g : I \to \tilde{X}$ is a unique lift of f initial point $\tilde{x}_0$. Let $\tilde{x}_1$ be the terminal point of the lifted path. Then

$$p_*(\pi_1(\tilde{X},\tilde{x}_1)) = [\alpha^{-1}]\, p_*(\pi_1(\tilde{X},\tilde{x}_0))\, [\alpha]$$

(3) Let $H = p_*(\pi_1(\bar{X},\bar{x}_0)$. Define a map

$$\begin{aligned} \phi : \frac{\pi_1(X,x_0)}{H} &\to p^{-1}(x_0) \\ [f] + H &\mapsto \bar{f}(1) \end{aligned}$$

Here $\bar{f}$ is a lift of the path f. We claim that ϕ is well-defined. Given $[f] \in \pi_1(X,x_0)$ and $[h] \in H$, let $\bar{h}$ be a loop in $\bar{X}$ based at $\bar{x}_0$. Thus, $(\bar{h} \cdot \bar{f})(1) = \bar{f}(1)$. This shows that ϕ is well-defined. We claim that ϕ is a bijection. Since $\bar{X}$ is path-connected, for any $\bar{x} \in p^{-1}(x_0)$, there exists a path $\bar{g}$ from $\bar{x}_0$ to $\bar{x}$, and it must project to a loop g in X based at x_0. Thus, ϕ is surjective. Now suppose

$$\phi([f] + H) = \phi([f'] + H)$$

Then $\bar{f}(1) = \bar{f}'(1)$, and so the path $f \cdot (f')^{-1}$ lifts to a loop in $\bar{X}$ based at $\bar{x}_0$, i.e., $[f][f']^{-1} \in H$. This shows that ϕ is connected.

(4) This follows from (3).

This completes the proof. □

Example 7.5. Consider the covering $p : \mathbb{S}^n \to \mathbb{RP}^n$. For $n \geq 2$, $\mathbb{S}^n$ is path-connected and simply-connected. Hence, Proposition 7.4 implies that

$$\pi_1(\mathbb{RP}^n, x_0) \to p^{-1}(x_0)$$

as sets. Since $|p^{-1}(x_0)| = 2$ and there is a unique group of order 2, it follows that

$$\pi_1(\mathbb{RP}^n, x_0) \cong \mathbb{Z}/2\mathbb{Z}$$

for $n \geq 2$. For $n = 1$, we have that $\mathbb{RP}^1 \cong \mathbb{S}^1$. Hence, $\pi_1(\mathbb{RP}^1, x_0) \cong \mathbb{Z}$.

Previously, we considered a path in the unit interval I within X and lifted it to a corresponding path in the covering space $\bar{X}$. We now extend this concept by studying the lifting of paths in X from an arbitrary connected space Y. Let $p : (\bar{X}, \bar{x}_0) \to (X, x_0)$ be a covering space. Let (Y, y_0) be a topological space and let $f : (Y, y_0) \to (X, x_0)$ be a pointed continuous map. We seek to determine conditions under which there exists a map $\phi : (Y, y_0) \to (X, \bar{x}_0)$ such that the following diagram commutes:

$$\begin{tikzcd}
 & (\bar{X}, \bar{x}_0) \arrow[d, "p"] \\
(Y, y_0) \arrow[ur, "\bar{\phi}"] \arrow[r, "\phi"] & (X, \bar{x}_0)
\end{tikzcd}$$

If ϕ exists, we say that ϕ can be lifted to $\bar{X}$. We refer to ϕ as a lifting of ϕ. Note that if ϕ exists, then the following commutative diagram of group homomorphisms holds:

$$\begin{tikzcd}
 & \pi_1(\bar{X}, \bar{x}_0) \arrow[dd, "p_*"] \\
\pi_1(Y, y_0) \arrow[ur, "\bar{\phi}_*"] \arrow[dr, "\phi_*"] & \\
 & \pi_1(X, x_0)
\end{tikzcd}$$

Since p_* is injective, for the diagram to commute it is necessary that

$$\phi_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(\bar{X}, \bar{x}_0))$$

This condition is also sufficient.

Proposition 7.6. *Let $p : (\bar{X}, \bar{x}_0) \to (X, x_0)$ be a covering map. Let (Y, y_0) be a connected and locally path-connected space. Given a pointed continuous map $\phi : (Y, y_0) \to (X, x_0)$, there exists a lifting $\phi : (Y, y_0) \to (\bar{X}, \bar{x}_0)$ if and only if*

$$\phi_*(\pi_1(Y, y_0)) \subseteq p_*(\pi_1(\bar{X}, \bar{x}_0))$$

Proof. Skipped. □

7.1. **Covering Space Automorphisms.** An automorphism of a covering map is an isomorphism from a covering space to itself. An automorphism of a covering map interchanges points in the fiber point in the base space. The set of all automorphisms of forms a group under composition and can be interpreted as the symmetries of the covering space.

Remark 7.7. *Automorphisms of a covering map are also called deck transformations.*

Using Proposition 7.1 and Proposition 7.6, we first establish various properties of the morphisms of a covering map.

Corollary 7.8. *Let $(\bar{X}_1, p_1)$ and $(\bar{X}_2, p_2)$ be covering spaces of a topological space (X, x_0) such that $p_1(\bar{x}_1) = p_2(\bar{x}_2) = x_0$*

1. *Let ϕ_0 and ϕ_1 be homomorphisms of $(\bar{X}_1, p_1)$ into $(\bar{X}_2, p_2)$. If there exists a point $x \in \bar{X}_1$ such that $\phi_0(x) = \phi_1(x)$, then $\phi_0 = \phi_1$.*
2. *There exists a morphism ϕ of $(\bar{X}_1, p_1)$ into $(\bar{X}_2, p_2)$ such that $\phi(\bar{x}_1) = \bar{x}_2$ if and only if*
$$p_{1*}(\pi_1(\bar{X}_1, \bar{x}_1)) \subseteq p_{2*}(\pi_1(\bar{X}_2, \bar{x}_2))$$
3. *The morphism in (2) is an isomorphism if and only if*
$$p_{1*}(\pi_1(\bar{X}_1, \bar{x}_1)) = p_{2*}(\pi_1(\bar{X}_2, \bar{x}_2))$$
4. *$(\bar{X}_1, p_1)$ and $(\bar{X}_2, p_2)$ are isomorphic if and only if the subgroups $p_{1*}(\pi_1(\widetilde{X}_1, \widetilde{x}_1))$ and $p_{2*}(\pi_1(\widetilde{X}_2, \widetilde{x}_2))$ of $\pi_1(X, x_0)$ belong to the same conjugacy class.*

Proof. The proof is given below:

1. This follows from Proposition 7.1.
2. This is a special case of Proposition 7.6.
3. This follows from (2).
4. This follows (3) and Proposition 7.4(3).

This completes the proof. □

We can now specialize to the case of an automorphism of a covering map.

Corollary 7.9. *Let $p : (\bar{X}, \bar{x}_0) \to (X, x_0)$ be a covering map.*

1. *If ϕ is an automorphism $(\bar{X}, \bar{x}_0)$ and ϕ is not the identity map then ϕ has no fixed points.*
2. *Let $\bar{x}_1, \bar{x}_2 \in p^{-1}(x_0)$. There exists an automorphism $\phi \in \mathrm{Aut}(\bar{X}, p)$ such that $\phi(\bar{x}_1) = \bar{x}_2$ if and only if*
$$p_*(\pi_1(\bar{X}, \bar{x}_1)) = p_*(\pi_1(\bar{X}, \bar{x}_2))$$

Proof. The proof is given below:

1. This follows from Corollary 7.8(1).
2. This follows from Corollary 7.8(3).

This completes the proof. □

8. Action of Fundamental Group on Fibers

We have seen in Proposition 7.4 if $p : (\bar{X}, \bar{x}_0) \to (X, x_0)$ is a covering map, then if $\bar{X}$ is path-connected and simply-connected, then
$$\pi_1(X, x_0) \cong p^{-1}(x_0)$$

as sets. We provide another perspective on this bijection of sets by observing that $\pi_1(X, x_0)$ acts naturally on $p^{-1}(x_0)$. For any point $\bar{x} \in p^{-1}(x_0)$ and any $[\alpha] \in \pi_1(X, x_0)$, define $\bar{x} \cdot [\alpha] \in p^{-1}(x)$ as follows: let $\bar{\alpha}$ be the lift of α to $\bar{X}$ starting at $\bar{x}$, so that $p_*(\bar{\alpha}) = \alpha$. Then define $\bar{x} \cdot [\alpha]$ to be the terminal point of the path class $\bar{\alpha}$.

Remark 8.1. *It can be verified that the action defined above is well-defined.*

It follows from the definition that:

(1) $\bar{x} \cdot [c_{x_0}] = \bar{x}$
(2) $(\bar{x} \cdot \alpha) \cdot \beta = \bar{x} \cdot (\alpha\beta)$

Therefore, this defines a right group action of $\pi_1(X, x_0)$ on the set $p^{-1}(x_0)$.

Proposition 8.2. *Let $p : (\bar{X}, \bar{x}_0) \to (X, x_0)$ be a covering map. If $\bar{X}$ is path-connected, the action of $\pi_1(X, x_0)$ on $p^{-1}(x_0)$ is transitive. As a right $\pi_1(X, x_0)$-space, we have*

$$p^{-1}(x_0) \cong \frac{\pi_1(X, x)}{p_*(\pi_1(\bar{X}, \bar{x}_0))}$$

Proof. Let $\bar{x}, \bar{y} \in p^{-1}(x_0)$. Since $\bar{X}$ is path-connected, there exists a path $\bar{\alpha}$ in $\bar{X}$ with the initial point $\bar{x}$ and terminal point $\bar{y}$. Let $[\alpha] = [p_*(\bar{\alpha})]$. We have $\bar{x} \cdot [\alpha] = \bar{y}$. This shows that the action is transitive. Note that the isotropy subgroup of any $\bar{x}_0$ is the set.

$$\{[\alpha] \in \pi_1(X, x) \mid \bar{x}_0 \cdot [\alpha] = \bar{x}_0\} \cong p_*(\pi_1(X, \bar{x}_0))$$

The desired isomorphism of $\pi_1(X, x_0)$-sets follows by the orbit-stabilizer theorem. □

In fact, the automorphism group of the covering space, denoted as $\mathrm{Aut}(\bar{X}, p)$, acts on the fiber $p^{-1}(x_0)$ as a right $\pi_1(X, x)$-space. This action is compatible with the group action of $\pi_1(X, x_0)$ on the fiber. Indeed, let $\phi \in \mathrm{Aut}(\bar{X}, p)$, any point $\bar{x} \in p^{-1}(x_0)$, and any $[\alpha] \in \pi_1(X, x_0)$. Lift α to a path $\bar{\alpha}$ in $\bar{X}$ with initial point $\bar{x}$, such that $p_*(\bar{\alpha}) = \alpha$. Note that $\bar{x} \cdot [\alpha]$ is the terminal point of $\bar{\alpha}$. Now consider the path $\phi \circ \bar{\alpha}$ in $\bar{X}$. Its initial point is $\phi(\bar{x})$ and the terminal point is $\phi(\bar{x} \cdot [\alpha])$. Observe that:

$$p(\phi \circ \bar{\alpha}) = (p \circ \phi)(\bar{\alpha}) = p(\bar{\alpha}) = \alpha.$$

This implies that $\phi \circ \bar{\alpha}$ is also a lifting of α. By definition, $(\phi(\bar{x})) \cdot [\alpha]$ is the terminal point of $\phi \circ \bar{\alpha}$. Therefore, we have

$$\phi(\bar{x} \cdot [\alpha]) = \phi(\bar{x}) \cdot [\alpha]$$

We now state an important result relating automorphisms of covering spaces to automorphisms of the fiber.

Proposition 8.3. *Let $p : (\bar{X}, \bar{x}_0) \to (X, x_0)$ be a covering map.*

(1) $\mathrm{Aut}(\bar{X}, p)$ *is naturally isomorphic to the group of automorphisms of the set $p^{-1}(x_0)$ considered as a right $\pi_1(X, x)$-set.*
(2) *The automorphism group* $\mathrm{Aut}(\bar{X}, p)$ *is isomorphic to the quotient group*

$$\frac{N(p_*(\pi_1(\bar{X}, \bar{x}_0))}{p_*(\pi_1(\bar{X}, \bar{x}_0))},$$

where $N(p_*(\pi_1(\bar{X}, \bar{x}_0))$ denotes the normalizer of $\pi_1(\bar{X}, \bar{x}_0))$ in $\pi_1(X, x_0)$.

Proof. The proof is given below:

(1) Note that if ϕ is an automorphism of $\bar{X}$, then $\phi|_{p^{-1}(x)}$ is an automorphism of the fiber $p^{-1}(x_0)$. We will prove that the map

$$\phi \mapsto \phi|_{p^{-1}(x_0)}$$

is bijective. Suppose $\phi|_{p^{-1}(x_0)} = \psi|_{p^{-1}(x_0)}$. This implies that $(\phi \circ \psi^{-1})|_{p^{-1}(x)} = \mathrm{Id}_{p^{-1}(x_0)}$. Since automorphisms of covering spaces have no fixed points unless they are the identity (Corollary 7.8)(1)), it follows that $\phi \circ \psi^{-1} = \mathrm{Id}_{(\bar{X},p)}$, and thus $\phi = \psi$. This shows the map is injective. If ϕ is an automorphism of the fiber $p^{-1}(x_0)$ such that $\phi(\bar{x}_0) = \bar{x}_1$, where $\bar{x}_1 \in p^{-1}(x_0)$. Then $p_*(\pi_1(\bar{X}, \bar{x}_0)) = p_*(\pi_1(\bar{X}, \bar{x}_1))$. By Corollary 7.9(2), there exists an automorphism $\psi \in \mathrm{Aut}(X, \bar{p})$ such that $\psi(\bar{x}_0) = \bar{x}_1$. This shows the map is surjective.

(2) This follows from (1) and the group-theoretic fact that if Z is a transtive G-set and H is the isotropy subgroup of some $z \in Z$. Then the automorphism group $\mathrm{Aut}(Z)$ is isomorphic to the quotient group

$$\mathrm{Aut}(Z) \cong \frac{N(H)}{H}$$

This completes the proof. □

We now state two important corollaries of the previous result:

Corollary 8.4. *Let $p : (\bar{X}, \bar{x}_0) \to (X, x_0)$ be a covering map.*

(1) *If $p_*(\pi_1(\bar{X}, \bar{x}_0))$ is a normal subgroup of $\pi_1(X, x_0)$, then*

$$\mathrm{Aut}(\bar{X}, p) \cong \frac{\pi_1(X, x_0)}{p_*(\pi_1(\bar{X}, \bar{x}_0))}$$

(2) *If $\bar{X}$ is simply-connected then*

$$\mathrm{Aut}(\bar{X}, p) \cong \pi_1(X, x_0)$$

Proof. (1) follows at once from Proposition 8.3. (2) also follows from (1) since $N(\{c_{x_0}\} = \pi_1(X, x_0)$. □

Corollary 8.4 provides key insights into the structure of the automorphism group of a covering space.

(1) The first part shows that when the image of the induced map on the fundamental group $p_*(\pi_1(\bar{X}, \bar{x}_0))$ is a normal subgroup of $\pi_1(X, x_0)$, the automorphism group of the covering space is isomorphic to the quotient of the fundamental group of the base space by this normal subgroup. Moreover, for any $\bar{x} \in p^{-1}(x_0)$, we have

$$p_*(\pi_1(X, \bar{x}_0)) \cong p_*(\pi_1(X, \bar{x}))$$

since there is only one conjugacy class of $p_*(\pi_1(X, \bar{x}))$. Covering spaces that satisfy this property are called *regular/normal* covering spaces.

(2) The second part of the corollary, which applies when $\bar{X}$ is simply-connected, shows that the automorphism group of such a covering space is isomorphic to the fundamental group of the base space. Covering spaces that are simply connected are called *universal* covering spaces

9. Classification of Covering Spaces

If $p : (\bar{X}, \bar{x}_0) \to (X, x_0)$ is a covering map, we have proven that a covering space $(\bar{X}, p)$ is determined up to isomorphism by the conjugacy class of the subgroup $p_*(\pi_1(X, \bar{x}_0))$ of $\pi_1(X, x_0)$ (Corollary 7.9). Now, we address the inverse question:

Suppose (X, x_0) is a topological space and we are given a conjugacy class of subgroups of $\pi_1(X, x)$. Does there exist a topological space $(\bar{X}, \bar{x}_0)$ and a covering map $p : (\bar{X}, \bar{x}_0) \to (X, x_0)$ such that $p_*(\pi_1(\bar{X}, \bar{x}))$ belongs to the given conjugacy class?

We will see that the properties of regular and universal covering spaces are closely related to this question.

Proposition 9.1. *Let (X, x_0) be a topological space that is connected, locally path-connected, and semi-locally simply connected. Then, given any conjugacy class of subgroups of $\pi_1(X, x_0)$, there exists a topological space $(\bar{X}, \bar{x}_0)$ and a covering map $p : (\bar{X}, \bar{x}_0) \to (X, x_0)$ such that $p_*(\pi_1(\bar{X}, \bar{x}))$ belongs to the given conjugacy class.*

Remark 9.2. *A topological space (X, x_0) is semi-locally simply connected if every $x \in X$ has a neighborhood U_x such that the homomorphism*

$$\pi_1(U_x, x) \to \pi_1(X, x)$$

is trivial. That is, every loop in U_x can be contracted to x within X. Note that U need not be simply connected since every loop in U may not be contractible within U. For this reason, a space can be semi-locally simply connected without being locally simply connected. The definition of the latter is obvious. It turns out that (X, x_0) has a universal cover if and only if (X, x_0) is connected, locally path-connected, and semi-locally simply connected. See [Lee10; Hat02] for details. Universal covering spaces are called universal because they satisfy the following property: let $q : (\bar{Y}, \bar{y}_0) \to (X, x_0)$ be a covering map such that $(\bar{Y}, \bar{y}_0)$ is a universal covering space. Then for any other covering space $p : (\bar{X}, \bar{x}_0) \to (X, x_0)$, there exists a unique covering map

$$\phi : (Y, y_0) \to (\bar{X}, \bar{x}_0)$$

such that the following diagram commutes:

$$\begin{tikzcd}
(Y, y_0) \arrow[rr, "\phi"] \arrow[dr, "q"'] & & (\bar{X}, \bar{x}_0) \arrow[dl, "p"] \\
& (X, x_0) &
\end{tikzcd}$$

This follow at once from Corollary 7.8(2).

Proof. (Proposition 9.1) The assumptions on (X, x_0) imply that there exists a universal covering space, (Y, y_0), for (X, x_0). Let $q : (Y, y_0) \to (X, x_0)$ denote the corresponding (universal) covering map. We know the following facts:

(1) $\pi_1(X, x_0)$ acts freely and transitively on the set $q^{-1}(x_0)$.
(2) $\mathrm{Aut}(Y, q) \cong \pi_1(X, x_0)$.

Choose a subgroup $G \subseteq \pi_1(X, x_0)$ that lies in the given conjugacy class. Consider the following subgroup:

$$H := \{\phi : \mathrm{Aut}(Y, q) \mid \text{there exists } \alpha_\phi \in G \text{ such that } \phi(y) = y \cdot [\alpha] \in p^{-1}(x_0)\}$$

Note that $G \cong H$ under the correspondence $\phi \mapsto \alpha_\phi$. Since H is a subgroup of $\mathrm{Aut}(Y, q)$, it is a satisfies the hypothesis of Proposition 6.7. Hence, the quotient map $r : Y \to Y/G := \bar{X}$ is a covering map. The universal property of universal covering spaces (Remark 9.2) implies that we have a commutative diagram:

$$\begin{tikzcd}
(Y, y_0) \arrow[dr, "r"] \arrow[dd, "q"'] & \\
& (\bar{X}, \bar{x}_0) \arrow[dl, dashed, "p"'] \\
(Y, x_0) &
\end{tikzcd}$$

Here $p : \bar{X} \to X$ is a map induced by q and $\bar{x}_0 = r(y_0) \in p^{-1}(x_0)$. It is not hard to verify that $p : \bar{X} \to X$ is a covering map. Thus, the group $\pi_1(X, x_0)$ acts transitively on the right of the set $p^{-1}(x_0)$. Since Y is simply-connected, we have $\mathrm{Aut}(Y, r) \cong \pi_1(\bar{X}, \bar{x}_0)$. We claim that $\mathrm{Aut}(Y, r) = H$. Clearly, $H \subseteq \mathrm{Aut}(Y, r)$. Suppose $y_1, y_2 \in Y$, and let $\phi \in G$ be such that $\phi(y_1) = y_2$. Since ϕ is a covering transformation, we can choose an automorphism $\psi \in \mathrm{Aut}(Y, p)$ such that $\psi(y_1) = y_2$. It follows that $\phi = \psi$. Hence, $\mathrm{Aut}(Y, p) \subseteq G$, and therefore $G = \mathrm{Aut}(Y, p)$[13]. Hence, we have

$$\mathrm{Aut}(Y, r) \cong H \cong G.$$

So p_* maps $\pi_1(\bar{X}, \bar{x}_0)$ onto G. This completes the proof. □

Remark 9.3. *Proposition 9.1 proves the additional fact that if a group acts on a simply-connected space X such that the group action satisfies the hypotheses in Proposition 6.7, then the quotient map $p : X \to X/G$ is a regular covering map, and*

$$\mathrm{Aut}(X/G, p) \cong \pi_1(X, x_0) \cong G$$

We have shown that there is a one-to-one correspondence:

$$\{\text{Conjugacy classes of } \pi_1(X, x_0)\} \longleftrightarrow \{\text{Covering maps } p : (\bar{X}, \bar{x}_0) \to (X, x_0)\}$$

provided that (X, x_0) is connected, locally path-connected, and semi-locally simply connected. Since the universal covering space is connected, we in fact have a one-to-one correspondence for connected covering maps. Let us now consider some examples to illustrate this correspondence.

Example 9.4. (**Coverings of $\mathbb{S}^1$**) Sice $\mathbb{R}$ is simply connected, the covering map $p : \mathbb{R} \to \mathbb{S}^1$ is a universal covering map. We know that $\pi_1(\mathbb{S}^1, x_0) \cong \mathbb{Z}$. Every connected covering space of $\mathbb{S}^1$ corresponds to a subgroup of $\mathbb{Z}$. Every non-trivial subgroup of $\mathbb{Z}$ is of the form $n\mathbb{Z}$ for some integer $n \geq 1$. Note that the index of $n\mathbb{Z}$ is n. For each $n \geq 1$, Proposition 9.1 implies that there exists a unique (up to isomorphism) connected n-fold covering space of $\mathbb{S}^1$, which can be described as the quotient $\mathbb{R}/n\mathbb{Z} \cong \mathbb{S}^1$. The associated covering map is the n-th power map on $\mathbb{S}^1$.

[13]This shows that $r : Y \to Y/G$ is a regular covering map.

Part 3. Categorical Nuances

The category **hTop** is the appropriate framework for studying homotopy theory. However, not all concepts from the category **Top** carry over directly to **hTop**. For instance, we have the following pushout diagram in **Top**:

$$\begin{tikzcd} \mathbb{S}^{n-1} \arrow[r] \arrow[d, hook] & \{*\} \arrow[d] \\ \mathbb{D}^n \arrow[r] & \mathbb{S}^n \end{tikzcd}$$

On the other hand, we also have the pushout diagram in **Top**:

$$\begin{tikzcd} \mathbb{S}^{n-1} \arrow[r] \arrow[d] & \{*\} \arrow[d] \\ \{*\} \arrow[r] & \{*\} \end{tikzcd}$$

Therefore, even though $\mathbb{D}^n$ is homotopy equivalent to $\{*\}$, the two pushouts are not homotopy equivalent. Therefore, contrary to expectation, the pushout diagrams in **hTop** are not the same. This example suggests that further analysis and applications of the homotopy notion require a certain amount of formal (categorical) considerations. In this section, we discuss some basic constructions of a categorical nature. More advanced constructions such as homotopy pullbacks and homotopy pushouts will be discussed later on if needed.

10. Cones & Suspensions

In this section, we discuss the the categorical constructions of cones and suspensions.

10.1. **Cone & Suspension.** Let $I = [0,1] \subseteq \mathbb{R}$. The space $X \times I$ is called a cylinder over X, and the subspaces $X \times \{0\}$, $X \times \{1\}$ are the bottom and top "bases". Now we will construct new spaces out of the cylinder $X \times I$.

Definition 10.1. Let X be a topological space. The cone of X is the quotient space:

$$CX = X \times I/(X \times \{0\})$$

Remark 10.2. *CX has a natural basepoint given by the collaposed space $X \times \{0\}$. Hence, we have a functor*

$$C : \mathbf{Top} \to \mathbf{Top}_*$$

Indeed, if $f : X \to Y$ is a continuous map, we have a continuous map $f \times id_I : X \times I \to Y \times Y$ and if we define $C(f)$ to be the map

$$C(f) : CX \to CY, \qquad [x,t] \mapsto [f(x), t],$$

We have $Cf \circ q_X = q_Y \circ (f \times \mathrm{id}_I)$ where q_X, q_Y are quotient maps defining CX and CY.

$$\begin{tikzcd} X \times I \arrow[r, "f \times id_I"] \arrow[d, "q_X"] & Y \times I \arrow[d, "q_Y"'] \\ CX \arrow[r, "C(f)"] & CY \end{tikzcd}$$

The universal property of quotient topology implies that Cf is continuous.

The cone of a topological space is always a contractible space.

Proposition 10.3. *Let $X \in \textbf{Top}$. Then CX is contractible.*

Proof. A homotopy between the identity on CX and the map to the basepoint is given by:

$$\begin{aligned} F : CX \times I &\to CX, \\ ([x,t],s) &\mapsto [x,(1-s)t] \end{aligned}$$

This completes the proof. ☐

The motivation for introducing the cone of a topological space is given by the following proposition:

Proposition 10.4. *Let $X, Y \in \textbf{Top}$. A map $f : X \to Y$ is nullhomotopic if and only if it extends to a map $\overline{f} : CX \to Y$.*

Proof. Consider a continuous map $H : X \times [0,1] \to Y$ with $H(\cdot, 0) = f(\cdot)$. Note that $H(x,1)$ is constant for all $x \in X$ if and only if $X \times \{1\}$ is contained in a fiber of H, which in turn, by the universal property of quotient spaces, says that H factors uniquely through the canonical quotient map $X \times [0,1] \to CX$. This proves the claim. ☐

Remark 10.5. *Proposition 10.4 implies that a continuous map $f : \mathbb{S}^n \to X$ is nullhomotopic if and only if f extends to a continuous map $\overline{f} : \mathbb{D}^{n+1} \to X$. This is because $C\mathbb{S}^n \cong \mathbb{D}^{n+1}$*

We now define the suspension of a topological space.

Definition 10.6. Let $X \in \textbf{Top}$. The suspension of X is the quotient space:

$$SX = X \times I/(X \times \{0\}, X \times \{1\})$$

Remark 10.7. *S defines a a functor $S : \textbf{Top} \to \textbf{Top}$ This follows by a similar reasoning that cone is a functor.*

Example 10.8. The suspension of $\mathbb{S}^0 = \{x, x_1\}$ consists of two lines (one over each point in $\mathbb{S}^0$) joined at 0 and 1, giving $\mathbb{S}^1$. In fact,

$$\mathbb{S}^{n+1} \cong S\mathbb{S}^n$$

in general. To see this, WLOG, replace $I = [0,1]$ by $I = [-1,1]$. Define

$$f : \mathbb{S}^n \times [-1,1] \to \mathbb{S}^{n+1}$$

by

$$f((x, \ldots, x_n), t) = (x \cdot \sqrt{1-t^2}, \ldots, x_n \cdot \sqrt{1-t^2}, t)$$

It is clear that f is continuous and surjective. Moreover, f agrees on the fibers of $S\mathbb{S}^n$. Hence, f descends to a continuous bijection $\widetilde{f}$ from $S\mathbb{S}^n$ to $\mathbb{S}^{n+1}$. Since $S\mathbb{S}^n$ is compact and $\mathbb{S}^{n+1}$ is Hausdorff, $\widetilde{f}$ is a homemorphism.

$$\begin{tikzcd} \mathbb{S}^n \times [-1,1] \arrow[d, "q"] \arrow[dr, "f"] & \\ S\mathbb{S}^n \arrow[r, dotted, "\widetilde{f}"'] & \mathbb{S}^{n+1} \end{tikzcd}$$

11. Compact Open Topology, Path & Loop Spaces

11.1. **Compact Open Topology.** If $(X, x_0) \in \mathbf{Top}_*$, note that $\pi_1(X, x_0)$ is, in particular, a space of continuous functions. Hence, we would like to discuss what appropriate topology to put on the function space of continuous maps between topological spaces.

Definition 11.1. Let $X, Y \in \mathbf{Top}$ and let $\mathrm{C}(X, Y)$ denote the set of of continuous maps $X \to Y$. $\mathrm{C}(X, Y)$ carries a natural topology, called the compact-open topology, generated by a subbasis formed by the sets of the form

$$B(K, U) = \{f : X \to Y \mid f(K) \subseteq U\}$$

where $K \subseteq X$ is a compact set and $U \subseteq Y$ is an open set.

Remark 11.2. *The topological space given by this compact-open topology will be denoted by* $\mathrm{Maps}(X, Y)$*.*

Remark 11.3. *For a map $f : X \to Y$, one can form a typical basis open neighborhood by choosing compact subsets $K_1, \ldots, K_n \subseteq X$ and small open sets $U_i \subseteq X$ with $f(K_i) \subseteq U_i$ to get a neighborhood O_f of f,*

$$O_f = B(K_1, U_1) \cap \cdots \cap B(K_n, U_n).$$

The collection of all such sets forms a basis for the compact-open topology.

What is the motivation behind the definition of the compact-open topology? If X is compact Hausdorff and Y a metric space, then one can consider the supremum norm on $(C(X, Y), \|\cdot\|_\infty)$. It can be checked that in this case $\mathrm{Maps}(X, Y) = (C(X, Y), \|\cdot\|_\infty)$. We prove a slightly more genral claim:

Proposition 11.4. *Let $X, Y \in$ **Top** such that (Y, d) is a metric space. The compact-open topology and the topology of uniform convergence on compact sets coincide on $C(X, Y)$.*

Proof. We first prove that the topology of compact convergence is finer than the compact-open topology. Let $B(K, U)$ be a subbasis element for the compact-open topology, and let $f \in B(K, U)$. Because f is continuous, $f(K)$ is a compact subset of U. Therefore, we can choose $\varepsilon > 0$ so that the ε-neighborhood of $f(K)$ is contained in U. Then, as desired,

$$E_K(f, \varepsilon) \subseteq B(C, U).$$

Here

$$E_K(f, \varepsilon) = \{g \in C(X, Y) \mid \|f - g\|_{\infty, K} < \varepsilon\}$$

is a basis element of the topology of compact convergence. We now prove that the compact-open topology is finer than the topology of compact convergence. Let $f \in C(X, Y)$ and consider $E_K(f, \varepsilon)$ for some $\varepsilon > 0$. Every $x \in X$ has a neighborhood V_x such that $f(\overline{V_x})$ lies in an open set $U_{f(x)}$ of Y having diameter less than ε. For example, choose V_x so that $f(V_x)$ lies in the $\varepsilon/4$-neighborhood of $f(x)$. Then $f(\overline{V_x})$ lies in the $\varepsilon/3$-neighborhood of $f(x)$, which has diameter at most $2\varepsilon/3$. Cover K by finitely many such sets V_x, say for $x = x_1, \ldots, x_n$. Let $K_x = \overline{V_x} \cap K$. Then K_x is compact, and the basis element

$$B(K_{x_1}, U_{x_1}) \cap \cdots \cap B(K_{x_n}, U_{x_n})$$

contains f and lies in $E_K(f, \varepsilon)$, as desired. $\square$

Remark 11.5. *If Y is not a metric space, we need to redefine the notion of proximity between maps. Suppose $f, g \in \mathrm{Maps}(X, Y)$ are two continuous maps. Let $K \subseteq X$ be a compact subset and $U \subseteq Y$ be an open subset such that $f(K) \subseteq U$. Assume that Y is Hausdorff, which ensures that closed sets in Y behave well under continuous maps. Since $f(K)$ is compact and Y is Hausdorff, $f(K)$ is closed in Y, and intuitively, small perturbations of $f(K)$ should remain within U. Thus, to define the topology on $\mathrm{Maps}(X, Y)$, we say that a neighborhood of f is the set of maps $g \in \mathrm{Maps}(X, Y)$ such that $g(K) \subseteq U$. This formalizes the notion that g is 'close' to f if it maps the compact set K into the same open set U that contains $f(K)$.*

Proposition 11.6. *(**Exponential Law**) Let $X, Y, Z \in \mathbf{Top}$. If X is Hausdorff and Y is locally compact, then*

$$\varphi : \mathrm{Maps}(X \times Y, Z) \to \mathrm{Maps}(X, \mathrm{Maps}(Y, Z)), \quad \varphi(g)(x)(y) = g(x, y)$$

is a continuous bijection.

Proof. We first show that φ is well-defined. Suppose g is continuous, and choose an arbitrary sub-basis open set $B(K, U)$ in $\mathrm{Maps}(Y, Z)$. Choose $x \in \varphi(g)^{-1}(B(K, U))$, so $g(\{x\} \times K) \subseteq U$. Since K is compact and g is continuous, there are open sets $V \ni x$ and $W \supseteq K$ such that $g(V \times W) \subseteq U$. Then V is a neighborhood of x with $\varphi(g)(V) \subseteq B(K, U)$, showing that $\varphi(g)^{-1}(B(K, U))$ is open. Hence, φ is well-defined. φ is obviously an injection. We now show that φ is continuous and surjective.

We first show that φ is continuous. Let $g : X \times Y \to Z$ be a continuous map. Let $K_1 \subseteq X$ and $K_2 \subseteq Y$ be compact subsets, $U \subseteq Z$ be an open set. Let

$$B(K_1, B(K_2, U)) = \{g : X \to \mathrm{Maps}(Y, Z) \mid g(K_1)(K_2) \subseteq U\}$$

be an open neighborhood of $\varphi(g)$. Then $[K_1 \times K_2, U]$ is an open neighborhood of g in $\mathrm{Maps}(X \times Y, Z)$[14] such that

$$\varphi(B(K_1 \times K_2, U)) \subseteq B(K_1, B(K_2, U))$$

This shows that φ is continuous. We now show that φ is surjective. Let $f : X \to \mathrm{Maps}(Y, Z)$ be a continuous map. Let $(x, y) \in X \times Y$ and W be an open neighborhood of $\varphi^{-1}(f)(x, y)$ in Z. We find neighborhoods $x \in U \subseteq X$ and $y \in V \subseteq Y$ such that $\varphi^{-1}(f)(U \times V) \subseteq W$. Since $f(x) : Y \to Z$ is continuous, $f(x)^{-1}(W)$ is an open neighborhood of y in Y. Since Y is locally compact, there exists a compact set $K \subseteq f(x)^{-1}(W)$ such that $y \in \mathrm{Int}(K) \subseteq K$. Then $[K, W]$ is an open neighborhood of $f(y)$, and since $f : X \to \mathrm{Maps}(Y, Z)$ is continuous at x, there exists an open neighborhood $x \in U \subseteq X$ such that $f(U) \subseteq [K, W]$. This implies that $F(U \times K) \subseteq W$. We can take $V = \mathrm{Int}(K)$, and thus we have $F(U \times V) \subseteq W$. $\square$

Remark 11.7. *If X is locally compact Hausdorff, the continuous bijection in Proposition 11.6 is in fact a homeomorphism.*

Corollary 11.8. *Let $X, Y \in \mathbf{Top}$ such that X is locally compact and Y is Hausdorff. The evaluation map*

$$\begin{aligned} \mathrm{Ev}_{X,Y} : X \times \mathrm{Maps}(X, Y) &\to Y, \\ (x, f) &\mapsto f(x). \end{aligned}$$

is continuous.

[14] We need that X is Hausdorff here. See Lemma XII.5.1 (a) of Dugundji's *Topology*.

Proof. We take for granted the statement that Y is Hausdorff implies that $\mathrm{Maps}(X,Y)$ is Hausdorff. Proposition 11.6 implies there is a continuous bijection:

$$\mathrm{Maps}(\mathrm{Maps}(X,Y)\times X, Y) \cong \mathrm{Maps}(\mathrm{Maps}(X,Y), \mathrm{Maps}(X,Y))$$

The inverse image of $\mathrm{Id}_{\mathrm{Maps}(X,Y)}$ is $\mathrm{Ev}_{X,Y}$. □

Remark 11.9. *Here is an important observation. If $X, Y \in \mathbf{Top}$, then a homotopy between two maps $f, g : X \to Y$ as an element of $\mathrm{Maps}(X \times I, Y)$. Based on Proposition 11.6, it is possible to reinterpret a homotopy between two maps $f, g : X \to Y$ as an element of $\mathrm{Maps}(X, \mathrm{Maps}(I, Y))$ or $\mathrm{Maps}(I, \mathrm{Maps}(X, Y))$. The latter says that a homotopy is a path in $\mathrm{Maps}(X, Y)$.*

Proposition 11.10. *If $X, Y \in \mathbf{Top}$ are locally compact Hausdorff spaces, then the function*

$$\Phi_{X,Y,Z} : \mathrm{Maps}(X,Y) \times \mathrm{Maps}(Y,Z) \to \mathrm{Maps}(X,Z)$$

given by composition is continuous.

Proof. By Proposition 11.6 we have the bijection

$$\mathrm{Maps}(\mathrm{Maps}(X,Y)\times\mathrm{Maps}(Y,Z), \mathrm{Maps}(X,Z)) \cong \mathrm{Maps}(\mathrm{Maps}(X,Y)\times\mathrm{Maps}(Y,Z), \times X, Z)$$

Hence, $\Phi_{X,Y,Z}$ is continuous if and only if the image of $\Phi_{X,Y,Z}$, denoted $\Phi'_{X,Y,Z}$, under the exponential law is continuous. Let $(f,g) \in \mathrm{Maps}(X,Y)\times\mathrm{Maps}(Y,Z)$ and $x \in X$. We have

$$\Phi'_{X,Y,Z}((f,g),x) = (T(f,g))(x) = f(g(x)).$$

We can decompose $\Phi'_{X,Y,Z}$ as the following composition:

$$\mathrm{Maps}(Y,Z)\times\mathrm{Maps}(X,Y)\times X \xrightarrow{(f,g,x)\mapsto(f,g(x))} \mathrm{Maps}(Y,Z)\times Y \xrightarrow{(g,y)\mapsto g(y)} Z.$$

The first map is just $\mathrm{Id}_{\mathrm{Maps}(Y,Z)} \times \mathrm{Ev}_{X,Y}$ and the second map is $\mathrm{Ev}_{Y,Z}$. Both these maps are continuous by Corollary 11.8. The claim follows. □

11.2. **Path & Loop Spaces.** We can consider special instances of the function space discussed above to define loop spaces. For instance, if $X \in \mathbf{Top}$, the space $\Lambda(X) = \mathrm{Maps}(\mathbb{S}^1, X) \in \mathbf{Top}$ is the free loop space of X. Similarly, the space $P(X) = \mathrm{Maps}(I, X) \in \mathbf{Top}$ is the free path space of X.

Remark 11.11. *If $X = I = [0,1]$, Y is locally compact and Z is a topological space, then Proposition 11.6 reads*

$$\begin{aligned}\mathrm{Maps}(Y, \mathrm{Maps}(I, Z)) &\cong \mathrm{Maps}(I \times Y, Z)\\ &\cong \mathrm{Maps}(I, \mathrm{Maps}(Y, Z))\end{aligned}$$

This is called the cylinder-free path adjunction. This is because $I \times Y$ is a cylinder on Y and $(\mathrm{Maps}(I, Z)))$ is the path space on Z. Note that $\mathrm{Maps}(I \times Y, Z) = [Y, Z]$.

We now make the following definition:

Definition 11.12. Let $(X, x_0) \in \mathbf{Top}_*$.

(1) The path space $P(X, x_0) \in \mathbf{Top}_*$ of (X, x_0) is the pointed space given by

$$P(X, x_0) = \{\gamma \in P(X) \mid \gamma(0) = x_0\}.$$

with the constant path c_x at x as the base point.

(2) The loop space $\Omega(X, x_0) \in \mathbf{Top}_*$ of (X, x_0) is the pointed space
$$\Omega(X, x_0) = \{\gamma \in P(X) \mid \gamma(0) = x_0 = \gamma(1)\}$$
with the constant loop c_x at x as the base point.

Remark 11.13. *Note that $\Omega(X, x_0)$ consists of pointed loops $(\mathbb{S}^1, *) \to (X, x_0)$. Moreover, note that $P(X, x_0)$ can be thought of as a pullback:*
$$\begin{tikzcd} P(X,x_0) \arrow[r] \arrow[d] & X^I \arrow[d, "\mathrm{Ev}_0"] \\ \{x_0\} \arrow[r, hook] & X \end{tikzcd}$$

Proposition 11.14. *Let $X \in \mathbf{Top}$. The path space, $P(X, x_0)$, is contractible.*

Proof. A homotopy between the identity on $P(X, x_0)$ and the map to the basepoint (the constant path) is given by:
$$\begin{aligned} F : P(X, x_0) \times I &\to P(X, x_0), \\ (\gamma, s) &\mapsto (t \mapsto \gamma((1-s)t)) . \end{aligned}$$
This completes the proof. □

Remark 11.15. *We discuss some applications of function space* $\mathrm{Maps}(X, Y)$ *to establish some basic facts:*

(1) *A point in X can be identified with a map $x : * \to X$ sending the unique point $*$ to x. Hence, we have a bijective correspondence*
$$X \cong \mathrm{Maps}(*, X).$$
(2) *In the case of a pointed space, we have a bijective correspondence*
$$\begin{aligned} (X, x_0) &\cong \mathrm{Maps}\left((\{*, *'\}, *'), (X, x_0)\right) \\ &\cong \mathrm{Maps}\left((\mathbb{S}^0, 1), (X, x_0)\right). \end{aligned}$$
(3) *Note that we have*
$$[X, Y] \cong \pi_0(\mathrm{Maps}(X, Y))$$
(4) *Let X, Y be locally compact Hausdorff spaces. Consider the continuous function*
$$T : \mathrm{Maps}(X, Y) \times \mathrm{Maps}(Y, Z) \to \mathrm{Maps}(X, Z)$$
Hence, T induces a map
$$\begin{aligned} [X, Y] \times [Y, Z] &= \pi_0(\mathrm{Maps}(X, Y)) \times \pi_0(\mathrm{Maps}(Y, Z)) \\ &= \pi_0(\mathrm{Maps}(X, Y) \times \mathrm{Maps}(Y, Z)) \\ &\to \pi_0(\mathrm{Maps}(X, Z)) \\ &= [X, Z] \end{aligned}$$
In particular, a continuous function $f : X \to Y$ induces a map
$$f^\# : [Y, Z] \to [X, Z]$$
and a continuous function $g : Y \to Z$ induces a map
$$g_\# : [X, Y] \to [X, Z].$$

(5) *Let $f : X \to Y$ be a homotopy equivalence with homotopy inverse and $g : Y \to X$. Using (3), we have two induced maps*

$$f^{\#} : [Y, Z] \to [X, Z] \quad f_{\#} : [Z, X] \to [Z, Y]$$

The maps $g^{\#}$ and $g_{\#}$ are inverses of f^* and $f_{\#}$ respectively. Hence, we have a bijection of sets

$$[Y, Z] \cong [X, Z], \qquad [Z, X] \cong [Z, Y]$$

12. Smash Products

We introduce the notion of a smash product that *forces us to take basepoints seriously.* The need for the smash product arises based on the need to consider the pointed analog of $\mathrm{Maps}(\cdot, \cdot)$.

Definition 12.1. Let $(X, x_0), (Y, y_0) \in \mathbf{Top}_*$. The pointed space $\mathrm{Maps}((X, x_0), (Y, y_0))$ is defined to be subspace of $\mathrm{Maps}(x_0, y_0)$ consisting of pointed maps, along with the natural basepoint given by constant map $X \to y_0$.

Remark 12.2. *We have $[(X, x_0), (Y, y_0)] = \pi_0(\mathrm{Maps}((X, x_0), (Y, y_0)))$.*

We now define the smash product:

Definition 12.3. Let $(X, x_0), (Y, y_0) \in \mathbf{Top}_*$. The smash product is defined as the quotient space

$$(X, x_0) \wedge (Y, y_0) = (X \times Y, (x_0, y_0))/(X \vee Y)$$

Remark 12.4. *The wedge sum $X \vee Y$ of two pointed spaces is naturally a pointed subspace of $(X, x_0) \times (Y, y_0)$. For pointed spaces (X, x_0) and (Y, y_0), the pointed product $(X \times Y, (x_0, y_0))$ comes naturally with an inclusion map of (X, x_0) given by*

$$\begin{aligned} (X, x_0) &\to (X \times Y, (x_0, y_0)), \\ x &\mapsto (x, y_0). \end{aligned}$$

There is a similar map $(Y, y_0) \to (X \times Y, (x_0, y_0))$. Since $X \vee Y$ is a pushout in $\mathbf{Top}_*$, we obtain a pointed map $X \vee Y \to (X \times Y, (x_0, y_0))$ which yields the desired inclusion.

Remark 12.5. *It can be checked that the smash product defines a functor*

$$\wedge : \mathbf{Top}_* \to \mathbf{Top}_*$$

The motivation behind the definition of a smash product is to extend Proposition 11.6 to $\mathbf{Top}_*$. Let $(X, x_0), (Y, y_0), (Z, z_0) \in \mathbf{Top}_*$. Since y_0 and z_0 are basepoints in Y and Z, respectively, then $\mathrm{Maps}((Y, y_0), (X, x_0))$ has a basepoint given by the constant function $X \to z_0$. We want a map $f : (X, x_0) \to \mathrm{Maps}((Y, y_0), (Z, z_0))$ to preserve basepoints, meaning that it must satisfy

$$f(x_0)(y) = z_0 \quad \text{for all } y \in Y.$$

Additionally, for any $x \in X$, the map $f(x) : Y \to Z$ must also preserve basepoints, i.e.,

$$f(x)(y_0) = z_0 \quad \text{for all } x \in X.$$

Therefore, if Proposition 11.6 is to be extend to $\mathbf{Top}_*$, then a a map $f : X \times Y \to Z$ f must be constant on

$$(\{x_0\} \times Y) \cup (X \times \{y_0\}),$$

sending it to z_0. This is exactly how we have defined the smash product, which yields the following result:

Proposition 12.6. *Let $(X, x_0), (Y, y_0), (Z, z_0) \in \mathbf{Top}_*$. If Y is locally compact and X is Hausdorff, then the smash product satsifes the pointed version of the exponential law. That is we have a continuous bijection:*

$$\mathrm{Maps}_*((X, x_0) \wedge (Y, y_0), (Z, z)) \cong \mathrm{Maps}_*((X, x_0), \mathrm{Maps}((Y, y_0), (Z, z)))$$

Proof. Clear. Invoke the discussion above, and note that Proposition 11.6 descends to yield the desired result. □

Remark 12.7. *If X is locally compact Hausdorff, the continuous bijection in Proposition 12.6 is in fact a homeomorphism.*

Remark 12.8. *Let M, N be locally compact Hausdorff spaces. Then their one-point compactifications M_∞, N_∞ are compact Hausdorff spaces, and each is equipped with a canonical basepoint. We continue to write (M_∞, ∞_M) as M_∞. The product $M \times N$ is locally compact Hausdorff and we have the basic relation*

$$(M \times N)_\infty \cong M_\infty \wedge N_\infty.$$

Indeed, there is canonical continuous map

$$u : M_\infty \times N_\infty \to (M \times N)_\infty$$

which maps $M \times N \subseteq M_\infty \times N_\infty$ via the identity onto $M \times N \subseteq (M \times Y)_\infty$ and maps $M_\infty \times \{\infty_N\} \cup \{\infty_M\} \times N_\infty$ to $\{\infty_{M \times N}\}$. Therefore it induces a continuous bijection

$$u' : M_\infty \wedge N_\infty \to (M \times Y)_\infty$$

on the quotient space $M_\infty \wedge N_\infty$ of $M_\infty \times N_\infty$. This space is comapct, therefore u' is a homeomorphism.

Example 12.9. Each $(\mathbb{S}^n, *)$ is a pointed topological space. We have

$$(\mathbb{S}^n, *) = (\mathbb{S}^1, *) \wedge (\mathbb{S}^1, *)$$

Note that $(\mathbb{S}^1, *) \times (\mathbb{S}^1, *)$ is a torus. Visualizing the torus as quotient of a square with endpoints identified appropriately, $\mathbb{S}^1 \vee \mathbb{S}^1$ corresponds to the to the boundary of the square. The smash product identifies all these boundary points to a single point, yielding $(\mathbb{S}^2, *)$. More generally, we have

$$(\mathbb{S}^n, *) = (\mathbb{S}^1, *) \wedge \cdots \wedge (\mathbb{S}^1, *)$$

Indeed,

$$\begin{aligned}(\mathbb{S}^{m+n}, *) &\cong (\mathbb{R}^{m+n})_\infty \\ &= (\mathbb{R}^m \times \mathbb{R}^n)_\infty \\ &\cong (\mathbb{R}^m)_\infty \wedge (\mathbb{R}^n)_\infty \cong (\mathbb{S}^m, *) \wedge (\mathbb{S}^n, *).\end{aligned}$$

Example 12.10. Let $(X, x_0) \in \mathbf{Top}_*$. We can define the reduced cone of (X, x_0) as

$$\widetilde{C}(X, x_0) = (X, x_0) \times (I, 0)/((X, x_0) \times \{x_0\} \cup \{*\} \times (I, 0)).$$

Essentially by definition,

$$\widetilde{C}(X, x_0) \cong (X, x_0) \wedge (I, 0)$$

We can also define the notion of a reduced suspension.

Definition 12.11. Let $(X, x_0) \in \mathbf{Top}_*$. The reduced suspension $\Sigma(X, x_0) \in \mathbf{Top}_*$ is the pointed space

$$\Sigma(X, x_0) = ((X, x_0) \times (\mathbb{S}^1, *)) / \left(\{x_0\} \times (\mathbb{S}^1, *) \cup (X, x_0) \times \{*\}\right),$$

where the base point is given by the collapsed subspace.

Remark 12.12. *Using the quotient map $I \to I/\partial I \cong \mathbb{S}^1$, an alternative description of the reduced suspension $\Sigma(X, x_0)$ is given by*

$$\Sigma(X, x_0) = ((X, x_0) \times (I, 0)) / \left(\{x_0\} \times I \cup X \times \{0, 1\}\right),$$

Example 12.13. Let $(X, x_0) \in \mathbf{Top}_*$. We have

$$\Sigma(X, x_0) = (X, x_0) \wedge (\mathbb{S}^1, *)$$

Indeed, consider the quotient map $f : (I, 0) \to (\mathbb{S}^1, *)$ given by $f(t) = e^{2\pi i t}$, and the diagram:

$$\begin{tikzcd}
(X, x_0) \times (I, 0) \arrow[r, "1\times f"] \arrow[d, "p"'] & (X, x_0) \times (\mathbb{S}^1, *) \arrow[d, "q"] \\
\Sigma(X, x_0) & (X, x_0) \wedge (\mathbb{S}^1, *)
\end{tikzcd}$$

It can be checked that We show that $1 \times f$ is a quotient map. The characteristic property of the quotient topology now implies that

$$\Sigma(X, x_0) \cong (X, x_0) \wedge (\mathbb{S}^1, *)$$

Remark 12.14. *Along with Example 12.9, the previous examples readily implies that we have*

$$\Sigma(\mathbb{S}^n, *) = (\mathbb{S}^{n+1}, *)$$

Corollary 12.15. *Let $(X, x_0), (Y, y_0) \in \mathbf{Top}_*$ such that X is Hausdorff. Then there is a continuous bijective correspondence*

$$\mathrm{Maps}_*(\Sigma(X, x_0), (Y, y_0)) \cong \mathrm{Maps}_*((X, x_0), \Omega(Y, y_0))$$

Passing to π_0, we have

$$[\Sigma(X, x_0), (Y, y_0)] \cong [(X, x_0), \Omega(Y, y_0)].$$

Proof. This follows from Proposition 12.6 and that Remark 12.14. □

13. Compactly Generated Spaces

The bijection in Proposition 11.6 relies on the fact that Y is locally compact. A number of topological spaces in homotopy theory are non-locally finite CW complexes. Fundamental examples include $\mathbb{RP}^\infty$ and $\mathbb{CP}^\infty$. We now look at a category of topological spaces where we expect homotopy theoretic propositions to be true without additional assumptions.

13.1. **Compactly Generated Spaces.** Informally, a compactly generated space is a topological space whose topology is determined by all continuous maps from arbitrary compact spaces.

Definition 13.1. Let $X \in \mathbf{Top}$. A subset $A \subseteq X$ is called k-closed in X if, for any compact Hausdorff space K and continuous map $f : K \to X$, the preimage $f^{-1}(A) \subseteq K$ is closed in K.

The collection of k-closed subsets of X forms a topology, which contains the original topology of X. Let kX denote the topological space whose underlying set is that of X, but equipped with the topology of k-closed subsets of X. Because the k-topology contains the original topology on X, the identity function $\mathrm{Id} : kX \to X$ is continuous.

Definition 13.2. Let $X \in \mathbf{Top}$. X is compactly generated (CG) if $\mathrm{Id} : kX \to X$ is a homeomorphism.

Let $\mathbf{CG}$ denote the full subcategory of $\mathbf{Top}$ consisting of compactly generated spaces. Let's discuss categorical properties:

Proposition 13.3. *Let* $X \in \mathbf{Top}$.

(1) *The k-ification is a functor.*
(2) *For any space X, the map $k^2X \to kX$ is a homeomorphism. Hence, $k^2X \cong kX$.*
(3) *The k-ification functor is right adjoint to the forgetful functor. That is,*

$$\mathrm{Hom}_{\mathbf{CG}}(X, kY) = \mathrm{Hom}_{\mathbf{Top}}(X, Y)$$

for all $X \in \mathbf{CG}$ and $Y \in \mathbf{Top}$.
(4) *The k-ification functor commutes with limits. Hence, limits exist in* $\mathbf{CG}$.
(5) *Disjoint unions of compactly generated spaces are compactly generated. Quotients of compactly generated spaces by equivalence relations are compactly generated.*
(6) *Colimits exist in* $\mathbf{CG}$ *and can simply be computed in* $\mathbf{Top}$.

Proof. The proof is given below:

(1) Suppose $f : X \to Y$ is any continuous map and $A \subseteq Y$ is compactly closed. For any map $u : K \to X$, the set $u^{-1}(f^{-1}(A))$ is closed in K. Thus, $f^{-1}(A)$ is compactly closed in X. This means that $f : kX \to kY$ is continuous.
(2) Given a compact Hausdorff space X and a (set) map $f : K \to X$, the map f is continuous if and only if $f : X \to kX$ is continuous. So the compactly closed sets of X are the same as the compactly closed sets of kX. In other words, $kx \cong k^2X$.
(3) It suffices to show that $f : X \to Y$ is continuous if and only if $\bar{f} : X \to kY$ is continuous. Since the k-ification topology is finer, we assume that f is continuous and show that $\bar{f}$ is continuous. But $k(f) : kX \to kY$ is continuous and $kX \cong X$.
(4) This follows from (3) and categorical arguments. Indeed, we have:

$$\begin{aligned}\mathrm{Hom}_{\mathbf{CG}}(X, k(\varprojlim_i Y_i)) &\cong \mathrm{Hom}_{\mathbf{Top}}(X, \varprojlim_i Y_i) \\ &\cong \varprojlim_i \mathrm{Hom}_{\mathbf{Top}}(X, Y_i) \\ &= \varprojlim_i \mathrm{Hom}_{\mathbf{CG}}(X, kY_i) = \mathrm{Hom}_{\mathbf{CG}}(X, \varprojlim_i kY_i)\end{aligned}$$

for all $X \in \mathbf{CG}$ and $Y \in \mathbf{Top}$. Hence,

$$k(\varprojlim_i Y_i) = \varprojlim_i kY_i$$

(5) Let $X = \coprod_i X_i$ such that each X_i is compactly generated. Let $A \subseteq X$ be k-closed. Then A has the form $\coprod_i A_i$, where $A_i = A \cap X_i$, and it is sufficient to check that A_i is closed in X_i. As X_i is CG, it is enough to check that A_i is k-closed in X_i. Consider a map $f : K \to X_i$. Then the composite $i \circ f : K \to X_i \hookrightarrow X)$ is continuous and
$$f^{-1}(A_i) = (i \circ f')^{-1}(A),$$
which is closed because A is k-closed in X. Now let X be compactly generated and let $q : X \to Y$ a quotient map. Since X is compactly generated, q induces a continuous map $\tilde{q} : X \to kY$ as shown below:
$$\begin{tikzcd} X \arrow[r, "\tilde{q}"] \arrow[rr, bend left, "q"] & kY \arrow[r, "\mathrm{Id}"] & Y \end{tikzcd}$$
Let $A \subseteq Y$ be a k-open subset of Y. Hence, $\mathrm{Id}^{-1}(A) \subseteq kY$ is open in kY. Then the preimage
$$q^{-1}(A) = (\mathrm{Id} \circ \tilde{q})^{-1}(A) = \tilde{q}^{-1}(\mathrm{Id}^{-1}(A)) \subseteq X$$
is open in X since $\tilde{q} : X \to kY$ is continuous. Therefore, $A \subseteq Y$ is open in Y since q is a quotient map.

(6) Colimits in **Top** can be constructed by taking disjoint unions and quotients. The colimit of compactly generated spaces in the category **Top** is a compactly generated space. Thus, it is also the colimit in **CG**.

This completes the proof. □

Remark 13.4. *In Proposition 13.3(3) we have shown that $f : X \to Y$ is continuous if and only if $f : X \to kY$ is continuous for all $X \in$ **CG** and $Y \in$ **Top**. This can be summarized such that following diagram commutes:*
$$\begin{tikzcd} kY \arrow[r, "Id"] & Y \\ X \arrow[u, dotted, "\bar{f}"] \arrow[ur, "f"] & \end{tikzcd}$$
Note that $\bar{f}$ has the same underlying function as f. This exhibits $kY \to Y$ as the 'closest approximation' of Y by a CG space.

Proposition 13.5. *Every locally compact Hausdorff space and CW-complex is CG.*

Proof. Let X be a locally compact space assume $f^{-1}(A) \subseteq K$ is closed for every compact Hausdorff space, K. We show A is closed by showing that A^c is open. Let $x \in A^c$. By local compactness, there exists a compact neighbourhood of x, say K_x. Let U_x be an open neighbourhood of x such that $x \in U_x \subseteq K_x$. Because $K_x \cap A$ is closed by hypothesis (consider the inclusion map $i_x : K_x \to X$), we have that $(K_x \cap A)^c$ is open. Therefore,
$$(K_x \cap A)^c \cap U_x = U_x^c := V_x$$
is an open neighbourhood of x not intersecting A. We have
$$A^c = \bigcup_{x \in A^c} V_x,$$
and therefore A^c is open. A CW complex is a colimit constructed by consideting closed disks. Since closed disks are in **CG** and **CG** is closed under taking colimits, every CW complex is in **CG**. □

Corollary 13.6. *Let $X \in \mathbf{Top}$. Then $X \in \mathbf{CG}$ if and only if X is a quotient space of a locally compact space.*

Proof. Let $X \in \mathbf{CG}$. The converse follows from Proposition 13.3. Consider the following collection:

$$\mathcal{K} = \{f_K(K) \mid f_K : K \to X \text{ is continuous and } K \text{ compact Hausdorff}\}$$

Let $Y = \bigoplus_{f_K(K)\in\mathcal{K}} f_K(K)$ where each $f_K(K) \in \mathcal{K}$ has the subspace topology inherited from X. Then Y is a locally compact space. Let

$$q : Y \to X$$

be the map maps each $f_K(K)$ onto the corresponding compact subset $f_K(K) \subseteq X$ by the identity map. We claim that the quotient topology, τ_q, generated by this mapping coincides with the original topology, τ, on X. Clearly, $\tau \subseteq \tau_q$ since q is a continuous map. Let $U \in \tau_q$. Let $g : L \to X$ be continuous such that L is compact. Since $U \in \tau_q$, we have that $q^{-1}(U)$ is open in Y. Since $g(L)$ is open in Y, it follows that $q^{-1}(U) \cap g(L)$ is open in Y. But $q^{-1}(U) \cap g(L) = g^{-1}(U)$. Thus, $g^{-1}(U)$ is open in L and consequently, $U \in \tau$. □

Remark 13.7. *Limits in $\mathbf{CG}$ need not coincide with limits in $\mathbf{Top}$. Let $X, Y \in \mathbf{CG}$ such that $X = \mathbb{R}\setminus\{1, 1/2, 1/3, \dots\}$ with the subspace topology, and let $Y = \mathbb{R}/\mathbb{Z}$ with the quotient topology. $X \in \mathbf{CG}$ since X is a CW complex[15] and $Y \in \mathbf{CG}$ by Corollary 13.6. In fact, Y is also a CW complex since Y is an infinite bouqet of circles. However, $X \times Y$ is not compactly generated. Let*

$$A = \bigcup_{i=1}^{\infty} A_i = \bigcup_{i,j=1}^{\infty} \left\{\left(\frac{1}{i} + \frac{a_i}{j}, i + \frac{0.5}{j}\right) \in X \times Y : j \in \mathbb{N}\right\}, \quad a_i = \left(\frac{1}{i} - \frac{1}{i+1}\right)10^{-i}.$$

The closure of A closure contains $(0,0)$. Hence, A is not closed. But for any compact subset $K \subseteq X \times Y$, the set $A \cap K$ has only finitely many points. This is because for fixed $i \in \mathbb{N}$, there are only finitely many $j \in \mathbb{N}$, and also there can be only finitely many i. Hence A k-closed. This shows that $X \times Y$ is not in $\mathbf{CG}$. See [Eng89] for details.

Remark 13.8. *We have*

$$\mathbf{CW} \subsetneq \mathbf{CG} \subsetneq \mathbf{Top}$$

as inclusion of categories. The inclusions are in general strict. Indeed, the Hawaiian earring is in $\mathbf{CG}$ since it is compact and hence locally compact. However, we have already seen that it admits no CW decomposition. For the inclusion $\mathbf{CG} \subsetneq \mathbf{Top}$, consider the example in Remark 13.7.

We can now discuss the mapping spaces with the notion of a compactly generated space in place. We need to modify the definition given in the previous section a bit since we deal with compact Hausdorff spaces in this section.

Definition 13.9. Let $C_0(X, Y)$ be the set of continuous functions from X to Y with the compact-open topology that is generated by a subbasis formed by the sets of the form

$$B(u, K, U) = \{f : K \to Y \mid f(u(K)) \subseteq U,\ u : K \to X \text{ is cts. s.t } K \text{ is cpt. Hausdorff}\}$$

We define $C(X, Y) = kC_0(X, Y)$.

[15]Right?

Remark 13.10. *If $X, Y \in \mathbf{CG}$, then $X \times Y$ might not be in $\mathbf{CG}$. See Remark 13.7. In this case, we can consider $k(X \times Y)$. Below, we write $k(X \times Y)$ as $X \times_k Y$.*

Remark 13.11. *If $X \in \mathbf{CG}$ and $Y \in \mathbf{Top}$ is locally compact, it turns our that $X \times Y \in \mathbf{CG}$. Since $X \in \mathbf{CG}$, we have $X = Z/\sim$ such that Z is locally compact by Corollary 13.6. In other words, we have a quotient map $q : Z \to X$. Consider the map*

$$q \times \mathrm{Id}_Y : Z \times Y \to X \times Y$$

It is a standard fact that $q \times \mathrm{Id}_Y$ is a quotient map since Y is assumed to be locally compact. It is clear that

$$X \times Y = \frac{Z}{\sim} \times Y = \frac{Z \times Y}{\sim'},$$

where $(z, y) \sim' (z', y')$ if and only if $z \sim z'$. In other words, we have "$\sim' = \sim \times \mathrm{Id}$". Here we have implicitly used the fact that the bijection of sets

$$X \times Y \cong \frac{Z}{\sim} \times Y \cong \frac{Z \times Y}{\sim'}$$

is in fact a homeomorphism in $\mathbf{Top}$ essentially because the product topology (left hand side) and the quotient topology (right hand side) are the same. Since $Z \times Y$ is locally compact, the claim follows from Corollary 13.6.

Proposition 13.12. *Let $X, Y, Z \in \mathbf{CG}$.*

1. *For $X, Y \in \mathbf{CG}$, $C(X, \cdot)$ is a covariant functor from $\mathbf{CG}$ to $\mathbf{Sets}$. Similarly, $C(\cdot, Y)$ is a contravariant functor from $\mathbf{CG}$ to $\mathbf{Sets}$.*
2. *The evaluation map*

$$\mathrm{Ev}_{X,Y} : X \times C(X, Y) \to Y$$

and the injection map

$$i_{X,Y} : Y \to C(X \times_k C(X, Y))$$

are continuous.

3. *(**Exponential Law**) The map*

$$\varphi : C(X \times_k Y, Z) \to C(X, C(Y, Z)),$$

as discussed in Proposition 11.6 is a homeomorphism.

4. *The composition map*

$$\Phi_{X,Y,Z} : C(X, Y) \times_k C(Y, Z) \to C(X, Z)$$

is continuous.

Proof. The proof is given below:

1. We prove the the covariant case. It suffices to check that $C_0(X, \cdot)$ is a covariant functor. We have to check that if $g : Y \to Z$ is a continuous map, then $g_* = C_0(X, Y) \to C_0(X, Z)$ is continuous. But we have

$$(g_*)^{-1} B(u, K, U) = B(u, K, g^{-1}(U))$$

The claim follows.

(2) It suffices to show that $Y \to C_0(X, X \times_k Y)$ or equivalently that $i^{-1}B(u,K,U)$ is open in Y. As $Y \in \mathbf{CG}$, it is equivalent to check that $v^{-1}i^{-1}B(u,K,U)$ is open in L for every test map $v : L \to Y$, where L is a compact Hausdorff space. Note that $u \times v : K \times_k L \to X \times_k Y$ is a test map, so $(u \times v)^{-1}(U)$ is open in $K \times_k L$. By the Tube Lemma, the set

$$\{b \in L : K \times \{b\} \subseteq (u \times v)^{-1}(U)\}$$

is open in L. It is easy to check that this set is the same as $v^{-1}\mathrm{inj}^{-1}B(u,K,U)$, which completes the proof.

Consider an open set $U \subseteq Y$, and a map $u : K \to X \times_k C(X,Y)$. We show that $V = u^{-1}\mathrm{Ev}^{-1}(U)$ is open in K. Let $v : K \to X$ and $w : K \to C(X,Y)$ be the two components of u, so

$$V = \{a \in K : w(a)(v(a)) \in U\}.$$

Suppose that $a \in V$. As $w(a) \circ v : K \to Y$ is continuous, we can choose a compact neighbourhood L of a in K such that $w(a)(v(L)) \subseteq U$. This means that $w(a) \in B(v,L,U) \subseteq C(X,Y)$. As $w : K \to C(X,Y)$ is continuous, the set $N = w^{-1}(B(v,L,U))$ is a neighbourhood of a in K. If $b \in N \cap L$, then $w(b)(v(b)) \in w(b)(v(L)) \subseteq U$, so $b \in V$. Thus, the neighbourhood $N \cap L$ of a is contained in V. This shows that V is open, as required.

(3) We first show that it is a bijection at the level of sets. If $f : X \to C(Y,Z)$ is continuous, then its image

$$X \times_k Y \xrightarrow{f \times \mathrm{Id}} C(Y,Z) \times_k Y \xrightarrow{\mathrm{Ev_{Y,Z}}} Z$$

is continuous. On the other hand, if $g : X \times_{\mathrm{CG}} Y \to Z$ is continuous, then its image

$$X \xrightarrow{\mathrm{inj}_{X,Y}} C(Y, X \times_k Y) \xrightarrow{\mathrm{Ev_{X,Y}}} Y$$

is continuous. This shows that the exponential map is bijection. Moreover, if $W \in \mathbf{CG}$ we have bijections:

$$\begin{aligned} C(W, C(X, C(Y,Z))) &\cong C(W \times_k X, C(Y,Z)) \\ &\cong C(W \times_k X \times_k Y, Z) \\ &\cong C(W, C(X \times_k Y, Z)). \end{aligned}$$

This means that $C(X, C(Y,Z))$ and $C(X \times_k Y, Z)$ represent the same contravariant functor and the claim now follows by Yoneda's Lemma.

(4) The proof is similar to Proposition 11.10.

This completes the proof. □

Thus, we have obtained a category **CG** that contains all locally compact Hausdorff spaces, CW-complexes, admits all limits and colimits, and is Cartesian closed.

13.2. **Weakly Hausdorff Spaces.** The category **CG** still contains some bad topological spaces, like the Sierpinski space. These do not satisfy the Hausdorff condition and we would like to exclude them by imposing a Hausdorff like condition.

Definition 13.13. Let $X \in \mathbf{Top}$. Then X is weakly Hausdorff (WH) if for every compact Hausdorff space K and every continuous map $u : K \to X$, the image $u(K) \subseteq X$ is closed in X.

Example 13.14. If X is a Hausdorff space, then X is weakly-Hausdorff since $u(K)$ is compact and thus closed in X. Every CW-complex is Hausdorff, hence in particular weakly Hausdorff.

Proposition 13.15. *Let X be a weakly Hausdorff topological space.*

(1) *Any finer topology on X is still weakly Hausdorff. In particular, kX is weakly Hausdorff.*
(2) *Any subspace of X is weakly Hausdorff.*

Proof. The proof is given below:

(1) Let x be the set X equipped with a topology containing the original topology, i.e., the identity function $\mathrm{Id} : x \to X$ is continuous. For any compact Hausdorff space K and continuous map $u : K \to x$, the composite $\mathrm{Id} \circ u : K \to X$ is continuous, and so its image $(\mathrm{Id} \circ u)(K) \subseteq X$ is closed in X. Thus, $u(K) = \mathrm{Id}^{-1}((\mathrm{Id} \circ u)(K))$ is closed in x.
(2) Let $i : A \hookrightarrow X$ be the inclusion of a subspace in X. For any compact Hausdorff space K and continuous map $u : K \to A$, the composite $i \circ u : K \to X$ is continuous, and so its image $(i \circ u)(K) \subseteq X$ is closed in X, and thus in A as well.

This completes the proof. □

Let **CGWH** denote the full subcategory of **CG** consisting of compactly generated weakly Hausdorff spaces. We have

$$\mathbf{CW} \subsetneq \mathbf{CGWH} \subsetneq \mathbf{CG} \subsetneq \mathbf{Top}$$

as inclusion of categories. The inclusion $\mathbf{CGWH} \subsetneq \mathbf{CG}$ is strict since the Sierpinski space is in **CG** but not in **CGWH**. Similarly, the inclusion $\mathbf{CW} \subsetneq \mathbf{CGWH}$ is strict. Simply consider the Hawaiian earring.

Proposition 13.16. *Let $X \in \mathbf{CG}$. Then $X \in \mathbf{CGWH}$ if and only if the diagonal subspace $\Delta_X = \{(x, x) \mid x \in X\}$ is k-closed in $X \times_k X$.*

Proof. Suppose that X is weakly Hausdorff. First, observe that every one-point set $\{x\} \subseteq X$ is certainly a continuous image of a compact Hausdorff space and thus is closed in X, so X is T_1. Next, consider a test map $u = (v, w) : K \to X \times_k X$. It will be enough to show that the set $u^{-1}(\Delta_X) = \{a \in K : v(a) = w(a)\}$ is closed in K. Suppose that $a \notin u^{-1}(\Delta_X)$, so $v(a) \neq w(a)$. Then the set

$$U = \{b : v(b) \neq w(a)\}$$

is an open neighbourhood of a (because $\{w(a)\}$ is closed in X). Now K is compact Hausdorff and therefore regular, so there is an open neighbourhood V of a in K such that $\overline{V} \subseteq U$, or equivalently $w(a) \notin v(\overline{V})$. This means that a lies in the set

$$W = w^{-1}(v(\overline{V})^c).$$

The weak Hausdorff condition implies that $v(\overline{V})$ is closed in X, and thus W is open in K. We claim that $(V \cap W) \cap u^{-1}(\Delta_X) = \emptyset$. Indeed, if $b \in V \cap W$, then $v(b) \in v(\overline{V})$ but $w(b) \in v(\overline{V})^c$ by the definition of W, so $v(b) \neq w(b)$, which implies $u(b) = (v(b), w(b)) \notin \Delta_X$. This shows that $u^{-1}(\Delta_X)$ is closed in K, as required.

Conversely, suppose that Δ_X is k-closed in $X \times_k X$. Let $u : K \to X$ be a test map. Given any other test map $v : L \to X$, we define

$$M = \{(a, b) \in K \times L : u(a) = v(b)\} \subseteq K \times L.$$

This can also be described as $(u \times v)^{-1}(\Delta_X)$, so it is closed in $K \times L$ and thus compact. It follows that the projection $\pi_L(M)$ is compact and hence closed in L. However, it is easy to see that $\pi_L(M) = v^{-1}(u(K))$. This shows that $u(K)$ is k-closed in X, and hence closed. This means that X is weakly Hausdorff. □

Remark 13.17. *Proposition 13.16 is an important characterization of weakly Hausdorff spaces. In* **Top***, the criteria in Proposition 13.16 is exactly the characterization of a Hausdorff space. That is $X \in$ **Top** is Hausdorff if and only if $\Delta_X = \{(x, x) \mid x \in X\}$ is closed in $X \times X$. Here $X \times X$ is the product in* **Top***.*

We now have two additional functors. The first is the forgetful functor from **CGWH** to **CG**. Another is weak-Hausdorffification, dented as h, from **CG** to **CGWH**. We need to construct this functor h. We fist need some additional facts about wtaking quotients in **CG**.

Lemma 13.18. *Let $X, Y \in$ **CG** and let $\sim$ be an equivalence relation on X.*

(1) *We have*
$$(X \times_k Y)/(\sim \times \mathrm{Id}) \cong (X/\sim) \times_k Y$$
(2) *Let q be the map*
$$q : X \times_k X \to (X/\sim) \times_k (X/\sim)$$
*The set $q^{-1}(\Delta_{X/\sim}) \subseteq X \times_k X$ is closed if and only if $X/\sim \in$ **CGWH**.*

Proof. The proof is given below:

(1) The standard map
$$f : X \times_k Y \to (X/\sim) \times_k Y$$
respects the relation $\sim \times \mathrm{Id}$ and thus factors as
$$\overline{f} : (X \times_k Y)/(\sim \times \mathrm{Id}) \to (X/\sim) \times_k Y$$
Let g_1 be the projection map.
$$g_1 : (X/\sim) \times_k Y \to (X \times_k Y)/(\sim \times \mathrm{Id})$$
Using the exponential law, we get a map
$$g_2 : X \to C\left(Y, (X \times_k Y)/(\sim \times \mathrm{Id})\right)$$
This respects $\sim$ on the level of sets, and so factors to give a map
$$\overline{g}_2 : X/\sim \to C\left(Y, (X \times_k Y)/(\sim \times \mathrm{Id})\right)$$
Using the exponential law again, we get a map
$$\overline{g} : X/\sim \times_k Y \to (X \times_k Y)/(\sim \times \mathrm{Id})$$
$\overline{f}$ and $\overline{g}$ are clearly inverses.

(2) By applying (1) twice, we have
$$(X \times_k X)/(\sim \times \sim) \cong (X/\sim) \times_k (X/\sim)$$
Thus, $\Delta_{X/\sim}$ is closed if and only if $q^{-1}(\Delta_{X/\sim})$ is closed if and only if $X/\sim$ is in **CGWH**.

This completes the proof. □

Proposition 13.19. *There exists a functor $h : \mathbf{CG} \to \mathbf{CGWH}$ that is a left adjoint to the forgetful functor $\mathbf{CGWH} \to \mathbf{CG}$. That is,*

$$\mathrm{Hom}_{\mathbf{CG}}(h(X), Y) = \mathrm{Hom}_{\mathbf{CGWH}}(X, Y)$$

for all $X \in \mathbf{CG}$ and $Y \in \mathbf{CGWH}$.

Proof. We first construct h. For $X \in \mathbf{CG}$, consider the smallest equivalence relation on $X \times_k X$ that is closed. We can take the intersection of all closed equivalence relations. Then $hX := X/\sim \in \mathbf{CGWH}$ and there is a natural projection map $p\colon X \to X/\sim$. We now show that h is left-adjoint to the forgetful functor. It suffices to show that every $f : X \to Y$ factors through $X \to hX$. Since $Y \in \mathbf{CGWH}$ $\Delta_Y \subseteq Y \times_k Y$ is closed and hence $f^{-1}(\Delta_Y)$ is closed in $X \times_k Y$. This is an equivalence relation that contains $\sim$ since it is closed. Thus $X \to Y$ respects $\sim$ and factors through $X \to hX$. Moreover, h is a functor since for $f : X \to Y$ such that $X, Y \in \mathbf{CG}$, we have $X \to Y \to hY$ which in turn gives $hX \to hY$. □

Remark 13.20. *The functors discussed are summarized in the diagram below:*

$$\begin{tikzcd}
\mathbf{Top} \arrow[r, bend left, "\mathrm{k}"] \arrow[r, bend right, "\mathrm{Forgetful}"'] & \mathbf{CG} \arrow[rr, bend left, "\mathrm{h}"] & & \mathbf{CGWH} \arrow[ll, bend left, "\mathrm{Forgetful}"]
\end{tikzcd}$$

Corollary 13.21. *The following properties hold:*

(1) *Limits exist in $\mathbf{CGWH}$ and can simply be computed in $\mathbf{CG}$. In fact, small colimits in $\mathbf{CGWH}$ can be computed in $\mathbf{CG}$.*
(2) *h commutes with colimits. In particular, colimits in exist in $\mathbf{CGWH}$ and are obtained by applying h to the colimit in $\mathbf{CG}$. In particular, the category $\mathbf{CGWH}$ is admits small colimits exist because $\mathbf{CG}$ admits admits small colimits.*
(3) *For $X \in \mathbf{CG}$ and $Y \in \mathbf{CGWH}$, $C(X, Y) \in \mathbf{CGWH}$. Hence, $\mathbf{CGWH}$ is Cartesian closed.*

Proof. (Sketch) The proof is given below:

(1) (Sketch) This is because an arbitrary product in $\mathbf{CG}$ of CGWH spaces is still WH, and so is an equalizer in $\mathbf{CG}$ of two maps.
(2) This follows since h is left adjoint to the forgetful functor.
(3) Define

$$\mathrm{Ev}_x : C(X, Y) \cong \{x\} \times C(X, Y) \hookrightarrow X \times C(X, Y) \xrightarrow{\mathrm{Ev}} Y$$

We have

$$\Delta_{C(X,Y)} = \bigcap_{x \in X} (\mathrm{Ev}_x \times \mathrm{Ev}_x)^{-1}(\Delta_Y)$$

This is closed. Hence, $C(X, Y) \in \mathbf{CGWH}$,

This completes the proof. □

Thus, we have obtained a category **CGWH** that contains all locally compact Hausdorff spaces, CW-complexes, admits all limits and colimits, and is Cartesian closed.

Remark 13.22. *All results about $\mathrm{Maps}(X, Y)$ that hold under the hypothesis of locally compact and Hausdorff hold without any additional assumptions in $\mathrm{C}(X, Y)$.*

Part 4. Fibrations

We adopt the following conventions from now on:

(1) We will assume that we work in the category **CGWH**.
(2) Abusing notation, we will write **CGWH** as **Top**.
(3) We will write $X \times_k Y$ simply as $X \times Y$.

These assumptions will allow the theory of fibrations to be developed without further restrictions.

14. Fibrations

Fibrations play a fundamental role in homotopy theory. In a sense, fibrations can be thought of as 'homotopically nice projections,' a notion made precise below. We will introduce two types of fibrations - the Hurewicz fibrations and Serre fibrations - which are both obtained by imposing certain homotopy lifting properties. Prominent examples of fibrations are fiber covering spaces and fiber bundles, which are introduced in the next section. These fibrations provide powerful tools for understanding the relationships between the base and total spaces, and they allow us to analyze the homotopy type of complex spaces by studying simpler ones.

14.1. Definition & Examples.

Definition 14.1. Let $X, E \in \mathbf{Top}$. A continuous surjective map $p : E \to X$ satisfies the homotopy lifting for $A \in \mathbf{Top}$ if for any homotopy $H : A \times I \to X$ and map $f : A \times \{0\} \to E$, there exists a homotopy $\widetilde{H} : A \times I \to E$ such that the following diagram commutes:

$$\begin{tikzcd} A \times \{0\} \arrow[r, "H_0"] \arrow[d, "i_0"] & E \arrow[d, "p"] \\ A \times I \arrow[r, "H"] \arrow[ur, dotted, "\widetilde{H}"] & X \end{tikzcd}$$

(1) We say a continuous surjective map $p : E \to X$ is a Hurewicz fibration if it satisfies the homotopy lifting property for any $A \in \mathbf{Top}$.
(2) We say a continuous surjective map $p : E \to X$ is a Serre fibration if it satisfies the homotopy lifting property for any $I^n \in \mathbf{Top}$ for each $n \geq 0$.

Remark 14.2. *Clearly, a Hurewicz fibration is a Serre fibration. It is clear that a fibration is a generalization of the notion of a covering space since covering spaces satisfy the homotopy lifting property.*

Remark 14.3. *A continuous surjective map $p : E \to X$ satisfies the homotopy lifting property for $A \in \mathbf{Top}$ if and only if the following diagram commutes:*

$$\begin{tikzcd} A \arrow[drr, bend left, "H"] \arrow[dr, dotted, "\widetilde{H}"] \arrow[ddr, bend right, "f"'] & & \\ & E^I \arrow[r, "p_*"] \arrow[d, "\mathrm{Ev}_0"] & X^I \arrow[d, "\mathrm{Ev}_0"] \\ & E \arrow[r, "p"] & X \end{tikzcd}$$

Here Ev_0 *is the evaluation at* 0 *map and* X^I *denotes* $\mathrm{Maps}(I, X)$*.*

Remark 14.4. *If* $p : E \to X$ *is a fibration and* $x \in X$*, then* $F_x := p^{-1}(x) \subseteq E$ *is called the fiber of* p *over* x*. We write*

$$F_x \to E \to X$$

Example 14.5. Let's look at some examples of fibrations:

(1) For any $X \in \mathbf{Top}$, the unique map $X \to *$ is a Hurewicz fibration. This is clear.
(2) Any projection $p : X \times Y \to X$ is a Hurewicz fibration. For $A \in \mathbf{Top}$, let $H : A \times I \to X$ be a homotopy such that H_0 lifts to a map $A \to X \times Y$. We can define $\widetilde{H}$ by

$$\begin{aligned} \widetilde{H} : A \times I &\to X \times Y, \\ (a, t) &\mapsto (H(a, t), H(a, 0)). \end{aligned}$$

It is clear that $\widetilde{H}$ satisfies the definition.
(3) A homeomorphism $f : X \to Y$ is a Hurewicz fibration since we can simply define $\widetilde{H} = f^{-1} \circ H$.
(4) Consider the evaluation map

$$\begin{aligned} \mathrm{Ev}_{0,1} : \mathrm{Maps}(I, X) &\to X \times X \\ \gamma &\mapsto (\gamma(0), \gamma(1)) \end{aligned}$$

We show that $\mathrm{Ev}_{0,1}$ is a Hurewicz fibration. Consider the diagram:

$$\begin{tikzcd} A \cong A \times \{0\} \arrow[r, "H_0"] \arrow[d, "i_0"] & \mathrm{Maps}(I, X) \arrow[d, "\mathrm{Ev}_{0,1}"] \\ A \times I \arrow[r, "H"] \arrow[ur, dotted, "\widetilde{H}"] & X \times X \end{tikzcd}$$

Equivalently, we are given a continuous map

$$\varphi : (A \times \{0\} \times I) \cup (A \times I \times \{0, 1\}) \to X$$

which we wish to extend to $A \times I \times I$. But

$$(\{0\} \times I) \cup (I \times \{0, 1\}) := J_1 \subseteq I^2$$

is a retract of I^2. The argument is similar to Example 2.17. Hence so is $A \times J_1$ of $A \times I \times I$. Therefore, we can simply pre-compose φ with the retraction

$$r : A \times I \times I \to A \times J_1$$

to find the required extension.

Proposition 14.6. *The following statements are true:*

(1) *The composition of Hurewicz fibrations is a Hurewicz fibration.*
(2) *The product of Hurewicz fibrations is a Hurewicz fibration.*
(3) *The pullback of a Hurewicz fibration is a Hurewicz fibration.*

(4) *(**Universal Test Space**) Let $p : E \to X$ be a continuous surjective map and let N_p be the following pullback:*

$$\begin{tikzcd}
N_p \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & X^I \arrow[d, "\mathrm{Ev}_0"] \\
E \arrow[r, "p"] & X
\end{tikzcd}$$

Here

$$N_p := E \times_X X^I = \{(e, \gamma) \in E \times X^I \mid p(e) = \gamma(0)\}.$$

*If p satisfies the homotopy lifting property for $N_p \in$ **Top**, then $p : E \to X$ is a Hurewicz fibration.*

Proof. The proof is given below:

(1) Let $p_1 : E_1 \to X$ and $p_2 : E_2 \to E_1$ be fibrations and consider the following diagram:

$$\begin{tikzcd}
A \arrow[drrr, bend left, "H"] \arrow[dr, dotted, "\widetilde{\widetilde{H}}"'] \arrow[drr, dotted, "\widetilde{H}"] \arrow[ddr, bend right, "f"'] & & & \\
& E_2^I \arrow[r, "(p_2)_*"'] \arrow[d, "\mathrm{Ev}_0"] & E_1^I \arrow[r, "(p_1)_*"'] \arrow[d, "\mathrm{Ev}_0"] & X^I \arrow[d, "\mathrm{Ev}_0"'] \\
& E_2 \arrow[r, "p_2"] & E_1 \arrow[r, "p_1"] & X
\end{tikzcd}$$

Since p_1 is a Hurewicz fibration, $\widetilde{H}$ exists to make the right-hand side of the diagram commute. Since p_2 is a Hurewicz fibration, $\widetilde{\widetilde{H}}$ exists to make the left-hand side of the diagram commute. The claim follows.

(2) This is clear. The same argument as in covering space theory applies here.

(3) Let $p : E \to X$ be a Hurewicz fibration and consider a pullback square:

$$\begin{tikzcd}
E' \arrow[r] \arrow[d, "q"] & E \arrow[d, "p"'] \\
X' \arrow[r] & X
\end{tikzcd}$$

Consider the diagram below:

$$\begin{tikzcd}
A \times \{0\} \arrow[r, "f"] \arrow[d, "i_0"'] & E' \arrow[r] \arrow[d, dashed, "q"] & E \arrow[d, "p"'] \\
A \times I \arrow[r, "H"] \arrow[ur, dotted, "\widetilde{H}"] \arrow[urr, dotted] & X' \arrow[r] & X
\end{tikzcd}$$

The unmarked dotted arrow from $A \times I$ to E can be completed since p is a Hurewicz fibration. The fact that the square is a pulback square then implies the existence of H.

(4) Consider the following diagram:

$$\begin{tikzcd}
A \arrow[dr, dotted, "q"] \arrow[drrr, bend left, "H"] \arrow[ddrr, bend right, "f"'] & & & \\
& N_p \arrow[dr, dotted, "s"] \arrow[drr, bend left, "\pi_2"] \arrow[dr, bend right, "\pi_1"', shift right=0] \arrow[ddr, bend right, "\pi_1"'] & & \\
& & E^I \arrow[r, "p_*"'] \arrow[d, "\mathrm{Ev}_0"] & X^I \arrow[d, "\mathrm{Ev}_0"'] \\
& & E \arrow[r, "p"] & X
\end{tikzcd}$$

Suppose p satisfies the homotopy lifting property for N_p. Then the map s exists as in the diagram. By the universal property of pullbacks, there exists a map $q : Y \to N_p$ such that the diagram commutes. Then $s \circ q :\to E^I$ solves the problem.

This completes the proof. □

Remark 14.7. *We give N_p the subspace topology with respect to the compact open topology. We say that N_p is a universal test space $p : E \to X$.*

14.2. **Mapping Path Space.** N_p is an instance of the construction of a mapping path space which we now describe.

Definition 14.8. Let $f : X \to Y$ be a continuous map. The mapping math space is the topological space

$$N_f = X \times_Y Y^I = \{(x, \gamma) \in X \times Y^I \mid f(x) = \gamma(0)\}.$$

We give N_f the subspace topology with respect to the compact open topology.

Note that N_f is defined as a pullback:

$$\begin{tikzcd}
N_f \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & Y^I \arrow[d, "\mathrm{Ev}_0"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

We can now use the mapping path space construction in Proposition 14.6(4) to argue that any continuous map $f : X \to Y$ can be decomposed as a composition of a homotopy equivalence and a Hurewicz fibration.

Proposition 14.9. *Let $f : X \to Y$ be a continuous map. Then f can be decomposed as*

$$\begin{tikzcd}
X \arrow[r, "i"'] \arrow[rr, bend left, "f"] & N_f \arrow[r, "p"'] & Y
\end{tikzcd}$$

where i is a homotopy equivalence and p is a Hurewicz fibration.

Proof. We have $X \subseteq N_f$ via mapping $x \mapsto (x, c_{f(x)})$, where $c_{f(x)}$ is the constant path based at the image of x under f. Call this map i as in the diagram above. Define

$$\begin{aligned} p \colon N_f &\to Y \\ (x, \gamma) &\mapsto \gamma(1) \end{aligned}$$

Clearly, $f = p \circ i$. We first show that i is a homotopy equivalence. Let $\pi_1 : N_f \to X$ be the projection onto X. Then $\pi_1 \circ i = \mathrm{Id}_X$ and we have a homotopy

$$H : N_f \times I \to N_f$$
$$((x, \gamma), t) \mapsto (x, s \mapsto \gamma((1-t)s))$$

from $i \circ \pi_1$ to Id_{N_f}. We now check that p is a Hurewicz fibration. Consider the following diagram:

$$\begin{tikzcd} A \times \{0\} \arrow[r, "H_0"] \arrow[d, "i_0"] & N_f \arrow[d, "p"] \\ A \times I \arrow[r, "H"] & Y \end{tikzcd}$$

First note that we have the following commutative diagram:

$$\begin{tikzcd} A \cong A \times \{0\} \arrow[r, "H_0"] \arrow[d, "i_0"] \arrow[drr, "\mathrm{Id}"] & N_f \arrow[r, "\pi_1"] & X \\ A \times I \arrow[rr, "\pi_A"] & & A \cong A \times \{0\} \arrow[u, "\pi_1 \circ H_0"'] \end{tikzcd}$$

If we write $H_0(a) = (I(a), J(a))$, then $\pi_A \circ (\pi \circ H_0)(a, t) = I(a)$. Hence, we identify $\pi_A \circ (\pi_1 \circ H_0)$ with I. Moreover, using Example 14.5(4), we have the following commutative diagram:

$$\begin{tikzcd} A \times \{0\} \arrow[r, "H_0"] \arrow[d, "i_0"] & N_f \arrow[r, "\pi_2"] & Y^I \arrow[d, "\mathrm{Ev}_{0,1}"] \\ A \times I \arrow[urr, "K"] \arrow[rr, "(f \circ I, H)"] & & Y \times Y \end{tikzcd}$$

Hence, we can define $\widetilde{H}(a, t) = (I(a, t), K(a, t))$. The image of $\widetilde{H}$ is in N_f. This is because

$$f(I(a)) = \mathrm{Ev}_0(K(a, t))$$

Moreover, the intended diagram commutes since

$$(p \circ \tilde{H})(a, t) = K(a, 1) = \mathrm{Ev}_1 K(a, t) = H(a, t),$$
$$\tilde{H} \circ i_0(a) = \tilde{H}(a, 0) = H_0(a).$$

This completes the proof. $\square$

Motivated by Proposition 14.9 we can make the following definition of the homotopy fiber of any arbitrary continuous map $f : X \to Y$.

Definition 14.10. Let $f : X \to Y$ be a continuous map. Let $p : N_f \to Y$ denote the map as in Proposition 14.9. The homotopy fiber of f over $y_0 \in Y$ is

$$\mathrm{hFiber}_f(y_0) := p^{-1}(f) = \{(x, \gamma) \mid \gamma(0) = f(x), \gamma(1) = y_0\}$$

Remark 14.11. *For each $y_0 \in Y$, note that there is a canonical map from the fiber of f over x to the homotopy fiber of f over x:*

$$f^{-1}(y_0) \to \mathrm{hFiber}_f(y_0)$$
$$x \mapsto (x, c_{f(x)})$$

Thus, the fiber sits in the homotopy fiber while the homotopy fiber can be thought of as a 'relaxed' version of the fiber: a point of the homotopy fiber is a pair (x, γ) *consisting of* $x \in X$ *together with a path* γ *in* Y *'witnessing' that* x *'lies in the fiber up to homotopy.'*

Remark 14.12. *Let* $f : X \to Y$ *be a fibration. We check that the canonical map*

$$f^{-1}(y_0) \to \mathrm{hFiber}_f(y_0)$$

is a homotopy equivalence in this case. Define a homotopy

$$\begin{aligned} H : N_f \times I &\to Y \\ ((x, \gamma), t) &\mapsto \gamma(t) \end{aligned}$$

Note that $H_0(x, \gamma) = \gamma(0) = f(x)$, *and* H_0 *lifts through* f *by* $\bar{H}_0 : N_f \to X$, $\bar{H}_0(x, \gamma) = x$. *That is,* $f \circ \bar{H}_0 = H_0$. *Because* $X \to Y$ *is a fibration, there is a full lift*

$$\bar{H} : N_f \times I \to X$$

of H *through* f. *In other words,* $\bar{H}_t$ *satisfies the following equation:*

$$f(\bar{H}_t(x, \gamma)) = \gamma(t)$$

Now restrict everything to the fibers. Let

$$\begin{aligned} h_t : \mathrm{hFiber}_f(y_0) &\to \mathrm{hFiber}_f(y_0) \\ (x, \gamma) &\mapsto \left(\bar{H}_t(x, \gamma), \gamma_{|[t,1]}\right) \end{aligned}$$

Then h_0 *is the identity, whereas* $h_1(x, \gamma) = (\bar{H}_1(x, \gamma), c_{y_0})$ *is in the image of* $i : f^{-1}(y_0) \to \mathrm{hFiber}_f(y_0)$. *Now that* h_t *is a homotopy between* $i \circ h_1$ *and the identity, while the restriction of* h_t *is a homotopy between* $h_1 \circ i$ *and the identity. This verifies the assertion.*

14.3. **Fiber Homotopy Equivalence.** It is important to study fibrations over a given base space $X \in \mathbf{Top}$, working in the category of spaces over X which we denote as $\mathbf{Top}_X$. An object in $\mathbf{Top}_X$ is a continuous map $p : E \to X$. Moreover, a morphism in $\mathbf{Top}_X$ is a commutative diagram

$$\begin{tikzcd} E_1 \arrow[rr, "f"] \arrow[dr, "p_1"'] & & E_2 \arrow[dl, "p_2"] \\ & X & \end{tikzcd}$$

A homotopy in $\mathbf{Top}_X$ is commutative diagram

$$\begin{tikzcd} E_1 \times I \arrow[rr, "H"] \arrow[dr, "p_1"'] & & E_2 \arrow[dl, "p_2"] \\ & X & \end{tikzcd}$$

such that for all $t \in I$, we have the following commutative diagram:

$$\begin{tikzcd} E_1 \times \{t\} \arrow[rr, "H|_{E_1 \times \{t\}}"] \arrow[dr, "p_1|_{E_1 \times \{t\}}"'] & & E_2 \arrow[dl, "p_2"] \\ & X & \end{tikzcd}$$

Definition 14.13. Let $X \in \mathbf{Top}$ and $E_1, E_2 \in \mathbf{Top}_X$. An object $f : E_1 \to E_2$ in $\mathbf{Top}_X$ is homotopy equivalent if there exists another object $g : E_2 \to E_1$ in $\mathbf{Top}_X$ such that

$$g \circ f \sim \mathrm{Id}_{E_1}, \qquad f \circ g \sim \mathrm{Id}_{E_2},$$

in $\mathbf{Top}_X$. The maps f and g are called fibre homotopy equivalences.

The following result will be useful later on:

Proposition 14.14. *Let $X \in \mathbf{Top}$. Let $p_1 : E_1 \to X$ and $p_2 : E_2 \to X$ be fibrations in $\mathbf{Top}_X$. Let $f : E_1 \to E_2$ be a map such that $p_2 \circ f = p_1$. Suppose that f is a homotopy equivalence in $\mathbf{Top}$. Then f is a fiber homotopy equivalence in $\mathbf{Top}_X$.*

Proof. The proof is skipped. □

14.4. **Characterization of Fibrations.** We end with a criterion that allows us to recognize Hurewicz fibrations. The criterion will also allow us to deduce that covering spaces and fiber bundles over nice spaces are Hurewicz fibrations.

Definition 14.15. Let $\mathcal{U}$ be an open cover of $X \in \mathbf{Top}$. We say that $\mathcal{U}$ is numerable if there are maps $\lambda_U : X \to I$ for each $U \in \mathcal{U}$ such that $\lambda_U^{-1}((0,1]) = U$.

Proposition 14.16. *Let $p : E \to X$ be a continuous surjective map and let $\mathcal{U}$ be a locally finite numerable open cover of X. Then p is a Hurewicz fibration if and only if $p|_U : p^{-1}(U) \to U$ is a Hurewicz fibration for every $U \in \mathcal{U}$.*

Proof. The proof can be found in [May99]. We will see in Proposition 15.4 that fiber bundles are Serre fibrations. This suffices for most purposes. □

15. Fibre & Principal Bundles

In this section, we discuss fiber bundles providing the key definitions required to introduce some important examples of interest. Our primary interest in fiber bundles arises from the fact that important examples of Serre fibrations are given by fiber bundles.

15.1. Definitions.

Definition 15.1. Let $E, X, F \in \mathbf{Top}$. A continuous surjective map $p : E \to X$ is a F-fibre bundle if it satisfies the following conditions:

(1) There is an open cover $\{U_\alpha\}_\alpha$
(2) There are homeoomorphisms $\varphi_\alpha : p^{-1}(U_\alpha) \to U_\alpha \times F$ such that the following diagram commutes:

$$\begin{tikzcd} p^{-1}(U_\alpha) \arrow[r, "\varphi_\alpha"] \arrow[d, "p"'] & U_\alpha \times F \arrow[dl, "\mathrm{pr}_1"] \\ U_\alpha & \end{tikzcd}$$

Remark 15.2. *If $p : E \to X$ is a continuous surjective map, we will henceforth use the term fibre bundle to refer to a F-fiber bundle when the fiber F is clear from context.*

Example 15.3. Here is a basic list of examples of fibre bundles:

(1) A trivial fibre bundle is of the form $F \times X$ with fibre X. It is clear that this is a fibre bundle since $p : F \times X \to X$ is a continuous surjective map and the following diagram commutes:

$$\begin{tikzcd} X \times F \arrow[r, "\mathrm{Id}"] \arrow[d, "p"'] & X \times F \arrow[dl, "\mathrm{pr}_1"] \\ X & \end{tikzcd}$$

(2) Let $p : E \to X$ be a covering space with a discrete fibres, F. Then $p : E \to X$ is a fibre bundle with fibre F.

(3) Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$. Let $p : E \to X$ be a rank n $\mathbb{K}$-vector bundle. Then $p : E \to X$ is a fibre bundle with fibre $\mathbb{K}^n$.

Why are we interested in fiber bundles in homotopy theory? We demonstrate that a fiber bundle is a Serre fibration.

Proposition 15.4. *Let $p : E \to X$ be a fibre bundle. Then $p : E \to X$ is a Serre fibration.*

Proof. Consider the following diagram:

$$\begin{tikzcd} I^n \times \{0\} \arrow[r, "H_0"] \arrow[d, "i_0"] & E \arrow[d, "p"'] \\ I^n \times I \arrow[r, "H"] & X \end{tikzcd}$$

Since $\pi : E \to B$ is a fiber bundle, there exists an open covering by subsets U_α such that $p^{-1}(U_\alpha) \cong U_\alpha \times F$ (over U_α). We can cover I^{n+1} by the open subsets $H^{-1}(U_\alpha)$. Since I^{n+1} is compact, the Lebesgue number lemma implies there exists a $k \in \mathbb{N}$ such that, for any sequence $(j_1, \ldots, j_n)$ of numbers $0 \le j_1, \ldots, j_n \le k-1$, the small cube

$$\left[\frac{j_1}{k}, \frac{j_1+1}{k}\right] \times \cdots \times \left[\frac{j_n}{k}, \frac{j_n+1}{k}\right]$$

is mapped by H into an open set $U_\alpha \subseteq X$. We construct the lift H incrementally, one cube at a time. Thus, we may assume that no further subdivision of I^{n+1} is necessary and that H maps I^{n+1} entirely into some U_α. Moreover, we are given H_0 defined on $I^n \times \{0\}$ that can be extended onto $\partial I^n \times I$. Consequently, we can also assume that p is the trivial fiber bundle $U_\alpha \times F$. Thus, we can construct such a lift as

$$\widetilde{H} : I^{n+1} \to U_\alpha \times F; \quad (x_1, \ldots, x_{n+1}) \mapsto \big(H(x_1, \ldots, x_{n+1}), f(x_1, \ldots, x_n)\big),$$

where f is the composition $I^{n+1} \to I^n \times \{0\} \cup \partial I^n \times I \to I^n \times \{0\} \to U_\alpha \times F \to F$. Here the first map is a deformation retraction. $\square$

Remark 15.5. *Proposition 14.16 implies that fibre bundles with paracompact base spaces are Hurewicz fibrations. Recall that a paracompact space is a topological space in which every open cover has an open refinement that is locally finite. Moreover, every open cover on a paracompact space can be shown to be numerable by working with the existence of bump functions guaranteed to exist by Urysohn's lemma.*

Let us consider some examples. We construct examples of fiber bundles (and hence Serre fibrations) via group actions. In fact, the examples we construct will be principal bundles, which are specific instances of fiber bundles. To proceed, we first define the notion of a principal bundle.

Definition 15.6. Let $p : E \to X$ be a fibre bundle with a topological group, G, as its fibre. Then $p : E \to X$ is a principal G-bundle if the following hold:

(1) There is a continuous, free group action $E \times G \to E$,
(2) For each $x \in X$, the action of G preserves the fibre E_x and the orbit map $G \to E_x$ is a homeomorphism,
(3) The locally trivalizing cover $\{U_\alpha, \varphi_\alpha\}_\alpha$ is such that each φ is G-equivariant. That is,
$$\varphi_\alpha(e \cdot g) = \varphi_\alpha(e) \cdot g$$
The group G is called the structure group of the principal G-bundle.

The examples of principal G-bundles we construct will be derived from the category of smooth manifolds. Consequently, the remainder of this section is adapted for the category of smooth manifolds. We will use the following important result:

Proposition 15.7. *Let G be a Lie group and M be a smooth manifold. A smooth, free, properly disctontnuous action of G on M induces a smooth manifold structure on M/G such that the map $M \to M/G$ is principle G-bundle.*

Proof. The proof is skipped. $\square$

Example 15.8. (**Hopf Fibrations**) We discuss the all important example of Hopf fibrations over $k = \mathbb{R}, \mathbb{C}, \mathbb{H}$.

(1) Let $\mathbb{S}^n \subseteq \mathbb{R}^{n+1}$. Let $\mathbb{Z}_2$ acts on $\mathbb{S}^n$ via
$$\mathbb{S}^n \times \mathbb{Z}_2 \to \mathbb{S}^{2n+1}, \quad (w, \pm 1) \mapsto \pm w.$$
The action is free. Since $\mathbb{Z}_2$ is compact, the action is proper as well. Proposition 15.7 implies that the $\mathbb{S}^n/\mathbb{Z}_2$ is principal $\mathbb{Z}_2$-bundle. In fact, we have
$$\mathbb{Z}_2 \to \mathbb{S}^n \to \mathbb{S}^n/\mathbb{Z}_2 \cong \mathbb{RP}^n$$
is a principal $\mathbb{Z}_2$-bundle called the real Hopf bundle. By letting $n \to \infty$, we get:
$$\mathbb{Z}_2 \to \mathbb{S}^\infty \to \mathbb{RP}^\infty$$
(2) Let $\mathbb{S}^{2n+1} \subseteq \mathbb{C}^{n+1}$ be a sphere of odd dimension. Let $\mathbb{S}^1 \cong \mathrm{U}(1) \subseteq \mathbb{C}$ acts on $\mathbb{S}^{2n+1}$ via
$$\mathbb{S}^{2n+1} \times \mathbb{S}^1 \to \mathbb{S}^{2n+1}, \quad (w, z) \mapsto wz.$$
The action is free. Since $\mathbb{S}^1$ is compact, the action is proper as well. Proposition 15.7 implies that the $\mathbb{S}^{2n+1}/\,\mathrm{U}(1)$ is principal $\mathbb{S}^1$-bundle. Hence,
$$\mathbb{S}^1 \to \mathbb{S}^{2n+1} \to \mathbb{S}^{2n+1}/\,\mathrm{U}(1) \cong \mathbb{CP}^n$$
is a principal $\mathbb{S}^1$-bundle called the complex Hopf bundle. By letting $n \to \infty$, we get:
$$\mathbb{S}^1 \to \mathbb{S}^\infty \to \mathbb{CP}^\infty$$
(3) Let $\mathbb{S}^3 \subseteq \mathbb{H}$. and $\mathbb{S}^{4n+3} \subseteq \mathbb{H}^{n+1}$. An argument as in (2) shows that
$$\mathbb{S}^3 \to \mathbb{S}^{4n+3} \to \mathbb{S}^{4n+3}/\mathbb{S}^3 \cong \mathbb{HP}^n$$
is a principal $\mathbb{S}^3$-bundle called the quarternionic Hopf bundle. By letting $n \to \infty$, we get:
$$\mathbb{S}^3 \to \mathbb{S}^\infty \to \mathbb{HP}^\infty$$

Remark 15.9. *For $n = 1$, the Hopf fibrations reduce to:*

$$\mathbb{S}^0 \to \mathbb{S}^1 \to \mathbb{S}^1,$$
$$\mathbb{S}^1 \to \mathbb{S}^3 \to \mathbb{S}^2,$$
$$\mathbb{S}^3 \to \mathbb{S}^7 \to \mathbb{S}^4.$$

There is also an octonionic fibration:

$$\mathbb{S}^7 \to \mathbb{S}^{15} \to \mathbb{S}^8,$$

but there are no higher octonionic versions of the Hopf fibrations.

Let G be a Lie group and $H \subseteq G$ is a closed Lie subgroup. The natural of H on G by right multiplication is smooth, free and proper. Hence, Proposition 15.7 implies that G/H is a H-principal bundle.

Example 15.10. (Homoegenous Spaces) The following is a list of examples of homogenous space which are principal bundles.

(1) Consider $\mathrm{O}(n-1)$ a closed subgroup acting naturally on $\mathrm{O}(n)$. Note that
$$\mathrm{O}(n)/\,\mathrm{O}(n-1) \cong \mathbb{S}^{n-1}$$
This follows from the standard transitive action of $\mathrm{O}(n)$ on $\mathbb{S}^{n-1}$, the orbit-stabilizer theorem and the characteristic property of smooth submersions. Hence,
$$\mathrm{O}(n-1) \to \mathrm{O}(n) \to \mathbb{S}^{n-1}$$
is a principal $\mathrm{O}(n-1)$-bundle.

(2) Consider $\mathrm{SO}(n-1)$ a closed subgroup acting naturally on $\mathrm{SO}(n)$. Note that
$$\mathrm{SO}(n)/\,\mathrm{SO}(n-1) \cong \mathbb{S}^{n-1}$$
This follows from the standard transitive acton of $\mathrm{SO}(n)$ on $\mathbb{S}^{n-1}$, the orbit-stabilizer theorem and the characteristic property of smooth submersions. Hence,
$$\mathrm{SO}(n-1) \to \mathrm{SO}(n) \to \mathbb{S}^{n-1}$$
is a principal $\mathrm{SO}(n-1)$-bundle.

(3) Consider $\mathrm{U}(n-1)$ a closed subgroup acting naturally on $\mathrm{U}(n)$. Note that
$$\mathrm{U}(n)/\,\mathrm{U}(n-1) \cong \mathbb{S}^{2n-1}$$
This follows from the standard transitive acton of $\mathrm{U}(n)$ on $\mathbb{S}^{2n-1}$, the orbit-stabilizer theorem and the characteristic property of smooth submersions. Hence,
$$\mathrm{U}(n-1) \to \mathrm{U}(n) \to \mathbb{S}^{2n-1}$$
is a principal $\mathrm{U}(n-1)$-bundle.

We can generalize the above examples by introducing the notion of Stiefel manifolds.

Definition 15.11. Let $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$. A k-frame in $\mathbb{K}^n$ is an ordered orthonormal set of vectors $\{v_1, \ldots, v_k\} \subseteq \mathbb{K}^n$[16]. The set of all k-frames, $V_k(\mathbb{K}^n)$, is called the Stiefel manifold.

Remark 15.12. *It can be verified that $V_k(\mathbb{K}^n)$ is a compact smooth manifold. Note that we have the following identifications:*

(1) $V_1(\mathbb{R}^n) \cong \mathbb{S}^{n-1}$

[16]Here we take the standard inner product.

(2) $V_1(\mathbb{C}^n) \cong \mathbb{S}^{2n-1}$
(3) $V_n(\mathbb{R}^n) \cong \mathrm{O}(n)$
(4) $V_n(\mathbb{C}^n) \cong \mathrm{U}(n)$

Example 15.13. Consider $\mathrm{O}(n-k)$ a closed subgroup acting naturally on $\mathrm{O}(n)$. Note that

$$\mathrm{O}(n)/\,\mathrm{O}(n-k) \cong V_k(\mathbb{R}^n)$$

The group $\mathrm{O}(n)$ acts on the set $V_k(\mathbb{R}^n)$ via

$$A \cdot (v_1, \ldots, v_k) = (Av_1, \ldots, Av_k).$$

Since the vectors $v_1, \ldots, v_k$ can be completed to form an orthonormal basis of $\mathbb{R}^n$, and $\mathrm{O}(n)$ acts transitively on orthonormal bases, it follows that the action of $\mathrm{O}(n)$ on $V_k(\mathbb{R}^n)$ is also transitive. The isotropy group of the point

$$p = (e_1, \ldots, e_k) \in V_k(\mathbb{R}^n)$$

is given by

$$\mathrm{O}(n)_p = \left\{ \begin{pmatrix} E_k & 0 \\ 0 & A \end{pmatrix} \;\middle|\; A \in \mathrm{O}(n-k) \right\} \cong \mathrm{O}(n-k).$$

The characteristic property of smooth submersions now implies that

$$V_k(\mathbb{R}^n) = O(n)/O(n-k).$$

as smooth manifolds. The discussion of homogenous spaces implies that

$$\mathrm{O}(n-k) \to \mathrm{O}(n) \to V_k(\mathbb{R}^n)$$

is a principal $\mathrm{O}(n-k)$-bundle.

Remark 15.14. *Similarly, it can be shown that*

$$V_k(\mathbb{C}^n) = \mathrm{U}(n)/\,\mathrm{U}(n-k),$$
$$V_k(\mathbb{H}^n) = \mathrm{Sp}(n)/\,\mathrm{Sp}(n-k).$$

Hence, we have additional examples:

$$\mathrm{U}(n-k) \to \mathrm{U}(n) \to V_k(\mathbb{C}^n),$$
$$\mathrm{Sp}(n-k) \to \mathrm{Sp}(n) \to V_k(\mathbb{H}^n).$$

We can also define the notion of a Grassmannian that can be used to generated additional principal G-bundles.

Example 15.15. Let $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$. The set of all k-dimensional subspaces of $\mathbb{K}^n$, $G_k(\mathbb{K}^n)$, is called the Grassmannian.

There is a natural surjection

$$p : V_k(\mathbb{K}^n) \longrightarrow G_k(\mathbb{K}^n)$$
$$\{v_1, \ldots, v_n\} \mapsto \mathrm{span}\{v_1, \ldots, v_n\}.$$

The fact that p is onto follows from the Gram-Schmidt procedure. Thus, $G_k(\mathbb{K}^n)$ is a topological space endowed with the quotient topology via p.

Example 15.16. (Sketch) Note that $\mathrm{O}(k)$ acts on $V_k(\mathbb{R}^n)$ smoothly, freely and properly discontinuously. We have that

$$V_k(\mathbb{R}^n)/\,\mathrm{O}(k) \cong G_k(\mathbb{R}^n)$$

Hence,

$$\mathrm{O}(k) \to V_k(\mathbb{R}^n) \to G_k(\mathbb{R}^n)$$

is a principal $\mathrm{O}(k)$-bundle. If we let $n \to \infty$, we get:

$$\mathrm{O}(k) \to V_k(\mathbb{R}^\infty) \to G_k(\mathbb{R}^\infty)$$

Here $G_k(\mathbb{R}^\infty)$ is the infinite Grassmannian.

Remark 15.17. *Similarly, it can be shown that we have the following examples:*

$$\mathrm{U}(k) \to V_k(\mathbb{C}^n) \to G_k(\mathbb{C}^n)$$

If we let $n \to \infty$, we get:

$$\mathrm{U}(k) \to V_k(\mathbb{C}^\infty) \to G_k(\mathbb{C}^\infty)$$

16. Based Fibrations

Fibrations discussed above are called unbased fibrations. We can now define pointed fibrations.

Definition 16.1. Let $(E, e_0), (X, x_0) \in \mathbf{Top}_*$. A pointed in $p : (E, e_0) \to (X, x_0)$ is a based fibration if all the relevant maps in Definition 14.1 are pointed maps.

Remark 16.2. *$F = p^{-1}(x_0)$ is called the pointed fiber of p over x. We have a sequence*

$$F \xrightarrow{i} E \xrightarrow{p} X$$

Remark 16.3. *Let $(X, x_0), (Y, y_0) \in \mathbf{Top}_*$. If $f : (X, x_0) \to (Y, y_0)$ is a pointed map, we redefine N_f to be*

$$N_f = X \times_f Y^I : \{(x, \gamma) \in X \times Y^I \mid f(x) = \gamma(1)\}.$$

The proof in Proposition 14.9 goes through and f can be decomposed as:

$$\begin{tikzcd} (X, x_0) \arrow[r, "i"'] \arrow[rr, bend left, "f"] & N_f \arrow[r, "p"'] & (Y, y_0) \end{tikzcd}$$

Here i is a homotopy equivalence as defined in Proposition 14.9 and p is a (Hurewicz) fibration such that

$$\begin{aligned} p\colon N_f &\to Y \\ (x, \gamma) &\mapsto \gamma(0) \end{aligned}$$

Example 16.4. Let $(X, x_0) \in \mathbf{Top}_*$ and let $f : \{x_0\} \hookrightarrow X$ denote the inclusion of a singleton. In this case, $N_f \cong P(X, x_0)$. Here N_f is as redefined in Remark 16.3. We have

$$\begin{tikzcd} \{x_0\} \arrow[r] \arrow[rr, bend left, "f"] & P(X, x_0) \arrow[r, "p"'] & (X, x_0) \end{tikzcd}$$

Here p is a fibration called the path space fibration. Note that we have $p^{-1}(x_0) = \Omega(X, x_0)$. Hence, we have the following sequence

$$\Omega(X, x_0) \to P(X, x_0) \to (X, x_0).$$

Remark 16.5. *We shall only use the phrase fibration when working with a based fibration.*

We would to generate a long exact sequence of homotopy groups associated to a continuous map. Here is the strategy. Let $f : X \to Y$ be a pointed continuous map of pointed topological spaces. Using Remark 16.3, we can decomopose f as:

$$\begin{tikzcd} X \arrow[r, "i"'] \arrow[rr, bend left, "f"] & N_f \arrow[r, "p'"'] & Y \end{tikzcd}$$

Consider the homotopy fiber:

$$\mathrm{hFiber}_f(y_0) := (p')^{-1}(y) = \{(x, \gamma) \in X \times P(Y, y_0) \mid \gamma(1) = f(x), \gamma(0) = y_0\}$$

For brevity, we write $\mathrm{hFiber}_f(y_0)$ and hFib_f. Note that hFib_f is a pullback:

$$\begin{tikzcd} \mathrm{hFib}_f \arrow[r] \arrow[d] & P(Y, y_0) \arrow[d, "\mathrm{Ev}_1"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

Therefore, Proposition 14.6 implies that the projection $p : \mathrm{hFib}_f \to X$ is a fibration. We get a sequence

$$\mathrm{hFib}_f \xrightarrow{p} X \xrightarrow{f} Y$$

We would like to iterate this construction. Since $x \in X_0$ is the basepoint of X, note that the fiber of the fibration p is

$$p^{-1}(x_0) = \{x_0\} \times \{\gamma \in P(Y, y_0) \mid \gamma(1) = f(x_0) = y_0, \gamma(0) = y_0\} \cong \Omega(Y, y_0)$$

Hence, the usual fiber of p over x_0 can be identified with $\Omega(Y, y_0)$. As before, we have an inclusion of $p^{-1}(x_0)$ into the homotopy fiber hFib_p. Since p is a fibration, this inclusion is a homotopy equivalence by Remark 14.12. Hence, we have a sequence

$$\begin{tikzcd} \Omega(Y, y_0) \arrow[r, "i"] \arrow[d, "\cong"] & \mathrm{hFib}_f \arrow[r, "p"] \arrow[d, equal] & X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[d, equal] \\ \mathrm{hFib}_p \arrow[r, "\mathrm{proj}"] & \mathrm{hFib}_f \arrow[r, "p"] & X \arrow[r, "f"] & Y \end{tikzcd}$$

Here i is the inclusion mapping $\gamma \to (x_0, \gamma)$. The left most square commutes by construction. Hence, the diagram above commutes in $\mathbf{Top}_*$. How shall we extend the sequence? The answer is given by the following result:

Lemma 16.6. *Let $(E, e_0), (X, x_0) \in \mathbf{Top}_*$ and let $f : (E, e_0) \to (X, x_0)$ be a pointed map. Let $F = p^{-1}(x_0)$ and*

$$F \xrightarrow{g} E \xrightarrow{f} X$$

be the associated fiber sequence. The homotopy fibre hFiber_g *of g is homotopy equivalent to $\Omega(X, x_0)$.*

Proof. Using Remark 16.3, decompose f as:

$$\begin{tikzcd}
(E, e_0) \arrow[rr, "i"] \arrow[dr, "f"'] & & N_f \arrow[dl, "p"] \\
& (X, x_0) &
\end{tikzcd}$$

Since i is a homotopy equivalence and f, p are fibrations, Proposition 14.14 implies that i is in fact a fiber homotopy equivalence. It follows that

$$i|_F : F \to \mathrm{hFiber}_f$$

is a homotopy equivalence. From the discussion above hFiber_f is defined as a pullback and the projection $\mathrm{hFiber}_f \to E$ is a fibration. We can furthermore decompose g as:

$$F \to N_g \to E$$

We know that N_g is also defined as a pullback, the map $F \to N_g$ is a homotopy equivalence and $N_g \to E$ is a fibration. The fiber of the fibration $N_g \to E$ is hFiber_g and the fiber of the fibration $\mathrm{hFiber}_f \to E$ is $\Omega(X, x_0)$. All in all, we have the following diagram:

$$\begin{tikzcd}
\mathrm{hFiber}_g \arrow[r, hook] \arrow[dd, dotted] & N_g \arrow[d, "\cong"] \arrow[dr, "\text{fib.}"] & \\
& F \arrow[r, "g"] \arrow[d, "\cong"] & E \\
\Omega(X, x_0) \arrow[r, hook] & \mathrm{hFiber}_f \arrow[ur, "\text{fib.}"'] &
\end{tikzcd}$$

Since the map from $N_g \to \mathrm{hFiber}_f$, Proposition 14.14 implies that the map is in fact a fiber homotopy equivalence over E. In particular, the map restricts to a homotopy equivalnce between hFiber_g and $\Omega(X, x_0)$. $\square$

We can now use Lemma 16.6 to continue to construction of the sequence.

$$\begin{tikzcd}
\Omega(X, x_0) \arrow[r, "\Omega f"] \arrow[d, "j' \circ \mathrm{inv}"] & \Omega(Y, y_0) \arrow[r, "i"] \arrow[d, "j"] & \mathrm{hFib}_f \arrow[r, "p"] \arrow[d, equal] & X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[d, equal] \\
\mathrm{hFib}_i \arrow[r, "\mathrm{proj}'"] & \mathrm{hFib}_p \arrow[r, "\mathrm{proj}"] & \mathrm{hFib}_f \arrow[r, "p"] & X \arrow[r, "f"] & Y
\end{tikzcd}$$

Here j is the homotopy equivalence discussed above and j' that exists by Lemma 16.6. Moreover, inv is the map

$$\mathrm{inv} : \Omega(X, x_0) \to \Omega(X, x_0),$$
$$\gamma \mapsto \gamma^{-1}.$$

Since $\mathrm{hFib}_p \cong \Omega(Y, y_0)$, we identify proj' to be simply the projection onto $\Omega(Y, y_0)$. We claim that the diagram above commutes in in $\mathbf{hTop}_*$. The first and second squares are clearly commutative. The third square commutes as discussed above. It suffices to consider the left most square. Let

$$k = \mathrm{proj}' \circ (j' \circ \mathrm{inv})$$

We claim that $k \sim j \circ \Omega f$. Note that we have

$$k[\gamma] = (c_{y_0}, [\gamma^{-1}])$$
$$j \circ \Omega f[\gamma] = ([f \circ \gamma], c_{x_0})$$

The desired homotopy is given by

$$H([\gamma], t) = (f(\gamma|_{[t,1]}]), [\gamma^{-1}|_{[0,t]}])$$

Iterating the above construction, we get the following sequence in $\mathbf{hTop}_*$:

$$\begin{tikzcd}
\cdots \arrow[r] & \Omega(\mathrm{hFib}_f) \arrow[r, "\Omega p_1"] \arrow[d, "\cong"] & \Omega(X, x_0) \arrow[r, "\Omega f"] \arrow[d, "\cong"] & \Omega(Y, y_0) \arrow[r, "i"] \arrow[d, "\cong"] & \mathrm{hFib}_f \arrow[r, "p"] \arrow[d, equal] & X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[d, equal] \\
\cdots \arrow[r] & \mathrm{hFib}_{\Omega f} \arrow[r] & \mathrm{hFib}_i \arrow[r] & \mathrm{hFib}_p \arrow[r] & \mathrm{hFib}_f \arrow[r] & X \arrow[r] & Y
\end{tikzcd}$$

For each pair of adjacent maps, the first is the inclusion of the homotopy fibre of the next, up to homotopy equivalence. What now? For a fixed Y, w can take the homotopy classes of maps $[Y, -]_*$, where $\cdot$ is a space in the sequence above. We need the following lemma and a definition.

Definition 16.7. A sequence of functions of pointed sets

$$(A, a) \xrightarrow{f} (B, b) \xrightarrow{g} (C, c)$$

is exact if $f(A) = g^{-1}(c)$.

Lemma 16.8. *Let $(E, e_0), (X, x_0), (Z, z_0) \in \mathbf{Top}_*$. Let $p : (E, e_0) \to (X, x_0)$ be a fibration and let $F = p^{-1}(x_0)$. The sequence*

$$F \xrightarrow{i} E \xrightarrow{p} X$$

induces an exact sequence of sets[17]*:*

$$[Z, F]_* \xrightarrow{i_\#} [Z, E]_* \xrightarrow{p_\#} [Z, X]_*$$

Proof. Let $[g] \in [Z, F]_*$. Then

$$\begin{aligned} p_\# \circ i_\#([g]) : Z &\to X \\ y &\mapsto x_0 \end{aligned}$$

and so

$$i_\#([Z, F]_*) \subseteq p_\#^{-1}([c_{x_0}])$$

where c_{x_0} is the constant map $E \to x_0$. Now, let $[f] \in p_\#^{-1}([c_{x_0}])$. So $f : Z \to E$ is such that

$$p_\#([f]) = [p \circ f] = [c_{x_0}]$$

That is $p \circ f$ is homotopic to c_{x_0}. Let $G : Z \times I \to X$ be the corresponding homotopy. Now define $H : Z \times I \to E$ via the homotopy lifting property as in the following commutative

[17]A sequence of functions of pointed sets $(A, a) \xrightarrow{f} (B, b) \xrightarrow{g} (C, c)$is exact if $f(A) = g^{-1}(c)$.

diagram.

$$\begin{tikzcd}
Z \times \{0\} \arrow[r, "f"] \arrow[d, "i_0"] & E \arrow[d, "p"] \\
Z \times I \arrow[r, "G"] \arrow[ur, dotted, "H"] & X
\end{tikzcd}$$

Then

$$p \circ H(z,1) = G(z,1) = c_{x_0}$$

Hence $H(Z,1) \subseteq F$. So $z \mapsto H(z,1)$ can be restricted to a map $f' : Z \to F$. But $H(z,0) = f(z)$, so we have

$$f \cong i \circ f'$$

That is, $[f] = i_{\#}([f'])$ and so $[f] \in i_{\#}([Y,F])$. This completes the proof. □

Let's now use Lemma 16.8 to get the following result:

Proposition 16.9. *(**Exact Puppe Sequence**) Let $(X,x_0),(Y,y_0) \in \mathbf{Top}_*$ and let $f : (X,x_0) \to (Y,y_0)$ be a pointed continuous map. The sequence*

$$\cdots \longrightarrow \Omega(\mathrm{hFib}_f) \xrightarrow{\Omega p_1} \Omega(X,x_0) \xrightarrow{\Omega f} \Omega(Y,y_0) \xrightarrow{i} \mathrm{hFib}_f \xrightarrow{p} X \xrightarrow{f} Y$$

is exact.

Proof. Let $(Z,z_0) \in \mathbf{Top}_*$. First consider

$$\mathrm{hFib} \xrightarrow{p} X \xrightarrow{f} Y$$

Instead consider the sequence

$$\mathrm{hFib}_f \xrightarrow{i} N_f \xrightarrow{p'} Y$$

Here the map p' is an honest fibration and hFib_f is the fibre of p'. For $Y \in \mathbf{Top}_*$, we can apply Lemma 16.6 to get an exact sequence of sets:

$$[Z, \mathrm{hFib}_f]_* \to [Z, N_f]_* \to [Z, Y]_*$$

However, note that $[Z, N_f]_* \cong [Z, X]_*$ since $P(Y,y_0)$ is contractible. Hence, we find that the sequence

$$\mathrm{hFib} \xrightarrow{p} X \xrightarrow{f} Y$$

is exact. Moreover,

$$\Omega^k(\mathrm{hFib}) \xrightarrow{p} \Omega^k(X,x_0) \xrightarrow{f} \Omega^k(Y,y_0)$$

is exact for each $k \geq 1$. This is because the sequence

$$[Z, \Omega^k(\mathrm{hFib})]_* \to [Z, \Omega^k(X,x_0)] \to [Z, \Omega^k(Y,y_0)]$$

can be written as

$$[\Sigma^k Z, \mathrm{hFib}]_* \to [\Sigma^k Z, (X,x_0)]_* \to [\Sigma^k Z, (Y,y_0)]_*$$

which is know to be exact. Hence, the given long exact sequence is an exact sequence. □

Part 5. Cofibrations

Part 6. Higher Homotopy Groups

As before, we adopt the following conventions from now on:

(1) We will assume that we work in the category **CGWH**.
(2) Abusing notation, we will write **CGWH** as **Top**.
(3) We will write $X \times_k Y$ simply as $X \times Y$.

Let's finally get to higher homotopy groups.

17. Definitions

In this section, we generalize the definition of the first homotopy group.

Definition 17.1. Let $(X, x_0) \in \mathbf{Top}_*$ be a path-connected pointed topological space. The n-th homotopy group of (X, x_0), denoted as $\pi_n(X, x_0)$, is defined as

$$\pi_n(X, x_0) = [(\mathbb{S}^n, *), (X, x_0)] := [\mathbb{S}^n, X]_*$$

X is or n-connected if $\pi_k(X, x_0)$ for $1 \leq k \leq n$. We say that X is weakly contractible or ∞-connected if $\pi_k(X, x_0) = 0$ for all $k \in \mathbb{N}$.

Remark 17.2. *Note that*

$$(I^n/\partial I^n, \partial I^n/\partial I^n) \cong (\mathbb{S}^n, *)$$

Hence, we have the following commutative diagram:

$$\begin{tikzcd}
(I^n, \partial I^n) \arrow[r, "f"] \arrow[d, "q"] & (X, x_0) \\
(I^n/\partial I^n, \partial I^n/\partial I^n) \arrow[ur, "g"] &
\end{tikzcd}$$

Equivalently, $\pi_n(X, x_0)$ consists of homotopy classes of maps $f : I^n \to X$ for which ∂I^n is mapped onto x_0. This is because the properties of the quotient topology imply that every f in the diagram above uniquely factors through a g in the same diagram.

Remark 17.3. *If $n = 0$, then $\pi_0(X, x_0)$ is the set of connected components of X. Indeed, we have $I^0 = \{*\}$ and $\partial I^0 = \emptyset$. Hence,*

$$\pi_0(X, x_0) = \{[y] \mid y \in X\}$$

Moreover, $[y] \sim [y']$ if and only if there is a path between y and y'. Hence, $\pi_0(X, x_0)$ consists of homotopy classes of maps from a point into the space X.

Proposition 17.4. *Let $(X, x_0) \in \mathbf{Top}_*$. Then there is a bijection of pointed sets:*

$$\pi_{n-1}(\Omega(X, x_0), c_{x_0}) \cong \pi_n(X, x_0)$$

for each $n \geq 1$. Here c_{x_0} is the constant loop at x_0.

Proof. We have:

$$\begin{aligned}
\pi_{n-1}(\Omega(X, x_0), c_{x_0}) &= \pi_0(\mathrm{Maps}((\mathbb{S}^{n-1}, *), \Omega(X, x_0))) \\
&\cong \pi_0(\mathrm{Maps}(\Sigma(\mathbb{S}^{n-1}, *), (X, x_0))) \\
&\cong \pi_0(\mathrm{Maps}((\mathbb{S}^n, *), (X, x_0))) \\
&= \pi_n(X, x_0).
\end{aligned}$$

for each $n \geq 1$. $\square$

Remark 17.5. *We can also make the following computation:*

$$\begin{aligned}\pi_n(X,x_0) &\cong \pi_0(\mathrm{Maps}((\mathbb{S}^n,*),(X,x_0)))\\ &\cong \pi_0(\mathrm{Maps}((\mathbb{S}^1,*)\wedge(\mathbb{S}^{n-1},*),(X,x_0)))\\ &\cong \pi_0(\mathrm{Maps}((\mathbb{S}^1,*),\Omega_{,}^{n-1}(X,x_0)))\\ &:= \pi_1(\Omega_{,}^{n-1}(X,x_0)).\end{aligned}$$

for each $n \geq 1$. Here we denote

$$\Omega_{,}^{n-1}(X,x_0) := [(\mathbb{S}^{n-1},*),(X,x_0)]_*$$

Note that we have

$$\begin{aligned}(\Omega^{n-1}(X,x_0),c_{x_0}) &\cong [(\mathbb{S}^0,*),\Omega^{n-1}(X,x_0)]\\ &\cong [\Sigma^{n-1}(\mathbb{S}^0,*),(X,x_0)]\\ &\cong [(\mathbb{S}^{n-1},*),(X,x_0)]\\ &:= \Omega_{,}^{n-1}(X,x_0).\end{aligned}$$

This identification follows since $\Omega^{n-1}(X,x_0)$ is a pointed topological space with the constant loop as the base point. Hence, we have

$$\pi_n(X,x_0) = \pi_1(\Omega^{n-1}(X,x_0),c_{x_0})$$

for $n \geq 1$.

Proposition 17.6. *Let $(X,x_0) \in \mathbf{Top}_*$. The set $\pi_n(X,x_0)$ forms a group for $n \geq 2$.*

Proof. This follows immediately from the fact that $\pi_n(X,x_0) = \pi_1(\Omega^{n-1}(X,x_0),c_{x_0})$ and $\pi_1(\cdot)$ is a group. We can also give a more direct argument. Consider the following map:

$$(f+g)(s_1,s_2,\ldots,s_n) = \begin{cases} f(2s_1,s_2,\ldots,s_n) & \text{if } 0 \leq s_1 \leq \frac{1}{2}\\ g(2s_1-1,s_2,\ldots,s_n) & \text{if } \frac{1}{2} \leq s_1 \leq 1\end{cases}$$

Note that since only the first coordinate is involved in this operation, the same argument used to prove that $\pi_1(X,x_0)$ is a group is valid here as well. In particular, the identity element is the constant map taking all of I^n to x_0 and the inverse element is given by

$$-f(s_1,s_2,\ldots,s_n) = f(1-s_1,s_2,\ldots,s_n).$$

This completes the proof. $\square$

The additive notation for the group operation is used because $\pi_n(X,x_0)$ is abelian for $n \geq 2$.

Lemma 17.7. ***(Eckmann–Hilton Argument)*** *Let X be a set equipped with two binary operations, $\circ$ and $\otimes$, such that:*

- *$\circ$ and $\otimes$ are both unital. That is, there are identity elements $1_\circ$ and $1_\otimes$ such that*

$$1_\circ \circ a = a = a \circ 1_\circ$$
$$1_\otimes \otimes a = a = a \otimes 1_\otimes$$

  for all $a \in X$.
- *We have,*

$$(a \otimes b) \circ (c \otimes d) = (a \circ c) \otimes (b \circ d),$$

  for all $a,b,c,d \in X$.

Then $\circ$ and $\otimes$ are the same and in fact commutative and associative.

Proof. Observe that the units of the two operations coincide:

$$1_\circ = 1_\circ \circ 1_\circ = (1_\otimes \otimes 1_\circ) \circ (1_\circ \otimes 1_\otimes) = (1_\otimes \circ 1_\circ) \otimes (1_\circ \circ 1_\otimes) = 1_\otimes \otimes 1_\otimes = 1_\otimes.$$

We denote the common identity as 1. Now, let $a, b \in X$. Then

$$a \circ b = (1 \otimes a) \circ (b \otimes 1) = (1 \circ b) \otimes (a \circ 1) = b \otimes a = (b \circ 1) \otimes (1 \circ a) = (b \otimes 1) \circ (1 \otimes a) = b \circ a.$$

This establishes that the two operations coincide and are commutative. For associativity,

$$(a \otimes b) \otimes c = (a \otimes b) \otimes (1 \otimes c) = (a \otimes 1) \otimes (b \otimes c) = a \otimes (b \otimes c).$$

This completes the proof. □

Proposition 17.8. *Let $(X, x_0) \in \mathbf{Top}_*$. If $n \geq 2$, $\pi_n(X, x_0)$ is an abelian group.*

Proof. We use Lemma 17.7. To use Lemma 17.7, we define an alternative binary operation. Let $[f], [g] \in \pi_n(X, x_0)$. Then define $[f] \times [g]$ to be the homotopy class of the map $f \times g$ defined by

$$(f \times g)(t_1, \dots, t_n) = \begin{cases} f(t_1, 2t_2, t_3, \dots, t_n) & \text{if } t_2 \in [0, 1/2], \\ g(t_1, 2t_2 - 1, t_3, \dots, t_n) & \text{if } t_2 \in [1/2, 1]. \end{cases}$$

It is clear that $\times$ is a well-defined operation on $\pi_n(X, x_0)$. Moreover, $\times$ is a unital operation with the identity element given by the constant map taking all of I^n to x_0. To make use of Lemma 17.7, it remains to prove that for any $[f], [g], [h], [k] \in \pi_n(X, x_0)$,

$$([f] \times [g]) + ([h] \times [k]) = ([f] + [h]) \times ([g] + [k]).$$

The left-hand side is defined to be the homotopy class of

$$(([f] \times [g]) + ([h] \times [k]))(t_1, \dots, t_n) = \begin{cases} f(2t_1, 2t_2, t_3, \dots, t_n) & \text{if } t_1 \leq \frac{1}{2},\ t_2 \leq \frac{1}{2}, \\ g(2t_1, 2t_2 - 1, t_3, \dots, t_n) & \text{if } t_1 \leq \frac{1}{2},\ t_2 \geq \frac{1}{2}, \\ h(2t_1 - 1, 2t_2, t_3, \dots, t_n) & \text{if } t_1 \geq \frac{1}{2},\ t_2 \leq \frac{1}{2}, \\ k(2t_1 - 1, 2t_2 - 1, t_3, \dots, t_n) & \text{if } t_1 \geq \frac{1}{2},\ t_2 \geq \frac{1}{2}. \end{cases}$$

The right-hand side is the homotopy class of

$$(([f] + [h]) \times ([g] + [k]))(t_1, \dots, t_n) = \begin{cases} f(2t_1, 2t_2, t_3, \dots, t_n) & \text{if } t_1 \leq \frac{1}{2},\ t_2 \leq \frac{1}{2}, \\ h(2t_1 - 1, 2t_2, t_3, \dots, t_n) & \text{if } t_1 \geq \frac{1}{2},\ t_2 \leq \frac{1}{2}, \\ g(2t_1, 2t_2 - 1, t_3, \dots, t_n) & \text{if } t_1 \leq \frac{1}{2},\ t_2 \geq \frac{1}{2}, \\ k(2t_1 - 1, 2t_2 - 1, t_3, \dots, t_n) & \text{if } t_1 \geq \frac{1}{2},\ t_2 \geq \frac{1}{2}. \end{cases}$$

Both these maps are exactly the same map. By Lemma 17.7, $+$ is commutative, so $\pi_n(X, x_0)$ is abelian for $n \geq 2$. □

Remark 17.9. *The proof of Proposition 17.8 makes it clear why $\pi_1(X, x_0)$ need not be abelian. We simply do not have "enough space" in $[0, 1]$ to carry out the same argument.*

We end this section is devoted to discussing various properties of higher homotopy groups that are analogous to those of the fundamental group.

Proposition 17.10. *Let $(X, x_0) \in \mathbf{Top}_*$. The following are some properties of $\pi_n(X, x_0)$ for $n \geq 1$.*

(1) *For each $x_0' \in X$, such that $x_0' \in \pi_0(x)$, we have*
$$\pi_n(X, x_0) \cong \pi_n(X, x')$$
(2) *For $n \geq 2$, π_n is a functor from* **Top**$_*$ *to* **Ab***, the category abelian of groups.*
(3) *For each $n \geq 1$, π_n preserves products. If $(Y, y) \in$* **Top**$_*$ *then*
$$\pi_n(X \times Y, (x_0, y_0)) \cong \pi_n(X, x_0) \cong \pi_n(Y, y_0)$$
That is, π_n preserves products for $n \geq 1$.
(4) *If $f : (X, x_0) \to (Y, y_0)$ is a pointed homotopy equivalence, then the induced homomorphism $f_* : \pi_n(X, x_0) \to \pi_n(Y, y_0)$ is an isomorphism.*
(5) *If $(\widetilde{X}, \tilde{x}_0) \in$* **Top** *and $p : \widetilde{X} \to X$ is a covering map, then $p_* : \pi_n(\widetilde{X}, \tilde{x}_0) \to \pi_n(X, p(\tilde{x}_0))$ is an isomorphism for all $n \geq 2$.*

Proof. The proof is given below:

(1) This follows because
$$\pi_n(X, x_0) \cong \pi_1(\Omega^{n-1}(X, x_0), c_{x_0}) \cong \pi_1(\Omega^{n-1}(X, x_0'), c_{x_0'}) \cong \pi_n(X, x_0')$$
Here we have used the fact that $\Omega^{n-1}(X, x_0)$ and $\Omega^{n-1}(X, x_0')$ are homeomorphic topological spaces.
(2) Let $\phi : (X, x_0) \to (Y, y_0)$ be a continuous map. If $f \sim g$, then $\varphi \circ f \sim \varphi \circ g$ as before. Hence, the induced map $\phi_* : \pi_n(X, x_0) \to \pi_n(Y, y)$ is well-defined. Moreover, from the definition of the group operation on π_n, it is clear that we have
$$\varphi \circ (f + g) = (\varphi \circ f) + (\varphi \circ g)$$
Thus, $\varphi_*([f + g]) = \varphi_*([f]) + \varphi_*([g])$. Hence, φ_* is a group homomorphism.
(3) The proof in the case of π_1 goes through as before.
(4) Let $g : (Y, y_0) \to (X, x_0)$ be an inverse pointed homotopy equivalence so that we have:
$$g \circ f \sim \mathrm{Id}_X \text{ rel } x_0 \quad \text{and} \quad f \circ g \sim \mathrm{Id}_Y \text{ rel } y_0.$$
Homotopy invariance gives[18] $g_* \circ f_* = (g \circ f)_* = \mathrm{Id}_{\pi_n(X,x_0)}$, and similarly $f_* \circ g_* = \mathrm{Id}_{\pi_n(Y,y_0)}$.
(5) First, we show that p_* is surjective. Let $x_0 = p(\tilde{x}_0)$ and consider $f : (\mathbb{S}^n, *) \to (X, x_0)$. Since $n \geq 2$, we have $\pi_1(\mathbb{S}^n, *) = 0$, so
$$f_*(\pi_1(\mathbb{S}^n, *)) = \{0\} \subseteq p_*(\pi_1(\tilde{X}, \tilde{x})).$$
By the path lifting criterion, f admits a lift to $(X, \tilde{x}_0)$. That is there exists $\tilde{f} : (\mathbb{S}^n, *) \to (\tilde{X}, \tilde{x}_0)$ such that $p \circ \tilde{f} = f$. Then $[f] = [p \circ f] = p_*([f])$. Next, we show that p_* is injective. Suppose $[\tilde{f}] \in \ker p_*$. So $[p \circ \tilde{f}] = 0$. Let $p \circ \tilde{f} = f$. Then $f \sim c_{x_0}$ via some homotopy
$$H_t : (\mathbb{S}^n, *) \to (X, x_0)$$
with $\varphi_1 = f$ and $\varphi_0 = c_{x_0}$. By the homotopy lifting criterion, there is a unique $\tilde{H}_t : (\mathbb{S}^n, *) \to (\tilde{X}, \tilde{x}_0)$ with $p \circ \tilde{H}_t = H_t$. Then we have $p \circ \tilde{H}_1 = H_1 = f$ and $p \circ \tilde{H}_0 = H_0 = c_{x_0}$, so by the uniqueness of lifts, we must have $\tilde{H}_1 = \tilde{f}$ and $\tilde{H}_0 = c_{\tilde{x}_0}$. Then H_t is a homotopy between $\tilde{f}$ and $c_{\tilde{x}}$. So $[\tilde{f}] = 0$. Thus, p_* is injective.

[18] It is easy to see that homotopic maps induce identical homotopic maps and hence identical maps on homotopy groups.

This completes the proof. □

Remark 17.11. *Proposition 17.10(5) can be interpreted as mentioning that covering spaces cannot be used to compute higher homotopy groups!*

How does one compute higher homotopy groups? In general, this is a difficult problem. But we can at the very least state some trivial calculations for definition and basic properties:

Proposition 17.12. *The following are calculations of some higher homotopy groups:*

(1) *If $X = \{\bullet\}$ is a one-point space, then $\pi_n(X) = \{0\}$, is the trivial abelian group for $n \geq 2$.*
(2) *If (X, x_0) is contractible, then $\pi_n(X, x_0) = \{0\}$ is the trivial abelian group for $n \geq 2$.*
(3) *We have*
$$\pi_n(\mathbb{S}^1) = \begin{cases} \mathbb{Z}, & \text{if } n = 0, 1, \\ 0, & \text{if } n \geq 2 \end{cases}$$
(4) *We have*
$$\pi_n\Big(\underbrace{\mathbb{S}^1 \times \cdots \times \mathbb{S}^1}_{k\text{-times}}, (*_1, \cdots, *_k)\Big) = \{0\}$$
for $n \geq 2$.
(5) *We have $\pi_n(\mathbb{RP}^k, *) \cong \pi_n(\mathbb{S}^k, *)$ for $n \geq 2$.*
(6) *We have $\pi_n(\mathbb{RP}^\infty, *) = 0$ for $n \geq 2$.*

Proof. The proof is given below:

(1) A one-point space has only the constant loop. Hence, each higher homotopy group is trivial.
(2) This follows from the Proposition 17.10(5) and (1) above.
(3) Consider $\mathbb{S}^1$ with its universal covering map $p : \mathbb{R} \to \mathbb{S}^1$. If $n \geq 2$, we have
$$\pi_n(\mathbb{S}^1, *) = \pi_n(\mathbb{R}, 0) = 0$$
We already know the result for $n = 0, 1$.
(4) This follows from (3) and Proposition 17.10(3). We can also apply a covering space argument as in (3).
(5) This is because $\mathbb{S}^n \to \mathbb{RP}^n$ is a covering map.
(6) This is because $\mathbb{S}^\infty \to \mathbb{RP}^\infty$ is a covering map and $\mathbb{S}^\infty$ is contractible.

This completes the proof. □

18. Cellular Approximation

Homotopy theory of CW complexes is more tractable. For instance, a key results includes the cellular approximation, which allows for approximating maps by cellular ones. We prove the cellular approximation theorem in this section. Since CW complexes are built inductively, the following strategy will not come as a surprise. Given a map $f : X \to Y$ of CW complexes, we will try to deform f cell by cell into a cellular map. As an important building block for the proof of the theorem, there is the following case of a single cell.

Lemma 18.1. *Let $X, Y \in$ **Top** such that we have a pushout diagram*

$$\begin{tikzcd}
\mathbb{S}^{n-1} \arrow[r] \arrow[d] & X \arrow[d] \\
\mathbb{D}^n \arrow[r, "\chi"] & Y
\end{tikzcd}$$

Any map $f : (\mathbb{D}^m, \mathbb{S}^{m-1}) \to (Y, X)$ with $m < n$ is homotopic relative to $\mathbb{S}^{m-1}$ to a map g satisfying $g(\mathbb{D}^m) \subseteq X$.

Let us describe the strategy of the proof. The attaching map $\chi : \mathbb{D}^n \to Y$ restricts to a homeomorphism $\chi_{\mathrm{Int}(\mathbb{D}^n)}$. Hence, we identify $\mathrm{Int}(\mathbb{D}^n) \subseteq Y$. We show that we can construct a homotopy relative to $\mathbb{S}^{m-1}$ such that $f \simeq h$ and $0 \notin h(\mathbb{D}^m)$. Here 0 is the origin of $\mathrm{Int}(\mathbb{D}^n)$. To see that this is enough, consider $Y \setminus \{0\}$. The inclusion $i : X \to Y \setminus \{0\}$ is the inclusion of a strong deformation retraction $r : Y \setminus \{0\} \to X$ induced by collapsing $\mathrm{Int}(\mathbb{D}^n) \setminus \{0\}$ onto $\mathbb{S}^{n-1}$. Hence, we have

$$\mathrm{Id}_{Y-\{0\}} \simeq i \circ r$$

relative to X which induces the desired relative homotopy

$$h = \mathrm{Id}_{Y-\{0\}} \circ h \simeq i \circ r \circ h = g$$

relative to $\mathbb{S}^{m-1}$. Putting these two homotopies together, we conclude that $f \simeq g$ relative to $\mathbb{S}^{m-1}$.

Proof. We induct on n. Let $n = 1, m = 0$. In this case, $\mathbb{S}^{m-1} = \emptyset$ and $\mathbb{D}^m = \{*\}$. A map

$$f : (\{*\}, \emptyset) \to (Y, X)$$

is essentially the same as a point $y \in Y$. For some $x \in X$, there is a path $\omega : I \to Y$ with $\omega(0) = y$ and $\omega(1) = x \in X$. This path defines the desired homotopy. Assume the claim has been prove for $n - 1$. We list the following consequences of our inductive assumption:

(1) Any map $\mathbb{S}^k \to \mathbb{S}^{n-1}$ for $k < n-1$ is homotopic to a constant map. Indeed, apply the inductive hypothesis to for $n - 1$ to the standard map $(\mathbb{D}^k, \mathbb{S}^{k-1}) \to (\mathbb{D}^{n-1} \coprod_{\mathbb{S}^{n-2}} *, *)$.
(2) Any map $\mathbb{S}^k \to \mathbb{S}^{n-1} \times (a, b)$ for $k < n-1$ is homotopic to a constant map. This follows from (1) and that (a, b) is contractible.
(3) Any map $\mathbb{S}^k \to \mathbb{S}^{n-1} \times (a, b)$ for $k < n-1$ can be extended $\mathbb{D}^k$. This follows from (2) and Proposition 10.4.

We now construct a homotopy $f \simeq h : \mathbb{D}^m \to Y$ relative to $\mathbb{S}^{m-1}$ such that $0 \notin h(\mathbb{D}^m)$.Consider the subsets

$$U_0 = \{x \in \mathbb{D}^n \mid \|x\| < 2/3\},$$
$$V_0 = \{x \in \mathbb{D}^n \mid \|x\| > 1/3\}.$$

and define two subsets of Y by setting $U = \chi(U_0)$ and $V = X \coprod_{\mathbb{S}^{n-1}} \chi(V_0)$. Note that

$$U \cap V \cong \mathbb{S}^{n-1} \times (1/3, 2/3).$$

We construct a homotopy $f \simeq h$ such that the image of h entirely lies in V, and hence avoids the point $0 \in Y$. WLOG replace $(\mathbb{D}^m, \mathbb{S}^{m-1})$ by $(I^m, \partial I^m)$. Note that $\{U, V\}$ is an open cover for Y. Pulling back the open cover of Y along f induces an open cover

$f^{-1}(U), f^{-1}(V)$ of of I^m. The Lebesgue number lemma implies there exists a $N > 0$ such that the image of each m-cube

$$I^m_{k_1,\dots,k_m} = \left[\frac{k_1}{N}, \frac{k_1+1}{N}\right] \times \cdots \times \left[\frac{k_m}{N}, \frac{k_m+1}{N}\right], \quad 0 \leq k_i < N$$

under f is contained in either U or V. We construct homotopies to modify f only on those $I^m_{k_1,\dots,k_m}$ which are not entirely mapped to V. We define a filtration on I^m,

$$\partial I^m \subseteq Z^{(-1)} \subseteq Z^{(0)} \subseteq \cdots \subseteq Z^{(m)} = I^m.$$

Let J^{-1} be the index set for all l-dimensional sub-cubes, $0 \leq l \leq m$, of all cubes $I^m_{k_1,\dots,k_m}$ which are already completely mapped to V by f. Let us denote the l-dimensional sub-cube corresponding to such an index $\varphi \in J^{-1}$ by I^l_φ. We then set

$$Z^{(-1)} = \bigcup_{\varphi \in J^{-1}} I^l_\varphi,$$

and it follows from our assumption on f that $\partial I^m \subseteq Z^{(-1)}$. We now take care of the remaining sub-cubes, and this will be done by induction over the dimension of these sub-cubes. For each $0 \leq k \leq m$, let J_k be the index set for all k-dimensional sub-cubes, I^k_φ for $\varphi \in J_k$, of the cubes $I^m_{k_1,\dots,k_m}$ which satisfy $f(I^k_\varphi) \nsubseteq V$. Set

$$Z^{(k)} = Z^{(k-1)} \coprod \bigcup_{\varphi \in J_k} I^k_\varphi.$$

This defines a filtration for X. We now want to inductively construct maps $h_k : Z^{(k)} \to Y$, $k \geq -1$, such that:

- The map h_{-1} is obtained from f by restriction.
- The map h_k sends the cubes I^k_φ to $U \cap V$ for all $\varphi \in J_k$ and $k \geq 0$.
- The map h_k extends h_{k-1}, i.e., we have $h_k|_{Z^{(k-1)}} = h_{k-1}$ for all $k \geq 0$.

For h_0, note that $Z^{(0)}$ is obtained from $Z^{(-1)}$ by possibly adding some vertices which are mapped to U. For each such vertex, choose a path to a point in $U \cap V$. This defines h_0. Inductively, assume that h_{k-1} has already been constructed. For each $\varphi \in J_k$, we have that $h_{k-1}(\partial I^\varphi_k) \subseteq U \cap V$. Since $U \cap V \cong \mathbb{S}^{n-1} \times (1/3, 2/3)$, the induction hypothesis and (3) above implies that we can find extensions as indicated in the following diagram:

$$\begin{tikzcd}
\partial I^k_\varphi \arrow[r, "h_{k-1}"] \arrow[d] & U \cap V \\
I^k_\varphi \arrow[ur, dotted, "h_{k,\varphi}"'] &
\end{tikzcd}$$

These maps $h_{k,\varphi}$ and h_{k-1} can be assembled together in order to define a map $h_k : Z^{(k)} \to Y$ with the desired properties. If we set $h = h_m : I^m = X^{(m)} \to Y$, then we have $h(I^m) \subseteq V$. Hence, it suffices to show that $f \cong h$ relative to ∂I^m. We show that in fact we construct such a homotopy relative to $Z^{(-1)}$. Both maps f and h coincide on $Z^{(-1)}$. Moreover, the restrictions of both maps to $I^m - Z^{(-1)}$ can be considered as maps taking values in U. But, U is homeomorphic to an open n-disc, hence convex, so that the two restrictions are homotopic via linear homotopies. This homotopy, together with the constant homotopy on $Z^{(-1)}$, can be assembled together to give the desired homotopy $f \sim h$ relative to $Z^{(-1)}$. $\square$

We now state and prove the cellular approximation theorem.

Proposition 18.2. *Let (X, A) be a finite-dimensional CW pair, let Y be a CW complex. If $f : X \to Y$ is a continuous map such that $f|_A : A \to Y$ is a cellular map, then f is homotopic to a cellular map $g : X \to Y$ relative to A. In particular, any map of CW complexes is homotopic to a cellular map.*

Proof. We have a filtration

$$A = X^{(-1)} \subseteq X^{(0)} \subseteq X^{(1)} \subseteq \ldots X^{(m)} = X,$$

Similarly, we also have a filtration for Y. Let $g_{-1} = f$. We construct maps $g_n : X \to Y$ and homotopies $g_{n-1} \simeq g_n$ such that

(1) The map g_n sends the relative n-cells to $Y^{(n)}$.
(2) The homotopy $g_{n-1} \simeq g_n$ is relative to $X^{(n-1)}$.

We proceed by induction. Let J_n denoting the set of relative n-cells. We have a pushout diagram:

$$\begin{tikzcd} \coprod_{\sigma \in J_n} \partial \mathbb{D}^n_\sigma \arrow[r] \arrow[d] & X^{(n-1)} \arrow[d] \\ \coprod_{\sigma \in J_n} \mathbb{D}^n_\sigma \arrow[r] & X^{(n)}. \end{tikzcd}$$

For each such cell $\mathbb{D}^n_\sigma$ such that $g_{n-1}(\mathbb{D}^n_\sigma)$ is not contained in $Y^{(n-1)}$, there is a finite relative subcomplex Y' with

$$Y^{(n)} \subseteq Y' \subseteq Y$$

such that $g_{n-1}(\mathbb{D}^n_\sigma) \subseteq Y'$. Choose a cell of maximal dimension in Y' which has a nontrivial intersection with $g_{n-1}(\mathbb{D}^n_\sigma)$. By Lemma 18.1, this cell can be avoided up to relative homotopy. Repeating this finitely many times and gluing the relative homotopies together, we obtain a homotopy $H_{n,\sigma} : g_{n-1} \simeq g_{n,\sigma} : \mathbb{D}^n_\sigma \to Y$ relative to $\partial \mathbb{D}^n_\sigma$ such that $g_{n,\sigma}(\mathbb{D}^n_\sigma) \subseteq Y^{(n)}$. Recalling that $X^{(n)} \times [0, 1]$ carries the quotient topology with respect to the map

$$\left(X^{(n-1)} \sqcup \left(\coprod_{\sigma \in J_n} \mathbb{D}^n_\sigma \right) \right) \times [0, 1] \longrightarrow X^{(n)} \times [0, 1].$$

We can glue the homotopies $H_{n,\sigma}$, the constant homotopies on $g_{n-1} : \mathbb{D}^n_\sigma \to Y$ for all n-cells with $g_{n-1}(e^n_\sigma) \subseteq Y^{(n)}$, and the constant homotopy on $g_{n-1}|_{X^{(n-1)}}$ together in order to obtain a homotopy

$$\widetilde{H}_n : X^{(n)} \times [0, 1] \to Y.$$

relative to $X^{(n)}$. Since $X^{(n)} \to X$ is a cofibation, we obtain a homotopy

$$H_n : X \times [0, 1] \to Y.$$

which admits a solution since the inclusion $X^{(n)} \to X$ is a cofibration. We set $g_n = H_n(-, 1)$. Since $X^{(m)} = X$, it suffices to compose the finitely many homotopies H_k, $0 \leq k \leq n$, to obtain a homotopy $H : f \simeq g = g_n$ relative to A such that $g : X \to Y$ is a cellular map. □

Remark 18.3. *Proposition 18.2 can be extended to the case where X is infinite-dimensional.*

19. Relative Homotopy Groups

We define relative homotopy groups. We also state and prove the long exact sequence in homotopy groups, which is a crucial tool for computations.

Definition 19.1. Let $(X, A, x_0) \in \mathbf{Top}_2^*$ such that A contains the basepoint x_0. For $n \geq 1$, the n-th relative homotopy group, denoted as $\pi_n(X, A, x_0)$, is defined as

$$\pi_n(X, A, x_0) = [(\mathbb{D}^n, \mathbb{S}^{n-1}, *), (X, A, x_0)]$$

We say (X, A, x_0) is n-connected if $\pi_k(X, A, x_0)$ for $0 \leq k \leq n$.

One can think about relative homotopy groups differently. Consider the relative path space

$$P(X, A, x_0) = \{\gamma \in P(X, x_0) \mid \gamma(1) \in A\}.$$

Note that $P(X, A, x_0)$ is a based topological sapce with basepoint the constant path at $I \to x_0$. Consider $\pi_{n-1}(P(X, A, x_0), c_{x_0})$ An element of $\pi_{n-1}(P(X, A, x_0), c_{x_0})$ is a map $(\mathbb{S}^{n-1}, *) \to P(X, A, c_{x_0})$ up to homotopy that sends $*$ to c_{x_0}. Equivalently, it is a map $(\mathbb{S}^{n-1}, *) \times I \to X$ up to homotopy that:

1. Maps $(\mathbb{S}^{n-1}, *) \times \{0\}$ to x_0, as every path starts at x_0.
2. Maps $\{*\} \times I$ to c_{x_0}
3. Maps $(\mathbb{S}^{n-1}, *) \times \{1\}$ into A as paths end in A.

This is nothing but the data of a map

$$(\mathbb{D}^n, \mathbb{S}^{n-1}, *) \to (X, A, x_0)$$

defined up to homotopy. Hence,

$$\pi_n(X, A, x_0) = \pi_{n-1}(P(X, A, x_0), c_{x_0})$$

This shows that $\pi_n(X, A, x_0)$ is a group for $n \geq 2$ and an abelian group for $n \geq 3$. We have a long exact sequence in relative homotopy groups:

Proposition 19.2. *Let $(X, A, x_0) \in \mathbf{Top}_*^2$ such that A is a closed subspace. Then there is an exact sequence of homotopy groups:*

$$\cdots \to \pi_n(A, x_0) \to \pi_n(X, x_0) \to \pi_n(X, A, x_0) \to \pi_{n-1}(A, x_0) \to \cdots$$

Proof. Consider the inclusion map $i : A \to X$. The homotopy fiber of i is $P(X, A, x_0)$. Note that

$$\begin{aligned} [(\mathbb{S}^0, *), \Omega^n(P(X, A, x_0), c_{x_0})] &\cong [\Sigma^n(\mathbb{S}^0, *), (P(X, A, x_0), c_{x_0})] \\ &\cong [(\mathbb{S}^n, *), (P(X, A, x_0), c_{x_0})] \\ &= \pi_n(P(X, A, x_0), c_{x_0}) = \pi_{n+1}(X, A, x_0).d \end{aligned}$$

The claim now follows by letting $(Z, z_0) = (\mathbb{S}^0, *)$ and applying $[(\mathbb{S}^0, *), -]$ to the exact sequence in Proposition 16.9. □

Remark 19.3. *Using Proposition 19.2 and some algebraic manipulations, one can show if $(B \subset A \subset X) \in \mathbf{Top}_*^3$ such that $B \subseteq A \subseteq A$ and B contains the base point x_0, then there is a long exact sequence:*

$$\cdots \to \pi_n(A, B, x_0) \to \pi_n(X, B, x_0) \to \pi_n(X, A, x_0) \to \pi_{n-1}(A, B, x_0) \to \cdots$$

20. Freudenthal's Suspension Theorem

The purpose of this section is to prove Freudenthal's suspension theorem. We first state a notion of the excision theorem for homotopy groups. Recall that a remarkable fact about homology groups is that the relative homology groups satisfy the excision property. However, this is not the case for relative homotopy groups. However, there is a version of excision that holds for CW complexes that have rather strong connectedness properties.

Proposition 20.1. *Let X be a CW complex such that $X = A \cup B$ and $A \cap B$ is non-empty and connected. If $(A, A \cap B)$ is k-connected, $(B, A \cap B)$ is l-connected, and $i : (A, A \cap B) \to (X, B)$ is the inclusion map, then for any $x_0 \in A \cap B$, the induced map*

$$i_* : \pi_n(A, A \cap B, x_0) \to \pi_n(X, B, x_0)$$

is an isomorphism when $n < k + l$ and is a surjection when $n = k + l$.

Proof. The proof is lengthy and technical. See [May99; Hat02]. □

We now state and prove Freudenthal's Suspension Theorem:

Proposition 20.2. *Let (X, x_0) be an $(k-1)$-connected pointed CW complex. For any map $f : \mathbb{S}^k \to (X, x_0)$, consider its suspension,*

$$\Sigma f : \Sigma \mathbb{S}^n = \mathbb{S}^{n+1} \to \Sigma(X, x_0).$$

The assignment

$$\begin{gathered} \pi_n((X, x_0)) \to \pi_{n+1}(\Sigma(X, x_0)) \\ [f] \mapsto [\Sigma f] \end{gathered}$$

is a homomorphism which is an isomorphism for $n < 2k - 1$ and a surjection for $n = 2k - 1$.

Proof. We can think of ΣX as two copies of CX, which we call C^+X and C^-X, identified along their bases. Define j to be the composition of the following three maps drawn in the diagram below, each of which is induced by the obvious inclusion maps.

$$\pi_n(X, x_0) \xrightarrow{\cong} \pi_{n+1}(C^+X, X, x_0) \to \pi_{n+1}(\Sigma X, C^-X, x_0) \xrightarrow{\cong} \pi_{n+1}(\Sigma X, x_0)$$

For any n, the leftmost and rightmost maps are isomorphisms because of the long exact sequences of the CW pairs (C^+X, X) and $(\Sigma X, C^-X)$, respectively, since $\pi_n(C^\pm X)$ is always trivial. Also, when X is $(k-1)$-connected, the CW pair $(C^\pm X, X)$ is k-connected by the long exact sequence of the pair $(C^\pm X, X)$. This allows us to apply Proposition 20.1, so the middle map in the diagram above is an isomorphism when $n + 1 < 2k$ and a surjection when $n + 1 = 2k$. □

Remark 20.3. *Let X be a k-connected CW complex. For any $n \in \mathbb{N}$, consider the sequence of maps:*

$$\cdots \to \pi_n(X, x_0) \to \pi_{n+1}(\Sigma X, x_0) \to \pi_{n+2}(\Sigma^2 X, x_0) \to \dots$$

Since X is an k-connected CW complex, the Proposition 20.2 implies that us that $\pi_n(X, x_0) \cong \pi_{n+1}(\Sigma X, x_0)$ $n < 2k + 1$. We make the following observations:

(1) *In particular,*

$$\pi_n(\Sigma X, x_0) = \pi_{n-1}(X, x_0) = 0$$

if $0 < n \leq k + 1$. If X is 0-connected (path-connected) then ΣX is also 0-connected (path-connected). Hence, we have that

$$X \text{ is } k\text{-connected} \Longrightarrow \Sigma X \text{ is } (k+1)\text{-connected}$$

More generally, we have that $\Sigma^n X$ is $(k+n)$-connected for any $n \in \mathbb{N}$.

(2) *Let $N(k,n) = n - 1 - 2k$, and observe that when $i > N(k,n)$, we have $n + i < 2(k+i)+1$. Thus, the groups $\pi_{n+i}(\Sigma^i X)$ are isomorphic for all $i > N(k,n)$. Let $N = N(n,k)$, and define $\pi_{n+N}(\Sigma^N X, x_0)$ as the n-th stable homotopy group of X.*

*More generally, the n-th stable homotopy group of any $X \in$ **Top**$_*$,*

$$\pi_n^s(X) := \varinjlim_{k \in \mathbb{N}} \pi_{n+k}(\Sigma^k X, x_0).$$

Observe that since ΣX is always 0-connected, we do not actually need X to be 0-connected, so every space X has n-th stable homotopy groups for all $n \in \mathbb{N}$. Moreover, Proposition 20.2 proves this colimit is realized after finitely many elements along the sequence. This is the start of the subject of stable homotopy theory.

21. Some Computations

The purpose of this section is to compute the homotopy groups. Our main computational tool will be the long exact sequence associated with a fibration, as proved below. We begin by considering a basic example.

Example 21.1. (**Spheres**) Let $k \geq 2$. We compute $\pi_n(\mathbb{S}^k, *)$.

(1) Let $1 \leq n < k$. Let $\mathbb{S}^n \to \mathbb{S}^k$ be a continuous map. WLOG, we can assume that f is a cellular map by Proposition 18.2. If $\mathbb{S}^k$ is given the standard cellular structure with a 0-cell and a k-cell, then the n-th skeleton of $\mathbb{S}^k$ for $n < k$ is simply the 0-cell. Therefore, $f : \mathbb{S}^n \to \mathbb{S}^k$ is homotopic to the constant map. Hence, $\pi_n(\mathbb{S}^k, *) = 0$ for $1 \leq n < k$.

(2) Let $n = k$. Since $\mathbb{S}^n$ is $(n-1)$-connected by (1), Proposition 20.2 implies that $\pi_j(\mathbb{S}^n, *) \to \pi_{j+1}(\mathbb{S}^{n+1}, *)$ is an isomorphism for $j < 2n - 1$. Therefore,

$$\pi_n(\mathbb{S}^n, *) \to \pi_{n+1}(\mathbb{S}^{n+1}, *)$$

for $n \geq 2$. Moreover, Proposition 20.2 implies that $\mathbb{Z} \cong \pi_1(\mathbb{S}^1, *) \to \pi_2(\mathbb{S}^2, *)$ is surjective. We will show that in Example 21.5 that in fact this map is an isomorphism since $\pi_2(\mathbb{S}^2, *) \cong \mathbb{Z}$. Therefore, we have

$$\mathbb{Z} \cong \pi_2(\mathbb{S}^2, *) \cong \pi_3(\mathbb{S}^3, *) \cong \pi_4(\mathbb{S}^4, *) \cong \cdot$$

That is, $\pi_n(\mathbb{S}^n, *) \cong \mathbb{Z}$.

Hence, for $k \geq 2$ we have

$$\pi_n(\mathbb{S}^k) = \begin{cases} 0, & \text{if } 1 \leq n < k, \\ \mathbb{Z}, & \text{if } n = k. \end{cases}$$

Remark 21.2. *Example 21.1 implies that it remains to compute $\pi_n(\mathbb{S}^k)$ for $n > k$. This turns out to be a very difficult problem.*

Remark 21.3. *We can now also argue that homotopy groups are not a perfect topological invariant. Consider*

$$X = \mathbb{S}^2 \times \mathbb{RP}^3, \quad Y = \mathbb{RP}^2 \times \mathbb{S}^3.$$

By Example 21.1, we have:

$$\pi_1(X) \cong \pi_1(\mathbb{S}^2) \times \pi_1(\mathbb{RP}^3) \cong 0 \times \mathbb{Z}_2 \cong \mathbb{Z}_2$$
$$\pi_1(Y) \cong \pi_1(\mathbb{RP}^2) \times \pi_1(\mathbb{S}^3) \cong \mathbb{Z}_2 \times 0 \cong \mathbb{Z}_2$$

The universal cover of both X and Y is homeomorphic to $\mathbb{S}^5$. Hence,

$$\pi_n(X) \cong \pi_n(Y)$$

for $n \geq 2$. Hence, X and Y have same homotopy groups. However, X and Y are not homotopy equivalent. Indeed,

$$H_5(X) = \mathbb{Z}, \quad H_5(Y) = 0$$

since X is compact and orientable, and Y is compact and non-orientable.

We now establish the long exact sequence associated with a fibration. Notably, we can leverage the properties of fibrations along with the exact Puppe sequence (Proposition 16.9) to derive a long exact sequence of homotopy groups.

Proposition 21.4. *Let $(E, e_0), (X, x_0) \in$ **Top**$_*$ and let $p : (E, e_0) \to (X, x_0)$ be a based fibration. Let $F = p^{-1}(x_0)$. Then we have an exact sequence of homotopy groups:*

$$\cdots \to \pi_n(F, e_0) \to \pi_n(E, e_0) \to \pi_n(X, x_0) \to \pi_{n-1}(F, e_0) \to \pi_{n-1}(E, e_0) \to \pi_{n-1}(X, e_0) \to \cdots$$

Proof. Since p is a fibration, we have that the homotopy fiber of p is homotopy equivalent to F. Observe that

$$\begin{aligned}
[(\mathbb{S}^0, *), \Omega^n(X, x_0)]_* &\cong [\Sigma^n(\mathbb{S}^0, *), (X, x_0)]_* \\
&\cong [(\mathbb{S}^n, *), (X, x_0)]_* \\
&= \pi_n(X, x_0)
\end{aligned}$$

The claim now follows by letting $(Z, z_0) = (\mathbb{S}^0, *)$ and applying $[(\mathbb{S}^0, *), -]$ to the exact sequence in Proposition 16.9. $\square$

We can perform further computations by exploiting the long exact sequence of homotopy groups associated to fibrations.

Example 21.5. (**Hopf Fibration**) Consider the Hopf Fibration:

$$\mathbb{S}^1 \to \mathbb{S}^3 \to \mathbb{S}^2$$

Barring π_0, the long exact sequence reads:

$$\cdots \to 0 \to \pi_3(\mathbb{S}^3) \to \pi_3(\mathbb{S}^2) \to 0 \to 0 \to \pi_2(\mathbb{S}^2) \to \mathbb{Z} \to 0 \to 0$$

Hence,

$$\begin{aligned}
&\pi_2(\mathbb{S}^2, *) \cong \mathbb{Z} \\
&\pi_n(\mathbb{S}^3, *) \cong \pi_n(\mathbb{S}^2, *), \quad n \geq 2
\end{aligned}$$

In particular,

$$\pi_3(\mathbb{S}^2, *) \cong \mathbb{Z}$$

Remark 21.6. *The computation in Example 21.5 is independent of the computation in Example 21.1. This shows that the claim $\pi_n(\mathbb{S}^n, *) \cong \mathbb{Z}$ made in Example 21.1 is correct.*

Example 21.7. Consider the fibration

$$\mathbb{S}^1 \to \mathbb{S}^\infty \to \mathbb{CP}^\infty$$

Barring π_0, the long exact sequence reads:

$$\cdots 0 \to \pi_3(\mathbb{CP}^\infty) \to 0 \to \pi_2(\mathbb{CP}^\infty) \to \mathbb{Z} \to 0 \to \pi_1(\mathbb{CP}^\infty) \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}$$

Hence,

$$\pi_n(\mathbb{CP}^\infty) \cong \begin{cases} \mathbb{Z} & k = 0, 2 \\ 0 & \text{otherwise} \end{cases}$$

Example 21.8. (**Special Orhthogonal Group**) Consider the fibration:

$$\mathrm{SO}(k-1) \to \mathrm{SO}(k) \to \mathbb{S}^{k-1}$$

We have a long exact sequence of homotopy groups:

$$\cdots \to \pi_n(\mathrm{SO}(k-1)) \to \pi_n(\mathrm{SO}(k)) \to \pi_n(\mathbb{S}^{k-1}) \to \pi_{n-1}(\mathrm{SO}(k-1)) \to \pi_{n-1}(\mathrm{SO}(k)) \to \pi_{n-1}(\mathbb{S}^{k-1}) \to \cdots$$

For $n \leq k-3$, we have

$$\cdots \to \pi_n(\mathrm{SO}(k-1)) \to \pi_n(\mathrm{SO}(k)) \to 0 \to \cdots$$

This implies that

$$\pi_n(\mathrm{SO}(k)) \cong \pi_n(\mathrm{SO}(k-1))$$

for $k \geq n+3$. This isomorphism doesn't hold for $k = n+2$ ($n = k-2$). Indeed, if $k = 3$ and $n = 1$, we have

$$\mathbb{Z}_2 \cong \pi_1(\mathbb{RP}^3) \cong \pi_1(\mathrm{SO}(3)) \not\cong \pi_1(\mathrm{SO}(2)) \cong \pi_1(\mathbb{S}^1) \cong \mathbb{Z}$$

In any case if $k \geq n+3$, we have

$$\pi_n(\mathrm{SO}(k)) \cong \pi_n(\mathrm{SO}(k-1)) \cong \pi_n(\mathrm{SO}(n+2))$$

In particular, we have

$$\pi_0(\mathrm{SO}(k)) \cong \mathbb{Z}, \qquad \pi_1(\mathrm{SO}(k)) \cong \begin{cases} 0 & k = 1 \\ \mathbb{Z} & k = 2 \\ \mathbb{Z}_2 & k = 3 \\ \mathbb{Z}_2 & k \geq 4 \end{cases}$$

Example 21.9. (**Unitary Group**) Consider the fibration:

$$\mathrm{U}(k-1) \to \mathrm{U}(k) \to \mathbb{S}^{2k-1}$$

We have a long exact sequence of homotopy groups:

$$\cdots \to \pi_n(\mathrm{U}(k-1)) \to \pi_n(\mathrm{U}(k)) \to \pi_n(\mathbb{S}^{2k-1}) \to \pi_{n-1}(\mathrm{U}(k-1)) \to \pi_{n-1}(\mathrm{U}(k)) \to \pi_{n-1}(\mathbb{S}^{2k-1}) \to \cdots$$

For $n \leq 2k-3$, we have

$$\cdots \to \pi_n(\mathrm{U}(k-1)) \to \pi_n(\mathrm{U}(k)) \to 0 \to \cdots$$

This implies that

$$\pi_n(\mathrm{U}(k)) \cong \pi_n(\mathrm{U}(k-1))$$

for $n \leq 2k-3$. This isomorphism doesn't hold for $n = 2k-1$. Indeed, if $k = 2$ and $n = 3$, we have

$$\begin{aligned}\mathbb{Z} &\cong \pi_3(\mathbb{S}^3) \\ &\cong \pi_3(\mathrm{SU}(2)) \\ &\cong \pi_3(\mathrm{SU}(2)) \times \pi_3(\mathbb{S}^1) \\ &\cong \pi_3(\mathrm{SU}(2) \times \mathbb{S}^1) \\ &\cong \pi_3(\mathrm{U}(2)) \not\cong \pi_3(\mathrm{U}(1)) \\ &\cong \pi_3(\mathbb{S}^1) \cong 0.\end{aligned}$$

In any case if $k \geq \lceil (n+3)/2 \rceil$, we have

$$\pi_n(\mathrm{U}(k)) \cong \pi_n(\mathrm{U}(k-1)) \cong \pi_n(\mathrm{U}(\lceil (n+3)/2 \rceil - 1))$$

In particular, we have

$$\pi_0(\mathrm{U}(k)) \cong \mathbb{Z}, \qquad \pi_1(\mathrm{U}(k)) \cong \begin{cases} \mathbb{Z} & k = 1 \\ \mathbb{Z} & k \geq 2 \end{cases} \qquad \pi_2(\mathrm{U}(k)) \cong \begin{cases} 0 & k = 1 \\ 0 & k = 2 \\ 0 & k \geq 3 \end{cases} \qquad \pi_3(\mathrm{U}(k)) \cong \begin{cases} 0 & k = 1 \\ \mathbb{Z} & k = 2 \\ \mathbb{Z} & k \geq 3 \end{cases}$$

22. Classification of Principal G-bundles

We can use homotopy theory to provide a homotopy-theoretic classification of principal G-bundles. More precisely, we show that the functor taking a topological space to the set of principal G-bundles on it is representable in the homotopy category. Recall that a principal G-bundle is defined as follows:

Definition 22.1. Let $E, X \in \mathbf{Top}$ such that $p : E \to X$ be a fibre bundle with a topological group, G, as its fibre. Then $p : E \to X$ is a principal G-bundle if the following hold:

(1) There is a continuous, free group action $E \times G \to E$,
(2) For each $x \in X$, the action of G preserves the fibre E_x and the orbit map $G \to E_x$ is a homeomorphism,
(3) The locally trivalizing cover $\{U_\alpha, \varphi_\alpha\}_\alpha$ is such that each φ is G-equivariant. That is,
$$\varphi_\alpha(e \cdot g) = \varphi_\alpha(e) \cdot g$$
The group G is called the structure group of the principal G-bundle.

We now define the notion of morphisms of principal G-bundles.

Definition 22.2. Let $p_i : E_i \to X_i$ be two principal G-bundles. A morphism of principal G-bundles is given by a pair of smooth functions $f : E_1 \to E_2$ and $g : X_1 \to X_2$ such f is a G-equivariant map and the diagram

$$\begin{tikzcd} E_1 \arrow[r, "f"] \arrow[d, "1"'] & E_2 \arrow[d, "p_2"] \\ X_1 \arrow[r, "g"'] & X_2 \end{tikzcd}$$

commutes. A morphism of principal G-bundles is an isomorphism of principal G-bundles if f, g are diffeomorphisms.

Remark 22.3. *If $p_i : E_i \to X_i$ be two principal G-bundles for $i = 1, 2$, any G-equivariant map $f : E_1 \to E_2$ defines a morphism of principal G-bundles. This is because*

$$f(e_1 \cdot g) = g(e_1) \cdot g$$

implies that f maps fibers of E_1 to fibers of E_2. Hence, defining

$$\begin{aligned} g : X_1 &\to X_2 \\ x_1 &\mapsto p_2(f(e_1)) \end{aligned}$$

is well-defined for any choice of $e_1 \in p_1^{-1}(x_1)$ and uniquely determines the base map $g : X_1 \to X_2$.

Remark 22.4. *An important special case we will consider is when $X_1 = X_2 = X$. In this case, $g = \mathrm{Id}_X$.*

Before stating the classification theorem, we need to introduce some constructions of principal G-bundles, with the first important one being the pullback construction.

Example 22.5. The following are examples of constructions of principal G-bundles.

(1) Let $X_1, X_2 \in \mathbf{Top}$ and let $p_i : E_i \to X_i$ be a principal G-bundles for $i = 1, 2$. Then

$$p_1 \times p_2 : E_1 \times E_2 \to X_1 \times X_2$$

is a principal G-bundle. Indeed, $p_1 \times p_2$ is a fibre bundle. This is in my other notes. The group action on $E_1 \times E_2$ is given by the diagonal action:

$$(e_1, e_2) \cdot g = (e_1 \cdot g, e_2 \cdot g)$$

It is clear that Definition 22.1 is satisfied.

(2) Let $X \in \mathbf{Top}$ and let $p : E \to X$ be a principal G-bundle. If $X' \subseteq X$ is a subspace of X, then

$$p|_{X'} : p^{-1}(X') \to X'$$

is a principal G-bundle. Indeed, $p|_{X'}$ is a fibre bundle. This is in my other notes. Moreover, Definition 22.1 is easily satisfied.

(3) **(Pullback)** Let $X, Y \in \mathbf{Top}$ and let $p : E \to Y$ be a principal G-bundle. Consider a continuous map $f : X \to Y$. We construct a principal G-bundle $f^*p : f^*E \to X$ that fits into the following commutative diagram:

$$\begin{tikzcd} f^*E \arrow[r, "\pi_2"] \arrow[d, "f^*p"'] & E \arrow[d, "p"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

Consider

$$f^*E := \{(x, e) \in X \times E \mid f(x) = p(e)\} := X \times_Y E$$

We endow f^*E with the subspace topology of the product topology. Let f^*p and π_2 be projections onto first and second factors respectively. Consider the product principal G-bundle.

$$\mathrm{Id}_X \times p : X \times E \to X \times Y$$

Consider the graph of f:

$$\Gamma_f = \{(x, y) \in X \times Y \mid y = f(x)\} \subseteq X \times Y$$

Note that we have

$$(x,e) \in (\mathrm{Id}_X \times p)^{-1}(\Gamma_f) \iff f(x) = p(e).$$

Hence, the inverse image of Γ_f is f^*E. This shows that f^*E is a principal G-bundle. Uniqueness follows from categorical nonsense.

We now prove the important fact the pullbacks of principal G-bundles along homotopic maps are isomorphic.

Proposition 22.6. *Let $X, Y \in$ **Top** be paracompact Hausdorff topological spaces. Let $p : E \to Y$ be a principal G-bundle. If $h_0, h_1 : X \to Y$ are homotopic maps, then $h_0^*(E) \cong h_1^*(E)$.*

Remark 22.7. *We will invoke the following facts on principal G-bundles in the proof of Proposition 22.6.*

(1) *Every morphism of principal G-bundles is an isomorphism.*
(2) *There is a bijection between morphisms of G-bundles $E_1 \to E_2$ over a common base and global sections of the associated bundle $E_1 \times_G E_2$[19] over X with fiber E_2.*

These constructions and facts are covered in my other notes.

Proof. Consider a homotopy

$$H : X \times I \to Y$$

such that $H_0 = h_0$ and $H_1 = h_1$. Pulling back $p : E \to Y$ along H, we get a principal G-bundle $H^*E \to X \times I$ such that

$$H^*E|_{X\times\{0\}} = h_0^*(E)$$
$$H^*E|_{X\times\{1\}} = h_1^*(E)$$

Hence, it suffices to sow that for any principal G-bundle $q : F \to X \times I$, the restrictions $q|_{X\times\{0\}}$ and $q|_{X\times\{1\}}$ are isomorphic. Denote the restrictions as

$$q_0 : F_0 \to X \times \{0\} \cong X$$
$$q_1 : F_1 \to X \times \{1\} \cong X$$

It suffices to prove that $F \cong F_0 \times I$ as prinicpal G-bundles over $X \times I$, since then restriction to $X \times \{1\}$ gives the isomorphism

$$F|_{X\times\{1\}} \equiv F_1 \cong (F_0 \times I)|_{X\times\{1\}} \equiv F_0.$$

It suffices to find a global section of $F \times_G (F_0 \times I) \to X \times I$. Now, $F \times_G (F_0 \times I)$ has a section over $X \times \{0\}$, since

$$F|_{X\times\{0\}} \cong F_0 \cong F_0 \times I|_{X\times\{0\}}$$

Since X is paracompact Hausdorff, $X \times I$ is paracompact Hausdorff. Hence,

$$F \times_G (F_0 \times I) \to X \times I$$

is a fibration by Remark 15.5. The claim now follows from the homotopy lifting property of fibrations.

$$\begin{tikzcd}
X \times \{0\} \arrow[r] \arrow[d] & F \times_G (F_0 \times I) \arrow[d] \\
X \times I \arrow[r, "\mathrm{Id}"] \arrow[ur, dashed] & X \times I
\end{tikzcd}$$

[19]Here E_2 is endowed with the left action $g \cdot e_2 := e_2 \cdot g^{-1}$

This completes the proof. □

Let $\mathscr{P}(X,G)$ denote the set of isomorphism classes of principal G-bundles over X, and let $\mathscr{P}(G)$ denote the isomorphism classes of all principal G-bundles. The assignment

$$\begin{aligned}\mathbf{Top} &\to \mathscr{P}(G)\\ X &\mapsto \mathscr{P}(X,G)\end{aligned}$$

is a contravariant (set-valued) functor. Indeed, this follows from the pullback construction. Proposition 22.6 states that the functor actually descends to the homotopy category:

$$\mathbf{Top} \to \mathscr{P}(G)$$

The homotopy theoretic classification of principal G-bundles argues that this functor is representable. We restrict ourselves to the category CW complexes. The representability below can be generalized to other categories. We need the following facts about CW complexes.

Lemma 22.8. *Let (X,Y) be a CW pair and let $p : E \to X$ be a fiber bundle with fiber F. Assume that $p_k(F) = 0$ for each k such that $X \setminus Y$ has cells of dimension $k+1$.*

(1) *Every map $f : Y \to F$ extends to a map $\bar{f} : X \to F$.*
(2) *Every section $s \in \Gamma(Y,E)$ can be extended to a global section $\bar{s} \in \Gamma(X,E)$. In particular (taking $Y = \emptyset$) $p : E \to X$ admits global sections if F is k-connected where $k = \dim(X)$.*

Remark 22.9. *Let $p : E \to X$ be fiber bundle with fiber F. We will invoke the following facts in the proof of Lemma 22.8.*

(1) *There is a bijection between local sections defined over a locally trivial cover U_α and smooth maps $U_\alpha \to F$*

This is proved in my other notes.

Proof. The proof is given below:

(1) We have a relative CW-complex structure:

$$Y \subseteq Z^{(-1)} \subseteq Z^{(0)} \subseteq \cdots \subseteq Z^{(m)} = X.$$

We induct on k. Assume that f has been extended to $Z^{(k)}$. The base case $k = -1$ follows by assumption. For each $k+1$-cell $\mathbb{D}^{k+1} \subseteq X$ with attaching map $\varphi : \mathbb{S}^k \to Z^{(k)}$, the composition

$$f \circ \varphi : \mathbb{S}^k \to F$$

is nullhomotopic by assumption. Hence $f \circ \varphi$ can be extended to $\mathbb{D}^{k+1}$ and hence to $Z^{(k)} \cup_\varphi \mathbb{D}^{k+1}$. Extending f in this way for each $k+1$-cell completes the induction.

(2) If $E = X \times F$, the claim follows from (1) and Remark 22.9. Generally, we proceed as above by induction on k. Assume a section has been extended to $Z^{(k)}$, so $s \in \Gamma(Z^{(k)}, E)$. Given a $k+1$ cell $\mathbb{D}_{k+1}$ of X, we can subdivide $\mathbb{D}_{k+1} \cong I^{k+1}$ into sufficiently small cubes and reduce to the case $\mathbb{D}_{k+1} \subseteq U_\alpha$, where U_α is a locally trivial open set. The claim follows as above.

This completes the proof. □

We now state and prove the main result regarding classification of principal G-bundles.

Proposition 22.10. *Let X be a CW-complex. For a topological group, G, $p_G : EG \to BG$ be a principal G-bundle such that EG is weakly contractible. There is a bijective correspondence*

$$\Phi : [X, BG] \to \mathscr{P}(X, G)$$
$$[f] \mapsto [f^* p_G]$$

BG is called the classifying space for principal G-bundles.

Remark 22.11. *We will invoke the following facts on principal G-bundles in the proof of Proposition 22.10.*

(1) *There is a bijection between morphisms of G-bundles $E_1 \to X_1$ and $E_2 \to X_2$ and global sections of the associated bundle $E_1 \times_G E_2$[20] over X_1 with fiber E_2.*

This is covered in my other notes.

Proof. Φ is well-defined by Proposition 22.6. We first show Φ is surjective. Suppose $p : E \to X$ is a principal G-bundle. We need to find $f : B \to BG$ and a principal G-bunle morphism $\hat{f} : E \to EG$ such that the following diagram commutes:

$$\begin{tikzcd}
E \cong f^*(EG) \arrow[r, "\bar{f}"] \arrow[d, "p"] & EG \arrow[d, "p_G"] \\
X \arrow[r, "f"'] & BG
\end{tikzcd}$$

This is equivalent to the existence of a global section of the associated bundle $E \times_G EG \to X$ with fiber EG. Since EG is weakly contractible, such a section exists by Lemma 22.8(2). We now show Φ is injective. Suppose that $f_0, f_1 : B \to BG$ are two maps such that $E_0 := f_0^*(EG) \cong f_1^*(EG) := E_1$. Let $p_0 : E_0 \to X$ and $p_1 : E_1 \to X$. We show that $f_0 \sim f_1$. We have a commutative diagram:

$$\begin{tikzcd}
E_0 \times \{0,1\} \arrow[r, "(\bar{f}_0, \bar{f}_1)"] \arrow[d, "p_0 \times \mathrm{Id}"] & EG \arrow[d, "p_G"] \\
X \times \{0,1\} \arrow[r, "(f_0, f_0)"] & BG
\end{tikzcd}$$

We extend it to a commutative diagram:

$$\begin{tikzcd}
E_0 \times I \arrow[r, "\bar{H}"] \arrow[d, "p_0 \times \mathrm{Id}"] & EG \arrow[d, "p_G"] \\
X \times I \arrow[r, "H"] & BG
\end{tikzcd}$$

This yields the desired homotopy $H : X \times I$ between f_0 and g_1. This is equivalent to finding a section of the associated bundle bundle $(E_0 \times I) \times_G EG \to X \times I$ with fiber EG. We have already have a section of the associated bundle $(E_0 \times \{0,1\}) \times_G EG \to X \times \{0,1\}$. Under the obvious inclusion

$$(E_0 \times \{0,1\}) \times_G EG \subseteq (E_0 \times I) \times_G EG,$$

this section can be regarded as a section of $(E_0 \times I) \times_G EG \to X \times I$ over $X \times \{0,1\}$. Since EG is weakly contractible, the section can be extended via Lemma 22.8(2). $\square$

[20]Here E_2 is endowed with the left action $g \cdot e_2 := e_2 \cdot g^{-1}$

The question remains: how does one construct the universal bundle $EG \to BG$? We will not present a general construction ; rather, we will explicitly find such universal bundles for specific examples. However, we can prove that such a universal bundle is defined uniquely up to homotopy.

Proposition 22.12. *Let G be a topological group. Then a universal principal G-bundle $p_G : EG \to BG$ such that EG is weakly contractible exists. Moreover, the construction is functorial in the sense that a continuous group homomorphism $\mu : G \to H$ induces a bundle map*

$$\begin{tikzcd} EG \arrow[r, "E\mu"] \arrow[d, "p_G"] & EH \arrow[d, "p_H"] \\ BG \arrow[r, "B\mu"] & BH \end{tikzcd}$$

Furthermore, the classifying space BG is unique up to homotopy.

Proof. (Sketch) There is a general construction due to Milnor of BG associated to any locally compact topological group G. We don't discuss it here. We first show that BG is unique up to homotopy. Assume we are given two universal principal G-bundles

$$p_G : EG \to BG$$
$$p_{G'} : EG' \to BG'$$

By regarding each as a universal principal G-bundle for the other principal G-bundle, we obtain the following commutative diagram:

$$\begin{tikzcd} EG \arrow[r, "\bar{f}"] \arrow[d, "p_G"] & EG' \arrow[r, "\bar{f}"] \arrow[d, "p_{G'}"] & EG \arrow[d, "p_G"] \\ BG \arrow[r, "g"] & BG' \arrow[r, "f"] & BG \end{tikzcd}$$

By Proposition 22.10, $f \circ g = \mathrm{Id}_{BG}$ and $g \circ f = \mathrm{Id}_{BG'}$. This shows uniqueness up to homotopy. Functoriality is clear. $\square$

How does one construct the classifying space BG? Note that if $EG \to BG$ is a principal G-bundle, then G acts freely on EG such that $BG \cong EG/G$. Hence, it suffices to find a weakly contractible space EG on which G acts freely.

Example 22.13. The following is a list of examples of some classifying spaces:

(1) Let $G = \mathbb{Z}$. We can take $EG = \mathbb{R}$ since $\mathbb{Z}$ acts freely on $\mathbb{R}$ by translations. Hence, we have

$$\begin{aligned} B\mathbb{Z} &\cong E\mathbb{Z}/\mathbb{Z} \\ &\cong \mathbb{R}/\mathbb{Z} \cong \mathbb{S}^1 \end{aligned}$$

(2) Let $G = \mathbb{Z}^n$. We can take $EG = \mathbb{R}^n$ since $\mathbb{Z}^n$ acts freely on $\mathbb{R}^n$ by translations. Hence, we have

$$\begin{aligned} B\mathbb{Z}^n &\cong E\mathbb{Z}^n/\mathbb{Z}^n \\ &\cong \mathbb{R}^n/\mathbb{Z}^n \cong \underbrace{\mathbb{S}^1 \times \cdots \times \mathbb{S}^1}_{n\text{-times}} \end{aligned}$$

(3) Let $G = \mathbb{Z}_2$. We can take $EG = \mathbb{S}^\infty$ since $\mathbb{Z}_2$ acts freely on $\mathbb{S}^\infty$ and $\mathbb{S}^\infty$ is contractible. Hence, we have

$$\begin{aligned} B\mathbb{Z}_2 &\cong E\mathbb{Z}_2/\mathbb{Z}_2 \\ &\cong \mathbb{S}^\infty/\mathbb{Z}_2 \cong \mathbb{RP}^\infty \end{aligned}$$

(4) Let $G = \mathbb{S}^1$. We can take $EG = \mathbb{S}^\infty$ since $\mathbb{S}^1$ acts freely on $\mathbb{S}^\infty$ and $\mathbb{S}^\infty$ is contractible. Hence, we have

$$\begin{aligned} B\mathbb{S}^1 &\cong E\mathbb{S}^1/\mathbb{S}^1 \\ &\cong \mathbb{S}^\infty/\mathbb{S}^1 \cong \mathbb{CP}^\infty \end{aligned}$$

(5) Let $G = \mathrm{O}(k)$. It can be checked that $V_k(\mathbb{R}^\infty)$ is contractible. Hence, we can take $EG = V_k(\mathbb{R}^\infty)$ since $\mathrm{O}(k)$ acts freely on $V_k(\mathbb{R}^\infty)$. Hence, we have

$$\begin{aligned} B\,\mathrm{O}(k) &\cong E\,\mathrm{O}(k)/\,\mathrm{O}(k) \\ &\cong V_k(\mathbb{R}^\infty)/\,\mathrm{O}(k) \cong G_k(\mathbb{R}^\infty) \end{aligned}$$

(6) Let $G = \mathrm{U}(k)$. It can be checked that $V_k(\mathbb{C}^\infty)$ is contractible. Hence, we can take $EG = V_k(\mathbb{C}^\infty)$ since $\mathrm{U}(k)$ acts freely on $V_k(\mathbb{C}^\infty)$. Hence, we have

$$\begin{aligned} B\,\mathrm{U}(k) &\cong E\,\mathrm{U}(k)/\,\mathrm{U}(k) \\ &\cong V_k(\mathbb{C}^\infty)/\,\mathrm{U}(k) \\ &\cong G_k(\mathbb{C}^\infty) \end{aligned}$$

Remark 22.14. *The following observation is quite useful. Since $EG \to BG$ is a principal G-bundle, the long exact sequence in homotopy associated to a fibration reads:*

$$\cdots \to \pi_{n+1}(BG) \to \pi_n(G) \to \pi_n(EG) \to \pi_n(BG) \to \pi_{n-1}(G) \to \dots$$

As EG is weakly contractible, $\pi_n(EG) = 0$ for $n > 0$. Hence, we see that

$$\pi_{n+1}(BG) \cong \pi_n(G)$$

for $n \geq 1$.

23. Eilenberg-Maclane Spaces

We can use homotopy theory to show that the singular cohomology functor is representable in the homotopy category. If G is an abelian group, assume there exists a topological space Z_n such that

$$H^n(X, G) = [X, Z_n]$$

for all topological spaces and all $X \in \mathbf{Top}$. If $X = \mathbb{S}^k$, note that

$$\pi_k(Z_n) = [\mathbb{S}^k, Z_n] = H^n(\mathbb{S}^k, G) = \begin{cases} G & \text{if } n = 0, k \\ 0 & \text{otherwise} \end{cases}$$

Hence, we see that $\pi_k(Z_n)$ is non-trivial for exactly one value of $k \in \mathbb{N}$. This motivates the following definition.

Definition 23.1. Let $X \in \mathbf{Top}, G \in \mathbf{Grp}$ If X has only one non-trivial homotopy group such that

$$\pi_n(X) \cong G$$

for some $n \in \mathbb{N}$, then X is called an Eilenberg-MacLane space.

A generic Eilenberg-Maclane space is denoted as $K(G,n)$. The question remains: how does one construct an Eilenberg-Maclane space $K(G,n)$. We will not present a general existence and uniqueness argument; rather, we will explicitly find $K(G,n)$ for specific examples. We first discuss a link between classifying spaces and Eilenberg-Maclane spaces for discrete groups:

Proposition 23.2. *Let G be a discrete abelian group. Then $BG \cong K(G,1)$.*

Proof. Since G is discrete, we have

$$\pi_n(G) = \begin{cases} G & \text{if } n = 0 \\ 0 & \text{otherwise} \end{cases}$$

By Remark 22.14, we have $\pi_n(BG) = 0$ for $n \geq 2$. Since $EG \to BG$ is a universal covering map with discrete fibers G, covering space theory implies that $\pi_1(BG) \cong G$. This proves the claim. $\square$

Example 23.3. The following is a list of Eilenberg-Maclane space:

(1) $\mathbb{S}^1$ is a model of $K(\mathbb{Z},1)$. This follows from Proposition 23.2 and that $B\mathbb{Z} \cong \mathbb{S}^1$.
(2) $\mathbb{RP}^\infty$ is a model of $K(\mathbb{Z}_2,1)$. This follows from Proposition 23.2 and that $B\mathbb{Z}_2 \cong \mathbb{RP}^\infty$.
(3) More generally, the cyclic group $\mathbb{Z}/m$ acts on $\mathbb{S}^\infty$ when thought of as a direct limit of spheres in complex vector spaces $\mathbb{C}^n$, where the action is by multiplication of each coordinate by $e^{2\pi i/m}$. We have a principal $\mathbb{Z}/m$-bundle:
$$\mathbb{Z}/m \to \mathbb{S}^\infty \to \mathbb{S}^\infty/\mathbb{Z}/m$$
The quotient $S^\infty/\mathbb{Z}/m$ is called the infinite-dimensional lens space, which is a $K(\mathbb{Z}/m,1)$.
(4) $\mathbb{CP}^\infty$ is a model of $K(\mathbb{Z},2)$. This follows from Example 21.7.
(5) The wedge sum of n-circles is a model space for $K(F_n,1)$, where F_n is the free group on n generators. Clearly, we have
$$\pi_1\left(\bigvee_{i=1}^n \mathbb{S}^1\right) \cong F_n$$
Moreover, the higher homotopy groups of a wedge sum of n-circles vanish since its universal covering space, which is the Cayley graph on n generators, is contractible. By Proposition 23.2, we also have
$$BF_n \cong \bigvee_{i=1}^n \mathbb{S}^1$$

The following is a basic list of properties of $K(G,1)$.

Proposition 23.4. *Let G be a group.*

(1) *If G' is another group, we have*
$$K(G,n) \times K(G',n) \cong K(G \times G',n)$$
(2) *$K(G,1)$ exists for any finitely-generated abelian group.*
(3) *If $X \cong K(G,n)$, then $\Omega X \cong K(G,n-1)$.*

Proof. The proof is given below:

(1) This is clear π_n is a functor that preserves products.
(2) We have constructed a $K(\mathbb{Z},1)$ and a $K(\mathbb{Z}/m,1)$ for each $m \geq 2$. Thus, we can construct a $K(G,1)$ for any finitely generated G by (1).
(3) This follows because $\pi_n(\Omega X) \cong \pi_{n+1}(X)$

This completes the proof. □

Remark 23.5. *If G is a finitely-generated abelian group, then if G has torsion, then the $K(G,1)$ contains an infinite-dimensional lens space in the product. Since a $K(G,1)$ is unique up to homotopy equivalence (assumed without proof), a finite-dimensional $K(G,1)$ cannot exist if G is finitely generated and has torsion.*

We now use homotopy theory to show that the singular cohomology functor is representable in the homotopy category in terms of Eilenberg-Maclane spaces. We will invoke the definition of reduced cohomology. We first prove the following lemma:

Lemma 23.6. *Let h^* be an unreduced cohomology theory with $\mathbb{Z}$ coefficients defined as a collection of functors*

$$h^n : \mathbf{CW}^2 \to \mathbf{Ab}$$

If $h^n(*;\mathbb{Z}) \cong 0$ for $n \neq 0$, then there exists a natural isomorphism

$$h^n(X,A) \cong H^n(X,A;G)$$

for all CW-pairs (X,A) and for all $n \geq 1$, where $G := h^0(*;\mathbb{Z}) \in \mathbf{Ab}$.

Proof. (Sketch) The proof is similar to the proof for homology theories defined on $\mathbf{CW}^2$. (**??**). See [Hat02] for the difference that needs to be accounted for. □

Remark 23.7. *There is also a version of Lemma 23.6 for reduced cohomology.*

We now prove the desired result:

Proposition 23.8. *Let H^* be an unreduced singular cohomology theory with $\mathbb{Z}$ coefficients defined as a collection of functors*

$$H^n : \mathbf{CW}^2_* \to \mathbf{Ab}$$

There exists a natural isomorphism

$$T_n : H^n(X;G) \to [X, K(G,n)]_*$$

for all $X \in \mathbf{CW}_*$ and any abelian group G for all $n \geq 1$.

Proof. Using Proposition 23.4 we have that $\Omega K(G,n) \cong K(G,n-1)$. Define the functors

$$\begin{aligned} L^n : \mathbf{CW}_* &\to \mathbf{Ab} \\ X &\mapsto [X, K(G,n)]_* \end{aligned}$$

We claim that these functors define a reduced cohomology theory on $\mathbf{CW}_*$.

(1) **(Homotopy invariance)** A map $f : X \to Y$ induces a map

$$f^* : [Y, K(G,n)]_* \to [X, K(G,n)]_*$$

which depends only on the basepoint-preserving homotopy class. It can be checked that f^* is indeed a homomorphism by replacing $K(G,n)$ with $\Omega K(G,n+1)$.

(2) **(Wedge sum axiom)** Let $i_\alpha : X_\alpha \hookrightarrow \bigvee_{\alpha\in A} X_\alpha$ be the inclusion. We want to show that the map

$$\prod_{\alpha\in A} i_\alpha^* : \Big[\bigvee_{\alpha\in A} X_\alpha, K(G,n)\Big]_* \to \prod_{\alpha\in A}[X_\alpha, K(G,n)]_*$$

is an isomorphism for all n. This follows from Remark 2.14.

(3) (**Suspension Axiom**) We have

$$\begin{aligned} L^{n+1}(\Sigma X) &= [\Sigma X, K(G,n+1)]_* \\ &= [X, \Omega K(G,n+1)]_* \\ &= [X, K(G,n)]_* \\ &= L^n(X) \end{aligned}$$

for all n. Hence, the suspension axiom holds.

(4) (**Long Exact Sequence**) This follows from the coexact Puppe sequence (which is not included in the notes for now).

Hence, we have an reduced cohomology theory. The reduced version of Lemma 23.6 shows that there exists natural isomorphism

$$T_n : H^n(X;G) \to [X, K(G,n)]_*$$

This completes the proof. □

Remark 23.9. *In the proof Proposition 23.8 we used the fact that the family of spaces* $\{K(G,n)\}_{n\geq 0}$ *for a fixed* $G \in \mathbf{Ab}$ *is such that*

$$K(G,n) \cong \Omega K(G,n+1)$$

We say that $\{K(G,n)\}_{n\geq 0}$ *is an* Ω*-spectrum. This suggests that* Ω*-spectrum can be used to defined cohomology theories. This is the start of the study of spectra in stable homotopy theory.*

Part 7. Serre Spectral Sequence

24. Construction

The Serre spectral sequence is a powerful computational tool in algebraic topology that arises in the study of the homology and cohomology of fibrations. It allows one to relate the (co)homology of the total space of a fibration to that of its base and fiber, often turning otherwise intractable computations into manageable ones. We present the Serre spectral sequence and illustrate its use through examples and applications. We will treat the general theory of spectral sequences largely as a black box, relying on established results without reproving them here.[21]

Remark 24.1. *There is a version of the Serre spectral sequence for both homology and cohomology. In these notes, we focus on the cohomological version, as it is the one most commonly used in practice. The homological version is very similar in structure and can be invoked when needed. For further details, the reader is referred to [Hat04].*

[21] Some of these general results are developed in more detail in my other notes.