Particle Astrophysics and Cosmology – Exercise sheet 2

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Wednesday 23 October To be submitted in next Wednesday lecture.

Homework

2.1 Getting proficient with the Friedmann equations and its solutions [5 points]

The Friedmann equations are the solution of Einstein's field equations for the Robertson-Walker metric

$$\mathrm{ds}^2 = -\mathrm{dt}^2 + \mathrm{a}^2(\mathrm{t}) \left[\frac{\mathrm{dr}^2}{1 - \mathrm{kr}^2} + \mathrm{r}^2 \mathrm{d}\theta^2 + \mathrm{r}^2 \sin^2\theta^2 \mathrm{d}\phi^2 \right] .$$

The first Friedmann equation reads

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \frac{8\pi G}{3}\rho ,$$

where G is the gravitational constant, k the curvature parameter, ρ the density, and a the scale factor of the universe. Often $\frac{\dot{a}}{a}$ is denoted as H, the Hubble parameter.

(a) The critical density is defined as $\rho_c = \frac{3H^2}{8\pi G}$. Rewrite the first Friedmann equation using this to obtain a relation between the spatial curvature $K = k/a^2$, the density and the critical density. How does the nature of the spatial curvature (positive, negative, zero) depend on the density and critical density?

(b) The critical density at the current time can be obtained by replacing H with the Hubble constant H_0. Calculate the value of the critical density and compare it to the density of earth's atmosphere, the density of the sun, and the best laboratory vacuum.

(c) How many hydrogen atoms per m^3 does the critical density correspond to?

(d) In general relativity, energy and momentum conservation is expressed in the *energy continuity equation*:

$$\dot{\rho} = -3(\rho + p)\frac{\dot{a}}{a}$$

For matter $p \approx 0$, for radiation $p = \rho/3$ and for vacuum energy for which $\rho = const.$ Solve the *energy continuity equation* for these cases by making an Ansatz of $\frac{\mathrm{d}(\rho\, \mathrm{a}^{\mathrm{n}})}{\mathrm{dt}} = 0$.

2.2 Age of the Universe [5 points]

Calculate (analytically) the age of a flat universe if radiation is neglected and it is presently made up of matter with $\Omega_{\mathrm{mat}} = 0.3$ and vacuum energy with $\Omega_{\mathrm{vac}} = 0.7$. You may use the following integral identity:

$$\int_0^1 \sqrt{\frac{x}{(1-\alpha) + \alpha x^3}} dx = \frac{2}{3\sqrt{\alpha}} \log\left(\frac{1 + \sqrt{\alpha}}{\sqrt{1-\alpha}}\right)$$

To use this you will need to set the current scale parameter as $a_0 = 1$, then integrate from the scale parameter value at the start of the universe, $a = 0$, to the current value $a = 1$.

2.3 Friedmann Cosmology [5 points]

The Friedmann equation for a flat universe ($k = 0$) with no dark-energy ($\Lambda = 0$) reads:

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho}{3} \tag{1}$$

(a) Using the ansatz $\rho = \rho_0(a_0/a)^n$, where a_0 and ρ_0 are the values at the current time, t_0, find an expression for the elapsed time, t, required to reach a density, ρ_1. You may also want to use $H_0 = 8\pi G\rho_0/3$.

(b) Now consider two such universes, one matter dominated and the other radiation dominated. For the same initial starting conditions (a_0, ρ_0, t_0), calculate the ratio of the elapsed times, $t_{\rm mat}/t_{\rm rad}$, required for each universe to reach the same density ρ_1.

2.4 Evolution of the Universe [5 points]

Assuming that the universe is flat,

(a) at what redshift did the matter density begin to dominate the expansion of the universe? Use today's values of $\Omega_{\rm mat} = 0.314$ and $\Omega_{\rm rad} = 10^{-4}$ and neglect the contribution to the energy density from vacuum energy.

(b) What is the minimium value of $\Omega_{\rm vac}$ that will result in an accelerated expansion today? What is the value of $\Omega_{\rm vac}$ needed to achieve acceleration at a redshift of $z = 0.5$? Neglect the contribution to the energy density from radiation.