Kinematics

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September 8, 2020

Welcome

Lecture Recordings This lecture will be recorded. By attending the live sessions, you agree to the recording, and you understand that your image, voice, and name may be disclosed to classmates

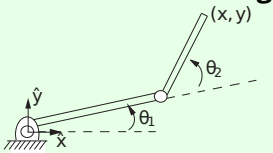
During the lecture
I will mute the audience
Ask questions through 'Chat'

Outline

Outline

Robots

A robot is a set of **rigid bodies** connected by a set of **joints**



Robots

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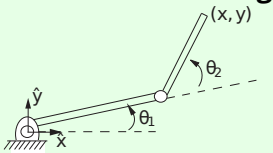

Rigid-body

A rigid-body is a solid object in which deformation can be neglected

Configuration Space

Configuration $\mathbf{q}$
a set of quantities that provides a complete specification of the position of every point of the robot

Degree-of-freedom $\mathcal{N}$
the smallest number that is needed to represent the configuration space

(a.k.a. generalized coordinates, joint-space, state space)

Configuration Space

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Degree-of-freedom $\mathcal{N}$
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The first thing to describe your robot

$$\mathbf{q} = \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_{\mathcal{N}} \end{bmatrix}$$

Configuration Space

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a set of quantities that provides a complete specification of the position of every point of the robot

Degree-of-freedom $\mathcal{N}$
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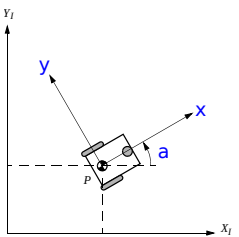
Example (Mobile Robots)



Origin: P
Configuration: x-y position, orientation

$$\mathbf{q} = \begin{bmatrix} x \\ y \\ a \end{bmatrix}$$

Configuration Space

Configuration q
a set of quantities that provides a complete specification of the position of every point of the robot

Degree-of-freedom $\mathcal{N}$
the smallest number that is needed to represent the configuration space

Example (Quadrotor)

Origin: P
Configuration:
x, y, z, roll, pitch, yaw

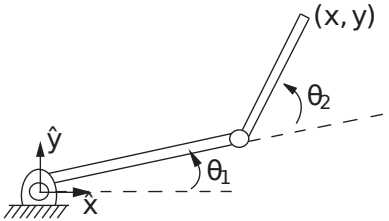
$$\mathbf{q} = \begin{bmatrix} x \\ y \\ a \\ \rho \\ \theta \\ \psi \end{bmatrix}$$

Configuration Space

Configuration q
a set of quantities that provides a complete specification of the position of every point of the robot

Degree-of-freedom $\mathcal{N}$
the smallest number that is needed to represent the configuration space

Example (Planar Robot Arm)



Origin: **0**
Configuration: joint positions

$$\mathbf{q} = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}$$

Task Space

Task space

space in which the robot's task can be naturally expressed
may not be the same with the configuration space

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{\mathcal{P}} \end{bmatrix}$$

$\mathcal{P}$ is the dimensionality of the task

a robot can have more than one task space

Task space

space in which the robot's task can be naturally expressed
may not be the same with the configuration space

Example (Manipulation)

move hand from position A to position B

- Configuration space
  - 7 joint position $\mathbf{q} \in \mathbb{R}^7$
- Task space
  - end-effector (hand) position $\mathbf{x} \in \mathbb{R}^3$

Task space

space in which the robot's task can be naturally expressed
may not be the same with the configuration space

Example (Quadruped Locomotion)

walking from point A to point B

- Configuration space
  - Base position and orientation
  - 12 Joint positions $\mathbf{q} \in \mathbb{R}^{18}$
- Task Space
  - Body position $\mathbf{x}_{body} \in \mathbb{R}^3$
  - Feet position $\mathbf{x}_{feet} \in \mathbb{R}^3$

Positions/Translations

Displacement from a reference point

$$\mathbf{p} = \begin{bmatrix} \mathbf{p}_x \\ \mathbf{p}_y \end{bmatrix} \in \mathbb{R}^2 \qquad \mathbf{p} = \begin{bmatrix} \mathbf{p}_x \\ \mathbf{p}_y \\ \mathbf{p}_z \end{bmatrix} \in \mathbb{R}^3$$

Positions/Translations

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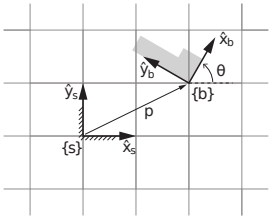
Example (2D displacement)



Reference frame {s}
Robot frame {b}

Orientations

- **angles**: rotation with respect to a fixed coordinate system
- **unit quaternions**: alternative representation in 3D
- **rotation matrices**: matrices that rotate vectors

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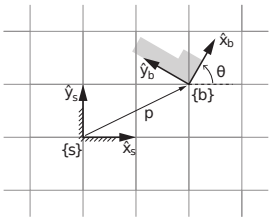
Example (Orientations in 2D)



Angle: θ
(w.r.t. x-axis)

$$\mathbf{R} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

Orientations

- **angles**: rotation with respect to a fixed coordinate system
- **unit quaternions**: alternative representation in 3D
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Rotation Matrices

$$\mathbf{R} = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix} \in \mathbb{SO}(2) \qquad \mathbf{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \in \mathbb{SO}(3)$$

$\mathbb{SO}(2)$ and $\mathbb{SO}(3)$: **special orthogonal** group in $\mathbb{R}^2$ and $\mathbb{R}^3$

Homogeneous Transformation Matrix
Represent orientation and position of a rigid body

$$\mathbf{T} = \begin{bmatrix} \mathbf{R} & \mathbf{p} \\ \mathbf{0} & 1 \end{bmatrix} \in \mathbb{SE}(.) : \text{ special euclidean group}$$

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Representations in $\mathbb{R}^2$ and $\mathbb{R}^3$

$$\mathbf{T} = \begin{bmatrix} r_{11} & r_{12} & p_x \\ r_{21} & r_{22} & p_y \\ 0 & 0 & 1 \end{bmatrix} \in \mathbb{SE}(2) \qquad \mathbf{T} = \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \in \mathbb{SE}(3)$$

Properties of Homogeneous Transformation

Question: When do I use it?

- to represent the position and orientation of a rigid body
- to change the reference frame
- to displace a vector or frame

Properties of Homogeneous Transformation

The inverse of a transformation matrix is a transformation matrix

$$\mathbf{T}^{-1} = \begin{bmatrix} \mathbf{R} & \mathbf{p} \\ \mathbf{0} & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{R}^\top & -\mathbf{R}^\top \mathbf{p} \\ \mathbf{0} & 1 \end{bmatrix}$$

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The multiplication of transformation matrices is associative

$$(\mathbf{T}_1 \mathbf{T}_2)\mathbf{T}_3 = \mathbf{T}_1(\mathbf{T}_2 \mathbf{T}_3)$$

Outline

Kinematics is a geometric description of motion

- Positions, velocities, accelerations
- Visually observable

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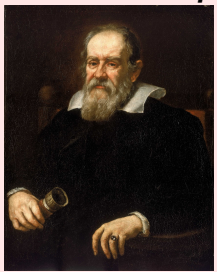
Galileo's Experiments on the Tower of Pisa



Question: Given current configurations, what is the task space?
Configurations provide a complete specification of the position of every point of the robot. How do we compute it.

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Forward Kinematics
Relates the configuration space to the task space

Forward Kinematics at Position Level

$$\mathbf{x} = \mathcal{F}(\mathbf{q})$$

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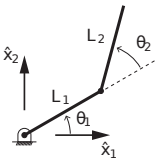
Example (2D planar arm)



configuration $\boldsymbol{\theta} = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}$

task $\mathbf{x} = \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix}$

Forward Kinematics at Position Level

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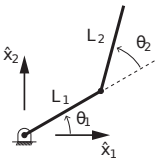

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$$\text{task } \mathbf{x} = \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix}$$

$$\mathbf{x}_1 = L_1 \cos(\theta_1) + L_2 \cos(\theta_1 + \theta_2)$$

$$\mathbf{x}_2 = L_1 \sin(\theta_1) + L_2 \sin(\theta_1 + \theta_2)$$

Forward Kinematics using Homogenous Transformation Matrix

The product of two transformation matrices is a transformation matrix

$$\mathbf{T}_1\mathbf{T}_2 = \begin{bmatrix} \mathbf{R}_1\mathbf{R}_2 & \mathbf{R}_1\mathbf{p}_2 + \mathbf{p}_1 \\ \mathbf{0} & 1 \end{bmatrix}$$

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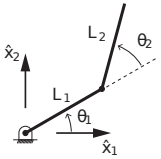

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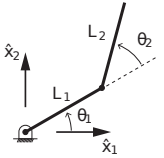

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$$\mathbf{T}_{01} = \begin{bmatrix} \cos(\theta_1) & -\sin(\theta_1) & 0 \\ \sin(\theta_1) & \cos(\theta_1) & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

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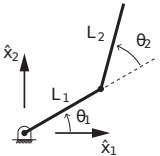
Example (2D planar arm)



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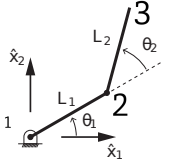

configuration $\boldsymbol{\theta} = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}$

task $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

points?

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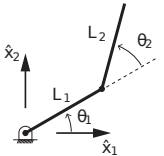

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$$\mathbf{T}_{03} = \mathbf{T}_{01}\mathbf{T}_{12}\mathbf{T}_{23}$$

Question: What about velocity?
Velocity is the time derivative of position

Forward Kinematics

$$\frac{\partial \mathbf{x}}{\partial t} = \frac{\partial \mathcal{F}(\mathbf{q})}{\partial t}$$ (time derivative of forward kineamtics)

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$$\dot{\mathbf{x}} = \frac{\partial \mathcal{F}(\mathbf{q})}{\partial \mathbf{q}} \frac{\partial \mathbf{q}}{\partial t}$$ (chain rule)

Forward Kinematics

$$\frac{\partial \mathbf{x}}{\partial t} = \frac{\partial \mathcal{F}(\mathbf{q})}{\partial t} \qquad \text{(time derivative of forward kineamtics)}$$

$$\dot{\mathbf{x}} = \frac{\partial \mathcal{F}(\mathbf{q})}{\partial \mathbf{q}} \frac{\partial \mathbf{q}}{\partial t} \qquad \text{(chain rule)}$$

$$= \begin{bmatrix} \frac{\partial \mathcal{F}_1(\mathbf{q})}{\partial q_1} & \frac{\partial \mathcal{F}_1(\mathbf{q})}{\partial q_2} & \cdots & \frac{\partial \mathcal{F}_1(\mathbf{q})}{\partial q_\mathcal{N}} \\ \frac{\partial \mathcal{F}_2(\mathbf{q})}{\partial q_1} & \frac{\partial \mathcal{F}_2(\mathbf{q})}{\partial q_2} & \cdots & \frac{\partial \mathcal{F}_2(\mathbf{q})}{\partial q_\mathcal{N}} \\ \vdots & & & \\ \frac{\partial \mathcal{F}_\mathcal{P}(\mathbf{q})}{\partial q_1} & \frac{\partial \mathcal{F}_\mathcal{P}(\mathbf{q})}{\partial q_2} & \cdots & \frac{\partial \mathcal{F}_\mathcal{P}(\mathbf{q})}{\partial q_\mathcal{N}} \end{bmatrix} \dot{\mathbf{q}}$$

(partial derivative of a vector over a vector gives you a matrix)

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(partial derivative of a vector over a vector gives you a matrix)

$$= \mathbf{J}(\mathbf{q})\dot{\mathbf{q}}$$

$$\frac{\partial \mathbf{x}}{\partial t} = \frac{\partial \mathcal{F}(\mathbf{q})}{\partial t} \qquad \text{(time derivative of forward kineamtics)}$$

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(partial derivative of a vector over a vector gives you a matrix)

$$= \mathbf{J}(\mathbf{q})\dot{\mathbf{q}}$$

where $\mathbf{J} \in \mathbb{R}^{\mathcal{P} \times \mathcal{N}}$ is called the **jacobian** matrix

Forward Kinematics at Velocity Level

$$\dot{\mathbf{x}} = \mathbf{J}(\mathbf{q})\dot{\mathbf{q}}$$

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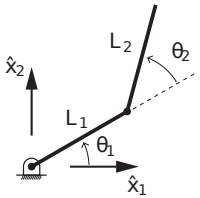
Let $\mathbf{J}_i$ be the i^{th} row of $\mathbf{J}$.

$$\dot{\mathbf{x}}_1 = \mathbf{J}_1\dot{\boldsymbol{\theta}}$$
$$\dot{\mathbf{x}}_2 = \mathbf{J}_2\dot{\boldsymbol{\theta}}$$
$$\vdots$$
$$\dot{\mathbf{x}}_{\mathcal{P}} = \mathbf{J}_{\mathcal{P}}\dot{\boldsymbol{\theta}}$$

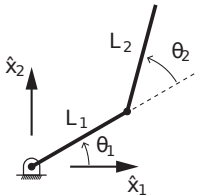
Example (2D planar arm)



configuration $\boldsymbol{\theta} = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}$

task $\mathbf{x} = \begin{bmatrix} \mathsf{x}_1 \\ \mathsf{x}_2 \end{bmatrix}$

Example (2D planar arm)



configuration $\boldsymbol{\theta} = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix}$

task $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

Take time-derivative on both sides

$$\dot{x}_1 = -L_1\dot{\theta}_1 \sin(\theta_1) - L_2(\dot{\theta}_1 + \dot{\theta}_2)\sin(\theta_1 + \theta_2)$$
$$\dot{x}_2 = L_1\dot{\theta}_1 \cos(\theta_1) + L_2(\dot{\theta}_1 + \dot{\theta}_2)\cos(\theta_1 + \theta_2)$$

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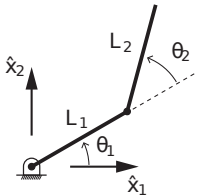

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Rearrange into the form of $\dot{\mathbf{x}} = \mathbf{J}\dot{\mathbf{q}}$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -L_1 \sin(\theta_1) - L_2 \sin(\theta_1 + \theta_2), & -L_2 \sin(\theta_1 + \theta_2) \\ L_1 \cos(\theta_1) + L_2 \cos(\theta_1 + \theta_2), & L_2 \cos(\theta_1 + \theta_2) \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

Forward Kinematics at Acceleration Level

$$\frac{\partial \dot{\mathbf{x}}}{\partial t} = \frac{\partial \left[\mathbf{J}\dot{\mathbf{q}}\right]}{\partial t}$$

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Forward Kinematics at Acceleration Level

$$\begin{aligned}\frac{\partial \dot{\mathbf{x}}}{\partial t} &= \frac{\partial \left[\mathbf{J}\dot{\mathbf{q}}\right]}{\partial t} \\ \ddot{\mathbf{x}} &= \mathbf{J}\frac{\partial \dot{\mathbf{q}}}{\partial t} + \frac{\partial \mathbf{J}}{\partial t}\dot{\mathbf{q}} \\ &= \mathbf{J}\ddot{\mathbf{q}} + \dot{\mathbf{J}}\dot{\mathbf{q}}\end{aligned}$$

Forward Kinematics

Question: Too much math?

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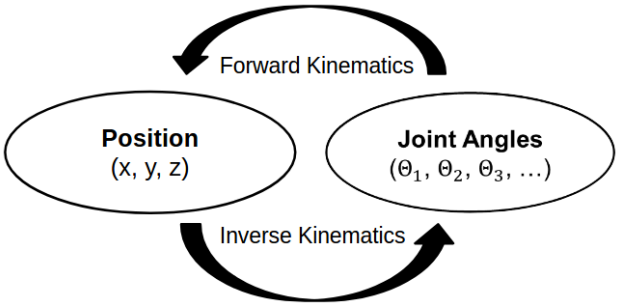
most robotics library offers forward kinematics and Jacobian

Outline

Inverse Kinematics



Question: Given the task-space velocities, what are the configuration space velocities?

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Question: We know $\dot{\mathbf{x}} = \mathbf{J}\dot{\mathbf{q}}$, can we do $\dot{\mathbf{q}} = \mathbf{J}^{-1}\dot{\mathbf{x}}$?
No! $\mathbf{J} \in \mathbb{R}^{\mathcal{P} \times \mathcal{N}}$ is rank deficient, and the inverse does not exist

Inverse Kinemaitcs

$$\dot{\mathbf{x}} = \mathbf{J}\dot{\mathbf{q}} \quad \text{(forward kinematics)}$$

$$\Downarrow$$

$\dot{\mathbf{q}} = \mathbf{J}^{\dagger}\dot{\mathbf{x}}$ where $\mathbf{J}^{\dagger}$ is the **pseudo inverse** of $\mathbf{J}$

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$\dot{\mathbf{q}} = \mathbf{J}^{\dagger}\dot{\mathbf{x}}$ where $\mathbf{J}^{\dagger}$ is the **pseudo inverse** of $\mathbf{J}$

Pseudo-inverse has properties that are similar to inverse

$$\begin{cases} \mathbf{J}^{\dagger} \text{ exists for all } \mathbf{J} \\ \mathbf{J}\mathbf{J}^{\dagger} = \mathbf{I} \\ \mathbf{J}\mathbf{J}^{\dagger}\mathbf{J} = \mathbf{J} \end{cases}$$

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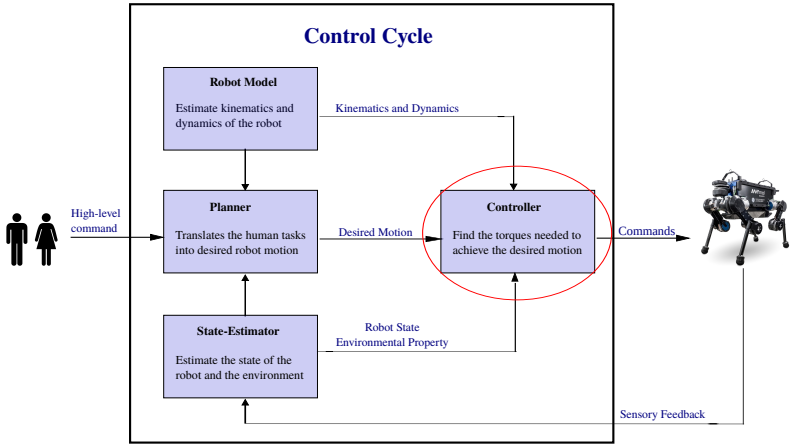
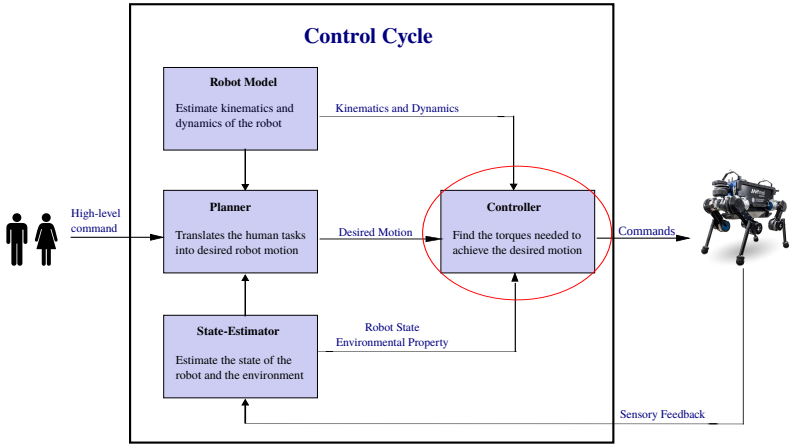
- most tools (matlab, python, c++) have a function to calculate pseudo inverse
- If $\mathbf{J}$ is full row rank, $\mathbf{J}^{\dagger} = \mathbf{J}(\mathbf{J}^{\top}\mathbf{J})^{-1}$ has an analytical solution

Controllers



Controllers



Commands

- positions
- velocities
- torques

Inverse Kinematics Controller

Desired position $\mathbf{x}^* \in \mathbb{R}^3$

$$\mathbf{x}^* = \begin{bmatrix} \mathbf{x}_x^* \\ \mathbf{x}_y^* \\ \mathbf{x}_z^* \end{bmatrix}$$

provided by the planner

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current position $\mathbf{x} \in \mathbb{R}^3$

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calculated by forward kinematics

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calculated by forward kinematics

Question: How to move from $\mathbf{x}$ to $\mathbf{x}^*$
The desired task space velocity: $\dot{\mathbf{x}} = \mathbf{x}^* - \mathbf{x}$
The desired configuration space velocity: $\dot{\mathbf{q}} = \mathbf{J}^\dagger \dot{\mathbf{x}}$

Matlab Example

Example (Kinova Jaco arm)

Move hand from position A to position B
$\mathbf{q} \in \mathbb{R}^7$: joints
$\mathbf{x} \in \mathbb{R}^3$: hand position
$\mathbf{J} \in \mathbb{R}^{3\times 7}$: Jacobian

Blue: initial
Red: target
Green: trajectory

Outline

What is Redundancy?

[0] © Ministry of Silly Walks

What is Redundancy?

Redundancy
More degree-of-freedom than what is required for your task
There are more than one solutions to achieve the desired task

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Redundancy
More degree-of-freedom than what is required for your task
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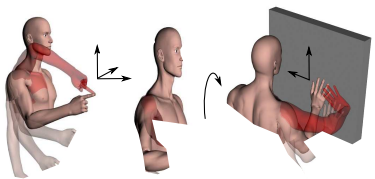
Example

- keep your finger tip and the same position and move your elbows



- joint-space velocity but no task-space velocity

Redundancy

Mathematically....

Redundancy

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The nullspace of a matrix $\mathbf{A}$ consists of all the vectors $\mathbf{x}$ such that $\mathbf{A}\mathbf{x} = \mathbf{0}$

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Nullspace
The nullspace of a matrix $\mathbf{A}$ consists of all the vectors $\mathbf{x}$ such that $\mathbf{A}\mathbf{x} = \mathbf{0}$

so... $\mathbf{J}\dot{\mathbf{q}}^0 = \mathbf{0}$ for some $\dot{\mathbf{q}}^0$

Inverse Kinemaitcs Controller with Redudnacy Resolution

$\dot{\mathbf{q}} = \mathbf{J}^{\dagger}\mathbf{x} + \mathbf{N}\dot{\mathbf{q}}^{0}$ where

$\mathbf{N} = \mathbf{I} - \mathbf{J}^{\dagger}\mathbf{J}$ is the nullspace projection matrix

Inverse Kinemaitcs Controller with Redudnacy Resolution

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A projection matrix $\mathbf{N}$ projects a vector onto its subspace

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  $\mathbf{J}\mathbf{N}\dot{\mathbf{q}}^{0} = \mathbf{J}(\mathbf{I} - \mathbf{J}^{\dagger}\mathbf{J})\dot{\mathbf{q}}^{0} = (\mathbf{J} - \mathbf{J}\mathbf{J}^{\dagger}\mathbf{J})\dot{\mathbf{q}}^{0} = (\mathbf{J} - \mathbf{J})\dot{\mathbf{q}}^{0} = \mathbf{0}$

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Move the redundant dimension to some default positions

$$\dot{\mathbf{q}}^0 = \mathbf{q}^0 - \mathbf{q}$$

$\mathbf{q}^0$ is the home configuration

$\mathbf{q}$ is the current configuration

Matlab Example

Without Redundancy Resolution

With Redundancy Resolution