COHOMOLOGY

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Abstract. These notes cover various aspects of singular cohomology within the framework of algebraic topology. I compiled them with the intention of revisiting cohomology as a foundation for exploring more advanced topics such as K-Theory, characteristic classes, and cobordism, etc. If you come across any typos, please send corrections to junaida@umd.edu or junaid.aftab1994@gmail.com.

Contents

1. Why Cohomology?

Let X be a topological space and G be an abelian group. The following is a list of some reasons to study cohomology:

(1) Cohomology groups with G-coefficients can be 'summed up' to yield a direct sum decomposition:

$$H^*(X;G) = \bigoplus_{n\geq 0} H^n(X;G)$$

It turns out that $H^*(X)$ support a natural graded algebra structure induced by the cup product. This suggests that cohomology is a stronger topological invariant than homology: indeed, the classic example is that $\mathbb{CP}^2$ and $\mathbb{S}^2 \vee \mathbb{S}^4$ have isomorphic homology and cohomology groups, but different cohomology rings.

(2) Categorically, cohomology is a better invariant than homology in that singular cohomology for 'sufficiently nice' topological spaces is a representable functor.

(3) If we restrict to smooth manifolds, then there is a geometric interpretation of singular cohomology. This is given by the fact that singular cohomology is isomorphic to de-Rham cohomology, and the latter is defined in terms of differential forms on a smooth manifold. The k-th de Rham cohomology group measures the extent to which closed k-forms are exact k-forms on a smooth manifold. Since it is always possible to express a closed k-form as an exact k-form *locally*, we see that de Rham cohomology captures the general philosophy that a cohomology theory answers the following meta-question:

When can we promote local solutions of a problem to global solutions?

Part 1. Singular Cohomology

2. Singular Cohomology

In parallel with the theory of singular homology, we develop the theory of singular cohomology. Let G be an abelian group and let X be a topological space with a singular chain complex $(C_\bullet, \partial_\bullet)$ of abelian groups:

$$\cdots \xrightarrow{\partial_{n+1}} C_n(X) \xrightarrow{\partial_n} C_{n-1}(X) \xrightarrow{\partial_{n-1}} C_{n-2}(X) \xrightarrow{\partial_{n-2}} \cdots .$$

Consider $C_n^*(X) = \mathrm{Hom}(C_n(X), G)$, the group of singular n co-chains of X with G-coefficients. This defines the dual chain complex:

$$\xleftarrow{\partial_{n+1}^*} C_n^*(X) \xleftarrow{\partial_n^*} C_{n-1}^*(X) \xleftarrow{\partial_{n-1}^*} C_{n-2}^*(X) \xleftarrow{\partial_{n-2}^*} \cdots .$$

Remark 2.1. *We write $(C^\bullet, \partial^\bullet)$ for the above diagram which is called a singular co-chain complex. We often abbreviate $(C^\bullet, \partial^\bullet)$ as $C^\bullet$. We write C^n for $C_n^* = \mathrm{Hom}(C_n, G)$. Moreover, we shall also write the boundary map ∂_{n+1}^* as δ^n for the boundary map.*

The boundary maps are $\partial_n^* : C_{n-1}^* \longrightarrow C_n^*$ defined as:

$$(\partial_n^* \psi)(\alpha) = (\psi \circ \partial_n)(\alpha) \quad \psi \in C_{n-1}^*,\ \alpha \in C_n.$$

Note that the boundary map are such that $\partial_{n+1}^* \circ \partial_n^* = 0$ for $n \in \mathbb{Z}$. Indeed,

$$(\partial_{n+1}^* \circ \partial_n^*)(\psi) = \psi(\partial_{n+1} \circ \partial_n) = 0 \qquad \psi \in C_{n-1}^*$$

We can now make the following definition:

Definition 2.2. Let G be an abelian group, and let $(C_\bullet, \partial_\bullet)$ be a chain complex of free abelian groups. The n-th cohomology group of $(C_\bullet, \partial_\bullet)$ with G-coefficients is defined as

$$H^n((C_\bullet, \partial_\bullet); G) := H_n((C^\bullet, \partial^\bullet); G)$$

Elements of $\ker \partial_{n+1}^*$ are called n-cocycles, and elements of $\mathrm{Im}\ \partial_n^*$ are called n-coboundaries. We shall write $Z^n(X)$ for $\ker \partial_{n+1}^* = \ker \delta^n$ and $B^n(X)$ for $\mathrm{Im}\,\partial_n^* = \ker \delta^{n-1}$.

Remark 2.3. *Recall that chain complexes of abelian groups for a category, $\mathbf{Chain_{Ab}}$. The dual category, $\mathbf{Chain_{Ab}^{Op}}$, is called the category of co-chain complexes of abelian groups. Singular co-chain complexes are elements of $\mathbf{Chain_{Ab}^{Op}}$. It can be checked that both $\mathbf{Chain_{Ab}}$ and $\mathbf{Chain_{Ab}^{Op}}$ are abelian categories. Thus, all results that hold for $\mathbf{Chain_{Ab}}$, or singular chain complexes in particular continue to hold in $\mathbf{Chain_{Ab}^{Op}}$, or singular co-chain co-chain complexes in particular. For instance, we have various diagram-chasing lemmas such as the five lemma, the nine lemma, and the snake lemma. We shall not repeat these details in these notes. In any case, the proofs are similar to those discussed in the context of homology.*

Proposition 2.4. *Singular cohomology with coefficients in G defines a contravariant functor* **Top** *to* **Ab**.

Proof. Recall that if If $f : X \to Y$ is a continuous map, we have induced chain maps $f_n : C_n(X) \to C_n(Y)$ satisfying $f_n \circ \partial_{n+1} = \partial_n' \circ f_{n+1}$ for each $n \geq 0$.

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1}(X) \arrow[r, "\partial_{n+1}"] \arrow[d, "f_{n+1}"] & C_n(X) \arrow[r, "\partial_n"] \arrow[d, "f_n"] & C_{n-1}(X) \arrow[r] \arrow[d, "f_{n-1}"] & \cdots \\
\cdots \arrow[r] & C_{n+1}(Y) \arrow[r, "\partial_{n+1}'"] & C_n(Y) \arrow[r, "\partial_n'"] & C_{n-1}(Y) \arrow[r] & \cdots
\end{tikzcd}$$

Apply the $\text{Hom}(-, G)$ functor, we get maps such that

$$f^n : C^n(Y; G) \to C^n(X; G)$$

defined such that

$$f^n(\gamma)(\sigma) = \gamma(f_n(\sigma)) = \gamma(f \circ \sigma)$$

for $\gamma : C_n(Y) \to G$ and $\sigma : \Delta^n \to X$ a singular n-simplex in X. We claim that

$$\delta^n \circ f^n = f^{n+1} \circ \delta^{n'}$$

$$\begin{tikzcd}
\cdots & C^{n+1}(X,G) \arrow[l] & C^n(X,G) \arrow[l, "\delta^n"'] & C^{n-1}(X,G) \arrow[l, "\delta^{n-1}"'] & \cdots \arrow[l] \\
\cdots & C^{n+1}(Y,G) \arrow[l] \arrow[u, "f^{n+1}"] & C^n(Y,G) \arrow[l, "\delta^{n'}"] \arrow[u, "f^n"] & C^{n-1}(Y,G) \arrow[l, "\delta^{n-1'}"] \arrow[u, "f^{n-1}"] & \cdots \arrow[l]
\end{tikzcd}$$

Indeed, we have

$$(\delta^n \circ f^n)(\gamma) = \partial_{n+1} \circ f_n(\gamma) = \partial'_{n+1} \circ f_{n+1}(\gamma) = f^{n+1} \circ \delta^{n'}(\gamma)$$

for $\gamma : C_n(Y) \to G$. If $\gamma \in Z^n(Y)$ then we claim that $f^n(\gamma) \in Z^n(X)$. Indeed,

$$\delta^n(f^n(\sigma)) = f^{n+1}(\delta^{n'}(\sigma)) = f^{n+1}(0) = 0.$$

for $\gamma : C_n(Y) \to G$. Similarly, if $\gamma \in B^n(Y)$ then $f^n(\gamma) \in B^n(X)$. Thus f_n induces a map $H_n(f) : H_n(Y, G) \to H_n(X, G)$. One easily sees that

$$H^n(\text{Id}_X) = \text{Id}_{H^n(X)}$$

and that if $f : X \to Y$ and $g : Y \to Z$ then

$$H^n(g \circ f) = H^n(f) \circ H^n(g)$$

This completes the proof. □

3. Ext Functor

We now discuss the Ext (derived) functor, which arises as a derived functor of $\text{Hom}(-, G)$, and plays a crucial role in the formulation of the Universal Coefficient Theorem for singular cohomology.

Remark 3.1. *We work with commutative rings below. Hence, we don't make any distinction between the categories of left R-modules and right R-modules. We use the generic phrase 'R-module' to refer to a left/right R-module.*

In the category of R-modules, recall that the $\text{Hom}(X, -)$ functor defines a covariant functor from the category of R-modules to itself. If M is an R-module, then

$$\text{Hom}(X, -)(M) = \text{Hom}(X, M).$$

Moreover, if $f : M \to M'$ is a morphism of R-modules, then the functor acts on morphisms by

$$\text{Hom}(X, -)(f) : \text{Hom}(X, M) \longrightarrow \text{Hom}(X, M')$$

defined. It can be checked that $\text{Hom}(X, -)$ is a left exact functor. However, $\text{Hom}(X, -)$ is not a right exact functor in general.

Example 3.2. The functor $\mathrm{Hom}(X, -)$ is not a right exact functor in general. Let $R = \mathbb{Z}$. Consider the short exact sequence of abelian groups:

$$0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0.$$

Apply the functor $\mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}, -)$ to this sequence. We obtain:

$$0 \longrightarrow \mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) \xrightarrow{(\cdot 2)_*} \mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) \longrightarrow \mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})$$

The resulting sequence is:

$$0 \to 0 \to 0 \to \mathbb{Z}/2\mathbb{Z},$$

which is not an exact since $\mathbb{Z}/2\mathbb{Z} \to 0$ is not a surjective function.

Definition 3.3. Let R be a ring and let X be a R-module. The i-th Ext functor is the i-th left derived functor of $\mathrm{Hom}(X, -) := h_X$. It is denoted as

$$\mathrm{Ext}^i_I(X, -)$$

Remark 3.4. *The subscript I denotes that we have taken an injective resolution.*

By definition, $\mathrm{Ext}^i_I(X, -)$ is computed as follows: for an R-module Y take any injective resolution

$$0 \to Y \to I_0 \to I_1 \to \cdots$$

and form the co-chain complex:

$$\mathrm{Hom}(X, I_0) \to \mathrm{Hom}(X, I_1) \to \cdots .$$

For each integer i, $\mathrm{Ext}^i(X, Y)$ is the homology of this co-chain complex at position i:

$$\mathrm{Ext}^i_I(X, Y) = H_i(\mathrm{Hom}(X, I_i)^\bullet)$$

Similarly, we can consider the contravariant Hom functor and consider its right derived functor. Since it is a contravariant functor, we take projective resolutions now.

Definition 3.5. Let R be a ring and let Y be a R-module. The i-th Ext functor is the i-th left derived functor of $\mathrm{Hom}(-, Y) := h^Y$. It is denoted as

$$\mathrm{Ext}^i_P(-, Y)$$

Remark 3.6. *The subscript P denotes that we have taken a projective resolution.*

By definition, $\mathrm{Ext}^i_P(-, Y)$ is computed as follows: for an R-module X take any projective resolution

$$\cdots \to P^1 \to P^0 \to X \to 0,$$

and form the co-chain complex:

$$\mathrm{Hom}(P^0, Y) \to \mathrm{Hom}(P^1, Y) \to \cdots .$$

Then $\mathrm{Ext}^i_P(X, Y)$ is the homology of this co-chain complex at position i:

$$\mathrm{Ext}^i_P(X, Y) = H_i(\mathrm{Hom}(P^i, Y)^\bullet)$$

The left exact $\mathrm{Hom}(-, -)$ functor can be thought of as a bifunctor which is covariant in the second variable and contravariant in the first variable. The discussion above seemingly provides us with with two different strategies to compute the Ext functor. Fortunately, it turns out that we can use either strategy as formalized by the following proposition.

Proposition 3.7. ***(Balancing Ext)*** *Let X, Y be R-modules. Then*

$$\mathrm{Ext}^i_P(X,Y) \cong \mathrm{Ext}^i_I(X,Y)$$

for each $i \geq 0$.

Proof. See [Wei94] for a proof. □

Therefore, one can work with either strategy mentioned above. Therefore, we can now unambiguously write $\mathrm{Ext}^i(X,Y)$.

Proposition 3.8. *The Ext functor satisfies the following properties:*

(1) $\mathrm{Ext}^0(X,Y) \cong \mathrm{Hom}(X,Y)$ *for all R-modules X, Y..*
(2) *If X is a projective R-module, then* $\mathrm{Ext}^i(X,Y) = 0$ *for all $i \geq 1$*
(3) *If Y is an injective R-module, then* $\mathrm{Ext}^i(X,Y) = 0$ *for all $i \geq 1$*
(4) *Any $f : X_1 \to X_2$ induces a morphism*

$$f^{*,i} : \mathrm{Ext}^i(X_2,Y) \longrightarrow \mathrm{Ext}^i(X_1,Y)$$

for each $i \geq 0$.

(5) *Any $g : Y_1 \to Y_2$ induces a morphism*

$$g^i_* : \mathrm{Ext}^i(X,Y_1) \longrightarrow \mathrm{Ext}^i(X,Y_2)$$

for each $i \geq 0$.

(6) *Any short exact sequence $0 \to Y_1 \xrightarrow{\phi} Y_2 \xrightarrow{\psi} Y_3 \to 0$ induces a long exact sequence:*

$$0 \to \mathrm{Ext}^0(X,Y_1) \to \mathrm{Ext}^0(X,Y_2) \to \mathrm{Ext}^0(X,Y_3) \to \mathrm{Ext}^1(X,Y_1) \to \mathrm{Ext}^2(X,Y_2) \to \cdots$$

(7) *Any short exact sequence $0 \to X_1 \xrightarrow{\phi} X_2 \xrightarrow{\psi} X_3 \to 0$ induces a long exact sequence:*

$$0 \to \mathrm{Ext}^0(X_3,Y) \to \mathrm{Ext}^0(X_2,Y) \to \mathrm{Ext}^0(X_1,Y) \to \mathrm{Ext}^1(X_3,Y) \to \mathrm{Ext}^2(X_2,Y) \to \cdots$$

Proof. (1), (2) and (3) all follow from general properties of derived functors (**??**). For (4) Let $P_1^\bullet$ be a projective resolution of X_1 and $P_2^\bullet$ be a projective resolution of X_2. General properties about resolutions implies that f lifts to a chain map $\varphi^\bullet : P_1^\bullet \longrightarrow P_2^\bullet$. Then, $\varphi^\bullet$ induces a morphism of chain complexes $\mathrm{Hom}(P_2^\bullet, Y) \longrightarrow \mathrm{Hom}(P_1^\bullet, Y)$ which, in turn, induces a morphism:

$$f^{*,i} : \mathrm{Ext}^i(X_2,Y) \longrightarrow \mathrm{Ext}^i(X_1,Y)$$

for each $i \geq 0$. For (5), let $P^\bullet$ be a projective resolution of X. Then, there is a morphism of chain complexes $\beta^\bullet : \mathrm{Hom}(P^\bullet, Y_1) \longrightarrow \mathrm{Hom}(P^\bullet, Y_2)$ induced by g, which, in turn, induces a morphism:

$$g^i_* : \mathrm{Ext}^i(X,Y_1) \longrightarrow \mathrm{Ext}^i(X,Y_2)$$

for each $i \geq 0$. For (6), let $P^\bullet$ be a projective resolution of X. Then there is an induced short exact sequence of chain complexes:

$$0 \to \mathrm{Hom}(P^\bullet, Y_1) \to \mathrm{Hom}(P^\bullet, Y_2) \to \mathrm{Hom}(P^\bullet, Y_3) \to 0$$

because each module P^i is projective. Indeed, at each degree i, P^i this sequence is

$$0 \to \mathrm{Hom}(P^i, Y_1) \to \mathrm{Hom}(P^i, Y_2) \to \mathrm{Hom}(P^i, Y_3) \to 0$$

obtained by applying the functor $\mathrm{Hom}(P^i, -)$, which is exact as P^i is projective. It is then easily checked that this gives a short exact sequence of chain complexes. Thus, applying the long exact sequence in homology produces the required long exact sequence. For (7), Let $P^\bullet$ be a projective resolution of X_1 and let $Q^\bullet$ be a projective resolution of X_3. By the horseshoe lemma (**??**), there exists a projective resolution $R^\bullet$ of X_2 and a short exact sequence of chain complexes

$$0 \to P^\bullet \to R^\bullet \to Q^\bullet \to 0,$$

Since Q^i is projective, applying $\mathrm{Hom}(-, Y)$ yields

$$0 \to \mathrm{Hom}(Q^i, Y) \to \mathrm{Hom}(R^i, Y) \to \mathrm{Hom}(P^i, Y) \to 0$$

for each $i \geq 0$. It follows that there is a s.e.s. of cochain complexes

$$0 \to \mathrm{Hom}(Q^\bullet, Y) \to \mathrm{Hom}(R^\bullet, Y) \to \mathrm{Hom}(P^\bullet, Y) \to 0.$$

The associated long exact sequence in cohomology is the required long exact sequence. □

The above proposition show that the Ext groups 'measure' and 'repair' the non-exactness of the functors $\mathrm{Hom}(-, Y)$ and $\mathrm{Hom}(X, -)$. Let us now specialize to $R = \mathbb{Z}$. In what follows, let G be a fixed abelian group.

Lemma 3.9. *For any abelian group A, we have that*

$$\mathrm{Ext}^n(A, G) = 0 \textit{ if } n > 1,$$

Proof. Any abelian group, A, admits a two-step free resolution.

$$0 \to F_1 \to F_0 \to A \to 0$$

Thus, $\mathrm{Ext}^n(A, G) = 0$ if $n > 1$. □

Remark 3.10. *Only* $\mathrm{Ext}^1(A, G)$ *encodes interesting information for abelian groups. We write* $\mathrm{Ext}(A, G) := \mathrm{Ext}^1(A, G)$.

Proposition 3.11. *The Ext functor satisfies the following properties:*

(1) $\mathrm{Ext}\left(\bigoplus_i A_i, G\right) \cong \prod_i \mathrm{Ext}(A_i, G)$.
(2) *If A is free, then* $\mathrm{Ext}(A, G) = 0$.
(3) $\mathrm{Ext}(\mathbb{Z}/n\mathbb{Z}, G) = G/nG$.
(4) *If H is a finitely generated abelian group, then:*

$$\mathrm{Ext}(H, G) = \mathrm{Ext}(\mathrm{Torsion}(H), G) = \mathit{Torsion}(H) \otimes_{\mathbb{Z}} G$$

(5) *For a short exact sequence: $0 \to A \to A' \to A'' \to 0$ of abelian groups, there is a natural exact sequence:*

$$0 \to \mathrm{Hom}(A'', G) \to \mathrm{Hom}(A', G) \to \mathrm{Hom}(A, G) \to \mathrm{Ext}(A'', G) \to \mathrm{Ext}(A', G) \to \mathrm{Ext}(A, G) \to 0$$

Proof. The proof is given below:

(1) This follows from the identity,

$$\mathrm{Hom}\left(\bigoplus_i A_i, G\right) = \prod_i \mathrm{Hom}(A_i, G),$$

and noting that taking direct sums of projective resolutions of A_i forms a projective resolution for $\bigoplus_i A^i$, and that homology commutes with arbitrary direct products.
(2) If A is free, then

$$0 \to A \to A \to 0$$

is a projective resolution of A, so $\mathrm{Ext}(A, G) = 0$.

(3) Consider the projective resolution of $\mathbb{Z}/n\mathbb{Z}$ given by

$$0 \to \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$$

dualize it and use the fact that $\mathrm{Hom}(\mathbb{Z}, G) \cong G$ to conclude that $\mathrm{Ext}(\mathbb{Z}/n\mathbb{Z}, G) = G/nG$.

(4) This follows at once from the previous statement.

(5) This follows from Proposition 3.8.

This completes the proof. □

Remark 3.12. *The discussion above implies has dealt with the case of $\mathbb{Z}$-modules (abelian groups). The general case can be more involved. For instance, consider $\mathbb{Z}_2$ as a $\mathbb{Z}_4$-module. Let $\mathbb{Z}_4 \xrightarrow{q} \mathbb{Z}_2$ denote the quotient map. Let $\mathbb{Z}_4 \xrightarrow{\times 2} \mathbb{Z}_4$ denote multiplication by 2. $\mathbb{Z}_2$ has the following free resolution over $\mathbb{Z}_4$:*

$$\cdots \xrightarrow{\times 2} \mathbb{Z}_4 \xrightarrow{\times 2} \mathbb{Z}_4 \xrightarrow{\times 2} \mathbb{Z}_4 \xrightarrow{\times 2} \mathbb{Z}_4 \xrightarrow{q} \mathbb{Z}_2 \to 0.$$

Since $\mathrm{Hom}_{\mathbb{Z}_4}(\mathbb{Z}_4, \mathbb{Z}_2) \cong \mathbb{Z}_2$ (by mapping the generator of $\mathbb{Z}_4$ to either 0 or 1), the dual of $\times 2 : \mathbb{Z}_4 \to \mathbb{Z}_4$ is simply the zero map. Hence, we have the dual sequence

$$0 \to \mathbb{Z}_2 \to \mathbb{Z}_2 \xrightarrow{0} \mathbb{Z}_2 \xrightarrow{0} \mathbb{Z}_2 \xrightarrow{0} \mathbb{Z}_2 \to \cdots$$

Consider the truncated sequence

$$\mathbb{Z}_2 \xrightarrow{0} \mathbb{Z}_2 \xrightarrow{0} \mathbb{Z}_2 \xrightarrow{0} \mathbb{Z}_2 \to \cdots$$

The homology of this complex is $\mathbb{Z}_2$ for every degree. Hence,

$$\mathrm{Ext}^n_{\mathbb{Z}_4}(\mathbb{Z}_2, \mathbb{Z}_2) \cong \mathbb{Z}_2$$

is nonzero for all $n \in \mathbb{N}$. This is stark contrast Remark 3.10.

Remark 3.13. *The name Ext comes from the phrase extension. We say X is an extension of A by B if*

$$0 \to B \to X \to A \to 0$$

is exact. Given A and B, there is always the trivial extension $X = A \oplus B$, corresponding to the isomorphism class of the split exact sequence. It can be shown that isomorphism classes of extensions of A by B are in 1-1 correspondence with elements of $\mathrm{Ext}^1(A, B)$, with the trivial extension corresponding to 0.

4. Universal Coefficient Theorem

Recall the construction of singular cohomology in Section 2. Since everything is determined in terms of $(C_\bullet, \partial_\bullet)$, can we compute cohomology groups using information about homology groups? The answer is a qualified yes. This is the universal coefficient theorem (UCT) for cohomology, which we now discuss. We first motivate the statement of UCT. As a first guess, we might think that

$$H^n(C_\bullet; G) := H_n(C^\bullet; G) \cong \mathrm{Hom}(H_n(C_\bullet), G)$$

This turns out to be almost true. We indeed have a natural map:

$$\varphi : H^n(C_\bullet, G) \longrightarrow \mathrm{Hom}(H_n(C_\bullet), G).$$

Denote $Z_n = \ker \partial_n \subseteq C_n$ and $B_n = \mathrm{Im}\, \partial_{n+1} \subseteq C_n$. We have $B_n \subseteq Z_n$. A class in $H^n(C^\bullet; G)$ is represented by a homomorphism $\phi : C_n \to G$ such that $\partial^*_{n+1}\phi = 0$. That is, $\phi\partial_{n+1} = 0$, or in words, ϕ vanishes on B_n. The restriction $\phi_0 = \phi|Z_n$ then induces a quotient homomorphism

$$\bar{\phi}_0 : Z_n/B_n \to G,$$

an element of $\mathrm{Hom}(H_n(C_\bullet), G)$. If ϕ is in $\mathrm{Im}\,\partial_n^*$, say $\phi = \psi\partial_n$ for some $\psi \in C_{n-1}^*$, then ϕ is zero on Z_n since $\partial_n \circ \partial_{n+1} = 0$. So $\phi_0 = 0$ and hence also $\bar{\phi}_0 = 0$.

$$\begin{tikzcd}
\cdots \arrow[r, "\partial_{n+2}"] & C_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[dr, dashed, "\phi\circ\partial_{n+1}"'] & C_n \arrow[r, "\partial_n"] \arrow[d, "\phi"] & C_{n-1} \arrow[r, "\partial_{n-1}"] \arrow[dl, dashed, "\psi"] & \cdots \\
 & & G & &
\end{tikzcd}$$

Thus, there is a well-defined quotient map

$$h : H^n(C_\bullet, G) \to \mathrm{Hom}(H_n(C), G)$$

sending the cohomology class of $[\phi]$ to $\bar{\phi}_0$. Obviously h is a homomorphism.

Proposition 4.1. ***(Universal Coefficient Theorem)*** *If a chain complex $(C_\bullet, \partial_\bullet)$ of free abelian groups has homology groups $H_n(C_\bullet)$, then the cohomology groups $H^n(C_\bullet; G)$ of the cochain complex* $\mathrm{Hom}(C_n, G)$ *are determined by the short exact sequence:*

$$0 \longrightarrow \mathrm{Ext}(H_{n-1}(C_\bullet), G) \longrightarrow H^n(C_\bullet; G) \xrightarrow{h} \mathrm{Hom}(H_n(C_\bullet), G) \longrightarrow 0$$

Proof. We first show that h is surjective. Consider the short exact sequence:

$$(*) \qquad \begin{tikzcd}
0 \arrow[r] & Z_n \arrow[r, hook, "i"'] & C_n \arrow[l, bend right, dashed, "p"'] \arrow[r, "\partial_n"] & B_{n-1} \arrow[r] & 0
\end{tikzcd}$$

Since B_{n-1} is a free abelian group, the short exact sequence splits. Thus there is a homomorphism $p : C_n \to Z_n$ that restricts to the identity on Z_n. That is $p \circ i = \mathrm{Id}_{Z_n}$. Composing with p gives a way of extending homomorphisms

$$\phi_0 : Z_n \to G$$

to homomorphisms

$$\phi = \phi_0 \circ p : C_n \to G$$

In particular, this extends homomorphisms $Z_n \to G$ that vanish on B_n to homomorphisms $C_n \to G$ that still vanish on B_n, or in other words, it extends homomorphisms $H_n(C_\bullet) \to G$ to elements of $\ker\ \partial_{n+1}^*$. Thus we have a homomorphism

$$\mathrm{Hom}(H_n(C_\bullet), G) \to \ker\ \partial_{n+1}^*$$

Composing this with the quotient map $\ker\ \partial_{n+1}^* \to H^n(C_\bullet; G)$ gives a homomorphism from

$$\mathrm{Hom}(H_n(C_\bullet), G) \to H^n(C_\bullet; G)$$

If we follow this map by h we get the identity map on $\mathrm{Hom}(H_n(C_\bullet), G)$ since the effect of composing with h is simply to undo the effect of extending homomorphisms via p. This shows that h is surjective. We now analyze kernel of h. We have:

$$
\begin{tikzcd}
 & \vdots & \vdots & \vdots & \\
0 & Z_{n+1}^* \arrow[l] \arrow[u] & C_{n+1}^* \arrow[l] \arrow[u] & B_n^* \arrow[l] \arrow[u] & 0 \arrow[l] \\
0 & Z_n^* \arrow[l] \arrow[u, "0"] & C_n^* \arrow[l] \arrow[u, "\partial_{n+1}^*"] & B_{n-1}^* \arrow[l] \arrow[u, "0"] & 0 \arrow[l] \\
0 & Z_{n-1}^* \arrow[l] \arrow[u, "0"] & C_{n-1}^* \arrow[l] \arrow[u, "\partial_n^*"] & B_{n-2}^* \arrow[l] \arrow[u, "0"] & 0, \arrow[l] \\
 & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] &
\end{tikzcd}
$$

The above complex follows by applying the $\mathrm{Hom}(-, G)$ functor to the split short exact sequences in $(*)$. Since each such short exact sequence is split, the resulting chain complex has rows consisting of split short exact sequences. The associated long exact sequence of homology groups has the form:

$$\cdots \longleftarrow B_n^* \longleftarrow Z_n^* \xleftarrow{i_n^*} H^n(C_\bullet, G) \longleftarrow B_{n-1}^* \longleftarrow Z_{n-1}^* \xleftarrow{i_{n-1}^*} \cdots$$

The connecting morphism $i_n^* : Z_n^* \to B_n^*$ are in fact the dual of the inclusion map $i_n : B_n \to Z_n$[1]. One takes an element of Z_n^*, pulls it back to C_n^*, applies ∂_{n+1}^* to get an element of C_{n+1}^*, then pulls this back to B_n^*. The first of these steps extends a homomorphism $\phi_0 : Z_n \to G$ to $\phi : C_n \to G$, the second step composes this ϕ with ∂_{n+1} to yield a map $C_{n+1} \to G$ and the third step undoes this composition and restricts ϕ to B_n[2]. The net effect is just to restrict ϕ_0 from Z_n to B_n. A long exact sequence can always be broken up into short exact sequences, yielding:

$$0 \longrightarrow \text{coker } i_{n-1}^* \longrightarrow H^n(C_\bullet; G) \longrightarrow \ker i_{n-1}^* \longrightarrow 0$$

ker i_{n-1}^* can be identified with $\mathrm{Hom}(H_n(C_\bullet), G)$. This is because elements of ker i_n^* are homomorphisms $Z_n \to G$ that vanish on the subgroup B_n, and such homomorphisms are the same as homomorphisms $Z_n/B_n \to G$. Under this identification of ker i_n^* with $\mathrm{Hom}(H_n(C_\bullet), G)$, the map $H_n(C_\bullet; G) \to$ ker i_n^* becomes the map h considered earlier. Hence:

$$0 \longrightarrow \text{coker } i_{n-1}^* \longrightarrow H^n(C_\bullet; G) \xrightarrow{h} \mathrm{Hom}(H_n(C_\bullet), G) \longrightarrow 0$$

What about coker i_{n-1}^*? Consider the short exact sequence:

$$0 \longrightarrow B_{n-1} \xrightarrow{i_{n-1}} Z_{n-1} \longrightarrow H_{n-1}(C_\bullet) \longrightarrow 0$$

The exact sequence above is a free resolution of $H_{n-1}(C_\bullet)$. There is an associated natural exact sequence:

$$0 \to H_{n-1}^*(C_\bullet) \to Z_{n-1}^* \to B_{n-1}^* \to \mathrm{Ext}(H_{n-1}(C_\bullet), G) \to \mathrm{Ext}(Z_{n-1}, G) \to \mathrm{Ext}(B_{n-1}, G) \to 0$$

[1]Hence the use of the suggestive notation.

[2]Go back to the short exact sequence analyzed to prove surjectivity to see why the third step has this effect.

We see that coker i^*_{n-1} is the first cohomology group of the free resolution of $H_{n-1}(C_\bullet)$. This group is $\mathrm{Ext}(H_{n-1}(C_\bullet, G), G)$, proving the claim. $\square$

Remark 4.2. *We have*

$$H^n(X;G) = \mathrm{Hom}_{\mathbb{Z}}(H_n(X;\mathbb{Z}),G) \oplus \mathrm{Ext}(H_{n-1}(X;\mathbb{Z}),G).$$

This is because in the course of the proof of Proposition 4.1, we constructed a morphism

$$\mathrm{Hom}(H_n(C_\bullet),G) \to H^n(C_\bullet;G)$$

that ensures that the sequence splits.

Corollary 4.3. *Let $(C_\bullet, \partial_\bullet)$ be a chain complex so that its $\mathbb{Z}$-homology groups are finitely generated. Let $T_n = Torsion(H_n)$. We have*

$$0 \to T_{n-1} \to H^n(C_\bullet;\mathbb{Z}) \to H_n/T_n \to 0$$

This sequence splits[3]*, so:*

$$H^n(C^\bullet;\mathbb{Z}) \cong T_{n-1} \oplus H_n/T_n.$$

Proof. Clear. $\square$

Let us now derive some immediate consequences of the UCT:

(1) If $n = 0$, we have
$$H^0(X;G) = \mathrm{Hom}_{\mathbb{Z}}(H_0(X),G) \cong \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^{\#\text{path components}},G)$$
(2) If $n = 1$, the Ext-term vanishes since $H_0(X)$ is free, so we get:
$$H^1(X;G) = \mathrm{Hom}_{\mathbb{Z}}(H_1(X),G)$$

Remark 4.4. *There is also a universal coefficient theorem for cohomology where $\mathbb{Z}$ is replaced by a PID, R and G is a R-module. In this case, we have*

$$0 \to \mathrm{Ext}^1_R(H_{n-1}(X;R),G) \to H^n(X;G) \xrightarrow{h} \mathrm{Hom}_R(H_n(X;R),G) \to 0.$$

This comes from first establishing that Ext^n_R vanishes for $n \geq 2$ for when R is a PID, and then going through a proof for universal coefficient theorem as above.

5. Eilenberg-Steenrod Axioms

We have defined singular cohomology. There are many other cohomology theories: sheaf cohomology, Cech cohomology, etc. All these cohomology theories satisfy the Eilenberg-Steenrod axioms. The purpose of this section is to state these axioms and prove that singular cohomology satisfies these axioms.

Definition 5.1. (**Eilenberg-Steenrod Axioms**) Let G be an abelian group. A (unreduced) cohomology theory consists of

(1) A family of functors $H^n : \mathbf{Top}^2 \to \mathbf{Ab}$ for $n \geq 0$, and
(2) A family of natural transformations $\gamma^n : H^n \to H^{n+1} \circ p$, where p is the functor sending (X,A) to $(A,\varnothing)$ and $f : (X,A) \to (Y,B)$ to $f|_B : (A,\varnothing) \to (B,\varnothing)$.

such that the following axioms are satisfied:

[3] Since H_n/T_n is free and hence projective.

(a) (Homotopy invariance) If $f, g : (X, A) \to (Y, B)$ are homotopic maps, then the induced maps
$$H^n(f), H^n(g) : H^n(X, A) \to H^n(Y, B)$$
are such that $H^n(f) = H^n(g)$ for $n \geq 0$. In other words, H^n may be regarded as a functor from **hTop** to **Ab**.

(b) (Long exact sequence) For every pair (X, A), the inclusions
$$(A, \emptyset) \overset{i}{\hookrightarrow} (X, \emptyset) \overset{j}{\hookrightarrow} (X, A)$$
give rise to a long exact sequence
$$\cdots \longrightarrow H^n(X, A) \xrightarrow{j_n^*} H^n(X) \xrightarrow{i_n^*} H^n(A) \xrightarrow{\gamma^n} H^{n+1}(X, A) \xrightarrow{j_{n+1}^*} H^{n+1}(X) \xrightarrow{i_{n+1}^*} H^{n+1}(A) \longrightarrow \cdots$$

(c) (Excision) If $Z \subseteq A \subseteq X$ are topological spaces such that $\overline{Z} \subseteq \mathrm{Int}(A)$, the inclusion of pairs $(X \setminus Z, A \setminus Z) \subseteq (X, A)$ induces isomorphisms
$$H^n(X \setminus Z, A \setminus Z) \to H^n(X, A)$$
for all $n \geq 0$.

(d) (Multiplicativity) If $X = \coprod_\alpha X_\alpha$ and $A = \coprod_\alpha A_\alpha$ is the disjoint union of a family of topological spaces X_α, then
$$H^n(X, A) = \prod_\alpha H^n(X_\alpha, A_\alpha)$$
for each $n \geq 0$.

Additionally, if a cohomology theory satisfies the following additional axiom

(e) (Dimension Axiom) For any one-point set $X = \{\bullet\}$,
$$H^n(X) = \begin{cases} G & \text{if } n = 0 \\ 0 & \text{otherwise,} \end{cases}$$
the the cohomology theory is called an ordinary cohomology theory.

The purpose of the remainder of this section is to show that singular cohomology satisfies the Eilenberg-Steenrod axioms.

5.1. **Relative cohomology groups.** We first contruct the relative cohomology group that shall allow us to construct the appropriate functors from **Top** to **Ab**. We apply the $\mathrm{Hom}(-, G)$ functor to the the relative singular chain complex to get
$$C^n(X, A; G) := \mathrm{Hom}(C_n(X, A), G).$$
The group $C^n(X, A; G)$ can be identified with functions from the set of n-simplices in X to G that vanish on simplices in A, so we have a natural inclusion
$$C^n(X, A; G) \hookrightarrow C^n(X; G)$$
The relative coboundary maps
$$\overline{\delta}^n : C^n(X, A; G) \to C^{n+1}(X, A; G)$$
are obtained by restricting δ^n. We have a co-chain complex $(C^\bullet(X, A), \overline{\delta}^\bullet)$.

Definition 5.2. Let $A \subseteq X$ be a subspace of a topological space X. The n-th relative cohomology group, $H^n(X, A)$, is the n-th homology group of the chain complex $(C^\bullet(X, A), \overline{\delta}^\bullet)$. That is:

$$H^n(X, A) = \frac{\text{Ker } \overline{\delta}^n}{\text{Im } \overline{\delta}^{n+1}}$$

Similar to Proposition 2.4, it is easily checked that each H^n is a functor from $\mathbf{Top}^2$ to $\mathbf{Ab}$. This effectively checkes the first two conditons in the definition of the Eilenberg-Steenrod axioms.

Remark 5.3. *Since the cohomology of the empty set is trivial for all $n \geq 0$, we have:*

$$H^n(X, \emptyset) = H^n(X), \quad \forall n \geq 0.$$

Remark 5.4. *Universal coefficient theorem continues to hold true for relative cohomology. The proof is identical as the one given before.*

We now prove that singular cohomology satisfies the long exact sequence axiom. The importance of the long exact sequence axiom is that is allows us to compute cohomology groups of various spaces in using an ‘inductive’ and/or ‘bottom-up’ approach. Applying by the $\text{Hom}(-, G)$ functor to the short exact sequence,

$$0 \to C_n(A) \xrightarrow{i_n} C_n(X) \xrightarrow{j_n} C_n(X, A) \to 0,$$

we get another short exact sequence[4]

$$0 \leftarrow C^n(A; G) \xleftarrow{i^*} C^n(X; G) \xleftarrow{j^*} C^n(X, A; G) \leftarrow 0.$$

Since i_n and j_n commute with the boundary maps, it follows that i_n^* and j_n^* commute with co-boundary maps. So we obtain a short exact sequence of cochain complexes:

$$0 \leftarrow C^\bullet(A; G) \xleftarrow{i^*} C^\bullet(X; G) \xleftarrow{j^*} C^\bullet(X, A; G) \leftarrow 0.$$

By taking the associated long exact sequence of homology groups, we get the long exact sequence for the cohomology groups of the pair (X, A):

$$\cdots \longrightarrow H^n(X, A; G) \xrightarrow{j_n^*} H^n(X; G) \xrightarrow{i_n^*} H^n(A; G) \xrightarrow{\gamma^n} H^{n+1}(X, A; G) \xrightarrow{j_{n+1}^*} H^{n+1}(X; G) \xrightarrow{i_{n+1}^*} \cdots$$

This shows that the long exact sequence axiom is satified.

5.2. Homotopy Invariance.

We now show that singular cohomology satisfies the homotopy invariance property.

Proposition 5.5. *(**Homotopy Invariance**) Let X, Y be topological spaces, and let G be an abelian group. If $f \simeq g : X \to Y$ are homotopic maps, then*

$$H^n(f) = H^n(g) : H^n(Y, G) \to H^n(X, G).$$

Proof. Recall from the proof of the similar statement for homology that a chain homotopy between $C_\bullet(X, A; G)$ and $C_\bullet(Y, B; G)$ is given by a prism operator

$$T_n : C_n(X, A; G) \to C_{n+1}(Y, B; G)$$

satisfying

$$f_n - g_n = T_{n-1} \circ \partial_n + \partial'_{n+1} \circ T_n$$

[4] $\text{Hom}(-, G)$ is only a left exact functor in general. But it can be checked in this case that the resulting sequence is both left exact and right exact.

with f_n and g_n being the induced maps on singular chain complexes. The claim about cohomology follows by applying the Hom$(-.G)$ functor to the prism operator to get

$$T^n : C^{n+1}(Y, B; G) \to C^n(X, A; G)$$

which satisfies

$$f^n - g^n = \partial_n^* \circ T^{n-1} + T^n \circ \partial_{n+1}^{*'}.$$

Hence, we have a chain homotopy between $C^\bullet(X, A; G)$ and $C^\bullet(Y, B; G)$. It is now a standard fact that a chain homotopy induces the same maps on homology groups. Hence,

$$H^n(f) = H^n(g)$$

for each $n \geq 0$. □

Corollary 5.6. *If X is contractible, then $H^n(X) = 0$ for all $n \geq 1$.*

Proof. Immediate from the homotopy invariance of singular cohomology and that $H^n(\{*\}) = 0$ for $n \geq 1$. □

5.3. **Excision.** We now prove that singular cohomology satisfies the excision axiom. The important of the excision axiom is that if $A \subseteq X$ and n-cochains are "sufficiently inside" of A, we can cut A out without affecting the relative cohomology groups $H^n(X, A)$. Here is the formal statement we'd like to prove in this section:

Proposition 5.7. *(**Excision**) Given a topological space X, suppose that $Z \subset A \subset X$, with $\overline{Z} \subseteq int(A)$. Then the inclusion of pairs $i : (X \setminus Z, A \setminus Z) \hookrightarrow (X, A)$ induces isomorphisms:*

$$i^n : H^n(X, A; G) \to H^n(X \setminus Z, A \setminus Z; G)$$

for all n. Equivalently, if A and B are subsets of X with $X = int(A) \cup int(B)$, then the inclusion map $(B, A \cap B) \hookrightarrow (X, A)$ induces isomorphisms in cohomology.

Proof. Excision for singular homology implies that left and right maps in the diagram below are isomorphisms.

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ext}(H_{n-1}(X, A), G) \arrow[r] & H^n(X, A; G) \arrow[r] & \mathrm{Hom}(H_n(X, A), G) \arrow[r] & 0 \\
0 \arrow[r] & \mathrm{Ext}(H_{n-1}(X \setminus Z, A \setminus Z), G) \arrow[r] \arrow[u] & H^n(X \setminus Z, A \setminus Z; G) \arrow[r] \arrow[u] & \mathrm{Hom}(H_n(X \setminus Z, A \setminus Z), G) \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

The five-lemma then implies that the middle map is also an isomorphism. This completes the proof. □

5.4. **Dimension Axiom.** Let $X = \{*\}$ be a single point space. By Proposition 4.1, we have:

$$H^n(X; G) = \mathrm{Hom}(H_n(X), G) \oplus \mathrm{Ext}(H_{n-1}(X), G).$$

Since

$$H_n(X) \cong \begin{cases} \mathbb{Z}, & \text{if } n = 0 \\ 0, & \text{otherwise} \end{cases}$$

we get

$$\mathrm{Hom}(H_n(X), G) \cong \begin{cases} G, & \text{if } n = 0 \\ 0, & \text{otherwise} \end{cases}.$$

Furthermore, since $H_n(X)$ is free for all n, we also have that $\operatorname{Ext}(H_{n-1}(X), G) = 0,$ for all n. Therefore,

$$H^n(X;G) = \begin{cases} G, & \text{if } n = 0 \\ 0, & \text{otherwise} \end{cases}.$$

5.5. **Multiplicativity Axiom.** The multiplicativityaxiom is easily seen to hold using the universal coefficient theorem in relative cohomology. Let $X = \coprod_\alpha X_\alpha$ and $A = \coprod_\alpha A_\alpha$. We have:

$$\begin{aligned} H^n(X, A; G) &= \operatorname{Ext}(H_{n-1}(X, A); G) \oplus \operatorname{Hom}(H_n(X, A); G) \\ &= \operatorname{Ext}(H_{n-1}(\coprod_\alpha X_\alpha, \coprod_\alpha A_\alpha); G) \oplus \operatorname{Hom}(H_n(\coprod_\alpha X_\alpha, \coprod_\alpha A_\alpha); G) \\ &= \operatorname{Ext}(\bigoplus_\alpha H_{n-1}(X_\alpha, A_\alpha); G) \oplus \operatorname{Hom}(\bigoplus_\alpha H_n(X_\alpha, A_\alpha); G) \\ &= \prod_\alpha \operatorname{Ext}(H_{n-1}(X_\alpha, A_\alpha); G) \oplus \prod_\alpha \operatorname{Hom}(H_n(X_\alpha, A_\alpha); G) \\ &= \prod_\alpha \operatorname{Ext}(H_{n-1}(X_\alpha, A_\alpha); G) \oplus \operatorname{Hom}(H_n(X_\alpha, A_\alpha); G) = \prod_\alpha H^n(X_\alpha, A_\alpha; G) \end{aligned}$$

Hence, we see that singular cohomology satisfies the Eilenberg-Steenrod axioms.

Remark 5.8. *The Mayer-Viertoris sequence is a formal consequence of the Eilenberg-Steenrod axioms. Therefore, we have that Mayer-Viertoris holds for singular cohomology: if X be a topological space, and A and B are open subsets of X such that $X = int(A) \cup int(B)$, then there is a long exact sequence of cohomology groups:*

$$\cdots \to H^n(X;G) \xrightarrow{\psi} H^n(A;G) \oplus H^n(B;G) \xrightarrow{\phi} H^n(A \cap B; G) \to \cdots$$

We also have a Mayer-Vietoris sequence in relative cohomology groups.

Remark 5.9. *We can also define reduced cohomology. Consider the augmented singular chain complex for X:*

$$\cdots \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

where $\varepsilon(\sum_i n_i x_i) = \sum_i n_i$. After applying the $\operatorname{Hom}(-, G)$ *functor, we get the augmented co-chain complex:*

$$\xleftarrow{\partial_3^*} C_2^* \xleftarrow{\partial_2^*} C_1^* \xleftarrow{\partial_1^*} C_0^* \xleftarrow{\varepsilon^*} \mathbb{Z} \leftarrow 0$$

Note that since $\epsilon \circ \partial_1 = 0$, we get by by applying the $\operatorname{Hom}(-, G)$ *functor that $\partial_1^* \circ \epsilon^* = 0$. The cohomology of this augmented cochain complex is the* reduced cohomology *of X with G-coefficients, denoted by $\widetilde{H}^n(X;G)$. It follows by definition that*

$$\widetilde{H}^n(X;G) = H^n(X;G) \quad n > 0$$

and by the universal coefficient theorem (applied to the augmented chain complex), we get

$$\widetilde{H}^0(X;G) = \operatorname{Hom}(\widetilde{H}^0(X), G).$$

Remark 5.10. *If (X, A) is a good pair, then the long exact sequence in reduced cohomology holds true. This is because the analogous result is a formal consequence of the Eilenberg-Steenrod axioms.*

$$\cdots \to H^n(X, A; G) \to \widetilde{H}^n(X; G) \to \widetilde{H}^n(A; G) \to H^{n+1}(X, A; G) \to \cdots$$

In particular, if $A = \{*\}$ is a point in X, we get that

$$\widetilde{H}^n(X; G) \cong H^n(X, x_0; G)$$

for $n \geq 1$. Moreover, we have

$$H^n(X, A; G) \cong H^n(X/A; G)$$

for all $n \in \mathbb{N}$. The proof is the same as in the homology case since it is a formal consequence of the Eilenberg-Steenrod axioms and the hypothesis on the space. Also, if each X_α is path-connected, we have

$$\widetilde{H}^n(\bigvee_\alpha X_\alpha) = \prod_\alpha \widetilde{H}^n(X_\alpha)$$

for $n \geq 0$. Once again, the proof is similar to the proof in the case of singular homology. We also have a Mayer-Vietoris sequence in reduced cohomology.

Remark 5.11. *We can define simplicial cohomology and cellular cohomology in exactly the same way as expected. As expexted, simplicial cohomology and cellular cohomology are isomorphic to singular cohomology.*

6. Examples

The purpose of this section is to compute cohomology groups of some topological spaces. We begin by looking at some specific examples.

Example 6.1. (**Contractible Spaces**) Let X be a contractible topological space. We have:

$$H^n(X; G) = \begin{cases} G, & \text{if } n = 0 \\ 0, & \text{otherwise} \end{cases}.$$

This follows immediately by the homotopy invariance of cohomology groups since X homotopy equivalent to a point.

Example 6.2. (**Spheres**) Let $X = \mathbb{S}^n$. Then we have

$$H_k(\mathbb{S}^n\ \mathbb{Z}) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z}, & \text{if } k = n = 0 \\ \mathbb{Z}, & \text{if } k = n > 0,\ k = 0, n > 0 \\ 0, & \text{otherwise} \end{cases}$$

Since, $\mathbb{Z}$ and $\mathbb{Z} \oplus \mathbb{Z}$ are free-abelian groups, the Ext term in the UCT for cohomology vanishes for each k. Hence,

$$H^k(\mathbb{S}^n, G) \cong \mathrm{Hom}(H_k(\mathbb{S}^n, G), \mathbb{Z}) \cong \begin{cases} G \oplus G, & \text{if } k = n = 0 \\ G, & \text{if } k = n > 0,\ k = 0, n > 0 \\ 0, & \text{otherwise} \end{cases}.$$

for each $k \geq 0$.

Remark 6.3. *We can also compute the cohomology groups of* $\mathbb{S}^n$ *by using the above Mayer-Vietoris sequence. Cover* $\mathbb{S}^n$ *by two open sets* $A = \mathbb{S}^n \setminus \{N\}$ *and* $B = \mathbb{S}^n \setminus \{S\}$*, where* N *and* S *are the North and South poles of* S^n*. Then we have*

$$A \cap B \simeq \mathbb{S}^{n-1} \qquad A \simeq B \simeq \mathbb{R}^n$$

Thus, by the Mayer-Vietoris sequence for reduced cohomology, homotopy invariance, and induction, we get:

$$\widetilde{H}^k(\mathbb{S}^n; G) \cong \widetilde{H}_{k-n}(\mathbb{S}^0; G) \cong \begin{cases} G, & \textit{if } k = n \\ 0, & \textit{otherwise} \end{cases}$$

for each $k \geq 0$*.*

Example 6.4. (**Mobius Band**) Let M denote the Mobius band. Since M is homotopic to $\mathbb{S}^1$. we have,

$$H^k(M, G) \cong \begin{cases} G, & \text{if } k = 0, 1 \\ 0, & \text{otherwise} \end{cases}.$$

for each $k \geq 0$.

Example 6.5. (**Torus**) Let $X = \mathbb{S}^1 \times \mathbb{S}^1$. Recall that we have,

$$H_n(\mathbb{S}^1 \times \mathbb{S}^1, \mathbb{Z}) \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & \text{for } n = 1 \\ \mathbb{Z} & \text{for } n = 0, 2 \\ 0 & \text{for } n \geq 3 \end{cases}$$

Since, $\mathbb{Z}$ and $\mathbb{Z} \oplus \mathbb{Z}$ are free-abelian groups, the Ext term in the UCT for cohomology vanishes for each k. Hence,

$$H^n(\mathbb{S}^1 \times \mathbb{S}^1, G) \cong \mathrm{Hom}_{\mathbb{Z}}(H_n(\mathbb{S}^1 \times \mathbb{S}^1, \mathbb{Z}), G) \cong \begin{cases} G \oplus G & \text{for } n = 1 \\ G & \text{for } n = 0, 2 \\ 0 & \text{for } n \geq 3 \end{cases}$$

Example 6.6. (**Klein Bottle**) Let $X = K$ be the Klein bottle. Recall that we have,

$$H_n(K\ \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{for } n = 0 \\ \mathbb{Z} \oplus \mathbb{Z}_2 & \text{for } n = 1 \\ 0 & \text{for } n \geq 2 \end{cases}$$

Note that we have,

$$\begin{aligned} &\mathrm{Ext}(H_0(K, \mathbb{Z}), G) = 0, \\ &\mathrm{Ext}(H_1(K, \mathbb{Z}), G) \cong \mathrm{Ext}(\mathbb{Z}_2, G) \cong G/2G \end{aligned}$$

Therefore, we have

$$\begin{aligned} H^n(K, G) &\cong \mathrm{Hom}_{\mathbb{Z}}\left(H_n(K, \mathbb{Z}), G\right) \oplus \mathrm{Ext}\left(H_{n-1}(K, \mathbb{Z}), G\right) \\ &\cong \begin{cases} G, & \text{for } n = 0, \\ G \oplus G/2G, & \text{for } n = 1, \\ G/2G, & \text{for } n = 2, \\ 0, & \text{for } n \geq 3. \end{cases} \end{aligned}$$

The case $G = \mathbb{Z}_2$ is important. Then,

$$H^n(K\ \mathbb{Z}_2) \cong \begin{cases} \mathbb{Z}_2 & \text{for } n = 0, 2, \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{for } n = 1 \\ 0 & \text{for } n \geq 3 \end{cases}$$

Example 6.7. (**Real Projective Space**) Let $X = \mathbb{RP}^n$. Recall that we have,

$$H_k(\mathbb{RP}^n, \mathbb{Z}) = \begin{cases} \mathbb{Z}_2 & \text{if } k \text{ is odd, } 0 < k < n, \\ \mathbb{Z} & \text{if } k = 0, n \text{ is odd,} \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_2, G) \cong G_2 = \{g \in G \mid 2g = 0\}$ and $\mathrm{Ext}(\mathbb{Z}_2, G) \cong G/2G$. If n is odd, we have:

$$H^k(\mathbb{RP}^n, G) \cong \begin{cases} G & \text{if } k = 0, n, \\ G/2G & \text{if } k \text{ is even, } 0 < k < n, \\ G_2 & \text{if } k \text{ is odd, } 0 < k < n, \\ 0 & \text{otherwise.} \end{cases}$$

If n is even, we have:

$$H^k(\mathbb{RP}^n, G) \cong \begin{cases} G & \text{if } k = 0, \\ G/2G & \text{if } k \text{ is even, } 0 < k \leq n, \\ G_2 & \text{if } k \text{ is odd, } 0 < k < n, \\ 0 & \text{otherwise.} \end{cases}$$

Remark 6.8. *The case $G = \mathbb{Z}_2$ in Example 6.7 is important. We have*

$$H^k(\mathbb{RP}^n\ \mathbb{Z}_2) \cong \begin{cases} \mathbb{Z}_2 & k = 0, \cdots, n, \\ 0 & \textit{otherwise.} \end{cases}$$

Example 6.9. (**Complex Projective Space**) Let $X = \mathbb{CP}^n$. Recall that we have,

$$H_k(\mathbb{CP}^n\ \mathbb{Z}) = \begin{cases} \mathbb{Z}, & \text{if } k = 0, 2, 4, \dots, 2n \\ 0, & \text{otherwise.} \end{cases}$$

Since $\mathbb{Z}$ is a free abelian group, all the Ext terms in the UCT for cohomology vanish. Hence,

$$H^k(\mathbb{CP}^n, G) \cong \mathrm{Hom}_{\mathbb{Z}}(H_k(\mathbb{CP}^n, \mathbb{Z}), G) \cong \begin{cases} G, & \text{if } k = 0, 2, 4, \dots, 2n \\ 0, & \text{otherwise.} \end{cases}$$

Part 2. **de-Rham Cohomology**

This section assumes knowledge of smooth manifolds theory.

7. de-Rham Cohomology

7.1. **Definitions.** Singular cohomology is quite abstract and somewhat useless unless we develop algebraic computational tools to compute singular cohomology. We now discuss de Rham cohomology for smooth manifolds. This cohomology theory is quite geometric because it is phrased in terms of differential forms on smooth manifolds. Let M be a smooth manifold. Since

$$d^2 = d \circ d : \Omega^{k-1}(M) \xrightarrow{d} \Omega^k(M) \xrightarrow{d} \Omega^{k+1}(M)$$

is the zero operator for every $k \geq 1$, we have

$$\operatorname{im}\left(d : \Omega^{k-1}(M) \to \Omega^k(M)\right) \subseteq \ker\left(d : \Omega^k(M) \to \Omega^{k+1}(M)\right).$$

Thus, $\operatorname{im} d$ is a subspace of $\ker d$ for all $k \geq 1$.

Remark 7.1. *Let M be a smooth n-dimensional manifold. For convenience, we set $\Omega^k(M) = \{0\}$ for all $k < 0$ and $k > n$. Moreover, we set*

$$d = 0 : \Omega^k(M) \to \Omega^{k+1}(M)$$

for all $k < 0$ and $k \geq n$. Then the inclusion above holds for all $k \in \mathbb{Z}$.

Definition 7.2. Let M be a smooth manifold. The quotient vector space

$$H^k_{\mathrm{dR}}(M) = \frac{\ker(d : \Omega^k(M) \to \Omega^{k+1}(M))}{\operatorname{im}(d : \Omega^{k-1}(M) \to \Omega^k(M))} = \frac{\{\omega \in \Omega^k(M) : d\omega = 0\}}{\{d\omega : \omega \in \Omega^{k-1}(M)\}}$$

is the k-th de Rham cohomology group of M.

Let M be a smooth manifold. A form $\omega \in \Omega^k(M)$ is closed if $d\omega = 0$ and exact if there exists a $(k-1)$-form $\tau \in \Omega^{k-1}(M)$ for which $d\tau = \omega$. Since $d \circ d = 0$, every exact form is closed. hus,

$$H^k_{\mathrm{dR}}(M) = \frac{\{\text{closed } k\text{-forms in } M\}}{\{\text{exact } k\text{-forms in } M\}}$$

This suggests that Definition 7.2 measures the failure of closed forms to be exact forms. Indeed, every closed form need not be exact:

Example 7.3. Consider the 1-form on $\mathbb{R}^2 \setminus \{0\}$ defined by:

$$\omega = \frac{x\,dy - y\,dx}{x^2 + y^2}$$

Then,

$$\begin{aligned} d\omega &= \frac{(dx \wedge dy - dy \wedge dx)(x^2 + y^2) - (2x\,dx + 2y\,dy)(x\,dy - y\,dx)}{(x^2 + y^2)^2} \\ &= \frac{2(x^2 + y^2)\,dx \wedge dy - (2x^2\,dx \wedge dy - 2y^2\,dy \wedge dx)}{(x^2 + y^2)^2} = 0 \end{aligned}$$

So, ω is closed. But writing ω in polar coordinates and integrating around a circle centered at 0 in $\mathbb{R}^2 \setminus \{0\}$ gives

$$\int_{\mathbb{S}^1} \omega = 2\pi.$$

If $\omega = d\eta$ were exact, Stokes' theorem would imply

$$0 = \int_{\emptyset} \eta = \int_{\partial \mathbb{S}^1} \eta = \int_{\mathbb{S}^1} d\eta = \int_{\mathbb{S}^1} \omega = 2\pi.$$

Hence, ω is not exact.

Remark 7.4. *Elements $H^k(M)$ are equivalence classes of k-forms. Given $\omega \in \ker(d : \Omega^k(M) \to \Omega^{k+1}(M))$, we denote the equivalence class by*

$$[\omega] = \{\omega + d\tau \in \Omega^k(M) : \tau \in \Omega^{k-1}(M)\}.$$

Therefore, $H^k_{dR}(M)$ is a vector space that classifies the closed k-forms in M up to exact forms.

7.2. **Properties of de Rham cohomology.** We now discuss several algebraic properties of de Rham cohomology, which are analogous to the properties of singular cohomology for general topological spaces. We first show that the de Rham cohomology defines a contravariant functor from the category of smooth manifolds, **Man**, to the category of abelian groups, **Ab**.

Proposition 7.5. *Let M, N be smooth manifolds and let $F : M \to N$ be a smooth map. For each $k \in \mathbb{Z}$, let $F^* : \Omega^k(N) \to \Omega^k(M)$ be the pullback map.*

1. *For each $k \in \mathbb{Z}$, F^* descends to a linear map $F^\# : H^k_{dR}(N) \to H^k_{dR}(M)$ between the de Rham cohomology groups given by $F^\#[\omega] = [F^*\omega]$.*
2. *(**Functoriality**) For each $k \in \mathbb{Z}$, $H^k_{dR} : \mathbf{Man} \to \mathbf{Ab}$ is a contravariant functor.*

Proof. We shall use the fact that the exterior derivative commutes with pullbacks. The proof is given below:

1. Let ω is a closed form. Then
$$d(F^*\omega) = F^*(d\omega) = 0$$
Hence, $F^*\omega$ is also closed a form. This shows that $F^*\omega$ restricts to a map
$$F^* : \{\text{closed } k-\text{forms on } N\} \to \{\text{closed } k-\text{forms on } M\}$$
Now let $\omega = d\eta$ be an exact form. Then
$$F^*\omega = F^*(d\eta) = d(F^*\eta),$$
Hence, $F^*\omega$ is also an exact form. This shows that F^* descends to a well-defined map map
$$F^\# : H^k_{\mathrm{dR}}(N) \to H^k_{\mathrm{dR}}(M)$$
given by $F^\#[\omega] = [F^*\omega]$.
2. This follows from (1).

This completes the proof. □

Proposition 7.6. *(**de Rham Cohomolgy of Disjoint Unions**) Let $\{M_j\}_{j\in J}$ be a countable collection of smooth n-dimensional manifolds. Let $M = \bigsqcup_{j\in J} M_j$. For each $k \in \mathbb{Z}$, the inclusion maps $i_j : M_j \hookrightarrow M$ induce an isomorphism*

$$H^k_{dR}(M) \cong \prod_{j\in J} H^k_{dR}(M_j)$$

Proof. The pullback maps $i_j^*: H^k(M) \to H^k(M_j)$ induce an isomorphism from

$$i: H^k(M) \to \prod_{j\in J} H^k(M_j), \qquad i(\omega) \mapsto \left(i_j^*(\omega)\right)_{j\in J} = \left(\omega|_{M_j}\right)_{j\in J}$$

This map is injective because any smooth k-form whose restriction to each M_j is zero must itself be zero, and it is surjective because giving an arbitrary smooth k-form on each M_j defines one on M. □

We now discuss the homotopy invariance of de Rham cohomology, allowing us to show that de Rham cohomology is a topological invariant. If M and N are smooth manifolds, and $F, G: M \to N$ are smooth maps, we shall show homotopy invariance by constructing a co-chain homotopy between $F^\#$ and $G^\#$ which are given by linear maps

$$h_k : \Omega^k(N) \to \Omega^{k-1}(M)$$

for each $k \in \mathbb{Z}$ such that

$$F^\#(\omega) - G^\#(\omega) = d(h_k\omega) - h_{k+1}(d\omega)$$

for each $\omega \in \Omega^k(N)$ and $k \in \mathbb{Z}$.

$$\begin{tikzcd}
\cdots \arrow[r, "d"] & \Omega^{k-1}(N) \arrow[r, "d"] \arrow[d, "F^\#-G^\#"] & \Omega^k(N) \arrow[r, "d"] \arrow[d, "F^\#-G^\#"] \arrow[dl, "h_k"'] & \Omega^{k+1}(N) \arrow[r, "d"] \arrow[d, "F^\#-G^\#"] \arrow[dl, "h_{k+1}"'] & \cdots \\
\cdots \arrow[r, "d"] & \Omega^{k-1}(M) \arrow[r, "d"] & \Omega^k(M) \arrow[r, "d"] & \Omega^{k+1}(M) \arrow[r, "d"] & \cdots
\end{tikzcd}$$

The key to our proof of homotopy invariance is to construct a homotopy operator first in the following special case. Let M be a smooth manifold, and for each $t \in I$, let

$$i_t : M \to M \times I$$

be the map $i_t(x) = (x, t)$. We first construct a co-chain homotopy between $i_0^\#$ and $i_1^\#$.

Lemma 7.7. *Let M be a smooth n-dimensional manifold. There exists a co-chain homotopy between the two maps $i_0^\#$ and $i_1^\#$.*

Proof. For each $k \in \mathbb{Z}$, we need to define a linear map

$$h_k : \Omega^k(M \times I) \to \Omega^{k-1}(M)$$

such that

$$(*) \qquad h_{k+1}(d\omega) + d(h_k\omega) = i_1^\#(\omega) - i_0^\#(\omega)$$

for each $\omega \in \Omega^k(M \times I)$. Let S be the vector field on $M \times \mathbb{R}$ given by $S(p, s) = (0, \frac{\partial}{\partial s}\big|_s)$. Given $\omega \in \Omega^k(M \times I)$, define $h_k(\omega) \in \Omega^{k-1}(M)$ by

$$h_k(\omega) = \int_0^1 i_t^\#(S \lrcorner \omega)\, dt.$$

We shall verify the formula in $(*)$ in local coordinates. For $p \in M$, let $U = (x^1, \cdots, x^n)$ denote a co-ordinate chart containing then. Then $U \times \mathbb{R} = (x^1, \cdots, x^n, s)$ is a co-cordinate chart containing (p, s) for each $s \in \mathbb{R}$. In coordinates:

$$\omega = \sum_I \omega_I^1(x, s)\, ds \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k} + \sum_J \omega_I^2(x, s)\, dx^{j_1} \wedge \cdots \wedge dx^{j_k}$$

where I, J range over all increasing k-multi-indices over $\{1, \cdots, n\}$. We have,

$$S \lrcorner \omega = \sum_I \omega_I^1(x,s)\, dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

$$i_t^{\#}(S \lrcorner \omega) = i_t^{\#}\left(\sum_I \omega_I^1(x,s)\, dx^{i_1} \wedge \cdots \wedge dx^{i_k}\right) = \sum_I \omega_I^1(x,s)\, dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

We have,

$$\begin{aligned} d(h_k\omega) &= d\int_0^1 i_t^{\#}(S \lrcorner \omega)\, dt \\ &= d\int_0^1 \left(\sum_I \omega_I^1(x,t)\, dx^{i_1} \wedge \cdots \wedge dx^{i_k}\right) dt = \sum_I \int_0^1 \left(\frac{\partial \omega_I^1(x,t)}{\partial x^j}\, dx^j \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}\right) dt. \end{aligned}$$

We now compute $h_{k+1}(d\omega)$. Here d is the exterior derivative on $M \times I$. First note that,

$$d\omega = \sum_I \frac{\partial \omega_I^1(x,s)}{\partial x^j}\, dx^j \wedge dt \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k} + \sum_J \frac{\partial \omega_I^2(x,s)}{\partial x^l}\, dx^l \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_k} + \sum_J \frac{\partial \omega_I^2(x,s)}{\partial s}\, ds \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_k}$$

We now find $h_{k+1}(d\omega)$, which is given by the expression:

$$h_{k+1}(d\omega) = \int_0^1 i_t^{\#}(S \lrcorner d\omega) dt$$

We have,

$$S \lrcorner d\omega = \sum_J \frac{\partial \omega_I^2(x,s)}{\partial s}\, ds \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_k} - \sum_I \frac{\partial \omega_I^1(x,s)}{\partial x^j}\, dt \wedge dx^j \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

Therefore, we have,

$$\begin{aligned} h_{k+1}(d\omega) &= \int_0^1 i_t^{\#}(S \lrcorner d\omega) dt \\ &= \int_0^1 \left(\sum_J \frac{\partial \omega_I^2(x,t)}{\partial s}\, dx^{j_1} \wedge \cdots \wedge dx^{j_k} - \sum_I \frac{\partial \omega_I^1(x,t)}{\partial x^j}\, dx^j \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}\right) dt \end{aligned}$$

We have,

$$d(h_k\omega) + h_{k+1}(d\omega) = \int_0^1 \left(\sum_J \frac{\partial \omega_I^2(x,t)}{\partial s}\, dx^{j_1} \wedge \cdots \wedge dx^{j_k}\right) dt$$

Noting that,

$$i_t^{\#}(\omega) = \sum_J \omega_I^2(x,t)\, dx^{j_1} \wedge \cdots \wedge dx^{j_k},$$

we have,

$$\frac{d i_t^{\#}(\omega)}{dt} = \sum_J \frac{\partial \omega_I^2(x,t)}{\partial t}\, dx^{j_1} \wedge \cdots \wedge dx^{j_k}$$

As a result, we have,

$$d(h_k\omega) + h_{k+1}(d\omega) = \int_0^1 \frac{d i_t^{\#}(\omega)}{dt}\, dt = i_1^{\#}(\omega) - i_0^{\#}(\omega)$$

Hence, $(*)$ holds in every co-ordinate chart. This proves the claim. □

Proposition 7.8. *Let M and N be smooth manifolds. If $F, G : M \to N$ are smoothly homotopic smooth maps, then the induced cohomology maps $F^*, G^* : H^k_{dR}(N) \to H^k_{dR}(M)$ are equal for each $k \in \mathbb{Z}$.*

Proof. There exists a homotopy $H : M \times I \to N$ from F to G such that $F = H \circ i_0, G = H \circ i_1$ We have,

$$F^{\#} = (H \circ i_0)^{\#} = i_0^{\#} \circ H^*,$$
$$G^{\#} = (H \circ i_1)^{\#} = i_1^{\#} \circ H^*.$$

By Lemma 7.7, we know the maps $i_0^{\#}$ and $i_1^{\#}$ are equal from $H^k_{\mathrm{dR}}(M \times I)$ to $H^k_{\mathrm{dR}}(M)$ for each $k \in \mathbb{Z}$. Therefore,

$$F^{\#} = (H \circ i_0)^{\#} = i_0^{\#} \circ H^* = i_1^{\#} \circ H^* = G^{\#}$$

This proves the claim. □

Corollary 7.9. ***(Smooth Homotopy Invariance)*** *Let M and N be smoothly homotopy equivlaent smooth manifolds. Then*

$$H^k_{dR}(M) \cong H^k_{dR}(N)$$

for each $k \in \mathbb{Z}$.

Proof. Let $F : M \to N$ and $G : N \to M$ be smooth maps such that

$$G \circ F \simeq \mathrm{Id}_M$$
$$F \circ G \simeq \mathrm{Id}_N$$

We have,

$$(G \circ F)^{\#} = F^{\#} \circ G^{\#} = \mathrm{Id}_M^{\#} = \mathrm{Id}_{H^k(M)},$$
$$(F \circ G)^{\#} = G^{\#} \circ F^{\#} = \mathrm{Id}_N^{\#} = \mathrm{Id}_{H^k(N)}.$$

Since $\mathrm{Id}_M^{\#}$ is a surjective map, then $F^{\#}$ is surjective. Moreover, since $\mathrm{Id}_N^{\#}$ is an injective map, then $F^{\#}$ is an injective map. Hence, $F^{\#}$ is a linear map bijection, and hence an isomorphism. Hence, we have

$$H^k_{\mathrm{dR}}(M) \cong H^k_{\mathrm{dR}}(N)$$

for each $k \in \mathbb{Z}$. □

It is clear that if $M = \{*\}$, then $H^k_{\mathrm{dR}}(M) = 0$ for all $k > 0$. We will verify this explicitly later on. If M is a star-like manifold, then by smooth homotopy invariance, $H^k_{\mathrm{dR}}(M) = 0$ for all $k > 0$ since M is contractible. This is immediately implies that the famous Poincaré lemma which states that if U is an open star-shaped subset of $\mathbb{R}^n$, then every closed form on U is exact. A consequence of the Poincaré lemma is that every closed form on a smooth manifold, M, is locally exact. This suggests that the the obstruction of solving the equation

$$d\eta = \omega,$$

is connected to a global problem. This hints that the de Rham cohomology group is not affected by the differential structure that is of local nature. This is made precise below:

Corollary 7.10. ***(Topological Invariance of de Rham Cohomology)*** *If M and N are homotopy equivalent,*

$$H^k_{dR}(M) \cong H^k_{dR}(N)$$

for each $k \in \mathbb{Z}$.

Proof. By Whitney's approximation theorem, every topological homotopy equivalence can be approximate is homotopic to a smooth homotopy equivalence. The result then follows from Corollary 7.9. □

7.3. **Mayer–Vietoris Sequence.** Suppose M is a smooth manifold, and let U and V be open subsets of M such that $U \cup V = M$. The main goal of using the Mayer-Vietoris Sequence is to compute $H^k_{\mathrm{dR}}(M)$ in terms of $H^k_{\mathrm{dR}}(U)$, $H^k_{\mathrm{dR}}(V)$, and $H^k_{\mathrm{dR}}(U \cap V)$ where $\{U, V\}$ is an open cover of M. We have the following inclusions:

$$\begin{tikzcd} & U \arrow[dr, hook, "k"] & \\ U \cap V \arrow[ur, hook, "i"] \arrow[dr, hook, "j"'] & & M \\ & V \arrow[ur, hook, "l"'] & \end{tikzcd}$$

For each $k \in \mathbb{Z}$, these inclusion induce pullback maps on differential forms

$$\begin{tikzcd} & \Omega^k(U) \arrow[dl, "i^*"'] & \\ \Omega^k(U \cap V) & & \Omega^k(M) \arrow[ul, "k^*"'] \arrow[dl, "l^*"] \\ & \Omega^k(V) \arrow[ul, "j^*"] & \end{tikzcd}$$

Note that these pullbacks are in fact just restrictions. If we take some $\omega \in \Omega^k(M)$ and apply the map $k^* \oplus \ell^*$, we get

$$k^* \oplus \ell^*(\omega) : \Omega^k(M) \to \Omega^k(U) \oplus \Omega^k(V), \qquad k^* \oplus \ell^*(\omega) = (k^*\omega, \ell^*\omega) = (\omega|_U, \omega|_V)$$

Furthermore, if we take $(\omega, \eta) \in \Omega^p(U) \oplus \Omega^p(V)$ and apply the map $i^* - j^*$, we have

$$i^* - j^* : \Omega^k(U) \oplus \Omega^k(V) \to \Omega^k(U \cap V) \qquad (i^* - j^*)(\omega, \eta) = \omega\big|_{U \cap V} - \eta\big|_{U \cap V}$$

In other words, we have the following diagram

$$\begin{tikzcd} & \Omega^k(U) \arrow[dl, "i^*"] & \\ \Omega^k(U \cap V) & \Omega^k(U) \oplus \Omega^k(V) \arrow[l, "i^* - j^*"'] & \Omega^k(M) \arrow[ul, "k^*"'] \arrow[l, "k^* \oplus l^*"'] \arrow[dl, "l^*"] \\ & \Omega^k(V) \arrow[ul, "j^*"'] & \end{tikzcd}$$

Proposition 7.11. *(**Mayer–Vietoris Sequence**) Let M be a smooth manifold, and let U, V be open subsets of M such that $M = U \cup V$. For each $k \in \mathbb{Z}$, there is a linear map $\delta^k : H^k_{dR}(U \cap V) \to H^{k+1}_{dR}(M)$ such that the following sequence, called the Mayer-Vietoris sequence for the open cover $\{U, V\}$, is exact:*

$$\cdots \xrightarrow{\delta^{k-1}} H^k_{dR}(M) \xrightarrow{k^\# \oplus l^\#} H^k_{dR}(U) \oplus H^k_{dR}(V) \xrightarrow{i^\# - j^\#} H^k_{dR}(U \cap V) \xrightarrow{\delta^k} H^{k+1}_{dR}(M) \xrightarrow{k^\# \oplus l^\#} \cdots$$

Proof. Consider the following sequence:

$$0 \to \Omega^k(M) \xrightarrow{k^* \oplus l^*} \Omega^k(U) \oplus \Omega^k(V) \xrightarrow{i^* - j^*} \Omega^k(U \cap U) \to 0$$

We show that this sequence is a short exact sequence. We first show that $k^* \oplus l^*$ is injective. Suppose that $\sigma \in \Omega^p(M)$ satisfies

$$(k^* \oplus l^*)(\sigma) = (\sigma|_U,\, \sigma|_V) = (0, 0)$$

This means that the restrictions of σ to U and V are both zero. Since $\{U, V\}$ is an open cover of M, this implies that σ is zero. We now show exactness at $\Omega^k(U) \oplus \Omega^k(V)$. Note that

$$(i^* - j^*) \circ (k^* \oplus l^*)(\sigma) = (i \circ j)(\sigma|_U, \sigma|_V) = \sigma|_{U \cap V} - \sigma|_{U \cap V} = 0,$$

This shows that $\mathrm{Im}(k^* \oplus l^*) \subseteq \ker(i^* - j^*)$. Conversely, suppose we are given $(\alpha, \alpha') \in \Omega^k(U) \oplus \Omega^k(V)$ such that $(i^* \circ j^*)(\alpha, \alpha') = 0$. This means that $\alpha|_{U \cap V} = \alpha'|_{U \cap V}$. So there is $\sigma \in \Omega^k(M)$ defined by

$$\sigma = \begin{cases} \alpha \text{ on } U, \\ \alpha' \text{ on } V. \end{cases}$$

Clearly, $(\alpha, \alpha') = (k \oplus l)(\sigma)$. So $\ker(i^* - j^*) \subseteq \mathrm{im}(k^* \oplus l^*)$. We now show that $i^* - j^*$ is surjective. Let $\omega \in \Omega^k(U \cap U)$. Let $\{\varphi, \psi\}$ be a smooth partition of unity subordinate to the open cover $\{U, V\}$ of M. Define $\alpha \in \Omega^k(U)$ and $\alpha' \in \Omega^k(V)$ by

$$\alpha = \begin{cases} \psi\omega \text{ on } U \cap V, \\ 0 \text{ on } U \setminus \operatorname{supp} \psi \end{cases} \qquad \alpha' = \begin{cases} -\varphi\omega \text{ on } U \cap V, \\ 0 \text{ on } U \setminus \operatorname{supp} \varphi \end{cases}$$

We have

$$(i^* - j^*)(\alpha, \alpha') = \alpha|_{U \cap V} - \alpha'|_{U \cap V} = \psi\omega - (-\varphi\omega) = (\psi - \varphi)\omega = \omega.$$

Hence, the sequence is indeed a short exact sequence. Because pullback maps commute with the exterior derivative, above short exact sequence induces the following this will show that we have the following short exact sequence:

$$0 \to H^k_{\mathrm{dR}}(M) \xrightarrow{k^\# \oplus l^\#} H^k_{\mathrm{dR}}(U) \oplus H^k_{\mathrm{dR}}(V) \xrightarrow{i^\# - j^\#} H^k_{\mathrm{dR}}(U \cap U) \to 0$$

Since this is true for each $k \in \mathbb{Z}$ we get a short exact sequence of co-chain complexes involing the de-Rham cohomology groups. The Mayer–Vietoris theorem then a formal consequence of the snake lemma. $\square$

The snake lemma defines the connecting morphism

$$H^k_{\mathrm{dR}}(U \cap V) \xrightarrow{\delta^k} H^{k+1}_{\mathrm{dR}}(M)$$

A characterization of the connecting homomorphism is given in the proof of the snake lemma. Recalling it and adapting it to our case, we have that $\delta^k[\omega] = [\sigma]$, provided there exists $(\alpha, \alpha) \in \Omega^k(U) \oplus \Omega^k(V)$ such that

$$i^*\alpha - j^*\alpha' = \omega, \quad (k^*\sigma, l^*\sigma) = (d\alpha, d\alpha').$$

α, α' can be defined as in Proposition 7.11 to satisfy the first equation. Given such forms (α, α'), the fact that ω is closed implies that $d\alpha = d\alpha'$ on $U \cap V$. Thus, there is a smooth $(k+1)$-form σ on M that is equal to $d\alpha$ on U and $d\alpha'$ on V, and it satisfies the second equation.

7.4. **de Rham cohomology in Degrees Zero & One.** It is quite easy to characterize the de Rham cohomology in degree zero.

Proposition 7.12. *Let M is a connected smooth manifold. Then $H^0_{dR}(M)$ is equal to the space of constant functions. Therefore,*

$$H^0_{dR}(M) \cong \mathbb{R}$$

Proof. Note that

$$H^0_{\mathrm{dR}}(M) \cong \{\text{closed 0 forms on } M\} \cong \{f \in C^\infty(M) \mid df = 0\}$$

Since M is connected, $df = 0$ if and only if f is constant real-valued function. Therefore,

$$H^0_{\mathrm{dR}}(M) \cong \mathbb{R}$$

This completes the proof. □

Corollary 7.13. *Let M be a smooth manifold. Then*

$$H^0_{dR}(M) \cong \mathbb{R}^{|J|},$$

where $|J|$ is the number of connected components of M.

Proof. We have,

$$M = \coprod_{j \in J} M_j,$$

where each M_j is a connected component of M and J is at most countably infinite. By Proposition 7.6 and Proposition 7.12, we have,

$$H^0_{\mathrm{dR}}(M) \cong \prod_{j\in J} H^0_{\mathrm{dR}}(M_j \cong) \prod_{j \in J} \mathbb{R} \cong \mathbb{R}^{|J|}$$

This completes the proof. □

Another case in which we can say quite a lot about de Rham cohomology is in degree one. Let $\mathrm{Hom}(\pi_1(M,p), \mathbb{R})$ denote the set of group homomorphisms from $\pi_1(M,p)$ to the additive group $\mathbb{R}$. We define a linear map $\Phi\colon H^1_{\mathrm{dR}}(M) \to \mathrm{Hom}(\pi_1(M,p), \mathbb{R})$ as follows: given a cohomology class $[\omega] \in H^1_{\mathrm{dR}}(M)$, define $\Phi([\omega])\colon \pi_1(M,p) \to \mathbb{R}$ by

$$\Phi([\omega])([\gamma]) = \int_\gamma \omega,$$

where $[\gamma]$ is any path homotopy class in $\pi_1(M,p)$, and γ is any piecewise smooth curve representing the same path class.

Proposition 7.14. *Suppose M is a connected smooth manifold. For each $q \in M$, the linear map $\Phi\colon H^1_{\mathrm{dR}}(M) \to \mathrm{Hom}(\pi_1(M,p), \mathbb{R})$ is well defined and injective.*

Proof. (Sketch) Given $[\gamma] \in \pi_1(M,p)$, it follows from the Whitney approximation theorem that there is some smooth closed curve segment $\widetilde{\gamma}$ in the same path class as γ. We use without proof the fact that

$$\int_{\widetilde{\gamma}} \omega = \int_{\widetilde{\widetilde{\gamma}}} \omega$$

for every closed forms, ω and every other smooth closed curve $\widetilde{\widetilde{\gamma}}$ in the same path class as γ. If $\widetilde{\omega}$ is another smooth 1-form in the same cohomology class as ω, then $\widetilde{\omega} - \omega = df$ for some smooth function f, which implies

$$\int_{\widetilde{\gamma}} \widetilde{\omega} - \int_{\widetilde{\gamma}} \omega = \int_{\widetilde{\gamma}} df = f(q) - f(q) = 0.$$

Thus, Φ is well defined. It follows from properties of the line integral that $\Phi([\omega])$ is a group homomorphism from $\pi_1(M,p)$ to $\mathbb{R}$, and that Φ itself is a linear map. Suppose $\Phi([\omega])$ is the zero homomorphism. This means that $\int_{\widetilde{\gamma}} \omega = 0$ for every piecewise smooth closed curve $\widetilde{\gamma}$ with basepoint q. If γ is a piecewise smooth closed curve starting at some other point $q_0 \in M$, we can choose a piecewise smooth curve α from q to q_0, so that the path product $\alpha \cdot \gamma \cdot \overline{\alpha}$ is a closed curve based at q. It then follows that

$$0 = \int_{\alpha\cdot\gamma\cdot\overline{\alpha}} \omega = \int_{\alpha} \omega + \int_{\gamma} \omega + \int_{\overline{\alpha}} \omega = \int_{\alpha} \omega + \int_{\gamma} \omega - \int_{\alpha} \omega = \int_{\gamma} \omega.$$

Thus, ω is conservative and therefore exact. ☐

Corollary 7.15. *If M is a connected smooth manifold with finite fundamental group, then $H^1_{\mathrm{dR}}(M) = 0$.*

Proof. There are no nontrivial homomorphisms from a finite group to $\mathbb{R}$. The claim follows from Proposition 7.14. ☐

Remark 7.16. *If M is a connected smooth manifold whose fundamental group is a torsion group, then $H^1_{\mathrm{dR}}(M) = 0$. This is because $\mathbb{R}$ has no torsion elements. Hence, $\mathrm{Hom}(\pi_1(M,p), \mathbb{R}) = 0$ in this case.*

8. Examples & Applications

We discuss some example computations of de Rham cohomology.

Example 8.1. (**0-Dimensions**) Let M be a 0-dimensional smooth manifold. We have,

$$M \cong \coprod_{i\in I} \{*\}$$

where $|I|$ is the cardinality[5] of M. Then

$$H^k_{\mathrm{dR}}(M) = \begin{cases} \mathbb{R}^{|I|}, & \text{if } k = 0 \\ 0, & \text{otherwise} \end{cases}.$$

where $|I|$ is the cardinality of M. This follows at once from Proposition 7.6 and Proposition 7.12.

[5] $|I|$ is at most countably infinite.

Example 8.2. (**Contractible Manifolds**) Let M be contractible manifold. Then,

$$H^k_{\mathrm{dR}}(M) = \begin{cases} \mathbb{R}^{|J|}, & \text{if } k = 0 \\ 0, & \text{otherwise} \end{cases}.$$

where $|J|$ is the number of connected components of M. This follows immediately from Example 8.1 and Corollary 7.13.

Remark 8.3. *If M is a star-like manifold, then by homotopy invariance, $H^k_{dR}(M) = 0$ for all $k > 0$ since M is contractible. This is immediately implies that the famous Poincaré lemma which states that if U is an open star-shaped subset of $\mathbb{R}^n$, then every closed form on U is exact. A consequencee of the Poincaré lemma is that every closed form on a smooth manifold, M, is locally exact.*

Example 8.4. (**Circle**) Let's compute the de-Rham cohomology of $\mathbb{S}^1$. Clearly, $H^0_{\mathrm{dR}}(\mathbb{S}^1) = \mathbb{R}$ since $\mathbb{S}^1$ is connected. Write $\mathbb{S}^1 = U \cup V$, where U, V represent the 'nothern hemisphere' and 'southern heisphere'. U, V are contractible and $U \cap V \cong \{\pm 1\}$. The Mayer-Vietoris theorem implies

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathbb{R} \oplus \mathbb{R} \longrightarrow \mathbb{R} \oplus \mathbb{R} \longrightarrow H^1_{\mathrm{dR}}(\mathbb{S}^1) \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots$$

This clearly implies that $H^k_{\mathrm{dR}}(\mathbb{S}^1) = 0$ for $k > 2$. We can immediately conclude via exactness that $H^1_{\mathrm{dR}}(\mathbb{S}^1) = \mathbb{R}$. Hence,

$$H^k_{\mathrm{dR}}(\mathbb{S}^1) = \begin{cases} \mathbb{R}, & \text{if } k = 0, 1 \\ 0, & \text{otherwise} \end{cases}.$$

We can compute the generator for $H^1_{\mathrm{dR}}(\mathbb{S}^1)$. The generator of is the angular 1-form $d\theta$. Notice that $d\theta$ is not globally defined on the circle since it is a multiple-valued function. Therefore, $d\theta$ is not zero in cohomology and generates $H^1_{\mathrm{dR}}(\mathbb{S}^1)$.

Example 8.5. (**Spheres**) Let's compute the de Rham cohomology of $\mathbb{S}^n$ for $n \geq 1$. We proceed by induction on k to show that

$$H^k_{\mathrm{dR}}(\mathbb{S}^n) = \begin{cases} \mathbb{R}, & \text{if } k = 0, n \\ 0, & \text{otherwise} \end{cases}.$$

We have verified the claim for $k = 1$ in Example 8.4. Now assume the claim is true for $n - 1$. Let $U = \mathbb{S}^n \setminus \{N\}$ and $V = \mathbb{S}^n \setminus \{S\}$. We have

$$U \cap V \simeq \mathbb{S}^{n-1} \quad U \simeq V \simeq \mathbb{R}^n$$

The Mayer-Vietoris sequence implies

$$\cdots \to 0 \to H^{k-1}_{\mathrm{dR}}(\mathbb{S}^{n-1}) \to H^k_{\mathrm{dR}}(\mathbb{S}^n) \to 0 \to \cdots$$

This implies that $H^{k-1}_{\mathrm{dR}}(\mathbb{S}^{n-1}) \cong H^k_{\mathrm{dR}}(\mathbb{S}^n)$. The claim now follows via induction and Example 8.4.

Example 8.6. (**Punctured Euclidean Space**) Let $p \in \mathbb{R}^n$ for $n \geq 2$. WLOG, we can assume that $p = 0$. We have

$$H^k_{\mathrm{dR}}(\mathbb{R}^n \setminus \{p\}) = \begin{cases} \mathbb{R}, & \text{if } k = 0, n-1 \\ 0, & \text{otherwise} \end{cases}.$$

Indeed, the inclusion $\mathbb{S}^{n-1} \to \mathbb{R}^n$ is a homotopy equivalence. The claim now follows from Example 8.5.

We can now discuss some elementary applications of de-Rham cohomology. We can now prove the topological invariance of the dimension of smooth manifolds.

Proposition 8.7. *If $m \neq n$, then $\mathbb{R}^n$ is not homeomorphic to $\mathbb{R}^m$. In particular, if M be a topological n-manifold then its dimension is uniquely determined.*

Proof. Assume that $\mathbb{R}^m \cong \mathbb{R}^n$. If $f : \mathbb{R}^n \to \mathbb{R}^m$ is a homeomorphism, then $f : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^m \setminus \{f(0)\}$ is a homeomorphism. So,

$$H^k_{\mathrm{dR}}(\mathbb{R}^n \setminus \{0\}) = H^k_{\mathrm{dR}}(\mathbb{R}^m \setminus \{f(0)\}),$$

for each $k \in \mathbb{Z}$. But $\mathbb{R}^n \setminus \{0\} \cong \mathbb{S}^{n-1}$ and $\mathbb{R}^m \setminus \{f(0)\} \cong \mathbb{S}^{m-1}$. So,

$$H^k_{\mathrm{dR}}(\mathbb{S}^{m-1}) = H^k_{\mathrm{dR}}(\mathbb{S}^{n-1})$$

for each $k \in \mathbb{Z}$. This is a contradiction by Example 8.5. The claim for a topological manifold follows by working in co-ordinate charts. $\square$

We can also show that the rank of the de-Rham cohomology groups is finite for *most manifolds*. We first need a definition:

Definition 8.8. Let M be a smooth n-manifold and $\{U_\alpha\}_{\alpha \in \Lambda}$ an open cover of M. We say $\{U_\alpha\}_{\alpha \in \Lambda}$ is a good cover if for any finite subset $I = \{\alpha_1, \dots, \alpha_k\} \subseteq \Lambda$ of indices, the intersection

$$U_I := U_{\alpha_1} \cap U_{\alpha_2} \cap \cdots \cap U_{\alpha_k}$$

is either empty or diffeomorphic to $\mathbb{R}^n$.

Remark 8.9. *By using the theory of geodesically convex neighborhoods in Riemannian geometry, one can show that any open cover of any smooth manifold M admits a refinement which is a good cover. In particular, if M is compact, then M admits a good cover which contains only finitely many open sets. See [BT13].*

Proposition 8.10. *Let M be a smooth n-manifold. If M admits a finite good cover, $\dim H^k_{dR}(M) < \infty$ for each $k \in \mathbb{Z}$.*

Proof. We proceed by induction on the number of sets in a finite good cover of M. If M admits a good cover that contains only one open set, then that open set has to be M itself. In this case, M is diffeomorphic to $\mathbb{R}^n$, and the conclusion follows. Now suppose the theorem holds for any manifold that admits a good cover containing $k-1$ open sets. Let M be a manifold with a good cover $\{U_1, \dots, U_k\}$. We denote

$$U = U_1 \cup \cdots \cup U_{k-1} \quad \text{and} \quad V = U_k.$$

Then $U \cap V$ admits a finite good cover $\{U_1 \cap U_k, \dots, U_{k-1} \cap U_k\}$. By the induction hypothesis, all the de Rham cohomology groups of U, V, and $U \cap V$ are finite-dimensional. Now consider the Mayer-Vietoris sequence:

$$\cdots \to H^{k-1}_{\mathrm{dR}}(U \cap V) \xrightarrow{\delta^{k-1}} H^k_{\mathrm{dR}}(M) \xrightarrow{\alpha^k} H^k_{\mathrm{dR}}(U) \oplus H^k_{\mathrm{dR}}(V) \to \cdots .$$

The conclusion follows since

$$\dim \mathrm{Im}(\alpha_k) \leq \dim H^k_{\mathrm{dR}}(U) \oplus H^k_{\mathrm{dR}}(V) < \infty,$$
$$\dim \ker(\alpha_k) = \dim \mathrm{Im}(\delta_{k-1}) \leq \dim H^{k-1}_{\mathrm{dR}}(U \cap V) < \infty.$$

This completes the proof. $\square$

Corollary 8.11. *If M is a compact manifold (or M is homotopy equivalent to a compact manifold), then* $\dim H^k_{dR}(M) < \infty$ *for all* $k \in \mathbb{Z}$.

Proof. This follows from Proposition 8.10. $\square$

9. Compactly Supported de-Rham Cohomology

Let M be an orientable smooth manifold. Integration is a pairing between compactly supported forms and oriented manifolds. This observation motivates that $H^n_{\mathrm{dR}}(M)$ is important for studying orientations on M. Unfortunately, if M is non-compact, the integration of a n-form is not nicely defined unless the differential form is compactly supported. This observation motivates the study of de-Rham cohomology with compact support.

Definition 9.1. Let M be a smooth n-manifold and let $\omega \in \Omega^k(M)$. The **support** of ω is

$$\mathrm{supp}(\omega) = \{p \in M \mid \omega_p \neq 0\}.$$

ω is compactly supported if $\mathrm{supp}(\omega)$ is a compact set.

We set,

$$\Omega^k_c(M) = \{\omega \in \Omega^k(M) \mid \omega \text{ is compactly supported}\},$$

be the set of all compactly supported smooth k-forms. Clearly, the following facts are true:

(1) if ω_1, ω_2 are compactly supported k-forms, so is $c_1\omega_1 + c_2\omega_2$;
(2) if ω is compactly supported, then $d\omega$ is also compactly supported.

So $\Omega^k_c(M)$ are real vector spaces for each $k \in \mathbb{Z}$, and the exterior derivative makes these vector spaces a co-chain complex:

$$0 \to \Omega^0_c(M) \xrightarrow{d} \Omega^1_c(M) \xrightarrow{d} \Omega^2_c(M) \xrightarrow{d} \Omega^3_c(M) \xrightarrow{d} \cdots .$$

Definition 9.2. Let M be a smooth manifold. The quotient vector space

$$H^k_{\mathrm{dR,c}}(M) = \frac{\ker(d : \Omega^k_c(M) \to \Omega^{k+1}_c(M))}{\mathrm{im}(d : \Omega^{k-1}_c(M) \to \Omega^k_c(M))} = \frac{\{\omega \in \Omega^k_c(M) : d\omega = 0\}}{\{d\omega : \omega \in \Omega^{k-1}_c(M)\}}$$

is the k-th de Rham cohomology group with compact support of M.

Example 9.3. Let M be a smooth manifold. For $k = 0$, by definition

$$H^k_{\mathrm{dR,c}}(M) = \{f \in C^\infty(M) \mid df = 0 \text{ and } \mathrm{supp}(f) \text{ is compact}\}.$$

But $df = 0$ if and only if f is locally constant, i.e., f is constant on each connected component. Moreover, a locally constant compactly supported function has to be zero on any non-compact connected component. So we conclude

$$H^0_{\mathrm{dR,c}}(M) \cong \mathbb{R}^{m_c},$$

where m_c is the number of *compact* connected components of M. In particular,

$$H^0_{\mathrm{dR,c}}(\mathrm{pt}) = \mathbb{R}, \quad \text{and} \quad H^0_{\mathrm{dR,c}}(\mathbb{R}^n) = 0$$

for all $n \geq 1$.

Remark 9.4. *Since $\mathbb{R}^n$ is homotopy equivalent to $\{pt\}$, we conclude that $H^0_{\mathrm{dR,c}}(M)$ is* no longer *a homotopy invariant.*

We now discuss the analog of the Mayer-Vietoris sequence for the compactly supported case. If $F : M \to N$ is a smooth map between smooth manifolds, note that by definition,

$$\operatorname{supp}(F^*\omega) \subseteq F^{-1}(\operatorname{supp}(\omega)).$$

So if $\omega \in \Omega^k_c(N)$, in general we may have $F^*\omega \notin \Omega^k_c(M)$. Hence, we cannot expect to pull back compactly-supported cohomology classes on N to compactly-supported cohomology classes on M!

Remark 9.5. *If $F : M \to N$ is proper map, then the pull-back $F^*\omega$ of a compactly supported differential form $\omega \in \Omega^k_c(N)$ is still compactly supported. In this case, we have an induced map:*

$$F^* : H^k_{\mathrm{dR,c}}(N) \to H^k_{\mathrm{dR,c}}(M)$$

In this case, one can prove that if $F_0, F_1 : M \to N$ are proper smooth maps that are properly homotopic, then the induced maps are equal:

$$F_1^* = F_2^* : H^k_{\mathrm{dR,c}}(N) \to H^k_{\mathrm{dR,c}}(M).$$

Note that any homeomorphism is proper. So, in particular, the compactly supported de Rham cohomology groups are still topological invariants up to homeomorphisms. That is, if $M \cong N$ as smooth manifolds, then

$$H^k_c(M) = H^k_c(N)$$

for each $k \in \mathbb{Z}$. We have aready seen that compactly supported de Rham cohomology groups is not a topological invariant up to homotopy equivalence.

So how do we prove an analog of the Mayer-Vietoris sequence for the compactly supported case. Note that we can now instead pushforward compactly supported differential forms and hence cohomology classes. If $U \subseteq M$ is an open set, the inclusion $i : U \hookrightarrow M$ induces a map

$$i_* : \Omega^n_c(U) \to \Omega^n_c(M)$$

that sends a compactly supported differential form on U to the same differential form extended by zero outside of U.

Lemma 9.6. *For each $k \in \mathbb{Z}$, the map i_* commutes with the exterior derivative.*

Proof. For $\omega \in \Omega^n_c(U)$, we have $d\omega \in \Omega^{n+1}_c(U)$. Thus, applying $(i_* \circ d)$ to ω results in $d\omega$ extended by zero outside of U. If we first apply i_*, we obtain

$$i_*(\omega) = \begin{cases} 0, & \text{on } M \setminus U, \\ \omega, & \text{on } U. \end{cases}$$

Taking the exterior derivative, we get

$$d(i_*\omega) = \begin{cases} 0, & \text{on } M \setminus U, \\ d\omega, & \text{on } U. \end{cases}$$

Thus, i_* commutes with d. That is, $i_* \circ d = d \circ i_*$. □

Lemma 9.6 allows us to establish the following version of the Mayer-Vietoris sequence for the compactly supported case.

Proposition 9.7. *Let M be a smooth n-manifold and let $U, V \subseteq M$ be open sets such that $U \cup V = M$. Then there exists linear maps $\delta_k^c : H_c^k(M) \to H_c^{k+1}(U \cap V)$ so that the following sequence is exact:*

$$\cdots \to H_c^k(U \cap V) \to H_c^k(U) \oplus H_c^k(V) \to H_c^k(M) \xrightarrow{\delta_k^c} H_c^{k+1}(U \cap V) \to \cdots$$

Proof. The proof is so much like the original Mayer-Vietoris proof, and it involves a diagram chase. We omit details. □

Example 9.8. We compute $H^k_{\mathrm{dR,c}}(\mathbb{R}^n)$ for $k < n$. We have seen $H^0_{\mathrm{dR,c}}(\mathbb{R}^n) = 0$. Now we show that

$$H^k_{\mathrm{dR,c}}(\mathbb{R}^n) = 0$$

for $1 \le k < n$. We identify $\mathbb{R}^n$ with then open set $\mathbb{S}^n - \{N\}$. Then we get an inclusion map

$$\iota : \mathbb{R}^n \to \mathbb{S}^n,$$

(1) Let $k = 1$. Let $\omega \in \Omega_c^1(\mathbb{R}^n)$ such that $d\omega = 0$. Since d commutes with i as seen above, we have that $\iota_*\omega \in \Omega_c^1(\mathbb{S}^n)$ such that $d(\iota_*\omega) = 0$. Since[6]

$$H^1_{\mathrm{dR,c}}(\mathbb{S}^n) = H^1_{\mathrm{dR}}(\mathbb{S}^n) = 0$$

there exists $\eta \in \Omega^0(\mathbb{S}^n) = C_c^\infty(\mathbb{S}^n)$ such that $\iota_*\omega = d\eta$. Noting that $\iota_*\omega$ is supported in $\mathbb{S}^n - U$ for open set U containing N, we have $d\eta = \iota^*\omega = 0$ on U. This implies that $\eta|_U \equiv c$ for some constant $c \in \mathbb{R}$. It follows that if we take $\tilde{\eta} = \eta - c$, then $\tilde{\eta} \in \Omega_c^0(\mathbb{S}^n - \{N\}) = \Omega_c^0(\mathbb{R}^n)$ and $d\tilde{\eta} = \omega$.

(2) Let $k > 1$. Let $\omega \in \Omega_k^c(\mathbb{R}^n)$ such that $d\omega = 0$. As above, $\iota_*\omega \in \Omega_k^c(\mathbb{R}^n)$ such that $d(\iota_*\omega) = 0$, and $\iota_*\omega$ is supported in $\mathbb{S}^n - U$ for open set U containing N. Since[7]

$$H^k_{\mathrm{dR,c}}(\mathbb{S}^n) = H^k_{\mathrm{dR}}(\mathbb{S}^n) = 0$$

there exists $\eta \in \Omega^{k-1}(\mathbb{S}^n)$ such that $\iota_*\omega = d\eta$. By shrinking the neighborhood U of N, we can assume that U is contractible. Then the fact that $d\eta = \iota_*\omega = 0$ in U implies that there exists a $\mu \in \Omega_c^{k-2}(U)$ such that $\eta|_U = d\mu$. Now pick a bump function ψ on $\mathbb{S}^n$ which compactly supported in U that equals 1 on N. Then

$$\tilde{\eta} = \eta - d(\psi\mu) \in \Omega_c^{k-1}(\mathbb{S}^n)$$

and $\tilde{\eta} = 0$ near N. By construction, $d\tilde{\eta} = d\eta = \omega$.

9.1. **Top Degree Cohomology.** We now set up the machinery to argue that the degree k de Rham cohomology with compact support is related to the orientation of smooth manifolds. First, an example:

Example 9.9. Let's compute $H^1_{\mathrm{dR,c}}(\mathbb{R})$. Consider the integration map

$$\int_{\mathbb{R}} : \Omega_c^1(\mathbb{R}) \to \mathbb{R}, \quad \omega \mapsto \int_{\mathbb{R}} \omega.$$

This map is clearly linear and surjective. Moreover, if $\omega = df$ is a compactly supported exact form, then

$$\int_{-\infty}^{\infty} df\, dx = \int_{-R}^{R} \frac{df}{dx}\, dx = f(R) - f(-R),$$

[6] Note that $k = 1 < n$

[7] Once again, note that $k < 1 < n$

for each $R > 0$. Since $f \in C_c^\infty(\mathbb{R})$, $f(R) = f(-R) = 0$ for R large enough. So it induces a surjective linear map

$$\int_{\mathbb{R}} : H^1_{\mathrm{dR,c}}(\mathbb{R}) \to \mathbb{R}.$$

Moreover, if $\int_{\mathbb{R}} f(t)\,dt = 0$, where $f \in C_c^\infty(\mathbb{R})$, then consider the function

$$g(t) = \int_{-\infty}^{t} f(\tau)\,d\tau$$

Clearly, g is smooth. If we choose $T > 0$ and $R < 0$ large enough, we get

$$F(T) = \int_{-\infty}^{T} f(t)\,dt = \int_{-\infty}^{\infty} f(t)\,dt = 0.$$
$$F(R) = \int_{-\infty}^{R} f(t)\,dt = \int_{-\infty}^{R} 0\,dt = 0.$$

Hence, $g \in C_c^\infty(\mathbb{R})$. Since $dg = f$, we have $[f(t)dt]$ in $H^1_{\mathrm{dR,c}}(\mathbb{R})$. Thus, $\int_{\mathbb{R}}$ is an isomorphism between $H_c^1(\mathbb{R})$ and $\mathbb{R}$, i.e.,

$$H_c^1(\mathbb{R}) \cong \mathbb{R}.$$

The same method as in Example 9.9 works generally. Let M be a connected, oriented n-manifold, and let $\omega \in \Omega_c^n(M)$ be a compactly supported n-form. Then ω is closed, and we have defined the integral $\int_M \omega$. So we get a map

$$\int_M : \Omega_c^n(M) \to \mathbb{R}, \quad \omega \mapsto \int_M \omega.$$

Suppose $\omega = d\eta$ for some $\eta \in \Omega_c^{n-1}(M)$. We can take a compact set $K \subseteq M$ such that $\mathrm{supp}(\eta) \subseteq K$. By Stokes' theorem,

$$\int_M \omega = \int_M d\eta = \int_K d\eta = \int_{\partial K} \eta = 0.$$

Thus, $\int_M$ induces a linear map

$$\int_M : H^n_{\mathrm{dR,c}}(M) \to \mathbb{R}, \quad [\omega] \mapsto \int_M \omega$$

Proposition 9.10. *Let M be an oriented smooth n-manifold. Then the map $\int_M : H^n_{\mathrm{dR,c}}(M) \to \mathbb{R}$ is surjective.*

Proof. Fix a n-form (a volume form) ω on M. For any $c \in \mathbb{R}$, one can find a smooth function f that is compactly supported in a coordinate chart U, such that $\int_U f\omega = c$. □

We can prove the following corollary based on Proposition 9.10:

Corollary 9.11. *The following statements are true:*

(1) *If $\omega \in \Omega^n(\mathbb{S}^n)$ and $\int_{\mathbb{S}^n} \omega = 0$, then ω is exact.*
(2) *We have*

$$H^k_{\mathrm{dR,c}}(\mathbb{R}^n) = \begin{cases} \mathbb{R}, & k = n, \\ 0, & k \neq n. \end{cases}$$

(3) *Let M be a smootn n-manifold. if M admits a finite good cover, then* $\dim H^k_{\mathrm{dR,c}}(M) < \infty$ *for all $k \in \mathbb{Z}$.*

Proof. The proof is given below:

(1) Note that
$$H^n_{\mathrm{dR}}(\mathbb{S}^n) = H^n_{\mathrm{dR,c}}(\mathbb{S}^n) \cong \mathbb{R}$$
Hence, the map in Proposition 9.10 is in fact a linear isomorphism. In other words, if $\int_{\mathbb{S}^n} \omega = 0$, then $[\omega] = 0$, i.e., ω is exact.

(2) Example 9.9 proves the case $n = 1$ and Example 9.8 takes care of the case $1 \leq k < n$ for $n \geq 2$. We discuss the case $k = n \geq 2$. It suffices to show that the surjective linear map
$$\int_{\mathbb{R}^n} : H^n_{\mathrm{dR,c}}(\mathbb{R}^n) \to \mathbb{R}, \quad [\omega] \mapsto \int_{\mathbb{R}^n} \omega$$
is in fact an isomorphism. We show that the map is injective. Assume that $\int_{\mathbb{R}^n} \omega = 0$ for some $\omega \in \Omega^n_c(\mathbb{R}^n)$. Automatically, we have $d\omega = 0$. As before, consider the inclusion map $\iota : \mathbb{R}^n \to \mathbb{S}^n$. Then $\iota_*\omega \in \Omega^n(S^n)$. Since
$$\int_{\mathbb{S}^n} \iota_*\omega = \int_{\mathbb{R}^n} \omega = 0,$$
by (1), we see $\iota_*\omega = d\eta$ for some $\eta \in \Omega^{n-1}(S^n)$. The rest of the proof is similar to that of Example 9.8(2).

(3) We can use Mayer-Vertoris sequence compactly supported de Rham cohomology and induction and the number of open sets in a good cover. The same as the proof for the ordinary de Rham cohomology.

This completes the proof. □

We now reach the punchline for this section. We argue that Proposition 9.10 is, in fact, a linear isomorphism if the underlying smooth manifold is connected and orientable.

Proposition 9.12. *Let M be a smooth connected orientable n-manifold. The map in Proposition 9.10 is an isomorphism. In particular,*
$$H^n_{\mathrm{dR,c}}(M) \cong \mathbb{R}$$

Proof. In Proposition 9.10, we have already checked that the map is a surjective linear isomorphism. We check that it is injective. Let $\omega \in \Omega^c_c(M)$ such that $\int_M \omega = 0$. Since M is connected and $\mathrm{supp}(\omega)$ is compact, we can take a connected compact set $\mathrm{supp}(\omega) \subseteq K_\omega$. If we can cover K_ω by a good cover which contains only one chart, then Corollary 9.11(2) implies that $\omega = d\mu$ for some $\mu \in \Omega^{n-1}_c(M)$. We can now proceed by induction. Suppose the claim is true if K_ω can be covered by $k-1$ 'good charts,' and suppose $\omega \in \Omega^n_c(M)$ satisfies the property that K_ω admits a good cover $\{U_1, \ldots, U_k\}$. There exists one U_i, say U_k for simplicity, such that both $U = U_1 \cup \cdots \cup U_{k-1}$ and $V = U_k$ are connected[8]. Pick a partition of unity $\{\rho_U, \rho_V\}$ of $U \cup V$ subordinate to the cover $\{U, V\}$, and let $\omega|_U = \rho_U \omega$, $\omega|_V = \rho_V \omega$. Since K_ω is connected, $U \cap V \neq \emptyset$. We pick an n-form ω_0 compactly supported in $U \cap V$ so that
$$\int_M \omega_0 = \int_M \omega|_U.$$

Then $\omega|_U - \omega_0$ is compactly supported in U, which is connected and admits a good cover of $k-1$ good charts, and
$$\int_M (\omega|_U - \omega_0) = 0.$$

[8]This needs proof.

So by the induction hypothesis,

$$\omega_U - \omega_0 = d\eta|_U$$

for some $\eta_U \in \Omega_c^{n-1}(M)$. Similarly,

$$\int_M (\omega|_V + \omega_0) = -\int_M \omega|_U + \int_M \omega_0 = 0$$

implies

$$\omega_V + \omega_0 = d\eta|_V$$

for some $\eta|_V \in \Omega_c^{n-1}(M)$. It follows that

$$\omega = \omega_U + \omega_V = d(\eta_U + \eta_V),$$

where $\eta|_U + \eta|_V \in \Omega_c^{n-1}(M)$. This completes the proof. □

10. de-Rham's Theorem

10.1. Smooth Singular Homology.

10.2. Proof of De-Rham's Theorem.

10.3. Applications.

Part 3. Products & Duality

11. Cup Product

Let's revert back to singular cohomology. However, we will keep on referring to de Rham cohomology for some down-to-earth motivation. We have worked with coefficients G, where G is some abelian group. We now show that if we take $G = R$ to be a commutative ring R, then the singular cohomology with coefficients in R also forms a ring under the cup product operation. First, let's define the algebraic object over which we define the ring structure.

Definition 11.1. Let X be a topological space, and let R be a commutative ring. The total cohomology of X with coefficients in R is given by

$$H^\bullet(X; R) := \bigoplus_{n\geq 0} H^n(X; R).$$

Our aim is to make $H^\bullet(X; R)$ into a graded ring when R is a commutative ring. We shall do this by first making

$$C^\bullet(X; R) := \bigoplus_{n\geq 0} C^n(X; R)$$

into a graded ring, and then showing that the ring structure descends to cohomology. This will be done by introducing a cup product structure on $C^\bullet(X; R)$.

Example 11.2. We first discuss the special case of de Rham cohomology. The advantage here is that we can directly work at the de Rham cohomology groups. Let M be a smooth manifold, and let $\omega \in \Omega^k(M)$, $\eta \in \Omega^l(M)$ be closed forms. If $[\omega] = [\omega']$ and $[\eta] = [\eta']$, we have

$$\omega = \omega' + d\alpha, \qquad \eta = \eta' + d\beta$$

for $\alpha \in \Omega^{k-1}(M)$ and $\beta \in \Omega^{l-1}(M)$. Note that we have

$$\begin{aligned} \omega \wedge \eta &= (\omega' + d\alpha) \wedge (\eta' + d\beta) \\ &= \omega' \wedge \eta' + \omega' \wedge d\beta + d\alpha \wedge \eta' + d(\alpha \wedge \beta) \\ &= \omega' \wedge \eta' - d\beta \wedge \omega' + d\alpha \wedge \eta' + d(\alpha \wedge \beta) \\ &= \omega' \wedge \eta' - d(\beta \wedge \omega') + d(\alpha \wedge \eta') + d(\alpha \wedge \beta) \\ &= \omega' \wedge \eta' - d(\beta \wedge \omega' + \alpha \wedge \eta' + \alpha \wedge \beta). \end{aligned}$$

Hence, $[\omega \wedge \eta] = [\omega' \wedge \eta']$. This shows that the wedge product

$$\wedge : \Omega^k(M) \times \Omega^l(M) \to \Omega^{k+l}(M)$$

descends to a well-defined bilinear map

$$\begin{aligned} \smile\colon\ H^k_{\mathrm{dR}}(M) \times H^l_{\mathrm{dR}}(M) &\to H^{k+l}_{\mathrm{dR}}(M), \\ [\omega] \smile [\eta] &\mapsto [\omega \wedge \eta]. \end{aligned}$$

This is called the cup product in de-Rham cohomology.

Let's now move back to the singular cohomology case and define the cup product. We first define it at the level of $C^\bullet(X; R)$.

Definition 11.3. Let $\phi \in C^k(X; R)$ and $\psi \in C^l(X; R)$. The cup product $\phi \smile \psi \in C^{k+l}(X; R)$ is defined by:

$$(\phi \smile \psi)(\sigma : \Delta^{k+l} \to X) = \phi(\sigma|_{[v_0,\ldots,v_k]}) \cdot \psi(\sigma|_{[v_k,\ldots,v_{k+l}]})$$

where '$\cdot$' denotes the multiplication in the ring R.

Remark 11.4. *Technically, these restricted maps in Definition 11.3 have the wrong domains; they aren't the standard k, l-simplices. But we just pre-compose with the 'obvious' maps from the standard simplices. We shall not do this below.*

The cup product extends by linearity to define a function $C^k(X;R) \times C^l(X;R) \to C^{k+l}(X;R)$ by

$$\left(\sum_i \phi_i\right) \smile \left(\sum_j \psi_j\right) := \sum_{i,j} \phi_i \smile \psi_j.$$

Let us first check this gives us a ring structure.

Lemma 11.5. *Let X be a topological space and let R be a commutative ring. Then $C^\bullet(X;R)$ is a graded ring under the cup product. If R has an identity then $C^\bullet(X;R)$ also has an identity.*

Proof. Suppose $\phi \in C^k(X;R)$ and $\psi, \gamma \in C^l(X;R)$. We claim that $\phi \smile (\psi + \gamma) = \phi \smile \psi + \phi \smile \gamma$. For this, take $\sigma : \Delta^{k+l} \to X$. Then

$$\begin{aligned}(\phi \smile (\psi+\gamma))(\sigma) &= \phi(\sigma_{[v_0,\ldots,v_k]}) \cdot (\psi+\gamma)(\sigma_{[v_k,\ldots,v_{k+l}]}) \\ &= \phi(\sigma_{[v_0,\ldots,v_k]}) \cdot \psi(\sigma_{[v_k,\ldots,v_{k+l}]}) + \phi(\sigma_{[v_0,\ldots,v_k]})) \cdot \gamma(\sigma_{[v_k,\ldots,v_{k+l}]}) \\ &= \phi \smile \psi(\sigma_{[v_k,\ldots,v_{k+l}]}) + \phi \smile \gamma(\sigma_{[v_k,\ldots,v_{k+l}]}).\end{aligned}$$

A similar computation shows that $(\phi+\psi) \smile \gamma = \phi \smile \gamma + \psi \smile \gamma$. Associativity follows by a similar computation. Let 1_R denote the identity in R. Define a cochain $\nu \in C^0(X;R)$ by $\nu(x) = 1_R \quad \forall x \in X$ and extend by linearity. It is clear that

$$\nu \smile \phi = \phi = \phi \smile \nu$$

for any $\phi \in C^n(X;R)$ and any $n \geq 0$. Thus, $C^\bullet(X;R)$ is indeed a graded ring. $\square$

Unfortunately, the ring structure on $C^\bullet(X;R)$ is not very useful, as it is too "large" and almost impossible to compute. However, as we will now see, the total cohomology $H^\bullet(X;R)$ also inherits a ring structure, and this structure is much nicer. We need the following result:

Lemma 11.6. *Let $\phi \in C^k(X;R)$ and $\psi \in C^l(X;R)$*

$$\delta^{k+l}(\phi \smile \psi) = \delta^k \phi \smile \psi + (-1)^k \phi \smile \delta^l \psi$$

Proof. For $\sigma : \Delta^{k+l+1} \to X$, we have

$$(\delta^k \phi \smile \psi)(\sigma) = \sum_{i=0}^{k} (-1)^i \phi(\sigma_{[v_0,\ldots,\widehat{v}_i,\ldots,v_{k+1}]}) \cdot \psi(\sigma_{[v_{k+1},\ldots,v_{k+l+1}]}),$$

$$(-1)^k (\phi \smile \delta\psi)(\sigma) = \sum_{i=k}^{k+l+1} (-1)^i \phi(\sigma_{[v_0,\ldots,v_k]}) \cdot \psi(\sigma_{[v_k,\ldots,\widehat{v}_i,\ldots,v_{k+l+1}]}).$$

When we add these two expressions, the last term of the first sum cancels with the first term of the second sum, and the remaining terms are exactly $\delta^{k+l}(\phi \smile \psi)(\sigma) = (\phi \smile \psi)(\partial_{k+l+1}\sigma)$ since

$$\partial_{k+l+1}\sigma = \sum_{i=0}^{k+l+1} (-1)^i \sigma_{[v_0,\ldots,\widehat{v}_i,\ldots,v_{k+l+1}]}.$$

This completes the proof. $\square$

Corollary 11.7. *The following statements are true:*

1. *If $\phi \in C^k(X;R)$ and $\psi \in C^l(X;R)$ are cocycles, then $\delta^{k+l}(\phi \smile \psi) = 0$.*
2. *If $\phi \in C^k(X;R)$ and $\psi \in C^l(X;R)$ are such that one of ϕ or ψ is a cocycle and the other a coboundary, then $\phi \smile \psi$ is a coboundary.*

Proof. The proof is given below:

1. Since $\delta^k\phi = 0$ and $\delta^l\psi = 0$, we have that that
$$\delta^{k+l}(\phi \smile \psi) = \delta^k\phi \smile \psi + (-1)^k\phi \smile \delta^l\psi = 0$$
2. Say $\delta^k\phi = 0$ and $\psi = \delta^{l-1}\eta$. Then
$$\delta^{k+l-1}(\phi \smile \eta) = (-1)^k\phi \smile \delta^{l-1}\eta = (-1)^k\phi \smile \psi$$
The other case is similar.

This completes the proof □

It follows that we get an induced cup product on cohomology:

$$\begin{aligned} \smile : H^k(X;R) \times H^l(X;R) &\to H^{k+l}(X;R) \\ [\phi] \times [\psi] &\mapsto [\phi \smile \psi] \end{aligned}$$

Well-definedness follows from Corollary 11.7. Indeed, if $[\phi] = [\phi']$ and $[\psi] = [\psi']$, then

$$\phi = \phi' + \alpha, \quad \psi = \psi' + \beta$$

where α, β are co-chains. We have

$$\begin{aligned} \phi \smile \psi &= (\phi' + \alpha) \smile (\psi' + \beta) \\ &= \phi' \smile \psi' + (\phi' \smile \beta + \alpha \smile \psi' + \alpha \smile \beta) \end{aligned}$$

Corollary 11.7 implies that the term in paranthesis is a coboundary. Hence,

$$[\phi \smile \psi] = [\phi' \smile \psi']$$

The operation is distributive and associative since it is so on the co-chain level. If R has an identity element, then there is an identity element for the cup product, namely the class $[1] \in H^0(X;R)$ defined by the 0-cocycle taking the value 1_R on each singular 0-simplex. Considering the cup product as an operation on the direct sum of all cohomology groups, we get a (graded) ring structure on $H^\bullet(X;R)$.

Definition 11.8. Let X be a topological space and let R be a commutative ring. The cohomology ring of X is the graded ring

$$H^\bullet(X;R) := \left(\bigoplus_{n \geq 0} H^n(X;R), \smile \right)$$

with respect to the cup product operation. If R has an identity, then so does $H^\bullet(X;R)$.

Remark 11.9. *We can also define the relative cup product. The cup product on cochains*

$$C^k(X;R) \times C^l(X;R) \to C^{k+l}(X;R)$$

restricts to cup products

$$C^k(X, A; R) \times C^l(X; R) \to C^{k+l}(X, A; R),$$
$$C^k(X, A; R) \times C^l(X, A; R) \to C^{k+l}(X, A; R),$$
$$C^k(X; R) \times C^l(X, A; R) \to C^{k+l}(X, A; R).$$

since $C^i(X, A; R)$ can be regarded as the set of cochains vanishing on chains in A, and if φ or ψ vanishes on chains in A, then so does $\varphi \smile \psi$. So there exist relative cup products:

$$H^k(X, A; R) \times H^l(X; R) \to H^{k+l}(X, A; R),$$
$$H^k(X, A; R) \times H^l(X, A; R) \to H^{k+l}(X, A; R),$$
$$H^k(X; R) \times H^l(X, A; R) \to H^{k+l}(X, A; R).$$

In particular, if A is a point, we get a cup product on the reduced cohomology $\widetilde{H}^*(X; R)$. More generally, we can define

$$H^k(X, A; R) \times H^\ell(X, B; R) \to H^{k+\ell}(X, A \cup B; R)$$

when A and B are open subsets of X or sub-complexes of the CW complex X.

Normally, no one computes cohomology rings using the definition of the cup product, as this can be quite tedious for the most part. However, we compute a couple of basic examples:

Example 11.10. (**Spheres**) Let $X = \mathbb{S}^n$ for $n \geq 1$ and $R = \mathbb{Z}$. We have

$$H^k(\mathbb{S}^n; \mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & \text{if } k = 0, n \\ 0, & \text{otherwise} \end{cases}.$$

The generating element in $H^0(\mathbb{S}^n; \mathbb{Z})$ is the identity element. We label the generators of $H^0(\mathbb{S}^n; \mathbb{Z}))$ and $H^n(\mathbb{S}^n; \mathbb{Z})$ as 1 and x respectively. We have the following relations

$$1 \smile 1 = 1, \quad 1 \smile x = x, \quad x \smile 1 = x, \quad x \smile x = 0$$

The last relation is true since $H^{2n}(\mathbb{S}^n; \mathbb{Z}) = 0$. Hence, we have

$$H^*(\mathbb{S}^n; \mathbb{Z}) \cong \frac{\mathbb{Z}[x]}{\langle x^2 \rangle} = \mathbb{Z}[x]/(x^2) \cong \Lambda_{\mathbb{Z}}[x]$$

Here $\Lambda_{\mathbb{Z}}[x]$ is the exterior algebra on two generator over $\mathbb{Z}$.

Remark 11.11. *We can define a cup product for simplicial cohomology by the same formula as for singular cohomology. It can be checked that the isomorphism between simplicial and singular cohomology respects cup products. Hence, we can compute cup products using simplicial cohomology.*

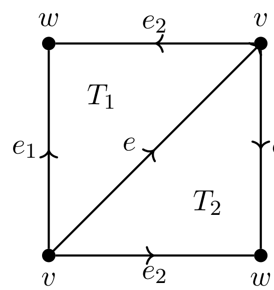
Example 11.12. (**Real Projective Plane**) Let $X = \mathbb{RP}^2$ and $R = \mathbb{Z}_2$. We have

$$H^k(\mathbb{RP}^2; \mathbb{Z}_2) \cong \begin{cases} \mathbb{Z}_2 & k = 0, 1, 2, \\ 0 & \text{otherwise.} \end{cases}$$

Let α be the generator of $H^1(\mathbb{RP}^2; \mathbb{Z}_2)$. Consider

$$\alpha^2 := \alpha \smile \alpha \in H^2(\mathbb{RP}^2; \mathbb{Z}_2).$$

We claim that $\alpha^2 \neq 0$, so α^2 is in fact the generator of $H^2(\mathbb{RP}^2;\mathbb{Z}_2) \cong \mathbb{Z}_2$. Consider the cell structure on $\mathbb{RP}^2$ shown in the figure below. The 2-cell T_1 is attached by the word $e_1 e_2^{-1} e^{-1}$, and the 2-cell T_2 is attached by the word $e_2 e_1^{-1} e^{-1}$.



Since α is a generator of $H^1(\mathbb{RP}^2;\mathbb{Z}/2\mathbb{Z}) \cong \mathrm{Hom}_{\mathbb{Z}}(H_1(\mathbb{RP}^2;\mathbb{Z}),\mathbb{Z}/2\mathbb{Z})$, it is represented by a cocycle

$$\varphi : C_1(\mathbb{RP}^2) \to \mathbb{Z}/2\mathbb{Z}$$

with $\varphi(e) = 1$, where e represents the generator of $H_1(\mathbb{RP}^2;\mathbb{Z}) \cong \mathbb{Z}_2$. The co-cycle condition for φ translates into the identities:

$$0 = (\delta\varphi)(T_1) = \varphi(\partial T_1) = \varphi(e_1) - \varphi(e_2) - \varphi(e),$$

$$0 = (\delta\varphi)(T_2) = \varphi(\partial T_2) = \varphi(e_2) - \varphi(e_1) - \varphi(e_1).$$

As $\varphi(e) = 1$, we may WLOG take $\varphi(e_1) = 1$ and $\varphi(e_2) = 0$. Note that α^2 is represented by $\varphi \smile \varphi$, and we have:

$$(\varphi \smile \varphi)(T_1) = \varphi(e_1) \cdot \varphi(e) = 1.$$

Similarly,

$$(\varphi \smile \varphi)(T_2) = \varphi(e_2) \cdot \varphi(e) = 0.$$

Since the generator of $C_2(\mathbb{RP}^2)$ is $T_1 + T_2$, and we have

$$(\varphi \smile \varphi)(T_1 + T_2) = (\varphi \smile \varphi)(T_1) + (\varphi \smile \varphi)(T_2) = 1 + 0 = 1,$$

it follows that $\alpha^2 = [\varphi \smile \varphi]$ is the generator of $H^2(\mathbb{RP}^2;\mathbb{Z}/2\mathbb{Z})$. Let I denote the ideal generated by the relations. Hence, we have

$$H^*(\mathbb{RP}^n;\mathbb{Z}_2) = \frac{\mathbb{Z}_2[x]}{I} \cong \mathbb{Z}_2[x]/(x^3)$$

Let's prove some important facts about the cup product.

Proposition 11.13. *Let X, Y be topological spaces and let let $f : X \to Y$ be a continuous map. For each $n \in \mathbb{Z}$, the induced maps*

$$f_n^* = H^n(Y;R) \to H^n(X;R)$$

are ring homomorphisms. That is,

$$f_n^*(\alpha \smile \beta) = f^*(\alpha) \smile f^*(\beta)$$

for each $\alpha, \beta \in H^k(Y;R)$.

Proof. It suffices to show the following co-chain formula:

$$f^{\#}(\varphi \smile \psi) = f^{\#}(\varphi) \smile f^{\#}(\psi).$$

For $\varphi \in C^k(Y;\mathbb{R})$ and $\psi \in C^l(Y;\mathbb{R})$, we have:

$$\begin{aligned}
(f^{\#}\varphi \smile f^{\#}\psi)(\sigma : \Delta^{k+l} \to X) &= (f^{\#}\varphi)(\sigma|_{[v_0,\dots,v_k]}) \cdot (f^{\#}\psi)(\sigma|_{[v_k,\dots,v_{k+l}]}) \\
&= \varphi((f^{\#}\sigma)|_{[v_0,\dots,v_k]}) \cdot \psi((f^{\#}\sigma)|_{[v_k,\dots,v_{k+l}]}) \\
&= (\varphi \smile \psi)(f^{\#}\sigma) \\
&= (f^{\#}(\varphi \smile \psi))(\sigma).
\end{aligned}$$

This completes the proof. □

Corollary 11.14. *If $f : X \to Y$ is a continuous map, then there is a ring homomorphism*

$$f^* : H^*(Y;R) \to H^*(X;R).$$

Proof. We have

$$H^*(Y;R) = \bigoplus_{n\geq 0} H^n(Y;R), \quad H^*(X;R) = \bigoplus_{n\geq 0} H^n(X;R)$$

If we define f^* such that $f^*|_{H^n(Y;R)} = f_n$, the claim follows via Proposition 11.13. □

Remark 11.15. *The discussion above implies that the operation of taking the cohomology ring is a (contravariant) functor from* **Top** *to* **CRing**.

Example 11.16. The isomorphisms

$$H^*\left(\coprod_\alpha X_\alpha; R\right) \cong \prod_\alpha H^*(X_\alpha;R)$$

whose coordinates are induced by the inclusions $i_\alpha : X_\alpha \hookrightarrow \coprod_\alpha X_\alpha$, is a ring isomorphism with respect to the coordinatewise multiplication in a ring product, since each coordinate function i_α^* is a ring homomorphism. Similarly, the group isomorphism

$$H^*(\bigvee_\alpha X_\alpha; R) \cong \prod_\alpha H^*(X_\alpha;R)$$

is a ring isomorphism.

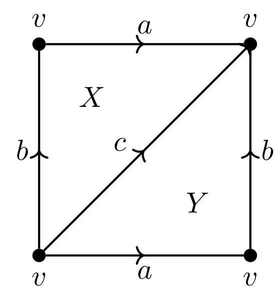
We now show that the cup product is graded anti-commutative.

Proposition 11.17. *Let X be a topological space and let R be a commutative ring. Let $\alpha \in H^k(X;R)$ and $\beta \in H^l(X;R)$. We have*

$$\alpha \smile \beta = (-1)^{kl}\beta \smile \alpha$$

Proof. See [Hat02]. □

Example 11.18. Let $X = \mathbb{S}^1 \times \mathbb{S}^1 = T^2$. We can use Proposition 11.17 to compute $H^*(T^2; R)$. Consider the following simplicial complex structure on T^2:



The generator $1 \in H^0(T^2;\mathbb{Z})$ is the unit. By examining the dimensions of the other generators, the only non-identity generators which could multiply together and give something non-zero are the generators of $H^1(T^2;\mathbb{Z})$. Let $\alpha, \beta \in H^1(T^2;\mathbb{Z})$ be generators of $H^1(T^2;\mathbb{Z})$. We compute

$$\alpha \smile \alpha, \quad \alpha \smile \beta, \quad \beta \smile \alpha, \quad \beta \smile \beta.$$

By Proposition 11.17, we must have $\alpha \smile \alpha = \beta \smile \beta = 0$. But let's verify it explicitly. α is represented by a cocycle

$$\varphi_\alpha : C_1(T^2) \to \mathbb{Z}$$

with $\varphi_\alpha(a) = 1, \varphi_\alpha(b) = 0$. Here a, b are generators of $H_1(T^2;\mathbb{Z})$. The co-cycle condition for φ translates into the identities:

$$0 = (\delta\varphi_\alpha)(X) = \varphi_\alpha(X) = \varphi_\alpha(a) - \varphi_\alpha(c) + \varphi_\alpha(b),$$

$$0 = (\delta\varphi_\alpha)(Y) = \varphi_\alpha(Y) = \varphi_\alpha(b) - \varphi_\alpha(c) + \varphi_\alpha(a).$$

As $\varphi_\alpha(a) = 1, \varphi_\alpha(b) = 0$, we must have $\varphi_\alpha(c) = 1$. Note that α^2 is represented by $\varphi \smile \varphi$, and we have:

$$(\varphi_\alpha \smile \varphi_\alpha)(X) = \varphi_\alpha(b) \cdot \varphi_\alpha(a) = 1.$$

$$(\varphi_\alpha \smile \varphi_\alpha)(Y) = \varphi_\alpha(a) \cdot \varphi_\alpha(b) = 1.$$

Hence, $\varphi_\alpha \smile \varphi_\alpha = 0$. This shows that $\alpha \smile \alpha = 0$. If we choose β to be represented by a cocycle

$$\varphi_\beta : C_1(T^2) \to \mathbb{Z}$$

with $\varphi_\beta(b) = 1, \varphi_\beta(a) = 0$, we similarly have $\beta \smile \beta = 0$. We now compute $\alpha \smile \beta$. Note that $\alpha \smile \beta$ is represented by $\varphi_\alpha \smile \varphi_\beta$. We have

$$(\varphi_\alpha \smile \varphi_\beta)(X) = \varphi_\alpha(b) \cdot \varphi_\alpha(a) = 0.$$

$$(\varphi_\alpha \smile \varphi_\beta)(Y) = \varphi_\alpha(a) \cdot \varphi_\alpha(b) = 1.$$

Since the generator of $C_2(T^2)$ is $X + Y$, and $(\varphi_\alpha \smile \varphi_\beta)(X + Y) = 1$, it follows that $\alpha \smile \beta$ is the generator of $H^2(T^2;\mathbb{Z})$. By Proposition 11.17, we have $\beta \smile \alpha = -\alpha \smile \beta$. Hence, we have

$$H^*(T^2;\mathbb{Z}) \cong \frac{\mathbb{Z}[x, y]}{\langle x^2, y^2, xy + yx\rangle} \cong \Lambda_{\mathbb{Z}}[x, y]$$

Here $\Lambda_{\mathbb{Z}}[x, y]$ is the exterior algebra on two generator over $\mathbb{Z}$.

12. Poincaré Duality for Smooth Manifolds

We discuss Poincaré duality for smooth, oriented, n-manifolds in this section. We can prove this special case by leveraging de Rham cohomology. Using Stokes' theorem, Poincaré duality for smooth, oriented n-manifolds asserts that there is a non-degenerate pairing between de Rham cohomology groups:

$$H^k_{\mathrm{dR}}(M) \times H^{n-k}_{\mathrm{dR,c}}(M) \to \mathbb{R}, \quad ([\alpha],[\beta]) \mapsto \int_M \alpha \wedge \beta.$$

It is easily checked that the pairing defined above is well-defined. The pairing above can be equivalently defined as a linear map from $H^k_{\mathrm{dR}}(M)$ to $(H^{n-k}_{\mathrm{dR,c}}(M))^*$. We show that this linear map is an isomorphism.

Proposition 12.1. *Let M be a smooth, oriented, n-manifold that admits a good finite cover. Then*

$$H^k_{dR}(M) \cong (H^{n-k}_{dR,c}(M))^*$$

for each $0 \leq k \leq n$.

13. Poincaré Duality

Poincaré duality is a fundamental result in algebraic topology that relates the homology and cohomology groups of an orientable closed manifold. It states that for an n-dimensional orientable manifold M, there exists an isomorphism

$$H_k(M;\mathbb{Z}) \cong H^{n-k}_c(M;\mathbb{Z})$$

This duality provides deep insights into the topology of manifolds, constraining their possible homology groups and aiding in the computation of topological invariants. It also plays a crucial role in intersection theory. Before defining Poincaré duality, we need to define the notation of a fundamental class. In order to define a fundamental class, we need to define the notation of an orientation.

References

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