

BANACH & C^* -ALGEBRAS

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ABSTRACT. This document comprises notes on Banach algebras and C^* -algebras. A portion of these notes was taken during my participation in the Groundwork for Operator Algebras Lecture Series (GOALS) workshop at IPAM, UCLA. If you come across any typos, please send corrections to junaid.aftab1994@gmail.com.

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Remark 0.1. Unless otherwise specified, we work over $\mathbb{K} = \mathbb{C}$, the field of complex numbers in these notes.

Part I. Banach Algebras

Banach algebras represent one of the most general forms of operator algebras. They provide a natural framework for exploring classical topics in functional analysis, including spectral theory. Fundamental notions such as the spectrum, resolvent set, and spectral radius can be rigorously analyzed within this setting, enabling a deeper understanding of operator behavior on Hilbert spaces and beyond. Moreover, C^* -algebras and von Neumann algebras constitute important subclasses of Banach algebras. Consequently, the study of Banach algebras serves as a foundational step in the broader investigation of operator algebras.

I. DEFINITIONS & EXAMPLES

We begin with a detailed discussion of definitions and examples in Banach algebra theory. If \mathcal{H} is a Hilbert space, note that the Banach space, $\mathcal{B}(\mathcal{H})$, of bounded linear operators on \mathcal{H} has more structure than that of a Banach space. Indeed, if $T, S \in \mathcal{B}(\mathcal{H})$, then $T \circ S \in \mathcal{B}(\mathcal{H})$ such that

$$\|T \circ S\| \leq \|T\| \|S\|$$

This observation motivates the definition of a Banach algebra. We will proceed in multiple steps.

Definition 1.1. Let V be a complex vector space.

- (1) V is an **algebra** if the underlying abelian group admits a multiplication operation

$$\begin{aligned} \cdot : V \times V &\rightarrow V, \\ (x, y) &\mapsto x \cdot y \end{aligned}$$

endowing V with a ring structure compatible with the given scalar multiplication. That is,

- (i) For all $x, y \in V$, the product $x \cdot y \in V$.
- (ii) For all $x, y, z \in V$ and all scalars $\alpha, \beta \in \mathbb{C}$,

$$\begin{aligned} (\alpha x + \beta y) \cdot z &= \alpha(x \cdot z) + \beta(y \cdot z), \\ x \cdot (\alpha y + \beta z) &= \alpha(x \cdot y) + \beta(x \cdot z). \end{aligned}$$

- (iii) Multiplication distributes over addition:

$$x \cdot (y + z) = x \cdot y + x \cdot z, \quad (x + y) \cdot z = x \cdot z + y \cdot z.$$

- (iv) For all $\lambda \in \mathbb{C}$ and $x, y \in V$,

$$(\lambda x) \cdot y = \lambda(x \cdot y) = x \cdot (\lambda y).$$

- (v) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ for all $x, y, z \in V$.

- (2) V is a **normed algebra** if it is equipped a norm function $\|\cdot\| : V \rightarrow [0, \infty)$ that is submultiplicative: for all $x, y \in V$,

$$\|x \cdot y\| \leq \|x\| \|y\|.$$

Remark 1.2. Let V be a (normed) algebra.

- (1) We say V is a **unital (normed) algebra** if it admits an identity element. That is, there exists a $e \in V$ such that $e \cdot x = x \cdot e = x$ for all $x \in V$. It is a simple exercise to check that the identity in a unital algebra is unique. The proof is identical from group theory.
- (2) If V is a (normed) algebra, we say V is an **abelian (normed) algebra** if $x \cdot y = y \cdot x$ for all $x, y \in V$

Remark 1.3. From now on, we abbreviate the ring multiplication operator $x \cdot y$ as simply xy for x, y in an algebra.

If V is a normed algebra, then the norm induces a metric on V which in turn induces a topology on V called the norm topology. Here is a sample proposition:

Lemma 1.4. *Let V be a normed algebra. Addition, scalar multiplication and multiplication are continuous in the norm topology on V .*

Proof. Let's consider the multiplication operation. Let $x_n \rightarrow x$ and $y_n \rightarrow y$. The sub-multiplicativity of the norm implies that

$$\|xy - x_n y_n\| \leq \|xy - xy_n\| + \|xy_n - x_n y_n\| \leq \|x\|\|y - y_n\| + \|x - x_n\|\|y_n\| \rightarrow 0.$$

It is clear that addition and scalar multiplication are continuous. \square

Definition 1.5. A **Banach algebra**, A , is a normed algebra that is complete in the metric topology induced by the norm.

In other words, a Banach algebra, A , is a Banach space endowed with a sub-multiplicative operation making X into a normed algebra. A unital Banach algebra is defined similarly.

Remark 1.6. If A is unital Banach algebra with identity element $e \in A$, then note that we have

$$\|e\| = \|ee\| \leq \|e\|\|e\|$$

Hence, $\|e\| \geq 1$. In a C^* -algebra, we will see that we automatically have $\|e\| = 1$.

Definition 1.7. Let A be an algebra over \mathbb{C} . A subset $B \subseteq A$ is called a **subalgebra** of A if B is closed under addition, scalar multiplication and multiplication.

If A is a normed algebra, then any subalgebra $B \subseteq A$, equipped with the same norm, is also a normed algebra. Furthermore, the closure of B in the norm topology of A remains a normed algebra. In particular, if A is a Banach algebra, then the closure of any subalgebra $B \subseteq A$ is again a Banach algebra. This observation generates a long list of examples:

Example 1.8. The following is a list of some basic examples of Banach algebras:

- (1) Let S be a set, and let $\ell^\infty(S)$ be the collection of all bounded complex-valued functions on S . Then $\ell^\infty(S)$ is a Banach algebra with respect to the usual pointwise operations defined as follows:

$$\begin{aligned} (f + g)(s) &:= f(s) + g(s), \\ (fg)(s) &:= f(s)g(s), \\ (\lambda f)(s) &:= \lambda f(s) \end{aligned}$$

for all $s \in S$. It is a commutative unital Banach algebra with identity given by the constant function $x \mapsto 1$. The norm is given by

$$\|f\|_\infty = \sup_{s \in S} |f(s)|.$$

- (2) Let X be a locally compact Hausdorff space, X . Consider the space of bounded continuous complex-valued functions on X .

$$C_b(X) = \{f : X \rightarrow \mathbb{F} \mid f \text{ is continuous and bounded}\},$$

It can be checked that $C_b(X)$ is a closed subalgebra of $\ell^\infty(X)$. Hence, $C_b(X)$ is a unital commutative Banach algebra.

- (3) Let X be a locally compact Hausdorff space, X . Consider the space of continuous complex-valued functions on X that vanish at infinity.

$$C_0(X) = \{f \in C(X) : \forall \varepsilon > 0 \exists \text{ compact } K \subseteq X \text{ with } |f(x)| < \varepsilon \text{ for } x \notin K\},$$

It can be checked that $C_0(X)$ is a closed subalgebra of $C_b(X)$. Hence, is a commutative Banach algebra. It is unital if and only if X is compact.

- (4) Let \mathcal{H} be a Hilbert space. Then $\mathcal{B}(\mathcal{H})$ is a (generally non-commutative) unital Banach algebra with multiplication operation given by composition, identity given by the identity operator, and norm given by the operator norm. In particular, if $\mathcal{H} = \mathbb{C}^n$ then $M_n(\mathbb{C})$ is a Banach algebra under matrix multiplication and the operator norm.

Remark 1.9. If X is locally compact and Hausdorff, then $C_b(X)$ contains many functions due to Urysohn's Lemma. If X is locally compact but not compact, then $C_b(X)$ is likely non-separable. Let $X = \mathbb{R}$. Consider the subset $K \subseteq C_b(\mathbb{R})$ consisting of functions that are either 0 or 1 at the integers. There is an uncountable subset S of K such that:

$$\|f - g\| \geq 1, \quad \text{whenever } f, g \in S \text{ with } f \neq g.$$

Given a countable subset B of $C_b(\mathbb{R})$, it follows that there is an $s \in S$ that is at least $\frac{1}{2}$ distance away from every element of B . Thus, B is not dense in $C_b(\mathbb{R})$.

Remark 1.10. If X is a locally compact Hausdorff space, we have that $C(X)$ is also a Banach algebra. In general, we have the following inclusions of Banach algebras:

$$C_0(X) \subsetneq C_b(X) \subsetneq C(X)$$

If X is a compact Hausdorff space, then

$$C_0(X) = C_b(X) = C(X)$$

Definition 1.11. Let A and B be Banach algebras. A **morphism** $\phi : A \rightarrow B$ is a continuous \mathbb{C} -linear, multiplicative map. Moreover, ϕ is an **isometric isomorphism** if it has a continuous inverse and ϕ is an isometry.

Remark 1.12. If A, B are unital Banach algebras, then a morphism $\phi : A \rightarrow B$ is unital morphism if $\phi(e_A) = e_B$.

Example 1.13. Let X be a compact topological space and let $Y \subseteq X$ be a compact subspace. Then the restriction of functions is a momorphism of Banach algebras from $C(X)$ to $C(Y)$. This includes the special case when $Y = \{x\}$, consisting of a single element. In this case, $C(Y) \cong \mathbb{C}$, and the restriction is the evaluation homomorphism $\delta_x : C(X) \rightarrow \mathbb{C}$ mapping f to $f(x)$.

Commutative Banach algebras are much easier to study. In spite of the fact that we are most interested in algebras modeling $\mathcal{B}(\mathcal{H})$ for some Hilbert space \mathcal{H} — which are not commutative — the theory of commutative Banach algebras plays a very important role in the sequel. We discuss an important example of a commutative Banach algebras to end this section:

Example 1.14. Consider $L^1(\mathbb{R})$, the Banach space of Lebesgue integrable complex-valued functions on \mathbb{R} . The convolution of two functions f and g in $L^1(\mathbb{R})$ is defined as

$$(f * g)(x) := \int_{\mathbb{R}} f(x - y)g(y) dy.$$

$L^1(\mathbb{R})$ is a commutative Banach algebra. It is easy to check that the $+$ and $*$ operations on $L^1(\mathbb{R})$ satisfy the axioms of an abelian algebra. That is, we have,

$$\begin{aligned}(f * g) * h &= f * (g * h) \\ f * (g + h) &= f * g + f * h \\ (f + g) * h &= f * h + g * h \\ f * g &= g * f\end{aligned}$$

Moreover, $*$ is sub-multiplicative. Indeed, we have,

$$\begin{aligned}\|f * g\|_1 &= \int_{\mathbb{R}} \int_{\mathbb{R}} |f(x-y)| |g(y)| dy dx \\ &\leq \int_{\mathbb{R}} \int_{\mathbb{R}} |f(x-y)| |g(y)| dx dy \\ &= \int_{\mathbb{R}} \left[\int_{\mathbb{R}} |f(x-y)| dx \right] |g(y)| dy \\ &= \|f\|_1 \int_{\mathbb{R}} |g(y)| dy = \|f\|_1 \|g\|_1\end{aligned}$$

$L^1(\mathbb{R})$ is a non-unital Banach algebra. Assume that there is a $f \in L^1(\mathbb{R})$ is a unit. For each $g \in L^1(\mathbb{R})$, we have by the convolution theorem,

$$\hat{g} = \widehat{f * g} = \hat{f} \hat{g}$$

But if $g \in L^1(\mathbb{R})$ is a Gaussian, then \hat{g} doesn't vanish. Hence, $\hat{f} \equiv 1$. But the Riemann Lebesgue lemma implies that $\hat{f} \in C_0(\mathbb{R})$. This is a contradiction.

Remark 1.15. *The convolution on $L^1(\mathbb{R})$ is first defined on the dense subspace $C_c(\mathbb{R})$, and subsequently be extended by continuity to $L^1(\mathbb{R})$.*

Example 1.16. Let $\mathbb{D} = \{z \in \mathbb{C} : |z| \leq 1\}$ be the open unit disk. Let $\mathcal{O}(\mathbb{D}) \subseteq C(\mathbb{D})$ be the subalgebra of holomorphic functions on \mathbb{D}° . By Morera's Theorem, the uniform limit of holomorphic functions is holomorphic, which implies that $\mathcal{O}(\mathbb{D})$ is a closed subalgebra of $C(\mathbb{D})$. Let $\mathcal{A}(\mathbb{D})$ be the image of $\mathcal{O}(\mathbb{D})$ in $C(\mathbb{S}^1)$ via the restriction map. Since the map $g \mapsto g|_{\mathbb{S}^1}$ is isometric by the Maximum Modulus Principle, $\mathcal{A}(\mathbb{D})$ is complete and therefore closed in $C(\mathbb{S}^1)$. Hence, $\mathcal{A}(\mathbb{D})$ is a closed subalgebra of $C(\mathbb{S}^1)$ and forms a Banach algebra, commonly referred to as the disk algebra. It consists of those functions in $C(\mathbb{S}^1)$ that have holomorphic extensions to \mathbb{D} .

We conclude this section by briefly discussing the notion of ideals in a Banach algebra.

Definition 1.17. Let A be a Banach algebra. A subspace $I \subseteq A$ is called a **left ideal** if for any $a \in I$, it follows that $ab \in I$ for all $b \in A$. A **right ideal** is defined similarly.

Proposition 1.18. *Let A be a Banach algebra and let $I \subseteq A$ be a closed two-sided left proper ideal. The quotient space A/I is a Banach algebra with respect to the quotient norm.*

Proof. It is a standard fact from Banach space theory that A/I is Banach space with the quotient norm:

$$\|a + I\|_{A/I} = \inf_{m \in I} \|a + m\|_A.$$

We only show that the quotient norm is sub-multiplicative. For given $a, b \in A$ and for every $\epsilon > 0$, by the definition of the quotient norm, there exist $m, n \in I$ such that

$$\|a + m\|_A \leq \|a + I\|_{A/I} + \epsilon, \quad \|b + n\|_A \leq \|b + I\|_{A/I} + \epsilon$$

Since $(a + m)(b + n) \in ab + I$, we have

$$\begin{aligned}\|ab + I\|_{A/I} &\leq \|(a + m)(b + n)\|_A \\ &\leq \|a + m\|_A \|b + n\|_A \\ &\leq \|a + I\|_{A/I} \|b + I\|_{A/I} + \epsilon(\|a + I\|_{A/I} + \|b + I\|_{A/I} + \epsilon).\end{aligned}$$

Since this holds for every $\epsilon > 0$, claim follows. \square

Remark 1.19. *Proposition 1.18 can be easily extended to the case when I is a right ideal. If A is a commutative Banach algebra, then the distinction between right and left ideals is unnecessary.*

Example 1.20. Let A be a Banach algebra.

- (1) There are two trivial ideals, the one consisting of the zero element and the one consisting of A itself.
- (2) If A is unital, any ideal, I , that contains the unit element e is equal to A .

Definition 1.21. Let A be a Banach algebra.

- (1) If A is unital, a left (resp. right) maximal ideal if it is not contained in any other non-left (resp. right) trivial ideal.
- (2) If A is non-unital, a left (resp. right) ideal $I \subseteq A$ is called regular if A/I contains a unit.

Zorn's lemma implies that maximal ideals always exist.

Proposition 1.22. *Let A be a unital Banach algebra. Any non-trivial left (resp. right) ideal is a subset of a left (resp. right) maximal ideal.*

Proof. Denote the ideal by I . A partial order among left (resp. right) ideals containing I is established through inclusion. Consider any chain of such non-trivial left (resp. right) ideals I_α , i.e., for any $\alpha \neq \beta$, either $I_\alpha \subseteq I_\beta$ or $I_\beta \subseteq I_\alpha$. We claim that $U = \bigcup_\alpha I_\alpha$ contains I , is a left (resp. right) ideal, and hence an upper bound. Clearly, U is a subspace since all I_β are subspaces. Any $x \in U$ is in some I_α , and if $y \in A$ is any element, we have that $xy \in I_\alpha$ and thus in U . Since e is not in any of the I_β , it is not in U , and hence U is non-trivial. That $I \subseteq U$ is evident. By Zorn's lemma, there exists a maximal element M , i.e., M is an ideal such that whenever V is an ideal that contains I and M , then $V = M$. Hence, M is a maximal ideal. \square

Corollary 1.23. *Let A be a unital Banach algebra and let $I \subseteq A$ be a left (resp. right) ideal. Then \bar{I} is a left (resp. right) ideal. In particular any maximal ideal is closed.*

Proof. This follows from the continuity of multiplication and the non-triviality of the ideal's closure. \square

Remark 1.24. *By Zorn's lemma, every non-unital Banach algebra contains a maximal regular ideal.*

2. UNITIZATION

When a Banach algebra, A , does not contain a unit, we can always add one, as follows. Form the vector space $A' := A \oplus \mathbb{C}$ and make this into an algebra by means of

$$(a, \lambda)(b, \mu) := (ab + \lambda b + \mu a, \lambda\mu)$$

for each $a, b \in A$ and $\lambda, \mu \in \mathbb{C}$. In other words, $(0, 1) \in A'$, which can be identified with $1 \in \mathbb{C}$, is the identity, e' , in A' . Furthermore, we can define a norm on A' by

$$\|(a, \lambda)\|_{A'} := \|a\|_A + |\lambda|$$

In particular, $\|e'\| = |1| = 1$. We have,

$$\begin{aligned}\|(a, \lambda)(b, \mu)\|_{A'} &\leq \|a\|_A\|b\|_A + |\lambda|\|b\|_A + |\mu|\|a\|_A + |\lambda||\mu| \\ &= \|(a, \lambda)\|_{A'}\|(b, \mu)\|_{A'}\end{aligned}$$

Hence, A' is a unital Banach algebra containing A . It is clear that the embedding of A into A' is linear and isometric and that A sits inside of A' as a closed ideal¹. Hence, we can in a way think of A' as the smallest unital Banach algebra in which A sits as an ideal. We denote the identity element in A' as e' .

Remark 2.1. *If A already has a unit e , then the algebra A' is isomorphic to the direct sum $A \oplus \mathbb{C}$ of the algebras A and \mathbb{C} , where we define multiplication component-wise. The isomorphism from A' to $A \oplus \mathbb{C}$ is given by*

$$(a, \lambda) \mapsto (a + \lambda e, \lambda)$$

As a crucial example, we compute the unitization $C_0(X)'$. We recall the one-point compactification X_∞ of the space X . Let ∞ denote a new point, and define $X_\infty = X \cup \{\infty\}$, where X_∞ is X with an additional point. A set $U \subseteq X_\infty$ is open if it is either open in X or contains ∞ and the set $X \setminus U$ is compact in X . Every continuous function in $C(X_\infty)$ restricts to a continuous function on X , so $C_0(X)$ can be identified with the subspace of continuous functions on X_∞ that vanish at ∞ .

Proposition 2.2. *Let X be a locally compact non-compact Hausdorff space. There is a topological (non-isometric) isomorphism of Banach algebras $C(X_\infty) \cong C_0(X)'$.*

Proof. Extending every $f \in C_0(X)$ by zero to X_∞ , we consider $C_0(X)$ as a subspace of $C(X_\infty)$. Define $\psi : C_0(X)' \rightarrow C(X_\infty)$ by $\psi(f, \lambda) = f + \lambda 1_{X_\infty}$, where $1(x) = 1$ for all $x \in X_\infty$. Moreover, define $\phi : C(X_\infty) \rightarrow C_0(X)'$ by $\phi(f) = (f|_{C(X)}, 0)$. Then ψ is an isomorphism of algebras with inverse ϕ as can be easily checked. For the norms, we have

$$\|\psi(f, \lambda)\|_{C(X_\infty)} = \sup_{x \in X_\infty} |f(x) + \lambda| \leq \sup_{x \in X} |f(x)| + |\lambda| = \|(f, \lambda)\|_{C(X)'},$$

This shows that ψ is continuous. It can also be checked that ϕ is continuous. \square

3. SPECTRUM

In this section, we consider the notion of the spectrum of an element in a Banach algebra. The spectrum of an element in a Banach algebra generalizes the notion of eigenvalues of a matrix. For each element in a Banach algebra, we show that its spectrum is a non-empty, compact set. We then define the spectral radius of an element of a Banach algebra, and we prove the spectral radius formula.

Definition 3.1. Let A be a unital Banach algebra with unit e . An element $x \in A$ is called **invertible** if there exists $y \in A$ such that

$$xy = yx = e$$

The element y is called the inverse of x , denoted x^{-1} .

Remark 3.2. *It can be easily checked that the inverse of an element is unique. Let $\text{GL}(A)$ denote the set of invertible elements of A . Then $\text{GL}(A)$ forms a group, and $(xy)^{-1} = y^{-1}x^{-1}$ for $x, y \in \text{GL}(A)$.*

Clearly, we need A to have a unit in order to define the notion of invertibility.

¹In fact, as a maximal ideal, given that its co-dimension is equal to 1.

Definition 3.3. Let A be a unital Banach algebra with unit e . The **resolvent of a in A** , denoted as $\rho_A(a)$, is the set,

$$\rho_A(a) = \{z \in \mathbb{C} \mid a - ze \text{ such that } a - ze \text{ is invertible}\}$$

The **spectrum of a in A** , denoted as $\sigma_A(a)$, is the set,

$$\sigma_A(a) = \{z \in \mathbb{C} \mid a - ze \text{ such that } a - ze \text{ is not invertible}\}$$

Clearly, we have, $\sigma_A(a) = \rho_A(a)^c$ for each $a \in A$. When A has no unit, the resolvent and the spectrum are defined through the embedding of A in its unitization $A' = A \oplus \mathbb{C}$. Hence, we define $\sigma_A(a) = \sigma_{A'}(a)$ for each $a \in A$.

Lemma 3.4. Let A be a non-unital Banach algebra and let A' be its unitization. Then

$$\sigma_{A'}(a) = \sigma_A(a) \cup \{0\}$$

Proof. Let e' denote the identity in A' . Then $0 \in \sigma_A(x)$ because otherwise $e' = x^{-1}x \in A$, a contradiction. Thus, $\sigma_A(x) \cup \{0\} = \sigma_A(x) := \sigma_{A'}(x)$. \square

Example 3.5. Here is a list of basic examples of the spectrum of some concrete Banach algebras.

- (1) When $A = M_n(\mathbb{C})$ is the algebra of $n \times n$ matrices, the spectrum of $a \in A$ is just the set of eigenvalues of a .
- (2) When $A = C(X)$ for some compact topological space, the spectrum of $f \in A$ is just the range of f . Indeed,

$$\begin{aligned}\sigma_A(f) &= \{z \in \mathbb{C} \mid f - z1 \text{ is not invertible}\} \\ &= \{z \in \mathbb{C} \mid f - z1 = 0 \text{ for some } x \in X\} \\ &= \{z \in \mathbb{C} \mid z = f(x) \text{ for some } x \in X\} = \text{Range}(f)\end{aligned}$$

- (3) Let X be a locally compact, non-compact topological space and let $A = C_0(X)$. We have that $\text{Range}(f) \cup \{0\} \subseteq \sigma_{A'}(f)$. If $\lambda \neq 0$ such that $\lambda \notin f(X)$, then consider

$$g(x) = \frac{f(x)}{\lambda(\lambda - f(x))}$$

$g(x)$ is continuous on X . Moreover, because $f \in C(X)$ implies $g \in C_0(X)$. It is easily verified that

$$\left(\frac{1}{\lambda} + g(x)\right)(\lambda - f(x)) = 1$$

Hence,

$$\sigma_{A'}(f) = \text{Range}(f) \cup \{0\}$$

This is consistent with the result we would get by applying Lemma 3.4 to $C_0(X)$.

We can establish numerous algebraic properties of the spectrum. Here is a sample proposition:

Proposition 3.6. Let A, B be unital Banach algebras.

- (1) If $a, b \in A$, then $\sigma_A(ab) = \sigma_A(ba)$.
- (2) If $a, u \in A$, and u is invertible, then $\sigma_A(uau^{-1}) = \sigma_A(a)$.
- (3) If $\phi : A \rightarrow B$ is a unital morphism, then $\sigma_B(\phi(a)) \subseteq \sigma_A(a)$

Proof. The proof is given below:

- (i) Indeed, let $0 \neq \lambda \in \rho_A(ab)$ and set $u := (ab - \lambda)^{-1}$. Hence $abu = uab = 1 + \lambda u$, and from this we obtain

$$\begin{aligned} (ba - \lambda)(bu - 1) &= \lambda \\ (bu - 1)(ba - \lambda) &= \lambda. \end{aligned}$$

Thus $ba - \lambda$ is invertible, and so $\lambda \in \rho_A(ba)$.

- (2) Note that we have

$$uau^{-1} - \lambda e = uau^{-1} - \lambda ueu^{-1} = u(a - \lambda e)u^{-1}$$

It is then clear that $uau^{-1} - \lambda e$ is invertible if and only if $u(a - \lambda e)u^{-1}$ is invertible. The claim follows.

- (3) Let $\lambda \in \sigma_B(\phi(a))$. Then

$$\phi(a) - \lambda e_B = \phi(a - \lambda e_A)$$

is not invertible. Since ϕ preserves invertibility, $a - \lambda e_A$ is not invertible. Hence, $\lambda \in \sigma_A(a)$.

This completes the proof. \square

Remark 3.7. *Invertibility sometimes depends on the algebra in which one allows the inverse to exist. For instance, if $A \subseteq B$ is a unital subalgebra of a unital Banach algebra B , then by Proposition 3.6(3), $\rho_A(a) \subseteq \rho_B(a)$, but the containment may be strict. For example, consider the Banach subalgebra $\mathcal{A}(\mathbb{D}) \subseteq C(\mathbb{T})$. Here $\mathcal{A}(\mathbb{D})$ is the disk algebra. The function $f(z) = z$ in $C(\mathbb{T})$ is invertible in $C(\mathbb{T})$ but not in $A(\mathbb{D})$, since its inverse would have to be $1/z$, which has a singularity at the origin. This phenomenon does not occur for unital C^* -algebras (see Lemma 9.2).*

The rest of the section is devoted to proving some crucial properties of the spectrum of an element in a Banach algebra. We first prove some important properties about invertible elements in a Banach algebra.

Lemma 3.8. *Let A be a unital Banach algebra, and let $a \in A$.*

- (1) *If $\|a\| < 1$, then $a - e$ is invertible with inverse $\sum_{n=0}^{\infty} a^n$.*
- (2) *For each $z \in \mathbb{C} \setminus \{0\}$, $(a - ze)^{-1}$ always exists when $|z| > \|a\|$.*
- (3) *The group $\text{GL}(A)$ is an open set.*
- (4) *The inversion map $a \mapsto a^{-1}$ of $\text{GL}(A)$ is a homeomorphism.*
- (5) *If $a \in \text{GL}(A)$, then $\sigma_A(a^{-1}) = \{\lambda^{-1} \mid \lambda \in \sigma_A(a)\}$*

Proof. The proof is given below:

- (1) We first show that the sum is a Cauchy sequence. Indeed, for $n > m$, we have

$$\left\| \sum_{k=0}^n a^k - \sum_{k=0}^m a^k \right\| = \left\| \sum_{k=m+1}^n a^k \right\| \leq \sum_{k=m+1}^n \|a^k\| \leq \sum_{k=m+1}^n \|a\|^k$$

The sum on the right converges to zero by the theory of the geometric series. Since A is complete, the sum $\sum_{k=0}^{\infty} a^k$ is a well-defined element of A . Now compute

$$\sum_{k=0}^n a^k (a - e) = \sum_{k=0}^n (a^k - a^{k+1}) = e - a^{n+1}.$$

Hence

$$\left\| e - \sum_{k=0}^n a^k (a - e) \right\| = \|a^{n+1}\| \leq \|a\|^{n+1}.$$

which goes to 0 for $n \rightarrow \infty$, as $\|a\| < 1$ by assumption. Thus

$$\left(\lim_{n \rightarrow \infty} \sum_{k=0}^n a^k (a - e) \right) = e.$$

By a similar argument,

$$(a - e) \left(\lim_{n \rightarrow \infty} \sum_{k=0}^n a^k \right) = e.$$

So that, by continuity of multiplication in a Banach algebra, one finally has

$$\sum_{k=0}^{\infty} a^k := \lim_{n \rightarrow \infty} \sum_{k=0}^n a^k = (a - e)^{-1}.$$

(2) Note that

$$(a - ze) = z(z^{-1}a - e)$$

Since $|z| > \|a\|$, we have that $|z^{-1}a| < 1$. The claim follows from (1).

(3) Given $a \in \text{GL}(A)$, let $b \in A$ for which $\|b\| < \|a^{-1}\|^{-1}$. Observe that we have,

$$\|a^{-1}b\| \leq \|a^{-1}\| \|b\| < 1$$

Hence

$$a + b = a(e + a^{-1}b)$$

has an inverse, namely

$$(e + a^{-1}b)^{-1}a^{-1}$$

which exists by (1). It follows that

$$\{y \in A \mid \|x - y\| < \|a^{-1}\|^{-1}\} \subseteq \text{GL}(A).$$

Indeed, if $\|y - x\| \leq \|a^{-1}\|^{-1}$, then if $b = y - x$, then $y = x + b$ is invertible from above. This shows that $\text{GL}(A)$ is open in A .

(4) Let $a \in \text{GL}(A)$. Let b such that $\|a - b\| < \frac{1}{2}\|a^{-1}\|^{-1}$. Note that,

$$\|b^{-1}\| - \|a^{-1}\| \leq \|b^{-1} - a^{-1}\| = \|b^{-1}(a - b)a^{-1}\| \leq \frac{1}{2}\|b^{-1}\|,$$

Hence, $\|b^{-1}\| \leq 2\|a^{-1}\|$. If $\varepsilon \in (0, 1)$, we can now choose b such that $\|a - b\| < \frac{\varepsilon}{2}\|a^{-1}\|^{-2}$. Since the estimate above continues to hold, we have

$$\|b^{-1} - a^{-1}\| \leq \|b^{-1}(a - b)a^{-1}\| \leq 2\|a^{-1}\|^2\|b - a\| \leq \varepsilon$$

This shows that the inversion map is continuous. Since the inversion map is its own inverse, the claim follows.

(5) If $\lambda \neq 0$, then note that

$$a - \lambda e = a\lambda(e\lambda^{-1} - a^{-1}) = -a\lambda(a^{-1} - e\lambda^{-1})$$

Hence, if a is invertible and $\lambda \neq 0$, then $a - \lambda e$ is invertible if and only if $a^{-1} - \lambda^{-1}e$ is invertible. The claim follows from this observation.

This completes the proof. □

Example 3.9. Let A be a Banach algebra. Here is a list of basic examples of the topological group $\text{GL}(A)$:

- (i) Let $A = M_n(\mathbb{C})$. Then the unit group $\text{GL}(A)$ is the group of invertible matrices. The continuity of the determinant function in this case gives another proof that A^\times is open.

- (2) Let $A = C(X)$ for a compact Hausdorff space X . Then the unit group $\text{GL}(A)$ consists of all $f \in C(X)$ with $f(x) \neq 0$ for every $x \in X$.

Proposition 3.10. *Let A be a unital Banach algebra, and let $a \in A$. The spectrum of a is a compact set contained in the unit ball of radius $\|a\|$ in \mathbb{C} .*

Proof. Lemma 3.8(2) implies that

$$\sigma_A(a) \subseteq \{z \in \mathbb{C} \mid |z| \leq \|a\|\}$$

We show that $\sigma_A(a)$ is compact by showing that $\sigma_A(a)$ is a closed set. Given $a \in A$, we define a function f_a by

$$\begin{aligned} f_a : \mathbb{C} &\rightarrow A, \\ z &\mapsto a - ze \end{aligned}$$

Clearly, f_a is a continuous function. Because $\text{GL}(A)$ is open in A by Lemma 3.8(3), it follows that $f_a^{-1}(\text{GL}(A))$ is open in \mathbb{C} . But

$$f_a^{-1}(\text{GL}(A)) = \{z \in \mathbb{C} \mid \text{such that } a - ze \text{ is invertible}\} = \rho_A(a)$$

Hence, $\sigma_A(a) = \rho_A(a)^c$ is a closed set. This shows that $\sigma_A(a)$ is a compact set. \square

Is the spectrum of an element non-empty? The answer is yes, and we now prove it. Since the spectrum is a generalization of the study of eigenvalues of a complex-valued matrix, it is expected that some complex analysis will be required to prove the claim.

Definition 3.11. Let A be a unital Banach algebra, and let $W \subseteq \mathbb{C}$. A function $f : W \rightarrow A$ is a **Banach-algebra valued holomorphic function**

$$\frac{\partial f}{\partial z}(z_0) := \lim_{z \in W, z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists for each $z_0 \in W$.

Remark 3.12. Major results from single-valued complex analysis continue to hold, such as Cauchy's integral formula or Liouville's theorem. However, we don't include these results here.

Proposition 3.13. *Let A be a Banach algebra and let $a \in A$. Then $\sigma_A(a) \neq \emptyset$.*

Proof. WLOG, assume that $a \neq 0$ because if $a = 0$, then $\sigma_A(0) = \{0\}$. Consider the function,

$$\begin{aligned} f : \rho_A(a) &\rightarrow A \\ z &\mapsto (a - ze)^{-1} \end{aligned}$$

We show that f is a Banach space-valued holomorphic function. For $\lambda \neq \mu \in \rho_A(a) = \mathbb{C}$, we have

$$\begin{aligned} (\lambda a - e)^{-1} &= (\lambda a - e)^{-1}(\mu a - e)(\mu a - e)^{-1} \\ &= (\lambda a - e)^{-1}((\mu - \lambda)e + \lambda a - e)(\mu a - e)^{-1} \\ &= ((\mu - \lambda)(\lambda a - e)^{-1} + e)(\mu a - e)^{-1} = (\mu - \lambda)(\lambda a - e)^{-1}(\mu a - e)^{-1} + (\mu a - e)^{-1} \end{aligned}$$

Therefore, we have,

$$\frac{f(\lambda) - f(\mu)}{\lambda - \mu} = -(\lambda a - e)^{-1}(\mu a - e)^{-1}$$

Therefore,

$$\lim_{\mu \rightarrow \lambda} \frac{f(\lambda) - f(\mu)}{\lambda - \mu} = -(\lambda a - e)^{-2} = -f(\lambda)^2$$

This shows that f is a Banach space-valued holomorphic function. If $|\lambda| > \|a\|$, we have,

$$\begin{aligned} f(\lambda) &= (\lambda a - e)^{-1} \\ &= \lambda^{-1} \left(e - \frac{a}{\lambda} \right)^{-1} = \frac{1}{\lambda} \sum_{n=0}^{\infty} \left(\frac{a}{\lambda} \right)^n = \frac{1}{\lambda} e + \frac{1}{\lambda^2} a + \dots. \end{aligned}$$

For $r > \|a\|$, let Γ_r denote a contour that is a circle of radius r . Since

$$\frac{1}{2\pi i} \int_{\Gamma_r} \lambda^m d\lambda = \delta_{m,-1},$$

we have,

$$a^n = \frac{1}{2\pi i} \int_{\Gamma_r} \lambda^n f(\lambda) d\lambda$$

for each $n \geq 0$. If $\sigma_A(a) = \emptyset$, then $\rho_A(a) = \mathbb{C}$ and f is an entire function. Moreover, f is bounded. Indeed, if $|\lambda| > 2\|a\|$, then

$$\|f(\lambda)\| \leq \frac{1}{|\lambda|} \frac{1}{1 - \frac{\|a\|}{|\lambda|}} \leq \frac{1}{|\lambda| - \|a\|} \leq \frac{1}{\|a\|}$$

Hence, f is constant. But then we would have²

$$0 \leq \|e\| = \|a^0\| = \frac{1}{2\pi} \int_{\Gamma_r} \|f(\lambda)\| d\lambda \leq \frac{M_r}{2\pi} \int_{\Gamma_r} d\lambda = 0$$

where $M_r = \max_{\lambda \in \Gamma_r} \|f(\lambda)\|$. Hence, $e = 0$, a contradiction. Hence, $\sigma_A(a)$ is non-empty for each $a \neq 0$. \square

Definition 3.14. Let A be a unital Banach algebra and let $a \in A$. The **spectral radius** of $a \in A$ is defined as

$$r(a) := \sup\{|z| \mid z \in \sigma_A(a)\}.$$

Example 3.15. Let X be a compact Hausdorff space and let $A = C(X)$. Then $r(f) = \|f\|_\infty$.

Remark 3.16. Let $A = M_2(\mathbb{C})$. Consider the family of matrices $\{A_t \mid t \in \mathbb{R}^+\}$ such that

$$A_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

Note that $\sigma_A(A_t) = \{1\}$ and $r(A_t) = \{1\}$. A is a C^* -algebra, and we can compute the norm of A_t as,

$$\|A_t\|^2 = r(A_t^T A_t)$$

Note that,

$$A_t^T A_t = \begin{pmatrix} 1 & t \\ t & 1+t^2 \end{pmatrix}$$

Note that $r(A_t^T A_t) = 1/2(2 + t^2 + t\sqrt{4 + t^2})$. We see that in this case, $r(A_t) < \|A_t\|$ for $t > 0$. Hence, we have that the spectral radius can be less than the norm of an element.

We end with a non-example of a Banach algebra.

Example 3.17. Let $A = \mathbb{C}[x]$, the algebra of complex polynomials in one variable, x . Let $w \in \mathbb{C}$ and $p \in \mathbb{C}[x]$ be a non-constant polynomial. Then $p - w$ is a non-constant polynomial, and so has a zero z_w by the fundamental theorem of algebra. This means that $p(z_w) = w$. Hence, p is surjective. That is, $\sigma_A(p) = \mathbb{C}$. Hence, $A = \mathbb{C}[x]$ is not a Banach algebra since $\sigma_A(a)$ is a non-compact set.

²Here we assume that a Banach space valued integral exists and is well-defined.

Remark 3.18. Similarly, $B = \mathbb{C}(x)$, the field of quotients for $A = \mathbb{C}[x]$ is not a Banach algebra.

4. POLYNOMIAL FUNCTIONAL CALCULUS

In this section, we develop the polynomial functional calculus for elements of a unital Banach algebra. Given an element a in a unital Banach algebra, A , one can form polynomials in a and its adjoint a^* , which provide a natural way to construct new elements of A from a . This construction is fundamental in understanding the spectral and structural properties of elements in unital Banach algebra, and it serves as the first step toward the more general continuous and holomorphic functional calculi that will be discussed later for C^* -algebras.

Remark 4.1. There is also holomorphic functional calculus for unital Banach algebra that is not discussed in the notes.

Proposition 4.2. Let A be a unital Banach algebra, and let $a \in A$.

(1) (**Polynomial Spectral Mapping Theorem**) For a polynomial $p(z)$ on $\mathbb{C}[z]$, define $p(\sigma_A(a))$ as $\{p(z) \mid z \in \sigma_A(a)\}$. Then

$$p(\sigma_A(a)) = \sigma_A(p(a)).$$

(2) (**Spectral Radius Formula**) We have,

$$r(a) = \lim_{n \rightarrow \infty} \|a^n\|^{1/n}$$

Proof. The proof is given below:

(1) We may suppose that p is not constant. If $\mu \subseteq \mathbb{C}$, there are elements $\lambda_0, \dots, \lambda_n \in \mathbb{C}$, where $\lambda_0 \neq 0$, such that

$$p(z) - \mu = \lambda_0(z - \lambda_1) \cdots (z - \lambda_n),$$

This follows because \mathbb{C} is algebraically closed. Therefore,

$$p(a) - \mu e = \lambda_0(a - \lambda_1 e) \cdots (a - \lambda_n e).$$

Note that $p(a) - \mu$ is invertible if and only if all $a - \lambda_1, \dots, a - \lambda_n$ are invertible. Therefore, we have

$$\begin{aligned} p(a) - \mu e \text{ is not invertible} &\iff \text{at least one of } a - \lambda_i e \text{ is not invertible} \\ &\iff \lambda_i \in \sigma_A(a) \text{ for some } i \\ &\iff \mu = p(\lambda_i) \text{ for some } \lambda_i \in \sigma_A(a) \end{aligned}$$

The last statement follows because non-invertibility of $p(a) - \mu e$ is equivalent to the existence of some root λ_i of $p(z) - \mu$ that lies in $\sigma_A(a)$. Because each λ_i satisfies $p(\lambda_i) = \mu$, we have

$$p(a) - \mu e \text{ is not invertible} \iff \exists \lambda_i \in \sigma_A(a) \text{ such that } p(\lambda_i) = \mu.$$

The claim now follows.

(2) If $\lambda \in \sigma_A(a)$ and $n \in \mathbb{N}$, (1) implies that we have $\lambda^n \in \sigma_A(a^n)$. Therefore,

$$|\lambda^n| \leq r(a^n) \leq \|a^n\|$$

Thus $|\lambda| \leq \|a^n\|^{1/n}$. Taking supremum as λ ranges over $\sigma_A(a)$ yields

$$r(a) \leq \liminf_{n \rightarrow \infty} \|a^n\|^{1/n}$$

Let M_r as in [Proposition 3.10](#). Using the formula for a^n as in [Proposition 3.10](#), we have $\|a^n\| \leq r^{n+1} M_r^n$. Thus,

$$\limsup_{n \rightarrow \infty} \|a^n\|^{\frac{1}{n}} \lim_{n \rightarrow \infty} r^{\frac{n+1}{n}} M_r^{\frac{1}{n}} = r$$

for $r > \|a\|$. Since $|\lambda| > r(a)$ implies that $\lambda \in \rho_A(a)$, it follows that if $r, r' > r(a)$, then Γ_r and $\Gamma_{r'}$ are homotopic in $\rho_A(a)$. Thus, the formula for a^n in [Proposition 3.10](#) holds for all $r > r(a)$. It follows that

$$\limsup_{n \rightarrow \infty} \|a^n\|^{\frac{1}{n}} \leq r(a).$$

This completes the proof. \square

Remark 4.3. Consider the algebra homomorphism $\pi : \mathbb{C}[z] \rightarrow A$ such that $\pi(1) = e$ and $\pi(z) = a$ for some $a \in A$. We then have,

$$\pi\left(\sum_{i=0}^n c_i z^i\right) = \sum_{i=0}^n c_i a^i$$

This homomorphism is called the polynomial functional calculus for a . Hence, [Proposition 3.10\(i\)](#) is a statement about the spectrum of the polynomial functional calculus for a .

Corollary 4.4 (Gelfand-Mazur). Let A be a unital Banach algebra in which every non-zero element is invertible. Then A is isomorphic to \mathbb{C} .

Proof. Let $a \in A$. Since $\sigma_A(a) \neq \emptyset$, there is some $\lambda_a \in \mathbb{C}$ such that $a - \lambda_a e$ is not invertible. By assumption, $a - \lambda_a e = 0$. Hence, $a = \lambda_a e$. Define the map,

$$\begin{aligned} h : A &\rightarrow \mathbb{C} \\ a &\mapsto \lambda_a \end{aligned}$$

In order to show that h is injective, it suffices to show that the representation $a = \lambda_a e$ is unique. If not, note that

$$\lambda_a e = \mu e \iff (\lambda_a - \mu)e = 0 \iff \lambda_a = \mu.$$

The last implication follows since if $\lambda_a \neq \mu$, then $e = 0$, which is a contradiction. Hence, h is injective. Given $\lambda \in \mathbb{C}$, note that $h(\lambda e) = \lambda$ since $\sigma_A(\lambda e) = \{\lambda\}$. Hence, h is surjective. It is easy to check that h is an algebra homomorphism. Indeed, if $x = \lambda_a e$ and $y = \lambda_b e$, uniqueness of representations of elements of A implies that

$$\begin{aligned} x + y &= (\lambda_a + \lambda_b)e \implies h(x + y) = \lambda_a + \lambda_b, \\ xy &= (\lambda_a \lambda_b)e \implies h(xy) = \lambda_a \lambda_b, \\ \alpha &= (\alpha \lambda_a)e \implies h(\alpha x) = \alpha \lambda_a, \quad \alpha \in \mathbb{C} \end{aligned}$$

This completes the proof. \square

Remark 4.5. If we choose $e \in A$ such that $\|e\| = 1$, then A is isometrically isomorphic to \mathbb{C} in [Corollary 4.4](#).

Remark 4.6. Most of the results presented above hold for a non-unital Banach algebra by passing to its unitization. For example, if A is a non-unital algebra, A' is its unitization, $a \in A$ and $p \in \mathbb{C}[z]$, then

$$\sigma_A(p(x)) := \sigma_{A'}(p(x)) = p(\sigma_{A'}(x)) := p(\sigma_A(x))$$

by the proof of the polynomial spectral mapping theorem.

5. GELFAND TRANSFORM

The Gelfand transform is a fundamental tool in the theory of Banach algebras, providing a powerful method for analyzing the structure of commutative algebras. It is crucial in the study of the spectrum of operators and in representation theory. The Gelfand transform also plays a central role in understanding the duality between a Banach algebra and its space of continuous functions. The Gelfand transform is also the first step in establishing that the category of C^* -algebras is equivalent to the category of locally

compact Hausdorff spaces. How should one study an arbitrary unital commutative Banach algebra? A key tenet of modern mathematics is that an abstract object should be studied by examining the algebra of continuous functions on it.

Definition 5.1. Let A be a Banach algebra. The **character space**, denoted as \widehat{A} , is the set of all non-zero linear maps, $\omega : A \rightarrow \mathbb{C}$ that are also group homomorphisms. Each such ω is called a **character**.

Lemma 5.2. Let A be a unital Banach algebra, and let \widehat{A} be its character space. The following statements are true:

- (1) If $\omega \in \widehat{A}$, then $\omega(e) = 1$.
- (2) If $x \in \text{GL}(A)$, then $\omega(x) \neq 0$.
- (3) If $\omega \in \widehat{A}$, then

$$|\omega(a)| \leq \|a\|$$

for each $a \in A$. In particular, ω is continuous.

- (4) If A is commutative, then there is a bijective correspondence between \widehat{A} and the set of all two-sided maximal ideals in A (which are closed in A).
- (5) For every $\lambda \in \sigma_A(a)$, there is a character $\omega \in \widehat{A}$ such that $\omega_\lambda(a) = \lambda$.

Proof. The proof is given below:

- (1) Let $a \in A$ such that $\omega(a) \neq 0$. Then,

$$\omega(a) = \omega(ae) = \omega(a)\omega(e)$$

This implies that

$$\omega(a)(1 - \omega(e)) = 0$$

Since $\omega(a) \neq 0$, we must have that $\omega(e) = 1$.

- (2) We have

$$\omega(x^{-1})\omega(x) = \omega(x^{-1}x) = \omega(e) = 1$$

Hence $\omega(x) \neq 0$.

- (3) For each $a \in A$, we have that $\sigma_A(a) \subseteq B(0, \|a\|)$. Therefore, if $|z| > \|a\|$, then $a - ze$ is invertible. Hence,

$$\omega(a) - z = \omega(a) - z\omega(e) = \omega(a - ze) \neq 0$$

Therefore, if $|z| > \|a\|$, then we cannot have that $\omega(a) = z$. Hence,

$$|\omega(a)| \leq \|a\|$$

In particular, this implies that ω is continuous.

- (4) Let $\omega \in \widehat{A}$. Clearly, $\ker \omega$ is a closed subspace since ω is continuous. Since ω is multiplicative, $\ker \omega$ is a two-sided ideal. Since the kernel of every linear map into \mathbb{C} has codimension one, $\ker \omega$ is a maximal ideal. Conversely, let I be a two-sided maximal ideal of A . Proposition 1.18 implies that A/I is a Banach algebra. It is a standard algebraic fact that every element in A/I is invertible. Hence, Corollary 4.4 implies that $A/I \cong \mathbb{C}$. Hence, there is a homomorphism $\psi : A/I \rightarrow \mathbb{C}$. We can define a map $\omega : A \rightarrow \mathbb{C}$ by $\omega = \psi \circ \tau$, where τ is the canonical projection map. This map is clearly linear, since τ and ψ are. Also,

$$\begin{aligned} \omega(a)\omega(b) &= \psi(\tau(a))\psi(\tau(b)) \\ &= \psi(\tau(a)\tau(b)) \\ &= \psi(\tau(ab)) \\ &= \omega(ab), \end{aligned}$$

Therefore, ω is multiplicative; it is nonzero because $\omega(b) \neq 0$ for each $b \notin I$. Hence $\omega \in \widehat{A}$. Finally, $I \subseteq \ker(\omega)$ since $I = \ker(\tau)$; but if $b \notin I$ we know that $\omega(b) \neq 0$. Hence, $I = \ker \omega$. This also shows that I is closed.

- (5) Since $a - \lambda e$ is not invertible, it generates a proper ideal $I = \langle a - \lambda e \rangle$ in A . I is contained in a maximal ideal, M , which is the kernel of some character $\omega_\lambda \in \widehat{A}$. Moreover,

$$\omega_\lambda(a) = \lambda \iff \lambda - ae \in \ker \omega_\lambda \iff \lambda - ae \in M$$

The last condition is true. Hence, $\omega_\lambda(a) = \lambda$.

This completes the proof. \square

Remark 5.3. Let A be a non-unital Banach algebra. In this case, the proof of [Lemma 5.2](#) can be modified. For instance, we can still show that for each $\omega \in \widehat{A}$, we have that $\|\omega\| \leq 1$. For every $a \in A$ and $n \in \mathbb{N}$, we have

$$|\omega(a)| = |\omega(a^n)|^{1/n} \leq \|\omega\|^{1/n} \|a^n\|^{1/n}$$

Thus

$$|\omega(a)| \leq \limsup_{n \rightarrow \infty} \|\omega\|^{1/n} \|a^n\|^{1/n} = r(a) \leq \|a\|^3.$$

Therefore, $\|\omega\| \leq 1$. Recall that a left (resp. right) ideal $I \subseteq A$ is called regular if A/I contains a unit. We can show that there is bijection between \widehat{A} and the set of all regular two-sided maximal ideals. The proof in [Proposition 3.10\(3\)](#) now goes through since we can apply the Gelfand-Mazur theorem under the assumption that I is a maximal regular ideal.

Remark 5.4. If A is a commutative Banach algebra, [Lemma 5.2](#) implies that $\widehat{A} \neq \emptyset$ since Zorn's lemma guarantees the existence of at least one non-zero maximal ideal if A is unital, and the existence of a non-zero maximal regular ideal if A is non-unital. In order to produce a non-trivial maximal ideal, we have to assume that A is a commutative Banach algebra. Indeed, the statement above does not hold if $A = M(2, \mathbb{C})$. This follows because $M(2, \mathbb{C})$ is a simple ring.

If A is a (unital) commutative Banach algebra, [Lemma 5.2](#) also implies that

$$\widehat{A} \subseteq A^*,$$

where A^* is the dual space of A . Recall that A^* is usually considered endowed with the weak-* topology. Hence, the topology on \widehat{A} is the relative weak-* topology. In fact, we can say a bit more about the topological properties of \widehat{A} .

Proposition 5.5. Let A be a unital commutative Banach algebra. Then \widehat{A} is a compact Hausdorff subspace of A^* in the relative weak-* topology on \widehat{A} .

Proof. Clearly, \widehat{A} is a weak-* Hausdorff space since A^* is a weak-* Hausdorff space. We now show that \widehat{A} is weak-* closed. Let $(\omega_n)_{n \in \mathbb{N}} \subseteq \widehat{A}$ such that $\omega_n \rightarrow \omega$ in the weak-* topology for some $\omega \in A$. That is, $\omega_n(a) \rightarrow \omega(a)$ for all $a \in A$. We have,

$$\begin{aligned} |\omega(ab) - \omega(a)\omega(b)| &= |\omega(ab) - \omega_n(ab) + \omega_n(a)\omega_n(b) - \omega(a)\omega(b)| \\ &\leq |\omega(ab) - \omega_n(ab)| + |\omega_n(a)\omega_n(b) - \omega(a)\omega(b)| \\ &\leq |\omega(ab) - \omega_n(ab)| + |(\omega_n(a) - \omega(a))\omega_n(b) + \omega(a)(\omega_n(b) - \omega(b))| \\ &\leq |\omega(ab) - \omega_n(ab)| + |\omega_n(a) - \omega(a)|\|b\| + \|a\|\|\omega_n(b) - \omega(b)\|. \end{aligned}$$

³Note that the spectral radius is defined by passing to the unitization of A .

Since $\omega_n \rightarrow \omega$ in the weak-* topology, we obtain that

$$|\omega(ab) - \omega(a)\omega(b)| = 0$$

Hence, $\omega \in \widehat{A}$. Hence, \widehat{A} is weak-* closed. By [Lemma 5.2\(2\)](#), we have \widehat{A} is contained in the unit ball in A^* . By the Banach-Alaoglu theorem, the unit ball in A^* is weak-* compact. Since, \widehat{A} is weak-* closed set of a weak-* compact set, \widehat{A} is a weak-* compact set since A^* is Hausdorff. \square

The motivation behind the Gelfand transform is that a Banach algebra, A , should be studied by invoking the principle of duality: elements in a Banach algebra can be studied can by studying the collection of evaluation maps. More precisely, we consider the map

$$\begin{aligned} \widetilde{\Gamma} : A &\rightarrow A^{**} \\ a &\mapsto \widetilde{\Gamma}(a)(\phi) = \phi(a) \end{aligned}$$

When $\omega \in \widehat{A}$, this defines $\widetilde{\Gamma}(a)$ as a function on \widehat{A} for each $a \in A$. The definition of the weak-* topology implies that $\widetilde{\Gamma}(a) \in C(\widehat{A})$.

Definition 5.6. Let A be a unital commutative Banach algebra. The map Γ defined as,

$$\begin{aligned} \Gamma : A &\rightarrow C(\widehat{A}) \\ a &\mapsto \Gamma(a)(\omega) = \omega(a) \end{aligned}$$

is the **Gelfand transform**.

Proposition 5.7. Let A be a unital commutative Banach algebra, and let $\Gamma : A \rightarrow C(\widehat{A})$ denote the Gelfand transform.

- (1) The Gelfand transform is an algebra homomorphism.
- (2) The spectrum of $a \in A$ is:

$$\sigma_A(a) = \{\omega(a) \mid \omega \in \widehat{A}\}$$

- (3) The Gelfand transform is a contraction, that is,

$$\|\Gamma(a)\|_\infty \leq \|a\|.$$

for each $a \in A$.

Proof. The proof is given below:

- (1) This is clear.
- (2) For $a \in A$ and $z \in \mathbb{C}$, consider the element $a - ze$. If $a - ze$ is invertible, then $\omega(a - ze) \neq 0$ for each $\omega \in \widehat{A}$. If $a - ze$ is not invertible, then $a - ze$ is contained in a proper maximal ideal of A . Invoking [Lemma 5.2\(3\)](#), we have that there is a $\omega \in \widehat{A}$ such that $\omega(a - ze) = 0$. Hence $a - ze$ is invertible if and only if $\omega(a - ze) \neq 0$ for all $\omega \in \widehat{A}$ if and only if $\omega(a) \neq \omega(z)$ for all $\omega \in \widehat{A}$. Hence,

$$\rho_A(a) = \{z \in \mathbb{C} \mid z \neq \omega(a) \text{ for all } \omega \in \widehat{A}\}$$

Taking the complement, we have,

$$\begin{aligned} \sigma_A(a) &= \{z \in \mathbb{C} \mid z = \omega(a) \text{ for some } \omega \in \widehat{A}\} \\ &= \{\omega(a) \mid \omega \in \widehat{A}\} \end{aligned}$$

- (3) We have,

$$\|\Gamma(a)\|_\infty = \sup_{\omega \in \widehat{A}} |\omega(a)| = r(a) \leq \|a\|$$

This completes the proof. \square

Remark 5.8. Note that $x \in \text{GL}(A)$ if and only if $\Gamma(a)$ never vanishes. Observe that

$$\begin{aligned} a \text{ is not invertible} &\iff I = \langle a \rangle \text{ is proper} \\ &\iff a \text{ is contained in a maximal ideal} \\ &\iff \omega(a) = 0 \text{ for some } \omega \in \widehat{A} \\ &\iff \Gamma(a) \text{ has a zero.} \end{aligned}$$

There are examples non-unital Banach algebras. It turns out that we can extend the results discussed to the case of non-unital Banach algebras.

Proposition 5.9. Let A be a non-unital commutative Banach algebra and let A' denote its unitization. Let Γ denote the Gelfand transform as in [Proposition 5.7](#).

(1) We have,

$$\widehat{A}' = \widehat{A} \cup \{\phi_0\},$$

where we have define $\phi_0((a, \lambda)) = \lambda$

(2) \widehat{A} is a locally compact, Hausdorff space.

(3) The Gelfand transform is a contractive, algebra homomorphism into $C_0(\widehat{A})$.

(4) The spectrum of $a \in A$ is

$$\sigma_A(a) = \{\omega(a) \mid \omega \in \widehat{A}\} \cup \{0\}$$

for each $a \in A$.

Proof. (Sketch) The proof is given below:

- (1) It is clear that there can be no other elements of \widehat{A}' . If $\phi \in \widehat{A}'$, then the restriction of ϕ to A has a unique multiplicative extension to A' (see [Proposition 7.5](#)), unless it identically vanishes on A . In the latter case, ϕ_0 is clearly the only possibility.
- (2) Since \widehat{A}' is a compact Hausdorff space, $\widehat{A} = \widehat{A}' \setminus \{\phi_0\}$ is a locally compact, Hausdorff space by the characterization of locally compact, Hausdorff spaces.
- (3) Note that $\Gamma(a)(\phi_0) = \phi_0(a) = 0$ This means that $\Gamma(A) \subseteq C_0(\widehat{A})$.
- (4) We have,

$$\sigma_A(a) := \sigma_{A'}(a) = \{\omega(a) \mid \omega \in \widehat{A}'\} = \{\omega(a) \mid \omega \in \widehat{A}\} \cup \{0\}$$

This completes the proof. \square

Remark 5.10. If A is a (unital) commutative Banach algebra, we have proved that the Gelfand transform is a contractive (hence injective) algebra homomorphism. The Gelfand transform becomes an isometric *-isomorphism if A is a commutative C^* -algebra.

We end with some examples. We first compute $\widehat{C(X)}$ when X is a compact Hausdorff space.

Example 5.11. Let X be a compact Hausdorff space. We find $\widehat{C(X)}$. By [Lemma 5.2\(3\)](#), it suffices to compute the set of maximal ideals of $C(X)$. For each $x \in X$, define I_x by

$$I_x = \{f \in C(X) : f(x) = 0\}$$

Let I be a proper ideal in $C(X)$. We claim that $I \subseteq I_x$ for some $x \in X$. Assume this is not the case. Then for each $x \in X$, we can find a $f_x \in I$ such that $f_x \notin I_x$. That is, $f_x(x) \neq 0$. Since f_x is continuous there

is a open neighbourhood $x \in U_x$ that $f_x|_{U_x} \neq 0$. Since X is compact, the open cover $(U_x)_{x \in X}$ admits a finite sub-cover. Hence, we can find $x_1, \dots, x_n \in X$ such that $X = \cup_{i=1}^n U_{x_i}$. Consider the function

$$f(x) = \sum_{i=1}^n |f_{x_i}(x)|^2 = \sum_{i=1}^n \overline{f_{x_i}(x)} f_{x_i}(x)$$

Clearly, $f \in I$. But by construction $f > 0$ on X . Hence, f is invertible, so that I contains an invertible element, contradicting that I is a proper ideal. Hence, we have that for every proper ideal, I , there exists a $x_I \in I$ such that $I \subseteq I_{x_I}$. Moreover, let $x \neq y$. Since X is compact and Hausdorff, X is a normal space. By Urysohn's lemma, there exists $f, g \in C(X)$ such that $f|_{\{x\}} = 0$, $f|_{\{y\}} = 1$, and $g|_{\{y\}} = 1$, $g|_{\{x\}} = 0$. This shows that $I_x \subsetneq I_y$ and $I_y \subsetneq I_x$. Hence, we can conclude that the set of maximal ideals, \mathcal{M} , is of the form

$$\mathcal{M} = \{I_x \mid x \in X\}$$

In particular, we can conclude that $\widehat{C(X)} = X$ as sets. In fact the map

$$\begin{aligned} \varphi : X &\rightarrow \widehat{C(X)} \\ x &\mapsto \text{Ev}_x \end{aligned}$$

is a homeomorphism. It is clear that each $\text{Ev}_x \in \widehat{C(X)}$, and $\text{Ev}_x \neq \text{Ev}_y$ for $x \neq y$ as above. If $x_\alpha \rightarrow x$, then $f(x_\alpha) \rightarrow f(x)$ for each $f \in C(X)$, which implies that $\text{Ev}_{x_\alpha} \rightarrow \text{Ev}_x$ in the weak-*topology. This shows that φ is a continuous injection. It is also surjective by the discussion above. Since these spaces are compact Hausdorff, the claim follows.

In fact, one can generalize Example 5.11.

Example 5.12. Let X be a locally compact Hausdorff space. We claim that

$$\begin{aligned} \varphi : X &\rightarrow \widehat{C_0(X)} \\ x &\mapsto \text{Ev}_x \end{aligned}$$

is a homeomorphism. Continuity follows as in Example 5.11. Injectivity follows as in Example 5.11 by applying Uryson's lemma to X_∞ . We show φ is onto. Let $\omega \in \widehat{C_0(X)}$. By the Riesz representation theorem, there exists a positive Radon measure μ on X such that

$$\omega(f) = \int_X f(x) d\mu(x) \quad \text{for all } f \in C_0(X).$$

Thus, we have

$$0 = \omega((\overline{f - \omega(f)})(f - \omega(f))) = \int_X |f(x) - \omega(f)|^2 d\mu(x).$$

This means that, for every $f \in C_0(X)$, f equals the constant function $\omega(f)$ μ -almost everywhere. Hence, there is a point $x_0 \in X$ such that

$$\omega(f) = f(x_0) \quad \text{for all } f \in C_0(X).$$

In other words, $\omega = \text{Ev}_{x_0}$. One can φ to a continuous and bijective map from X_∞ onto $\widehat{A'}$. This is a homeomorphism as in Example 5.11. Hence, so is φ .

Part 2. C^* -Algebras I

Linear algebra studies linear operators which are linear maps $T : \mathbb{C}^n \rightarrow \mathbb{C}^n$. If $T, S : \mathbb{C}^n \rightarrow \mathbb{C}^n$ are two such linear operators, then $T \circ S \neq S \circ T$ in general. This is perhaps the first instance where one encounters the phenomenon of ‘non-commutativity’ in mathematics. More generally, functional analysis studies infinite-dimensional linear spaces with additional analytic structures. A key example is that of a Hilbert space. If \mathcal{H} is an infinite-dimensional Hilbert space, we consider the Hilbert space of bounded/continuous linear operator on \mathcal{H} :

$$\mathcal{B}(\mathcal{H}) := \{T : H \rightarrow H \mid T \text{ is linear and bounded}\},$$

Once again, elements of $\mathcal{B}(\mathcal{H})$ are non-commutative, in general. The phenomenon of non-commutativity is prevalent in different topics of mathematics and physics. Examples include quantum physics, linear algebra, representation theory of groups, etc. The theory of operator algebras (including C^* -algebras) attempts to capture the essence of non-commutativity. The pioneers of operator algebras, Francis Murray and John von Neumann, wrote in their very first article on operator algebras⁴ in 1936 that

“various aspects of the quantum mechanical formalism suggest strongly the elucidation of this subject.”

Hence, the study of operator algebras can be considered an essential part of *non-commutative mathematics*, which is also called *quantum mathematics*⁵ more colloquially and popularly.

Why C^* -algebras, though? C^* -algebras can be thought of as a non-commutative or quantum version of topology. This is the content of a result of Gelfand and Naimark (to be proved later):

Theorem 5.13. (Gelfand & Naimark) Let A be a C^* -algebra. Then A is commutative if and only if A is isometrically $*$ -isomorphic to $C_0(X)$ for some locally compact Hausdorff topological space, X .

Corollary 5.14. Let A be a unital C^* -algebra. Then A is commutative if and only if A is isometrically $*$ -isomorphic to $C(X)$ for some compact Hausdorff topological space, X .

Hence, any locally compact Hausdorff topological space gives rise to a commutative C^* -algebra. On the other hand, any commutative C^* -algebra is exactly of this form. In this sense, commutative C^* -algebras correspond to “commutative-topology,” and we may view the theory of non-commutative C^* -algebras as a kind of “non-commutative topology.” This duality is also the basis for other topics in non-commutative/quantum mathematics. Refer to Table 1.

Classical Mathematics	Quantum Mathematics
Groups	Quantum groups
Cohomology	Quantum Cohomology
Topology	C^* -algebras
Differential Geometry	Non Commutative Geometry
Probability Theory	Free Probability
Information Theory	Quantum Information Theory

Correspondence of some topics in *classical mathematics* and *quantum mathematics*.

⁴It was an article on von Neumann algebras

⁵Thus, phrases such as *quantum groups*, *quantum cohomology*, etc., are used to describe objects studied in *quantum mathematics*.

6. DEFINITIONS & EXAMPLES

We first define $*$ -algebras. The $*$ -operation on a algebra is analogous to taking adjoints in complex matrix algebras.

Definition 6.1. Let V be an \mathbb{C} -algebra. Then V is a **$*$ -algebra** if there is a map $* : A \rightarrow A$ is an antiautomorphism and an involution. More precisely, $*$ is required to satisfy the following properties:

- (1) $(x + y)^* = x^* + y^*$,
- (2) $(xy)^* = y^*x^*$,
- (3) $(x^*)^* = x$,
- (4) $(\lambda x)^* = \bar{\lambda}x^*$

for all $x, y \in V$ and $\lambda \in \mathbb{C}$.

A Banach $*$ -algebra is a Banach algebra with a $*$ map.

Definition 6.2. A **Banach $*$ -algebra** is a Banach algebra that is a $*$ -algebra. A morphism $\phi : A \rightarrow B$ between two Banach $*$ -algebra A and B is a morphism of the underlying Banach algebras that that is $*$ -preserving. That is, $\phi(a^*) = \phi(a)^*$ for each $a \in A$.

We discuss some basic properties of Banach $*$ -algebra.

Proposition 6.3. Let A be a unital Banach $*$ -algebra.

- (1) $e = e^*$.
- (2) If $x \in \underline{\text{GL}(A)}$, then $x^* \in \text{GL}(A)$, and $(x^*)^{-1} = (x^{-1})^*$.
- (3) $\sigma_A(a^*) = \sigma_A(a)$ for each $a \in A$.

Proof. The proof is given below:

- (1) By definition, we have $e^*e = e^*$. Applying $*$, this implies $e^* = e^*e = e$.
- (2) This is clear.
- (3) By (1)

$$a^* - \lambda e = a^* - (\bar{\lambda}e)^* = (a - \bar{\lambda}e)^*$$

The claim follows by (2) now.

This completes the proof. \square

We have seen that the canonical example of a non-commutative Banach algebra if $\mathcal{B}(\mathcal{H})$, where \mathcal{H} is some Hilbert space. It turns out that $\mathcal{B}(\mathcal{H})$ has even more structure. Recall that the if $T \in \mathcal{B}(\mathcal{H})$, the adjoint of T , denoted as T^* , is in $\mathcal{B}(\mathcal{H})$ defined by the property

$$\langle \Psi, T^* \Phi \rangle_{\mathcal{H}} := \langle T\Psi, \Phi \rangle_{\mathcal{H}}$$

for all $\Psi, \Phi \in \mathcal{H}$. For $T, S \in \mathcal{B}(\mathcal{H})$ and $\lambda \in \mathbb{C}$, it is well-known that the adjoint operator satisfies the following properties:

- (1) $(T + S)^* = T^* + S^*$;
- (2) $T^{**} = T$;
- (3) $(TS)^* = S^*T^*$;
- (4) $(\lambda T)^* = \bar{\lambda}T^*$.

Remark 6.4. Since the adjoint operation, denoted as $*$, is idempotent ($T^{**} = T$), we say that $*$ is an involution.

This makes $\mathcal{B}(\mathcal{H})$ into a Banach $*$ -algebra. How does the adjoint operation interact with the norm? Pick $\Psi \in H$, and use the Cauchy-Schwarz inequality to estimate

$$\|T\Psi\|^2 = \langle T\Psi, T\Psi \rangle = \langle \Psi, T^*T\Psi \rangle \leq \|\Psi\| \|T^*T\Psi\| \leq \|T^*T\| \|\Psi\|^2.$$

Using the definition of the operator norm and the sub-multiplicative nature of the operator norm, we infer that

$$\|T\|^2 \leq \|T^*T\| \leq \|T^*\| \|T\|. \quad (1)$$

This implies to $\|T\| \leq \|T^*\|$. Replacing T by T^* and the fact that $T^{**} = T$ implies $\|T^*\| \leq \|T\|$. Hence,

$$\|T^*\| = \|T\|,$$

for each $T \in \mathcal{B}(\mathcal{H})$. Substituting this in Equation (1), we derive the crucial property

$$\|T^*T\| = \|T\|^2 \quad (2)$$

for each $T \in \mathcal{B}(\mathcal{H})$. This is called that C^* identity. This discussion motivates the definition of a C^* -algebra.

Definition 6.5. A C^* -algebra, A , is a Banach $*$ -algebra that is a $*$ -algebra that satisfies the C^* identity:

$$\|a^*a\| = \|a\|^2$$

for each $a \in A$. A morphism $\phi : A \rightarrow B$ between two C^* algebras A and B is a morphism of the underlying Banach $*$ -algebras.

Example 6.6. The following is a list of some basics examples of C^* algebras:

- (1) As discussed above, $\mathcal{B}(\mathcal{H})$ is a C^* algebra for any Hilbert space, \mathcal{H} .
- (2) If X is a locally compact Hausdorff topological space, then $C_0(X)$ is a C^* algebra with involution given by complex conjugation. This is an example of a non-unital C^* algebra.
- (3) If X is a locally compact Hausdorff topological space, $C_b(X)$ is also a C^* algebra with involution given by complex conjugation.

Example 6.7. Let A be a C^* -algebra and $F \subseteq A$. The sub- C^* -algebra of A generated by F , denoted $C^*(F)$, is the smallest sub- C^* -algebra of A containing F ; equivalently, it is the intersection of all sub- C^* -algebras of A that contain F . For each $n \in \mathbb{N}$, define

$$W_n = \{x_1 x_2 \cdots x_n : x_j \in F \cup F^*\},$$

where $F^* = \{x^* : x \in F\}$, and set $W = \bigcup_{n=1}^{\infty} W_n$. The set W consists of all words in $F \cup F^*$; it satisfies $W = W^*$ and is closed under multiplication. Hence, the linear span of W forms a $*$ -subalgebra of A , and its norm closure in A is $C^*(F)$. That is,

$$C^*(F) = \overline{\text{span } W},$$

When $a_1, a_2, \dots, a_n \in A$, we write

$$C^*(a_1, a_2, \dots, a_n) := C^*(\{a_1, a_2, \dots, a_n\}).$$

Example 6.8. Let A be a C^* -algebra. A left (or right) ideal I of A is an ideal of the underlying Banach algebra that⁶. The quotient A/I is a C^* -algebra and the canonical quotient map $\pi : A \rightarrow A/I$ is a $*$ -homomorphism. A C^* -algebra A is called simple if the only ideals in A are the trivial ones, $\{0\}$ and A itself.

⁶It turns out that every (two-sided) ideal in a C^* -algebra is automatically self-adjoint and hence forms a sub- C^* -algebra.

Example 6.9. Let $\varphi : A \rightarrow B$ be a $*$ -homomorphism between C^* -algebras. The kernel, $\ker(\varphi)$, of φ is an ideal in A , and the image, $\text{im}(\varphi) = \varphi(A)$, is a sub- C^* -algebra of B . By the first isomorphism theorem, there exists a unique $*$ -homomorphism $\tilde{\varphi} : A/\ker(\varphi) \rightarrow B$ such that the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & B \\ \downarrow \pi & \nearrow \tilde{\varphi} & \\ A/\ker(\varphi) & & \end{array}$$

Remark 6.10. Example 6.8 and Example 6.9 apply equally well to general Banach algebras. More interesting examples and constructions will be discussed later on.

Analogous to the algebraic characterization of operators in $\mathcal{B}(\mathcal{H})$, we identify the following special class of elements a in a C^* -algebra.

Definition 6.11. Let A be a C^* -algebra and let $a \in A$.

- (1) a is normal if $a^*a = aa^*$,
- (2) a is self-adjoint if $a = a^*$,
- (3) a is a projection if $a = a^* = a^2$,
- (4) If A is unital, a is a unitary if $a^*a = aa^* = e$,
- (5) If A is unital, a is an isometry if $a^*a = e$,

Proposition 6.12. Let A be a non-zero C^* -algebra. The following is a list of some elementary properties of a C^* -algebra.

- (1) $\|a\| = \|a^*\|$ for each $a \in A$,
- (2) A Banach $*$ -algebra is a C^* -algebra if and only if $\|a\|^2 \leq \|a^*a\|$ for each $a \in A$
- (3) If A is unital and e is the identity element, then $\|e\| = 1$.
- (4) Any element a in a C^* -algebra is the sum of two self-adjoint operators.
- (5) If A is unital and $a \in A$ is unitary, then $\sigma_A(a) \subseteq \mathbb{S}^1$.
- (6) If A is unital, and $a \in A$ is a self-adjoint element, then $r(a) = \|a\|$.
- (7) If A is unital, and $a \in A$ is a normal element, then $r(a) = \|a\|$.
- (8) If A is unital, then the norm on A is unique.
- (9) A linear map between C^* -algebras is $*$ -preserving if and only if it maps self-adjoint elements to self-adjoint elements.
- (10) Let A, B be two unital C^* algebras. A $*$ -homomorphism $\pi : A \rightarrow B$ is contractive (i.e. $\|\pi(a)\| \leq \|a\|$ for each $a \in A$) and hence continuous. Moreover, a $*$ -isomorphism between C^* algebras is isometric.
- (11) If $\phi \in \widehat{A}$ and $a \in A$ is self-adjoint, then $\phi(a) \in \mathbb{R}$.
- (12) If $\phi \in \widehat{A}$, then ϕ is $*$ -preserving with $\|\phi\| = 1$.
- (13) If $\phi \in \widehat{A}$, then $\phi(a^*a) \geq 0$.
- (14) If A is unital, and $\phi \in \widehat{A}$, and a is a unitary, then $|\phi(a)| = 1$.

Proof. The proof is given below:

- (1) We use the C^* -identity. We have,

$$\|a\|^2 = \|aa^*\| \leq \|a\|\|a^*\|$$

Hence, we have $\|a\| \leq \|a^*\|$. Similarly, we have $\|a^*\| \leq \|a^{**}\| = \|a\|$. The result now follows.

- (2) If A is a Banach $*$ -algebra satisfying the given assumption, we have,

$$\|a\|^2 \leq \|a^*a\| \leq \|a^*\|\|a\| \leq \|a\|\|a\|$$

The last inequality follows since (1) holds in a Banach $*$ -algebra. Hence, we have an equality above, which implies that the C^* identity holds. The converse is clear.

- (3) The C^* identity implies that

$$\|e\| = \|ee\| = \|e^*e\| = \|e\|^2$$

Hence, $\|e\| = 1$ ⁷.

- (4) Simply note that, $a = \operatorname{Re}(a) + i \operatorname{Im}(a)$, where

$$\operatorname{Re}(a) = \frac{1}{2}(a + a^*) \quad \operatorname{Im}(a) = \frac{1}{2i}(a - a^*)$$

It is clear that $\operatorname{Re}(a)$ and $\operatorname{Im}(a)$ are self-adjoint elements.

- (5) Recall that for any invertible element, a , in a Banach algebra, we have

$$\sigma_A(a^{-1}) = \{\lambda^{-1} \mid \lambda \in \sigma_A(a)\}$$

Since a and a^* are unitary, we have,

$$\|a\|^2 = \|a^*a\| = \|e\| = 1 = \|e\| = \|aa^*\| = \|a^*\|^2,$$

Hence, $\|a\| = \|a^*\| = 1$. Therefore, if $\lambda \in \sigma_A(a)$, we have $|\lambda| \leq 1$. Similarly, for any $\lambda \in \sigma_A(a)$, we have that $\lambda^{-1} \in \sigma_A(a^{-1}) = \sigma_A(a^*)$. Hence, $|\lambda^{-1}| \leq 1$. Hence, $|\lambda| = 1$.

- (6) The C^* -identity implies that

$$\|a\|^2 = \|a^*a\| = \|a^2\|$$

Repeated use of the C^* identity implies that,

$$\|a\|^{2^n} = \|a^{2^n}\|$$

Hence,

$$r(a) = \lim_{n \rightarrow \infty} \|a^n\|^{\frac{1}{n}} = \lim_{n \rightarrow \infty} \|a^{2^n}\|^{\frac{1}{2^n}} = \lim_{n \rightarrow \infty} \|a\|^{\frac{2^n}{2^n}} = \|a\|.$$

- (7) The C^* identity implies that,

$$\|a^2\|^2 = \|(a^2)(a^2)^*\| = \|(a^*a)^*(a^*a)\| = \|a^*a\|^2 = (\|a\|^2)^2$$

holds. The remaining argument is same as in (8).

- (8) If $a \in A$ is any element, then,

$$\|a\|^2 = \|a^*a\| = r(a^*a)$$

Hence, we see that the spectral radius *intrinsically* determines the norm of any element in a C^* algebra. Hence, the norm is uniquely defined.

- (9) A $*$ -preserving map clearly sends self-adjoint elements to self-adjoint elements. Conversely, let $\phi : A \rightarrow B$ be a linear map that maps self-adjoint elements to self-adjoint elements. Using (5), we have,

$$\phi(a) = \phi(\operatorname{Re}(a)) + i\phi(\operatorname{Im}(a))$$

$$\phi(a^*) = \phi(\operatorname{Re}(a)) - i\phi(\operatorname{Im}(a)).$$

⁷Note that $\|e\| \neq 0$ since $e \neq 0$ since A is a non-zero C^* algebra.

Since $\text{Re}(a)$ and $\text{Im}(a)$ are self-adjoint, $\phi(\text{Re}(a))$ and $\phi(\text{Im}(a))$ are self-adjoint by assumption. So

$$\begin{aligned}\phi(a)^* &= \phi(\text{Re}(a) + i\text{Im}(a))^* \\ &= (\phi(\text{Re}(a)) + i\phi(\text{Im}(a)))^* \\ &= \phi(\text{Re}(a)) - i\phi(\text{Im}(a)) \\ &= \phi(a^*).\end{aligned}$$

- (10) Let $a \in A$. Then a^*a is a normal element in A , which means $\|a^*a\| = r(a^*a)$ by (9). By Proposition 3.6(3), we have, $r(\pi(a^*a)) \leq r(a^*a)$. Hence,

$$\begin{aligned}\|a\|^2 &= \|a^*a\| \\ &= r(a^*a) \\ &\geq r(\pi(a^*a)) \\ &= r(\pi(a^*)\pi(a)) \\ &= \|\pi(a)^*\pi(a)\| = \|\pi(a)\|^2.\end{aligned}$$

If π is a $*$ -isomorphism, then a symmetric argument shows the inequality above is an equality.

- (11) Let $a = a^*$, and let $\phi(a) = \alpha + i\beta \in \mathbb{C}$. Note that $\phi(a) + i\lambda = \phi(a + i\lambda)$ and $|\phi(a) + i\lambda| \leq \|a + i\lambda\|$ for all $\lambda \in \mathbb{R}$. Hence:

$$\begin{aligned}\alpha^2 + (\lambda + \beta)^2 &= |\phi(a) + i\lambda|^2 \\ &\leq \|a + i\lambda\|^2 \\ &= \|(a + i\lambda)^*(a + i\lambda)\| \\ &= \|a^2 + \lambda^2\| \\ &\leq \|a\|^2 + \lambda^2\end{aligned}$$

Thus, $\alpha^2 + 2\lambda\beta + \beta^2 \leq \|a\|^2$, for all $\lambda \in \mathbb{R}$, which implies $\beta = 0$.

- (12) We already know that $\|\phi\| \leq 1$. But $\phi(e) = 1$ implies that $\|\phi\| = 1$. The $*$ -preserving condition follows from (10) and (12).

- (13) We have,

$$\phi(a^*a) = \phi(a^*)\phi(a) = \overline{\phi(a)}\phi(a) \geq 0$$

The last equality follows by (13).

- (14) We have,

$$|\phi(a)|^2 = \overline{\phi(a)}\phi(a) = \phi(a^*)\phi(a) = \phi(a^*a) = \phi(e) = 1$$

This completes the proof. \square

Let's end with a non-example of a C^* -algebra.

Example 6.13. Let $\mathcal{A}(\mathbb{D})$ be the disk algebra. It is clear that $\mathcal{A}(\mathbb{D})$ is a Banach $*$ -algebra with the $*$ -operation given by $f^*(z) := \overline{f(\bar{z})}$. We show that $\mathcal{A}(\mathbb{D})$ is not a C^* -algebra. Let $f(z) = e^{iz} \in \mathcal{A}(\mathbb{D})$. We have $f^*(z) = e^{-i\bar{z}}$. For $z \in \mathbb{D}$

$$f^*f(z) := f^*(z)f(z) = e^{-i\bar{z}}e^{iz} = e^{i(z-\bar{z})} = 1$$

Therefore $\|f^*f\| = 1$. Since

$$\|f\|^2 = \sup\{|e^{iz}|^2 : z \in \mathbb{D}\} = \sup\{e^{-i\bar{z}+iz} : z \in \mathbb{D}\} = \sup\{e^{-2\text{Im } z} : z \in \mathbb{D}\} = \sup\{e^{-2b} : b \in [-1, 1]\},$$

we have $\|f\|^2 = e^2$ and the C^* -identity is not satisfied.

7. UNITIZATION

Not all C^* -algebras possess a multiplicative identity (or unit). A standard example of a non-unital C^* -algebra is $C_0(\mathbb{R})$, the algebra of continuous complex-valued functions on \mathbb{R} that vanish at infinity, equipped with pointwise operations and the supremum norm. Since no nonzero constant function belongs to $C_0(\mathbb{R})$, it lacks a multiplicative unit. In such cases, it is often convenient to adjoin a unit artificially. This leads to the unitization of a non-unital C^* -algebra A , denoted A' .

Definition 7.1. Let A be a non-unital C^* -algebra. The (smallest) unital C^* -algebra containing A is called its **unitization**, A' , defined A' as $A' := A \oplus \mathbb{C}$ with algebraic operations given by

$$\begin{aligned} (a, \alpha) \cdot (b, \beta) &= (ab + \alpha b + \beta a, \alpha\beta) \\ (a, \alpha)^* &= (a^*, \bar{\alpha}) \\ \|(a, \alpha)\| &= \sup_{b \in A, \|b\| \leq 1} \|ab + \alpha b\| \end{aligned}$$

Remark 7.2. Consider the map $\Phi : A \rightarrow \mathcal{B}(B)$ such that $\Phi(a) = L_a$, where L_a is the left-multiplication operator on A . It is easy to check that Φ is a $*$ -homomorphism. We have that $\|L_a\| = \|a\|$. Indeed, if $x \in A$ such that $\|x\| \leq 1$, then $\|ax\| \leq \|a\|\|x\|$. Hence,

$$\|L_a\| = \sup_{\|x\| \leq 1} \|ax\| \leq \sup_{\|x\| \leq 1} \|a\|\|x\| = \|a\|$$

If $x = a^*/\|a\|$, then $\|x\| = 1$, and $L_a(x) = \|aa^*/a\| = \|a\|$. This shows that Φ is an isometric $*$ -homomorphism. Hence, A can be identified with a $*$ -subalgebra of $\mathcal{B}(A)$. If we identify $a \in A$ with the left multiplication operator $L_a \in \mathcal{B}(A)$, and we identify (a, α) with the operator $L_a + \alpha \text{Id}_A$, then the norm on A' is the norm induced from $\mathcal{B}(A)$ on the $*$ -subalgebra $\langle L_a, \text{Id}_A \mid a \in A \rangle$.

Remark 7.3. We have already seen that the norm on a C^* algebra is unique. Thus, the norm defined above is the ‘right’ choice.

Let’s verify that A' is indeed a unital C^* -algebra. The unit is $(0, 1)$, and clearly A' is a $*$ -algebra. It is a norm by the remark made above. Moreover, note that the identification $a \mapsto L_a$ is isometric. Indeed, using the C^* -identity in A , we have for any nonzero $a \in A$,

$$\|a\| = \left\| a \left(\frac{a^*}{\|a\|} \right) \right\| \leq \sup_{\|b\| \leq 1} \|ab\| \leq \|a\| \sup_{\|b\| \leq 1} \|b\| = \|a\|.$$

So, $\|(a, 0)\|_{A'} = \|a\|_A$, and the embedding of A into A' is isometric. Since $\mathcal{B}(A)$ is complete

$$\{L_a + \alpha \text{id}_A : a \in A, \alpha \in \mathbb{C}\}$$

is complete since it is a closed subspace of $\mathcal{B}(A)$. Hence, A' is a Banach algebra. It remains to show that the given norm satisfies the C^* -identity. To that end, we compute for $a \in A$ and $\alpha \in \mathbb{C}$,

$$\begin{aligned} \|(a, \alpha)\|^2 &= \sup_{\|b\| \leq 1} \|ab + \alpha b\|^2 \\ &= \sup_{\|b\| \leq 1} \|b^*(a^*a + \alpha a^* + \bar{\alpha}a + |\alpha|^2 \text{Id}_A)b\| \\ &\leq \sup_{\|b\| \leq 1} \|a^*a + \alpha a^* + \bar{\alpha}a + |\alpha|^2 \text{Id}_A\| \\ &= \|(a, \alpha)^*(a, \alpha)\| \leq \|(a, \alpha)^*\| \|(a, \alpha)\|. \end{aligned}$$

So $\|(a, \alpha)\| \leq \|(a, \alpha)^*\|$, and a symmetric argument yields $\|(a, \alpha)^*\| = \|(a, \alpha)\|$. Then the above inequality gives

$$\|(a, \alpha)\|^2 \leq \|(a, \alpha)^*(a, \alpha)\| \leq \|(a, \alpha)\|^2.$$

This proves the C^* identity in A' .

Remark 7.4. Properties of unital C^* algebras discussed in [Proposition 6.12](#) can be extended to the case of non-unital C^* algebras by passing to the unitization of non-unital C^* algebras.

We have a short exact sequence

$$0 \longrightarrow A \xrightarrow{i} A' \begin{array}{c} \xrightarrow{\pi_A} \\ \xleftarrow{\lambda} \end{array} \mathbb{C} \longrightarrow 0$$

Here $\pi_A(a, z) = z$ and $\lambda(z) = ze_A$. One notable advantage of the unitization construction is that any $*$ -homomorphism between non-unital C^* -algebras admits a unique and canonical extension to their unitizations. This property ensures that the process of adjoining a unit is functorial, making unitizations particularly convenient for categorical and structural analyses of C^* -algebras.

Proposition 7.5. Let A, B be non-unital C^* -algebras and $\pi : A \rightarrow B$ a $*$ -homomorphism. Then there is a unique extension of π to a unital $*$ -homomorphism $\pi' : A' \rightarrow B'$ given by

$$\pi'(a + \lambda e'_A) = \pi(a) + \lambda e'_B$$

In other words, we have the following commutative diagram:

$$\begin{array}{ccccc} A & \xrightarrow{\iota_A} & A' & \longrightarrow & \mathbb{C} \\ \pi \downarrow & & \downarrow \pi' & & \parallel \\ B & \xrightarrow{\iota_B} & B' & \longrightarrow & \mathbb{C} \end{array}$$

Proof. We just need to check that the given formula is a $*$ -homomorphism. Linearity and $*$ -preserving are immediate. For $a, b \in A$ and $\lambda, \eta \in \mathbb{C}$, we compute

$$\begin{aligned} \pi'(a + \lambda e'_A) \pi'(b + \eta e'_A) &= (\pi(a) + \lambda e'_B)(\pi(b) + \eta e'_B) \\ &= \pi(ab) + \lambda \pi(b) + \eta \pi(a) + \lambda \eta e'_B \\ &= \pi'(ab + \lambda b + \eta a + \lambda \eta e'_A). \end{aligned}$$

Uniqueness is forced by the fact that we require π' to be linear and $e'_A \mapsto e'_B$. \square

Remark 7.6. We did not actually use the fact that π was $*$ -preserving in the proof. Indeed, it suffices to assume that π is linear and multiplicative map. Moreover, essentially the same proof works if A and B are assumed to be Banach algebras.

Remark 7.7. If A in a unital C^* -algebra with unit e_A , then $p := e'_A - e_A$ is a projection in A' , and

$$A' = \{a + \lambda p : a \in A, \lambda \in \mathbb{C}\}.$$

The map

$$\begin{aligned} A \oplus \mathbb{C} &\longrightarrow A', \\ (a, \lambda) &\longmapsto a + \lambda p \end{aligned}$$

is a $*$ -isomorphism, and hence $A' \cong A \oplus \mathbb{C}$ as a direct sum of C^* -algebras. We omit the verification of the details.

8. GELFAND-NAIMARK THEOREM

We prove the Gelfand-Naimark Theorem for commutative C^* -algebras.

Theorem 8.1. (*Gelfand & Naimark*) Let A be a C^* -algebra. Then A is commutative if and only if A is isometrically $*$ -isomorphic to $C_0(X)$ for some locally compact Hausdorff topological space, X .

We have already observed that if A is considered as a Banach $*$ -algebra, the Gelfand transform,

$$\Gamma : A \rightarrow C_0(\widehat{A})$$

defines a contractive (and hence continuous) algebra homomorphism. The purpose of the remainder of the section is to show that Γ is $*$ -preserving surjective isometry.

Proof. We first show that Γ is an isometry. Since A is commutative, every element in A is normal. Hence,

$$\|\Gamma(a)\|_\infty = \sup_{\omega \in \widehat{A}} |\omega(a)| = r(a) = \|a\|$$

The last equality follows from [Proposition 6.12\(8\)](#). Hence, Γ is isometric and injective. We now show that Γ is $*$ -preserving. By [Proposition 6.12\(10\)](#), it suffices to show that Γ maps self-adjoint elements to self-adjoint elements. But if $a \in A$ is any self-adjoint element, we have that

$$\sigma_A(a) = \{\omega(a) \mid \omega \in \widehat{A}\} \subseteq \mathbb{R}$$

by [Proposition 6.12\(12\)](#). Hence, $\text{range}(\Gamma(a)) \subseteq \mathbb{R}$, which means $\Gamma(a) = \overline{\Gamma(a)}$ is self-adjoint. This shows that Γ is $*$ -preserving. We now show that Γ is surjective. We have that $\Gamma(A)$ is a $*$ -subalgebra of $C_0(\widehat{A})$. $\Gamma(A)$ separates points. Indeed, if $\omega_1 \neq \omega_2 \in \widehat{A}$, there is some $a \in A$ such that $\omega_1(a) \neq \omega_2(a)$. Hence, $\Gamma(a)(\omega_1) \neq \Gamma(a)(\omega_2)$. Moreover, $\Gamma(A)$ vanishes no-where. Indeed, if $\omega \in \widehat{A}$, since ω is non-zero, then there is a $a \in A$ such that $\Gamma(a)(\omega) \neq 0$. The Stone–Weierstrass theorem⁸ now implies that $\overline{\Gamma(A)} = C_0(\widehat{A})$. But since A is a closed set of itself Γ is a linear isometry and $C_0(\widehat{A})$ is a normed space, general Banach space theory shows that $\Gamma(A)$ is in fact closed⁹. Hence, $\Gamma(A) = C_0(\widehat{A})$. \square

9. CONTINUOUS FUNCTIONAL CALCULUS

Recall that in [Proposition 4.2](#) we characterized the spectrum of an element of a Banach algebra obtained by applying a polynomial function to an element of a Banach algebra. This is an example of polynomial functional calculus, which deals with the study of polynomials of functions of elements in a Banach algebra. Note that polynomial functional calculus can be extended to holomorphic functional calculus in a Banach algebra. In this section, we establish a more general functional calculus for C^* -algebras: continuous functional calculus, which is a functional calculus which allows the application of a continuous function to normal elements of a C^* -algebra.

9.1. Motivation. We already know how to apply polynomial functional calculus for polynomials defined on the spectrum of any element a of a Banach algebra. If we wish to extend polynomial functional calculus to continuous functions defined on spectrum, it seems obvious to approximate a continuous function by polynomials according to the Stone-Weierstrass theorem, and insert the element into these polynomials and to show that this sequence of elements converges in A . In particular, we shall approximate continuous functions on the spectrum of an element by Laurent polynomials, i.e., by polynomials of the form

$$p(z, \bar{z}) = \sum_{k,l=0}^N c_{k,l} z^k \bar{z}^l \quad c_{k,l} \in \mathbb{C}$$

⁸The complex version for locally compact Hausdorff spaces.

⁹The claim is that the image of a closed set under a linear isometry from a Banach space to a normed vector space is closed.

Here, \bar{z} denotes the complex conjugation, which is an involution on the complex numbers. To be able to insert a in place of z in this kind of polynomial, Banach $*$ -algebras are considered, i.e., Banach algebras that also have an involution $*$, and a^* is inserted in place of \bar{z} . In order to obtain a homomorphism

$$\mathbb{C}[z, \bar{z}] \rightarrow A,$$

a restriction to normal elements, i.e., elements with $a^*a = aa^*$, is necessary, as the polynomial ring $\mathbb{C}[z, \bar{z}]$ is commutative. If

$$(p_n(z, \bar{z}))_n$$

is a sequence of polynomials that converges uniformly on the spectrum of a to a continuous function f , the convergence of the sequence

$$(p_n(a, a^*))_{n \in \mathbb{N}}$$

in A to an element $f(a)$ must be ensured. A detailed analysis of this convergence problem shows that it is necessary to resort to C^* -algebras. These considerations lead to the so-called continuous functional calculus.

9.2. Construction. Let A be a C^* algebra. For any $a \in A$ that is a normal element, we write $C^*(a)$ for the C^* -algebra generated by a . This can be identified as the norm closure of the set of all polynomials in a, a^* with zero constant term, i.e.,

$$C^*(a) = \overline{\{p(a, a^*) \mid p \in \mathbb{C}[z_1, z_2], p(0, 0) = 0\}}.$$

When A is unital with unit e , $C^*(a, e)$ can be identified with the closure of the set of all polynomials on a, a^*

$$C^*(a, e) = \overline{\{p(a, a^*) \mid p \in \mathbb{C}[z_1, z_2]\}}.$$

Moreover, we denote as $C(\sigma_A(a))$ the C^* -algebra of continuous functions on $\sigma_A(a)$, the spectrum of a . Note that all of these C^* algebras are commutative.

Proposition 9.1. *Let A be a unital C^* algebra and let a be a normal element of A . Then there exists a unique $*$ -isometric algebra isomorphism*

$$\Phi_a: C(\sigma_A(a)) \rightarrow C^*(e, a)$$

with $\Phi_a(1_{\sigma_A(a)}) = e$ for $1_{\sigma_A(a)}(z) = 1$ and $\Phi_a(\text{Id}_{\sigma_A(a)}) = a$ for the identity. The mapping Φ_a is called the continuous functional calculus of the normal element a . Usually it is suggestively set $f(a) := \Phi_a(f)$.

We first need to prove the following lemma.

Lemma 9.2. *Let A be a unital C^* -algebra, and let B be a C^* -subalgebra containing the identity of A . Then for all $b \in B$, we have*

$$\sigma_B(b) = \sigma_A(b).$$

Proof. Clearly, we have that have

$$\sigma_A(b) \subseteq \sigma_B(b),$$

and the reverse inclusion will follow if we can show that b is invertible in A , implies that it is already invertible in B . First assume that $b = b^*$. Let,

$$E = C^*(b, b^{-1}) \subseteq A,$$

$$D = C^*(e, b) \subseteq E \cap B.$$

We show that $E = D$. This readily implies that $b^{-1} \in B$. Since $(b^{-1})^* = b^{-1}$, E is the closure of the algebra generated by $\{b, b^{-1}\}$. In particular, E is a unital commutative C^* -algebra. Hence,

$$E \cong C(\widehat{E})$$

Let $D' \subseteq C(\widehat{E})$ denote the image of D under the Gelfand transform. D' is a closed¹⁰ $*$ -subalgebra of $C(\widehat{E})$. Furthermore, if $\phi, \psi \in \widehat{E}$ are such that $\phi \neq \psi$, we must have $\phi(a) \neq \psi(a)$ since otherwise ϕ, ψ agree on a, a^{-1} and hence everywhere. Since $\text{Ev}_a \in D'$, D' separates the points of \widehat{E} . By Stone-Weierstrass, $\overline{D'} = C(\widehat{E})$. Since D' is also closed, it follows that $D' = C(\widehat{E})$. Hence,

$$C(\widehat{D}) \cong D' = C(\widehat{E})$$

Hence, $\widehat{D} = \widehat{E}$, which in turn implies that $D = E$. If b is not necessarily self-adjoint, note that b^*b is invertible in A since b is invertible in A . Then

$$(b^*b)^{-1}b^*b = e,$$

and

$$b^{-1} = (b^*b)^{-1}b^*.$$

By the argument above, $(b^*b)^{-1} \in B$. Hence, $b^{-1} \in B$. \square

Proof. (Proposition 9.1) We first show existence. Let $B = C^*(a, e)$. Since B is a unital commutative C^* -algebra, the Gelfand transform gives us an isometric $*$ -isomorphism

$$\Psi : B \rightarrow C(\widehat{B}).$$

The key argument is to explicitly identify \widehat{B} for $B = C^*(a, e)$. Note that any non-zero, linear and multiplicative map on B is uniquely defined by its action on a . Since $\sigma_A(a) = \{\omega(a) \mid \omega \in \widehat{A}\}$, the map

$$\begin{aligned} \tau : \widehat{B} &\rightarrow \sigma_A(a) \\ \omega &\mapsto \omega(a) \end{aligned}$$

is a continuous bijection. Since \widehat{B} and $\sigma_A(a)$ are compact and Hausdorff, τ is a homeomorphism. Then we get an isometric $*$ -isomorphism

$$\Theta : C(\sigma_A(a)) \rightarrow C(\widehat{B})$$

by

$$\Theta(f)(\omega) = f(\tau(\omega)) = f(\omega(a))$$

for $f \in C(\sigma_A(a))$, $\omega \in \widehat{B}$. The desired conclusion follows by letting $\Phi = \Psi^{-1} \circ \Theta$. Indeed, note that $\Theta(\text{Id}_{\sigma_A(a)})(\omega) = \tau(\omega) = \omega(a)$. Therefore, $\Theta(\text{Id}_{\sigma_A(a)}) = \text{Ev}_a$. Hence, $\Psi^{-1} \circ \Theta(\text{Id}_{\sigma_A(a)}) = a$ as required. We now show uniqueness. Since $\Phi_a(1_{\sigma_A(a)})$ and $\Phi_a(\text{Id}_{\sigma_A(a)})$ are fixed, Φ_a is already uniquely defined for all Laurent polynomials since Φ_a is a $*$ -homomorphism. These polynomials form a dense subalgebra of $C(\sigma_A(a))$ by the Stone-Weierstrass theorem. Thus Φ_a is unique. \square

Proposition 9.1 shows that continuous functional calculus can be used to reduce some abstract problems involving normal elements of a (not necessarily commutative!) C^* -algebra to problems about function algebras.

Remark 9.3. In what follows, we shall write $f(a)$ for $\Phi_a(f)$ from time to time.

Corollary 9.4. Let A be a unital C^* -algebra A , and let a be a normal element of A . Let $f \in C(\sigma_A(a))$. We have the following:

- (1) $f(a) \in A$ is normal
- (2) (**Spectral Mapping Theorem**) $f(\sigma_A(a)) = \sigma_A(f(a))$

Proof. The proof is given below:

¹⁰[Junaid:Why is it closed? Is D a $*$ -closed subalgebra of E ?]

(1) We have,

$$f(a)^* f(a) = \Phi_a(f)^* \Phi_a(f) = \Phi_a(\overline{f} f) = \Phi_a(f \overline{f}) = \Phi_a(f) \Phi_a(f)^* = f(a) f(a)^*$$

(2) Since $f(a) \in C^*(a, e)$ and Ψ_a is a $*$ -isometric isomorphism, we have

$$\sigma_A(f(a)) := \sigma_A(\Phi_a(f)) = \sigma_A(f) = f(\sigma_A(a)).$$

The last equality follows since the spectrum of f is simply its range.

This completes the proof. \square

We now present some applications of [Proposition 9.1](#) and [Corollary 9.4](#). We first present a complete characterization of self-adjoint, unitary and projection elements in a unital C^* -algebra.

Corollary 9.5. *Let A be a unital C^* -algebra, and let $a \in A$ be a normal element. Then:*

- (1) *a is self-adjoint if and only if $\sigma_A(a) \subseteq \mathbb{R}$.*
- (2) *a is unitary if and only if $\sigma_A(a) \subseteq \mathbb{S}^1$.*
- (3) *a is a projection if and only if $\sigma_A(a) \subseteq \{0, 1\}$.*

Proof. The proof is given below:

- (1) If a is self-adjoint, then [Proposition 5.7\(2\)](#) [Proposition 6.12\(ii\)](#) imply that

$$\sigma_A(a) = \{\omega(a) \mid \omega \in \widehat{A}\} \subseteq \mathbb{R}.$$

Conversely, assume that $\sigma_A(a) \subseteq \mathbb{R}$. Then,

$$a^* = \Phi_a(\text{Id}_{\sigma_A(a)})^* = \Phi_a(\overline{\text{Id}_{\sigma_A(a)}}) = \Phi_a(\text{Id}_{\sigma_A(a)}) = a.$$

Hence, a is self-adjoint.

- (2) The forward direction was proved in [Proposition 6.12\(5\)](#). Conversely, assume that $\sigma_A(a) \subseteq \mathbb{S}^1$. Then,

$$a^* a = \Phi_a(\text{Id}_{\sigma_A(a)})^* \Phi_a(\text{Id}_{\sigma_A(a)}) = \Phi_a(\overline{\text{Id}_{\sigma_A(a)}}) \text{Id}_{\sigma_A(a)} = \Phi_a(1_{\sigma_A(a)}) = e$$

Similarly, $aa^* = e$. Hence, a is a unitary.

- (3) If a is a projection, then $a^2 = a$. By [Corollary 9.4\(2\)](#), we have

$$\{0\} = \sigma_A(0) = \sigma_A(a^2 - a) = \{\lambda^2 - \lambda \mid \lambda \in \sigma(A)\}$$

This shows that $\sigma_A(a) \subseteq \{0, 1\}$. The converse follows as in (2).

This completes the proof. \square

Here is another application:

Corollary 9.6. *Let A be a unital C^* -algebra. If $a \in A$ is unitary such that $\sigma_A(a) \subsetneq \mathbb{S}^1$, then there exists a self-adjoint $b \in A$ such that $a = e^{ib}$.*

Proof. WLOG, we can assume that $-1 \notin \sigma_A(a)$. Let $\ln : \mathbb{C} \setminus (-\infty, 0] \rightarrow \mathbb{C}$ denote the principal branch of the logarithm function. Note that we have $e^{\ln(\cdot)} = \text{Id}_{\sigma_A(a)}(\cdot)$. Since $|\text{Id}_{\sigma_A(a)}| = 1$, the real part of \ln restricted to $\sigma_A(a)$ vanishes. Hence, $\ln|_{\sigma_A(a)} = ih$ for some real-valued function $h \in C(\sigma_A(a))$. Let $b = \Phi_a(h)$. Since h is real-valued, b is self-adjoint. Moreover, we have

$$a = \Phi_a(\text{Id}_{\sigma_A(a)}) = \Phi_a(e^{ih}) = e^{i\Phi_a(h)} = e^{ib}.$$

This completes the proof. \square

Here is another application:

Corollary 9.7. Let A, B be unital C^* -algebras and let $\varphi : A \rightarrow B$ be a unital $*$ -homomorphism. If $a \in A$ is normal element, then for every $f \in C(\sigma_A(a))$, we have $\varphi(f(a)) = f(\varphi(a))$.

Proof. Note that $\varphi(a)$ is normal and $\sigma_B(\varphi(a)) \subseteq \sigma_A(a)$, and so the restriction of f to $\sigma_B(a)$ is continuous. Define two unital $*$ -homomorphisms $\Phi_1, \Phi_2 : C(\sigma_A(a)) \rightarrow B$ by

$$\Phi_1(f) := \varphi(\Phi_a(f)) = \varphi(f(a)),$$

and

$$\Phi_2(f) := \Phi_{\varphi(a)}(f|_{\sigma_B(\varphi(a))}) = f|_{\sigma_B(\varphi(a))}(\varphi(a)).$$

It is easy to check that they both map $1_{\sigma_A(a)}$ and $\text{Id}_{\sigma_A(a)}$ to 1_B and $\varphi(a)$, respectively. Thus they agree on all polynomials of two variables z and \bar{z} over $\sigma_A(a)$ and since they are continuous, they agree on all of $C(\sigma_A(a))$. \square

10. POSITIVE ELEMENTS & STATES

10.1. Positive Elements. Continuous Functional Calculus (CFC) is a powerful tool for manipulating normal elements of a C^* -algebra. Granted, every element of a non-commutative C^* -algebra is not normal. Nonetheless, we can associate a self-adjoint element for every element in a non-commutative C^* -algebra, allowing us to spread the influence of the functional calculus to an entire non-commutative C^* -algebra. In this section, we discuss how positive elements can be defined via CFC.

Definition 10.1. Let A be a C^* -algebra. A self-adjoint element $a \in A$ is **positive** if $\sigma_A(a) \subseteq \mathbb{R}^+$. The subset of positive elements is denoted by A_+ , and if $a \in A_+$ we write $a \geq 0$.

Example 10.2. Let $A = C_0(X)$ for some locally compact topological space. Positive elements in are non-negative real-valued functions.

We now discuss applications of CFC to produce new elements in a C^* -algebra.

Proposition 10.3. Let A be a unital C^* -algebra.

- (1) If $a \in A$ is a self-adjoint element, then a can be written uniquely as

$$a = a_+ - a_- \quad a_\mp a_\pm = 0$$

for $a_\pm \in A_+$. The elements a_\pm are called the positive/negative parts of a .

- (2) Every $a \in A_+$ and $n \geq 1$, there exists a unique element $b \in A_+$ such that $a = b^n$. The element b is called the n -th root of a .

Proof. The proof is given below:

- (1) Consider the functions

$$f^+(t) = \max(t, 0), \quad f^-(t) = -\min(t, 0)$$

We have that $f^\pm \in C(\sigma_A(a))$ and $f^+(t) - f^-(t) = t^{\text{n}}$. By CFC, we have that,

$$a = a_+ - a_-$$

such that $a_\pm = \Psi_a(f^\pm)$. Since f^\pm are non-negative real-valued functions, we have that $\sigma_A(a_\pm) \subseteq \mathbb{R}^+$ by the spectral mapping theorem. Hence, a_\pm are positive elements. Moreover, note that $f^+ \cdot f^- = 0$. Hence, $a^\mp a^\pm = 0$ by CFC.

ⁱⁱSince a is self-adjoint, we can assume that $t \in \mathbb{R}$.

(2) For each $n \in \mathbb{N}$, consider the function

$$\begin{aligned} f_n : \mathbb{R}^+ &\rightarrow \mathbb{R}^+ \\ x &\mapsto \sqrt[n]{x} \end{aligned}$$

which is a continuous function on $\sigma_A(a) \subseteq \mathbb{R}^+$. Define, $b = \Phi_a(f_n)$. Then

$$b^n = \Phi_a(f_n)^n = \Phi_a(f_n^n) = \Phi_a(\text{Id}_{\sigma_A(a)}) = a$$

by CFC. By the spectral mapping theorem. we have

$$\sigma_A(b) = \sigma_A(f_n(a)) = f_n(\sigma_A(a)) \subseteq \mathbb{R}^+$$

i.e., b is positive. If $c \in A_+$ is another positive element such that $c^n = a = b^n$, then

$$c = \Phi_a(f_n) = \Phi_a(f_n) = b$$

by uniqueness of the inverse of the square root function defined on $\sigma_A(a)$. This proves uniqueness.

This completes the proof. \square

If X is compact and $f \in C(X)_+$, then notice that $|f(x) - t| \leq t$ for every real number $t \geq \|f\|$. Conversely, if $|f(x) - t| \leq t$ for some $t \geq \|f\|$, then $f(x) \geq 0$ for all x and so $f \geq 0$. These observations are behind some of the statements in the next result.

Lemma 10.4. *Let A be a unital C^* -algebra, and let a be a self-adjoint element. Then the following are equivalent.*

- (1) $a \geq 0$
- (2) $\|\alpha e - a\| \leq \alpha$ for all $\alpha \geq \|a\|$
- (3) $\|\alpha e - a\| \leq \alpha$ for some $\alpha \geq \|a\|$.

Proof. We first prove (1) implies (2). Since $a \geq 0$, $a = a^*$. Hence, $C^*(a)$ is abelian. Recall that $C(\sigma_A(a)) \cong C^*(a)$, such that the identity function in $C(\sigma_A(a))$ corresponds to a . Since $a \geq 0$, we have that the identity function is in $C(\sigma_A(a))_+$. Since $\sigma_A(a)$ is compact, the discussion preceding the statement of the proposition then implies that (2) is true if we take f to be the identity function. Clearly, (2) implies (3). (3) implies (1) follows from an argument similar to that that implies (1) implies (2). \square

[Lemma 10.4](#) gives us a nice characterization of positive elements. We have the following corollaries.

Corollary 10.5. *Let A be a unital C^* -algebra.*

- (1) *Then A_+ is closed.*
- (2) *If $a, b \in A_+$, then $a + b \in A_+$.*
- (3) *If $a, b \in A_+$ and a and b commute, then $ab \in A_+$.*
- (4) *a is positive if and only if $a = b^*b$ for some $b \in A$.*

Proof. The proof is given below:

- (1) Suppose $(a_n) \in A_+$ converges to $a \in A$. Then

$$\|a_n^* - a^*\| = \|a_n - a\| \rightarrow 0,$$

and so $(a_n^*) = (a_n) \in A_+$ converges to a^* . Hence $a^* = a$. Moreover, we have that $(\|a_n\|)$ converges to $\|a\|$. By [Lemma 10.4](#)

$$\|\|a_n\|e - a_n\| \leq \|a_n\|$$

for each $n \in \mathbb{N}$. Hence,

$$\|\|a\|e - a\| \leq \|a\|$$

for each $n \in \mathbb{N}$. By [Lemma 10.4](#), we have that $a \geq 0$.

- (2) Clearly, $a + b$ is self-adjoint. It suffices to assume that $\|a\|, \|b\| \leq 1^{12}$. But

$$\left\| 1 - \frac{1}{2}(a + b) \right\| = \frac{1}{2} \left\| (1 - a) + (1 - b) \right\| \leq 1$$

by [Lemma 10.4](#). Hence, $\frac{1}{2}(a + b) \geq 0$ by [Lemma 10.4](#), which implies that $a + b \geq 0$.

- (3) Note that,

$$(a + b)^2 = a^2 + 2ab + b^2$$

By the spectral mapping theorem and (2) above, $a^2, b^2, (a + b)^2 \in A_+$. Hence, $2ab \in A_+$ which in turn implies that $ab \in A_+$.

- (4) The forward implication follows from [Proposition 10.3\(2\)](#) by simply taking the square root of a . Conversely, clearly a is self-adjoint. We show that $a = b^*b$ implies that $\sigma_A(a) \subseteq \mathbb{R}^+$. Note that $a = b^*b$. Hence, we shall apply CFC to $a = b^*b$. Define

$$f(t) = \begin{cases} \sqrt{t} & \text{if } t \geq 0, \\ 0 & \text{otherwise} \end{cases} \quad g(t) = \begin{cases} 0 & \text{if } t \geq 0, \\ \sqrt{-t} & \text{otherwise.} \end{cases}$$

Then for all $t \in \mathbb{R}$, we have $f(t)g(t) = 0$ and $f(t)^2 - g(t)^2 = t$. Since f and g both vanish at 0, we get self-adjoint elements of $u = \Phi_{b^*b}(f)$ and $v = \Phi_{b^*b}(g)$ of A such that

$$\Phi_{b^*b}(g)\Phi_{b^*b}(f) = \Phi_{b^*b}(g)\Phi_{b^*b}(f) = 0, \quad \Phi_{b^*b}(f)^2 - \Phi_{b^*b}(g)^2 = b^*b$$

But then

$$\Phi_{b^*b}(g)(\Phi_{b^*b}(f)^2 - \Phi_{b^*b}(g)^2)\Phi_{b^*b}(g) = -\Phi_{b^*b}(g)^4.$$

Thus¹³

$$\sigma_A((bv)^*bv) = \sigma_A(-\Phi_{b^*b}(g)^4) \subseteq (-\infty, 0].$$

Thus $-\Phi_{b^*b}(g)^4 = 0$. Since v is self-adjoint, this means $\Phi_{b^*b}(g) = 0$. But then $b^*b = \Phi_{b^*b}^2(f)$, and $\Phi_{b^*b}^2(f)$ is positive by the spectral mapping theorem for normal elements.

This completes the proof. □

Remark 10.6. Note that [Corollary 10.5\(3\)](#) is not true in general. Indeed, let

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$A, B \in M_2(\mathbb{C})$ are positive. However, AB is not positive since

$$AB = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$$

is not even self-adjoint.

Remark 10.7. All claims about positive elements are true in a non-unital C^* -algebra. These facts can be proved by passing to the unitization of a non-unital C^* -algebra.

¹²This because a positive scalar of a positive element is a positive element

¹³Here we use the result that if $a \in A$ and $\sigma_A(a^*a) \subseteq (-\infty, 0]$, then $a = 0$.

10.2. States. We discuss states in this section. These will be useful in the GNS construction to be discussed later on.

Definition 10.8. Let A be a C^* -algebra. A **state** on A is a linear functional such that:

- (1) φ is positive. That is, $\varphi(A_+) \subseteq \mathbb{R}^+$. Equivalently, $\varphi(a^*a) \geq 0$ for each $a \in A$
- (2) The norm of φ is one. That is,

$$\|\varphi\| := \sup\{|\varphi(a)| : \|a\| = 1\} = 1.$$

The subset $S(A) \subseteq A_{\leq 1}^*$ consisting of states is called the **state space**.

Proposition 10.9. Let A be a C^* -algebra, and let φ be a positive linear functional on A .

- (1) φ is $*$ -preserving. That is,

$$\varphi(a^*) = \overline{\varphi(a)}$$

- (2) (**Cauchy-Schwartz**) For $a, b \in A$, we have,

$$|\varphi(b^*a)|^2 \leq \varphi(b^*b)\varphi(a^*a)$$

such that we have equality if and only if $\varphi(a^*b) = \varphi(b^*a)$.

- (3) If A is unital, $\varphi(e) = \|\varphi\|$.

Proof. The proof is given below:

- (1) Note that [Proposition 6.12](#) that it suffices to show that φ maps self-adjoint elements to self-adjoint elements. If $a \in A$ is a self-adjoint element, then $a = a_+ - a_-$ for some $a_{\pm} \in A_+$. Note that

$$\varphi(a) = \varphi(a_+) - \varphi(a_-) \in \mathbb{R},$$

since φ is a positive linear functional. The claim follows.

- (2) Let $\lambda \in \mathbb{C}$. Since φ is positive, it follows that

$$\varphi((\lambda a + b)^*(\lambda a + b)) = |\lambda|^2 \varphi(a^*a) + \bar{\lambda} \varphi(a^*b) + \lambda \varphi(b^*a) + \varphi(b^*b) \geq 0.$$

Because this expression must be real for all $\lambda \in \mathbb{C}$, it follows that $\overline{\varphi(a^*b)} = \varphi(b^*a)$. Hence, we get the inequality,

$$|\lambda|^2 \varphi(a^*a) + 2\operatorname{Re}(\lambda \varphi(b^*a)) + \varphi(b^*b) \geq 0.$$

Let $\gamma \in \mathbb{S}^1$ such that $\gamma \varphi(b^*a) = |\varphi(b^*a)|$. Given $t \in \mathbb{R}$, put $\lambda = t\gamma$. Hence, we get the inequality,

$$t^2 \varphi(a^*a) + 2t|\varphi(b^*a)| + \varphi(b^*b) \geq 0.$$

As we can do this for any $t \in \mathbb{R}$ and this is a real quadratic, for this to be always non-negative we need $b^2 \leq 4ac$, i.e.,

$$4|\varphi(b^*a)|^2 \leq 4\varphi(a^*a)\varphi(b^*b)$$

The desired result now follows.

- (3) Clearly, $\varphi(e) \leq \|\varphi\|$. Using $b = e$ in (2), we have

$$|\varphi(a)|^2 \leq \varphi(a^*a)\varphi(e) \leq \|\varphi\| \|a^*a\| \varphi(e) = \|\varphi\| \|a\|^2 \varphi(e)$$

Taking the supremum over all a of norm one, we get

$$\|\varphi\|^2 \leq \|\varphi\| \varphi(e)$$

It follows that

$$\|\varphi\| \leq \varphi(e).$$

Hence, $\varphi(e) = \|\varphi\|$.

This completes the proof. □

II. REPRESENTATIONS OF C^* -ALGEBRAS

Definition II.1. Let A be a C^* -algebra. A **representation** of A is a pair, (π, \mathcal{H}) , where \mathcal{H} is a Hilbert space and $\pi : A \rightarrow \mathcal{B}(\mathcal{H})$ is a $*$ -homomorphism.

Definition II.2. Let A be a C^* -algebra. Two representations of A , (π_1, \mathcal{H}_1) , (π_2, \mathcal{H}_2) , are **unitarily equivalent** if there is a unitary operator $U : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ such that

$$U\pi_1(a) = \pi_2(a)U,$$

for all a in A . In this case, we write $(\pi_1, \mathcal{H}_1) \sim_U (\pi_2, \mathcal{H}_2)$ or $\pi_1 \sim_U \pi_2$.

Remark II.3. We usually write that π is a representation of A on \mathcal{H} .

An important problem in representation theory is to understand irreducible representations of an abstract mathematical object.

Definition II.4. Let A be a C^* -algebra and let (π, \mathcal{H}) be a representation of A . A subspace $\mathcal{N} \subseteq \mathcal{H}$ is said to be **invariant** if $\pi(a)\mathcal{N} \subseteq \mathcal{N}$, for all a in A . The representation (π, \mathcal{H}) is said to be **irreducible** if the only invariant subspaces of $\{0\}$ and \mathcal{H} .

An important technique in representation theory is to construct ‘larger’ representations from ‘smaller’ representations by means of algebraic operation.

Definition II.5. Let A be a C^* -algebra and $(\pi_\iota, \mathcal{H}_\iota)$, $\iota \in I$, be a collection of representations of A . The **direct sum representation** is $(\bigoplus_{\iota \in I} \pi_\iota, \bigoplus_{\iota \in I} \mathcal{H}_\iota)$, where $\bigoplus_{\iota \in I} \mathcal{H}_\iota$ consists of tuples, $x = (x_\iota)_{\iota \in I}$ satisfying $\sum_{\iota \in I} \|x_\iota\|^2 < \infty$ and

$$\left(\bigoplus_{\iota \in I} \pi_\iota(a)x \right)_\iota = \pi_\iota(a)x_\iota, \quad \iota \in I.$$

Proposition II.6. Let A be a C^* -algebra, and let (π, \mathcal{H}) be a representation of A . A closed subspace $\mathcal{N} \subseteq \mathcal{H}$ is invariant if and only if \mathcal{N}^\perp is invariant.

Proof. Assume that $\mathcal{N} \subseteq \mathcal{H}$ is a closed invariant subspace. Let x in \mathcal{N}^\perp , $y \in \mathcal{N}$ and $a \in A$. We have

$$\langle \pi(a)x, y \rangle = \langle x, \pi(a)^*y \rangle = \langle x, \pi(a^*)y \rangle = 0,$$

since $\pi(a^*)\mathcal{N} \subseteq \mathcal{N}$. Hence, $\pi(a)x \in \mathcal{N}^\perp$, showing that \mathcal{N}^\perp is invariant. The converse follows from the observation that $(\mathcal{N}^\perp)^\perp = \mathcal{N}$. \square

The previous proposition shows that it is possible to define two representations of A by simply restricting the operators to either \mathcal{N} or \mathcal{N}^\perp . Moreover, the direct sum of these two representations is unitarily equivalent to the original representation. That is, we have

$$(\pi, \mathcal{H}) \sim_U (\pi|_{\mathcal{N}}, \mathcal{N}) \oplus (\pi|_{\mathcal{N}^\perp}, \mathcal{N}^\perp).$$

We now discuss a characterization irreducible representations of a C^* -algebra.

Definition II.7. Let A be a C^* -algebra, and let (π, \mathcal{H}) be a representation of A . $x \in \mathcal{H}$ is **cyclic** if

$$\overline{\text{span}\{\pi(a)x \mid a \in A\}} = \mathcal{H}.$$

We say that the representation is a **cyclic representation** if each non-zero vector in \mathcal{H} is cyclic.

Remark II.8. A representation, (π, \mathcal{H}) of a C^* -algebra A is non-degenerate if the only $x \in \mathcal{H}$ such that $\pi(a)x = 0$ for all a in A is $x = 0$. Otherwise, the representation is degenerate. It can be easily showed that a representation (π, \mathcal{H}) of a unital C^* -algebra is non-degenerate if and only if $\pi(a) = I_{\mathcal{H}}$ implies that $a = e$.

Proposition II.9. *Let A be a C^* -algebra, and let (π, \mathcal{H}) be a non-degenerate representation of A . (π, \mathcal{H}) is an irreducible representation if and only if the representation (π, \mathcal{H}) is cyclic.*

Proof. Assume that (π, \mathcal{H}) is irreducible. Let x be a non-zero vector, then

$$\text{span}\{\pi(a)x \mid a \in A\}$$

is an invariant subspace and its closure is a closed invariant subspace. If it is \mathcal{H} , then the representation is degenerate, which is impossible. Otherwise, it must be \mathcal{H} , meaning that x is a cyclic vector for π . Conversely, suppose that (π, \mathcal{H}) is non-degenerate, but reducible. Let \mathcal{N} be a proper closed invariant subspace which is neither \mathcal{H} nor \mathcal{H}^\perp . If x is any non-zero vector in \mathcal{N} , then

$$\text{Span}\{\pi(a)x \mid a \in A\} \subseteq \mathcal{N}$$

Hence, it cannot be dense in \mathcal{H} . This shows that (π, \mathcal{H}) is not a cyclic representation. \square

We end this section give a more useful criterion for a representation to be reducible.

Proposition II.10. (Schur's Lemma) *Let A be a C^* -algebra, and let (π, \mathcal{H}) be a non-degenerate representation of A . Then (π, \mathcal{H}) is irreducible if and only if the only positive operators which commute with its image are scalar multiples of the identity operator.*

Proof. First assume that (π, \mathcal{H}) is a reducible representation. Let $\mathcal{N} \subseteq \mathcal{H}$ be a non-trivial proper closed invariant subspace of \mathcal{H} . Let p be the orthogonal projection onto \mathcal{N} . We have that $p = p^*$ and $\sigma_A(p) \in \{0, 1\}$, which means that p is positive. We check that it commutes with $\pi(a)$, for any a in A . If $x \in \mathcal{N}$, we know that $\pi(a)x \in \mathcal{N}$ and so

$$(p\pi(a))x = p(\pi(a)x) = \pi(a)x = \pi(a)(px) = (\pi(a)p)x.$$

On the other hand, if x is in \mathcal{N}^\perp , then so is $\pi(a)x$ and

$$(p\pi(a))x = p(\pi(a)x) = 0 = \pi(a)(0) = \pi(a)(px) = (\pi(a)p)x.$$

Since every vector in \mathcal{H} is the sum of two as above, we see that

$$p\pi(a) = \pi(a)p$$

for each $a \in A$. As both \mathcal{N} and \mathcal{N}^\perp are non-empty, this operator is not a scalar multiple of the identity operator on \mathcal{H} . Conversely, suppose that p is a positive that is not a scalar multiple of the identity operator on \mathcal{H} , but commutes with every element of $\pi(a)$ for each $a \in A$. We must have that

$$\sigma_A(p) = \{0, 1\}$$

We may then find non-zero continuous functions¹⁴ f, g on $\sigma_A(p)$ whose product is zero. By CFC, $f(p)$ and $g(p)$ are well-defined operators in $\mathcal{B}(\mathcal{H})$. Since f is non-zero, the operator $\Phi_p(f)$ is non-zero. Let \mathcal{N} denote the closure of its range, which is a non-zero subspace of \mathcal{H} . On the other hand, $\Phi_p(g)$ is also a non-zero operator, but it is zero on the range of $\Phi_p(f)$ and hence on \mathcal{N} . This implies that \mathcal{N} is a proper subspace of \mathcal{H} . Note that $\Phi_a(f)$ commutes with $\pi(a)$ for each $a \in A$. Let $a \in A$. Indeed, for any $\epsilon > 0$, we may find a polynomial $q(x) \in C(\sigma_A(p))$ such that

$$\|f - q\|_\infty < \epsilon$$

in $C(\sigma_A(p))$. This means that

$$\|\Phi_p(f) - \Phi_p(q)\| < \epsilon$$

¹⁴Use Uryshon's lemma or simply consider bump functions.

It is clear that $\Phi_p(q)$ will commute with $\pi(a)$, since p commutes with $\pi(a)$. A triangle inequality type argument now shows that $\Phi_p(f)$ must also commute with $\pi(a)$. Finally, we claim that \mathcal{N} is invariant under $\pi(a)$. In fact, it suffices to check that the range of $\Phi_p(f)$ is invariant. But if $x \in \mathcal{H}$, we have

$$\pi(a)(\Phi_p(f)x) = \pi(a)\Phi_p(f)x = \Phi_p(f)\pi(a)x \in \Phi_p(f)\mathcal{H},$$

Hence, \mathcal{N} is a proper, closed invariant subspace of \mathcal{H} . This completes the proof. \square

12. GELFAND-NAIMARK-SEGAL CONSTRUCTION

In this section, we explicitly construct a representation of a given C^* -algebra. The basic idea is that multiplication allows one to see the elements of a C^* -algebra acting as linear transformations of itself. The problem is, of course, that the C^* -algebra does not usually have the structure of a Hilbert space. To produce an inner product or bilinear form, we use the linear functionals on the C^* -algebra in a clever way, leading to the Gelfand-Naimark-Segal (GNS) construction. First consider the following example.

Example 12.1. Let A be a unital C^* -algebra, and $(\pi, \mathcal{B}(\mathcal{H}))$ be a non-degenerate representation of A . Let $x \in \mathcal{H}$ such that $\|x\| = 1$, and consider the linear functional

$$\varphi(a) := \langle \pi(a)x, x \rangle$$

on A . Note that,

$$\begin{aligned} \varphi(a^*a) &= \langle \pi(a^*a)x, x \rangle \\ &= \langle \pi(a^*)\pi(a)x, x \rangle \\ &= \langle \pi(a)^\dagger\pi(a)x, x \rangle \\ &= \langle \pi(a)x, \pi(a)x \rangle \geq 0 \end{aligned}$$

Here \dagger denotes the Hilbert space adjoint operator. Moreover, note that,

$$\begin{aligned} \|\varphi(a)\| &= |\langle \pi(a)x, x \rangle| \\ &\leq \|\pi(a)x\|_{\mathcal{H}} \|x\|_{\mathcal{H}} \\ &\leq \|\pi(a)\|_{\mathcal{B}(\mathcal{H})} \|x\|_{\mathcal{H}} \|x\|_{\mathcal{H}} \\ &= \|\pi(a)\|_{\mathcal{B}(\mathcal{H})} \|x\|_{\mathcal{H}}^2 \\ &\leq \|a\|_A \|x\|_{\mathcal{H}}^2 = \|a\|_A \end{aligned}$$

The last inequality follows since π is a $*$ -algebra which are known to be contractive. Hence, $\|\varphi\| \leq 1$. But note that,

$$\varphi(e) = \langle \pi(e)x, x \rangle = \langle I_{\mathcal{H}}x, x \rangle = \langle x, x \rangle = 1.$$

Hence, $\|\varphi\| = 1$. This shows that φ is a state.

Example 12.1 implies that we can associate a state corresponding to a non-degenerate representation of a C^* -algebra. We now show that the converse is true as well, which is the GNS construction.

Proposition 12.2. (GNS Construction) *Let A be a C^* -algebra. If φ is any state on A , there is a non-degenerate representation $(\pi_\varphi, \mathcal{H}_\varphi)$ and a unit vector $x_\varphi \in \mathcal{H}_\varphi$ such that*

$$\varphi(a) = \langle \pi_\varphi(a)x_\varphi, x_\varphi \rangle_\varphi$$

for any $a \in A$. Moreover, the representation is unique up to unitary equivalence. That is, if $\pi : A \rightarrow \mathcal{B}(\mathcal{H})$ is another representation with cyclic unit vector $x \in \mathcal{H}$ satisfying

$$\varphi(a) = \langle \pi(a)x, x \rangle$$

for all $a \in A$, then there exists a unique unitary $U : \mathcal{H}_\varphi \rightarrow \mathcal{H}$ such that

$$Ux_\varphi = x$$

and

$$U\pi_\varphi(a) = \pi(a)U$$

for all $a \in A$.

Proof. If φ is a state, then φ defines a sesqui-linear form. Define the set,

$$\mathcal{N}_\varphi = \{a \in A \mid \varphi(a^*a) = 0\}$$

We first show that φ is a closed left ideal. Clearly, φ is closed since φ is continuous¹⁵. Moreover, \mathcal{N}_φ is a vector subspace. Indeed, if $a, b \in \mathcal{N}_\varphi$, and $\lambda \in \mathbb{C}$, then

$$\varphi((\lambda a + b)^*(\lambda a + b)) = |\lambda|^2\varphi(a^*a) + \lambda\varphi(b^*a) + \bar{\lambda}\varphi(a^*b) + \varphi(b^*b) = 0$$

The first and fourth terms are zero by assumption. The second and third terms are zero by Proposition 10.9(2). If $a \in \mathcal{N}_\varphi$ and $b \in A$, consider the functional

$$\psi(c) = \varphi(a^*ca)$$

for any $c \in A$. This is clearly another positive linear functional, and so we have

$$\|\psi\| = \psi(1) = \varphi(a^*a) \tag{*}$$

Then we have

$$0 \leq \varphi((ba)^*ba) = \varphi(a^*b^*ba) = \psi(b^*b) \leq \|\psi\|\|b^*b\| = \varphi(a^*a)\|b\|^2 = 0$$

This shows that $ba \in \mathcal{N}_\varphi$. Hence, \mathcal{N}_φ is a closed left ideal. Consider the map,

$$\begin{aligned} (,) : A \times A &\rightarrow \mathbb{C} \\ (a, b) &\mapsto \varphi(a^*b) \end{aligned}$$

Our discussion above shows that the map defined above is a sesquilinear form. Indeed, it is easy to check that the map is linear in the second argument. Moreover, it is conjugate symmetric since,

$$(a, b) = \varphi(a^*b) = \varphi((ba^*)^*) = \overline{\varphi(ba^*)} = \overline{(b, a)}$$

But it might not be an inner product since $\mathcal{N}_\varphi \neq \emptyset$. Consider $V_\varphi = A/\mathcal{N}_\varphi$. We can now define an *honest* inner product on V_φ by the formula:

$$\langle a + \mathcal{N}_\varphi, b + \mathcal{N}_\varphi \rangle_\varphi := \varphi(b^*a).$$

To see that this is well-defined, note that, for any $x, y \in \mathcal{N}_\varphi$ and $a, b \in A$,

$$\begin{aligned} \varphi((b + y)^*(a + x)) &= \varphi(b^*a + b^*x + y^*a + y^*x) \\ &= \varphi(b^*a) + \varphi(b^*x) + \varphi(y^*a) + \varphi(y^*x) \\ &= \varphi(b^*a). \end{aligned}$$

The last equality follows since \mathcal{N}_φ is a left ideal. Let \mathcal{H}_φ to be the completion of V_φ with respect to the norm induced by $\langle \cdot, \cdot \rangle_\varphi$. The action of A on A by left multiplication induces a representation $\pi_\varphi : A \rightarrow \mathcal{B}(\mathcal{H}_\varphi)$ given by left multiplication:

$$\pi_\varphi(a)(b + \mathcal{N}_\varphi) = ab + \mathcal{N}_\varphi.$$

¹⁵Note that φ is the composition of the maps $a \mapsto (a, a^*) \mapsto \varphi(a^*a)$ which is indeed continuous.

The fact that \mathcal{N}_φ is an ideal ensures that $\pi_\varphi(a)$ is a well-defined map from V_φ to V_φ for each $a \in A$. Moreover, the map $\pi_\varphi(a)$ is bounded from V_φ to V_φ for each $a \in A$. Indeed,

$$\begin{aligned}\|\pi_\varphi(a)\|^2 &= \sup\{\langle \pi_\varphi(a)(x + \mathcal{N}_\varphi), \pi_\varphi(a)(x + \mathcal{N}_\varphi) \rangle_\varphi \mid x \in A, \varphi(x^*x) = 1\} \\ &= \sup\{\varphi((ax)^*(ax)) \mid x \in A, \varphi(x^*x) = 1\} \\ &= \sup\{\varphi(x^*a^*ax) \mid x \in A, \varphi(x^*x) = 1\} \\ &\leq \sup\{\varphi(x^*x)\|a^*a\| \mid x \in A, \varphi(x^*x) = 1\} \\ &= \|a^*a\| = \|a\|^2.\end{aligned}$$

We have used the information in (*). It is clear that $\pi_\varphi(a)$ is linear and multiplicative from V_φ to V_φ for each $a \in A$. We check that $\pi_\varphi(a)$ is $*$ -preserving from V_φ to V_φ for each $a \in A$. For $a, b, c \in A$, we have

$$\begin{aligned}\langle \pi_\varphi(a^*)b + \mathcal{N}_\varphi, c + \mathcal{N}_\varphi \rangle_\varphi &= \langle a^*b + \mathcal{N}_\varphi, c + \mathcal{N}_\varphi \rangle_\varphi \\ &= \varphi(c^*(a^*b)) \\ &= \varphi((ac)^*b) \\ &= \langle b + \mathcal{N}_\varphi, ac + \mathcal{N}_\varphi \rangle_\varphi \\ &= \langle b + \mathcal{N}_\varphi, \pi_\varphi(a)c + \mathcal{N}_\varphi \rangle_\varphi \\ &= \langle \pi_\varphi(a^*)b + \mathcal{N}_\varphi, c + \mathcal{N}_\varphi \rangle_\varphi.\end{aligned}$$

Since this holds for arbitrary b and $c \in A$, we conclude that $\pi_\varphi(a)^* = \pi_\varphi(a)^*$. A standard density argument now shows that the map $\pi_\varphi : A \rightarrow \mathcal{B}(\mathcal{H}_\varphi)$ is a well-defined, linear, multiplicative and $*$ -preserving map. Consider the vector $x_\varphi = 1 + \mathcal{N}_\varphi$. Note that,

$$\begin{aligned}\|x_\varphi\| &= \|1 + \mathcal{N}_\varphi\| \\ &= \varphi(1^*1)^{1/2} \\ &= \varphi(1)^{1/2} \\ &= 1.\end{aligned}$$

If b is any element of A , it is clear that

$$\pi_\varphi(b)x_\varphi = b \cdot 1 + \mathcal{N}_\varphi = b + \mathcal{N}_\varphi.$$

It follows then that $\pi_\varphi(A)x_\varphi$ contains A/\mathcal{N}_φ and is therefore dense in H_φ . Note that we have,

$$\langle \pi_\varphi(a)x_\varphi, x_\varphi \rangle_\varphi = \langle a + \mathcal{N}_\varphi, 1 + \mathcal{N}_\varphi \rangle = \varphi(a)$$

for each $a \in A$. This proves existence. Let $\pi : A \rightarrow \mathcal{B}(\mathcal{H})$ be another representation with cyclic unit vector $x \in \mathcal{H}$ satisfying

$$\varphi(a) = \langle \pi(a)x, x \rangle$$

for all $a \in A$. Let V denote the set

$$V = \{\pi(a)x : a \in A\} \subseteq \mathcal{H}.$$

Define

$$\begin{aligned}U : V_\varphi &\rightarrow V, \\ \pi_\varphi(a)x_\varphi &\mapsto \pi(a)x\end{aligned}$$

We first show that U is well-defined. Assume that $\pi_\varphi(a)x_\varphi = \pi_\varphi(b)x_\varphi$. We must show that $\pi(a)x = \pi(b)x$. Certainly, using that π is a $*$ -morphism:

$$\begin{aligned}
\langle \pi(a-b)x, \pi(a-b)x \rangle &= \langle \pi((a-b)^*(a-b))x, x \rangle \\
&= \varphi((a-b)^*(a-b)) \\
&= \langle \pi_\varphi((a-b)^*(a-b))x_\varphi, x_\varphi \rangle_\varphi \\
&= \langle \pi_\varphi(a-b)x_\varphi, \pi_\varphi(a-b)x_\varphi \rangle_\varphi \\
&= \|\pi_\varphi(a-b)x\|_\varphi^2 = 0.
\end{aligned}$$

Hence, we have $\pi(a)x = \pi(b)x$. This shows that U is a well-defined map. We now check that U is an injective map. Assume that $\pi(a)x = \pi(b)x$ for some $a, b \in A$. Invoking the above calculation, we have,

$$\|\pi_\varphi(a-b)x\|_\varphi^2 = \|\pi(a-b)\|^2$$

Hence, $\pi_\varphi(a)x = \pi_\varphi(b)x$, implying that U is injective. By a simple inspection, U is surjective. Hence, U is a bijection. To see that U preserves the inner product, notice that for $a, b \in A$, the following holds:

$$\begin{aligned}
\langle U(\pi_\varphi(a)x_\varphi), U(\pi(b)x_\varphi) \rangle &= \langle \pi(a)x, \pi(b)x \rangle \\
&= \langle \pi(b^*a)x, x \rangle \\
&= \varphi(b^*a).
\end{aligned}$$

A verbatim argument shows that

$$\langle \pi_\varphi(a)x_\varphi, \pi(b)x_\varphi \rangle = \varphi(b^*a)$$

This implies that U preserves inner products. Since U preserves inner products, it follows that U is also bounded. Moreover, it is a simple matter to check that U is a linear map. Hence, we see that

$$\begin{aligned}
U : V_\varphi &\rightarrow V \\
\pi_\varphi(a)x_\varphi &\mapsto \pi(a)x
\end{aligned}$$

is a unitary map. Using the density of V_φ in \mathcal{H}_φ (and that of V in \mathcal{H}), we may uniquely extend U to a bounded linear operator

$$\tilde{U} : \mathcal{H}_\varphi \rightarrow \mathcal{H}$$

Simple density arguments can be used to show that \tilde{U} is surjective, preserves the inner product and is a unitary equivalence. The claim follows at once if we replace \tilde{U} by U . \square

Remark 12.3. A state, φ , is called a pure state if φ cannot be written as a non-trivial convex combination of states. With a bit more work, one can show that the representation constructed in [Proposition 12.2](#) is irreducible if and only if φ is a pure state.

13. GELFAND-NAIMARK-SEGAL THEOREM

The study of C^* -algebras is motivated by the prime example of norm closed $*$ -algebras of operators on Hilbert space. With this in mind, it is natural to find ways that a given abstract C^* -algebra may act as operators on Hilbert space. Such an object is called a representation of a C^* -algebra, and the study of representations of a C^* -algebra leads to the proof of the Gelfand-Naimark-Segal theorem:

Theorem 13.1. (Gelfand, Naimark & Segal) Let A be a C^* -algebra. Then A is isometrically $*$ -isomorphic to a $*$ -closed subalgebra of $\mathcal{B}(\mathcal{H})$ for some Hilbert space, \mathcal{H} .

In order to prove [Theorem 13.1](#), we will take the direct sum of *a lot* of the representations given in [Proposition 12.2](#) to produce a faithful representation of a C^* -algebra. We must take into account one caveat, though. [Proposition 12.2](#) is proved under the assumption that there exists a state defined on A . In order to take a direct sum of *a lot* of representations given in [Proposition 12.2](#), we need to show the existence of *a lot* of states on A .

Lemma 13.2. *Let A be a unital C^* -algebra.*

- (1) *If φ is a bounded linear functional on A which satisfies*

$$1 = \|\varphi\| = \varphi(e),$$

then φ is a state.

- (2) *Let a be a non-zero, self-adjoint (hence normal) element of A . Then there is a state ψ on A such that*

$$|\psi(a)| = \|a\|.$$

Proof. The proof is given below:

- (1) Skipped.

- (2) Let $B = C^*(a, e) \subseteq A$. Let $\lambda = r(a)$ ¹⁶. Let $\text{Ev}_\lambda : C(\sigma_A(a)) \rightarrow \mathbb{C}$ be given by evaluation at λ . Since Ev_λ is a character on $C(\sigma_A(a))$, it is, in particular, a state on $C(\sigma_A(a))$. Since $B \cong C^*(a, e)$, we have furnished a state on B . Since B is a closed subspace of A , the Hahn-Banach theorem allows us to extend it to a linear functional $\psi \in A^*$ with the same norm (i.e., $\|\psi\| = 1$). As

$$\psi(1) = \text{Ev}_\lambda(1) = 1,$$

(1) tells us that ψ is also a state. Since the Gelfand transform takes a to the function $f(z) = z$, it follows that

$$|\psi(a)| = |\lambda| = r(a) = \|a\|$$

The last equality follows since a is self-adjoint.

This completes the proof. □

We are now ready to prove [Theorem 13.1](#).

Proof. ([Theorem 13.1](#)) Let $D \subseteq A$ be a dense subset. For each $a \in D$, let $\varphi_a \in S(A)$ be such that $|\varphi_a(a^*a)| = \|a^*a\| = \|a\|^2$. Let $(\pi_a, \mathcal{H}_a, x_{\varphi_a})$ representation constructed in [Proposition 12.2](#). Consider the direct sum representation:

$$\pi := \bigoplus_{a \in D} \pi_a : A \rightarrow \mathcal{B} \left(\bigoplus_{a \in D} \mathcal{H}_a \right) := \mathcal{B}(\mathcal{H}).$$

Assume $a \neq 0 \in A$ such that $\pi(a) = 0$. Then

$$\|\pi_a(a)x_{\varphi_a}\|^2 = \langle \pi_a(a^*a)x_{\varphi_a}, x_{\varphi_a} \rangle_{\varphi_a} = \varphi_a(a^*a) = \|a^*a\| \neq 0$$

This shows that π is non-zero, a contradiction. Hence, π is injective. It is clear that π is a $*$ -homomorphism since each π_a is a faithful $*$ -homomorphism. □

Remark 13.3. *With a bit more work, one can show that each state constructed in [Lemma 13.2](#) can be taken to be a pure state. Hence, π in the proof of [Theorem 13.1](#) can be taken to be a direct sum of irreducible representations of A .*

¹⁶We know this exists since $\sigma_A(a)$ is a non-empty, closed subset.