CHARACTERISTIC CLASSES

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Abstract. These are notes on characteristic classes. I compiled them during my graduate studies at the University of Maryland while participating in a reading course under the guidance of Dr. Jonathan Rosenberg. If you notice any typos or errors, please feel free to send corrections to junaida@umd.edu or junaid.aftab1994@gmail.com.

Contents

1. Why Characteristic Classes?

Characteristic classes are fundamental invariants in the study of vector bundles (and G-principal bundles), playing a central role in topology, differential geometry, and mathematical physics. At a high level, they associate to each vector bundle a cohomology class of the base space, capturing geometric and topological information about how the bundle is "twisted" over the space. More formally, characteristic classes are defined as follows for vector bundles.

Definition 1.1. Let $E \to X$ be a (continuous, smooth) vector bundle and let A be an abelian group. For each $k \in \mathbb{N}$, a characteristic class, s_k, is an assignment to each $E \to X$ a cohomology class $s_k(E) \in H^k(X; A)$. This assignment is natural in the sense that if $f : Y \to X$ is a (continuous, smooth) map, then

$$s_k(f^*E) = f_\#(s_k(E)) \in H^k(Y; A)$$

Here $f^*(E)$ is the pullback bundle and $f_\# : H^k(X; A) \to H^k(Y; A)$ is the induced map on cohomology groups.

Remark 1.2. *Characteristic classes for principal G-bundles can be defined analogously. We will mostly move back and forth between vector bundles and principal G-bundles as necessary.*

There are four important examples of characteristic classes:

(1) Stiefel-Whitney class. These are defined for real vector bundles.
(2) Chern Class. These are defined for complex vector bundles.

(3) Pontryagin Class. These are defined for real vector bundles.
(4) Euler Class. These are defined for oriented real vector bundles.

Characteristic Class	Defined For	Cohomology Coefficients
Stiefel–Whitney Class	Real vector bundle	$H^n(X;\mathbb{Z}_2)$
Chern Class	Complex vector bundle	$H^{2n}(X;\mathbb{Z})$
Euler Class	Oriented real vector bundle	$H^n(X;\mathbb{Z})$
Pontryagin Class	Real vector bundle	$H^{4n}(X;\mathbb{Z})$

Characteristic classes have many uses, particularly in topology, geometry, and physics. They provide deep insights into the structure of vector bundles and play a critical role in understanding various mathematical and physical phenomena. Some key applications include:

(1) Quantifying obstructions to existece of certain structures such as global sections, orientation, spin strcutre etc.
(2) Distinguishing homotopically distinct maps and serving as essential tools in the study of generalized cohomology theories such as K-theory and (co)bordism.
(3) Pairing a product of characteristic classes with the fundamental class defines a characteristic number which are (co)bordism invariants, and in many cases, the set of characteristic numbers forms a complete and computable (co)bordism invariant.
(4) Playing a central role in condensed matter physics in the classification of topological phases of matter, such as topological insulators and quantum Hall systems. In these systems, quantized physical observables reflect underlying topological invariants.

Part 1. Stiefel-Whitney Classes

2. Axiomatic Definition

In what follows, let X be a connected topological space. We introduce Stiefel-Whitney classes via the axiomatic approach. In particular, we introduce four axioms which characterize the Stiefel-Whitney classes.

Definition 2.1. Let X be a paracompact connected topological space and let $\pi : E \to X$ be a $\mathbb{R}$-vector bundle. The k-th Stiefel Whitney class, w_k, is a cohomology class contained in $H^k(X;\mathbb{Z}_2)$ satisfying:

(1) $w_0 = 1 \in H^0(X;\mathbb{Z}_2) \cong \mathbb{Z}_2$.
(2) $w_k(E) = 0$ for $k > n$ if $E \to X$ has rank n.
(3) (Naturality) If $\pi_i : E_i \to X_i$ are $\mathbb{R}$-vector bundles, and there $f : X_2 \to X_1$ is a continuous map that lies over a vector bundle morphism $g : E_1 \to E_2$, then
$$w_k(X_2) = f_\#(w_k(X_1)) \in H^*(X_2;\mathbb{Z}_2)$$
for $k \in \mathbb{N}$.
(4) (Whitney Sum Formula) If $\pi_i : E_i \to X$ two $\mathbb{R}$-vector bundles, then
$$w_k(E_1 \oplus E_2) = \sum_{i=0}^{k} w_i(E_1) \smile w_{k-i}(E_2)$$
Here $\smile$ denotes the cup product.
(5) If γ_1^1 is the tautological line bundle over $\mathbb{RP}^1$, then $w_1(\gamma_1^1) \neq 0$.

It is not immediately apparent that cohomology classes satisfying Definition 2.1 can be defined. The existence and uniqueness of cohomology classes satisfying these axioms will be established later. For the time being, let us examine some immediate consequences of the axioms:

Proposition 2.2. *Let $\pi_i : E_i \to X$ be $\mathbb{R}$-vector bundles of rank n_i for $i = 1, 2$. Let ε denote the trivial $\mathbb{R}$-vector bundle of rank n.*

(1) *If $E_1 \cong E_2$, then*
$$w_k(E_1) = w_k(E_2)$$
for $k \in \mathbb{N} \cup \{0\}$.

(2) *If $E_1 \cong \varepsilon^{n_1}$, then $w_k(E_1) = 0$ for $k \in \mathbb{N}$.*

(3) *If $E_1 \cong \varepsilon^{n_1}$, then*
$$w_k(E_1 \oplus E_2) = w_k(E_2)$$
for $k \in \mathbb{N} \cup \{0\}$.

(4) *If E_1 and E_2 are (eventually) stably equivalent, that is if there exist $j_1, j_2 \in \mathbb{N}$ such that $E_1 \oplus \varepsilon^{j_1} \cong E_2 \oplus \varepsilon^{j_2}$, then*
$$w_k(E_1) = w_k(E_2)$$
for $k \in \mathbb{N} \cup \{0\}$.

Proof. The proof is given below:

(1) This follows directly from (3) in Definition 2.1.

(2) Observe that $X \times \mathbb{R}^{n_1}$ arises as the pullback of the rank-n_1 trivial vector bundle over a point along the constant map $X \to *$. By Definition 2.1(3), the Stiefel–Whitney classes of this bundle vanish, as $H^n(*, \mathbb{Z}_2) = 0$ for $n > 1$. The claim thus follows.

(3) This follows from (2) and (4) in Definition 2.1. Indeed, we have
$$w_k(E_1 \oplus E_2) = 1 \smile w_k(E_2) + \sum_{i=1}^{k} 0 \smile w_{k-i}(E_2) = w_k(E_2)$$
for $k \in \mathbb{N}$.

(4) This follows from (3).

This completes the proof. □

Remark 2.3. *Proposition 2.2(1) implies that the Stiefel-Whitney classes are topological invariants. While it is generally difficult to determine whether two $\mathbb{R}$-vector bundles E and F are isomorphic, their Stiefel-Whitney classes are often easier to compute. If $w_k(E) \neq w_k(F)$ for some $k \in \mathbb{N}$, it follows that E and F are not isomorphic. However, Stiefel-Whitney classes are not perfect topological invariants. This is easily seen via Proposition 2.2(2): two trivial $\mathbb{R}$-vector bundles of different rank have the same Stiefel-Whitney classes.*

We now define the total Stiefel-Whitney class. If X is a topological space, let $H^{\Pi}(X; \mathbb{Z}_2)$ denote the set of all formal sums $\sum_i a_i$, where $a_i \in H^i(X; \mathbb{Z}_2)$. It is important to note that $H^{\Pi}(X; \mathbb{Z}_2)$ can be endowed with a ring structure, where the product is defined as follows:
$$\left(\sum_i a_i\right)\left(\sum_i b_i\right) = \sum_i \sum_{j=0}^{i} a_j \smile b_{i-j}.$$

Lemma 2.4. *Let X be a paracompact topological space. The collection of all infinite series*

$$a = \sum_i a_i \in H^{\Pi}(X; \mathbb{Z}/2)$$

with the leading term 1 forms a commutative group under multiplication.

Proof. This product is commutative, as we are working in $\mathbb{Z}_2$, and it is also associative. The formula for the inverse of a, denoted as $\bar{a}$, can be deduced from the following formal computation:

$$\begin{aligned}
\bar{a} &= \left(1 + \sum_i a_i\right)^{-1} \\
&= 1 + \left(\sum_i a_i\right) + \left(\sum_i a_i\right)^2 + \left(\sum_i a_i\right)^3 + \cdots \\
&= 1 + a_1 + (a_1^2 + a_2) + (a_1^3 + 2a_1a_2 + a_3) + \cdots = 1 + a_1 + (a_1^2 + a_2) + (a_1^3 + a_3) + \cdots
\end{aligned}$$

This completes the proof. □

This leads to the following definition.

Definition 2.5. Let $\pi : E \to X$ be a $\mathbb{R}$-vector bundle of rank n. The total Stiefel-Whitney class is defined to be the following element of $H^{\Pi}(B; \mathbb{Z}_2)$:

$$w(E) = \sum_{k=0}^{n} w_k(E)$$

Remark 2.6. *We end with the following observations:*

(1) *The Whitney sum formula can be succinctly expressed as*

$$w(E_1 \oplus E_2) = w(E_1)w(E_2),$$

(2) *If E_1, E_2 are $\mathbb{R}$-vector bundles such that $E_1 \oplus E_2 \cong \varepsilon^n$, then $w(E_2) = \bar{w}(E_1)$ since $w(\varepsilon^n) = 1$. In particular, $w(E_2)$ can be computed recursively via the Whitney sum formula. Indeed, we have*

$$w_k(E_2) = w_1(E_1)w_{k-1}(E_2) + w_2(E_1)w_{k-2}(E_2) + \cdots + w_{k-1}(E_1)w_1(E_2) + w_k(E_1)$$

for $k \geq 1$. We have

$$\begin{aligned}
w_1(E_2) &= w_1(E_1) \\
w_2(E_2) &= w_1^2(E_1) + w_2(E_1) \\
w_3(E_2) &= w_1^3(E_1) + w_3(E_1) \\
w_3(E_2) &= w_1^4(E_1) + w_1^2(E_1)w_2(E_1) + w_2^2(E_1) + w_4(E_1) \qquad (1)
\end{aligned}$$

and so on.

(3) *Let M be a smooth manifold. Consider the tangent bundle $TM \to M$ and the normal bundle $NM \to M$. Then*

$$w_k(TM) = \bar{w}_k(NM)$$

for each $k \in \mathbb{N}$. This follows from (2).

3. First Computations

Let's compute the total Stiefel-Whitney classes in some tractable cases.

Example 3.1. Consider $T\mathbb{S}^n$. Note that $N\mathbb{S}^n$ is a trivial vector bundle since it has a nowhere vanishing section given by the unit outward normal to the sphere, $\mathbf{x}/\|\mathbf{x}\|$. Hence, $w_k(N\mathbb{S}^n) = 0$ for $k \geq 1$. Since $T\mathbb{S}^n \oplus N\mathbb{S}^n \cong \varepsilon^{n+1}$, we have $w_k(T\mathbb{S}^n) = 0$ since $\bar{w}_k(N\mathbb{S}^n) = 0$ for $k \geq 1$ by Equation (1).

Remark 3.2. *Example 3.1 shows that the non-zero Stiefel-Whitney classes of a non-trivial $\mathbb{R}$-vector bundle can all be non zero. Indeed, we have that $T\mathbb{S}^2$ is non-trivial but $w_k(T\mathbb{S}^2) = 0$ for $k \geq 1$.*

Example 3.3.

4. Obstructions via Stiefel Whitney Classes

Our first application of characteristic classes will be to obstructing certain structures.

4.1. **Existence of Sections.**

Stiefel-Whitney classes can allow us to determine when a vector bundle admits linearly independent sections.

Proposition 4.1. *Let X be a paracompact topological space and let $E \to X$ be a rank n $\mathbb{R}$-vector bundle. If $E \to X$ possesses a no-where vanishing section, then $w_n(E) = 0$. More generally, if $E \to X$ admits l linearly independent sections, then $w_k(E) = 0$ for $k \geq n-l+1$.*

Proof. Since X admits l linearly independent sections, we can construct a rank l $\mathbb{R}$-subbundle of $E \to X$, denoted as F, such that $F \cong \varepsilon^l$. Since X is paracompact, $E \to X$ admits an inner product. Using the inner product, we can write E as $E = F \oplus F^\perp$, where $F^\perp$ is a rank $n-l$ $\mathbb{R}$-vector bundle. We have,

$$\begin{aligned} w_k(E) &= w_k(\varepsilon^l \oplus F^\perp) \\ &= \sum_{i=0}^{k} w_i(\varepsilon^l) \smile w_{k-i}(F^\perp) \\ &= w_0(\varepsilon^l) \smile w_k(F^\perp) \\ &= w_k(F^\perp) \\ &= 1 \smile w_k(F^\perp) = w_k(F^\perp) \end{aligned}$$

Since $F^\perp$ has rank $n-l$, $w_k(E) = 0$ if $k \geq n-l+1$ by Definition 2.1(2). $\square$

Remark 4.2. *Proposition 4.1 implies that if $w_{n-l}(E) \neq 0$, then E has at most l linearly independent sections.*

5. Existence of Stiefel-Whitney Classes