HOMOLOGY

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Abstract. These are notes on algebraic topology covering singular homology. I took these notes during graduate school to better understand singular homology. The purpose of the notes is to discuss the theoretical foundations of singular homology, focusing on examples from low dimensions (≤ 2) and simple examples in arbitrary dimensions usually covered in a first year graduate course. There may be typos; please send corrections to junaid.aftab1994@gmail.com.

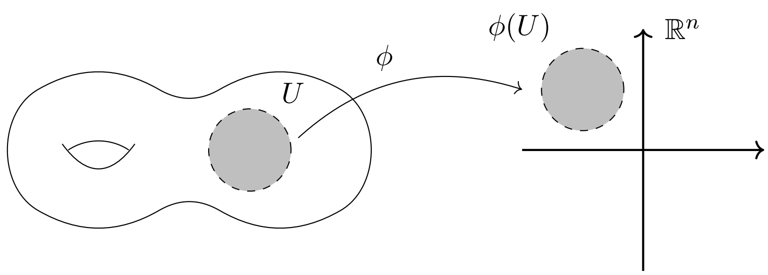
Contents

Part 1. Preliminaries

1. Topological Manifolds

One of the primary class of spaces we aim to understand using topological invariants in algebraic topology are topological manifolds. These are spaces that locally resemble Euclidean space. Familiar examples include plane curves such as circles and parabolas, as well as surfaces like spheres and tori.

Definition 1.1. Let X be a topological space. X is a topological n-manifold if X is a second-countable, Hausdorff space that is locally homemorphic to $\mathbb{R}^n$. That is, each point of X is contained in a coordinate chart, which is a pair (U, ϕ), where U is an open subset of X and $\phi : U \to \phi(U) \subseteq \mathbb{R}^n$ is a homeomorphism from U to an open subset $\phi(U)$ of $\mathbb{R}^n$.



Remark 1.2. *The number n is attached to a single chart and might apriori depend on the chart itself. This turns out to be not the case. This result is called the invariance of dimension and will be proved later.*

Remark 1.3. *A collection of charts (U_α, ϕ_α) such that $\bigcup_\alpha U_\alpha = M$ is an atlas for X.*

We discuss the implications of the conditions imposed in Definition 1.1. Since a topological manifold is locally Euclidean, it is easy to see that it inherits a number of properties of Euclidean space locally. For instance, we have the following:

Proposition 1.4. *Let X be a topological n-manifold. Then X is locally compact, locally path-connected and locally contractible.*

Proof. Every point of X has a neighborhood homeomorphic to the open unit ball in $\mathbb{R}^n$. Each open ball in $\mathbb{R}^n$ is locally compact, locally compact and locally path-connected, locally contractible. The claim follows. □

The locally Euclidean condition does not impose any topological properties at the global level. The second-countablity and Hausdorff conditions account for this detail. Intuitively, Hausdorff spaces have 'enough open sets.' This ensures that familiar properties hold: for example, in a Hausdorff space, finite subsets are closed, limits of convergent sequences are unique etc. Moreover, this condition also excludes certain pathological examples like the line with two origins etc. On the other hand, second-countable spaces 'don't have too many open sets that are required to cover the space.' The following is a sample global topological property of a topological n-manifold.

Proposition 1.5. *Let X be a topological n-manifold. X has a countable basis of precompact coordinate balls.*

Proof. First consider the special case in which X can be covered by a single chart. Suppose $\varphi : M \to U \subseteq \mathbb{R}^n$ is a global coordinate chart. Let

$$\mathcal{B} = \{B_r(x) : x \in \mathbb{Q},\ x \in \mathbb{Q}^n\ B_{r'}(x) \subseteq U \text{ for some } r' < r\}$$

Each $B_r(x) \in \mathcal{B}$ is pre-compact in U, and it is easy to check that $\mathcal{B}$ is a countable basis for the topology of U. Because φ is a homeomorphism, it follows that $\varphi^{-1}(\mathcal{B})$ is a countable basis for X, consisting of pre-compact coordinate balls. More generally, each $p \in M$ is in the domain of a coordinate chart. Since X is second-countable, X is covered by countably many coordinate charts $\{(U_i, \varphi_i)\}_{i=1}^{\infty}$. By the argument in the preceding paragraph, each U_i has a countable basis of coordinate balls that are pre-compact in U_i. If $V \subseteq U_i$ is one of these balls, then the closure of V in U_i is compact, and because X is Hausdorff, it is closed in X. It follows that the closure of V in X is the same as its closure in U_i, so V is precompact in X as well. Clearly, the union of all these countable bases is a countable basis for X. $\square$

Example 1.6. The following is a list of examples of topological manifolds.

(1) $\mathbb{R}^n$ is a topological n-manifold. $\mathbb{R}^n$ is covered by a single chart $(\mathbb{R}^n, \mathrm{Id}_{\mathbb{R}^n})$, where $\mathrm{Id}_{\mathbb{R}^n} : \mathbb{R}^n \to \mathbb{R}^n$ is the identity map.

(2) (**Spheres**) The unit n-sphere, $\mathbb{S}^n$, is Hausdorff and second-countable because it is a topological subspace of $\mathbb{R}^{n+1}$. For each $1 \leq i \leq n+1$, consider the sets:

$$U_i^+ = \{(u^1, \cdots, u^{n+1}) \in \mathbb{R}^{n+1} \mid u^i > 0\}$$
$$U_i^- = \{(u^1, \cdots, u^{n+1}) \in \mathbb{R}^{n+1} \mid u^i < 0\}$$

Let $f : \mathbb{B}^n \to \mathbb{R}$ be the continuous function defined by

$$f(x) = \sqrt{1 - \|u\|^2}$$

For each $1 \leq i \leq n$, $U_i^{\pm} \cap \mathbb{S}^n$ is the graph of the function

$$u^i = \pm f(u^1, \cdots, \widehat{u^i}, \cdots, u^{n+1}),$$

where the hat indicates that u^i is omitted. Thus, each subset $U_i^{\pm} \cap \mathbb{S}^n$ is locally Euclidean of dimension n, and the maps $\phi_i \colon U_i^{\pm} \cap \mathbb{S}^n \to \mathbb{B}^n$ given by

$$\phi_i(u^1, \cdots, u^{n+1}) = (u^1, \cdots, \widehat{u^i}, \cdots, u^{n+1})$$

defines the desired homemorphism.

(3) (**Real Projective Space**) The real projective space, $\mathbb{RP}^n$, is defined as the quotient space, $\mathbb{RP}^n = (\mathbb{R}^{n+1} \setminus \{0\})/\sim$ with the equivalence relation

$$x \sim y \text{ in } \mathbb{R}^{n+1} \setminus \{0\} \iff x = \lambda y \text{ for some } \lambda \in \mathbb{R}^\times$$

It is made into a topological space by giving it the quotient topology via the map

$$\pi \colon \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{RP}^n,$$

where $[x] := \pi(x) = \operatorname{span}\{x\}$. It can be easily checked that

$$\mathbb{RP}^n \cong \mathbb{S}^n/\sim$$

where $\sim$ is the equivalence relation on $\mathbb{S}^n$ such that $x \sim -x$ (i.e., antipodal points are identified). We check that $\mathbb{RP}^n$ is both second-countable and Hausdorff:

(a) Consider the quotient map: $q : \mathbb{S}^n \to \mathbb{S}^n/\sim$ Note that q is an open map. Indeed for any open subset $V \subseteq \mathbb{S}^n$, we have:

$$q^{-1}(q(V)) = V \cup -V,$$

Since $\mathbb{S}^n$ is second-countable, $\mathbb{RP}^n \cong \mathbb{S}^n/\sim$ is also second-countable as q is an open map.

(b) If $[x], [y] \in \mathbb{S}^n/\sim$, then one can choose $\varepsilon > 0$ small enough that

$$U = \mathbb{B}(x, \varepsilon) \cap \mathbb{S}^n$$
$$V = \mathbb{B}(y, \varepsilon) \cap \mathbb{S}^n$$

are open sets in $\mathbb{S}^n$ such that $\pm U, \pm V$ are pairwise disjoint. Since,

$$q^{-1}(q(U)) = U \cup -U$$
$$q^{-1}(q(V)) = V \cup -V$$

$q^{-1}(q(U))$ and $q^{-1}(q(V))$ are open disjoint subsets of $\mathbb{S}^n/\sim$ containing $[x]$ and $[y]$. Hence, $\mathbb{RP}^n \cong \mathbb{S}^n/\sim$ is Hausdorff.

For each $1 \leq i \leq n+1$, consider the sets:

$$\widetilde{U}_i = \{(u^1, \cdots, u^{n+1}) \in \mathbb{R}^{n+1} \mid u^i \neq 0\}$$

Let $U_i = \pi(\widetilde{U}_i)$. By properties of the quotient topology, U_i is an open subset of $\mathbb{R}^n$. Consider the map $\phi_i \colon U_i \to \mathbb{R}^n$ defined as:

$$\phi_i\left([u]\right) = \left(\frac{u^1}{u^i}, \cdots, \frac{u^{i-1}}{u^i}, 1, \frac{u^{i+1}}{u^i}, \cdots, \frac{u^{n+1}}{u^i}\right).$$

This map is well-defined because its value is unchanged by multiplying x by a nonzero constant. By properties of the quotient topology, ϕ_i is continuous. In fact, ϕ_i is a homeomorphism because it has a continuous inverse given by

$$\phi_i^{-1}(u^1, \cdots, u^n) = [u^1, \cdots, u^{i-1}, 1, u^i, \cdots, u^n];$$

This shows that $\mathbb{RP}^n$ is locally Euclidean of dimension n.

(4) (**Tori**) For a positive integer $n \geq 2$, the $(n-1)$-torus is the product space

$$\mathbb{S}^1 \times \cdots \times \mathbb{S}^1$$

It is clear that a product of topological manofolds is a topolopgical manifold. Hence, T is topological n-manifold since $\mathbb{S}^1$ is a 1-manifold.

Remark 1.7. *For $n \geq 2$, we usually abbreviate the n-torus as $\mathbb{T}^{n-1}$.*

Sets such as closed intervals in $\mathbb{R}$ and closed balls in $\mathbb{R}^n$ fail to be both topological manifolds since they 'have a boundary of sorts.' We make precise the notion of a topological manifold with boundary.

Definition 1.8. Let X be a topological space. X is a topological n-manifold with boundary if X is a second countable, Hausdorff space such that each point $x \in M$ is contained in a coordinate chart, (U, ϕ), such that:

(1) (**Interior Chart**) Either $\phi : U \to \phi(U) \subseteq \mathbb{R}^n$ is a homeomorphism from U to an open subset $\phi(U)$ of $\mathbb{R}^n$.
(2) (**Boundary Chart**) Or $\phi : U \to \phi(U) \subseteq \mathbb{H}^n$ is a homeomorphism from U to an open subset $\phi(U)$ of $\mathbb{H}^n$, the upper-half plane, such that $\phi(x) \cap \partial\mathbb{H}^n \neq \emptyset$.

A point $p \in M$ is called an interior point of X if it is in the domain of some interior chart or a boundary chart (U, ϕ) such that $\phi(U) \cap \partial\mathbb{H}^n = \emptyset$. It is a boundary point of X if it is in the domain of a boundary chart that sends p to $\partial\mathbb{H}^n$. The boundary of X (the set of all its boundary points) is denoted by ∂M; similarly, its interior, the set of all its interior points, is denoted by $\mathrm{Int}(M)$.

Remark 1.9. *A point $p \in M$ might apriori simultaneously be a boundary point and an interior point, meaning that there is one interior chart whose domain contains p, and another boundary chart that sends p to $\partial\mathbb{H}^n$. This turns out not to be the case. This result is called the invariance of boundary and will be proved later.*

Example 1.10. (Sketch) The following is a list of basic examples of a topological manifold with boundary.

(1) $\overline{\mathbb{B}^n}$ is smooth n-manifold with boundary. One can prove this by definition. We skip details.
(2) If X is a n-dimesnional manifold with boundary, then ∂M is a $(n-1)$-dimensional manifold without boundary. We skip details.

2. CW Complexes

2.1. **Definitions.** Another primary class of spaces we aim to understand using topological invariants in algebraic topology are topological manifolds. An arbitrary topological space, X, can be difficult to visualize and analyze. We shall focus mostly focus on the subcategory of topological spaces that can be constructed inductively using open cells. This will be category of CW-complexes. This approach will allow us to meaningfully study a lot of topological spaces.

Definition 2.1. An open n-cell is a topological space that is homeomorphic to the open unit ball $\mathbb{B}^n$. A closed n-cell is a topological space homeomorphic to $\mathbb{D}^n$.

Remark 2.2. *We will only use the phrase n-cell when the context is clear.*

New topological spaces can be constructed from old topological spaces by attaching an n-cell. Let X be a topological space. Suppose there is a map $\varphi : \mathbb{S}^{n-1} \to X$ a map. One can form a new topological space, $X \coprod_\varphi \mathbb{D}^n$, from the disjoint union $X \coprod \mathbb{D}^n$ by identifying each $\varphi(x) \in \mathbb{S}^{n-1}$ with $\varphi(x) \in X$ for all $x \in \mathbb{S}^{n-1}$, and equipping the resulting set with the quotient topology. The map φ is called the characteristic map. We refer to the space $X \coprod_\varphi \mathbb{D}^n$ as being obtained from X by 'attaching an n-cell', and call $\varphi : \mathbb{S}^{n-1} \to X$

the attaching map. Using the the universal properties of the disjoint union and quotient topology, we have the following commutative diagram.

$$\begin{tikzcd}
\mathbb{S}^{n-1} \arrow[r, "\varphi"] \arrow[d, hook, "\iota"] & X \arrow[d] \arrow[ddr, bend left, "f_1"] & \\
\mathbb{D}^n \arrow[r] \arrow[drr, bend right, "f_2"] & X \coprod_{\varphi} \mathbb{D}^n \arrow[dr, dotted, "f"] & \\
& & Y
\end{tikzcd}$$

Remark 2.3. *In fact, this shows that $X \coprod_{\varphi} \mathbb{D}^n$ is a pushout in* **Top**.

One can also attach more than one n-cell. Let $\{\mathbb{D}_i^n\}_{i \in I_n}$ be a collection of n-cells and let $\varphi_i^n : \mathbb{S}_i^{n-1} \to X$ be a collection of continuous maps. One can form a new topological space, $X \coprod_{\substack{i \in I_n \\ \varphi_i}} \mathbb{D}_i^n$, by attaching the aforementioned collection of n-cells using the rule prescribed above. Once again, we have a commutative diagram:

$$\begin{tikzcd}
\coprod_{\substack{i \in I_n \\ \varphi_i^n}} \mathbb{S}_i^{n-1} \arrow[r, "f"] \arrow[d, hook, "\iota"] & X \arrow[d] \\
\coprod_{\substack{i \in I_n \\ \varphi_i^n}} \mathbb{D}_i^n \arrow[r] & X \coprod_{\substack{i \in I_n \\ \varphi_i^n}} \mathbb{D}_i^n
\end{tikzcd}$$

Remark 2.4. *This shows that $X \coprod_{\substack{i \in I_n \\ \varphi_i^n}} \mathbb{D}_i^n$ is a pushout in* **Top**.

Definition 2.5. Let X be a topological space. A CW decomposition of X is a sequence of subspaces

$$X^0 \subseteq X^1 \subseteq X^2 \subseteq \cdots \subseteq X^n \subseteq \cdots \quad n \in \mathbb{N},$$

of X such that the following three conditions are satisfied:

(1) The space X^0 is discrete.
(2) The space X^n is obtained from X^{n-1} by attaching a (possibly) infinite number of n-cells $\{\mathbb{D}_i^n\}_{i \in I_n}$ via attaching maps $\varphi_i : \mathbb{S}_i^{n-1} \to X^{n-1}$.
(3) The topology of X is compatible with quotient topology on X that makes the

$$\coprod_{n \in \mathbb{N}} X^n \to X$$

continuous. In other words, $A \subseteq X$ is open if and only if $A \cap X^n$ is open for all $n \geq 0$.

Remark 2.6. *If X admits a CW decomposition, then it can be easily checked that X is a colimit of $\{X^n\}_{n \in \mathbb{N} \cup \{0\}}$. In particular, X is the colimit of the diagram*

$$X^0 \xrightarrow{j_0} X^1 \to \cdots \to X^n \xrightarrow{j_n} X^{n+1} \to \cdots$$

in **Top**. *Here j_n is the inclusion of X_n into X_{n+1}.*

We can now define the following categories:

(1) **CW** is the category of whose objects are topological spaces that admit a CW structure and morphisms between CW complexes are cellular continuous maps. That is, $f(X^n) \subseteq Y^n$ for each $n \geq 0$ where f is a continuous map. In other words, if X and Y are CW complexes and we have a commutative diagram

$$\begin{tikzcd} X^0 \arrow[r] \arrow[d] & X^1 \arrow[r] \arrow[d] & \cdots \arrow[r] & X^n \arrow[r] \arrow[d] & X^{n+1} \arrow[r] \arrow[d] & \cdots \\ Y^0 \arrow[r] & Y^1 \arrow[r] & \cdots \arrow[r] & Y^n \arrow[r] & Y^{n+1} \arrow[r] & \cdots \end{tikzcd}$$

then, on forming the colimits, we obtain an induced map $f : X \to Y$ which is a cellular map.

(2) $\mathbf{CW}_*$ is the category of pointed CW complexes defined analogously to $\mathbf{Top}_*$.

(3) $\mathbf{CW}^2$ is the category of pairs of CW complexes defined analogously to $\mathbf{Top}^2$.

Remark 2.7. *The categories $\mathbf{CW}_*$ and $\mathbf{CW}^2$ are defined similarly.*

Each cell $\mathbb{D}_i^n$ has its characteristic map Φ_i^n, which is by definition the composition of continuous maps:

$$\begin{tikzcd} \mathbb{D}_i^n \arrow[r, hook] \arrow[rrr, bend left, "\Phi_i^n"] & X^{n-1} \coprod \mathbb{D}_i^n \arrow[r, two heads] & X^{n-1} \coprod_{\varphi} \mathbb{D}_i^n \arrow[r, hook] & X \end{tikzcd}$$

Proposition 2.8. *Let X be a topological space with a CW decomposition. $A \subseteq X$ is open if and only if $(\Phi_i^n)^{-1}(\mathbb{D}_i^n)$ is continuous for each $i \in I_n$ and $n \in \mathbb{N}$. In particular, X is a quotient space of $\coprod_{\substack{n\in\mathbb{N} \\ i \in I_n}} \mathbb{D}_i^n$*

Proof. The forward implication is clear. Conversely, suppose $(\Phi_i^n)^{-1}(\mathbb{D}_i^n)$ is open in $\mathbb{D}_i^n$ for each for each $i \in I_n$ and $n \in \mathbb{N}$. Suppose by induction on n that $A \cap X^{n-1}$ is open in X^{n-1}. Since $(\Phi_i^n)^{-1}(\mathbb{D}_i^n)$ is open in $\mathbb{D}_i^n$ for all $i \in I_n$, then $A \cap X^n$ is open in X^n by the definition of the quotient topology on X^n. The last implication is clear by definition. $\square$

Definition 2.9. Let X be a topological space. X is a CW complex if X admits a CW decomposition satisfying the following two properties:

(1) The closure of each open cell is contained in a union of finitely many cells.

(2) The topology of X is coherent with $\{\{\mathbb{D}_i^n\}_{i\in I_n} : n \in \mathbb{N}\}$[1].

A CW complex is finite (or finite-dimensional) if there are only finitely many cells involved. Every finite CW decomposition is automatically a finite CW complex. In fact, every locally finite CW decomposition is automatically a CW complex as we show below.

Proposition 2.10. *Let X be a topological space endowed with a CW decomposition. If $\{\{\mathbb{D}_i^n\}_{i\in I_n} : n \in \mathbb{N}\}$ is a locally finite collection, then X is a CW complex.*

Proof. By assumption, every point $\mathbb{D}_i^n$ has a neighborhood that intersects only finitely many cells. Since $\mathbb{D}_i^n$ is compact, it is covered by finitely many such neighborhoods. This readily implies (1) in Definition 2.9. Suppose $A \subseteq X$ is a subset such that $A \cap \mathbb{D}_i^n$ is closed for each $i \in I_n$ and $n \in \mathbb{N}$. Given $x \in X \setminus A$, let W_x be a neighborhood of x that intersects

[1]That is, $A \subseteq X$ is open/closed if and only if $A \cap \overline{\mathbb{D}_i^n}$ is open/closed for each $i \in I_n$ and $n \in \mathbb{N}$.

the closures of only finitely many cells, say $\mathbb{D}_1^{n_1}, \ldots, \mathbb{D}_k^{n_k}$. Since $A \setminus \mathbb{D}_j^{n_j}$ is closed in $\mathbb{D}_j^{n_j}$ and thus in X, it follows that

$$W \setminus A = W \setminus (A \cap \mathbb{D}_1^{n_1}) \cup \ldots \cup \left(A \cap \mathbb{D}_k^{n_k}\right)$$

is a neighborhood of x contained in $X \setminus A$. Thus $X \setminus A$ is open, so A is closed. This readily implies (2) in Definition 2.9. □

2.2. **Examples.** In the examples that follows, we will not explicitly check that condition (3) in Definition 2.5 is satisfied. It should be straightforward to do verify these claims, though.

Example 2.11. Let $N = (0, \ldots, 0, 1)$ in $\mathbb{S}^n$. Consider the map $\sigma_N : \mathbb{S}^n \setminus \{N\} \to \mathbb{R}^n$ by [2]

$$\sigma_N(u^1, \ldots, u^{n+1}) = \left(\frac{u^1}{1 - u^{n+1}}, \ldots, \frac{u^n}{1 - u^{n+1}}\right)$$

Similarly, consider $\beta_N : \mathbb{R}^n \to \mathbb{S}^n \setminus \{N\}$

$$\beta_N(u^1, \ldots, u^n) = \left(\frac{2u^1}{|u|^2 + 1}, \cdots, \frac{2u^n}{|u|^2 + 1}, \frac{|u|^2 - 1}{|u|^2 + 1}\right).$$

It is easy to check that σ_N, β_N are inverses of each other. Hence, $\mathbb{R}^n \cong \mathbb{S}^n \setminus \{N\}$. The map σ_N is called the stereographic projection. $\mathbb{S}^n$ can now be given a CW structure with one 0-cell ($\mathbb{D}^0$) and one n-cell ($\mathbb{D}^n$). The attaching map for the n-cell is $\varphi : \mathbb{S}^{n-1} = \partial\mathbb{D}^n \to \{*\}$.

Example 2.12. $\mathbb{S}^n$ can be given a different CW structure with two k-cells in each dimension for $0 \leq k \leq n$. Let $X^0 = \mathbb{S}^0 = \{\mathbb{D}_1^0, \mathbb{D}_2^0\}$. Then $X^1 = \mathbb{S}^1$ where the two 1-cells $\mathbb{D}_1^1, \mathbb{D}_2^1$ are attached to the 0-cells by homeomorphisms on their boundary. Similarly, two 2-cells can be attached to $X^1 = \mathbb{S}^1$ by homeomorphism on their boundary, giving $X^2 = \mathbb{S}^2$. Proceed inductively.

Example 2.13. There are natural inclusions

$$\mathbb{S}^0 \subseteq \mathbb{S}^1 \subseteq \cdots \subseteq \mathbb{S}^n \subseteq \cdots \subseteq$$

We can then define $\mathbb{S}^\infty = \varinjlim_{n \in \mathbb{N}} \mathbb{S}^n$. If $\mathbb{S}^n$ is given a CW structure as in Example 2.12 for each $n \geq 0$, then $\mathbb{S}^\infty$ is a CW complex as well. Note that $\mathbb{S}^\infty$ is a colimit of the $\mathbb{S}^n$'s for $n \geq 0$.

Example 2.14. Consider $\mathbb{RP}^n$ as the quotient of $\mathbb{S}^n$ with anti-podal points identified. An easy observation shows that $\mathbb{RP}^n$ is a quotient of $\mathbb{D}^n$ by the relation $x \sim -x$ on the boundary $\mathbb{S}^{n-1}$[3]. Thus, $\mathbb{RP}^n$ can be obtained from $\mathbb{RP}^{n-1}$ by attaching a one cell.

$$\begin{tikzcd}
\mathbb{S}^{n-1} \arrow[r, hook] \arrow[d, two heads] & \mathbb{D}^n \arrow[d, two heads] \\
\mathbb{RP}^{n-1} \arrow[r, hook] & \mathbb{RP}^n
\end{tikzcd}$$

Thus $\mathbb{RP}^n$ can be built as a CW complex with a single cell in each dimension $\leq n$.

[2]Let $x = (x^1, \ldots, x^{n+1}) \in \mathbb{S}^n \setminus \{N\}$. The line through N and x is parameterized by

$$u^1 = x^1 t, \ldots, u^n = x^n t, u^{n+1} = (x^{n+1} - 1)t + 1$$

The intersection of this line with $u^{n+1} = 0$ occurs when $t = \frac{1}{1 - x^{n+1}}$. Hence, the intersection point is $(\sigma_N(x), 0)$, as desired. Therefore, $\sigma_N(x)$ is the intersection of the line through N and x with the $\mathbb{R}^n$ plane.

[3]It is easy to check that these identifications are consistent with out discussion of the real projective plane, which is $\mathbb{RP}^2$.

Example 2.15. There are natural inclusions

$$\mathbb{RP}^0 \subseteq \mathbb{RP}^1 \subseteq \cdots \subseteq \mathbb{RP}^n \subseteq \cdots \subseteq$$

We can then define $\mathbb{RP}^\infty = \varinjlim_{n\in\mathbb{N}} \mathbb{RP}^n$. Note that $\mathbb{RP}^\infty$ is a colimit of the $\mathbb{RP}^n$'s for $n \geq 0$. We can define $\mathbb{CP}^\infty$ similarly to $\mathbb{RP}^\infty$.

Example 2.16. The complex projective space, $\mathbb{CP}^n$, is defined as the quotient space $\mathbb{CP}^n = (\mathbb{C}^{n+1} \setminus \{0\})/\sim$ with the equivalence relation $x \sim y$ in $\mathbb{C}^{n+1} \setminus \{0\}$ if and only if $x = \lambda y$ for some $\lambda \neq 0$. Note that there is a map

$$\mathbb{D}^{2n} \to \mathbb{CP}^n$$

$$(z_0, \dots, z_{n-1}) \mapsto [z_0, \dots, z_{n-1}, \sqrt{1 - \|z\|}]$$

The boundary of $\mathbb{D}^{2n}$ (where $\sqrt{1-\|z\|} = 0$) is sent to $\mathbb{CP}^{n-1}$. In this way, $\mathbb{CP}^n$ is obtained from $\mathbb{CP}^{n-1}$ by attaching one $2n$-cell. So $\mathbb{CP}^n$ has a CW structure with one cell in each even dimension $0, 2, \dots, 2n$.

Example 2.17. There are natural inclusions

$$\mathbb{CP}^0 \subseteq \mathbb{CP}^1 \subseteq \cdots \subseteq \mathbb{CP}^n \subseteq \cdots \subseteq$$

We can then define $\mathbb{CP}^\infty = \varinjlim_{n\in\mathbb{N}} \mathbb{CP}^n$ as before.

Let's discuss some 2-dimensional examples. It is well-known that compact, connected 2-dimensional manifolds are classified into the following types:

(1) $\mathbb{S}^2$,
(2) A connected sum of g-tori $\mathbb{T}$ (or a g-hold torus) for $g \geq 2$,
(3) A connected sum of g-projective spaces $\mathbb{RP}^2$, for $g \geq 2$.

We have already discussed a CW-structure on $\mathbb{S}^2$. We discuss examples of the other 2-manifolds below:

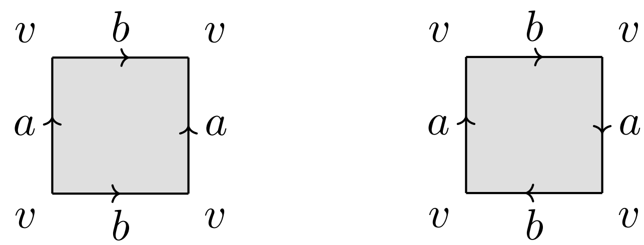
Example 2.18. Consider $X = \mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ (the 1-torus) or $\mathbb{RP}^2$ (the real projective plane). Both spaces can be constructed as quotients of a rectangle by identifying edges according to specific rules: for the torus, opposite edges are identified in the same direction, while for $\mathbb{RP}^2$, one pair of opposite edges are identified normally and the other pair with reversed orientation. These identification diagrams offer a convenient way to visualize the topology of each space. Each space admits a natural CW complex structure with the following cells:

(1) a single 0-cell representing the vertex of the rectangle,
(2) two 1-cells corresponding to the edges of the rectangle,
(3) a single 2-cell which is attached via a continuous map from the boundary circle $\mathbb{S}^1$ into the 1-skeleton.

Example 2.19. For $g \geq 1$, a model for a connected sum of g copies of the torus $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ is denoted by M_g, and is known as an orientable surface of genus g. The surface M_g can be constructed by taking a polygon with $4g$ sides and identifying its edges in pairs according to the word

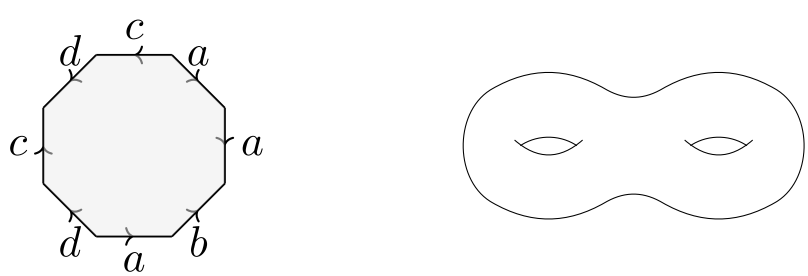
$$a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1},$$

which encodes the edge identifications that yield a closed orientable surface. Each pair a_i, a_i^{-1} and b_i, b_i^{-1} contributes a 'handle,' so M_g can be visualized as a torus with g holes, or a g-holed doughnut. This construction endows M_g with a natural CW complex structure consisting of:



$X = \mathbb{T}$ $X = \mathbb{RP}^2$

(1) a single 0-cell where all loops based on the edges are attached;
(2) $2g$ 1-cells corresponding to the edges of the polygon;
(3) a single 2-cell attached along the loop described by the edge word above.

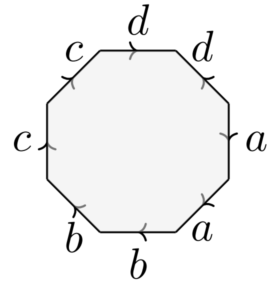


Example 2.20. For $g \geq 2$, a model for the connected sum of g copies of the real projective plane $\mathbb{RP}^2$ is denoted by N_g, and is known as a non-orientable surface of genus g. The surface N_g can be constructed from a polygon with g sides by identifying the edges according to the word

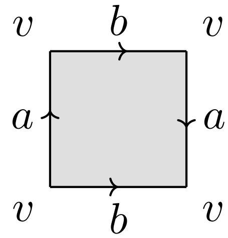
$$a_1 a_1 \cdots a_g a_g,$$

where each pair $a_i a_i$ represents an edge identification. This construction yields a closed surface that is non-orientable and has genus g. The surface N_g admits a CW complex structure consisting of:

(1) a single 0-cell to which all loops are attached;
(2) g 1-cells corresponding to the edges of the polygon;
(3) a single 2-cell attached via a loop following the word $a_1 a_1 \cdots a_g a_g$.



Remark 2.21. *N_2 is usually called a Klein bottle. Another model for the Klein bottle is given by the CW structure shown below:*



It can be checked that both models are homeomorphic.

2.3. **Properties.** A sub-complex of X is a subspace $Y \subseteq X$ that is a union of open cells of X, such that if Y contains a cell, it also contains its closure. It follows immediately that the union and the intersection of any collection of sub-complexes are themselves sub-complexes. Examples of a sub-complexes would be be the subspaces X^n for $n \geq 0$ in the definition of a CW complex.

Proposition 2.22. *Suppose X is a CW complex and Y is a sub-complex of X. Then Y is closed in X, and with the subspace topology and the cell decomposition that it inherits from X, it is a CW complex.*

Proof. Let $\mathbb{B}^n \subseteq Y$ denote such an open n-cell in Y. Since $\overline{\mathbb{B}^n} \subseteq Y$, the finitely many cells of X that have nontrivial intersections with $\mathbb{B}^n$ must also be cells of Y. So condition (1) in Definition 2.9 is automatically satisfied by Y. In addition, any characteristic map $\varphi : \overline{\mathbb{B}^n} \to X$ for $\mathbb{D}^n$ in X also serves as a characteristic map for $\overline{\mathbb{B}^n}$ in Y. Suppose $A \subseteq Y$ is a subset such that $A \cap \mathbb{D}^n$ is closed in $\mathbb{D}^n$ for every n-cell $\mathbb{D}^n$ contained in Y. Let $\mathbb{D}^n$ be a n-cell of X that is not contained in Y. We know that $\mathbb{D}^n \setminus \mathbb{B}^n$ is contained in the union of finitely many open cells of X; some of these, say $\mathbb{B}_1^{n_1}, \ldots, \mathbb{B}_k^{n_k}$, might be contained in Y. Then $\overline{\mathbb{B}_1^{n_1}} \cup \ldots \cup \overline{\mathbb{B}_k^{n_k}} \subseteq Y$, and

$$A \cap \mathbb{D}^n = A \cap (\overline{\mathbb{B}_1^{n_1}} \cup \ldots \cup \overline{\mathbb{B}_k^{n_k}}) \cap \mathbb{D}^n = ((A \cap \overline{\mathbb{B}_1^{n_1}}) \cup \cdots \cup (A \cap \overline{\mathbb{B}_k^{n_k}})) \cap \mathbb{D}^n$$

which is closed in $\mathbb{D}^n$. It follows that A is closed in X and therefore in Y. This implies (2) in Definition 2.9. Hence Y is a CW complex. Taking $A = Y$ shows that Y is closed. □

Proposition 2.23. *The following is a list of some categorical/topological properties of CW complexes.*

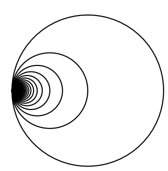
(1) *If A is a subcomplex of X, then the inclusion $\iota : A \hookrightarrow X$ is a cellular map.*
(2) *If A is a subcomplex of X, then X/A is a CW complex such that the quotient map $X \to X/A$ is a cellular map.*
(3) *If X and Y are finite CW complexes, then $X \times Y$ is a CW complex.*
(4) *The closure of each cell in a CW complex is contained in a finite subcomplex.*
(5) *A subset of a CW complex is compact if and only if it is closed and contained in a finite subcomplex.*
(6) *A CW complex is compact if and only if it is a finite complex.*
(7) *A CW complex is locally compact if and only if it is locally finite.*
(8) *A CW complex is locally path-connected.*
(9) *A CW complex is a T_1, normal space. Hence, a CW complex is a Hausdorff space.*

Proof. (**Sketch**) The proof of some of the properties is given below:

(1) The proof is skipped, but it is clear given the definition of a sub-complex.
(2) The proof is skipped.
(3) The proof is skipped.
(4) Let $\mathbb{D}^n$ be an n-cell of a CW complex. We prove the claim by induction on n. If $n = 0$, then $\overline{\mathbb{D}^0} = \mathbb{D}^0$ is itself a finite subcomplex. Assume the claim is true for every cell of dimension less than n. By (1) in Definition 2.9, $\overline{\mathbb{D}^n} \setminus \mathbb{D}^n$ is contained in the union of finitely many cells of lower dimension, each of which is contained in a finite subcomplex by the inductive hypothesis. The claim now follows by taking a union of these these finite subcomplexes together with $\mathbb{D}^n$.
(5) Every finite subcomplex $Y \subseteq X$ is compact because it is the union of finitely many closed cells. Thus, if $K \subseteq X$ is closed and contained in a finite subcomplex, it is also compact. Conversely, suppose $K \subseteq X$ is compact. If K intersects infinitely many cells, by choosing one point of K in each such cell, we obtain an infinite discrete subset of K, which is impossible. Therefore, K is contained in the union of finitely many cells, and thus in a finite subcomplex by (1).
(6) This follows from (5).
(7) This essentially follows from (5).
(8) Consider the spaces $X^n \subseteq X$[4]. We induct on $n \in \mathbb{N}$. X^0 is obviously locally path-connected. If X^{n-1} is locally path-connected then X^n is also locally path-connected since it is the the quotient of the disjoint union of X^{n-1} and a bunch of n-cells which are locally path-connected. Therefore, $\coprod_{n \in \mathbb{N}} X_n$ is locally path-connected. Since
$$\coprod_{n \in \mathbb{N}} X_n \to X$$
is a quotient map, X is locally-path connected.
(9) See [Hat02] for a proof.

This completes the proof. ☐

Remark 2.24. *Every topological space is not a CW complex. Consider the Hawaiian earring, X:*



The easiest way to see the Hawaiian earring has no CW decomposition is using information about the first homology group. If X were a CW-complex, then it would have to be a finite CW-complex by Proposition 2.23(6) since it is compact. Since every finite CW-complex has finitely generated homology, it suffices to show that the homology of X is not finitely generated. Observe that for any $n \in \mathbb{N}$, X has a retract which is a wedge of n circles - namely, the union of n of the circles that make up X (the retraction just maps all the other circles to the origin). The first homology group of a wedge of n circles is $\mathbb{Z}^n$, which cannot be generated by fewer than n elements. It follows that $H_1(X)$ cannot be generated by fewer

[4]We will use the following facts from general topology. A disjoint union of locally path-connected spaces is locally path-connected. Moreover, a quotient of a locally path-connected space is locally path-connected.

than n elements for any $n \in \mathbb{N}$, and thus cannot be finitely generated. We have

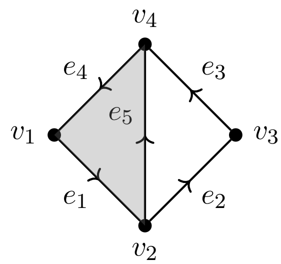
$$\mathbf{CW} \subsetneq \mathbf{Top}$$

as inclusion of categories.

Part 2. Simplicial & Singular Homology

3. What is Homology?

Heuristically, homology measures the existence of holes in a topological space. For instance, consider the topological space, X, shown below:



Intuitively, the boundary of an edge in the diagram can be thought of as a formal difference between the 'target' and the 'source'. So, the boundary of e_1 is given by $v_2 - v_1$. Moreover, let us define a chain of paths to be a formal sum of edges. For instance, we have the chains

$$\begin{aligned} c_1 &= e_1 + e_5 + e_4 \\ c_2 &= e_2 + e_3 + e_5^{-1} \\ c_3 &= e_1 + e_2 + e_3 + e_4 \end{aligned}$$

Here e_5^{-1} denotes the that edge e_5 is traversed in the opposite direction. In this terminology, we say there is a loop in X if the boundary of a formal sum of edges vanish. For instance, the boundary of c_1, c_2, c_3 vanishes. However, the loop c_1 can be shrunk to a point by deforming the path $e_4 + e_1$ to e_5 by continuously moving it within the interior of c_1. In our terminology, this can be detected by the fact that c_1 is the boundary of the solid triangle v_1, v_2, v_4. On the other hand, c_2 cannot be shrunk to a point since the triangle v_2, v_3, v_4 is hollow. Hence, we expect that there is one hole in X. The first homology group shall detect the presence of such a hole.

Remark 3.1. *The above intuition can be made precise by Hurewicz theorem which states that*

$$H_1(X) \cong \frac{\pi_1(X)}{[\pi_1(X), \pi_1(X)]}$$

That is $H_1(X)$ is isomorphic to the abelianization of $\pi_1(X)$, the fundamental group of X which quite literally is a measure of holes in a topological space.

More generally, the n-th homology group measures the existence of n-dimensional holes in a topological space, X. We consider an n-dimensional loop in $L \subseteq X$ which are reasonably such that that $L \cong \mathbb{S}^n$. For some n-dimensional loops $L \subseteq X$, there might be an $(n+1)$-dimensional disc $D \subseteq X$ such that $D \cong \mathbb{B}^{n+1}$ and L is the boundary of D. A n-dimensional hole will then be a n- dimensional loop that is not the boundary of a n-dimensional disc.

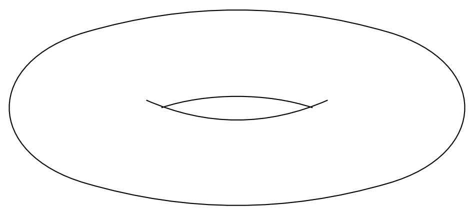
The n-th homology group will then be a measure of the existence of such n-dimensional holes.

Remark 3.2. *The phrase n-dimensional is only used informally in this section. Moreover, the statement above is only meant for intuition and should be taken with a grain of salt. In general, there is only a group homomorphism*

$$\pi_n(X) \to H_n(X),$$

for $n \geq 2$ if X is path-connected.

Example 3.3. The remarks made above precisely hold in one one important case. Consider a hollow doughnut-shaped surface, D, considered as a subset of $\mathbb{R}^3$.



Intuitively, there are two 1-dimensional holes in D. We will see that the first homology group (with $\mathbb{Z}$-coefficients) will be $\mathbb{Z} \oplus \mathbb{Z}$. The fact that there is a direct sum of two copies of $\mathbb{Z}$ is a measure of the existence of two and only two 1-dimensional holes. Note that D is hollow. Hence, we expect that there is 2-dimensional hole in D - a region in space with empty volume! We will see that the second homology group (once again with $\mathbb{Z}$-coefficients) will be $\mathbb{Z}$ since there is one and only one 2-dimensional hole.

4. Simplicial Homology

Let X be a topological space. In this section, we will define simplicial homology. Simplicial homology has the advantage of being computationally tractable since it can be used when a topological space can be triangulated. Indeed, we will define it in terms of Δ-complexes which will serve as basic building block of our triangulation.

Definition 4.1. Let $[v_0, v_1, \ldots, v_n]$ be an ordered tuple in $\mathbb{R}^m$.

(1) $[v_0, v_1, \ldots, v_n] \subseteq \mathbb{R}^m$ is said to be affinely independent if the set

$$\{v_1 - v_0, v_2 - v_0, \ldots, v_n - v_0\}$$

is linearly independent[5].

(2) Given an affinely independent ordered tuple $[v_0, v_1, \ldots, v_n] \subseteq \mathbb{R}^m$, the n-simplex generated by $[v_0, v_1, \ldots, v_n]$ is the convex span in $\mathbb{R}^m$ of the $n+1$ points $v_0, \ldots, v_n$:

$$\operatorname{conv}[v_0, v_1, \ldots, v_n] = \left\{ x = \sum_{i=0}^{n} t_i v_i \in \mathbb{R}^m \mid t_i \geq 0, \sum_i t_i = 1 \right\},$$

We call the points v_i the vertices of the n-simplex $[v_0, v_1, \ldots, v_n]$.

(3) Given an n-simplex $\operatorname{conv}[v_0, v_1, \ldots, v_n]$, the face opposite to v_i is the $(n-1)$-simplex:

$$\operatorname{conv}[v_0, \ldots, \widehat{v_i}, \ldots, v_n] := \{x \in \operatorname{conv}[v_0, v_1, \ldots, v_n] \mid t_i = 0\}.$$

The boundary of an n-simplex is the union of its faces.

[5]Thus necessarily $n \leq m$

Geometrically, one can think of an n-simplex as the smallest convex subset containing $v_0, \dots, v_n$ such that the points do not lie in a hyperplane of dimension less than n. As an example consider the standard n-simplex:

Definition 4.2. The standard simplex, $\Delta^n \subseteq \mathbb{R}^{n+1}$, is

$$\Delta^n = \left\{ (t_0, \dots, t_n) \in \mathbb{R}^{n+1} \mid t_i \geq 0, \sum_i t_i = 1 \right\}.$$

Remark 4.3. *The standard simplex allows one to induce coordinates on all n-simplices by sending $e_i \mapsto v_i$ inducing a map of simplicies:*

$$(t_0, \dots, t_n) \mapsto \sum_{i=0}^{n} t_i v_i$$

$(t_0, \dots, t_n)$ *are are called barycentric coordinates.*

Definition 4.4. A Δ-complex structure on a topological space X is a collection of maps $\{\sigma_j^n : \Delta^n \to X\}_{n \geq 0}^{j \in J_n}$ such that:

(1) The restriction $\sigma_j^n|_{\mathrm{Int}(\Delta^n)}$ is injective, and each point of X is in the image of exactly one such $\sigma_j^n|_{\mathrm{Int}(\Delta^n)}$.
(2) Restriction of each σ_j^n to a face of Δ^n is one of the maps $\sigma_k^{n-1} : \Delta^{n-1} \to X$.
(3) A set $A \subseteq X$ is open if and only if $(\sigma_j^n)^{-1}(A)$ is open in X for each σ_j^n.

Remark 4.5. *In what follows, we shall identify a $\sigma_j^n : \Delta^n \to X$ with a n-simplex $[v_0, \cdots, v_n]$.*

Our goal is to define the simplicial homology groups of a Δ-complex structure on a topological space, X. Let $\Delta_n(X)$ be the free abelian group with basis the open n-simplices of X. Elements of $\Delta_n(X)$ are called n-chains. These can be written as finite formal sums

$$\sum_{j \in J_n} n_j \sigma_j^n \qquad n_j \in \mathbb{Z}$$

Definition 4.6. Let X be a topological space with a Δ-complex structure. The boundary operator

$$\partial_n^\Delta : \Delta_n(X) \to \Delta_{n-1}(X)$$

is defined on each basis element of $\Delta_n(X)$ as:

$$\partial_n^\Delta [v_0, \dots, v_n] = \sum_{i=0}^{n} (-1)^i [v_0, \dots, \widehat{v_i}, \dots, v_n]$$

We say that

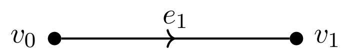
$$\partial_n^\Delta \Big(\sum_{j \in J_n} n_j \sigma_j^n \Big) \in \Delta_{n-1}(X)$$

is the boundary of $\sum_{j \in J_n} n_i \sigma_j^n$ in X.

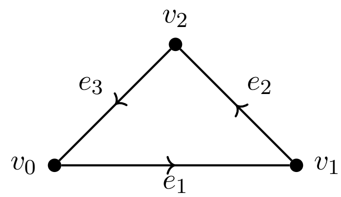
Remark 4.7. *Note that the boundary of an n-simplex in X is a $\mathbb{Z}$-linear combination (with coefficients ± 1) of $n-1$-simplices. This provides one motivation as to why we consider $\mathbb{Z}$-linear combinations of n-simplices. Moreover, heuristically the signs are inserted to take orientations into account, so that all the faces of a simplex are coherently oriented. See Example 4.8.*

Example 4.8. The following are examples of boundaries some standard simplexes.

(1) Consider $X = \Delta^1$. Then $\partial_1^\Delta[v_0, v_1] = [v_1] - [v_0]$



(2) Consider $X = \Delta^2$. Then $\partial_2^\Delta[v_0, v_1, v_2] = [v_1, v_2] - [v_0, v_2] + [v_0, v_1]$



Lemma 4.9. *Let X be a topological space with a Δ-complex structure. The map,*

$$\partial_{n-1}^\Delta \circ \partial_n^\Delta : \Delta_n(X) \xrightarrow{\partial_n^\Delta} \Delta_{n-1}(X) \xrightarrow{\partial_{n-1}^\Delta} \Delta_{n-2}(X)$$

is zero for each $n \geq 0$.

Proof. Note that:

$$\sum_{0 \leq j < i \leq n} (-1)^i(-1)^j[v_0, \cdots, \widehat{v_j}, \cdots, \widehat{v_i}, \cdots, v_n] + \sum_{0 \leq i < j \leq n} (-1)^i(-1)^{j-1}[v_0, \cdots, \widehat{v_i}, \cdots, \widehat{v_j}, \cdots, v_n] = 0$$

The latter two summations cancel since after switching i and j in the second sum, it becomes the negative of the first. □

Remark 4.10. *Note that $\Delta^1 \in \ker \partial_1^\Delta$ if and only if $v_0 = v_1$. In this case, Δ^1 can be thought of as a circle or a 1-loop. Indeed, this observation motivates the the observation that n-loops in X correspond to elements of $\ker \partial_n^\Delta$ for each $n \geq 1$. Moreover, the condition $\partial_n^\Delta \circ \partial_{n+1}^\Delta = 0$ is the observation that the boundary of a $\mathbb{Z}$-linear combination of $(n+1)$-simplicies is a n-loop.*

Let $C_n^\Delta(X) = \Delta_n(X)$ for each $n \geq 0$. Purely algebraically, we have a sequence of homomorphisms of abelian groups:

$$\cdots \xrightarrow{\partial_{n+1}^\Delta} C_n^\Delta(X) \xrightarrow{\partial_n^\Delta} C_{n-1}^\Delta(X) \xrightarrow{\partial_{n-1}^\Delta} C_{n-2}^\Delta(X) \xrightarrow{\partial_{n-2}^\Delta} \cdots$$

The boundary map $\partial_n^\Delta : C_n^\Delta(X) \longrightarrow C_{n-1}^\Delta(X)$ is such that

$$\partial_{n-1}^\Delta \circ \partial_n^\Delta = 0$$

That is,

$$\mathrm{im}(\partial_{n+1}^\Delta) \subseteq \ker(\partial_n^\Delta)$$

Elements of $\ker(\partial_n^\Delta)$ are called n-cycles (or n-loops) and elements of $\mathrm{im}(\partial_{n+1}^\Delta)$ are called n-boundaries.

Definition 4.11. Let X be a topological space with a Δ-complex structure. The n-th simplicial homology group of X with $\mathbb{Z}$-coefficients of the associated chain complex $(C_n^\Delta(X), \partial_n^\Delta)_{n \in \mathbb{N}}$ is

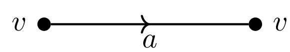
$$H_n^\Delta(X; \mathbb{Z}) = \frac{\ker(\partial_n^\Delta)}{\mathrm{im}(\partial_{n+1}^\Delta)}$$

$H_n^\Delta(X; \mathbb{Z})$ is called the n-th simplicial homology group of X.

Remark 4.12. *In what follows, we will not explicitly verify Definition 4.4(3). For instance, we will not explicitly verify that the Δ-complex structure on the circle, $\mathbb{S}^1$, in Example 4.13(1) is compatible with the topology on $\mathbb{S}^1$. Similarly, Example 4.13(2)-(8) we will not explicitly verify that the Δ-complex structure is compatible with the underlying quotient topology. It should be straightforward to do verify these claims, though.*

Example 4.13. We compute simplicial homology groups of various topological spaces below.

(1) **(Circle)** Consider $X = \mathbb{S}^1$ with a Δ-complex structure with a single 1-simplex and a single 0-simplex.



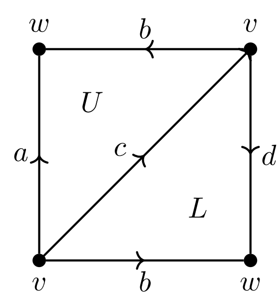
We have a chain complex of the following form:

$$\cdots \longrightarrow 0 \xrightarrow{\partial_2^\Delta} \mathbb{Z} \xrightarrow{\partial_1^\Delta} \mathbb{Z} \xrightarrow{\partial_0^\Delta} 0.$$

Here ∂_1^Δ is the zero map. Therefore, we have:

$$H_n^\Delta(\mathbb{S}^1, \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } n = 0, 1 \\ 0 & \text{otherwise} \end{cases}$$

(2) (**Mobius Band**) Consider $X = M$, the Mobius band. A Δ-complex structure on M is pictured below.



We have a complex of the following form:

$$\cdots 0 \xrightarrow{\partial_3^\Delta} \mathbb{Z}^{\oplus 2} \xrightarrow{\partial_2^\Delta} \mathbb{Z}^{\oplus 4} \xrightarrow{\partial_1^\Delta} \mathbb{Z}^{\oplus 2} \xrightarrow{\partial_0^\Delta} 0.$$

We have

$$\partial_1^\Delta a = \partial_1^\Delta b = \partial_1^\Delta d = w - v$$
$$\partial_1^\Delta c = 0$$

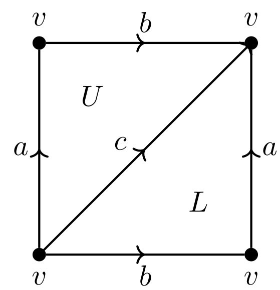
Hence $\mathrm{Im}\ \partial_1^\Delta \cong \mathbb{Z}$, implying that $H_0^\Delta(X) \cong \mathbb{Z}^{\oplus 2}/\mathbb{Z} \cong \mathbb{Z}$. Also

$$\partial_2^\Delta U = a - b - c$$
$$\partial_2^\Delta L = b - d - c$$

This implies ∂_2^Δ is injective. Hence $H_2^\Delta(X) \cong 0$. A basis for $\ker \partial_1^\Delta$ is $\{x = a-d, y = b-d, z = c\}$. Hence $\ker \partial_1^\Delta \cong \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}$. A basis for $\mathrm{Im}\, \partial_2^\Delta$ is $\{x-y-z, y-z\}$. An equivalent basis is $\{x, y - z\}$. Hence $H_1^\Delta(X) \cong \mathbb{Z}$.

$$H_n^\Delta(M,\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{for } n = 0,1 \\ 0 & \text{for } n \geq 2 \end{cases}$$

(3) (**Torus**) Consider the $X = \mathbb{T}$, the torus, with the Δ-complex structure is pictured below having one vertex, three edges a, b, and c, and two 2-simplices U and L[6].



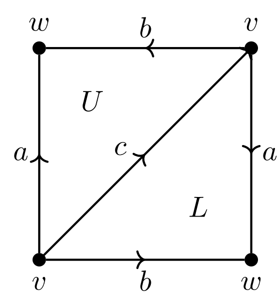
We have a complex of the following form:

$$\cdots 0 \xrightarrow{\partial_3^\Delta} \mathbb{Z}^{\oplus 2} \xrightarrow{\partial_2^\Delta} \mathbb{Z}^{\oplus 3} \xrightarrow{\partial_1^\Delta} \mathbb{Z} \xrightarrow{\partial_0^\Delta} 0.$$

As in the previous example, $\partial_1^\Delta = 0$. Also $\partial_2^\Delta U = a + b - c = \partial_2^\Delta L$. Since $\partial_1^\Delta = 0$, $H_0^\Delta(T) \cong \mathbb{Z}$ Since $\{a, b, a+b-c\}$ is a valid basis for $\mathbb{Z}^{\oplus 3}$, it follows that $H_1^\Delta(T) \cong \mathbb{Z}^2$ with basis the homology classes $[a]$ and $[b]$. Since there are no 3-simplices, $H_2^\Delta(T)$ is equal to $\ker \partial_2^\Delta$, which is infinite cyclic generated by $U - L$. Thus,

$$H_n^\Delta(T,\mathbb{Z}) \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & \text{for } n = 1 \\ \mathbb{Z} & \text{for } n = 0,2 \\ 0 & \text{for } n \geq 3 \end{cases}$$

(4) (**Real Projective Plane**) Consider $X = \mathbb{RP}^2$. The delta complex structure is pictured below.



We have a complex of the following form:

$$\cdots 0 \xrightarrow{\partial_3^\Delta} \mathbb{Z}^{\oplus 2} \xrightarrow{\partial_2^\Delta} \mathbb{Z}^{\oplus 3} \xrightarrow{\partial_1^\Delta} \mathbb{Z}^{\oplus 2} \xrightarrow{\partial_0^\Delta} 0.$$

We have

$$\partial_1^\Delta b = \partial_1^\Delta a = w - v \qquad \partial^\Delta c = 0.$$

Hence $\operatorname{Im} \partial_1^\Delta \cong \mathbb{Z}$, implying that $H_0^\Delta(X) = \mathbb{Z}^{\oplus 2}/\mathbb{Z} \cong \mathbb{Z}$. Also

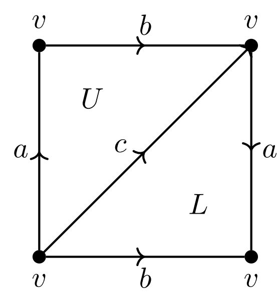
$$\partial_2^\Delta U = a - b - c \qquad \partial_2^\Delta L = b - a - c$$

[6]We use the notation $\mathbb{T}$ is homemorphic to $\mathbb{S}^1 \times \mathbb{S}^1$ the donught-shaped surface in Section 3.

This implies ∂_2^Δ is injective. Hence $H_2^\Delta(X) \cong 0$. A basis for $\ker \partial_1^\Delta$ is $\{x = a-b, y = c\}$. Hence $\ker \partial_1^\Delta \cong \mathbb{Z}\oplus\mathbb{Z}$. A basis for $\mathrm{Im}\partial_2^\Delta$ is $\{x-y, -x-y\}$. An equivalent basis is $\{x-y, 2y\}$. Hence $H_1^\Delta(X) \cong \mathbb{Z}_2$.

$$H_n^\Delta(\mathbb{RP}^2, \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{for } n = 0 \\ \mathbb{Z}_2 & \text{for } n = 1 \\ 0 & \text{for } n \geq 2 \end{cases}$$

(5) **(Klein Bottle)** Consider $X = K$, the Klein bottle, with the Δ-complex structure is pictured below having one vertex, three edges a, b and c, and two 2-simplices U and L.:



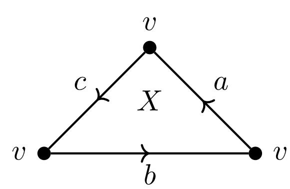
We have a complex of the following form:

$$\cdots 0 \xrightarrow{\partial_3^\Delta} \mathbb{Z}^{\oplus 2} \xrightarrow{\partial_2^\Delta} \mathbb{Z}^{\oplus 3} \xrightarrow{\partial_1^\Delta} \mathbb{Z} \xrightarrow{\partial_0^\Delta} 0.$$

Clearly, $\partial_1^\Delta = 0$. $\partial_2^\Delta U = a+b-c$ and $\partial_2^\Delta L = a-b+c$. Since $\partial_1^\Delta = 0$, $H_0^\Delta(K) \cong \mathbb{Z}$. We have $\mathrm{Im}(\partial_2^\Delta) = \mathrm{span}\{2a, a+b-c\}$. Since $\{a, a+b-c, c\}$ is a valid basis for $\mathbb{Z}^{\oplus 3}$, it follows that $H_1^\Delta(K) \cong \mathbb{Z}\oplus\mathbb{Z}_2$. Since there are no 3-simplices, $H_2^\Delta(K)$ is equal to $\ker \partial_2^\Delta$, which is easily seen to be trivial. Thus,

$$H_n^\Delta(K, \mathbb{Z}) \cong \begin{cases} \mathbb{Z}\oplus\mathbb{Z}_2 & \text{for } n = 1 \\ \mathbb{Z} & \text{for } n = 0 \\ 0 & \text{for } n \geq 2 \end{cases}$$

(6) **(Triangular Parachute)** Let X be a triangular parachute obtained from Δ^2 by identifying its three vertices to a single point.



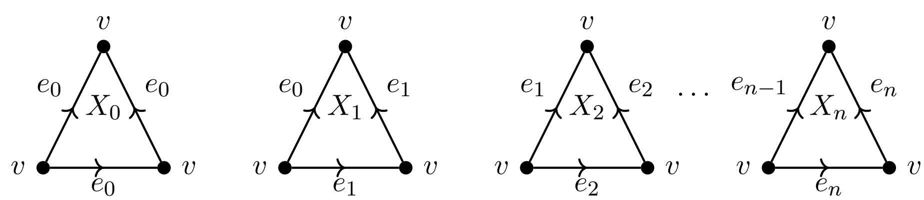
We have 1 face, 3 edges, and 1 vertex so that $\Delta^2(X)$, $\Delta^0(X) \cong \mathbb{Z}$, $\Delta^1(X) \cong \mathbb{Z}^3$. Note that

$$\begin{aligned} \partial_2^\Delta(X) &= b + a - c \\ \partial_1^\Delta(a) = \partial_1^\Delta(b) &= \partial_1^\Delta(c) = \partial_0^\Delta(v) = 0 \end{aligned}$$

Hence ker $\partial_2^{\Delta} = 0$, ker $\partial_1^{\Delta} = \mathbb{Z}^3$, ker $\partial_0^{\Delta} = \mathbb{Z}$. On the other hand, $\operatorname{Im} \partial_2^{\Delta} = \mathbb{Z}$ as the subgroup $\langle b + a - c \rangle$ is free on one generator. Hence we have,

$$H_n^{\Delta}(X, \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{for } n = 0 \\ \mathbb{Z}^{\oplus 2} & \text{for } n = 1 \\ 0 & \text{for } n \geq 2 \end{cases}$$

(7) Let X be the topological space obtained obtained from $n+1$ 2-simplices $\Delta_0^2, \ldots, \Delta_n^2$ by identifying all three edges of Δ_0^2 to a single edge, and for $i > 0$ identifying the edges $[v_0, v_1]$ and $[v_1, v_2]$ of Δ_i^2 to a single edge and the edge $[v_0, v_2]$ to the edge $[v_0, v_1]$ of Δ_{i-1}^2.



We have 1 vertex, $n+1$ edges, and $n+1$ faces so that $\Delta_0(X) \cong \mathbb{Z}$, $\Delta_1(X), \Delta_2(X) \cong \mathbb{Z}^{n+1}$. We have a complex of the following form:

$$\cdots 0 \xrightarrow{\partial_3^{\Delta}} \mathbb{Z}^{\oplus n+1} \xrightarrow{\partial_2^{\Delta}} \mathbb{Z}^{\oplus n+1} \xrightarrow{\partial_1^{\Delta}} \mathbb{Z}^{\oplus 1} \xrightarrow{\partial_0^{\Delta}} 0.$$

Clearly, $\partial_0^{\Delta} = 0$ and Im $\partial_1^{\Delta} = 0$. Hence $H_0^{\Delta}(X) \cong \mathbb{Z}$. Let's compute Im ∂_2. Note that:

$$\partial_2 X_i = \begin{cases} e_0 & \text{if } i = 0 \\ 2e_i - e_{i-1} & \text{if } i > 1 \end{cases}$$

It is clear that a basis for Im $\partial_2 = \{e_0\} \cup \{2e_i - e_{i-1} : 1 \leq i \leq n\}$. Note that in $H_1^{\Delta}(X) = \ker \partial_1/\text{Im } \partial_2$, we set $e_0 = 0$ and $2e_i - e_{i-1} = 0$ so that $e_0 = 0$, $2e_i = e_{i-1}$. This implies that

$$2e_1 = e_0 = 0 \quad 2^2 e_2 = e_0 = 0 \quad \cdots \quad 2^k e_k = e_0 = 0$$

so that Therefore:

$$H_1^{\Delta}(X) \cong \mathbb{Z}^{n+1}/(\mathbb{Z} \times 2\mathbb{Z} \times \cdots \times 2^n\mathbb{Z}) \cong \mathbb{Z}_2 \times \mathbb{Z}_4 \times \cdots \times \mathbb{Z}_{2^n}$$

It is easy to see that $\ker \partial_2^{\Delta} = 0$. Hence $H_2^{\Delta}(X) = 0$. Therefore, we have:

$$H_n^{\Delta}(X, \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{for } n = 0 \\ \mathbb{Z}_2 \times \mathbb{Z}_4 \times \cdots \times \mathbb{Z}_{2^n} & \text{for } n = 1 \\ 0 & \text{for } n \geq 2 \end{cases}$$

(8) Let X_n be obtained from an n-simplex by identifying all faces of the same dimension. Since there is only one k-simplex for each $k \leq n$, we see that $\Delta^k(X_n) \cong \mathbb{Z}$ for $k \leq n$.

Choose a generator σ_k for each of these. Note that the restriction of σ_k to a $(k-1)$-dimensional face will just be σ_{k-1}. Thus,

$$\partial_k^{\Delta}\sigma_k = \sum_{i=0}^{k}(-1)^i\sigma_{k-1} = \begin{cases} 0 & \text{if } k = 0, \\ 0 & \text{if } k \leq n, \text{ and } k \text{ is odd}, \\ \sigma_{k-1} & \text{if } k \leq n, \text{ and } k \text{ is even}, \\ 0 & \text{if } k > n. \end{cases}$$

Therefore:

$$\ker(\partial_k^{\Delta}) = \begin{cases} \mathbb{Z} & \text{if } k = 0, \\ \mathbb{Z} & \text{if } k \leq n, \text{ and } k \text{ is odd}, \\ 0 & \text{if } k \leq n, \text{ and } k \text{ is even}, \\ 0 & \text{if } k > n. \end{cases} \qquad \mathrm{Im}(\partial_k) = \begin{cases} 0 & \text{if } k = 0, \\ 0 & \text{if } k \leq n, \text{ and } k \text{ is odd}, \\ \mathbb{Z} & \text{if } k \leq n, \text{ and } k \text{ is even}, \\ 0 & \text{if } k > n. \end{cases}$$

Hence:

$$H_k^{\Delta}(X_n, \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } k = 0, \\ \mathbb{Z} & \text{if } k = n, \text{ and } n \text{ is odd}, \\ 0 & \text{else}. \end{cases}$$

5. Homological Algebra

In this section we take an algebraic detour and introduce the basics of homological algebra by discussing exact sequences, chain complexes and some diagram chasing lemmas. We work in the category **Ab**, the category of abelian groups. However, the discussion applies verbatim in $\mathbf{Mod}_R$, the category of R-modules[7].

5.1. **Exact Sequences.** We first discuss the fundamental concept of exact sequences. Exact sequences are a central tool in homological algebra and algebraic topology, providing a way to relate algebraic invariants of different objects. They encode how one algebraic object maps into another and help detect kernels and images of homomorphisms, which are essential for defining and computing homology and cohomology groups.

Definition 5.1. A sequence

$$A \xrightarrow{f} B \xrightarrow{g} C$$

of two homomorphisms of abelian groups is said to be exact at B if $\mathrm{im}\, f = \ker g$. More generally, a sequence

$$\ldots \to A_{n+1} \xrightarrow{f_{n+1}} A_n \xrightarrow{f_n} A_{n-1} \to \ldots \quad n \in \mathbb{Z}$$

is said to be exact if it is exact at A_n for each $n \in \mathbb{Z}$. Such a sequence is called a long exact sequence of abelian groups.

The following is an important special case:

Definition 5.2. A short exact sequence of abelian groups is an exact sequence of the form

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0.$$

that is exact in each degree.

[7] In general, in any abelian category.

Example 5.3. Using the notion of exactness, we can rephrase familiar definitions from basic algebra. Suppose $f : A \to B$ is a homomorphism of abelian groups.

(1) f is injective if and only if $0 \to A \xrightarrow{f} B$ is exact. Indeed, the sequence is exact at A if and only if $\ker f = 0$ if and only if f is injective.
(2) f is surjective if and only if $A \xrightarrow{f} B \to 0$ is exact. Indeed, the sequence is exact at B if and only if im $f = B$ if and only if f is surjective.
(3) f is an isomorphism if and only if $0 \to A \xrightarrow{f} B \to 0$ is exact. This follows from the two statements above.

Remark 5.4. *Functors in* **Ab** *(or in any abelian category) can preserve algebraic structure in different ways. A functor $\mathscr{F}$ is said to be left exact if it sends exact sequences of the form*

$$0 \to A \to B \to C$$

to exact sequences

$$0 \to \mathscr{F}(A) \to \mathscr{F}(B) \to \mathscr{F}(C).$$

Similarly, a functor is said to be right exact *if it sends exact sequences of the form*

$$A \to B \to C \to 0$$

to exact sequences

$$\mathscr{F}(A) \to \mathscr{F}(B) \to \mathscr{F}(C) \to 0.$$

*A functor is called exact if it is both left exact and right exact. The notion of exact functors is used in the appendix (**??**) and later on.*

5.2. **Chain Complexes & Chain Homotopy.** We now define the notions of chain complexes and chain homotopy. Chain complexes provide the algebraic framework for computing homology, encoding sequences of abelian groups connected by boundary maps. Chain homotopy, on the other hand, allows us to compare chain maps up to a 'deformation,' playing a crucial role in establishing when two chain complexes have the same homological properties.

Definition 5.5. A chain complex is a sequence of abelian groups and homomorphisms

$$\cdots \xrightarrow{\partial_{n+2}} C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \cdots$$

for $n \in \mathbb{Z}$ which satisfies $\partial_n \circ \partial_{n+1} = 0,$ for each $n \in \mathbb{Z}$. That is,

$$\text{im } \partial_{n+1} \subseteq \ker \partial_n \quad \iff \quad \partial_n \circ \partial_{n+1} = 0$$

We refer to the entire complex as $(C_\bullet, \partial_\bullet)$ or sometimes just $C_\bullet$. The maps ∂_n are called the boundary operators of the chain complex.

Example 5.6. Let X be a topological space. The chain complex

$$(C_n^\Delta(X), \partial_n^\Delta)_{n\in\mathbb{N}}$$

encountered in Section 4 is a chain complex. We call this the simplicial chain complex. Note that in this example the abelian groups are all zero for negative subscripts; this, however, is not part of the definition in general.

Remark 5.7. *Elements of* $\ker \partial_n$ *are called n-chains and elements of* $\operatorname{im} \partial_{n+1}$ *are called n-boundaries.*

We now define the notion of a chain map between chain complexes.

Definition 5.8. Let $(C_\bullet, \partial_\bullet)$ and $(C'_\bullet, \partial'_\bullet)$ be chain complexes of abelian groups. A chain map between $(C_\bullet, \partial_\bullet)$ and $(C'_\bullet, \partial'_\bullet)$ is a sequence of group homomorphisms $f_n : C_n \to C'_n$ for $n \in \mathbb{Z}$ such that the diagram commutes:

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, "f_{n+1}"] & C_n \arrow[r, "\partial_n"] \arrow[d, "f_n"] & C_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \cdots \\
\cdots \arrow[r] & C'_{n+1} \arrow[r, "\partial'_{n+1}"] & C'_n \arrow[r, "\partial'_n"] & C'_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

Proposition 5.9. *Chain complexes of abelian groups form a category, denoted as* $\mathbf{Chain_{Ab}}$.

Proof. Objects in $\mathbf{Chain_{Ab}}$ are chain complexes of abelian groups and a morphism between chain complexes of abelian groups is a chain map. If $(C_\bullet, \partial_\bullet)$, $(C'_\bullet, \partial'_\bullet)$ and $(C''_\bullet, \partial''_\bullet)$ are chain complexes such that $f_\bullet : C_\bullet \to C'_\bullet$ and $g_\bullet : C'_\bullet \to C''_\bullet$ are two chain maps. Then

$$(g \circ f)_\bullet : C_\bullet \to C''_\bullet$$

is the chain map given by $(g \circ f)_n = g_n \circ f_n$. This is indeed a valid chain map as the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, "f_{n+1}"] & C_n \arrow[r, "\partial_n"] \arrow[d, "f_n"] & C_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \cdots \\
\cdots \arrow[r] & C'_{n+1} \arrow[r, "\partial'_{n+1}"] \arrow[d, "g_{n+1}"] & C'_n \arrow[r, "\partial'_n"] \arrow[d, "g_n"] & C'_{n-1} \arrow[r] \arrow[d, "g_{n-1}"] & \cdots \\
\cdots \arrow[r] & C''_{n+1} \arrow[r, "\partial''_{n+1}"] & C''_n \arrow[r, "\partial''_{n+1}"] & C''_{n-1} \arrow[r, "\partial''_{n+1}"] & \cdots
\end{tikzcd}$$

commutes essentially by construction as can be easily checked. This defines the composition of two chain maps. Moreover, the identity chain map

$$\mathrm{Id} : C_\bullet \to C_\bullet$$

is the chain map given by $\mathrm{Id}_n = \mathrm{Id}_{C_n}$ where Id_{C_n} is the identity homomorphism from C_n to C_n.

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, "\mathrm{Id}_{n+1}"] & C_n \arrow[r, "\partial_n"] \arrow[d, "\mathrm{Id}_n"] & C_{n-1} \arrow[r] \arrow[d, "\mathrm{Id}_{n-1}"] & \cdots \\
\cdots \arrow[r] & C_{n+1} \arrow[r, "\partial'_{n+1}"] & C_n \arrow[r, "\partial'_n"] & C_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

All that is required is check that composition of chain maps satisfies the associativity property and the composition of a chain map with the identity chain map yields the original chain map. All these are routine checks. □

We now define the notion of a chain homotopy between chain complexes.

Definition 5.10. Suppose $(C_\bullet, \partial)$ and $(C'_\bullet, \partial')$ are two chain complexes with chain maps $f_\bullet, g_\bullet$. A chain homotopy between $f_\bullet, g_\bullet$ is a series of maps $T_n : C_n$ and C_{n+1} such that

$$\begin{aligned}
f_n - g_n &= \partial'_{n+1} T_n + T_{n-1} \partial_n \qquad && n \geq 1 \\
T_0 \circ \partial_1 &= f_0 - g_0 && n = 0
\end{aligned}$$

That is, the following diagram commutes

$$
\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[dl, "T_{n+1}"'] \arrow[d, "f_{n+1}"', "g_{n+1}"] & C_n \arrow[r, "\partial_n"] \arrow[dl, "T_n"] \arrow[d, "f_n"', "g_n"] & C_{n-1} \arrow[r] \arrow[dl, "T_{n-1}"'] \arrow[d, "f_{n-1}"', "g_{n-1}"] & \cdots \arrow[dl, "T_{n-2}"] \\
\cdots \arrow[r] & C'_{n+1} \arrow[r, "\partial'_{n+1}"'] & C'_n \arrow[r, "\partial'_n"'] & C'_{n-1} \arrow[r] & \cdots
\end{tikzcd}
$$

Remark 5.11. *In Section 7, we will provide a geometric intuition behind the definition of a chain homotopy.*

Proposition 5.12. *Let $(C_\bullet, \partial)$ and $(C'_\bullet, \partial')$ be two chain complexes. The relation of chain homotopy between these chain complexes is an equivalence relation.*

Proof. Let $f_\bullet$ be a chain map. Define $T_n : C_n \to C'_{n+1}$ to be the zero map. Then

$$T_{n-1}\partial_n + \partial'_{n+1}T_n = 0 = f - f,$$

so f is chain homotopic to itself. Assume $f_\bullet$ is chain homotopic to $g_\bullet$. That is, consider maps $f_n, g_n : C_n \to C'_n$ such that there is a $T_n : C_n \to C'_{n+1}$ such that

$$f_n - g_n = T_{n-1}\partial_n + \partial'_{n+1}T_n$$

Then

$$g_n - f_n = -(f_n - g_n) = -T_{n-1}\partial_n + \partial'_{n+1}T_n = (-T_{n-1})\partial_n + \partial'_{n+1}(-T_n)$$

Therefore $g_\bullet$ is chain homotopic to $f_\bullet$. Finally, suppose $f_\bullet$ is chain homotopic to $g_\bullet$ and that $g_\bullet$ is chain homotopic to $h_\bullet$. Then there exist maps $T_n, S_n : C_n \to C'_{n+1}$ such that

$$
\begin{aligned}
f_n - g_n &= T_{n-1}\partial_n + \partial'_{n+1}T_n \\
g_n - h_n &= S_{n-1}\partial_n + \partial'_{n+1}S_n
\end{aligned}
$$

Adding equations, we get

$$f_n - h_n = (T_{n-1} + S_{n-1})\partial_n + \partial'_{n+1}(T_n + S_n)$$

Thus, $f_\bullet$ is chain homotopic to $h_\bullet$. This proves the claim. □

5.3. **Homology of a Chain Complex.** Given a chain complex, $(C_\bullet, \partial_\bullet)$, the condition $\partial_n \circ \partial_{n+1} = 0$ implies that

$$\operatorname{im} \partial_{n+1} \subseteq \ker \partial_n$$

This motivates the following definition:

Definition 5.13. Let $(C_\bullet, \partial_\bullet)$ be a chain complex. The n-th homology group is defined as

$$H_n(C_\bullet) := \frac{\ker \partial_n}{\operatorname{im}\, \partial_{n+1}}.$$

Example 5.14. Let's compute the homology of some chain complexes:

(1) Consider the chain complex

$$C_\bullet : \cdots \to 0 \to \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\partial_2} \mathbb{Z} \xrightarrow{\partial_1} 0 \to \cdots,$$

where the chain groups are given by

$$\begin{aligned} C_1 &= \mathbb{Z}, \\ C_2 &= \mathbb{Z} \oplus \mathbb{Z}, \\ C_n &= 0 \quad \text{for } n \neq 1, 2. \end{aligned}$$

The homomorphism ∂_2 is defined by $\partial_2(x, y) = 3x + 3y$. Note that we have the following

$$\ker \partial_n \cong \begin{cases} \mathbb{Z}, & \text{if } n = 1 \text{ or } n = 2, \\ 0, & \text{if } n \neq 1, 2. \end{cases}$$

Similarly, we have

$$\operatorname{im} \partial_n \cong \begin{cases} 3\mathbb{Z}, & \text{if } n = 2, \\ 0, & \text{if } n \neq 2. \end{cases}$$

Therefore, the homology of the chain complex is given as:

$$H_n(C_\bullet) \cong \begin{cases} \mathbb{Z}_3, & \text{if } n = 1, \\ \mathbb{Z}, & \text{if } n = 2, \\ 0, & \text{if } n \neq 1, 2. \end{cases}$$

(2) Consider the chain complex:

$$\cdots \xrightarrow{\partial} \mathbb{Z}/8\mathbb{Z} \xrightarrow{\partial} \mathbb{Z}/8\mathbb{Z} \xrightarrow{\partial} \mathbb{Z}/8\mathbb{Z} \xrightarrow{\partial} \mathbb{Z}/8\mathbb{Z} \to 0 \to 0 \to \cdots$$

where $C_n = \mathbb{Z}/8\mathbb{Z}$ for $n \leq 0$ and $C^n = 0$ for $n > 0$ and the map ∂ is given by $x \bmod 8 \mapsto 4x \bmod 8$. It is easy to see that

$$\begin{aligned} \ker \partial &= \{\overline{0}, \overline{2}, \overline{4}, \overline{6}\} \cong \mathbb{Z}/4\mathbb{Z} \\ \operatorname{im} \partial &= \{\overline{0}, \overline{4}\} \cong \mathbb{Z}/2\mathbb{Z} \end{aligned}$$

for $n < 0$. Hence,

$$H_n(C_\bullet) \cong \frac{\mathbb{Z}/4\mathbb{Z}}{\mathbb{Z}/2\mathbb{Z}} \cong \mathbb{Z}/2\mathbb{Z}$$

Trivially, $H_n(C_\bullet) \cong 0$ for $n > 0$ and $H_0(C_\bullet) \cong \mathbb{Z}/4\mathbb{Z}$.

Proposition 5.15. *Let $(C, \partial_\bullet), (C', \partial'_\bullet)$ be chain complexes and let $f_\bullet, g_\bullet$ be chain maps between the chain complexes. If there is a chain homotopy $f_\bullet$ and $g_\bullet$, then the induced maps in homology are equal, i.e., we have:*

$$H_n(f) = H_n(g) : H_n(C_\bullet) \to H_n(C'_\bullet)$$

Proof. Let $(T_n)_{n \geq 1}$ be the sequence of maps defining a chain homotopy. Let $[c] \in H_n(C)$. If $n = 0$, we have

$$H_0(f)([c]) = [f_0(c)] = [g_0(c) + \partial_1 T_0(c)] = [g_0(c)] = H_0(g)([c])$$

For $n \geq 1$, we have:

$$\begin{aligned} H_n(f)([c]) &= [f_n(c)] \\ &= [g_n(c) + \partial'_{n+1} T_n(c) + T_{n-1} \partial_n(c)] \\ &= [g_n(c)] \\ &= H_n(g)([c]) \end{aligned}$$

The third equality uses that c is a n-cycle and that a homology class is not changed if we add a n-boundary. The claim follows. $\square$

Proposition 5.16. *There is a functor $H_n : \mathbf{Chain}_{\mathbf{Ab}} \to \mathbf{Ab}$ that associates to a chain complex over abelian groups its n-th homology group.*

Proof. Consider a chain map between chain complexes given by the following diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, "f_{n+1}"] & C_n \arrow[r, "\partial_n"] \arrow[d, "f_n"] & C_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \cdots \\
\cdots \arrow[r] & C'_{n+1} \arrow[r, "\partial'_{n+1}"] & C'_n \arrow[r, "\partial'_n"] & C'_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

The relation $f_n\partial_{n+1} = \partial'_{n+1}f_{n+1}$ implies that f_n takes n-cycles to takes n-cycles for each $n \in \mathbb{N}$. This is because if $\partial_n c = 0$, then

$$\partial'_n(f_n(c)) = f_{n-1}(\partial_n c) = 0$$

Also, f_n takes n boundaries to n-boundaries since

$$f_n(\partial_{n+1}c) = \partial'_{n+1}(f_{n+1}c)$$

Hence f_n descends to a homomorphism

$$H_n(f) : H_n(C_\bullet) \to H_n(C'_\bullet)$$

It remains to check that $H_n(g \circ f) = H_n(g) \circ H_n(f)$ and that $H_n(\mathrm{Id}_X) = \mathrm{Id}_{H_n(X)}$. Both of these are immediate from the definitions. $\square$

5.4. **Diagram Chasing Lemmas.** In this section, we discuss some important diagram chasing lemmas in homological algebra. In particular, we discuss the short-five lemma, four-lemma, five-lemma and the snake lemma. The snake lemma allows us to construct a long exact sequence of homology[8] of pairs of topological spaces. We end our discussion by discussing the braid lemma. The braid lemma will allow us to construct a long exact sequence of homology of triples of topological spaces.

Proposition 5.17. ***(Short Five Lemma)*** *Consider the the diagram below:*

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "\alpha"'] & B \arrow[r, "g"] \arrow[d, "\beta"'] & C \arrow[r] \arrow[d, "\gamma"'] & 0 \\
0 \arrow[r] & A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' \arrow[r] & 0
\end{tikzcd}$$

be a commutative diagram with exact rows of abelian groups. We have the following:

1. *If α and γ are injective homomorphisms, then β is an injective homomororphism.*
2. *If α and γ are surjective homomorphisms, then β is a surjective homorophism.*
3. *If α and γ are isomorphisms, then β is an isomorphism.*

Proof. The proof of (2) is similar to that of (1). (3) follows from (1) and (2). We prove (1). Let $b \in B$ such that $\beta(b) = 0$. Clearly, $g'(\beta(b)) = 0$. But by the commutativity of the right-most square, we also have that $\gamma(g(b)) = 0$. Since γ is injective, $g(b) = 0$. Hence $b \in \ker g = \operatorname{im} f$. Therefore, there is a $a \in A$ such that $f(a) = b$. By the commutativity of

[8] And cohomology later on.

the left-most square, we must have that $f'(\alpha(a)) = 0$. Since f' and α are injective, $a = 0$. Hence, $b = 0$.

$$\begin{tikzcd}
a \arrow[r, mapsto, "f"] \arrow[d, mapsto, "\alpha"] & b \arrow[r, mapsto, "g"] \arrow[d, mapsto, "\beta"] & g(b) \arrow[d, mapsto, "\gamma"'] \\
\alpha(a) \arrow[r, mapsto, "f'"] & 0 \arrow[r, mapsto, "g'"] & 0
\end{tikzcd}$$

This completes the proof. $\square$

Proposition 5.18. ***(Four & Five Lemmas)*** *Consider the the diagram below:*

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\alpha"'] & B \arrow[r, "g"] \arrow[d, "\beta"'] & C \arrow[r, "h"] \arrow[d, "\mu"'] & D \arrow[r, "i"] \arrow[d, "\gamma"'] & E \arrow[d, "\delta"'] \\
A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' \arrow[r, "h'"] & D' \arrow[r, "i'"] & E'
\end{tikzcd}$$

be a commutative diagram with exact rows of abelian groups. We have the following:

1. ***(Four Lemma I)*** *If α is a surjective homomorphism and β and γ are injective homomorphisms, then μ is an injective homomorphism.*
2. ***(Four Lemma II)*** *If δ is an injective homomorphism and β and γ are surjective homomorphisms, then μ is a surjective homomorphism.*
3. ***(Five Lemma)*** *If α is a surjective homomorphism, δ is an injective homomorphism, and β and γ are isomorphisms, then μ is an isomorphism.*

Proof. (3) follows from (1) and (2) and the proof of (2) is similar to that of (1). We only prove (1). Let $c \in C$ such that $\mu(c) = 0$. Note that: $\gamma \circ h(c) = h' \circ \mu(c) = 0$ Since γ is an injective homomorphism, $h(c) = 0$. Therefore, there exists a $b \in B$ such that $g(b) = c$. Note that $g' \circ \beta(b) = \mu \circ g(c) = 0$ Hence, there exists a $a' \in A$ such that $f'(a') = \beta(b)$. Since α is a surjective homomorphism, α, there exists a $a \in A$ such that $\alpha(a) = a'$. Since β is an injective homomorphism, we must have that $f(a) = b$. But then $c = g(b) = f \circ g(a) = 0$.

$$\begin{tikzcd}
a \arrow[r, mapsto, "f"'] \arrow[rr, bend left, "0"] \arrow[d, mapsto, "\alpha"] & b \arrow[r, mapsto, "g"'] \arrow[d, mapsto, "\beta"] & c \arrow[r, mapsto, "h"'] \arrow[d, mapsto, "\mu"] & h(c) \arrow[d, mapsto, "\gamma"] \\
a' \arrow[r, mapsto, "f'"] & \beta(b) \arrow[r, mapsto, "g'"] & 0 \arrow[r, mapsto, "h'"] & 0
\end{tikzcd}$$

This completes the proof. $\square$

Where are we headed? We would like to consider a short exact sequence of chain complexes. A short exact sequence of chain complexes is a commutative diagram of the form:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & A_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, "i_{n+1}"] & A_n \arrow[r, "\partial_n"] \arrow[d, "i_n"] & A_{n-1} \arrow[r, "\partial_{n-1}"] \arrow[d, "i_{n-1}"] & \cdots \\
\cdots \arrow[r] & B_{n+1} \arrow[r, "\partial'_{n+1}"] \arrow[d, "j_{n+1}"] & B_n \arrow[r, "\partial'_n"] \arrow[d, "j_n"] & B_{n-1} \arrow[r, "\partial'_{n-1}"] \arrow[d, "j_{n-1}"] & \cdots \\
\cdots \arrow[r] & C_{n+1} \arrow[r, "\partial''_{n+1}"] \arrow[d] & C_n \arrow[r, "\partial''_n"] \arrow[d] & C_{n-1} \arrow[r, "\partial''_{n-1}"] \arrow[d] & \cdots \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

We abbreviate the diagram as

$$0_\bullet \longrightarrow A_\bullet \xrightarrow{i_\bullet} B_\bullet \xrightarrow{j_\bullet} C_\bullet \longrightarrow 0_\bullet$$

The snake lemma will be a statement about short exact sequences in $\mathbf{Chain_{Ab}}$. We can define the category of short exact sequence of chain complexes, $\mathbf{Chain_{Ab}^{Exact}}$. Objects in $\mathbf{Chain_{Ab}^{Exact}}$ are short exact sequences of chain complexes. A morphism between short exact sequence of chain complexes is a diagram

$$\begin{tikzcd}
0_\bullet \arrow[r] & A_\bullet \arrow[r, "i_\bullet"] \arrow[d, "f_\bullet"] & B_\bullet \arrow[r, "j_\bullet"] \arrow[d, "g_\bullet"] & C_\bullet \arrow[r] \arrow[d, "h_\bullet"] & 0_\bullet \\
0_\bullet \arrow[r] & A'_\bullet \arrow[r, "i'_\bullet"] & B'_\bullet \arrow[r, "j'_\bullet"] & C'_\bullet \arrow[r] & 0_\bullet
\end{tikzcd}$$

such that $f_\bullet, g_\bullet, h_\bullet$ are chain maps. We will not go through the pain of writing the diagram out explicitly.

Proposition 5.19. *(**Snake Lemma**) Consider a short exact sequence in* $\mathbf{Chain_{Ab}}$*:*

$$0_\bullet \to A_\bullet \xrightarrow{i_\bullet} B_\bullet \xrightarrow{j_\bullet} C_\bullet \to 0_\bullet$$

For each $n \geq 1$, there exist connecting morphisms

$$\delta_n : H_n(C_\bullet) \to H_{n-1}(A_\bullet)$$

such that there is a long exact sequence of homology groups:

$$\begin{tikzcd}
\cdots \arrow[r] & H_{n+1}(B_\bullet) \arrow[r] & H_{n+1}(C_\bullet) \arrow[dll, "\delta_{n+1}"] \\
H_n(A_\bullet) \arrow[r] & H_n(B_\bullet) \arrow[r] & H_n(C_\bullet) \arrow[dll, "\delta_n"] \\
H_{n-1}(A_\bullet) \arrow[r] & H_{n-1}(B_\bullet) \arrow[r] & \cdots
\end{tikzcd}$$

In fact, the above construction defines a functor from $\mathbf{Chain_{Ab}^{Exact}}$ *to* $\mathbf{Ab^{Long}}$*, the category of long exact sequences of abelian groups.*

Proof. The short exact sequence of chain complexes can be drawn more explicitly as:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & A_{n+1} & \xrightarrow{\partial_{n+1}} & A_n & \xrightarrow{\partial_n} & A_{n-1} & \xrightarrow{\partial_{n-1}} & \cdots \\
 & & \downarrow{\scriptstyle i_{n+1}} & & \downarrow{\scriptstyle i_n} & & \downarrow{\scriptstyle i_{n-1}} & & \\
\cdots & \longrightarrow & B_{n+1} & \xrightarrow{\partial'_{n+1}} & B_n & \xrightarrow{\partial'_n} & B_{n-1} & \xrightarrow{\partial'_{n-1}} & \cdots \\
 & & \downarrow{\scriptstyle j_{n+1}} & & \downarrow{\scriptstyle j_n} & & \downarrow{\scriptstyle j_{n-1}} & & \\
\cdots & \longrightarrow & C_{n+1} & \xrightarrow{\partial''_{n+1}} & C_n & \xrightarrow{\partial''_n} & C_{n-1} & \xrightarrow{\partial''_{n-1}} & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Since $i_\bullet$ and $j_\bullet$ are chain maps, the homology functor maps induce maps on the homology groups which we write as $H_n(i)$ and $H_n(j)$ for each $n \in \mathbb{N}$. The connecting morphisms δ_n are constructed as follows: let $c \in C_n$ be a cycle representative for $[\alpha] \in H_n(C)$. Then, since j_n is surjective, there exists $b \in B_n$ such that $c = j_n(b)$. Therefore, we have that $\partial'_n(b) \in B_{n-1}$. By the commutativity of the diagram, we know that

$$j_{n-1}(\partial''_n(b)) = \partial''_n(j_{n-1}(b)) = \partial''_n(c) = 0,$$

since c is a cycle. Therefore, $\partial'_n(b) \in \ker j_{n-1} = \operatorname{im} i_{n-1}$. So, there exists a (unique, since i_{n-1} is injective) $a \in A_{n-1}$ with $\partial'_n(b) = i_{n-1}(a)$. We show that a is a cycle. Note that

$$i_{n-2}(\partial_{n-1}(a)) = \partial'_{n-1}(i_{n-1}(a)) = \partial'_{n-1}(\partial'_n(b)) = 0$$

Since i_{n-2} is injective, this implies that $\partial_{n-1}(a) = 0$. Finally, we define $\delta_n([\alpha]) = [a] \in H_{n-1}(A)$. We have to show that this assignment is independent of all choices.

(1) Suppose we choose $b' \in B_n$ such that $j_n(b') = c$. Then, $b' - b \in \ker j_n = \operatorname{im} i_n$. So, there exists $a' \in A_n$ such that $b' - b = i_n(a')$. Therefore,

$$\begin{aligned}
\partial'_n(b') &= \partial'_n(b) + \partial'_n(i_n(a')) \\
&= \partial(b) + i_{n-1}(\partial_n(a')) \\
&= i_{n-1}(a) + i_{n-1}(\partial_n(a')) \\
&= i_{n-1}(a + \partial_n(a')).
\end{aligned}$$

So we see that changing b to b' amounts to changing a by a homologous cycle $a + \partial_n(a')$.

(2) If instead of c we use $c + \partial''_{n+1}(c')$ for some $c' \in C_{n+1}$. But then, $c' = j_{n+1}(b')$ for some $b' \in B_{n+1}$. So,

$$\begin{aligned}
c + \partial''_{n+1}(c') &= c + \partial''_{n+1}(j_{n+1}(b')) \\
&= c + j_n(\partial'_{n+1}(b')) \\
&= j_n(b + \partial'_{n+1}(b'))
\end{aligned}$$

Then b will be replaced by $b + \partial'_{n+1}(b')$, which leaves $\partial'_n(b)$ unchanged, hence a unchanged.

We now prove that the following sequence is exact:

$$\cdots \longrightarrow H_n(A) \xrightarrow{H_n(i)} H_n(B) \xrightarrow{H_n(j)} H_n(C) \xrightarrow{\delta_n} H_{n-1}(A) \xrightarrow{H_{n-1}(i_{n-1})} H_{n-1}(B) \longrightarrow \cdots$$

(1) im $H_n(i) \subseteq \ker H_n(j)$. This is immediate since $j_n \circ i_n = 0$ implies $H_n(j) \circ H_n(i) = 0$.
(2) im $H_n(j) \subseteq \ker \delta_n$. We have $\delta_n \circ H_n(j) = 0$ since in this case $\partial'_n b = 0$ in the definition of δ_n.
(3) im $\delta_n \subseteq \ker H_{n-1}(i_{n-1})$. This follows because $H_{n-1}(i_{n-1}) \circ \delta_{n-1,*}$ takes $[c]$ to $[\partial'_n b] = 0$.
(4) $\ker H_{n-1}(i_{n-1}) \subseteq$ im δ_n. Given a cycle $a \in A_{n-1}$ such that $i_{n-1}(a) = \partial'_n b$ for some $b \in B_n$, $j_n(b)$ is a cycle since $\partial''_n j_n(b) = j_{n-1}(\partial'_n b) = j_{n-1} i_{n-1}(a) = 0$, and δ_n takes $[j_n(b)]$ to $[a]$.
(5) $\ker H_n(j) \subseteq$ im $H_n(i)$. A homology class in $\ker H_n(j)$ is represented by a cycle $b \in B_n$ with $j_n(b)$ a boundary, so $j_n(b) = \partial''_{n+1} c'$ for some $c' \in C_{n+1}$. Since j_{n+1} is surjective, $c' = j_{n+1}(b')$ for some $b' \in B_{n+1}$. We have

$$\begin{aligned} j_n(b - \partial'_{n+1} b') &= j_n(b) - j_n(\partial'_{n+1} b') \\ &= j_n(b) - \partial''_{n+1} j_{n+1}(b') = 0 \end{aligned}$$

since $\partial''_{n+1} j_{n+1}(b') = \partial'_{n+1} c' = j_n(b)$. So $b - \partial'_{n+1} b' = i_n(a)$ for some $a \in A_n$. This a is a cycle since

$$\begin{aligned} i_{n-1}(\partial_n a) &= \partial'_n i_n(a) \\ &= \partial'_n(b - \partial'_{n+1} b') \\ &= \partial'_n b = 0 \end{aligned}$$

and i_{n-1} is injective. Thus $H_n(i)[a] = [b - \partial'_{n+1} b'] = [b]$, showing that $H_n(i)$ maps onto $\ker H_n(j)$.
(6) $\ker \delta_n \subseteq$ im $H_n(j)$. Given a cycle $c \in C_n$ such that $\delta_n(c) = 0$, we have that $\delta_n(c) = \partial_n(a)$ for some $a \in A_n$. Let $b \in B_n$ be the element constructed in the definition of δ_n. The element $b - i_n(a)$ is a cycle since

$$\begin{aligned} \partial'_n(b - i_n(a)) &= \partial'_n b - \partial'_n i_n(a) \\ &= \partial'_n b - i_{n-1}(\partial_n a) \\ &= \partial'_n b - i_{n-1}(\delta_n(c)) = 0 \end{aligned}$$

Note that:

$$j_n(b - i_n(a)) = j_n(b) - j_n i_n(a') = j_n(b) = c$$

So $H_n(j)$ maps $[b - i_n(a)]$ to $[c]$.

We now show that the above construction defines a functor from $\mathbf{Chain}^{\mathbf{Exact}}_{\mathbf{Ab}}$ to $\mathbf{Ab}^{\mathbf{Long}}$. Consider the following diagram in $\mathbf{Chain}^{\mathbf{Exact}}_{\mathbf{Ab}}$:

$$\begin{tikzcd} 0_\bullet \arrow[r] & A_\bullet \arrow[r, "i_\bullet"] \arrow[d, "f_\bullet"] & B_\bullet \arrow[r, "j_\bullet"] \arrow[d, "g_\bullet"] & C_\bullet \arrow[r] \arrow[d, "h_\bullet"] & 0_\bullet \\ 0_\bullet \arrow[r] & A'_\bullet \arrow[r, "i'_\bullet"] & B'_\bullet \arrow[r, "j'_\bullet"] & C'_\bullet \arrow[r] & 0_\bullet \end{tikzcd}$$

We show that induces the following commutative diagram.

$$\begin{tikzcd} \cdots \arrow[r] & H_n(A) \arrow[r, "H_n(i)"] \arrow[d, "H_n(f_n)"] & H_n(B) \arrow[r, "H_n(j)"] \arrow[d, "H_n(g_n)"] & H_n(C) \arrow[r, "\delta_n"] \arrow[d, "H_n(h_n)"] & H_{n-1}(A) \arrow[r, "H_{n-1}(i_{n-1})"] \arrow[d, "H_{n-1}(f_{n-1})"] & H_{n-1}(B) \arrow[r] \arrow[d, "H_{n-1}(g_{n-1})"] & \cdots \\ \cdots \arrow[r] & H_n(A') \arrow[r, "H_n(i'_n)"] & H_n(B') \arrow[r, "H_n(j'_n)"] & H_n(C') \arrow[r, "\delta'_n"] & H_{n-1}(A') \arrow[r, "H_n(i'_{n-1})"] & H_{n-1}(B') \arrow[r] & \cdots \end{tikzcd}$$

The commutativity of the first two squares and the last square is obvious since n-th homology is a functor. It suffices to check that the the diagram

$$\begin{tikzcd}
H_n(C) \arrow[r, "\delta_n"] \arrow[d, "H_n(h_n)"'] & H_{n-1}(A) \arrow[d, "H_{n-1}(f_{n-1})"] \\
H_n(C') \arrow[r, "\delta'_n"] & H_{n-1}(A')
\end{tikzcd}$$

is commutative. Recall that the map $\delta_n : H_n(C) \to H_{n-1}(A)$ was defined by $\delta_n[c] = [a]$ where $c = j_n(b)$ and $i_{n-1}(a) = \partial'_n b$. Consider $h_n(c) \in C'_n$. Note that

$$h_n(c) = h_n(j_n(b)) = j'_n(g_n(b))$$
$$i'_{n-1}(f_{n-1}(a)) = g_{n-1}(i_{n-1}(a)) = g_{n-1}(\partial'_n(b)) = d'_n(g_n(b)).$$

Here d'_n is the map from B'_n to B'_{n-1}. Hence,

$$[\delta'_n h_n(c))] = [f_{n-1}(a)] = [f_{n-1}\delta_n(c)]$$

This shows that the construction defines a functor from the $\mathbf{Chain}^{\mathbf{Exact}}_{\mathbf{Ab}}$ to $\mathbf{Ab}^{\mathbf{Long}}$. $\square$

Proposition 5.20. *(**Braid Lemma**) Suppose three long exact sequences and a chain complex we have a commutative diagram. Then the chain complex is also a long exact sequence*

$$\begin{tikzcd}
\cdots \arrow[rr] \arrow[dr] & & A \arrow[rr, "g_1"] \arrow[dr, "f_1"] & & D \arrow[rr, "h_3"] \arrow[dr, "g_2"] & & G \arrow[rr, "j_4"] \arrow[dr, "h_4"] & & J \arrow[dr] & \\
\cdots & O \arrow[ur, "g_0"] \arrow[dr, "j_0"] & & C \arrow[ur, "h_2"] \arrow[dr, "f_2"] & & F \arrow[ur, "j_3"] \arrow[dr, "g_3"] & & I \arrow[ur, "f_5"] \arrow[dr, "h_5"] & & \cdots \\
\cdots \arrow[rr] \arrow[ur] & & B \arrow[rr, "j_1"] \arrow[ur, "h_1"] & & E \arrow[rr, "f_3"] \arrow[ur, "j_2"] & & H \arrow[rr, "g_4"] \arrow[ur, "f_4"] & & K \arrow[ur] &
\end{tikzcd}$$

Proof. WLOG, assume that the f maps describe the chain complex. By symmetry, it suffices to show exactness at C, E and H:

(1) $\ker(f_2) \subseteq \operatorname{im}(f_1)$: Let $x \in \ker(f_2)$. Then $0 = f_2(x) = j_2 f_2(x) = g_2 h_2(x)$ by commutativity. It follows that $h_2(x) \in \ker(g_2) = \operatorname{im}(g_1)$. So $\exists x_1 \in A$ such that $g_1(x_1) = h_2(x)$. By commutativity, $g_1(x_1) = h_2 f_1(x_1)$. So we have that $0 = g_1(x_1) - h_2(x) = h_2(f_1(x_1) - x)$. Let $x_2 := f_1(x_1) - x \in \ker(h_2) = \operatorname{im}(h_1)$. Then $\exists x_3 \in B$ such that $h_1(x_3) = x_2$. Note that

$$j_1(x_3) = f_2 h_1(x_3) = f_2(x_2) = f_2(f_1(x_1) - x) = 0,$$

where the last equality follows from $f_2 \circ f_1 = 0$ and $f_2(x) = 0$. We therefore have that $x_3 \in \ker(j_1) = \operatorname{im}(j_0)$. So there exists $x_4 \in O$ such that $j_0(x_4) = x_3$. Consider$g_0(x_4)$. It satisfies

$$f_1 g_0(x_4) = h_1 j_0(x_4) = h_1(x_3) = x_2 = f_1(x_1) - x$$

Therefore, we have $x = f_1(x_1 - g_0(x_4))$. This shows that $x \in \operatorname{im}(f_1)$.

$$\begin{tikzcd}
 & g_0(x_4), x_1 \arrow[rr, mapsto, "g_1"] \arrow[dr, mapsto, "f_1"] & & h_2(x) \arrow[dr, mapsto, "g_2"] & \\
x_4 \arrow[ur, mapsto, "g_0"] \arrow[dr, mapsto, "j_0"] & & x_2, x \arrow[ur, mapsto, "h_2"] \arrow[dr, mapsto, "f_2"] & & 0 \\
 & x_3 \arrow[ur, mapsto, "h_1"] \arrow[rr, mapsto, "j_1"] & & 0 \arrow[ur, mapsto, "j_2"] &
\end{tikzcd}$$

(2) $\ker(f_3) \subseteq \operatorname{im}(f_2)$: Let $x \in E$ be such that $f_3(x) = 0$. By commutativity, $g_3 j_2(x) = 0$, so $j_2(x) \in \ker(g_3) = \operatorname{im}(g_2)$. Then $\exists x_1 \in D$ such that $g_2(x_1) = j_2(x)$. It satisfies $h_3(x_1) = j_3 g_2(x_1) = j_3 j_2(x) = 0$, as (j_i) is a chain complex. So $x_1 \in \ker(h_3) = \operatorname{im}(h_2)$. Therefore, there exists $x_2 \in C$ such that $h_2(x_2) = x_1$. This element is such that $j_2 f_2(x_2) = g_2 h_2(x_2) = g_2(x_1) = j_2(x)$. We therefore have $j_2(f_2(x_2) - x) = 0$. Let $x_3 := f_2(x_2) - x$. Then $x_3 \in \ker(j_2) = \operatorname{im}(j_1)$. Let $x_4 \in B$ be such that $j_1(x_4) = x_3$. x_4 is such that $f_2 h_1(x_4) = j_1(x_4) = x_3 = f_2(x_2) - x$. Finally, we see that $x = f_2(x_2 - h_1(x_4))$, so $x \in \operatorname{im}(f_2)$ as required.

$$\begin{tikzcd}
 & & x_1 \arrow[rr, mapsto, "h_3"] \arrow[dr, mapsto, "g_2"] & & 0 \\
 & h_1(x_4), x_2 \arrow[ur, mapsto, "h_2"] \arrow[dr, mapsto, "f_2"] & & 0, j_2(x) \arrow[ur, mapsto, "j_3"] \arrow[dr, mapsto, "g_3"] & \\
x_4 \arrow[ur, mapsto] \arrow[rr, mapsto, "j_1"] & & x_3, x \arrow[ur, mapsto, "j_2"] \arrow[rr, mapsto, "f_3"] & & 0
\end{tikzcd}$$

(3) $\ker(f_4) \subseteq \operatorname{im}(f_3)$: Let $x \in H$ be such that $f_4(x) = 0$. Then $0 = h_5 f_4(x) = g_4(x)$. So $x \in \ker(g_4) = \operatorname{im}(g_3)$. Let $x_1 \in F$ be such that $g_3(x_1) = x$. Then $h_4 j_3(x_1) = f_4 g_3(x_1) = f_4(x) = 0$. So $j_3(x_1) \in \ker(h_4) = \operatorname{im}(h_3)$. Let $x_2 \in D$ be such that $h_3(x_2) = j_3(x_1)$. Then $j_3(x_1) = j_2 g_2(x_2)$, such that $x_3 := g_2(x_2) - x_1 \in \ker(j_3) = \operatorname{im}(j_2)$. Let $x_4 \in E$ be such that $j_2(x_4) = x_3$. Then $f_3(x_4) = g_3 j_2(x_4) = g_3(x_3) = g_3(g_2(x_2) - x_1) = -g_3(x_1) = -x$. Therefore, $x = f_3(-x_4)$, and $x \in \operatorname{im}(f_3)$ as required.

$$\begin{tikzcd}
x_2 \arrow[rr, mapsto, "h_3"] \arrow[dr, mapsto, "g_2"] & & j_3(x_1) \arrow[dr, mapsto, "h_4"] & & \\
 & x_3, x_1 \arrow[ur, mapsto, "j_3"] \arrow[dr, mapsto, "g_3"] & & 0 \arrow[dr, mapsto, "h_5"] & \\
x_4 \arrow[ur, mapsto, "j_2"] \arrow[rr, mapsto, "f_3"] & & -x, x \arrow[ur, mapsto, "f_4"] \arrow[rr, mapsto, "g_4"] & & 0
\end{tikzcd}$$

This completes the proof. $\square$

6. Singular Homology

In this section, we define singular homology. Singular homology is difficult to compute, but singular homology has nice theoretical properties which allows us to prove a host of properties about a homology theory. It can be checked that simplicial homology and singular homology is coincide as we will do later on. Hence, simplicial homology provides a computational tool to compute homology, and singular homology provides a theoretical tool to study homology theoretically.

Definition 6.1. Let X be a topological space. A singular n-simplex is a continuous map $\sigma : \Delta^n \to X$.

Example 6.2. Since Δ^0 is a point, a 0-simplex in X is simply a point in X. Since Δ^1 is a closed interval, a 1-simplex is a path in X.

Remark 6.3. *The phrase 'singular' is used here to express the idea that σ need not be an embedding or a homeomorphism but can have 'singularities' where its image does not look at all like a simplex. All that is required is that σ be continuous.*

Let X be a topological space and let $C_n(X)$ be the free abelian group with basis the set of singular n-simplices in X:

$$C_n(X) = \left\{ \sum_{i=0}^{n} n_i\sigma_i : n_i \in \mathbb{Z}, \sigma_i : \Delta^n \to X \text{ continuous} \right\}$$

where each formal sum $\sum_{i=0}^{n} n_i\sigma_i$ is finite, i.e., all but finitely many n_i are zero. Elements of $C_n(X)$, called n-chains.

Remark 6.4. *Let $\sigma : \Delta^n \to X$ be a singular n-simplex in X. If we restrict σ to one of the faces of Δ^n, we get a continuous map from an $n-1$-simplex into X. Is this a singular $n-1$ simplex? While any face of Δ^n is an $(n-1)$-simplex, it is not the* standard *$(n-1)$-simplex, since the domain is wrong. Thus, strictly speaking, the restriction of an n-simplex σ in X to a face is not actually a singular $(n-1)$-simplex in X, since it is not a continuous map from Δ^{n-1} into X. This issue can be avoided as follows. Consider the map:*

$$d_i^n : \Delta^{n-1} \to \Delta^n, \quad (t_0, \dots, t_{n-1}) \mapsto (t_0, \dots, t_{i-1}, 0, t_i, \dots, t_n),$$

for each $0 \le i \le n$. The image $d_i(\Delta^{n-1}) \subseteq \Delta^n$ can be identified with the the i-th face of Δ^n. If $\sigma : \Delta^n \to X$ is a singular n-simplex, then composition $\sigma \circ d_i$ is then a singular $(n-1)$-simplex in X. For the most part, however, we shall ignore this pedantic issue because after all, it's clear what we mean. For the most part, we shall use the notation $\sigma|_{[v_0,\dots,\widehat{v_i},\dots,v_n]}$ to refer to the map d_i.

Definition 6.5. Let X be a topological space and let $C_n(X)$ be the free abelian group with basis the set of singular n-simplices in X. The n-th boundary map

$$\partial_n : C_n(X) \to C_{n-1}(X)$$

is defined on the basis of $C_n(X)$ by the formula

$$\partial_n(\sigma) = \sum_{i=0}^{n} (-1)^i \sigma \circ d_i$$

Lemma 6.6. *Let X be a topological space. The composition*

$$\partial_{n-1} \circ \partial_n : C_n(X) \xrightarrow{\partial_n} C_{n-1}(X) \xrightarrow{\partial_{n-1}} C_{n-2}(X)$$

is the zero map.

Proof. The crucial observation about the maps d_i's we need is that for every $n \geq 2$ and every $0 \leq j < i \leq n$, we have:

$$d_i^n \circ d_j^{n-1} = d_j^n \circ d_{i-1}^{n-1} : \Delta^{n-2} \to \Delta^n$$

Indeed, it is easy to verify that both maps are given by

$$(t_0, t_1, \ldots, t_{n-2}) \mapsto (t_0, \ldots, t_{j-1}, 0, t_j, \ldots, t_{i-1}, 0, t_i, \ldots, t_{n-2})$$

Note that

$$\begin{aligned}
\partial_{n-1} \circ \partial_n(\sigma) &= \sum_{j=0}^{n-1} \sum_{i=0}^{n} (-1)^{i+j} \sigma \circ d_i^n \circ d_j^{n-1} \\
&= \sum_{j=0}^{n-1} \sum_{i=0}^{j} (-1)^{i+j} d_i^n \circ d_j^{n-1} + \sum_{j=0}^{n-1} \sum_{i=j+1}^{n} (-1)^{i+j} d_i^n \circ d_j^{n-1} \\
&= \sum_{j=0}^{n-1} \sum_{i=0}^{j} (-1)^{i+j} d_i^n \circ d_j^{n-1} + \sum_{j=0}^{n-1} \sum_{i=j+1}^{n} (-1)^{i+j} d_j^n \circ d_{i-1}^{n-1} \\
&= \sum_{j=0}^{n-1} \sum_{i=0}^{j} (-1)^{i+j} d_i^n \circ d_j^{n-1} + \sum_{j=0}^{n-1} \sum_{i=j}^{n-1} (-1)^{i+j+1} d_j^n \circ d_i^{n-1} \\
&= \sum_{j=0}^{n-1} \sum_{i=0}^{j} (-1)^{i+j} d_i^n \circ d_j^{n-1} - \sum_{j=0}^{n-1} \sum_{i=j}^{n-1} (-1)^{i+j} d_j^n \circ d_i^{n-1}
\end{aligned}$$

The second last equality follows by a shift of the inner summation index in the second nested sum. If we now interchange the roles of i and j in the second sum, the two nested sums cancel. □

Remark 6.7. *In what follows, we shall write the boundary operator as*

$$\partial_n(\sigma) = \sum_{i=0}^{n} (-1)^i \sigma|_{[v_0, \ldots, \widehat{v_i}, \ldots, v_n]}$$

Note that:

$$\sum_{j<i} (-1)^i (-1)^j \sigma \mid_{[v_0, \cdots, \widehat{v_j}, \cdots, \widehat{v_i}, \cdots, v_n]} + \sum_{j>i} (-1)^i (-1)^{j-1} \sigma \mid_{[v_0, \cdots, \widehat{v_i}, \cdots, \widehat{v_j}, \cdots, v_n]} = 0.$$

The latter two summations cancel since after switching i *and* j *in the second sum, it becomes the negative of the first.*

Purely algebraically, we have a sequence of homomorphisms of abelian groups:

$$\cdots \xrightarrow{\partial_{n+1}} C_n(X) \xrightarrow{\partial_n} C_{n-1}(X) \xrightarrow{\partial_{n-1}} C_{n-2}(X) \xrightarrow{\partial_{n-2}} \cdots .$$

The boundary map $\partial_n : C_n(X) \longrightarrow C(X)_{n-1}$ is such that

$$\partial_n \circ \partial_{n+1} = 0$$

That is:

$$\operatorname{im}(\partial_{n+1}) \subseteq \ker(\partial_n)$$

A sequence $(C_n(X), \partial_n)_{n\in\mathbb{N}}$ satisfying these properties is is called a singular chain complex. Elements of $\ker(\partial_n)$ are called (singular) n-cycles and elements of $\mathrm{im}(\partial_{n+1})$ are called (singular) n-boundaries.

Definition 6.8. Let X be a topological space. The n-th homology of the chain complex $(C_n(X), \partial_n)_{n\in\mathbb{N}}$ is

$$H_n(X;\mathbb{Z}) = \frac{\ker(\partial_n)}{\mathrm{im}(\partial_{n+1})}$$

$H_n(X)$ is called the n-th singular homology group of X with $\mathbb{Z}$ coefficients.

Calculation with singular homology is difficult because each C_n is generally a free abelian group on uncountably many generators! Eventually, however, we will show that simplicial homology and singular homology are isomorphic.

Remark 6.9. *We will also introduce cellular homology which is isomorphic to singular homology and is amenable to computation.*

Here is a trivial computation:

Example 6.10. (**Singular Homology of a Point**) If X is a single point, then there is exactly one map $\Delta_n \to X$, and it is continuous, so $C_n(X) = \mathbb{Z}$ for all n. Moreover,

$$\partial_n(\sigma_n) = \sum_{i=0}^{n-1}(-1)^i\sigma_{n-1} = \begin{cases} 0 & \text{for } n \text{ odd} \\ \sigma_{n-1} & \text{for } n \text{ even} \end{cases}$$

We end up with:

$$\cdots \xrightarrow{\cong} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{\cong} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \to 0$$

Thus, we can quotient out to get the homology:

$$H_n(X;\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{for } n = 0 \\ 0 & \text{for } n \geq 1 \end{cases}$$

On the other hand, singular homology is much nicer theoretically, because we don't have to worry about choosing a Δ-complex structure, so it provides a convenient tool to prove various properties about a homology theory. For instance:

Proposition 6.11. *Let X be a topological space.*

(1) *Let $(X_\alpha)_{\alpha\in A}$ are the path-connected components of X. Then,*

$$H_n(X;\mathbb{Z}) \cong \bigoplus_{\alpha\in A} H_n(X_\alpha;\mathbb{Z})$$

(2) *(0**-th Singular Homology Groups**) If X is non-empty and path-connected, then $H_0(X) \cong \mathbb{Z}$. Hence, for any space X, $H_0(X;\mathbb{Z})$ is a direct sum of $\mathbb{Z}$'s, one for each path-component of X.*

Proof. The proof is given below:

(1) Since Δ^n is path-connected, and an n-simplex $\sigma : \Delta^n \to X$ is a continuous map, we have that $\mathrm{im}(\sigma) \subseteq X_\alpha$ for some α. Therefore, we get a decomposition:

$$C_n(X) \cong \bigoplus_{\alpha} C_n(X_\alpha).$$

The boundary maps preserve this decomposition, i.e., $\partial_n(C_n(X_\alpha)) \subseteq C_{n-1}(X_\alpha)$. Hence $\ker(\partial_n)$ and $\mathrm{im}(\partial_{n+1})$ split similarly as direct sums, and the result follows.

(2) By definition, $H_0(X;\mathbb{Z}) = C_0(X)/\operatorname{im}\partial_1$. Define a homomorphism

$$\begin{aligned} \varepsilon : C_0(X) &\to \mathbb{Z} \\ \sum_{i=0}^{n} n_i\sigma_i &\mapsto \sum_{i=0}^{n} n_i \end{aligned}$$

This is obviously surjective if X is non-empty. We claim that $\ker\varepsilon = \operatorname{im}\partial_1$ if X is path-connected. Observe first that $\operatorname{im}\partial_1 \subseteq \ker\varepsilon$ since for a singular 1-simplex $\sigma : \Delta^1 \to X$, we have

$$\varepsilon\partial_1(\sigma) = \varepsilon(\sigma\big|_{[v_1]} - \sigma\big|_{[v_0]}) = 1 - 1 = 0$$

For the reverse inclusion, $\ker\varepsilon \subseteq \operatorname{im}\partial_1$, suppose $\varepsilon\left(\sum_{i=0}^{n} n_i\sigma_i\right) = 0$, so $\sum_{i=0}^{n} n_i = 0$. The σ_i's are singular 0-simplices, which are simply points of X. Choose a path $\tau_i : I \to X$ from a basepoint, x_0, to $\sigma_i(v_0)$, and let σ_0 be the singular 0-simplex with image x_0. We can view τ_i as a singular 1-simplex, a map $\tau_i : [v_0, v_1] \to X$, and then we have $\partial\tau_i = \sigma_i - \sigma_0$. Hence,

$$\partial\left(\sum_{i=0}^{n} n_i\tau_i\right) = \sum_{i=0}^{n} n_i\sigma_i - \sum_{i=0}^{n} n_i\sigma_0 = \sum_{i=0}^{n} n_i\sigma_i,$$

since $\sum_{i=0}^{n} n_i = 0$. Thus, $\sum_{i=0}^{n} n_i\sigma_i$ is a boundary, which shows that $\ker\varepsilon \subseteq \operatorname{im}\partial_1$. Hence, ε induces an isomorphism $H_0(X;\mathbb{Z}) \cong \mathbb{Z}$.

This completes the proof. □

Remark 6.12. *It is often very convenient to have a slightly modified version of homology for which a point has trivial homology groups in all dimensions, including zero. This is done by defining the* reduced homology *groups $\widetilde{H}_n(X)$ to be the homology groups of the augmented chain complex:*

$$\cdots \to C_2(X) \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\epsilon} \mathbb{Z} \to 0$$

where

$$\epsilon\left(\sum_{i=0}^{n} n_i\sigma_i\right) = \sum_{i=0}^{n} n_i$$

Since $\epsilon \circ \partial_1 = 0$, ϵ vanishes on $\operatorname{im}\partial_1$ and hence induces a map $H_0(X) \to \mathbb{Z}$ with kernel $\widetilde{H}_0(X)$, so

$$H_0(X;\mathbb{Z}) \cong \widetilde{H}_0(X;\mathbb{Z}) \oplus \mathbb{Z}$$

Why all the fuss about singular homology? Singular homology defines a functor from **Top** to **Ab**. Thus, singular homology yields an invariant that can distinguish spaces. More importantly, it provides a systematic and general way to study topological spaces using algebraic methods. Unlike simplicial homology, which require specific decompositions, singular homology applies to all topological spaces, making it a powerful and flexible theoretical tool in algebraic topology.

Proposition 6.13. *$H_n : \textbf{Top} \to \textbf{Ab}$ is a covariant functor for each $n \geq 0$.*

Proof. Let X, Y be topological spaces and let $f : X \to Y$ be a continuous map. Then, we have a sequence of induced homomorphisms:

$$\begin{aligned} f_n : C_n(X) &\to C_n(Y) \\ \sigma &\mapsto f \circ \sigma \end{aligned}$$

Extending linearly gives a group homomorphism.

$$f_n\left(\sum_{i=0}^{n} n_i\sigma_i\right) = \left(\sum_{i=0}^{n} n_i f_n\sigma_i\right) = \left(\sum_{i=0}^{n} n_i f \circ \sigma_i\right)$$

Consider the following diagram:

$$\begin{tikzcd} \cdots \arrow[r] & C_{n+1}(X) \arrow[r, "\partial_{n+1}"] \arrow[d, "f_{n+1}"] & C_n(X) \arrow[r, "\partial_n"] \arrow[d, "f_n"] & C_{n-1}(X) \arrow[r] \arrow[d, "f_{n-1}"] & \cdots \\ \cdots \arrow[r] & C_{n+1}(Y) \arrow[r, "\partial'_{n+1}"] & C_n(Y) \arrow[r, "\partial'_n"] & C_{n-1}(Y) \arrow[r] & \cdots \end{tikzcd}$$

This diagram commutes because:

$$\begin{aligned} f_n(\partial_{n+1}\sigma) &= f_n\left(\sum_{i=0}^{n}(-1)^i \sigma_\alpha|_{[v_0,\ldots,\widehat{v_i},\ldots,v_n]}\right) \\ &= \sum_{i=0}^{n}(-1)^i f_n\sigma_\alpha|_{[v_0,\ldots,\widehat{v_i},\ldots,v_n]} \\ &= \sum_{i=0}^{n}(-1)^i f \circ \sigma_\alpha|_{[v_0,\ldots,\widehat{v_i},\ldots,v_n]} = \partial'_{n+1}(f_{n+1}\sigma). \end{aligned}$$

Hence, we have a functor from **Top** to $\mathbf{Chain_{Ab}}$. By Proposition 5.16, we have a functor from $\mathbf{Chain_{Ab}}$ to **Ab**. Composing the two functors yields the desired functor. □

7. The Eilenberg-Steenrod Axioms

We have met two homology theories: simplicial homology and singular homology. Later on, we will discuss cellular homology. In fact, there are many other homology theories in mathematics. Eilenberg and Steenrod united the different homology theories by laying out a set of axioms that all homology theories satisfy.

Definition 7.1. (**Eilenberg-Steenrod Axioms**) A homology theory with $\mathbb{Z}$ coefficients consists of

(1) A family of functors $H_n : \mathbf{Top}^2 \to \mathbf{Ab}$ for $n \geq 0$, and
(2) A family of natural transformations $\delta_n : H_n \to H_{n-1} \circ p$, where p is the functor sending (X, A) to $(A, \varnothing)$ and $f : (X, A) \to (Y, B)$ to $f|_A : (A, \varnothing) \to (B, \varnothing)$.

such that the following axioms are satisfied:

(a) (Homotopy invariance) If $f, g : (X, A) \to (Y, B)$ are homotopic maps, then the induced maps

$$H_n(f), H_n(g) : H_n(X, A; \mathbb{Z}) \to H_n(Y, B; \mathbb{Z})$$

are such that $H_n(f) = H_n(g)$ for $n \geq 0$[9].

[9]In other words, H_n may be regarded as a functor from **hTop** to **Ab**.

(b) (Long exact sequence) The inclusions
$$(A, \emptyset) \hookrightarrow (X, \emptyset) \hookrightarrow (X, A)$$
give rise to a long exact sequence
$$\cdots \longrightarrow H_{n+1}(X; \mathbb{Z}) \longrightarrow H_{n+1}(X, A; \mathbb{Z}) \xrightarrow{\delta_{n+1}} H_n(A; \mathbb{Z}) \longrightarrow H_n(X; \mathbb{Z}) \longrightarrow \cdots$$

(c) (Excision) If $Z \subseteq A \subseteq X$ are topological spaces such that $\overline{Z} \subseteq \mathrm{Int}(A)$, the inclusion of pairs $(X \setminus Z, A \setminus Z) \subseteq (X, A)$ induces isomorphisms
$$H_n(X \setminus Z, A \setminus Z; \mathbb{Z}) \to H_n(X, A; \mathbb{Z})$$
for all $n \geq 0$.

(d) (Additivity) If $X = \coprod_\alpha X_\alpha$ is the disjoint union of a family of topological spaces X_α, then
$$H_n(X; \mathbb{Z}) = \bigoplus_\alpha H_n(X_\alpha; \mathbb{Z})$$
for each $n \in \mathbb{N}$.

Additionally, if a homology theory satisfies the following additional axiom

(e) (Dimension Axiom) For any one-point set $X = \{\bullet\}$,
$$H_n(X; \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } n = 0 \\ 0 & \text{otherwise,} \end{cases}$$
the the homology theory is called an ordinary homology theory with $\mathbb{Z}$ coefficients.

7.1. **Relative Homology.** We first introduce the notion of relative homology functors to make sense of the family of functors in Definition 7.1:
$$H_n : \mathbf{Top}^2 \to \mathbf{Ab}$$
Given $(X, A) \in \mathbf{Top}^2$, we have $C_n(A) \subseteq C_n(X)$ such that ∂_n restricts to a map from $C_n(A)$ to $C_{n-1}(A)$. Therefore, we can consider a chain complex $(C_\bullet(A), \partial_\bullet|_A)$ which is a sub-complex [10] of the chain complex $(C_\bullet, \partial_\bullet)$ The chain complex $(C_\bullet(A), \partial_\bullet|_A)$ is usually drawn as:
$$\cdots \longrightarrow C_2(A) \xrightarrow{\partial_2|_A} C_1(A) \xrightarrow{\partial_1|_A} C_0(A)$$
Note that $C_n(A)$ is an abelian subgroup of $C_n(X)$. Hence, we e can consider quotient group
$$C_n(X, A) = \frac{C_n(X)}{C_n(A)}$$
Since the boundary map
$$\partial_n : C_n(X) \to C_{n-1}(X)$$
takes $C_n(A)$ to $C_{n-1}(A)$, it induces a quotient boundary map
$$\overline{\partial}_n : C_n(X, A) \to C_{n-1}(X, A)$$
Since $\partial_{n+1} \circ \partial_n = 0$ on $C_n(X)$, we have that $\overline{\partial}_{n+1} \circ \overline{\partial}_n = 0$ on $C_n(X, A)$. Therefore, we get a chain complex $(C_\bullet(X, A), \overline{\partial}_\bullet)$ The chain complex is usually drawn as:
$$\cdots \longrightarrow C_2(X, A) \xrightarrow{\overline{\partial}_2} C_1(X, A) \xrightarrow{\overline{\partial}_1} C_0(X, A)$$

[10] Given a chain complex $(C_\bullet, \partial_\bullet)$, a subcomplex of $(C_\bullet, \partial_\bullet)$ is given by a family of subgroups $C'_n \subseteq C_n$ such that the boundary operator $\partial'_n : C_n \to C_{n-1}$ restricts to a homomorphism $C'_n \to C'_{n-1}$ for all n.

The above discussion implies that the construction of relative singular chain complexes defines a functor from $\mathbf{Top}^2$ to $\mathbf{Chain_{Ab}}$.

Definition 7.2. Let $(X, A) \in \mathbf{Top}^2$. The n-th relative homology group with $\mathbb{Z}$ coefficients, $H_n(X, A)$, is the n-th homology group of the chain complex $(C_\bullet(X, A), \overline{\partial}_\bullet)$. That is:

$$H_n(X, A; \mathbb{Z}) = \frac{\text{Ker } \overline{\partial}_n}{\text{Im } \overline{\partial}_{n+1}}$$

Remark 7.3. *It is clear that the n-th relative homology group with $\mathbb{Z}$ coefficients defines a functor from $\mathbf{Top}^2$ to $\mathbf{Ab}$.*

Remark 7.4. *Since the homology of the empty set is trivial for all $n \geq 0$, we have:*

$$H_n(X, \emptyset; \mathbb{Z}) = H_n(X; \mathbb{Z}), \quad \forall n \geq 0.$$

By considering the definition of the relative boundary map we see that: x

(1) Elements of $H_n(X, A; \mathbb{Z})$ are represented by relative n-cycles: n-chains $\alpha \in C_n(X)$ such that $\partial_n(\alpha) \in C_{n-1}(A)$.
(2) A relative n-cycle, α, is trivial in $H_n(X, A; \mathbb{Z})$ iff it is a relative n-boundary: $\alpha = \partial_{n+1}(\beta) + \gamma$ for some $\beta \in C_{n+1}(X)$ and $\gamma \in C_n(A)$.

In Example 6.10, we have already seen that singular homology with $\mathbb{Z}$ coefficients satisfies the dimension axiom. Moreover, an argument similar to that given in Proposition 6.11(a) shows that singular homology satisfies the additivity axiom. The purpose of the remainder of this section is to show that singular homology satisfies the homotopy invariance, long exact sequence and excision axioms. Hence, singular homology is an ordinary homology theory.

7.2. **Homotopy Invariance of Singular Homology.** We prove that singular homology groups satisfy the homotopy invariance axiom. We content ourselves to give a proof in the absolute case. The proof in the relative homology case is similar. In order to prove this statement, we we will make use of the notion of a chain homotopy between chain complexes as introduced in Section 5.

Remark 7.5. *What does this definition of a chain homotopy mean geometrically? Let h be a homotopy between maps f, g from X to Y. Consider a 1-chain, a, in X. Then $f(a), g(b)$ are 1-chains in Y. The homotopy h maps the endpoints of $f(a)$ to the endpoints of $g(a)$. Let's look at the boundary of $h(a)$ in the diagram below (Figure 3). If we read counterclockwise starting at the bottom right, we see:*

$$\partial_2 h(a) = g(a) - \delta_+ - f(a) + \delta_-$$

What is $\delta_+ - \delta_-$? It is $h(\partial_1 a)$! Hence:

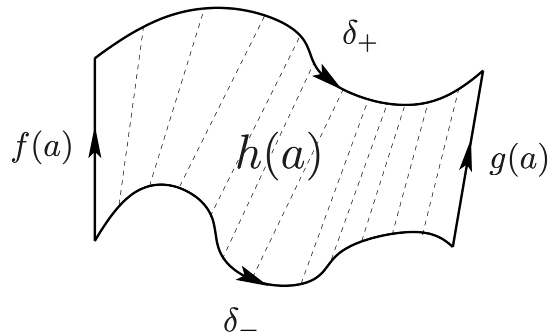
$$\partial_2 h(a) = g(a) - f(a) - h(\partial_1 a)$$

Hence the definition of a chain homotopy mimics the notion of a homotopy at the level of chain complexes.

Proposition 7.6. *Let X and Y be topological spaces. If $f, g : X \to Y$ are two homotopic maps, then*

$$H_n(f) = H_n(g) : H_n(X; \mathbb{Z}) \to H_n(Y; \mathbb{Z})$$

for each $n \geq 0$.



The image is taken from [Alu21].

Remark 7.7. *Before providing the proof, we discuss the idea behind the proof. The essential ingredient is a procedure for subdividing $\Delta^n \times I$ into simplices. In $\Delta^n \times I$, let*

$$\Delta^n \times 0 = [v_0, \ldots, v_n]$$
$$\Delta^n \times 1 = [w_0, \ldots, w_n]$$

where v_i and w_i have the same image under the projection $\Delta^n \times I \to \Delta^n$. We can pass from $[v_0, \ldots, v_n]$ to $[w_0, \ldots, w_n]$ by interpolating a sequence of n simplices, each obtained from the preceding one by moving one vertex v_i up to w_i, starting with v_n and working backwards to v_0. For instance,

$$[v_0, \ldots, v_i, w_{i+1}, \ldots, w_n]$$

moves up to

$$[v_0, \ldots, v_{i-1}, w_i, \ldots, w_n]$$

The region between these two n simplices is exactly the $(n+1)$ simplex

$$[v_0, \ldots, v_i, w_i, \ldots, w_n]$$

Lemma 7.8. *$\Delta^n \times I$ is the union of $n+1$ copies of Δ^{n+1}.*

Proof. For $i = -1, 0, \ldots, n-1$, let $g_i : \Delta^n \to I$ denote the map

$$g_i(s_0, s_1, \ldots, s_n) = \sum_{i<j} s_j.$$

Let $G_i \subseteq \Delta^n \times I$ denote the graph of g_i. Then G_i is homeomorphic to Δ^n via the projection $\Delta^n \times I \to \Delta^n$ onto the first factor. Let us now label the vertices at the "bottom" (i.e., $\Delta^n \times \{0\}$) of $\Delta^n \times I$ by $v_0, v_1, \ldots, v_n$ and those at the "top" (i.e., $\Delta^n \times \{1\}$) by $w_0, w_1, \ldots, w_n$. Then G_i is the n-simplex

$$G_i = [v_0, \ldots, v_i, w_{i+1}, \ldots, w_n].$$

Since G_i lies below G_{i-1} as $g_i \leq g_{i-1}$, it follows that the region between G_i and G_{i-1} is the $(n+1)$-simplex $[v_0, \ldots, v_i, w_i, \ldots, w_n]$; this is indeed an $(n+1)$-simplex as w_i is not in G_i and hence not in the n-simplex $[v_0, \ldots, v_i, w_i, \ldots, w_n]$. Since

$$0 = g_n \leq g_{n-1} \leq \ldots \leq g_0 \leq g_{-1} = 1,$$

we see that $\Delta^n \times I$ is the union of the regions between the G_i, and hence the union of $n+1$ different $(n+1)$-simplices $[v_0, \ldots, v_i, w_i, \ldots, w_n]$, each intersecting the next in an n-simplex face. $\square$

Proof. (Proposition 7.6) Given a homotopy $F : X \times I \to Y$ from f to g and a singular simplex $\sigma : \Delta^n \to X$, we can form the composition

$$F \circ (\sigma \times \mathrm{Id}_I) : \Delta^n \times I \to X \times I \to Y$$

Using this, we can define *prism operators* $P_n : C_n(X) \to C_{n+1}(Y)$ by the following formula:

$$P_n\sigma = \sum_{i=0}^{n+1} (-1)^i F \circ (\sigma \times 1)|_{[v_0,\cdots,v_i,w_i,\cdots,w_n]}.$$

The prism operator is our proposed chain homotopy. A simple computation shows that we have

$$f_n - g_n = \partial'_{n+1} P_n + P_{n-1}\partial_n$$

Indeed:

$$\begin{aligned}\partial'_{n+1} P_n(\sigma) &= \sum_{j \le i} (-1)^i (-1)^j F \circ (\sigma \times 1)|_{[v_0,\ldots,\widehat{v_j},\ldots,v_i,w_i,\ldots,w_n]} \\ &\quad + \sum_{j \ge i} (-1)^i (-1)^{j+1} F \circ (\sigma \times 1)|_{[v_0,\ldots,v_i,w_i,\ldots,\widehat{w_j},\ldots,w_n]}\end{aligned}$$

The terms with $i = j$ in the two sums cancel except for

$$\begin{aligned} F \circ (\sigma \times 1)\big|_{[\widehat{v_0},w_0,\ldots,w_n]} &= g \circ \sigma = g_n(\sigma), \\ -F \circ (\sigma \times 1)\big|_{[v_0,\ldots,v_n,\widehat{w_n}]} &= -f \circ \sigma = -f_n(\sigma). \end{aligned}$$

The terms with $i \neq j$ are exactly $-P_{n-1}\partial_n(\sigma)$. Hence, the sequence of maps $(P_n)_{n \ge 0}$ defines a chain homotopy. The claim follows by invoking Proposition 5.15. $\square$

Corollary 7.9. *If X is contractible, then $H_n(X;\mathbb{Z}) = 0$ for all $n > 0$.*

Proof. Immediate from the previous corollary and that $H_n(\{*\}) = 0$ for $n \ge 1$. $\square$

7.3. **Long Exact Sequence in Singular Homology.** We now prove that singular homology satisfies the long exact sequence axiom. The importance of the long exact sequence axiom is that is allows us to compute homology groups of various spaces in using an 'inductive' and/or 'bottom-up' approach, as we shall see in various examples later on. We have a short exact sequence of chain complexes:

$$0_\bullet \longrightarrow (C_\bullet(A), \partial_\bullet|_A) \xrightarrow{i_\bullet} (C_\bullet, \partial_\bullet) \xrightarrow{j_\bullet} (C_\bullet(X,A), \overline{\partial}_\bullet) \longrightarrow 0_\bullet$$

Invoking the snake lemma (Proposition 5.19), we have the following long exact sequence is homology associated to the pair of spaces (X, A):

$$\cdots \longrightarrow H_{n+1}(X;\mathbb{Z}) \longrightarrow H_{n+1}(X,A;\mathbb{Z}) \xrightarrow{\delta_{n+1}} H_n(A) \longrightarrow H_n(X;\mathbb{Z}) \longrightarrow \cdots$$

Remark 7.10. *The boundary map $\delta_n : H_n(X,A;\mathbb{Z}) \to H_{n-1}(A;\mathbb{Z})$ has a very simple description: if a class $[\alpha] \in H_n(X,A;\mathbb{Z})$ is represented by a relative cycle α, then $\delta_n[\alpha]$ is the class of the cycle $\delta_n\alpha$ in $H_{n-1}(A;\mathbb{Z})$.*

Remark 7.11. *An easy generalization of the long exact sequence of a pair (X, A) is the long exact sequence of a triple $(X, A, B) \in \mathbf{Top}^3$. Indeed, we have $(X, A), (X, B), (A, B) \in \mathbf{Top}^2$. The three long exact sequences assemble in the following diagram:*

$$\begin{tikzcd}[column sep=small]
H_{n+2}(X;\mathbb{Z}) \arrow[rr, dashed, "g_1"] \arrow[dr, "f_1"] & & H_{n+2}(X,A;\mathbb{Z}) \arrow[rr, Rightarrow] \arrow[dr, dashed, "g_2"] & & H_{n+1}(A,B;\mathbb{Z}) \arrow[rr, dashed, "j_4"] \arrow[dr, Rightarrow] & & H_n(B;\mathbb{Z}) \\
& H_{n+2}(X,B;\mathbb{Z}) \arrow[ur, Rightarrow] \arrow[dr, "f_2"] & & H_{n+1}(A;\mathbb{Z}) \arrow[ur, dashed, "j_3"] \arrow[dr, dashed, "g_3"] & & H_{n+1}(X,B;\mathbb{Z}) \arrow[ur, "f_5"] \arrow[dr, Rightarrow] & \\
H_{n+2}(A,B;\mathbb{Z}) \arrow[rr, dashed, "j_1"] \arrow[ur, Rightarrow] & & H_{n+1}(B;\mathbb{Z}) \arrow[rr, "f_3"] \arrow[ur, dashed, "j_2"] & & H_{n+1}(X;\mathbb{Z}) \arrow[rr, dashed, "g_4"] \arrow[ur, "f_4"] & & H_{n+1}(X,A;\mathbb{Z})
\end{tikzcd}$$

The braid lemma (Proposition 5.20) implies that the chain complex labeled with $\Rightarrow$ arrows is a chain complex. This is the desired long exact sequence in homology generated by (X, A, B).

7.4. **Excision in Singular Homology.** We now prove that singular homology satisfies the excision axiom. The important of the excision axiom is that if $A \subseteq X$ if n-chains are "sufficiently inside" of A, we can cut A out without affecting the relative homology groups $H_n(X, A; \mathbb{Z})$. Here is the formal statement we'd like to prove in this section:

Proposition 7.12. *Suppose $Z \subseteq A \subseteq X$ are topological spaces such that $\overline{Z} \subseteq \operatorname{Int}(A)$. Then there is an inclusion of the pair $(X \setminus Z, A \setminus Z) \subseteq (X, A)$, and the induced map*

$$H_n(X \setminus Z, A \setminus Z; \mathbb{Z}) \to H_n(X, A; \mathbb{Z})$$

is an isomorphism for all $n \geq 0$. Equivalently, for subspaces $A, B \subseteq X$ whose interiors cover X, the inclusion $(B, A \cap B) \hookrightarrow (X, A)$ induces isomorphisms

$$H_n(B, A \cap B; \mathbb{Z}) \cong H_n(X, A; \mathbb{Z})$$

Remark 7.13. *To see that the two statements of the Excision Theorem are equivalent, just take $B = X \setminus Z$ (or $Z = X \setminus B$). Then $A \cap B = A \setminus Z$, and the condition $\overline{Z} \subset \operatorname{int}(A)$ is equivalent to $X = \operatorname{int}(A) \cup \operatorname{int}(B)$.*

Let's provide some intuition. Assume that $X = \operatorname{int}(A) \cup \operatorname{int}(B)$. We expect $H_n(X, A)$ remains unchanged if we cut A out. This argument works when all chains are belong either to A or B. But if a chain doesn't entirely lie entirely in A or B, then we have a problem. The solution is given by the method of barycentric subdivision: replace the 'large' chains with 'small' chains by subdividing. We formalize this intuition in Proposition 7.15. We first prove a lemma.

Lemma 7.14. *Let $S = [v_0, v_1, \dots, v_n]$ denote an n-simplex in some Euclidean space. Then if $x, y \in S$, one has*

$$\|x - y\| \leq \max_{0 \leq i \leq n} \|v_i - y\|$$

Hence[11]

$$\operatorname{diam} S = \max_{0 \leq i,j \leq n} \|v_i - v_j\|.$$

If b is the barycenter of S

$$b = \sum_{i=0}^{n} \frac{1}{n+1} v_i,$$

[11]The diameter of a simplex is the maximum Euclidean distance between any two of its points.

then

$$\|b - v_i\| \leq \frac{n}{n+1} \operatorname{diam} S.$$

Proof. Let $x, y \in S$, and write $x = \sum_{i=0}^{n} s_i v_i$ with $\sum_{i=0}^{n} s_i = 1$. Then

$$\|x - y\| \leq \sum_{i=0}^{n} s_i \|v_i - y\| \leq \max_{0 \leq i \leq n} \|v_i - y\|.$$

This shows in particular $\|y - v_i\| \leq \max_{0 \leq j \leq n} \|v_i - v_j\|$ for each $0 \leq i \leq n$. Hence, $\|x - y\| \leq \max_{0 \leq i \leq n} \|v_i - y\|$. If b is the barycenter, we have

$$\begin{aligned} \|b - v_i\| &= \left\| \frac{1}{n+1} \sum_{j=0}^{n} v_j - v_i \right\| \\ &\leq \frac{1}{n+1} \sum_{i=0}^{n} \|v_j - v_i\| \\ &\leq \frac{n}{n+1} \max_{0 \leq i,j \leq n} \|v_j - v_i\| = \frac{n}{n+1} \operatorname{diam} S \end{aligned}$$

This completes the proof. □

We now formalize the idea that subdividing a 'large' singular chains into a union of 'small' singular chains. Let $U = \{U_j\}$ be a collection of subspaces of X whose interiors form an open cover of X, and let $C_n^U(X)$ be the subgroup of $C_n(X)$ consisting of chains $\sum_i n_i \sigma_i$ such that each σ_i has image contained in some set in the cover U. The boundary map ∂_n takes $C_n^U(X)$ to $C_{n-1}^U(X)$, so the groups $C_U^n(X)$ form a chain complex. We denote the homology groups of this chain complex by $H_n^U(X)$.

Proposition 7.15. *Consider the chain map* $\iota_\bullet : C_\bullet^U(X) \hookrightarrow C_\bullet(X)$ *such that* i_n *is the inclusion map for each* $n \geq 0$*. There is a chain map* $\rho : C_\bullet(X) \to C_\bullet^U(X)$ *such that* $\iota \circ \rho$ *and* $\rho \circ \iota$ *are chain homotopic to the identity.*

Proof. See [Hat02] for the the proof. □

We can now prove the excision theorem:

Proof. (Proposition 7.12) Assume that $X = A \cup B$. WLOG, assume that A and B are open sets. We have

$$C_n^U(X) = C_n(A) + C_n(B) \qquad C_n(A \cap B) = C_n(A) \cap C_n(B)$$

Therefore, we have

$$\frac{C_n(B)}{C_n(A \cap B)} = \frac{C_n(B)}{C_n(A) \cap C_n(B)} \cong \frac{C_n(A) + C_n(B)}{C_n(A)} \cong \frac{C_n^U(X)}{C_n(A)}$$

All the maps appearing in the proof of Proposition 7.15 take chains in A to chains in A. So these maps induce quotient maps when we factor out chains in A and the quotient maps satisfy all the corresponding formulas in the proof of Proposition 7.15. There, Proposition 7.15 implies that the inclusion

$$C_n^U(X)/C_n(A) \hookrightarrow C_n(X)/C_n(A)$$

induces an isomorphism on homology. Since

$$C_n^U(X)/C_n(A) = \frac{C_n(B)}{C_n(A \cap B)},$$

we have that

$$H_n(B, A \cap B; \mathbb{Z}) \cong H_n(X, A; \mathbb{Z})$$

for each $n \geq 0$. This completes the proof. □

8. Relative Homology

We discuss relative homology in more detail in this section. We start with a useful lemma.

Lemma 8.1. *Let $A \subseteq X$ be topological spaces. Consider an exact sequence of abelian groups:*

$$A \to B \to C \to D \to E$$

1. *$C = 0$ if and only if the map $A \to B$ is surjective and $D \to E$ is injective.*
2. *For a pair of spaces $(X, A) \in \mathbf{Top}^2$, the inclusion $A \hookrightarrow X$ induces isomorphisms on all homology groups if and only if $H_n(X, A; \mathbb{Z}) = 0$ for all $n \geq 0$.*

Proof. The proof is as follows:

1. Let $\alpha, \beta, \gamma, \delta$ be the corresponding maps. By exactness,
$$\mathrm{Im}(\alpha) = \ker(\beta), \quad \mathrm{Im}(\beta) = \ker(\gamma), \quad \mathrm{Im}(\gamma) = \ker(\delta).$$
Note that α is surjective iff $\ker(\beta) = B$ iff $\mathrm{Im}(\beta) = 0$, and δ is injective iff $\mathrm{Im}(\gamma) = 0$ iff $\ker(\gamma) = C$. Putting both together, α is surjective and δ is injective iff $C = 0$, since $\mathrm{Im}(\beta) = \ker(\gamma)$.
2. Consider the following part of the the long exact sequence in homology:
$$\cdots \longrightarrow H_{n+1}(A; \mathbb{Z}) \longrightarrow H_{n+1}(X; \mathbb{Z}) \longrightarrow H_{n+1}(X, A; \mathbb{Z}) \longrightarrow H_n(A; \mathbb{Z}) \longrightarrow H_n(X; \mathbb{Z}) \longrightarrow \cdots$$
The maps $H_n(A; \mathbb{Z}) \to H_n(X; \mathbb{Z})$ are isomorphisms for all $n \geq 0$ if and only if they are both injective and surjective for all $n \geq 0$. By re-indexing, this is true if and only if the leftmost map in our five-term exact sequence is surjective and the rightmost map is injective for all $n \geq 0$. But (1), this is true if and only if the middle group vanishes for all $n \geq 0$.

This completes the proof. □

Remark 8.2. *As per Lemma 8.1, we can think of $H_n(X, A; \mathbb{Z})$ as measuring the failure of the induced morphism $H_n(A; \mathbb{Z}) \to H_n(X; \mathbb{Z})$ to be an isomorphism for each $n \geq 0$.*

Based on Lemma 8.1, we can characterize relative homology groups for $n = 0, 1$.

Proposition 8.3. *Let $A \subseteq X$ be topological spaces.*

1. *$H_0(X, A; \mathbb{Z}) = 0$ if and only if A meets each path-component of X.*
2. *$H_1(X, A; \mathbb{Z}) = 0$ if and only if $H_1(A; \mathbb{Z}) \to H_1(X; \mathbb{Z})$ is surjective and each path-component of X contains at most one path-component of A.*
3. *Let (X, x) be a pointed topological space. Then*
$$H_n(X, x; \mathbb{Z}) \cong H_n(X; \mathbb{Z}) \cong \widetilde{H}_n(X; \mathbb{Z})$$
for each $n \geq 1$.

Proof. The proof is given below:

(1) We first prove the special case that if X is a non-empty *path-connected* space and $A \subseteq X$, then $H_0(X, A; \mathbb{Z}) = 0$ if and only if A is not-empty. Consider the end of the long exact sequence for the pair $(X, A; \mathbb{Z})$:
$$H_0(A; \mathbb{Z}) \to H_0(X; \mathbb{Z}) \to H_0(X, A; \mathbb{Z}) \to 0$$
Note that $H_0(X; \mathbb{Z}) \cong \mathbb{Z}$. If A is empty, the sequence is,
$$0 \to \mathbb{Z} \to H_0(X, A; \mathbb{Z}) \to 0$$
Since the map from $\mathbb{Z}$ to $H_0(X, A; \mathbb{Z})$ is injective, $H_0(X, A; \mathbb{Z})$ must be non-zero. If A is non-empty, pick a point $a \in A$ and consider the homology class $[a] \in H_0(A; \mathbb{Z})$. The image of $[a]$ under
$$H_0(A; \mathbb{Z}) \to H_0(X; \mathbb{Z})$$
is the homology class of a point, which generates the co-domain. So $H_0(A; \mathbb{Z}) \to H_0(X; \mathbb{Z})$ is onto. Hence
$$H_0(A; \mathbb{Z}) \to H_0(X, A; \mathbb{Z})$$
is onto as well implying that and $H_0(X, A; \mathbb{Z}) = 0$. More generally, suppose X has multiple connected components. Assume that A meets each path component of X. If X_i is a component of X, then $H_0(A \cap X_i; \mathbb{Z}) \to H_0(X_i; \mathbb{Z})$ is surjective. But then
$$H_0(A; \mathbb{Z}) = \bigoplus_i H_0(A \cap X_i; \mathbb{Z}) \to \bigoplus_i H_0(X_i; \mathbb{Z}) = H_0(X; \mathbb{Z})$$
is surjective. Therefore, So $H_0(X, A; \mathbb{Z}) = 0$. Conversely, if A does not meet a component of X, say X_j, then $H_0(X_j, A; \mathbb{Z}) \neq 0$. But then $H_0(X_j, A; \mathbb{Z}) \neq 0$ is a direct summand of $H_0(X, A; \mathbb{Z})$. Hence $H_0(X, A; \mathbb{Z})$ must be non-zero.

(2) If $H_1(X, A; \mathbb{Z}) = 0$, then $H_1(A; \mathbb{Z}) \to H_1(X; \mathbb{Z})$ is onto and $H_0(A; \mathbb{Z}) \to H_0(X; \mathbb{Z})$ is injective by Lemma 8.1. This last statement can't be true if some path component X_i of X contains multiple components of A because then $H_0(A \cap X_i) \cong \mathbb{Z}^n$ for some $n \geq 2$ while $H_0(X_i) = \mathbb{Z}$. So then
$$H_0(A \cap X_i) \to H_0(X_i)$$
can't be one-to-one, and the same follows for
$$H_0(A; \mathbb{Z}) \to H_0(X; \mathbb{Z})$$
If $H_1(A; \mathbb{Z}) \to H_1(X; \mathbb{Z})$ is onto, then the kernel of the map $H_1(X; \mathbb{Z}) \to H_1(X, A; \mathbb{Z})$ is $H_1(X; \mathbb{Z})$. So the map $H_1(X; \mathbb{Z}) \to H_1(X, A; \mathbb{Z})$ is the 0 map. Similarly, if each component of X contains at most one component of A, then $H_0(A; \mathbb{Z}) \to H_0(X; \mathbb{Z})$ is injective. So its kernel is 0, so the image of $H_1(X, A; \mathbb{Z}) \to H_0(A; \mathbb{Z})$ is 0. But then by exactness, $0 = H_1(X, A; \mathbb{Z})$.

(3) If $n \geq 2$, then $H_n(X; \mathbb{Z}) = 0$ and $H_{n-1}(x; \mathbb{Z}) = 0$, and thus we immediately see $H_n(X, x) \cong H_n(X; \mathbb{Z})$ by inspecting the long exact sequence in relative homology. For $n = 1$, consider the following part of the long exact sequence in relative homology:
$$0 \to H_1(X; \mathbb{Z}) \to H_1(X, x; \mathbb{Z}) \to H_0(X; \mathbb{Z}) \cong \mathbb{Z} \to H_0(X; \mathbb{Z}) \to H_0(X, x; \mathbb{Z}) \to 0$$
Proposition 8.3(1) readily implies that
$$0 \to H_1(X; \mathbb{Z}) \to H_1(X, x; \mathbb{Z})$$

is surjective if and only if

$$H_0(X;\mathbb{Z}) \cong \mathbb{Z} \to H_0(X;\mathbb{Z})$$

is injective if and only if it is not-the zero map. The last equivalence follows from the observation that $H_0(X;\mathbb{Z})$ is a free abelian group. If it were the zero map, the map $H_0(X;\mathbb{Z}) \to H_0(X,x;\mathbb{Z})$ will be injective. However, this is not the case since the point $x \in X$ defines a generator $\langle x \rangle$ of $H_0(X;\mathbb{Z})$ that is is in the kernel of the map $H_0(X;\mathbb{Z}) \to H_0(X,x)$. Therefore, the claim is true for $n = 1$ as well.

This completes the proof. □

Definition 8.4. Let (X, A) be in $\mathbf{Top^2}$. If $A \subseteq X$ is a closed subspace such that there exists a neighborhood V of X such that A is a strong deformation retract of V, we say that $(X, A;\mathbb{Z})$ is a good pair.

The next proposition allows us to identify an alternative way to think about relative homology in most cases of interest.

Proposition 8.5. *Let (X, A) be a good pair. Then*

$$H_n(X, A;\mathbb{Z}) \cong H_n(X/A, A/A;\mathbb{Z}) \cong \widetilde{H}_n(X/A;\mathbb{Z})$$

for all $n \geq 0$.

Proof. Consider the following diagram:

$$\begin{array}{ccccc} H_n(X,A;\mathbb{Z}) & \longrightarrow & H_n(X,V;\mathbb{Z}) & \longleftarrow & H_n(X-A,V-A;\mathbb{Z}) \\ \downarrow & & \downarrow & & \downarrow \\ H_n(X/A,A/A;\mathbb{Z}) & \longrightarrow & H_n(X/A,V/A;\mathbb{Z}) & \longleftarrow & H_n(X/A-A/A,V/A-A/A;\mathbb{Z}) \end{array}$$

The upper left horizontal map is an isomorphism since in the long exact sequence of the triple (X, V, A) (Remark 7.11) the groups $H_n(V, A)$ are zero for all $n \geq 0$, because a deformation retraction of V onto A gives a homotopy equivalence of pairs $(V, A) \simeq (A, A)$, and $H_n(A, A) = 0$. The deformation retraction of V onto A induces a deformation retraction of V/A onto A/A, so the same argument shows that the lower left horizontal map is an isomorphism as well. The other two horizontal maps are isomorphisms directly from excision. The right-hand vertical map is an isomorphism. It follows that the left-hand vertical arrow also is an isomorphism. □

Corollary 8.6. *If $(X, A;\mathbb{Z})$ is a good pair, then there is an exact sequence:*

$$\cdots \to \widetilde{H}_n(A;\mathbb{Z}) \to \widetilde{H}_n(X;\mathbb{Z}) \to \widetilde{H}_n(X/A;\mathbb{Z}) \to \widetilde{H}_{n-1}(A;\mathbb{Z}) \to \widetilde{H}_{n-1}(X;\mathbb{Z}) \to \cdots$$

Proof. This is clear. □

Corollary 8.7. *Let $(X_\alpha, x_\alpha)_{\alpha\in I}$ be a collection of good pairs in $\mathbf{Top}_*$. Let $X = \bigvee_{\alpha\in I} X_\alpha$ with the basepoint $x = (x_\alpha)_{\alpha\in I}$ in $\mathbf{Top}_*$. Then*

$$\widetilde{H}_n(X;\mathbb{Z}) \cong \bigoplus_{\alpha\in I} \widetilde{H}_n(X_\alpha;\mathbb{Z})$$

for $n \geq 1$.

Proof. Since $(X_\alpha, x_\alpha)_{\alpha\in I}$ be a collection of good pairs, (X, x) is also a good pair. We have:

$$\begin{aligned}
\widetilde{H}_n(X;\mathbb{Z}) &= \widetilde{H}_n\left(\coprod_{\alpha\in I} X_\alpha \Big/ \coprod_{\alpha\in I}\{x_\alpha\};\mathbb{Z}\right)\\
&\cong H_n\left(\coprod_{\alpha\in I} X_\alpha, \coprod_{\alpha\in I}\{x_\alpha\};\mathbb{Z}\right)\\
&\cong \bigoplus_{\alpha\in I} H_n(X_\alpha, x_\alpha;\mathbb{Z})\\
&\cong \bigoplus_{\alpha\in I} \widetilde{H}_n(X_\alpha;\mathbb{Z}).
\end{aligned}$$

The first and third equivalences follow by Proposition 8.3. The second equivalence follows by observing that the additivity axiom holds in $\mathbf{Top}^2$ as can be checked. □

Example 8.8. (**Homology of Spheres**) We now are now in a position to compute the reduced homology groups of spheres. The reduced homology groups of spheres are given as:

$$\widetilde{H}_k(\mathbb{S}^n;\mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & \text{if } k = n\\ 0, & \text{if } k \neq 0 \end{cases}$$

Since $(\mathbb{D}^n, \mathbb{S}^{n-1})$ is a good pair and $\mathbb{D}^n/\mathbb{S}^{n-1} \cong \mathbb{S}^n$, the long exact sequence in relative reduced homology yields:

$$\cdots \to \widetilde{H}_k(\mathbb{S}^{n-1};\mathbb{Z}) \to \widetilde{H}_k(\mathbb{D}^n;\mathbb{Z}) \to \widetilde{H}_k(\mathbb{S}^n;\mathbb{Z}) \to \widetilde{H}_{k-1}(\mathbb{S}^{n-1};\mathbb{Z}) \to \widetilde{H}_{k-1}(\mathbb{D}^n;\mathbb{Z}) \to \cdots$$

Since $\mathbb{D}^n$ is contractible, $\widetilde{H}_k(\mathbb{D}^n;\mathbb{Z}) = 0$ for $k \geq 0$. Therefore,

$$\widetilde{H}_k(\mathbb{S}^{n-1};\mathbb{Z}) \to 0 \to \widetilde{H}_k(\mathbb{S}^n;\mathbb{Z}) \to \widetilde{H}_{k-1}(\mathbb{S}^{n-1};\mathbb{Z}) \to 0 \to \cdots$$

Hence, we have:

$$\widetilde{H}_k(\mathbb{S}^n;\mathbb{Z}) \cong \widetilde{H}_{k-1}(\mathbb{S}^n;\mathbb{Z})$$

The result now follows via induction and the observation that

$$\widetilde{H}_0(\mathbb{S}^0;\mathbb{Z}) \cong \mathbb{Z} \qquad \widetilde{H}_k(\mathbb{S}^0;\mathbb{Z}) \cong 0 \quad k > 0$$

The computation above readily implies the following:

$$H_k(\mathbb{S}^n;\mathbb{Z}) \cong \begin{cases} \mathbb{Z}\oplus\mathbb{Z}, & \text{if } k = 0, n = 0\\ \mathbb{Z} & \text{if } k = 0, n > 1\\ \mathbb{Z} & \text{if } k = n > 0\\ 0 & \text{otherwise} \end{cases}$$

Corollary 8.9. *Let $m \neq n$.*

1. *$\mathbb{S}^m$ and $\mathbb{S}^n$ are not homotopy equivalent.*
2. *$\mathbb{R}^m$ and $\mathbb{R}^n$ are not homeomorphic.*
3. *If $U \subseteq \mathbb{R}^m$ and $V \subseteq \mathbb{R}^n$ are non-empty homeomorphic open sets, then $m = n$.*

Proof. The proof is given below:

1. This follows from Example 8.8 since the homology groups are not isomorphic for $m \neq n$.

(2) If m or n is zero, this is clear. So let $m, n > 0$. Assume we have a homeomorphism $f : \mathbb{R}^m \to \mathbb{R}^n$. WLOG assume that $f(0) = 0$. This restricts to a homeomorphism $\mathbb{R}^m \setminus \{0\} \to \mathbb{R}^n \setminus \{f(0)\}$. But these spaces are homotopy equivalent to spheres of different dimension, yielding a contradiction.

(3) For all $x \in U$ and for all $k \in \mathbb{Z}$, we have

$$H_k(U, U \setminus \{x\}) \cong H_k(\mathbb{R}^m, \mathbb{R}^m \setminus \{x\})$$

by the Excision Theorem. Combining this with the long exact sequence for the reduced homology of $(\mathbb{R}^m, \mathbb{R}^m \setminus \{x\})$ and the fact that $\mathbb{R}^m \setminus \{x\}$ is homotopy equivalent to $\mathbb{S}^{m-1}$, we obtain for all $x \in U$ and all $k \in \mathbb{Z}$:

$$H_k(U, U \setminus \{x\}) \cong H_k(\mathbb{R}^m, \mathbb{R}^m \setminus \{x\}) \cong \widetilde{H}_{k-1}(\mathbb{R}^m \setminus \{x\}) \cong \begin{cases} \mathbb{Z}, & k = m \\ 0, & k \neq m. \end{cases}$$

Similarly,

$$H_k(V, V \setminus \{x\}) \cong H_k(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\}) \cong \widetilde{H}_{k-1}(\mathbb{R}^n \setminus \{x\}) \cong \begin{cases} \mathbb{Z}, & k = n \\ 0, & k \neq n. \end{cases}$$

If U, V are homeomorphic via $f : U \to V$, then

$$H_k(U, U \setminus \{x\}) \cong H_k(V, V \setminus \{f(x)\})$$

The claim follows by comparing homology groups.

This completes the proof. □

Remark 8.10. *If X is a topological space, $x \in X$, and $U \subseteq X$ is an open neighborhood of x, then for all $n \in \mathbb{Z}$, the Excision Theorem yields that*

$$H_n(X, X \setminus \{x\}) \cong H_n(U, U \setminus \{x\}).$$

In particular, for all $n \in \mathbb{Z}$, the group $H_n(X, X \setminus \{x\})$ depends only on the topology of a neighborhood of x. Therefore, these homology groups are called the local homology groups of X at x.

Using the discussion about the homology groups of spheres and Corollary 8.9, we can prove the invariance of dimension (Remark 1.2) and the invariance of boundary (Remark 1.9) results about topological manifolds.

Proof. Let X be a topological n-manifold.

(1) Working in local coordinate charts, this follows from Corollary 8.9(2).
(2) Skipped.

This completes the proof. □

Example 8.11. Let $A \subseteq X$ be a finite set of points in X. We compute $H_n(X, A; \mathbb{Z})$ in the two cases:

(1) Let $X = \mathbb{S}^2$. Assume $|A| = k$ for $k \geq 1$. Since A is assumed to be non-empty, Proposition 8.3 implies $H_0(\mathbb{S}^2, A; \mathbb{Z}) = 0$. The long exact sequence in relative homology implies we have:

$$\cdots \longrightarrow H_{n+1}(A; \mathbb{Z}) \longrightarrow H_{n+1}(\mathbb{S}^2; \mathbb{Z}) \longrightarrow H_{n+1}(\mathbb{S}^2, A; \mathbb{Z}) \longrightarrow H_n(A; \mathbb{Z}) \longrightarrow H_n(\mathbb{S}^2; \mathbb{Z}) \longrightarrow \cdots$$

Noting that,

$$H_1(\mathbb{S}^2;\mathbb{Z}) = 0 \qquad H_0(\mathbb{S}^2;\mathbb{Z}) = \mathbb{Z} \qquad H_0(A;\mathbb{Z};\mathbb{Z}) = \mathbb{Z}^k,$$

the right most end of the long exact sequence becomes

$$\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow H_1(\mathbb{S}^2, A;\mathbb{Z}) \longrightarrow \mathbb{Z}^k \longrightarrow \mathbb{Z} \longrightarrow 0 \longrightarrow 0$$

Since $\mathbb{Z}$ is a free abelian group, the sequence above splits and implies that

$$\mathbb{Z}^k \cong H_1(\mathbb{S}^2, A;\mathbb{Z}) \oplus \mathbb{Z}.$$

Hence,

$$H_1(\mathbb{S}^2, A;\mathbb{Z}) \cong \mathbb{Z}^{k-1}.$$

For $n \geq 2$, $H_n(A;\mathbb{Z}) = 0$ implies that we have the sequence

$$\cdots \longrightarrow 0 \longrightarrow H_{n+1}(\mathbb{S}^2;\mathbb{Z}) \longrightarrow H_{n+1}(\mathbb{S}^2, A;\mathbb{Z}) \longrightarrow 0 \longrightarrow H_n(\mathbb{S}^2;\mathbb{Z}) \longrightarrow H_n(\mathbb{S}^2, A;\mathbb{Z}) \longrightarrow \cdots$$

By Lemma 8.1, $H_n(\mathbb{S}^2) \to H_n(\mathbb{S}^2, A)$ is surjective. But the map is also injective. Hence by exactness and the first isomorphism theorem, Therefore,

$$H_n(\mathbb{S}^2, A;\mathbb{Z}) \cong H_n(\mathbb{S}^2;\mathbb{Z}).$$

Hence, we have:

$$H_n(\mathbb{S}^2, A;\mathbb{Z}) \cong \begin{cases} 0 & \text{if } n \geq 3 \\ \mathbb{Z} & \text{if } n = 2 \\ \mathbb{Z}^{k-1} & \text{if } n = 1 \\ 0 & \text{if } n = 0 \end{cases}$$

(2) Let $X = \mathbb{S}^1 \times \mathbb{S}^1$. It can be checked that $\mathbb{S}^1 \times \mathbb{S}^1$ is homemorphic to the torus, T, considered in Section 4. As in (1), $H_0(\mathbb{S}^1 \times \mathbb{S}^1, A;\mathbb{Z}) = 0$ and $H_n(\mathbb{S}^1 \times \mathbb{S}^1, A;\mathbb{Z}) \cong H_n(\mathbb{S}^1 \times \mathbb{S}^1)$ for $n \geq 2$. For $n = 1$, noting that,

$$H_1(\mathbb{S}^1 \times \mathbb{S}^1;\mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z} \qquad H_0(\mathbb{S}^1 \times \mathbb{S}^1;\mathbb{Z}) = \mathbb{Z} \qquad H_0(A;\mathbb{Z}) = \mathbb{Z}^k$$

the right most end of the long exact sequence in Homology becomes

$$\cdots \longrightarrow 0 \longrightarrow \mathbb{Z} \oplus \mathbb{Z} \longrightarrow H_1(\mathbb{S}^1 \times \mathbb{S}^1, A;\mathbb{Z}) \longrightarrow \mathbb{Z}^k \longrightarrow \mathbb{Z} \longrightarrow 0 \longrightarrow 0$$

The reduced homology version of the sequence above is

$$\cdots \longrightarrow 0 \longrightarrow \mathbb{Z} \oplus \mathbb{Z} \longrightarrow \widetilde{H}_1(\mathbb{S}^1 \times \mathbb{S}^1, A;\mathbb{Z}) \longrightarrow \mathbb{Z}^{k-1} \longrightarrow 0$$

Since $\mathbb{Z}$ is a free $\mathbb{Z}$-module, the sequence above splits and implies that

$$H_1(\mathbb{S}^1 \times \mathbb{S}^1, A;\mathbb{Z}) \cong \widetilde{H}_1(\mathbb{S}^1 \times \mathbb{S}^1, A) \cong \mathbb{Z}^{k+1}.$$

Hence, we have:

$$H_n(\mathbb{S}^1 \times \mathbb{S}^1, A;\mathbb{Z}) \cong \begin{cases} 0 & \text{if } n \geq 3 \\ \mathbb{Z} & \text{if } n = 2 \\ \mathbb{Z}^{k+1} & \text{if } n = 1 \\ 0 & \text{if } n = 0 \end{cases} x$$

Example 8.12. We compute $H_1(\mathbb{R}, \mathbb{Q})$. We have the following exact sequence in homology,

$$\cdots \to H_n(\mathbb{Q}) \to H_n(\mathbb{R}) \to H_n(\mathbb{R}, \mathbb{Q}) \to H_{n-1}(\mathbb{Q}) \to \cdots$$

Since $\mathbb{Q}$ is a totally disconnected set, every point $q \in \mathbb{Q}$ is a path-component. Hence, we have

$$H_n(\mathbb{Q}) = \begin{cases} \bigoplus_{\alpha\in\mathbb{Q}} \mathbb{Z} & \text{if } n = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Since $\mathbb{R}$ is contractible, the long-exact sequence on the right becomes

$$0 \to H_1(\mathbb{R}, \mathbb{Q}) \to \bigoplus_{\alpha\in\mathbb{Q}} \mathbb{Z} \to \mathbb{Z} \to 0.$$

This implies that the map $H_1(\mathbb{R}, \mathbb{Q}) \to \bigoplus_{\alpha\in Q} \mathbb{Z}$ is injective, and since subgroups of free groups are free, $H_1(\mathbb{R}, \mathbb{Q})$ is a free abelian group. Since $\mathbb{Z}$ is a free $\mathbb{Z}$-module, we have

$$\bigoplus_{\alpha\in\mathbb{Q}} \mathbb{Z} \cong H_1(\mathbb{R}, \mathbb{Q}) \oplus \mathbb{Z} \Rightarrow H_1(\mathbb{R}, \mathbb{Q}) \cong \bigoplus_{\alpha\in\mathbb{Q}} \mathbb{Z}$$

If $\sigma_q : \Delta^0 \to \mathbb{Q}$, the set $\{\sigma_0 - \sigma_q \mid q \in Q\}$ is a basis for $H_1(\mathbb{R}, \mathbb{Q})$.

9. Equivalence of Simplicial Homology & Singular Homology

Let X be a topological space that admits a Δ-complex structure. We say that a subspace $A \subseteq X$ admits a Δ-subcomplex structure on X if A is a union of simplicies of X. Relative simplicial homology group can be defined in the same way as relative (singular) homology groups. That is, the n-th relative simplicial homology group, $H_n^\Delta(X, A; \mathbb{Z})$, is the n-th homology group of the chain complex:

$$\cdots \longrightarrow \Delta_2(X)/\Delta_2(A) \xrightarrow{\overline{\partial}_2^\Delta} \Delta_1(X)/\Delta_1(A) \xrightarrow{\overline{\partial}_1^\Delta} \Delta_0(X; \mathbb{Z})/\Delta_0(A)$$

That is:

$$H_n^\Delta(X, A; \mathbb{Z}) = \frac{\operatorname{Ker} \overline{\partial}_n^\Delta}{\operatorname{Im} \overline{\partial}_{n+1}^\Delta}$$

As before, this yields a long exact sequence of simplicial homology groups for the pair $(X, A; \mathbb{Z})$ by the same algebraic argument as for singular homology. We now show that the simplicial homology groups of X corresponding to any Δ-complex structure on X coincides with its singular homology groups of X.

Proposition 9.1. *Let X be a topological space that admits a Δ-complex structure and let A be a Δ-subcomplex of X. The inclusion map*

$$\Delta_n(X, A) \hookrightarrow C_n(X, A)$$

induces an isomorphism

$$H_n^\Delta(X, A; \mathbb{Z}) \cong H_n(X, A; \mathbb{Z})$$

for each $n \geq 0$.

Remark 9.2. *Taking $A = \emptyset$, we obtain the equivalence of absolute singular and simplicial homology.*

Our strategy will be to proceed by induction X_k^Δ consisting of all simplices of dimension k or less.

Proof. We proceed in multiple steps:

(1) First suppose that X is finite dimensional. That is, $X_m^\Delta = \emptyset$ for $m \geq n$ for some $n \in \mathbb{N}$. Assume that $A = \emptyset$. Consider the following diagram:

$$\begin{array}{ccccccccc} H_{n+1}^\Delta(X_k^\Delta, X_{k-1}^\Delta\ Z) & \longrightarrow & H_n^\Delta(X_{k-1}^\Delta\ Z) & \longrightarrow & H_n^\Delta(X_k^\Delta\ Z) & \longrightarrow & H_n^\Delta(X_k^\Delta, X_{k-1}^\Delta\ Z) & \longrightarrow & H_{n-1}^\Delta(X_{k-1}^\Delta\ Z) \\ \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle f_4} & & \downarrow{\scriptstyle f_5} \\ H_{n+1}(X_k^\Delta, X_{k-1}^\Delta\ Z) & \longrightarrow & H_n(X_{k-1}^\Delta\ Z) & \longrightarrow & H_n(X_k^\Delta\ Z) & \longrightarrow & H_n(X_k^\Delta, X_{k-1}^\Delta\ Z) & \longrightarrow & H_{n-1}(X_{k-1}^\Delta\ Z) \end{array}$$

Note that $\Delta_k(X_k^\Delta, X_{k-1}^\Delta)$ is a free abelian group generated by k-simplices and $\Delta_n(X_k^\Delta, X_{k-1}^\Delta) = \emptyset$ for $n \neq k$. Therefore, we have:

$$\Delta_k(X_k^\Delta, X_{k-1}^\Delta) = \begin{cases} \text{free abelian group generated by } k\text{-simplices} & \text{if } n = k \\ \emptyset & \text{if } n \neq k \end{cases}$$

A simple calculation shows that:

$$H_n^\Delta(X_k^\Delta, X_{k-1}^\Delta\ Z) = \begin{cases} \mathbb{Z}^{\#k-\text{simplicies}} & \text{if } n = k \\ 0 & \text{if } n \neq k \end{cases}$$

It is easy to check that $(X_k^\Delta, X_{k-1}^\Delta)$ is a good pair and

$$X_k^\Delta / X_{k-1}^\Delta = \bigvee_{i=1}^{\#k-\text{simplicies}} \mathbb{S}^k$$

Therefore, Corollary 8.7 implies

$$H_n(X_k^\Delta, X_{k-1}^\Delta\ Z) = \begin{cases} \mathbb{Z}^{\#k-\text{simplicies}} & \text{if } n = k \\ 0 & \text{if } n \neq k \end{cases}$$

Therefore, both f_1 and f_4 are isomorphisms. An induction argument shows that f_2 and f_5 are isomorphisms. The five lemma (Proposition 5.18) then implies that f_3 is an isomorphism.

(2) Suppose that X is possibly infinite-dimensional. Assume that $A = \emptyset$. Note that a compact set $C \subseteq X$ can meet only finitely many open simplices of X. If not, C would contain an infinite sequence of points x_i, each lying in a different open simplex. Then the sets

$$U_i = X - \bigcup_{i \neq j} \{x_j\}$$

which are open since their pre-images under the characteristic maps of all the simplices are clearly open, form an open cover of C with no finite sub-cover. This can be applied to show the map

$$H_n^\Delta(X; \mathbb{Z}) \to H_n(X; \mathbb{Z})$$

is bijective. For surjectivity, let $[c] \in H_n^\Delta(X; \mathbb{Z})$. Choose a representative n-cycle, α, of $[c]$. Now α is a linear combination of finitely many singular simplices with compact images, meeting only finitely many open simplices of X. Hence, α in X_k^Δ for some k. We have shown that

$$H_n(X_k^\Delta\ Z) \cong H_n^\Delta(X_k^\Delta\ Z)$$

So there exists a n-cycle $v \in \Delta_n(X_k^\Delta)$ such that $[v]$ gets mapped to $[c]$. This proves surjectivity. Injectivity is similar so we omit details.

(3) Now consider the general case where $A \neq \emptyset$. Consider the following diagram:

$$\begin{array}{ccccccccc} H_{n+1}^\Delta(X,A;\mathbb{Z}) & \longrightarrow & H_n^\Delta(A;\mathbb{Z}) & \longrightarrow & H_n^\Delta(X;\mathbb{Z}) & \longrightarrow & H_n^\Delta(X,A;\mathbb{Z}) & \longrightarrow & H_{n-1}^\Delta(A;\mathbb{Z}) \\ \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle f_4} & & \downarrow{\scriptstyle f_5} \\ H_{n+1}(X,A;\mathbb{Z}) & \longrightarrow & H_n(A;\mathbb{Z}) & \longrightarrow & H_n(X;\mathbb{Z}) & \longrightarrow & H_n(X,A;\mathbb{Z}) & \longrightarrow & H_{n-1}(A;\mathbb{Z}) \end{array}$$

By (2), f_2, f_3, f_5 are isomorphisms. The claim now follows by induction and the five-lemma.

This completes the proof. $\square$

Example 9.3. Let $X = \mathbb{S}^1 \times \mathbb{S}^1$. Note that X is homemorphic to the torus, T, considered in Section 4. Hence, Proposition 9.1 implies that

$$H_n(X;\mathbb{Z}) = H_n^\Delta(T;\mathbb{Z}) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & \text{for } n = 1 \\ \mathbb{Z} & \text{for } n = 0, 2 \\ 0 & \text{for } n \geq 3 \end{cases}$$

Consider $Y = \mathbb{S}^1 \vee \mathbb{S}^1 \vee \mathbb{S}^2$. We have

$$H_n(Y) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & \text{for } n = 1 \\ \mathbb{Z} & \text{for } n = 0, 2 \\ 0 & \text{for } n \geq 3 \end{cases}$$

We see that

$$H_n(X;\mathbb{Z}) = H_n(Y)$$

for $n \geq 0$. It can be checked that the covering spaces of X and Y have different homology groups. Hence, X and Y are not homotopy equivalent. Therefore, homology groups might not be able to distinguish topological spaces that are not homotopy equivalent.

Part 3. Computations & Applications

10. Mayer-Vietoris Sequence

In addition to the long exact sequence of homology groups for a pair (X, A), there is another sort of long exact sequence, known as a Mayer–Vietoris sequence, which is equally powerful but is sometimes more convenient to use. The Mayer–Vietoris sequence is also applied frequently in induction arguments, where one might know that a certain statement is true for A, B, and $A \cap B$ by induction and then deduce that it is true for $A \cup B$ by the exact sequence[12].

[12] Mayer–Vietoris sequence can also be thought of as an abelianization of the Seifert Van Kampen Theorem.

Lemma 10.1. *(**Barrett-Whitehead Lemma**) Suppose we have the following commutative diagram of abelian groups, where the two rows are exact:*

$$\begin{tikzcd}
\cdots \arrow[r] & A_n \arrow[r, "i_n"] \arrow[d, "f_n"] & B_n \arrow[r, "j_n"] \arrow[d, "g_n"] & C_n \arrow[r, "k_n"] \arrow[d, "h_n"] & A_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \cdots \\
\cdots \arrow[r] & A'_n \arrow[r, "i'_n"] & B'_n \arrow[r, "j'_n"] & C'_n \arrow[r, "k'_n"] & A'_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

Assume each map $h_n : C_n \to C_{n-1}$ is an isomorphism. Then there is a long exact sequence

$$\cdots \longrightarrow A_n \xrightarrow{(i_n, f_n)} B_n \oplus A'_n \xrightarrow{g_n - i'_n} B'_n \xrightarrow{k_n h_n^{-1} j'_n} A_{n-1} \longrightarrow \cdots$$

Proof. The proof is by a diagram chase. We omit details. $\square$

Proposition 10.2. *(**Mayer-Viertoris Sequence**) Let $X_1, X_2 \subseteq X$ be open sets such that $X = X_1 \cup X_2$. Let*

$$i_1 : X_0 \hookrightarrow X_1, \qquad i_2 : X_1 \hookrightarrow X$$

denote inclusions for $i = 1, 2$. Then there is a long exact sequence

$$\cdots \to H_n(X_1 \cap X_2; \mathbb{Z}) \to H_n(X_1; \mathbb{Z}) \oplus H_n(X_2; \mathbb{Z}) \to H_n(X; \mathbb{Z}) \to H_{n-1}(X_1 \cap X_2; \mathbb{Z}) \to \cdots$$

Proof. We have the following diagram:

$$\begin{tikzcd}
(X_1 \cap X_2, \emptyset) \arrow[r, "i_1"] \arrow[d, "i_2"] & (X_1, \emptyset) \arrow[r, "f"] \arrow[d, "j_1"] & (X_1, X_1 \cap X_2) \arrow[d, "h"] \\
(X_2, \emptyset) \arrow[r, "j_2"] & (X, \emptyset) \arrow[r, "g"] & (X, X_2)
\end{tikzcd}$$

Applying Remark 7.11 yields the following diagram:

$$\begin{tikzcd}
\cdots \arrow[r] & H_n(X_1 \cap X_2; \mathbb{Z}) \arrow[r, "H_n(i_n)"] \arrow[d, "H_n(f_n)"'] & H_n(X_1; \mathbb{Z}) \arrow[r, "H_n(j_n)"] \arrow[d, "H_n(g_n)"'] & H_n(X_1, X_1 \cap X_2; \mathbb{Z}) \arrow[r, "\delta_n"] \arrow[d, "H_n(h_n)"'] & H_n(X_1 \cap X_2; \mathbb{Z}) \arrow[r] \arrow[d, "H_n(i_2)"] & \cdots \\
\cdots \arrow[r] & H_n(X_2; \mathbb{Z}) \arrow[r, "H_n(i'_n)"] & H_n(X; \mathbb{Z}) \arrow[r, "H_n(j'_n)"] & H_n(X, X_2; \mathbb{Z}) \arrow[r, "\delta'_n"] & H_n(X_2; \mathbb{Z}) \arrow[r] & \cdots
\end{tikzcd}$$

The excision axioms implies that $H_n(h_n)$ is an isomorphism for each $n \geq 0$. Lemma 10.1 then implies the existence of the desired long exact sequence. $\square$

Remark 10.3. *By using augmented chain complexes, we also obtain a corresponding Mayer-Vietoris sequence for the reduced homology groups. We omit details.*

Remark 10.4. *We can also use the Mayer-Viertoris sequence to compute the homology groups of sphere. Indeed, consider the following argument. Let $X = \mathbb{S}^n$, $A = \mathbb{S}^n \setminus \{S\}$, and $B = \mathbb{S}^n \setminus \{N\}$, where S and N are the south pole and north pole, respectively. Then*

$$A \simeq \mathbb{R}^n \quad B \simeq \mathbb{R}^n \quad A \cap B \simeq \mathbb{S}^{n-1}$$

From the Mayer-Vietoris sequence for reduced homology groups, we get $\widetilde{H}_k(\mathbb{S}^n) \simeq \widetilde{H}_{k-1}(\mathbb{S}^{n-1})$ for all i. By induction, we find as before:

$$\widetilde{H}_k(\mathbb{S}^n; \mathbb{Z}) \simeq \begin{cases} \mathbb{Z}, & k = n \\ 0, & k \neq n. \end{cases}$$

Proposition 10.5. ***(Suspension Theorem)***[13] *Let X be a topological space and let SX be its suspension. We have*

$$\widetilde{H}_n(X;\mathbb{Z}) \cong \widetilde{H}_{n+1}(SX;\mathbb{Z})$$

for $n \geq -1$.

Proof. For $n = -1$, $\widetilde{H}_{-1}(X;\mathbb{Z})$ is the trivial group. Since SX is path-connected, $\widetilde{H}_0(SX;\mathbb{Z})$ is also the trivial group. Let $n \geq 0$. Let P, Q denote the collapsed spaces $X \times \{0\}$ and $X \times \{1\}$ respectively. Let $A = SX - \{P\}$ and let $B = SX - \{Q\}$. Each of A and B are homeomorphic to the cone space

$$CX = (X \times I)/(X \times \{0\})$$

By the Mayer-Vietoris sequence for reduced homology, since $A \cap B = X \times (0,1)$, we obtain the exact sequence

$$\cdots \to \widetilde{H}_{n+1}(A;\mathbb{Z}) \oplus \widetilde{H}_{n+1}(B;\mathbb{Z}) \to \widetilde{H}_{n+1}(SX;\mathbb{Z}) \to \widetilde{H}_n(A \cap B;\mathbb{Z}) \to \widetilde{H}_n(A;\mathbb{Z}) \oplus \widetilde{H}_n(B;\mathbb{Z}) \to \cdots$$

for all n. Note that CX is contractible[14]. Moreover, $X \times (0,1)$ deformation retracts down to X. Hence, the sequence simplifies to:

$$\cdots \to 0 \to \widetilde{H}_{n+1}(SX;\mathbb{Z}) \to \widetilde{H}_n(X;\mathbb{Z}) \to 0 \to \cdots$$

This proves the claim. □

11. Degree Theory

We focus on the application of homology theory to degree theory (and fixed point theory) in this section. We shall see that we can use homology theory to study continuous maps between spheres. This allows us to introduce and study the notion of degree of a map between spheres. Moreover, degree of maps between spheres will play a fundamental role in our being able to compute cellular homology, which we discuss in the next section.

Definition 11.1. The degree of a continuous map $f : \mathbb{S}^n \to \mathbb{S}^n$ is defined as:

$$\deg f := H_n(f)(1)$$

where

$$H_n(f) : H_n(\mathbb{S}^n;\mathbb{Z}) \cong \mathbb{Z} \to H_n(\mathbb{S}^n;\mathbb{Z}) \cong \mathbb{Z}$$

is the homomorphism induced by f in homology, and $1 \in \mathbb{Z}$ denotes the generator.

In what follows, we write $H_n(f)$ as f_n.

Proposition 11.2. *Here are some basic properties of degree:*

(1) $\deg(\mathrm{Id}_{\mathbb{S}^n}) = 1$.
(2) *If f is not surjective, then* $\deg(f) = 0$.
(3) *If $f \simeq g$ are homotopic maps, then* $\deg(f) = \deg(g)$.
(4) $\deg(g \circ f) = \deg(g) \cdot \deg(f)$.
(5) *If f is a homotopy equivalence, then* $\deg(f) = \pm 1$.

Proof. The proof is given below:

(1) This follows because $(\mathrm{Id}_{\mathbb{S}^n})_n = \mathrm{Id}_{\mathbb{Z}}$ which is multiplication by the integer 1.

[13] Suspension is defined formally later on.

[14] Indeed, the homotopy $h_t(x,s) = (x,(1-t)s)$ continuously shrinks CX down to its vertex point.

(2) Indeed, if f is not surjective, there is some $y \notin \mathrm{Im}(f)$. Then we can factor f in the following way:
$$\mathbb{S}^n \to \mathbb{S}^n \setminus \{y\} \to \mathbb{S}^n$$
Since $\mathbb{S}^n \setminus \{y\} \simeq \mathbb{R}^n$ is contractible, $H_n(\mathbb{S}^n \setminus \{y\}) = 0$. Therefore, $f_n = 0$, so $\deg(f) = 0$.
(3) This follows because because if f and g are homotopic, then $f_n = g_n$ for $n \geq 0$.
(4) This is clear.
(5) By definition, there exists a map $g : \mathbb{S}^n \to \mathbb{S}^n$ so that $g \circ f \simeq \mathrm{Id}_{\mathbb{S}^n}$ and $f \circ g \simeq \mathrm{Id}_{\mathbb{S}^n}$. The claim follows directly from previous results, since $f \circ g \simeq \mathrm{id}_{\mathbb{S}^n}$ implies that $\deg(f) \cdot \deg(g) = \deg(\mathrm{Id}_{\mathbb{S}^n}) = 1$.

This completes the proof. □

We now prove a far less obvious result:

Proposition 11.3. *Let $n \geq 1$ and let $A \in O(n+1)$ denote an orthogonal linear transformation. Set $f := A|_{\mathbb{S}^n}$. Then $\deg(f) = \det(A) = \pm 1$.*

Proof. The group $O(n+1)$ has two connected components, $O^+(n+1)$ and $O^-(n+1)$, distinguished by det: $O(n+1) \to \{+1, -1\}$. By homotopy invariance, it suffices to check the result for one such A in each component. Note that $I_{n+1} \in O^+(n+1)$ has degree 1. Hence, every $A \in O^+(n+1)$ has degree one. Consider $O^-(n+1)$. We take A to be reflection in a hyperplane $H \subseteq \mathbb{R}^{n+1}$. WLOG, we can assume the hyperplane is $\mathbb{R}^n \times \{0\} \subset \mathbb{R}^{n+1}$ and
$$A(x_0, x_1, \ldots, x_n) = (x_0, x_1, \ldots, -x_n).$$

The upper and lower hemispheres U and L of $\mathbb{S}^n$ can be regarded as singular n-simplices, via their standard homeomorphisms with Δ^n [15]. Then the generator of $H_n(\mathbb{S}^n)$ is $[U - L]$. The reflection map r maps the cycle $U - L$ to $L - U = -(U - L)$. So
$$H_n(r)([U - L]) = [L - U] = [-(U - L)] = -1 \cdot [U - L],$$
so $\deg(r) = -1$. □

Corollary 11.4. *If $a : \mathbb{S}^n \to \mathbb{S}^n$ is the antipodal map, then $\deg(a) = (-1)^{n+1}$.*

Proof. Note that a is a composition of $n+1$ reflections since there are $n+1$ coordinates in x, each changing sign by an individual reflection. From above, we know that the composition of maps leads to multiplication of degrees. □

This immediately gives a proof of the following famous result.

Proposition 11.5. ***(Hairy Ball Theorem)*** *$\mathbb{S}^n$ has a continuous vector field of nonzero tangent vectors if and only if n is odd.*

Proof. Suppose $x \mapsto v(x)$ is a tangent vector field on $\mathbb{S}^n$, assigning to a vector $x \in \mathbb{S}^n$ the vector $v(x)$ tangent to $\mathbb{S}^n$ at x. Regarding $v(x)$ as a vector at the origin instead of at x, we have that $x \perp v(x)$ for each $x \in \mathbb{S}^n$. If $v(x) \neq 0$ for all x, we may normalize so that $|v(x)| = 1$ for all x by replacing $v(x)$ by $v(x)/|v(x)|$. Consider
$$W : \mathbb{S}^n \times I \to \mathbb{S}^n,$$
$$(x, t) \mapsto (\cos(\pi t))x + (\sin(\pi t))v(x).$$

[15] Here we are implicitly using the result that if $C \subseteq \mathbb{R}^n$ is a compact convex subset with nonempty interior, then C is a closed n-cell (and its interior is an open n-cell). That is, $C \cong \mathbb{D}^n$.

Note that W indeed takes values in $\mathbb{S}^n$ since $x \perp v(x)$ for each $x \in \mathbb{S}^n$. Hence, we obtain a homotopy from the identity map on $\mathbb{S}^n$ to the antipodal map on $\mathbb{S}^n$. This implies that $(-1)^{n+1} = 1$ and n must be odd. Conversely, if n is odd, say $n = 2k - 1$, we can define $v : \mathbb{R}^{2k} \to \mathbb{R}^{2k}$ by

$$v(x_1, x_2, \ldots, x_{2k-1}, x_{2k}) = (-x_2, x_1, \ldots, -x_{2k}, x_{2k-1})$$

Then $v(x) \perp x$ for each $x \in \mathbb{S}^n$. Hence, v is a vector field on $\mathbb{S}^n$, and it is non-vanishing since $|v(x)| = 1$ for all $x \in \mathbb{S}^n$. □

11.1. **Applications to Fixed Point Theory.** Let's note some results in fixed point theory that can be easily proved using degree theory.

(1) If $f : \mathbb{S}^n \to \mathbb{S}^n$ has no fixed points, then $\deg(f) = (-1)^{n+1}$. Since $f(x) \neq x$, the segment

$$(1-t)f(x) + t(-x)$$

from $-x$ to $f(x)$ does not pass through the origin in $\mathbb{R}^{n+1}$. So we can normalize to obtain a homotopy:

$$\begin{aligned} g_t(x) : \mathbb{S}^n &\to \mathbb{S}^n \\ x &\mapsto \frac{(1-t)f(x) + t(-x)}{\|(1-t)f(x) + t(-x)\|} \end{aligned}$$

Note that this homotopy is well defined since $(1-t)f(x) - tx \neq 0$ for any $x \in \mathbb{S}^n$ and $t \in [0,1]$, because $f(x) \neq x$ for all x. Then g_t is a homotopy from f to a, the antipodal map, and the result follows.

(2) (**Brouwer's Fixed Point Theorem**) Let $n \geq 1$ and suppose that a continuous map $f : \mathbb{D}^n \to \mathbb{D}^n$ has no fixed points. Since $\mathbb{D}^n \cong \mathbb{S}^n_+$, where $\mathbb{S}^n_+$ is the northern hemisphere, f can be thought of a map from $\mathbb{S}^n_+$ to $\mathbb{S}^n_+$. Now we can extend f to a map on $\mathbb{S}^n$ as follows. We define

$$g(x) = \begin{cases} f(x), & \text{if } x \in \mathbb{S}^n_+, \\ f \circ r(x), & \text{if } x \in \mathbb{S}^n_- \end{cases}$$

where $r(x)$ is reflection about the plane through the equator and $\mathbb{S}^n_+$ is the southern hemisphere. It is clear that $g(x)$ is a continuous function; furthermore $g(x)$ has no fixed points. By (1), g is homotopic to the antipodal map on $\mathbb{S}^n$ that has degree $(-1)^{n+1}$. Clearly, g is not surjective[16]. It follows that $\deg g = 0$. But $(-1)^{n+1} \neq 0$ for $n \geq 1$. Hence, every continuous map $f : \mathbb{D}^n \to \mathbb{D}^n$ has a fixed point.

(3) Consider a continuous map $f : \mathbb{S}^{2n} \to \mathbb{S}^{2n}$. Then either f or $-f$ must have a fixed point. If f and $-f$ don't have fixed points, then f and $-f$ are homotopic to the antipodal map. Thus f and $-f$ have degree -1. But both f and $-f$ both cannot have the same degree. Hence there is a point $x \in \mathbb{S}^{2n}$ such that $f(x) = \pm x$.

(4) Consider a continuous map $f : \mathbb{S}^n \to \mathbb{S}^n$ of degree zero. Then there exist points $x, y \in \mathbb{S}^n$ for such that $f(x) = x$ and $f(y) = -y$. If not, then f and $-f$ are homotopic to the antipodal map. But then degree of $\pm f$ is $0 \neq \pm(-1)^{n+1}$. Hence, there are points $x, y \in \mathbb{S}^n$ for such that $f(x) = x$ and $f(y) = -y$.

[16]Because no point in the southern hemisphere is in the image.

Remark 11.6. *Brouwer's fixed point thereom can be derived without resorting to degree theory. First, note that for* $n \geq 1$ *there does not exist a continuous map* $r : \mathbb{D}^n \to \partial\mathbb{D}^n$ *such that* $r(x) = x$ *for* $x \in \partial\mathbb{D}^n$. *Assume by contradiction that there exists a retraction* $r : \mathbb{D}^n \to \partial\mathbb{D}^n = \mathbb{S}^{n-1}$. *Then, if* $i : \mathbb{S}^{n-1} \to \mathbb{D}^n$ *is the inclusion, we have* $r \circ i = Id_{\mathbb{S}^{n-1}}$.

$$\begin{tikzcd} \mathbb{S}^{n-1} \arrow[rr, "Id"] \arrow[dr, "i"'] & & \mathbb{S}^{n-1} \\ & \mathbb{D}^n \arrow[ur, "r"'] & \end{tikzcd}$$

If $n > 1$, *we then have:*

$$\begin{tikzcd} \mathbb{Z} \cong H_{n-1}(\mathbb{S}^{n-1};\mathbb{Z}) \arrow[rr, "Id"] \arrow[dr, "0"] & & \mathbb{H}_{n-1}(\mathbb{S}^{n-1};\mathbb{Z}) \cong \mathbb{Z} \\ & H_{n-1}(\mathbb{D}^n;\mathbb{Z}) = 0 \arrow[ur, "0"] & \end{tikzcd}$$

This is a contradiction. A similar argument can be made in the case $n = 0$. *We can now prove Brouwer's fixed point thereom: let* $f : \mathbb{D}^n \to \mathbb{D}^n$ *be a continuous map. Assume by contradiction that* $f(x) \neq x$ *for all* $x \in \mathbb{D}^n$. *Then, we may define a function* $r : \mathbb{D}^n \to \mathbb{S}^{n-1}$ *in the following way. Let* $x \in \mathbb{D}^n$ *and let* $[f(x), x)$ *denote the (unique) ray based at* $f(x)$ *passing through* x. *Define* $r(x)$ *to be the unique element in* $([f(x), x)) \cap \partial\mathbb{D}^n \setminus \{f(x)\}$. *Then,* r *is continuous and is a retraction* $\mathbb{D}^n \to \partial\mathbb{D}^n$, *a contradiction.*

Remark 11.7. *Here is a cute application of (3) above. Consider* $g : \mathbb{RP}^{2n} \to \mathbb{RP}^{2n}$. *By covering space theory, the map* g *lifts to a map* $\hat{g} : \mathbb{RP}^{2n} \to \mathbb{S}^{2n}$. *Define* f *as in the diagram below:*

$$\begin{tikzcd} \mathbb{S}^{2n} \arrow[r, dashed, "f"] \arrow[d] & \mathbb{S}^{2n} \arrow[d] \\ \mathbb{RP}^{2n} \arrow[r, "g"'] \arrow[ur, dashed, "\tilde{g}"] & \mathbb{RP}^{2n} \end{tikzcd}$$

Choose $x \in \mathbb{S}^{2n}$ *such that* $f(x) = \pm x$. *Then the point* $[x] \in \mathbb{RP}^{2n}$ *is a fixed point for* g[17]. *We have just seen that any linear transformation of* $\mathbb{R}^{2n+1}$ *has a real eigenvalue! The analogous result is not necessarily true for* $\mathbb{R}^{2n}$ *for* $n \geq 1$. *Consider the following matrix:*

$$R = \begin{pmatrix} R_{\theta_1} & 0 & \dots & 0 \\ 0 & R_{\theta_1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & R_{\theta_n} \end{pmatrix}$$

Here $\mathbb{R}_{\theta_i}$ *is a 2-by-2 matrix that represents a rotation through angle* θ_i. *Clearly,* R *maps* $\mathbb{S}^{2n-1}$ *to* $\mathbb{S}^{2n-1}$, *but it has no fixed point since each* R_{θ_i} *has no (real) eigenvector.*

[17] For $m \geq$, recall that $\mathbb{RP}^m = \mathbb{S}^{m+1}/\sim$ where we identify antipodal points in $\mathbb{S}^m$. Covering space theory tells us that $\mathbb{S}^m$ is the universal covering space of $\mathbb{RP}^m$. Hence, the map g lifts since $\mathbb{S}^{m+1}$ is simply-connected.

Remark 11.8. *Here is a cute application of (4) above. Let $F : \mathbb{D}^n \to \mathbb{D}^n$ be a continuous function such that $F(x) \neq 0$ for $x \in \mathbb{D}^n$. We can then consider the map*

$$G(x) = \frac{F(x)}{\|F(x)\|}$$

Note that G maps $\mathbb{S}^{n-1}$ to $\mathbb{S}^{n-1}$. Moreover, G has degree zero since $\mathbb{D}^n$ has trivial homology. Hence, there are points $x, y \in \mathbb{S}^{n-1}$ such that $G(x) = x$ and $G(y) = -y$. If we think of F defining a vector field on $\mathbb{D}^n$, then this meas that there exists a point on$\mathbb{S}^{n-1}$ where the vector field points radially outward and another point on $\mathbb{S}^{n-1}$ where where the vector field points radially inward.

Remark 11.9. *Consider a continuous map $f : \mathbb{S}^n \to \mathbb{S}^n$ for $n \geq 1$. If f doesn't have a fixed point, then f is homotopic to the antipodal map. If n is odd, then the linear transformation represented by the matrix*

$$R_t = \begin{pmatrix} R_{t\pi} & 0 & \dots & 0 \\ 0 & R_{t\pi} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & R_{t\pi} \end{pmatrix}$$

is homotopy between the antipodal map and the identity map. If n is even, then we can apply the argument above to the "equatorial sphere" $\mathbb{S}^{n-1} \subset \mathbb{S}^n$. In any case, we see that if f is homotopic to continuous map with a fixed point.

11.2. **Local Degrees.** How to compute degrees, though? We describe a technique for computing degrees which can be applied to most maps that arise in practice. Assume $f : \mathbb{S}^n \to \mathbb{S}^n$ is surjective, and that f has the property that there exists some $y \in \mathrm{Im}(\mathbb{S}^n)$ so that $f^{-1}(y)$ is a finite number of points, say $f^{-1}(y) = \{x_1, x_2, \dots, x_m\}$. Let U_i be a neighborhood of x_i such that all U_i's get mapped to some neighborhood V of y. So

$$f(U_i \setminus \{x_i\}) \subseteq V \setminus \{y\}$$

As f is continuous, we can choose the U_i's to be disjoint. Let $f|_{x_i} : U_i \to V$ be the restriction of f to U_i, with the induced homomorphism

$$f_n|_{x_i} : H_n(U_i, U_i \setminus \{x_i\}; \mathbb{Z}) \to H_n(V, V \setminus \{y\}; \mathbb{Z}).$$

Definition 11.10. Let $f : \mathbb{S}^n \to \mathbb{S}^n$ be a continuous map. The local degree of f at x_i, $\deg(f|_{x_i})$, to be the degree of $f_n|_{x_i}$.

We have the following result:

Lemma 11.11. *The degree of f equals the sum of local degrees at points in a generic finite fiber, that is,*

$$\deg(f) = \sum_{i=1}^{m} \deg(f|_{x_i}).$$

Proof. Consider the commutative diagram:

$$\begin{tikzcd}
 & \mathbb{Z} \cong H_n(U_i, U_i \setminus x_i; \mathbb{Z}) \arrow[r, "f_n|_{x_i}"] \arrow[d, "k_i"] \arrow[dl, "\cong"'] & \mathbb{Z} \cong H_n(V, V \setminus y; \mathbb{Z}) \arrow[d, "\cong"] \\
\mathbb{Z} \cong H_n(\mathbb{S}^n, \mathbb{S}^n \setminus x_i; \mathbb{Z}) & H_n(\mathbb{S}^n, \mathbb{S}^n - f^{-1}(y); \mathbb{Z}) \arrow[l, "p_i"'] \arrow[r, "f_n"] & H_n(\mathbb{S}^n, \mathbb{S}^n \setminus y; \mathbb{Z}) \\
 & \mathbb{Z} \cong \widetilde{H}_n(\mathbb{S}^n; \mathbb{Z}) \arrow[r, "f_n"] \arrow[u, "j"'] \arrow[ul, "\cong"] & \mathbb{Z} \cong \widetilde{H}_n(\mathbb{S}^n; \mathbb{Z}) \arrow[u, "\cong"']
\end{tikzcd}$$

The two isomorphisms in the upper half of the diagram come from excision, while the lower two isomorphisms come from exact sequences of pairs. The maps k_i and p_i are induced by inclusions, so the triangles and squares commute. The map j comes from the long exact sequence in homology. By excision,

$$H_n(\mathbb{S}^n, \mathbb{S}^n - f^{-1}(y); \mathbb{Z}) \cong \bigoplus_{i=1}^{m} H_n(U_i, U_i \setminus x_i; \mathbb{Z}) \cong \mathbb{Z}^m$$

The map p_i is the projection onto the i-th summand. We have:

$$k_i(1) = (0, \dots, 0, 1, 0, \dots, 0)$$

where the entry 1 is in the ith place. Also, $p_i \circ j(1) = 1$ for all i, so

$$j(1) = (1, 1, \dots, 1) = \sum_{i=1}^{m} k_i(1).$$

Commutativity of the upper square says that the middle f_n takes $k_i(1)$ to $\deg f\,|_{x_i}$, hence the sum $\sum_i k_i(1) = j(1)$ is taken to $\sum_i \deg f|_{x_i}$. The commutativity of the lower square gives:

$$\deg f = f_n j(1) = f_n\left(\sum_{i=1}^{m} k_i(1)\right) = \sum_{i=1}^{m} \deg f|_{x_i}.$$

This completes the proof. □

12. Cellular Homology

We define the cellular homology of a CW complex X in terms of a given cell structure, then we show that it coincides with the singular homology, so it is in fact independent on the cell structure. Cellular homology is very useful for computations. Before discussing cellular homology, we compute the relative homology groups of a topological space, X, that can be given the structure of a CW complex.

Lemma 12.1. *Let X be a topological space that can be endowed with the structure of a CW complex. Then:*

(1) *The relative romology $H_k(X^n, X^{n-1}; \mathbb{Z})$ is given by:*

$$H_k(X^n, X^{n-1}; \mathbb{Z}) = \begin{cases} 0, & \text{if } k \neq n \\ \mathbb{Z}^{\#n\text{-cells}}, & \text{if } k = n. \end{cases}$$

for $k \geq 1$.

(2) *$H_k(X^n; \mathbb{Z}) = 0$ if $k > n \geq 1$. In particular, if X is finite dimensional, then $H_k(X; \mathbb{Z}) = 0$ if $k > \dim(X)$.*

(3) *The inclusion* $i : X^n \hookrightarrow X$ *induces an isomorphism* $H_k(X^n;\mathbb{Z}) \cong H_k(X)$ *if* $k < n$.

Proof. The proof is given below:

(1) Since (X^n, X^{n-1}) is a good pair, we have:
$$\begin{aligned} H_k(X^n, X^{n-1};\mathbb{Z}) &\cong \widetilde{H}_k(X^n/X^{n-1};\mathbb{Z}) \\ &= H_k(X^n/X^{n-1};\mathbb{Z}) \\ &\cong \bigvee_{i=1}^{\#n\text{-cells}} \mathbb{S}^n \cong \begin{cases} 0, & \text{if } k \neq n, \\ \mathbb{Z}^{\#n\text{-cells}}, & \text{if } k = n. \end{cases} \end{aligned}$$

(2) Since (X^n, X^{n-1}) is a good pair for each $n \geq 1$, we can consider the following portion of the long exact sequence:
$$H_{k+1}(X^n, X^{n-1};\mathbb{Z}) \longrightarrow H_k(X^{n-1};\mathbb{Z}) \longrightarrow H_k(X^n;\mathbb{Z}) \longrightarrow H_k(X^n, X^{n-1};\mathbb{Z})$$
If $k+1 \neq n$ and $k \neq n$, we have from (1) we have that $H_{k+1}(X^n, X^{n-1};\mathbb{Z}) = 0$ and $H_k(X^n, X^{n-1}) = 0$. Thus
$$H_k(X^{n-1};\mathbb{Z}) \cong H_k(X^n;\mathbb{Z})$$
Hence, if $k > n$ (so in particular, $n \neq k+1$ and $n \neq k$), we get by iteration that
$$H_k(X^n;\mathbb{Z}) \xrightarrow{\cong} H_k(X^{n-1};\mathbb{Z}) \xrightarrow{\cong} \cdots \xrightarrow{\cong} H_k(X^0;\mathbb{Z})$$
Note that X^0 is just a collection of points, so $H_k(X^0;\mathbb{Z}) = 0$. Thus, when $k > n \geq 1$, we have $H_k(X^n;\mathbb{Z}) = 0$ as desired.

(3) We only prove the statement for finite-dimensional CW complexes. Let $k < n$, and consider the following portion of the long exact sequence:
$$\cdots \to H_{k+1}(X^{n+1}, X^n;\mathbb{Z}) \to H_k(X^n;\mathbb{Z}) \to H_k(X^{n+1};\mathbb{Z}) \to H_k(X^{n+1}, X^n;\mathbb{Z}) \to \cdots$$
Since $k < n$, we have $k+1 \neq n+1$ and $k \neq n+1$, so by part (1), we get that $H_{k+1}(X^{n+1}, X^n;\mathbb{Z}) = 0$ and $H_k(X^{n+1}, X^n;\mathbb{Z}) = 0$. Thus, $H_k(X^n) \cong H_k(X^{n+1};\mathbb{Z})$. By repeated iteration, we obtain:
$$H_k(X^n;\mathbb{Z}) \cong H_k(X^{n+1};\mathbb{Z}) \cong H_k(X^{n+2};\mathbb{Z}) \cong \cdots \cong H_k(X^{n+l};\mathbb{Z}) = H_k(X;\mathbb{Z}),$$
where l is such that $X^{n+l} = X$ since we assumed X is finite dimensional. See [Hat02] for the case when X is infinite-dimensional.

This completes the proof. □

In what follows we define the cellular homology of a CW complex, X, in terms of a given cell structure, then we show that it coincides with the singular homology.

Definition 12.2. The cellular homology $H^{\mathrm{CW}}(X)$ of a CW complex X is the homology of the cellular chain complex $(C_*(X), d_*)$ indexed by the cells of X, i.e.,
$$C_n(X) := H_n(X^n, X^{n-1};\mathbb{Z}) = \mathbb{Z}^{\#n\text{-cells}},$$
and with differentials $d_n : C_n(X) \to C_{n-1}(X)$ defined by the following diagram: d_n etc. are defined in the obvious way to make the diagram commute. It is easy to check that $d_{n+1} \circ d_n = 0$ since the composition of these two maps induces two successive maps in one of the diagonal exact sequences.

$$\begin{tikzcd}
 & & & H_n(X^{n+1},X^n;\mathbb{Z})=0 & \\
0=H_n(X^{n-1};\mathbb{Z}) \arrow[dr] & & H_n(X^{n+1};\mathbb{Z})\cong H_n(X;\mathbb{Z}) \arrow[ur] & & \\
 & H_n(X^n;\mathbb{Z}) \arrow[ur, "i_n"] \arrow[dr, "j_n"] & & & \\
H_{n+1}(X^{n+1},X^n;\mathbb{Z}) \arrow[ur, "\partial_{n+1}"] \arrow[rr, "d_{n+1}"] & & H_n(X^n,X^{n-1};\mathbb{Z}) \arrow[rr, "d_n"] \arrow[dr, "\partial_n"] & & H_{n-1}(X^{n-1},X^{n-2};\mathbb{Z}) \\
 & & & H_{n-1}(X^{n-1};\mathbb{Z}) \arrow[ur, "j_{n-1}"] & \\
 & & H_{n-1}(X^{n-2};\mathbb{Z})=0 \arrow[ur] & &
\end{tikzcd}$$

Proposition 12.3. *Let X be a topological space that admits a CW-complex structure. We have:*

$$H_n^{\mathrm{CW}}(X) \cong H_n(X;\mathbb{Z})$$

for all n, where $H_n(X;\mathbb{Z})$ is the singular homology of X.

Proof. Since $H_n(X^{n+1},X^n;\mathbb{Z})=0$ and $H_n(X;\mathbb{Z})\cong H_n(X^{n+1};\mathbb{Z})$, we get from the diagram above that

$$H_n(X;\mathbb{Z}) \cong \frac{H_n(X^n;\mathbb{Z})}{\ker i_n} \cong \frac{H_n(X^n;\mathbb{Z})}{\operatorname{Im}\partial_{n+1}}.$$

Now, $H_n(X^n;\mathbb{Z}) \cong \operatorname{Im} j_n \cong \ker\partial_n \cong \ker d_n$. The first isomorphism comes from j_n being injective, while the second follows by exactness. Finally, $\ker\partial_n = \ker d_n$ since $d_n = j_{n-1}\circ\partial_n$ and j_{n-1} is injective. Also, we have $\operatorname{Im}\partial_{n+1} = \operatorname{Im} d_{n+1}$. Indeed, $d_{n+1} = j_n\circ\partial_{n+1}$ and j_n is injective. Altogether, we have

$$H_n(X;\mathbb{Z}) \cong \frac{\ker d_n}{\operatorname{Im} d_{n+1}} = H_n^{\mathrm{CW}}(X).$$

This completes the proof. □

Let's make some observations which are immediate:

(1) If X has no n-cells, then $H_n(X;\mathbb{Z}) = 0$. Indeed, in this case we have $C_n = H_n(X^n,X^{n-1};\mathbb{Z}) = 0$. Therefore, $H_n^{\mathrm{CW}}(X;\mathbb{Z}) = 0$.
(2) If X is connected and has a single 0-cell, then $d_1 : C_1 \to C_0$ is the zero map. Indeed, since X contains only a single 0-cell, $C_0 = \mathbb{Z}$. Also, since X is connected, $H_0(X) = \mathbb{Z}$. So, by the above theorem, $\mathbb{Z} = H_0(X;\mathbb{Z}) = \ker d_0/\operatorname{Im} d_1 = \mathbb{Z}/\operatorname{Im} d_1$. This implies that $\operatorname{Im} d_1 = 0$, so d_1 is the zero map as desired.

If X has no cells in adjacent dimensions, then $d_n = 0$ for all n, and $H_n(X;\mathbb{Z}) \cong \mathbb{Z}^{\#n\text{-cells}}$ for all n. Indeed, in this case, all maps d_n vanish. So for any n, $H_n^{\mathrm{CW}}(X) \cong C_n \cong \mathbb{Z}^{\#n\text{-cells}}$. Let's look at two examples:

Example 12.4. When $n > 1$, $\mathbb{S}^n \times \mathbb{S}^n$ has one 0-cell, two n-cells, and one $2n$-cell. Since $n > 1$, these cells are not in adjacent dimensions. Hence:

$$H_k(\mathbb{S}^n \times \mathbb{S}^n;\mathbb{Z}) = \begin{cases} \mathbb{Z}, & \text{if } k = 0, 2n \\ \mathbb{Z}^2, & \text{if } k = n \\ 0, & \text{otherwise.} \end{cases}$$

Example 12.5. Recall that $\mathbb{CP}^n$ has one cell in each even dimension $0, 2, 4, \ldots, 2n$. So $\mathbb{CP}^n$ has no two cells in adjacent dimensions. Hence:

$$H_k(\mathbb{CP}^n; \mathbb{Z}) = \begin{cases} \mathbb{Z}, & \text{if } k = 0, 2, 4, \ldots, 2n \\ 0, & \text{otherwise.} \end{cases}$$

We next discuss how to compute, in general, the maps

$$d_n : C_n(X) = \mathbb{Z}^{\#n\text{-cells}} \to C_{n-1}(X) = \mathbb{Z}^{\#(n-1)\text{-cells}}$$

of the cellular chain complex. Let us consider the n-cells $\{e_n^\alpha\}_\alpha$ as the basis for $C_n(X)$ and the $(n-1)$-cells $\{e_{n-1}^\beta\}_\beta$ as the basis for $C_{n-1}(X)$. In particular, we can write:

$$d_n(e_n^\alpha) = \sum_\beta d_{\alpha,\beta} \cdot e_{n-1}^\beta \qquad d_{\alpha,\beta} \in \mathbb{Z},$$

Proposition 12.6. ***(Cellular Boundary Formula)*** *The coefficient $d_{\alpha,\beta}$ is equal to the degree of the map $\Delta_{\alpha,\beta} : \mathbb{S}_\alpha^{n-1} \to \mathbb{S}_\beta^{n-1}$ defined by the composition:*

$$\mathbb{S}_\alpha^{n-1} = \partial \mathbb{D}_\alpha^n \xrightarrow{\varphi_\alpha^n} X^{n-1} = X^{n-2} \cup_\gamma \mathbb{D}_\gamma^{n-1} \xrightarrow{\textit{collapse}} X^{n-1}/(X^{n-2} \cup_{\gamma \neq \beta} \mathbb{D}_\gamma^{n-1}) = \mathbb{S}_\beta^{n-1},$$

where φ_α^n is the attaching map of $\mathbb{D}_\alpha^n$, and the collapsing map sends $X^{n-2} \cup_{\gamma \neq \beta} \mathbb{D}_\gamma^{n-1}$ to a point.

Proof. We will proceed with the proof by chasing the following diagram:

$$\begin{tikzcd}
H_n(\mathbb{D}_\alpha^n, \mathbb{S}_\alpha^{n-1}; \mathbb{Z}) \arrow[r, "\cong"] \arrow[d, "(\Phi_\alpha^n)_*"'] & \widetilde{H}_n(\mathbb{S}_\alpha^{n-1}; \mathbb{Z}) \arrow[rr, "(\Delta_{\alpha,\beta})_*"] \arrow[d, "(\phi_\alpha^n)_*"'] & & \widetilde{H}_n(\mathbb{S}_\beta^{n-1}; \mathbb{Z}) \\
H_n(X^n, X^{n-1}; \mathbb{Z}) \arrow[r, "\partial_n"] \arrow[dr, "d_n"'] & \widetilde{H}_{n-1}(X^{n-1}; \mathbb{Z}) \arrow[rr, "q_*"] \arrow[d, "j_{n-1}"] & & \widetilde{H}_{n-1}(X^{n-1}/X^{n-2}; \mathbb{Z}) \arrow[u, "q_{\beta,*}"'] \arrow[d, "\cong"'] \\
& H_{n-1}(X^{n-1}, X^{n-2}; \mathbb{Z}) \arrow[rr, "\cong"'] & & H_{n-1}(X^{n-1}/X^{n-2}, X^{n-2}/X^{n-2}; \mathbb{Z})
\end{tikzcd}$$

The maps are as follows:

1. Φ_α^n is the characteristic map of the cell e_α^n, and ϕ_α^n is its attaching map.
2. The map
$$q_* : \widetilde{H}_{n-1}(X^{n-1}; \mathbb{Z}) \to \widetilde{H}_{n-1}(X^{n-1}/X^{n-2}; \mathbb{Z}) = \bigoplus_\beta \widetilde{H}_{n-1}(\mathbb{D}_\beta^{n-1}/\partial \mathbb{D}_\beta^{n-1}; \mathbb{Z})$$
is induced by the quotient map $q : X^{n-1} \to X^{n-1}/X^{n-2}$.
3. $q_\beta : X^{n-1}/X^{n-2} \to \mathbb{S}_\beta^{n-1}$ collapses the complement of the cell e_β^{n-1} to a point, the resulting quotient sphere being identified with $\mathbb{S}_\beta^{n-1} = \mathbb{D}_\beta^{n-1}/\partial \mathbb{D}_\beta^{n-1}$ via the characteristic map Φ_β^{n-1}.
4. $\Delta_{\alpha\beta} : \mathbb{S}_\alpha^{n-1} \to \mathbb{S}_\beta^{n-1}$ is the composition $q_\beta \circ q \circ \phi_\alpha^n$, i.e., the attaching map of e_α^n followed by the quotient map $X^{n-1} \to \mathbb{S}_\beta^{n-1}$ collapsing the complement of $\mathbb{D}_\beta^{n-1}$ in X^{n-1} to a point.

The top left-hand square commutes by naturality of the long-exact sequence in reduced homology. The top right-hand square commutes by the definition of $\Delta_{\alpha,\beta}$. The bottom left-hand triangle commutes by definition of d_n . The bottom right-hand square commutes due to the relationship between reduced and relative homology. The map $(\Phi^n_\alpha)_*$ takes the generator $[\mathbb{D}^n_\alpha] \in H_n(\mathbb{D}^n_\alpha, \mathbb{S}^{n-1}_\alpha)$ to a generator of the $\mathbb{Z}$-summand of $H_n(X^n, X^{n-1})$ corresponding to $\mathbb{D}^n_\alpha$, i.e.,

$$(\Phi^n_\alpha)_*([\mathbb{D}^n_\alpha]) = \mathbb{D}^n_\alpha$$

Since the top left square and the bottom left triangle both commute, this gives that

$$\sum_\beta d_{\alpha,\beta}\mathbb{D}^{n-1}_\beta = d_n(\mathbb{D}^n_\alpha) = d_n \circ (\Phi^\alpha_n)_*([\mathbb{D}^n_\alpha]) = j_{n-1} \circ (\phi^n_\alpha)_*([\mathbb{D}^n_\alpha]).$$

Here we have implicitly identified $H_n(\mathbb{D}^n_\alpha, \mathbb{S}^{n-1}_\alpha)$ with $H_n(\mathbb{S}^{n-1}_\alpha)$. Looking to the bottom right square, recall that since X is a CW complex, (X^n, X^{n-1}) is a good pair. This gives the isomorphism

$$\begin{aligned} H_{n-1}(X^{n-1}, X^{n-2}; \mathbb{Z}) &\cong \widetilde{H}_{n-1}(X^{n-1}/X^{n-2}; \mathbb{Z}) \\ &\cong H_{n-1}(X^{n-1}/X^{n-2}, X^{n-2}/X^{n-2}; \mathbb{Z}). \end{aligned}$$

Notice that the map q_β, collapsing all the $n-1$ cells of X to the $n-1$ cell $\mathbb{S}^{n-1}_\beta$, induces the map $q_{\beta,*}$, which projects linear combinations of $\{\mathbb{D}^{n-1}_{\beta'}\}$ onto its summand of $\mathbb{D}^{n-1}_\beta$. Therefore, the value of $d_n(\mathbb{D}^n_i)$ is going to be the sum of the projections $q_{\beta',*}$ on the $n-1$ dimensional cells e^{n-1}_β. In other words:

$$\sum_\beta d_{\alpha,\beta}\mathbb{D}^\beta_\beta = d_n(\mathbb{D}^\alpha_n) = \sum_\beta q_{\beta*} \circ q_* \circ (\phi^\alpha_n)_* \circ [\mathbb{D}^\alpha_n].$$

As noted before, we have defined $(\Delta_{\alpha\beta})_* = q_{\beta*} \circ q_* \circ (\phi^\alpha_n)_*$. The result now follows. $\square$

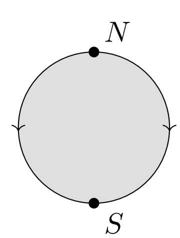
Example 12.7. Let $X = \mathbb{S}^2$. We $\mathbb{S}^2$ with $\mathbb{D}^2/\sim$ such that

$$(x, y) \sim (x', y') = x',\ y = |y'|$$

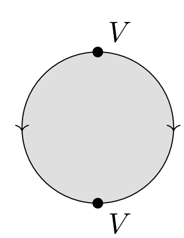
This induces a cell decomposition into one 2-cell, the image of the interior, one 1-cell, the image of $\mathbb{S}^1 \setminus \{(0,1), (0,-1)\}$, and two 0-cells, the images of $(0,1)$ and $(0,1)$ which are N and S. Let $A = \{N, S\}$. Since A is a sub-complex, X/A inherits a CW complex structure with one one 2-cell, one 1-cell and one 0-cell. We have

$$0 \to \mathbb{Z} \xrightarrow{d_2} \mathbb{Z} \xrightarrow{d_1} \mathbb{Z} \to 0$$

Since X/A is connected as has a single 0-cell, $d_1 \equiv 0$. The attaching map of the two-cell in



$X = \mathbb{S}^2$



$X = \mathbb{S}^2/\{N, S\}$

either case can be identified with the map:

$$\phi_{1,2}(e^{\phi i}) = \begin{cases} e^{i\phi} & 0 \leq \phi \leq \pi \\ e^{-i\phi} & \pi \leq \phi \leq 2\pi \end{cases}$$

The map has degree 0. Hence, $d_2 \equiv 0$. As a result, we have

$$H_n(\mathbb{S}^2/\{N, S\}; \mathbb{Z}) = \begin{cases} \mathbb{Z}, & \text{if } n = 0, 1, 2 \\ 0, & \text{otherwise.} \end{cases}$$

Example 12.8. Recall that $\mathbb{RP}^n$ has a CW structure with one k-cell $\mathbb{D}^k$ in each dimension $0 \leq k \leq n$. The attaching map for $\mathbb{D}^k$ is the standard 2-fold covering map $\phi : \mathbb{S}^{k-1} \to \mathbb{RP}^{k-1}$ identifying a point and its antipodal point in $\mathbb{S}^{k-1}$. To compute the boundary map d_k, we compute the degree of the composition

$$f : \mathbb{S}^{k-1} \to \mathbb{RP}^{k-1} \to \frac{\mathbb{RP}^{k-1}}{\mathbb{RP}^{k-2}} = \mathbb{S}^{k-1}$$

We consider a neighborhood V of y and the two neighborhoods U_1 and U_2 given to exist by the local homeomorphism property of f. One of the homeomorphisms is the identity map and the other homemorphisms is the anti-podal map. Then by the local degree formula implies

$$d_k = 1 + (-1)^k$$

It follows that

$$d_k = \begin{cases} 0 & \text{if } k \text{ is odd,} \\ 2 & \text{if } k \text{ is even,} \end{cases}$$

and therefore we obtain that

$$H_k(\mathbb{RP}^n; \mathbb{Z}) = \begin{cases} \mathbb{Z}_2 & \text{if } k \text{ is odd, } 0 < k < n, \\ \mathbb{Z} & \text{if } k = 0, n \text{ is odd,} \\ 0 & \text{otherwise.} \end{cases}$$

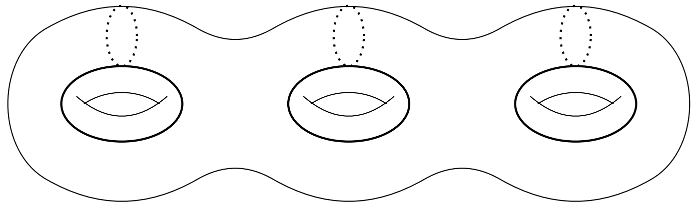
Example 12.9. Let M_g be the closed oriented surface of genus g, with its usual CW structure: one 0-cell, $2g$ 1-cells $\{a_1, b_1, \ldots, a_g, b_g\}$, and one 2-cell attached by the product of commutators $[a_1, b_1] \cdot \ldots \cdot [a_g, b_g]$. The associated cellular chain complex of M_g is:

$$0 \xrightarrow{d_3} \mathbb{Z} \xrightarrow{d_2} \mathbb{Z}^{2g} \xrightarrow{d_1} \mathbb{Z} \xrightarrow{d_0} 0$$

Since M_g is connected and has only one 0-cell, we get that $d_1 = 0$. We claim that d_2 is also the zero map. As the attaching map sends the generator to $a_1 b_1 a_1^{-1} b_1^{-1} \ldots a_g b_g a_g^{-1} b_g^{-1}$, when we collapse all 1-cells (except a_i) to a point, the word defining the attaching map $a_1 b_1 a_1^{-1} b_1^{-1} \ldots agbga_g^{-1} b_g^{-1}$ reduces to $a_i a_i^{-1}$. Hence, the coefficient$d_{ea_i} = 1 - 1 = 0$. Altogether, $d_2(e) = 0$. So the homology groups of M_g are given by

$$H_n(M_g; \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{for } n = 0, 2, \\ \mathbb{Z}^{2g} & \text{for } n = 1, \\ 0 & \text{otherwise.} \end{cases}$$

For $g = 3$, see the figure below to visualize the $2g = 6$ generators of $H_1(M_3)$:



13. Euler Characteristic

Definition 13.1. Let X be a finite CW complex of dimension n. The Euler characteristic of X is defined as:

$$\chi(X) = \sum_{i=0}^{n} (-1)^i \cdot \# n - \text{cells} = \sum_{i=0}^{n} (-1)^i \cdot \# \operatorname{rank}(C_i^{\text{CW}})$$

Here C_i^{CW} is the i-th abelian group in the chain complex that determines cellular homology. We show that the Euler characteristic does not depend on the cell structure chosen for the space X. As we will see below, this is not the case.

Proposition 13.2. *The Euler characteristic can be computed as:*

$$\chi(X) = \sum_{i=0}^{n} (-1)^i \cdot \operatorname{rank}(H_i^{\text{CW}}(X; \mathbb{Z}))$$

In particular, $\chi(X)$ is independent of the chosen cell structure on X.

Proof. We use the following notation: $B_i = \operatorname{im}(d_{i+1})$, $Z_i = \ker(d_i)$, and $H_i^{\text{CW}} = Z_i/B_i$. The additivity of rank yields that

$$\operatorname{rank}(C_i) = \operatorname{rank}(Z_i) + \operatorname{rank}(B_{i-1})$$
$$\operatorname{rank}(Z_i) = \operatorname{rank}(B_i) + \operatorname{rank}(H_i^{\text{CW}})$$

Substitute the second equality into the first, multiply the resulting equality by $(-1)^i$, and sum over i to get that

$$\chi(X) = \sum_{i=0}^{n} (-1)^i \cdot \operatorname{rank}(H_i^{\text{CW}})$$

Since cellular homology is isomorphic to singular homology and the latter is homotopy invariant, the result follows.. $\square$

Proposition 13.3. *Let X, Y be finite-dimensional CW complexes and let*

$$\chi(X) = \sum_{i=0}^{n} (-1)^i a_i$$
$$\chi(Y) = \sum_{j=0}^{m} (-1)^j b_j$$

Here a_i is the number of i-cells in X. Similarly, b_j is the number of j-cells in B The Euler characteristic enjoys some nice properties:

(1) $\chi(X \times Y) = \chi(X) \times \chi(Y)$

(2) *If $X = A \cup B$ such that A, B are sub-complexes of X. Then*

$$\chi(X) = \chi(A) + \chi(B) - \chi(A \cap B)$$

(3) *If $p : \widetilde{X} \to X$ is an n-sheeted covering space, then*

$$\chi(\widetilde{X}) = n\chi(X)$$

Proof. The proof is given below:

(1) For any index k, k-cells in $X \times Y$ are created by considering products of r-cells and $k-r$ cells from X and Y respectively where $0 \leq r \leq k$. Hence the number of k-cells is

$$\sum_{r=0}^{k} a_r b_{k-r}$$

Therefore,

$$\chi(X) \times \chi(Y) = \left(\sum_{i=0}^{n}(-1)^i a_i\right) \times \left(\sum_{j=0}^{m}(-1)^j b_j\right) = \sum_{k=0}^{m+n}(-1)^k \sum_{r=0}^{k} a_r b_{k-r} = \chi(X \times Y)$$

(2) Let a_i^A denote the number of i-cells in A. Similarly, let a_i^B be the number of i-cells in B. Similarly, let $a_i^{A\cap B}$ be the number of i-cells in $A \cap B$. We have

$$a_i = a_i^A + a_i^B - a_i^{A\cap B}$$

for $i = 1, \cdots, n$. Therefore, we have,

$$\begin{aligned}\chi(X) &= \sum_{i=0}^{n}(-1)^i a_i \\ &= \sum_{i=0}^{n}(-1)^i a_i^A + \sum_{i=0}^{n}(-1)^i a_i^B - \sum_{i=0}^{n}(-1)^i a_i^{A\cap B} \\ &= \chi(A) + \chi(B) - \chi(A \cap B)\end{aligned}$$

(3) Recall that if $\mathbb{D}_\alpha^k$ is a k-cell in X, then $\widetilde{X}$ has n k-cells. Therefore, it is clear that

$$\chi(\widetilde{X}) = n\chi(X)$$

This completes the proof. □

Example 13.4. Let M_g be the oriented surface of genus g, and let N_g be the oriented surface of genus g. We have

$$\chi_{M_g} = 2 - 2g$$
$$\chi_{N_g} = 2 - g$$

Thus all the M_g, N_g are distinguished from each other by their Euler characteristics. There are only the relations

$$\chi(M_g) = \chi(N_{2g})$$

14. Tor Functor

We now discuss the Tor (derived) functor which will play an important role in the discussion of homology with coefficients.

Remark 14.1. *We work with commutative rings below. Hence, we don't make any distinction between the categories of left R-modules and right R-modules. We use the generic phrase 'R-module' to refer to a left/right R-module.*

Recall that the tensor product, $\otimes_R$, defines a functor from the category of R-modules to itself such that if N is a R-module, then

$$- \otimes_R M(N) = N \otimes_R M$$

Moreover, if $f : N \to N'$ is a R-module morphism, then

$$- \otimes_R M(f) : N \otimes_R M \xrightarrow{f \otimes_R \mathrm{Id}_M} N' \otimes_R M$$

It can be checked that $- \otimes_R M$ is a right exact functor. However, $- \otimes_R M$ is not a left exact functor in general.

Example 14.2. The functor $- \otimes_R M$ need not be left exact functor. Let $R = \mathbb{Z}$. Consider the sequence:

$$0 \to \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z}$$

Here $\cdot n$ is the multiplication by n map. Let $M = \mathbb{Z}/n\mathbb{Z}$ we obtain a map:

$$\mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \xrightarrow{\cdot n \otimes_{\mathbb{Z}} \mathrm{Id}_{\mathbb{Z}/n\mathbb{Z}}} \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/n\mathbb{Z}$$

However, this is the zero map since we have

$$\cdot n \otimes_{\mathbb{Z}} \mathrm{Id}_{\mathbb{Z}/n\mathbb{Z}}(1 \otimes_{\mathbb{Z}} \overline{m}) = n \otimes_{\mathbb{Z}} \overline{m} = 1 \otimes_{\mathbb{Z}} \overline{nm} = 0.$$

The zero map is not injective.

Remark 14.3. *A R-module M is called flat if $- \otimes_R M$ is a left exact functor. If M is a projective R-module, then $- \otimes_R M$ is a left exact functor. This follows because a projective R-module is a direct summand of a free R-module, a free R-module is a flat module and that a R-module is flat if and only if each summad is a flat R-module.*

Since the $- \otimes_R M$ functor is a right exact functor which in general is not a left exact functor, we can consider its left derived functor.

Definition 14.4. Let R be a ring and let M be a R-module. The i-th Tor functor is the i-th left derived functor of $- \otimes_R M$. It is denoted as

$$\mathrm{Tor}_i^R(-, M)$$

By definition, $\mathrm{Tor}_i^R(-, M)$ is computed as follows. If N is a R-module, take any projective resolution

$$\cdots \to P^1 \to P^0 \to N \to 0,$$

and form the chain complex:

$$\cdots \to P^2 \otimes_R M \to P^1 \otimes_R M \to P^0 \otimes_R M$$

Then $\mathrm{Tor}_i^R(N, M)$ is the homology of this complex at position i.

$$\mathrm{Tor}_i^R(N, M) = H_i((P^i \otimes_R M)_\bullet)$$

Remark 14.5. *General results about derived functors show that the homology is independent of the choice of the projective resolution.*

If R is a commutative ring and M is a R-module, we can define another functor $M \otimes_R -$. The definition is similar to that of the functor defined above. It can also be checked that $M \otimes_R -$ is right exact functor that is, in general, not left exact. Hence, we can attempt to construct a left-derived functor associated to $M \otimes_R -$ as above. We label that derived functor $\overline{\mathrm{Tor}}_i^R(M, -)$. We have the following result:

Proposition 14.6. ***(Balanacing Tor)*** *Let R be a commutative ring and let M be a R-module. Then*

$$\mathrm{Tor}_i^R(-, M) \cong \overline{\mathrm{Tor}}_i^R(M, -)$$

That is, for every R-module N, we have

$$\mathrm{Tor}_i^R(N, M) \cong \overline{\mathrm{Tor}}_i^R(M, N)$$

Proof. See [Wei94] for a proof. □

Remark 14.7. *In light of Proposition 14.6, we can identify Tor_i^R with $\overline{\mathrm{Tor}}_i^R$ for each $i \geq 0$. This allows us to compute projective resolutions of either N or M to compute $\mathrm{Tor}_i^R(N, M)$ for each $i \geq 0$.*

Proposition 14.8. *Let R be a commutative ring and let M be a R-module. The Tor functor satisfies the following properties:*

1. $\mathrm{Tor}_0^R(N, M) \cong N \otimes_R M$ *for any R-modules M, N.*
2. *If N is a projective R-module, then $\mathrm{Tor}_i^R(N, M) = 0$ for all $i \geq 1$*
3. *Any $f : N_1 \to N_2$ R-module homomorphism induces a morphism*
$$f_*^i : \mathrm{Tor}_i^R(N_1, M) \longrightarrow \mathrm{Tor}_i^R(N_2, M)$$
for each $i \geq 0$.
4. *Any short exact sequence $0 \to N_1 \xrightarrow{\phi} N_2 \xrightarrow{\psi} N_3 \to 0$ of R-modules induces a long exact sequence:*

$$\cdots \to \mathrm{Tor}_1^R(N_1, M) \to \mathrm{Tor}_1^R(N_2, M) \to \mathrm{Tor}_1^R(N_3, M) \to N_1 \otimes_R M \to N_2 \otimes_R M \to N_3 \otimes_R M \to 0$$

Proof. (1) and (2) follow from general properties of derived functors (**??**). For (3), let $P_1^\bullet$ be a projective resolution of N_1 and $P_2^\bullet$ be a projective resolution of N_2. General properties about projective resolutions imply that that f lifts to a chain map $\varphi^\bullet : P_1^\bullet \longrightarrow P_2^\bullet$. Then, $\varphi^\bullet$ induces a morphism of chain complexes $P_1^\bullet \otimes_R M \longrightarrow P_2^\bullet \otimes_R M$ which, in turn, induces a morphism:

$$f_*^i : \mathrm{Tor}_i^R(N_1, M) \longrightarrow \mathrm{Tor}_i^R(N_2, M)$$

for each $i \geq 0$. For (4), let $P^\bullet$ be a projective resolution of M. Then there is an induced short exact sequence of chain complexes:

$$0 \to N_1 \otimes_R P^\bullet \to N_2 \otimes_R P^\bullet \to N_3 \otimes_R P^\bullet \to 0$$

because each module P^i is projective. Applying the long exact sequence in homology produces the required long exact sequence. □

We now specialize to the category of $\mathbb{Z}$-modules. In what follows, we fix G to be an abelian group. We have the following result:

Lemma 14.9. *For any abelian group A, we have $\mathrm{Tor}_i^{\mathbb{Z}}(A, G) = 0 \quad$ if $i > 1$.*

Proof. Recall that any abelian group, A, admits a two-step free resolution.

$$0 \to F_1 \to F_0 \to A \to 0$$

Thus, $\mathrm{Tor}_i^{\mathbb{Z}}(A, G) = 0$ if $i > 1$. $\square$

Remark 14.10. *Only* $\mathrm{Tor}_1(-, G)$ *encodes any interesting information. In what follows, we adopt the notation:* $\mathrm{Tor}(-, G) := \mathrm{Tor}_1^{\mathbb{Z}}(-, G)$.

Proposition 14.11. *If* $R = \mathbb{Z}$*, the Tor functor satisfies the following properties:*

(1) $\mathrm{Tor}\left(\bigoplus_i A_i, G\right) \cong \bigoplus_i \mathrm{Tor}(A_i, G)$.
(2) *If* A *is a free abelian group, then* $\mathrm{Tor}(A, G) = 0$.
(3) $\mathrm{Tor}(\mathbb{Z}/n\mathbb{Z}, G) \cong \ker(G \xrightarrow{n} G)$.
(4) *For a short exact sequence:* $0 \to B \to C \to D \to 0$ *of abelian groups, there is a natural exact sequence:*

$$0 \to \mathrm{Tor}(B, G) \to \mathrm{Tor}(C, G) \to \mathrm{Tor}(D, G) \to B \otimes_R G \to C \otimes_R G \to D \otimes_R G \to 0.$$

Proof. The proof is given below:

(1) This follows from the identity,

$$\left(\bigoplus_i A_i\right) \otimes_{\mathbb{Z}} G = \bigoplus_i (A_i \otimes_{\mathbb{Z}} G)$$

and noting that taking direct sums of projective resolutions of A_i forms a projective resolution for $\bigoplus_i A^i$, and that homology commutes with direct sums.
(2) If A is free, then $0 \to A \to A \to 0$ is a projective resolution of A, so $\mathrm{Tor}(A, G) = 0$.
(3) The exact sequence $0 \to \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$ is a projective resolution of $\mathbb{Z}/n\mathbb{Z}$. Tensoring with G and dropping the right-most term yields the complex:

$$G \cong \mathbb{Z} \otimes_R G \xrightarrow{\cdot n \otimes_R 1_G} G \cong \mathbb{Z} \otimes_R G \to 0,$$

Thus, $\mathrm{Tor}(\mathbb{Z}/n\mathbb{Z}, G) = \ker(G \xrightarrow{n} G)$.
(4) This follows from Proposition 14.8(4).

This completes the proof. $\square$

15. Homology with Coefficients & Universal Coefficient Theorem

In this section, we discuss homology with coefficients and the universal coefficient theorem in homology.

Definition 15.1. Let G be an abelian group and X a topological space. The homology of X with G-coefficients, denoted $H_n(X; G)$ for $n \in \mathbb{N}$, is the homology of the chain complex:

$$C_\bullet(X; G) = C_\bullet(X) \otimes_{\mathbb{Z}} G$$

consisting of finite formal sums $\sum_i \eta_i \cdot \sigma_i$ with $\eta_i \in G$, and with boundary maps given by

$$\partial_n^G := \partial_n \otimes_{\mathbb{Z}} \mathrm{Id}_G\,.$$

Remark 15.2. *Since* ∂_n *satisfies* $\partial_n \circ \partial_{n+1} = 0$*, it follows that* $\partial_n^G \circ \partial_{n+1}^G = 0$*. Hence,* $(C_\bullet(X); G, \partial_\bullet^G)$ *is indeed a chain complex.*

We can construct versions of the usual modified homology groups (relative, reduced, etc.) in the most natural way.

(1) **(Relative homology with G-coefficients)** Consider the augmented chain complex:
$$C_1(X;G) \to C_0(X;G) \to G \to 0$$
where $\epsilon(\sum_i \eta_i \sigma_i) = \sum_i \eta_i \in G$. Reduced homology with G-coefficients is defined as the homology of the augmented chain complex.

(2) **(Relative chain Complex with G-coefficients)** Define relative chains with G-coefficients by:
$$C_n(X, A; G) := C_n(X;G)/C_n(A;G),$$
Consider the chain complex:
$$C_1(X, A; G) \to C_0(X, A; G) \to 0$$
The relative homology with G-coefficients is defined as the homology of the augmeneted chain complex.

(3) **(Cellular homology with G-coefficients)** We can build cellular homology with G-coefficients by defining
$$C_n^G(X) = H_n(X_n, X_{i-1}; G) \cong G^{(\text{number of } i\text{-cells})}$$
The cellular boundary maps are given by:
$$d_n^G(e_n^\alpha) = \sum_\beta d_{\alpha\beta} e_{i-1}^\beta,$$
where $d_{\alpha\beta}$ is as before the degree of a map $\Delta_{\alpha\beta} : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$. As it is the case for integers, we get an isomorphism:
$$H_n^{\mathrm{CW}}(X;G) \cong H_n(X;G)$$

Example 15.3. Let's look at some examples:

(1) By studying the chain complex with G-coefficients, it follows that:
$$H_n(\{*\};G) = \begin{cases} G & \text{if } n = 0, \\ 0 & \text{if } n \neq 0. \end{cases}$$

(2) (Sketch) The homology of a sphere as before by induction and using the long exact sequence of the pair $(\mathbb{D}^n, \mathbb{S}^n)$ to be:
$$H_n(\mathbb{S}^n;G) = \begin{cases} G & \text{if } i = 0, n, \\ 0 & \text{otherwise.} \end{cases}$$

The result will follow more easily from our discussion of the universal coefficient theorem below.

We now prove an important theorem that relates how homology with different coefficients are connected. Changing the coefficient group in homology can alter the resulting invariants, and understanding this relationship is essential for computations and deeper theoretical insights. The theorem we present provides a precise mechanism—often involving universal coefficient theorems—for translating between homology groups with various coefficients.

Proposition 15.4. ***(Universal Coefficient Theorem)*** *If a chain complex $(C_\bullet, \partial_\bullet)$ of free abelian groups has homology groups $H_n(C_\bullet)$, then the homology groups $H_n(C_\bullet;G)$ are determined by the short exact sequence:*
$$0 \to H_n(C_\bullet) \otimes_{\mathbb{Z}} G \to H_n(C_\bullet;G) \xrightarrow{h} \mathrm{Tor}(H_{n-1}(C_\bullet), G) \to 0$$

Proof. We have $B_n = \operatorname{im}\partial_{n+1} \subseteq Z_n = \ker\partial_n \subseteq C_n$. Since $\partial_n|_{Z_n} = 0$ and $\partial_{n-1}|_{B_{n-1}} = 0$, we have the following short exact sequence of chain complexes:

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & Z_{n+1} \arrow[r, hook, "i_{n+1}"] \arrow[d, "\partial_{n+1}=0"] & C_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, "\partial_{n+1}"] & B_n \arrow[r] \arrow[d, "\partial_n=0"] & 0 \\
0 \arrow[r] & Z_n \arrow[r, hook, "i_n"] \arrow[d, "\partial_n=0"] & C_n \arrow[r, "\partial_n"] \arrow[d, "\partial_n"] & B_{n-1} \arrow[r] \arrow[d, "\partial_{n-1}=0"] & 0 \\
0 \arrow[r] & Z_{n-1} \arrow[r, hook, "i_{n-1}"] \arrow[d] & C_{n-1} \arrow[r, "\partial_{n-1}"] \arrow[d] & B_{n-2} \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

Since C_n is a free abelian group, Z_n, B_n are also free abelian groups. Hence, $\otimes_{\mathbb{Z}}$ is an exact functor if applied to each row. Hence, we have a short exact sequence of chain complexes:

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & Z_{n+1} \otimes_{\mathbb{Z}} G \arrow[r, hook, "i_{n+1}\otimes_{\mathbb{Z}}\mathrm{Id}_G"] \arrow[d, "0"] & C_{n+1} \otimes_{\mathbb{Z}} G \arrow[r, "\partial_{n+1}\otimes_{\mathbb{Z}}\mathrm{Id}_G"] \arrow[d, "\partial_{n+1}\otimes_{\mathbb{Z}}\mathrm{Id}_G"] & B_n \otimes_{\mathbb{Z}} G \arrow[r] \arrow[d, "0"] & 0 \\
0 \arrow[r] & Z_n \otimes_{\mathbb{Z}} G \arrow[r, hook, "i_n\otimes_{\mathbb{Z}}\mathrm{Id}_G"] \arrow[d, "0"] & C_n \otimes_{\mathbb{Z}} G \arrow[r, "\partial_n\otimes_{\mathbb{Z}}\mathrm{Id}_G"] \arrow[d, "\partial_n\otimes_{\mathbb{Z}}\mathrm{Id}_G"] & B_{n-1} \otimes_{\mathbb{Z}} G \arrow[r] \arrow[d, "0"] & 0 \\
0 \arrow[r] & Z_{n-1} \otimes_{\mathbb{Z}} G \arrow[r, hook, "i_{n-1}\otimes_{\mathbb{Z}}\mathrm{Id}_G"] \arrow[d] & C_{n-1} \otimes_{\mathbb{Z}} G \arrow[r, "\partial_{n-1}\otimes_{\mathbb{Z}}\mathrm{Id}_G"] \arrow[d] & B_{n-2} \otimes_{\mathbb{Z}} G \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

By the snake lemma (Proposition 5.19) we have a connecting homomorphism

$$\Delta_n : B_n \otimes_{\mathbb{Z}} G \to Z_n \otimes_{\mathbb{Z}} G$$

Following the construction of the connecting homomorphism in the proof of the snake lemma, we see that $\Delta_n = j_n \otimes_{\mathbb{Z}} \mathrm{Id}_M$ where $j_n : B_n \to Z_n$ is the inclusion $B_n \subseteq Z_n$. Additionally, we get an exact sequence

$$\cdots \to H_n(B \otimes_{\mathbb{Z}} G) \xrightarrow{(j_n\otimes_{\mathbb{Z}}\mathrm{Id}_G)_*} H_n(Z \otimes_{\mathbb{Z}} G) \xrightarrow{(i_n\otimes_{\mathbb{Z}}\mathrm{Id}_G)_*} H_n(C \otimes_{\mathbb{Z}} G) \xrightarrow{(\partial_n\otimes_{\mathbb{Z}}\mathrm{Id}_G)_*} H_{n-1}(B \otimes_{\mathbb{Z}} G) \to \cdots$$

Since the chain complexes $Z \otimes_{\mathbb{Z}} G$ and $B \otimes_{\mathbb{Z}} G$ have null boundary maps, we deduce that the sequence

$$\cdots \to B_n \otimes_{\mathbb{Z}} G \xrightarrow{j_n\otimes_{\mathbb{Z}}\mathrm{Id}_G} Z_n \otimes_{\mathbb{Z}} G \xrightarrow{i_n\otimes_{\mathbb{Z}}\mathrm{Id}_G} H_n(C \otimes_{\mathbb{Z}} G) \xrightarrow{(\partial_n\otimes_{\mathbb{Z}}\mathrm{Id}_G)_*} B_{n-1} \otimes_{\mathbb{Z}} G \to \cdots$$

We claim that this sequence is natural. Suppose that we have another chain complex $(C'_\bullet, \partial'_\bullet)$ and a chain map $f : C_\bullet \to C'_\bullet$. We want to see that the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & B_n(C)\otimes_{\mathbb{Z}} M \arrow[r, "j_n\otimes_{\mathbb{Z}}\mathrm{Id}_G"] \arrow[d, "f_n\otimes_{\mathbb{Z}}\mathrm{Id}_M"] & Z_n(C)\otimes_{\mathbb{Z}} M \arrow[r, "i_n\otimes_{\mathbb{Z}}\mathrm{Id}_G"] \arrow[d, "f_n\otimes_{\mathbb{Z}}\mathrm{Id}_M"] & H_n(C\otimes_{\mathbb{Z}} M) \arrow[r, "(\partial_n\otimes_{\mathbb{Z}}\mathrm{Id}_G)_*"] \arrow[d, "(f_n\otimes_{\mathbb{Z}}\mathrm{Id}_M)_*"] & B_{n-1}(C)\otimes_{\mathbb{Z}} M \arrow[r] \arrow[d, "f_n\otimes_{\mathbb{Z}}\mathrm{Id}_M"] & \cdots \\
\cdots \arrow[r] & B_n(C')\otimes_{\mathbb{Z}} M \arrow[r, "j'_n\otimes_{\mathbb{Z}}\mathrm{Id}_G"] & Z_n(C')\otimes_{\mathbb{Z}} M \arrow[r, "i'_n\otimes_{\mathbb{Z}}\mathrm{Id}_G"] & H_n(C'\otimes_{\mathbb{Z}} M) \arrow[r, "(\partial'_n\otimes_{\mathbb{Z}}\mathrm{Id}_G)_*"] & B_{n-1}(C')\otimes_{\mathbb{Z}} M \arrow[r] & \cdots
\end{tikzcd}$$

commutes. It suffices to show that the three squares commute. For the left-most square, given an n-boundary b_n, its image $f_n(b_n)$ is also an n-boundary, hence $f_n \circ j_n(b_n) = f_n(b_n) = j'_n \circ f_n(b_n)$. Therefore, we deduce that

$$(f_n \otimes_{\mathbb{Z}} \mathrm{Id}_G) \circ (j_n \otimes_{\mathbb{Z}} \mathrm{Id}_G) = (j'_n \otimes_{\mathbb{Z}} \mathrm{Id}_G) \circ (f_n \otimes_{\mathbb{Z}} \mathrm{Id}_G).$$

The middle square commutes for similar reasons. Finally, the right most square commutes since

$$(f_n \otimes_{\mathbb{Z}} \mathrm{Id}_G) \circ (\partial_n \otimes_{\mathbb{Z}} \mathrm{Id}_G) = (\partial'_n \otimes_{\mathbb{Z}} \mathrm{Id}_G) \circ (f_n \otimes_{\mathbb{Z}} \mathrm{Id}_G).$$

From this discussion, we deduce the natural short exact sequences:

$$0 \to \mathrm{coker}(j_n \otimes_{\mathbb{Z}} \mathrm{Id}_G) \xrightarrow{i_n\otimes_{\mathbb{Z}}\mathrm{Id}} H_n(C \otimes_{\mathbb{Z}} G) \xrightarrow{(\partial_n\otimes_{\mathbb{Z}}\mathrm{Id}_G)_*} \ker(j_{n-1} \otimes_{\mathbb{Z}} \mathrm{Id}_G) \to 0.$$

Consider the projections $\pi_n : Z_n \to H_n(C)$, and the short exact sequence

$$0 \to B_n \xrightarrow{j_n} Z_n \xrightarrow{\pi_n} H_n(C) \to 0.$$

Note that these sequences are free resolutions of the homology modules $H_n(C)$. Hence, by properties of Tor, we have

$$0 \to \mathrm{Tor}(H_n(C), G) \to B_n \otimes_{\mathbb{Z}} G \xrightarrow{\iota_n\otimes\mathrm{Id}_G} C_n \otimes_{\mathbb{Z}} G \xrightarrow{\pi_n\otimes\mathrm{Id}} H_n(C) \otimes_{\mathbb{Z}} G \to 0.$$

We can now identify $\ker(j_{n-1} \otimes_{\mathbb{Z}} \mathrm{Id}_G)$ as $\mathrm{Tor}(H_{n-1}(C), G)$ and $\mathrm{coker}(j_n \otimes_{\mathbb{Z}} \mathrm{Id}_G)$ as $H_n(C) \otimes_{\mathbb{Z}} G$. The claim follows. □

Remark 15.5. *The sequence in Proposition 15.4 splits. This is because in the beginning of the proof, we considered the the exact sequences*

$$0 \to Z_n \xrightarrow{i_n} C_n \xrightarrow{\partial_n} B_{n-1} \to 0,$$

This sequence splits since B_{n-1} is a free abelian group. Hence, there must exist group homomorphisms $l_n : C_n \to Z_n$ such that $l_n \circ i_n = \mathrm{Id}$: $Z_n \to Z_n$ for all $n \in \mathbb{Z}$. Hence the compositions $(\pi_n \circ l_n)_{n\in\mathbb{N}}$ induce maps in homology

$$((\pi_n \circ l_n) \otimes_{\mathbb{Z}} \mathrm{Id}_G)_* : H_n(C \otimes_{\mathbb{Z}} G) \to H_n(C) \otimes_{\mathbb{Z}} G,$$

which are such that

$$((\pi_n \circ l_n) \otimes_{\mathbb{Z}} \mathrm{Id}_G)_* \circ ((i_n)_* \otimes_{\mathbb{Z}} \mathrm{Id}_G) = (\mathrm{Id}_G \otimes_{\mathbb{Z}} \mathrm{Id}_G)_*.$$

As a consequence, the sequence splits and we obtain

$$H_n(C_\bullet \otimes_{\mathbb{Z}} G) \cong (H_n(C_\bullet) \otimes_{\mathbb{Z}} G) \oplus \mathrm{Tor}(H_{n-1}(C), G).$$

Remark 15.6. *There is also a universal coefficient theorem for homology where $\mathbb{Z}$ is replaced by a PID, R and G is a R-module. In this case, we have*

$$0 \longrightarrow H_n(C_\bullet) \otimes_R G \longrightarrow H_n(C_\bullet; G) \xrightarrow{h} \mathrm{Tor}_1^R(H_{n-1}(C_\bullet), G) \longrightarrow 0$$

This comes from first establishing that Tor_i^R vanishes for $i \geq 2$ for when R is a PID, and then going through a proof for universal coefficient theorem essentially as above.

Example 15.7. Suppose $X = K$ is the Klein bottle, and $G = \mathbb{Z}/4$. Recall that $H_1(K;\mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}/2$, and $H_2(K;\mathbb{Z}) = 0$, so:

$$\begin{aligned} H_2(K;\mathbb{Z}/4) &= (H_2(K;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}/4) \oplus \mathrm{Tor}(H_1(K), \mathbb{Z}/4) \\ &= \mathrm{Tor}(\mathbb{Z}, \mathbb{Z}/4) \oplus \mathrm{Tor}(\mathbb{Z}/2, \mathbb{Z}/4) \\ &= 0 \oplus \mathbb{Z}/2 \\ &= \mathbb{Z}/2. \end{aligned}$$

Example 15.8. Let $X = \mathbb{RP}^n$ and $G = Z/2\mathbb{Z}$. Recall that we have

$$H_k(\mathbb{RP}^n;\mathbb{Z}) = \begin{cases} \mathbb{Z}/2\mathbb{Z} & \text{if } k \text{ is odd, } 0 < k < n, \\ \mathbb{Z} & \text{if } k = 0, n \text{ is odd,} \\ 0 & \text{otherwise.} \end{cases}$$

We compute $H_k(\mathbb{RP}^n;\mathbb{Z}/2\mathbb{Z})$. We consider multiple cases. For $k = 0$, we have:

$$H_0(\mathbb{RP}^n;\mathbb{Z}/2\mathbb{Z}) \cong H_0(\mathbb{RP}^n;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} = \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z}.$$

For $k = 1$, we have:

$$\begin{aligned} H_1(\mathbb{RP}^n;\mathbb{Z}/2\mathbb{Z}) &\cong H_1(\mathbb{RP}^n;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \oplus \mathrm{Tor}(H_0(\mathbb{RP}^n), \mathbb{Z}/2\mathbb{Z}) \\ &= (\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}) \oplus \mathrm{Tor}(\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \\ &= \mathbb{Z}/2\mathbb{Z} \oplus 0 = \mathbb{Z}/2\mathbb{Z}. \end{aligned}$$

For $1 < k < n$, such that k is an odd integer, we have

$$\begin{aligned} H_k(\mathbb{RP}^n;\mathbb{Z}/2\mathbb{Z}) &\cong (H_k(\mathbb{RP}^n;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}) \oplus \mathrm{Tor}(H_{k-1}(\mathbb{RP}^n;\mathbb{Z}), \mathbb{Z}/2\mathbb{Z}) \\ &= (\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}) \oplus \mathrm{Tor}(0, \mathbb{Z}/2\mathbb{Z}) \\ &= \mathbb{Z}/2\mathbb{Z} \end{aligned}$$

For $1 < k < n$, such that k is an even integer, we have

$$\begin{aligned} H_k(\mathbb{RP}^n;\mathbb{Z}/2\mathbb{Z}) &\cong (H_k(\mathbb{RP}^n;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}) \oplus \mathrm{Tor}(H_{k-1}(\mathbb{RP}^n;\mathbb{Z}), \mathbb{Z}/2\mathbb{Z}) \\ &= (0 \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}) \oplus \mathrm{Tor}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \\ &= \mathbb{Z}/2\mathbb{Z} \end{aligned}$$

For $k = n$ even, we have

$$H_k(\mathbb{RP}^n;\mathbb{Z}/2\mathbb{Z}) = (0 \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}) \oplus \mathrm{Tor}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$$

If $k = n$ is odd, we have

$$H_k(\mathbb{RP}^n;\mathbb{Z}/2\mathbb{Z}) = (\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}) \oplus \mathrm{Tor}(0, \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$$

All in all, we have

$$H_k(\mathbb{RP}^n;\mathbb{Z}/2\mathbb{Z}) = \begin{cases} \mathbb{Z}/2\mathbb{Z} & \text{if } k = 0, \cdots, n \\ 0 & \text{otherwise.} \end{cases}$$

16. Equivalence of Homology Theories

We have encountered various homology theories, including singular, simplicial, and cellular homology, and have seen that they all coincide in specific cases. For instance, if a topological space admits a Δ-complex structure, the singular and simplicial homologies coincide. Similarly, if a topological space admits a CW-complex structure, the singular and cellular homologies coincide. We now demonstrate that this is a specific instance of a more general principle: homology theories are uniquely determined on well-behaved topological spaces, particularly within the category of CW pairs.

Proposition 16.1. *Let h_* be a homology theory in the sense of Definition 7.1 with $\mathbb{Z}$ coefficients defined as a collection of functors*

$$h_n : \mathbf{CW}^2 \to \mathbf{Ab}$$

If $h_n(*;\mathbb{Z}) \cong 0$ for $n \neq 0$, then there exists a natural isomorphism

$$h_n(X, A) \cong H_n(X, A; G)$$

for all CW-pairs (X, A) and for all $n \geq 1$, where $G := h_0(*;\mathbb{Z}) \in \mathbf{Ab}$.

Proof. Since (X, A) is a good pair, we have an isomorphism

$$h_n(X, A; \mathbb{Z}) \cong \tilde{h}_n(X/A; \mathbb{Z})$$

for all $n \geq 0$. This is a formal consequence of Eilenberg-Steenrod axioms that we have verified for singular homology. Hence, we only need to check the absolute case. Just as for singular homology, we have

$$h_n^{\mathrm{CW}}(X; \mathbb{Z}) \cong h_n(X; \mathbb{Z})$$

The hypothesis that $h_n(*;\mathbb{Z}) = 0$ for $n \neq 0$ is used here. The long exact sequences of h_* homology groups for the pairs (X_n, X_{n-1}) give rise to a cellular chain complex.

$$\cdots \to h_n^{\mathrm{CW}}(X_n, X_{n-1}; \mathbb{Z}) \xrightarrow{d_n} h_{n-1}^{\mathrm{CW}}(X_{n-1}, X_{n-2}; \mathbb{Z}) \to \cdots$$

We also have

$$\cdots \to H_n^{\mathrm{CW}}(X_n, X_{n-1}; G) \xrightarrow{\partial_n} H_{n-1}^{\mathrm{CW}}(X_{n-1}, X_{n-2}; G) \to \cdots$$

The individual groups are isomorphic, since

$$h_n^{\mathrm{CW}}(X_n, X_{n-1}; \mathbb{Z}) \cong G^{\#n\text{-cells}} \cong H_n^{\mathrm{CW}}(X_n, X_{n-1}; G).$$

Thus, it remains to show that $d_n = \partial_n$ for $n \geq 1$. For $n = 1$, we can pass from X to S^2X since suspension is a natural isomorphism in any homology theory. S^2X has no 1-cells, so immediately $d_1 = 0 = \partial_1$. Now let $n > 1$. The calculation of cellular boundary maps d_n for $n > 1$ in terms of degrees of certain maps between spheres works equally well for h_*, where degree now means degree with respect to the h_* theory. But a map

$$f : \mathbb{S}^n \to \mathbb{S}^n$$

of degree m in the usual sense is simply multiplication by m on $H_n(\mathbb{S}^n; G) \cong G \cong h_n(\mathbb{S}^n; G)$. The claim follows. $\square$

Part 4. Singular Cohomology

17. Singular Cohomology

In parallel with the theory of singular homology, we develop the theory of singular cohomology. Let G be an abelian group and let X be a topological space with a singular chain complex $(C_\bullet, \partial_\bullet)$ of abelian groups:

$$\cdots \xrightarrow{\partial_{n+1}} C_n(X) \xrightarrow{\partial_n} C_{n-1}(X) \xrightarrow{\partial_{n-1}} C_{n-2}(X) \xrightarrow{\partial_{n-2}} \cdots .$$

Consider $C_n^*(X) = \mathrm{Hom}(C_n(X), G)$, the group of singular n co-chains of X with G-coefficients. This defines the dual chain complex:

$$\xleftarrow{\partial_{n+1}^*} C_n^*(X) \xleftarrow{\partial_n^*} C_{n-1}^*(X) \xleftarrow{\partial_{n-1}^*} C_{n-2}^*(X) \xleftarrow{\partial_{n-2}^*} \cdots .$$

Remark 17.1. *We write $(C^\bullet, \partial^\bullet)$ for the above diagram which is called a singular co-chain complex. We often abbreviate $(C^\bullet, \partial^\bullet)$ as $C^\bullet$. We write C^n for $C_n^* = \mathrm{Hom}(C_n, G)$. Moreover, we shall also write the boundary map ∂_{n+1}^* as δ^n for the boundary map.*

The boundary maps are $\partial_n^* : C_{n-1}^* \longrightarrow C_n^*$ defined as:

$$(\partial_n^* \psi)(\alpha) = (\psi \circ \partial_n)(\alpha) \quad \psi \in C_{n-1}^*,\ \alpha \in C_n.$$

Note that the boundary map are such that $\partial_{n+1}^* \circ \partial_n^* = 0$ for $n \in \mathbb{Z}$. Indeed,

$$(\partial_{n+1}^* \circ \partial_n^*)(\psi) = \psi(\partial_{n+1} \circ \partial_n) = 0 \qquad \psi \in C_{n-1}^*$$

We can now make the following definition:

Definition 17.2. Let G be an abelian group, and let $(C_\bullet, \partial_\bullet)$ be a chain complex of free abelian groups. The n-th cohomology group of $(C_\bullet, \partial_\bullet)$ with G-coefficients is defined as

$$H^n((C_\bullet, \partial_\bullet); G) := H_n((C^\bullet, \partial^\bullet); G)$$

Elements of $\ker\, \partial_{n+1}^*$ are called n-cocycles, and elements of $\mathrm{Im}\ \partial_n^*$ are called n-coboundaries. We shall write $Z^n(X)$ for $\ker \partial_{n+1}^* = \ker \delta^n$ and $B^n(X)$ for $\mathrm{Im}\, \partial_n^* = \ker \delta^{n-1}$.

Remark 17.3. *Recall that chain complexes of abelian groups for a category,* $\mathbf{Chain_{Ab}}$. *The dual category,* $\mathbf{Chain_{Ab}^{Op}}$, *is called the category of co-chain complexes of abelian groups. Singular co-chain complexes are elements of* $\mathbf{Chain_{Ab}^{Op}}$. *It can be checked that both* $\mathbf{Chain_{Ab}}$ *and* $\mathbf{Chain_{Ab}^{Op}}$ *are abelian categories. Thus, all results that hold for* $\mathbf{Chain_{Ab}}$, *or singular chain complexes in particular continue to hold in* $\mathbf{Chain_{Ab}^{Op}}$, *or singular co-chain co-chain complexes in particular. For instance, we have various diagram-chasing lemmas such as the five lemma, the nine lemma, and the snake lemma. We shall not repeat these details in these notes. In any case, the proofs are similar to those discussed in the context of homology.*

Proposition 17.4. *Singular cohomology with coefficients in G defines a contravariant functor* **Top** *to* **Ab**.

Proof. Recall that if If $f : X \to Y$ is a continuous map, we have induced chain maps $f_n : C_n(X) \to C_n(Y)$ satisfying $f_n \circ \partial_{n+1} = \partial_n' \circ f_{n+1}$ for each $n \geq 0$.

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1}(X) \arrow[r, "\partial_{n+1}"] \arrow[d, "f_{n+1}"] & C_n(X) \arrow[r, "\partial_n"] \arrow[d, "f_n"] & C_{n-1}(X) \arrow[r] \arrow[d, "f_{n-1}"] & \cdots \\
\cdots \arrow[r] & C_{n+1}(Y) \arrow[r, "\partial_{n+1}'"] & C_n(Y) \arrow[r, "\partial_n'"] & C_{n-1}(Y) \arrow[r] & \cdots
\end{tikzcd}$$

Apply the $\operatorname{Hom}(-, G)$ functor, we get maps such that

$$f^n : C^n(Y; G) \to C^n(X; G)$$

defined such that

$$f^n(\gamma)(\sigma) = \gamma(f_n(\sigma)) = \gamma(f \circ \sigma)$$

for $\gamma : C_n(Y) \to G$ and $\sigma : \Delta^n \to X$ a singular n-simplex in X. We claim that

$$\delta^n \circ f^n = f^{n+1} \circ \delta^{n'}$$

$$\begin{tikzcd}
\cdots & C^{n+1}(X,G) \arrow[l] & C^n(X,G) \arrow[l, "\delta^n"'] & C^{n-1}(X,G) \arrow[l, "\delta^{n-1}"'] & \cdots \arrow[l] \\
\cdots & C^{n+1}(Y,G) \arrow[l] \arrow[u, "f^{n+1}"] & C^n(Y,G) \arrow[l, "\delta^{n'}"] \arrow[u, "f^n"] & C^{n-1}(Y,G) \arrow[l, "\delta^{n-1'}"] \arrow[u, "f^{n-1}"] & \cdots \arrow[l]
\end{tikzcd}$$

Indeed, we have

$$(\delta^n \circ f^n)(\gamma) = \partial_{n+1} \circ f_n(\gamma) = \partial'_{n+1} \circ f_{n+1}(\gamma) = f^{n+1} \circ \delta^{n'}(\gamma)$$

for $\gamma : C_n(Y) \to G$. If $\gamma \in Z^n(Y)$ then we claim that $f^n(\gamma) \in Z^n(X)$. Indeed,

$$\delta^n(f^n(\sigma)) = f^{n+1}(\delta^{n'}(\sigma)) = f^{n+1}(0) = 0.$$

for $\gamma : C_n(Y) \to G$. Similarly, if $\gamma \in B^n(Y)$ then $f^n(\gamma) \in B^n(X)$. Thus f_n induces a map $H_n(f) : H_n(Y, G) \to H_n(X, G)$. One easily sees that

$$H^n(\mathrm{Id}_X) = \mathrm{Id}_{H^n(X)}$$

and that if $f : X \to Y$ and $g : Y \to Z$ then

$$H^n(g \circ f) = H^n(f) \circ H^n(g)$$

This completes the proof. □

18. Ext Functor

We now discuss the Ext (derived) functor, which arises as a derived functor of $\operatorname{Hom}(-, G)$, and plays a crucial role in the formulation of the Universal Coefficient Theorem for singular cohomology.

Remark 18.1. *We work with commutative rings below. Hence, we don't make any distinction between the categories of left R-modules and right R-modules. We use the generic phrase 'R-module' to refer to a left/right R-module.*

In the category of R-modules, recall that the $\operatorname{Hom}(X, -)$ functor defines a covariant functor from the category of R-modules to itself. If M is an R-module, then

$$\operatorname{Hom}(X, -)(M) = \operatorname{Hom}(X, M).$$

Moreover, if $f : M \to M'$ is a morphism of R-modules, then the functor acts on morphisms by

$$\operatorname{Hom}(X, -)(f) : \operatorname{Hom}(X, M) \longrightarrow \operatorname{Hom}(X, M')$$

defined. It can be checked that $\operatorname{Hom}(X, -)$ is a left exact functor. However, $\operatorname{Hom}(X, -)$ is not a right exact functor in general.

Example 18.2. The functor $\mathrm{Hom}(X, -)$ is not a right exact functor in general. Let $R = \mathbb{Z}$. Consider the short exact sequence of abelian groups:

$$0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0.$$

Apply the functor $\mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}, -)$ to this sequence. We obtain:

$$0 \longrightarrow \mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) \xrightarrow{(\cdot 2)_*} \mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) \longrightarrow \mathrm{Hom}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})$$

The resulting sequence is:

$$0 \to 0 \to 0 \to \mathbb{Z}/2\mathbb{Z},$$

which is not an exact since $\mathbb{Z}/2\mathbb{Z} \to 0$ is not a surjective function.

Definition 18.3. Let R be a ring and let X be a R-module. The i-th Ext functor is the i-th left derived functor of $\mathrm{Hom}(X, -) := h_X$. It is denoted as

$$\mathrm{Ext}^i_I(X, -)$$

Remark 18.4. *The subscript I denotes that we have taken an injective resolution.*

By definition, $\mathrm{Ext}^i_I(X, -)$ is computed as follows: for an R-module Y take any injective resolution

$$0 \to Y \to I_0 \to I_1 \to \cdots$$

and form the co-chain complex:

$$\mathrm{Hom}(X, I_0) \to \mathrm{Hom}(X, I_1) \to \cdots .$$

For each integer i, $\mathrm{Ext}^i(X, Y)$ is the homology of this co-chain complex at position i:

$$\mathrm{Ext}^i_I(X, Y) = H_i(\mathrm{Hom}(X, I_i)^\bullet)$$

Similarly, we can consider the contravariant Hom functor and consider its right derived functor. Since it is a contravariant functor, we take projective resolutions now.

Definition 18.5. Let R be a ring and let Y be a R-module. The i-th Ext functor is the i-th left derived functor of $\mathrm{Hom}(-, Y) := h^Y$. It is denoted as

$$\mathrm{Ext}^i_P(-, Y)$$

Remark 18.6. *The subscript P denotes that we have taken a projective resolution.*

By definition, $\mathrm{Ext}^i_P(-, Y)$ is computed as follows: for an R-module X take any projective resolution

$$\cdots \to P^1 \to P^0 \to X \to 0,$$

and form the co-chain complex:

$$\mathrm{Hom}(P^0, Y) \to \mathrm{Hom}(P^1, Y) \to \cdots .$$

Then $\mathrm{Ext}^i_P(X, Y)$ is the homology of this co-chain complex at position i:

$$\mathrm{Ext}^i_P(X, Y) = H_i(\mathrm{Hom}(P^i, Y)^\bullet)$$

The left exact $\mathrm{Hom}(-, -)$ functor can be thought of as a bifunctor which is covariant in the second variable and contravariant in the first variable. The discussion above seemingly provides us with with two different strategies to compute the Ext functor. Fortunately, it turns out that we can use either strategy as formalized by the following proposition.

Proposition 18.7. *(**Balancing Ext**) Let X, Y be R-modules. Then*

$$\mathrm{Ext}^i_P(X,Y) \cong \mathrm{Ext}^i_I(X,Y)$$

for each $i \geq 0$.

Proof. See [Wei94] for a proof. □

Therefore, one can work with either strategy mentioned above. Therefore, we can now unambiguously write $\mathrm{Ext}^i(X,Y)$.

Proposition 18.8. *The Ext functor satisfies the following properties:*

(1) $\mathrm{Ext}^0(X,Y) \cong \mathrm{Hom}(X,Y)$ *for all R-modules X, Y..*
(2) *If X is a projective R-module, then $\mathrm{Ext}^i(X,Y) = 0$ for all $i \geq 1$*
(3) *If Y is an injective R-module, then $\mathrm{Ext}^i(X,Y) = 0$ for all $i \geq 1$*
(4) *Any $f : X_1 \to X_2$ induces a morphism*
$$f^{*,i} : \mathrm{Ext}^i(X_2, Y) \longrightarrow \mathrm{Ext}^i(X_1, Y)$$
for each $i \geq 0$.
(5) *Any $g : Y_1 \to Y_2$ induces a morphism*
$$g^i_* : \mathrm{Ext}^i(X, Y_1) \longrightarrow \mathrm{Ext}^i(X, Y_2)$$
for each $i \geq 0$.
(6) *Any short exact sequence $0 \to Y_1 \xrightarrow{\phi} Y_2 \xrightarrow{\psi} Y_3 \to 0$ induces a long exact sequence:*
$$0 \to \mathrm{Ext}^0(X,Y_1) \to \mathrm{Ext}^0(X,Y_2) \to \mathrm{Ext}^0(X,Y_3) \to \mathrm{Ext}^1(X,Y_1) \to \mathrm{Ext}^2(X,Y_2) \to \cdots$$
(7) *Any short exact sequence $0 \to X_1 \xrightarrow{\phi} X_2 \xrightarrow{\psi} X_3 \to 0$ induces a long exact sequence:*
$$0 \to \mathrm{Ext}^0(X_3,Y) \to \mathrm{Ext}^0(X_2,Y) \to \mathrm{Ext}^0(X_1,Y) \to \mathrm{Ext}^1(X_3,Y) \to \mathrm{Ext}^2(X_2,Y) \to \cdots$$

Proof. (1), (2) and (3) all follow from general properties of derived functors (**??**). For (4) Let $P_1^\bullet$ be a projective resolution of X_1 and $P_2^\bullet$ be a projective resolution of X_2. General properties about resolutions implies that f lifts to a chain map $\varphi^\bullet : P_1^\bullet \longrightarrow P_2^\bullet$. Then, $\varphi^\bullet$ induces a morphism of chain complexes $\mathrm{Hom}(P_2^\bullet, Y) \longrightarrow \mathrm{Hom}(P_1^\bullet, Y)$ which, in turn, induces a morphism:
$$f^{*,i} : \mathrm{Ext}^i(X_2, Y) \longrightarrow \mathrm{Ext}^i(X_1, Y)$$
for each $i \geq 0$. For (5), let $P^\bullet$ be a projective resolution of X. Then, there is a morphism of chain complexes $\beta^\bullet : \mathrm{Hom}(P^\bullet, Y_1) \longrightarrow \mathrm{Hom}(P^\bullet, Y_2)$ induced by g, which, in turn, induces a morphism:
$$g^i_* : \mathrm{Ext}^i(X, Y_1) \longrightarrow \mathrm{Ext}^i(X, Y_2)$$
for each $i \geq 0$. For (6), let $P^\bullet$ be a projective resolution of X. Then there is an induced short exact sequence of chain complexes:
$$0 \to \mathrm{Hom}(P^\bullet, Y_1) \to \mathrm{Hom}(P^\bullet, Y_2) \to \mathrm{Hom}(P^\bullet, Y_3) \to 0$$
because each module P^i is projective. Indeed, at each degree i, P^i this sequence is
$$0 \to \mathrm{Hom}(P^i, Y_1) \to \mathrm{Hom}(P^i, Y_2) \to \mathrm{Hom}(P^i, Y_3) \to 0$$
obtained by applying the functor $\mathrm{Hom}(P^i, -)$, which is exact as P^i is projective. It is then easily checked that this gives a short exact sequence of chain complexes. Thus, applying the long exact sequence in homology produces the required long exact sequence. For (7), Let $P^\bullet$ be a projective resolution of X_1 and let $Q^\bullet$ be a projective resolution of X_3. By

the horseshoe lemma (**??**), there exists a projective resolution $R^\bullet$ of X_2 and a short exact sequence of chain complexes

$$0 \to P^\bullet \to R^\bullet \to Q^\bullet \to 0,$$

Since Q^i is projective, applying $\mathrm{Hom}(-, Y)$ yields

$$0 \to \mathrm{Hom}(Q^i, Y) \to \mathrm{Hom}(R^i, Y) \to \mathrm{Hom}(P^i, Y) \to 0$$

for each $i \geq 0$. It follows that there is a s.e.s. of cochain complexes

$$0 \to \mathrm{Hom}(Q^\bullet, Y) \to \mathrm{Hom}(R^\bullet, Y) \to \mathrm{Hom}(P^\bullet, Y) \to 0.$$

The associated long exact sequence in cohomology is the required long exact sequence. □

The above proposition show that the Ext groups 'measure' and 'repair' the non-exactness of the functors $\mathrm{Hom}(-, Y)$ and $\mathrm{Hom}(X, -)$. Let us now specialize to $R = \mathbb{Z}$. In what follows, let G be a fixed abelian group.

Lemma 18.9. *For any abelian group A, we have that*

$$\mathrm{Ext}^n(A, G) = 0 \text{ if } n > 1,$$

Proof. Any abelian group, A, admits a two-step free resolution.

$$0 \to F_1 \to F_0 \to A \to 0$$

Thus, $\mathrm{Ext}^n(A, G) = 0$ if $n > 1$. □

Remark 18.10. *Only* $\mathrm{Ext}^1(A, G)$ *encodes interesting information for abelian groups. We write* $\mathrm{Ext}(A, G) := \mathrm{Ext}^1(A, G)$.

Proposition 18.11. *The Ext functor satisfies the following properties:*

(1) $\mathrm{Ext}\left(\bigoplus_i A_i, G\right) \cong \prod_i \mathrm{Ext}(A_i, G)$.
(2) *If A is free, then* $\mathrm{Ext}(A, G) = 0$.
(3) $\mathrm{Ext}(\mathbb{Z}/n\mathbb{Z}, G) = G/nG$.
(4) *If H is a finitely generated abelian group, then:*

$$\mathrm{Ext}(H, G) = \mathrm{Ext}(\mathrm{Torsion}(H), G) = \mathit{Torsion}(H) \otimes_{\mathbb{Z}} G$$

(5) *For a short exact sequence: $0 \to A \to A' \to A'' \to 0$ of abelian groups, there is a natural exact sequence:*

$$0 \to \mathrm{Hom}(A'', G) \to \mathrm{Hom}(A', G) \to \mathrm{Hom}(A, G) \to \mathrm{Ext}(A'', G) \to \mathrm{Ext}(A', G) \to \mathrm{Ext}(A, G) \to 0$$

Proof. The proof is given below:

(1) This follows from the identity,

$$\mathrm{Hom}\left(\bigoplus_i A_i, G\right) = \prod_i \mathrm{Hom}(A_i, G),$$

and noting that taking direct sums of projective resolutions of A_i forms a projective resolution for $\bigoplus_i A^i$, and that homology commutes with arbitrary direct products.
(2) If A is free, then

$$0 \to A \to A \to 0$$

is a projective resolution of A, so $\mathrm{Ext}(A, G) = 0$.

(3) Consider the projective resolution of $\mathbb{Z}/n\mathbb{Z}$ given by

$$0 \to \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$$

dualize it and use the fact that $\mathrm{Hom}(\mathbb{Z}, G) \cong G$ to conclude that $\mathrm{Ext}(\mathbb{Z}/n\mathbb{Z}, G) = G/nG$.

(4) This follows at once from the previous statement.

(5) This follows from Proposition 18.8.

This completes the proof. $\square$

Remark 18.12. *The discussion above implies has dealt with the case of $\mathbb{Z}$-modules (abelian groups). The general case can be more involved. For instance, consider $\mathbb{Z}_2$ as a $\mathbb{Z}_4$-module. Let $\mathbb{Z}_4 \xrightarrow{q} \mathbb{Z}_2$ denote the quotient map. Let $\mathbb{Z}_4 \xrightarrow{\times 2} \mathbb{Z}_4$ denote multiplication by 2. $\mathbb{Z}_2$ has the following free resolution over $\mathbb{Z}_4$:*

$$\cdots \xrightarrow{\times 2} \mathbb{Z}_4 \xrightarrow{\times 2} \mathbb{Z}_4 \xrightarrow{\times 2} \mathbb{Z}_4 \xrightarrow{\times 2} \mathbb{Z}_4 \xrightarrow{q} \mathbb{Z}_2 \to 0.$$

Since $\mathrm{Hom}_{\mathbb{Z}_4}(\mathbb{Z}_4, \mathbb{Z}_2) \cong \mathbb{Z}_2$ (by mapping the generator of $\mathbb{Z}_4$ to either 0 or 1), the dual of $\times 2 : \mathbb{Z}_4 \to \mathbb{Z}_4$ is simply the zero map. Hence, we have the dual sequence

$$0 \to \mathbb{Z}_2 \to \mathbb{Z}_2 \xrightarrow{0} \mathbb{Z}_2 \xrightarrow{0} \mathbb{Z}_2 \xrightarrow{0} \mathbb{Z}_2 \to \cdots$$

Consider the truncated sequence

$$\mathbb{Z}_2 \xrightarrow{0} \mathbb{Z}_2 \xrightarrow{0} \mathbb{Z}_2 \xrightarrow{0} \mathbb{Z}_2 \to \cdots$$

The homology of this complex is $\mathbb{Z}_2$ for every degree. Hence,

$$\mathrm{Ext}^n_{\mathbb{Z}_4}(\mathbb{Z}_2, \mathbb{Z}_2) \cong \mathbb{Z}_2$$

is nonzero for all $n \in \mathbb{N}$. This is stark contrast Remark 18.10.

Remark 18.13. *The name Ext comes from the phrase extension. We say X is an extension of A by B if*

$$0 \to B \to X \to A \to 0$$

is exact. Given A and B, there is always the trivial extension $X = A \oplus B$, corresponding to the isomorphism class of the split exact sequence. It can be shown that isomorphism classes of extensions of A by B are in 1-1 correspondence with elements of $\mathrm{Ext}^1(A, B)$, with the trivial extension corresponding to 0.

19. Universal Coefficient Theorem

Recall the construction of singular cohomology in Section 17. Since everything is determined in terms of $(C_\bullet, \partial_\bullet)$, can we compute cohomology groups using information about homology groups? The answer is a qualified yes. This is the universal coefficient theorem (UCT) for cohomology, which we now discuss. We first motivate the statement of UCT. As a first guess, we might think that

$$H^n(C_\bullet; G) := H_n(C^\bullet; G) \cong \mathrm{Hom}(H_n(C_\bullet), G)$$

This turns out to be almost true. We indeed have a natural map:

$$\varphi : H^n(C_\bullet, G) \longrightarrow \mathrm{Hom}(H_n(C_\bullet), G).$$

Denote $Z_n = \ker \partial_n \subseteq C_n$ and $B_n = \mathrm{Im}\, \partial_{n+1} \subseteq C_n$. We have $B_n \subseteq Z_n$. A class in $H^n(C^\bullet; G)$ is represented by a homomorphism $\phi : C_n \to G$ such that $\partial^*_{n+1}\phi = 0$. That is, $\phi\partial_{n+1} = 0$, or in words, ϕ vanishes on B_n. The restriction $\phi_0 = \phi|Z_n$ then induces a quotient homomorphism

$$\bar{\phi}_0 : Z_n/B_n \to G,$$

an element of $\operatorname{Hom}(H_n(C_\bullet), G)$. If ϕ is in $\operatorname{Im}\partial_n^*$, say $\phi = \psi\partial_n$ for some $\psi \in C_{n-1}^*$, then ϕ is zero on Z_n since $\partial_n \circ \partial_{n+1} = 0$. So $\phi_0 = 0$ and hence also $\bar{\phi}_0 = 0$.

$$\begin{tikzcd}
\cdots \arrow[r, "\partial_{n+2}"] & C_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[dr, dashed, "\phi\circ\partial_{n+1}"'] & C_n \arrow[r, "\partial_n"] \arrow[d, "\phi"] & C_{n-1} \arrow[r, "\partial_{n-1}"] \arrow[dl, dashed, "\psi"] & \cdots \\
 & & G & &
\end{tikzcd}$$

Thus, there is a well-defined quotient map

$$h : H^n(C_\bullet, G) \to \operatorname{Hom}(H_n(C), G)$$

sending the cohomology class of $[\phi]$ to $\bar{\phi}_0$. Obviously h is a homomorphism.

Proposition 19.1. ***(Universal Coefficient Theorem)*** *If a chain complex $(C_\bullet, \partial_\bullet)$ of free abelian groups has homology groups $H_n(C_\bullet)$, then the cohomology groups $H^n(C_\bullet; G)$ of the cochain complex* $\operatorname{Hom}(C_n, G)$ *are determined by the short exact sequence:*

$$0 \longrightarrow \operatorname{Ext}(H_{n-1}(C_\bullet), G) \longrightarrow H^n(C_\bullet; G) \xrightarrow{h} \operatorname{Hom}(H_n(C_\bullet), G) \longrightarrow 0$$

Proof. We first show that h is surjective. Consider the short exact sequence:

(*)
$$\begin{tikzcd}
0 \arrow[r] & Z_n \arrow[r, hook, "i"'] & C_n \arrow[l, dashed, bend right, "p"'] \arrow[r, "\partial_n"] & B_{n-1} \arrow[r] & 0
\end{tikzcd}$$

Since B_{n-1} is a free abelian group, the short exact sequence splits. Thus there is a homomorphism $p : C_n \to Z_n$ that restricts to the identity on Z_n. That is $p \circ i = \mathrm{Id}_{Z_n}$. Composing with p gives a way of extending homomorphisms

$$\phi_0 : Z_n \to G$$

to homomorphisms

$$\phi = \phi_0 \circ p : C_n \to G$$

In particular, this extends homomorphisms $Z_n \to G$ that vanish on B_n to homomorphisms $C_n \to G$ that still vanish on B_n, or in other words, it extends homomorphisms $H_n(C_\bullet) \to G$ to elements of $\ker\ \partial_{n+1}^*$. Thus we have a homomorphism

$$\operatorname{Hom}(H_n(C_\bullet), G) \to \ker\ \partial_{n+1}^*$$

Composing this with the quotient map $\ker\ \partial_{n+1}^* \to H^n(C_\bullet; G)$ gives a homomorphism from

$$\operatorname{Hom}(H_n(C_\bullet), G) \to H^n(C_\bullet; G)$$

If we follow this map by h we get the identity map on $\operatorname{Hom}(H_n(C_\bullet), G)$ since the effect of composing with h is simply to undo the effect of extending homomorphisms via p. This shows that h is surjective. We now analyze kernel of h. We have:

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \\
0 & Z^*_{n+1} \arrow[l] \arrow[u] & C^*_{n+1} \arrow[l] \arrow[u] & B^*_n \arrow[l] \arrow[u] & 0 \arrow[l] \\
0 & Z^*_n \arrow[l] \arrow[u, "0"] & C^*_n \arrow[l] \arrow[u, "\partial^*_{n+1}"] & B^*_{n-1} \arrow[l] \arrow[u, "0"] & 0 \arrow[l] \\
0 & Z^*_{n-1} \arrow[l] \arrow[u, "0"] & C^*_{n-1} \arrow[l] \arrow[u, "\partial^*_n"] & B^*_{n-2} \arrow[l] \arrow[u, "0"] & 0, \arrow[l] \\
 & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] &
\end{tikzcd}$$

The above complex follows by applying the $\mathrm{Hom}(-, G)$ functor to the split short exact sequences in $(*)$. Since each such short exact sequence is split, the resulting chain complex has rows consisting of split short exact sequences. The associated long exact sequence of homology groups has the form:

$$\cdots \longleftarrow B^*_n \longleftarrow Z^*_n \xleftarrow{i^*_n} H^n(C_\bullet, G) \longleftarrow B^*_{n-1} \longleftarrow Z^*_{n-1} \xleftarrow{i^*_{n-1}} \cdots$$

The connecting morphism $i^*_n : Z^*_n \to B^*_n$ are in fact the dual of the inclusion map $i_n : B_n \to Z_n$[18]. One takes an element of Z^*_n, pulls it back to C^*_n, applies ∂^*_{n+1} to get an element of C^*_{n+1}, then pulls this back to B^*_n. The first of these steps extends a homomorphism $\phi_0 : Z_n \to G$ to $\phi : C_n \to G$, the second step composes this ϕ with ∂_{n+1} to yield a map $C_{n+1} \to G$ and the third step undoes this composition and restricts ϕ to B_n[19]. The net effect is just to restrict ϕ_0 from Z_n to B_n. A long exact sequence can always be broken up into short exact sequences, yielding:

$$0 \longrightarrow \text{coker } i^*_{n-1} \longrightarrow H^n(C_\bullet; G) \longrightarrow \ker i^*_{n-1} \longrightarrow 0$$

ker i^*_{n-1} can be identified with $\mathrm{Hom}(H_n(C_\bullet), G)$. This is because elements of ker i^*_n are homomorphisms $Z_n \to G$ that vanish on the subgroup B_n, and such homomorphisms are the same as homomorphisms $Z_n/B_n \to G$. Under this identification of ker i^*_n with $\mathrm{Hom}(H_n(C_\bullet), G)$, the map $H_n(C_\bullet; G) \to$ ker i^*_n becomes the map h considered earlier. Hence:

$$0 \longrightarrow \text{coker } i^*_{n-1} \longrightarrow H^n(C_\bullet; G) \xrightarrow{h} \mathrm{Hom}(H_n(C_\bullet), G) \longrightarrow 0$$

What about coker i^*_{n-1}? Consider the short exact sequence:

$$0 \longrightarrow B_{n-1} \xrightarrow{i_{n-1}} Z_{n-1} \longrightarrow H_{n-1}(C_\bullet) \longrightarrow 0$$

The exact sequence above is a free resolution of $H_{n-1}(C_\bullet)$. There is an associated natural exact sequence:

$$0 \to H^*_{n-1}(C_\bullet) \to Z^*_{n-1} \to B^*_{n-1} \to \mathrm{Ext}(H_{n-1}(C_\bullet), G) \to \mathrm{Ext}(Z_{n-1}, G) \to \mathrm{Ext}(B_{n-1}, G) \to 0$$

[18] Hence the use of the suggestive notation.

[19] Go back to the short exact sequence analyzed to prove surjectivity to see why the third step has this effect.

We see that coker i_{n-1}^* is the first cohomology group of the free resolution of $H_{n-1}(C_\bullet)$. This group is $\mathrm{Ext}(H_{n-1}(C_\bullet, G), G)$, proving the claim. □

Remark 19.2. *We have*

$$H^n(X;G) = \mathrm{Hom}_{\mathbb{Z}}(H_n(X;\mathbb{Z}), G) \oplus \mathrm{Ext}(H_{n-1}(X;\mathbb{Z}), G).$$

This is because in the course of the proof of Proposition 19.1, we constructed a morphism

$$\mathrm{Hom}(H_n(C_\bullet), G) \to H^n(C_\bullet; G)$$

that ensures that the sequence splits.

Corollary 19.3. *Let $(C_\bullet, \partial_\bullet)$ be a chain complex so that its $\mathbb{Z}$-homology groups are finitely generated. Let $T_n = Torsion(H_n)$. We have*

$$0 \to T_{n-1} \to H^n(C_\bullet; \mathbb{Z}) \to H_n/T_n \to 0$$

This sequence splits[20]*, so:*

$$H^n(C^\bullet; \mathbb{Z}) \cong T_{n-1} \oplus H_n/T_n.$$

Proof. Clear. □

Let us now derive some immediate consequences of the UCT:

(1) If $n = 0$, we have

$$H^0(X;G) = \mathrm{Hom}_{\mathbb{Z}}(H_0(X), G) \cong \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^{\#\text{path components}}, G)$$

(2) If $n = 1$, the Ext-term vanishes since $H_0(X)$ is free, so we get:

$$H^1(X;G) = \mathrm{Hom}_{\mathbb{Z}}(H_1(X), G)$$

Remark 19.4. *There is also a universal coefficient theorem for cohomology where $\mathbb{Z}$ is replaced by a PID, R and G is a R-module. In this case, we have*

$$0 \to \mathrm{Ext}^1_R(H_{n-1}(X;R), G) \to H^n(X;G) \xrightarrow{h} \mathrm{Hom}_R(H_n(X;R), G) \to 0.$$

This comes from first establishing that Ext^n_R vanishes for $n \geq 2$ for when R is a PID, and then going through a proof for universal coefficient theorem as above.

20. Eilenberg-Steenrod Axioms

We have defined singular cohomology. There are many other cohomology theories: sheaf cohomology, Cech cohomology, etc. All these cohomology theories satisfy the Eilenberg-Steenrod axioms. The purpose of this section is to state these axioms and prove that singular cohomology satisfies these axioms.

Definition 20.1. (**Eilenberg-Steenrod Axioms**) Let G be an abelian group. A (unreduced) cohomology theory consists of

(1) A family of functors $H^n : \mathbf{Top}^2 \to \mathbf{Ab}$ for $n \geq 0$, and
(2) A family of natural transformations $\gamma^n : H^n \to H^{n+1} \circ p$, where p is the functor sending (X, A) to $(A, \varnothing)$ and $f : (X, A) \to (Y, B)$ to $f|_B : (A, \varnothing) \to (B, \varnothing)$.

such that the following axioms are satisfied:

[20] Since H_n/T_n is free and hence projective.

(a) (Homotopy invariance) If $f, g : (X, A) \to (Y, B)$ are homotopic maps, then the induced maps

$$H^n(f), H^n(g) : H^n(X, A) \to H^n(Y, B)$$

are such that $H^n(f) = H^n(g)$ for $n \geq 0$. In other words, H^n may be regarded as a functor from **hTop** to **Ab**.

(b) (Long exact sequence) For every pair (X, A), the inclusions

$$(A, \emptyset) \overset{i}{\hookrightarrow} (X, \emptyset) \overset{j}{\hookrightarrow} (X, A)$$

give rise to a long exact sequence

$$\cdots \longrightarrow H^n(X, A) \xrightarrow{j_n^*} H^n(X) \xrightarrow{i_n^*} H^n(A) \xrightarrow{\gamma^n} H^{n+1}(X, A) \xrightarrow{j_{n+1}^*} H^{n+1}(X) \xrightarrow{i_{n+1}^*} H^{n+1}(A) \longrightarrow \cdots$$

(c) (Excision) If $Z \subseteq A \subseteq X$ are topological spaces such that $\overline{Z} \subseteq \mathrm{Int}(A)$, the inclusion of pairs $(X \setminus Z, A \setminus Z) \subseteq (X, A)$ induces isomorphisms

$$H^n(X \setminus Z, A \setminus Z) \to H^n(X, A)$$

for all $n \geq 0$.

(d) (Multiplicativity) If $X = \coprod_\alpha X_\alpha$ and $A = \coprod_\alpha A_\alpha$ is the disjoint union of a family of topological spaces X_α, then

$$H^n(X, A) = \prod_\alpha H^n(X_\alpha, A_\alpha)$$

for each $n \geq 0$.

Additionally, if a cohomology theory satisfies the following additional axiom

(e) (Dimension Axiom) For any one-point set $X = \{\bullet\}$,

$$H^n(X) = \begin{cases} G & \text{if } n = 0 \\ 0 & \text{otherwise,} \end{cases}$$

the the cohomology theory is called an ordinary cohomology theory.

The purpose of the remainder of this section is to show that singular cohomology satisfies the Eilenberg-Steenrod axioms.

20.1. **Relative cohomology groups.** We first contruct the relative cohomology group that shall allow us to construct the appropriate functors from **Top** to **Ab**. We apply the $\mathrm{Hom}(-, G)$ functor to the the relative singular chain complex to get

$$C^n(X, A; G) := \mathrm{Hom}(C_n(X, A), G).$$

The group $C^n(X, A; G)$ can be identified with functions from the set of n-simplices in X to G that vanish on simplices in A, so we have a natural inclusion

$$C^n(X, A; G) \hookrightarrow C^n(X; G)$$

The relative coboundary maps

$$\overline{\delta}^n : C^n(X, A; G) \to C^{n+1}(X, A; G)$$

are obtained by restricting δ^n. We have a co-chain complex $(C^\bullet(X, A), \overline{\delta}^\bullet)$.

Definition 20.2. Let $A \subseteq X$ be a subspace of a topological space X. The n-th relative cohomology group, $H^n(X, A)$, is the n-th homology group of the chain complex $(C^\bullet(X, A), \overline{\delta}^\bullet)$. That is:

$$H^n(X, A) = \frac{\text{Ker } \overline{\delta}^n}{\text{Im } \overline{\delta}^{n+1}}$$

Similar to Proposition 17.4, it is easily checked that each H^n is a functor from $\mathbf{Top}^2$ to $\mathbf{Ab}$. This effectively checkes the first two conditons in the definition of the Eilenberg-Steenrod axioms.

Remark 20.3. *Since the cohomology of the empty set is trivial for all $n \geq 0$, we have:*

$$H^n(X, \emptyset) = H^n(X), \quad \forall n \geq 0.$$

Remark 20.4. *Universal coefficient theorem continues to hold true for relative cohomology. The proof is identical as the one given before.*

We now prove that singular cohomology satisfies the long exact sequence axiom. The importance of the long exact sequence axiom is that is allows us to compute cohomology groups of various spaces in using an 'inductive' and/or 'bottom-up' approach. Applying by the $\text{Hom}(-, G)$ functor to the short exact sequence,

$$0 \to C_n(A) \xrightarrow{i_n} C_n(X) \xrightarrow{j_n} C_n(X, A) \to 0,$$

we get another short exact sequence[21]

$$0 \leftarrow C^n(A; G) \xleftarrow{i^*} C^n(X; G) \xleftarrow{j^*} C^n(X, A; G) \leftarrow 0.$$

Since i_n and j_n commute with the boundary maps, it follows that i_n^* and j_n^* commute with co-boundary maps. So we obtain a short exact sequence of cochain complexes:

$$0 \leftarrow C^\bullet(A; G) \xleftarrow{i^*} C^\bullet(X; G) \xleftarrow{j^*} C^\bullet(X, A; G) \leftarrow 0.$$

By taking the associated long exact sequence of homology groups, we get the long exact sequence for the cohomology groups of the pair (X, A):

$$\cdots \longrightarrow H^n(X, A; G) \xrightarrow{j_n^*} H^n(X; G) \xrightarrow{i_n^*} H^n(A; G) \xrightarrow{\gamma^n} H^{n+1}(X, A; G) \xrightarrow{j_{n+1}^*} H^{n+1}(X; G) \xrightarrow{i_{n+1}^*} \cdots$$

This shows that the long exact sequence axiom is satified.

20.2. **Homotopy Invariance.** We now show that singular cohomology satisfies the homotopy invariance property.

Proposition 20.5. *(**Homotopy Invariance**) Let X, Y be topological spaces, and let G be an abelian group. If $f \simeq g : X \to Y$ are homotopic maps, then*

$$H^n(f) = H^n(g) : H^n(Y, G) \to H^n(X, G).$$

Proof. Recall from the proof of the similar statement for homology that a chain homotopy between $C_\bullet(X, A; G)$ and $C_\bullet(Y, B; G)$ is given by a prism operator

$$T_n : C_n(X, A; G) \to C_{n+1}(Y, B; G)$$

satisfying

$$f_n - g_n = T_{n-1} \circ \partial_n + \partial'_{n+1} \circ T_n$$

[21] $\text{Hom}(-, G)$ is only a left exact functor in general. But it can be checked in this case that the resulting sequence is both left exact and right exact.

with f_n and g_n being the induced maps on singular chain complexes. The claim about cohomology follows by applying the $\mathrm{Hom}(-.G)$ functor to the prism operator to get

$$T^n : C^{n+1}(Y, B; G) \to C^n(X, A; G)$$

which satisfies

$$f^n - g^n = \partial_n^* \circ T^{n-1} + T^n \circ \partial_{n+1}^{*'}.$$

Hence, we have a chain homotopy between $C^\bullet(X, A; G)$ and $C^\bullet(Y, B; G)$. It is now a standard fact that a chain homotopy induces the same maps on homology groups. Hence,

$$H^n(f) = H^n(g)$$

for each $n \geq 0$. $\square$

Corollary 20.6. *If X is contractible, then $H^n(X) = 0$ for all $n \geq 1$.*

Proof. Immediate from the homotopy invariance of singular cohomology and that $H^n(\{*\}) = 0$ for $n \geq 1$. $\square$

20.3. **Excision.** We now prove that singular cohomology satisfies the excision axiom. The important of the excision axiom is that if $A \subseteq X$ and n-cochains are "sufficiently inside" of A, we can cut A out without affecting the relative cohomology groups $H^n(X, A)$. Here is the formal statement we'd like to prove in this section:

Proposition 20.7. *(**Excision**) Given a topological space X, suppose that $Z \subset A \subset X$, with $\overline{Z} \subseteq int(A)$. Then the inclusion of pairs $i : (X \setminus Z, A \setminus Z) \hookrightarrow (X, A)$ induces isomorphisms:*

$$i^n : H^n(X, A; G) \to H^n(X \setminus Z, A \setminus Z; G)$$

for all n. Equivalently, if A and B are subsets of X with $X = int(A) \cup int(B)$, then the inclusion map $(B, A \cap B) \hookrightarrow (X, A)$ induces isomorphisms in cohomology.

Proof. Excision for singular homology implies that left and right maps in the diagram below are isomorphisms.

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ext}(H_{n-1}(X, A), G) \arrow[r] & H^n(X, A; G) \arrow[r] & \mathrm{Hom}(H_n(X, A), G) \arrow[r] & 0 \\
0 \arrow[r] & \mathrm{Ext}(H_{n-1}(X \setminus Z, A \setminus Z), G) \arrow[r] \arrow[u] & H^n(X \setminus Z, A \setminus Z; G) \arrow[r] \arrow[u] & \mathrm{Hom}(H_n(X \setminus Z, A \setminus Z), G) \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

The five-lemma then implies that the middle map is also an isomorphism. This completes the proof. $\square$

20.4. **Dimension Axiom.** Let $X = \{*\}$ be a single point space. By Proposition 19.1, we have:

$$H^n(X; G) = \mathrm{Hom}(H_n(X), G) \oplus \mathrm{Ext}(H_{n-1}(X), G).$$

Since

$$H_n(X) \cong \begin{cases} \mathbb{Z}, & \text{if } n = 0 \\ 0, & \text{otherwise} \end{cases}$$

we get

$$\mathrm{Hom}(H_n(X), G) \cong \begin{cases} G, & \text{if } n = 0 \\ 0, & \text{otherwise} \end{cases}.$$

Furthermore, since $H_n(X)$ is free for all n, we also have that $\operatorname{Ext}(H_{n-1}(X), G) = 0$, for all n. Therefore,

$$H^n(X;G) = \begin{cases} G, & \text{if } n = 0 \\ 0, & \text{otherwise} \end{cases}.$$

20.5. **Multiplicativity Axiom.** The multiplicativityaxiom is easily seen to hold using the universal coefficient theorem in relative cohomology. Let $X = \coprod_\alpha X_\alpha$ and $A = \coprod_\alpha A_\alpha$. We have:

$$\begin{aligned}
H^n(X, A; G) &= \operatorname{Ext}(H_{n-1}(X, A); G) \oplus \operatorname{Hom}(H_n(X, A); G) \\
&= \operatorname{Ext}(H_{n-1}(\coprod_\alpha X_\alpha, \coprod_\alpha A_\alpha); G) \oplus \operatorname{Hom}(H_n(\coprod_\alpha X_\alpha, \coprod_\alpha A_\alpha); G) \\
&= \operatorname{Ext}(\bigoplus_\alpha H_{n-1}(X_\alpha, A_\alpha); G) \oplus \operatorname{Hom}(\bigoplus_\alpha H_n(X_\alpha, A_\alpha); G) \\
&= \prod_\alpha \operatorname{Ext}(H_{n-1}(X_\alpha, A_\alpha); G) \oplus \prod_\alpha \operatorname{Hom}(H_n(X_\alpha, A_\alpha); G) \\
&= \prod_\alpha \operatorname{Ext}(H_{n-1}(X_\alpha, A_\alpha); G) \oplus \operatorname{Hom}(H_n(X_\alpha, A_\alpha); G) = \prod_\alpha H^n(X_\alpha, A_\alpha; G)
\end{aligned}$$

Hence, we see that singular cohomology satisfies the Eilenberg-Steenrod axioms.

Remark 20.8. *The Mayer-Viertoris sequence is a formal consequence of the Eilenberg-Steenrod axioms. Therefore, we have that Mayer-Viertoris holds for singular cohomology: if X be a topological space, and A and B are open subsets of X such that $X = int(A) \cup int(B)$, then there is a long exact sequence of cohomology groups:*

$$\cdots \to H^n(X;G) \xrightarrow{\psi} H^n(A;G) \oplus H^n(B;G) \xrightarrow{\phi} H^n(A \cap B;G) \to \cdots$$

We also have a Mayer-Vietoris sequence in relative cohomology groups.

Remark 20.9. *We can also define reduced cohomology. Consider the augmented singular chain complex for X:*

$$\cdots \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

where $\varepsilon(\sum_i n_i x_i) = \sum_i n_i$. After applying the $\operatorname{Hom}(-, G)$ *functor, we get the augmented co-chain complex:*

$$\xleftarrow{\partial_3^*} C_2^* \xleftarrow{\partial_2^*} C_1^* \xleftarrow{\partial_1^*} C_0^* \xleftarrow{\varepsilon^*} \mathbb{Z} \leftarrow 0$$

Note that since $\epsilon \circ \partial_1 = 0$, we get by by applying the $\operatorname{Hom}(-, G)$ *functor that $\partial_1^* \circ \epsilon^* = 0$. The cohomology of this augmented cochain complex is the* reduced cohomology *of X with G-coefficients, denoted by $\widetilde{H}^n(X;G)$. It follows by definition that*

$$\widetilde{H}^n(X;G) = H^n(X;G) \quad n > 0$$

and by the universal coefficient theorem (applied to the augmented chain complex), we get

$$\widetilde{H}^0(X;G) = \operatorname{Hom}(\widetilde{H}^0(X), G).$$

Remark 20.10. *If (X, A) is a good pair, then the long exact sequence in reduced cohomology holds true. This is because the analogous result is a formal consequence of the Eilenberg-Steenrod axioms.*

$$\cdots \to H^n(X, A; G) \to \widetilde{H}^n(X; G) \to \widetilde{H}^n(A; G) \to H^{n+1}(X, A; G) \to \cdots$$

In particular, if $A = \{*\}$ is a point in X, we get that

$$\widetilde{H}^n(X; G) \cong H^n(X, x_0; G)$$

for $n \geq 1$. Moreover, we have

$$H^n(X, A; G) \cong H^n(X/A; G)$$

for all $n \in \mathbb{N}$. The proof is the same as in the homology case since it is a formal consequence of the Eilenberg-Steenrod axioms and the hypothesis on the space. Also, if each X_α is path-connected, we have

$$\widetilde{H}^n(\bigvee_\alpha X_\alpha) = \prod_\alpha \widetilde{H}^n(X_\alpha)$$

for $n \geq 0$. Once again, the proof is similar to the proof in the case of singular homology. We also have a Mayer-Vietoris sequence in reduced cohomology.

Remark 20.11. *We can define simplicial cohomology and cellular cohomology in exactly the same way as expected. As expexted, simplicial cohomology and cellular cohomology are isomorphic to singular cohomology.*

21. Examples

The purpose of this section is to compute cohomology groups of some topological spaces. We begin by looking at some specific examples.

Example 21.1. (**Contractible Spaces**) Let X be a contractible topological space. We have:

$$H^n(X; G) = \begin{cases} G, & \text{if } n = 0 \\ 0, & \text{otherwise} \end{cases}.$$

This follows immediately by the homotopy invariance of cohomology groups since X homotopy equivalent to a point.

Example 21.2. (**Spheres**) Let $X = \mathbb{S}^n$. Then we have

$$H_k(\mathbb{S}^n\ \mathbb{Z}) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z}, & \text{if } k = n = 0 \\ \mathbb{Z}, & \text{if } k = n > 0,\ k = 0, n > 0 \\ 0, & \text{otherwise} \end{cases}$$

Since, $\mathbb{Z}$ and $\mathbb{Z} \oplus \mathbb{Z}$ are free-abelian groups, the Ext term in the UCT for cohomology vanishes for each k. Hence,

$$H^k(\mathbb{S}^n, G) \cong \text{Hom}(H_k(\mathbb{S}^n, G), \mathbb{Z}) \cong \begin{cases} G \oplus G, & \text{if } k = n = 0 \\ G, & \text{if } k = n > 0,\ k = 0, n > 0 \\ 0, & \text{otherwise} \end{cases}.$$

for each $k \geq 0$.

Remark 21.3. *We can also compute the cohomology groups of $\mathbb{S}^n$ by using the above Mayer-Vietoris sequence. Cover $\mathbb{S}^n$ by two open sets $A = \mathbb{S}^n \setminus \{N\}$ and $B = \mathbb{S}^n \setminus \{S\}$, where N and S are the North and South poles of S^n. Then we have*

$$A \cap B \simeq \mathbb{S}^{n-1} \qquad A \simeq B \simeq \mathbb{R}^n$$

Thus, by the Mayer-Vietoris sequence for reduced cohomology, homotopy invariance, and induction, we get:

$$\widetilde{H}^k(\mathbb{S}^n; G) \cong \widetilde{H}_{k-n}(\mathbb{S}^0; G) \cong \begin{cases} G, & \text{if } k = n \\ 0, & \text{otherwise} \end{cases}$$

for each $k \geq 0$.

Example 21.4. (**Mobius Band**) Let M denote the Mobius band. Since M is homotopic to $\mathbb{S}^1$. we have,

$$H^k(M, G) \cong \begin{cases} G, & \text{if } k = 0, 1 \\ 0, & \text{otherwise} \end{cases}.$$

for each $k \geq 0$.

Example 21.5. (**Torus**) Let $X = \mathbb{S}^1 \times \mathbb{S}^1$. Recall that we have,

$$H_n(\mathbb{S}^1 \times \mathbb{S}^1, \mathbb{Z}) \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & \text{for } n = 1 \\ \mathbb{Z} & \text{for } n = 0, 2 \\ 0 & \text{for } n \geq 3 \end{cases}$$

Since, $\mathbb{Z}$ and $\mathbb{Z} \oplus \mathbb{Z}$ are free-abelian groups, the Ext term in the UCT for cohomology vanishes for each k. Hence,

$$H^n(\mathbb{S}^1 \times \mathbb{S}^1, G) \cong \mathrm{Hom}_{\mathbb{Z}}(H_n(\mathbb{S}^1 \times \mathbb{S}^1, \mathbb{Z}), G) \cong \begin{cases} G \oplus G & \text{for } n = 1 \\ G & \text{for } n = 0, 2 \\ 0 & \text{for } n \geq 3 \end{cases}$$

Example 21.6. (**Klein Bottle**) Let $X = K$ be the Klein bottle. Recall that we have,

$$H_n(K\ \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{for } n = 0 \\ \mathbb{Z} \oplus \mathbb{Z}_2 & \text{for } n = 1 \\ 0 & \text{for } n \geq 2 \end{cases}$$

Note that we have,

$$\begin{aligned} &\mathrm{Ext}(H_0(K, \mathbb{Z}), G) = 0, \\ &\mathrm{Ext}(H_1(K, \mathbb{Z}), G) \cong \mathrm{Ext}(\mathbb{Z}_2, G) \cong G/2G \end{aligned}$$

Therefore, we have

$$\begin{aligned} H^n(K, G) &\cong \mathrm{Hom}_{\mathbb{Z}}\left(H_n(K, \mathbb{Z}), G\right) \oplus \mathrm{Ext}\left(H_{n-1}(K, \mathbb{Z}), G\right) \\ &\cong \begin{cases} G, & \text{for } n = 0, \\ G \oplus G/2G, & \text{for } n = 1, \\ G/2G, & \text{for } n = 2, \\ 0, & \text{for } n \geq 3. \end{cases} \end{aligned}$$

The case $G = \mathbb{Z}_2$ is important. Then,

$$H^n(K\ \mathbb{Z}_2) \cong \begin{cases} \mathbb{Z}_2 & \text{for } n = 0, 2, \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{for } n = 1 \\ 0 & \text{for } n \geq 3 \end{cases}$$

Example 21.7. (**Real Projective Space**) Let $X = \mathbb{RP}^n$. Recall that we have,

$$H_k(\mathbb{RP}^n, \mathbb{Z}) = \begin{cases} \mathbb{Z}_2 & \text{if } k \text{ is odd, } 0 < k < n, \\ \mathbb{Z} & \text{if } k = 0, n \text{ is odd,} \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_2, G) \cong G_2 = \{g \in G \mid 2g = 0\}$ and $\mathrm{Ext}(\mathbb{Z}_2, G) \cong G/2G$. If n is odd, we have:

$$H^k(\mathbb{RP}^n, G) \cong \begin{cases} G & \text{if } k = 0, n, \\ G/2G & \text{if } k \text{ is even, } 0 < k < n, \\ G_2 & \text{if } k \text{ is odd, } 0 < k < n, \\ 0 & \text{otherwise.} \end{cases}$$

If n is even, we have:

$$H^k(\mathbb{RP}^n, G) \cong \begin{cases} G & \text{if } k = 0, \\ G/2G & \text{if } k \text{ is even, } 0 < k \leq n, \\ G_2 & \text{if } k \text{ is odd, } 0 < k < n, \\ 0 & \text{otherwise.} \end{cases}$$

Remark 21.8. *The case $G = \mathbb{Z}_2$ in Example 21.7 is important. We have*

$$H^k(\mathbb{RP}^n\ \mathbb{Z}_2) \cong \begin{cases} \mathbb{Z}_2 & k = 0, \cdots, n, \\ 0 & \textit{otherwise.} \end{cases}$$

Example 21.9. (**Complex Projective Space**) Let $X = \mathbb{CP}^n$. Recall that we have,

$$H_k(\mathbb{CP}^n\ \mathbb{Z}) = \begin{cases} \mathbb{Z}, & \text{if } k = 0, 2, 4, \ldots, 2n \\ 0, & \text{otherwise.} \end{cases}$$

Since $\mathbb{Z}$ is a free abelian group, all the Ext terms in the UCT for cohomology vanish. Hence,

$$H^k(\mathbb{CP}^n, G) \cong \mathrm{Hom}_{\mathbb{Z}}(H_k(\mathbb{CP}^n, \mathbb{Z}), G) \cong \begin{cases} G, & \text{if } k = 0, 2, 4, \ldots, 2n \\ 0, & \text{otherwise.} \end{cases}$$

Part 5. de-Rham Cohomology

This section assumes knowledge of smooth manifolds theory.

22. de-Rham Cohomology

22.1. **Definitions.** Singular cohomology is quite abstract and somewhat useless unless we develop algebraic computational tools to compute singular cohomology. We now discuss de Rham cohomology for smooth manifolds. This cohomology theory is quite geometric because it is phrased in terms of differential forms on smooth manifolds. Let M be a smooth manifold. Since

$$d^2 = d \circ d : \Omega^{k-1}(M) \xrightarrow{d} \Omega^k(M) \xrightarrow{d} \Omega^{k+1}(M)$$

is the zero operator for every $k \geq 1$, we have

$$\operatorname{im}\left(d : \Omega^{k-1}(M) \to \Omega^k(M)\right) \subseteq \ker\left(d : \Omega^k(M) \to \Omega^{k+1}(M)\right).$$

Thus, $\operatorname{im} d$ is a subspace of $\ker d$ for all $k \geq 1$.

Remark 22.1. *Let M be a smooth n-dimensional manifold. For convenience, we set $\Omega^k(M) = \{0\}$ for all $k < 0$ and $k > n$. Moreover, we set*

$$d = 0 : \Omega^k(M) \to \Omega^{k+1}(M)$$

for all $k < 0$ and $k \geq n$. Then the inclusion above holds for all $k \in \mathbb{Z}$.

Definition 22.2. Let M be a smooth manifold. The quotient vector space

$$H^k_{\mathrm{dR}}(M) = \frac{\ker(d : \Omega^k(M) \to \Omega^{k+1}(M))}{\operatorname{im}(d : \Omega^{k-1}(M) \to \Omega^k(M))} = \frac{\{\omega \in \Omega^k(M) : d\omega = 0\}}{\{d\omega : \omega \in \Omega^{k-1}(M)\}}$$

is the k-th de Rham cohomology group of M.

Let M be a smooth manifold. A form $\omega \in \Omega^k(M)$ is closed if $d\omega = 0$ and exact if there exists a $(k-1)$-form $\tau \in \Omega^{k-1}(M)$ for which $d\tau = \omega$. Since $d \circ d = 0$, every exact form is closed. hus,

$$H^k_{\mathrm{dR}}(M) = \frac{\{\text{closed } k\text{-forms in } M\}}{\{\text{exact } k\text{-forms in } M\}}$$

This suggests that Definition 22.2 measures the failure of closed forms to be exact forms. Indeed, every closed form need not be exact:

Example 22.3. Consider the 1-form on $\mathbb{R}^2 \setminus \{0\}$ defined by:

$$\omega = \frac{x\,dy - y\,dx}{x^2 + y^2}$$

Then,

$$\begin{aligned} d\omega &= \frac{(dx \wedge dy - dy \wedge dx)(x^2 + y^2) - (2x\,dx + 2y\,dy)(x\,dy - y\,dx)}{(x^2 + y^2)^2} \\ &= \frac{2(x^2 + y^2)\,dx \wedge dy - (2x^2\,dx \wedge dy - 2y^2\,dy \wedge dx)}{(x^2 + y^2)^2} = 0 \end{aligned}$$

So, ω is closed. But writing ω in polar coordinates and integrating around a circle centered at 0 in $\mathbb{R}^2 \setminus \{0\}$ gives

$$\int_{\mathbb{S}^1} \omega = 2\pi.$$

If $\omega = d\eta$ were exact, Stokes' theorem would imply

$$0 = \int_{\emptyset} \eta = \int_{\partial \mathbb{S}^1} \eta = \int_{\mathbb{S}^1} d\eta = \int_{\mathbb{S}^1} \omega = 2\pi.$$

Hence, ω is not exact.

Remark 22.4. *Elements $H^k(M)$ are equivalence classes of k-forms. Given $\omega \in \ker(d : \Omega^k(M) \to \Omega^{k+1}(M))$, we denote the equivalence class by*

$$[\omega] = \{\omega + d\tau \in \Omega^k(M) : \tau \in \Omega^{k-1}(M)\}.$$

Therefore, $H^k_{dR}(M)$ is a vector space that classifies the closed k-forms in M up to exact forms.

22.2. **Properties of de Rham cohomology.** We now discuss several algebraic properties of de Rham cohomology, which are analogous to the properties of singular cohomology for general topological spaces. We first show that the de Rham cohomology defines a contravariant functor from the category of smooth manifolds, **Man**, to the category of abelian groups, **Ab**.

Proposition 22.5. *Let M, N be smooth manifolds and let $F : M \to N$ be a smooth map. For each $k \in \mathbb{Z}$, let $F^* : \Omega^k(N) \to \Omega^k(M)$ be the pullback map.*

(1) *For each $k \in \mathbb{Z}$, F^* descends to a linear map $F^\# : H^k_{dR}(N) \to H^k_{dR}(M)$ between the de Rham cohomology groups given by $F^\#[\omega] = [F^*\omega]$.*
(2) *(**Functoriality**) For each $k \in \mathbb{Z}$, $H^k_{dR} : \mathbf{Man} \to \mathbf{Ab}$ is a contravariant functor.*

Proof. We shall use the fact that the exterior derivative commutes with pullbacks. The proof is given below:

(1) Let ω is a closed form. Then

$$d(F^*\omega) = F^*(d\omega) = 0$$

Hence, $F^*\omega$ is also closed a form. This shows that $F^*\omega$ restricts to a map

$$F^* : \{\text{closed } k - \text{forms on } N\} \to \{\text{closed } k - \text{forms on } M\}$$

Now let $\omega = d\eta$ be an exact form. Then

$$F^*\omega = F^*(d\eta) = d(F^*\eta),$$

Hence, $F^*\omega$ is also an exact form. This shows that F^* descends to a well-defined map map

$$F^\# : H^k_{\mathrm{dR}}(N) \to H^k_{\mathrm{dR}}(M)$$

given by $F^\#[\omega] = [F^*\omega]$.
(2) This follows from (1).

This completes the proof. $\square$

Proposition 22.6. *(**de Rham Cohomolgy of Disjoint Unions**) Let $\{M_j\}_{j\in J}$ be a countable collection of smooth n-dimensional manifolds. Let $M = \bigsqcup_{j \in J} M_j$. For each $k \in \mathbb{Z}$, the inclusion maps $i_j : M_j \hookrightarrow M$ induce an isomorphism*

$$H^k_{dR}(M) \cong \prod_{j\in J} H^k_{dR}(M_j)$$

Proof. The pullback maps $i_j^*: H^k(M) \to H^k(M_j)$ induce an isomorphism from

$$i : H^k(M) \to \prod_{j\in J} H^k(M_j), \qquad i(\omega) \mapsto \left(i_j^*(\omega)\right)_{j\in J} = \left(\omega|_{M_j}\right)_{j\in J}$$

This map is injective because any smooth k-form whose restriction to each M_j is zero must itself be zero, and it is surjective because giving an arbitrary smooth k-form on each M_j defines one on M. $\square$

We now discuss the homotopy invariance of de Rham cohomology, allowing us to show that de Rham cohomology is a topological invariant. If M and N are smooth manifolds, and $F, G : M \to N$ are smooth maps, we shall show homotopy invariance by constructing a co-chain homotopy between $F^\#$ and $G^\#$ which are given by linear maps

$$h_k : \Omega^k(N) \to \Omega^{k-1}(M)$$

for each $k \in \mathbb{Z}$ such that

$$F^\#(\omega) - G^\#(\omega) = d(h_k\omega) - h_{k+1}(d\omega)$$

for each $\omega \in \Omega^k(N)$ and $k \in \mathbb{Z}$.

$$\begin{tikzcd}
\cdots \arrow[r, "d"] & \Omega^{k-1}(N) \arrow[r, "d"] \arrow[d, "F^\# - G^\#"] & \Omega^k(N) \arrow[r, "d"] \arrow[d, "F^\# - G^\#"] \arrow[dl, "h_k"'] & \Omega^{k+1}(N) \arrow[r, "d"] \arrow[d, "F^\# - G^\#"] \arrow[dl, "h_{k+1}"'] & \cdots \\
\cdots \arrow[r, "d"] & \Omega^{k-1}(M) \arrow[r, "d"] & \Omega^k(M) \arrow[r, "d"] & \Omega^{k+1}(M) \arrow[r, "d"] & \cdots
\end{tikzcd}$$

The key to our proof of homotopy invariance is to construct a homotopy operator first in the following special case. Let M be a smooth manifold, and for each $t \in I$, let

$$i_t : M \to M \times I$$

be the map $i_t(x) = (x, t)$. We first construct a co-chain homotopy between $i_0^\#$ and $i_1^\#$.

Lemma 22.7. *Let M be a smooth n-dimensional manifold. There exists a co-chain homotopy between the two maps $i_0^\#$ and $i_1^\#$.*

Proof. For each $k \in \mathbb{Z}$, we need to define a linear map

$$h_k : \Omega^k(M \times I) \to \Omega^{k-1}(M)$$

such that

$$(*) \qquad h_{k+1}(d\omega) + d(h_k\omega) = i_1^\#(\omega) - i_0^\#(\omega)$$

for each $\omega \in \Omega^k(M \times I)$. Let S be the vector field on $M \times \mathbb{R}$ given by $S(p, s) = (0, \frac{\partial}{\partial s}\big|_s)$. Given $\omega \in \Omega^k(M \times I)$, define $h_k(\omega) \in \Omega^{k-1}(M)$ by

$$h_k(\omega) = \int_0^1 i_t^\#(S \lrcorner \omega)\, dt.$$

We shall verify the formula in $(*)$ in local coordinates. For $p \in M$, let $U = (x^1, \cdots, x^n)$ denote a co-ordinate chart containing then. Then $U \times \mathbb{R} = (x^1, \cdots, x^n, s)$ is a co-cordinate chart containing (p, s) for each $s \in \mathbb{R}$. In coordinates:

$$\omega = \sum_I \omega_I^1(x, s)\, ds \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k} + \sum_J \omega_I^2(x, s)\, dx^{j_1} \wedge \cdots \wedge dx^{j_k}$$

where I, J range over all increasing k-multi-indices over $\{1, \cdots, n\}$. We have,

$$S \lrcorner \omega = \sum_I \omega_I^1(x,s)\, dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

$$i_t^{\#}(S \lrcorner \omega) = i_t^{\#}\left(\sum_I \omega_I^1(x,s)\, dx^{i_1} \wedge \cdots \wedge dx^{i_k}\right) = \sum_I \omega_I^1(x,s)\, dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

We have,

$$\begin{aligned} d(h_k\omega) &= d\int_0^1 i_t^{\#}(S \lrcorner \omega)\, dt \\ &= d\int_0^1 \left(\sum_I \omega_I^1(x,t)\, dx^{i_1} \wedge \cdots \wedge dx^{i_k}\right) dt = \sum_I \int_0^1 \left(\frac{\partial \omega_I^1(x,t)}{\partial x^j}\, dx^j \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}\right) dt. \end{aligned}$$

We now compute $h_{k+1}(d\omega)$. Here d is the exterior derivative on $M \times I$. First note that,

$$d\omega = \sum_I \frac{\partial \omega_I^1(x,s)}{\partial x^j}\, dx^j \wedge dt \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k} + \sum_J \frac{\partial \omega_I^2(x,s)}{\partial x^l}\, dx^l \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_k} + \sum_J \frac{\partial \omega_I^2(x,s)}{\partial s}\, ds \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_k}$$

We now find $h_{k+1}(d\omega)$, which is given by the expression:

$$h_{k+1}(d\omega) = \int_0^1 i_t^{\#}(S \lrcorner d\omega) dt$$

We have,

$$S \lrcorner d\omega = \sum_J \frac{\partial \omega_I^2(x,s)}{\partial s}\, ds \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_k} - \sum_I \frac{\partial \omega_I^1(x,s)}{\partial x^j}\, dt \wedge dx^j \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

Therefore, we have,

$$\begin{aligned} h_{k+1}(d\omega) &= \int_0^1 i_t^{\#}(S \lrcorner d\omega) dt \\ &= \int_0^1 \left(\sum_J \frac{\partial \omega_I^2(x,t)}{\partial s}\, dx^{j_1} \wedge \cdots \wedge dx^{j_k} - \sum_I \frac{\partial \omega_I^1(x,t)}{\partial x^j}\, dx^j \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}\right) dt \end{aligned}$$

We have,

$$d(h_k\omega) + h_{k+1}(d\omega) = \int_0^1 \left(\sum_J \frac{\partial \omega_I^2(x,t)}{\partial s}\, dx^{j_1} \wedge \cdots \wedge dx^{j_k}\right) dt$$

Noting that,

$$i_t^{\#}(\omega) = \sum_J \omega_I^2(x,t)\, dx^{j_1} \wedge \cdots \wedge dx^{j_k},$$

we have,

$$\frac{d i_t^{\#}(\omega)}{dt} = \sum_J \frac{\partial \omega_I^2(x,t)}{\partial t}\, dx^{j_1} \wedge \cdots \wedge dx^{j_k}$$

As a result, we have,

$$d(h_k\omega) + h_{k+1}(d\omega) = \int_0^1 \frac{d i_t^{\#}(\omega)}{dt}\, dt = i_1^{\#}(\omega) - i_0^{\#}(\omega)$$

Hence, $(*)$ holds in every co-ordinate chart. This proves the claim. $\square$

Proposition 22.8. *Let M and N be smooth manifolds. If $F, G : M \to N$ are smoothly homotopic smooth maps, then the induced cohomology maps $F^*, G^* : H^k_{dR}(N) \to H^k_{dR}(M)$ are equal for each $k \in \mathbb{Z}$.*

Proof. There exists a homotopy $H : M \times I \to N$ from F to G such that $F = H \circ i_0, G = H \circ i_1$ We have,

$$F^{\#} = (H \circ i_0)^{\#} = i_0^{\#} \circ H^*,$$
$$G^{\#} = (H \circ i_1)^{\#} = i_1^{\#} \circ H^*.$$

By Lemma 22.7, we know the maps $i_0^{\#}$ and $i_1^{\#}$ are equal from $H^k_{\mathrm{dR}}(M \times I)$ to $H^k_{\mathrm{dR}}(M)$ for each $k \in \mathbb{Z}$. Therefore,

$$F^{\#} = (H \circ i_0)^{\#} = i_0^{\#} \circ H^* = i_1^{\#} \circ H^* = G^{\#}$$

This proves the claim. $\square$

Corollary 22.9. *(**Smooth Homotopy Invariance**) Let M and N be smoothly homotopy equivlaent smooth manifolds. Then*

$$H^k_{dR}(M) \cong H^k_{dR}(N)$$

for each $k \in \mathbb{Z}$.

Proof. Let $F : M \to N$ and $G : N \to M$ be smooth maps such that

$$G \circ F \simeq \mathrm{Id}_M$$
$$F \circ G \simeq \mathrm{Id}_N$$

We have,

$$(G \circ F)^{\#} = F^{\#} \circ G^{\#} = \mathrm{Id}_M^{\#} = \mathrm{Id}_{H^k(M)},$$
$$(F \circ G)^{\#} = G^{\#} \circ F^{\#} = \mathrm{Id}_N^{\#} = \mathrm{Id}_{H^k(N)}.$$

Since $\mathrm{Id}_M^{\#}$ is a surjective map, then $F^{\#}$ is surjective. Moreover, since $\mathrm{Id}_N^{\#}$ is an injective map, then $F^{\#}$ is an injective map. Hence, $F^{\#}$ is a linear map bijection, and hence an isomorphism. Hence, we have

$$H^k_{\mathrm{dR}}(M) \cong H^k_{\mathrm{dR}}(N)$$

for each $k \in \mathbb{Z}$. $\square$

It is clear that if $M = \{*\}$, then $H^k_{\mathrm{dR}}(M) = 0$ for all $k > 0$. We will verify this explicitly later on. If M is a star-like manifold, then by smooth homotopy invariance, $H^k_{\mathrm{dR}}(M) = 0$ for all $k > 0$ since M is contractible. This is immediately implies that the famous Poincaré lemma which states that if U is an open star-shaped subset of $\mathbb{R}^n$, then every closed form on U is exact. A consequence of the Poincaré lemma is that every closed form on a smooth manifold, M, is locally exact. This suggests that the the obstruction of solving the equation

$$d\eta = \omega,$$

is connected to a global problem. This hints that the de Rham cohomology group is not affected by the differential structure that is of local nature. This is made precise below:

Corollary 22.10. *(**Topological Invariance of de Rham Cohomology**) If M and N are homotopy equivalent,*
$$H^k_{dR}(M) \cong H^k_{dR}(N)$$
for each $k \in \mathbb{Z}$.

Proof. By Whitney's approximation theorem, every topological homotopy equivalence can be approximate is homotopic to a smooth homotopy equivalence. The result then follows from Corollary 22.9. $\square$

22.3. **Mayer–Vietoris Sequence.** Suppose M is a smooth manifold, and let U and V be open subsets of M such that $U \cup V = M$. The main goal of using the Mayer-Vietoris Sequence is to compute $H^k_{\mathrm{dR}}(M)$ in terms of $H^k_{\mathrm{dR}}(U)$, $H^k_{\mathrm{dR}}(V)$, and $H^k_{\mathrm{dR}}(U \cap V)$ where $\{U, V\}$ is an open cover of M. We have the following inclusions:

$$\begin{tikzcd} & U \arrow[dr, hook, "k"] & \\ U \cap V \arrow[ur, hook, "i"] \arrow[dr, hook, "j"'] & & M \\ & V \arrow[ur, hook, "l"'] & \end{tikzcd}$$

For each $k \in \mathbb{Z}$, these inclusion induce pullback maps on differential forms

$$\begin{tikzcd} & \Omega^k(U) \arrow[dl, "i^*"'] & \\ \Omega^k(U \cap V) & & \Omega^k(M) \arrow[ul, "k^*"'] \arrow[dl, "l^*"] \\ & \Omega^k(V) \arrow[ul, "j^*"] & \end{tikzcd}$$

Note that these pullbacks are in fact just restrictions. If we take some $\omega \in \Omega^k(M)$ and apply the map $k^* \oplus \ell^*$, we get

$$k^* \oplus \ell^*(\omega) : \Omega^k(M) \to \Omega^k(U) \oplus \Omega^k(V), \qquad k^* \oplus \ell^*(\omega) = (k^*\omega, \ell^*\omega) = (\omega|_U, \omega|_V)$$

Furthermore, if we take $(\omega, \eta) \in \Omega^p(U) \oplus \Omega^p(V)$ and apply the map $i^* - j^*$, we have

$$i^* - j^* : \Omega^k(U) \oplus \Omega^k(V) \to \Omega^k(U \cap V) \qquad (i^* - j^*)(\omega, \eta) = \omega\big|_{U\cap V} - \eta\big|_{U \cap V}$$

In other words, we have the following diagram

$$\begin{tikzcd} & \Omega^k(U) \arrow[dl, "i^*"] & \\ \Omega^k(U \cap V) & \Omega^k(U) \oplus \Omega^k(V) \arrow[l, "i^*-j^*"'] & \Omega^k(M) \arrow[l, "k^*\oplus l^*"'] \arrow[ul, "k^*"'] \arrow[dl, "l^*"] \\ & \Omega^k(V) \arrow[ul, "j^*"] & \end{tikzcd}$$

Proposition 22.11. *(**Mayer–Vietoris Sequence**) Let M be a smooth manifold, and let U, V be open subsets of M such that $M = U \cup V$. For each $k \in \mathbb{Z}$, there is a linear map $\delta^k : H^k_{dR}(U \cap V) \to H^{k+1}_{dR}(M)$ such that the following sequence, called the Mayer-Vietoris sequence for the open cover $\{U, V\}$, is exact:*

$$\cdots \xrightarrow{\delta^{k-1}} H^k_{dR}(M) \xrightarrow{k^{\#}\oplus l^{\#}} H^k_{dR}(U) \oplus H^k_{dR}(V) \xrightarrow{i^{\#}-j^{\#}} H^k_{dR}(U \cap V) \xrightarrow{\delta^k} H^{k+1}_{dR}(M) \xrightarrow{k^{\#}\oplus l^{\#}} \cdots$$

Proof. Consider the following sequence:

$$0 \to \Omega^k(M) \xrightarrow{k^*\oplus l^*} \Omega^k(U) \oplus \Omega^k(V) \xrightarrow{i^*-j^*} \Omega^k(U \cap U) \to 0$$

We show that this sequence is a short exact sequence. We first show that $k^* \oplus l^*$ is injective. Suppose that $\sigma \in \Omega^p(M)$ satisfies

$$(k^* \oplus l^*)(\sigma) = (\sigma|_U,\, \sigma|_V) = (0, 0)$$

This means that the restrictions of σ to U and V are both zero. Since $\{U, V\}$ is an open cover of M, this implies that σ is zero. We now show exactness at $\Omega^k(U) \oplus \Omega^k(V)$. Note that

$$(i^* - j^*) \circ (k^* \oplus l^*)(\sigma) = (i \circ j)(\sigma|_U, \sigma|_V) = \sigma|_{U\cap V} - \sigma|_{U\cap V} = 0,$$

This shows that $\operatorname{Im}(k^* \oplus l^*) \subseteq \ker(i^* - j^*)$. Conversely, suppose we are given $(\alpha, \alpha') \in \Omega^k(U) \oplus \Omega^k(V)$ such that $(i^* \circ j^*)(\alpha, \alpha') = 0$. This means that $\alpha|_{U\cap V} = \alpha'|_{U\cap V}$. So there is $\sigma \in \Omega^k(M)$ defined by

$$\sigma = \begin{cases} \alpha \text{ on } U, \\ \alpha' \text{ on } V. \end{cases}$$

Clearly, $(\alpha, \alpha') = (k \oplus l)(\sigma)$. So $\ker(i^* - j^*) \subseteq \operatorname{im}(k^* \oplus l^*)$. We now show that $i^* - j^*$ is surjective. Let $\omega \in \Omega^k(U \cap U)$. Let $\{\varphi, \psi\}$ be a smooth partition of unity subordinate to the open cover $\{U, V\}$ of M. Define $\alpha \in \Omega^k(U)$ and $\alpha' \in \Omega^k(V)$ by

$$\alpha = \begin{cases} \psi\omega \text{ on } U \cap V, \\ 0 \text{ on } U \setminus \operatorname{supp}\psi \end{cases} \qquad \alpha' = \begin{cases} -\varphi\omega \text{ on } U \cap V, \\ 0 \text{ on } U \setminus \operatorname{supp}\varphi \end{cases}$$

We have

$$(i^* - j^*)(\alpha, \alpha') = \alpha|_{U\cap V} - \alpha'|_{U\cap V} = \psi\omega - (-\varphi\omega) = (\psi - \varphi)\omega = \omega.$$

Hence, the sequence is indeed a short exact sequence. Because pullback maps commute with the exterior derivative, above short exact sequence induces the following this will show that we have the following short exact sequence:

$$0 \to H^k_{\mathrm{dR}}(M) \xrightarrow{k^{\#}\oplus l^{\#}} H^k_{\mathrm{dR}}(U) \oplus H^k_{\mathrm{dR}}(V) \xrightarrow{i^{\#}-j^{\#}} H^k_{\mathrm{dR}}(U \cap U) \to 0$$

Since this is true for each $k \in \mathbb{Z}$ we get a short exact sequence of co-chain complexes involing the de-Rham cohomology groups. The Mayer–Vietoris theorem then a formal consequence of the snake lemma. □

The snake lemma defines the connecting morphism

$$H^k_{\mathrm{dR}}(U \cap V) \xrightarrow{\delta^k} H^{k+1}_{\mathrm{dR}}(M)$$

A characterization of the connecting homomorphism is given in the proof of the snake lemma. Recalling it and adapting it to our case, we have that $\delta^k[\omega] = [\sigma]$, provided there exists $(\alpha, \alpha) \in \Omega^k(U) \oplus \Omega^k(V)$ such that

$$i^*\alpha - j^*\alpha' = \omega, \quad (k^*\sigma, l^*\sigma) = (d\alpha, d\alpha').$$

α, α' can be defined as in Proposition 22.11 to satisfy the first equation. Given such forms (α, α'), the fact that ω is closed implies that $d\alpha = d\alpha'$ on $U \cap V$. Thus, there is a smooth $(k+1)$-form σ on M that is equal to $d\alpha$ on U and $d\alpha'$ on V, and it satisfies the second equation.

22.4. **de Rham cohomology in Degrees Zero & One.** It is quite easy to characterize the de Rham cohomology in degree zero.

Proposition 22.12. *Let M is a connected smooth manifold. Then $H^0_{dR}(M)$ is equal to the space of constant functions. Therefore,*

$$H^0_{dR}(M) \cong \mathbb{R}$$

Proof. Note that

$$H^0_{\mathrm{dR}}(M) \cong \{\text{closed 0 forms on } M\} \cong \{f \in C^\infty(M) \mid df = 0\}$$

Since M is connected, $df = 0$ if and only if f is constant real-valued function. Therefore,

$$H^0_{\mathrm{dR}}(M) \cong \mathbb{R}$$

This completes the proof. □

Corollary 22.13. *Let M be a smooth manifold. Then*

$$H^0_{dR}(M) \cong \mathbb{R}^{|J|},$$

where $|J|$ is the number of connected components of M.

Proof. We have,

$$M = \coprod_{j \in J} M_j,$$

where each M_j is a connected component of M and J is at most countably infinite. By Proposition 22.6 and Proposition 22.12, we have,

$$H^0_{\mathrm{dR}}(M) \cong \prod_{j \in J} H^0_{\mathrm{dR}}(M_j \cong) \prod_{j \in J} \mathbb{R} \cong \mathbb{R}^{|J|}$$

This completes the proof. □

Another case in which we can say quite a lot about de Rham cohomology is in degree one. Let $\mathrm{Hom}(\pi_1(M,p), \mathbb{R})$ denote the set of group homomorphisms from $\pi_1(M,p)$ to the additive group $\mathbb{R}$. We define a linear map $\Phi\colon H^1_{\mathrm{dR}}(M) \to \mathrm{Hom}(\pi_1(M,p), \mathbb{R})$ as follows: given a cohomology class $[\omega] \in H^1_{\mathrm{dR}}(M)$, define $\Phi([\omega])\colon \pi_1(M,p) \to \mathbb{R}$ by

$$\Phi([\omega])([\gamma]) = \int_\gamma \omega,$$

where $[\gamma]$ is any path homotopy class in $\pi_1(M,p)$, and γ is any piecewise smooth curve representing the same path class.

Proposition 22.14. *Suppose M is a connected smooth manifold. For each $q \in M$, the linear map $\Phi\colon H^1_{\mathrm{dR}}(M) \to \mathrm{Hom}(\pi_1(M,p), \mathbb{R})$ is well defined and injective.*

Proof. (Sketch) Given $[\gamma] \in \pi_1(M,p)$, it follows from the Whitney approximation theorem that there is some smooth closed curve segment $\widetilde{\gamma}$ in the same path class as γ. We use without proof the fact that

$$\int_{\widetilde{\gamma}} \omega = \int_{\widetilde{\widetilde{\gamma}}} \omega$$

for every closed forms, ω and every other smooth closed curve $\widetilde{\widetilde{\gamma}}$ in the same path class as γ. If $\widetilde{\omega}$ is another smooth 1-form in the same cohomology class as ω, then $\widetilde{\omega} - \omega = df$ for some smooth function f, which implies

$$\int_{\widetilde{\gamma}} \widetilde{\omega} - \int_{\widetilde{\gamma}} \omega = \int_{\widetilde{\gamma}} df = f(q) - f(q) = 0.$$

Thus, Φ is well defined. It follows from properties of the line integral that $\Phi([\omega])$ is a group homomorphism from $\pi_1(M,p)$ to $\mathbb{R}$, and that Φ itself is a linear map. Suppose $\Phi([\omega])$ is the zero homomorphism. This means that $\int_{\widetilde{\gamma}} \omega = 0$ for every piecewise smooth closed curve $\widetilde{\gamma}$ with basepoint q. If γ is a piecewise smooth closed curve starting at some other point $q_0 \in M$, we can choose a piecewise smooth curve α from q to q_0, so that the path product $\alpha \cdot \gamma \cdot \overline{\alpha}$ is a closed curve based at q. It then follows that

$$0 = \int_{\alpha\cdot\gamma\cdot\overline{\alpha}} \omega = \int_{\alpha} \omega + \int_{\gamma} \omega + \int_{\overline{\alpha}} \omega = \int_{\alpha} \omega + \int_{\gamma} \omega - \int_{\alpha} \omega = \int_{\gamma} \omega.$$

Thus, ω is conservative and therefore exact. □

Corollary 22.15. *If M is a connected smooth manifold with finite fundamental group, then $H^1_{\mathrm{dR}}(M) = 0$.*

Proof. There are no nontrivial homomorphisms from a finite group to $\mathbb{R}$. The claim follows from Proposition 22.14. □

Remark 22.16. *If M is a connected smooth manifold whose fundamental group is a torsion group, then $H^1_{\mathrm{dR}}(M) = 0$. This is because $\mathbb{R}$ has no torsion elements. Hence,* $\mathrm{Hom}(\pi_1(M,p),\mathbb{R}) = 0$ *in this case.*

23. Examples & Applications

We discuss some example computations of de Rham cohomology.

Example 23.1. (**0-Dimensions**) Let M be a 0-dimensional smooth manifold. We have,

$$M \cong \coprod_{i\in I} \{*\}$$

where $|I|$ is the cardinality[22] of M. Then

$$H^k_{\mathrm{dR}}(M) = \begin{cases} \mathbb{R}^{|I|}, & \text{if } k = 0 \\ 0, & \text{otherwise} \end{cases}.$$

where $|I|$ is the cardinality of M. This follows at once from Proposition 22.6 and Proposition 22.12.

[22] $|I|$ is at most countably infinite.

Example 23.2. (**Contractible Manifolds**) Let M be contractible manifold. Then,

$$H^k_{\text{dR}}(M) = \begin{cases} \mathbb{R}^{|J|}, & \text{if } k = 0 \\ 0, & \text{otherwise} \end{cases}.$$

where $|J|$ is the number of connected components of M. This follows immediately from Example 23.1 and Corollary 22.13.

Remark 23.3. *If M is a star-like manifold, then by homotopy invariance, $H^k_{dR}(M) = 0$ for all $k > 0$ since M is contractible. This is immediately implies that the famous Poincaré lemma which states that if U is an open star-shaped subset of $\mathbb{R}^n$, then every closed form on U is exact. A consequencee of the Poincaré lemma is that every closed form on a smooth manifold, M, is locally exact.*

Example 23.4. (**Circle**) Let's compute the de-Rham cohomology of $\mathbb{S}^1$. Clearly, $H^0_{\text{dR}}(\mathbb{S}^1) = \mathbb{R}$ since $\mathbb{S}^1$ is connected. Write $\mathbb{S}^1 = U \cup V$, where U, V represent the 'nothern hemisphere' and 'southern heisphere'. U, V are contractible and $U \cap V \cong \{\pm 1\}$. The Mayer-Vietoris theorem implies

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathbb{R} \oplus \mathbb{R} \longrightarrow \mathbb{R} \oplus \mathbb{R} \longrightarrow H^1_{\text{dR}}(\mathbb{S}^1) \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots$$

This clearly implies that $H^k_{\text{dR}}(\mathbb{S}^1) = 0$ for $k > 2$. We can immediately conclude via exactness that $H^1_{\text{dR}}(\mathbb{S}^1) = \mathbb{R}$. Hence,

$$H^k_{\text{dR}}(\mathbb{S}^1) = \begin{cases} \mathbb{R}, & \text{if } k = 0, 1 \\ 0, & \text{otherwise} \end{cases}.$$

We can compute the generator for $H^1_{\text{dR}}(\mathbb{S}^1)$. The generator of is the angular 1-form $d\theta$. Notice that $d\theta$ is not globally defined on the circle since it is a multiple-valued function. Therefore, $d\theta$ is not zero in cohomology and generates $H^1_{\text{dR}}(\mathbb{S}^1)$.

Example 23.5. (**Spheres**) Let's compute the de Rham cohomology of $\mathbb{S}^n$ for $n \geq 1$. We proceed by induction on k to show that

$$H^k_{\text{dR}}(\mathbb{S}^n) = \begin{cases} \mathbb{R}, & \text{if } k = 0, n \\ 0, & \text{otherwise} \end{cases}.$$

We have verified the claim for $k = 1$ in Example 23.4. Now assume the claim is true for $n - 1$. Let $U = \mathbb{S}^n \setminus \{N\}$ and $V = \mathbb{S}^n \setminus \{S\}$. We have

$$U \cap V \simeq \mathbb{S}^{n-1} \quad U \simeq V \simeq \mathbb{R}^n$$

The Mayer-Vietoris sequence implies

$$\cdots \to 0 \to H^{k-1}_{\text{dR}}(\mathbb{S}^{n-1}) \to H^k_{\text{dR}}(\mathbb{S}^n) \to 0 \to \cdots$$

This implies that $H^{k-1}_{\text{dR}}(\mathbb{S}^{n-1}) \cong H^k_{\text{dR}}(\mathbb{S}^n)$. The claim now follows via induction and Example 23.4.

Example 23.6. (**Punctured Euclidean Space**) Let $p \in \mathbb{R}^n$ for $n \geq 2$. WLOG, we can assume that $p = 0$. We have

$$H^k_{\text{dR}}(\mathbb{R}^n \setminus \{p\}) = \begin{cases} \mathbb{R}, & \text{if } k = 0, n-1 \\ 0, & \text{otherwise} \end{cases}.$$

Indeed, the inclusion $\mathbb{S}^{n-1} \to \mathbb{R}^n$ is a homotopy equivalence. The claim now follows from Example 23.5.

We can now discuss some elementary applications of de-Rham cohomology. We can now prove the topological invariance of the dimension of smooth manifolds.

Proposition 23.7. *If $m \neq n$, then $\mathbb{R}^n$ is not homeomorphic to $\mathbb{R}^m$. In particular, if M be a topological n-manifold then its dimension is uniquely determined.*

Proof. Assume that $\mathbb{R}^m \cong \mathbb{R}^n$. If $f : \mathbb{R}^n \to \mathbb{R}^m$ is a homeomorphism, then $f : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^m \setminus \{f(0)\}$ is a homeomorphism. So,

$$H^k_{\mathrm{dR}}(\mathbb{R}^n \setminus \{0\}) = H^k_{\mathrm{dR}}(\mathbb{R}^m \setminus \{f(0)\}),$$

for each $k \in \mathbb{Z}$. But $\mathbb{R}^n \setminus \{0\} \cong \mathbb{S}^{n-1}$ and $\mathbb{R}^m \setminus \{f(0)\} \cong \mathbb{S}^{m-1}$. So,

$$H^k_{\mathrm{dR}}(\mathbb{S}^{m-1}) = H^k_{\mathrm{dR}}(\mathbb{S}^{n-1})$$

for each $k \in \mathbb{Z}$. This is a contradiction by Example 23.5. The claim for a topological manifold follows by working in co-ordinate charts. □

We can also show that the rank of the de-Rham cohomology groups is finite for *most manifolds*. We first need a definition:

Definition 23.8. Let M be a smooth n-manifold and $\{U_\alpha\}_{\alpha \in \Lambda}$ an open cover of M. We say $\{U_\alpha\}_{\alpha \in \Lambda}$ is a good cover if for any finite subset $I = \{\alpha_1, \dots, \alpha_k\} \subseteq \Lambda$ of indices, the intersection

$$U_I := U_{\alpha_1} \cap U_{\alpha_2} \cap \cdots \cap U_{\alpha_k}$$

is either empty or diffeomorphic to $\mathbb{R}^n$.

Remark 23.9. *By using the theory of geodesically convex neighborhoods in Riemannian geometry, one can show that any open cover of any smooth manifold M admits a refinement which is a good cover. In particular, if M is compact, then M admits a good cover which contains only finitely many open sets. See [BT13].*

Proposition 23.10. *Let M be a smooth n-manifold. If M admits a finite good cover,* $\dim H^k_{dR}(M) < \infty$ *for each $k \in \mathbb{Z}$.*

Proof. We proceed by induction on the number of sets in a finite good cover of M. If M admits a good cover that contains only one open set, then that open set has to be M itself. In this case, M is diffeomorphic to $\mathbb{R}^n$, and the conclusion follows. Now suppose the theorem holds for any manifold that admits a good cover containing $k-1$ open sets. Let M be a manifold with a good cover $\{U_1, \dots, U_k\}$. We denote

$$U = U_1 \cup \cdots \cup U_{k-1} \quad \text{and} \quad V = U_k.$$

Then $U \cap V$ admits a finite good cover $\{U_1 \cap U_k, \dots, U_{k-1} \cap U_k\}$. By the induction hypothesis, all the de Rham cohomology groups of U, V, and $U \cap V$ are finite-dimensional. Now consider the Mayer-Vietoris sequence:

$$\cdots \to H^{k-1}_{\mathrm{dR}}(U \cap V) \xrightarrow{\delta^{k-1}} H^k_{\mathrm{dR}}(M) \xrightarrow{\alpha^k} H^k_{\mathrm{dR}}(U) \oplus H^k_{\mathrm{dR}}(V) \to \cdots .$$

The conclusion follows since

$$\begin{aligned}
\dim \mathrm{Im}(\alpha_k) &\leq \dim H^k_{\mathrm{dR}}(U) \oplus H^k_{\mathrm{dR}}(V) < \infty, \\
\dim \ker(\alpha_k) = \dim \mathrm{Im}(\delta_{k-1}) &\leq \dim H^{k-1}_{\mathrm{dR}}(U \cap V) < \infty.
\end{aligned}$$

This completes the proof. □

Corollary 23.11. *If M is a compact manifold (or M is homotopy equivalent to a compact manifold), then* $\dim H^k_{dR}(M) < \infty$ *for all* $k \in \mathbb{Z}$.

Proof. This follows from Proposition 23.10. □

24. Compactly Supported de-Rham Cohomology

Let M be an orientable smooth manifold. Integration is a pairing between compactly supported forms and oriented manifolds. This observation motivates that $H^n_{\mathrm{dR}}(M)$ is important for studying orientations on M. Unfortunately, if M is non-compact, the integration of a n-form is not nicely defined unless the differential form is compactly supported. This observation motivates the study of de-Rham cohomology with compact support.

Definition 24.1. Let M be a smooth n-manifold and let $\omega \in \Omega^k(M)$. The **support** of ω is

$$\mathrm{supp}(\omega) = \{p \in M \mid \omega_p \neq 0\}.$$

ω is compactly supported if $\mathrm{supp}(\omega)$ is a compact set.

We set,

$$\Omega^k_c(M) = \{\omega \in \Omega^k(M) \mid \omega \text{ is compactly supported}\},$$

be the set of all compactly supported smooth k-forms. Clearly, the following facts are true:

(1) if ω_1, ω_2 are compactly supported k-forms, so is $c_1\omega_1 + c_2\omega_2$;
(2) if ω is compactly supported, then $d\omega$ is also compactly supported.

So $\Omega^k_c(M)$ are real vector spaces for each $k \in \mathbb{Z}$, and the exterior derivative makes these vector spaces a co-chain complex:

$$0 \to \Omega^0_c(M) \xrightarrow{d} \Omega^1_c(M) \xrightarrow{d} \Omega^2_c(M) \xrightarrow{d} \Omega^3_c(M) \xrightarrow{d} \cdots.$$

Definition 24.2. Let M be a smooth manifold. The quotient vector space

$$H^k_{\mathrm{dR,c}}(M) = \frac{\ker(d : \Omega^k_c(M) \to \Omega^{k+1}_c(M))}{\mathrm{im}(d : \Omega^{k-1}_c(M) \to \Omega^k_c(M))} = \frac{\{\omega \in \Omega^k_c(M) : d\omega = 0\}}{\{d\omega : \omega \in \Omega^{k-1}_c(M)\}}$$

is the k-th de Rham cohomology group with compact support of M.

Example 24.3. Let M be a smooth manifold. For $k = 0$, by definition

$$H^k_{\mathrm{dR,c}}(M) = \{f \in C^\infty(M) \mid df = 0 \text{ and } \mathrm{supp}(f) \text{ is compact}\}.$$

But $df = 0$ if and only if f is locally constant, i.e., f is constant on each connected component. Moreover, a locally constant compactly supported function has to be zero on any non-compact connected component. So we conclude

$$H^0_{\mathrm{dR,c}}(M) \cong \mathbb{R}^{m_c},$$

where m_c is the number of *compact* connected components of M. In particular,

$$H^0_{\mathrm{dR,c}}(\mathrm{pt}) = \mathbb{R}, \quad \text{and} \quad H^0_{\mathrm{dR,c}}(\mathbb{R}^n) = 0$$

for all $n \geq 1$.

Remark 24.4. *Since $\mathbb{R}^n$ is homotopy equivalent to $\{pt\}$, we conclude that $H^0_{\mathrm{dR,c}}(M)$ is* no longer *a homotopy invariant.*

We now discuss the analog of the Mayer-Vietoris sequence for the compactly supported case. If $F : M \to N$ is a smooth map between smooth manifolds, note that by definition,

$$\operatorname{supp}(F^*\omega) \subseteq F^{-1}(\operatorname{supp}(\omega)).$$

So if $\omega \in \Omega_c^k(N)$, in general we may have $F^*\omega \notin \Omega_c^k(M)$. Hence, we cannot expect to pull back compactly-supported cohomology classes on N to compactly-supported cohomology classes on M!

Remark 24.5. *If $F : M \to N$ is proper map, then the pull-back $F^*\omega$ of a compactly supported differential form $\omega \in \Omega_c^k(N)$ is still compactly supported. In this case, we have an induced map:*

$$F^* : H^k_{\mathrm{dR,c}}(N) \to H^k_{\mathrm{dR,c}}(M)$$

In this case, one can prove that if $F_0, F_1 : M \to N$ are proper smooth maps that are properly homotopic, then the induced maps are equal:

$$F_1^* = F_2^* : H^k_{\mathrm{dR,c}}(N) \to H^k_{\mathrm{dR,c}}(M).$$

Note that any homeomorphism is proper. So, in particular, the compactly supported de Rham cohomology groups are still topological invariants up to homeomorphisms. That is, if $M \cong N$ as smooth manifolds, then

$$H_c^k(M) = H_c^k(N)$$

for each $k \in \mathbb{Z}$. We have aready seen that compactly supported de Rham cohomology groups is not a topological invariant up to homotopy equivalence.

So how do we prove an analog of the Mayer-Vietoris sequence for the compactly supported case. Note that we can now instead pushforward compactly supported differential forms and hence cohomology classes. If $U \subseteq M$ is an open set, the inclusion $i : U \hookrightarrow M$ induces a map

$$i_* : \Omega_c^n(U) \to \Omega_c^n(M)$$

that sends a compactly supported differential form on U to the same differential form extended by zero outside of U.

Lemma 24.6. *For each $k \in \mathbb{Z}$, the map i_* commutes with the exterior derivative.*

Proof. For $\omega \in \Omega_c^n(U)$, we have $d\omega \in \Omega_c^{n+1}(U)$. Thus, applying $(i_* \circ d)$ to ω results in $d\omega$ extended by zero outside of U. If we first apply i_*, we obtain

$$i_*(\omega) = \begin{cases} 0, & \text{on } M \setminus U, \\ \omega, & \text{on } U. \end{cases}$$

Taking the exterior derivative, we get

$$d(i_*\omega) = \begin{cases} 0, & \text{on } M \setminus U, \\ d\omega, & \text{on } U. \end{cases}$$

Thus, i_* commutes with d. That is, $i_* \circ d = d \circ i_*$. □

Lemma 24.6 allows us to establish the following version of the Mayer-Vietoris sequence for the compactly supported case.

Proposition 24.7. *Let M be a smooth n-manifold and let $U, V \subseteq M$ be open sets such that $U \cup V = M$. Then there exists linear maps $\delta_k^c : H_c^k(M) \to H_c^{k+1}(U \cap V)$ so that the following sequence is exact:*

$$\cdots \to H_c^k(U \cap V) \to H_c^k(U) \oplus H_c^k(V) \to H_c^k(M) \xrightarrow{\delta_k^c} H_c^{k+1}(U \cap V) \to \cdots$$

Proof. The proof is so much like the original Mayer-Vietoris proof, and it involves a diagram chase. We omit details. □

Example 24.8. We compute $H^k_{\mathrm{dR,c}}(\mathbb{R}^n)$ for $k < n$. We have seen $H^0_{\mathrm{dR,c}}(\mathbb{R}^n) = 0$. Now we show that

$$H^k_{\mathrm{dR,c}}(\mathbb{R}^n) = 0$$

for $1 \leq k < n$. We identify $\mathbb{R}^n$ with then open set $\mathbb{S}^n - \{N\}$. Then we get an inclusion map

$$\iota : \mathbb{R}^n \to \mathbb{S}^n,$$

(1) Let $k = 1$. Let $\omega \in \Omega_c^1(\mathbb{R}^n)$ such that $d\omega = 0$. Since d commutes with i as seen above, we have that $\iota_*\omega \in \Omega_c^1(\mathbb{S}^n)$ such that $d(\iota_*\omega) = 0$. Since[23]

$$H^1_{\mathrm{dR,c}}(\mathbb{S}^n) = H^1_{\mathrm{dR}}(\mathbb{S}^n) = 0$$

there exists $\eta \in \Omega^0(\mathbb{S}^n) = C_c^\infty(\mathbb{S}^n)$ such that $\iota_*\omega = d\eta$. Noting that $\iota_*\omega$ is supported in $\mathbb{S}^n - U$ for open set U containing N, we have $d\eta = \iota^*\omega = 0$ on U. This implies that $\eta|_U \equiv c$ for some constant $c \in \mathbb{R}$. It follows that if we take $\tilde{\eta} = \eta - c$, then $\tilde{\eta} \in \Omega_c^0(\mathbb{S}^n - \{N\}) = \Omega_c^0(\mathbb{R}^n)$ and $d\tilde{\eta} = \omega$.

(2) Let $k > 1$. Let $\omega \in \Omega_k^c(\mathbb{R}^n)$ such that $d\omega = 0$. As above, $\iota_*\omega \in \Omega_k^c(\mathbb{R}^n)$ such that $d(\iota_*\omega) = 0$, and $\iota_*\omega$ is supported in $\mathbb{S}^n - U$ for open set U containing N. Since[24]

$$H^k_{\mathrm{dR,c}}(\mathbb{S}^n) = H^k_{\mathrm{dR}}(\mathbb{S}^n) = 0$$

there exists $\eta \in \Omega^{k-1}(\mathbb{S}^n)$ such that $\iota_*\omega = d\eta$. By shrinking the neighborhood U of N, we can assume that U is contractible. Then the fact that $d\eta = \iota_*\omega = 0$ in U implies that there exists a $\mu \in \Omega_c^{k-2}(U)$ such that $\eta|_U = d\mu$. Now pick a bump function ψ on $\mathbb{S}^n$ which compactly supported in U that equals 1 on N. Then

$$\tilde{\eta} = \eta - d(\psi\mu) \in \Omega_c^{k-1}(\mathbb{S}^n)$$

and $\tilde{\eta} = 0$ near N. By construction, $d\tilde{\eta} = d\eta = \omega$.

24.1. **Top Degree Cohomology.** We now set up the machinery to argue that the degree k de Rham cohomology with compact support is related to the orientation of smooth manifolds. First, an example:

Example 24.9. Let's compute $H^1_{\mathrm{dR,c}}(\mathbb{R})$. Consider the integration map

$$\int_{\mathbb{R}} : \Omega_c^1(\mathbb{R}) \to \mathbb{R}, \quad \omega \mapsto \int_{\mathbb{R}} \omega.$$

This map is clearly linear and surjective. Moreover, if $\omega = df$ is a compactly supported exact form, then

$$\int_{-\infty}^{\infty} df\, dx = \int_{-R}^{R} \frac{df}{dx}\, dx = f(R) - f(-R),$$

[23] Note that $k = 1 < n$

[24] Once again, note that $k < 1 < n$

for each $R > 0$. Since $f \in C_c^\infty(\mathbb{R})$, $f(R) = f(-R) = 0$ for R large enough. So it induces a surjective linear map

$$\int_{\mathbb{R}} : H^1_{\mathrm{dR,c}}(\mathbb{R}) \to \mathbb{R}.$$

Moreover, if $\int_{\mathbb{R}} f(t)\,dt = 0$, where $f \in C_c^\infty(\mathbb{R})$, then consider the function

$$g(t) = \int_{-\infty}^{t} f(\tau)\,d\tau$$

Clearly, g is smooth. If we choose $T > 0$ and $R < 0$ large enough, we get

$$F(T) = \int_{-\infty}^{T} f(t)\,dt = \int_{-\infty}^{\infty} f(t)\,dt = 0.$$
$$F(R) = \int_{-\infty}^{R} f(t)\,dt = \int_{-\infty}^{R} 0\,dt = 0.$$

Hence, $g \in C_c^\infty(\mathbb{R})$. Since $dg = f$, we have $[f(t)dt]$ in $H^1_{\mathrm{dR,c}}(\mathbb{R})$. Thus, $\int_{\mathbb{R}}$ is an isomorphism between $H^1_c(\mathbb{R})$ and $\mathbb{R}$, i.e.,

$$H^1_c(\mathbb{R}) \cong \mathbb{R}.$$

The same method as in Example 24.9 works generally. Let M be a connected, oriented n-manifold, and let $\omega \in \Omega^n_c(M)$ be a compactly supported n-form. Then ω is closed, and we have defined the integral $\int_M \omega$. So we get a map

$$\int_M : \Omega^n_c(M) \to \mathbb{R}, \quad \omega \mapsto \int_M \omega.$$

Suppose $\omega = d\eta$ for some $\eta \in \Omega^{n-1}_c(M)$. We can take a compact set $K \subseteq M$ such that $\mathrm{supp}(\eta) \subseteq K$. By Stokes' theorem,

$$\int_M \omega = \int_M d\eta = \int_K d\eta = \int_{\partial K} \eta = 0.$$

Thus, $\int_M$ induces a linear map

$$\int_M : H^n_{\mathrm{dR,c}}(M) \to \mathbb{R}, \quad [\omega] \mapsto \int_M \omega$$

Proposition 24.10. *Let M be an oriented smooth n-manifold. Then the map $\int_M$: $H^n_{\mathrm{dR,c}}(M) \to \mathbb{R}$ is surjective.*

Proof. Fix a n-form (a volume form) ω on M. For any $c \in \mathbb{R}$, one can find a smooth function f that is compactly supported in a coordinate chart U, such that $\int_U f\omega = c$. ☐

We can prove the following corollary based on Proposition 24.10:

Corollary 24.11. *The following statements are true:*

(1) *If $\omega \in \Omega^n(\mathbb{S}^n)$ and $\int_{\mathbb{S}^n} \omega = 0$, then ω is exact.*
(2) *We have*

$$H^k_{\mathrm{dR,c}}(\mathbb{R}^n) = \begin{cases} \mathbb{R}, & k = n, \\ 0, & k \neq n. \end{cases}$$

(3) *Let M be a smootn n-manifold. if M admits a finite good cover, then* $\dim H^k_{\mathrm{dR,c}}(M) < \infty$ *for all $k \in \mathbb{Z}$.*

Proof. The proof is given below:

(1) Note that
$$H^n_{\mathrm{dR}}(\mathbb{S}^n) = H^n_{\mathrm{dR,c}}(\mathbb{S}^n) \cong \mathbb{R}$$
Hence, the map in Proposition 24.10 is in fact a linear isomorphism. In other words, if $\int_{\mathbb{S}^n} \omega = 0$, then $[\omega] = 0$, i.e., ω is exact.

(2) Example 24.9 proves the case $n = 1$ and Example 24.8 takes care of the case $1 \leq k < n$ for $n \geq 2$. We discuss the case $k = n \geq 2$. It suffices to show that the surjective linear map
$$\int_{\mathbb{R}^n} : H^n_{\mathrm{dR,c}}(\mathbb{R}^n) \to \mathbb{R}, \quad [\omega] \mapsto \int_{\mathbb{R}^n} \omega$$
is in fact an isomorphism. We show that the map is injective. Assume that $\int_{\mathbb{R}^n} \omega = 0$ for some $\omega \in \Omega^n_c(\mathbb{R}^n)$. Automatically, we have $d\omega = 0$. As before, consider the inclusion map $\iota : \mathbb{R}^n \to \mathbb{S}^n$. Then $\iota_* \omega \in \Omega^n(S^n)$. Since
$$\int_{\mathbb{S}^n} \iota_* \omega = \int_{\mathbb{R}^n} \omega = 0,$$
by (1), we see $\iota_* \omega = d\eta$ for some $\eta \in \Omega^{n-1}(S^n)$. The rest of the proof is similar to that of Example 24.8(2).

(3) We can use Mayer-Vertoris sequence compactly supported de Rham cohomology and induction and the number of open sets in a good cover. The same as the proof for the ordinary de Rham cohomology.

This completes the proof. □

We now reach the punchline for this section. We argue that Proposition 24.10 is, in fact, a linear isomorphism if the underlying smooth manifold is connected and orientable.

Proposition 24.12. *Let M be a smooth connected orientable n-manifold. The map in Proposition 24.10 is an isomorphism. In particular,*
$$H^n_{\mathrm{dR,c}}(M) \cong \mathbb{R}$$

Proof. In Proposition 24.10, we have already checked that the map is a surjective linear isomorphism. We check that it is injective. Let $\omega \in \Omega^c_c(M)$ such that $\int_M \omega = 0$. Since M is connected and $\mathrm{supp}(\omega)$ is compact, we can take a connected compact set $\mathrm{supp}(\omega) \subseteq K_\omega$. If we can cover K_ω by a good cover which contains only one chart, then Corollary 24.11(2) implies that $\omega = d\mu$ for some $\mu \in \Omega^{n-1}_c(M)$. We can now proceed by induction. Suppose the claim is true if K_ω can be covered by $k-1$ 'good charts,' and suppose $\omega \in \Omega^n_c(M)$ satisfies the property that K_ω admits a good cover $\{U_1, \ldots, U_k\}$. There exists one U_i, say U_k for simplicity, such that both $U = U_1 \cup \cdots \cup U_{k-1}$ and $V = U_k$ are connected[25]. Pick a partition of unity $\{\rho_U, \rho_V\}$ of $U \cup V$ subordinate to the cover $\{U, V\}$, and let $\omega|_U = \rho_U \omega$, $\omega|_V = \rho_V \omega$. Since K_ω is connected, $U \cap V \neq \emptyset$. We pick an n-form ω_0 compactly supported in $U \cap V$ so that
$$\int_M \omega_0 = \int_M \omega|_U.$$

[25]This needs proof.

Then $\omega|_U - \omega_0$ is compactly supported in U, which is connected and admits a good cover of $k-1$ good charts, and

$$\int_M (\omega|_U - \omega_0) = 0.$$

So by the induction hypothesis,

$$\omega_U - \omega_0 = d\eta|_U$$

for some $\eta_U \in \Omega_c^{n-1}(M)$. Similarly,

$$\int_M (\omega|_V + \omega_0) = -\int_M \omega|_U + \int_M \omega_0 = 0$$

implies

$$\omega_V + \omega_0 = d\eta|_V$$

for some $\eta|_V \in \Omega_c^{n-1}(M)$. It follows that

$$\omega = \omega_U + \omega_V = d(\eta_U + \eta_V),$$

where $\eta|_U + \eta|_V \in \Omega_c^{n-1}(M)$. This completes the proof. $\square$

25. de-Rham's Theorem

25.1. **Smooth Singular Homology.**

25.2. **Proof of De-Rham's Theorem.**

25.3. **Applications.**

Part 6. Products & Duality

26. Cup Product

Let's revert back to singular cohomology. However, we will keep on referring to de Rham cohomology for some down-to-earth motivation. We have worked with coefficients G, where G is some abelian group. We now show that if we take $G = R$ to be a commutative ring R, then the singular cohomology with coefficients in R also forms a ring under the cup product operation. First, let's define the algebraic object over which we define the ring structure.

Definition 26.1. Let X be a topological space, and let R be a commutative ring. The total cohomology of X with coefficients in R is given by

$$H^\bullet(X;R) := \bigoplus_{n\geq 0} H^n(X;R).$$

Our aim is to make $H^\bullet(X;R)$ into a graded ring when R is a commutative ring. We shall do this by first making

$$C^\bullet(X;R) := \bigoplus_{n\geq 0} C^n(X;R)$$

into a graded ring, and then showing that the ring structure descends to cohomology. This will be done by introducing a cup product structure on $C^\bullet(X;R)$.

Example 26.2. We first discuss the special case of de Rham cohomology. The advantage here is that we can directly work at the de Rham cohomology groups. Let M be a smooth manifold, and let $\omega \in \Omega^k(M)$, $\eta \in \Omega^l(M)$ be closed forms. If $[\omega] = [\omega']$ and $[\eta] = [\eta']$, we have

$$\omega = \omega' + d\alpha, \qquad \eta = \eta' + d\beta$$

for $\alpha \in \Omega^{k-1}(M)$ and $\beta \in \Omega^{l-1}(M)$. Note that we have

$$\begin{aligned}
\omega \wedge \eta &= (\omega' + d\alpha) \wedge (\eta' + d\beta) \\
&= \omega' \wedge \eta' + \omega' \wedge d\beta + d\alpha \wedge \eta' + d(\alpha \wedge \beta) \\
&= \omega' \wedge \eta' - d\beta \wedge \omega' + d\alpha \wedge \eta' + d(\alpha \wedge \beta) \\
&= \omega' \wedge \eta' - d(\beta \wedge \omega') + d(\alpha \wedge \eta') + d(\alpha \wedge \beta) \\
&= \omega' \wedge \eta' - d(\beta \wedge \omega' + \alpha \wedge \eta' + \alpha \wedge \beta).
\end{aligned}$$

Hence, $[\omega \wedge \eta] = [\omega' \wedge \eta']$. This shows that the wedge product

$$\wedge : \Omega^k(M) \times \Omega^l(M) \to \Omega^{k+l}(M)$$

descends to a well-defined bilinear map

$$\begin{aligned}
\smile :\ & H^k_{\mathrm{dR}}(M) \times H^l_{\mathrm{dR}}(M) \to H^{k+l}_{\mathrm{dR}}(M), \\
& [\omega] \smile [\eta] \mapsto [\omega \wedge \eta].
\end{aligned}$$

This is called the cup product in de-Rham cohomology.

Let's now move back to the singular cohomology case and define the cup product. We first define it at the level of $C^\bullet(X;R)$.

Definition 26.3. Let $\phi \in C^k(X;R)$ and $\psi \in C^l(X;R)$. The cup product $\phi \smile \psi \in C^{k+l}(X;R)$ is defined by:

$$(\phi \smile \psi)(\sigma : \Delta^{k+l} \to X) = \phi(\sigma|_{[v_0,\dots,v_k]}) \cdot \psi(\sigma|_{[v_k,\dots,v_{k+l}]})$$

where '$\cdot$' denotes the multiplication in the ring R.

Remark 26.4. *Technically, these restricted maps in Definition 26.3 have the wrong domains; they aren't the standard k, l-simplices. But we just pre-compose with the 'obvious' maps from the standard simplices. We shall not do this below.*

The cup product extends by linearity to define a function $C^k(X;R) \times C^l(X;R) \to C^{k+l}(X;R)$ by

$$\left(\sum_i \phi_i\right) \smile \left(\sum_j \psi_j\right) := \sum_{i,j} \phi_i \smile \psi_j.$$

Let us first check this gives us a ring structure.

Lemma 26.5. *Let X be a topological space and let R be a commutative ring. Then $C^\bullet(X;R)$ is a graded ring under the cup product. If R has an identity then $C^\bullet(X;R)$ also has an identity.*

Proof. Suppose $\phi \in C^k(X;R)$ and $\psi, \gamma \in C^l(X;R)$. We claim that $\phi \smile (\psi + \gamma) = \phi \smile \psi + \phi \smile \gamma$. For this, take $\sigma : \Delta^{k+l} \to X$. Then

$$\begin{aligned}(\phi \smile (\psi+\gamma))(\sigma) &= \phi(\sigma_{[v_0,\dots,v_k]}) \cdot (\psi+\gamma)(\sigma_{[v_k,\dots,v_{k+l}]}) \\ &= \phi(\sigma_{[v_0,\dots,v_k]}) \cdot \psi(\sigma_{[v_k,\dots,v_{k+l}]}) + \phi(\sigma_{[v_0,\dots,v_k]})) \cdot \gamma(\sigma_{[v_k,\dots,v_{k+l}]}) \\ &= \phi \smile \psi(\sigma_{[v_k,\dots,v_{k+l}]}) + \phi \smile \gamma(\sigma_{[v_k,\dots,v_{k+l}]}).\end{aligned}$$

A similar computation shows that $(\phi+\psi) \smile \gamma = \phi \smile \gamma + \psi \smile \gamma$. Associativity follows by a similar computation. Let 1_R denote the identity in R. Define a cochain $\nu \in C^0(X;R)$ by $\nu(x) = 1_R \quad \forall x \in X$ and extend by linearity. It is clear that

$$\nu \smile \phi = \phi = \phi \smile \nu$$

for any $\phi \in C^n(X;R)$ and any $n \geq 0$. Thus, $C^\bullet(X;R)$ is indeed a graded ring. ∎

Unfortunately, the ring structure on $C^\bullet(X;R)$ is not very useful, as it is too "large" and almost impossible to compute. However, as we will now see, the total cohomology $H^\bullet(X;R)$ also inherits a ring structure, and this structure is much nicer. We need the following result:

Lemma 26.6. *Let $\phi \in C^k(X;R)$ and $\psi \in C^l(X;R)$*

$$\delta^{k+l}(\phi \smile \psi) = \delta^k\phi \smile \psi + (-1)^k \phi \smile \delta^l \psi$$

Proof. For $\sigma : \Delta^{k+l+1} \to X$, we have

$$\begin{aligned}(\delta^k\phi \smile \psi)(\sigma) &= \sum_{i=0}^{k} (-1)^i \phi(\sigma_{[v_0,\dots,\widehat{v}_i,\dots,v_{k+1}]}) \cdot \psi(\sigma_{[v_{k+1},\dots,v_{k+l+1}]}), \\ (-1)^k(\phi \smile \delta\psi)(\sigma) &= \sum_{i=k}^{k+l+1} (-1)^i \phi(\sigma_{[v_0,\dots,v_k]}) \cdot \psi(\sigma_{[v_k,\dots,\widehat{v}_i,\dots,v_{k+l+1}]}).\end{aligned}$$

When we add these two expressions, the last term of the first sum cancels with the first term of the second sum, and the remaining terms are exactly $\delta^{k+l}(\phi \smile \psi)(\sigma) = (\phi \smile \psi)(\partial_{k+l+1}\sigma)$ since

$$\partial_{k+l+1}\sigma = \sum_{i=0}^{k+l+1} (-1)^i \sigma_{[v_0,\dots,\widehat{v}_i,\dots,v_{k+l+1}]}.$$

This completes the proof. ∎

Corollary 26.7. *The following statements are true:*

(1) *If $\phi \in C^k(X;R)$ and $\psi \in C^l(X;R)$ are cocycles, then $\delta^{k+l}(\phi \smile \psi) = 0$.*
(2) *If $\phi \in C^k(X;R)$ and $\psi \in C^l(X;R)$ are such that one of ϕ or ψ is a cocycle and the other a coboundary, then $\phi \smile \psi$ is a coboundary.*

Proof. The proof is given below:

(1) Since $\delta^k \phi = 0$ and $\delta^l \psi = 0$, we have that that
$$\delta^{k+l}(\phi \smile \psi) = \delta^k \phi \smile \psi + (-1)^k \phi \smile \delta^l \psi = 0$$
(2) Say $\delta^k \phi = 0$ and $\psi = \delta^{l-1}\eta$. Then
$$\delta^{k+l-1}(\phi \smile \eta) = (-1)^k \phi \smile \delta^{l-1}\eta = (-1)^k \phi \smile \psi$$
The other case is similar.

This completes the proof □

It follows that we get an induced cup product on cohomology:
$$\begin{aligned} \smile : H^k(X;R) \times H^l(X;R) &\to H^{k+l}(X;R) \\ [\phi] \times [\psi] &\mapsto [\phi \smile \psi] \end{aligned}$$
Well-definedness follows from Corollary 26.7. Indeed, if $[\phi] = [\phi']$ and $[\psi] = [\psi']$, then
$$\phi = \phi' + \alpha, \quad \psi = \psi' + \beta$$
where α, β are co-chains. We have
$$\begin{aligned} \phi \smile \psi &= (\phi' + \alpha) \smile (\psi' + \beta) \\ &= \phi' \smile \psi' + (\phi' \smile \beta + \alpha \smile \psi' + \alpha \smile \beta) \end{aligned}$$
Corollary 26.7 implies that the term in paranthesis is a coboundary. Hence,
$$[\phi \smile \psi] = [\phi' \smile \psi']$$
The operation is distributive and associative since it is so on the co-chain level. If R has an identity element, then there is an identity element for the cup product, namely the class $[1] \in H^0(X;R)$ defined by the 0-cocycle taking the value 1_R on each singular 0-simplex. Considering the cup product as an operation on the direct sum of all cohomology groups, we get a (graded) ring structure on $H^\bullet(X;R)$.

Definition 26.8. Let X be a topological space and let R be a commutative ring. The cohomology ring of X is the graded ring
$$H^\bullet(X;R) := \left(\bigoplus_{n \geq 0} H^n(X;R), \smile \right)$$
with respect to the cup product operation. If R has an identity, then so does $H^\bullet(X;R)$.

Remark 26.9. *We can also define the relative cup product. The cup product on cochains*
$$C^k(X;R) \times C^l(X;R) \to C^{k+l}(X;R)$$

restricts to cup products

$$C^k(X,A;R)\times C^l(X;R)\to C^{k+l}(X,A;R),$$
$$C^k(X,A;R)\times C^l(X,A;R)\to C^{k+l}(X,A;R),$$
$$C^k(X;R)\times C^l(X,A;R)\to C^{k+l}(X,A;R).$$

since $C^i(X,A;R)$ can be regarded as the set of cochains vanishing on chains in A, and if φ or ψ vanishes on chains in A, then so does $\varphi\smile\psi$. So there exist relative cup products:

$$H^k(X,A;R)\times H^l(X;R)\to H^{k+l}(X,A;R),$$
$$H^k(X,A;R)\times H^l(X,A;R)\to H^{k+l}(X,A;R),$$
$$H^k(X;R)\times H^l(X,A;R)\to H^{k+l}(X,A;R).$$

In particular, if A is a point, we get a cup product on the reduced cohomology $\widetilde{H}^*(X;R)$. More generally, we can define

$$H^k(X,A;R)\times H^\ell(X,B;R)\to H^{k+\ell}(X,A\cup B;R)$$

when A and B are open subsets of X or sub-complexes of the CW complex X.

Normally, no one computes cohomology rings using the definition of the cup product, as this can be quite tedious for the most part. However, we compute a couple of basic examples:

Example 26.10. (**Spheres**) Let $X=\mathbb{S}^n$ for $n\geq 1$ and $R=\mathbb{Z}$. We have

$$H^k(\mathbb{S}^n;\mathbb{Z})\cong\begin{cases}\mathbb{Z}, & \text{if } k=0,n\\ 0, & \text{otherwise}\end{cases}.$$

The generating element in $H^0(\mathbb{S}^n;\mathbb{Z})$ is the identity element. We label the generators of $H^0(\mathbb{S}^n;\mathbb{Z}))$ and $H^n(\mathbb{S}^n;\mathbb{Z})$ as 1 and x respectively. We have the following relations

$$1\smile 1=1,\quad 1\smile x=x,\quad x\smile 1=x,\quad x\smile x=0$$

The last relation is true since $H^{2n}(\mathbb{S}^n;\mathbb{Z})=0$. Hence, we have

$$H^*(\mathbb{S}^n;\mathbb{Z})\cong\frac{\mathbb{Z}[x]}{\langle x^2\rangle}=\mathbb{Z}[x]/(x^2)\cong\Lambda_{\mathbb{Z}}[x]$$

Here $\Lambda_{\mathbb{Z}}[x]$ is the exterior algebra on two generator over $\mathbb{Z}$.

Remark 26.11. *We can define a cup product for simplicial cohomology by the same formula as for singular cohomology. It can be checked that the isomorphism between simplicial and singular cohomology respects cup products. Hence, we can compute cup products using simplicial cohomology.*

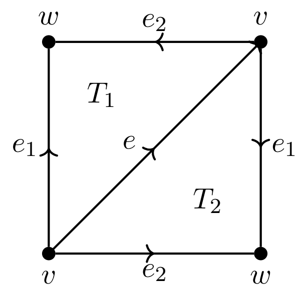
Example 26.12. (**Real Projective Plane**) Let $X=\mathbb{RP}^2$ and $R=\mathbb{Z}_2$. We have

$$H^k(\mathbb{RP}^2;\mathbb{Z}_2)\cong\begin{cases}\mathbb{Z}_2 & k=0,1,2,\\ 0 & \text{otherwise.}\end{cases}$$

Let α be the generator of $H^1(\mathbb{RP}^2;\mathbb{Z}_2)$. Consider

$$\alpha^2:=\alpha\smile\alpha\in H^2(\mathbb{RP}^2;\mathbb{Z}_2).$$

We claim that $\alpha^2 \neq 0$, so α^2 is in fact the generator of $H^2(\mathbb{RP}^2; \mathbb{Z}_2) \cong \mathbb{Z}_2$. Consider the cell structure on $\mathbb{RP}^2$ shown in the figure below. The 2-cell T_1 is attached by the word $e_1 e_2^{-1} e^{-1}$, and the 2-cell T_2 is attached by the word $e_2 e_1^{-1} e^{-1}$.



Since α is a generator of $H^1(\mathbb{RP}^2; \mathbb{Z}/2\mathbb{Z}) \cong \mathrm{Hom}_{\mathbb{Z}}(H_1(\mathbb{RP}^2; \mathbb{Z}), \mathbb{Z}/2\mathbb{Z})$, it is represented by a cocycle

$$\varphi : C_1(\mathbb{RP}^2) \to \mathbb{Z}/2\mathbb{Z}$$

with $\varphi(e) = 1$, where e represents the generator of $H_1(\mathbb{RP}^2; \mathbb{Z}) \cong \mathbb{Z}_2$. The co-cycle condition for φ translates into the identities:

$$0 = (\delta\varphi)(T_1) = \varphi(\partial T_1) = \varphi(e_1) - \varphi(e_2) - \varphi(e),$$

$$0 = (\delta\varphi)(T_2) = \varphi(\partial T_2) = \varphi(e_2) - \varphi(e_1) - \varphi(e_1).$$

As $\varphi(e) = 1$, we may WLOG take $\varphi(e_1) = 1$ and $\varphi(e_2) = 0$. Note that α^2 is represented by $\varphi \smile \varphi$, and we have:

$$(\varphi \smile \varphi)(T_1) = \varphi(e_1) \cdot \varphi(e) = 1.$$

Similarly,

$$(\varphi \smile \varphi)(T_2) = \varphi(e_2) \cdot \varphi(e) = 0.$$

Since the generator of $C_2(\mathbb{RP}^2)$ is $T_1 + T_2$, and we have

$$(\varphi \smile \varphi)(T_1 + T_2) = (\varphi \smile \varphi)(T_1) + (\varphi \smile \varphi)(T_2) = 1 + 0 = 1,$$

it follows that $\alpha^2 = [\varphi \smile \varphi]$ is the generator of $H^2(\mathbb{RP}^2; \mathbb{Z}/2\mathbb{Z})$. Let I denote the ideal generated by the relations. Hence, we have

$$H^*(\mathbb{RP}^n; \mathbb{Z}_2) = \frac{\mathbb{Z}_2[x]}{I} \cong \mathbb{Z}_2[x]/(x^3)$$

Let's prove some important facts about the cup product.

Proposition 26.13. *Let X, Y be topological spaces and let let $f : X \to Y$ be a continuous map. For each $n \in \mathbb{Z}$, the induced maps*

$$f_n^* = H^n(Y; R) \to H^n(X; R)$$

are ring homomorphisms. That is,

$$f_n^*(\alpha \smile \beta) = f^*(\alpha) \smile f^*(\beta)$$

for each $\alpha, \beta \in H^k(Y; R)$.

Proof. It suffices to show the following co-chain formula:

$$f^{\#}(\varphi \smile \psi) = f^{\#}(\varphi) \smile f^{\#}(\psi).$$

For $\varphi \in C^k(Y;\mathbb{R})$ and $\psi \in C^l(Y;\mathbb{R})$, we have:

$$\begin{aligned}(f^{\#}\varphi \smile f^{\#}\psi)(\sigma : \Delta^{k+l} \to X) &= (f^{\#}\varphi)(\sigma|_{[v_0,\dots,v_k]}) \cdot (f^{\#}\psi)(\sigma|_{[v_k,\dots,v_{k+l}]}) \\ &= \varphi((f^{\#}\sigma)|_{[v_0,\dots,v_k]}) \cdot \psi((f^{\#}\sigma)|_{[v_k,\dots,v_{k+l}]}) \\ &= (\varphi \smile \psi)(f^{\#}\sigma) \\ &= (f^{\#}(\varphi \smile \psi))(\sigma).\end{aligned}$$

This completes the proof. □

Corollary 26.14. *If $f : X \to Y$ is a continuous map, then there is a ring homomorphism*

$$f^* : H^*(Y;R) \to H^*(X;R).$$

Proof. We have

$$H^*(Y;R) = \bigoplus_{n\geq 0} H^n(Y;R), \quad H^*(X;R) = \bigoplus_{n\geq 0} H^n(X;R)$$

If we define f^* such that $f^*|_{H^n(Y;R)} = f_n$, the claim follows via Proposition 26.13. □

Remark 26.15. *The discussion above implies that the operation of taking the cohomology ring is a (contravariant) functor from* **Top** *to* **CRing***.*

Example 26.16. The isomorphisms

$$H^*\left(\coprod_\alpha X_\alpha; R\right) \cong \prod_\alpha H^*(X_\alpha;R)$$

whose coordinates are induced by the inclusions $i_\alpha : X_\alpha \hookrightarrow \coprod_\alpha X_\alpha$, is a ring isomorphism with respect to the coordinatewise multiplication in a ring product, since each coordinate function i_α^* is a ring homomorphism. Similarly, the group isomorphism

$$H^*(\bigvee_\alpha X_\alpha; R) \cong \prod_\alpha H^*(X_\alpha;R)$$

is a ring isomorphism.

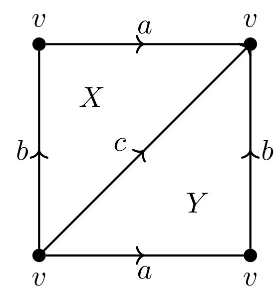
We now show that the cup product is graded anti-commutative.

Proposition 26.17. *Let X be a topological space and let R be a commutative ring. Let $\alpha \in H^k(X;R)$ and $\beta \in H^l(X;R)$. We have*

$$\alpha \smile \beta = (-1)^{kl}\beta \smile \alpha$$

Proof. See [Hat02]. □

Example 26.18. Let $X = \mathbb{S}^1 \times \mathbb{S}^1 = T^2$. We can use Proposition 26.17 to compute $H^*(T^2; R)$. Consider the following simplicial complex structure on T^2:



The generator $1 \in H^0(T^2;\mathbb{Z})$ is the unit. By examining the dimensions of the other generators, the only non-identity generators which could multiply together and give something non-zero are the generators of $H^1(T^2;\mathbb{Z})$. Let $\alpha, \beta \in H^1(T^2;\mathbb{Z})$ be generators of $H^1(T^2;\mathbb{Z})$. We compute

$$\alpha \smile \alpha, \quad \alpha \smile \beta, \quad \beta \smile \alpha, \quad \beta \smile \beta.$$

By Proposition 26.17, we must have $\alpha \smile \alpha = \beta \smile \beta = 0$. But let's verify it explicitly. α is represented by a cocycle

$$\varphi_\alpha : C_1(T^2) \to \mathbb{Z}$$

with $\varphi_\alpha(a) = 1, \varphi_\alpha(b) = 0$. Here a, b are generators of $H_1(T^2;\mathbb{Z})$. The co-cycle condition for φ translates into the identities:

$$0 = (\delta\varphi_\alpha)(X) = \varphi_\alpha(X) = \varphi_\alpha(a) - \varphi_\alpha(c) + \varphi_\alpha(b),$$

$$0 = (\delta\varphi_\alpha)(Y) = \varphi_\alpha(Y) = \varphi_\alpha(b) - \varphi_\alpha(c) + \varphi_\alpha(a).$$

As $\varphi_\alpha(a) = 1, \varphi_\alpha(b) = 0$, we must have $\varphi_\alpha(c) = 1$. Note that α^2 is represented by $\varphi \smile \varphi$, and we have:

$$(\varphi_\alpha \smile \varphi_\alpha)(X) = \varphi_\alpha(b) \cdot \varphi_\alpha(a) = 1.$$

$$(\varphi_\alpha \smile \varphi_\alpha)(Y) = \varphi_\alpha(a) \cdot \varphi_\alpha(b) = 1.$$

Hence, $\varphi_\alpha \smile \varphi_\alpha = 0$. This shows that $\alpha \smile \alpha = 0$. If we choose β to be represented by a cocycle

$$\varphi_\beta : C_1(T^2) \to \mathbb{Z}$$

with $\varphi_\beta(b) = 1, \varphi_\beta(a) = 0$, we similarly have $\beta \smile \beta = 0$. We now compute $\alpha \smile \beta$. Note that $\alpha \smile \beta$ is represented by $\varphi_\alpha \smile \varphi_\beta$. We have

$$(\varphi_\alpha \smile \varphi_\beta)(X) = \varphi_\alpha(b) \cdot \varphi_\alpha(a) = 0.$$

$$(\varphi_\alpha \smile \varphi_\beta)(Y) = \varphi_\alpha(a) \cdot \varphi_\alpha(b) = 1.$$

Since the generator of $C_2(T^2)$ is $X + Y$, and $(\varphi_\alpha \smile \varphi_\beta)(X + Y) = 1$, it follows that $\alpha \smile \beta$ is the generator of $H^2(T^2;\mathbb{Z})$. By Proposition 26.17, we have $\beta \smile \alpha = -\alpha \smile \beta$. Hence, we have

$$H^*(T^2;\mathbb{Z}) \cong \frac{\mathbb{Z}[x, y]}{\langle x^2, y^2, xy + yx\rangle} \cong \Lambda_{\mathbb{Z}}[x, y]$$

Here $\Lambda_{\mathbb{Z}}[x, y]$ is the exterior algebra on two generator over $\mathbb{Z}$.

27. Poincaré Duality for Smooth Manifolds

We discuss Poincaré duality for smooth, oriented, n-manifolds in this section. We can prove this special case by leveraging de Rham cohomology. Using Stokes' theorem, Poincaré duality for smooth, oriented n-manifolds asserts that there is a non-degenerate pairing between de Rham cohomology groups:

$$H^k_{\mathrm{dR}}(M) \times H^{n-k}_{\mathrm{dR,c}}(M) \to \mathbb{R}, \quad ([\alpha],[\beta]) \mapsto \int_M \alpha \wedge \beta.$$

It is easily checked that the pairing defined above is well-defined. The pairing above can be equivalently defined as a linear map from $H^k_{\mathrm{dR}}(M)$ to $(H^{n-k}_{\mathrm{dR,c}}(M))^*$. We show that this linear map is an isomorphism.

Proposition 27.1. *Let M be a smooth, oriented, n-manifold that admits a good finite cover. Then*

$$H^k_{dR}(M) \cong (H^{n-k}_{dR,c}(M))^*$$

for each $0 \leq k \leq n$.

28. Poincaré Duality

Poincaré duality is a fundamental result in algebraic topology that relates the homology and cohomology groups of an orientable closed manifold. It states that for an n-dimensional orientable manifold M, there exists an isomorphism

$$H_k(M;\mathbb{Z}) \cong H^{n-k}_c(M;\mathbb{Z})$$

This duality provides deep insights into the topology of manifolds, constraining their possible homology groups and aiding in the computation of topological invariants. It also plays a crucial role in intersection theory. Before defining Poincaré duality, we need to define the notation of a fundamental class. In order to define a fundamental class, we need to define the notation of an orientation.

References

[Hat02] Allen Hatcher. *Algebraic Topology.* Cambridge University Press, 2002 (cit. on pp. 12, 43, 60, 113).

[Alu21] Paolo Aluffi. *Algebra: chapter 0.* Vol. 104. American Mathematical Soc., 2021 (cit. on p. 40).

[Wei94] Charles A. Weibel. *An introduction to homological algebra.* 38. Cambridge university press, 1994 (cit. on pp. 68, 78).

[BT13] Raoul Bott and Loring W Tu. *Differential forms in algebraic topology.* Vol. 82. Springer Science & Business Media, 2013 (cit. on p. 101).