Assignment 2

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Question

If A and B are square matrices of the same order such that AB=BA, then prove by induction that $AB^{\,n}=B^{\,n}A$.
Further, prove that $(AB)^n=A^{\,n}B^{\,n}$ for all nϵN.

Solution

A and B are square matrices of the same order such that AB=BA

To prove: P(n):$AB^n=B^nA$,nϵN

Using the principle of mathematical induction

For n=1, we have:

P(1):$AB=BA$ [Given]

$AB^1=B^1A$

i,e the result is true for n=1.

Let the result be true for n=k.

P(k):$AB^k=B^kA$ (1)

Now, we prove that the result is true for n=k+1.

$AB^{k+1} = AB^k.B = (B^k A)B$ (By(1))

$= B^k (AB)$ (By Associative law)

$= B^k (BA)$ (AB=BA (given))

$= (B^k B)A$ (By Associative law)

$= B^{k+1} A$

i,e the result is true for n=k+1.

Thus, by the principle of mathematical induction , we have $AB^n = B^n A$,nϵN.

Now, we prove that $(AB)^n = A^n B^n$ for all nϵN

For n=1, we have:

$(AB)^1 = A^1 B^1 = AB$

i,e the result is true for n=1.

Let it be true for n=k.

$(AB)^k = A^k B^k$ (2)

Now we prove that the result is true for n=k+1.

$(AB)^{k+1} = (AB)^k.(AB) = (A^k B^k)(AB).$ (By (2))

$= A^k (B^k A)B$ (By Associative law)

$= A^k (AB^k)B$ (AB^n=B^{n}A, nϵN)

$= (A^k A).(B^k B)$ (By Associative law)

$= A^{k+1} B^{k+1}$

i,e the result is true for n=k+1.

Therefore, by the principle of mathematical induction , we have $(AB)^n = A^n B^n$,

for all natural numbers.

Question

Let A=$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$,show that $(aI + bA)^n = a^n I + n a^{n-1} bA$,where I is tha identity matrix of order 2 and nϵN.

Solution

Using principle of mathematical induction
For n=1,we have

$P(1){:}(aI+bA)=aI+a^0 bA=aI+bA$

i,e result is true for n=1

Let it be true for n=k

i,e $P(k){:}(aI+bA)^k = a^k I + k a^{k-1} bA$

Now,we have to prove the result is true for n=k+1.

$$(aI+bA)^{k+1} = (aI+bA)^k (aI+bA)$$

$$= (a^k I + k a^{k-1} bA)(aI+bA)$$

$$= a^{k+1} I + k a^k bAI + a^k bIA + k a^{k-1} b^2 A^2$$

$$= a^{k+1} I + (k+1) a^k bA + k a^{k-1} b^2 A^2 \tag{1}$$

Now, $A^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$

Therfore,eqn(1) becomes
$(aI+bA)^{k+1} = a^{k+1} I + (k+1) a^k bA$

Hence proved result is true for n=k+1

Therefore, by principle of mathematical induction we have $(aI\ +\ bA)^n = a^n I + na^{n-1} bA$