COMPLEX GEOMETRY

JUNAID AFTAB

Abstract. These are notes on complex geometry. I wrote them during graduate school learning the subject. The language of manifolds, bundles, sheaves, and sheaf cohomology is used freely throughout. There may be typos; please send corrections to junaid.aftab1994@gmail.com.

Contents

Part 1. Preliminaries

1. Several Complex Variables

A complex manifold is modeled as a topological space where each open subset resembles an open subset of $\mathbb{C}^n$. Therefore, we first study some fundamentals of several complex variables theory.

1.1. Definitions.

A complex differentiable function $f : \mathbb{C}^n \to \mathbb{C}$ is called analytic or holomorphic. Recall that if $n = 1$, a holomorphic function in one variable admits a local representation in terms of convergent power series. The purpose of this section is to discuss the case $n > 1$ and to elucidate both the similarities and differences across the cases.

Remark 1.1. *Let $p \in \mathbb{C}^n$. We will find it convenient to consider open discs with respect to the supremum norm in $\mathbb{C}^n$:*

$$\Delta_\varepsilon(p) := \{z \in \mathbb{C}^n : |z^k - p^k| < \varepsilon \text{ for } k = 1, 2, \dots, n\}$$

Here $|\cdot|$ is the usual metric on $\mathbb{C}$. Such a $\Delta_\varepsilon(p)$ is called a polydisc of radius ε around p. Let $\langle , \rangle_{\mathbb{C}^n}$ denote the inner product on $\mathbb{C}^n$ and let $\|\cdot\|_{\mathbb{C}^n}$ denote the corresponding norm. We denote the open ball of radius ε around $p \in \mathbb{C}^n$ as

$$\mathbb{B}_\varepsilon(p) := \{z \in \mathbb{C}^n \mid \|z - a\|_{\mathbb{C}^n} < \varepsilon\}$$

If $n = 1$, we denote $\mathbb{B}_\varepsilon(p)$ as $\mathbb{D}_\varepsilon(p)$. Note that we have,

$$\Delta_\varepsilon(p) = \mathbb{D}_\varepsilon(p^1) \times \cdots \times \mathbb{D}_\varepsilon(p^n)$$

The notion of a holomorphic function of one variable can be extended in a straightforward way.

Definition 1.2. Let $U \subseteq \mathbb{C}^n$ be an open subset. A function $f : U \to \mathbb{C}$ is a **holomorphic** in U if for each $p = (p^1, \ldots, p^n) \in U$, it it continuous at p and the partial derivatives

$$\frac{\partial f}{\partial z^j}(p) = \lim_{\xi \to 0} \frac{f(p^1, \ldots, p^j + \xi + \cdots, p^n) - f(p^1, \ldots, p^n)}{\xi}, \qquad \xi \in \mathbb{C} \setminus \{0\}$$

exist for each $j \in \{1, \ldots, n\}$. The limit over some punctured polydisc centered at the origin in $\mathbb{C}$.

Remark 1.3. *More generally, a vector-valued function $f : U \to \mathbb{C}^k$ is said to be holomorphic if each of its component functions is holomorphic.*

Remark 1.4. *If $n = 1$, it can be easily shown that the continuity assumption can be removed from Definition 1.2. In fact, the continuity assumption can be removed if $n > 1$: Hartog (1906) proved that a function that has complex partial derivatives at every point of an open subset of $\mathbb{C}^n$ is automatically continuous. The proof is involved.*

In one complex variable, there are several equivalent ways to characterize holomorphic functions. There are similar equivalent characterizations for holomorphic functions of several variables. We first generalize Cauchy's integral formula to several variables:

Lemma 1.5. *Let $\Delta_\varepsilon(p)$ be polydisc in $\mathbb{C}^n$. Let $f : \overline{\Delta_\varepsilon(p)} \to \mathbb{C}$ be a continuous function such that f is holomorphic with respect to every single component z^i at any point of $\Delta_\varepsilon(p)$. Then for any $z \in \Delta_\varepsilon(p)$, we have:*

$$f(z) = \frac{1}{(2\pi i)^n} \int_{|\zeta^1 - p^1| = \varepsilon} \cdots \int_{|\zeta^n - p^n| = \varepsilon} \frac{f(\zeta^1, \ldots, \zeta^n)}{(\zeta^1 - z^1) \cdots (\zeta^n - z^n)} \, d\zeta^1 \cdots d\zeta^n.$$

Proof. Repeated application of the Cauchy integral formula in one variable yields

$$\begin{aligned} f(z^1, \ldots, z^n) &= \frac{1}{2\pi i} \int_{|\zeta^n - p^n| = \varepsilon} \frac{f(z^1, \ldots, z^{n-1}, \zeta^n)}{\zeta^n - z^n} \, d\zeta^n \\ &= \frac{1}{(2\pi i)^2} \int_{|\zeta^n - p^n| = \varepsilon} \int_{|\zeta^{n-1} - p^{n-1}| = \varepsilon} \frac{f(z^1, \ldots, \zeta^{n-1}, \zeta^n)}{(\zeta^n - z^n)(\zeta^{n-1} - z^{n-1})} \, d\zeta^{n-1} d\zeta^n \\ &\vdots \\ &= \frac{1}{(2\pi i)^n} \int_{|\zeta^n - p^n| = \varepsilon} \cdots \int_{|\zeta^1 - p^1| = \varepsilon} \frac{f(\zeta^1, \ldots, \zeta^n)}{(\zeta^n - z^n) \cdots (\zeta^1 - z^1)} \, d\zeta^1 \cdots d\zeta^n \end{aligned}$$

This completes the proof. ☐

Remark 1.6. *If we let $\partial \Delta_\varepsilon(p^i) = \{\zeta \in \mathbb{C} \mid |\zeta^i - p^i| = \varepsilon\}$ and*

$$\Gamma_\varepsilon(p) := \partial \Delta_\varepsilon(p^1) \times \cdots \times \partial \Delta_\varepsilon(p^n)$$

Fubini's theorem implies that the integral in Lemma 1.5 can be written as

$$f(z) = \frac{1}{(2\pi i)^n} \int_{\Gamma_\varepsilon(p)} \frac{f(\zeta^1, \ldots, \zeta^n)}{(\zeta^1 - z^1) \cdots (\zeta^n - z^n)} \, d\zeta^1 \cdots d\zeta^n.$$

Lemma 1.5 implies an important point about holomorphic functions in several variables. The value of f on $\Delta_\varepsilon(p)$ is completely determined by the values of f on the set $\Gamma_\varepsilon(p)$, which is much smaller than the boundary of the polydisc $\partial\Delta_\varepsilon(p)$!

Proposition 1.7. ***(Osgood's Lemma)*** *Let $U \subseteq \mathbb{C}^n$ be an open set and $f : U \to \mathbb{C}$ is a continuous function. The following are equivalent:*

(1) *f is holomorphic.*
(2) *If $f(z) = u(z)+iv(z)$, then f is smooth and satisfies the following Cauchy–Riemann equations:*
$$\frac{\partial u}{\partial x^j} = \frac{\partial v}{\partial y^j}, \quad \frac{\partial u}{\partial y^j} = -\frac{\partial v}{\partial x^j}, \quad j = 1, \ldots, n,$$
where $z^j = x^j + iy^j$.
(3) *For each $p = (p^1, \ldots, p^n) \in U$, there exists a neighborhood of p in U on which f is equal to the sum of an absolutely convergent power series of the form*
$$f(z) = \sum_{k_1,\ldots,k_n=0}^{\infty} a_{k_1\ldots k_n}(z^1 - p^1)^{k_1} \cdots (z^n - p^n)^{k_n}.$$

Proof. The proof is given below:

- (1) $\iff$ (2): Assume (1) is true. Because f is holomorphic in each variable separately, complex variable theory shows that it satisfies the Cauchy–Riemann equations with respect to each variable. If $p \in U$, let $\varepsilon > 0$ such that $\overline{\Delta_\varepsilon(p)} \subseteq U$. Smoothness now follows from Lemma 1.5. This is because we can repeatedly differentiate under the integral sign because $\overline{\Delta_\varepsilon(p)}$ is compact and the integrand is smooth. Hence, (1) is also true. Conversely, if (2) is true, then it is certainly continuous, and complex variable theory implies that it has a complex derivative with respect to each variable. Hence, (1) is also true.
- (1) $\iff$ (3): Assume (1) (and hence (2)) is true. Note that
$$\frac{1}{\zeta^j - z^j} = \frac{1}{(\zeta^j - p^j) - (z^j - p^j)} = \frac{1}{\zeta^j - p^j} \cdot \frac{1}{1 - \frac{z^j - p^j}{\zeta^j - p^j}},$$
Since $\frac{|z^j - p^j|}{|\zeta^j - p^j|} < 1$ on the domain of integration in the integral in Lemma 1.5, we can expand the last fraction on the right in a power series to obtain
$$\frac{1}{\zeta^j - z^j} = \frac{1}{\zeta^j - p^j} \sum_{k=0}^{\infty} \left(\frac{z^j - p^j}{\zeta^j - p^j}\right)^k,$$
This power series converges uniformly and absolutely for z^j in any closed polydisk $\Delta_{\varepsilon'}(p^j)$ with $0 < \varepsilon' < \varepsilon$ by comparison with the geometric series. Inserting this formula for each variable, we conclude that f can be expanded in a power series with coefficients
$$a_{k_1\ldots k_n} = \frac{1}{(2\pi i)^n} \int_{\Gamma_\varepsilon(p)} \frac{f(\zeta^1, \ldots, \zeta^n)}{(\zeta^n - p^n)^{k_n+1} \cdots (\zeta^1 - p^1)^{k_1+1}} \, d\zeta^1 \cdots d\zeta^n.$$
Assume (3) is true. Then Weierstrass' M-test implies that f is continuous, and complex variable theory implies that f has partial derivatives with respect to each z^j.

This completes the proof. $\square$

Remark 1.8. *We could have alternatively defined $f : U \to \mathbb{C}$ to be holomorphic if and only if f admits a convergent power series about each point in U. Proposition 1.7 then implies that f is holomorphic in this new sense if and only if f is holomorphic in each variable separately. There is no analogue of this result for real variables. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function*

$$f(x,y) = \begin{cases} \frac{2x^2y+y^3}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

This function is everywhere continuous and has well-defined partial derivatives with respect to x and y everywhere (including at the origin), but it is not differentiable at the origin. Indeed, we have,

$$\lim_{x\to 0} \frac{f(x,mx) - f(0,0)}{x-0} = \lim_{x\to 0} \frac{2mx^3 + m^3x^3}{x(x^2+m^2x^2)} = \lim_{x\to 0} \frac{x^3(2m+m^3)}{x^3(1+m^2)} = \frac{2m+m^3}{1+m^2}$$

Clearly, the limits are different for different values of m.

Remark 1.9. *As in one variable, we define the Wirtinger operators*

$$\frac{\partial}{\partial z^k} := \frac{1}{2}\left(\frac{\partial}{\partial x^k} - i\frac{\partial}{\partial y^k}\right),$$
$$\frac{\partial}{\partial \overline{z^k}} := \frac{1}{2}\left(\frac{\partial}{\partial x^k} + i\frac{\partial}{\partial y^k}\right).$$

An alternative definition is to say that a continuously differentiable function $f : U \to \mathbb{C}$ is holomorphic if it satisfies the Cauchy–Riemann equations

$$\frac{\partial f}{\partial \overline{z}^k} = 0 \text{ for } k = 1, 2, \ldots, n.$$

This follows readily from Proposition 1.10(3).

1.2. **Properties.** We now prove some properties of holomorphic functions of several variables that extend the properties of holomorphic functions of one variable.

Proposition 1.10. *Let $U \subseteq \mathbb{C}^n$ and let $f : U \to \mathbb{C}$ be a holomorphic function.*

(1) *If $g\colon U \to \mathbb{C}$ is a holomorphic function, then $f \pm g$ fg are holomorphic on U and f/g is holomorphic on $U \setminus g^{-1}(0)$.*
(2) *Let $W \subseteq \mathbb{C}^m$ be open. If $g\colon W \to U$ is a holomorphic, then $f \circ g$ is holomorphic.*
(3) *We have*
$$\frac{\partial f}{\partial z^j} = \frac{\partial f}{\partial x^j} = \frac{1}{i}\frac{\partial f}{\partial y^j}.$$
(4) *If $p \in U$ and $\Delta_\varepsilon(p) \subseteq U$, then the power series representation of f is given explicitly by the following formula*
$$f(z) = \sum_{k_1,\ldots,k_n=0}^{\infty} \frac{1}{k_1!\cdots k_n!} \frac{\partial^{k_1+\cdots+k_n} f(p)}{(\partial z^1)^{k_1}\cdots(\partial z^n)^{k_n}} (z^1 - p^1)^{k_1}\cdots(z^n - p^n)^{k_n}.$$
(5) ***(Cauchy Estimate)*** *If $p \in U$ and $\Delta_\varepsilon(p) \subseteq U$, then*
$$\left| \frac{\partial^{k_1+\cdots+k_n} f(p)}{(\partial z^1)^{k_1}\cdots(\partial z^n)^{k_n}} \right| \leq \|f\|_\infty \frac{k_1!\cdots k_n!}{\varepsilon^{k_1+\cdots+\varepsilon_n}},$$
where $\|f\|_\infty$ is the bounded on $|f|$ on $\Delta_\varepsilon(p)$.

(6) ***(Identity Theorem)*** *If U is connected and $g : U \to \mathbb{C}$ is anther holomorphic function that agrees with f on a non-empty open subset of $V \subseteq U$, then $f = g$ on U.*
(7) ***(Liouville's Theorem)*** *If $U = \mathbb{C}^n$ and f is bounded, then f is constant.*
(8) ***(Maximum Principle)*** *If $|f|$ attains a maximum value at some point in U, then f is constant.*
(9) *Let $f_k \colon U \to \mathbb{C}$ be a sequence of holomorphic functions that converge uniformly on compact subsets of U to a function $f \colon U \to \mathbb{C}$. Then f is holomorphic.*

Remark 1.11. *Recall that Proposition 1.10(7) is false for real-analytic functions.*

Proof. The proof is given below:
(1) This is clear.
(2) Certainly $f \circ g$ is continuous. Let $z = (z^1, \dots, z^m)$ denote the coordinates on W, and $w = (w^1, \dots, w^n)$ those on U. Then $w^j = g_j(z^1, \dots, z^m)$. By the chain rule, we have
$$\frac{\partial (f \circ g)}{\partial \bar{z}^k} = \sum_j \left(\frac{\partial g}{\partial w^j} \frac{\partial f_j}{\partial \bar{z}^k} + \frac{\partial g}{\partial \bar{w}^j} \frac{\partial \bar{f}_j}{\partial \bar{z}^k} \right) = 0,$$
This is zero because $\frac{\partial f_j}{\partial \bar{z}^k} = 0$ and $\frac{\partial g}{\partial \bar{w}^j} = 0$ by Remark 1.9. This is sufficient to infer that that $f \circ g$ is holomorphic.
(3) Compute the limits:
$$\frac{\partial f}{\partial z^j}(p) = \lim_{h \to 0,\, h \in \mathbb{R}} \frac{f(p^1, \dots, p^j + h, \dots, p^n) - f(p^1, \dots, p^n)}{h} = \frac{\partial f}{\partial x^j}(p),$$
$$\frac{\partial f}{\partial z^j}(p) = \lim_{k \to 0,\, k \in \mathbb{R}} \frac{f(p^1, \dots, p^j + ik, \dots, p^n) - f(p^1, \dots, p^n)}{ik} = \frac{1}{i} \frac{\partial f}{\partial y^j}(p).$$
Now simply note that the limits are equal.
(4) Simply differentiate the expression in Proposition 1.7(3) repeatedly term-by-term and evaluate at $z = p$ to determine the coefficients $a_{k_1, \cdots, k_n}$. This is justified by results concerning power series in several variables.
(5) Note that we have
$$\begin{aligned} a_{k_1 \cdots k_n} &= \frac{1}{k_1! \cdots k_n!} \frac{\partial^{k_1 + \cdots + k_n} f(p)}{(\partial z^1)^{k_1} \cdots (\partial z^n)^{k_n}} \\ &= \frac{1}{(2\pi i)^n} \int_{\Gamma_\varepsilon(p)} \frac{f(\zeta^1, \dots, \zeta^n)}{(\zeta^n - p^n)^{k_n + 1} \cdots (\zeta^1 - p^1)^{k_1 + 1}} \, d\zeta^1 \cdots d\zeta^n. \end{aligned}$$
From the obvious bounds on the integrand of this integral, it follows that
$$|a_{k_1 \cdots k_n}| \leq \frac{\|f\|_\infty}{\varepsilon^{k_1 + \cdots + \varepsilon_n}}$$
The Cauchy estimate now follows.
(6) Set $h = f - g$, so $h \equiv 0$ on a nonempty open subset $V \subseteq U$. Let
$$W = \{ z \in U \mid h \text{ and all its partial derivatives vanish at } z \}.$$
Then U is nonempty because $U_0 \subseteq U$. Let $z \in U$ be a limit point of W. There is a sequence of points $z^j \in W$ converging to z. Hence, all partial derivatives of h vanish at each z^j. By continuity, they also vanish at z. Hence, $z \in U$ implying that W is closed in U. Suppose $z \in W$. By (1) h is equal to a convergent power series

in a neighborhood of z such that every term in the series is zero. Thus, W is open in U. Since W is cl-open and U is connected, the claim follows.

(7) Given any point $z \in \mathbb{C}^n$, define the function

$$g(\zeta) = f(\zeta z), \qquad \zeta \in \mathbb{C}$$

g is a bounded holomorphic function on $\mathbb{C}$. By Liouville's theorem from compelx variable theory, g is constant. Hence,

$$f(z) = g(1) = g(0) = f(0)$$

Since $z \in \mathbb{C}^n$ is arbitrary, f is constant.

(8) Suppose $|f|$ attains a maximum value at $z' \in U$. Consider the set,

$$W = \{z \in U \mid f(z) = f(z')\}.$$

Clearly, W is non-empty and closed. Given $z \in W$, choose $\varepsilon > 0$ such that $\Delta_\varepsilon(z) \subseteq U$. For each $w \in \mathbb{C}^n$ with $|w| = 1$, consider the function

$$g(\zeta) = f(z + \zeta w)$$

g is holomorphic on the disk $\mathbb{D}_\varepsilon(0) \subseteq \mathbb{C}$ and achieves its maximum modulus at $\zeta = 0$. By the maximum principle from complex variable theory, g is constant. Since w is arbitrary, this shows f is constant on $\Delta_\varepsilon(z)$. Thus, W is open. Since U is connected, $W = U$. Hence, $f \equiv f(z')$ on U.

(9) Given $p \in U$, choose $\varepsilon > 0$ such that $\overline{\Delta_\varepsilon}(p) \subset U$. For all $z \in \Delta_\varepsilon(p)$, we can apply the Cauchy integral formula to f_k, and uniform convergence guarantees that

$$f(z) = \lim_{k\to\infty} \frac{1}{(2\pi i)^n} \int_{|\zeta^n - p^n| = r} \cdots \int_{|\zeta^1 - p^1| = r} \frac{f_k(\zeta^1, \ldots, \zeta^n)}{(\zeta^n - z^n)\cdots(\zeta^1 - z^1)}\, d\zeta^1 \cdots d\zeta^n$$

$$= \frac{1}{(2\pi i)^n} \int_{|\zeta^n - p^n| = r} \cdots \int_{|\zeta^1 - p^1| = r} \frac{f(\zeta^1, \ldots, \zeta^n)}{(\zeta^n - z^n)\cdots(\zeta^1 - z^1)}\, d\zeta^1 \cdots d\zeta^n.$$

Clearly, f is holomorphic.

This completes the proof. □

Remark 1.12. *Proposition 1.10 conveys that holomorphic functions are quite rigid. Indeed, here is an implication of Proposition 1.10. Let $f : \mathbb{C} \to \mathbb{H}$[1]. We claim that f is constant. Note that $g(z) := e^{if(z)}$ is holomorphic on $\mathbb{C}$ with $\|g\|_\infty \leq 1$[2]. So g is a constant by Liouville's Theorem and hence f is a constant as well.*

So far, all these facts about holomorphic functions of several variables have been straightforward generalizations of standard facts about holomorphic functions of one variable. The next result, however, is radically different from anything in the one-variable theory.

Proposition 1.13. *(**Hartog's Extension Theorem**) Let $n \geq 2$, and let $U = \Delta_\varepsilon(p) \setminus \overline{\Delta_{\varepsilon'}(p)}$ for some $p \in \mathbb{C}^n$ and $0 < \varepsilon' < \varepsilon$. Every holomorphic function $f \colon U \to \mathbb{C}$ has a unique extension to a holomorphic function on all of $\Delta_\varepsilon(p)$.*

[1] $\mathbb{H}$ is the upper half plane.

[2] If $f(z) = a_z + ib_z$, then $e^{if(z)} = e^{ia_z - b_z}$. Hence $|e^{if(z)}| = e^{-b_z} < 1$ since $b_z > 0$.

Proof. WLOG, we may assume that $p = 0$. Choose any $\delta > 0$ such that $\varepsilon' < \delta < \varepsilon$. As long as $\varepsilon' < |z^2| < \varepsilon$, the function $z^1 \mapsto f(z^1, \dots, z^n)$ is holomorphic on $\Delta_\varepsilon(0) \subseteq \mathbb{C}$. Cauchy's integral formula shows that

$$f(z^1, \dots, z^n) = \frac{1}{2\pi i} \int_{|\zeta|=\delta} \frac{f(\zeta, z^2, \dots, z^n)}{\zeta - z^1}\, d\zeta.$$

This formula actually makes sense for all $(z^1, \dots, z^n) \in \Delta_\delta(0)$ because the integration contour is contained in U and it defines a holomorphic function f_1 there by differentiation under the integral sign. Because f_1 agrees with f on the open subset of $\Delta_\delta(0)$ where $\varepsilon' < |z^2| < \delta$, the identity theorem shows that it agrees on the entire connected set $\Delta_\delta(0) \setminus \Delta_{\varepsilon'}(0)$. Thus we can define a holomorphic function on all of $\Delta_\varepsilon(0)$ by letting it be equal to f on U and to f_1 on $\Delta_\delta(0)$. Uniqueness follows from the identity theorem. □

Remark 1.14. *In the language of sheaves of holomorphic functions, Proposition 1.13 states that the map $\mathscr{O}_{\mathbb{C}^n}(\Delta_\varepsilon(p)) \to \mathscr{O}(\Delta_\varepsilon(p) \setminus \Delta_{\varepsilon'}(p))$ is bijective.*

Remark 1.15. *Proposition 1.13 is false if $n = 1$. Let $f(z) = 1/z$ on an annular region centered at the origin. Then f is holomorphic on the annular region but f is not holomorphic on large unit disk defining the annular region.*

Proposition 1.13 implies that singularities of holomorphic functions in two or more variables are never isolated. Similarly, it implies that the zeros of a holomorphic function in two or more variables are never isolated. If not, then let f have an isolated zero at p. Then $1/f$ would have an isolated singularity, which is a contradiction.

Remark 1.16. *A holomorphic function of one complex variable may have isolated singularities. Simply consider $f(z) = 1/z$ on $\mathbb{C}$.*

Remark 1.17. *The zero of a holomorphic function of one complex variable are always isolated. The claim is obviously true if $f \in \mathbb{C}[z]$. Generally, if f is a holomorphic function on an open set and f has a zero of multiplicity k at $a \in U$, then we have*

$$f(z) = \sum_{i=k}^{\infty} \frac{f^{(i)}(a)}{i!}(z-a)^i = (z-a)^k \sum_{i=k}^{\infty} \frac{f^{(i)}(a)}{i!}(z-a)^{i-k} := (z-a)^k g(z), \quad 0 \le |z-a| < r$$

for some $r \in \mathbb{R}$ such that the open disk is contained in U. Since $g \neq 0$ on $0 < |z-a| < r$, we have that $f \neq 0$ on $0 < |z-a| < r$.

We end this section by proving the Schwarz lemma:

Proposition 1.18. *(**Schwarz Lemma**) Let $\Delta_\varepsilon(0)$ be a polydisc and let $f : U \to \mathbb{C}$ be a holomorphic function such that $\overline{\Delta_\varepsilon(0)} \subseteq U$. Assume that f non-trivial monomials of degree $< k$ do not occur in the power series expansion of f. If $|f(z)| \le C$ on $\overline{\Delta_\varepsilon(0)}$ can be bounded from then*

$$|f(z)| \le C|z|^k \varepsilon^{-k}$$

for all $z \in \overline{\Delta_\varepsilon(0)}$.

Proof. Let $\mathbb{D}_\varepsilon(0)$ be a unit ball in $\mathbb{C}$. Fix $0 \neq z \in \Delta_\varepsilon(0)$. Define

$$g_z(w) = w^{-k} f(wz/|z|), \quad w \in \mathbb{D}_\varepsilon(0).$$

Then $|g_z(w)| \leq C\varepsilon'^{-k}$ for $|w| = \varepsilon' < \varepsilon$. The maximum principle implies that $|g_z(w)| \leq C\varepsilon'^{-k}$ for $w \in \mathbb{D}_{\varepsilon'}(0)$. If $|z| = \varepsilon' < \varepsilon$, we have,

$$|z|^{-k}|f(z)| = |g_z(|z|)| \leq C\varepsilon'^{-k}$$

Since $\varepsilon' < \varepsilon$ is arbitrary, we have

$$|z|^{-k}|f(z)| = |g_z(|z|)| \leq C\varepsilon^{-k}$$

This completes the proof. □

2. Complexification & Complex Structures

2.1. **Complexification.** We begin by discussing the technique of complexification, which allows us to *complexify* a $\mathbb{R}$-vector space.

Definition 2.1. If V is a $\mathbb{R}$-vector space, we define the **complexification** of V, denoted by $V^{\mathbb{C}}$, to be the $\mathbb{C}$-vector space $V \oplus V$ with scalar multiplication by complex numbers defined as follows:

$$(a+ib)(u,v) = (au - bv, av + bu) \quad \text{for } a+ib \in \mathbb{C}.$$

Remark 2.2. *$V^{\mathbb{C}}$ is then a $\mathbb{C}$-vector space over $\mathbb{C}$.*

The map $V \to V^{\mathbb{C}}$ given by $u \mapsto (u,0)$ is a $\mathbb{R}$-linear isomorphism from V onto the (real) subspace $V \oplus \{0\} \subseteq V^{\mathbb{C}}$. We identify V with its image under this map. We can write

$$(u,v) = u + iv$$

and we can think of $V^{\mathbb{C}}$ as consisting of the set of all linear combinations of elements of V with complex coefficients. If $\dim_{\mathbb{R}} V = n$ and $\{b_1, \ldots, b_n\}$ is any basis for V (over $\mathbb{R}$), then

$$\{(b_1, 0), \ldots, (b_n, 0)\}$$

is a basis for $V^{\mathbb{C}}$ over $\mathbb{C}$. Hence $\dim_{\mathbb{C}} V^{\mathbb{C}} = n$. On the other hand,

$$\{(b_1, 0), \ldots, (b_n, 0), (0, ib_1), \ldots, (0, ib_n)\}$$

is a basis for $V^{\mathbb{C}}$ over $\mathbb{R}$. Hence $\dim_{\mathbb{R}} V^{\mathbb{C}} = 2n$.

Definition 2.3. If $L\colon V \to W$ is a linear map between $\mathbb{R}$-vector spaces, the **complexification** of L is the $\mathbb{C}$-linear map

$$\begin{aligned} L^{\mathbb{C}}\colon V^{\mathbb{C}} &\to W^{\mathbb{C}}, \\ u + iv &\mapsto L(u) + iL(v) \end{aligned}$$

For $k = \mathbb{R}, \mathbb{C}$ if $\mathbf{Vec}_k$ is the category of finite-dimensional k-vector spaces, then complexification can be thought of as a functor,

$$\mathscr{F} : \mathbf{Vec}_{\mathbb{R}} \to \mathbf{Vec}_{\mathbb{C}}$$

such that $\mathscr{F}(V) = V^{\mathbb{C}}$ and $\mathscr{F}(L) = L^{\mathbb{C}}$. Clearly, $\mathscr{F}(\mathrm{Id}_V) = \mathrm{Id}_{V^{\mathbb{C}}}$. Moreover if $L_1 : V_1 \to W_1$ and $L_2 : V_2 \to W_2$ are $\mathbb{R}$-linear maps, then

$$\begin{aligned} \mathscr{F}(L_2 \circ L_1)(u+iv) &= (L_2 \circ L_1)^{\mathbb{C}}(u+iv) \\ &= L_2(L_1(u)) + iL_2(L_1(v)) \\ &= (L_2^{\mathbb{C}} \circ L_1^{\mathbb{C}})(u+iv) \\ &= (\mathscr{F}(L_2) \circ \mathscr{F}(L_1))(u+iv) \end{aligned}$$

Hence, $\mathscr{F}(L_2 \circ L_1) = \mathscr{F}(L_2) \circ \mathscr{F}(L_1)$.

Remark 2.4. *There is another way to think about complexification. If V is a $\mathbb{R}$-vector space, we can consider the $(V')^{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$. $(V')^{\mathbb{C}}$ is a $\mathbb{C}$-vector space with the usual addition and with scalar multiplication defined by*

$$\alpha(v \otimes \beta) = v \otimes (\alpha\beta), \qquad v \in V \ \alpha, \beta \in \mathbb{C}$$

Consider the map

$$\begin{aligned} \phi \colon V^{\mathbb{C}} &\to (V')^{\mathbb{C}} \\ (u, v) &\mapsto u \otimes_{\mathbb{R}} 1 + v \otimes_{\mathbb{R}} i \end{aligned}$$

The map is $\mathbb{C}$-linear. Indeed, ϕ is additive because $\otimes_{\mathbb{R}}$ is bilinear. Moreover, we have,

$$\begin{aligned} \phi((a+ib)(u,v)) &= \phi(au - bv, av + bu) \\ &= (au - bv) \otimes_{\mathbb{R}} 1 + (av + bu) \otimes_{\mathbb{R}} i \\ &= u \otimes_{\mathbb{R}} (a + ib) + v \otimes_{\mathbb{R}} i(a + ib) \\ &= (a + ib)(u \otimes_{\mathbb{R}} 1 + v \otimes_{\mathbb{R}} i) \\ &= (a + ib)\phi(u, v) \end{aligned}$$

Hence, ϕ is $\mathbb{C}$-linear. ϕ is surjective. Indeed if $v \otimes (a + ib) \in (V')^{\mathbb{C}}$, then

$$v \otimes_{\mathbb{R}} (a + ib) = av \otimes_{\mathbb{R}} 1 + bv \otimes_{\mathbb{R}} i$$

implies that $\phi(av, bv) = v \otimes (a + ib)$. Since $V^{\mathbb{C}}$ and $(V')^{\mathbb{C}}$ are finite-dimensional $\mathbb{C}$-vector spaces, ϕ is a $\mathbb{C}$-linear isomorphism.

2.2. **Complex Structures.** We now discuss complex structures. A complex structure is a property of a $\mathbb{R}$-vector space that allows it to be treated as a $\mathbb{C}$-vector space. To motivate this, consider the following: Let V be a $\mathbb{C}$-vector space. Scalar multiplication by i defines a linear map $v \mapsto iv$ on the underlying $\mathbb{R}$-vector space, which squares to $-I$. By ignoring the $\mathbb{C}$-vector space structure, we can think of J as a $\mathbb{R}$-linear map satisfying $J \circ J = -\mathrm{Id}$.

Definition 2.5. Let V be a $\mathbb{R}$-vector space. A **complex structure** on V is a $\mathbb{R}$-linear map $J \colon V \to V$ satisfying $J \circ J = -\mathrm{Id}_V$.

Example 2.6. Let $V = \mathbb{C}^n$. Then

$$\mathbb{C}^n \cong \mathbb{R}^{2n} = \{(x_1, \ldots, x_n, y_1, \ldots, y_n) \mid x_i, y_i \in \mathbb{R}\}$$

and the complex structure $J^{\mathbb{C}^n}$ is given by

$$J^{\mathbb{C}^n}(x_1, \ldots, x_n, y_1, \ldots, y_n) = (-y_1, \ldots, -y_n, x_1, \ldots, x_n).$$

Lemma 2.7. *Let V be a $\mathbb{R}$-vector space. If J is a complex structure on V, then V admits in a natural way the structure of a $\mathbb{C}$-vector space.*

Proof. Define scalar multiplication by $\mathbb{C}$ on V by

$$(a + ib)v = av + bJ(v),$$

where $a, b \in \mathbb{R}$. The $\mathbb{R}$-linearity of J and the assumption $J^2 = -\mathrm{Id}$ yield

$$((a + ib)(c + id))v = (a + ib)((c + id) \cdot v),$$

This completes the proof. $\square$

Remark 2.8. *If V is a $\mathbb{R}$-linear vector space admitting a complex structure, J, then the $\mathbb{C}$-linear vector space structure on V is denoted as (V, J). Combined with our discussion of complexification, we see that complex structures on $\mathbb{R}$-vector spaces and $\mathbb{C}$-vector spaces are equivalent notions.*

Proposition 2.9. *Let V be a $\mathbb{R}$-vector space. If J is a complex structure V, then $V^{\mathbb{C}}$ has an eigenspace decomposition of the form*

$$\begin{aligned} V^{\mathbb{C}} &= V_{(1,0)} \oplus V_{(0,1)} \\ &= \left\{ \frac{v - iJv}{2} \mid v \in V \right\} \bigoplus \left\{ \frac{v + iJv}{2} \mid v \in V \right\}. \end{aligned}$$

where $V_{1,0}$ is the i-eigenspace of $J^{\mathbb{C}}$ and $V_{0,1}$ is the $-i$-eigenspace of J. Moreover, $V_{(1,0)} \cong V_{(0,1)}$ as $\mathbb{R}$-linear vector spaces.

Proof. Given $v \in V^{\mathbb{C}}$, define v' and v'' by the formulas

$$v' = \frac{1}{2}(v - iJv), \quad w'' = \frac{1}{2}(v + iJv).$$

A simple computation show that

$$J^{\mathbb{C}} v' = iv' \quad J^{\mathbb{C}} v'' = -iv''$$

Because $v = v' + v''$, this shows that $V^{\mathbb{C}} = V_{(1,0)} + V_{(0,1)}$. A non-zero vector cannot be an eigenvector with two different eigenvalues, so $V_{(1,0)} \cap V_{(0,1)} = \{0\}$, which shows that the sum is direct. Conjugation is a bijective $\mathbb{R}-$linear map from $V^{\mathbb{C}}$ to itself and it interchanges $V_{(1,0)}$ and $V_{(0,1)}$. This shows that $V_{(1,0)} \cong V_{(0,1)}$ as $\mathbb{R}$-linear vector spaces. □

Lemma 2.10. *Let V be a finite-dimensional $\mathbb{R}$-vector space. If V admits a complex structure, then $\dim_{\mathbb{R}} V$ is even.*

Proof. If V admits a complex structure, then Proposition 2.9 implies that $\dim_{\mathbb{R}} V_{(1,0)} = \dim_{\mathbb{R}} V_{(0,1)}$. In turn, this implies that $\dim_{\mathbb{C}} V_{(1,0)} = \dim_{\mathbb{C}} V_{(0,1)}$. This follows because since $V_{(1,0)}, V_{(0,1)}$ are $\mathbb{C}$-vector spaces, we have that $\dim_{\mathbb{C}} V_{(1,0)}, V_{(0,1)}$ is half that of $\dim_{\mathbb{R}} V_{(1,0)}, V_{(0,1)}$. Hence, $\dim_{\mathbb{C}} V^{\mathbb{C}}$ is even. Since $\dim_{\mathbb{R}} V = \dim_{\mathbb{C}} V^{\mathbb{C}}$, the result follows. □

We close this section with an important observation that will be useful later on:

Proposition 2.11. *Let V be a $\mathbb{R}$-vector space such that $\dim_{\mathbb{R}} V = n$. Then*

(1) *We have*

$$(V^{\mathbb{C}})^* = (V^*)^{\mathbb{C}}.$$

(2) *We have*

$$(V^*)^{\mathbb{C}} \cong ((V^*, J^*))_{(1,0)} \oplus ((V^*, J^*))^*_{(0,1)}$$

Here J^* is a complex structure on V^* we view (V^*, J^*) as a $\mathbb{C}$-linear space via Lemma 2.7.

Proof. The proof is given below:

(1) Note that we have

$$\begin{aligned}(V^{\mathbb{C}})^* &= \mathrm{Hom}_{\mathbb{C}}(V^{\mathbb{C}}, \mathbb{C}) \\ &= \mathrm{Hom}_{\mathbb{C}}(V \otimes_{\mathbb{R}} \mathbb{C}, \mathbb{C}) \\ &\cong \mathrm{Hom}_{\mathbb{R}}(V, \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}, \mathbb{C})) \\ &\cong \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C}) \cong \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} = (V^*)^{\mathbb{C}}\end{aligned}$$

The second isomorphism follows from the tensor-hom adjunction since V is a $(\mathbb{R}, \mathbb{C})$-vector space. The isomorphism $\mathrm{Hom}_{\mathbb{C}}(\mathbb{C}, \mathbb{C}) \cong \mathbb{C}$ is clear.

(2) First note that V^* has a complex structure given by the $\mathbb{R}$-linear map

$$\begin{aligned}J^* : V^* &\to V^*, \\ \varphi &\mapsto \varphi \circ J\end{aligned}$$

The isomorphism

$$(V^*)^{\mathbb{C}} \cong (V_{(1,0)})^* \oplus (V_{(0,1)})^*$$

follows from (1) and Proposition 2.9. It suffices to prove that $(V_{(1,0)})^* \cong (V^*, J^*)_{(1,0)}$. Note that there exists a $\mathbb{C}$-linear isomorphism

$$\begin{aligned}(V, J) &\to V_{(1,0)}, \\ v &\mapsto \frac{1}{2}(v - iJv).\end{aligned}$$

Here V is viewed as a $\mathbb{C}$-linear space via Lemma 2.7. This induces the $\mathbb{C}$-linear isomorphism $\mathrm{Hom}_{\mathbb{C}}(V_{(1,0)}, \mathbb{C}) \cong \mathrm{Hom}_{\mathbb{C}}((V, J), \mathbb{C})$. Therefore, we have

$$\begin{aligned}(V^*, J^*)_{(1,0)} &= \{\varphi \in \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C}) \mid J^*(\varphi) = i\varphi\} \\ &= \{\varphi \in \mathrm{Hom}_{\mathbb{R}}(V, \mathbb{C}) \mid \varphi \circ J = i\varphi\} \\ &\simeq \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C}) \simeq \mathrm{Hom}_{\mathbb{C}}(V_{(1,0)}, \mathbb{C}) = (V_{(1,0)})^*.\end{aligned}$$

This completes the proof. □

3. Multilinear & Alternating Maps

3.1. Multilinear Maps.

We first discuss multi-linear maps.

Definition 3.1. For $i = 1, \cdots, k$, let V_i be $\mathbb{R}$-vector spaces, and let W be any other $\mathbb{R}$-vector space. A map

$$\omega\colon \underbrace{V_1 \times \cdots \times V_k}_{k-\text{times}} \to \mathbb{R}$$

is said to be k-multi-linear if it is linear as a function of each variable separately when the others are held fixed. For each i, this means:

$$\omega(v_1, \ldots, av_i + a'v_i', \ldots, v_k) = a\omega(v_1, \ldots, v_i, \ldots, v_k) + a'\omega(v_1, \ldots, v_i', \ldots, v_k),$$

where $a, a' \in \mathbb{R}$ and $v_i, v_i' \in V_i$.

Remark 3.2. *It can be easily checked that the set of k-multi-linear maps is a $\mathbb{R}$-linear space, which we denote as $L(V_1, \ldots, V_k; W)$. A k-multi-linear map is also called a k-tensor.*

When $W = \mathbb{R}$, we can characterize the vector space of k-multi-linear maps as follows:

Proposition 3.3. *For $i = 1, \cdots, k$, let V_i be $\mathbb{R}$-vector spaces such that $\dim_{\mathbb{R}} = n_i$. There is a canonical isomorphism:*

$$V_1^* \otimes \cdots \otimes V_k^* \cong L(V_1, \dots, V_k; \mathbb{R}),$$

Hence,

$$\dim_{\mathbb{R}} L(V_1, \dots, V_k; \mathbb{R}) = n_1 \cdots n_k$$

Proof. Define a map $\Phi\colon V_1 \times \cdots \times V_k \to L(V_1, \dots, V_k; \mathbb{R})$ such that

$$\Phi(\omega^1, \dots, \omega^k)(v_1, \dots, v_k) = \omega^1(v_1) \cdots \omega^k(v_k).$$

The expression on the right depends linearly on each v_i, so $\Phi(\omega_1, \dots, \omega_k)$ is indeed an element of the space $L(V_1, \dots, V_k; \mathbb{R})$. It is easy to check that Φ is k-multi-linear as a function of $\omega_1, \dots, \omega_k$. By the characteristic property on tensor products, it descends uniquely to a linear map $\overline{\Phi} : V_1^* \otimes \cdots \otimes V_k^* \to L(V_1, \dots, V_k; \mathbb{R})$, which satisfies

$$\overline{\Phi}(\omega_1 \otimes \cdots \otimes \omega_k)(v_1, \dots, v_k) = \omega_1(v_1) \cdots \omega_k(v_k).$$

The linear map $\overline{\Phi}$ takes the basis of $V_1^* \otimes \cdots \otimes V_k^*$ to the basis for $L(V_1, \dots, V_k; \mathbb{R})$. So it is an isomorphism. $\square$

Remark 3.4. *From now on, assume that $V_i = V$, $\dim_{\mathbb{R}} V = n$ for each i and $W = \mathbb{R}$.*

3.2. **Alternating Maps.** We now specialize to the case of alternating k-multi-linear maps.

Definition 3.5. A map

$$\omega\colon \underbrace{V \times \cdots \times V}_{k-\text{times}} \to \mathbb{R}$$

is said to be an alternating k-multi-linear map if it is k-multi-linear and that for every pair of distinct indices i, j, it satisfies

$$\omega(v_1, \dots, v_i, \dots, v_j, \dots, v_k) = -\omega(v_1, \dots, v_j, \dots, v_i, \dots, v_k).$$

Remark 3.6. *If ω is an alternating k-multi-linear map, then the effect of an arbitrary permutation $\sigma \in S_k$ of its arguments is given by*

$$\omega(v_{\sigma(1)}, \dots, v_{\sigma(k)}) = \operatorname{sgn}(\sigma)\omega(v_1, \dots, v_k),$$

where $\operatorname{sgn}(\sigma)$ represents the sign of the permutation $\sigma \in S_k$. This follows from repeated applications of the definition of an alternating k-multi-linear map and the definition of $\operatorname{sgn}(\sigma)$ for $\sigma \in S_k$.

Remark 3.7. *It can be easily checked that the set of alternating k-multi-linear maps is a $\mathbb{R}$-linear space, which we denote as $A(V_1, \dots, V_k; W)$. An alternating k-multi-linear map is also called a k-form.*

Example 3.8. The determinant is a $\mathbb{R}$-valued multilinear n-form in $\mathbb{R}^n$. That is, the map $\det\colon (\mathbb{R}^n)^n \to \mathbb{R}$ is a an alternating n-multi-linear map.

The following lemma gives a different alternative characterization for the alternating condition:

Lemma 3.9. *Let V be a $\mathbb{R}$-vector space such that $\dim_{\mathbb{R}} V = n$. Let ω be a k-multi-linear map. The following are equivalent:*

1. *ω is an alternating k-multi-linear map.*
2. *$\omega(v_1, \dots, v_k) = 0$ whenever the k-tuple $(v_1, \dots, v_k)$ is linearly dependent.*

(3) *ω gives the value zero whenever two of its arguments are equal:*

$$\omega(v_1, \dots, w, \dots, w, \dots, v_k) = 0.$$

Proof. (1),(2) imply (3) are immediate. We complete the proof by showing that (3) implies both (1) and (2). Assume that ω satisfies (3). For any vectors $v_1, \dots, v_k$, the hypothesis implies

$$\begin{aligned} 0 &= \omega(v_1, \dots, v_i + v_j, \dots, v_i + v_j, \dots, v_k) \\ &= \omega(v_1, \dots, v_i, \dots, v_i, \dots, v_k) + \omega(v_1, \dots, v_i, \dots, v_j, \dots, v_k) \\ &\quad + \omega(v_1, \dots, v_j, \dots, v_i, \dots, v_k) + \omega(v_1, \dots, v_j, \dots, v_j, \dots, v_k) \\ &= \omega(v_1, \dots, v_i, \dots, v_j, \dots, v_k) + \omega(v_1, \dots, v_j, \dots, v_i, \dots, v_k). \end{aligned}$$

Thus, ω is an alternating k-multi-linear map. Hence, (3) implies (1). If $(v_1, \dots, v_k)$ is a linearly dependent k-tuple, then one of the v_i's can be written as a linear combination of the others. For simplicity, let us assume that $v_k = \sum_{j=1}^{k-1} a_j v_j$. Since ω is k-multi-linear, we have,

$$\omega(v_1, \dots, v_k) = \sum_{j=1}^{k-1} a_j \omega(v_1, \dots, v_{k-1}, v_j).$$

In each of these terms, ω has two identical arguments, so every term is zero. Hence, (3) implies (2). $\square$

Remark 3.10. *In what follows, let $\{E_i\}_{i=1}^n$ denote a basis for V, and let $\{\varepsilon^j\}_{j=1}^n$ denote the dual basis for V^*.*

Note that Proposition 3.3 implies that the set

$$\{\varepsilon^{i_1} \otimes \cdots \varepsilon^{i_k} \mid 1 \leq i_1, \cdots, i_k, \leq n\}$$

is a basis for $L(V, \cdots, V; \mathbb{R})$. We would like to find a basis for $A(V, \cdots, V; \mathbb{R})$, the subspace of alternating k-multi-linear maps. Preempting the discussion in the next section, we write $A(V, \cdots, V; \mathbb{R})$ as $\Lambda^k(V)$ in the remainder of this section. We first define a collection of alternating k-multi-linear maps on V that generalize the determinant function on $\mathbb{R}^n$.

Definition 3.11. Let V be a $\mathbb{R}$-vector space and let $\dim_{\mathbb{R}} V = n$. For each multi-index $I = (i_1, \dots, i_k)$ of length k such that $1 \leq i_1, \dots, i_k \leq n$, define an alternating k-multi-linear map as follows:

$$\varepsilon^I : \underbrace{V \times \cdots \times V}_{k\text{-times}} \to \mathbb{R}$$

$$(v_1, \dots, v_k) \mapsto \det \begin{pmatrix} \varepsilon^{i_1}(v_1) & \cdots & \varepsilon^{i_1}(v_k) \\ \vdots & \ddots & \vdots \\ \varepsilon^{i_k}(v_1) & \cdots & \varepsilon^{i_k}(v_k) \end{pmatrix} = \det \begin{pmatrix} v_1^{i_1} & \cdots & v_k^{i_1} \\ \vdots & \ddots & \vdots \\ v_1^{i_k} & \cdots & v_k^{i_k} \end{pmatrix}$$

Because the determinant changes sign whenever two columns are interchanged, it is clear that ε^I in Definition 3.11 is an alternating k-multi-linear map.

Remark 3.12. *We introduce some notation. If $I = (i_1, \cdots, i_k)$ is a multi-index and $\sigma \in S_k$ is a permutation of $\{1, \dots, k\}$, we write I_σ for the following multi-index:*

$$I_\sigma = (i_{\sigma(1)}, \dots, i_{\sigma(k)}).$$

Note that $I_{\sigma\tau} = (I_\sigma)_\tau$ for $\sigma, \tau \in S_k$. If I and J are multi-indices of length k, we define δ^I_J as follows:

$$\delta^I_J = \det \begin{pmatrix} \delta^{i_1}_{j_1} & \delta^{i_1}_{j_2} & \cdots & \delta^{i_1}_{j_k} \\ \delta^{i_2}_{j_1} & \delta^{i_2}_{j_2} & \cdots & \delta^{i_2}_{j_k} \\ \vdots & \vdots & \ddots & \vdots \\ \delta^{i_k}_{j_1} & \delta^{i_k}_{j_2} & \cdots & \delta^{i_k}_{j_k} \end{pmatrix}$$

We have

$$\delta^J_I = \begin{cases} \operatorname{sgn}(\sigma) & \textit{if neither } I \textit{ nor } J \textit{ has a repeated index and } J = I_\sigma \textit{ for some } \sigma \in S_k, \\ 0 & \textit{if } I \textit{ or } J \textit{ has a repeated index or } J \textit{ is not a permutation of } I. \end{cases}$$

Lemma 3.13. *The following statements are true:*

1. *If I has a repeated index, then $\varepsilon^I = 0$.*
2. *If $J = I_\sigma$ for some $\sigma \in S_k$, then $\varepsilon^I = sgn(\sigma)\varepsilon^J$.*
3. *The result of evaluating ε^I on a sequence of basis vectors is*

$$\varepsilon^I(E_{j_1}, \ldots, E_{j_k}) = \delta^I_J$$

Proof. If I has a repeated index, then for any vectors $v_1, \ldots, v_k$, the determinant in the definition of ε^I has two identical rows and thus is equal to zero, which proves (1). On the other hand, if J is obtained from I by interchanging two indices, then the corresponding determinants have opposite signs; this implies (2). Finally, (3) follows immediately from the definition of ε^I. □

The importance of the k-alternating tensors ε^I, for an increasing multi-index I, is given by the following proposition:

Proposition 3.14. *For each positive integer $0 \le k \le n$, the collection of k-alternating tensors*

$$\mathcal{E} = \{\varepsilon^I : I \textit{ is an increasing multi-index of length } k\}$$

is a basis for $\Lambda^k(V^*)$. Therefore,

$$\dim \Lambda^k(V^*) = \binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

If $k > n$, then $\dim \Lambda^k(V^*) = 0$.

Remark 3.15. *A multi-index $I = (i_1, \ldots, i_k)$ is said to be increasing if $i_1 < \ldots < i_k$.*

Proof. The fact that $\Lambda^k(V^*)$ is the trivial vector space when $k > n$ follows immediately from Lemma 3.13, since every k-tuple of vectors is linearly dependent in that case. So let $k \le n$. To show that $\mathcal{E}$ spans $\Lambda^k(V^*)$, let $\alpha \in \Lambda^k(V^*)$ be arbitrary. For each multi-index $I = (i_1, \ldots, i_k)$ (not necessarily increasing), define a real number α_I by

$$\alpha_I = \alpha(E_{i_1}, \ldots, E_{i_k}).$$

Lemma 3.13 implies:

$$\sum_I \alpha_I \varepsilon^I(E_{j_1}, \ldots, E_{j_k}) = \sum_I \alpha_I \delta^I_J = \alpha_J = \alpha(E_{j_1}, \ldots, E_{j_k}).$$

Thus, $\sum_I \alpha_I \varepsilon^I = \alpha$, so $\mathcal{E}$ spans $\Lambda^k(V^*)$. To show that $\mathcal{E}$ is a linearly independent set, suppose the identity

$$\sum_I \alpha_I \varepsilon^I = 0$$

holds for some coefficients α_I. Let J be any increasing multi-index. Applying both sides of the identity to the vectors $E_{j_1}, \ldots, E_{j_k}$ and using Lemma 3.13, we get

$$0 = \sum_I \alpha_I \varepsilon^I (E_{j_1}, \ldots, E_{j_k}) = \alpha_J.$$

Thus, each coefficient α_J is zero. □

3.3. **Wedge Product.** We can go a step further and define a product operation for alternating k-multi-linear maps. Recall that there is a product operation on the $\mathbb{R}$-linear space of k-multi-linear maps. If

$$\omega : \underbrace{V \times \cdots \times V}_{k-\text{times}} \to \mathbb{R}$$
$$\gamma : \underbrace{V \times \cdots \times V}_{l-\text{times}} \to \mathbb{R}$$

multi-linear maps we can define their product $\omega \otimes \gamma$ to be a $(k+l)k$-multi-linear map

$$\omega \otimes \gamma : \underbrace{V \times \cdots \times V}_{(k+l)-\text{times}} \to \mathbb{R}$$

such that

$$\omega \otimes \gamma(v_1, \ldots, v_k, v_{k+1}, \ldots, v_{k+l}) = \omega(v_1, \ldots, v_k)\gamma(v_{k+1}, \ldots, v_{k+l})$$

Now consider the case of alternating multi-linear maps. Given $\alpha \in \Lambda^k(V^*)$ and $\beta \in \Lambda^l(V^*)$, we define their wedge product (or exterior product) to be the following element in $\Lambda^{k+l}(V^*)$:

$$\alpha \wedge \beta = \frac{(k+l)!}{k!l!} \operatorname{Alt}(\alpha \otimes \beta) =$$

If ω is a k-multi-linear map, then $\operatorname{Alt}(\omega)$ is defined as

$$\operatorname{Alt}(\omega)(v_1, \ldots, v_k) = \frac{1}{k!} \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma)\omega(v_{\sigma(1)}, \ldots, v_{\sigma(k)}).$$

It is easy to verify that Alt defines a projection operator:

$$\operatorname{Alt} : L(V, \cdots, V; \mathbb{R}) \to A(V, \cdots, V; \mathbb{R})$$

The mysterious coefficient is motivated by statement of Lemma 3.16:

Lemma 3.16. *For any multi-indices $I = (i_1, \ldots, i_k)$ and $J = (j_1, \ldots, j_l)$,*

$$\varepsilon^I \wedge \varepsilon^J = \varepsilon^{IJ},$$

where $IJ = (i_1, \ldots, i_k, j_1, \ldots, j_l)$ is obtained by concatenating I and J.

Proof. By multilinearity, it suffices to show that

$$\varepsilon^I \wedge \varepsilon^J \left(E_{p_1}, \cdots, E_{p_{k+1}}\right) = \varepsilon^{IJ} \left(E_{p_1}, \cdots, E_{p_{k+1}}\right)$$

We consider several cases.

- If $P = (p_1, \ldots, p_k, q_1, \ldots, q_l)$ has a repeated index. In this case, both sides above of are zero.
- If P contains an index that does not appear in either I or J, the right-hand side is zero. Similarly, each term in the expansion of the left-hand side involves either I or J evaluated on a sequence of basis vectors that is not a permutation of I or J, respectively, so the left-hand side is also zero.
- If $P = IJ$ and P has no repeated indices, the right-hand side of is equal to 1. So we need to show that the left-hand side is also equal to 1. By definition,

$$\varepsilon^I \wedge \varepsilon^J \left(E_{p_1}, \cdots, E_{p_{k+1}}\right) = \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} \operatorname{sgn}(\sigma)\, \varepsilon^I \left(E_{\sigma(1)}, \cdots, E_{\sigma(k)}\right) \varepsilon^J \left(E_{\sigma(k+1)}, \cdots, E_{\sigma(k+l)}\right).$$

  By Lemma 3.13, the only terms in the sum above that give nonzero values are those in which σ permutes the first k indices and the last l indices of P separately. In other words, σ must be of the form $\sigma = \tau\eta$, where $\tau \in S_k$ acts by permuting $\{1, \ldots, k\}$ and $\eta \in S_l$ acts by permuting $\{k+1, \ldots, k+l\}$. Since $\operatorname{sgn}(\sigma) = \operatorname{sgn}(\tau)\operatorname{sgn}(\eta)$, we have:

$$\begin{aligned}\varepsilon^I \wedge \varepsilon^J \left(E_{p_1}, \cdots E_{p_{k+l}}\right) &= \frac{1}{k!l!} \sum_{\tau \in S_k\ \eta \in S_l} \operatorname{sgn}(\tau)\operatorname{sgn}(\sigma)\, \varepsilon^I \left(E_{p_{\tau(1)}}, \cdots, E_{p_{\tau(k)}}\right) \varepsilon^J \left(E_{p_{k+\eta(1)}}, \cdots, E_{p_{k+\eta(l)}}\right)\\ &= \varepsilon^I (E_{p_1}, \cdots E_{p_k}) \varepsilon^J (E_{p_{k+1}}, \cdots E_{p_{k+l}}) = 1\end{aligned}$$

- If P is a permutation of IJ and has no repeated indices, applying a permutation to P brings us back to the above case. Since the effect of the permutation is to multiply both sides of

$$\varepsilon^I \wedge \varepsilon^J \left(E_{p_1}, \cdots, E_{p_{k+1}}\right) = \varepsilon^{IJ} \left(E_{p_1}, \cdots, E_{p_{k+1}}\right)$$

  by the same sign, the result holds in this case as well.

This completes the proof. □

Proposition 3.17. *Let V be a $\mathbb{R}$-vector space such that $\dim_{\mathbb{R}} V = n$. Let α, β, γ be alternating k-multi-linear maps on V. The wedge product satisfies the following properties:*

(1) *(Bilinearlity) For $a, b \in \mathbb{R}$,*

$$(a\alpha + b\beta) \wedge \gamma = a\alpha \wedge \gamma + b\beta \wedge \gamma$$

(2) *(Associativity)*

$$(\alpha \wedge \beta) \wedge \gamma = \alpha \wedge (\beta \wedge \gamma).$$

(3) *(Graded Anti-Commutivity) If $\alpha \in \Lambda^k(V^*)$ and $\beta \in \Lambda^l(V^*)$,*

$$\alpha \wedge \beta = (-1)^{kl}\, \beta \wedge \alpha$$

(4) *If $I = (i_1, \ldots, i_k)$ is any multi-index, then*

$$\varepsilon^{i_1} \wedge \ldots \wedge \varepsilon^{i_k} = \varepsilon^I$$

Proof. (1) follows immediately from the definition, because the tensor product is bilinear and Alt is linear. By Lemma 3.16,

$$(\varepsilon^I \wedge \varepsilon^J) \wedge \varepsilon^K = \varepsilon^{IJ} \wedge \varepsilon^K = \varepsilon^{IJK} = \varepsilon^I \wedge \varepsilon^{JK} = \varepsilon^I \wedge (\varepsilon^J \wedge \varepsilon^K)$$

The general case follows from bilinearity. Similarly, using Lemma 3.16 again, we get

$$\varepsilon^I \wedge \varepsilon^J = \varepsilon^{IJ} = \operatorname{sgn}(\sigma)\, \varepsilon^{JI} = \operatorname{sgn}(\sigma)\, \varepsilon^J \wedge \varepsilon^I,$$

where σ is the permutation that sends IJ to JI. It is easy to check that $\operatorname{sgn}(\sigma) = (-1)^{kl}$, because σ can be decomposed as a composition of kl transpositions (each index of I must be moved past each of the indices of J). (3) then follows from bilinearity. (4) is an immediate consequence of Lemma 3.13. $\square$

Proposition 3.18. *The wedge product is the unique associative, bilinear map*

$$\Lambda^k(V^*) \times \Lambda^l(V^*) \to \Lambda^{k+l}(V^*)$$

satisfying (1)-(4) in Proposition 3.17.

Proof. If we take $\omega \in \Lambda^k(V^*)$ and $\eta \in \Lambda^l(V^*)$ then we can expand in the usual basis as $\omega = \sum_I \omega_I \varepsilon^I$, $\eta = \sum_J \eta_J \varepsilon^J$. Taking $*$ to be a map satisfying these four properties. By bilinearity,

$$\omega * \eta = \left(\sum_I \omega_I \varepsilon^I\right) * \left(\sum_J \eta_J \varepsilon^J\right) = \sum_I \sum_J \omega_I \eta_J \left(\varepsilon^I * \varepsilon^J\right)$$

Using associativity and (4) in Proposition 3.17

$$\varepsilon^I * \varepsilon^J = (\varepsilon^{i_1} * \cdots * \varepsilon^{i_k}) * (\varepsilon^{j_1} * \cdots * \varepsilon^{j_l}) = \varepsilon^{i_1} * \cdots * \varepsilon^{i_k} * \varepsilon^{j_1} * \cdots * \varepsilon^{j_l} = \varepsilon^{IJ} = \varepsilon^I \wedge \varepsilon^J$$

This proves the claim. $\square$

Remark 3.19. *We can now define a $\mathbb{R}$-vector space $\Lambda(V)$*

$$\Lambda(V^*) = \bigoplus_{k=0}^{n} \Lambda^k(V^*),$$

Clearly, $\dim_{\mathbb{R}} \Lambda(V) = 2^n$. *The wedge product turns $\Lambda(V)$ into an associative algebra, called the exterior algebra of V^*. This algebra is not commutative, but it is graded-anticommutative in the sense that if $\alpha \in \Lambda^k(V^*), \beta \in \Lambda^l(V^*)$, then $\alpha \wedge \beta \in \Lambda^{k+l}(V^*)$ and*

$$\alpha \wedge \beta = (-1)^{kl} \beta \wedge \alpha$$

4. Exterior Algebra

4.1. **Definitions.** Let V be an $\mathbb{R}$-vector space such that $\dim_{\mathbb{R}} V = n$. In the previous section we have constructed a graded associative algebra, $\Lambda(V^*)$, such that if $\alpha \in \Lambda(V^*)$, then $\alpha \wedge \alpha = 0$. Since V is finite-dimensional, we can identify V with $(V^*)^*$. Hence, we formally have

$$\Lambda(V) = \Lambda((V^*)^*) = \bigoplus_{k=0}^{n} \Lambda^k((V^*)^*)$$

Exploiting Proposition 3.3 If an element $\omega \in \Lambda((V^*)^*)$ is an alternating k-multi-linear map that can be identified with an element of $\underbrace{V \otimes \cdots \otimes V}_{k-\text{times}}$, we must have that $\omega \otimes \omega$ under this identification. This motivates the following definition:

Definition 4.1. Let V be a $\mathbb{R}$-vector space. Consider the tensor algebra $T(V)$:

$$T(V) = \bigoplus_{k=0}^{\infty} T^k V = \mathbb{R} \oplus V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \cdots .$$

The **exterior algebra** of V is the quotient algebra $\overline{\Lambda}(V) = T(V)/I$, where I is the two-sided ideal generated by all elements of the form $v \otimes v$ such that $v \in V$.

In analogy with the definition of the wedge product, we write an arbitrary element in $\overline{\Lambda}(V)$ is written as

$$v_1 \overline{\wedge} \cdots \overline{\wedge} v_k := \overline{v_1 \otimes \cdots \otimes v_k}$$

for some $k \geq 0$. Moreover, in analogy with the analogous construction in the previous section, we can define the following subspace of $\overline{\Lambda}(V)$:

$$\overline{\Lambda}^k(V) = \mathrm{Span}_{\mathbb{R}} = \{v_1 \overline{\wedge} \cdots \overline{\wedge} v_k \mid 1, i_1, \cdots, i_k \leq n\} \subseteq \Lambda(V)$$

We have the following basic properties:

Lemma 4.2. *Let V be a $\mathbb{R}$-vector space.*

(1) *We have $v \overline{\wedge} w = -w \overline{\wedge} v$ for each $v, w \in V$.*
(2) *For $k > n$, $\overline{\Lambda}^k(V) = 0$.*

Proof. The proof is given below:

(1) By construction $v \overline{\wedge} v = 0$ for all $v \in V$. Consequently, if $v, w \in V$ we have

$$\begin{aligned} 0 &= (v + w) \overline{\wedge} (v + w) \\ &= v \overline{\wedge} v + v \overline{\wedge} w + w \overline{\wedge} v + w \overline{\wedge} w \\ &= v \overline{\wedge} w + w \overline{\wedge} v. \end{aligned}$$

(2) Consider $v_1 \overline{\wedge} \cdots \overline{\wedge} v_k \in \Lambda^k(V)$. We can write each v_i as $v_i = c_i^j E_j$. We then have

$$\begin{aligned} v_1 \overline{\wedge} \cdots \overline{\wedge} v_k &= \left(\sum_{j=1}^n c_1^j E_j \right) \overline{\wedge} \cdots \overline{\wedge} \left(\sum_{j=1}^n c_k^j E_j \right) \\ &= \sum_{j_1, \cdots, j_k = 1}^n c_1^{j_1} \cdots c_k^{j_k} E_{j_1} \overline{\wedge} \cdots \overline{\wedge} E_{j_k}. \end{aligned}$$

Consider the term $E_{j_1} \overline{\wedge} \cdots \overline{\wedge} E_{j_k}$. Since $k > n$ we must have that $E_{j_l} = E_{j_k}$ for some $l \neq k$. Hence,

$$E_{j_1} \overline{\wedge} \cdots \overline{\wedge} E_{j_k} = 0$$

by (1). Since each summand is zero, we have that $v_1 \overline{\wedge} \cdots \overline{\wedge} v_k = 0$.

This completes the proof. □

Lemma 4.2 implies that we get an analogous direct sum decomposition

$$\overline{\Lambda}(V) = \bigoplus_{k=0}^n \overline{\Lambda}^k(V)$$

As discussed at the start of this section, the motivation behind the definition of the exterior algebra is to generalize the construction of the previous section. This is indeed the case as shown by Proposition 4.5. We first introduce a definition and a lemma.

Definition 4.3. Let V, W $\mathbb{R}$-vector spaces. A **pairing** between V and W is a bilinear map

$$\langle , \rangle : V \times W \to \mathbb{R}$$

A pairing is **perfect** if for every $v \in V$, there is a $w \in W$ such that $\langle v, w \rangle \neq 0$, and vice versa.

Lemma 4.4. *Let V and W be finite-dimensional $\mathbb{R}$-vector spaces The following statements are equivalent:*

(1) *There exists an isomorphism* $W \cong V^*$,
(2) *There exists a perfect pairing* $\langle , \rangle : V \times W \to \mathbb{R}$.

Proof. Assume that (1) is true. There is an obvious perfect pairing between V and V^*, given by

$$\langle , \rangle : V \times V^* \to \mathbb{R}, \quad \langle v, \varphi \rangle = \varphi(v)$$

Composing this with the obvious map $V \times W \to V \times V^*$, we see that (1) implies (2). Now assume that (2) is true. Given $w \in W$, we get a linear map $\varphi : V \to \mathbb{R}$ by sending v to $\varphi(v) = \langle v, w \rangle$. It is easy to see that this gives us a linear map $W \to V^*$, and since we have a perfect pairing, this map is injective. Since the vector spaces are finite-dimensional, we have that the linear map $W \to V^*$ is bijective, an isomorphism. Hence, (2) implies (1). □

We now prove the desired result.

Proposition 4.5. *Let* V *be a* $\mathbb{R}$*-vector space such that* $\dim_{\mathbb{R}} V = n$. *For each* $0 \leq k \leq n$, *we have*

$$\Lambda^k(V^*) \cong (\overline{\Lambda}^k(V))^* \cong \overline{\Lambda}^k(V^*)$$

Proof. (Sketch) Consider the following map:

$$\begin{aligned}\langle \cdot, \cdot \rangle : \overline{\Lambda}^k(V) \times \Lambda^k(V^*) \\ (v_1 \wedge \cdots \wedge v_k, \varepsilon^I) \mapsto \varepsilon^I(v_1 \wedge \cdots \wedge v_k)\end{aligned}$$

This is a valid map since any alternating k-multi-linear map in $\Lambda^k(V^*)$ is a linear combination of ε^I. It is clear that $\langle \cdot, \cdot \rangle$ is a pairing. Moreover, it is also a perfect pairing. The first isomorphism follows from Lemma 4.4. The second isomorphism is given by:

$$\varphi^1 \wedge \cdots \wedge \varphi^k \mapsto \left(v_1 \wedge \cdots \wedge v_k \mapsto \det\left(\varphi^i(v_j)\right)\right).$$

It can be easily checked that this is an isomorphism. □

Remark 4.6. *Based on Proposition 4.5 we can now write* $\overline{\Lambda}$ *as* Λ.

4.2. **Complexification of Exterior Algebra.** We now study the complexififcation of $\Lambda(V)$. We first prove a basic lemma.

Lemma 4.7. *Let* V *be a* $\mathbb{R}$*-vector space such that* $\dim_{\mathbb{R}} = n$. *Furthermore, assume that* $V = W_1 \oplus W_2$, *where* W_1, W_2 *are* $\mathbb{R}$*-vector subspaces of* V *such that* $\dim_{\mathbb{R}} W_i = m_i$. *Then*

(1) *We have*
$$(\Lambda^k V)^{\mathbb{C}} \cong \Lambda^k V^{\mathbb{C}}$$
(2) *We have*
$$\Lambda(W_1 \oplus W_2) \cong \Lambda W_1 \otimes_{\mathbb{R}} \Lambda W_2$$
(3) *For each* $0 \leq k \leq n$, *we have*
$$\Lambda^k(W_1 \oplus W_2) \cong \bigoplus_{p+q=k} (\Lambda^p W_1 \otimes_{\mathbb{R}} \Lambda^q W_2)$$

Proof. The proof is given below:

(1) First fix some $z \in \mathbb{C}$ and consider the the map
$$\begin{aligned}\underbrace{V \times \cdots \times V}_{k-\text{times}} \to \Lambda^k V^{\mathbb{C}} \\ (v_1, \ldots, v_k) \mapsto z \cdot (v_1 \otimes 1) \wedge \cdots \wedge (v_k \otimes 1).\end{aligned}$$

This map is an alternating k-multi-linear map. Hence it descends to an $\mathbb{R}$-linear map

$$\begin{aligned}\phi_z : \Lambda^k V &\to \Lambda^k(V^{\mathbb{C}}) \\ v_1 \wedge \cdots \wedge v_k &\mapsto z \cdot (v_1 \otimes 1) \wedge \cdots \wedge (v_k \otimes 1).\end{aligned}$$

Here we have used the universal property of the tensor product and the k-th exterior power. Now, we define an $\mathbb{R}$-bilinear map

$$\begin{aligned}T : \Lambda^k V \times \mathbb{C} &\to \Lambda^k(V^{\mathbb{C}}) \\ (\omega, z) &\mapsto \phi_z(\omega).\end{aligned}$$

Since T is $\mathbb{R}$-bilinear, it lifts to an $\mathbb{R}$-linear map

$$\begin{aligned}\overline{T} : \Lambda^k V \otimes_{\mathbb{R}} \mathbb{C} \to (\Lambda^k V)^{\mathbb{C}} &\to \Lambda^k(V^{\mathbb{C}}) \\ (\omega \otimes z) &\mapsto \phi_z(\omega).\end{aligned}$$

The map $\overline{T}$ is defined as

$$\overline{T}\big((v_1 \wedge \ldots \wedge v_k) \otimes z\big) = z \cdot (v_1 \otimes 1) \wedge \ldots \wedge (v_k \otimes 1)$$

It is easy to that T is $\mathbb{C}$-linear isomorphism.

(2) (Sketch) We take for granted the statement that $\Lambda(-)$ is a functor from the category of $\mathbb{R}$-vector spaces to the category of graded-commutative $\mathbb{R}$-algebras, such that $\Lambda(-)$ is left adjoint to the functor that takes the degree 1 part. Since $\Lambda(-)$ is is left adjoint, it preserves colimits, in particular direct sums. Hence,

$$\Lambda(W_1 \oplus W_2) \cong \Lambda W_1 \otimes_{\mathbb{R}} \Lambda W_2$$

(3) This follows from (2). Indeed, (2) implies

$$\begin{aligned}\bigoplus_{k=0}^{n} \Lambda^k(W_1 \oplus W_2) &= \Lambda(W_1 \oplus W_2) \\ &= \Lambda W_1 \otimes_{\mathbb{R}} \Lambda W_2 \\ &= \left(\bigoplus_{p=0}^{m_1} \Lambda^p W_1\right) \otimes_{\mathbb{R}} \left(\bigoplus_{q=0}^{m_2} \Lambda^q W_2\right) = \bigoplus_{p,q=0}^{m_1,m_2} \Lambda^p W_1 \otimes_{\mathbb{R}} \Lambda^q\end{aligned}$$

Recall the Vandermode identity

$$\binom{n}{k} = \binom{m_1 + m_2}{k} = \sum_{l=0}^{k} \binom{m_1}{l}\binom{m_2}{k-l} = \sum_{p+q=k} \binom{m_1}{p}\binom{m_2}{q} W_2$$

If we identify the degree k-component of each side of the equation with a vector subspace of dimension $\binom{n}{k}$, we have, by comparing the degree k-component, that

$$\Lambda^k(W_1 \oplus W_2) = \bigoplus_{p+q=k} \Lambda^p W_1 \otimes_{\mathbb{R}} \Lambda^q W_2$$

This completes the proof $\square$

Corollary 4.8. *Let V be a $\mathbb{R}$-vector space such that $\dim_{\mathbb{R}} = n$. Then*

$$(\Lambda^k V)^{\mathbb{C}} = \bigoplus_{p+q=k} (\Lambda^p V_{(1,0)} \otimes_{\mathbb{C}} \Lambda^q V_{(0,1)})$$

Proof. Using Lemma 4.7, we can do the following computation:

$$\begin{aligned}(\Lambda^k V)^{\mathbb{C}} &= \Lambda^k V^{\mathbb{C}} \\ &= \Lambda^k (V_{(1,0)} \oplus V_{(0,1)}) \\ &= \bigoplus_{p+q=k} \left(\Lambda^p V_{(1,0)} \otimes_{\mathbb{C}} \Lambda^q V_{(0,1)}\right).\end{aligned}$$

This completes the proof. □

We will use Lemma 4.7 as a starting point for the study of differential forms on a complex manifold.

Part 2. Complex Manifolds

Complex manifolds are topological spaces locally modeled on open subsets in $\mathbb{C}^n$ with holomorphic transition functions. Complex manifolds are closely related to smooth manifolds, yet they exhibit notable distinctions in several aspects. The global counterparts of the similarities and differences between differentiable and holomorphic functions arise within the framework of complex manifold theory.

5. Definitions & Examples

5.1. Definitions.

We first provide definitions.

Definition 5.1. Let X be a second-countable, Hausdorff, connected topological space. X is a **complex manifold of dimension** n if there is a there exists a collection $\{(U_i, \varphi_i)\}_{i\in I}$ such that:

(1) U_i are open sets cover X,
(2) Each $\varphi_i : U_i \to D_i$, where D_i is an open subset of some $\mathbb{C}^n$, is a homemorphism,
(3) The transition functions

$$\varphi_i \circ \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j),$$

are holomorphic.

Such a collection is called a **holomorphic atlas** for X. Each (U_i, ε_i) is called a **coordinate chart**.

Remark 5.2. *Since all holomorphic functions are smooth, a holomorphic atlas is also a smooth atlas and thus determines a unique smooth structure on X. Thus, every complex manifold is also a smooth manifold in a canonical way.*

Philosophically, a complex n-manifold locally resembles $\mathbb{C}^n$. This local resemblance enables the extension of many constructions valid in $\mathbb{C}^n$ to a complex manifold. For instance, this framework allows us to define holomorphic functions on X.

Definition 5.3. Let X be a complex manifold. A **holomorphic function** on X is a function $f : X \to \mathbb{C}$ such that $f \circ \varphi_i^{-1} : \varphi_i(U_i) \to \mathbb{C}$ is holomorphic for some holomorphic chart (φ_i, U_i).

Remark 5.4. *It can be easily shown that if f is holomorphic, then f is holomorphic with respect to any holomorphic chart.*

We can now package all holomorphic functions on a complex manifold into the structure of a sheaf.

Definition 5.5. Let X be a complex manifold. The **structure sheaf on** X, denoted by $\mathscr{O}_X$, is the sheaf of holomorphic functions on X defined such that for any open subset $U \subseteq X$ we have

$$\mathscr{O}_X(U) = \{f : U \to \mathbb{C} \mid f \text{ is a holomorphic function}\}.$$

Remark 5.6. *It is clear from the definition that via a holomorphic chart (U, φ) with $p \in U$ and $\varphi(p) = 0 \in \mathbb{C}^n$, the stalk $\mathscr{O}_{X,p}$ is isomorphic to $\mathscr{O}_{\mathbb{C}^n,0}$.*

We define holomorphic functions between complex manifolds analogously to how smooth functions are defined between smooth manifolds.

Definition 5.7. Let X and Y be complex manifolds. A **holomorphic map** from X to Y is a continuous function $F : X \to Y$ with the property that for every $p \in X$ there exist holomorphic coordinate charts (U, φ) for X and (V, ψ) for Y such that

- $p \in U$ and $F(p) \in V$
- $F(U) \subseteq V$
- The composite map $\psi \circ F \circ \varphi^{-1}$ is holomorphic as a map from $\varphi(U)$ to $\psi(V)$.

The function $\widehat{F} := \psi \circ F \circ \varphi^{-1}$ is called the coordinate representation of f with respect to the given holomorphic coordinates.

Remark 5.8. *If $F : X \to Y$ is a bijective holomorphic map with holomorphic inverse, then we say that F is a biholomorphism.*

The fundamental difference between complex and differentiable manifolds becomes manifest is given by the following proposition:

Proposition 5.9. *Let X be a compact connected complex manifold.*

1. *Any global holomorphic function on X is constant. That is, $\mathscr{O}_X(X) \cong \mathbb{C}$.*
2. *If $\dim X \geq 2$ and $p \in X$, then $\mathscr{O}_X(X) = \mathscr{O}_X(X \setminus \{p\}) = \mathbb{C}$.*

Proof. The proof is given below:

1. Since X is compact, $|f| : X \to \mathbb{R}$ attains its maximum at some point $p \in X$. If (U_i, φ_i) is a holomorphic chart with $p \in U_i$, then $f \circ \varphi_i^{-1}$ is constant due to the maximum principle on $\varphi_i(U_i) \subseteq \mathbb{C}^n$. Hence, f is constant on U_i. Since X is connected, this shows that f must be constant[3]. Thus, $\mathscr{O}_X(X) \cong \mathbb{C}$.
2. This follows from (1) and Proposition 1.13.

This completes the proof. ☐

Here is another key difference between smooth manifolds and complex manifolds. A smooth manifold can always be covered by open subsets diffeomorphic to $\mathbb{R}^n$. In contrast, a complex manifold cannot be covered by open subsets biholomorphic to $\mathbb{C}^n$. This is because of the following proposition:

Proposition 5.10. *The unit ball $\mathbb{B}^{2n} \subseteq \mathbb{C}^n$ is not biholomorphic to $\mathbb{C}^n$.*

Proof. We know that $\mathbb{B}^{2n}$ and $\mathbb{C}^n$ are diffeomorphic. If $F : \mathbb{C}^n \to \mathbb{B}^{2n}$ is any holomorphic map, each of its coefficient functions is a bounded holomorphic function on $\mathbb{C}^n$ and therefore constant by Liouville's theorem. Thus, there is no biholomorphism between $\mathbb{B}^{2n}$ and $\mathbb{C}^n$. ☐

[3]A locally continuous functions on a connected space that is constant on an open set is constant.

The definitions of topological manifolds, smooth manifolds, and complex manifolds have the same structure. Hence, we now introduce a convenient framework that includes smooth manifolds, complex manifolds, and many other kinds of spaces.

Definition 5.11. Let X be a connected topological space. A **geometric structure** on X, denoted as $\mathscr{G}$, is a sub-sheaf of the sheaf of continuous functions on X such that the abelian groups $\mathscr{G}(U) \subseteq \mathscr{C}(U)$ contain all constant functions for each open set $U \subseteq X$. The pair $(X, \mathscr{G})$ is called a **geometric space**.

We have already discussed various examples of geometric spaces. In order to view a complex manifold as a geometric space, we need to define the notion of a morphism of geometric spaces.

Definition 5.12. Let $(X, \mathscr{G}_X)$ and $(Y, \mathscr{G}_Y)$ be geometric spaces. A **morphism of geometric spaces** is a continuous map $f : X \to Y$ such that whenever $U \subseteq Y$ is open, and $g \in \mathscr{G}_Y(U)$, the composition $g \circ f$ belongs to $\mathscr{G}_X(f^{-1}(U))$.

Example 5.13. Let $X = \mathbb{C}^n$ and $Y = \mathbb{C}^m$ and let $\mathscr{O}_X, \mathscr{O}_Y$ be the sheafs of holomorphic functions. A morphism $f : (\mathbb{C}^n, \mathscr{O}_X) \to (\mathbb{C}^m, \mathscr{O}_Y)$ is the same as a holomorphic mapping $f : \mathbb{C}^n \to \mathbb{C}^m$. This is because a continuous map $f : \mathbb{C}^n \to \mathbb{C}^m$ is holomorphic if and only if it preserves complex-valued holomorphic functions. The forward direction is clear since a composition of holomorphic functions is holomorphic functions. The reverse direction follows easily since the hypothesis implies that each coordinate function of f is holomorphic.

Example 5.14. If $(X, \mathscr{G}_X)$ is a geometric space, then any open subset $U \subseteq X$ inherits a geometric structure $(U, \mathscr{G}_X|_U)$, where $\mathscr{G}_X|_U$ is the restriction sheaf. With this definition, the natural inclusion map $(U, \mathscr{G}|_U) \to (X, \mathscr{G}_X)$ becomes a morphism of geometric spaces.

Remark 5.15. *For a morphism $f : (X, \mathscr{G}_X) \to (Y, \mathscr{G}_Y)$ of geometric spaces, we typically write*

$$f_\# : \mathscr{G}_Y(U) \to \mathscr{G}_X(f^{-1}(U))$$

for the induced homomorphisms. We say that f is an isomorphism if it has an inverse that is also a morphism. This means that $f : X \to Y$ should be a homeomorphism, and that each map $f_\# : \mathscr{G}_Y(U) \to \mathscr{G}_X(f^{-1}(U))$ should be an isomorphism.

We can now give an alternative definition of a complex manifold.

Definition 5.16. A **complex manifold of dimension** n is a geometric space $(X, \mathscr{G}_X)$ such that each $p \in X$ has an open neighborhood $U \subseteq X$, such that $(U, \mathscr{G}_X|_U) \cong (D, \mathscr{O}|_D)$for some open subset $D \subseteq \mathbb{C}^n$.

Proposition 5.17. *Let X be a topological space. X is a complex manifold in the sense of Definition 5.16 if and only if X is a complex manifold in the sense of Definition 5.1.*

Proof. Let $(X, \mathscr{G}_X)$ be a complex manifold in the sense of Definition 5.16. We can find for each $p \in X$ an open neighborhood $p \in U_p$, together with an isomorphism of geometric spaces $\varphi_p : (U_p, \mathscr{G}_X|_{U_p}) \to (D_p, \mathscr{O}|_D)$, for $D_p \subseteq \mathbb{C}^n$ open. The transition maps are clearly biholomorphic. This defines a cordinates atlas on X. Conversely, let X be a complex manifold in the sense of Definition 5.1. Let $\{(U_i, \varphi_i)\}_{i \in I}$ be a holomorphic atlas. For $U \subseteq X$ open, set

$$\mathscr{G}_X(U) = \{f \in \mathscr{C}(U) \mid (f|_{U \cap U_i}) \circ \varphi_i^{-1} \text{ is holomorphic on } \varphi_i(U \cap U_i) \text{ for all } i \in I\}.$$

This makes sense because the transition functions are biholomorphic. It is easy to see that $\mathscr{O}_X$ is a sub-sheaf of $\mathscr{C}$. Hence, $(X, \mathscr{G}_X)$ is a geometric space. It is also a complex manifold, because every point has an open neighborhood (namely one of the U_i) that is isomorphic to an open subset of $\mathbb{C}^n$. $\square$

Remark 5.18. *Is the dimension uniquely defined in Definition 5.1 or Definition 5.16? This is indeed the case. Let $x \in X$ such that x is contained in a coordinate chart (U, ϕ) such that $\phi(x) = 0$. Consider*

$$\mathscr{O}_x = \varinjlim_{x \in U} \mathscr{O}(U)$$

Let $\mathscr{O}_n$ denote the stalk at 0 of the sheaf of holomorphic functions on $\mathbb{C}^n$. Clearly,

$$\mathscr{O}_x \cong \mathscr{O}_n \cong \mathbb{C}[[z^1, \cdots, z^n]]$$

$\mathscr{O}_n$ is a local ring. Indeed, the unique maximal ideal is

$$\mathfrak{m}_n = \{f \in \mathscr{O}_n \mid f(0) = 0\};$$

If $f \in \mathscr{O}_n$ satisfies $f(0) \neq 0$, then f^{-1} is holomorphic in a neighborhood of the origin, and therefore $f^{-1} \in \mathscr{O}_n$. Clearly,

$$\mathscr{O}_n / \mathfrak{m}_n \cong \mathbb{C}$$

The integer n can be recovered from the ring $\mathscr{O}_n$. This is because Krull's dimension theory implies we have

$$\begin{aligned} n &= \dim(\mathbb{C}[[z^1, \dots, z^n]]) \\ &= \dim\left(\mathbb{C}[[z^1, \dots, z^n]]/\mathfrak{m}_n\right) + \mathrm{height}(\mathfrak{m}_n) \\ &= \dim(\mathbb{C}) + \mathrm{height}(\mathfrak{m}_n) \\ &= \mathrm{height}(\mathfrak{m}_n) \end{aligned}$$

This shows that $n = \dim X$ is well-defined.

Remark 5.19. *It follows that the function $x \mapsto \dim X$ is locally constant. Hence, if X is connected, the dimension is the same at each point, and the common value is called the dimension of the complex manifold X, denoted by $\dim X$, as we have done above.*

5.2. **Examples.** We now discuss several examples.

Example 5.20. $\mathbb{C}^n$ has a holomorphic structure determined by the holomorphic atlas consisting of the single coordinate chart $(\mathbb{C}^n, \mathrm{Id}_{\mathbb{C}^n})$. Similarly, the holomorphic structure on every open subset $U \subseteq \mathbb{C}^n$ is defined by the single chart (U, Id_U).

Example 5.21. (**Complex Projective Space**) For any $n \in \mathbb{N}$, the complex projective space, $\mathbb{CP}^n$ of dimension n is the set of complex 1-dimensional subspaces of $\mathbb{C}^{n+1}$, which we can identify with the quotient of $\mathbb{C}^{n+1} \setminus \{0\}$ by the equivalence relation defined by

$$w \sim w' \iff w' = \lambda w \text{ for some } \lambda \in \mathbb{C}^\times$$

We endow $\mathbb{CP}^n$ with the quotient topology. The equivalence class of $w \in \mathbb{C}^{n+1} \setminus \{0\}$ is denoted by $[w]$. Points of $\mathbb{CP}^n$ can be described through their homogeneous coordinates $[w^0, w^1, \dots, w^n]$. For each $\alpha = 0, \dots, n$, let $U_\alpha \subseteq \mathbb{CP}^n$ be the open subset $U_\alpha = \{[w] \in \mathbb{CP}^n : w^\alpha \neq 0\}$, and define a map

$$\begin{aligned} \varphi_\alpha \colon U_\alpha &\to \mathbb{C}^n \\ [w^0, \dots, w^n] &\mapsto \left(\frac{w^0}{w^\alpha}, \dots, \frac{w^{\alpha-1}}{w^\alpha}, \frac{w^{\alpha+1}}{w^\alpha}, \dots, \frac{w^n}{w^\alpha}\right) \end{aligned}$$

It is continuous by the characteristic property of the quotient topology, and it is a homeomorphism because it has a continuous inverse given by

$$\varphi_\alpha^{-1}(z^1, \ldots, z^n) = [z^1, \ldots, z^{\alpha-1}, 1, z^\alpha, \ldots, z^n].$$

Thus each $(U_\alpha, \varphi_\alpha)$ is a coordinate chart, called affine coordinates for $\mathbb{CP}^n$. This makes $\mathbb{CP}^n$ into a topological manifold. For $\alpha < \beta$, the transition function between these charts can be computed explicitly as

$$\varphi_\alpha \circ \varphi_\beta^{-1}(z^1, \ldots, z^n) = \left(\frac{z^1}{z^\alpha}, \ldots, \frac{\widehat{z^\alpha}}{z^\alpha}, \ldots, \frac{1}{z^\alpha}, \ldots, \frac{z^n}{z^\alpha} \right),$$

where the hat indicates that the term in position α is omitted, and the $1/z^\alpha$ term is in position β. These transition functions are all holomorphic. It can be checked that $\mathbb{CP}^n$ is Haudorff and second-countable. Hence, $\mathbb{CP}^n$ is a complex manifold of dimension n.

Remark 5.22. *Note that $\mathbb{CP}^n$ is compact and connected because it is the image of the surjective continuous map*

$$q : \mathbb{S}^{2n+1} \to \mathbb{CP}^n$$

given by $q(w^0, \ldots, w^n) = [w^0, \ldots, w^n]$, where $\mathbb{S}^{2n+1}$ is the set of unit vectors in $\mathbb{C}^{n+1}$. Moreover, we have,

$$\mathbb{CP}^n \cong \mathbb{S}^{2n+1}/\mathbb{S}^1$$

Remark 5.23. *Note that the quotient map $q : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{CP}^n$ is holomorphic. Indeed, if $w \in \mathbb{C}^{n+1} \setminus \{0\}$ such that $w^\alpha \neq 0$, then let $U_\alpha \subseteq \mathbb{CP}^n$ and $q^{-1}(U_\alpha) \subseteq \mathbb{C}^{n+1} \setminus \{0\}$ be open sets containing w and $[w]$ respectively. Clearly, $q(q^{-1}(U_\alpha)) \subseteq U_\alpha$. Moreover, we have,*

$$\varphi_\alpha \circ q \circ \mathrm{Id}^{-1}_{\mathbb{C}^{n+1}\setminus\{0\}}(z^0, \cdots, z^n) = \varphi_\alpha[z^0, \cdots, z^n] = \left(\frac{z^0}{z^\alpha}, \ldots, \frac{z^{\alpha-1}}{z^\alpha}, \frac{z^{\alpha+1}}{z^\alpha}, \ldots, \frac{z^n}{z^\alpha} \right).$$

which is holomorphic. Hence, q is holomorphic. Therefore, if $f : \mathbb{CP}^n \to \mathbb{C}$ is any holomorphic function, then $g_f := f \circ q$ is also a holomorphic function such that g_f is scale-invariant, i.e, $g_f(\lambda \cdot) = g_f(\cdot)$ for each $\lambda \in \mathbb{C}^\times$. In fact, the converse is true as well. Let $f : \mathbb{CP}^n \to \mathbb{C}$ be a continuous such that $g_f := f \circ q$ is holomorphic and g_f is scale-invariant. Then if (U_β, φ_β) is a chart on $\mathbb{CP}^n$, then,

$$\begin{aligned} f \circ \varphi_\beta^{-1}(z^1, \cdots, z^n) &= f[z^1, \cdots, z^{\alpha-1}, 1, z^{\alpha+1}, \cdots, z^n] \\ &= f \circ q(z^1, \cdots, z^{\alpha-1}, 1, z^{\alpha+1}, \cdots, z^n) \\ &= g_f(z^1, \cdots, z^{\alpha-1}, 1, z^{\alpha+1}, \cdots, z^n) \end{aligned}$$

is holomorphic. Therefore we have that the corresponding sheaf on $\mathbb{CP}^n$ is given by

$$\begin{aligned} \mathscr{G}_X(U) &= \{ f \in \mathscr{C}(U) \mid (f|_{U \cap U_i}) \circ \varphi_i^{-1} \textit{ is holomorphic on } \varphi_i(U \cap U_i) \textit{ for all } i \in I \} \\ &= \{ f \in \mathscr{C}(U) \mid g_f = f \circ q \textit{ is holomorphic and } g_f \textit{ is scale invariant} \} \end{aligned}$$

Example 5.24. (**Complex Lie Groups**) A complex Lie group, G, is a complex manifold that is also a group such that the map $(x, y) \mapsto x \cdot y^{-1}$ is holomorphic. Examples of complex Lie groups are provided by $\mathrm{GL}(n, \mathbb{C})$, $\mathrm{SL}(n, \mathbb{C})$, and $\mathrm{Sp}(n, \mathbb{C})$. They are certainly not abelian for $n > 1$.

Remark 5.25. *Note that certain classical groups like* $\mathrm{U}(n, \mathbb{C})$ *are often not complex complex Lie groups, but just ordinary real Lie groups. The easiest way of proving that* $\mathrm{U}(n, \mathbb{C})$ *is not a complex Lie group consists in using the fact that its Lie algebra* $\mathfrak{u}(n)$ *is not a complex Lie algebra. We have*

$$\mathfrak{u}(n) = \{A \in M_n(\mathbb{C}) \mid A = -A^*\}$$

Unless $A = 0$*, if* $A \in \mathfrak{u}(n)$*, then* $iA \notin \mathfrak{u}(n)$*.*

Another method to construct complex manifolds is to consider the quotient space when a group acts by automorphisms on the complex manifold. The next two propositions allow us to construct new complex manifolds in this manner.

Proposition 5.26. *Let* Y *be a connected complex manifold and* $\pi\colon X \to Y$ *be a covering map. Then* X *is a complex manifold and has a unique holomorphic atlas such that* π *is a holomorphic covering map.*

Proof. (**Sketch**) We know from smooth manifold theory that X is a topological manifold and has a unique smooth structure such that π is a smooth covering map. We can define holomorphic charts on X as follows: Given a point $p \in X$, let U be an evenly covered neighborhood of $\pi(p)$. After shrinking U if necessary, we can find a holomorphic coordinate map $\varphi\colon U \to \mathbb{C}^n$. Let $\widetilde{U}$ be the connected component of $\pi^{-1}(U)$ containing p, and define $\widetilde{\varphi} = \varphi \circ \pi\colon \widetilde{U} \to \mathbb{C}^n$. When two such charts $(U, \widetilde{\varphi})$ and $(V, \widetilde{\psi})$ overlap, in a neighborhood of each point the transition function can be expressed as

$$\widetilde{\psi}^{-1} \circ \widetilde{\varphi} = \psi^{-1} \circ \varphi,$$

which in this case is holomorphic. Clearly, this makes π into a local biholomorphism. ☐

Proposition 5.27. *(**Holomorphic Quotient Manifold Theorem**) Let* X *be a complex manifold and* Γ *is a discrete (complex) Lie group acting holomorphically*[4]*, freely*[5]*, and properly*[6] *on* X*. Then the quotient space* X/Γ *has a unique complex manifold structure such that the quotient map* $q\colon X \to X/\Gamma$ *is a holomorphic (normal) covering map.*

Proof. (**Sketch**) We know from smooth manifold theory that that X/Γ has a unique smooth manifold structure such that q is a smooth (normal) covering map. To define a complex manifold structure on X/Γ, let $U \subseteq X/\Gamma$ be any evenly covered open set, and choose a smooth local section $\sigma\colon U \to X$. Because X is a complex manifold, $\sigma(U)$ has a covering by holomorphic charts $(U_\alpha, \varphi_\alpha)$, and for each such chart we can define $(\sigma^{-1}(U_\alpha), \varphi_\alpha \circ \sigma)$ as a chart for X/Γ. For a fixed local section σ, all of these charts are holomorphically compatible with each other. If $\tilde{\sigma}\colon U \to X$ is any other local section, there is an element $g \in \Gamma$ such that $\tilde{\sigma}(x) = g \cdot \sigma(x)$ for all $x \in U$; and the fact that $x \mapsto g \cdot x$ is a biholomorphism of X with inverse $x \mapsto g^{-1} \cdot x$ guarantees that the charts obtained from $\tilde{\sigma}$ will be holomorphically compatible with those obtained from σ. ☐

Corollary 5.28. *Suppose* G *is a connected complex Lie group and* $\Gamma \subseteq G$ *is a discrete subgroup.The left coset space* G/Γ *is a complex manifold, and the quotient map* $\pi\colon G \to G/\Gamma$ *is a holomorphic (normal) covering map.*

Proof. This follows from Proposition 5.27 since the action automatically satisfies assumptions in Proposition 5.27. ☐

[4]The action Γ on X is holomorphic if the map $x \mapsto g \cdot x$ is holomorphic for each $g \in \Gamma$.
[5]An action of Γ on X is free if $g \cdot x = x$ for some $g \in \Gamma$ and $x \in X$ implies g is the identity.
[6]The action of Γ on X is proper if the map $\Gamma \times X \to X \times X$ given by $(g, x) \mapsto (g \cdot x, x)$ is a proper map.

Remark 5.29. *It is clear that the quotient maps in Proposition 5.26, Proposition 5.27 and Corollary 5.28 are local biholomorphisms. An argument similar to that in Remark 5.23 shows that,*

$$\mathscr{O}_{G/\Gamma}(U) = \left\{ f \in \mathscr{O}_G\left(q^{-1}(U)\right) \mid f \circ \gamma = f \text{ for every } \gamma \in \Gamma \right\}.$$

Example 5.30. (**Complex Tori**) Suppose V is an n-dimensional complex vector space, considered as an abelian complex Lie group. A lattice $\Lambda \subseteq V$ is a subgroup $\Lambda \subseteq V$ generated by taking $\mathbb{Z}$-linear combinations of $2n$ $\mathbb{R}$-linearly independent vectors $v_1, \dots, v_{2n}$. By Corollary 5.28 V/Λ is an n-dimensional complex Lie group, called a complex torus.

Remark 5.31. *When $n = 0$, V/Λ is a single point. When $n > 0$, we can think of V as a $2n$ real-vector space with the the real-linear isomorphism*

$$A\colon \mathbb{R}^{2n} \to V, \quad A(x^1, \dots, x^{2n}) = \sum_{j=1}^{2n} x^j v_j$$

This map descends to a map $\widetilde{A}$:

$$\begin{tikzcd}
\mathbb{R}^{2n} \arrow[r, "A"] \arrow[d, two heads, "q"'] & V \arrow[r, two heads] & V/\Lambda \\
\mathbb{R}^{2n}/\mathbb{Z}^{2n} \arrow[urr, dotted, "\widetilde{A}"'] & &
\end{tikzcd}$$

Since q is a smooth covering map and hence a smooth submersion and the map $\mathbb{R}^{2n} \to V/\Lambda$ is smooth, we have that $\widetilde{A}$ is a smooth map. Since $\widetilde{A}$ is bijective and a local diffeomorphism (because the maps q and $\mathbb{R}^{2n} \to V/\Lambda$ are local diffeomorphisms) , we have

$$V/\Lambda \cong \mathbb{R}^{2n}/\mathbb{Z}^{2n} \cong (\mathbb{R}/\mathbb{Z})^{2n} \cong \underbrace{\mathbb{S}^1 \times \cdots \times \mathbb{S}^1}_{2n \text{ times}}$$

as smooth manifolds. Thus, the complex tori defined by different lattices are all diffeomorphic to each other. Moreover, this argument also shows that the complex tori are compact connected smooth manifolds.

Example 5.32. (**Hopf Manifold**) There is a diffeomorphism

$$\varphi\colon \mathbb{S}^{2n-1} \times \mathbb{R} \to \mathbb{C}^n \setminus \{0\}$$

given by $\varphi(z^1, \cdots, z^n, t) = (e^t z^1, \cdots, e^t z^n)$. Let $\mathbb{Z}$ naturally acts on $\mathbb{S}^{2n-1} \times \mathbb{R}$, by letting

$$m \cdot (z^1, \cdots, z^n, t) = (z^1, \cdots, z^n, t + m)$$

for $m \in \mathbb{Z}$. Clearly, the resulting quotient space is diffeomorphic to $\mathbb{S}^{2n-1} \times \mathbb{S}^1$. The diffeomorphism φ allows us to transfer the action of $\mathbb{Z}$ to an action of $\mathbb{Z}$ on $\mathbb{C}^2 \setminus \{0\}$. Explicitly, it is given by the formula

$$m \cdot (z^1, \cdots, z^n) = (e^m z^n, \cdots, e^m z^n)$$

$\mathbb{Z}$ acts by biholomorphisms and the action is clearly free and properly discontinuous. Hence, $\mathbb{C}^n \setminus \{0\}/\mathbb{Z}$ is a complex manifold called the Hopf manifold. By construction, it is diffeomorphic to $\mathbb{S}^{2n-1} \times \mathbb{S}^1$ (and hence compact).

Remark 5.33. *A general version of Corollary 5.28 holds. If G is a complex Lie group and H is a (closed) complex Lie subgroup of G acting on G holomorphically, freely, and properly, then the quotient G/H is a complex manifold, and the quotient map $\pi : G \to G/H$ is holomorphic.*

Example 5.34. (**Complex Grassmanian**) Let $\mathrm{Gr}_k(\mathbb{C}^n)$ denote the set of k-dimensional subspaces in $\mathbb{C}^n$. $\mathrm{GL}(n, \mathbb{C})$ acts transitively on $\mathrm{Gr}_k(\mathbb{C}^n)$. Indeed, if $U, U' \in \mathrm{Gr}_k(\mathbb{C}^n)$, we can let $\mathscr{B}$ and $\mathscr{B}'$ be basis for $\mathbb{C}^n$ obtained by extending a basis for U and U' respectively. A change of basis matrix from $\mathscr{B}$ to $\mathscr{B}'$ then maps U to U'. The isotropy subgroup of $U = \langle e_1, \ldots, e_k\rangle$ is

$$H = \begin{pmatrix} *_k & * \\ 0 & *_{n-k} \end{pmatrix}$$

Here $*_k$ is a k by k matrix and $*_{n-k}$ is a $(n-k)$ by $(n-k)$ matrix. Thus $\mathrm{Gr}_k(\mathbb{C}^n)$ is the coset space $\mathrm{GL}(n, \mathbb{C})/H$. Both $\mathrm{GL}(n, \mathbb{C})$ and H are complex Lie groups (open subsets of some $\mathbb{C}^N$). Hence, $\mathrm{Gr}_k(\mathbb{C}^n)$ is a complex manifold by Remark 5.33.

Remark 5.35. *The Grassmanian is compact. Indeed, observing that we can choose orthonormal bases of subspaces, we have that $\mathrm{U}(n, \mathbb{C})$ acts continuously and transitively on $\mathrm{G}_k(\mathbb{C}^n)$ and we have*

$$\mathrm{Gr}_k(\mathbb{C}^n) = \frac{\mathrm{U}(n, \mathbb{C})}{\mathrm{U}(k, \mathbb{C}) \times \mathrm{U}(n-k, \mathbb{C})}$$

Since $\mathrm{U}(m, \mathbb{C})$ is compact for all $m \in \mathbb{N}$, we have that $\mathrm{Gr}_k(\mathbb{C}^n)$ is compact.

Remark 5.36. *Another important class of complex manifolds is that of holomorphic vector bundles. The general theory of holomorphic vector bundles is more or less the same as for smooth vector bundles. Smooth vector bundles are discussed in the Riemannian geometry notes. The language of (holomorphic) vector bundles will be used throughout the rest of the notes.*

6. Tangent Vectors & Tangent Bundle

6.1. **Tangent Vectors.**

Recall that on smooth manifolds, we can make sense of calculus by introducing the tangent space at a point, which serves as a 'linear model' for the manifold near that point. We now extend this concept to the case of complex manifolds. If $X = \mathbb{R}^n$, $T_p\mathbb{R}^n$ denote the tangent space at $p \in \mathbb{R}^n$. Recall that, we have

$$T_p\mathbb{R}^n \cong \mathrm{Der}_p\mathbb{R}^n,$$

where $\mathrm{Der}_p\mathbb{R}^n$ is the space of linear operators $D_p : C^\infty_{\mathbb{R}}(\mathbb{R}^n) \to \mathbb{R}$ with the property

$$D_p(fg) = f(p)D_p(g) + g(p)D_p(f)$$

for all $f, g \in C^\infty_{\mathbb{R}}(\mathbb{R}^n)$. Here $\mathscr{C}^\infty_{\mathbb{R}}(\mathbb{R}^n)$ denotes sheaf of $\mathbb{R}$-valued smooth functions on $\mathbb{R}^n$. It suffices to replace $\mathscr{C}^\infty_{\mathbb{R}}(\mathbb{R}^n)$ by $\mathscr{C}^\infty_{p,\mathbb{R}}$ in the definition of $\mathrm{Der}_p\mathbb{R}^n$ since $\mathrm{Der}_p\mathbb{R}^n$ is a local operator. If X is a smooth manifold, we have

$$T_pX = \mathrm{Der}_pX$$

If X is a complex manifold we can now use the complexification functor to define the tangent space to a point $p \in X$. Indeed, we make the following definition.

Definition 6.1. Let X be a complex manifold. The **complex tangent space** at $p \in X$ is given by $(T_pX)^{\mathbb{C}} := T_pX \otimes_{\mathbb{R}} \mathbb{C}$

We set $\mathrm{Der}_p^{\mathbb{C}}(X) \cong \mathrm{Der}_p(X) \otimes_{\mathbb{R}} \mathbb{C}$. What is $\mathrm{Der}_p^{\mathbb{C}}(X)$ concretely? This is just the complexification of $\mathrm{Der}_p(X)$. Morally, we can think of elements of $\mathrm{Der}_p^{\mathbb{C}}(X)$ as the set of all linear combinations of elements of $\mathrm{Der}_p(X)$ with complex coefficients. Hence, elements of $\mathrm{Der}_p(X)$ are of the form $X_p + iY_p$ such that $X_p, Y_p \in \mathrm{Der}_p(X)$. We can think of elements of $\mathrm{Der}_p^{\mathbb{C}}(X)$ as $\mathbb{C}$-linear derivations on $\mathscr{C}^\infty_{p,\mathbb{C}}$, where $\mathscr{C}^\infty_{p,\mathbb{C}}$ is the stalk of the sheaf of complex-valued *smooth* functions on X, denoted as $\mathscr{C}^\infty_{\mathbb{C}}$. We have

$$\begin{aligned}(T_pX)^{\mathbb{C}} &= T_pX \otimes_{\mathbb{R}} \mathbb{C} \\ &\cong \mathrm{Der}_p(X) \otimes_{\mathbb{R}} \mathbb{C} \\ &\cong \mathrm{Der}_p^{\mathbb{C}}(X)\end{aligned}$$

If X, Y be smooth manifolds and $F : X \to Y$ is a smooth map, recall that for $p \in X$, the differential of F is the map

$$dF_p : T_pX \to (T_{F(p)}Y)^{\mathbb{R}}$$

defined by $dF_p(D_p)(f) = D_p(f \circ F)$ for $D_p \in T_pX$ and $f \in C^\infty_{\mathbb{R}}(Y)$. This leads to the following definition.

Definition 6.2. Let X and Y be complex manifolds and let $F : X \to Y$ be a holomorphic map. The complex differential of F at p - denoted as $(dF_p)^{\mathbb{C}}$ - is the complexification of the linear map dF_p.

6.2. **Computations in Coordinates.** We discuss how to do computations with tangent vectors in local coordinates. We first discuss the smooth manifold case. Suppose X is a n-dimensional smooth manifold and let (U, ϕ) be a smooth coordinate chart on X. Write the local coordinate as $(x^1, \cdots, x^n)$. Recall that $d\phi_p : T_pX \to T_{\phi(p)}\mathbb{R}^n$ is an isomorphism. The derivations

$$\left.\frac{\partial^{\mathbb{R}^n}}{\partial x^1}\right|_{\phi(p)}, \cdots, \left.\frac{\partial^{\mathbb{R}^n}}{\partial x^n}\right|_{\phi(p)}$$

form a basis for $T_p\mathbb{R}^n$. Therefore, the preimages of these vectors under the isomorphism $d\phi_p$ form a basis for T_pX. We write these basis vectors as $\frac{\partial^X}{\partial x^1}|_p, \cdots, \frac{\partial^X}{\partial x^n}|_p$ and we have:

$$\begin{aligned}\left.\frac{\partial^X}{\partial x^i}\right|_p &= (d\phi_p)^{-1}\left(\left.\frac{\partial^{\mathbb{R}^n}}{\partial x^i}\right|_{\phi(p)}\right) \\ &= (d\phi^{-1})_{\phi(p)}\left(\left.\frac{\partial^{\mathbb{R}^n}}{\partial x^i}\right|_{\phi(p)}\right)\end{aligned}$$

We see that $\frac{\partial^X}{\partial x^i}|_p$ acts on a function $f \in C^\infty_{\mathbb{R}}(U)$ by

$$\begin{aligned}\left.\frac{\partial^X f}{\partial x^i}\right|_p &= (d\phi^{-1})_{\phi(p)}\left(\left.\frac{\partial^{\mathbb{R}^n}}{\partial x^i}\right|_{\phi(p)}\right) f \\ &= \left.\frac{\partial^{\mathbb{R}^n}(f \circ \phi^{-1})}{\partial x^1}\right|_{\phi(p)} \\ &:= \left.\frac{\partial^{\mathbb{R}^n}(\hat{f})}{\partial x^1}\right|_{\phi(p)}\end{aligned}$$

where $\hat{f} = f \circ \phi^{-1}$ is the coordinate representation of f. In other words, $\frac{\partial^X}{\partial x^i}|_p$ is just the derivation that takes the ith partial derivative of the coordinate representation of f at the coordinate representation of p. It is important to note that these computations depend on the chart ϕ of choice. A tangent vector $D_p \in T_pM$ can be written uniquely as a linear combination

$$D_p = D_p^i \frac{\partial^X}{\partial x^i}\bigg|_p$$

Now let X be a complex manifold of dimension n. Then X can be thought of as a smooth manifold of dimension $2n$. Let (U, ϕ) be a holomorphic coordinate chart. Write local coordinates as $(z^1, \cdots, z^n)$. We have $z^i = x^i + iy^i$. The (real) tangent space at the point p is:

$$(T_pX)^{\mathbb{R}} = \mathrm{Span}_{\mathbb{R}} \left\langle \frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}, \frac{\partial}{\partial y^1}, \dots, \frac{\partial}{\partial y^n} \right\rangle.$$

The vector fields $\frac{\partial}{\partial x^j}$ and $\frac{\partial}{\partial y^j}$ are interpreted as smooth vector fields on $U \subseteq X$. These vector fields are called complex coordinate vector fields. The complexified tangent space is

$$\begin{aligned} (T_pX)^{\mathbb{C}} &= \mathrm{Span}_{\mathbb{C}} \left\langle \frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}, \frac{\partial}{\partial y^1}, \dots, \frac{\partial}{\partial y^n} \right\rangle \\ &= \mathrm{Span}_{\mathbb{C}} \left\langle \frac{\partial}{\partial z^1}, \dots, \frac{\partial}{\partial z^n}, \frac{\partial}{\partial \bar{z}^1}, \dots, \frac{\partial}{\partial \bar{z}^n} \right\rangle. \end{aligned}$$

Here the alternative basis in the second line is again given by

$$\frac{\partial}{\partial z^j} = \frac{1}{2}\left(\frac{\partial}{\partial x^j} - i\frac{\partial}{\partial y^j}\right) \quad \text{and} \quad \frac{\partial}{\partial \bar{z}^j} = \frac{1}{2}\left(\frac{\partial}{\partial x^j} + i\frac{\partial}{\partial y^j}\right).$$

Remark 6.3. *If $X = \mathbb{C}^n$ and f is a holomorphic function on $\mathbb{C}^n$, recall that*

$$\frac{1}{2}\left(\frac{\partial}{\partial x^j} - i\frac{\partial}{\partial y^j}\right) f = \frac{1}{2}\left(\frac{\partial f}{\partial x^j} - i\frac{\partial f}{\partial y^j}\right) = \frac{1}{2}\left(\frac{\partial f}{\partial z^j} + \frac{\partial f}{\partial z^j}\right) = \frac{\partial f}{\partial z^j}$$

This motivates the consideration of the alternative basis for $(T_pX)^{\mathbb{C}}$.

This define a smooth local complex frame $\left\{\frac{\partial}{\partial z^j}, \frac{\partial}{\partial \bar{z}^j}\right\}_{j=1}^n$ for $(T_pX)^{\mathbb{C}}$. Consider the following two subspaces:

$$(T_pX)_{(1,0)} = \mathbb{C}\left\langle \frac{\partial}{\partial z^1}, \dots, \frac{\partial}{\partial z^n} \right\rangle, \qquad (T_pX)_{(0,1)} = \mathbb{C}\left\langle \frac{\partial}{\partial \bar{z}^1}, \dots, \frac{\partial}{\partial \bar{z}^n} \right\rangle$$

It can be easily checked that $T^{\mathbb{C}}_{p,\pm 1}X$ is the $\pm i$ eigenspace of the complex structure on $(T_pX)^{\mathbb{C}}$ induced by multiplication by i. Therefore, Proposition 2.9 implies that we have a direct sum decomposition

$$(T_pX)^{\mathbb{C}} = (T_pX)_{(1,0)} \oplus (T_pX)_{(0,1)}$$

Remark 6.4. *$(T_pX)^{\mathbb{C}}_{(1,0)}$ is called the holomorphic tangent space and $(T_pX)^{\mathbb{C}}_{(0,1)}$ is called the anti-holomorphic tangent space.*

Remark 6.5. *If $f\colon X \to \mathbb{C}$ is a smooth function in a co-ordinate chart $(z^1, \dots, z^n)$, then f is holomorphic if and only if $\frac{\partial f}{\partial \bar{z}^j} \equiv 0$ on U for $j = 1, \dots, n$. This follows readily from Remark 1.9.*

We now discuss how to compute the differential of a smooth map between complex manifolds in coordinates.

Proposition 6.6. *Let X and Y be complex manifolds and $F\colon X \to Y$ be a holomorphic map. Let $p \in X$ and let $(dF_p)^{\mathbb{C}}$ be the complexified differential from $(T_pX)^{\mathbb{C}}$ to $(T_pY)^{\mathbb{C}}$. Let $z^j = x^j + iy^j$ be local holomorphic coordinates for X in a neighborhood of p, and $w^j = u^j + iv^j$ for N in a neighborhood of $F(p)$. We have:*

$$(dF_p)^{\mathbb{C}}\left(\frac{\partial}{\partial z^j}\bigg|_p\right) = \frac{\partial F^k}{\partial z^j}(p)\frac{\partial}{\partial w^k}\bigg|_{F(p)} + \frac{\partial \overline{F}^k}{\partial \bar{z}^j}(p)\frac{\partial}{\partial \bar{w}^k}\bigg|_{F(p)},$$
$$(dF_p)^{\mathbb{C}}\left(\frac{\partial}{\partial \bar{z}^j}\bigg|_p\right) = \frac{\partial F^k}{\partial \bar{z}^j}(p)\frac{\partial}{\partial w^k}\bigg|_{F(p)} + \frac{\partial \overline{F}^k}{\partial \bar{z}^j}(p)\frac{\partial}{\partial \bar{w}^k}\bigg|_{F(p)}.$$

Proof. (Sketch) We use the standard coordinate formula for the differential of a smooth map between smooth manifolds without providing a proof. Write F as $F = U + iV$. Considering X and Y as smooth manifolds, we have the usual coordinate formula for dF_p:

$$dF_p\left(\frac{\partial}{\partial x^j}\bigg|_p\right) = \frac{\partial U^k}{\partial x^j}(p)\frac{\partial}{\partial u^k}\bigg|_{F(p)} + \frac{\partial V^k}{\partial x^j}(p)\frac{\partial}{\partial v^k}\bigg|_{F(p)},$$
$$dF_p\left(\frac{\partial}{\partial y^j}\bigg|_p\right) = \frac{\partial U^k}{\partial y^j}(p)\frac{\partial}{\partial u^k}\bigg|_{F(p)} + \frac{\partial V^k}{\partial y^j}(p)\frac{\partial}{\partial v^k}\bigg|_{F(p)}.$$

We transform this formula into holomorphic coordinates. Using the definitions of $\frac{\partial}{\partial z^j}$ and $\frac{\partial}{\partial \bar{z}^j}$, we obtain:

$$(dF_p)^{\mathbb{C}}\left(\frac{\partial}{\partial z^j}\bigg|_p\right) = \frac{\partial U^k}{\partial z^j}(p)\frac{\partial}{\partial u^k}\bigg|_{F(p)} + \frac{\partial V^k}{\partial z^j}(p)\frac{\partial}{\partial v^k}\bigg|_{F(p)},$$
$$(dF_p)^{\mathbb{C}}\left(\frac{\partial}{\partial \bar{z}^j}\bigg|_p\right) = \frac{\partial U^k}{\partial \bar{z}^j}(p)\frac{\partial}{\partial u^k}\bigg|_{F(p)} + \frac{\partial V^k}{\partial \bar{z}^j}(p)\frac{\partial}{\partial v^k}\bigg|_{F(p)}.$$

Now substitute $\frac{\partial}{\partial u^k} = \frac{\partial}{\partial w^k} + \frac{\partial}{\partial \overline{w}^k}$ and $\frac{\partial}{\partial v^k} = i\left(\frac{\partial}{\partial w^k} - \frac{\partial}{\partial \overline{w}^k}\right)$ and collect terms:

$$(dF_p)^{\mathbb{C}}\left(\frac{\partial}{\partial z^j}\bigg|_p\right) = \left(\frac{\partial U^k}{\partial z^j}(p) + i\frac{\partial V^k}{\partial z^j}(p)\right)\frac{\partial}{\partial w^k}\bigg|_{F(p)} + \left(\frac{\partial U^k}{\partial z^j}(p) - i\frac{\partial V^k}{\partial z^j}(p)\right)\frac{\partial}{\partial \bar{w}^k}\bigg|_{F(p)},$$
$$(dF_p)^{\mathbb{C}}\left(\frac{\partial}{\partial \bar{z}^j}\bigg|_p\right) = \left(\frac{\partial U^k}{\partial \bar{z}^j}(p) + i\frac{\partial V^k}{\partial \bar{z}^j}(p)\right)\frac{\partial}{\partial w^k}\bigg|_{F(p)} + \left(\frac{\partial U^k}{\partial \bar{z}^j}(p) - i\frac{\partial V^k}{\partial \bar{z}^j}(p)\right)\frac{\partial}{\partial \bar{w}^k}\bigg|_{F(p)}.$$

The desired formulas now follow. □

Corollary 6.7. ***(Chain Rule in Coordinates)*** *Let X and Y be complex manifolds and let $F\colon X \to Y$ is a holomorphic map, and $h\colon Y \to \mathbb{C}$ is a holomorphic function. In terms of local holomorphic coordinates (z^j) for M and (w^k) for N, we have:*

$$\frac{\partial(h \circ F)}{\partial z^j} = \frac{\partial h}{\partial w^k}\frac{\partial F^k}{\partial z^j} + \frac{\partial h}{\partial \bar{w}^k}\frac{\partial \overline{F}^k}{\partial z^j},$$
$$\frac{\partial(h \circ F)}{\partial \bar{z}^j} = \frac{\partial h}{\partial w^k}\frac{\partial F^k}{\partial \bar{z}^j} + \frac{\partial h}{\partial \bar{w}^k}\frac{\partial \overline{F}^k}{\partial \bar{z}^j}.$$

Proof. This follows from Proposition 6.6 upon noting that the value of $\frac{\partial(h\circ F)}{\partial z^j}$ at $p \in X$ is equal to the $\frac{\partial}{\partial w}$ component of $(d(h \circ F)_p)^{\mathbb{C}}(\frac{\partial}{\partial z^j}|_p)$. But this expression is $(dh_{F(p)} \circ dF_p)^{\mathbb{C}}(\frac{\partial}{\partial z^j}|_p)$. The formula for $\frac{\partial(h\circ F)}{\partial z^j}$ at $p \in X$ now follows by invoking the formulas in Proposition 6.6. A similar argument applies to the $\frac{\partial(h\circ F)}{\partial \bar{z}^j}$ derivative $p \in X$. □

6.3. **Tangent Bundle.** We can now construct the tangent bundle associated with a complex manifold. Let X be a complex n-manifold. Let $\pi\colon TX \to X$ be the smooth rank-$2n$ tangent bundle over the underlying smooth manifold structure on X. Define the complexification of TX to be the set

$$(TX)^{\mathbb{C}} = \coprod_{p\in X} (T_pX)^{\mathbb{C}}$$

together with the obvious projection $\pi_{\mathbb{C}}\colon (TX)^{\mathbb{C}} \to X$. For each smooth local trivialization $\Phi\colon \pi^{-1}(U) \to U \times \mathbb{R}^{2n}$, we define a local trivialization $\Phi^{\mathbb{C}}\colon (\pi^{-1})^{\mathbb{C}}(U) \to U \times \mathbb{C}^{2n}$ by

$$\Phi^{\mathbb{C}}(\xi) = \big(\pi^{\mathbb{C}}(\xi), (\Phi|_{TX_{\pi^{\mathbb{C}}(\xi)}})^{\mathbb{C}}(\xi)\big).$$

Wherever two such trivializations (U, Φ) and (V, Ψ) overlap, we can write

$$\Psi \circ \Phi^{-1}(p, v) = (p, \tau(p)v)$$

for some smooth transition function $\tau\colon U \cap V \to \mathrm{GL}(2n, \mathbb{R})$. Clearly the transition function from $\Phi_{\mathbb{C}}$ to $\Psi_{\mathbb{C}}$ is the same:

$$\Psi^{\mathbb{C}} \circ (\Phi^{-1})^{\mathbb{C}}(p, v) = (p, \tau(p)v),$$

where now τ is considered as a map into $\mathrm{GL}(2n, \mathbb{C})$. It follows from the vector bundle chart lemma [Lee12] (adapted in the obvious way for holomorphic vector bundles) that $\pi^{\mathbb{C}}\colon (TX)^{\mathbb{C}} \to X$ has a unique structure as a smooth rank-$2n$ holomorphic vector bundle, with the maps constructed above as smooth local trivializations.

Definition 6.8. Let X be a complex manifold. A **holomorphic vector field** is a section of $(TX)^{\mathbb{C}}$.

A holomorphic vector field can be written as $Z = X + iY$, where X, Y are smooth vector fields. Z acts on a holomorphic function $f = u + iv$ by

$$Zf = Xf + iYf = (Xu + iXv + Yu + iYv) = (Xu + Yu) + i(Xv + Yv)$$

The Lie bracket operation can be extended to pairs of smooth holomorphic vector fields by complex bilinearity:

$$[X_1 + iY_1, X_2 + iY_2] = ([X_1, X_2] - [Y_1, Y_2]) + i([X_1, Y_2] + [Y_2, X_1]).$$

Example 6.9. Let $X = \mathbb{C}^n$. We have the following facts:

(1) We have

$$T\mathbb{C}^n = \coprod_{p\in\mathbb{C}^n} T_p\mathbb{C}^n = \coprod_{p\in\mathbb{C}^n} (\mathbb{C}^n)_{\mathbb{R}} = (\mathbb{C}^n)_{\mathbb{R}} \times \mathbb{C}^n,$$
$$(T\mathbb{C}^n)^{\mathbb{C}} = \coprod_{p\in\mathbb{C}^n} (T_p\mathbb{C}^n)^{\mathbb{C}} = \coprod_{p\in\mathbb{C}^n} \mathbb{C}^n = \mathbb{C}^n \times \mathbb{C}^n.$$

(2) The complexified tangent bundle $T^{\mathbb{C}}\mathbb{C}^n$ splits as:

$$\begin{aligned}(T\mathbb{C}^n)^{\mathbb{C}} &= T_{(1,0)}\mathbb{C}^n \oplus T_{(0,1)}\mathbb{C}^n \\ &= \operatorname{Span}_{\mathbb{C}}\left\langle \frac{\partial}{\partial z^1}, \dots, \frac{\partial}{\partial z^n} \right\rangle \oplus \operatorname{Span}_{\mathbb{C}}\left\langle \frac{\partial}{\partial \overline{z}^1}, \dots, \frac{\partial}{\partial \overline{z}^n} \right\rangle.\end{aligned}$$

(3) $T\mathbb{C}^n$ has a canonical complex structure $J_{\mathbb{C}^n}$, which satisfies:

$$J_{\mathbb{C}^n}\frac{\partial}{\partial x^j} = \frac{\partial}{\partial y^j}, \quad J_{\mathbb{C}^n}\frac{\partial}{\partial y^j} = -\frac{\partial}{\partial x^j}.$$

(4) For an open subset $U \subseteq \mathbb{C}^n$, a smooth function $F\colon U \to \mathbb{C}^m$ is holomorphic if and only if the following relation holds for all $p \in U$:

$$DF(p) \circ J_{\mathbb{C}^n} = J_{\mathbb{C}^m} \circ DF(p).$$

for all $p \in U$. This follows from the following computation:

$$\begin{aligned}DF\left(J_{\mathbb{C}^n}\frac{\partial}{\partial \overline{z}^j}\right) - J_{\mathbb{C}^m}\left(DF\frac{\partial}{\partial \overline{z}^j}\right) &= DF\left(-i\frac{\partial}{\partial \overline{z}^j}\right) - J_{\mathbb{C}^m}\left(DF\frac{\partial}{\partial \overline{z}^j}\right) \\ &= -i\frac{\partial F^k}{\partial \overline{z}^j}\frac{\partial}{\partial w^k} - i\frac{\partial F^k}{\partial \overline{z}^j}\frac{\partial}{\partial \overline{w}^k} - J_{\mathbb{C}^m}\frac{\partial F^k}{\partial \overline{z}^j}\frac{\partial}{\partial w^k} - J_{\mathbb{C}^m}\frac{\partial F^k}{\partial \overline{z}^j}\frac{\partial}{\partial \overline{w}^k} \\ &= -2i\frac{\partial F^k}{\partial \overline{z}^j}\frac{\partial}{\partial w^k}.\end{aligned}$$

If X is an arbitrary complex manifold, we now argue that $(TX)^{\mathbb{C}}$ has a canonical complex structure that induces a canonical decomposition of $(TX)^{\mathbb{C}}$:

Proposition 6.10. *Let X be a complex n-manifold.*

(1) *There is a canonical complex structure on TX, denoted by $J_X : (TX)^{\mathbb{C}} \to (TX)^{\mathbb{C}}$.*
(2) *There are smooth sub-bundles $(TX)_{(1,0)}, (TX)_{(0,1)} \subseteq (TX)^{\mathbb{C}}$ whose fibers at each point are the i-eigenspace and $(-i)$-eigenspace of J_X, respectively, such that we have:*

$$(TX)^{\mathbb{C}} = (TX)_{(1,0)} \oplus (TX)_{(0,1)}$$

Proof. The proof is given below:

(1) Given $p \in X$, choose a holomorphic coordinate chart (U, φ) on a neighborhood of p, and define $J_X : TX|_U \to TX|_U$ by

$$J_X = D\varphi^{-1} \circ J_{\mathbb{C}^n} \circ D\varphi.$$

Wherever two holomorphic charts (U, φ) and (V, ψ) overlap, the transition map $\psi \circ \varphi^{-1}$ is a holomorphic map between open subsets of $\mathbb{C}^n$, so its differential commutes with $J_{\mathbb{C}^n}$ as in Example 6.9. Therefore,

$$\begin{aligned}D\psi^{-1} \circ J_{\mathbb{C}^n} \circ D\psi &= D\psi^{-1} \circ J_{\mathbb{C}^n} \circ (D\psi \circ D\varphi^{-1}) \circ D\varphi \\ &= D\psi^{-1} \circ (D\psi \circ D\varphi^{-1}) \circ J_{\mathbb{C}^n} \circ D\varphi \\ &= D\varphi^{-1} \circ J_{\mathbb{C}^n} \circ D\varphi.\end{aligned}$$

So J_X is well-defined. The fact that it satisfies $J_X \circ J_X = -\mathrm{Id}$ follows from the corresponding fact for $J_{\mathbb{C}^n}$.

(2) (Sketch) This follows since we have the decomposition

$$(T_pX)^{\mathbb{C}} = (T_pX)_{(1,0)} \oplus (T_pX)_{(0,1)}$$

for each $p \in X$.

This completes the proof. □

7. Cotangent Bundle & Differential Forms

7.1. **Smooth Differential Forms.** We first recall the notion of differential forms on a smooth manifold.

Definition 7.1. Let X be a smooth manifold. A **differential k-form** is a smooth map $\sigma : X \to \Lambda^k T^*X$ such that $\pi \circ \sigma = \mathrm{Id}_X$, where π denotes the projection map from $\Lambda^k T^*X$ onto X and

$$\Lambda^k T^*X = \coprod_{p \in X} \Lambda^k(T_p^*X).$$

is the k-th exterior bundle.

Remark 7.2. *It can be checked that $\Lambda^k(T^*M)$ has the structure of a smooth manifold of dimension $n + \binom{n}{k}$. Moreover, it can be shown that $\Lambda^k T^*X$ has the structure of a smooth vector bundle over X.*

Remark 7.3. *It can be checked that the set of smooth differential k-forms is a $\mathbb{R}$-vector space. We denote the vector space of smooth differential k-forms by $\Omega^k(X)$.*

We can use the discussion in Section 4 to define the wedge product of two differential forms in a pointwise manner:

$$(\omega \wedge \eta)_p = \omega_p \wedge \eta_p, \quad p \in X$$

Thus, if $\omega \in \Omega^k(M)$ and $\eta \in \Omega^l(X)$, then $\omega \wedge \eta \in \Omega^{k+l}(X)$.

Remark 7.4. *A 0-form is just a continuous real-valued function. If f is a 0-form and η is a k-form, we interpret the wedge product $f \wedge \eta$ to mean the ordinary product $f\eta$.*

If we define

$$\Omega^*(X) = \bigoplus_{k=0}^{n} \Omega^k(X),$$

then the wedge product turns $\Omega^*(X)$ into an associative, anti-commutative graded algebra. In any smooth chart $(U, (x^i))$ on X, a smooth k-form ω can be written locally as

$$\omega = \sum_I \omega_I \, dx^{i_1} \wedge \ldots \wedge dx^{i_k},$$

where the coefficients ω_I are smooth functions defined on the coordinate domain. Lemma 3.13 implies

$$dx^{i_1} \wedge \ldots \wedge dx^{i_k} \left(\frac{\partial}{\partial x^{j_1}}, \ldots, \frac{\partial}{\partial x^{j_k}} \right) = \delta^I_J,$$

Thus, the component functions ω_I of the k-form ω are determined by

$$\omega_I = \omega \left(\frac{\partial}{\partial x^{i_1}}, \ldots, \frac{\partial}{\partial x^{i_k}} \right).$$

The great thing about differential forms is that we can pullback differential forms. Let X, Y be smooth manifolds. Given a smooth map $F : X \to Y$ and a differential form

$\omega \in \Omega^k(Y)$, the pullback along F, denoted as $F^*\omega$, gives a differential form $\Omega^k(X)$. We can describe it by its action on tangent vectors:

$$(F^*\omega)_p(v_1, \ldots, v_k) = \omega_{F(p)}(dF_p(v_1), \ldots, dF_p(v_k))$$

where dF_p is the differential at p. The pullback satisfies some nice properties:

Proposition 7.5. *Let X, Y be smooth manifolds, and let $F : X \to Y$ be a smooth function. The pullback satisfies the following properties:*

(1) *$F\colon \Omega^k(Y) \to \Omega^k(X)$ is linear over $\mathbb{R}$ for each k.*
(2) *$F^*(\omega \wedge \eta) = F^*(\omega) \wedge F^*(\eta)$.*
(3) *In any smooth chart,*

$$F\left(\sum_I \omega_I \, dy^{i_1} \wedge \ldots \wedge dy^{i_k}\right) = \sum_I (\omega_I \circ F) \, d(y^{i_1} \circ F) \wedge \ldots \wedge d(y^{i_k} \circ F).$$

Proof. The proof is given below:

(1) This is clear.
(2) It suffices to prove the claim pointwise:

$$\begin{aligned}(F^*(\omega \wedge \eta))_p(v_1, \ldots, v_k, w_1, \cdots w_l) &= (\omega \wedge \eta)_{F(p)}(dF_p(v_1), \ldots, dF_p(v_k), dF_p(w_1), \ldots, dF_p(w_l)), \\ &= \omega_{F(p)}(dF_p(v_1), \ldots, dF_p(v_k))\eta_{F(p)}\eta(dF_p(w_1), \ldots, dF_p(w_l)) \\ &= F^*(\omega_p)(v_1, \ldots, v_k) \wedge F^*(\eta_p)(w_1, \cdots w_l)\end{aligned}$$

Hence,

$$(F^*(\omega \wedge \eta)) = F^*(\omega_p) \wedge F^*(\eta_p).$$

(3) This follows from (1) and (2) and the observation that if η is a 0-form (a function), then $F^*(\eta) = (\eta \circ F)$ and:

$$F^*(\omega \wedge \eta) = (\eta \circ F)F^*(\omega)$$

This completes the proof. $\square$

Example 7.6. Let $\omega = dx \wedge dy$ on $\mathbb{R}^2$. Thinking of the transformation to polar coordinates $x = r\cos(\theta)$, $y = r\sin(\theta)$ as an expression for the identity map with respect to different coordinates on the domain and codomain, we obtain

$$\begin{aligned}dx \wedge dy &= d(r\cos(\theta)) \wedge d(r\sin(\theta)) \\ &= (\cos(\theta)\, dr - r\sin(\theta)\, d\theta) \wedge (\sin(\theta)\, dr + r\cos(\theta)\, d\theta) \\ &= r\, dr \wedge d\theta.\end{aligned}$$

Let's now recall the definition of the exterior derivative. For $0 \leq k \leq n$, the exterior derivative is an operator

$$d : \Omega^k(X) \to \Omega^{k+1}(X)$$

In a smooth chart $(U, (x^i))$ on X, if a smooth k-form ω can be written locally as

$$\omega = \sum_I \omega_I \, dx^{i_1} \wedge \ldots \wedge dx^{i_k},$$

then in local coordinates d is defined as follows:

$$d\omega = \sum_I \sum_j \frac{\partial \omega_I}{\partial x^j} \, dx^j \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_k}$$

Before we proceed, we give an equivalent but coordinate-free definition of the exterior derivative. Let's start with small k's to find out the invariant formula of $d\omega$. Let $U \subseteq X$ be an open subset of X. We have the following:

(1) Let $k = 0$. Then $\omega = f \in C^\infty(U)$, and we can regard df as a $C^\infty(U)$-linear map
$$df : \mathfrak{X}(U) \to C^\infty(U)$$
such that
$$df(X) = Xf.$$

(2) Let $k = 1$. We want to regard $d\omega$ as a $C^\infty(U)$-bilinear map
$$d\omega : \mathfrak{X}(U) \times \mathfrak{X}(U) \to C^\infty(U).$$
We write $\omega = \sum_i \omega_i dx^i$, $X = \sum_k X^k \partial_k$, and $Y = \sum_l Y^l \partial_l$. Then
$$\begin{aligned} d\omega(X,Y) &= \sum_{i,j,k,l} (\partial_j \omega_i) dx^j \wedge dx^i (X^k \partial_k, Y^l \partial_l) \\ &= \sum_{i,j} (\partial_j \omega_i) X^j Y^i - (\partial_j \omega_i) X^i Y^j \\ &= \sum_{i,j} X^j \partial_j(\omega_i Y^i) - \omega_i X^j \partial_j(Y^i) - Y^j \partial_j(\omega_i X^i) + \omega_i Y^j \partial_j(X^i) \\ &= X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y]). \end{aligned}$$
So we arrive at
$$d\omega(X,Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y]).$$

(3) Let $k = 2$. We want to regard $d\omega$ as a $C^\infty(U)$-bilinear map
$$d\omega : \mathfrak{X}(U) \times \mathfrak{X}(U) \times \mathfrak{X}(U) \to C^\infty(U).$$
By a tedious but similar computation as above, one can show that
$$d\omega(X,Y,Z) = X(\omega(Y,Z)) - Y(\omega(X,Z)) + Z(\omega(X,Y)) - \omega([X,Y],Z) + \omega([X,Z],Y) - \omega([Y,Z],X).$$

So we are naturally led to the following the invariant formula for d:

Proposition 7.7. *Let X be a smooth n-manifold. Let $0 \le k \le n$ and let $U \subseteq X$ be an open set of X. For any $\omega \in \Omega^k(U)$, the $(k+1)$-form $d\omega$, viewed as a $C^\infty(U)$-multilinear map*
$$d\omega : \underbrace{\mathfrak{X}(U) \times \cdots \times \mathfrak{X}(U)}_{(k+1)-\text{times}} \to C^\infty(U),$$
is given by the formula
$$\begin{aligned} d\omega(X_1, \ldots, X_{k+1}) := &\sum_i (-1)^{i-1} X_i \left(\omega(X_1, \ldots, \hat{X}_i, \ldots, X_{k+1}) \right) \\ &+ \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_{k+1}) \end{aligned}$$

Proof. The proof is skipped. $\square$

We end with some properties of the exterior derivative:

Proposition 7.8. *Let X be a smooth n-manifold. Let $0 \le k, l \le n$ and let $U \subseteq X$ be an open set of X. Suppose $\omega \in \Omega^k(U)$, $\eta \in \Omega^l(U)$, $X \in \mathfrak{X}(U)$.*

(1) $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$.
(2) $d \circ d = 0$.
(3) *Let Y be a smooth m-manifold and let $V \subseteq Y$ be an open set. Let $F : U \to V$ be a smooth map. Then*
$$d \circ F^* = F^* \circ d$$

Proof. The proof is given below:

(1) Since d is linear, it is enough to assume
$$\omega = f\, dx^{i_1} \wedge \cdots \wedge dx^{i_k}, \quad \eta = g\, dx^{j_1} \wedge \cdots \wedge dx^{j_l}$$
with indices set $I \cap J = \emptyset$. Then the formula follows from a direct computation:
$$\begin{aligned} d(\omega \wedge \eta) &= d(fg\, dx^{i_1} \wedge \cdots \wedge dx^{i_k} \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_l}) \\ &= \sum_i \partial_i(fg)\, dx^i \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k} \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_l} \\ &= \sum_i (\partial_i f)\, dx^i \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k} \wedge \eta + (-1)^k \omega \wedge \sum_i (\partial_i g)\, dx^i \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_l} \\ &= d\omega \wedge \eta + (-1)^k \omega \wedge d\eta. \end{aligned}$$

(2) We first check this for $k = 0$:
$$\begin{aligned} d(df)(X, Y) &= X(df(Y)) - Y(df(X)) - df([X, Y]) \\ &= X(Y(f)) - Y(X(f)) - [X, Y]f = 0 \end{aligned}$$
For $k > 0$, by linearity we may assume
$$\omega = f\, dx^1 \wedge \cdots \wedge dx^k$$
Since $ddf = 0$ and $ddx^i = 0$, we get
$$\begin{aligned} d(d\omega) &= d(df \wedge dx^1 \wedge \cdots \wedge dx^k) \\ &= d(df) \wedge dx^1 \wedge \cdots \wedge dx^k + \sum_i (-1)^i df \wedge dx^1 \wedge \cdots \wedge d(dx^i) \wedge \cdots \wedge dx^k = 0 \end{aligned}$$

(3) For $0 \le k \le m$, let $\omega \in \Omega^k * (V)$. For $k = 0$, $\omega = f \in C^\infty(V)$ and
$$(\varphi^* df)_p(X_p) = df_{F(p)}(dF_p(X_p)) = d(F^* f)_p(X_p).$$
In general, assume
$$\omega = f\, dx^1 \wedge \cdots \wedge dx^k$$
By Proposition 7.5, we have
$$\begin{aligned} F^*(d\omega) &= F^*(df \wedge dx^1 \wedge \cdots \wedge dx^k) \\ &= F^*(df) \wedge F^*(dx^1) \wedge \cdots \wedge F^*(dx^k) \\ &= d(F^* f) \wedge d(F^* x^1) \wedge \cdots \wedge d(F^* x^k) \\ &= d(F^* f\, d(F^* x^1) \wedge \cdots \wedge d(F^* x^k)) \\ &= d(F^* \omega). \end{aligned}$$

This completes the proof. $\square$

7.2. **Cotangent Bundle.** Let X be a complex n-manifold and let $\pi\colon T^*X \to X$ be the smooth rank-$2n$ cotangent bundle over the underlying smooth manifold structure on X. We first discuss the complexification of T^*X. The details are similar to the complexification of the underlying smooth tangent bundle, but we repeat the details anyway. Define the complexification of T^*X to be the set

$$(T^*X)^{\mathbb{C}} = \coprod_{p\in X} (T_p^*X)^{\mathbb{C}}$$

together with the obvious projection $\pi_{\mathbb{C}}\colon (T^*X)^{\mathbb{C}} \to X$. For each smooth local trivialization $\Phi\colon \pi^{-1}(U) \to U \times \mathbb{R}^{2n}$, we define a local trivialization $\Phi^{\mathbb{C}}\colon (\pi^{-1})^{\mathbb{C}}(U) \to U \times \mathbb{C}^{2n}$ by

$$\Phi^{\mathbb{C}}(\xi) = \big(\pi^{\mathbb{C}}(\xi), (\Phi|T^*X_{\pi^{\mathbb{C}}(\xi)})^{\mathbb{C}}(\xi)\big).$$

Wherever two such trivializations (U,Φ) and (V,Ψ) overlap, we can write

$$\Psi \circ \Phi^{-1}(p,v) = (p, \tau(p)v)$$

for some smooth transition function $\tau\colon U \cap V \to \mathrm{GL}(2n,\mathbb{R})$. Clearly the transition function from $\Phi_{\mathbb{C}}$ to $\Psi_{\mathbb{C}}$ is the same:

$$\Psi^{\mathbb{C}} \circ (\Phi^{-1})^{\mathbb{C}}(p,v) = (p,\tau(p)v),$$

where now τ is considered as a map into $\mathrm{GL}(2n,\mathbb{C})$. It follows from the vector bundle chart lemma that $\pi^{\mathbb{C}}\colon (T^*X)^{\mathbb{C}} \to M$ has a unique structure as a smooth rank-$2n$ holomorphic vector bundle, with the maps constructed above as smooth local trivializations.

Definition 7.9. Let X be a complex manifold. A **holomorphic 1-form** is a section of $(T^*X)^{\mathbb{C}}$.

Example 7.10. Let $X = \mathbb{C}^n$. With (x^j, y^j) as smooth global coordinates for $\mathbb{C}^n$, the smooth global coframe $\{dx^j, dy^j\}$ forms a coframe for $T^*\mathbb{C}^n$, and also for $(T^*\mathbb{C}^n)^{\mathbb{C}}$. Note that we have

$$\begin{aligned}(T^*\mathbb{C}^n)^{\mathbb{C}} &= \mathrm{Span}_{\mathbb{C}}\langle dx^1,\dots,dx^n,dy^1,\dots,dy^n\rangle \\ &= \mathrm{Span}_{\mathbb{C}}\langle dz^1,\dots,dz^n,d\overline{z}^1,\dots,d\overline{z}^n\rangle.\end{aligned}$$

Here we have defined

$$dz^j = dx^j + i\,dy^j, \qquad d\overline{z}^j = dx^j - i\,dy^j$$

If $f\colon U \to \mathbb{C}$ is a smooth function on an open subset $U \subseteq \mathbb{C}^n$, we can write

$$\begin{aligned} df &= \frac{\partial f}{\partial x^j}dx^j + \frac{\partial f}{\partial y^j}dy^j \\ &= \frac{\partial f}{\partial x^j}\left(\frac{dz^j + d\overline{z}^j}{2}\right) + \frac{\partial f}{\partial y^j}\left(\frac{dz^j - d\overline{z}^j}{2i}\right) \\ &= \frac{1}{2}\left(\frac{\partial f}{\partial x^j} - i\frac{\partial f}{\partial y^j}\right)dz^j + \frac{1}{2}\left(\frac{\partial f}{\partial x^j} + i\frac{\partial f}{\partial y^j}\right)d\overline{z}^j \\ &= \frac{\partial f}{\partial z^j}dz^j + \frac{\partial f}{\partial \overline{z}^j}d\overline{z}^j \end{aligned}$$

7.3. **Holomorphic differential Forms.** Let X be a complex n-manifold. We now discuss holomorphic k-forms for $0 \leq k \leq n$. For $0 \leq k \leq n$, let $(\Lambda^k T^*X)^{\mathbb{C}}$ be the complexification of $\Lambda^k T^*X$. As a set, we have

$$(\Lambda^k T^*X)^{\mathbb{C}} = \coprod_{p \in X} (\Lambda^k(T_p^*X))^{\mathbb{C}}$$

A holomorophic vector bundle strucuture on $(\Lambda^k T^*X)^{\mathbb{C}}$ can be constructed in much the same way as that of the complex tangent bundle and complex cotangent bundle, so we don't provide additional details. This allows us to define holomorphic k-forms:

Definition 7.11. Let X be a complex manifold. A **holomorphic k-form** is a section of $(\Lambda^k X)^{\mathbb{C}}$ for $0 \leq k \leq n$

Remark 7.12. *If $k = 0$, a holomorphic 0-form is just a holomorphic function on X. If $k = 1$, then note that $\Lambda^1 T^*X = T^*X$ is the cotangent bundle and $(\Lambda^1 T^*X)^{\mathbb{C}}$ is the complexified cotangent bundle $(T^*X)^{\mathbb{C}}$. A holomorphic 1-form was defined in the previous section.*

Let's discuss the decomposition of $(\Lambda^k X)^{\mathbb{C}}$. Using results from Section 4 we have that

$$\begin{aligned}
(\Lambda^k(T_p^*X))^{\mathbb{C}} &= \Lambda^k(T_p^*X)^{\mathbb{C}} \\
&= \Lambda^k((T_p^*X)_{(1,0)} \oplus (T_p^*X)_{(0,1)}) \\
&= \bigoplus_{p+q=k} \Lambda^p((T_p^*X)_{(1,0)}) \otimes_{\mathbb{C}} \Lambda^q((T_p^*X)_{(0,1)}) := \bigoplus_{p+q=k} \Lambda^{p,q}(T_p^*X)
\end{aligned}$$

As a result, we have

$$\begin{aligned}
(\Lambda^k T^*X)^{\mathbb{C}} &= \coprod_{p \in X} (\Lambda^k(T_p^*X))^{\mathbb{C}} \\
&= \coprod_{p \in X} \bigoplus_{p+q=k} \Lambda^p((T_p^*X)_{(1,0)}) \otimes_{\mathbb{C}} \Lambda^q((T_p^*X)_{(0,1)}) \\
&:= \coprod_{p \in X} \bigoplus_{p+q=k} \Lambda^{p,q}(T_p^*X) := \bigoplus_{p+q=k} \Lambda^{p,q}(T^*X)
\end{aligned}$$

Definition 7.13. Let X be a complex n-manifold. Let $0 \leq k \leq 2n$ and $0 \leq p, q \leq n$ such that $p + q = k$. A **holomorphic (p,q)-form** is a section σ of $\Lambda^{p,q}(T^*X)$.

If $U \subseteq X$ is an open set of X corresponding to a holomorphic atlas then it is then clear that the following collection of forms constitutes a smooth local frame for $\Lambda^{p,q}(T^*X)$:

$$\{dz^{j_1} \wedge \cdots \wedge dz^{j_p} \wedge d\overline{z}^{l_1} \wedge \cdots \wedge d\overline{z}^{l_q} : p + q = k,\ j_1 < \cdots < j_p,\ l_1 < \cdots < l_q\}.$$

Hence, in every local holomorphic coordinate chart $(U, (z^1, \ldots, z^n))$, $\sigma \in \Lambda^{p,q}(T^*X)$ can be expressed as

$$\sigma|_U = \sum_{\substack{j_1 < \cdots < j_p, \\ l_1 < \cdots < l_q}} dz^{j_1} \wedge \cdots \wedge dz^{j_p} \wedge d\overline{z}^{l_1} \wedge \cdots \wedge d\overline{z}^{l_q}$$

Remark 7.14. *We use the notation $\mathscr{E}^k(X)$ to denote the space of holomorphic sections of $(\Lambda^k T^*X)^{\mathbb{C}}$, and $\mathscr{E}^{p,q}(X)$ for the space of holomorphic sections of $\Lambda^{p,q}(T^*X)$.*

For each $0 \leq p, q \leq n$, we have projection operators

$$\pi^{p,q} \colon (\Lambda^k T^* X)^{\mathbb{C}} \to \Lambda^{p,q}(T^* X)$$

Using the definition and properties of the wedge product, $\wedge$, and the exterior derivative, d, from the smooth manifold case, we have the following proposition:

Proposition 7.15. *Let X be a complex n-manifold.*

(1) *Let $\alpha \in \mathscr{E}^{p,q}(X)$. Then*

$$d\big(\mathscr{E}^{p,q}(X)\big) \subseteq \mathscr{E}^{p+1,q}(X) \oplus \mathscr{E}^{p,q+1}(X).$$

(2) *For each $0 \leq p, q \leq n$, there exists Dolbeault operators*

$$\partial \colon \mathscr{E}^{p,q}(X) \to \mathscr{E}^{p+1,q}(X). \qquad \bar{\partial} \colon \mathscr{E}^{p,q}(X) \to \mathscr{E}^{p,q+1}(X)$$

such that

$$\partial = \pi^{p+1,q} \circ d, \quad \bar{\partial} = \pi^{p,q+1} \circ d.$$

(3) *If $\alpha \in \mathscr{E}^{p,q}(X)$ and $\beta \in \mathscr{E}^{p',q'}(X)$*

$$\overline{\alpha} \in \mathscr{E}^{q,p}(M).$$

$$\alpha \wedge \beta \in \mathscr{E}^{p+p',q+q'}(M).$$

(4) *If $\alpha \in \mathscr{E}^{k}(X)$ and $\beta \in \mathscr{E}^{l}(X)$, then*

$$\partial(\alpha \wedge \beta) = \partial\alpha \wedge \beta + (-1)^k \alpha \wedge \partial\beta,$$

$$\overline{\partial}(\alpha \wedge \beta) = \overline{\partial}\alpha \wedge \beta + (-1)^k \alpha \wedge \overline{\partial}\beta.$$

Proof. It suffices to work in local holomorphic coordinates.

(1) Choose holomorphic local coordinates $(z^1, \ldots, z^n)$ and write

$$\alpha = \sum\nolimits_{J,L} \alpha_{J,L}\, dz^{j_1} \wedge \cdots \wedge dz^{j_p} \wedge d\bar{z}^{l_1} \wedge \cdots \wedge d\bar{z}^{l_q},$$

where $J = (j_1, \ldots, j_p)$ and $L = (l_1, \ldots, l_q)$ are strictly increasing multi-indices. We have

$$d\alpha = \sum\nolimits_{J,L} \sum_r \left(\frac{\partial \alpha_{J,L}}{\partial z^r} dz^r + \frac{\partial \alpha_{J,L}}{\partial \bar{z}^r} d\bar{z}^r \right) \wedge dz^{j_1} \wedge \cdots \wedge dz^{j_p} \wedge d\bar{z}^{l_1} \wedge \cdots \wedge d\bar{z}^{l_q},$$

The claim follows.

(2) This follows from (1).
(3) This is clear.
(4) This follows from (1), (2) and (3).

This completes the proof. □

7.4. **Dolbeault Cohomology.** We now discuss Dolbeault cohomology. Dolbeault cohomology is a fundamental tool in complex geometry, used to study the structure of complex manifolds. We start off with a basic prototype of holomorphic differential forms that will allow us to define the Dolbeault cohomology.

Lemma 7.16. *Let X, Y be a complex manifolds and let $F : X \to Y$ be a holomorphic map.*

(1) *If* $\alpha \in \mathscr{E}^k(X)$*, then we have*

$$\begin{aligned} d\alpha &= \partial\alpha + \overline{\partial}\alpha \\ \overline{\partial\alpha} &= \overline{\partial}(\overline{\alpha}), \\ \partial \circ \partial\alpha &= \overline{\partial} \circ \overline{\partial}\alpha = 0, \\ \partial \circ \overline{\partial}\alpha &= -\overline{\partial} \circ \partial\alpha. \end{aligned}$$

(2) *If* $\alpha \in \mathscr{E}^{p,q}(Y)$*, then we have*

$$\begin{aligned} F^*(\mathscr{E}^{p,q}(Y)) &\subseteq \mathscr{E}^{p,q}(X), \\ F^*(\partial\alpha) &= \partial(F^*\alpha), \\ F^*(\overline{\partial}\alpha) &= \overline{\partial}(F^*\alpha). \end{aligned}$$

Proof. The proof is given below:

(1) WLOG, let α be a (p,q) form. The first identity follows from (1) and (2) in Proposition 7.15. The second identity follows from the definition of conjugation. Note that
$$0 = d(d\alpha) = (\partial + \overline{\partial})(\partial + \overline{\partial})\alpha = \partial \circ \partial\alpha + (\partial \circ \overline{\partial}\alpha + \overline{\partial} \circ \partial\alpha) + \overline{\partial} \circ \overline{\partial}\alpha.$$
On the right-hand side, the first term is in $\mathscr{E}^{p+2,q}(X)$, the term in parentheses is in $\mathscr{E}^{p+1,q+1}(X)$, and the last term is in $\mathscr{E}^{p,q+2}(X)$. Since these spaces intersect only in the zero form, each of those three terms must be zero. Hence, the third and fourth identities follow.

(2) Choose holomorphic local coordinates $(z^1, \dots, z^n)$ and $(w^1, \dots, w^m)$. Note that we have
$$\begin{aligned} F^* dw^j &= \frac{\partial F^j}{\partial z^l} dz^l, \\ F^* d\overline{w}^j &= \frac{\partial \overline{F}^j}{\partial \overline{z}^l} d\overline{z}^l. \end{aligned}$$
The first identity follows by a simple linearity argument. The second and third identities follow from the fact that F^* commutes with $\pi^{p+1,q}$, $\pi^{p,q+1}$ and d.

This completes the proof. $\square$

Lemma 7.16 allows us to define a set of biholomorphic invariants. Because $\overline{\partial} \circ \overline{\partial} = 0$, for each p we obtain a cochain complex known as the p-th Dolbeault complex:

$$0 \to \mathscr{E}^{p,0}(M) \xrightarrow{\overline{\partial}} \mathscr{E}^{p,1}(M) \xrightarrow{\overline{\partial}} \cdots \xrightarrow{\overline{\partial}} \mathscr{E}^{p,n}(M) \to 0.$$

Definition 7.17. Let X be a complex n-manifold. Let $0 \leq p, q \leq n$. The (p,q)**-Dolbeault cohomology group** is defined as

$$H^{p,q}(X) = \frac{\operatorname{Ker}(\overline{\partial} : \mathscr{E}^{p,q}(M) \to \mathscr{E}^{p,q+1}(M))}{\operatorname{Im}(\overline{\partial} : \mathscr{E}^{p,q-1}(M) \to \mathscr{E}^{p,q}(M))}$$

It follows from Lemma 7.16 that the construction of Dolbeault cohomology is functorial and that it is indeed a biholomorphic invariant.

Definition 7.18. Let X be a complex n-manifold and and let $0 \leq p, q \leq n$. The **Hodge numbers** of X are given by

$$h^{p,q}(X) = \dim H^{p,q}(X)$$

The Dolbeault cohomology groups measure the the extent to which a 'which $\overline{\partial}$-closed forms fail to be $\overline{\partial}$-exact.' The $\overline{\partial}$ Poincaré lemma states that a $\overline{\partial}$-closed form can always be locally written as a $\overline{\partial}$-exact form.

Proposition 7.19. ***($\overline{\partial}$-Poincaré Lemma)*** *Let X be a complex n-manifold and let $0 \leq p, q \leq n$. If $\omega \in \Omega^{p,q}(X)$ is a smooth form that satisfies $\overline{\partial}\omega = 0$ for $q \geq 1$, then in a neighborhood of each point there is a $\eta \in \Omega^{p,q-1}(X)$ that is a smooth form such that $\overline{\partial}\eta = \omega$.*

Remark 7.20. *Nore that Proposition 7.19 is about smooth (p, q) forms. This is because the proof works with functions*

Proof. $\square$