COSSERAT THEORIES: SHELLS, RODS AND POINTS

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Cosserat Theories: Shells, Rods and Points

by

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DEDICATION

This book is dedicated to my loving wife Laurel and my sons Adam and Daniel. They have created a beautiful family environment which has given me the necessary peace of mind to concentrate on my research.

IN MEMORIAL

I have had the great honor and pleasure to have worked for 21 years with Professor Paul M. Naghdi of the Department of Mechanical Engineering at the University of California at Berkeley. I began my work with him in August 1973 during my first quarter at Berkeley as a graduate student and I continued working with him as a colleague until his death on the 9th of July 1994. The idea for this book was conceived after Paul's death as an attempt to preserve his unique approach to the formulation of Cosserat theories. Although the notation and some of the results presented in this book are new, Paul has had such an influence on my thinking that I share all of the originality in this book with him. However, I accept full responsibility for any conceptual or typographical errors.

CONTENTS

PREFACE

Continuum mechanics provides a theoretical structure for analyzing the response of materials to mechanical and thermal loads. One of the beauties of continuum mechanics is that the fundamental balance laws (conservation of mass and balances of linear momentum, angular momentum, energy and entropy) are valid for all simple materials. Most of the modern research in continuum mechanics focuses on the development of constitutive equations which are used to characterize the response of a particular class of materials (e.g. invisicid fluids, viscous fluids, elastic solids, viscoelastic solids, elastic-plastic solids, elastic-viscoplastic solids, etc.).

Within the context of the purely mechanical three-dimensional theory, the conservation of mass and the balance of linear momentum are used to determine the mass density and the position of each material point in the continuum. Whereas, the balance of angular momentum and the notion of invariance under superposed rigid body motions are used to place restrictions on the constitutive equations. In this regard, it should be emphasized that these restrictions help eliminate fundamentals errors in specific constitutive assumptions, but they do not completely characterize the physics of a particular material. Ultimately, the validity of a set of constitutive equations depends on the ability of the person developing the equations to creatively synthesize the experimental data and to propose appropriate history-dependent variables and functional forms that capture the main physics of the material response.

The equations characterizing the three-dimensional theory are nonlinear partial differential equations that depend on three spatial coordinates and time. Therefore, in the analysis of a particular structure, it is necessary to satisfy these equations at each point in the structure as well as to satisfy appropriate conditions on the boundary of the structure. However, some structures have special geometrical properties that can be exploited to develop simplified theories. For example, a shell-like structure is "thin" in one of its spatial dimensions, a rod-like structure is "thin" in two of its spatial dimensions, and a point-like structure is "thin" in all three of its spatial dimensions. Consequently, under certain conditions, it is possible to model: a shell-like structure with equations that depend on only two spatial coordinates and time; a rod-like structure with equations that depend on only one spatial coordinate and time; and a point-like structure with equations that depend on time only.

In their classical paper (E. and F. Cosserat, 1909), the Cosserat brothers proposed new equations which generalized the notion of a continuum to include a set of director vectors at each material point in addition to the position vector. Although a three-dimensional Cosserat continuum is somewhat abstract, the director vectors for shells, rods and points are very physical. Specifically, within the context of the theory of elasticity: a single director is introduced in shell theory to characterize a material fiber through the thickness of the shell; two directors are introduced in rod theory to characterize two material fibers in the cross-section of the rod; and three directors are introduced in point theory to characterize three material fibers in the point-like structure.

Since the directors are general three-dimensional vectors, the Cosserat theories of shell, rods and points allow for more general deformation than the classical theories. In particular, these Cosserat theories are general enough to model all homogeneous deformations exactly. Moreover, they allow for extension and contraction of the material fibers, which can be of great significance in contact problems.

Many articles are available in the literature that describe aspects of the Cosserat theories. However, since these articles typically assume familiarity with tensor analysis in general curvilinear coordinates, they are unintelligible by many practicing structural engineers. Moreover, often the articles characterize the constitutive equations in terms of a general strain energy function and they do not propose specific functional forms, especially for nonlinear deformations.

One objective of this book is to attempt to remedy both of these problems. Specifically, here the theories are developed using a minimum of mathematics and the essential mathematical preliminaries are included. Also, an attempt has been made to provide specific constitutive equations for nonlinear elastic response which require specification of only typical material and geometric constants. Consequently, these equations can be used directly to formulate and solve nonlinear structural problems.

Another objective of this book is to present a unified approach to the development of these Cosserat theories. Specifically, four level of theory are unified: three-dimensional theory; two-dimensional shell theory; one-dimensional rod theory; and zero-dimensional point theory. The Cosserat approach to the development of these structural theories is particularly enlightening because it allows the development of theory that parallels the full three-dimensional development. In particular, the basic balance laws include the conservation of mass and the balance of linear momentum as well as: a single balance of director momentum for shell theory; two balances of director momentum for rod theory; and three balances of director momentum for point theory. Moreover, the balance of angular momentum and the notion of invariance under superposed rigid body motions are used to place restrictions on the constitutive equations. Also, the constitutive equations for elastic structures are developed in terms of a strain energy function so that the resulting equations preserve all of the fundamental properties of the equations of three-dimensional elasticity (e.g. an elastic material is an ideal material and the work done on the material from one state to another is path-independent).

The notation used in this book has been modified relative to that which appears in the literature in order to maximize the uniformity of all four levels of theory. This notation makes it easy to recognize similar concepts in each level of theory. Consequently, ideas that are well understood in the three-dimensional theory can be more easily generalized to the structural theories of shells, rods and points.

This book is intended for graduate students and researchers in structural mechanics who are unfamiliar with the beauty and power of the Cosserat theories. A number of example problems have been analyzed to help evaluate the validity of constitutive assumptions for shells, rods and points. Moreover, exercise problems have been included to help the reader develop essential technical skills that are required to understand and utilize the theory effectively. Also, the book is structured so that the reader can proceed from the preliminary background material directly to the study of either shells, rods or points.

CHAPTER 1

INTRODUCTION

1.1 The basic idea of a Cosserat model

In the usual simple three-dimensional continuum it is sufficient to describe the motion of the continuum by the position vector which identifies the location of each material point as a function of time. For the purely mechanical theory, the laws of conservation of mass and the balance of linear momentum are used to determine the present values of the mass density and this position vector. Also, the balance of angular momentum is used to place restrictions on the constitutive equations of the continuum (i.e. the symmetry of the stress tensor).

From a historical perspective Naghdi (1972, p.445) states that the concept of a directed media was introduced by Duhem (1893) and that the two French brothers E. and F. Cosserat (1909) were the first to present a systematic development of theories for directed continua. In the three-dimensional context, the motion of a directed continuum is characterized by the position vector as well as additional vector quantities called directors which are assigned to each material point. Consequently, in addition to the laws of conservation of mass and balance of linear momentum, it is necessary to introduce balances of director momentum which together are used to determine the present values of the mass density, the position vector and the director vector fields. The balance of angular momentum again places restrictions on the constitutive equations in a similar manner to the three-dimensional theory.

At first, a three-dimensional directed media may seem rather abstract because the physical meaning of the directors is not immediately apparent. However, this theory has been used quite successfully to describe the response of liquid crystals (Ericksen, 1961) which have found a number of applications in recent years. Also, the use of Cosserat theory to describe couple stresses was considered by Toupin (1964), and related micropolar theories were described by Eringen and Suhubi (1964), and Suhubi and Eringen (1964).

In this book attention will be focused on Cosserat theories for shells, rods and points. From the geometrical point of view, shells, rods and points are structures that are "thin" in one or more of their dimensions. Specifically, a shell is a structure that is "thin" in one of its dimensions so that it is essentially a curved surface in space with some small thickness. A rod is a structure that is "thin" in two of its dimensions so that it is essentially a space curve with some small cross-sectional area. Finally, a point-like

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structure is "thin" in all three of its dimensions so that it is essentially a point in space with some small finite volume.

It will be seen that for these Cosserat theories the directors have direct physical interpretations because they become models for material fibers in specific directions. In particular, the theory of a Cosserat shell models the shell structure as a surface with an additional director which can model the deformation of a material fiber through the thickness of the shell. In general, the fiber (director) can stretch in length and shear relative to the normal to the shell surface. The theory of a Cosserat rod models the rod structure as a space curve with two additional directors which can model material fibers in the cross-section of the rod. In general, these fibers can stretch in length and shear relative to each other and the plane normal to the space curve. Finally, the theory of a Cosserat point models a point-like structure as a point with three additional directors which are linearly independent and can model material fibers in a finite region that experiences general homogeneous deformation. It should also be mentioned that a hierarchy of theories of shells, rods and points with an arbitrary but finite number of directors have been developed (Ericksen and Truesdell, 1958; Naghdi, 1972; Green et al., 1974a,b; Green and Naghdi, 1991), but they will not be discussed here.

Due to the fact that theories of shells, rods, and points are not exact theories from the three-dimensional point of view, there are a number of different approaches to the formulations of these theories. Some developments are limited to linearized theories, others to small strain but large rotation theories, and still others attempt to formulate general nonlinear theories. One approach is to use asymptotic expansions of the kinematical variables and the equations of motion which are then truncated by neglecting specified orders of the expansions. Another approach is called the direct approach which models a directed media by introducing additional kinematical variables (directors) at each material point, together with additional balance laws. The Cosserat theories discussed in this book are nonlinear theories which are examples of this direct approach.

One of the main advantages of the Cosserat approach is that the balance laws of the directed media are formulated as integral balance laws that have similar fundamental properties to those of the full three-dimensional theory. Specifically, the Cosserat equations are inherently nonlinear (from both the kinematic and constitutive points of view) and they are properly invariant under superposed rigid body motions. Also, the balance laws are valid for structures that are composed of arbitrary material properties (e.g. solids or fluids) and the constitutive equations are developed in a similar manner to those in the three-dimensional theory. In particular, for structures that are fabricated using nonlinear anisotropic elastic materials, the constitutive equations are hyperelastic in the sense that the response functions are determined by derivatives of a strain energy function.

A comprehensive review of the Cosserat theory of shells can be found in the article by Naghdi (1972) and early developments of the Cosserat theory of rods are presented by Ericksen and Truesdell (1958), Green (1959), Suhubi (1968), and Green et al.

(1974a,b). Also, a more modern direct notation formulation of both the theories of shells and rods can be found in (Naghdi, 1982). Another review of rod theory is presented by Antman (1972) and a more modern discussion of nonlinear problems of elasticity for both shells and rods can be found in (Antman, 1995; Villaggio, 1997).

The work of Slawianowski (1974, 1975, 1982) seems to be the first to analyze the homogeneous deformation of zero-dimensional bodies. Later, Cohen (1981), Muncaster (1984a,b), and Cohen and Muncaster (1984, 1988) developed the theory of pseudo-rigid bodies which also analyzes homogeneous deformations. These works have similarity with the theory of a Cosserat point (Rubin, 1985a) which has been used to formulate the numerical solution of mechanical problems (Rubin, 1985a,b, 1986, 1987a) and two- and three-dimensional thermomechanical problems (Rubin, 1995). Similar ideas have been applied using Cosserat shell theory to formulate the numerical solution of spherically symmetric problems (Rubin, 1987b). Also, recently the theory of a Cosserat point has been used to develop numerical solutions for rod problems (Rubin, 2000).

Cosserat theories are being used to describe an increasing number of physical phenomena. For example, Green, Naghdi and their coworkers have developed Cosserat theories for many applications which include: shells (Naghdi, 1972); rods (Green et al., 1974a,b); fluid jets and sheets (Naghdi, 1979); electromagnetic effects in shells (Green and Naghdi, 1983) and rods (Green and Naghdi, 1985); turbulence (Marshall and Naghdi, 1989a,b); microcrack growth (Marshall et al., 1991); composite materials (Green and Naghdi, 1991); and a model of dislocations in three-dimensional plasticity theory (Naghdi and Srinivasa, 1993a,b). In addition, a computer search of the Compendex and Inspec data bases indicates that Cosserat theories are also being used to describe: finite elements (Simo and Fox, 1989), (Simo et al., 1989) and (Simo et al., 1990); shear-banding and liquefaction in granular materials (Vardoulakis, 1989); fracture of bone (Lakes et al., 1990); size effects in rocks (Sulem and Vardoulakis, 1990) and foams (Lakes, 1993); liquid bridges subjected to microgravity (Meseguer and Perales, 1992); grain rotations in granular media (Alehossein and Muhlhaus, 1994); suppression of localization in plasticity (De Borst, 1991; Iordache and Willam, 1995); fracture scaling parameters of inhomogeneous microstructure in composites (Cairns et al., 1995); micromechanics of inclusions (Cheng and He, 1995) and failure of welds (Craine and Newman, 1996).

1.2 A brief outline of the book

The theoretical foundation of Cosserat theories of shells, rods and points can be studied from a number of journal articles and reference books which may present the material using different notations and different theoretical approaches. One of the objectives of the present book is to present an introduction to this material using a unified formulation and a unified notation. It is hoped that this book will provide an easy reference for both graduate students and researchers to become familiar enough with

Cosserat theories that they can contribute to the increasing number of areas of application of Cosserat theories in mechanics today.

For simplicity, attention will be confined to purely mechanical Cosserat theories of shells, rods and points that are valid for general nonlinear elastic materials. Formulations of more general thermomechanical theories can be found in (Green and Naghdi, 1979a) for shells, in (Green and Naghdi, 1979b) for rods, and in (Green and Naghdi, 1991; and Rubin, 1995) for points.

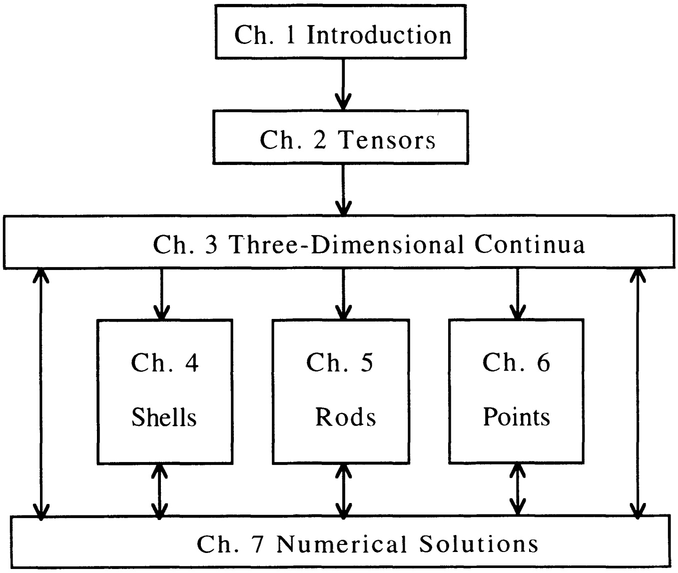

Fig. 1.2.1 A schematic flow chart of the contents of this book

Since the theories discussed here are nonlinear (both from the kinematic and constitutive points of view), a plate which is initially a flat surface can deform into a curved shell, and a beam which is initially a straight bar can deform into a curved rod. It therefore is quite natural to use convccted curvilinear coordinates to identify material points. Use of these coordinates requires some familiarity with tensor analysis in general coordinate systems. Most often the tensor equations are written in component forms which require knowledge of covariant and contravariant components of tensors, covariant differentiation, and Christoffel symbols. For this reason, the language and symbols commonly used in the development of Cosserat theories of shells and rods cause these theories to remain relatively inaccessible to a large number of people.

The main difference between curvilinear coordinates and rectangular Cartesian coordinates is that the base vectors of the curvilinear coordinate system are functions of the coordinates, whereas those of the rectangular Cartesian coordinate system are constants. The Christoffel symbols that appear in the definition of covariant differentiation are merely symbols that characterize various components of the derivatives of these curvilinear base vectors with respect to the coordinates. Consequently, by recognizing the fact that the curvilinear base vectors are functions of coordinates, it is possible to introduce all the necessary tensor operations using only the notion of partial differentiation, without the need to define the Christoffel symbols. This tends to cause the resulting expressions to appear less complicated and more familiar to many people. Of course, the two approaches yield identical equations when the component forms are fully expanded.

The mathematics of tensor analysis in general curvilinear coordinates is merely a necessary tool for describing the mechanics of shell and rod structures. This book will attempt to focus attention on the physical assumptions associated with the Cosserat models of shells, rods, and points, and on the structure of these theories from a mechanics point of view. Consequently, the book will emphasize only those mathematical operations which are essential. For example, the material on tensor analysis presented in sections 2.1 through 2.3 is essential for an understanding of the material in the remainder of the book, whereas the material on covariant differentiation and Christoffel symbols in section 2.4 is not essential. This later material is included to help the reader relate the present development to others which appear in the literature, and to serve as an introduction to other related mathematical topics for those readers so inclined.

Fig. 1.2.1 shows a schematic flow chart of the contents of this book. Chapter 2 deals with basic tensor operations. Chapter 3 reviews the mechanics of three-dimensional continua and sets the framework within which the Cosserat theories are developed. The next three chapters present the Cosserat theories of shells (chapters 4), rods (chapter 5), and points (chapter 6). Then, chapter 7 describes how the theory of a Cosserat point can be used to formulate the numerical solution of a number of problems in continuum mechanics. Also, appendix A provides additional information about tensor operations and appendix B provides useful results in cylindrical polar coordinates and spherical polar coordinates.

This order of presentation of the material is chosen for simplicity in developing the new features of these Cosserat theories. Specifically, in addition to the usual position vector of a material point, the Cosserat shell theory of chapter 4 introduces only a single director and a single director momentum equation. The Cosserat rod theory of chapter 5 introduces two directors and two director momentum equations, and the Cosserat point theory of chapter 6 introduces three directors and three director momentum equations. However, from the point of view of simplicity of the required solution procedures, the order of these theories is reversed. This is because the equations of motion are ordinary differential equations that depend on time only for the Cosserat point theory, they are partial differential equations that depend on time and only one spatial coordinate for the Cosserat rod theory, and they are partial differential equations that depend on time and two spatial coordinates for the Cosserat shell theory. Obviously, these equations of motion are still simpler than those of the three-dimensional theory, which are partial differential equations that depend on time and three spatial coordinates.

Three-dimensional Continua	
Position Vector	$\mathbf{x}^* = \mathbf{x}^*(\theta^i,t)$
Mass	$\dot{m}^* = 0 \;,\; m^* = \rho^* g^{1/2} = \rho_0^* \, G^{1/2} = m^*(\theta^i)$
Linear Momentum	$m^* \dot{\mathbf{v}}^* = m^* \mathbf{b}^* + \mathbf{t}^{*i}{}_{,i}$
Angular Momentum	$\mathbf{T}^* = \mathbf{T}^{*T} \;,\; g^{1/2}\, \mathbf{T}^* = \mathbf{t}^{*i} \otimes \mathbf{g}_i$

Cosserat Shell (two-dimensional continua)	
Position Vector	$\mathbf{x}^* = \mathbf{x}(\theta^\alpha,t) + \theta^3\, \mathbf{d}_3(\theta^\alpha,t)$
Mass	$\dot{m} = 0 \;,\; m = \rho\, a^{1/2} = \rho_0\, A^{1/2} = m(\theta^\alpha)$
Linear Momentum	$m\,(\dot{\mathbf{v}} + y^3\, \dot{\mathbf{w}}_3) = m\, \mathbf{b} + \mathbf{t}^\alpha{}_{,\alpha}$
Director Momentum	$m\,(y^3\, \dot{\mathbf{v}} + y^{33}\, \dot{\mathbf{w}}_3) = m\, \mathbf{b}^3 - \mathbf{t}^3 + \mathbf{m}^\alpha{}_{,\alpha}$
Angular Momentum	$\mathbf{T} = \mathbf{T}^T \;,\; a^{1/2}\, \mathbf{T} = \mathbf{t}^i \otimes \mathbf{d}_i + \mathbf{m}^\alpha \otimes \mathbf{d}_{3,\alpha}$

Cosserat Rod (one-dimensional continua)	
Position Vector	$\mathbf{x}^* = \mathbf{x}(\theta^3,t) + \theta^\alpha\, \mathbf{d}_\alpha(\theta^3,t)$
Mass	$\dot{m} = 0 \;,\; m = \rho\, d_{33}^{1/2} = \rho_0\, D_{33}^{1/2} = m(\theta^3)$
Linear Momentum	$m\,(\dot{\mathbf{v}} + y^\alpha\, \dot{\mathbf{w}}_\alpha) = m\, \mathbf{b} + \mathbf{t}^3{}_{,3}$
Director Momentum	$m\,(y^\alpha\, \dot{\mathbf{v}} + y^{\alpha\beta}\, \dot{\mathbf{w}}_\beta) = m\, \mathbf{b}^\alpha - \mathbf{t}^\alpha + \mathbf{m}^\alpha{}_{,3}$
Angular Momentum	$\mathbf{T} = \mathbf{T}^T \;,\; d_{33}^{1/2}\, \mathbf{T} = \mathbf{t}^i \otimes \mathbf{d}_i + \mathbf{m}^\alpha \otimes \mathbf{d}_{\alpha,3}$

Cosserat Point (zero-dimensional continua)	
Position Vector	$\mathbf{x}^* = \mathbf{x}(t) + \theta^i\, \mathbf{d}_i(t)$
Mass	$\dot{m} = 0 \;,\; m = \rho\, d^{1/2} = \rho_0\, D^{1/2} = \text{constant}$
Linear Momentum	$m\,(\dot{\mathbf{v}} + y^i\, \dot{\mathbf{w}}_i) = m\, \mathbf{b}$
Director Momentum	$m\,(y^i\, \dot{\mathbf{v}} + y^{ij}\, \dot{\mathbf{w}}_j) = m\, \mathbf{b}^i - \mathbf{t}^i$
Angular Momentum	$\mathbf{T} = \mathbf{T}^T \;,\; d^{1/2}\, \mathbf{T} = \mathbf{t}^i \otimes \mathbf{d}_i$

Table 1.2.1 Comparison of the equations of motion of four levels of theory.

Further in this regard, it is interesting to note that in some sense the theories discussed here form a closed loop. This is because the three-dimensional theory is reduced to shell theory which is further reduced to rod theory then to point theory. Finally, the theory of a Cosserat point closes the loop by using it to formulate a numerical solution procedure for three-dimensional problems in continuum mechanics (Rubin, 1995). The Cosserat point theory has also been used to formulate a numerical solution procedure for string theory (Rubin, 1987a) which is a simple theory of rods with no bending stiffness. Recently (Rubin, 2000) the Cosserat point theory has been used to formulate numerical solution procedures for rod theory. However, further research is needed to develop numerical solution procedures for shell theory.

It is expected that all readers will master the material on tensors in chapter 2 (sections 2.1 through 2.3) and in appendix A, and the material on three-dimensional continua in chapter 3. Chapters 4, 5 and 6 are structured so that they are totally independent. Consequently, those readers who are interested in only one of the topics covered in these chapters can move directly from chapter 3 to the chapter of interest. Chapter 7 integrates a number of ideas of all the previous chapters. The unified nature of the formulations of these theories and the notation is designed to make the structure of each theory appear as similar as possible to that of a three-dimensional continua (see Table 1.2.1). For example, the notion of the symmetry of the Cauchy stress tensor in the three-dimensional theory has an immediately recognizable counterpart in each of the theories of shells, rods and points.

The exercises that appear at the end of the book are identified by numbers like (E3.5-3). The first number E3.5 indicates that this is an exercise related to material in section 3.5. The second number 3 indicates that this is the third exercise related to the material in that section. Many of the exercises require the reader to prove the validity of various equations that appear in the text. These exercises are particularly useful for ensuring that the reader is developing proficiency with the necessary mathematical techniques required to follow the text and to use the material for other applications which are not directly covered in the text. Other exercises, such as some of those that appear in chapter 2, are designed to make the reader aware of related mathematical or physical topics that are not emphasized in the text.

Most of the material in this book emphasizes the structure of the theories that are developed. In particular, emphasis is focused on the physical phenomena that can be modeled using these theories and on the formulation of various constrained theories that can simplify the equations, while retaining relevant physical features of a particular problem. Moreover, due to the fact that most of the developments are fully nonlinear, there are very few nontrivial exact solutions. Consequently, it is expected that the solution of applied problems will usually require some numerical analysis.

The equations of motion cannot be solved, even numerically, until specific constitutive equations are supplied. Therefore, in order to help bridge the gap between purely theoretical developments and applications, sections in this book discuss specific constitutive equations. For the elastic response considered here the response functions are determined by derivatives of a strain energy function which can describe nonlinear elasticity and general anisotropy.

In this regard, it should be mentioned that even if the shell, rod, or point structure is fabricated using a homogeneous elastic material and the strain energy function for this material is known, the strain energy function for the shell, rod or point is not necessarily known. This is because the strain energy function in a theory that describes such a structure necessarily couples the influences of the geometry of the structure with the constitutive properties of the material which was used to fabricate it. For example, it is well known that for the simplest linear elastic Bernoulli-Euler beam, the resultant moment is related to the curvature by the product EI of Young's modulus of elasticity E (which is a material constant) and the second moment of area of the cross section I (which is a geometric constant). For a curved shell this coupling can be quite complicated.

Some progress has recently been made towards an understanding of the coupling of geometrical and material properties for general curved shells and rods. Specifically, restrictions on the strain energy function for shells (Naghdi and Rubin, 1995) and that for rods (Rubin, 1996) have been developed. These restrictions ensure that exact solutions of the equations for shells and rods are consistent with exact solutions of the three-dimensional equations for all homogeneous deformations. At present it is not known how to resolve this coupling for more general inhomogeneous deformations. Nevertheless, simple specific constitutive equations are proposed in sections of this book to provide a complete set of constitutive equations that can be used for applications. However, further research is needed in this area.

This book is mainly intended to provide an introduction to Cosserat theories of shells, rods and points which can be mastered by either self or guided study of the relevant chapters. Reference is made only to those articles and books that are particularly relevant to the ideas being discussed. Therefore, a number of excellent references are not listed, either because the approach taken in those references is different from the Cosserat approach or because they have been inadvertently omitted. As previously mentioned, a number of additional references can be found in the works of Naghdi (1972) and Antman (1972, 1995). Also, mention should be made of the classical works of Truesdell and Toupin (1960) and Truesdell and Noll (1965) on general continuum mechanics. Moreover, alternative formulations of shell theory and a number of solutions of shell equations can be found in the books by: Timoshenko and Woinowsky-Kreiger (1959), Novozhilov (1959), Gol'denveizer (1961), Kraus (1967), Flugge (1973), Mollmann (1981), Calladine (1983), Niordson (1985), Axelrad (1987), Libai and Simmonds (1988, 1998), and (Antman, 1995; Villaggio, 1997).

1.3 Notation

Throughout the text quantities which are first order tensors (vectors) or higher order tensors are denoted by bold faced symbols. The dot product operator ($\bullet$), cross product operator ($\times$) and tensor product operator ($\otimes$) are used in the usual way to define quantities like $\mathbf{a} \bullet \mathbf{b}$, $\mathbf{a} \times \mathbf{b}$, and $\mathbf{a} \otimes \mathbf{b}$, where $\mathbf{a}$ and $\mathbf{b}$ are vectors. Generalizations of these operators for higher order tensors are described in appendix A. Also, the usual summation convention is used for repeated indices, with Latin indices taking the values (1,2,3) and Greek indices taking the values (1,2). Often, to simplify formulas, a comma is used to denote partial differentiation with respect to the coordinates θ^i or θ^α so that

$$\mathbf{T}_{,i} = \frac{\partial \mathbf{T}}{\partial \theta^i} \quad , \quad \mathbf{T}_{,\alpha} = \frac{\partial \mathbf{T}}{\partial \theta^\alpha} \quad . \tag{1.3.1}$$

In order to make the notation for the Cosserat theories of shells, rods and points be similar to that of the three-dimensional theory, the same symbol is used for many quantities. However, to distinguish between the various theories, a superposed (*) is used to indicate that the quantity is related to the three-dimensional theory. Thus, for example, the symbol $\mathbf{x}^*$ denotes the position vector of a material point in the three-dimensional theory in chapter 3, whereas the symbol $\mathbf{x}$ denotes the position vector of a material point in the Cosserat theories of shells in chapter 4, rods in chapter 5 and points in chapter 6. Since it is expected that the equations in chapters 4, 5 and 6 will be compared most often with those of the three-dimensional theory in chapter 3 and not with themselves, it is not anticipated that the use of the same symbol $\mathbf{x}$ for the position vector in these chapters will cause any problem.

CHAPTER 2

BASIC TENSOR OPERATIONS IN CURVILINEAR COORDINATES

In continuum mechanics as well as in other branches of engineering and physics, it is necessary to develop mathematical models that describe phenomena observed in the physical world. Almost always it is necessary to describe the location of a material point in space relative to some fixed point and relative to some specified fixed axes. The specific choice of these axes remains arbitrary but it is usually guided by desire to simplify some aspect of the description of the material response. Obviously, the actual response of a given material to a specific loading must be independent of the particular choice of these fixed axes. Consequently, it is necessary to use mathematical tools that automatically ensure that predictions of the resulting mathematical equations are independent of the specific choice of these axes. Vectors and tensors are such mathematical tools.

The objective of the present section is to provide a brief practical introduction to tensors in curvilinear coordinates. Consequently, attention will be focused mainly on those tensor operations which are essential to develop the theories discussed in later sections. Some more general tensor operations have been described in appendix A and more complete mathematical descriptions of tensors in curvilinear coordinates, as they relate to mechanics, can be found in (Sokolnikoff, 1964; Eringen, 1967; Malvern, 1969). Also, a complete discussion of the application of tensors to shell theory can be found in (Naghdi, 1972).

2.1 Covariant and contravariant base vectors

By way of background it is first recalled that the position vector $\mathbf{x}^*$ of a material point Y is identified with the directed line from the fixed origin O to the point Y. For the developments in this book it is sufficient to confine attention to the three-dimensional Euclidean space. Throughout the book the superscript (*) is used to distinguish between quantities associated with the three-dimensional theory and similar quantities associated with the two- one- and zero-dimensional theories of shells, rods, and points, respectively.

Rectangular Cartesian coordinates are special independent coordinates x_i which are the components of the position vector $\mathbf{x}^*$ relative to a set of constant orthonormal base vectors $\mathbf{e}_i$. It then follows that $\mathbf{x}^*$ can be represented in the form

$$\mathbf{x}^* = x_i \, \mathbf{e}_i \ . \tag{2.1.1}$$

Moreover, since the base vectors $\mathbf{e}_i$ are independent of the coordinates x_i, it follows that these base vectors can also be calculated by differentiating the position vector $\mathbf{x}^*$

$$\mathbf{e}_i = \frac{\partial \mathbf{x}^*}{\partial x_i} \tag{2.1.2}$$

For general curvilinear coordinates, the notion of base vectors similar to (2.1.2) are generalized directly, but the position vector no longer admits the simple form (2.1.1). For this case, the position vector $\mathbf{x}^*$ is expressed as a function of three independent coordinates θ^i and time t

$$\mathbf{x}^* = \mathbf{x}^*(\theta^i,t) \ , \tag{2.1.3}$$

relative to any convenient set of base vectors. For example, $\mathbf{x}^*$ can be represented in the form

$$\mathbf{x}^* = x_m(\theta^i,t)\, \mathbf{e}_m \ , \tag{2.1.4}$$

where each of the rectangular Cartesian coordinates x_m is a function of θ^i and t. Now, the base vectors $\mathbf{g}_i$ associated with the representation (2.1.3) are defined by

$$\mathbf{g}_i = \frac{\partial \mathbf{x}^*}{\partial \theta^i} = \mathbf{x}^*{}_{,i} \ , \tag{2.1.5}$$

where a comma is used to denote partial differentiation with respect to the coordinates θ^i. Geometrically, these base vectors $\mathbf{g}_i$ represent tangent vectors to the three curves defined by varying one of the coordinates θ^i while holding the other two constant.

It is important to note that since θ^i are general coordinates, they need not have the dimensions of length. Consequently, the base vectors $\mathbf{g}_i$ need not be unitless. Moreover, $\mathbf{g}_i$ are functions of (θ^i,t) and in general are not orthonormal vectors. However, since θ^i are independent coordinates, the vectors $\mathbf{g}_i$ are linearly independent vectors and the coordinates θ^i can be arranged so that $\mathbf{g}_i$ form a right-handed set of vectors

$$g^{1/2} = \mathbf{g}_1 \times \mathbf{g}_2 \bullet \mathbf{g}_3 > 0 \ . \tag{2.1.6}$$

Using the chain rule of differentiation, the length squared of a line element $d\mathbf{x}^*$ at a fixed time t can be related to the elemental changes in the coordinates $d\theta^i$ by the expression

$$d\mathbf{x}^* \bullet d\mathbf{x}^* = \mathbf{x}^*{}_{,i}\, d\theta^i \bullet \mathbf{x}^*{}_{,j}\, d\theta^j = \mathbf{g}_i \bullet \mathbf{g}_j\, d\theta^i\, d\theta^j = g_{ij}\, d\theta^i\, d\theta^j \ , \tag{2.1.7}$$

where g_{ij} is called the metric of the space and is defined by

$$g_{ij} = \mathbf{g}_i \bullet \mathbf{g}_j = g_{ji} \ , \tag{2.1.8}$$

and the symbol ($\bullet$) denotes the usual dot (or scalar) product between two vectors (see appendix A). Also, it can be shown that the quantity g defined in (2.1.6) is equal to the determinant of the metric g_{ij}

$$g = \det\,(g_{ij}) \ . \tag{2.1.9}$$

Notice in the above definitions that the coordinates θ^i have superscripts for indices and the base vectors $\mathbf{g}_i$ have subscripts for indices. This is because for analysis in curvilinear coordinates it is necessary to distinguish between two types of bases: covariant bases which are formed using the covariant base vectors $\mathbf{g}_i$, and contravariant bases which are formed using reciprocal vectors $\mathbf{g}^i$ called contravariant base vectors. Specifically, the vectors $\mathbf{g}^i$ are defined by the cross-product operator such that

$$\mathbf{g}^1 = g^{-1/2} (\mathbf{g}_2 \times \mathbf{g}_3) \ , \ \ \mathbf{g}^2 = g^{-1/2} (\mathbf{g}_3 \times \mathbf{g}_1) \ , \ \ \mathbf{g}^3 = g^{-1/2} (\mathbf{g}_1 \times \mathbf{g}_2) \ . \tag{2.1.10}$$

Due to the properties of the cross-product, the covariant vectors $\mathbf{g}_i$ and the contravariant vectors $\mathbf{g}^i$ are biorthogonal sets of vectors satisfying the equations

$$\mathbf{g}_i \bullet \mathbf{g}^j = \delta_i^j \ , \tag{2.1.11}$$

where δ_i^j is the Kronecker delta taking the value 1 for (i=j) and 0 for (i≠j). Using the definitions (2.1.10) and the expansion of the vector triple product, it can be shown that

$$g^{-1/2} = \mathbf{g}^1 \times \mathbf{g}^2 \bullet \mathbf{g}^3 > 0 \ , \tag{2.1.12}$$

so that $\mathbf{g}^i$ form a set of right-handed linearly independent vectors. Also, the reciprocal metric g^{ij} is defined such that

$$g^{ij} = \mathbf{g}^i \bullet \mathbf{g}^j = g^{ji} \ . \tag{2.1.13}$$

Since $\mathbf{g}_i$ and $\mathbf{g}^i$ are both individually linearly independent sets of vectors, each of these sets spans the three-dimensional space so that both the sets $\mathbf{g}_i$ and $\mathbf{g}^i$ can be used as bases for vectors in three-dimensional. In particular, the covariant vectors $\mathbf{g}_i$ can be represented in terms of the contravariant vectors and vice versa such that

$$\mathbf{g}_i = g_{ij}\, \mathbf{g}^j \ , \ \ \mathbf{g}^i = g^{ij}\, \mathbf{g}_j \ . \tag{2.1.14}$$

It therefore follows that the metrics g_{ij} and g^{ij} act like shifters between the covariant and contravariant vectors.

Using the definitions (2.1.10) and the expansion

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \bullet \mathbf{c})\, \mathbf{b} - (\mathbf{a} \bullet \mathbf{c})\, \mathbf{b} \ , \tag{2.1.15}$$

of the vector triple product of the three vectors **a**, **b** and **c**, it can be shown that the covariant vectors $\mathbf{g}_i$ are related to cross products of the contravariant vectors $\mathbf{g}^i$ by the formulas

$$g^{-1/2}\, \mathbf{g}_1 = \mathbf{g}^2 \times \mathbf{g}^3 \ , \ \ g^{-1/2}\, \mathbf{g}_2 = \mathbf{g}^3 \times \mathbf{g}^1 \ , \ \ g^{-1/2}\, \mathbf{g}_3 = \mathbf{g}^1 \times \mathbf{g}^2 \ . \tag{2.1.16}$$

2.2 Base tensors and components of tensors

The covariant components v_i and contravariant components v^i of an arbitrary vector **v** are defined in the usual way by taking the inner product of **v** with the appropriate base vectors

$$v_i = \mathbf{v} \bullet \mathbf{g}_i \ , \ \ v^i = \mathbf{v} \bullet \mathbf{g}^i \ . \tag{2.2.1}$$

Moreover, in view of the biorthogonality of the covariant and contravariant base vectors, it follows that **v** can be represented in the equivalent forms

$$\mathbf{v} = v^i\, \mathbf{g}_i = v_i\, \mathbf{g}^i \ . \tag{2.2.2}$$

These equations express the fundamental property of a tensor that the tensor is independent of the basis with respect to which the components are evaluated. Also, it is emphasized that the components of a tensor depend explicitly on the choice of the basis.

A further discussion of these properties related to tensor transformation relations can be found in section A.6 of appendix A.

Here it is important to note that the covariant components are multiplied by the contravariant vectors and vice versa. Therefore, except for the special case of rectangular Cartesian coordinates and base vectors (2.1.1) and (2.1.2), the summation convention will be applied when subscripts associated with covariant quantities and superscripts associated with contravariant quantities have the same index.

A general second order tensor $\mathbf{T}$ has nine independent components which can be referred to a basis of nine tensors spanning the space of all second order tensors. Using tensor products of covariant and contravariant vectors, it is possible to form four different sets of base tensors of the forms

$$\mathbf{g}_i \otimes \mathbf{g}_j \ , \quad \mathbf{g}^i \otimes \mathbf{g}^j \ , \quad \mathbf{g}_i \otimes \mathbf{g}^j \ , \quad \mathbf{g}^i \otimes \mathbf{g}_j \ . \tag{2.2.3}$$

It then follows that the covariant component T_{ij}, the contravariant components T^{ij} and the mixed components $T_i{}^j$ and $T^i{}_j$ of $\mathbf{T}$ are defined by

$$\begin{aligned} T_{ij} &= \mathbf{T} \bullet (\mathbf{g}_i \otimes \mathbf{g}_j) \ , \quad T^{ij} = \mathbf{T} \bullet (\mathbf{g}^i \otimes \mathbf{g}^j) \ , \\ T_i{}^j &= \mathbf{T} \bullet (\mathbf{g}_i \otimes \mathbf{g}^j) \ , \quad T^i{}_j = \mathbf{T} \bullet (\mathbf{g}^i \otimes \mathbf{g}_j) \ , \end{aligned} \tag{2.2.4}$$

where use has been made of the dot product between higher order tensors (see appendix A). Thus, the tensor $\mathbf{T}$ can be represented in the equivalent forms

$$\mathbf{T} = T_{ij} \, (\mathbf{g}^i \otimes \mathbf{g}^j) = T^{ij} \, (\mathbf{g}_i \otimes \mathbf{g}_j) = T_i{}^j \, (\mathbf{g}^i \otimes \mathbf{g}_j) = T^i{}_j \, (\mathbf{g}_i \otimes \mathbf{g}^j) \ . \tag{2.2.5}$$

Furthermore, because of the nature of the mixed components, it is necessary to distinguish between the locations of the first and second indices when writing subscripts or superscripts.

As an example, it is of interest to consider the second order identity tensor $\mathbf{I}$ which has the properties that for an arbitrary vector $\mathbf{v}$

$$\mathbf{I} \, \mathbf{v} = \mathbf{v} \ , \quad \mathbf{v} \, \mathbf{I} = \mathbf{v} \ . \tag{2.2.6}$$

In view of the properties of the covariant and contravariant vectors, it can be shown that $\mathbf{I}$ can be written in the equivalent forms

$$\mathbf{I} = \mathbf{g}_i \otimes \mathbf{g}^i = \mathbf{g}^i \otimes \mathbf{g}_i \ . \tag{2.2.7}$$

Thus, the components of $\mathbf{I}$ become

$$\begin{aligned} g_{ij} &= \mathbf{I} \bullet (\mathbf{g}_i \otimes \mathbf{g}_j) \ , \quad g^{ij} = \mathbf{I} \bullet (\mathbf{g}^i \otimes \mathbf{g}^j) \ , \\ \delta_i{}^j &= \mathbf{I} \bullet (\mathbf{g}_i \otimes \mathbf{g}^j) \ , \quad \delta^i{}_j = \mathbf{I} \bullet (\mathbf{g}^i \otimes \mathbf{g}_j) \ . \end{aligned} \tag{2.2.8}$$

However, substitution of $(2.2.7)_1$ into $(2.2.8)_3$ yields the expression

$$\delta_i{}^j = (\mathbf{g}_m \otimes \mathbf{g}^m) \bullet (\mathbf{g}_i \otimes \mathbf{g}^j) = g_{im} \, g^{mj} \ . \tag{2.2.9}$$

Consequently, taking the determinant of (2.2.9) and using (2.1.9) yields the result

$$g^{-1} = \det \, (g^{ij}) \ . \tag{2.2.10}$$

Obviously, it is possible to generalize the definitions of tensor bases (2.2.3), components (2.2.4) and the representations (2.2.5) for tensors of general order M by taking a string of tensor products of M covariant or contravariant base vectors. For example, the covariant and contravariant bases for third order tensors are given by

$$\mathbf{g}_i \otimes \mathbf{g}_j \otimes \mathbf{g}_k \ , \quad \mathbf{g}^i \otimes \mathbf{g}^j \otimes \mathbf{g}^k \ , \tag{2.2.11}$$

and the covariant components T_{ijk} and contravariant components T^{ijk} of a third order tensor $\mathbf{T}$ are given by

$$T_{ijk} = \mathbf{T} \bullet (\mathbf{g}_i \otimes \mathbf{g}_j \otimes \mathbf{g}_k) \ , \quad T^{ijk} = \mathbf{T} \bullet (\mathbf{g}^i \otimes \mathbf{g}^j \otimes \mathbf{g}^k) \ . \tag{2.2.12}$$

Then, $\mathbf{T}$ admits the representations

$$\mathbf{T} = T_{ijk}\ (\mathbf{g}^i \otimes \mathbf{g}^j \otimes \mathbf{g}^k) = T^{ijk}\ (\mathbf{g}_i \otimes \mathbf{g}_j \otimes \mathbf{g}_k) \ . \tag{2.2.13}$$

Mixed components of $\mathbf{T}$ and representations of $\mathbf{T}$ in terms of these mixed components are determined in an obvious manner.

Using the previous definitions, it can be shown that the metrics g_{ij} and g^{ij} can be used to shift between covariant and contravariant components of a tensor. For example, with the help of the various representations of the second order tensor $\mathbf{T}$ defined in (2.2.5), the result (2.1.11), and the definitions (2.1.8) and (2.1.13), it follows that

$$T_{ij} = \mathbf{T} \bullet (\mathbf{g}_i \otimes \mathbf{g}_j) = T^{mn}\ (\mathbf{g}_m \otimes \mathbf{g}_n) \bullet (\mathbf{g}_i \otimes \mathbf{g}_j) = T^{mn}\ g_{mi}\ g_{nj} \ ,$$

$$T_{ij} = T_m{}^n\ (\mathbf{g}^m \otimes \mathbf{g}_n) \bullet (\mathbf{g}_i \otimes \mathbf{g}_j) = T_m{}^n\ \delta^m{}_i\ g_{nj} = T_i{}^n\ g_{nj} \ ,$$

$$T_{ij} = T^m{}_n\ (\mathbf{g}_m \otimes \mathbf{g}^n) \bullet (\mathbf{g}_i \otimes \mathbf{g}_j) = T^m{}_n\ g_{mi}\ \delta^n{}_j = T^m{}_j\ g_{mi} \ . \tag{2.2.14}$$

Since the coordinates θ^i need not have the dimensions of length, the vectors $\mathbf{g}_i$ and $\mathbf{g}^i$ are not necessarily unitless. Therefore, the components of an arbitrary tensor $\mathbf{T}$ relative to the base tensors associated with the vectors $\mathbf{g}_i$ or $\mathbf{g}^i$ will not necessarily have the same units as the physical tensor $\mathbf{T}$. However, it is always possible to refer $\mathbf{T}$ to an orthogonal tensor basis which is associated with a right-handed set of orthonormal base vectors $\mathbf{e}_i$. Then, the components $T_{<ij...rs>}$ of $\mathbf{T}$ relative to these base tensors

$$T_{<ij...rs>} = \mathbf{T} \bullet (\mathbf{e}_i \otimes \mathbf{e}_j \otimes ... \otimes \mathbf{e}_r \otimes \mathbf{e}_s) \ . \tag{2.2.15}$$

are called the physical components because they have the same dimensions as the physical quantity $\mathbf{T}$.

2.3 Basic tensor operations

In continuum mechanics it is often desirable to refer to a material point Y by either its location $\mathbf{x}^*$ in the present configuration at time t or its location $\mathbf{X}^*$ in an arbitrary fixed reference configuration. For general coordinates θ^i, the mapping from θ^i to $\mathbf{X}^*$ remains a function of time

$$\mathbf{X}^* = \mathbf{X}^*(\theta^i, t) \ . \tag{2.3.1}$$

However, for the special case when θ^i are convected Lagrangian coordinates, the mapping from θ^i to $\mathbf{X}^*$ is independent of time

$$\mathbf{X}^* = \mathbf{X}^*(\theta^i) \ . \tag{2.3.2}$$

For either case, it is possible to define covariant vectors $\mathbf{G}_i$ and contravariant vectors $\mathbf{G}^i$ associated with the reference position vector $\mathbf{X}^*$ by the formulas

$$\mathbf{G}_i = \frac{\partial \mathbf{X}^*}{\partial \theta^i} \ , \quad G^{1/2} = \mathbf{G}_1 \times \mathbf{G}_2 \bullet \mathbf{G}_3 \ , \quad \mathbf{G}_i \bullet \mathbf{G}^j = \delta_i^j \ ,$$

$$\mathbf{G}^1 = G^{-1/2}\,\mathbf{G}_2 \times \mathbf{G}_3 \ , \quad \mathbf{G}^2 = G^{-1/2}\,\mathbf{G}_3 \times \mathbf{G}_1 \ , \quad \mathbf{G}^3 = G^{-1/2}\,\mathbf{G}_1 \times \mathbf{G}_2 \ . \tag{2.3.3}$$

In the following, a number of tensor operations will be defined in terms of derivatives with respect to the present position $\mathbf{x}^*$ and the reference position $\mathbf{X}^*$ of a material point. To this end, the gradient of a scalar function f with respect to the present position $\mathbf{x}^*$ is a vector denoted by grad^* f, and the gradient of f with respect to the reference position $\mathbf{X}^*$ is a vector denoted by Grad^* f. These vectors can be conveniently expressed in terms of the contravariant base vectors in the forms

$$\text{grad}^*\, f = \partial f/\partial \mathbf{x}^* = \frac{\partial f}{\partial \theta^i}\,\mathbf{g}^i = f_{,i}\,\mathbf{g}^i \ ,$$

$$\text{Grad}^*\, f = \partial f/\partial \mathbf{X}^* = \frac{\partial f}{\partial \theta^i}\,\mathbf{G}^i = f_{,i}\,\mathbf{G}^i \ . \tag{2.3.4}$$

Furthermore, the gradients $\text{grad}^*\,\mathbf{T}$ and $\text{Grad}^*\,\mathbf{T}$ of an arbitrary tensor of order M (M≥1) are tensors of order M+1 that can be expressed in the forms

$$\text{grad}^*\,\mathbf{T} = \partial \mathbf{T}/\partial \mathbf{x}^* = \frac{\partial \mathbf{T}}{\partial \theta^i} \otimes \mathbf{g}^i = \mathbf{T}_{,i} \otimes \mathbf{g}^i \ ,$$

$$\text{Grad}^*\,\mathbf{T} = \partial \mathbf{T}/\partial \mathbf{X}^* = \frac{\partial \mathbf{T}}{\partial \theta^i} \otimes \mathbf{G}^i = \mathbf{T}_{,i} \otimes \mathbf{G}^i \ . \tag{2.3.5}$$

Using these definitions, it follows that

$$\partial \theta^i/\partial \mathbf{x}^* = \mathbf{g}^i \ , \quad \partial \theta^i/\partial \mathbf{X}^* = \mathbf{G}^i \ , \tag{2.3.6}$$

since the second order identity tensor $\mathbf{I}$ can be expressed using the chain rule of differentiation in the forms

$$\mathbf{I} = \partial \mathbf{x}^*/\partial \mathbf{x}^* = \frac{\partial \mathbf{x}^*}{\partial \theta^i} \otimes \partial \theta^i/\partial \mathbf{x}^* = \mathbf{g}_i \otimes \mathbf{g}^i \ ,$$

$$\mathbf{I} = \partial \mathbf{X}^*/\partial \mathbf{X}^* = \frac{\partial \mathbf{X}^*}{\partial \theta^i} \otimes \partial \theta^i/\partial \mathbf{X}^* = \mathbf{G}_i \otimes \mathbf{G}^i \ . \tag{2.3.7}$$

Also, it is noted for clarity that the gradient of a tensor function is written as a derivative with respect of $\mathbf{x}^*$ or $\mathbf{X}^*$ on a single line instead of as a fraction. This helps indicate that the gradient operator adds a tensor product on the right-hand side of the tensor being differentiated.

As a special case, it is possible to write the deformation gradient $\mathbf{F}^*$ from the reference configuration to the present configuration in the form

$$\mathbf{F}^* = \partial \mathbf{x}^*/\partial \mathbf{X}^* = \frac{\partial \mathbf{x}^*}{\partial \theta^i} \otimes \partial \theta^i/\partial \mathbf{X}^* = \mathbf{g}_i \otimes \mathbf{G}^i \ . \tag{2.3.8}$$

Then, the inverse $\mathbf{F}^{*-1}$, the transpose $\mathbf{F}^{*T}$ and the inverse transpose $\mathbf{F}^{*-T}$ of $\mathbf{F}^*$ can be expressed in the forms

$$\mathbf{F}^{*-1} = \mathbf{G}_i \otimes \mathbf{g}^i \ , \quad \mathbf{F}^{*T} = \mathbf{G}^i \otimes \mathbf{g}_i \ , \quad \mathbf{F}^{*-T} = \mathbf{g}^i \otimes \mathbf{G}_i \ . \tag{2.3.9}$$

Moreover, it can be shown that the determinant of $\mathbf{F}^*$ can be written as

$$\det \mathbf{F}^* = (\mathbf{g}_1 \times \mathbf{g}_2 \bullet \mathbf{g}_3)\,(\mathbf{G}^1 \times \mathbf{G}^2 \bullet \mathbf{G}^3) = g^{1/2}\,G^{-1/2} \ . \tag{2.3.10}$$

The divergence of a tensor function $\mathbf{T}$ of order M (M≥1) with respect to the present position $\mathbf{x}^*$ is a tensor of order M–1 denoted by $\text{div}^*\,\mathbf{T}$, and the divergence of $\mathbf{T}$ with respect to the reference position $\mathbf{X}^*$ is a tensor of order M–1 denoted by Div* $\mathbf{T}$. These tensors can be conveniently expressed in the forms

$$\text{div}^* \mathbf{T} = \mathbf{T},_j \bullet \mathbf{g}^j \ , \quad \text{Div}^* \mathbf{T} = \mathbf{T},_j \bullet \mathbf{G}^j . \tag{2.3.11}$$

However, since $\mathbf{T}$ is a tensor of order M it is necessary to differentiate M base tensors to evaluate the expression $\mathbf{T},_j$. Sometimes the amount of calculational effort can be reduced by rewriting the divergence operator in the alternative form

$$\text{div}^* \mathbf{T} = (\mathbf{T} \bullet \mathbf{g}^j),_j - \mathbf{T} \bullet \mathbf{g}^j,_j \ , \tag{2.3.12}$$

since $\mathbf{T} \bullet \mathbf{g}^j$ are tensors of order M–1 instead of order M. Moreover, it can be shown by differentiating (2.1.11) with respect to θ^j that

$$\mathbf{g}^j,_j = -(\mathbf{g}_i,_j \bullet \mathbf{g}^j)\, \mathbf{g}^i = -(\mathbf{g}_j,_i \bullet \mathbf{g}^j)\, \mathbf{g}^i \ , \tag{2.3.13}$$

where it has been assumed that the position vector $\mathbf{x}^*$ is sufficiently continuous so that

$$\mathbf{g}_i,_j = \frac{\partial \mathbf{g}_i}{\partial \theta^j} = \frac{\partial^2 \mathbf{x}^*}{\partial \theta^i \partial \theta^j} = \frac{\partial^2 \mathbf{x}^*}{\partial \theta^j \partial \theta^i} = \mathbf{g}_j,_i \ . \tag{2.3.14}$$

Also, by differentiating (2.1.12) with respect to θ^j and using the permutation properties of the scalar triple product, it can be shown that

$$(g^{1/2}),_j = g^{1/2}\, \mathbf{g}_m,_j \bullet \mathbf{g}^m \ , \quad \mathbf{g}^j,_j = -\, g^{-1/2}\, (g^{1/2}),_j\, \mathbf{g}^j \ , \quad (g^{1/2}\, \mathbf{g}^j),_j = 0 \ , \tag{2.3.15}$$

which can be used to express the divergence operator with respect to $\mathbf{x}^*$ in the simpler form

$$\text{div}^* \mathbf{T} = \mathbf{T},_j \bullet \mathbf{g}^j = g^{-1/2}\, (g^{1/2}\, \mathbf{T}\, \mathbf{g}^j),_j \ . \tag{2.3.16}$$

Similarly, the divergence operator with respect to $\mathbf{X}^*$ can be rewritten in the form

$$\text{Div}^* \mathbf{T} = \mathbf{T},_j \bullet \mathbf{G}^j = G^{-1/2}\, (G^{1/2}\, \mathbf{T}\, \mathbf{G}^j),_j \ . \tag{2.3.17}$$

For example, if $\mathbf{T}$ is a second order tensor, then the expressions (2.3.16) and (2.3.17) require differentiation of only three vectors as opposed to differentiation of the complete second order tensor.

The curl of a tensor $\mathbf{T}$ of order M (M≥1) with respect to the present position $\mathbf{x}^*$ is a tensor of order M denoted by curl* $\mathbf{T}$, and the curl of $\mathbf{T}$ with respect to the reference position $\mathbf{X}^*$ is a tensor of order M denoted by Curl* $\mathbf{T}$. These tensors can be expressed in the forms

$$\text{curl}^* \mathbf{T} = -\mathbf{T},_j \times \mathbf{g}^j \ , \quad \text{Curl}^* \mathbf{T} = -\mathbf{T},_j \times \mathbf{G}^j \ . \tag{2.3.18}$$

2.4 Covariant differentiation and Christoffel symbols

Since the base vectors $\mathbf{g}_i$ and $\mathbf{g}^i$ depend on the coordinates θ^i, it is necessary to differentiate these base vectors when deriving expressions for partial derivatives of tensors with respect to θ^i. In particular, using (2.2.2) the derivative of the vector $\mathbf{v}$ can be written in the forms

$$\mathbf{v},_i = (v^m\, \mathbf{g}_m),_i = v^m,_i\, \mathbf{g}_m + v^m\, \mathbf{g}_m,_i \ .$$

$$\mathbf{v},_i = (v_m\, \mathbf{g}^m),_i = v_m,_i\, \mathbf{g}^m + v_m\, \mathbf{g}^m,_i \ , \tag{2.4.1}$$

Then, the expression $(2.4.1)_1$ can be written in a simpler form by introducing the definitions

$$[mi,k] = \mathbf{g}_{m,i} \bullet \mathbf{g}_k = [im,k] \ , \quad \mathbf{g}_{m,i} = [mi,k]\, \mathbf{g}^k$$
$$\Gamma_{mi.}^{\ \ k} = \mathbf{g}_{m,i} \bullet \mathbf{g}^k = \Gamma_{im.}^{\ \ k} \ , \quad \mathbf{g}_{m,i} = \Gamma_{mi.}^{\ \ k}\, \mathbf{g}_k \ ,$$
$$\Gamma_{mi.}^{\ \ k} = [mi,n]\, g^{kn} \ , \quad [mi,k] = \Gamma_{mi.}^{\ \ n}\, g_{nk} \ , \tag{2.4.2}$$

of the Christoffel symbols $[mi,k]$ of the first kind and $\Gamma_{mi.}^{\ \ k}$ of the second kind. Moreover, it follows that

$$\mathbf{v}_{,i} = \left[v^k{}_{|i} \right] \mathbf{g}_k \ , \quad v^k{}_{|i} = \mathbf{v}_{,i} \bullet \mathbf{g}^k \ . \tag{2.4.3}$$

In (2.4.3), the operation $v^k{}_{|i}$ is called the covariant partial derivative of the contravariant components v^k of the vector $\mathbf{v}$ and it is defined in terms of the partial derivatives of v^k and the Christoffel symbols $\Gamma_{mi.}^{\ \ k}$ by

$$v^k{}_{|i} = v^k{}_{,i} + v^m\, \Gamma_{mi.}^{\ \ k} \ . \tag{2.4.4}$$

To write $(2.4.1)_2$ in a simpler form, it is necessary to derive expressions for the vectors $\mathbf{g}^m{}_{,i}$. This is done by differentiating the quantity $(\mathbf{g}_k \bullet \mathbf{g}^m)$ with respect to θ^i to deduce that

$$\mathbf{g}^m{}_{,i} = -\ \Gamma_{ki.}^{\ \ m}\, \mathbf{g}^k \ , \tag{2.4.5}$$

so that $(2.4.1)_2$ can be expressed in the form

$$\mathbf{v}_{,i} = \left[v_{k|i} \right] \mathbf{g}^k \ , \quad v_{k|i} = \mathbf{v}_{,i} \bullet \mathbf{g}_k \ , \tag{2.4.6}$$

where the covariant partial derivative $v_{k|i}$ of the covariant components v_k of the vector $\mathbf{v}$ is defined by

$$v_{k|i} = v_{k,i} - v_m\, \Gamma_{ki.}^{\ \ m} \ . \tag{2.4.7}$$

Recalling the definition (2.3.5) for the gradient of a tensor, it follows that the gradient of $\mathbf{v}$ with respect to $\mathbf{x}^*$ can be expressed in the forms

$$\text{grad}^*\, \mathbf{v} = \mathbf{v}_{,i} \otimes \mathbf{g}^i = \left[v_{k|i} \right] (\mathbf{g}^k \otimes \mathbf{g}^i) = \left[v^k{}_{|i} \right] (\mathbf{g}_k \otimes \mathbf{g}^i) \ . \tag{2.4.8}$$

In these formulas, it can be seen that $v_{k|i}$ are the covariant components and $v^k{}_{|i}$ are the mixed components of the second order tensor $\text{grad}^*\, \mathbf{v}$.

The covariant partial derivatives of components of higher order tensors can be defined in a similar manner. In particular, if $\mathbf{T}$ is a second order tensor with components T_{ij}, T^{ij}, $T_i{}^j$, $T^i{}_j$ then

$$\mathbf{T}_{,m} = T_{ij|m}\, (\mathbf{g}^i \otimes \mathbf{g}^j) = T^{ij}{}_{|m}\, (\mathbf{g}_i \otimes \mathbf{g}_j) = T_i{}^j{}_{|m}\, (\mathbf{g}^i \otimes \mathbf{g}_j) = T^i{}_{j|m}\, (\mathbf{g}_i \otimes \mathbf{g}^j) \ ,$$
$$T_{ij|m} = T_{ij,m} - T_{kj}\, \Gamma_{im.}^{\ \ k} - T_{ik}\, \Gamma_{jm.}^{\ \ k} \ ,$$
$$T^{ij}{}_{|m} = T^{ij}{}_{,m} + T^{kj}\, \Gamma_{km.}^{\ \ i} + T^{ik}\, \Gamma_{km.}^{\ \ j} \ ,$$
$$T_i{}^j{}_{|m} = T_i{}^j{}_{,m} - T_k{}^j\, \Gamma_{im.}^{\ \ k} + T_i{}^k\, \Gamma_{km.}^{\ \ j} \ ,$$
$$T^i{}_{j|m} = T^i{}_{j,m} + T^k{}_j\, \Gamma_{km.}^{\ \ i} - T^i{}_k\, \Gamma_{jm.}^{\ \ k} \ . \tag{2.4.9}$$

Similarly, it can be shown that $T_{ij|m}$ are the covariant components and $\{T^{ij}{}_{|m}, T_i{}^j{}_{|m}, T^i{}_{j|m}\}$ are the mixed components of the third order tensor $\text{grad}^*\, \mathbf{T}$.

CHAPTER 3

THREE-DIMENSIONAL CONTINUA

This chapter provides a brief review of the basic equations that describe the motion of a three-dimensional continua.

3.1 Configurations and motions

A body is considered to be a collection of material particles which are denoted by Y. The quantity Y can be any convenient label for the material particle that distinguishes one particle from another. For example, the particles can be labeled by number, color, or location. In continuum mechanics the abstract notion of a body, as denoted by the collection Y, is mapped in a one-to-one manner into a region of Euclidean 3-Space called a configuration of the body. The present configuration of the body is the region R^* of Euclidean 3-Space occupied by the body at the present time t. This region is assumed to be a smooth regular region that is bounded by the closed surface ∂R^*. Letting $\mathbf{x}^*$ be the position vector locating the place occupied by the particle Y at the time t, a motion of the body is specified by the vector function

$$\mathbf{x}^* = \bar{\mathbf{x}}^*(Y,t) \quad . \tag{3.1.1}$$

This function is assumed to be differentiable as many times as desired with respect to both Y and t, and it is also assumed to be invertible so that

$$Y = \bar{\mathbf{x}}^{*-1}(\mathbf{x}^*,t) = \tilde{Y}(\mathbf{x}^*,t) \quad . \tag{3.1.2}$$

Often it is convenient to refer everything concerning the body to a specific fixed configuration called the reference configuration. For example, since a homogeneous elastic solid has a unique volume and shape when it is unloaded, it is often convenient to specify one such unloaded configuration to be the reference configuration. Another common choice for the reference configuration is the initial configuration. Although these are common choices for the reference configuration, it should be emphasized that the choice of the reference configuration is arbitrary to the extent that it is a fixed configuration and that there exits a one-to-one mapping between the present configuration and the reference configuration. For any of these choices, the position of a material point Y in the reference configuration is denoted by the vector $\mathbf{X}^*$ which can be expressed in the form

$$\mathbf{X}^* = \bar{\mathbf{X}}^*(Y) \quad , \tag{3.1.3}$$

where the function $\bar{\mathbf{X}}^*(Y)$ is assumed to be invertible and differentiable as many times as desired. Specifically, the inverse mapping is given by

$$\mathrm{Y} = \bar{\mathbf{X}}^{*-1}(\mathbf{X}^*) = \hat{\mathrm{Y}}(\mathbf{X}^*) \quad . \tag{3.1.4}$$

It then follows from (3.1.1) and (3.1.4) that the present position $\mathbf{x}^*$ of a material point Y can be expressed as an alternative function of its location $\mathbf{X}^*$ in the reference configuration by the vector function

$$\mathbf{x}^* = \hat{\mathbf{x}}^*(\mathbf{X}^*,t) \quad . \tag{3.1.5}$$

Moreover, using the chain rule of differentiation, a material line element $d\mathbf{X}^*$ in the reference configuration is mapped to the material line element $d\mathbf{x}^*$ in the present configuration by the deformation gradient $\mathbf{F}^*$ such that

$$d\mathbf{x}^* = \mathbf{F}^* \, d\mathbf{X}^* \quad , \quad \mathbf{F}^* = \partial\hat{\mathbf{x}}/\partial\mathbf{X}^* \quad , \tag{3.1.6}$$

where $\mathbf{F}^*$ can also be expressed in the form (2.3.8). The mathematical condition that the mapping be invertible is that the determinant of $\mathbf{F}^*$ must be nonzero

$$J^* = \det \mathbf{F}^* \neq 0 \quad , \tag{3.1.7}$$

for all points in the region occupied by the body. However, if the present configuration coincides with the reference configuration, then $\mathbf{F}^*=\mathbf{I}$ and $J^*=1$. Consequently, in order to allow for this possibility it is further assumed that J^* is positive

$$J^* > 0 \quad . \tag{3.1.8}$$

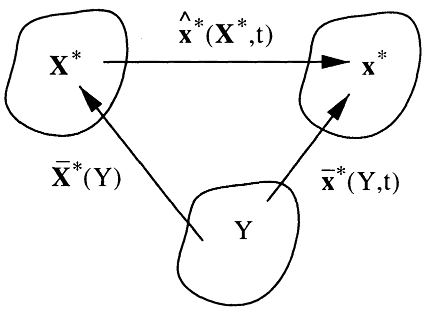

Fig. 3.1.1 Material description and reference and present configurations

Since the above mappings are invertible, a general scalar or tensor function f can be expressed in terms of three possible representations

$$f = \bar{f}(\mathrm{Y},t) = \hat{f}(\mathbf{X}^*,t) = \tilde{f}(\mathbf{x}^*,t) \quad . \tag{3.1.9}$$

For clarity in some instances, the value of a function and its specific functional form are distinguished by using an additional symbol over the usual symbol used for the value of the function. Specifically, an overbar $(\bar{\ })$ is used when the independent space variable is Y, a hat $(\hat{\ })$ is used when the independent space variable is $\mathbf{X}^*$, and a tilde $(\tilde{\ })$ is used when the independent space variable is $\mathbf{x}^*$. Moreover, $\bar{f}(\mathrm{Y},t)$ is called the material representation, $\hat{f}(\mathbf{X}^*,t)$ is called the Lagrangian or referential representation, and $\tilde{f}(\mathbf{x}^*,t)$ is called the Eulerian or spatial representation. The different mappings between the material description and the reference and present configurations are shown pictorially in Fig. 3.1.1.

In developing the balance laws of a simple continua, it is necessary to determine the absolute velocity $\mathbf{v}^*$ of a material point Y. This is accomplished by determining the time rate of change of the position of the material point Y relative to the fixed origin O. For this purpose it is convenient to define the material derivative of an arbitrary tensor function to be the time rate of change of the function holding Y fixed (i.e. the time rate of change of the function associated with a specific material point Y). This differential operator is denoted by the symbol $\dot{(\ )}$ over the function. In particular, the absolute velocity $\mathbf{v}^*$ of a material point Y is determined by the expression

$$\mathbf{v}^* = \dot{\mathbf{x}}^* = \left.\frac{\partial \bar{\mathbf{x}}^*}{\partial t}\right|_Y \ . \tag{3.1.10}$$

For a general function f with the representations (3.1.9), it is possible to use the chain rule of differentiation to write its material derivative in the forms

$$\dot{f} = \left.\frac{\partial \bar{f}}{\partial t}\right|_Y \ , \quad \dot{f} = \frac{\partial \hat{f}}{\partial t}\,\dot{t} + \partial \hat{f}/\partial \mathbf{X}^* \bullet \dot{\mathbf{X}}^* = \left.\frac{\partial \hat{f}}{\partial t}\right|_{\mathbf{X}^*} \ ,$$

$$\dot{f} = \frac{\partial \tilde{f}}{\partial t}\,\dot{t} + \partial \tilde{f}/\partial \mathbf{x}^* \bullet \dot{\mathbf{x}}^* = \frac{\partial \tilde{f}}{\partial t} + \partial \tilde{f}/\partial \mathbf{x}^* \bullet \mathbf{v}^* \ , \tag{3.1.11}$$

where use has been made of the fact that time t is an independent variable and that the mapping (3.1.3) is independent of time

$$\dot{t} = 1 \ , \quad \dot{\mathbf{X}}^* = 0 \ . \tag{3.1.12}$$

In this regard, it is important to emphasize that the physical meaning of the material derivative of a function always remains the same even though the chain rule of differentiation yields different representations (3.1.12) for the material derivative depending on whether the function is expressed in material, Lagrangian or Eulerian form.

3.2 Balance laws

The balance laws of a simple continua are a generalization of the equations of motion of a rigid body, which themselves are a generalization of Newton's equations of motion of a mass point. These balance laws are assumed to be valid over the whole region R^* (with closed boundary ∂R^*) occupied by the body as well as over any arbitrary material part P^* of R^* with closed boundary ∂P^*. The body is assumed to be sufficiently continuous that a mass density $\rho^*(\mathbf{x}^*,t)$ exists which expresses the mass per unit present volume of the body. Denoting dv^* as the element of volume in the present configuration, the conservation of mass can be expressed in the form

$$\frac{d}{dt}\int_{P^*} \rho^* \, dv^* = 0 \ . \tag{3.2.1}$$

The balance of linear momentum states that the rate of change of linear momentum of an arbitrary material part P^* of the body is equal to the total external force applied to that part of the body. These external forces are separated into two types: body forces which act at each point of the part P^* and surface tractions that act at each point of the surface

∂P^*. The body force per unit mass is denoted by the vector $\mathbf{b}^*(\mathbf{x}^*,t)$ and the surface traction per unit area is denoted by the stress vector $\mathbf{t}^*(\mathbf{x}^*,t\,;\,\mathbf{n}^*)$ which depends explicitly on the unit outward normal $\mathbf{n}^*$ to the surface ∂P^*. Using these definitions, the balance of linear momentum can be expressed as

$$\frac{d}{dt}\int_{P^*} \rho^*\, \mathbf{v}^*\, dv^* = \int_{P^*} \rho^*\, \mathbf{b}^*\, dv^* + \int_{\partial P^*} \mathbf{t}^*\, da^* \ , \tag{3.2.2}$$

where da^* is the element of area of ∂P^*. Also, using the usual argument, it can be shown by applying (3.2.2) to an elemental tetrahedron and taking the limit as the volume of the tetrahedron approaches zero, that the stress vector $\mathbf{t}^*$ is a linear function of $\mathbf{n}^*$ such that

$$\mathbf{t}^*(\mathbf{x}^*,t\,;\,\mathbf{n}^*) = \mathbf{T}^*(\mathbf{x}^*,t)\, \mathbf{n}^* \ . \tag{3.2.3}$$

In (3.2.3) the quantity $\mathbf{T}^*$ is a second order tensor called the Cauchy stress tensor that is explicitly independent of the normal $\mathbf{n}^*$. In this regard, it is noted that the stress tensor $\mathbf{T}^*$ represents the state of stress at a point $\mathbf{x}^*$ in the body, whereas the stress vector $\mathbf{t}^*$ represents the stress (force per unit present area) applied at the point $\mathbf{x}^*$ on the surface defined by the outward normal $\mathbf{n}^*$.

The balance of angular momentum states that the rate of change of angular momentum of an arbitrary material part P^* of a body is equal to the total external moment applied to that part of the body by the body force and the surface tractions. In this statement the angular momentum and the moment are referred to an arbitrary but fixed point. Letting $\mathbf{x}^*$ be the position vector relative to a fixed origin of an arbitrary point in P^*, the global form of the balance of angular momentum can be expressed as

$$\frac{d}{dt}\int_{P^*} \mathbf{x}^* \times \rho^*\, \mathbf{v}^*\, dv^* = \int_{P^*} \mathbf{x}^* \times \rho^*\, \mathbf{b}^*\, dv^* + \int_{\partial P^*} \mathbf{x}^* \times \mathbf{t}^*\, da^* \ . \tag{3.2.4}$$

For later convenience, it is desirable to develop local forms of the balance laws (3.2.1), (3.2.2) and (3.2.4). In order to develop these local equations, it is necessary to develop an expression for the time derivative of an integral over the material part P^* which can be changing with time t, and it is necessary to use the divergence theorem to convert an integral over the surface ∂P^* to an integral over the volume P^*.

To this end, it is noted that the material region P^* in the present configuration which depends on time can be mapped to the material region P_0^* in the reference configuration which does not depend on time. Specifically, the volume element dv^* in the present configuration can be expressed in terms of the coordinates θ^i by using (2.1.6) to write

$$dv^* = \frac{\partial \mathbf{x}^*}{\partial \theta^1}\, d\theta^1 \times \frac{\partial \mathbf{x}^*}{\partial \theta^2}\, d\theta^2 \bullet \frac{\partial \mathbf{x}^*}{\partial \theta^3}\, d\theta^3 = g^{1/2}\, d\theta^1\, d\theta^2\, d\theta^3 \ . \tag{3.2.5}$$

Similarly, using (2.3.3), the associated volume element dV^* in the reference configuration can be expressed in the form

$$dV^* = \frac{\partial \mathbf{X}^*}{\partial \theta^1}\, d\theta^1 \times \frac{\partial \mathbf{X}^*}{\partial \theta^2}\, d\theta^2 \bullet \frac{\partial \mathbf{X}^*}{\partial \theta^3}\, d\theta^3 = G^{1/2}\, d\theta^1\, d\theta^2\, d\theta^3 \ . \tag{3.2.6}$$

Thus, the dilatation J^* is a measure of volume change and is defined using (2.3.10) such that

$$J^* = \det \mathbf{F}^* = g^{1/2}\, G^{-1/2} \ , \quad dv^* = J^*\, dV^* \ , \tag{3.2.7}$$

where $\mathbf{F}^*$ is the deformation gradient (2.3.8). Consequently, the integral over the region P^* of an arbitrary function f can be expressed as an integral over the associated region P_0^* by the transformation

$$\int_{P^*} \tilde{f} dv^* = \int_{P_0^*} (\hat{f}\, \hat{J}^*)\, dV^* \ . \tag{3.2.8}$$

For definiteness, the function f is expressed in its Eulerian form when the integration is performed over the region P^*, and f and J^* are expressed in their Lagrangian forms when the integration is performed over the region P_0^* .

However, since P_0^* is a material region in the reference configuration it is independent of time. Thus, assuming continuity of the functions, the time differentiation and integration operators commute and it can be shown that

$$\frac{d}{dt}\int_{P^*} \tilde{f}\, dv^* = \frac{d}{dt}\int_{P_0^*} (\hat{f}\, \hat{J}^*)\, dV^* = \int_{P_0^*} \frac{d(\hat{f}\, \hat{J}^*)}{dt} dV^* = \int_{P_0^*} \overset{\bullet}{\overline{(\hat{f}\, \hat{J}^*)}}\, dV^*. \tag{3.2.9}$$

Here, it is noted that the time differentiation performed inside the integral over P_0^* corresponds to time differentiation holding the material point $\mathbf{X}^*$ so that it becomes material differentiation. Now, in order to complete the development, it is necessary to convert the integral on the right-hand side of (3.2.9) over P_0^* back to an integral over the region P^*. This is accomplished by recalling that the dilatation J^* is a function of the deformation gradient $\mathbf{F}^*$ so that the material derivative of J^* can be expressed using the chain rule of differentiation in the form

$$\dot{J}^* = \frac{\partial J^*}{\partial \mathbf{F}^*} \bullet \dot{\mathbf{F}}^* \ . \tag{3.2.10}$$

However, since $\mathbf{F}^*$ is nonsingular, it can be shown that

$$\frac{\partial J^*}{\partial \mathbf{F}^*} = J^* \, (\mathbf{F}^*)^{-T} \ . \tag{3.2.11}$$

Also, by using the expression (2.3.8) and taking the material derivative of $\mathbf{F}^*$ one obtains

$$\dot{\mathbf{F}}^* = \frac{\partial \hat{\mathbf{F}}^*}{\partial t} = \frac{\partial}{\partial t}(\partial \hat{\mathbf{x}}^*/\partial \mathbf{X}^*) = \partial^2 \hat{\mathbf{x}}^*/\partial t \partial \mathbf{X}^* = \partial^2 \hat{\mathbf{x}}^*/\partial \mathbf{X}^* \partial t$$

$$\dot{\mathbf{F}}^* = \partial(\partial \hat{\mathbf{x}}^*/\partial t)/\partial \mathbf{X}^* = \partial \hat{\mathbf{v}}^*/\partial \mathbf{X}^* = (\partial \tilde{\mathbf{v}}^*/\partial \mathbf{x}^*)\, (\partial \mathbf{x}^*/\partial \mathbf{X}^*)$$

$$\dot{\mathbf{F}}^* = \mathbf{L}^* \, \mathbf{F}^* \quad , \quad \mathbf{L}^* = \partial \mathbf{v}^*/\partial \mathbf{x}^* \ , \tag{3.2.12}$$

where $\mathbf{L}^*$ is the velocity gradient with respect to the present position $\mathbf{x}^*$. In this derivation, the material derivative and the derivative with respect to $\mathbf{X}^*$ commute because the $\mathbf{x}^*$ is expressed in its Lagrangian form with $\mathbf{X}^*$ and t being independent variables. It then follows from (3.2.10)-(3.2.12) that

$$\dot{J}^* = J^* \, (\mathbf{F}^*)^{-T} \bullet (\mathbf{L}^* \, \mathbf{F}^*) = J^* \, [(\mathbf{F}^*)^{-T}(\mathbf{F}^*)^T] \bullet \mathbf{L}^*$$

$$\dot{J}^* = J^* \, \mathbf{L}^* \bullet \mathbf{I} = J^* \, \mathrm{div}^* \, \mathbf{v}^* \ , \tag{3.2.13}$$

which allows (3.2.9) to be rewritten in the form

$$\frac{d}{dt}\int_{P^*} f\, dv^* = \int_{P^*} [\, \dot{f} + f\, \mathrm{div}^* \, \mathbf{v}^* \,]\, dv^* \ . \tag{3.2.14}$$

The general results (3.2.12) and (3.2.13) can also be obtained in terms of curvilinear coordinates by assuming that θ^i are a set of convected Lagrangian coordinates so that a material point Y is associated with constant values of θ^i. It then follows that the material derivative of θ^i vanishes

$$\dot{\theta}^i = 0 \ . \tag{3.2.15}$$

Now, since θ^i are Lagrangian coordinates it can be shown that

$$\dot{\mathbf{g}}_i = \frac{\partial}{\partial t}\left(\frac{\partial \mathbf{x}^*}{\partial \theta^i}\right) = \frac{\partial^2 \mathbf{x}^*}{\partial t \partial \theta^i} = \frac{\partial^2 \mathbf{x}^*}{\partial \theta^i \partial t} = \frac{\partial}{\partial \theta^i}\left(\frac{\partial \mathbf{x}^*}{\partial t}\right) = \frac{\partial \mathbf{v}^*}{\partial \theta^i} = \mathbf{v}^*{}_{,i} \ , \tag{3.2.16}$$

so that the velocity gradient $\mathbf{L}^*$ can be written in the form

$$\mathbf{L}^* = \mathbf{v}^*{}_{,i} \otimes \mathbf{g}^i = \dot{\mathbf{g}}_i \otimes \mathbf{g}^i \ , \quad \dot{\mathbf{g}}_i = \mathbf{L}^* \mathbf{g}_i \ . \tag{3.2.17}$$

Next, taking the material derivative of the expression (2.1.6) for $g^{1/2}$ and using the permutation properties of the scalar triple product between three vectors, it can be shown that

$$\overset{\bullet}{\overline{g^{1/2}}} = g^{1/2}\, \dot{\mathbf{g}}_i \bullet \mathbf{g}^i = g^{1/2}\, \mathbf{L}^* \bullet \mathbf{I} = g^{1/2} \operatorname{div}^* \mathbf{v}^* \ . \tag{3.2.18}$$

Consequently, since $G^{1/2}$ is independent of time when θ^i are Lagrangian coordinates, the results (3.2.13) follow from (3.2.7) and (3.2.18). Moreover, the result (3.2.12) follows by taking the material derivative of the expression (2.3.8) for the deformation gradient $\mathbf{F}^*$ and by using the formulas (3.2.17).

Next, the divergence theorem is recalled in the form

$$\int_{\partial P^*} f\, \mathbf{n}^*\, da^* = \int_{P^*} (\operatorname{div}^* f)\, dv^* \ , \tag{3.2.19}$$

where f is a scalar or vector function of $(\mathbf{x}^*,t)$ and $\mathbf{n}^*$ is the unit outward normal vector to the surface ∂P^*. For example, using (3.2.3) it can be shown that

$$\int_{\partial P^*} \mathbf{x}^* \times \mathbf{t}^*\, da^* = \int_{\partial P^*} (\mathbf{x}^* \times \mathbf{T}^*)\, \mathbf{n}^*\, da^* = \int_{P^*} \operatorname{div}^*(\mathbf{x}^* \times \mathbf{T}^*)\, dv^* \ . \tag{3.2.20}$$

But, the expression $\operatorname{div}^*(\mathbf{x}^* \times \mathbf{T}^*)$ can be expanded to obtain

$$\operatorname{div}^*(\mathbf{x}^* \times \mathbf{T}^*) = (\mathbf{x}^* \times \mathbf{T}^*)_{,j} \bullet \mathbf{g}^j = \mathbf{x}^*{}_{,j} \times (\mathbf{T}^*\mathbf{g}^j) + \mathbf{x}^* \times (\mathbf{T}^*{}_{,j} \bullet \mathbf{g}^j) \ ,$$
$$\operatorname{div}^*(\mathbf{x}^* \times \mathbf{T}^*) = \mathbf{g}_j \times (\mathbf{T}^*\mathbf{g}^j) + \mathbf{x}^* \times \operatorname{div}^* \mathbf{T}^* \ . \tag{3.2.21}$$

However, using the formulas (A.5.7) and (A.5.15) it follows that

$$\mathbf{g}_j \times (\mathbf{T}^*\mathbf{g}^j) = \boldsymbol{\varepsilon} \bullet [\mathbf{g}_j \otimes (\mathbf{T}^*\mathbf{g}^j)] = \boldsymbol{\varepsilon} \bullet [(\mathbf{g}_j \otimes \mathbf{g}^j)\, \mathbf{T}^{*T}] = \boldsymbol{\varepsilon} \bullet \mathbf{T}^{*T} \ , \tag{3.2.22}$$

where $\boldsymbol{\varepsilon}$ is the permutation symbol. Thus, (3.2.20) can be written in the form

$$\int_{\partial P^*} \mathbf{x}^* \times \mathbf{t}^*\, da^* = \int_{P^*} [\, \boldsymbol{\varepsilon} \bullet \mathbf{T}^{*T} + \mathbf{x}^* \times \operatorname{div}^* \mathbf{T}^* \,]\, dv^* \ . \tag{3.2.23}$$

Now, using the above expressions, the conservation of mass (3.2.1) can be rewritten as

$$\int_{P^*} [\, \dot{\rho}^* + \rho^* \operatorname{div}^* \mathbf{v}^*]\, dv^* = 0 \ . \tag{3.2.24}$$

Assuming that the integrand is a continuous function and that (3.2.24) holds for all parts P^* of the material region R^*, it can be shown that the integrand must vanish at each point $\mathbf{x}^*$, which yields the local form of the conservation of mass

$$\dot{\rho}^* + \rho^* \operatorname{div}^* \mathbf{v}^* = 0 \ . \tag{3.2.25}$$

The formula (3.2.25) is an Eulerian representation of the conservation of mass equation. An equivalent Lagrangian form of conservation of mass can be derived by defining the mass density ρ_0^* (mass per unit reference volume) in the reference configuration and by requiring the mass of a material part of the body to remain the same

$$\int_{P^*} \rho^* \, dv^* = \int_{P_0^*} \rho_0^* \, dV^* \ , \quad \rho_0^* = \rho_0^* (\mathbf{X}^*) \ . \tag{3.2.26}$$

Here, it is noted that the mass density ρ_0^* in the reference configuration is independent of time but can depend on the position $\mathbf{X}^*$. Next, by using (3.2.7), the integral over the region P^* can be converted to an integral over the region P_0^*. This leads to the local form of the Lagrangian expression of the conservation of mass

$$\rho^* J^* = \rho_0^* \ . \tag{3.2.27}$$

Then, the Eulerian form (3.2.25) follows directly by taking the material derivative of (3.2.27) and by using the expression (3.2.13).

An alternative, but equivalent form of the conservation of mass equation (3.2.27) that will be useful for the development of the Cosserat theories in the next chapters can be obtained by substituting (3.2.7) into (3.2.27) to deduce that

$$m^* = \rho^* g^{1/2} = \rho_0^* \, G^{1/2} = m^*(\theta^i) \ , \quad \dot{m}^* = 0 \ , \tag{3.2.28}$$

where m^* depends only on the convected coordinates θ^i and thus is independent of time. Also, it is noted that the units of m^* depend on the specification of the convected coordinates θ^i. However, m^* will have the units of mass per unit reference volume if each of θ^i has the units of length.

In each of the balance laws (3.2.2) and (3.2.4) a term appears which requires the evaluation of the time derivative of the integral over P^* of the mass density ρ^* times a function f. Therefore, it is desirable to develop a general expression for integrals of this type. To this end, the equation (3.2.14) is used to write

$$\frac{d}{dt} \int_{P^*} \rho^* \, f \, dv^* = \int_{P^*} [\, \rho^* \, \dot{f} + f \, \{\dot{\rho}^* + \rho^* \operatorname{div}^* \mathbf{v}^*\}] \, dv^* \ . \tag{3.2.29}$$

However, when the local form (3.2.25) of the conservation of mass holds this expression reduces to

$$\frac{d}{dt} \int_{P^*} \rho^* \, f \, dv^* = \int_{P^*} [\, \rho^* \, \dot{f} \,] \, dv^* \ . \tag{3.2.30}$$

This result can be easily understood by noting that an element of mass $dm^* = \rho^* dv^*$ of an elemental material part of the body is independent of time so that the time differentiation and integration operations commute when the integral is expressed in terms of mass instead of space.

Now, with the help of (3.2.19), (3.2.23) and (3.2.30), the local forms of the balances of linear momentum (3.2.2) and angular momentum (3.2.4) become

$$\rho^* \dot{\mathbf{v}}^* = \rho^* \mathbf{b}^* + \text{div}^* \mathbf{T}^* \ , \quad \boldsymbol{\varepsilon} \bullet \mathbf{T}^{*T} = 0 \ . \tag{3.2.31}$$

However, since the permutation tensor $\boldsymbol{\varepsilon}$ is skew-symmetric, it follows that the reduced form of the balance of angular momentum requires the Cauchy stress tensor $\mathbf{T}^*$ to be a symmetric tensor

$$\mathbf{T}^{*T} = \mathbf{T}^* \ . \tag{3.2.32}$$

For later reference, it is convenient to summarize these balance laws in the forms

$$\mathrm{m}^* = \rho^* \, g^{1/2} = \rho_0^* \, G^{1/2} = \mathrm{m}^*(\theta^i) \ , \quad \dot{\mathrm{m}}^* = 0 \ ,$$

$$\mathrm{m}^* \, \dot{\mathbf{v}}^* = \mathrm{m}^* \, \mathbf{b}^* + g^{1/2} \, \text{div}^* \, \mathbf{T}^* \ , \quad \mathbf{T}^{*T} = \mathbf{T}^* \ . \tag{3.2.33}$$

Alternatively, three vectors $\mathbf{t}^{*i}$ can be defined such that

$$\mathbf{t}^{*i} = g^{1/2} \, \mathbf{T}^* \, \mathbf{g}^i \ , \quad g^{1/2} \, \mathbf{T}^* = \mathbf{t}^{*i} \otimes \mathbf{g}_i \ , \tag{3.2.34}$$

and the expression (2.3.16) for the divergence operator can be used to summarize the balance laws in the forms

$$\mathrm{m}^* = \rho^* \, g^{1/2} = \rho_0^* \, G^{1/2} = \mathrm{m}^*(\theta^i) \ , \quad \dot{\mathrm{m}}^* = 0 \ ,$$

$$\mathrm{m}^* \, \dot{\mathbf{v}}^* = \mathrm{m}^* \, \mathbf{b}^* + \mathbf{t}^{*i}{}_{,i} \ , \quad \mathbf{T}^{*T} = \mathbf{T}^* \ . \tag{3.2.35}$$

In general, equations (3.2.33) or (3.2.35) represent a system of nonlinear partial differential equations which require specification of initial and boundary conditions. These balance laws are quite general because they are valid for all materials which can be modeled as simple continua. However, this system of equations is not complete because it represents a system of seven scalar equations to determine thirteen scalar unknowns $\{\rho^*, \mathbf{x}^*, \mathbf{T}^*\}$. The physical feature that is missing in this description is the characterization of the response of a particular material. Obviously, a fluid responds differently to some deformations than a solid. The equations that distinguish between responses of one material and another are called constitutive equations. Specifically, these constitutive equations specify the value of the stress tensor $\mathbf{T}^*$ associated with any possible deformation of the continuum.

Furthermore, it is noted that the reduced form of the balance of angular momentum $(3.2.33)_4$ places three restrictions on the constitutive equation for Cauchy stress $\mathbf{T}^*$ that must be satisfied for all possible motions of the continuum. Therefore, the balance of angular momentum has a different character from the other two balance laws because it is not used to determine the motion or deformation of the continuum. In contrast, the conservation of mass $(3.2.33)_1$ and the balance of linear momentum $(3.2.33)_3$ are used to determine the mass density ρ^* and the motion of the continuum through the functional form for $\mathbf{x}^*$.

3.3 Invariance under superposed rigid body motions

In order to motivate the basic character of the notion of invariance under superposed rigid body motion, it is sufficient to consider the simple problem of a point mass m that is attached to a rigid frame by an elastic spring of free-length L. The force of gravity per unit mass g acts in the negative $\mathbf{e}_1$ direction (see Fig. 3.3.1) and the tension in the spring T is a nonlinear function of the extension of the spring

$$T = \hat{T}(\ell - L) \quad , \quad \hat{T}(0) = 0 \ , \tag{3.3.1}$$

where ℓ is the current length of the spring. With respect to the fixed origin O, the location of the point A of the spring attached to the mass is denoted by $\mathbf{x} = x\,\mathbf{e}_1$ and the location of the point B of the spring attached to the rigid frame is denoted by $\mathbf{x}_B = x_B\,\mathbf{e}_1$ so that the current length of the spring ℓ is determined by the equation

$$\ell = |\,\mathbf{x} - \mathbf{x}_B\,| \ . \tag{3.3.2}$$

It then follows from the free-body diagram (Fig. 3.3.1) that the equation of motion of the mass can be written in the form

$$m\,\dot{\mathbf{v}} = m\,\mathbf{b} + \mathbf{F}_c \ , \tag{3.3.3}$$

where $\mathbf{v}$ is the absolute velocity of the mass, $\mathbf{b}$ is the body force due to gravity and $\mathbf{F}_c$ is the contact force due to the spring

$$\mathbf{v} = \dot{\mathbf{x}} \quad , \quad \mathbf{b} = -g\,\mathbf{e}_1 \quad , \quad \mathbf{F}_c = \hat{T}(\ell - L)\ \mathbf{e}_1 \ . \tag{3.3.4}$$

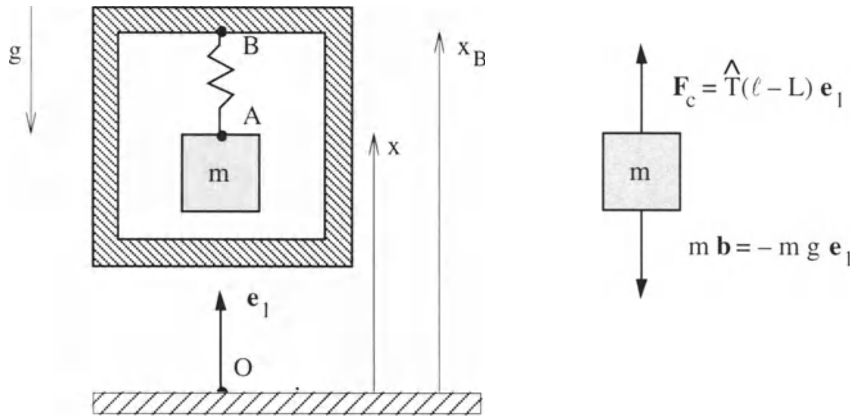

Fig. 3.3.1 Motion of a spring mass system.

Next, consider a new situation where the frame and the mass are subjected to a superposed rigid body motion (hereafter denoted by SRBM) characterized by the translation vector $\mathbf{c}(t)$ which can be a function of time (see Fig. 3.3.2). For this superposed motion, the new position of point A is denoted by $\mathbf{x}^+$ and the new position of point B is denoted by $\mathbf{x}_B^+$ such that

$$\mathbf{x}^+ = \mathbf{x} + \mathbf{c}(t) \quad , \quad \mathbf{x}_B^+ = \mathbf{x}_B + \mathbf{c}(t) \ . \tag{3.3.5}$$

Moreover, the mass is unaltered by the SRBM so the mass m^+ in the superposed configuration is given by

$$m^+ = m \ . \tag{3.3.6}$$

Now, using the transformations (3.3.5), (3.3.6) and the equation of motion (3.3.3), it follows that

$$m^+ \dot{\mathbf{v}}^+ = m\,\dot{\mathbf{v}} + m\,\ddot{\mathbf{c}} = [\,m\,\mathbf{b} + \mathbf{F}_c\,] + m\,[\,\dot{\mathbf{v}}^+ - \dot{\mathbf{v}}\,] \ ,$$

$$m^+ \dot{\mathbf{v}}^+ = m^+ [\,\dot{\mathbf{v}}^+ + (\mathbf{b} - \dot{\mathbf{v}}\,)] + \mathbf{F}_c \ , \tag{3.3.7}$$

where $\mathbf{v}^+ = \dot{\mathbf{x}}^+$ is the velocity of the mass in the superposed configuration. It presently will be shown that the body force and the contact force transform differently under SRBM. To this end, it is first noted that the new length of the spring ℓ^+ is defined by an equation similar to (3.2.33) in terms of $\mathbf{x}^+$ and $\mathbf{x}_B^+$ such that

$$\ell^+ = |\,\mathbf{x}^+ - \mathbf{x}_B^+\,| \ . \tag{3.3.8}$$

However, since the SRBM transforms all points in the system by the same translation vector $\mathbf{c}(t)$, it follows that the length of the spring is unaltered by the SRBM

$$\ell^+ = \ell \ . \tag{3.3.9}$$

Therefore, the contact force $\mathbf{F}_c$ which transforms to $\mathbf{F}_c^+$ in the superposed configuration also remains unaltered by the superposed translation

$$\mathbf{F}_c^+ = \hat{T}(\ell^+ - L)\,\mathbf{e}_1 = \hat{T}(\ell - L)\,\mathbf{e}_1 \ , \quad \mathbf{F}_c^+ = \mathbf{F}_c \ . \tag{3.3.10}$$

This result suggests that the force which causes the SRBM should be attributed to the body force. In particular, if it is assumed that the body force $\mathbf{b}$ transforms to $\mathbf{b}^+$ in the superposed configuration such that

$$\mathbf{b}^+ = \dot{\mathbf{v}}^+ + (\mathbf{b} - \dot{\mathbf{v}}) \ , \tag{3.3.11}$$

then the equation of motion of the mass (3.3.7) can be written in the form

$$m^+ \dot{\mathbf{v}}^+ = m^+ \mathbf{b}^+ + \mathbf{F}_c^+ \ . \tag{3.3.12}$$

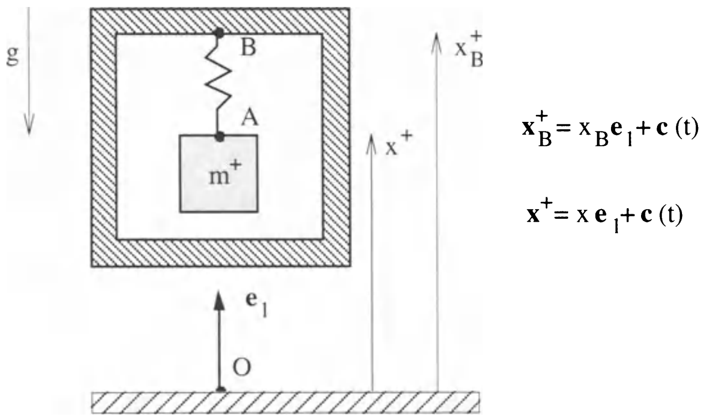

Fig. 3.3.2 Superposed Rigid Body Motion of a spring mass system.

Comparison of equations (3.3.3) and (3.3.12) reveals that the equation of motion of the mass remains form invariant under SRBM whenever the kinematic and kinetic quantities satisfy the transformation relations (3.3.5),(3.3.6),(3.3.10) and (3.3.11). This result is rather trivial when the SRBM is a superposed constant translational velocity ($\ddot{\mathbf{c}} = 0$) because it merely states the well known fact that it is impossible to distinguish between the influence of a force applied to a mass which is initially at rest and the influence of the same force applied to the same mass when it is initially moving with a constant translational velocity. However, this result is not trivial when the translation $\mathbf{c}(t)$ allows for accelerations. It will presently be shown in the generalized case of a three-dimensional continuum that invariance under SRBM places physical restrictions on proposed constitutive equations for stress.

The full generalization of the notion of SRBM requires two additional physical features. One feature is to include arbitrary rigid rotations of the continuum. The second feature is to include an arbitrary constant time shift. For example, consider a standard simple tension test of a cylindrical specimen which is used to determine the relationship between axial strain and axial stress of a particular material under consideration. For simplicity, let the test be performed with the specimen in the horizontal plane which is perpendicular to the force of gravity. Observations indicate that the stress-strain curve measured when the specimen is tested in one particular horizontal orientation is the same as that obtained when the specimen is rigidly rotated to another horizontal orientation and then tested with the same strain history. Moreover, observations indicate that it doesn't matter if the test is performed today or tomorrow.

The above discussion was presented to motivate the utility of considering the group of all motions that differ by a superposed rigid motion and a superposed constant time shift. To this end, suppose that under a SRBM the material point which is at $\mathbf{x}^*$ at time t in the present configuration moves to the location $\mathbf{x}^{*+}$ at time t^+

$$t^+ = t + a \ , \tag{3.3.13}$$

in the superposed configuration, where a is a constant. Throughout the text, quantities associated with the superposed configuration will be denoted using the same symbol as those associated with the present configuration, but with a superposed ($^+$). Thus, the vector function $\mathbf{x}^{*+}$ can be written in the Eulerian forms

$$\mathbf{x}^{*+} = \tilde{\tilde{\mathbf{x}}}^*(\mathbf{x}^*,t^+) = \tilde{\mathbf{x}}^*(\mathbf{x}^*,t) \ , \tag{3.3.14}$$

where the functional form $\tilde{\mathbf{x}}^*(\mathbf{x}^*,t)$ differs from $\tilde{\tilde{\mathbf{x}}}^*(\mathbf{x}^*,t^+)$ in that the constant a associated with the time shift has been absorbed into the functional form $\tilde{\mathbf{x}}^*(\mathbf{x}^*,t)$.

The condition that (3.3.14) represents the group of all SRBM requires the distance between any two material points in the superposed configuration to remain the same as the distance between the same two material points in the present configuration. Thus, letting $\mathbf{x}^*$ and $\mathbf{y}^*$ be two material points in the present configuration, and $\mathbf{x}^{*+}$ and $\mathbf{y}^{*+}$ be the same two material points in the superposed configuration, it follows that

$$| \mathbf{x}^{*+} - \mathbf{y}^{*+} | = | \mathbf{x}^* - \mathbf{y}^* | \ , \tag{3.3.15}$$

for all time t. In particular, it is important to emphasize that the condition (3.3.15) does not require the distance between these two points to remain constant in time (which would correspond only to a rigid motion of the continuum). Instead, the condition (3.3.15) requires the changing distance between the two points to remain the same in both the present and the superposed configurations. This condition is a generalization of the condition (3.3.9) which requires the length of the spring to remain the same in the simple example discussed previously.

It can be shown relatively easily that a necessary and sufficient condition for (3.3.15) to be valid is that the functional form (3.3.14) reduce to

$$\mathbf{x}^{*+} = \tilde{\mathbf{x}}^*(\mathbf{x}^*,t) = \mathbf{c}(t) + \mathbf{Q}(t)\,\mathbf{x}^* , \tag{3.3.16}$$

where $\mathbf{c}(t)$ is an arbitrary vector function of time only and $\mathbf{Q}(t)$ is an arbitrary orthogonal second order tensor function of time only

$$\mathbf{Q}\,\mathbf{Q}^T = \mathbf{Q}^T\,\mathbf{Q} = \mathbf{I} \ . \tag{3.3.17}$$

Obviously, if the distance between any two material points in the group of SRBM is a specific function of time, then the angle between any two material lines is another specific function of time, and the volume of a material region is still another specific function of time for all members of the group.

Since $\mathbf{Q}$ is an orthogonal tensor, its determinant is either plus one or minus one

$$\det\ \mathbf{Q}(t) = \pm 1 \ . \tag{3.3.18}$$

This means that if $\det \mathbf{Q} = -1$, then the transformation (3.3.16) includes all reflections of the body but it does not include the trivial transformation

$$\mathbf{x}^{*+} = \mathbf{x}^* \quad \text{for which} \quad \det \mathbf{Q} = +1 \ . \tag{3.3.19}$$

For this reason, attention is limited only to the subgroup of transformations (3.3.16) which includes the trivial transformation (3.3.19). Thus, in the following, the notion of SRBM will be limited to the transformation (3.3.16) together with the restriction that $\mathbf{Q}(t)$ is a proper orthogonal tensor function with positive determinant (3.3.19).

Among other results, it can be shown that under SRBM the covariant vectors $\mathbf{G}_i$ and $\mathbf{g}_i$, the contravariant vectors $\mathbf{G}^i$ and $\mathbf{g}^i$, the deformation gradient $\mathbf{F}^*$, the velocity gradient $\mathbf{L}^*$, the rate of deformation tensor $\mathbf{D}^*$, and the spin tensor $\mathbf{W}^*$, transform by the relations

$$\begin{gathered}
\mathbf{G}_i^+ = \mathbf{G}_i \ , \quad \mathbf{G}^{i+} = \mathbf{G}^i \ , \quad (G^{1/2})^+ = G^{1/2} \ , \\
\mathbf{g}_i^+ = \mathbf{Q}\,\mathbf{g}_i \ , \quad \mathbf{g}^{i+} = \mathbf{Q}\,\mathbf{g}^i \ , \quad (g^{1/2})^+ = g^{1/2} \ , \\
\mathbf{F}^{*+} = \mathbf{Q}\,\mathbf{F}^* \ , \quad \mathbf{L}^{*+} = \mathbf{Q}\,\mathbf{L}^*\,\mathbf{Q}^T + \boldsymbol{\Omega} \ , \\
\mathbf{D}^{*+} = \mathbf{Q}\,\mathbf{D}^*\,\mathbf{Q}^T \ , \quad \mathbf{W}^{*+} = \mathbf{Q}\,\mathbf{W}^*\,\mathbf{Q}^T + \boldsymbol{\Omega} \ ,
\end{gathered} \tag{3.3.20}$$

where $\mathbf{D}^*$ is the symmetric part of $\mathbf{L}^*$, and $\mathbf{W}^*$ is the skew symmetric part of $\mathbf{L}^*$, defined by

$$\mathbf{D}^* = \frac{1}{2}(\mathbf{L}^* + \mathbf{L}^{*T}) = \mathbf{D}^{*T} \ , \quad \mathbf{W}^* = \frac{1}{2}(\mathbf{L}^* - \mathbf{L}^{*T}) = -\,\mathbf{W}^{*T} \ . \tag{3.3.21}$$

Also, the second order tensor $\boldsymbol{\Omega}(t)$ is a function of time only which is determined by the derivative of $\mathbf{Q}$. Moreover, by differentiating (3.3.17) with respect to time, it can be shown that $\boldsymbol{\Omega}$ is a skew-symmetric tensor

$$\dot{\mathbf{Q}} = \mathbf{\Omega}(t)\,\mathbf{Q} \ , \quad \mathbf{\Omega}^T = -\mathbf{\Omega} \ . \tag{3.3.22}$$

The results (3.3.20) indicate that the transformations of some second order tensors under SRBM are unaffected by the superposed rigid rate of rotation $\mathbf{\Omega}$, whereas those of other second order tensors are influenced by $\mathbf{\Omega}$.

Moreover, it follows from (3.2.7), (3.3.19) and (3.3.20), that the dilatation J^* is unaltered by SRBM

$$J^{*+} = \det \mathbf{F}^{*+} = (\det \mathbf{Q})\,(\det \mathbf{F}^*) = J^* \ , \tag{3.3.23}$$

which is an expected result. Also, for later reference, it is necessary to determine how the normal $\mathbf{n}^*$ to a material surface transforms under SRBM. To this end, consider two distinct material line elements $d\mathbf{X}^{*1}$ and $d\mathbf{X}^{*2}$ and let dA^* be the element of area and $\mathbf{N}^*$ be the unit normal to the elemental surface of the parallelogram defined by these line elements such that

$$\mathbf{N}^*\,dA^* = d\mathbf{X}^{*1} \times d\mathbf{X}^{*2} \ . \tag{3.3.24}$$

The associated material line elements $d\mathbf{x}^{*1}$ and $d\mathbf{x}^{*2}$, element of area da^*, and unit normal $\mathbf{n}^*$ in the present configuration are given by

$$\mathbf{n}^*\,da^* = d\mathbf{x}^{*1} \times d\mathbf{x}^{*2} \ . \tag{3.3.25}$$

Next, recalling that the deformation gradient $\mathbf{F}^*$ maps material line elements from the reference to the present configurations

$$d\mathbf{x}^* = \mathbf{F}^*\,d\mathbf{X}^* \ , \tag{3.3.26}$$

it follows that

$$\mathbf{n}^*\,da^* = (\mathbf{F}^*\,d\mathbf{X}^{*1}) \times (\mathbf{F}^*\,d\mathbf{X}^{*2}) \ . \tag{3.3.27}$$

However, since $\mathbf{F}^*$ is nonsingular it can be shown that for any two vectors $\mathbf{a}$ and $\mathbf{b}$ [see equation (A.7.16)]

$$(\mathbf{F}^*\,\mathbf{a}) \times (\mathbf{F}^*\,\mathbf{b}) = J^*\,\mathbf{F}^{*-T}\,(\mathbf{a} \times \mathbf{b}) \ , \tag{3.3.28}$$

so that (3.3.27) yields

$$\mathbf{n}^*\,da^* = J^*\,\mathbf{F}^{*-T}\,(\mathbf{N}^*\,dA^*) \ . \tag{3.3.29}$$

Now, using this expression and the fact that the reference configuration is unaffected by SRBM, it follows that the unit normal $\mathbf{n}^{*+}$ and the element of area da^{*+} in the superposed configuration are given by

$$\mathbf{n}^{*+}\,da^{*+} = J^{*+}\,(\mathbf{F}^{*+})^{-T}\,(\mathbf{N}^*\,dA^*) = J^*\,\mathbf{Q}\,\mathbf{F}^{*-T}\,(\mathbf{N}^*\,dA^*)$$

$$\mathbf{n}^{*+}\,da^{*+} = \mathbf{Q}\,\mathbf{n}^*\,da^* \ . \tag{3.3.30}$$

Thus, by taking the inner product of (3.3.30) with itself, it can be shown that

$$\mathbf{n}^{*+} = \mathbf{Q}(t)\,\mathbf{n}^* \ , \quad da^{*+} = da^* \ , \tag{3.3.31}$$

which states that the unit normal to a material surface is merely rotated by SRBM and the element of area is unaltered by SRBM.

The results discussed above between equations (3.3.13) and (3.3.31) are purely kinematical conclusions that are derived for the group of SRBM. However, a discussion of invariance under SRBM requires additional statements about kinetic quantities like the body force and the stress. More specifically, the notion of invariance under SRBM defines a group of superposed configurations (*motions and states of the material*) which are considered to be physically equivalent. A member of this group is characterized by a specific SRBM and by specific values of all kinematic and kinetic quantities that influence the response of the material. In particular, for general continua these variables include various internal state variables (such as hardening variables in plasticity) that characterize the state of the material. For simplicity, the statement that a quantity transforms under SRBM according to a specific relationship will be used to describe the transformation relationships of the quantity in this group even though the quantity may be purely kinetic in nature. Furthermore, it will presently be shown that these invariance properties place important fundamental restrictions on constitutive equations for all continua.

The states of the material in this group are restricted by various kinetic assumptions. The first assumption is that the conservation of mass (3.2.1), and the balances of linear momentum (3.2.2) and angular momentum (3.3.4) remain form invariant in this group. It then follows that the local forms (3.2.35) of these conservation laws can be expressed in the superposed configuration by

$$m^{*+} = \rho^{*+}(g^{1/2})^{+} = \rho^{*}\, g^{1/2} = m^{*} \ ,$$

$$m^{*+}\, \dot{\mathbf{v}}^{*+} = m^{*+}\, \mathbf{b}^{*+} + \mathbf{t}^{*i+}{}_{,i} \ , \quad (\mathbf{T}^{*+})^{T} = \mathbf{T}^{*+} \ , \tag{3.3.32}$$

where $\mathbf{t}^{*i+}{}_{,i}$ is related to the divergence operator div^{*+} with respect to the position $\mathbf{x}^{*+}$ by the equation

$$\mathbf{t}^{*i+}{}_{,i} = \left[(g^{1/2})^{+}\, \mathbf{T}^{*+}\, \mathbf{g}^{i+}\right]_{,i} = (g^{1/2})^{+}\, \mathrm{div}^{*+}\, \mathbf{T}^{*+} \ . \tag{3.3.33}$$

Using the results (3.3.20), it is seen that the density of the body remain unaltered by SRBM

$$\rho^{*+} = \rho^{*} \ . \tag{3.3.34}$$

In contrast with equation $(3.3.32)_1$, the balance law $(3.3.32)_2$ introduces two unknown functions $\mathbf{b}^{*+}$ and $\mathbf{t}^{*i+}$ (or equivalently $\mathbf{T}^{*+}$) so that it is not possible to determine the proper transformation relations for the body force $\mathbf{b}^{*}$ and the stress tensor $\mathbf{T}^{*}$ without considering additional physical restrictions. To this end, it is recalled that the axial stress in the tension specimen discussed earlier is presumed to be insensitive to superposed rigid rotation. More specifically, although the orientation of the stress vector acting on the material cross-section of the specimen is expected to be influenced by the superposed rigid rotation, the component of this vector normal to the material cross-section is not expected to be influenced. In the more general situation presently under consideration, this suggests that the normal component of the stress vector acting on material surface is unaltered by SRBM so that

$$\mathbf{t}^{*+}(\mathbf{x}^{*+}, t^{+}; \mathbf{n}^{*+}) \bullet \mathbf{n}^{*+} = \mathbf{t}^{*}(\mathbf{x}^{*}, t\, ; \mathbf{n}^{*}) \bullet \mathbf{n}^{*} \ . \tag{3.3.35}$$

At this point, it should be emphasized that the condition (3.3.35) is a physical restriction that limits the states of the material which are considered to be physically equivalent in the group of superposed configurations characterized by invariance under SRBM. In particular, this condition is an independent kinetic assumption that cannot be derived directly by considering the kinematics (3.3.16) of SRBM alone. Moreover, the condition (3.3.35) is assumed to be valid for all continua including those whose responses are characterized by a number of internal state variables.

Next, using the fact that the normal transforms by the formula $(3.3.31)_1$, the equation (3.3.35) can be rewritten in the form

$$\left[\mathbf{t}^{*+}(\mathbf{x}^{*+},t^{+}; \mathbf{n}^{*+}) - \mathbf{Q}\, \mathbf{t}^{*}(\mathbf{x}^{*},t\,; \mathbf{n}^{*}) \right] \bullet \mathbf{n}^{*+} = 0 \ . \tag{3.3.36}$$

Since the coefficient of $\mathbf{n}^{*+}$ depends on $\mathbf{n}^{*+}$, it is not possible to conclude that this coefficient vanishes. However, using (3.2.3), the stress vector can be expressed in terms of the stress tensor which yields the equation

$$\left[\mathbf{T}^{*+}(\mathbf{x}^{*+},t^{+}) - \mathbf{Q}\, \mathbf{T}^{*}(\mathbf{x}^{*},t)\, \mathbf{Q}^{T} \right] \bullet (\mathbf{n}^{*+} \otimes \mathbf{n}^{*+}) = 0 \ , \tag{3.3.37}$$

that must be valid for all values of $\mathbf{n}^{*+}$. Now, in view of the balance of angular momentum (3.2.32) and $(3.3.32)_3$, it can be seen that the coefficient of (3.3.37) is a symmetric tensor that is independent of $\mathbf{n}^{*+}$. Consequently, it can be concluded that under SRBM the Cauchy stress tensor must transform by the formula

$$\mathbf{T}^{*+} = \mathbf{Q}\, \mathbf{T}^{*}(\mathbf{x}^{*},t)\, \mathbf{Q}^{T} \ . \tag{3.3.38}$$

Moreover, once the validity of (3.3.38) has been established, it can be shown that the stress vector $\mathbf{t}^{*}$ and the vectors $\mathbf{t}^{*i}$ defined in (3.2.34) transform by the formulas

$$\mathbf{t}^{*+} = \mathbf{Q}\, \mathbf{t}^{*} \ , \quad \mathbf{t}^{*i+} = \mathbf{Q}\, \mathbf{t}^{*i} \ . \tag{3.3.39}$$

Returning to the determination of the transformation relation for the body force $\mathbf{b}$, it is noted from (3.2.35) that

$$\mathbf{t}^{*i+}{}_{,i} = \mathbf{Q}\, \mathbf{t}^{*i}{}_{,i} = \mathbf{Q}\, m^{*}\, (\dot{\mathbf{v}}^{*} - \mathbf{b}^{*}) \ . \tag{3.3.40}$$

Thus, with the help of $(3.3.32)_1$, the balance of linear momentum in the superposed configuration $(3.3.32)_2$ yields the transformation relation for the body force of the form

$$\mathbf{b}^{*+} = \dot{\mathbf{v}}^{*+} + \mathbf{Q}\, (\mathbf{b}^{*} - \dot{\mathbf{v}}^{*}) \ , \tag{3.3.41}$$

which is a generalized version of the expression (3.3.11) developed for the spring mass system.

In summary, the superposed configurations form a group of *motions and states of the material* which are considered to be physically equivalent. The superposed rigid body motions (SRBM) in this group are characterized by

$$\mathbf{x}^{*+} = \tilde{\mathbf{x}}^{*}(\mathbf{x}^{*},t) = \mathbf{c}(t) + \mathbf{Q}(t)\, \mathbf{x}^{*} \ , \quad t^{+} = t + a \ ,$$
$$\mathbf{Q}\mathbf{Q}^{T} = \mathbf{Q}^{T}\mathbf{Q} = \mathbf{I} \ , \quad \det \mathbf{Q} = +1 \ , \tag{3.3.42}$$

and the transformation relations for density, Cauchy stress, m^{*}, $\mathbf{t}^{*i}$ and the body force in this group are

$$\rho^{*+} = \rho^{*} \ , \quad \mathbf{T}^{*+} = \mathbf{Q}\, \mathbf{T}^{*}(\mathbf{x}^{*},t)\, \mathbf{Q}^{T} \ ,$$
$$m^{*+} = m^{*} \ , \quad \mathbf{t}^{*i+} = \mathbf{Q}\, \mathbf{t}^{*i} \ , \quad \mathbf{b}^{*+} = \dot{\mathbf{v}}^{*+} + \mathbf{Q}\, (\mathbf{b}^{*} - \dot{\mathbf{v}}^{*}) \ . \tag{3.3.43}$$

Moreover, the transformation relations (3.3.43) are necessary and sufficient conditions for the conservation of mass, the balance of linear momentum, the balance of angular momentum, and the normal component of the traction vector to be form invariant under SRBM,

$$m^{*+} = m^{*} \ , \quad m^{*+}\, \dot{\mathbf{v}}^{*+} = m^{*+}\, \mathbf{b}^{*+} + \mathbf{t}^{*i+}{}_{,i} \ ,$$

$$(\mathbf{T}^{*+})^T = \mathbf{T}^{*+} \quad , \quad \mathbf{t}^+ \bullet \mathbf{n}^+ = \mathbf{t} \bullet \mathbf{n} \quad . \tag{3.3.44}$$

It will be shown in later sections that the condition $(3.3.43)_2$ places important physical restrictions on constitutive assumptions for stress.

Before closing this section, it is desirable to comment on the condition that constitutive equations must be invariant to a superposed constant time shift $(3.3.42)_2$. It is well known that materials like rubber can age in the sense that their material properties can change with time. This causes the response of the rubber to a specified loading path to change with time. For example, the response to a uniaxial tension experiment on a specimen of rubber performed today can be different from that obtained in an experiment done one year from today even if the specimen is kept at room temperature and stress-free during the one year period. This observation does not mean that some constitutive equations can be sensitive to a constant time shift. Instead, it means that the constitutive equations for such aging materials are not functions of the total deformation alone. In particular, such materials can be modeled by introducing additional internal state variables that evolve with time. Such variables are often characterized by evolution equations which are constitutive equations for their time rates of change. Consequently, the aging process can be associated with a change in the values of certain internal state variables which evolve during the year period.

In order to characterize the full invariance properties of such constitutive equations, it is necessary to supplement (3.3.43) by specifying transformation relations for these internal variables under SRBM. Usually, these variables are characterized by tensors whose magnitudes remain unaltered by SRBM. Consequently, states of the material in the group of superposed configurations which are considered to be physically equivalent will have the same state of aging. Therefore, although invariance under SRBM will require the constitutive equations to be explicitly independent of an arbitrary time shift, it will not preclude time dependence of material response through evolving internal state variables.

3.4 Mechanical power

For the purely mechanical theory it is convenient to define the notion of the mechanical power $\mathcal{P}^*$ due to the stress. This quantity is defined in terms of the rate of work $\mathcal{W}$ applied to the body by the body and contact forces

$$\mathcal{W} = \int_{P^*} \rho^* \, \mathbf{b}^* \bullet \mathbf{v}^* \, dv^* + \int_{\partial P^*} \mathbf{t}^* \bullet \mathbf{v}^* \, da^* \quad , \tag{3.4.1}$$

and the kinetic energy $\mathcal{K}$ of the body

$$\mathcal{K} = \int_{P^*} \frac{1}{2} \rho^* \, \mathbf{v}^* \bullet \mathbf{v}^* \, dv^* \quad . \tag{3.4.2}$$

Specifically, the mechanical power is defined by the global equation

$$\int_{P^*} \mathcal{P}^* \, dv^* = \mathcal{W} - \dot{\mathcal{K}} = \int_{P^*} \rho^* \, \mathbf{b}^* \bullet \mathbf{v}^* \, dv^* + \int_{\partial P^*} \mathbf{t}^* \bullet \mathbf{v}^* \, da^*$$

$$- \frac{d}{dt} \int_{P^*} \frac{1}{2} \rho^* \mathbf{v}^* \bullet \mathbf{v}^* \, dv^* \quad . \tag{3.4.3}$$

Next, assuming sufficient continuity of the functions and using the result that

$$\mathrm{div}^*(\mathbf{v}^* \bullet \mathbf{T}^*) = \mathbf{v}^* \bullet \mathrm{div}^* \mathbf{T}^* + \mathbf{T}^* \bullet \mathbf{L}^* \ , \tag{3.4.4}$$

it can be shown that the local form of equation (3.4.3) requires the mechanical power to be given by the expression

$$\mathcal{P}^* = \mathbf{T}^* \bullet \mathbf{L}^* \ . \tag{3.4.5}$$

Moreover, since the stress tensor must be symmetric, the skew-symmetric part $\mathbf{W}^*$ of $\mathbf{L}^*$ does not contribute to the expression (3.4.5) so that $\mathcal{P}^*$ reduces to

$$\mathcal{P}^* = \mathbf{T}^* \bullet \mathbf{D}^* \ . \tag{3.4.6}$$

Also, it can be shown using the transformation relations $(3.3.20)_5$ and (3.3.38), that the mechanical power is unaltered by SRBM

$$\mathcal{P}^{*+} = \mathcal{P}^* \ . \tag{3.4.7}$$

Further in this regard, it is noted that the rate of work $\mathcal{W}$ and the kinetic energy $\mathcal{K}$ are both altered by SRBM.

3.5 An alternative derivation of the balance laws

From the point of view presented previously, the condition that the balance laws remain form invariant under SRBM requires the kinematic and kinetic quantities to satisfy the transformation relations (3.3.16), (3.3.34), (3.3.38), (3.3.41), as well a number of other expressions [like (3.4.7)] that can be derived directly from these relations. In this regard, the notion of invariance under SRBM is a fundamental notion that causes intimate interconnections between the balance laws. To demonstrate the fundamental nature of invariance under SRBM, it will be shown that the global forms of the balance laws can be derived by assuming that the global form (3.4.3) of the expression for mechanical power remains form invariant, and that the local forms of the above transformation relations are valid. This means that the balance laws can be derived from a single scalar equation by demanding invariance for the class of all possible SRBM.

To this end, it is first noted that with respect to the superposed configuration, the global equation (3.4.3) can be written in the form

$$\int_{P^{*+}} \mathcal{P}^{*+} \, dv^{*+} = \int_{P^{*+}} \rho^{*+} \mathbf{b}^{*+} \bullet \mathbf{v}^{*+} \, dv^{*+} + \int_{\partial P^{*+}} \mathbf{t}^{*+} \bullet \mathbf{v}^{*+} \, da^{*+}$$
$$- \frac{d}{dt} \int_{P^{*+}} \frac{1}{2} \rho^{*+} \mathbf{v}^{*+} \bullet \mathbf{v}^{*+} \, dv^{*+} \quad . \tag{3.5.1}$$

Next, consider the special SRBM which is characterized by a superposed constant translational velocity with magnitude u and unit direction $\mathbf{u}$ so that

$$\mathbf{c}(t) = u\,\mathbf{u}\,t \ , \quad \dot{\mathbf{c}} = u\,\mathbf{u} \ , \quad \ddot{\mathbf{c}} = 0 \ , \quad \mathbf{u} \bullet \mathbf{u} = 1 \ ,$$
$$\mathbf{Q} = \mathbf{I} \ , \quad \dot{\mathbf{Q}} = 0 \ , \quad \ddot{\mathbf{Q}} = 0 \ ,$$
$$\mathbf{v}^{*+} = \mathbf{v}^* + u\,\mathbf{u} \ , \quad \dot{\mathbf{v}}^{*+} = \dot{\mathbf{v}}^* \ , \quad \mathbf{L}^{*+} = \mathbf{L}^* \ ,$$

$$\rho^{*+} = \rho^* \ , \ \ \mathbf{b}^{*+} = \mathbf{b}^* \ , \ \ \mathbf{t}^{*+} = \mathbf{t}^* \ , \ \ \mathbf{T}^{*+} = \mathbf{T}^* \ , \ \ \mathcal{P}^{*+} = \mathcal{P}^* \ . \tag{3.5.2}$$

Now, substituting (3.5.2) into (3.5.1), transforming the integrals over the superposed regions P^{*+} and ∂P^{*+} to the present regions P^* and ∂P^*, and subtracting the equation (3.4.3) from the result, yields the expression

$$\begin{aligned} \mathrm{u}\,\mathbf{u} \bullet \left[\frac{d}{dt}\int_{P^*} \rho^* \, \mathbf{v}^* \, dv^* - \int_{P^*} \rho^* \, \mathbf{b}^* \, dv^* - \int_{\partial P^*} \mathbf{t}^* \, da^*\right] & \\ + \frac{1}{2}\, \mathrm{u}^2 \left[\frac{d}{dt}\int_{P^*} \rho^* \, dv^*\right] = 0 \ , & \end{aligned} \tag{3.5.3}$$

which must be valid for all values of u and the unit vector $\mathbf{u}$. Moreover, since the coefficients in (3.5.3) are independent of u and $\mathbf{u}$, it follows that each of them must vanish. This procedure yields the global forms of conservation of mass (3.2.1) and the balance of linear momentum (3.2.2).

To derive the global form of the balance of angular momentum, it is convenient to consider a superposed constant rigid body rotation that is characterized by

$$\begin{gathered} \mathbf{c} = 0 \ , \ \dot{\mathbf{c}} = 0 \ , \ \ddot{\mathbf{c}} = 0 \ , \ \mathbf{Q} = \mathbf{Q}(t) \ , \ \dot{\mathbf{Q}} = \boldsymbol{\Omega}\,\mathbf{Q} \ , \ \dot{\boldsymbol{\Omega}} = 0 \ , \\ \mathbf{v}^{*+} = \mathbf{Q}\,\mathbf{v}^* + \boldsymbol{\Omega}\,\mathbf{Q}\,\mathbf{x}^* \ , \ \dot{\mathbf{v}}^{*+} = \mathbf{Q}\,\dot{\mathbf{v}}^* + 2\,\boldsymbol{\Omega}\,\mathbf{Q}\,\mathbf{v}^* + \boldsymbol{\Omega}^2\,\mathbf{Q}\,\mathbf{x}^* \ , \\ \mathbf{L}^{*+} = \mathbf{Q}\,\mathbf{L}^*\,\mathbf{Q}^T + \boldsymbol{\Omega} \ , \\ \rho^{*+} = \rho^* \ , \ \mathbf{b}^{*+} = \mathbf{Q}\,\mathbf{b}^* + 2\,\boldsymbol{\Omega}\,\mathbf{Q}\,\mathbf{v}^* + \boldsymbol{\Omega}^2\,\mathbf{Q}\,\mathbf{x}^* \ , \\ \mathbf{t}^{*+} = \mathbf{Q}\,\mathbf{t}^* \ , \ \mathbf{T}^{*+} = \mathbf{Q}\,\mathbf{T}^*\,\mathbf{Q}^T \ , \ \mathcal{P}^{*+} = \mathcal{P}^* \ . \end{gathered} \tag{3.5.4}$$

Thus, with the help of the definition (3.4.5) and the last of (3.5.4), it follows that

$$\mathcal{P}^{*+} = \mathbf{T}^{*+} \bullet \mathbf{L}^{*+} = \mathcal{P}^* + \mathbf{T}^* \bullet (\mathbf{Q}^T\boldsymbol{\Omega}\,\mathbf{Q}) = \mathcal{P}^* \ , \ \ \mathbf{T}^* \bullet (\mathbf{Q}^T\boldsymbol{\Omega}\,\mathbf{Q}) = 0 \ . \tag{3.5.5}$$

However, since $\mathbf{T}^*$ does not depend on $\boldsymbol{\Omega}$ and since $\boldsymbol{\Omega}$ is a skew-symmetric tensor, the quantity $\mathbf{Q}^T\boldsymbol{\Omega}\,\mathbf{Q}$ is also a skew-symmetric tensor so that (3.5.5) requires $\mathbf{T}^*$ to be a symmetric tensor, which yields the local form (3.2.32) of the balance of angular momentum. Furthermore, since $\boldsymbol{\Omega}$ is a skew symmetric tensor, it follows that for arbitrary vectors $\mathbf{a}$ and $\mathbf{b}$

$$\begin{gathered} (\mathbf{Q}\,\mathbf{a}) \times (\mathbf{Q}\,\mathbf{b}) = \det(\mathbf{Q})\,\mathbf{Q}^{-T}(\mathbf{a} \times \mathbf{b}) = \mathbf{Q}\,(\mathbf{a} \times \mathbf{b}) \ , \ \ \boldsymbol{\Omega}\,\mathbf{a} = \boldsymbol{\omega} \times \mathbf{a} \ , \\ (\boldsymbol{\Omega}\,\mathbf{a}) \bullet \mathbf{b} = (\boldsymbol{\omega} \times \mathbf{a}) \bullet \mathbf{b} = \boldsymbol{\omega} \bullet (\mathbf{a} \times \mathbf{b}) \ , \\ (\boldsymbol{\Omega}\,\mathbf{Q}\,\mathbf{a}) \bullet (\mathbf{Q}\,\mathbf{b}) = \boldsymbol{\omega} \bullet [(\mathbf{Q}\,\mathbf{a}) \times (\mathbf{Q}\,\mathbf{b})] = (\mathbf{Q}^T\boldsymbol{\omega}) \bullet (\mathbf{a} \times \mathbf{b}) \ , \\ (\boldsymbol{\Omega}\,\mathbf{a}) \bullet \mathbf{a} = 0 \ , \ \ \boldsymbol{\Omega}\,\boldsymbol{\omega} = \boldsymbol{\omega} \times \boldsymbol{\omega} = 0 \ , \end{gathered} \tag{3.5.6}$$

where $\boldsymbol{\omega}$ is the axial vector associated with $\boldsymbol{\Omega}$. Thus, using these results, it can be shown that

$$\begin{gathered} \mathbf{b}^{*+} \bullet \mathbf{v}^{*+} = \mathbf{b}^* \bullet \mathbf{v}^* + (\mathbf{Q}^T\boldsymbol{\omega}) \bullet (\mathbf{x}^* \times \mathbf{b}^*) + (\boldsymbol{\Omega}\,\mathbf{Q}\mathbf{x}^*) \bullet (\boldsymbol{\Omega}\,\mathbf{Q}\mathbf{v}^*) \ , \\ \mathbf{t}^{*+} \bullet \mathbf{v}^{*+} = \mathbf{t}^* \bullet \mathbf{v}^* + (\mathbf{Q}^T\boldsymbol{\omega}) \bullet (\mathbf{x}^* \times \mathbf{t}^*) \ , \\ \mathbf{v}^{*+} \bullet \mathbf{v}^{*+} = \mathbf{v}^* \bullet \mathbf{v}^* + 2(\mathbf{Q}^T\boldsymbol{\omega}) \bullet (\mathbf{x}^* \times \mathbf{v}^*) + (\boldsymbol{\Omega}\,\mathbf{Q}\mathbf{x}^*) \bullet (\boldsymbol{\Omega}\,\mathbf{Q}\mathbf{x}^*) \ . \end{gathered} \tag{3.5.7}$$

Now, substituting (3.5.4) and (3.5.7) into (3.5.1), transforming the integrals over the superposed region P^{*+} and ∂P^{*+} to the present regions P^* and ∂P^*, and subtracting the equation (3.4.3) from the result, yields the expression

$$(\mathbf{Q}^T\boldsymbol{\omega}) \bullet \left[\frac{d}{dt} \int_{P^*} \mathbf{x}^* \times \rho^* \mathbf{v}^* \, dv^* - \int_{P^*} \mathbf{x}^* \times \rho^* \mathbf{b}^* \, dv^* - \int_{\partial P^*} \mathbf{x}^* \times \mathbf{t}^* \, da^* \right]$$

$$+ (\dot{\mathbf{Q}}^T\boldsymbol{\omega}) \bullet \int_{P^*} \mathbf{x}^* \times \rho^* \mathbf{v}^* \, dv^* - \int_{P^*} \rho^* (\boldsymbol{\Omega}\,\mathbf{Q}\mathbf{x}^*) \bullet (\boldsymbol{\Omega}\,\mathbf{Q}\mathbf{v}^*) \, dv^*$$

$$+ \frac{d}{dt} \int_{P^*} \frac{1}{2} \rho^* (\boldsymbol{\Omega}\,\mathbf{Q}\mathbf{x}^*) \bullet (\boldsymbol{\Omega}\,\mathbf{Q}\mathbf{x}^*) \, dv^* = 0 \ . \tag{3.5.8}$$

However, with the help of (3.5.6) and the conservation of mass it can be shown that

$$\dot{\mathbf{Q}}^T\boldsymbol{\omega} = -\mathbf{Q}^T \boldsymbol{\Omega}\, \boldsymbol{\omega} = 0 \ ,$$

$$\frac{d}{dt} \int_{P^*} \frac{1}{2} \rho^* (\boldsymbol{\Omega}\,\mathbf{Q}\mathbf{x}^*) \bullet (\boldsymbol{\Omega}\,\mathbf{Q}\mathbf{x}^*) \, dv^* = \int_{P^*} \rho^* (\boldsymbol{\Omega}\,\mathbf{Q}\mathbf{x}^*) \bullet (\boldsymbol{\Omega}\,\mathbf{Q}\mathbf{v}^*) \, dv^* \ , \tag{3.5.9}$$

so that (3.5.8) reduces to

$$(\mathbf{Q}^T\boldsymbol{\omega}) \bullet \left[\frac{d}{dt} \int_{P^*} \mathbf{x}^* \times \rho^* \mathbf{v}^* \, dv^* - \int_{P^*} \mathbf{x}^* \times \rho^* \mathbf{b}^* \, dv^* - \int_{\partial P^*} \mathbf{x}^* \times \mathbf{t}^* \, da^* \right] . \tag{3.5.10}$$

Now, since (3.5.10) must be valid for all $\boldsymbol{\omega}$ and since the coefficient in the square brackets is independent of $\boldsymbol{\omega}$, it follows that (3.5.10) yields the global form of the balance of angular momentum (3.2.4).

In the above it has been shown that the global forms of the conservation of mass and the balances of linear and angular momentum are necessary conditions for the global expression (3.4.3) of mechanical power to remain form invariant under SRBM.

3.6 An averaged form of the balance of linear momentum

In the purely mechanical three-dimensional theory the balance of angular momentum places restrictions on the constitutive equations which require the Cauchy stress tensor to be symmetric. Also, the conservation of mass and the balance of linear momentum are used to determine the mass density and the position of each material point in the continuum. For the Cosserat theories that will be developed in the next chapters, alternative equations representing the conservation of mass and the balances of linear and angular momentum will be used in a similar manner to determine the mass density and the position of each material point. However, the Cosserat theories will introduce additional kinematical quantities called director vectors at each material point which also need to be determined by additional balance laws. In order to motivate the forms for these balance laws, it is convenient to consider an averaged form of the balance of linear momentum.

To this end, let $\phi(\theta^i)$ be a general weighting function that depends on the convected coordinates θ^i and is independent of time t so that

$$\phi = \phi(\theta^i) \ , \quad \dot{\phi} = 0 \ . \tag{3.6.1}$$

Now, multiplying the local form $(3.2.35)_2$ of linear momentum by the weighting function ϕ and using the expression (2.3.16) for the divergence operator and the definition (3.2.34), it can be deduced that

$$\phi m^* \dot{\mathbf{v}}^* = \phi m^* \mathbf{b}^* + \phi\, \mathbf{t}^{*j}{}_{,j} \ , \quad \phi m^* \dot{\mathbf{v}}^* = \phi m^* \mathbf{b}^* + (\phi\, \mathbf{t}^{*j})_{,j} - \mathbf{t}^{*j} \phi_{,j} \ ,$$

$$\phi m^* \dot{\mathbf{v}}^* = \phi m^* \mathbf{b}^* + g^{1/2}\,\mathrm{div}^*(\phi\,\mathbf{T}^*) - g^{1/2}\,\mathbf{T}^* \bullet \phi_{,j}\,\mathbf{g}^j \ . \tag{3.6.2}$$

Next, integrating (3.6.2) over the region P^* and using the divergence theorem (3.2.19) as well as the formula (3.2.30) derived from conservation of mass, it follows that

$$\frac{d}{dt}\int_{P^*} \phi\,\rho^* \mathbf{v}^*\,dv^* = \int_{P^*} [\phi\,\rho^* \mathbf{b}^* - g^{-1/2}\,\mathbf{t}^{*j}\,\phi_{,j}\,]\,dv^* + \int_{\partial P^*} \phi\,\mathbf{t}^*\,da^* \ , \tag{3.6.3}$$

where use has also been made of the expression (3.2.34). This equation represents an averaged form of the balance of linear momentum and it will be used later to motivate the forms for the balance laws of director momentum associated with the Cosserat theories discussed in the next chapters.

3.7 Anisotropic nonlinear elastic materials

In the previous sections kinematical expressions and balance laws were discussed which are valid for all materials that can be modeled as simple continua. Here, constitutive equations will be developed for general anisotropic nonlinear elastic materials. In continuum mechanics elastic materials are special materials that are considered to be ideal materials. For example, the response of an elastic material is insensitive to the rate at which it is loaded. Other features of an elastic material will be discussed presently.

To develop constitutive equations for an elastic material, it is convenient to characterize elastic materials by the following four assumptions:

Assumption 1: A strain energy Σ^* per unit mass exists for which

$$\rho^* \dot{\Sigma}^* = \mathcal{P}^* = \mathbf{T}^* \bullet \mathbf{D}^* \ . \tag{3.7.1}$$

Assumption 2: The strain energy Σ^* is a function of the deformation gradient $\mathbf{F}^*$ and the reference position $\mathbf{X}^*$ only

$$\Sigma^* = \tilde{\Sigma}^*(\mathbf{F}^*;\mathbf{X}^*) \ , \tag{3.7.2}$$

where dependence on the reference position $\mathbf{X}^*$ has been included to allow for the possibility that the material can be inhomogeneous in its reference configuration.

Assumption 3: The strain energy Σ^* is invariant under SRBM

$$\Sigma^{*+} = \Sigma^* \ . \tag{3.7.3}$$

Assumption 4: The Cauchy stress $\mathbf{T}^*$ is independent of the rate of deformation $\mathbf{L}^*$.

In order to explore the physical consequences of the assumption (3.7.1), it is convenient to define the total strain energy $\mathcal{U}$

$$\mathcal{U} = \int_{P^*} \rho^*\,\Sigma^*\,dv^* \ , \tag{3.7.4}$$

and to use the result (3.2.30) and (3.7.1) to deduce that

$$\dot{\mathcal{U}} = \int_{P^*} \rho^*\,\dot{\Sigma}^*\,dv^* = \int_{P^*} \mathcal{P}^*\,dv^* \ . \tag{3.7.5}$$

Thus, by substituting (3.7.5) into the mechanical power equation (3.4.3), it is possible to derive the following theorem:

$$\mathcal{W} = \dot{\mathcal{K}} + \dot{\mathcal{U}} \quad , \tag{3.7.6}$$

which states that for an elastic material the rate of work done on the body due to body forces and contact forces equals the rate of change of kinetic and strain energies. Since the strain energy Σ^* depends on the present configuration through the present value of $\mathbf{F}^*$ only [assumption (3.7.2)], the value of the strain energy Σ^* is independent of the particular loading path which caused $\mathbf{F}^*$. Consequently, the total work done on the body vanishes for any closed cycle in which the values of velocity $\mathbf{v}^*$ and the deformation gradient $\mathbf{F}^*$ are the same at the beginning and end of the cycle. Next, consider a special cycle which is composed of a loading path from one state A to another state B, followed by the reversal of this loading path. Then, in view of assumption 4, the work done on the body from A to B is fully recovered during the reverse loading from B to A. In this sense the elastic material is considered to be an ideal material.

The assumption (3.7.3) places restrictions on the functional form (3.7.2). To develop these restrictions, it is recalled that under SRBM $\mathbf{F}^{*+}=\mathbf{Q}\mathbf{F}^*$ so that (3.7.2) requires

$$\Sigma^{*+} = \tilde{\Sigma}^*(\mathbf{F}^{*+};\mathbf{X}^*) = \tilde{\Sigma}^*(\mathbf{Q}\mathbf{F}^*;\mathbf{X}^*) = \tilde{\Sigma}^*(\mathbf{F}^*;\mathbf{X}^*) \quad , \tag{3.7.7}$$

to hold for arbitrary proper orthogonal $\mathbf{Q}$. However, the polar decomposition theorem [see (A.7.14)] states that $\mathbf{F}^*$ can be separated multiplicatively into a rotation tensor $\mathbf{R}^*$ and positive definite symmetric stretch tensors $\mathbf{U}^*$ and $\mathbf{V}^*$ such that

$$\begin{gathered}
\mathbf{F}^* = \mathbf{R}^*\mathbf{U}^* = \mathbf{V}^*\mathbf{R}^* \quad , \\
\mathbf{C}^* = \mathbf{F}^{*T}\mathbf{F}^* = \mathbf{C}^{*T} \ , \ \ \mathbf{U}^{*T} = \mathbf{U}^* = (\mathbf{C}^*)^{1/2} \quad , \\
\mathbf{B}^* = \mathbf{F}^*\mathbf{F}^{*T} = \mathbf{B}^{*T} \ , \ \ \mathbf{V}^{*T} = \mathbf{V}^* = (\mathbf{B}^*)^{1/2} \ , \\
\mathbf{R}^{*T}\mathbf{R}^* = \mathbf{R}^*\mathbf{R}^{*T} = \mathbf{I} \ , \quad \det \mathbf{R}^* = 1 \quad ,
\end{gathered} \tag{3.7.8}$$

where $\mathbf{C}^*$ is the right Cauchy-Green deformation tensor and $\mathbf{B}^*$ is the left Cauchy-Green deformation tensor. Thus, the restriction (3.7.7) requires

$$\tilde{\Sigma}^*(\mathbf{F}^*;\mathbf{X}^*) = \tilde{\Sigma}^*(\mathbf{Q}\mathbf{F}^*;\mathbf{X}^*) = \tilde{\Sigma}^*(\mathbf{Q}\mathbf{R}^*\mathbf{U}^*;\mathbf{X}^*) \quad , \tag{3.7.9}$$

to hold for arbitrary values of the proper orthogonal tensor $\mathbf{Q}$. Since the deformation can be inhomogeneous, the rotation tensor $\mathbf{R}^*$ can be a function of position $\mathbf{X}^*$. However, for a given value of $\mathbf{X}^*$, say $\mathbf{X}_1^*$, it is possible to choose $\mathbf{Q}=\mathbf{R}^{*T}(\mathbf{X}_1^*)$ so that (3.7.9) yields

$$\tilde{\Sigma}^*(\mathbf{F}^*;\mathbf{X}^*) = \tilde{\Sigma}^*(\mathbf{R}^{*T}(\mathbf{X}_1^*)\mathbf{R}^*\mathbf{U}^*;\mathbf{X}^*) \quad . \tag{3.7.10}$$

Consequently, by evaluating (3.7.10) at $\mathbf{X}_1^*$, it can be shown that locally

$$\tilde{\Sigma}^*(\mathbf{F}^*;\mathbf{X}_1^*) = \tilde{\Sigma}^*(\mathbf{U}^*;\mathbf{X}_1^*) = \hat{\Sigma}^*(\mathbf{C}^*;\mathbf{X}_1^*) \quad , \tag{3.7.11}$$

where $\hat{\Sigma}^*$ is a new function of $\mathbf{C}^*$ and $\mathbf{X}^*$. Thus, a necessary condition for the strain energy Σ^* to be locally invariant under SRBM is that the strain energy function Σ^* must depend on the deformation gradient $\mathbf{F}^*$ only through its dependence on the deformation tensor $\mathbf{C}^*$. It is easy to see that this condition is also a sufficient condition because under SRBM

$$\mathbf{C}^{*+} = \mathbf{C}^* \quad . \tag{3.7.12}$$

However, since $\mathbf{X}_1^*$ is an arbitrary material point, it can be concluded that for each point $\mathbf{X}^*$, the strain energy Σ^* must depend on $\mathbf{F}^*$ only through its dependence on $\mathbf{C}^*$

$$\tilde{\Sigma}^*(\mathbf{F}^*;\mathbf{X}^*) = \hat{\Sigma}^*(\mathbf{C}^*;\mathbf{X}^*) \ . \tag{3.7.13}$$

Now, with the help of (3.2.12), (3.3.21) and (3.7.13), equation (3.7.1) yields

$$\mathbf{T}^* \bullet \mathbf{D}^* = \rho^* \frac{\partial \Sigma^*}{\partial \mathbf{C}^*} \bullet \dot{\mathbf{C}}^* = \rho^* \frac{\partial \Sigma^*}{\partial \mathbf{C}^*} \bullet 2\, \mathbf{F}^{*T}\mathbf{D}^*\mathbf{F}^* = 2\rho^* \, \mathbf{F}^* \frac{\partial \Sigma^*}{\partial \mathbf{C}^*} \mathbf{F}^{*T} \bullet \mathbf{D}^* \ ,$$

$$(\mathbf{T}^* - 2\rho^* \, \mathbf{F}^* \frac{\partial \Sigma^*}{\partial \mathbf{C}^*} \mathbf{F}^{*T}) \bullet \mathbf{D}^* = 0 \ . \tag{3.7.14}$$

Moreover, since the coefficient of $\mathbf{D}^*$ in (3.7.14) is independent of the rate $\mathbf{D}^*$ and is symmetric, it follows that for any fixed values of $\mathbf{F}^*$, $\mathbf{X}^*$, the coefficient of $\mathbf{D}^*$ is fixed and yet $\mathbf{D}^*$ can be an arbitrary symmetric tensor. Therefore, the necessary condition that (3.7.14) be valid for arbitrary motions is that the Cauchy stress be given by a derivative of the strain energy

$$\mathbf{T}^* = 2\rho^* \, \mathbf{F}^* \frac{\partial \Sigma^*}{\partial \mathbf{C}^*} \mathbf{F}^{*T} \ . \tag{3.7.15}$$

Furthermore, by defining the symmetric Piola-Kirchhoff stress $\mathbf{S}^*$ such that

$$\mathrm{J}^* \, \mathbf{T}^* = \mathbf{F}^* \, \mathbf{S}^* \, \mathbf{F}^* \ , \quad \mathbf{S}^{*T} = \mathbf{S}^* \ , \tag{3.7.16}$$

and by using the conservation of mass in the form (3.2.27), it follows that

$$\mathbf{S}^* = 2\, \rho_0^* \frac{\partial \Sigma^*}{\partial \mathbf{C}^*} \ . \tag{3.7.17}$$

Also, it is noted that the result (3.7.15) is properly invariant under SRBM because

$$\mathbf{S}^{*+} = \mathbf{S}^* \ . \tag{3.7.18}$$

Since the functional form (3.7.13) for Σ^* is trivially invariant under SRBM, Σ^* can be an arbitrary function of $\mathbf{C}^*$. This means that the strain energy function can describe anisotropic elastic materials since the response to a specified deformation relative to one material direction can be different from the response of the same deformation relative to another material direction.

Finally, use is made of the conservation of mass $(3.2.33)_1$, the definition (3.2.34) and the result (3.7.15), to deduce constitutive equations for the quantities $\mathbf{t}^{*i}$

$$\mathbf{t}^{*i} = 2\, \mathrm{m}^* \, \mathbf{F}^* \frac{\partial \Sigma^*}{\partial \mathbf{C}^*} \mathbf{G}^i \ . \tag{3.7.19}$$

3.8 Constraints

The notion of a mechanical constraint arises quite naturally in mechanics. For example, the stresses required to distort water or rubber are usually much smaller than those required to change the volume of these materials. Consequently, whenever part of the boundary of a body composed of these materials is either a free surface or is exposed to atmospheric pressure, then the body tends to distort rather than dilate. From the constitutive point of view, this means that the bulk modulus of each of these materials is much larger than its shear modulus. Moreover, this means that in many practical applications it is possible to model these materials as incompressible.

To model the response of an incompressible material, it is convenient to introduce the notion of a kinematical constraint which restricts all motions of these materials to be isochoric. Specifically, from (3.2.7) and (3.2.18), it follows that isochoric motions are characterized by the constraint

$$\mathbf{I} \cdot \mathbf{D}^* = 0 \ . \tag{3.8.1}$$

Another possible mechanical constraint naturally occurs in some fiber reinforced composite materials which have a relatively soft matrix material that is reinforced with strong fibers. If these fibers are all oriented in a particular direction, say **d**, then for certain applications it is possible to presume that the material is inextensible in the **d** direction so that

$$\mathbf{d} \cdot \mathbf{d} = \text{constant} \ . \tag{3.8.2}$$

Next, by taking the material derivative of (3.8.2) and using the fact that **d** is a material direction

$$\dot{\mathbf{d}} = \mathbf{L}^* \mathbf{d} \ , \tag{3.8.3}$$

it can be shown that the constraint (3.8.2) constrains the deformation so that

$$(\mathbf{d} \otimes \mathbf{d}) \cdot \mathbf{D}^* = 0 \ . \tag{3.8.4}$$

The expressions (3.8.1) and (3.8.4) suggest considering a class of mechanical constraints that can be represented in the form

$$\boldsymbol{\gamma}^* \cdot \mathbf{D}^* = 0 \ , \quad \boldsymbol{\gamma}^{*T} = \boldsymbol{\gamma}^* \ , \tag{3.8.5}$$

where $\boldsymbol{\gamma}^*$ is a symmetric second order tensor that characterizes the particular constraint under consideration. Moreover, it is assumed that under SRBM $\boldsymbol{\gamma}^*$ transforms by

$$\boldsymbol{\gamma}^{*+} = \mathbf{Q}\, \boldsymbol{\gamma}^*\, \mathbf{Q}^T \ , \tag{3.8.6}$$

so that the constraint equation $(3.8.5)_1$ remains properly invariant under SRBM.

Motivated by the work of Green, Naghdi and Trapp (1970), it is possible to develop a constitutive theory in the presence of mechanical constraints by making the following five assumptions:

(i) The Cauchy stress $\mathbf{T}^*$ separates additively into two parts

$$\mathbf{T}^* = \hat{\mathbf{T}}^* + \bar{\mathbf{T}}^* \ , \quad \hat{\mathbf{T}}^* = g^{-1/2}\, \hat{\mathbf{t}}^{*i} \otimes \mathbf{g}_i \ , \quad \bar{\mathbf{T}}^* = g^{-1/2}\, \bar{\mathbf{t}}^{*i} \otimes \mathbf{g}_i \ ,$$

$$\mathbf{t}^{*i} = \hat{\mathbf{t}}^{*i} + \bar{\mathbf{t}}^{*i} \ , \tag{3.8.7}$$

where $\hat{\mathbf{T}}^*$ and $\hat{\mathbf{t}}^{*i}$ are determined by constitutive equations that characterize the particular unconstrained material under consideration, and $\bar{\mathbf{T}}^*$ and $\bar{\mathbf{t}}^{*i}$ are the constraint responses.

(ii) The constraint responses $\bar{\mathbf{T}}^*$ and $\bar{\mathbf{t}}^{*i}$ are functions of $\mathbf{x}^*$ and t which are workless in the sense that

$$\bar{\mathbf{T}}^* \cdot \mathbf{D}^* = 0 \ , \tag{3.8.8}$$

for all possible motions of the constrained material.

(iii) Both parts $\hat{\mathbf{T}}^*$ and $\bar{\mathbf{T}}^*$ of the stress $\mathbf{T}^*$ are symmetric tensors

$$\hat{\mathbf{T}}^{*T} = \hat{\mathbf{T}}^* \ , \quad \bar{\mathbf{T}}^{*T} = \bar{\mathbf{T}}^* \ , \tag{3.8.9}$$

so that each of them satisfies the local form (3.2.32) of the balance of angular momentum.

(iv) Both parts $\hat{\mathbf{T}}^*$ and $\bar{\mathbf{T}}^*$ of the stress $\mathbf{T}^*$ transform under SRBM by

$$\hat{\mathbf{T}}^{*+} = \mathbf{Q}\,\hat{\mathbf{T}}^*\,\mathbf{Q}^T \ , \quad \bar{\mathbf{T}}^{*+} = \mathbf{Q}\,\bar{\mathbf{T}}^*\,\mathbf{Q}^T \ , \tag{3.8.10}$$

so that the stress $\mathbf{T}^*$ transforms by (3.3.38) and the expression (3.8.8) is properly invariant under SRBM.

(v) The tensors $\bar{\mathbf{T}}^*$ and $\boldsymbol{\gamma}^*$ are both independent of the rate $\mathbf{L}^*$.

Using a Lagrange multiplier $\gamma^*(\mathbf{x}^*,t)$ which is an arbitrary scalar function of $(\mathbf{x}^*,t)$, the equation (3.8.8), subject to the constraint (3.8.5), can be rewritten in the form

$$(\,\bar{\mathbf{T}}^* - g^{-1/2}\,\gamma^*\,\boldsymbol{\gamma}^*\,) \bullet \mathbf{D}^* = 0 \ . \tag{3.8.11}$$

Since at least one of the components of $\boldsymbol{\gamma}^*$ in (3.8.5) [say $\boldsymbol{\gamma}^* \bullet (\mathbf{g}_3 \otimes \mathbf{g}_3)$] is nonzero, the value of γ^* can be specified so that

$$\gamma^* = \frac{\bar{\mathbf{T}}^* \bullet (\mathbf{g}_3 \otimes \mathbf{g}_3)}{g^{-1/2}\,\boldsymbol{\gamma}^* \bullet (\mathbf{g}_3 \otimes \mathbf{g}_3)} \ . \tag{3.8.12}$$

It then follows that the coefficient of the component $[\mathbf{D}^* \bullet (\mathbf{g}_3 \otimes \mathbf{g}_3)]$ in (3.8.11) vanishes. Consequently, this component of $\mathbf{D}^*$ can be chosen to satisfy the constraint (3.8.5) for arbitrary values of the other components of $\mathbf{D}^*$. Moreover, since the coefficient of $\mathbf{D}^*$ in (3.8.11) is a symmetric tensor that is independent of rate, it follows that the constraint response $\bar{\mathbf{T}}^*$ must take the form

$$\bar{\mathbf{T}}^* = g^{-1/2}\,\gamma^*\,\boldsymbol{\gamma}^* \ , \quad \bar{\mathbf{t}}^{*i} = \gamma^*\,\boldsymbol{\gamma}^*\,\mathbf{g}^i \ , \tag{3.8.13}$$

where γ^* remains an arbitrary function of $(\mathbf{x}^*,t)$ that is determined by the equations of motion and the boundary conditions.

For the special case of the incompressibility constraint (3.8.1), $\boldsymbol{\gamma}^*$ equals the identity tensor $\mathbf{I}$ and the constraint response (3.8.13) causes the pressure p^* to be an arbitrary function of $(\mathbf{x}^*,t)$ since

$$p^* = -\frac{1}{3}\,\mathbf{T}^* \bullet \mathbf{I} \ , \quad p^* = \hat{p}^* + \bar{p}^* \ ,$$

$$\hat{p}^* = -\frac{1}{3}\hat{\mathbf{T}}^* \bullet \mathbf{I} \ , \quad \bar{p}^* = -\frac{1}{3}\bar{\mathbf{T}}^* \bullet \mathbf{I} = -\,g^{-1/2}\,\gamma^* \ . \tag{3.8.14}$$

In general, the constraint response $\bar{\mathbf{T}}^*$ influences the equation of linear momentum but not the conservation of mass or angular momentum equations (since it is a symmetric tensor). Therefore, if more than three independent kinematic constraints are imposed, then $\bar{\mathbf{T}}^*$ will have more than three independent components which are arbitrary functions of position and time. This means that even when appropriate boundary conditions are specified, it will not be possible to uniquely determine all components of $\bar{\mathbf{T}}^*$. However, it is reasonable to expect that the arbitrariness in $\bar{\mathbf{T}}^*$ that remains after the equations of motion and boundary conditions are satisfied will not influence the overall motion of the constrained continuum.

For example, consider the extreme case of a rigid body for which six independent constraints are imposed that require the rate of deformation tensor $\mathbf{D}^*$ to vanish at each point

$$\mathbf{D}^* = 0 \ . \tag{3.8.15}$$

Under these conditions, all six independent components of $\bar{\mathbf{T}}^*$ are arbitrary functions of position and time so that all six independent components of the total stress tensor $\mathbf{T}^*$ are also arbitrary functions of position and time. From rigid body dynamics it is well known that the six degrees of freedom of the rigid body (translational and rotational) are determined by the global forms of the balances of linear and angular momentum, once the total force and total moment (about a fixed point) applied to the body are specified. Moreover, an infinite number of body force fields $\mathbf{b}^*$ will yield specified values of the resultant force and moment due to body force. Similarly, an infinite number of surface traction fields $\mathbf{t}^*$ will yield specified values of the resultant force and moment due to surface tractions. Consequently, an infinite number of stress fields will be consistent with a given motion of the rigid body. In particular, the addition of an arbitrary constant symmetric tensor to $\mathbf{T}^*$ will not change the value of the divergence of $\mathbf{T}^*$ or the resultant force and moment due to surface tractions even though it will change the values of the traction vector $\mathbf{t}^*$ pointwise on the surface of the body.

Consequently, within the context of the rigid body model it is not possible to uniquely determine the detailed stress distribution in body that causes a particular motion. On the other hand, if the detailed stress distribution is needed in the body, then it is not appropriate to use the simple model of a rigid body.

For the special case of a constrained elastic material the part of the stress $\hat{\mathbf{T}}^*$ associated with a constitutive equation satisfies the condition (3.7.1) that the mechanical power due to this part of the stress is equal to the rate of change of the strain energy function. Then, $\hat{\mathbf{T}}^*$ is determined in terms of derivatives of the strain energy function by formulas of the type (3.7.15).

3.9 Initial and boundary conditions

In this section attention is confined to the discussion of initial and boundary conditions for the purely mechanical theory. In general, the number of initial conditions required and the type of boundary conditions required will depend on the specific type of material under consideration. However, it is possible to make some general observations that apply to all materials.

To this end, it is recalled that the local forms of conservation of mass $(3.2.33)_{1,2}$ and balance of linear momentum $(3.2.33)_3$ are partial differential equations which require both initial and boundary conditions. Specifically, the conservation of mass $(3.2.33)_2$ is first order in time with respect to density ρ^* so it is necessary to specify the initial value of density at each point of the body

$$\rho^*(\mathbf{x}^*,0) = \bar{\rho}(\mathbf{x}^*) \quad \text{on } P^* \quad \text{for } t = 0 \,. \tag{3.9.1}$$

Also the balance of linear momentum $(3.2.33)_3$ is second order in time with respect to position $\mathbf{x}^*$ so that it is necessary to specify the initial value of $\mathbf{x}^*$ and the initial value of the velocity $\mathbf{v}^*$ at each point of the body

$$\hat{\mathbf{x}}^*(\mathbf{X}^*,0) = \bar{\mathbf{x}}^*(\mathbf{X}^*) \ , \quad \hat{\mathbf{v}}^*(\mathbf{X}^*,0) = \tilde{\mathbf{v}}^*(\mathbf{x}^*,0) = \bar{\mathbf{v}}^*(\mathbf{x}^*) \quad \text{on } P^* \quad \text{for } t = 0 \,. \tag{3.9.2}$$

Guidance for determining the appropriate form of boundary conditions is usually obtained by considering the rate of work done by the stress vector. From (3.4.3) it is observed that $\mathbf{t}^* \bullet \mathbf{v}^*$ is the rate of work per unit present area done by the stress vector. At each point of the surface ∂P^* it is possible to define a right-handed orthogonal coordinate system with base vectors $\{ \mathbf{s}_1 , \mathbf{s}_2 , \mathbf{n}^* \}$, where $\mathbf{n}^*$ is the unit outward normal to ∂P^* and $\mathbf{s}_1$ and $\mathbf{s}_2$ are orthogonal vectors tangent to ∂P^*. Then, with reference to these base vectors, the expression $\mathbf{t}^* \bullet \mathbf{v}^*$ can be written in the form

$$\mathbf{t}^* \bullet \mathbf{v}^* = (\mathbf{t}^* \bullet \mathbf{s}_1)(\mathbf{v}^* \bullet \mathbf{s}_1) + (\mathbf{t}^* \bullet \mathbf{s}_2)(\mathbf{v}^* \bullet \mathbf{s}_2) + (\mathbf{t}^* \bullet \mathbf{n}^*)(\mathbf{v}^* \bullet \mathbf{n}^*) \quad \text{on } \partial P^* \,. \tag{3.9.3}$$

Using this representation it is possible to define four types of boundary conditions

Kinematic: All three components of the velocity are specified

$$(\mathbf{v}^* \bullet \mathbf{s}_1) \,,\; (\mathbf{v}^* \bullet \mathbf{s}_2) \,,\; (\mathbf{v}^* \bullet \mathbf{n}^*) \quad \text{specified on } \partial P^* \text{ for all } t \geq 0 \,, \tag{3.9.4}$$

Kinetic: All three components of the stress vector are specified

$$(\mathbf{t}^* \bullet \mathbf{s}_1) \,,\; (\mathbf{t}^* \bullet \mathbf{s}_2) \,,\; (\mathbf{t}^* \bullet \mathbf{n}) \quad \text{specified on } \partial P^* \text{ for all } t \geq 0 \,, \tag{3.9.5}$$

Mixed: Kinematic boundary conditions are specified on part of the boundary ∂P^* and kinetic boundary conditions are specified on the remaining part of the boundary.

Mixed-Mixed: Conjugate components of both the velocity and the stress vector are specified

$$\{(\mathbf{v}^* \bullet \mathbf{s}_1) \text{ or } (\mathbf{t}^* \bullet \mathbf{s}_1)\} \,,\; \{(\mathbf{v}^* \bullet \mathbf{s}_2) \text{ or } (\mathbf{t}^* \bullet \mathbf{s}_2)\} \,,$$
$$\{(\mathbf{v}^* \bullet \mathbf{n}^*) \text{ or } (\mathbf{t}^* \bullet \mathbf{n}^*)\} \quad \text{specified on } \partial P^* \text{ for all } t \geq 0 \,. \tag{3.9.6}$$

Essentially, the conjugate components $(\mathbf{t}^* \bullet \mathbf{s}_1),(\mathbf{t}^* \bullet \mathbf{s}_2),(\mathbf{t}^* \bullet \mathbf{n}^*)$ are the responses to the motions $(\mathbf{v}^* \bullet \mathbf{s}_1),(\mathbf{v}^* \bullet \mathbf{s}_2),(\mathbf{v}^* \bullet \mathbf{n}^*)$, respectively. Therefore, it is important to emphasize that, for example, both $(\mathbf{v}^* \bullet \mathbf{n}^*)$ and $(\mathbf{t}^* \bullet \mathbf{n}^*)$ cannot be specified at the same point of ∂P^* because this would mean that both the motion and the stress response can be specified independently of the material properties and geometry of the body. Notice also, that since the initial position of points on the boundary ∂P^* are specified by the initial condition $(3.9.2)_1$, the velocity boundary conditions (3.9.4) can be used to determine the position of the boundary for all time. This means that the kinematic boundary conditions (3.9.4) could also be characterized by specifying the position of points on the boundary for all time. Furthermore, for static problems the position vector will include a measure of arbitrariness if insufficient kinematic boundary conditions are supplied to specify the three translational and three rotational rigid-body degrees of freedom.

3.10 Material symmetry

In order to introduce the discussion of material symmetries of a nonlinear elastic material, it is convenient to first consider a series of simple tension tests that are performed on different specimens of the same material composition. In particular, it is of interest to choose these test specimens so that they are representative of different relative material orientations. If the stress responses of these different test specimens to the same axial strain are different, then the material is said to be anisotropic. On the other hand, if the stress responses of these different specimens are the same for all possible material orientations, then the material is said to be isotropic.

From a mathematical point of view, it is possible to compare the responses to general homogeneous deformations of different specimens with different relative material orientations. To this end, it is convenient to introduce a right handed orthonormal set of base vectors $\mathbf{M}_i^*$ defined relative to specified material orientations in the reference configuration such that

$$\mathbf{M}_i^* \cdot \mathbf{M}_j^* = \delta_{ij} \ , \quad \mathbf{M}_1^* \times \mathbf{M}_2^* \cdot \mathbf{M}_3^* = 1 \ , \quad \dot{\mathbf{M}}_i^* = 0 \ . \tag{3.10.1}$$

For example, these vectors can be chosen so that $\mathbf{M}_1^*$ is oriented in the direction $\mathbf{G}_1$ and $\mathbf{M}_2^*$ lies in the plane of $\mathbf{G}_1$ and $\mathbf{G}_2$ so that

$$\mathbf{M}_1^* = \frac{\mathbf{G}_1}{|\mathbf{G}_1|} \ , \quad \mathbf{M}_2^* = \mathbf{M}_3^* \times \mathbf{M}_1^* \ , \quad \mathbf{M}_3^* = \frac{\mathbf{G}_1 \times \mathbf{G}_2}{|\mathbf{G}_1 \times \mathbf{G}_2|} \ . \tag{3.10.2}$$

Then, a general homogeneous deformation $\mathbf{C}^*$ can be specified so that its components C_{ij}^* relative to the basis $\mathbf{M}_i^*$ are given by

$$C_{ij}^* = \mathbf{C}^* \cdot (\mathbf{M}_i^* \otimes \mathbf{M}_j^*) \ . \tag{3.10.3}$$

Next, consider another set of base vectors $\mathbf{M}_i^{*\prime}$ which are related to $\mathbf{M}_i^*$ by an arbitrary orthogonal tensor $\mathbf{H}^*$ such that

$$\mathbf{H}^* = \mathbf{M}_i^* \otimes \mathbf{M}_i^{*\prime} \ , \quad \mathbf{H}^{*T} \mathbf{H}^* = \mathbf{H}^* \mathbf{H}^{*T} = \mathbf{I} \ ,$$

$$\mathbf{M}_i^* = \mathbf{H}^* \mathbf{M}_i^{*\prime} \ , \quad \mathbf{M}_i^{*\prime} = \mathbf{H}^{*T} \mathbf{M}_i^* \ . \tag{3.10.4}$$

If $\det \mathbf{H}^* = 1$, then $\mathbf{H}^*$ merely rotates the basis $\mathbf{M}_i^*$ to a new material orientation, whereas if $\det \mathbf{H}^* = -1$, then $\mathbf{H}^*$ represents a reflection and a rotation of the basis. Moreover, since $\mathbf{H}^*$ is defined in terms of the two bases $\mathbf{M}_i^*$ and $\mathbf{M}_i^{*\prime}$, it can be shown that the components H_{ij}^* of $\mathbf{H}^*$ relative to $\mathbf{M}_i^*$ are the same as the components $H_{ij}^{*\prime}$ of $\mathbf{H}^*$ relative to $\mathbf{M}_i^{*\prime}$

$$H_{ij}^* = \mathbf{H}^* \cdot (\mathbf{M}_i^* \otimes \mathbf{M}_j^*) = \mathbf{M}_i^{*\prime} \cdot \mathbf{M}_j^* \ ,$$

$$H_{ij}^{*\prime} = \mathbf{H}^* \cdot (\mathbf{M}_i^{*\prime} \otimes \mathbf{M}_j^{*\prime}) = \mathbf{M}_i^{*\prime} \cdot \mathbf{M}_j^* \ . \tag{3.10.5}$$

Also, since $\mathbf{H}^*$ is an orthogonal tensor, the components H_{ij}^* satisfy the equations

$$H_{im}^* \, H_{jm}^* = \delta_{ij} \ , \quad H_{mi}^* \, H_{mj}^* = \delta_{ij} \ . \tag{3.10.6}$$

Now, it is possible to use the same reference configuration and define a different deformation tensor $\tilde{\mathbf{C}}^*$ by the formula

$$\tilde{\mathbf{C}}^* = C_{ij}^* (\mathbf{M}_i^{*\prime} \otimes \mathbf{M}_j^{*\prime}) = \mathbf{H}^{*T} \mathbf{C}^* \mathbf{H}^* \ . \tag{3.10.7}$$

Since the components of $\tilde{\mathbf{C}}^*$ relative to the basis $\mathbf{M}_i^{*\prime}$ are the same as the components of $\mathbf{C}^*$ relative to the basis $\mathbf{M}_i^*$, it follows that the deformation of a specimen oriented relative to $\mathbf{M}_i^*$ caused by $\mathbf{C}^*$ is the same as the deformation of a similar specimen oriented relative to $\mathbf{M}_i^{*\prime}$ caused by $\tilde{\mathbf{C}}^*$. Moreover, the components $\tilde{C}_{ij}^*$ of $\tilde{\mathbf{C}}^*$ relative to the basis $\mathbf{M}_i^*$ are given by

$$\tilde{C}_{ij}^* = \tilde{\mathbf{C}}^* \cdot (\mathbf{M}_i^* \otimes \mathbf{M}_j^*) = H_{mi}^* \, C_{mn}^* \, H_{nj}^* = H_{im}^{*T} \, C_{mn}^* \, H_{nj}^* \ , \tag{3.10.8}$$

which also shows that $\tilde{\mathbf{C}}^*$ is not the same tensor as $\mathbf{C}^*$.

In general, the strain energy function (3.7.13) depends also on the choice of the basis $\mathbf{M}_i^*$ because it really is a function of the components C_{ij}^* of $\mathbf{C}^*$ relative to this basis

$$\hat{\Sigma}^*(\mathbf{C}^*,\mathbf{X}^*) = \Sigma^*(\mathbf{C}^*,\mathbf{X}^*,\mathbf{M}_i^*) = \bar{\Sigma}^*(C_{ij}^*) \ , \tag{3.10.9}$$

where the dependence of $\bar{\Sigma}^*$ on $\mathbf{X}^*$ and $\mathbf{M}_i^*$ is retained but is not explicitly exhibited. Using the above definitions, the response of two specimens with different material orientations are said to be the same if the values of the strain energy function remains form invariant

$$\hat{\Sigma}^*(\mathbf{C}^*,\mathbf{X}^*) = \hat{\Sigma}^*(\tilde{\mathbf{C}}^*,\mathbf{X}^*) \quad \text{or} \quad \bar{\Sigma}^*(C_{ij}^*) = \bar{\Sigma}^*(\tilde{C}_{ij}^*) \ , \tag{3.10.10}$$

for arbitrary homogeneous deformations $\mathbf{C}^*$ (or C_{ij}^*). Using (3.10.7) and (3.10.8), this means that Σ^* satisfies the restrictions

$$\hat{\Sigma}^*(\mathbf{C}^*,\mathbf{X}^*) = \hat{\Sigma}^*(\mathbf{H}^{*T}\,\mathbf{C}^*\,\mathbf{H}^*,\mathbf{X}^*) \quad \text{or} \quad \bar{\Sigma}^*(C_{ij}^*) = \bar{\Sigma}^*(H_{im}^{*\,T}\,C_{mn}^*\,H_{nj}^*) \ , \tag{3.10.11}$$

for a specified class of $\mathbf{H}^*$. If the functional form for the strain energy is specified, then the restrictions (3.10.11) characterize the material symmetry group of orthogonal transformations $\mathbf{H}^*$ which satisfy the restrictions. On the other hand, if the material symmetry group of $\mathbf{H}^*$ is specified, then the equations (3.10.11) place restrictions on the possible forms for the strain energy function. For the case of crystalline materials, these symmetry groups can be related to the different crystal structures (see Green and Adkins, 1960).

<u>Case I (A General Anisotropic Material)</u>: If the material possesses no symmetry, then the symmetry group consists only of $\mathbf{H}^* = \mathbf{I}$ ($H_{ij}^* = \delta_{ij}$) and the strain energy becomes a general function of the form

$$\Sigma^* = \bar{\Sigma}^*(C_{11}^*\, , C_{22}^*\, , C_{33}^*\, , C_{12}^*\, , C_{13}^*\, , C_{23}^*) \ . \tag{3.10.12}$$

<u>Case II</u>: If the material possesses symmetry due to reflection about the plane defined by $\mathbf{M}_3^*$, then the symmetry group contains the term

$$H_{ij}^* = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \ , \tag{3.10.13}$$

and the restriction (3.10.11) requires $\bar{\Sigma}^*$ to satisfy the restriction

$$\bar{\Sigma}^*(C_{11}^*\, , C_{22}^*\, , C_{33}^*\, , C_{12}^*\, , C_{13}^*\, , C_{23}^*)$$
$$= \bar{\Sigma}^*(C_{11}^*\, , C_{22}^*\, , C_{33}^*\, , C_{12}^*\, , -C_{13}^*\, , -C_{23}^*) \ , \tag{3.10.14}$$

which must hold for all C_{ij}^* .

<u>Case III (An Orthotropic Material)</u>: If the material also possesses symmetry due to reflection about the plane defined by $\mathbf{M}_2^*$, then in addition to (3.10.13), the symmetry group contains the term

$$H_{ij}^* = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \ , \tag{3.10.15}$$

and the restriction (3.10.11) requires $\bar{\Sigma}^*$ to satisfy the additional restriction

$$\bar{\Sigma}^*(C_{11}^*\, , C_{22}^*\, , C_{33}^*\, , C_{12}^*\, , C_{13}^*\, , C_{23}^*)$$

$$= \bar{\Sigma}^*(C_{11}^*, C_{22}^*, C_{33}^*, -C_{12}^*, C_{13}^*, -C_{23}^*) \ , \tag{3.10.16}$$

which must hold for all C_{ij}^*. Next, by combining (3.10.13) and (3.10.15), it follows that the symmetry group automatically includes the term

$$H_{ij}^* = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \ , \tag{3.10.17}$$

which requires $\bar{\Sigma}^*$ to satisfy the restriction

$$\bar{\Sigma}^*(C_{11}^*, C_{22}^*, C_{33}^*, C_{12}^*, C_{13}^*, C_{23}^*)$$
$$= \bar{\Sigma}^*(C_{11}^*, C_{22}^*, C_{33}^*, -C_{12}^*, -C_{13}^*, C_{23}^*) \ . \tag{3.10.18}$$

This is the same restriction as would be obtained by taking

$$H_{ij}^* = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \ . \tag{3.10.19}$$

Consequently, if the material possesses symmetry due to reflections about the two planes defined by $\mathbf{M}_2^*$ and $\mathbf{M}_3^*$, then it automatically possesses symmetry due to reflection about the plane defined by $\mathbf{M}_1^*$ and the material is called orthotropic.

Case IV (An Isotropic Material): If the material possesses symmetry to all reflections and all rotations, then the symmetry group is the full orthogonal group and the material is called isotropic. For this case it can be shown that the strain energy function Σ^* can depend on $\mathbf{C}^*$ only through its three invariants.

3.11 Isotropic nonlinear elastic materials

For the case of an isotropic nonlinear elastic material the strain energy function Σ^* in (3.10.11) is invariant under the full orthogonal group for $\mathbf{H}^*$ and thus must be an isotropic function of $\mathbf{C}^*$. This means that it can depend on $\mathbf{C}^*$ only through its three invariants. Recalling that the principal invariants of $\mathbf{C}^*$ are the same as the invariants of $\mathbf{B}^*$ and are given by

$$I_1^* = \mathbf{C}^* \bullet \mathbf{I} = \mathbf{B}^* \bullet \mathbf{I} \ ,$$
$$I_2^* = \frac{1}{2}\left[(\mathbf{C}^* \bullet \mathbf{I})^2 - \mathbf{C}^* \bullet \mathbf{C}^{*T} \right] = \frac{1}{2}\left[(\mathbf{B}^* \bullet \mathbf{I})^2 - \mathbf{B}^* \bullet \mathbf{B}^{*T} \right] \ ,$$
$$I_3^* = \det \mathbf{C}^* = \det \mathbf{B}^* = J^{*2} \ , \tag{3.11.1}$$

it follows that for an isotropic elastic material the strain energy Σ^* can be expressed as an arbitrary function of the invariants I_1^*, I_2^*, I_3^*

$$\Sigma^* = \Sigma^*(I_1^*, I_2^*, I_3^*) \ . \tag{3.11.2}$$

Here, dependence on $\mathbf{X}^*$ is allowed for inhomogeneous materials but is not explicitly exhibited. Furthermore, it can be shown that

$$\dot{I}_1^* = \mathbf{I} \bullet \dot{\mathbf{C}}^* \ ,$$

$$\dot{I}_2^* = [\ (\mathbf{C}^* \bullet \mathbf{I})\,\mathbf{I} - \mathbf{C}^*\ ] \bullet \dot{\mathbf{C}}^*\ ,\quad \frac{\partial I_2^*}{\partial \mathbf{C}^*} = (\mathbf{C}^* \bullet \mathbf{I})\,\mathbf{I} - \mathbf{C}^*\ ,$$

$$\dot{I}_3^* = \frac{\partial I_3^*}{\partial \mathbf{C}^*} \bullet \dot{\mathbf{C}}^* = I_3^*\,\mathbf{C}^{*-1} \bullet \dot{\mathbf{C}}^*\ ,\quad \frac{\partial I_3^*}{\partial \mathbf{C}^*} = I_3^*\,\mathbf{C}^{*-1}\ . \tag{3.11.3}$$

Thus, with the help of the conservation of mass $(3.2.33)_1$ and the results (3.7.17) and (3.7.15) for elastic materials, it follows that the stresses $\mathbf{S}^*$ and $\mathbf{T}^*$ are given by

$$\mathbf{S}^* = 2\rho_0^*\left[\ \frac{\partial \Sigma^*}{\partial I_1^*} + (\mathbf{C}^* \bullet \mathbf{I})\,\frac{\partial \Sigma^*}{\partial I_2^*}\ \right]\mathbf{I} - 2\rho_0^*\left[\ \frac{\partial \Sigma^*}{\partial I_2^*}\ \right]\mathbf{C}^*$$
$$+ 2\rho_0^*\left[\ \frac{\partial \Sigma^*}{\partial I_3^*}\ \right] I_3^*\,\mathbf{C}^{*-1}\ ,$$

$$\mathbf{T}^* = 2\rho_0^*\,J^{*-1}\left[\ \frac{\partial \Sigma^*}{\partial I_1^*} + (\mathbf{B}^* \bullet \mathbf{I})\,\frac{\partial \Sigma^*}{\partial I_2^*}\ \right]\mathbf{B}^* - 2\rho_0^*\,J^{*-1}\left[\ \frac{\partial \Sigma^*}{\partial I_2^*}\ \right]\mathbf{B}^{*2}$$
$$+ 2\rho_0^*\left[\ \frac{\partial \Sigma^*}{\partial I_3^*}\ \right] J^*\,\mathbf{I}\ . \tag{3.11.4}$$

Also, with the help of (3.7.16), the pressure p^* defined in (3.8.14) is determined by

$$p^* = -\tfrac{1}{3}\,\mathbf{T}^* \bullet \mathbf{I} = -\tfrac{1}{3}\,J^{*-1}\,\mathbf{S}^* \bullet \mathbf{C}^*\ ,$$

$$p^* = -\tfrac{2}{3}\,\rho_0^*\,J^{*-1}\left[\ \frac{\partial \Sigma^*}{\partial I_1^*} + (\mathbf{B}^* \bullet \mathbf{I})\,\frac{\partial \Sigma^*}{\partial I_2^*}\ \right](\mathbf{B}^* \bullet \mathbf{I})$$
$$+ \tfrac{2}{3}\,\rho_0^*\,J^{*-1}\left[\ \frac{\partial \Sigma^*}{\partial I_2^*}\ \right](\mathbf{B}^* \bullet \mathbf{B}^*) - 2\rho_0^*\left[\ \frac{\partial \Sigma^*}{\partial I_3^*}\ \right] J^*\ . \tag{3.11.5}$$

Moreover, it can be shown that $\mathbf{S}^*$ and $\mathbf{T}^*$ can be represented in the forms

$$\mathbf{S}^* = -\,p^*\,J^{*-1}\,\mathbf{C}^{*-1} + \mathbf{S}^{*\prime}\ ,\quad \mathbf{S}^{*\prime} \bullet \mathbf{C}^* = 0\ ,$$
$$\mathbf{T}^* = -\,p^*\,\mathbf{I} + \mathbf{T}^{*\prime}\ ,\quad \mathbf{T}^{*\prime} \bullet \mathbf{I} = 0\ ,\quad \mathbf{T}^{*\prime} = J^{*-1}\,\mathbf{F}^*\,\mathbf{S}^{*\prime}\,\mathbf{F}^{*T}\ , \tag{3.11.6}$$

where $\mathbf{S}^{*\prime}$ and $\mathbf{T}^{*\prime}$ are given by

$$\mathbf{S}^{*\prime} = 2\rho_0^*\left[\ \frac{\partial \Sigma^*}{\partial I_1^*} + (\mathbf{C}^* \bullet \mathbf{I})\,\frac{\partial \Sigma^*}{\partial I_2^*}\ \right]\left[\mathbf{I} - \tfrac{1}{3}(\mathbf{C}^* \bullet \mathbf{I})\,\mathbf{C}^{*-1}\right]$$
$$- 2\rho_0\left[\ \frac{\partial \Sigma^*}{\partial I_2^*}\ \right]\left[\mathbf{C}^* - \tfrac{1}{3}(\mathbf{C}^{*2} \bullet \mathbf{I})\,\mathbf{C}^{*-1}\right]\ ,$$

$$\mathbf{T}^{*\prime} = 2\rho_0^*\,J^{*-1}\left[\ \frac{\partial \Sigma^*}{\partial I_1^*} + (\mathbf{B}^* \bullet \mathbf{I})\,\frac{\partial \Sigma^*}{\partial I_2^*}\ \right]\left[\mathbf{B}^* - \tfrac{1}{3}(\mathbf{B}^* \bullet \mathbf{I})\,\mathbf{I}\right]$$
$$- 2\rho_0^*\,J^{*-1}\left[\ \frac{\partial \Sigma^*}{\partial I_2^*}\ \right]\left[\mathbf{B}^{*2} - \tfrac{1}{3}(\mathbf{B}^{*2} \bullet \mathbf{I})\,\mathbf{I}\right]\ . \tag{3.11.7}$$

It is interesting to note that the pressure p^* in (3.11.5) depends on the derivatives of Σ^* with respect to all three invariants. This is because the invariants I_1^* and I_2^* are not pure measures of distortion. In this regard, it is possible to use the work of Flory (1961) to define the pure measures of distortion $\mathbf{C}^{*\prime}$ and $\mathbf{B}^{*\prime}$ such that

$$\mathbf{C}^{*\prime} = (\det \mathbf{C}^*)^{-1/3}\, \mathbf{C}^* \;\;,\;\; \det \mathbf{C}^{*\prime} = 1 \;\;,$$
$$\mathbf{B}^{*\prime} = (\det \mathbf{B}^*)^{-1/3}\, \mathbf{B}^* \;\;,\;\; \det \mathbf{C}^{*\prime} = 1 \;\;. \tag{3.11.8}$$

These unimodular tensors have only two independent invariants which can be taken to be

$$\alpha_1^* = \mathbf{C}^{*\prime} \bullet \mathbf{I} = \mathbf{B}^{*\prime} \bullet \mathbf{I} \;\;,\;\; \alpha_2^* = \mathbf{C}^{*\prime} \bullet \mathbf{C}^{*\prime} = \mathbf{B}^{*\prime} \bullet \mathbf{B}^{*\prime} \;\;. \tag{3.11.9}$$

Thus, a general isotropic nonlinear elastic material can be characterized by the alternative assumption that the strain energy function Σ^* depends on the pure measures of distortion α_1^* and α_2^* and on the pure measure of dilatation J^*, instead of on I_1^*, I_2^*, I_3^*, so that

$$\Sigma^* = \hat{\Sigma}^*(\alpha_1^*, \alpha_2^*, J^*) \;\;. \tag{3.11.10}$$

Then, with the help of the expressions

$$\dot{\alpha}_1^* = I_3^{*-1/3}\left[\mathbf{I} - \frac{1}{3}(\mathbf{C}^* \bullet \mathbf{I})\, \mathbf{C}^{*-1}\right] \bullet \dot{\mathbf{C}}^* \;\;,$$
$$\frac{\partial \alpha_1^*}{\partial \mathbf{C}^*} = I_3^{*-1/3}\left[\mathbf{I} - \frac{1}{3}(\mathbf{C}^* \bullet \mathbf{I})\, \mathbf{C}^{*-1}\right] \;\;,$$
$$\dot{\alpha}_2^* = 2I_3^{*-2/3}\left[\mathbf{C}^* - \frac{1}{3}(\mathbf{C}^{*2} \bullet \mathbf{I})\, \mathbf{C}^{*-1}\right] \bullet \dot{\mathbf{C}}^* \;\;,$$
$$\frac{\partial \alpha_2^*}{\partial \mathbf{C}^*} = 2I_3^{*-2/3}\left[\mathbf{C}^* - \frac{1}{3}(\mathbf{C}^{*2} \bullet \mathbf{I})\, \mathbf{C}^{*-1}\right] \;\;,$$
$$\dot{J}^* = \frac{1}{2} J^*\, \mathbf{C}^{*-1} \bullet \dot{\mathbf{C}}^* \;\;,\;\; \frac{\partial J^*}{\partial \mathbf{C}^*} = \frac{1}{2} J^*\, \mathbf{C}^{*-1} \;\;. \tag{3.11.11}$$

it can be shown that

$$\frac{\partial \Sigma^*}{\partial \mathbf{C}^*} = \frac{1}{2} J^* \frac{\partial \hat{\Sigma}^*}{\partial J^*}\, \mathbf{C}^{*-1} + J^{*-2/3} \frac{\partial \hat{\Sigma}^*}{\partial \alpha_1^*}\left[\mathbf{I} - \frac{1}{3}(\mathbf{C}^* \bullet \mathbf{I})\, \mathbf{C}^{*-1}\right]$$
$$+\, 2J^{*-4/3} \frac{\partial \hat{\Sigma}^*}{\partial \alpha_2^*}\left[\mathbf{C}^* - \frac{1}{3}(\mathbf{C}^{*2} \bullet \mathbf{I})\, \mathbf{C}^{*-1}\right] \;\;,$$
$$p^* = -\rho_0^*\left[\frac{\partial \hat{\Sigma}^*}{\partial J^*}\right] \;\;,$$
$$\mathbf{S}^{*\prime} = 2\rho_0^*\left[\frac{\partial \hat{\Sigma}^*}{\partial \alpha_1^*}\right] I_3^{*-1/3}\left[\mathbf{I} - \frac{1}{3}(\mathbf{C}^* \bullet \mathbf{I})\, \mathbf{C}^{*-1}\right]$$
$$+\, 4\rho_0^*\left[\frac{\partial \hat{\Sigma}^*}{\partial \alpha_2^*}\right] I_3^{*-2/3}\left[\mathbf{C}^* - \frac{1}{3}(\mathbf{C}^{*2} \bullet \mathbf{I})\, \mathbf{C}^{*-1}\right] \;\;,$$
$$\mathbf{T}^{*\prime} = 2\rho_0^*\, J^{*-1}\left[\frac{\partial \hat{\Sigma}^*}{\partial \alpha_1^*}\right] I_3^{*-1/3}\left[\mathbf{B}^* - \frac{1}{3}(\mathbf{B}^* \bullet \mathbf{I})\, \mathbf{I}\right]$$

$$+ 4\rho_0^* J^{*-1} \left[\frac{\partial \hat{\Sigma}^*}{\partial \alpha_2^*} \right] I_3^{*-2/3} \left[\mathbf{B}^{*2} - \frac{1}{3}(\mathbf{B}^{*2} \bullet \mathbf{I})\, \mathbf{I} \right] \ . \tag{3.11.12}$$

Notice now that the pressure is related to the derivative of Σ^* with respect to the dilatation J^*, and the deviatoric stress $\mathbf{T}^{*\prime}$ is related to derivatives of Σ^* with respect to the distortional measures of deformation α_1^* and α_2^*, but it also depends on the dilatation J^*. However, this does not mean that the pressure is independent of (α_1^*, α_2^*) because the derivative $\partial\hat{\Sigma}^*/\partial J^*$ can retain dependence on (α_1^*, α_2^*). Moreover, it is observed that in the reference configuration $\mathbf{C}^*=\mathbf{B}^*=\mathbf{I}$ and $J^*=1$, so that $\mathbf{S}^{*\prime}$ and $\mathbf{T}^{*\prime}$ both vanish. Consequently, if the stress vanishes in the reference configuration, then the strain energy function must be restricted so that

$$\frac{\partial \hat{\Sigma}^*}{\partial J^*} = 0 \quad \text{for} \quad \alpha_1^* = \alpha_2^* = 3 \ \text{ and } J^* = 1 \ . \tag{3.11.13}$$

Significant advances in the theory of finite elasticity were made studying the response of natural rubber and modeling the material by a Neo-Hookean strain energy

$$\rho_0^* \Sigma^* = C_1 \,(I_1^* - 3) \ , \tag{3.11.14}$$

or a Mooney-Rivlin strain energy

$$\rho_0^* \Sigma^* = C_1 \,(I_1^* - 3) + C_2 \,(I_2^* - 3) \ , \tag{3.11.15}$$

where C_1 and C_2 are material constants (see Atkin and Fox, 1980; Ogden, 1984). Also, in these studies, rubber was modeled as an incompressible material.

A natural generalization of the Mooney-Rivlin strain energy for compressible materials is characterized by the strain energy function

$$\rho_0^* \Sigma^* = \mu_0^* \left[C_1 \,(\alpha_1^* - 3) + C_2 \,(\alpha_2^* - 3) \right] + f(J^*) \ , \tag{3.11.16}$$

where $f(J^*)$ is a function of dilatation J^* only that determines the pressure

$$p^* = - \frac{df(J^*)}{dJ^*} \ . \tag{3.11.17}$$

Moreover, following the work in (Rubin, 1987c), it can be shown that μ_0^* will be the zero-stress value of the shear modulus (associated with the linearized theory) if the constants C_1 and C_2 satisfy the restriction that

$$C_1 + 4\, C_2 = 1 \ . \tag{3.11.18}$$

Also, as a special case, the function $f(J^*)$ can be specified by the form

$$f(J^*) = K_0^* \left[(J^*-1) - \ln(J^*) \right] \ , \tag{3.11.19}$$

where K_0^* is the zero-pressure bulk modulus. Using this form, the pressure and the bulk modulus K^* become

$$p^* = K_0^* \left[\frac{1}{J^*} - 1 \right] \ , \quad K^* = - J^* \frac{dp^*}{dJ^*} = \frac{K_0^*}{J^*} \ , \tag{3.11.20}$$

which provide reasonably accurate modeling of the nonlinear response of many materials to relatively high pressures.

3.12 A small strain theory

For a small strain theory, it is sufficient to assume that the strain energy function Σ^* is a quadratic function of the Lagrangian strain $\mathbf{E}^*$ defined by

$$\mathbf{E}^* = \frac{1}{2}(\mathbf{C}^* - \mathbf{I}) \ , \tag{3.12.1}$$

such that

$$\rho_0^* \, \Sigma^* = \frac{1}{2}\mathbf{K}^* \bullet (\mathbf{E}^* \otimes \mathbf{E}^*) \ , \tag{3.12.2}$$

where $\mathbf{K}^*$ is a constant double symmetric fourth order tensor having the symmetries

$$\mathbf{K}^{*T} = \mathbf{K}^* \ , \quad {}^{LT}\mathbf{K}^* = \mathbf{K}^* \ , \quad \mathbf{K}^{*T(2)} = \mathbf{K}^* \ . \tag{3.12.3}$$

Moreover, it can be shown that the restrictions (3.12.3) cause $\mathbf{K}^*$ to have only 21 independent components. Now, using (3.12.2) it follows that

$$\rho_0^* \frac{\partial \Sigma^*}{\partial \mathbf{C}^*} = \frac{1}{2}\mathbf{K}^* \bullet \mathbf{E}^* \ , \tag{3.12.4}$$

so that (3.2.28) and (3.7.15) can be used to determine the constitutive equation for the stress $\mathbf{T}^*$ in the form

$$g^{1/2} \, \mathbf{T}^* = G^{1/2} \, \mathbf{F}^* \, (\mathbf{K}^* \bullet \mathbf{E}^*) \, \mathbf{F}^{*T} \ . \tag{3.12.5}$$

From the theoretical point of view, the constitutive equations (3.12.2) and (3.12.5) represent the results of a special constitutive assumption that is valid for large deformations and large rotations. However, since the strain energy function depends only quadratically on strain, the constitutive equation is not necessarily adequate to describe materials like rubber which can exhibit significant nonlinear stress-strain response. Nevertheless, these constitutive assumptions produce a reasonably simple theory that is properly invariant under SRBM.

Next, by referring $\mathbf{E}^*$ and $\mathbf{K}^*$ to the orthonormal basis $\mathbf{M}_i^*$ defined in (3.10.2), their components become

$$E_{ij}^* = \mathbf{E}^* \bullet (\mathbf{M}_i^* \otimes \mathbf{M}_j^*) \ , \quad K_{ijkl}^* = \mathbf{K}^* \bullet (\mathbf{M}_i^* \otimes \mathbf{M}_j^* \otimes \mathbf{M}_k^* \otimes \mathbf{M}_l^*) \ , \tag{3.12.6}$$

and the symmetry conditions (3.12.3) can be written in the forms

$$K_{ijlk}^* = K_{ijkl}^* \ , \quad K_{jikl}^* = K_{ijkl}^* \ , \quad K_{klij}^* = K_{ijkl}^* \ . \tag{3.12.7}$$

Moreover, since the tensor $\mathbf{H}^*$ in (3.10.4) is an orthogonal tensor, the components of $\mathbf{E}^*$ transform by an equation similar to (3.10.8) so the material symmetry conditions (3.10.11) require

$$K_{ijkl}^* \, E_{ij}^* \, E_{kl}^* = K_{mnrs}^* \left[H_{im}^* \, E_{ij}^* \, H_{jn}^*\right] \left[H_{kr}^* \, E_{kl}^* \, H_{ls}^*\right] \ ,$$

$$\left[K_{ijkl}^* - H_{im}^* \, H_{jn}^* \, H_{kr}^* \, H_{ls}^* \, K_{mnrs}^*\right] E_{ij}^* \, E_{kl}^* = 0 \ . \tag{3.12.8}$$

However, since the coefficient of the strains above is independent of the strains and is a double symmetric tensor having properties similar to (3.12.7), and since (3.12.8) must hold for all values of the strain components, it follows that

$$K_{ijkl}^* = H_{im}^* \, H_{jn}^* \, H_{kr}^* \, H_{ls}^* \, K_{mnrs}^* \ . \tag{3.12.9}$$

Thus, the material symmetry is characterized by the group of all orthogonal H^*_{ij} which satisfy the conditions (3.12.9). With reference to the cases I-IV described in section 3.10, it follows that (3.12.9) places restrictions on the components of K^*_{ijkl} when the symmetry group of H^*_{ij} is specified.

Case I (A General Anisotropic Material): If the material possesses no symmetry then the symmetry group consists only of $\mathbf{H}^* = \mathbf{I}$ ($H^*_{ij} = \delta_{ij}$). Then, the mere fact that K^*_{ijkl} is a double symmetric tensor causes the number of independent components to reduce from $81=3^4$ to 21 constants

$$\begin{pmatrix} K^*_{1111} & K^*_{1112} & K^*_{1113} & K^*_{1122} & K^*_{1123} & K^*_{1133} & K^*_{1212} \\ K^*_{1213} & K^*_{1222} & K^*_{1223} & K^*_{1233} & K^*_{1313} & K^*_{1322} & K^*_{1323} \\ K^*_{1333} & K^*_{2222} & K^*_{2223} & K^*_{2233} & K^*_{2323} & K^*_{2333} & K^*_{3333} \end{pmatrix} . \qquad (3.12.10)$$

Case II: If the material possesses symmetry due to reflection about the plane defined by $\mathbf{M}^*_3$, then the symmetry group contains the term (3.10.13) and the restriction (3.12.9) requires the vanishing of all components of K^*_{ijkl} in which the component 3 appears an odd number of times. Consequently, the number of independent components of K^*_{ijkl} reduces to 13 constants

$$\begin{pmatrix} K^*_{1111} & K^*_{1112} & K^*_{1122} & K^*_{1133} & K^*_{1212} & K^*_{1222} & K^*_{1233} \\ K^*_{1313} & K^*_{1323} & K^*_{2222} & K^*_{2233} & K^*_{2323} & K^*_{3333} & \end{pmatrix} . \qquad (3.12.11)$$

Case III (An Orthotropic Material): If the material also possesses symmetry due to reflection about the plane defined by $\mathbf{M}^*_2$, then in addition to (3.10.13), the symmetry group contains the term (3.10.15) and the restriction (3.12.9) requires the vanishing of all components of K^*_{ijkl} in which the component 2 appears an odd number of times. Consequently, the number of independent components of K^*_{ijkl} reduces to 9 constants

$$\begin{pmatrix} K^*_{1111} & K^*_{1122} & K^*_{1133} & K^*_{1212} & K^*_{1313} \\ K^*_{2222} & K^*_{2233} & K^*_{2323} & K^*_{3333} & \end{pmatrix} . \qquad (3.12.12)$$

Furthermore, it can be seen that the index 1 appears only an even number of times in the components (3.12.12) so that the symmetry group automatically includes the term (3.10.19). Consequently, if the material possesses symmetry due to reflections about the two planes defined by $\mathbf{M}^*_2$ and $\mathbf{M}^*_3$, then it automatically possesses symmetry due to reflection about the plane defined by $\mathbf{M}^*_1$ and the material is called orthotropic.

Case IV (An Isotropic Material): If the material possesses symmetry to all reflections and all rotations, then the symmetry group is the full orthogonal group and the material is called isotropic. For this case it can be shown that there are only two independent material constants and that K^*_{ijkl} can be written in the form

$$K^*_{1111} = K^*_{2222} = K^*_{3333} = K^* + \frac{4}{3}\mu^* , \quad K^*_{1122} = K^*_{1133} = K^*_{2233} = K^* - \frac{2}{3}\mu^* ,$$

$$K^*_{1212} = K^*_{1313} = K^*_{2323} = \mu^* ,$$

$$K^*_{ijkl} = (K^* - \frac{2}{3}\mu^*)(\delta_{ij}\delta_{kl}) + \mu^*(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) , \qquad (3.12.13)$$

where K^* is the bulk modulus and μ^* is the shear modulus. Using the expression (3.12.13), $\mathbf{K}^*$ and $\mathbf{K}^* \bullet \mathbf{E}^*$ can be expressed in direct notation such that

$$\mathbf{K}^* = (K^* - \frac{2}{3}\,\mu^*)\,(\mathbf{I} \otimes \mathbf{I})$$
$$+ \frac{1}{4}\,\mu^*\left[\,(\mathbf{M}_i^*\otimes\mathbf{M}_j^* + \mathbf{M}_j^*\otimes\mathbf{M}_i^*)\otimes(\mathbf{M}_i^*\otimes\mathbf{M}_j^* + \mathbf{M}_j^*\otimes\mathbf{M}_i^*)\,\right]\;,$$
$$\mathbf{K}^* \bullet \mathbf{E}^* = (K^* - \frac{2}{3}\,\mu^*)\,(\mathbf{E}^* \bullet \mathbf{I})\,\mathbf{I} + 2\mu^*\,\mathbf{E}^*\;. \tag{3.12.14}$$

	λ^*	μ^*	E^*	ν^*	K^*
λ^*,μ^*			$\frac{\mu^*(3\lambda^*+2\mu^*)}{\lambda^*+\mu^*}$	$\frac{\lambda^*}{2(\lambda^*+\mu^*)}$	$\frac{3\lambda^*+2\mu^*}{3}$
λ^*,ν^*		$\frac{\lambda^*(1-2\nu^*)}{2\nu^*}$	$\frac{\lambda^*(1+\nu^*)(1-2\nu^*)}{\nu^*}$		$\frac{\lambda^*(1+\nu^*)}{3\nu^*}$
λ^*,K^*		$\frac{3(K^*-\lambda^*)}{2}$	$\frac{9K^*(K^*-\lambda)}{3K^*-\lambda}$	$\frac{\lambda}{3K^*-\lambda^*}$	
μ^*,E^*	$\frac{\mu^*(2\mu^*-E^*)}{E^*-3\mu^*}$			$\frac{E^*-2\mu^*}{2\mu^*}$	$\frac{\mu^*E^*}{3(3\mu^*-E^*)}$
μ^*,ν^*	$\frac{2\mu^*\nu^*}{1-2\nu^*}$		$2\mu^*(1+\nu^*)$		$\frac{2\mu^*(1+\nu^*)}{3(1-2\nu^*)}$
μ,K^*	$\frac{3K^*-2\mu^*}{3}$		$\frac{9K^*\mu^*}{3K^*+\mu^*}$	$\frac{3K^*-2\mu^*}{2(3K^*+\mu^*)}$	
E^*,ν^*	$\frac{E^*\nu^*}{(1+\nu^*)(1-2\nu^*)}$	$\frac{E^*}{2(1+\nu^*)}$			$\frac{E^*}{3(1-2\nu^*)}$
E^*,K^*	$\frac{3K^*(3K^*-E^*)}{9K^*-E^*}$	$\frac{3E^*K^*}{9K^*-E^*}$		$\frac{3K^*-E^*}{6K^*}$	
ν^*,K^*	$\frac{3K^*\nu^*}{1+\nu^*}$	$\frac{3K^*(1-2\nu^*)}{2(1+\nu^*)}$	$3K^*(1-2\nu^*)$		

$$\mu^* = \frac{(E^*-3\lambda^*)+\sqrt{(E^*-3\lambda^*)^2+8\lambda^*E^*}}{4}\;,\qquad \nu^* = \frac{-(E^*+\lambda^*)+\sqrt{(E^*+\lambda^*)^2+8\lambda^{*2}}}{4\lambda^*}\;,$$
$$K^* = \frac{(3\lambda^*+E^*)+\sqrt{(3\lambda^*+E^*)^2-4\lambda^*E^*}}{6}$$

Table 3.12.1 Expressions connecting pairs of moduli for isotropic elastic materials. Here, λ^* is Lame's constant, μ^* is the shear modulus, E^* is Young's modulus, ν^* is Poisson's ratio, and K^* is the bulk modulus.

Then, the strain energy (3.12.2) and the stress (3.12.5) can also be expressed in the forms

$$\rho_0^* \Sigma^* = \frac{1}{2}(K^* - \frac{2}{3}\mu^*) (\mathbf{E}^* \bullet \mathbf{I})^2 + \mu^* (\mathbf{E}^* \bullet \mathbf{E}^*) \ ,$$

$$g^{1/2}\,\mathbf{T}^* = G^{1/2}\,\mathbf{F}^* \left[(K^* - \frac{2}{3}\,\mu^*) (\mathbf{E}^* \bullet \mathbf{I})\,\mathbf{I} + 2\mu^*\,\mathbf{E}^* \right] \mathbf{F}^{*T} \ . \qquad (3.12.15)$$

From physical considerations, it is reasonable to expect that any deformation of the body from its stress-free state should cause an increase in the strain energy function. Mathematically, this means that the strain energy function is positive definite

$$\Sigma^* > 0 \quad \text{for any} \ \mathbf{E}^* \neq 0 \ . \qquad (3.12.16)$$

Noting that the strain $\mathbf{E}^*$ can be separated into its spherical and deviatoric parts

$$\mathbf{E}^* = \frac{1}{3}(\mathbf{E}^* \bullet \mathbf{I})\ \mathbf{I} + \mathbf{E}^{*\prime} \ , \quad \mathbf{E}^{*\prime} \bullet \mathbf{I} = 0 \ , \qquad (3.12.17)$$

it follows that the strain energy can be rewritten in the alternative form

$$\rho_0^* \Sigma^* = \frac{1}{2} K^* (\mathbf{E}^* \bullet \mathbf{I})^2 + \mu^*\,\mathbf{E}^{*\prime} \bullet \mathbf{E}^{*\prime} \ . \qquad (3.12.18)$$

Now, since the terms $(\mathbf{E}^* \bullet \mathbf{I})$ and $\mathbf{E}^{*\prime} \bullet \mathbf{E}^{*\prime}$ are independent of each other, it can be deduced that the strain energy will be positive definite whenever

$$K^* > 0 \ , \quad \mu^* > 0 \ . \qquad (3.12.19)$$

Finally, it is noted that an isotropic elastic material can be characterized by any two of the following moduli: K^* (bulk modulus); μ^* (shear modulus); λ^* (Lame's constant); E^* (Young's modulus); or ν^* (Poisson's ratio); which are related by expressions that are recorded in Table 3.12.1. Furthermore, using Table 3.12.1, it can be shown that the restrictions (3.12.19) also require that

$$\lambda^* > 0 \ , \quad E^* > 0 \ , \quad -1 < \nu^* < \frac{1}{2} \ , \qquad (3.12.20)$$

3.13 Small deformations superimposed on a large deformation

In general, the equations of motion of an elastic material are nonlinear partial differential equations for which only a few exact analytical solutions are known. The notion of small deformations superimposed on a large deformation is used to develop approximate equations that are linear functions of the superimposed small deformations and therefore are simpler to solve. Such equations can be used to analyze vibrations of prestressed or rotating structures such as space satellites. Moreover, if the large deformation represents an actual solution of the equations of motion, then the small deformation equations can be used to analyze linear stability of the large deformation solution.

To develop these small deformation equations, the position vector $\mathbf{x}^*(\theta^i,t)$ is represented as an additive function of the large deformation $\hat{\mathbf{x}}^*(\theta^i,t)$ to be analyzed and the small displacement vector $\mathbf{u}^*(\theta^i,t)$ such that

$$\mathbf{x}^*(\theta^i,t) = \hat{\mathbf{x}}^*(\theta^i,t) + \mathbf{u}^*(\theta^i,t) \ . \qquad (3.13.1)$$

The displacement vector $\mathbf{u}^*$ is considered to be small in the sense that its magnitude and the magnitudes of its space and time derivatives are small enough that quadratic and higher order terms in these quantities can be neglected. Thus, for example

$$|\mathbf{u}^*|^2 << |\mathbf{u}^*| \ . \tag{3.13.2}$$

Of course, the value of $\mathbf{u}^*$ and its space and time derivatives must be appropriately normalized in order to express these inequalities in unitless forms.

Quantities other than the displacement vector $\mathbf{u}^*$ are separated additively into a part associated with the large deformation which is denoted by placing a hat ($\hat{}$) over the symbol, and a part associated with the small deformation which is denoted by placing a tilde ($\tilde{}$) over the same symbol. For example, in general, the body force associated with the large deformation to be analyzed can be nonzero. Thus, the body force $\mathbf{b}^*$ is represented in the form

$$\mathbf{b}^* = \hat{\mathbf{b}}^* + \tilde{\mathbf{b}}^* \ . \tag{3.13.3}$$

In order to develop the equations of motion of the small deformation associated with (3.2.35), it is necessary to substitute (3.13.1) into the constitutive equation (3.7.15) for the stress tensor $\mathbf{T}^*$ and use $(3.2.34)_1$, together with a Taylor series expansion, to develop expressions for the vectors $\mathbf{t}^{*i}$

$$\mathbf{t}^{*i} = \hat{\mathbf{t}}^{*i} + \tilde{\mathbf{t}}^{*i} \ . \tag{3.13.4}$$

To this end, it is noted that

$$\begin{gathered}
\mathbf{g}_i = \hat{\mathbf{g}}_i + \tilde{\mathbf{g}}_i \ , \quad \hat{\mathbf{g}}_i = \hat{\mathbf{x}}^*{}_{,i} \ , \quad \tilde{\mathbf{g}}_i = \mathbf{u}^*{}_{,i} \ . \\
\mathbf{F}^* = \hat{\mathbf{F}}^* + \tilde{\mathbf{F}}^* \ , \quad \hat{\mathbf{F}}^* = \hat{\mathbf{g}}_i \otimes \mathbf{G}^i \ , \quad \tilde{\mathbf{F}}^* = \mathbf{u}^*{}_{,i} \otimes \mathbf{G}^i \ , \\
\mathbf{C}^* = \hat{\mathbf{C}}^* + \tilde{\mathbf{C}}^* \ , \quad \hat{\mathbf{C}}^* = \hat{\mathbf{F}}^{*T}\hat{\mathbf{F}}^* \ , \quad \tilde{\mathbf{C}}^* = \hat{\mathbf{F}}^{*T}(\mathbf{u}^*{}_{,i} \otimes \mathbf{G}^i) + (\mathbf{G}^i \otimes \mathbf{u}^*{}_{,i})\hat{\mathbf{F}}^* \ , \\
\mathbf{E}^* = \hat{\mathbf{E}}^* + \tilde{\mathbf{E}}^* \ , \quad \hat{\mathbf{E}}^* = \frac{1}{2}(\hat{\mathbf{F}}^{*T}\hat{\mathbf{F}}^* - \mathbf{I}) \ , \quad \tilde{\mathbf{E}}^* = \frac{1}{2}\left[\hat{\mathbf{F}}^{*T}(\mathbf{u}^*{}_{,i} \otimes \mathbf{G}^i) + (\mathbf{G}^i \otimes \mathbf{u}^*{}_{,i})\hat{\mathbf{F}}^*\right] \ , \\
g^{1/2} = \hat{g}^{1/2}\left[1 + \hat{\mathbf{g}}^i \bullet \mathbf{u}^*{}_{,i}\right] \ .
\end{gathered} \tag{3.13.5}$$

Next, the conservation of mass $(3.2.35)_1$ can be written in the form

$$m^* = \rho_0^* G^{1/2} = \rho^* g^{1/2} = \hat{\rho}^* \hat{g}^{1/2} \ , \quad \rho^* = \hat{\rho}\left[1 + \hat{\mathbf{g}}^i \bullet \mathbf{u}^*{}_{,i}\right] \ , \tag{3.13.6}$$

so that the definition $(3.2.34)_1$ and the constitutive equation (3.7.15) can be used to rewrite $\mathbf{t}^{*i}$ in the forms

$$\mathbf{t}^{*i} = 2\, m^* \, \mathbf{F}^* \frac{\partial \Sigma^*}{\partial \mathbf{C}^*} \mathbf{G}^i \ . \tag{3.13.7}$$

Then, expanding Σ^* in a Taylor series and neglecting quadratic terms in the small deformation quantities yields

$$\frac{\partial \Sigma^*}{\partial \mathbf{C}^*} = \frac{\partial \Sigma^*}{\partial \hat{\mathbf{C}}^*} + \frac{\partial \Sigma^*}{\partial \tilde{\mathbf{C}}^*} \ , \tag{3.13.8}$$

where the first term on the right-hand side is evaluated with $\mathbf{C}^*=\hat{\mathbf{C}}^*$ and the second term is first order in the small deformation quantities. Specifically, when Σ^* is a general function of $\mathbf{C}^*$, then

$$\frac{\partial \Sigma^*}{\partial \tilde{\mathbf{C}}^*} = \frac{\partial^2 \Sigma^*}{\partial \hat{\mathbf{C}}^* \otimes \hat{\mathbf{C}}^*} \bullet \tilde{\mathbf{C}}^* \ , \tag{3.13.9}$$

where the derivatives of Σ^* are evaluated with $\mathbf{C}^*=\hat{\mathbf{C}}^*$. On the other hand, if the three-dimensional strain energy function Σ^* is a quadratic function of strain (3.12.2), then this equation yields

$$m^* \frac{\partial \Sigma^*}{\partial \tilde{\mathbf{C}}^*} = \frac{1}{2} G^{1/2} \mathbf{K}^* \bullet \tilde{\mathbf{E}}^* \ . \tag{3.13.10}$$

For either of these cases (3.13.8) can be used to expand the constitutive equations for $\mathbf{t}^{*i}$ to deduce that

$$\mathbf{t}^{*i} = \hat{\mathbf{t}}^{*i} + \tilde{\mathbf{t}}^{*i} \ , \quad \hat{\mathbf{t}}^{*i} = 2\, m^* \hat{\mathbf{F}}^* \frac{\partial \Sigma^*}{\partial \hat{\mathbf{C}}^*} \mathbf{G}^i \ ,$$
$$\tilde{\mathbf{t}}^{*i} = \left[\mathbf{u}^*{}_{,j} \otimes \hat{\mathbf{g}}^j\right] \hat{\mathbf{t}}^{*i} + \hat{\mathbf{F}}^* \left[2\, m^* \frac{\partial^2 \Sigma^*}{\partial \hat{\mathbf{C}}^* \otimes \hat{\mathbf{C}}^*} \bullet \tilde{\mathbf{C}}^* \right] \mathbf{G}^i \ . \tag{3.13.11}$$

Also, with the help of the results (3.13.5) and the definition $(3.34)_2$, the Cauchy stress $\mathbf{T}^*$ admits the representation

$$\mathbf{T}^* = \hat{\mathbf{T}}^* + \tilde{\mathbf{T}}^* \ , \quad \hat{\mathbf{T}}^* = \hat{g}^{-1/2} \left[\hat{\mathbf{t}}^{*i} \otimes \hat{\mathbf{g}}_i \right] \ ,$$
$$\tilde{\mathbf{T}}^* = \hat{g}^{-1/2} \left[\hat{\mathbf{t}}^{*i} \otimes \mathbf{u}^*{}_{,i} + \tilde{\mathbf{t}}^{*i} \otimes \hat{\mathbf{g}}_i \right] - \left[\hat{\mathbf{g}}^j \bullet \mathbf{u}^*{}_{,j} \right] \hat{\mathbf{T}}^* \ . \tag{3.13.12}$$

Next, substitution of (3.13.11) into (3.13.12) yields

$$\hat{\mathbf{T}}^* = 2\, \hat{\rho}\, \hat{\mathbf{F}}^* \left[\frac{\partial \Sigma^*}{\partial \hat{\mathbf{C}}^*} \right] \hat{\mathbf{F}}^{*T} \ ,$$
$$\tilde{\mathbf{T}}^* = 2\, \hat{\rho}\, \hat{\mathbf{F}}^* \left[2\, m^* \frac{\partial^2 \Sigma^*}{\partial \hat{\mathbf{C}}^* \otimes \hat{\mathbf{C}}^*} \bullet \tilde{\mathbf{C}}^* \right] \hat{\mathbf{F}}^{*T}$$
$$+ \left[\mathbf{u}^*{}_{,j} \otimes \hat{\mathbf{g}}^j \right] \hat{\mathbf{T}}^* + \hat{\mathbf{T}}^* \left[\hat{\mathbf{g}}^j \otimes \mathbf{u}^*{}_{,j} \right] - \left[\hat{\mathbf{g}}^j \bullet \mathbf{u}^*{}_{,j} \right] \hat{\mathbf{T}}^* \ , \tag{3.13.13}$$

which can be seen to be equivalent to a direct expansion of the constitutive equation (3.7.15). Moreover, since both the terms $\hat{\mathbf{T}}^*$ and $\tilde{\mathbf{T}}^*$ are symmetric tensors, the reduced form (3.2.32) of the balance of angular momentum is satisfied by the small deformation terms.

Using these expressions, the balance of linear momentum $(3.2.35)_3$ becomes

$$m^* \ddot{\mathbf{u}}^* - m^* \tilde{\mathbf{b}}^* - \tilde{\mathbf{t}}^{*i}{}_{,i} = - m^* \ddot{\hat{\mathbf{x}}}^* + m^* \hat{\mathbf{b}}^* + \hat{\mathbf{t}}^{*i}{}_{,i} \ . \tag{3.13.14}$$

With $\hat{\mathbf{x}}^*$, $\hat{\mathbf{b}}^*$ and $\tilde{\mathbf{b}}^*$ specified, these equations become linear equations to determine the displacement field $\mathbf{u}^*$. Moreover, the equations must be supplemented by initial and boundary conditions. Also, when $\hat{\mathbf{x}}^*$ is a solution of the nonlinear linear momentum equation, then the right hand side of (3.13.14) vanishes.

As a special case, consider a homogeneous material and let $\hat{\mathbf{x}}^*$ be associated with a static homogeneous solution such that

$$\hat{\mathbf{x}}^* = \hat{\mathbf{F}}^* \mathbf{X}^* \ , \quad \dot{\hat{\mathbf{F}}}^* = 0 \ , \quad \hat{\mathbf{b}}^* = 0 \ , \tag{3.13.15}$$

so that

$$\hat{\mathbf{t}}^{*i}{}_{,i} = 0 \ , \tag{3.13.16}$$

and the equation of linear momentum (3.13.14) reduces to

$$m^* \ddot{\mathbf{u}}^* = m^* \tilde{\mathbf{b}}^* + \tilde{\mathbf{t}}^{*i}{}_{,i} \ . \qquad (3.13.17)$$

In particular, it is noted that $\tilde{\mathbf{t}}^{*i}$ retains a dependence on the values $\hat{\mathbf{t}}^{*i}$ associated with the large deformation.

Moreover, for the fully linearized theory, $\hat{\mathbf{F}}^*=\mathbf{I}$ so that $\hat{\mathbf{t}}^{*i}$ vanishes. Then, the motion is determined by the equations (3.13.17) with

$$\tilde{\mathbf{E}}^* = \frac{1}{2}\left[\mathbf{u}^*{}_{,i}\otimes\mathbf{G}^i + \mathbf{G}^i\otimes\mathbf{u}^*{}_{,i}\right] \ , \ \tilde{\mathbf{t}}^{*i} = G^{1/2}\, \tilde{\mathbf{T}}^* \, \mathbf{G}^i \ , \ \tilde{\mathbf{T}}^* = \mathbf{K}^* \bullet \tilde{\mathbf{E}}^* \ . \qquad (3.13.18)$$

For the special case of an isotropic material, the constitutive equation for $\tilde{\mathbf{T}}^*$ reduces to

$$\tilde{\mathbf{T}}^* = (K^* - \frac{2}{3}\, \mu^*)\, (\tilde{\mathbf{E}}^* \bullet \mathbf{I})\, \mathbf{I} + 2\mu^* \, \tilde{\mathbf{E}}^* \ ,$$

$$\tilde{\mathbf{T}}^* = 2\mu^* \left[\left\{\frac{\nu^*}{1-2\nu^*}\right\}(\tilde{\mathbf{E}}^* \bullet \mathbf{I})\, \mathbf{I} + \tilde{\mathbf{E}}^* \right] \ . \qquad (3.13.19)$$

Also, for the linearized theory, quadratic terms in the displacements are neglected in the expression for the material derivative.

3.14 Pure bending of an orthotropic rectangular parallelepiped

In order to determine values of some of the constitutive constants for the inhomogeneous response of shells and rods to be discussed in later chapters, it is convenient to exhibit the displacement field of the linearized theory associated with pure bending of an orthotropic rectangular parallelepiped (e.g. Lekhnitskii, 1963, sec. 15). The parallelepiped is assumed to be made of a homogeneous linear elastic material, and in its reference configuration it occupies the rectangular region with respect to θ^i given by

$$|\,\theta^1\,| \le \frac{L}{2} \ , \ |\,\theta^2\,| \le \frac{W}{2} \ , \ |\,\theta^3\,| \le \frac{H}{2} \ , \qquad (3.14.1)$$

where L, W and H are constants having the units of length. Moreover, in its reference configuration the position vector $\mathbf{X}^*$ is specified by

$$\mathbf{X}^* = \theta^i \, \mathbf{e}_i \ , \qquad (3.14.2)$$

where $\mathbf{e}_i$ are fixed orthonormal base vectors of a rectangular Cartesian coordinate system. Also, the body force is assumed to vanish

$$\tilde{\mathbf{b}}^* = 0 \ . \qquad (3.14.3)$$

Then, the linearized equations of equilibrium and the constitutive equations associated with (3.13.17) and (3.13.18) reduce to

$$\tilde{\mathbf{t}}^{*j}{}_{,j} = 0 \ , \ \tilde{\mathbf{t}}^{*j} = \tilde{\mathbf{T}}^* \, \mathbf{e}_j \ , \ \tilde{\mathbf{T}}^* = \mathbf{K}^* \bullet \tilde{\mathbf{E}}^* \ , \qquad (3.14.4)$$

where $\mathbf{K}^*$ is a constant tensor. Next, referring $\mathbf{u}^*$, $\tilde{\mathbf{t}}^{*j}$, $\tilde{\mathbf{T}}^*$, $\tilde{\mathbf{E}}^*$ and $\mathbf{K}^*$ to the base vectors $\mathbf{e}_i$

$$\mathbf{u}^* = u^*_i \ \mathbf{e}_i \ , \ \tilde{\mathbf{t}}^{*j} = T^*_{ij} \ \mathbf{e}_i \ , \ \tilde{\mathbf{E}}^* = E^*_{ij} \ \mathbf{e}_i\otimes\mathbf{e}_j \ , \ E^*_{ij} = \frac{1}{2}(u^*_{i,j} + u^*_{j,i}) \ ,$$

$$\mathbf{K}^* = K^*_{ijkl} \ \mathbf{e}_i\otimes\mathbf{e}_j\otimes\mathbf{e}_k\otimes\mathbf{e}_l \ , \qquad (3.14.5)$$

the component form of the equilibrium equation $(3.14.4)_1$ becomes

$$T^*_{ij,j} = 0 \ . \tag{3.14.6}$$

Furthermore, for an orthotropic material relative to the basis $\mathbf{e}_i$, K^*_{ijkl} are given by (3.12.12) and the constitutive equation $(3.14.4)_3$ can be written in the expanded form

$$\begin{aligned}
T^*_{11} &= K^*_{1111}E^*_{11} + K^*_{1122}E^*_{22} + K^*_{1133}E^*_{33} \ , \\
T^*_{22} &= K^*_{1122}E^*_{11} + K^*_{2222}E^*_{22} + K^*_{2233}E^*_{33} \ , \\
T^*_{33} &= K^*_{1133}E^*_{11} + K^*_{2233}E^*_{22} + K^*_{3333}E^*_{33} \ , \\
T^*_{12} = 2\, K^*_{1212}E^*_{12} \ , \quad & T^*_{13} = 2\, K^*_{1313}E^*_{13} \ , \quad T^*_{23} = 2\, K^*_{2323}E^*_{23} \ .
\end{aligned} \tag{3.14.7}$$

The following considers superposition of six solutions, each of which corresponds to pure bending about a particular axis. For these solutions the stresses are given by

$$\begin{aligned}
T^*_{11} &= \left[\frac{12M_{21}}{W^3H}\right] \theta^2 + \left[\frac{12M_{31}}{WH^3}\right] \theta^3 \ , \quad T^*_{22} = \left[\frac{12M_{12}}{L^3H}\right] \theta^1 + \left[\frac{12M_{32}}{LH^3}\right] \theta^3 \ , \\
T^*_{33} &= \left[\frac{12M_{13}}{L^3W}\right] \theta^1 + \left[\frac{12M_{23}}{LW^3}\right] \theta^2 \ , \quad T^*_{12} = T^*_{13} = T^*_{23} = 0 \ ,
\end{aligned} \tag{3.14.8}$$

where $\{M_{12}, M_{13}, M_{21}, M_{23}, M_{31}, M_{32}\}$ have the units of moments. These expressions satisfy the equations of equilibrium (3.14.6) and boundary conditions in a Saint Venant sense with

$$\begin{aligned}
M_{12} &= \int_{-H/2}^{H/2} \int_{-L/2}^{L/2} \theta^1\, T^*_{22}\, d\theta^1\, d\theta^3 \ , \quad M_{13} = \int_{-W/2}^{W/2} \int_{-L/2}^{L/2} \theta^1\, T^*_{33}\, d\theta^1\, d\theta^2 \ , \\
M_{21} &= \int_{-H/2}^{H/2} \int_{-W/2}^{W/2} \theta^2\, T^*_{11}\, d\theta^2\, d\theta^3 \ , \quad M_{23} = \int_{-W/2}^{W/2} \int_{-L/2}^{L/2} \theta^2\, T^*_{33}\, d\theta^1\, d\theta^2 \ , \\
M_{31} &= \int_{-H/2}^{H/2} \int_{-W/2}^{W/2} \theta^3\, T^*_{11}\, d\theta^2\, d\theta^3 \ , \quad M_{32} = \int_{-H/2}^{H/2} \int_{-L/2}^{L/2} \theta^3\, T^*_{22}\, d\theta^1\, d\theta^3 .
\end{aligned} \tag{3.14.9}$$

Also, it can be seen from (3.14.8) that when only one of the moments M_{ij} is nonzero, then only one of the components of the stress is nonzero. Consequently, the lateral surfaces of the parallelepiped are stress free for each individual solution.

The solution for the displacement field is most conveniently represented by inverting the stress-strain relations (3.14.7) and writing

$$\begin{aligned}
E^*_{11} &= C^*_{1111}T^*_{11} + C^*_{1122}T^*_{22} + C^*_{1133}T^*_{33} \ , \\
E^*_{22} &= C^*_{1122}T^*_{11} + C^*_{2222}T^*_{22} + C^*_{2233}T^*_{33} \ , \\
E^*_{33} &= C^*_{1133}T^*_{11} + C^*_{2233}T^*_{22} + C^*_{3333}T^*_{33} \ , \\
& E^*_{12} = E^*_{13} = E^*_{23} = 0 \ ,
\end{aligned} \tag{3.14.10}$$

where the compliances C^*_{ijkl} are given by

$$\begin{aligned}
C^*_{1111} &= \frac{1}{C^*}\left[K^*_{2222}K^*_{3333} - K^*_{2233}K^*_{2233}\right] \ , \\
C^*_{1122} &= \frac{1}{C^*}\left[K^*_{1133}K^*_{2233} - K^*_{1122}K^*_{3333}\right] \ , \\
C^*_{1133} &= \frac{1}{C^*}\left[K^*_{1122}K^*_{2233} - K^*_{1133}K^*_{2222}\right] \ , \\
C^*_{2222} &= \frac{1}{C^*}\left[K^*_{1111}K^*_{3333} - K^*_{1133}K^*_{1133}\right] \ ,
\end{aligned}$$

$$C^*_{2233} = \frac{1}{C^*}\left[K^*_{1122}K^*_{1133} - K^*_{1111}K^*_{2233}\right] ,$$

$$C^*_{3333} = \frac{1}{C^*}\left[K^*_{1111}K^*_{2222} - K^*_{1122}K^*_{1122}\right] ,$$

$$\begin{aligned} C^* = K^*_{1111}&\left[K^*_{2222}K^*_{3333} - K^*_{2233}K^*_{2233}\right] \\ &+ K^*_{1122}\left[K^*_{1133}K^*_{2233} - K^*_{1122}K^*_{3333}\right] \\ &+ K^*_{1133}\left[K^*_{1122}K^*_{2233} - K^*_{1133}K^*_{2222}\right] . \end{aligned} \tag{3.14.11}$$

It can be easily shown that the stresses (3.14.8) satisfy the equations of equilibrium (3.14.6) and that the displacement field associated with the strains (3.14.10) can be written in the form

$$\begin{aligned} u^*_1 = C^*_{1111}&\left[\left\{\frac{12M_{21}}{W^3H}\right\}\theta^1\theta^2 + \left\{\frac{12M_{31}}{WH^3}\right\}\theta^1\theta^3\right] \\ &+ C^*_{1122}\left[\left\{\frac{12M_{12}}{L^3H}\right\}\frac{(\theta^1)^2}{2} + \left\{\frac{12M_{32}}{LH^3}\right\}\theta^1\theta^3\right] \\ &+ C^*_{1133}\left[\left\{\frac{12M_{13}}{L^3W}\right\}\frac{(\theta^1)^2}{2} + \left\{\frac{12M_{23}}{LW^3}\right\}\theta^1\theta^2\right] \\ &- \left[\frac{12M_{12}}{L^3H}\right]\left[C^*_{2222}\frac{(\theta^2)^2}{2} + C^*_{2233}\frac{(\theta^3)^2}{2}\right] \\ &- \left[\frac{12M_{13}}{L^3W}\right]\left[C^*_{2233}\frac{(\theta^2)^2}{2} + C^*_{3333}\frac{(\theta^3)^2}{2}\right] \\ &+ \left[\alpha^*_{12}\,\theta^2 + \alpha^*_{13}\,\theta^3 + c^*_1\right] , \end{aligned}$$

$$\begin{aligned} u^*_2 = C^*_{1122}&\left[\left\{\frac{12M_{21}}{W^3H}\right\}\frac{(\theta^2)^2}{2} + \left\{\frac{12M_{31}}{WH^3}\right\}\theta^2\theta^3\right] \\ &+ C^*_{2222}\left[\left\{\frac{12M_{12}}{L^3H}\right\}\theta^1\theta^2 + \left\{\frac{12M_{32}}{LH^3}\right\}\theta^2\theta^3\right] \\ &+ C^*_{2233}\left[\left\{\frac{12M_{13}}{L^3W}\right\}\theta^1\theta^2 + \left\{\frac{12M_{23}}{LW^3}\right\}\frac{(\theta^2)^2}{2}\right] \\ &- \left[\frac{12M_{21}}{W^3H}\right]\left[C^*_{1111}\frac{(\theta^1)^2}{2} + C^*_{1133}\frac{(\theta^3)^2}{2}\right] \\ &- \left[\frac{12M_{23}}{LW^3}\right]\left[C^*_{1133}\frac{(\theta^1)^2}{2} + C^*_{3333}\frac{(\theta^3)^2}{2}\right] \\ &+ \left[-\alpha^*_{12}\,\theta^1 + \alpha^*_{23}\,\theta^3 + c^*_2\right] , \end{aligned}$$

$$\begin{aligned} u^*_3 = C^*_{1133}&\left[\left\{\frac{12M_{21}}{W^3H}\right\}\theta^2\theta^3 + \left\{\frac{12M_{31}}{WH^3}\right\}\frac{(\theta^3)^2}{2}\right] \\ &+ C^*_{2233}\left[\left\{\frac{12M_{12}}{L^3H}\right\}\theta^1\theta^3 + \left\{\frac{12M_{32}}{LH^3}\right\}\frac{(\theta^3)^2}{2}\right] \end{aligned}$$

$$+ C^*_{3333} \left[\left\{\frac{12M_{13}}{L^3W}\right\}\theta^1\theta^3 + \left\{\frac{12M_{23}}{LW^3}\right\}\theta^2\theta^3\right]$$
$$- \left[\frac{12M_{31}}{WH^3}\right]\left[C^*_{1111}\frac{(\theta^1)^2}{2} + C^*_{1122}\frac{(\theta^2)^2}{2}\right]$$
$$- \left[\frac{12M_{32}}{LH^3}\right]\left[C^*_{1122}\frac{(\theta^1)^2}{2} + C^*_{2222}\frac{(\theta^2)^2}{2}\right]$$
$$+ \left[-\alpha^*_{13}\,\theta^1 - \alpha^*_{23}\,\theta^2 + c^*_3\right] , \tag{3.14.12}$$

where α^*_{12}, α^*_{13}, α^*_{23} are constants associated with small rigid body rotations and c^*_i are constants associated with rigid body translations.

For the special case of an isotropic material, equations (3.12.13) and Table 3.12.1 can be used to determine the usual expressions

$$C^*_{1111} = C^*_{2222} = C^*_{3333} = \frac{1}{E^*} , \quad C^*_{1122} = C^*_{1133} = C^*_{2233} = -\frac{\nu^*}{E^*} . \tag{3.14.13}$$

3.15 Torsion of an orthotropic rectangular parallelepiped

In order to determine values of additional constitutive constants for the inhomogeneous response of shells and rods to be discussed in later chapters, it is also convenient to exhibit the displacement field of the linearized theory associated with torsion of an orthotropic rectangular parallelepiped (e.g. Lekhnitskii, 1963, sec. 31). The formulation of the linearized theory is the same as that presented in equations (3.14.1)-(3.14.7) of the previous section.

The following considers superposition of three solutions, each of which corresponds to torsion about a particular axis. For these solutions the displacements are given by

$$u^*_1 = \omega_1\phi_1(\theta^2,\theta^3) + \omega_2\theta^2\theta^3 - \omega_3\theta^2\theta^3 ,$$
$$u^*_2 = -\omega_1\theta^1\theta^3 + \omega_2\phi_2(\theta^1,\theta^3) + \omega_3\theta^1\theta^3 ,$$
$$u^*_3 = \omega_1\theta^1\theta^2 - \omega_2\theta^1\theta^2 + \omega_3\phi_3(\theta^1,\theta^2) , \tag{3.15.1}$$

where ω_i are the constant twists per unit length (rad/m) about the $\mathbf{e}_i$ axes, respectively, and ϕ_i are the associated warping functions of the cross-sections that are orthogonal to $\mathbf{e}_i$, respectively. Using (3.14.5) and (3.14.8), the stresses associated with these displacements become

$$T^*_{11} = T^*_{22} = T^*_{33} = 0 ,$$
$$T^*_{12} = K^*_{1212}\left[\omega_1\{\phi_{1,2} - \theta^3\} + \omega_2\{\theta^3 + \phi_{2,1}\}\right] ,$$
$$T^*_{13} = K^*_{1313}\left[\omega_1\{\phi_{1,3} + \theta^2\} + \omega_3\{-\theta^2 + \phi_{3,1}\}\right] ,$$
$$T^*_{23} = K^*_{2323}\left[\omega_2\{\phi_{2,3} - \theta^1\} + \omega_3\{\theta^1 + \phi_{3,2}\}\right] . \tag{3.15.2}$$

These stresses will satisfy the equations of equilibrium (3.14.6) provided that

$$K^*_{1212}\,\phi_{1,22} + K^*_{1313}\,\phi_{1,33} = 0 , \quad K^*_{1212}\,\phi_{2,11} + K^*_{2323}\,\phi_{2,33} = 0 ,$$
$$K^*_{1313}\,\phi_{3,11} + K^*_{2323}\,\phi_{3,22} = 0 . \tag{3.15.3}$$

Moreover, these equations are solved subject to the boundary conditions associated with vanishing lateral stresses which require

$$[\phi_{1,2} - \theta^3]_{\theta^2=\pm W/2} = 0 \ , \quad [\phi_{1,3} + \theta^2]_{\theta^3=\pm H/2} = 0 \ ,$$
$$[\phi_{2,1} + \theta^3]_{\theta^1=\pm L/2} = 0 \ , \quad [\phi_{2,3} - \theta^1]_{\theta^3=\pm H/2} = 0 \ ,$$
$$[\phi_{3,1} - \theta^2]_{\theta^1=\pm L/2} = 0 \ , \quad [\phi_{3,2} + \theta^1]_{\theta^2=\pm W/2} = 0 \ . \tag{3.15.4}$$

The solutions of these equations can be written in the forms

$$\phi_1 = -\,\theta^2\theta^3 - \sum_{n=1}^{\infty} \Big[\frac{8\,(-1)^n\, a_2\, a_3^2}{H\, k_{1n}^3 \cosh\{\frac{k_{1n}W}{2a_2}\}} \Big] \sinh\{\frac{k_{1n}\theta^2}{a_2}\} \sin\{\frac{k_{1n}\theta^3}{a_3}\} \ ,$$
$$\phi_2 = \theta^1\theta^3 + \sum_{n=1}^{\infty} \Big[\frac{8\,(-1)^n\, b_1\, b_3^2}{H\, k_{2n}^3 \cosh\{\frac{k_{2n}L}{2b_1}\}} \Big] \sinh\{\frac{k_{2n}\theta^1}{b_1}\} \sin\{\frac{k_{2n}\theta^3}{b_3}\} \ ,$$
$$\phi_3 = -\,\theta^1\theta^2 - \sum_{n=1}^{\infty} \Big[\frac{8\,(-1)^n\, c_1\, c_2^2}{W\, k_{3n}^3 \cosh\{\frac{k_{3n}L}{2c_1}\}} \Big] \sinh\{\frac{k_{3n}\theta^1}{c_1}\} \sin\{\frac{k_{3n}\theta^2}{c_2}\} \ , \tag{3.15.5}$$

where the constants $\{ a_2, a_3, b_1, b_3, c_1, c_2 \}$ are defined by

$$a_2 = \frac{[K^*_{1212}]^{1/2}}{[\{K^*_{1212}\}^2 + \{K^*_{1313}\}^2]^{1/4}} \ , \quad a_3 = \frac{[K^*_{1313}]^{1/2}}{[\{K^*_{1212}\}^2 + \{K^*_{1313}\}^2]^{1/4}} \ ,$$
$$b_1 = \frac{[K^*_{1212}]^{1/2}}{[\{K^*_{1212}\}^2 + \{K^*_{2323}\}^2]^{1/4}} \ , \quad b_3 = \frac{[K^*_{2323}]^{1/2}}{[\{K^*_{1212}\}^2 + \{K^*_{2323}\}^2]^{1/4}} \ ,$$
$$c_1 = \frac{[K^*_{1313}]^{1/2}}{[\{K^*_{1313}\}^2 + \{K^*_{2323}\}^2]^{1/4}} \ , \quad c_2 = \frac{[K^*_{2323}]^{1/2}}{[\{K^*_{1313}\}^2 + \{K^*_{2323}\}^2]^{1/4}} \ . \tag{3.15.6}$$

It is easy to see that the expressions (3.15.5) satisfy the equations of equilibrium (3.15.4). Moreover, the boundary conditions $(3.15.4)_{2,4,6}$ are satisfied provided that the wave numbers $\{ k_{1n}, k_{2n}, k_{3n} \}$ are given by

$$k_{1n} = \frac{\pi(2n-1)a_3}{H} \ , \quad k_{2n} = \frac{\pi(2n-1)b_3}{H} \ , \quad k_{3n} = \frac{\pi(2n-1)c_2}{W} \ , \tag{3.15.7}$$

so that

$$\sin\{\frac{k_{1n}H}{2a_3}\} = -\,(-1)^n \ , \quad \cos\{\frac{k_{1n}H}{2a_3}\} = 0 \ , \quad \sin\{\frac{k_{2n}H}{2b_3}\} = -\,(-1)^n \ ,$$

$$\cos\left\{\frac{k_{2n}H}{2b_3}\right\} = 0 \ , \quad \sin\left\{\frac{k_{3n}W}{2c_2}\right\} = -(-1)^n \ , \quad \cos\left\{\frac{k_{3n}W}{2c_2}\right\} = 0 \ . \tag{3.15.8}$$

Also, with the help of the Fourier series expansions

$$2\theta^3 = -\sum_{n=1}^{\infty}\left[\frac{8\,(-1)^n\,a_3^2}{H\,k_{1n}^2}\right]\sin\left\{\frac{k_{1n}\theta^3}{a_3}\right\} = -\sum_{n=1}^{\infty}\left[\frac{8\,(-1)^n\,b_3^2}{H\,k_{2n}^2}\right]\sin\left\{\frac{k_{2n}\theta^3}{b_3}\right\} \ ,$$

$$2\theta^2 = -\sum_{n=1}^{\infty}\left[\frac{8\,(-1)^n\,c_2^2}{W\,k_{3n}^2}\right]\sin\left\{\frac{k_{3n}\theta^2}{c_2}\right\} \ , \tag{3.15.9}$$

it can be seen that the boundary conditions $(3.15.4)_{1,3,5}$ are satisfied.

Next, with the help of (3.15.2) and (3.15.5), the nonzero stresses become

$$T_{12}^* = -K_{1212}^*\omega_1\Big[2\theta^3 + \sum_{n=1}^{\infty}\Big[\frac{8\,(-1)^n\,a_3^2}{H\,k_{1n}^2\cosh\left\{\frac{k_{1n}W}{2a_2}\right\}}\Big]\cosh\left\{\frac{k_{1n}\theta^2}{a_2}\right\}\sin\left\{\frac{k_{1n}\theta^3}{a_3}\right\}\Big]$$

$$+ K_{1212}^*\omega_2\Big[2\theta^3 + \sum_{n=1}^{\infty}\Big[\frac{8\,(-1)^n\,b_3^2}{H\,k_{2n}^2\cosh\left\{\frac{k_{2n}L}{2b_1}\right\}}\Big]\cosh\left\{\frac{k_{2n}\theta^1}{b_1}\right\}\sin\left\{\frac{k_{2n}\theta^3}{b_3}\right\}\Big] \ ,$$

$$T_{13}^* = -K_{1313}^*\omega_1\Big[\sum_{n=1}^{\infty}\Big[\frac{8\,(-1)^n\,a_2\,a_3}{H\,k_{1n}^2\cosh\left\{\frac{k_{1n}W}{2a_2}\right\}}\Big]\sinh\left\{\frac{k_{1n}\theta^2}{a_2}\right\}\cos\left\{\frac{k_{1n}\theta^3}{a_3}\right\}\Big]$$

$$- K_{1313}^*\omega_3\Big[2\theta^2 + \sum_{n=1}^{\infty}\Big[\frac{8\,(-1)^n\,c_2^2}{W\,k_{3n}^2\cosh\left\{\frac{k_{3n}L}{2c_1}\right\}}\Big]\cosh\left\{\frac{k_{3n}\theta^1}{c_1}\right\}\sin\left\{\frac{k_{3n}\theta^2}{c_2}\right\}\Big] \ ,$$

$$T_{23}^* = K_{2323}^*\,\omega_2\Big[\sum_{n=1}^{\infty}\Big[\frac{8\,(-1)^n\,b_1\,b_3}{H\,k_{2n}^2\cosh\left\{\frac{k_{2n}L}{2b_1}\right\}}\Big]\sinh\left\{\frac{k_{2n}\theta^1}{b_1}\right\}\cos\left\{\frac{k_{2n}\theta^3}{b_3}\right\}\Big]$$

$$- K_{2323}^*\omega_3\Big[\sum_{n=1}^{\infty}\Big[\frac{8\,(-1)^n\,c_1\,c_2}{W\,k_{3n}^2\cosh\left\{\frac{k_{3n}L}{2c_1}\right\}}\Big]\sinh\left\{\frac{k_{3n}\theta^1}{c_1}\right\}\cos\left\{\frac{k_{3n}\theta^2}{c_2}\right\}\Big] \ . \tag{3.15.10}$$

These stresses cause moments M_i applied about the $\mathbf{e}_i$ axes on the cross-sections whose outward normals are in the $\mathbf{e}_i$ directions, respectively. Specifically,

$$M_1(\theta^1) = \int_{-H/2}^{H/2}\int_{-W/2}^{W/2}\left[\theta^2\,T_{13}^* - \theta^3\,T_{12}^*\right]d\theta^2\,d\theta^3 \ ,$$

$$M_2(\theta^2) = \int_{-H/2}^{H/2}\int_{-L/2}^{L/2}\left[\theta^3\,T_{12}^* - \theta^1\,T_{23}^*\right]d\theta^1\,d\theta^3 \ ,$$

$$M_3(\theta^3) = \int_{-W/2}^{W/2}\int_{-L/2}^{L/2}\left[\theta^1\,T_{23}^* - \theta^2\,T_{13}^*\right]d\theta^1\,d\theta^2 \ , \tag{3.15.11}$$

where it is noted that these moments are even functions of their arguments. In general, these moments are functions of two of the twists ω_i. However, when they are evaluated on the boundaries, they become functions of a single twist and the torsional rigidities B_i can be defined so that

$$M_1(L/2) = B_1^*\omega_1 \ , \quad M_2(W/2) = B_2^*\omega_2 \ , \quad M_3(H/2) = B_3^*\omega_3 \ , \tag{3.15.12}$$

where the values of B_i^* are given by

$$B_1^* = \frac{W^2H^2}{3}\left[K_{1212}^*K_{1313}^*\right]^{1/2} b^*(\eta_1) \ , \quad \eta_1 = \frac{W}{H}\left[\frac{K_{1313}^*}{K_{1212}^*}\right]^{1/2} ,$$

$$B_2^* = \frac{L^2H^2}{3}\left[K_{1212}^*K_{2323}^*\right]^{1/2} b^*(\eta_2) \ , \quad \eta_2 = \frac{L}{H}\left[\frac{K_{2323}^*}{K_{1212}^*}\right]^{1/2} ,$$

$$B_3^* = \frac{L^2W^2}{3}\left[K_{1313}^*K_{2323}^*\right]^{1/2} b^*(\eta_3) \ , \quad \eta_3 = \frac{L}{W}\left[\frac{K_{2323}^*}{K_{1313}^*}\right]^{1/2} ,$$

$$b^*(\eta) = \frac{1}{\eta}\left[1 - \frac{192}{\pi^5\eta}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^5}\tanh\left\{\frac{\pi(2n-1)\eta}{2}\right\}\right] . \tag{3.15.13}$$

Moreover, by expanding the hyperbolic tangent function in a series it can be shown (Gladwell, 1998), that the function $b^*(\eta)$ has the property that

$$b^*(\eta) = b^*(1/\eta) \ . \tag{3.15.14}$$

This result reflects the fact that the expression for the torsional rigidity must be invariant to the arbitrary choice of associating one coordinate with a trigonometric sine series and the other with a hyperbolic sine series. Also, it can be shown that $b^*(\eta)$ converges more rapidly than $b^*(1/\eta)$ for $\eta>1$. Of course, for the isotropic case the rigidities in (3.15.13) can be shown to be equivalent to the standard form (see Timoshenko and Goodier, p.278). Moreover, for later convenience, the following limiting values of B_i^* are recorded

$$B_1^* = \frac{WH^3}{3}\left[K_{1212}^*\right] \ \text{for} \ H \ll W \ , \quad B_2^* = \frac{LH^3}{3}\left[K_{1212}^*\right] \ \text{for} \ H \ll L \ ,$$

$$B_3^* = \frac{LW^3}{3}\left[K_{1313}^*\right] \ \text{for} \ W \ll L \ . \tag{3.15.15}$$

3.16 Forced shearing vibrations of an orthotropic rectangular parallelepiped

In order to determine values of the director inertia coefficients used to describe the dynamic response of shells, rods and points to be discussed in later chapters, it is convenient to exhibit exact solutions of the linearized theory associated with forced shearing vibrations of an orthotropic rectangular parallelepiped. The formulation of the linearized theory is the same as that presented in equations (3.14.1)-(3.14.7).

For these solutions the displacements are given by

$$u_1^* = A_{12}^* \sin(\omega_{12}^*t)\sin(k_2^*\theta^2) + A_{13}^* \sin(\omega_{13}^*t)\sin(k_3^*\theta^3) \ ,$$

$$u_2^* = A_{21}^* \sin(\omega_{21}^*t)\sin(k_1^*\theta^1) + A_{23}^* \sin(\omega_{23}^*t)\sin(k_3^*\theta^3) \ ,$$

$$u_3^* = A_{31}^* \sin(\omega_{31}^*t)\sin(k_1^*\theta^1) + A_{32}^* \sin(\omega_{32}^*t)\sin(k_2^*\theta^2) \ , \tag{3.16.1}$$

where the amplitudes A_{ij}^*, the frequencies ω_{ij}^* and the wave numbers k_i^* are constants. Next, using (3.2.28), (3.14.2), (3.14.5) and (3.14.7), it follows that

$$m^* = \rho_0^* \quad , \quad T_{11}^* = T_{22}^* = T_{33}^* = 0 \quad ,$$
$$T_{12}^* = K_{1212}^* \left[A_{12}^* k_2^* \sin(\omega_{12}^* t)\cos(k_2^*\theta^2) + A_{21}^* k_1^* \sin(\omega_{21}^* t)\cos(k_1^*\theta^1)\right] \; ,$$
$$T_{13}^* = K_{1313}^* \left[A_{13}^* k_3^* \sin(\omega_{13}^* t)\cos(k_3^*\theta^3) + A_{31}^* k_1^* \sin(\omega_{31}^* t)\cos(k_1^*\theta^1)\right] \; ,$$
$$T_{23}^* = K_{2323}^* \left[A_{23}^* k_3^* \sin(\omega_{23}^* t)\cos(k_3^*\theta^3) + A_{32}^* k_2^* \sin(\omega_{32}^* t)\cos(k_2^*\theta^2)\right] \; ,$$
$$\tilde{\mathbf{t}}^{*1} = T_{12}^* \mathbf{e}_2 + T_{13}^* \mathbf{e}_3 \; , \quad \tilde{\mathbf{t}}^{*2} = T_{12}^* \mathbf{e}_1 + T_{23}^* \mathbf{e}_3 \; , \quad \tilde{\mathbf{t}}^{*3} = T_{13}^* \mathbf{e}_1 + T_{23}^* \mathbf{e}_2 \; . \tag{3.16.2}$$

Then, in the absence of body force the equations of motion (3.13.17) are satisfied provided that the frequencies are given by

$$\omega_{12}^* = \left[\frac{K_{1212}^*}{\rho_0^*}\right]^{1/2} k_2^* \; , \quad \omega_{13}^* = \left[\frac{K_{1313}^*}{\rho_0^*}\right]^{1/2} k_3^* \; , \quad \omega_{21}^* = \left[\frac{K_{1212}^*}{\rho_0^*}\right]^{1/2} k_1^* \; ,$$
$$\omega_{23}^* = \left[\frac{K_{2323}^*}{\rho_0^*}\right]^{1/2} k_3^* \; , \quad \omega_{31}^* = \left[\frac{K_{1313}^*}{\rho_0^*}\right]^{1/2} k_1^* \; , \quad \omega_{32}^* = \left[\frac{K_{2323}^*}{\rho_0^*}\right]^{1/2} k_2^* \; . \tag{3.16.3}$$

Moreover, the wave numbers are determined by the boundary conditions

$$\cos\left\{\frac{k_1^* L}{2}\right\} = 0 \; , \quad k_1^* = \frac{(2n-1)\pi}{L} \quad \text{for } n=1,2,3,\ldots \; ,$$
$$\cos\left\{\frac{k_2^* W}{2}\right\} = 0 \; , \quad k_2^* = \frac{(2n-1)\pi}{W} \quad \text{for } n=1,2,3,\ldots \; ,$$
$$\cos\left\{\frac{k_3^* H}{2}\right\} = 0 \; , \quad k_3^* = \frac{(2n-1)\pi}{H} \quad \text{for } n=1,2,3,\ldots \; , \tag{3.16.4}$$

which cause

$$\tilde{\mathbf{t}}^{*1}(\pm L/2,\theta^2,\theta^3,t) = K_{1212}^* \left[A_{12}^* k_2^* \sin(\omega_{12}^* t)\cos(k_2^*\theta^2)\right] \mathbf{e}_2$$
$$+ K_{1313}^* \left[A_{13}^* k_3^* \sin(\omega_{13}^* t)\cos(k_3^*\theta^3)\right] \mathbf{e}_3 \; ,$$
$$\tilde{\mathbf{t}}^{*2}(\theta^1,\pm W/2,\theta^3,t) = K_{1212}^* \left[A_{21}^* k_1^* \sin(\omega_{21}^* t)\cos(k_1^*\theta^1)\right] \mathbf{e}_1$$
$$+ K_{2323}^* \left[A_{23}^* k_3^* \sin(\omega_{23}^* t)\cos(k_3^*\theta^3)\right] \mathbf{e}_3 \; ,$$
$$\tilde{\mathbf{t}}^{*3}(\theta^1,\theta^2,\pm H/2) = K_{1313}^* \left[A_{31}^* k_1^* \sin(\omega_{31}^* t)\cos(k_1^*\theta^1)\right] \mathbf{e}_1$$
$$+ K_{2323}^* \left[A_{32}^* k_2^* \sin(\omega_{32}^* t)\cos(k_2^*\theta^2)\right] \mathbf{e}_2 \; . \tag{3.16.5}$$

It can be seen that the displacements (3.16.1) correspond to six distinct vibrational modes. To interpret the physical nature of these modes, it is best to focus attention on a single mode. For example, let A_{13}^* be the only nonzero amplitude so that

$$u_1^* = A_{13}^* \sin(\omega_{13}^* t)\sin(k_3^*\theta^3) \; , \quad u_2^* = 0 \; , \quad u_3^* = 0 \; ,$$
$$\tilde{\mathbf{t}}^{*1} = K_{1313}^* \left[A_{13}^* k_3^* \sin(\omega_{13}^* t)\cos(k_3^*\theta^3)\right] \mathbf{e}_3 \; , \quad \tilde{\mathbf{t}}^{*2} = 0 \; ,$$
$$\tilde{\mathbf{t}}^{*3} = K_{1313}^* \left[A_{13}^* k_3^* \sin(\omega_{13}^* t)\cos(k_3^*\theta^3)\right] \mathbf{e}_1 \; . \tag{3.16.6}$$

For this mode, shearing occurs in the $\mathbf{e}_1$–$\mathbf{e}_3$ plane with motion only in the $\mathbf{e}_1$ direction. Also, the boundaries $\theta^2=\pm W/2$ and $\theta^3=\pm H/2$ of the parallelepiped are stress free and the shear stresses acting on the boundaries $\theta^1=\pm L/2$ are

$$\tilde{\mathbf{t}}^{*1}(\pm L/2,\theta^2,\theta^3,t) = K^*_{1313}\,[A^*_{13}k^*_3 \sin(\omega^*_{13}t)\cos(k^*_3\theta^3)]\,\mathbf{e}_3 \ . \qquad (3.16.7)$$

3.17 Free isochoric vibrations of an isotropic cube

In order to check the values of the director inertia coefficients used to describe the dynamic response of shells, rods and points to be discussed in later chapters, it is convenient to exhibit specific exact solutions of the linearized theory associated with free isochoric vibrations of an isotropic cube. Although the general case of free vibrations of a rectangular parallelepiped has been considered by Hutchinson and Zillmer (1983), it is sufficient here to consider the simpler case when the parallelepiped is a cube. The formulation of the linearized theory is the same as that presented in equations (3.14.1)-(3.14.7), but here the lengths of the parallelepiped are taken to be equal so that

$$L = W = H \ . \qquad (3.17.1)$$

Following the work in (Rubin, 1986), the displacement field is taken in the form

$$\begin{aligned}
u^*_1 &= A^*_{12}\sin(\omega^* t)\sin(k^*\theta^1)\cos(k^*\theta^2) + A^*_{13}\sin(\omega^* t)\sin(k^*\theta^1)\cos(k^*\theta^3) \ ,\\
u^*_2 &= -A^*_{12}\sin(\omega^* t)\cos(k^*\theta^1)\sin(k^*\theta^2) + A^*_{23}\sin(\omega^* t)\sin(k^*\theta^2)\cos(k^*\theta^3) \ ,\\
u^*_3 &= -A^*_{13}\sin(\omega^* t)\cos(k^*\theta^1)\sin(k^*\theta^3) - A^*_{23}\sin(\omega^* t)\cos(k^*\theta^2)\sin(k^*\theta^3) \ ,
\end{aligned} \qquad (3.17.2)$$

where the amplitudes A^*_{ij}, the frequency ω^* and the wave number k^* are constants. Next, using (3.2.28), (3.14.2), (3.14.5) and (3.14.7), it follows that the deformation is isochoric

$$\tilde{\mathbf{E}}^* \bullet \mathbf{I} = 0 \ , \qquad (3.17.3)$$

and that for an isotropic material

$$\begin{aligned}
m^* &= \rho^*_0 \ ,\\
\tilde{\mathbf{t}}^{*1} &= 2\mu^* k^*\,[A^*_{12}\cos(k^*\theta^2) + A^*_{13}\cos(k^*\theta^3)]\,\sin(\omega^* t)\cos(k^*\theta^1)\,\mathbf{e}_1 \ ,\\
\tilde{\mathbf{t}}^{*2} &= 2\mu^* k^*\,[-A^*_{12}\cos(k^*\theta^1) + A^*_{23}\cos(k^*\theta^3)]\,\sin(\omega^* t)\cos(k^*\theta^2)\,\mathbf{e}_2 \ ,\\
\tilde{\mathbf{t}}^{*3} &= -2\mu^* k^*\,[A^*_{13}\cos(k^*\theta^1) + A^*_{23}\cos(k^*\theta^2)]\,\sin(\omega^* t)\cos(k^*\theta^3)\,\mathbf{e}_3 \ .
\end{aligned} \qquad (3.17.4)$$

Then, in the absence of body force, the equations of motion (3.13.17) are satisfied provided that the frequency ω^* is given by

$$\omega^* = \left[\frac{2\mu^*}{\rho^*_0}\right]^{1/2} k^* \ . \qquad (3.17.5)$$

Moreover, the wave number is determined by the boundary condition

$$\cos\left\{\frac{k^* H}{2}\right\} = 0 \ , \quad k^* = \frac{(2n-1)\pi}{H} \quad \text{for } n=1,2,3,\ldots \ , \qquad (3.17.6)$$

which ensures that the cube remains stress free.

It can be seen that the displacements (3.17.1) correspond to three distinct vibrational modes. To interpret the physical nature of these modes it is best to focus attention on a single mode. For example, let A^*_{13} be the only nonzero amplitude so that

$$u_1^* = A_{13}^* \sin(\omega^* t) \sin(k^* \theta^1) \cos(k^* \theta^3) \ , \quad u_2^* = 0 \ ,$$
$$u_3^* = -A_{13}^* \sin(\omega^* t) \cos(k^* \theta^1) \sin(k^* \theta^3) \ ,$$
$$\tilde{\mathbf{t}}^{*1} = 2\mu^* k^* A_{13}^* \sin(\omega^* t) \cos(k^* \theta^1) \cos(k^* \theta^3) \mathbf{e}_1 \ , \quad \tilde{\mathbf{t}}^{*2} = 0 \ ,$$
$$\tilde{\mathbf{t}}^{*3} = -2\mu^* k^* A_{13}^* \sin(\omega^* t) \cos(k^* \theta^1) \cos(k^* \theta^3) \mathbf{e}_3 \ . \tag{3.17.7}$$

For this mode, extension and contraction occur out of phase in the $\mathbf{e}_1$ and $\mathbf{e}_3$ directions. Also, the stresses normal to the $\mathbf{e}_1$ and $\mathbf{e}_3$ planes are equal in magnitude and opposite in sign, with both stresses vanishing on the boundaries of the cube.

3.18 An orthotropic rectangular parallelepiped loaded by its own weight

In order to check the values of the assigned fields used to describe the effect of body force on shells, rods and points to be discussed in later chapters, it is convenient to exhibit specific exact solutions of the linearized theory associated with an orthotropic rectangular parallelepiped loaded by its own weight. The formulation of the linearized theory is the same as that presented in equations (3.14.1)-(3.14.7), except that here the body force is specified by the constant vector

$$\tilde{\mathbf{b}}^* = -g_i^* \mathbf{e}_i \ , \tag{3.18.1}$$

where g_i^* is the constant force of gravity per unit mass in the negative $\mathbf{e}_i$ direction. Also, since comparison with the theories of shells, rods and points may require taking the gravitation field to act in different directions, the solution presented here is a superposition of three separate solutions.

Taking the surfaces (θ^1=L/2), (θ^2=W/2) and (θ^3=H/2) to be free from surface tractions, it is easy to see that the stresses

$$T_{11}^* = \rho_0^* g_1^* (\theta^1 - \frac{L}{2}) \ , \quad T_{22}^* = \rho_0^* g_2^* (\theta^2 - \frac{W}{2}) \ , \quad T_{33}^* = \rho_0^* g_3^* (\theta^3 - \frac{H}{2}) \ ,$$
$$T_{12}^* = T_{13}^* = T_{23}^* = 0 \ , \quad \tilde{\mathbf{t}}^{*1} = \rho_0^* g_1^* (\theta^1 - \frac{L}{2}) \mathbf{e}_1 \ , \quad \tilde{\mathbf{t}}^{*2} = \rho_0^* g_2^* (\theta^2 - \frac{W}{2}) \mathbf{e}_2 \ ,$$
$$\tilde{\mathbf{t}}^{*3} = \rho_0^* g_3^* (\theta^3 - \frac{H}{2}) \mathbf{e}_3 \ , \tag{3.18.2}$$

satisfy the linearized equations of equilibrium

$$\rho_0^* \tilde{\mathbf{b}} + \tilde{\mathbf{t}}^{*i}{}_{,i} = 0 \ . \tag{3.18.3}$$

Now, the strains for an orthotropic material are given by the expressions (3.14.10), and the associated displacements can be obtained by integration to deduce that

$$u_1^* = \rho_0^* \Big[\frac{1}{2} C_{1111}^* g_1^* (\theta^1 - \frac{L}{2})^2 + C_{1122}^* g_2^* \theta^1 (\theta^2 - \frac{W}{2}) + C_{1133}^* g_3^* \theta^1 (\theta^3 - \frac{H}{2})$$
$$- \frac{1}{2} C_{1122}^* g_1^* (\theta^2)^2 - \frac{1}{2} C_{1133}^* g_1^* (\theta^3)^2\Big] + \Big[\alpha_{12}^* \theta^2 + \alpha_{13}^* \theta^3 + c_1^*\Big] \ ,$$
$$u_2^* = \rho_0^* \Big[C_{1122}^* g_1^* (\theta^1 - \frac{L}{2}) \theta^2 + \frac{1}{2} C_{2222}^* g_2^* (\theta^2 - \frac{W}{2})^2 + C_{2233}^* g_3^* \theta^2 (\theta^3 - \frac{H}{2})$$
$$- \frac{1}{2} C_{1122}^* g_2^* (\theta^1)^2 - \frac{1}{2} C_{2233}^* g_2^* (\theta^3)^2\Big] + \Big[-\alpha_{12}^* \theta^1 + \alpha_{23}^* \theta^3 + c_2^*\Big] \ ,$$

$$u_3^* = \rho_0^* \left[C_{1133}^* g_1^* (\theta^1 - \frac{L}{2})\theta^3 + C_{2233}^* g_2^* (\theta^2 - \frac{W}{2})\theta^3 + \frac{1}{2} C_{3333}^* g_3^* (\theta^3 - \frac{H}{2})^2 \right.$$
$$\left. - \frac{1}{2} C_{1133}^* g_3^* (\theta^1)^2 - \frac{1}{2} C_{2233}^* g_3^* (\theta^2)^2 \right] + \left[- \alpha_{13}^* \theta^1 - \alpha_{23}^* \theta^2 + c_3^*\right] \ , \quad (3.18.4)$$

where $\alpha_{12}^*, \alpha_{13}^*, \alpha_{23}^*$ are constants associated with small rigid body rotations and c_i^* are constants associated with rigid body translations. Moreover, for the special case of an isotropic material the compliances are given by (3.14.13).

In order to interpret the physical nature of this solution, it is best to focus on a single solution. For example, let g_3^* be the only nonzero value of g_i^* so that

$$\tilde{\mathbf{t}}^{*1} = 0 \ , \quad \tilde{\mathbf{t}}^{*2} = 0 \ , \quad \tilde{\mathbf{t}}^{*3} = \rho_0^* g_3^* (\theta^3 - \frac{H}{2}) \mathbf{e}_3 \ ,$$
$$u_1^* = \rho_0^* \left[C_{1133}^* g_3^* \theta^1 (\theta^3 - \frac{H}{2})\right] + \left[\alpha_{12}^* \theta^2 + \alpha_{13}^* \theta^3 + c_1^*\right] \ ,$$
$$u_2^* = \rho_0^* \left[C_{2233}^* g_3^* \theta^2 (\theta^3 - \frac{H}{2})\right] + \left[- \alpha_{12}^* \theta^1 + \alpha_{23}^* \theta^3 + c_2^*\right] \ ,$$
$$u_3^* = \rho_0^* \left[\frac{1}{2} C_{3333}^* g_3^* (\theta^3 - \frac{H}{2})^2 - \frac{1}{2} C_{1133}^* g_3^* (\theta^1)^2 - \frac{1}{2} C_{2233}^* g_3^* (\theta^2)^2 \right]$$
$$+ \left[- \alpha_{13}^* \theta^1 - \alpha_{23}^* \theta^2 + c_3^*\right] \ . \quad (3.18.5)$$

It follows that for this solution all surfaces of the parallelepiped are stress free except for the bottom surface (θ^3=–H/2), which has a constant compressive stress of magnitude $\rho_0^* g_3^* H$ applied to it to support the weight. The cross-sections (θ^3=constant) which were horizontal in the reference configuration have curvature in both the $\mathbf{e}_1$ and $\mathbf{e}_2$ directions due to the Poisson effect. Also, within the linear approximation, the unit normal vectors to the lateral surfaces ($\theta^1=\pm L/2$ and $\theta^2=\pm W/2$) are constant vectors tilted in the $\mathbf{e}_3$ direction. Thus, these lateral surfaces remain planar.

3.19 An isotropic circular cylinder loaded by its own weight

In order to check the influence of geometry on the assigned fields used to describe the effect of body force on shells to be discussed in later chapters, it is convenient to exhibit the solution of the linearized theory associated with an isotropic circular cylinder loaded by its own weight. The formulation of the linearized theory is the same as that presented in equations (3.14.1)-(3.14.7), except that here the body force is specified by the constant vector

$$\tilde{\mathbf{b}}^* = - g^* \mathbf{e}_z \ , \quad (3.19.1)$$

where g^* is the constant force of gravity per unit mass in the negative $\mathbf{e}_z$ direction of a cylindrical polar coordinates system with base vectors $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$.

In its reference configuration, the circular cylinder has internal radius R_1, external radius R_2, and height W. For later convenience, the position vector is given by

$$\mathbf{X}^* = R^* \mathbf{e}_r(\theta) + \theta^2 \mathbf{e}_z \ , \quad (3.19.2)$$

and the convected coordinates θ^1 and θ^3 are defined so that

$$\theta^1 = \theta \ , \quad R^* = R + \theta^3 \ , \quad R = \frac{1}{2}(R_1 + R_2) \ . \quad (3.19.3)$$

Thus, the cylindrical region is specified by

$$|\theta| \le \pi \ , \quad |\theta^2| \le \frac{W}{2} \ , \quad |\theta^3| \le \frac{H}{2} \ , \quad H = R_2 - R_1 \ . \tag{3.19.4}$$

It is possible to specialize one of the solutions presented in section 3.18 and to use appropriate coordinate transformations to obtain the desired solution. Alternatively, the solution can be obtained directly as follows. Taking the internal and external surfaces of the cylinder and the end (θ^2=W/2) to be free from surface tractions, it is easy to see that the stress $\tilde{\mathbf{T}}$ is given by

$$\tilde{\mathbf{T}} = \rho_0^* \ g^*(\theta^2 - \frac{W}{2}) \ \mathbf{e}_z \otimes \mathbf{e}_z \ . \tag{3.19.5}$$

Moreover, the position vector (3.19.2) can be used to determine the expressions

$$\mathbf{G}_1 = (R+\theta^3) \ \mathbf{e}_\theta \ , \quad \mathbf{G}_2 = \mathbf{e}_z \ , \quad \mathbf{G}_3 = \mathbf{e}_r \ ,$$

$$\mathbf{G}^1 = \frac{1}{R+\theta^3} \ \mathbf{e}_\theta \ , \quad \mathbf{G}^2 = \mathbf{e}_z \ , \quad \mathbf{G}^3 = \mathbf{e}_r \ , \quad G^{1/2} = R+\theta^3 \ . \tag{3.19.6}$$

Thus, with the help of (3.2.28) and (3.13.8), it follows that

$$m^* = \rho_0^* \ (R+\theta^3) \ , \quad \tilde{\mathbf{t}}^{*1} = \tilde{\mathbf{t}}^{*3} = 0 \ , \quad \tilde{\mathbf{t}}^{*2} = \rho_0^* \ g^*(\theta^2 - \frac{W}{2}) \ \mathbf{e}_z \ , \tag{3.19.7}$$

which can be seen to satisfy the equilibrium equation

$$m^* \ \tilde{\mathbf{b}} + \tilde{\mathbf{t}}^{*i}{}_{,i} = 0 \ . \tag{3.19.8}$$

Next, using Table 3.12.1, the constitutive equation (3.13.19) for an isotropic material can be inverted to obtain

$$\tilde{\mathbf{E}}^* = \frac{1}{E^*} \left[(1+\nu^*) \ \tilde{\mathbf{T}} - \nu^* \ (\tilde{\mathbf{T}} \bullet \mathbf{I}) \ \mathbf{I} \right] \ . \tag{3.19.9}$$

Then, with the help of the strain-displacement relations $(3.13.18)_1$, it can be shown that the associated displacement field becomes

$$\mathbf{u}^* = \frac{\rho_0^* g^*}{E^*} \Big[-\nu^* \ (\theta^2 - \frac{W}{2})(R+\theta^3) \ \mathbf{e}_r$$
$$+ \frac{1}{2} \left\{ [(\theta^2 - \frac{W}{2})^2 - W^2] + \nu^*[(R+\theta^3)^2 - R^2] \right\} \mathbf{e}_z \Big] \ , \tag{3.19.10}$$

where the constants corresponding to rigid-body rotation and translation have been specified with

$$\mathbf{u}^*(\theta^1, -W/2, 0) \bullet \mathbf{e}_z = 0 \ . \tag{3.19.11}$$

3.20 Plane strain free vibrations of an isotropic solid circular cylinder

The exact solution for the general case of free vibration of a circular cylinder is reported in (Love, 1944, chp. XII). For convenience, here a special case of that solution is considered for which the three-dimensional displacement $\mathbf{u}^*$ is taken in the form

$$\mathbf{u}^* = u^*(R^*, t) \ \mathbf{e}_r \ , \quad R^* = R + \theta^3 = \frac{H}{2} + \theta^3 \ , \quad |\theta^3| \le \frac{H}{2} \ , \tag{3.20.1}$$

where H is the radius of the cylinder, θ^3 is a coordinate in the radial direction and $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ are the unit base vectors of a cylindrical polar coordinate system. Using this representation, the linearized strain (3.13.18) and stress (3.13.19) become

$$\tilde{\mathbf{E}}^* = \frac{\partial u^*}{\partial R^*}\mathbf{e}_r\otimes\mathbf{e}_r + \frac{u^*}{R^*}\mathbf{e}_\theta\otimes\mathbf{e}_\theta \ ,$$

$$\tilde{\mathbf{T}}^* = \frac{2\mu^*}{(1-2\nu^*)}\Big[\Big\{(1-\nu^*)\frac{\partial u^*}{\partial R^*} + \nu^*\frac{u^*}{R^*}\Big\}\mathbf{e}_r\otimes\mathbf{e}_r + \Big\{\nu^*\frac{\partial u^*}{\partial R^*} + (1-\nu^*)\frac{u^*}{R^*}\Big\}\mathbf{e}_\theta\otimes\mathbf{e}_\theta$$

$$+ \nu^*\Big\{\frac{\partial u^*}{\partial R^*} + \frac{u^*}{R^*}\Big\}\mathbf{e}_z\otimes\mathbf{e}_z\Big] \ . \tag{3.20.2}$$

Then, in the absence of body force, the equation of motion (3.13.17) reduces to

$$\rho_0^* R^* \ddot{u}^* = \frac{2\mu^*(1-\nu^*)}{(1-2\nu^*)}\Big[\frac{\partial}{\partial R^*}\Big\{R^*\frac{\partial u^*}{\partial R^*}\Big\} - \frac{u^*}{R^*}\Big] \ . \tag{3.20.3}$$

Next, it can be shown that a solution of this equation which includes the origin (R^*=0) can be written in the form

$$u^* = J_1(x)\sin(\omega^* t) \ \ , \ \ x = \beta\frac{R^*}{H} \ , \tag{3.20.4}$$

where $J_1(x)$ is the Bessel function of first kind and order one. Also, the natural frequency ω^* is given by the expressions

$$\omega^* = \Big[\frac{2\mu^*}{\rho_0^*}\Big]^{1/2}\frac{\Omega^*}{H} \ , \ \ \Omega^* = \Big[\frac{1-\nu^*}{1-2\nu^*}\Big]^{1/2}\beta \ , \tag{3.20.5}$$

where β is determined by the condition that the radial stress vanishes on the boundary (R^*=H and x=β) so that

$$\Big[\frac{dJ_1(x)}{dx} + \Big\{\frac{\nu^*}{1-\nu^*}\Big\}\frac{J_1(x)}{x}\Big]_{x=\beta} = 0 \ . \tag{3.20.6}$$

Moreover, it can shown by using standard identities for Bessel functions that this frequency equation is a special case of the one given by Love (1944, sec. 201).

3.21 Dissipation inequality and material damping

The previous sections have limited attention to purely elastic response which exhibits no dissipation. Consequently, structures made from such materials exhibit the unrealistic feature that free vibrations persist forever. In order to eliminate this unphysical response, it is necessary to include a model for material damping. To this end, it is noted that within the context of the purely mechanical theory, it is possible to define the rate of material of dissipation $\mathcal{D}^*$ per unit present volume by the formula

$$\int_{P^*} \mathcal{D}^* \, dv^* = \mathcal{W} - \dot{\mathcal{K}} - \dot{\mathcal{U}} \geq 0 \ , \tag{3.21.1}$$

where $\mathcal{W}$, $\mathcal{K}$ and $\mathcal{U}$ are defined by (3.4.1), (3.4.2) and (3.7.4), respectively. In words, this equation means that the rate of material dissipation is equal to the rate of work done by body forces and surface tractions $\mathcal{W}$, minus the rates of change of kinetic energy $\mathcal{K}$, and strain energy $\mathcal{U}$. Moreover, it is assumed that the rate of material dissipation is nonnegative.

Next, with the help of the conservation of mass and the balances of linear and angular momentum, it can be shown that the local form of equation (3.21.1) becomes

$$\mathcal{D}^* = \mathbf{T}^* \bullet \mathbf{D}^* - \rho^* \dot{\Sigma}^* \geq 0 \ . \tag{3.21.2}$$

Moreover, in view of the assumption (3.7.1), it is seen that an elastic material is an ideal material since the rate of dissipation $\mathcal{D}^*$ vanishes. Consequently, the assumption that the rate of material dissipation is nonnegative requires that, for a given motion, the work done on a dissipative material is greater than that done on an ideal elastic material. Also, using the transformation relations $(3.3.20)_5$, (3.3.34), (3.3.38) and (3.7.3), it can be shown that $\mathcal{D}^*$ remains unaltered by SRBM

$$\mathcal{D}^{*+} = \mathcal{D}^* \ . \tag{3.21.3}$$

Now, a model for material dissipation can be developed by assuming that the stress $\mathbf{T}^*$ separates additively into three parts

$$\mathbf{T}^* = \hat{\mathbf{T}}^* + \bar{\mathbf{T}}^* + \check{\mathbf{T}}^* \ , \quad \hat{\mathbf{T}}^* = g^{-1/2}\, \hat{\mathbf{t}}^{*i} \otimes \mathbf{g}_i \ , \quad \bar{\mathbf{T}}^* = g^{-1/2}\, \bar{\mathbf{t}}^{*i} \otimes \mathbf{g}_i \ ,$$
$$\check{\mathbf{T}}^* = g^{-1/2}\, \check{\mathbf{t}}^{*i} \otimes \mathbf{g}_i \ , \quad \mathbf{t}^{*i} = \hat{\mathbf{t}}^{*i} + \bar{\mathbf{t}}^{*i} + \check{\mathbf{t}}^{*i} \ , \tag{3.21.4}$$

with $\hat{\mathbf{T}}^*$ and $\hat{\mathbf{t}}^{*i}$ being the parts associated with elastic deformation [which balance the rate of change in strain energy (3.7.1)]

$$\hat{\mathbf{T}}^* \bullet \mathbf{D}^* = \rho^* \dot{\Sigma}^* \ , \tag{3.21.5}$$

$\bar{\mathbf{T}}^*$ and $\bar{\mathbf{t}}^{*i}$ being the constraint responses [which do no work (3.8.8)]

$$\bar{\mathbf{T}}^* \bullet \mathbf{D}^* = 0 \ , \tag{3.21.6}$$

and $\check{\mathbf{T}}^*$ and $\check{\mathbf{t}}^{*i}$ being the parts due to material dissipation. Thus, the restriction (3.21.2) reduces to

$$\mathcal{D}^* = \check{\mathbf{T}}^* \bullet \mathbf{D}^* \geq 0 \ . \tag{3.21.7}$$

For viscous damping the stress $\check{\mathbf{T}}^*$ is assumed to be a function of $\mathbf{D}^*$

$$\check{\mathbf{T}}^* = \check{\mathbf{T}}^*(\mathbf{D}^*) \ . \tag{3.21.8}$$

However, invariance under SRBM requires this function to satisfy the restriction

$$\mathbf{Q}\, \check{\mathbf{T}}^*(\mathbf{D}^*)\, \mathbf{Q}^T = \check{\mathbf{T}}^*(\mathbf{Q}\, \mathbf{D}^*\, \mathbf{Q}^T) \ , \tag{3.21.9}$$

for all proper orthogonal tensors $\mathbf{Q}$. Consequently, $\check{\mathbf{T}}^*$ must be an isotropic function of its argument. Furthermore, for the simple case when $\check{\mathbf{T}}^*$ is a linear function of $\mathbf{D}^*$ it follows that

$$\check{\mathbf{T}}^* = \eta_1^*\, (\mathbf{D}^* \bullet \mathbf{I})\, \mathbf{I} + 2\eta_2^*\, \mathbf{D}^{*\prime} \ , \tag{3.21.10}$$

where η_1^* and η_2^* are material constants and the deviatoric tensor $\mathbf{D}^{*\prime}$ is a pure measure of rate of distortional deformation

$$\mathbf{D}^{*\prime} = \mathbf{D}^* - \frac{1}{3}(\mathbf{D}^* \bullet \mathbf{I})\, \mathbf{I} \ , \quad \mathbf{D}^{*\prime} \bullet \mathbf{I} = 0 \ . \tag{3.21.11}$$

Consequently, η_1^* is the viscosity to dilatational deformation rate, and η_2^* is the viscosity to distortional deformation. Also, it can be shown that the restriction (3.21.7) is satisfied for all motions provided that η_1^* and η_2^* are both nonnegative

$$\eta_1^* \geq 0 \;, \quad \eta_2^* \geq 0 \;. \tag{3.21.12}$$

Finally, it is noted that the viscosity constants η_1^* and η_2^* can be determined by attempting to match the rate of damping associated with free vibrations of a structure.

CHAPTER 4

COSSERAT SHELLS

4.1 Description of a shell structure

A shell-like structure or shell is a three-dimensional body that has special geometric features. Most importantly, the shell is a three-dimensional body that is considered to be "thin" in one of its dimensions (see Fig. 4.1.1). In particular, the shell is characterized by its major surfaces (bottom and top) and its lateral surface. From another point of view, the shell is considered to be a material surface $\mathcal{S}$ which has some finite thickness bounded by the major surfaces. If this surface $\mathcal{S}$ is flat, then the shell-like structure is called a plate, otherwise it is called a shell. Such shell-like structures appear in practice in many applications. For example, the floors, walls and roofs of many buildings are flat surfaces that can be modeled as plates, whereas the surfaces of an airplane body or car body, the human skull and the veins and arteries in the human circulatory system are curved surfaces that can be modeled as shells.

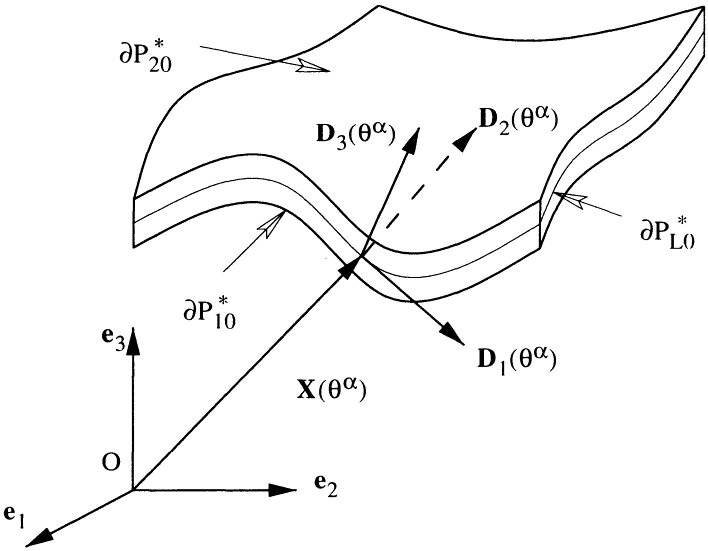

Fig. 4.1.1 A shell-like structure in its reference configuration

From a mathematical point of view it is necessary to clearly define in what sense the shell is considered to be "thin". To this end, it is convenient to consider the shell in its reference configuration and to denote the material surface by $\mathcal{S}_0$. Material points in the shell are located relative to a fixed origin O by the three-dimensional position vector $\mathbf{X}^*(\theta^i)$, and material points on the surface $\mathcal{S}_0$ are located by the position vector $\mathbf{X}(\theta^\alpha)$, which corresponds to the surface for which θ^3 vanishes

$$\mathbf{X} = \mathbf{X}(\theta^\alpha) = \mathbf{X}^*(\theta^\alpha,0) \ . \tag{4.1.1}$$

For convenience, throughout the text, Greek indices will take the values 1 and 2 only (α=1,2). The tangent vectors $\mathbf{A}_\alpha$ in the θ^α directions, the unit normal vector $\mathbf{A}_3$ to the surface $\mathcal{S}_0$, and the reciprocal vectors $\mathbf{A}^i$ are given by

$$\mathbf{A}_\alpha(\theta^\alpha) = \frac{\partial \mathbf{X}}{\partial \theta^\alpha} = \mathbf{X}_{,\alpha} \ , \quad \mathbf{A}_3 = A^{-1/2}\,(\mathbf{A}_1 \times \mathbf{A}_2) \ ,$$

$$A^{1/2} = \mathbf{A}_1 \times \mathbf{A}_2 \cdot \mathbf{A}_3 \ , \quad d\Sigma = A^{1/2}\, d\theta^1\, d\theta^2 \ ,$$

$$A^{1/2}\,\mathbf{A}^1 = \mathbf{A}_2 \times \mathbf{A}_3 \ , \quad A^{1/2}\,\mathbf{A}^2 = \mathbf{A}_3 \times \mathbf{A}_1 \ ,$$

$$\mathbf{A}^3 = \mathbf{A}_3 \ , \quad \mathbf{A}^i \cdot \mathbf{A}_j = \delta^i_j \ . \tag{4.1.2}$$

where $d\Sigma$ is the element of area on the surface $\mathcal{S}_0$.

In the following, it is also convenient to introduce a right-handed triad of linearly independent vectors $\mathbf{D}_i$ such that $\mathbf{D}_\alpha$ are equal to the tangent vectors $\mathbf{A}_\alpha$ and $\mathbf{D}_3$ always has a positive component normal to this surface

$$\mathbf{D}_\alpha(\theta^\alpha) = \mathbf{A}_\alpha \ , \quad D^{1/2} = \mathbf{D}_1 \times \mathbf{D}_2 \cdot \mathbf{D}_3 > 0 \ . \tag{4.1.3}$$

Since these vectors $\mathbf{D}_i$ are linearly independent, it can be shown that a finite region of three-dimensional space can be characterized by specifying the position vector $\mathbf{X}^*$ in the form

$$\mathbf{X}^*(\theta^i) = \mathbf{X}(\theta^\alpha) + \theta^3\, \mathbf{D}_3(\theta^\alpha) \ . \tag{4.1.4}$$

An arbitrary material part P_0^* of the shell is bounded by the major surfaces ∂P_{10}^* and ∂P_{20}^*, and the smooth lateral surface ∂P_{L0}^* (see Fig. 4.1). For definiteness, the major surfaces are defined by the functions $\xi_1(\theta^\alpha)$ and $\xi_2(\theta^\alpha)$, respectively, such that

$$\theta^3 = \xi_1(\theta^\alpha) \ \text{ on } \ \partial P_{10}^* \ , \quad \theta^3 = \xi_2(\theta^\alpha) \ \text{ on } \ \partial P_{20}^* \ . \tag{4.1.5}$$

In general, material lines in the thickness direction on this lateral surface ∂P_{L0}^* can be curved. This means that ∂P_{L0}^* is specified by a function $f^*(\theta^i)$ which depends on all three coordinates

$$f^*(\theta^i) = f^*(\theta^\alpha,\theta^3) = 0 \ \text{ and } \ \xi_1 \le \theta^3 \le \xi_2 \ \text{ on } \ \partial P_{L0}^* \ . \tag{4.1.6}$$

However, for simplicity, attention will be confined to shell-like bodies which have cylindrical lateral surfaces. For such shells it is convenient to consider the material part P_0 of the surface $\mathcal{S}_0$ which is contained within the three-dimensional region occupied by the part P_0^* of the shell, and to denote ∂P_0 as the smooth curve bounding the surface P_0. This boundary curve ∂P_0 is the intersection of the part P_0 with the lateral surface ∂P_{L0}^* and it can be characterized by the functional form

$$f(\theta^\alpha) = 0 \ \text{ on } \ \partial P_0 \ . \tag{4.1.7}$$

Assuming sufficient continuity for the implicit function theorem to hold, there exist functions $\hat{\theta}^\alpha(S)$ of the arclength parameter S which automatically satisfy the equation (4.1.7)

$$f(\hat{\theta}^\alpha(S)) = 0 \ . \tag{4.1.8}$$

The arclength parameter S is defined relative to the boundary curve ∂P_0 such that the tangent vector to this curve in the reference configuration is a unit vector

$$\frac{d\mathbf{X}}{dS} = \frac{d\hat{\theta}^\alpha}{dS} \mathbf{A}_\alpha \ , \quad \frac{d\mathbf{X}}{dS} \bullet \frac{d\mathbf{X}}{dS} = A_{\alpha\beta} \frac{d\hat{\theta}^\alpha}{dS} \frac{d\hat{\theta}^\beta}{dS} = 1 \quad \text{on } \partial P_0 \ , \tag{4.1.9}$$

where $A_{\alpha\beta}$ is the symmetric metric tensor associated with the surface $\mathcal{S}_0$ which is defined by

$$A_{\alpha\beta} = \mathbf{A}_\alpha \bullet \mathbf{A}_\beta = A_{\beta\alpha} \ . \tag{4.1.10}$$

It then follows that for such shells the cylindrical lateral surface ∂P^*_{L0} is defined by the union of the boundary curve ∂P_0 and the possible range of θ^3 such that

$$\theta^\alpha = \hat{\theta}^\alpha(S) \quad \text{and} \quad \xi_1 \le \theta^3 \le \xi_2 \quad \text{on} \quad \partial P^*_{L0} \ . \tag{4.1.11}$$

Next, it is of interest to determine the maximum extent of the finite shell-like region that can be described by the position vector (4.1.4). To answer this question, it is necessary to recall a fundamental property of the position vector. Specifically, the position vector is required to provide a one-to-one mapping between the convected coordinates θ^i which define a material point in the shell and the three-dimensional Euclidean space occupied by the shell. Mathematically, this means that the base vectors $\mathbf{G}_i$ associated with the representation (4.1.4) must be linearly independent

$$\mathbf{G}_i = \mathbf{X}^*{}_{,i} \ , \quad \mathbf{G}_\alpha = \mathbf{D}_\alpha + \theta^3 \, \mathbf{D}_{3,\alpha} \ , \quad \mathbf{G}_3 = \mathbf{D}_3 \ ,$$
$$G^{1/2} = \mathbf{G}_1 \times \mathbf{G}_2 \bullet \mathbf{G}_3 > 0 \ . \tag{4.1.12}$$

In order to write the expression for $G^{1/2}$ in a simplified form, it is convenient to introduce the tensor $\mathbf{\Lambda}$ associated with the derivatives of $\mathbf{D}_3$ such that

$$\mathbf{\Lambda} = \mathbf{D}_{3,\alpha} \otimes \mathbf{D}^\alpha \ , \quad \mathbf{D}_{3,\alpha} = \mathbf{\Lambda} \, \mathbf{D}_\alpha \ , \quad \mathbf{\Lambda} \, \mathbf{D}_3 = 0 \ , \tag{4.1.13}$$

where $\mathbf{D}^i$ are the reciprocal vectors of $\mathbf{D}_i$ defined by formulas of the type (2.1.6) and (2.1.10)

$$\mathbf{D}^1 = D^{-1/2} (\mathbf{D}_2 \times \mathbf{D}_3) \ , \quad \mathbf{D}^2 = D^{-1/2} (\mathbf{D}_3 \times \mathbf{D}_1) \ ,$$
$$\mathbf{D}^3 = D^{-1/2} (\mathbf{D}_1 \times \mathbf{D}_2) \ , \quad D^{1/2} = \mathbf{D}_1 \times \mathbf{D}_2 \bullet \mathbf{D}_3 > 0 \ ,$$
$$\mathbf{D}_i \bullet \mathbf{D}^j = \delta_i^j \ . \tag{4.1.14}$$

Using this definition of $\mathbf{\Lambda}$, the base vectors $\mathbf{G}_i$ can be written in the forms

$$\mathbf{G}_i = (\mathbf{I} + \theta^3 \, \mathbf{\Lambda}) \, \mathbf{D}_i \ . \tag{4.1.15}$$

Moreover, with the help of (A.7.16), (A.7.22) and (A.7.23), it can be shown that the reciprocal vectors $\mathbf{G}^i$ and the quantity $G^{1/2}$ are given by

$$\mathbf{G}^i = (\mathbf{I} + \theta^3 \, \mathbf{\Lambda})^{-T} \, \mathbf{D}^i \ , \quad G^{1/2} = D^{1/2} \det (\mathbf{I} + \theta^3 \, \mathbf{\Lambda}) \ ,$$
$$G^{1/2} = D^{1/2} \left[\ 1 + \theta^3(\mathbf{\Lambda} \bullet \mathbf{I}) + \frac{1}{2}(\theta^3)^2 \, \{(\mathbf{\Lambda} \bullet \mathbf{I})^2 - (\mathbf{\Lambda} \bullet \mathbf{\Lambda}^T)\} + (\theta^3)^3 \det \mathbf{\Lambda} \ \right] . \tag{4.1.16}$$

Furthermore, by introducing the zero vector $\mathbf{0}$ it follows that $\mathbf{\Lambda}$ can be expressed in the form

$$\mathbf{\Lambda} = \mathbf{D}_{3,\alpha} \otimes \mathbf{D}^{\alpha} + \mathbf{0} \otimes \mathbf{D}^3 \ , \tag{4.1.17}$$

which can be used to deduce that its determinant vanishes

$$\det \mathbf{\Lambda} = (\mathbf{D}_{3,1} \times \mathbf{D}_{3,2} \bullet \mathbf{0})\ (\mathbf{D}^1 \times \mathbf{D}^2 \bullet \mathbf{D}^3) = 0 \ . \tag{4.1.18}$$

It then follows that $G^{1/2}$ can be written as a quadratic function of θ^3

$$G^{1/2} = D^{1/2} \left[\ 1 + \theta^3(\mathbf{\Lambda} \bullet \mathbf{I}) + \frac{1}{2}(\theta^3)^2 \ \{(\mathbf{\Lambda} \bullet \mathbf{I})^2 - (\mathbf{\Lambda} \bullet \mathbf{\Lambda}^T)\}\ \right] \ . \tag{4.1.19}$$

Obviously, for $\mathbf{\Lambda} = 0$ and for small values of θ^3 when $\mathbf{\Lambda} \neq 0$, the expression for $G^{1/2}$ will remain positive. To determine the maximum possible range of θ^3 for which $G^{1/2}$ is positive, it is desirable to consider the following cases.

<u>Case I:</u> $(\mathbf{\Lambda} \bullet \mathbf{I})^2 = \mathbf{\Lambda} \bullet \mathbf{\Lambda}^T \neq 0$

For case I, the quadratic term in (4.1.19) vanishes and the bounds for θ^3 are determined by setting $G^{1/2}$ equal to zero to obtain

$$\begin{aligned} -\left(\frac{1}{\mathbf{\Lambda} \bullet \mathbf{I}}\right) = \xi_{\min} < \xi_1 < \xi_2 \quad &\text{for} \quad \mathbf{\Lambda} \bullet \mathbf{I} > 0 \ , \\ \xi_1 < \xi_2 < \xi_{\max} = -\left(\frac{1}{\mathbf{\Lambda} \bullet \mathbf{I}}\right) \quad &\text{for} \quad \mathbf{\Lambda} \bullet \mathbf{I} < 0 \ . \end{aligned} \tag{4.1.20}$$

For the remaining cases, the quadratic term does not vanish $[(\mathbf{\Lambda} \bullet \mathbf{I})^2 \neq \mathbf{\Lambda} \bullet \mathbf{\Lambda}^T]$ and it is convenient to determine the roots $\bar{\xi}$ and $\hat{\xi}$ of the quadratic equation obtained by setting $G^{1/2}$ equal to zero

$$\begin{aligned} \bar{\xi} &= \frac{-(\mathbf{\Lambda} \bullet \mathbf{I}) - [\,2\,\mathbf{\Lambda} \bullet \mathbf{\Lambda}^T - (\mathbf{\Lambda} \bullet \mathbf{I})^2\,]^{1/2}}{(\mathbf{\Lambda} \bullet \mathbf{I})^2 - \mathbf{\Lambda} \bullet \mathbf{\Lambda}^T} \ , \\ \hat{\xi} &= \frac{-(\mathbf{\Lambda} \bullet \mathbf{I}) + [\,2\,\mathbf{\Lambda} \bullet \mathbf{\Lambda}^T - (\mathbf{\Lambda} \bullet \mathbf{I})^2\,]^{1/2}}{(\mathbf{\Lambda} \bullet \mathbf{I})^2 - \mathbf{\Lambda} \bullet \mathbf{\Lambda}^T} \ . \end{aligned} \tag{4.1.21}$$

<u>Case II:</u> $[2\,\mathbf{\Lambda} \bullet \mathbf{\Lambda}^T - (\mathbf{\Lambda} \bullet \mathbf{I})^2] < 0$

For case II, the discriminant in (4.1.21) is negative so that $\bar{\xi}$ and $\hat{\xi}$ are imaginary numbers and $G^{1/2}$ will remain positive for any real value of θ^3

$$-\infty < \xi_1 < \xi_2 < \infty \quad \text{for} \quad 2\,\mathbf{\Lambda} \bullet \mathbf{\Lambda}^T - (\mathbf{\Lambda} \bullet \mathbf{I})^2 < 0 \ . \tag{4.1.22}$$

This means that the representation (4.1.4) of the position vector will be valid for a shell of arbitrary thickness.

<u>Case III:</u> $[2\,\mathbf{\Lambda} \bullet \mathbf{\Lambda}^T - (\mathbf{\Lambda} \bullet \mathbf{I})^2] > 0$

For case III, the discriminant in (4.1.21) is positive so that $\bar{\xi}$ and $\hat{\xi}$ are real numbers and the functions ξ_1 and ξ_2 defining the major surfaces are restricted to the range

$$\begin{gathered} \xi_{\min} = \mathrm{Min}\,(\bar{\xi}, \hat{\xi}) \ , \quad \xi_{\max} = \mathrm{Max}\,(\bar{\xi}, \hat{\xi}) \ , \\ \xi_{\min} < \xi_1 < \xi_2 < \xi_{\max} \quad \text{for} \quad \xi_{\min} < 0 \text{ and } \xi_{\max} > 0 \ , \\ \xi_1 < \xi_2 < \xi_{\min} \quad \text{for} \quad \xi_{\min} > 0 \ , \\ \xi_{\max} < \xi_1 < \xi_2 \quad \text{for} \quad \xi_{\max} < 0 \ . \end{gathered} \tag{4.1.23}$$

In particular, it is noted that the limits $\bar{\xi}$ and $\hat{\xi}$ depend explicitly on the particular choice of the surface $\mathcal{S}_0$ and on the specification of the vector $\mathbf{D}_3$.

In summary, a shell-like structure is considered to be "thin" if all material points in it can be mapped by the position vector (4.1.4), and if the magnitude of $(\theta^3 \boldsymbol{\Lambda})$ remains small. Moreover, the material part P_0^* of a shell in its reference configuration is defined by the three-dimensional position vector (4.1.4) with θ^α restricted to the region P_0 which is bounded by the smooth curve ∂P_0, and with the third coordinate θ^3 restricted to the range $[\xi_1,\xi_2]$. Also, the surface $\mathcal{S}_0$ associated with $\theta^3=0$ is called the reference surface of the shell and it is restricted to lie within the region P_0^* occupied by the shell so that

$$\xi_1 \le 0 \ , \quad 0 \le \xi_2 \ . \tag{4.1.24}$$

4.2 The Cosserat model of a shell

Within the context of the Cosserat theory a shell is modeled as a material surface $\mathcal{S}$ with some additional kinematic structure to provide limited information about deformation through the thickness of the shell. Specifically, with respect to the present configuration at time t, the material surface $\mathcal{S}$ is mapped by the two convected coordinates θ^α and the position vector (from a fixed origin) is denoted by

$$\mathbf{x} = \hat{\mathbf{x}}(\theta^\alpha,t) \ . \tag{4.2.1}$$

The tangent vectors $\mathbf{a}_\alpha$ in the θ^α directions, the unit normal vector $\mathbf{a}_3$ to the surface $\mathcal{S}$, and the reciprocal vectors $\mathbf{a}^i$ are given by

$$\mathbf{a}_\alpha(\theta^\alpha,t) = \frac{\partial \mathbf{x}}{\partial \theta^\alpha} = \mathbf{x},_\alpha \ , \quad \mathbf{a}_3(\theta^\alpha,t) = a^{-1/2}\,(\mathbf{a}_1 \times \mathbf{a}_2) \ ,$$
$$a^{1/2} = \mathbf{a}_1 \times \mathbf{a}_2 \bullet \mathbf{a}_3 \ , \quad d\sigma = a^{1/2}\, d\theta^1\, d\theta^2 \ ,$$
$$a^{1/2}\,\mathbf{a}^1 = \mathbf{a}_2 \times \mathbf{a}_3 \ , \quad a^{1/2}\,\mathbf{a}^2 = \mathbf{a}_3 \times \mathbf{a}_1 \ ,$$
$$\mathbf{a}^3 = \mathbf{a}_3 \ , \quad \mathbf{a}^i \bullet \mathbf{a}_j = \delta^i_j \ . \tag{4.2.2}$$

where $d\sigma$ is the element of area on the surface $\mathcal{S}$. In addition, the Cosserat theory endows the surface $\mathcal{S}$ with a director vector

$$\mathbf{d}_3 = \hat{\mathbf{d}}_3(\theta^\alpha,t) \ , \tag{4.2.3}$$

which is defined at each material point of the surface and is nowhere tangent to the surface. For convenience, a triad of linearly independent vectors $\mathbf{d}_i$ is introduced such that $\mathbf{d}_\alpha$ are equal to the tangent vectors $\mathbf{a}_\alpha$, and $\mathbf{d}_3$ always has a positive component normal to this surface

$$\mathbf{d}_\alpha(\theta^\alpha,t) = \mathbf{a}_\alpha \ , \quad d^{1/2} = \mathbf{d}_1 \times \mathbf{d}_2 \bullet \mathbf{d}_3 > 0 \ . \tag{4.2.4}$$

In a more general theory it is possible to introduce a finite set of N director vectors, but attention will be confined here to the simpler theory with a single director vector.

From a physical point of view, it is desirable to think of this director $\mathbf{d}_3$ as a vector that describes the deformation of a material fiber that is oriented through the thickness of the shell (see Fig. 4.2.1). In the general case of a single director, this material fiber is allowed to stretch along its length (i.e. the magnitude of $\mathbf{d}_3$ can change) and it is allowed to shear relative to the normal $\mathbf{a}_3$ of the surface $\mathcal{S}$.

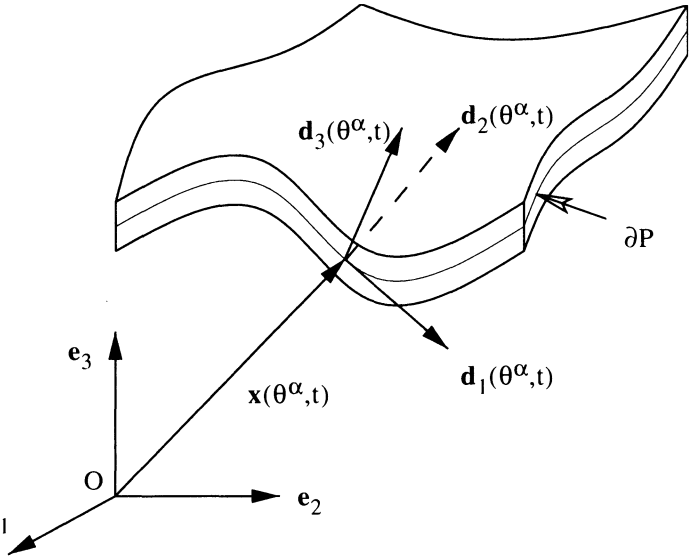

Fig. 4.2.1 The Cosserat model of a shell in its present configuration

The velocity vector $\mathbf{v}$ and the director velocities $\mathbf{w}_i$ derived from the functions (4.2.1), (4.2.3) and (4.2.4) are defined by

$$\mathbf{v} = \mathbf{v}(\theta^\alpha,t) = \dot{\mathbf{x}} \quad , \quad \mathbf{w}_i = \mathbf{w}_i(\theta^\alpha,t) = \dot{\mathbf{d}}_i \ , \tag{4.2.5}$$

where the superposed dot $(\dot{\ })$ denotes material time differentiation holding θ^α fixed. Furthermore, due to the fact that θ^α are considered to be convected Lagrangian coordinates, it follows that the director velocities $\mathbf{w}_\alpha$ are related to derivatives of $\mathbf{v}$ by the expressions

$$\mathbf{w}_\alpha = \mathbf{v}_{,\alpha} \ . \tag{4.2.6}$$

Motivated by the representation (4.1.4), it is natural to consider an associated kinematic assumption that the position vector $\mathbf{x}^*(\theta^i,t)$ of material points in the shell in its present configuration can be represented in the form

$$\mathbf{x}^*(\theta^i,t) = \mathbf{x}(\theta^\alpha,t) + \theta^3\, \mathbf{d}_3(\theta^\alpha,t) \ . \tag{4.2.7}$$

In contrast with the representation (4.1.4), which is always valid for a shell-like structure in its reference configuration, the representation (4.2.7) restricts material line elements through the thickness (θ^3) of the shell to remain straight. This kinematic assumption (4.2.7) will be used to motivate forms for certain quantities that appear in the balance laws of the Cosserat theory and it can be used to provide physical interpretation of results of the theory. However, it will be seen that, strictly speaking, the Cosserat theory is used to determine the vectors $\mathbf{x}$ and $\mathbf{d}_3$ which depend on only two space coordinates θ^α and time. Consequently, within the context of the Cosserat theory there is not necessarily a direct dependence on the assumption (4.2.7). Moreover, it will be seen later that the constitutive equations of the Cosserat theory do not enforce the kinematic assumption (4.2.7) to be valid pointwise through the shell's thickness.

In order to expand on physical interpretations based on the assumption (4.2.7), it is desirable to consider the base vectors $\mathbf{g}_i$ and the reciprocal vectors $\mathbf{d}^i$ defined by

$$\mathbf{g}_i = \mathbf{x}^*{}_{,i} \ , \quad \mathbf{g}_\alpha = \mathbf{d}_\alpha + \theta^3 \, \mathbf{d}_{3,\alpha} \ , \quad \mathbf{g}_3 = \mathbf{d}_3 \ ,$$
$$g^{1/2} = \mathbf{g}_1 \times \mathbf{g}_2 \cdot \mathbf{g}_3 \ , \quad \mathbf{d}^i \cdot \mathbf{d}_j = \delta^i_j \ . \tag{4.2.8}$$

Now, following the work of Naghdi (1982) and Naghdi and Rubin (1995), it is convenient to introduce the nonsingular second order tensor $\mathbf{F}$ such that

$$\mathbf{F} = \mathbf{d}_i \otimes \mathbf{D}^i \ , \quad \mathbf{F}^{-1} = \mathbf{D}_i \otimes \mathbf{d}^i \ , \quad \mathbf{d}_i = \mathbf{F} \, \mathbf{D}_i \ ,$$
$$\det \mathbf{F} = d^{1/2} \, D^{-1/2} \ , \tag{4.2.9}$$

and the second order tensor $\boldsymbol{\lambda}$ defined such that

$$\boldsymbol{\lambda} = \mathbf{F}^{-1}(\mathbf{d}_{3,\alpha} \otimes \mathbf{D}^\alpha) \ , \quad \mathbf{d}_{3,\alpha} = (\mathbf{F} \, \boldsymbol{\lambda}) \, \mathbf{D}_\alpha \ , \quad \boldsymbol{\lambda} \, \mathbf{D}_3 = 0 \ . \tag{4.2.10}$$

Thus, the base vectors $\mathbf{g}_i$ can be written in the alternative forms

$$\mathbf{g}_i = \mathbf{F}(\mathbf{I} + \theta^3 \, \boldsymbol{\lambda}) \, \mathbf{D}_i \ . \tag{4.2.11}$$

Also, the quantity $g^{1/2}$ becomes

$$g^{1/2} = [\mathbf{F}(\mathbf{I} + \theta^3 \, \boldsymbol{\lambda}) \, \mathbf{D}_1] \times [\mathbf{F}(\mathbf{I} + \theta^3 \, \boldsymbol{\lambda}) \, \mathbf{D}_2] \cdot [\mathbf{F}(\mathbf{I} + \theta^3 \, \boldsymbol{\lambda}) \, \mathbf{D}_3] \ ,$$
$$g^{1/2} = \det [\mathbf{F}(\mathbf{I} + \theta^3 \, \boldsymbol{\lambda})] \, D^{1/2} \left\{ [\mathbf{F}(\mathbf{I} + \theta^3 \, \boldsymbol{\lambda})]^{-T} \, \mathbf{D}^3 \right\} \cdot \left\{ \mathbf{F}(\mathbf{I} + \theta^3 \, \boldsymbol{\lambda}) \, \mathbf{D}_3 \right\} \ ,$$
$$g^{1/2} = d^{1/2} \det (\mathbf{I} + \theta^3 \, \boldsymbol{\lambda}) \ . \tag{4.2.12}$$

Next, with the help of (4.1.16) and the definition (2.3.8) of the three-dimensional deformation gradient $\mathbf{F}^*$, it follows that

$$\mathbf{F}^*(\theta^i,t) = \mathbf{F}(\mathbf{I} + \theta^3 \, \boldsymbol{\lambda}) \, (\mathbf{I} + \theta^3 \, \boldsymbol{\Lambda})^{-1} \ . \tag{4.2.13}$$

This expression shows that the tensor $\mathbf{F}$ is the value of $\mathbf{F}^*$ on the reference surface (θ^3=0) of the shell. Moreover, it will presently be shown that within the context of the kinematic assumption (4.2.7), the necessary and sufficient condition for the associated three-dimensional deformation to be homogeneous is

$$\mathbf{F}(\theta^\alpha,t) = \bar{\mathbf{F}}(t) \ , \tag{4.2.14}$$

where $\bar{\mathbf{F}}(t)$ is an arbitrary nonsingular tensor function of time only whose determinant is positive. To prove that this is a sufficient condition, use is made of the expressions $(4.1.13)_1$, $(4.2.9)_3$, $(4.2.10)_1$ and (4.2.14) to deduce that

$$\mathbf{d}_{3,\alpha} = (\mathbf{F}\mathbf{D}_3)_{,\alpha} = \mathbf{F}\mathbf{D}_{3,\alpha} \ , \quad \boldsymbol{\lambda} = \boldsymbol{\Lambda} \ . \tag{4.2.15}$$

Now, the expressions (4.2.14) and $(4.2.15)_2$ can be substituted into (4.2.13) to obtain

$$\mathbf{F}^*(\theta^i,t) = \bar{\mathbf{F}}(t) \ , \tag{4.2.16}$$

which proves that the three-dimensional deformation is homogeneous. On the other hand, if (4.2.16) is presumed, then (4.2.13) becomes

$$\bar{\mathbf{F}}(t) = \mathbf{F}(\mathbf{I} + \theta^3 \, \boldsymbol{\lambda}) \, (\mathbf{I} + \theta^3 \, \boldsymbol{\Lambda})^{-1} \ . \tag{4.2.17}$$

However, since the left-hand side of (4.2.17) is independent of θ^3, it follows by setting θ^3 equal to zero that $\mathbf{F}(\theta^\alpha,t)$ is independent of the coordinates θ^α

$$\mathbf{F}(\theta^\alpha,t) = \bar{\mathbf{F}}(t) \ , \tag{4.2.18}$$

which completes the proof. In summary, the associated three-dimensional deformation will be homogeneous if and only if $\mathbf{F}$ is independent of the coordinates θ^α (4.2.18).

Next, using the definition (3.1.6) of the three-dimensional deformation gradient $\mathbf{F}^*$, it can be shown by integration that for homogeneous deformations

$$\mathbf{x}^* = \mathbf{F}(t)\ \mathbf{X}^* + \mathbf{c}(t) \ , \tag{4.2.19}$$

where $\mathbf{c}(t)$ is an arbitrary vector function of time only representing translation of the shell. Then, substitution of the kinematic expression (4.1.4) for $\mathbf{X}^*$ into (4.2.19) and use of $(4.2.9)_3$ yields

$$\mathbf{x}^* = \left[\mathbf{F}(t)\ \mathbf{X}(\theta^\alpha) + \mathbf{c}(t)\right] + \theta^3\ \mathbf{F}(t)\ \mathbf{D}_3(\theta^\alpha) \ ,$$

$$\mathbf{x}^*(\theta^i,t) = \mathbf{x}(\theta^\alpha,t) + \theta^3\ \mathbf{d}_3(\theta^\alpha,t) \ , \tag{4.2.20}$$

which shows that the kinematic expression (4.2.7) is exact for homogeneous deformations.

4.3 Derivation of the balance laws from the three-dimensional theory

The global forms of the balance laws of the Cosserat theory of shells are similar to those of the three-dimensional theory in the sense that they include the notions of conservation of mass and balances of linear and angular momentum. Moreover, these equations are used to determine the current values of a mass density ρ and the position vector $\mathbf{x}$ of points on the surface $\mathcal{S}$ of the shell. Also, the balance of angular momentum places restrictions on the constitutive equations of the shell theory that are similar in nature to the restrictions (3.2.32) associated with the three-dimensional theory. However, in contrast with the three-dimensional theory, the Cosserat theory of shells introduces the additional kinematic quantity $\mathbf{d}_3$ at each point of the surface $\mathcal{S}$ of the shell which also must be determined by a balance law. Consequently, the Cosserat theory of shells requires an additional balance law called the balance of director momentum.

In this section it will be shown that the balance laws of the Cosserat theory can be developed by using the kinematic assumption (4.2.7) and the balance laws of the three-dimensional theory. Here, attention is focused on the development of the global forms of the balance laws. However, a simpler derivation of the local equations is provided in section 4.26 for convenience. It is noted that the three-dimensional region P_0^* (with boundary ∂P_0^*) of the shell-like structure in its reference configuration is mapped to the region P^* (with boundary ∂P^*) in the present configuration. Moreover, the major surfaces ∂P_{10}^* and ∂P_{20}^*, and the lateral surface ∂P_{L0}^* are mapped to the surfaces ∂P_1^*, ∂P_2^*, and ∂P_L^*, respectively, in the present configuration. Consequently, the boundary ∂P^* is the union of the boundaries ∂P_1^*, ∂P_2^*, and ∂P_L^*

$$\partial P^* = \partial P_1^* \cup \partial P_2^* \cup \partial P_L^* \ . \tag{4.3.1}$$

Also, since θ^i are convected coordinates, it follows from (4.1.5) and (4.1.7), (4.1.8) and (4.1.11) that

$$\theta^3 = \xi_1(\theta^\alpha) \quad \text{on } \partial P_1^* \ , \quad \theta^3 = \xi_2(\theta^\alpha) \quad \text{on } \partial P_2^* \ ,$$
$$f(\theta^\alpha) = 0 \quad \text{and} \quad \xi_1 \le \theta^3 \le \xi_2 \quad \text{on } \partial P_L^* \ , \tag{4.3.2}$$

where the function $f(\theta^\alpha)$ in $(4.3.2)_3$ defines the boundary curve ∂P of the surface $\mathcal{S}$ of the shell

$$f(\theta^\alpha) = 0 \quad \text{on } \partial P \ . \tag{4.3.3}$$

Now, with the help of the expression (3.2.5) for the element of volume dv^*, and the representation (4.3.2) of the region P^*, the total mass $(3.2.26)_1$ in P^* can be written as

$$\int_{P^*} \rho^* \, dv^* = \int_P \Big[\int_{\xi_1}^{\xi_2} \rho^* g^{1/2} \, d\theta^3 \Big] \, d\theta^1 \, d\theta^2 = \int_P \rho \, d\sigma \ , \tag{4.3.4}$$

where the expression $(4.2.2)_4$ for the element of area $d\sigma$ on $\mathcal{S}$ has been used. Also, the mass density ρ (mass per unit area $d\sigma$) has been defined so that it represents the integrated effect of the mass density through the thickness of the shell

$$m = \rho \, a^{1/2} = \int_{\xi_1}^{\xi_2} \rho^* g^{1/2} \, d\theta^3 = \int_{\xi_1}^{\xi_2} m^* \, d\theta^3 \ . \tag{4.3.5}$$

It then follows from (3.2.1) and (4.3.4), that the global form of conservation of mass of the Cosserat shell becomes

$$\frac{d}{dt} \int_P \rho \, d\sigma = 0 \ . \tag{4.3.6}$$

Also, it is noted that the units of m depend on the specification of the convected coordinates θ^α. However, m will have the units of mass per unit reference area if each of θ^α has the units of length.

Next, with the help of the kinematic assumption (4.2.7) and the definitions (4.2.5) for the velocity $\mathbf{v}$ and the director velocity $\mathbf{w}_3$, the three-dimensional balance of linear momentum (3.2.2) applied to the shell-like region P^* becomes

$$\frac{d}{dt} \int_P \Big[\int_{\xi_1}^{\xi_2} m^* \, (\mathbf{v} + \theta^3 \, \mathbf{w}_3) \, d\theta^3 \Big] \, d\theta^1 \, d\theta^2$$
$$= \int_P \Big[\int_{\xi_1}^{\xi_2} m^* \, \mathbf{b}^* \, d\theta^3 \Big] \, d\theta^1 \, d\theta^2$$
$$+ \int_{\partial P_1^*} \mathbf{t}^* \, da^* + \int_{\partial P_2^*} \mathbf{t}^* \, da^* + \int_{\partial P_L^*} \mathbf{t}^* \, da^* \ . \tag{4.3.7}$$

The main objective here is to rewrite the integrals in (4.3.7) as integrals over the region P and the boundary ∂P associated with the shell surface $\mathcal{S}$. To this end, it is noted that $\mathbf{v}$ and $\mathbf{w}_3$ are independent of the coordinate θ^3 so that it is convenient to define the director inertia coefficient y^3 by the expression

$$m \, y^3 = \int_{\xi_1}^{\xi_2} m^* \, \theta^3 \, d\theta^3 \ . \tag{4.3.8}$$

Then, the linear momentum in the part P^* of the shell can be written as

$$\int_{P^*} \rho^* \, \mathbf{v}^* \, dv^* = \int_P \rho \, (\mathbf{v} + y^3 \, \mathbf{w}_3) \, d\sigma \ . \tag{4.3.9}$$

Also, it is convenient to define the body force $\mathbf{b}_b$ per unit mass by the expression

$$m\,\mathbf{b}_b = \int_{\xi_1}^{\xi_2} m^* \,\mathbf{b}^*\, d\theta^3 \ , \tag{4.3.10}$$

so that the total body force applied to the part P^* of the shell can be rewritten as

$$\int_{P^*} \rho^*\, \mathbf{b}^*\, dv^* = \int_P \rho\, \mathbf{b}_b\, d\sigma \,. \tag{4.3.11}$$

The integrals in (4.3.7) over the major surfaces ∂P_1^* and ∂P_2^* of the shell can be expressed as integrals over the region P by using (4.3.2) to develop expressions for the unit outward normal $\mathbf{n}^*$ and the element of area da^* on these surfaces. In particular, on the major surface ∂P_2^* it follows that

On ∂P_2^*:

$$\mathbf{x}^* = \mathbf{x}^*(\theta^\alpha,\xi_2(\theta^\alpha),t) \ ,$$

$$\mathbf{n}^*\, da^* = \mathbf{x}^*{}_{,1} d\theta^1 \times \mathbf{x}^*{}_{,2} d\theta^2 = (\mathbf{g}_1+\xi_{2,1}\,\mathbf{g}_3)\times(\mathbf{g}_2+\xi_{2,2}\,\mathbf{g}_3)d\theta^1 d\theta^2$$

$$\mathbf{n}^*\, da^* = g^{1/2}\,(\mathbf{g}^3 - \xi_{2,\alpha}\,\mathbf{g}^\alpha)\, d\theta^1\, d\theta^2 \ , \quad da^* = g^{1/2}\,\alpha(\xi_2)\, d\theta^1\, d\theta^2 \ ,$$

$$\mathbf{n}^* = \frac{(\mathbf{g}^3 - \xi_{2,\alpha}\,\mathbf{g}^\alpha)}{\alpha(\xi_2)} \ , \quad \alpha(\xi_2) = |\, \mathbf{g}^3 - \xi_{2,\alpha}\,\mathbf{g}^\alpha\, | \ , \tag{4.3.12}$$

where all quantities in (4.3.12) are evaluated on the major surface $\theta^3=\xi_2(\theta^\alpha)$. Also, by recognizing that the outward normal to the major surface ∂P_1^* has a positive component in the negative $\mathbf{g}^3$ direction, it follows that

On ∂P_1^*:

$$\mathbf{x}^* = \mathbf{x}^*(\theta^\alpha,\xi_1(\theta^\alpha),t)$$

$$\mathbf{n}^*\, da^* = -\mathbf{x}^*{}_{,1} d\theta^1 \times \mathbf{x}^*{}_{,2} d\theta^2 = -(\mathbf{g}_1+\xi_{1,1}\,\mathbf{g}_3)\times(\mathbf{g}_2+\xi_{1,2}\,\mathbf{g}_3)d\theta^1 d\theta^2$$

$$\mathbf{n}^*\, da^* = -\, g^{1/2}\,(\mathbf{g}^3 - \xi_{1,\alpha}\,\mathbf{g}^\alpha)\, d\theta^1\, d\theta^2 \ , \quad da^* = g^{1/2}\,\alpha(\xi_1)\, d\theta^1\, d\theta^2 \ ,$$

$$\mathbf{n}^* = -\,\frac{(\mathbf{g}^3 - \xi_{1,\alpha}\,\mathbf{g}^\alpha)}{\alpha(\xi_1)} \ , \quad \alpha(\xi_1) = |\, \mathbf{g}^3 - \xi_{1,\alpha}\,\mathbf{g}^\alpha\, | \ , \tag{4.3.13}$$

where all quantities in (4.3.13) are evaluated on the major surface $\theta^3=\xi_1(\theta^\alpha)$. Now, using these expressions, the total force applied to the major surfaces of the shell can be expressed as

$$\int_{\partial P_1^*} \mathbf{t}^*\, da^* + \int_{\partial P_2^*} \mathbf{t}^*\, da^* = \int_P \rho\, \mathbf{b}_c\, d\sigma \ , \tag{4.3.14}$$

where $\mathbf{b}_c$ is the contact force per unit mass applied to the major surfaces defined by

$$m\,\mathbf{b}_c = \big[g^{1/2}\,\mathbf{T}^*(\mathbf{g}^3-\xi_{2,\alpha}\mathbf{g}^\alpha)\big]\Big|_{\theta^3=\xi_2} - \big[g^{1/2}\,\mathbf{T}^*(\mathbf{g}^3-\xi_{1,\alpha}\mathbf{g}^\alpha)\big]\Big|_{\theta^3=\xi_1} \ ,$$

$$m\,\mathbf{b}_c = \big[\mathbf{t}^{*3}-(\xi_{2,\alpha})\mathbf{t}^{*\alpha}\big]\Big|_{\theta^3=\xi_2} - \big[\mathbf{t}^{*3}-(\xi_{1,\alpha})\mathbf{t}^{*\alpha}\big]\Big|_{\theta^3=\xi_1} \ ,$$

$$m\,\mathbf{b}_c = \big[g^{1/2}\alpha(\xi_2)\,\mathbf{t}^*\,\big]\Big|_{\theta^3=\xi_2} + \big[g^{1/2}\alpha(\xi_1)\,\mathbf{t}^*\,\big]\Big|_{\theta^3=\xi_1} \ . \tag{4.3.15}$$

In the above, (3.2.34) has been used and the stress vector $\mathbf{t}^*$ on the major surfaces ($\theta^3=\xi_1$ and $\theta^3=\xi$) is defined by (3.2.3) in terms of the unit outward normal to each of these surfaces. Consequently, the term associated with the major surface ($\theta^3=\xi_1$) has a negative sign in $(4.3.15)_1$ and it has a positive sign in $(4.3.15)_2$.

To express the integral in (4.3.7) over the lateral surface ∂P_L^* as an integral over the boundary curve ∂P, it is first observed that with the help of (4.1.8) and (4.3.2), the position vector to points on ∂P can be expressed in the form

$$\mathbf{x} = \hat{\mathbf{x}}(\theta^\alpha(S),t) \quad \text{on } \partial P \ , \tag{4.3.16}$$

where S is the arclength parameter defined relative to the boundary curve ∂P_0 in the reference configuration (4.1.9). It then follows that the arclength parameter s of the boundary curve ∂P is defined such that the tangent vector to this curve is a unit vector in the present configuration

$$\frac{d\mathbf{x}}{ds} = \frac{d\hat{\theta}^\alpha}{dS}\frac{\partial\tilde{S}}{\partial s}\,\mathbf{a}_\alpha \ , \quad \frac{d\mathbf{x}}{ds}\bullet\frac{d\mathbf{x}}{ds} = a_{\alpha\beta}\frac{d\hat{\theta}^\alpha}{dS}\frac{d\hat{\theta}^\beta}{dS}\left(\frac{\partial\tilde{S}}{\partial s}\right)^2 = 1 \ , \tag{4.3.17}$$

where $a_{\alpha\beta}$ is the symmetric metric tensor associated with the surface $\mathcal{S}$, which is defined by

$$a_{\alpha\beta} = \mathbf{a}_\alpha \bullet \mathbf{a}_\beta = a_{\beta\alpha} \ . \tag{4.3.18}$$

Also, the arclength S is a function of (s,t) of the form

$$S = \tilde{S}(s,t) \ , \tag{4.3.19}$$

which is determined by the condition $(4.3.17)_2$. This means that on the boundary ∂P the values of the coordinates θ^α can be written as functions of s and t such that

$$\theta^\alpha = \tilde{\theta}^\alpha(s,t) \ , \quad f(\tilde{\theta}^\alpha(s,t)) = 0 \quad \text{on } \partial P \ . \tag{4.3.20}$$

Returning to the analysis of the lateral surface ∂P_L^*, it follows that

On ∂P_L^*: $\quad \mathbf{x}^* = \mathbf{x}^*(\tilde{\theta}^\alpha(s,t),\theta^3,t)$

$$\mathbf{n}^*\, da^* = \frac{\partial\mathbf{x}^*}{\partial s}\, ds \times \mathbf{x}^*{}_{,3}\, d\theta^3 = \frac{\partial\tilde{\theta}^\alpha}{\partial s}\,\mathbf{g}_\alpha \times \mathbf{g}_3\, ds\, d\theta^3$$

$$\mathbf{n}^*\, da^* = g^{1/2}\left[\frac{\partial\tilde{\theta}^2}{\partial s}\,\mathbf{g}^1 - \frac{\partial\tilde{\theta}^1}{\partial s}\,\mathbf{g}^2\right] d\theta^3\, ds \ . \tag{4.3.21}$$

Thus, with the help of (3.2.34), the force applied to the lateral surface can be written as

$$\int_{\partial P_L^*} \mathbf{t}^*\, da^* = \int_{\partial P}\left[\frac{\partial\tilde{\theta}^2}{\partial s}\int_{\xi_1}^{\xi_2}\mathbf{t}^{*1}\, d\theta^3 - \frac{\partial\tilde{\theta}^1}{\partial s}\int_{\xi_1}^{\xi_2}\mathbf{t}^{*2}\, d\theta^3\right] ds \ . \tag{4.3.22}$$

However, the position vector on the boundary curve ∂P and the unit outward normal $\mathbf{n}$ to ∂P which is tangent to the surface $\mathcal{S}$ are given by

$$\mathbf{x} = \mathbf{x}(\tilde{\theta}^\alpha(s,t),t) \ ,$$

$$\mathbf{n}\, ds = \frac{\partial\mathbf{x}}{\partial s}\, ds \times \mathbf{a}_3 = \frac{\partial\tilde{\theta}^\alpha}{\partial s}\,\mathbf{a}_\alpha \times \mathbf{a}_3\, ds = a^{1/2}\left[\frac{\partial\tilde{\theta}^2}{\partial s}\,\mathbf{a}^1 - \frac{\partial\tilde{\theta}^1}{\partial s}\,\mathbf{a}^2\right] ds \ ,$$

$$\mathbf{n} = n_i\,\mathbf{d}^i \ , \quad n_1 = \mathbf{n}\bullet\mathbf{d}_1 = a^{1/2}\frac{\partial\tilde{\theta}^2}{\partial s} \ , \quad n_2 = \mathbf{n}\bullet\mathbf{d}_2 = -\,a^{1/2}\frac{\partial\tilde{\theta}^1}{\partial s} \ ,$$

$$n_3 = \mathbf{n}\bullet\mathbf{d}_3 = a^{1/2}\left[\frac{\partial\tilde{\theta}^2}{\partial s}\,\mathbf{a}^1 - \frac{\partial\tilde{\theta}^1}{\partial s}\,\mathbf{a}^2\right]\bullet\mathbf{d}_3 \ . \tag{4.3.23}$$

Now, inspection of (3.2.34), (4.3.22) and (4.3.23) suggests that the vectors $\mathbf{t}^\alpha$ and the second order tensor $\mathbf{N}$ be defined by the expressions

$$\mathbf{t}^\alpha = a^{1/2}\, \mathbf{N}\, \mathbf{d}^\alpha = \int_{\xi_1}^{\xi_2} \mathbf{t}^{*\alpha}\, d\theta^3 \ ,$$

$$\mathbf{N} = a^{-1/2}\, \mathbf{t}^\alpha \otimes \mathbf{d}_\alpha \ , \quad \mathbf{N}\, \mathbf{a}_3 = 0 \ , \quad \mathbf{N}\, \mathbf{d}^3 = 0 \ . \tag{4.3.24}$$

Then, the resultant force applied to the lateral boundary of the shell can be expressed as

$$\int_{\partial P_L^*} \mathbf{t}^*\, da^* = \int_{\partial P} \mathbf{t}\, ds \ , \tag{4.3.25}$$

where $\mathbf{t}$ is the contact force per unit arclength ds applied to the boundary curve ∂P and $\mathbf{t}$ is related to the second order tensor $\mathbf{N}$ and the unit outward normal $\mathbf{n}$ by the expression

$$\mathbf{t} = \mathbf{N}\, \mathbf{n} \ . \tag{4.3.26}$$

Now, with the help of (3.2.2) and the expressions (4.3.9), (4.3.11), (4.3.14), (4.3.25), the balance of linear momentum associated with the Cosserat theory can be written in the global form

$$\frac{d}{dt}\int_P \rho\, (\mathbf{v} + y^3\, \mathbf{w}_3)\, d\sigma = \int_P \rho\, \mathbf{b}\, d\sigma + \int_{\partial P} \mathbf{t}\, ds \ , \tag{4.3.27}$$

where the external force $\mathbf{b}$ per unit mass applied to the shell is defined by

$$\mathbf{b} = \mathbf{b}_b + \mathbf{b}_c \ . \tag{4.3.28}$$

In this regard, it should be emphasized that the external force $\mathbf{b}$ is due to two different physical sources. One part $\mathbf{b}_b$ is due to the three-dimensional body force applied to the shell, and the other part $\mathbf{b}_c$ is due to contact forces applied to the major surfaces of the shell.

Before developing the balance of angular momentum, it is convenient to develop the balance of director momentum. Here, the balance of director momentum will be developed as an averaged first moment of the balance of linear momentum with respect to the thickness coordinate θ^3. Specifically, the averaged form (3.6.3) of the balance of linear momentum is applied to the shell-like region P^*, with the weighting function ϕ taken equal to the coordinate θ^3 so that with the help of the kinematic assumption (4.2.7), the expression (3.6.3) yields

$$\frac{d}{dt}\int_P \Big[\int_{\xi_1}^{\xi_2} m^*\, (\theta^3 \mathbf{v} + \theta^3\, \theta^3\, \mathbf{w}_3)\, d\theta^3 \Big]\, d\theta^1\, d\theta^2$$

$$= \int_P \Big[\int_{\xi_1}^{\xi_2} \{ m^*\, \theta^3 \mathbf{b}^* - \mathbf{t}^{*3} \}\, d\theta^3 \Big]\, d\theta^1\, d\theta^2$$

$$+ \int_{\partial P_1^*} \theta^3\, \mathbf{t}^*\, da^* + \int_{\partial P_2^*} \theta^3\, \mathbf{t}^*\, da^* + \int_{\partial P_L^*} \theta^3\, \mathbf{t}^*\, da^* \ . \tag{4.3.29}$$

Inspection of (4.3.29) indicates that it is convenient to introduce an additional director inertia coefficient y^{33} which is defined in terms of the second moment of the mass density through the thickness of the shell by the equation

$$m\, y^{33} = \int_{\xi_1}^{\xi_2} m^*\, \theta^3\, \theta^3\, d\theta^3 \ . \tag{4.3.30}$$

Consequently, the first moment of linear momentum can be expressed as

$$\int_{P^*} \rho^* \mathbf{v}^* \theta^3 \, dv^* = \int_P \rho \, (y^3 \, \mathbf{v} + y^{33} \, \mathbf{w}_3) \, d\sigma \ . \tag{4.3.31}$$

Also, it is convenient to define the external director couple $\mathbf{b}_b^3$ as the first moment of the body force per unit mass by the expression

$$m \, \mathbf{b}_b^3 = \int_{\xi_1}^{\xi_2} m^* \, \mathbf{b}^* \, \theta^3 \, d\theta^3 \ , \tag{4.3.32}$$

so that the first moment of the total body force applied to the part P^* of the shell can be rewritten as

$$\int_{P^*} \rho^* \, \mathbf{b}^* \, \theta^3 \, dv^* = \int_P \rho \, \mathbf{b}_b^3 \, d\sigma \ . \tag{4.3.33}$$

Furthermore, comparison of (4.3.7) and (4.3.29) indicates the presence of an extra term in (4.3.29) which is related to the integrated effect of the quantity $\mathbf{t}^{*3}$. Consequently, it is convenient to define the intrinsic director couple $\mathbf{t}^3$ per unit present area $d\sigma$ such that

$$\mathbf{t}^3 = \int_{\xi_1}^{\xi_2} \mathbf{t}^{*3} \, d\theta^3 \ , \quad \int_P \Big[\int_{\xi_1}^{\xi_2} \mathbf{t}^{*3} \, d\theta^3 \Big] \, d\theta^1 \, d\theta^2 = \int_P a^{-1/2} \, \mathbf{t}^3 \, d\sigma \ . \tag{4.3.34}$$

Next, attention is focused on the effect of the traction vectors on the major surfaces. In particular, with the help of the expressions (4.3.12) and (4.3.13), it follows that the first moment of the total force applied to the major surfaces can be written in the form

$$\int_{\partial P_1^*} \theta^3 \, \mathbf{t}^* \, da^* + \int_{\partial P_2^*} \theta^3 \, \mathbf{t}^* \, da^* = \int_P \rho \, \mathbf{b}_c^3 \, d\sigma \ , \tag{4.3.35}$$

where $\mathbf{b}_c^3$ is the first moment of the contact force per unit mass applied to the major surfaces defined by

$$m \, \mathbf{b}_c^3 = \big[g^{1/2} \xi_2 \mathbf{T}^* (\mathbf{g}^3 - \xi_{2,\alpha} \, \mathbf{g}^\alpha) \big] \Big|_{\theta^3 = \xi_2} - \big[g^{1/2} \xi_1 \mathbf{T}^* (\mathbf{g}^3 - \xi_{1,\alpha} \, \mathbf{g}^\alpha) \big] \Big|_{\theta^3 = \xi_1} \ ,$$

$$m \, \mathbf{b}_c^3 = \big[\xi_2 (\mathbf{t}^{*3} - \xi_{2,\alpha} \mathbf{t}^{*\alpha}) \big] \Big|_{\theta^3 = \xi_2} - \big[\xi_1 (\mathbf{t}^{*3} - \xi_{1,\alpha} \mathbf{t}^{*\alpha}) \big] \Big|_{\theta^3 = \xi_1} \ ,$$

$$m \, \mathbf{b}_c^3 = \big[g^{1/2} \alpha(\xi_2) \xi_2 \mathbf{t}^* \big] \Big|_{\theta^3 = \xi_2} + \big[g^{1/2} \alpha(\xi_1) \xi_1 \mathbf{t}^* \big] \Big|_{\theta^3 = \xi_1} \ . \tag{4.3.36}$$

Now, with the help of (3.2.34) and (4.3.21), the first moment of the traction vector on the lateral surface can be expressed in the form

$$\int_{\partial P_L^*} \theta^3 \, \mathbf{t}^* \, da^* = \int_{\partial P} \Big[\frac{\partial \tilde{\theta}^2}{\partial s} \int_{\xi_1}^{\xi_2} \theta^3 \, \mathbf{t}^{*1} \, d\theta^3 - \frac{\partial \tilde{\theta}^1}{\partial s} \int_{\xi_1}^{\xi_2} \theta^3 \, \mathbf{t}^{*2} \, d\theta^3 \Big] \, ds \ . \tag{4.3.37}$$

Thus, in view of the expression (4.3.23) for the unit outward normal $\mathbf{n}$ to the boundary curve ∂P, it is convenient to define the vectors $\mathbf{m}^\alpha$ and the second order tensor $\mathbf{M}$ by the expressions

$$\mathbf{m}^\alpha = a^{1/2} \, \mathbf{M} \, \mathbf{d}^\alpha = \int_{\xi_1}^{\xi_2} \theta^3 \, \mathbf{t}^{*\alpha} \, d\theta^3 \ ,$$

$$\mathbf{M} = a^{-1/2} \, \mathbf{m}^\alpha \otimes \mathbf{d}_\alpha \ , \quad \mathbf{M} \, \mathbf{a}_3 = 0 \ , \quad \mathbf{M} \, \mathbf{d}^3 = 0 \ . \tag{4.3.38}$$

Then, the contact moment applied to the lateral boundary of the shell can be expressed as

$$\int_{\partial P_L^*} \theta^3 \, \mathbf{t}^* \, da^* = \int_{\partial P} \mathbf{m}^3 \, ds \ , \tag{4.3.39}$$

where $\mathbf{m}^3$ is the contact moment per unit arclength ds applied to the boundary curve ∂P, and $\mathbf{m}^3$ is related to the tensor $\mathbf{M}$ and the unit outward normal $\mathbf{n}$ by the expression

$$\mathbf{m}^3 = \mathbf{M} \, \mathbf{n} \ . \tag{4.3.40}$$

Using the expressions (4.3.31), (4.3.33)-(4.3.35) and (4.3.39), the balance of director momentum (4.3.29) can be written in the simpler form

$$\frac{d}{dt}\int_P \rho \, (y^3 \, \mathbf{v} + y^{33} \, \mathbf{w}_3) \, d\sigma = \int_P [\rho \, \mathbf{b}^3 - a^{-1/2} \, \mathbf{t}^3 \,] \, d\sigma + \int_{\partial P} \mathbf{m}^3 \, ds \ , \tag{4.3.41}$$

where the external director couple $\mathbf{b}^3$ per unit mass applied to the shell is the sum of the part $\mathbf{b}_b^3$ due to body force and the part $\mathbf{b}_c^3$ due to contact forces on the major surfaces

$$\mathbf{b}^3 = \mathbf{b}_b^3 + \mathbf{b}_c^3 \ . \tag{4.3.42}$$

Here, it is of interest to note that the theoretical structure of the balance of director momentum (4.3.41) differs from that of the balance of linear momentum (4.3.27) due to the presence of the intrinsic director couple $\mathbf{t}^3$. In this sense, the averaged form (3.6.3) of the three-dimensional balance of linear momentum provided important theoretical guidance for motivating the form (4.3.41) of director momentum.

Returning to the analysis of angular momentum, it follows that when the three-dimensional form (3.2.4) for the balance of angular momentum is applied to the shell-like region P^* and the kinematic assumption (4.2.7) is used, it can be shown that

$$\begin{aligned}
\frac{d}{dt}\int_{P^*} \rho^* \, \Big[\mathbf{x} \times (\mathbf{v} + \theta^3 \, \mathbf{w}_3) + \mathbf{d}_3 \times (\theta^3 \, \mathbf{v} + \theta^3 \, \theta^3 \, \mathbf{w}_3) \Big] \, dv^* \\
= \int_{P^*} \rho^* \, \Big[\, (\mathbf{x} \times \mathbf{b}^*) + \ (\mathbf{d}_3 \times \theta^3 \, \mathbf{b}^*) \, \Big] \, dv^* \\
+ \int_{\partial P_1^*} \{(\mathbf{x} \times \mathbf{t}^*) + (\mathbf{d}_3 \times \theta^3 \, \mathbf{t}^*)\} \, da^* \\
+ \int_{\partial P_2^*} \{(\mathbf{x} \times \mathbf{t}^*) + (\mathbf{d}_3 \times \theta^3 \, \mathbf{t}^*)\} \, da^* \\
+ \int_{\partial P_L^*} \{(\mathbf{x} \times \mathbf{t}^*) + (\mathbf{d}_3 \times \theta^3 \, \mathbf{t}^*)\} \, da^* \ .
\end{aligned} \tag{4.3.43}$$

Moreover, the above definitions can be used to rewrite the global form of the balance of angular momentum of the Cosserat theory in the simpler form

$$\begin{aligned}
\frac{d}{dt}\int_P \rho \, \Big[\mathbf{x} \times (\mathbf{v} + y^3 \, \mathbf{w}_3) + \mathbf{d}_3 \times (y^3 \, \mathbf{v} + y^{33} \, \mathbf{w}_3)\Big] \, d\sigma \\
= \int_P [\mathbf{x} \times \rho \, \mathbf{b} + \mathbf{d}_3 \times \rho \, \mathbf{b}^3 \,] \, d\sigma + \int_{\partial P} \Big[\, \mathbf{x} \times \mathbf{t} + \mathbf{d}_3 \times \mathbf{m}^3 \, \Big] \, ds \ .
\end{aligned} \tag{4.3.44}$$

Inspection of (4.3.44) reveals that the intrinsic director couple $\mathbf{t}^3$ does not contribute to the balance of angular momentum even though it does contribute to the balance of director momentum.

Before closing this section, it is desirable to develop expressions for the rate of work $\mathcal{W}$ done on the shell and the kinetic energy $\mathcal{K}$ of the shell. To this end, the kinematic assumption (4.2.7) is used and the expressions (3.4.1) and (3.4.2) of the three-dimensional theory are evaluated for the shell-like region P^* to obtain

$$\mathcal{W} = \int_{P^*} \rho^* \, \mathbf{b}^* \bullet (\mathbf{v} + \theta^3 \, \mathbf{w}_3) \, dv^* + \int_{\partial P_1^*} \mathbf{t}^* \bullet (\mathbf{v} + \theta^3 \, \mathbf{w}_3) \, da^*$$

$$+ \int_{\partial P_2^*} \mathbf{t}^* \bullet (\mathbf{v} + \theta^3 \, \mathbf{w}_3) \, da^* + \int_{\partial P_L^*} \mathbf{t}^* \bullet (\mathbf{v} + \theta^3 \, \mathbf{w}_3) \, da^* ,$$

$$\mathcal{K} = \int_{P^*} \frac{1}{2} \rho^* \, (\mathbf{v} + \theta^3 \, \mathbf{w}_3) \bullet (\mathbf{v} + \theta^3 \, \mathbf{w}_3) \, dv^* \ . \tag{4.3.45}$$

Now, with the help of the above definitions these expressions can be written in the simpler forms

$$\mathcal{W} = \int_P \rho \, (\mathbf{b} \bullet \mathbf{v} + \mathbf{b}^3 \bullet \mathbf{w}_3) \, d\sigma + \int_{\partial P} (\mathbf{t} \bullet \mathbf{v} + \mathbf{m}^3 \bullet \mathbf{w}_3) \, ds \ ,$$

$$\mathcal{K} = \int_P \frac{1}{2} \rho \, (\mathbf{v} \bullet \mathbf{v} + 2 \, y^3 \, \mathbf{v} \bullet \mathbf{w}_3 + y^{33} \, \mathbf{w}_3 \bullet \mathbf{w}_3) \, d\sigma \ . \tag{4.3.46}$$

Also, it is noted that in view of the local three-dimensional form (3.2.28) of the conservation of mass, the director inertia coefficients y^3 in (4.3.8) and y^{33} in (4.3.30) are functions of the coordinates θ^α only and therefore are independent of time

$$y^3 = y^3(\theta^\alpha) \ , \quad \dot{y}^3 = 0 \ , \quad y^{33} = y^{33}(\theta^\alpha) \ , \quad \dot{y}^{33} = 0 \ . \tag{4.3.47}$$

Moreover, since the kinetic energy must be a nonnegative function of the velocities, it follows from the expression

$$\mathbf{v} \bullet \mathbf{v} + 2 \, y^3 \, \mathbf{v} \bullet \mathbf{w}_3 + y^{33} \, \mathbf{w}_3 \bullet \mathbf{w}_3 = (\mathbf{v} + y^3 \, \mathbf{w}_3) \bullet (\mathbf{v} + y^3 \, \mathbf{w}_3)$$

$$+ (y^{33} - y^3 \, y^3) \, \mathbf{w}_3 \bullet \mathbf{w}_3 \ , \tag{4.3.48}$$

that y^3 and y^{33} are further restricted by the condition that

$$y^{33} - y^3 \, y^3 \geq 0 \ . \tag{4.3.49}$$

4.4 Balance laws by the direct approach

In the previous section the global forms of conservation of mass and the balances of linear momentum, director momentum and angular momentum were developed by using the kinematic assumption (4.2.7) together with the balance laws of the three-dimensional theory. From the point of view of the full three-dimensional theory, the Cosserat theory of shells with a single director is an approximate theory whenever line elements through the thickness of the shell do not remain straight.

However, as will be seen in this section, the Cosserat theory of shells can be presented by a direct approach in which the theory is an exact nonlinear theory. From this perspective, the Cosserat theory is considered to be a *model* of a shell-like structure, and the balance laws are postulated without any direct connection with the three-dimensional theory. Specifically, the Cosserat theory models the shell as a surface $\mathcal{S}$ in the present configuration at time t. Material points on this surface are identified by constant values of the convected Lagrangian coordinates θ^α, and the kinematics of the Cosserat theory include the position vector $\mathbf{x}(\theta^\alpha,t)$ (relative to a fixed origin O) and the director vector $\mathbf{d}_3(\theta^\alpha,t)$ which are specified by the functional forms

$$\mathbf{x} = \hat{\mathbf{x}}(\theta^\alpha,t) \quad , \quad \mathbf{d}_3 = \hat{\mathbf{d}}_3(\theta^\alpha,t) \ . \tag{4.4.1}$$

Also, the velocity $\mathbf{v}$ and the director velocities $\mathbf{w}_i$ are defined by (4.2.5). Moreover, each material point is endowed with a mass density ρ per unit present area of $\mathcal{S}$ and director inertia coefficients y^3 and y^{33} which can depend on θ^α but are independent of time

$$\rho = \rho(\theta^\alpha,t) \quad , \quad y^3 = y^3(\theta^\alpha) \quad , \quad y^{33} = y^{33}(\theta^\alpha) \ . \tag{4.4.2}$$

An arbitrary material part of the Cosserat surface $\mathcal{S}$ is denoted by P and its regular closed boundary is denoted by ∂P. Each material point of P is subjected to a specific (per unit mass) external force $\mathbf{b}$ and a specific external director couple $\mathbf{b}^3$ which are due to body forces ($\mathbf{b}_b$ and $\mathbf{b}_b^3$) and contact forces ($\mathbf{b}_c$ and $\mathbf{b}_c^3$) applied to the major surfaces of the shell

$$\mathbf{b} = \mathbf{b}(\theta^\alpha,t) = \mathbf{b}_b(\theta^\alpha,t) + \mathbf{b}_c(\theta^\alpha,t) \ ,$$
$$\mathbf{b}^3 = \mathbf{b}^3(\theta^\alpha,t) = \mathbf{b}_b^3(\theta^\alpha,t) + \mathbf{b}_c^3(\theta^\alpha,t) \ . \tag{4.4.3}$$

In addition, each material point of P is subjected to an intrinsic director couple $a^{-1/2}\,\mathbf{t}^3$ per unit present area of $\mathcal{S}$

$$\mathbf{t}^3 = \mathbf{t}^3(\theta^\alpha,t) \ . \tag{4.4.4}$$

Also, material points of the boundary ∂P are subjected to a contact force $\mathbf{t}$ and a contact director couple $\mathbf{m}^3$ which are explicit functions of the unit outward normal $\mathbf{n}$ to ∂P in the surface $\mathcal{S}$

$$\mathbf{t} = \mathbf{t}(\theta^\alpha,t\,;\mathbf{n}) \quad , \quad \mathbf{m}^3 = \mathbf{m}^3(\theta^\alpha,t\,;\mathbf{n}) \ . \tag{4.4.5}$$

Using these definitions, the conservation of mass and the balances of linear momentum and director momentum are postulated in the forms

$$\frac{d}{dt}\int_P \rho \, d\sigma = 0 \ ,$$
$$\frac{d}{dt}\int_P \rho\,(\mathbf{v} + y^3\,\mathbf{w}_3)\, d\sigma = \int_P \rho\,\mathbf{b}\, d\sigma + \int_{\partial P} \mathbf{t}\, ds \ ,$$
$$\frac{d}{dt}\int_P \rho\,(y^3\,\mathbf{v} + y^{33}\,\mathbf{w}_3)\, d\sigma = \int_P [\rho\,\mathbf{b}^3 - a^{-1/2}\,\mathbf{t}^3\,]\, d\sigma + \int_{\partial P} \mathbf{m}^3\, ds \ , \tag{4.4.6}$$

where $d\sigma$ is the present element of area of P and ds is the present element of arclength of ∂P. Also, the balance of angular momentum about the fixed origin O is postulated in the form

$$\frac{d}{dt}\int_P \rho\,\big[\mathbf{x}\times(\mathbf{v} + y^3\,\mathbf{w}_3) + \mathbf{d}_3\times(y^3\,\mathbf{v} + y^{33}\,\mathbf{w}_3)\big]\, d\sigma$$
$$= \int_P [\mathbf{x}\times\rho\,\mathbf{b} + \mathbf{d}_3\times\rho\,\mathbf{b}^3\,]\, d\sigma + \int_{\partial P} [\,\mathbf{x}\times\mathbf{t} + \mathbf{d}_3\times\mathbf{m}^3\,]\, ds \ . \tag{4.4.7}$$

Using standard arguments in continuum mechanics (Naghdi, 1972, sec. 9), it can be shown that $\mathbf{t}$ and $\mathbf{m}^3$ are linear functions of the normal $\mathbf{n}$ so they can be expressed in terms of second order tensors $\mathbf{N}$ and $\mathbf{M}$ such that

$$\mathbf{t} = \mathbf{t}(\theta^\alpha,t\,;\mathbf{n}) = \mathbf{N}(\theta^\alpha,t)\,\mathbf{n} \quad , \quad \mathbf{m}^3 = \mathbf{m}^3(\theta^\alpha,t\,;\mathbf{n}) = \mathbf{M}(\theta^\alpha,t)\,\mathbf{n} \ . \tag{4.4.8}$$

Moreover, since the normal vector $\mathbf{n}$ lies in the surface $\mathcal{S}$, it follows that the tensors $\mathbf{N}$ and $\mathbf{M}$ are restricted by the equations

$$\mathbf{N}\,\mathbf{a}_3 = 0 \;\;,\;\; \mathbf{M}\,\mathbf{a}_3 = 0 \;\;, \tag{4.4.9}$$

where $\mathbf{a}_3$ is the unit normal vector to the surface $\mathcal{S}$. It then follows that the tensors $\mathbf{N}$ and $\mathbf{M}$ each have only six independent components so they can be expressed in terms of the vectors $\mathbf{n}^\alpha$ and $\mathbf{m}^\alpha$ by the equations

$$\begin{gathered}\mathbf{t}^\alpha = a^{1/2}\,\mathbf{N}\,\mathbf{d}^\alpha \;\;,\;\; \mathbf{N} = a^{-1/2}\,\mathbf{t}^\alpha \otimes \mathbf{d}_\alpha \;,\;\; \mathbf{N}\,\mathbf{d}^3 = 0 \;\;,\\ \mathbf{m}^\alpha = a^{1/2}\,\mathbf{M}\,\mathbf{d}^\alpha \;\;,\;\; \mathbf{M} = a^{-1/2}\,\mathbf{m}^\alpha \otimes \mathbf{d}_\alpha \;,\;\; \mathbf{M}\,\mathbf{d}^3 = 0 \;\;,\end{gathered} \tag{4.4.10}$$

where the triad $\mathbf{d}_i$ and the reciprocal vectors $\mathbf{d}^i$ are defined by (4.2.4), (4.2.5) and (4.2.8).

In order to develop the local forms of the balance laws (4.4.6) and (4.4.7), it is necessary to develop an expression for the time derivative of an integral over the material region P which changes with time, and it is necessary to use the divergence theorem to convert an integral over the boundary curve ∂P into an integral over the surface P.

To this end, it is noted that the material region P in the present configuration which depends on time can be mapped to the material region P_0 in a fixed reference configuration of the shell which does not depend on time. Specifically, the area element $d\sigma$ in the present configuration (4.2.2) can be related to the area element $d\Sigma$ in the reference configuration (4.1.2) by the expression

$$d\sigma = (a^{1/2}\,A^{-1/2})\,d\Sigma \;. \tag{4.4.11}$$

Thus, an integral over the material region P can be expressed as an integral over the corresponding material region P_0 such that

$$\frac{d}{dt}\int_P (\rho\,\phi)\,d\sigma = \frac{d}{dt}\int_{P_0} (\rho\,\phi\,a^{1/2})\,A^{-1/2}\,d\Sigma \;. \tag{4.4.12}$$

However, since P_0, $A^{1/2}$ and $d\Sigma$ are independent of time, the differentiation operation and the integration operation can be interchanged (assuming sufficient continuity) so that (4.4.12) can be rewritten in the form

$$\frac{d}{dt}\int_P (\rho\,\phi)\,d\sigma = \int_{P_0} \overset{\bullet}{\overline{(\rho\,\phi\,a^{1/2})}}\,A^{-1/2}\,d\Sigma \;. \tag{4.4.13}$$

Now, by taking the material derivative of the expression (4.2.2), it can be shown that

$$\overset{\bullet}{\overline{(a^{1/2})}} = a^{1/2}\,\mathrm{div}_s\,\mathbf{v} = a^{1/2}\,\mathbf{v}_{,\alpha} \bullet \mathbf{a}^\alpha \;\;, \tag{4.4.14}$$

where the divergence operator div_s of an arbitrary tensor function $\mathbf{T}(\theta^\alpha,t)$ with respect to the surface $\mathcal{S}$ is defined by the equation

$$\mathrm{div}_s\,\mathbf{T} = \mathbf{T}_{,\alpha} \bullet \mathbf{a}^\alpha \,. \tag{4.4.15}$$

It then follows that (4.4.13) can be rewritten in the form

$$\frac{d}{dt}\int_P (\rho\,\phi)\,d\sigma = \int_P \left[\,\rho\,\dot{\phi} + \phi\,\{\dot{\rho} + \rho\,\mathrm{div}_s\,\mathbf{v}\}\,\right]\,d\sigma \;. \tag{4.4.16}$$

Next, taking ϕ equal to unity in (4.4.16) and assuming sufficient continuity and that the conservation of mass $(4.4.6)_1$ holds for arbitrary material parts P, the local form of conservation of mass becomes

$$\dot{\rho} + \rho \, \mathrm{div}_s \, \mathbf{v} = 0 \ , \quad m = \rho \, a^{1/2} = m(\theta^\alpha) \ , \quad \dot{m} = 0 \ . \tag{4.4.17}$$

Also, equation (4.4.16) reduces to

$$\frac{d}{dt}\int_P (\rho \, \phi) \, d\sigma = \int_P [\rho \, \dot{\phi}] \, d\sigma \ . \tag{4.4.18}$$

Thus, the left-hand sides of the balance laws $(4.4.6)_{2,3}$ and (4.4.7) can be expressed as

$$\frac{d}{dt}\int_P \rho \, (\mathbf{v} + y^3 \, \mathbf{w}_3) \, d\sigma = \int_P \rho \, (\dot{\mathbf{v}} + y^3 \, \dot{\mathbf{w}}_3) \, d\sigma \ ,$$

$$\frac{d}{dt}\int_P \rho \, (y^3 \, \mathbf{v} + y^{33} \, \mathbf{w}_3) \, d\sigma = \int_P \rho \, (y^3 \, \dot{\mathbf{v}} + y^{33} \, \dot{\mathbf{w}}_3) \, d\sigma \ ,$$

$$\frac{d}{dt}\int_P \rho \left[\mathbf{x} \times (\mathbf{v} + y^3 \, \mathbf{w}_3) + \mathbf{d}_3 \times (y^3 \, \mathbf{v} + y^{33} \, \mathbf{w}_3)\right] d\sigma$$

$$= \int_P \rho \left[\mathbf{x} \times (\dot{\mathbf{v}} + y^3 \, \dot{\mathbf{w}}_3) + \mathbf{d}_3 \times (y^3 \, \dot{\mathbf{v}} + y^{33} \, \dot{\mathbf{w}}_3)\right] d\sigma \ . \tag{4.4.19}$$

Returning to the discussion of the divergence theorem, it is noted that for an arbitrary tensor function $\mathbf{T}(\theta^\alpha, t)$

$$\int_{\partial P} \mathbf{T} \, \mathbf{n} \, ds = \int_P \mathrm{div}_n \, \mathbf{T} \, d\sigma \ , \tag{4.4.20}$$

where the divergence operator div_n is defined by

$$\mathrm{div}_n \, \mathbf{T} = a^{-1/2} \, (a^{1/2} \, \mathbf{T} \bullet \mathbf{a}^\alpha)_{,\alpha} \ . \tag{4.4.21}$$

However, it can be shown that

$$(a^{1/2} \, \mathbf{a}^\alpha)_{,\alpha} = - \, a^{1/2} \, (\mathbf{a}_{3,\alpha} \bullet \mathbf{a}^\alpha) \, \mathbf{a}_3 \ , \tag{4.4.22}$$

so that (4.4.21) can be rewritten in the form

$$\mathrm{div}_n \, \mathbf{T} = \mathrm{div}_s \, \mathbf{T} - (\mathbf{a}_{3,\alpha} \bullet \mathbf{a}^\alpha) \, \mathbf{T} \bullet \mathbf{a}_3 \ . \tag{4.4.23}$$

Consequently, if the tensor $\mathbf{T}$ satisfies a restriction of the form $(4.4.9)_1$, then the two divergence operations are equivalent

$$\mathrm{div}_n \, \mathbf{T} = \mathrm{div}_s \, \mathbf{T} \quad \text{whenever} \quad \mathbf{T} \bullet \mathbf{a}_3 = 0 \ . \tag{4.4.24}$$

In particular, this is valid for the tensors $\mathbf{N}$ and $\mathbf{M}$, so the integrals over ∂P in (4.4.6) and (4.4.7) can be rewritten as integrals over the region P in the forms

$$\int_{\partial P} \mathbf{t} \, ds = \int_P \mathrm{div}_s \, \mathbf{N} \, d\sigma \ , \quad \int_{\partial P} \mathbf{m}^3 \, ds = \int_P \mathrm{div}_s \, \mathbf{M} \, d\sigma \ ,$$

$$\int_{\partial P} [\, \mathbf{x} \times \mathbf{t} + \mathbf{d}_3 \times \mathbf{m}^3 \,] \, ds = \int_P \big[\{ \mathbf{x} \times \mathrm{div}_s \, \mathbf{N} + \mathbf{d}_3 \times \mathrm{div}_s \, \mathbf{M} \}$$

$$+ \, \mathbf{d}_\alpha \times (\mathbf{N} \, \mathbf{d}^\alpha) + \mathbf{d}_{3,\alpha} \times (\mathbf{M} \, \mathbf{d}^\alpha)\big] \, d\sigma \ , \tag{4.4.25}$$

where use has been made of the fact that since $\mathbf{d}_\alpha = \mathbf{a}_\alpha$, it follows that

$$\mathbf{N} \, \mathbf{a}^\alpha = \mathbf{N} \, \mathbf{d}^\alpha \ , \quad \mathbf{M} \, \mathbf{a}^\alpha = \mathbf{M} \, \mathbf{d}^\alpha \ . \tag{4.4.26}$$

With the help of these results and again assuming sufficient continuity and that the balance laws $(4.4.6)_{2,3}$ and (4.4.7) hold for arbitrary material parts P, the local forms of the balances of linear and director momentum become

$$\rho\,(\dot{\mathbf{v}} + y^3\,\dot{\mathbf{w}}_3) = \rho\,\mathbf{b} + \mathrm{div}_s\,\mathbf{N}\ ,$$

$$\rho\,(y^3\,\dot{\mathbf{v}} + y^{33}\,\dot{\mathbf{w}}_3) = \rho\,\mathbf{b}^3 - a^{-1/2}\,\mathbf{t}^3 + \mathrm{div}_s\,\mathbf{M}\ , \tag{4.4.27}$$

and the local form of the balance of angular momentum reduces to

$$\mathbf{d}_\alpha \times (\mathbf{N}\,\mathbf{d}^\alpha) + \mathbf{d}_3 \times a^{-1/2}\,\mathbf{t}^3 + \mathbf{d}_{3,\alpha} \times (\mathbf{M}\,\mathbf{d}^\alpha) = 0\ . \tag{4.4.28}$$

Using (4.4.10), this equation can be written in the simpler form

$$\mathbf{t}^i \times \mathbf{d}_i + \mathbf{m}^\alpha \times \mathbf{d}_{3,\alpha} = 0\ , \tag{4.4.29}$$

where the order of the cross product has been inverted for later convenience. Next, recalling from appendix A that the permutation tensor $\boldsymbol{\varepsilon}$ has the property (A.5.15) that for any two vectors **a** and **b**

$$\mathbf{a} \times \mathbf{b} = \boldsymbol{\varepsilon} \bullet (\mathbf{a} \otimes \mathbf{b})\ , \tag{4.4.30}$$

it follows that the balance of angular momentum (4.4.29) can be rewritten as

$$\boldsymbol{\varepsilon} \bullet \left[\mathbf{t}^i \otimes \mathbf{d}_i + \mathbf{m}^\alpha \otimes \mathbf{d}_{3,\alpha}\right] = 0\ . \tag{4.4.31}$$

Thus, by defining the second order tensor **T**

$$a^{1/2}\,\mathbf{T} = \mathbf{t}^i \otimes \mathbf{d}_i + \mathbf{m}^\alpha \otimes \mathbf{d}_{3,\alpha}\ , \tag{4.4.32}$$

the reduced form of the balance of angular momentum (4.4.31) requires **T** to be a symmetric tensor

$$\mathbf{T}^T = \mathbf{T}\ , \tag{4.4.33}$$

which is similar to the result (3.2.32) associated with the three-dimensional theory.

By using (4.4.10), (4.4.21), (4.4.24) and (4.4.26), it can be shown that

$$a^{1/2}\,\mathrm{div}_s\,\mathbf{N} = \mathbf{t}^\alpha{}_{,\alpha}\ ,\quad a^{1/2}\,\mathrm{div}_s\,\mathbf{M} = \mathbf{m}^\alpha{}_{,\alpha}\ , \tag{4.4.34}$$

so that the local forms of the conservation of mass (4.4.17) and the balances of linear momentum and director momentum (4.4.27) can be summarized in the forms

$$m = \rho\,a^{1/2} = \rho_0\,A^{1/2} = m(\theta^\alpha) \quad \text{or} \quad \dot{\rho} + \rho\,\mathbf{v}_{,\alpha} \bullet \mathbf{a}^\alpha = 0\ ,$$

$$m\,(\dot{\mathbf{v}} + y^3\,\dot{\mathbf{w}}_3) = m\,\mathbf{b} + \mathbf{t}^\alpha{}_{,\alpha}\ ,$$

$$m\,(y^3\,\dot{\mathbf{v}} + y^{33}\,\dot{\mathbf{w}}_3) = m\,\mathbf{b}^3 - \mathbf{t}^3 + \mathbf{m}^\alpha{}_{,\alpha}\ , \tag{4.4.35}$$

where ρ_0 is the mass density per unit area of the reference surface in the reference configuration and the definition (4.4.15) has been used.

In general, equations (4.4.33) and (4.4.35) represent a system of nonlinear partial differential equations which require specification of initial and boundary conditions. These balance laws are quite general because they are valid for all materials. However, this system of equations is not complete because it represents a system of ten scalar equations to determine twenty-two scalar unknowns $\{\rho\ ,\ \mathbf{x}\ ,\ \mathbf{d}_3\ ,\ \mathbf{t}^i\ ,\ \mathbf{m}^\alpha\}$, once the external loads $\{\mathbf{b}\ ,\ \mathbf{b}^3\}$ have been specified. As in the three-dimensional theory, these equations must be supplemented by constitutive equations for the quantities $\{\mathbf{t}^i\ ,\ \mathbf{m}^\alpha\}$ (here it is tacitly assumed that the director inertias y^3 and y^{33} have been specified).

Furthermore, it is noted that the reduced form of the balance of angular momentum (4.4.33) places three restrictions on the constitutive equations that must be satisfied for all possible motions of the continuum. Therefore, the balance of angular momentum has a different character from the other three balance laws because it is not used to determine the motion or deformation of the continuum. In contrast, the conservation of mass $(4.4.35)_1$ and the balances of linear and director momentum $(4.4.35)_{2,3}$ are used to determine the mass density ρ and the motion of the continuum through the functional forms for $\{\mathbf{x}, \mathbf{d}_3\}$.

4.5 Invariance under superposed rigid body motions

Motivated by the relationships developed in section 4.3 and the discussion of superposed rigid body motions (SRBM) in section 3.3 for the three-dimensional theory, it is assumed that under SRBM the Cosserat shell is transformed from its present configuration occupying region P and boundary ∂P at time t to its superposed configuration occupying region P^+ and boundary ∂P^+ at time t^+ such that

$$t^+ = t + a \ , \quad \mathbf{x}^+ = \mathbf{c}(t) + \mathbf{Q}(t)\,\mathbf{x} \ , \quad \mathbf{d}_3^+ = \mathbf{Q}(t)\,\mathbf{d}_3 \ , \tag{4.5.1}$$

where a is a constant, $\mathbf{c}(t)$ is an arbitrary vector function of time only, and $\mathbf{Q}(t)$ is a proper orthogonal tensor function of time only

$$\mathbf{Q}^T\mathbf{Q} = \mathbf{I} \ , \quad \det \mathbf{Q} = +1 \ , \tag{4.5.2}$$

which is related to a skew-symmetric tensor $\mathbf{\Omega}(t)$ function of time only through the equations

$$\dot{\mathbf{Q}} = \mathbf{\Omega}\,\mathbf{Q} \ , \quad \mathbf{\Omega}^T = -\mathbf{\Omega} \ . \tag{4.5.3}$$

Throughout the text, quantities associated with the superposed configuration will be denoted using the same symbol as those associated with the present configuration but with a superposed $(^+)$. Next, using (4.5.1) it follows that since θ^α are convected coordinates they are unaffected by SRBM so that

$$\begin{gathered}
\mathbf{D}_i^+ = \mathbf{D}_i \ , \quad \mathbf{D}^{i+} = \mathbf{D}^i \ , \quad \mathbf{a}_i^+ = \mathbf{Q}\,\mathbf{a}_i \ , \quad \mathbf{a}^{i+} = \mathbf{Q}\,\mathbf{a}^i \ , \\
\mathbf{d}_i^+ = \mathbf{Q}\,\mathbf{d}_i \ , \quad \mathbf{d}^{i+} = \mathbf{Q}\,\mathbf{d}^i \ , \quad (a^{1/2})^+ = a^{1/2} \ , \quad (d\sigma)^+ = d\sigma \ , \\
(ds)^+ = ds \ , \quad \mathbf{n}^+ = \mathbf{Q}\,\mathbf{n} \ , \quad \mathbf{F}^+ = \mathbf{Q}\,\mathbf{F} \ , \quad \boldsymbol{\lambda}^+ = \boldsymbol{\lambda} \ .
\end{gathered} \tag{4.5.4}$$

Moreover, the various kinetic quantities are assumed to transform to their superposed values by the equations

$$\begin{gathered}
m^+ = m \ , \quad \rho^+ = \rho \ , \quad y^{3+} = y^3 \ , \quad y^{33+} = y^{33} \ , \\
\mathbf{t}^+ = \mathbf{Q}\,\mathbf{t} \ , \quad \mathbf{t}^{i+} = \mathbf{Q}\,\mathbf{t}^i \ , \quad \mathbf{N}^+ = \mathbf{Q}\,\mathbf{N}\,\mathbf{Q}^T \ , \\
\mathbf{m}^{3+} = \mathbf{Q}\,\mathbf{m}^3 \ , \quad \mathbf{m}^{\alpha+} = \mathbf{Q}\,\mathbf{m}^\alpha \ , \quad \mathbf{M}^+ = \mathbf{Q}\,\mathbf{M}\,\mathbf{Q}^T \ , \\
\mathbf{T}^+ = \mathbf{Q}\,\mathbf{T}\,\mathbf{Q}^T \ , \\
\mathbf{b}^+ = (\dot{\mathbf{v}}^+ + y^{3+}\,\dot{\mathbf{w}}_3^+) + \mathbf{Q}\,[\mathbf{b} - (\dot{\mathbf{v}} + y^3\,\dot{\mathbf{w}}_3)] \ , \\
\mathbf{b}_b^+ = (\dot{\mathbf{v}}^+ + y^{3+}\,\dot{\mathbf{w}}_3^+) + \mathbf{Q}\,[\mathbf{b}_b - (\dot{\mathbf{v}} + y^3\,\dot{\mathbf{w}}_3)] \ , \quad \mathbf{b}_c^+ = \mathbf{Q}\,\mathbf{b}_c \ ,
\end{gathered}$$

$$\mathbf{b}^{3+} = (y^{3+}\,\dot{\mathbf{v}}^{+} + y^{33+}\,\dot{\mathbf{w}}_3^{+}) + \mathbf{Q}\,[\mathbf{b}^3 - (y^3\,\dot{\mathbf{v}} + y^{33}\,\dot{\mathbf{w}}_3)] \ ,$$

$$\mathbf{b}_b^{3+} = (y^{3+}\,\dot{\mathbf{v}}^{+} + y^{33+}\,\dot{\mathbf{w}}_3^{+}) + \mathbf{Q}\,[\mathbf{b}_b^3 - (y^3\,\dot{\mathbf{v}} + y^{33}\,\dot{\mathbf{w}}_3)] \ , \quad \mathbf{b}_c^{3+} = \mathbf{Q}\,\mathbf{b}_c^3 \ . \tag{4.5.5}$$

Also, it can be shown using expressions of the form (3.5.4) and (3.5.6) that

$$\mathrm{div}_s^+\,\mathbf{v}^+ = (\mathbf{Q}\,\mathbf{v} + \dot{\mathbf{c}} + \mathbf{\Omega}\,\mathbf{Q}\,\mathbf{x})_{,\alpha} \bullet (\mathbf{Q}\,\mathbf{a}^\alpha) = \mathrm{div}_s\,\mathbf{v} \ ,$$

$$\mathrm{div}_s^+\,\mathbf{N}^+ = (\mathbf{Q}\,\mathbf{N}\,\mathbf{Q}^T)_{,\alpha} \bullet (\mathbf{Q}\,\mathbf{a}^\alpha) = \mathbf{Q}\,\mathrm{div}_s\,\mathbf{N} = \mathbf{Q}\,\left[a^{-1/2}\,\mathbf{t}^\alpha{}_{,\alpha}\right] ,$$

$$\mathrm{div}_s^+\,\mathbf{M}^+ = (\mathbf{Q}\,\mathbf{M}\,\mathbf{Q}^T)_{,\alpha} \bullet (\mathbf{Q}\,\mathbf{a}^\alpha) = \mathbf{Q}\,\mathrm{div}_s\,\mathbf{M} = \mathbf{Q}\,\left[a^{-1/2}\,\mathbf{m}^\alpha{}_{,\alpha}\right] . \tag{4.5.6}$$

Now, with the help of these results, it can be shown that the balance laws (4.4.33) and (4.4.35) remain form invariant under SRBM. Furthermore, it will be shown in a later section that these conditions place important physical restrictions on constitutive assumptions for the quantities $\{\mathbf{t}^i , \mathbf{m}^\alpha\}$.

4.6 Mechanical power

For the purely mechanical theory, it is convenient to define the notion of the mechanical power $\mathcal{P}$ due to the kinetic quantities $\{\mathbf{b} , \mathbf{b}^3 , \mathbf{t} , \mathbf{m}^3\}$ by the equation

$$\int_P \mathcal{P}\,d\sigma = \mathcal{W} - \dot{\mathcal{K}} = \int_P \rho\,(\mathbf{b} \bullet \mathbf{v} + \mathbf{b}^3 \bullet \mathbf{w}_3)\,d\sigma + \int_{\partial P} (\mathbf{t} \bullet \mathbf{v} + \mathbf{m}^3 \bullet \mathbf{w}_3)\,ds$$

$$- \frac{d}{dt} \int_P \frac{1}{2}\rho\,(\,\mathbf{v} \bullet \mathbf{v} + 2\,y^3\,\mathbf{v} \bullet \mathbf{w}_3 + y^{33}\,\mathbf{w}_3 \bullet \mathbf{w}_3\,)\,d\sigma \ , \tag{4.6.1}$$

where the expressions (4.3.46) have been used for the rate of work $\mathcal{W}$ applied to the shell and for the kinetic energy $\mathcal{K}$.

Next, assuming sufficient continuity of the functions and using the result that

$$a^{1/2}\,\mathrm{div}_s\,(\mathbf{v} \bullet \mathbf{N} + \mathbf{w}_3 \bullet \mathbf{M}) = \left[a^{1/2}\,\{\mathbf{v} \bullet (\mathbf{N}\,\mathbf{a}^\alpha) + \mathbf{w}_3 \bullet (\mathbf{M}\,\mathbf{a}^\alpha)\}\right]_{,\alpha}$$

$$= \left[\mathbf{v} \bullet \mathbf{t}^\alpha + \mathbf{w}_3 \bullet \mathbf{m}^\alpha\right]_{,\alpha}$$

$$= \mathbf{v} \bullet \mathbf{t}^\alpha{}_{,\alpha} + \mathbf{w}_3 \bullet \mathbf{m}^\alpha{}_{,\alpha} + (\mathbf{t}^\alpha \bullet \mathbf{w}_\alpha + \mathbf{m}^\alpha \bullet \mathbf{w}_{3,\alpha}) \ , \tag{4.6.2}$$

it can be shown that the local form of equation (4.6.1) requires the mechanical power to be given by the expression

$$a^{1/2}\,\mathcal{P} = \mathbf{t}^i \bullet \mathbf{w}_i + \mathbf{m}^\alpha \bullet \mathbf{w}_{3,\alpha} \ . \tag{4.6.3}$$

Also, it can be shown using the transformation relations of section 4.5 that under SRBM

$$\mathbf{w}_i^+ = \mathbf{Q}\,\mathbf{w}_i + \mathbf{\Omega}\,\mathbf{Q}\,\mathbf{d}_i \ , \quad \mathbf{w}_{3,\alpha}^+ = \mathbf{Q}\,\mathbf{w}_{3,\alpha} + \mathbf{\Omega}\,\mathbf{Q}\,\mathbf{d}_{3,\alpha} \ , \tag{4.6.4}$$

so that with the help of (3.5.6), the mechanical power in the superposed configuration becomes

$$(a^{1/2})^+\,\mathcal{P}^+ = a^{1/2}\,\mathcal{P} + (\mathbf{\Omega}\,\mathbf{Q}\,\mathbf{d}_i) \bullet (\mathbf{Q}\,\mathbf{t}^i) + (\mathbf{\Omega}\,\mathbf{Q}\,\mathbf{d}_{3,\alpha}) \bullet (\mathbf{Q}\,\mathbf{m}^\alpha) \ .$$

$$(a^{1/2})^+\,\mathcal{P}^+ = a^{1/2}\,\mathcal{P} + (\mathbf{Q}^T\boldsymbol{\omega}) \bullet (\mathbf{d}_i \times \mathbf{t}^i + \mathbf{d}_{3,\alpha} \times \mathbf{m}^\alpha) \ , \tag{4.6.5}$$

where $\boldsymbol{\omega}$ is the axial vector associated with $\mathbf{\Omega}$. However, using the local form (4.4.29) of the balance of angular momentum, it follows that the mechanical power is unaltered by SRBM

$$\mathcal{P}^+ = \mathcal{P} \ . \tag{4.6.6}$$

Before closing this section, it is desirable to rewrite the expression for mechanical power in an alternative form that is more similar to the expression (3.4.5) of the three-dimensional theory. To this end, it is convenient to define the second order tensor $\mathbf{L}$ by the expressions

$$\mathbf{L} = \mathbf{w}_i \otimes \mathbf{d}^i \ , \quad \mathbf{w}_i = \mathbf{L}\,\mathbf{d}_i \ . \tag{4.6.7}$$

It then follows from the definition (4.2.9) of the tensor $\mathbf{F}$ that

$$\dot{\mathbf{F}} = \mathbf{L}\,\mathbf{F} \ . \tag{4.6.8}$$

Consequently, comparison of these expressions with (3.2.12) suggests that $\mathbf{F}$ is similar to the three-dimensional deformation gradient and $\mathbf{L}$ is similar to the three-dimensional velocity gradient. Moreover, using the definition (4.2.10), it can be shown that

$$\mathbf{w}_{3,\alpha} = \mathbf{L}\,\mathbf{d}_{3,\alpha} + \mathbf{F}\,\dot{\boldsymbol{\lambda}}\,\mathbf{D}_\alpha \ . \tag{4.6.9}$$

Thus, using the definitions (4.4.10) and (4.4.32), the mechanical power can be rewritten in the form

$$\mathcal{P} = \mathbf{T} \bullet \mathbf{L} + a^{-1/2}\,(\mathbf{F}^T\mathbf{m}^\alpha \otimes \mathbf{D}_\alpha) \bullet \dot{\boldsymbol{\lambda}} \ . \tag{4.6.10}$$

Moreover, by separating $\mathbf{L}$ into its symmetric part $\mathbf{D}$ and its skew-symmetric part $\mathbf{W}$

$$\mathbf{L} = \mathbf{D} + \mathbf{W} \ ,$$
$$\mathbf{D} = \frac{1}{2}(\mathbf{L} + \mathbf{L}^T) = \mathbf{D}^T \ , \quad \mathbf{W} = \frac{1}{2}(\mathbf{L} - \mathbf{L}^T) = -\,\mathbf{W}^T \ , \tag{4.6.11}$$

and by using the result (4.4.33) of the balance of angular momentum that $\mathbf{T}$ is a symmetric tensor, it follows that the mechanical power reduces to

$$\mathcal{P} = \mathbf{T} \bullet \mathbf{D} + a^{-1/2}\,(\mathbf{F}^T\mathbf{m}^\alpha \otimes \mathbf{D}_\alpha) \bullet \dot{\boldsymbol{\lambda}} \ . \tag{4.6.12}$$

Next, it is noted that under SRBM the quantities $\mathbf{L}$, $\mathbf{D}$, $\mathbf{W}$ transform by

$$\mathbf{L}^+ = \mathbf{Q}\,\mathbf{L}\,\mathbf{Q}^T + \boldsymbol{\Omega} \ ,$$
$$\mathbf{D}^+ = \mathbf{Q}\,\mathbf{D}\,\mathbf{Q}^T, \quad \mathbf{W}^+ = \mathbf{Q}\,\mathbf{W}\,\mathbf{Q}^T + \boldsymbol{\Omega} \ , \tag{4.6.13}$$

and that $(\mathbf{F}^T\mathbf{m}^\alpha \otimes \mathbf{D}_\alpha)$ remains unaltered by SRBM so that the mechanical power is again shown to be unaltered by SRBM.

For future developments, it is convenient to introduce the second order tensor $\boldsymbol{\kappa}$ as a strain measure defined by

$$\boldsymbol{\kappa} = \boldsymbol{\lambda} - \boldsymbol{\Lambda} \ . \tag{4.6.14}$$

Moreover, in view of the definitions (4.1.13) and (4.2.10), it follows that $\boldsymbol{\kappa}\,\mathbf{D}_3$ vanishes so that $\boldsymbol{\kappa}$ can be characterized by the two vectors $\boldsymbol{\beta}_\alpha$ such that

$$\boldsymbol{\beta}_\alpha = \boldsymbol{\kappa}\,\mathbf{D}_\alpha = \mathbf{F}^{-1}\mathbf{d}_{3,\alpha} - \mathbf{D}_{3,\alpha} \ , \quad \boldsymbol{\kappa} = \boldsymbol{\beta}_\alpha \otimes \mathbf{D}^\alpha \ . \tag{4.6.15}$$

Then, the expression (4.6.10) for the mechanical power simplifies somewhat to become

$$\mathcal{P} = \mathbf{T} \bullet \mathbf{L} + a^{-1/2}\,(\mathbf{F}^T\mathbf{m}^\alpha) \bullet \dot{\boldsymbol{\beta}}_\alpha \ , \tag{4.6.16}$$

which with the help of the symmetry of $\mathbf{T}$ becomes

$$\mathcal{P} = \mathbf{T} \bullet \mathbf{D} + a^{-1/2}\,(\mathbf{F}^T\mathbf{m}^\alpha) \bullet \dot{\boldsymbol{\beta}}_\alpha \ , \tag{4.6.17}$$

Also, it can be shown that under SRBM the quantities $\boldsymbol{\kappa}$ and $\boldsymbol{\beta}_\alpha$ transform by

$$\boldsymbol{\kappa}^+ = \boldsymbol{\kappa} \ , \ \ \boldsymbol{\beta}^+_\alpha = \boldsymbol{\beta}_\alpha \ . \tag{4.6.18}$$

4.7 An alternative derivation of the balance laws

From the point of view presented previously, the condition that the balance laws remain form invariant under SRBM requires the kinematic and kinetic quantities to satisfy the transformation relations (4.5.1) and (4.5.5) as well a number of other expressions [like (4.5.4)] that can be derived directly from these relations. In this regard, the notion of invariance under SRBM is a fundamental notion that causes intimate interconnections between the balance laws. To demonstrate the fundamental nature of invariance under SRBM, it will be shown that the global forms of the conservation of mass and the balances of linear momentum and angular momentum can be derived by assuming that the global form (4.6.1) of the expression for mechanical power remains form invariant and that the local forms of the above transformation relations are valid. This means that these balance laws can be derived from a single scalar equation by demanding invariance for the class of all possible SRBM. However, this method does not produce the global or local form of the balance of director momentum.

To this end, it is first noted that with respect to the superposed configuration, the global equation (4.6.1) can be written in the form

$$\begin{aligned}\int_{P^+} \mathcal{P}^+ \, d\sigma^+ &= \int_{P^+} \rho^+ \, \mathbf{b}^+ \cdot \mathbf{v}^+ \, d\sigma^+ + \int_{\partial P^+} \mathbf{t}^+ \cdot \mathbf{v}^+ \, ds^+ \\ &+ \int_{P^+} \rho^+ \, \mathbf{b}^{3+} \cdot \mathbf{w}_3^+ \, d\sigma^+ + \int_{\partial P^+} \mathbf{m}^{3+} \cdot \mathbf{w}_3^+ \, ds^+ \\ &- \frac{d}{dt^+} \int_{P^+} \frac{1}{2} \rho^+ \, (\mathbf{v}^+ \cdot \mathbf{v}^+ + 2 \, y^{3+} \, \mathbf{v}^+ \cdot \mathbf{w}_3^+ + y^{33+} \, \mathbf{w}_3^+ \cdot \mathbf{w}_3^+) \, d\sigma^+ .\end{aligned} \tag{4.7.1}$$

Next, consider the special SRBM which is characterized by a superposed constant translational velocity with magnitude u and unit direction $\mathbf{u}$ so that

$$\begin{gathered}\mathbf{c}(t) = u \, \mathbf{u} \, t \ , \ \ \dot{\mathbf{c}} = u \, \mathbf{u} \ , \ \ \ddot{\mathbf{c}} = 0 \ , \ \ \mathbf{u} \cdot \mathbf{u} = 1 \ , \\ \mathbf{Q} = \mathbf{I} \ , \ \ \dot{\mathbf{Q}} = 0 \ , \ \ \ddot{\mathbf{Q}} = 0 \ , \ \ \mathbf{v}^+ = \mathbf{v} + u \, \mathbf{u} \ , \ \ \dot{\mathbf{v}}^+ = \dot{\mathbf{v}} \ , \\ \mathbf{F}^+ = \mathbf{F} \ , \ \ (a^{1/2})^+ = a^{1/2} \ , \ \ \boldsymbol{\beta}^+_\alpha = \boldsymbol{\beta}_\alpha \ , \ \ \mathbf{L}^+ = \mathbf{L} \ , \\ \mathbf{w}_i^+ = \mathbf{w}_i \ , \ \ \dot{\mathbf{w}}_3^+ = \dot{\mathbf{w}}_3 \ , \ \ \rho^+ = \rho \ , \ \ y^{3+} = y^3 \ , \ \ y^{33+} = y^{33} \ , \\ \mathbf{b}^+ = \mathbf{b} \ , \ \ \mathbf{b}^{3+} = \mathbf{b}^3 \ , \ \ \mathbf{t}^+ = \mathbf{t} \ , \ \ \mathbf{m}^{3+} = \mathbf{m}^3 \ , \ \ \mathbf{m}^{\alpha+} = \mathbf{m}^\alpha \ , \\ \mathbf{T}^+ = \mathbf{T} \ , \ \ \mathcal{P}^+ = \mathcal{P} \ .\end{gathered} \tag{4.7.2}$$

Now, substituting (4.7.2) into (4.7.1), transforming the integrals over the superposed regions P^+ and ∂P^+ to the present regions P and ∂P, and subtracting the equation (4.6.1) from the result, yields the expression

$$u \, \mathbf{u} \cdot \left[\frac{d}{dt} \int_P \rho \, (\mathbf{v} + y^3 \, \mathbf{w}_3) \, d\sigma - \int_P \rho \, \mathbf{b} \, d\sigma - \int_{\partial P} \mathbf{t} \, ds \right]$$

$$+\frac{1}{2}u^2\left[\frac{d}{dt}\int_P \rho\, d\sigma\right]=0 \ , \tag{4.7.3}$$

which must be valid for all values of u and the unit vector $\mathbf{u}$. Moreover, since the coefficients in (4.7.3) are independent of u and $\mathbf{u}$, it follows that each of them must vanish. This procedure yields the global forms of conservation of mass $(4.4.6)_1$ and balance of linear momentum $(4.4.6)_2$.

To derive the global form of the balance of angular momentum, it is convenient to consider a superposed constant rigid body rotation that is characterized by

$$\begin{gathered}
\mathbf{c}=0 \ , \ \dot{\mathbf{c}}=0 \ , \ \ddot{\mathbf{c}}=0 \ , \ \mathbf{Q}=\mathbf{Q}(t) \ , \ \dot{\mathbf{Q}}=\boldsymbol{\Omega}\,\mathbf{Q} \ , \ \dot{\boldsymbol{\Omega}}=0 \ , \\
\mathbf{v}^+=\mathbf{Q}\,\mathbf{v}+\boldsymbol{\Omega}\,\mathbf{Q}\,\mathbf{x} \ , \ \dot{\mathbf{v}}^+=\mathbf{Q}\,\dot{\mathbf{v}}+2\,\boldsymbol{\Omega}\,\mathbf{Q}\,\mathbf{v}+\boldsymbol{\Omega}^2\,\mathbf{Q}\,\mathbf{x} \ , \\
\mathbf{F}^+=\mathbf{Q}\,\mathbf{F} \ , \ (a^{1/2})^+=a^{1/2} \ , \ \boldsymbol{\beta}_\alpha^+=\boldsymbol{\beta}_\alpha \ , \ \mathbf{L}^+=\mathbf{Q}\,\mathbf{L}\,\mathbf{Q}^T+\boldsymbol{\Omega} \ , \\
\mathbf{w}_i^+=\mathbf{Q}\,\mathbf{w}_i+\boldsymbol{\Omega}\,\mathbf{Q}\,\mathbf{d}_i \ , \ \dot{\mathbf{w}}_3^+=\mathbf{Q}\,\dot{\mathbf{w}}_3+2\,\boldsymbol{\Omega}\,\mathbf{Q}\,\mathbf{w}_3+\boldsymbol{\Omega}^2\,\mathbf{Q}\,\mathbf{d}_3 \ , \\
\rho^+=\rho \ , \ y^{3+}=y^3 \ , \ y^{33+}=y^{33} \ , \\
\mathbf{b}^+=\mathbf{Q}\,\mathbf{b}+2\,\boldsymbol{\Omega}\,\mathbf{Q}\,(\mathbf{v}+y^3\,\mathbf{w}_3)+\boldsymbol{\Omega}^2\,\mathbf{Q}\,(\mathbf{x}+y^3\,\mathbf{d}_3) \ , \\
\mathbf{b}^{3+}=\mathbf{Q}\,\mathbf{b}^3+2\,\boldsymbol{\Omega}\,\mathbf{Q}\,(y^3\,\mathbf{v}+y^{33}\,\mathbf{w}_3)+\boldsymbol{\Omega}^2\,\mathbf{Q}\,(y^3\,\mathbf{x}+y^{33}\,\mathbf{d}_3) \ , \\
\mathbf{t}^+=\mathbf{Q}\,\mathbf{t} \ , \ \mathbf{m}^{3+}=\mathbf{Q}\,\mathbf{m}^3 \ , \ \mathbf{m}^{\alpha+}=\mathbf{Q}\,\mathbf{m}^\alpha \ , \ \mathbf{T}^+=\mathbf{Q}\,\mathbf{T}\,\mathbf{Q}^T \ , \\
\mathcal{P}^+=\mathcal{P} \ .
\end{gathered} \tag{4.7.4}$$

Thus, with the help of the definition (4.6.16) and the last of (4.7.4), it follows that

$$\begin{gathered}
\mathcal{P}^+=\mathbf{T}^+\bullet\mathbf{L}^+ + [a^{-1/2}\,\mathbf{F}^T\mathbf{m}^\alpha]^+\bullet\dot{\boldsymbol{\beta}}_\alpha^+=\mathcal{P}+\mathbf{T}\bullet(\mathbf{Q}^T\boldsymbol{\Omega}\,\mathbf{Q})=\mathcal{P} \ , \\
\mathbf{T}\bullet(\mathbf{Q}^T\boldsymbol{\Omega}\,\mathbf{Q})=0 \ .
\end{gathered} \tag{4.7.5}$$

However, since $\mathbf{T}$ does not depend on $\boldsymbol{\Omega}$ and since $\boldsymbol{\Omega}$ is a skew-symmetric tensor, the quantity $\mathbf{Q}^T\boldsymbol{\Omega}\,\mathbf{Q}$ is also a skew-symmetric tensor so that (4.7.5) requires $\mathbf{T}$ to be a symmetric tensor, which yields the local form (4.4.33) of the balance of angular momentum. Furthermore, using the results (3.5.6), it can be shown that

$$\begin{gathered}
\mathbf{b}^+\bullet\mathbf{v}^+ + \mathbf{b}^{3+}\bullet\mathbf{w}_3^+ = \mathbf{b}\bullet\mathbf{v}+\mathbf{b}^3\bullet\mathbf{w}_3+(\mathbf{Q}^T\boldsymbol{\omega})\bullet(\mathbf{x}\times\mathbf{b}+\mathbf{d}_3\times\mathbf{b}^3) \\
+(\boldsymbol{\Omega}\,\mathbf{Q}\mathbf{x})\bullet(\boldsymbol{\Omega}\,\mathbf{Q}\mathbf{v})+y^3\,(\boldsymbol{\Omega}\,\mathbf{Q}\mathbf{x})\bullet(\boldsymbol{\Omega}\,\mathbf{Q}\mathbf{w}_3) \\
+y^3\,(\boldsymbol{\Omega}\,\mathbf{Q}\mathbf{d}_3)\bullet(\boldsymbol{\Omega}\,\mathbf{Q}\mathbf{v})+y^{33}\,(\boldsymbol{\Omega}\,\mathbf{Q}\mathbf{d}_3)\bullet(\boldsymbol{\Omega}\,\mathbf{Q}\mathbf{w}_3) \ , \\
\mathbf{t}^+\bullet\mathbf{v}^+ + \mathbf{m}^{3+}\bullet\mathbf{w}_3^+ = \mathbf{t}\bullet\mathbf{v}+\mathbf{m}^3\bullet\mathbf{w}_3+(\mathbf{Q}^T\boldsymbol{\omega})\bullet(\mathbf{x}\times\mathbf{t}+\mathbf{d}_3\times\mathbf{m}^3) \ , \\
\mathcal{K}^+=\mathcal{K}+(\mathbf{Q}^T\boldsymbol{\omega})\bullet\int_P\left[\mathbf{x}\times\rho\,(\mathbf{v}+y^3\,\mathbf{w}_3)+\mathbf{d}_3\times\rho\,(y^3\,\mathbf{v}+y^{33}\,\mathbf{w}_3)\right]d\sigma \\
+\int_P\frac{1}{2}\rho\left[(\boldsymbol{\Omega}\,\mathbf{Q}\mathbf{x})\bullet(\boldsymbol{\Omega}\,\mathbf{Q}\mathbf{x})+2\,y^3\,(\boldsymbol{\Omega}\,\mathbf{Q}\mathbf{x})\bullet(\boldsymbol{\Omega}\,\mathbf{Q}\mathbf{d}_3)\right. \\
\left.+y^{33}\,(\boldsymbol{\Omega}\,\mathbf{Q}\mathbf{d}_3)\bullet(\boldsymbol{\Omega}\,\mathbf{Q}\mathbf{d}_3)\right]d\sigma \ ,
\end{gathered} \tag{4.7.6}$$

where the integrals over the superposed region P^+ and ∂P^+ have been transformed to integrals over the present regions P and ∂P. Also, with the help of (3.5.6), $(3.5.9)_1$ and the conservation of mass equation, it can be shown that

$$\dot{\mathcal{K}}^+ = \dot{\mathcal{K}} + (\mathbf{Q}^T\boldsymbol{\omega}) \bullet \frac{d}{dt}\int_P \left[\mathbf{x} \times \rho\ (\mathbf{v} + y^3\ \mathbf{w}_3) + \mathbf{d}_3 \times \rho\ (y^3\ \mathbf{v} + y^{33}\ \mathbf{w}_3)\right] d\sigma$$

$$+ \int_P \rho \left[(\boldsymbol{\Omega}\ \mathbf{Qx}) \bullet (\boldsymbol{\Omega}\ \mathbf{Qv}) + y^3\ (\boldsymbol{\Omega}\ \mathbf{Qx}) \bullet (\boldsymbol{\Omega}\ \mathbf{Qw}_3)\right.$$

$$\left. + y^3\ (\boldsymbol{\Omega}\ \mathbf{Qd}_3) \bullet (\boldsymbol{\Omega}\ \mathbf{Qv}) + y^{33}\ (\boldsymbol{\Omega}\ \mathbf{Qd}_3) \bullet (\boldsymbol{\Omega}\ \mathbf{Qw}_3)\right] d\sigma\ . \tag{4.7.7}$$

Now, substituting (4.7.6) and (4.7.7) into (4.7.1) and subtracting (4.6.1) from the result, yields the expression

$$(\mathbf{Q}^T\boldsymbol{\omega}) \bullet \left[\ \frac{d}{dt}\int_P \left[\mathbf{x} \times \rho\ (\mathbf{v} + y^3\ \mathbf{w}_3) + \mathbf{d}_3 \times \rho\ (y^3\ \mathbf{v} + y^{33}\ \mathbf{w}_3)\right] d\sigma\right.$$

$$\left. - \int_P (\mathbf{x} \times \rho\ \mathbf{b} + \mathbf{d}_3 \times \rho\ \mathbf{b}^3)\ d\sigma - \int_{\partial P} (\mathbf{x} \times \mathbf{t} + \mathbf{d}_3 \times \mathbf{m}^3)\ ds\right] = 0\ . \tag{4.7.8}$$

Furthermore, since (4.7.8) must be valid for all $\boldsymbol{\omega}$ and the coefficient in the square brackets is independent of $\boldsymbol{\omega}$, it follows that (4.7.8) yields the global form of the balance of angular momentum (4.4.7).

In the above, it has been shown that the global forms of the conservation of mass and the balances of linear and angular momentum are necessary conditions for the global expression (4.6.1) of mechanical power to remain form invariant under SRBM. However, this procedure does not produce the global or local form of the balance of director momentum.

4.8 Anisotropic nonlinear elastic shells

In the previous sections kinematical expressions and balance laws were discussed that are valid for shell-like structures that are made from arbitrary materials. Here, constitutive equations will be developed for shells that are composed of general anisotropic nonlinear elastic materials. Such elastic shells are considered to be ideal shells in the same sense that elastic materials are considered to be ideal materials in the three-dimensional theory. For example, the response of an elastic shell is insensitive to the rate of loading. Other fundamental features of elastic shells will be discussed presently.

The constitutive equations of elastic shells can be characterized by the following four assumptions:

Assumption 1: A strain energy Σ per unit mass exists for which

$$\rho\ \dot{\Sigma} = \mathcal{P} = \mathbf{T} \bullet \mathbf{D} + a^{-1/2}\ (\mathbf{F}^T\mathbf{m}^\alpha) \bullet \dot{\boldsymbol{\beta}}_\alpha\ . \tag{4.8.1}$$

Assumption 2: The strain energy Σ is a function of the deformation tensors $\mathbf{F}$ and $\boldsymbol{\beta}_\alpha$ and the convected coordinates θ^β

$$\Sigma = \tilde{\Sigma}\ (\mathbf{F}, \boldsymbol{\beta}_\alpha\ ;\ \theta^\beta)\ , \tag{4.8.2}$$

where dependence on θ^β is included to allow for the possibility that the shell can be composed of an inhomogeneous material or can have nonuniform geometrical properties in its reference configuration.

Assumption 3: The strain energy Σ is invariant under SRBM

$$\Sigma^+ = \Sigma \ . \tag{4.8.3}$$

Assumption 4: The kinetic quantities $\mathbf{T}$ and $\mathbf{M}$ are independent of the rates of deformations $\mathbf{L}$ and $\boldsymbol{\beta}_\alpha$.

In order to explore the physical consequences of the assumption (4.8.1), it is convenient to define the total strain energy $\mathcal{U}$ of the shell

$$\mathcal{U} = \int_P \rho\, \Sigma \, d\sigma \ , \tag{4.8.4}$$

and to use the result (4.4.18) and the assumption (4.8.1) to deduce that

$$\dot{\mathcal{U}} = \int_P \rho\, \dot{\Sigma} \, d\sigma = \int_P \mathcal{P} \, d\sigma \ . \tag{4.8.5}$$

Thus, by substituting (4.8.5) into the mechanical power equation (4.6.1), it is possible to derive the following theorem

$$\mathcal{W} = \dot{\mathcal{K}} + \dot{\mathcal{U}} \ , \tag{4.8.6}$$

which states that for an elastic shell the rate of work done on the shell due to external forces and couples and contact forces and couples, equals the rate of change of kinetic and strain energies. Since the strain energy Σ depends on the present configuration only through the present values of $\mathbf{F}$ and $\boldsymbol{\beta}_\alpha$ [assumption (4.8.2)], the value of the strain energy Σ is independent of the particular loading path which caused $\mathbf{F}$ and $\boldsymbol{\beta}_\alpha$. Consequently, the total work done on the body vanishes for any closed cycle in which the values of velocity $\mathbf{v}$, the director velocities $\mathbf{w}_i$, and deformation tensors $\mathbf{F}$ and $\boldsymbol{\beta}_\alpha$ are the same at the beginning and end of the cycle. Next, consider a special cycle which is composed of a loading path from one state A to another state B, followed by the reversal of this loading path. Then, in view of assumption 4, the work done on the body from A to B is fully recovered during the reverse loading from B to A. In this sense, the elastic shell is considered to be an ideal shell.

The assumption (4.8.3) places restrictions on the functional form (4.8.2). To develop these restrictions, it is recalled that under SRBM $\mathbf{F}^+ = \mathbf{QF}$ and $\boldsymbol{\beta}_\alpha^+ = \boldsymbol{\beta}_\alpha$ so that (4.8.2) requires

$$\Sigma^+ = \tilde{\Sigma}(\mathbf{F}^+, \boldsymbol{\beta}_\alpha^+; \theta^\beta) = \tilde{\Sigma}(\mathbf{QF}, \boldsymbol{\beta}_\alpha; \theta^\beta) = \tilde{\Sigma}(\mathbf{F}, \boldsymbol{\beta}_\alpha; \theta^\beta) \ , \tag{4.8.7}$$

to hold for arbitrary proper orthogonal $\mathbf{Q}$. However, the polar decomposition theorem states that $\mathbf{F}$ can be separated multiplicatively into a rotation tensor $\mathbf{R}$ and positive definite symmetric stretch tensors $\mathbf{U}$ and $\mathbf{V}$ such that

$$\mathbf{F} = \mathbf{RU} = \mathbf{VR} \ , \quad \mathbf{C} = \mathbf{F}^T\mathbf{F} = \mathbf{C}^T \ , \quad \mathbf{U}^T = \mathbf{U} = (\mathbf{C})^{1/2} \ ,$$
$$\mathbf{B} = \mathbf{F}\mathbf{F}^T = \mathbf{B}^T \ , \quad \mathbf{V}^T = \mathbf{V} = (\mathbf{B})^{1/2} \ , \quad \mathbf{R}^T\mathbf{R} = \mathbf{R}\mathbf{R}^T = \mathbf{I} \ , \quad \det \mathbf{R} = 1 \ , \tag{4.8.8}$$

where $\mathbf{C}$ and $\mathbf{B}$ are analogues of the right Cauchy-Green deformation tensor and the left Cauchy-Green deformation tensor in the three-dimensional theory. Following a similar argument to that associated with the three-dimensional theory (sec. 3.7), it can be shown that the strain energy function Σ can depend on $\mathbf{F}$ only through the deformation tensor $\mathbf{C}$ so that Σ necessarily must reduce to the form

$$\tilde{\Sigma}(\mathbf{F},\boldsymbol{\beta}_\alpha;\theta^\beta) = \hat{\Sigma}(\mathbf{C},\boldsymbol{\beta}_\alpha;\theta^\beta) \ . \tag{4.8.9}$$

Now, with the help of (4.8.1) and (4.8.9), it can be shown that

$$\mathbf{T}\bullet\mathbf{L} + a^{-1/2}(\mathbf{F}^T\mathbf{m}^\alpha)\bullet\dot{\boldsymbol{\beta}}_\alpha = 2\rho(\mathbf{F}\frac{\partial\Sigma}{\partial\mathbf{C}}\mathbf{F}^T)\bullet\mathbf{D} + \rho\frac{\partial\Sigma}{\partial\boldsymbol{\beta}_\alpha}\bullet\dot{\boldsymbol{\beta}}_\alpha \ ,$$

$$(\mathbf{T} - 2\rho\,\mathbf{F}\frac{\partial\Sigma}{\partial\mathbf{C}}\mathbf{F}^T)\bullet\mathbf{D} + (a^{-1/2}\,\mathbf{F}^T\mathbf{m}^\alpha - \rho\frac{\partial\Sigma}{\partial\boldsymbol{\beta}_\alpha})\bullet\dot{\boldsymbol{\beta}}_\alpha = 0 \ , \tag{4.8.10}$$

where use has been made of the expressions (4.6.8) and (4.6.13) to deduce that

$$\dot{\mathbf{C}} = 2\,\mathbf{F}^T\mathbf{D}\,\mathbf{F} \ . \tag{4.8.11}$$

In order to analyze the consequences of equation (4.8.10), it is noted that the coefficients of $\mathbf{D}$ and $\dot{\boldsymbol{\beta}}_\alpha$ are independent of the rates ($\mathbf{D}$, $\dot{\boldsymbol{\beta}}_\alpha$) and that the coefficient of $\mathbf{D}$ is also symmetric. Thus, for any fixed values of $\mathbf{F}$, $\boldsymbol{\beta}_\alpha$ and θ^β, the coefficients in (4.8.10) are fixed even though the rates $\mathbf{D}$ and $\dot{\boldsymbol{\beta}}_\alpha$ can be chosen arbitrarily. Therefore, the necessary condition that (4.8.10) be valid for arbitrary motions is that the kinetic quantities $\mathbf{T}$ and $\mathbf{m}^\alpha$ are given by derivatives of the strain energy

$$\mathbf{T} = 2\rho\,\mathbf{F}\frac{\partial\Sigma}{\partial\mathbf{C}}\mathbf{F}^T \ , \quad \mathbf{m}^\alpha = \mathrm{m}\,\mathbf{F}^{-T}\frac{\partial\Sigma}{\partial\boldsymbol{\beta}_\alpha} \ , \tag{4.8.12}$$

where m is given by $(4.4.35)_1$.

Next, by using the conservation of mass equation in the form $(4.4.35)_1$, it follows that comparison of $(4.8.12)_1$ with the expressions (3.7.16) and (3.7.17) associated with the three-dimensional theory suggests defining the symmetric tensor $\mathbf{S}$ such that

$$\mathbf{S} = 2\rho_0\frac{\partial\Sigma}{\partial\mathbf{C}} \ , \quad a^{1/2}\,\mathbf{T} = A^{1/2}\,\mathbf{F}\,\mathbf{S}\,\mathbf{F}^T \ . \tag{4.8.13}$$

This causes $\mathbf{S}$ to be an analogue of the symmetric Piola-Kirchhoff stress in the three-dimensional theory.

Notice that once a form is specified for the strain energy function Σ, then the definition (4.4.32) can be used to obtain

$$\mathbf{m}^\alpha = \mathrm{m}\,\mathbf{F}^{-T}\frac{\partial\Sigma}{\partial\boldsymbol{\beta}_\alpha} \ ,$$

$$\mathbf{t}^i = (a^{1/2}\,\mathbf{T} - \mathbf{m}^\alpha\otimes\mathbf{d}_{3,\alpha})\,\mathbf{d}^i = 2\,\mathrm{m}\,\mathbf{F}\frac{\partial\Sigma}{\partial\mathbf{C}}\mathbf{D}^i - \mathbf{m}^\alpha\,(\mathbf{d}_{3,\alpha}\bullet\mathbf{d}^i) \ . \tag{4.8.14}$$

These expressions determine the kinetic quantities that appear in the local forms (4.4.35) of the balance laws.

Before closing this section, it is desirable to emphasize an important distinction between constitutive equations for a material in the three-dimensional theory and constitutive equations for a Cosserat shell. Within the context of the three-dimensional theory, constitutive equations characterize the response of a material at each material point and are independent of the shape of the three-dimensional body constructed from the material. In contrast, within the context of the Cosserat shell theory, the constitutive equations necessarily couple influences of the geometry of the shell-like structure with those of the response of the three-dimensional material from which the shell is constructed. For example, consider the case of three shells that are constructed from the same homogeneous nonlinear elastic three-dimensional material. With respect to their stress-free reference configurations, let the first shell be a flat plate with uniform thickness, let the second shell be a flat plate with variable thickness, and let the third shell be an arbitrary curved surface with uniform thickness. Due to the homogeneity of the geometric properties of the first shell, the strain energy function can be taken to be independent of the coordinates θ^α. Whereas, the variable thickness of the second shell and the variable curvature of the third shell cause the strain energy functions for these shells to depend on the coordinates θ^α.

In general, the complicated coupling of material and geometrical properties of the shell must be modeled by the Cosserat constitutive equations. In this regard, it will be seen later that under conditions of homogeneous deformations some guidance can be deduced from the three-dimensional theory which helps separate these influences of material and geometric properties.

Furthermore, it is noted that in section 4.3 the theoretical structure of the balance laws for the Cosserat shell theory was developed from the three-dimensional theory by using the kinematic assumption (4.2.7). From this perspective, constitutive equations for the kinetic quantities (4.8.14) are directly related to integrals through the thickness of the shell of three-dimensional constitutive equations [see (4.3.24), (4.3.34) and (4.3.38)]. Even for elastic materials, these integrals usually cannot be evaluated exactly when large deformations are included. However, from the perspective of the direct approach of section 4.4, the balance laws of the Cosserat theory are postulated without any specific connection with the three-dimensional theory. Moreover, the mechanical power equation of section 4.6 and the constitutive equations of this section are also developed by a direct approach. This has the advantage that the constitutive equations are necessarily consistent with the balance laws of the Cosserat theory. In particular, the constitutive equations of nonlinear elastic shells retain the fundamental properties associated with the ideal character of an elastic material, as discussed previously. In contrast, if the constitutive equations for elastic shells were obtained by approximate integration of the three-dimensional constitutive equations, then special care would have to be imposed on the integration procedure to ensure that these fundamental characteristics of elastic materials are preserved.

4.9 Constraints

Mechanical constraints in the Cosserat theory of shells can be considered in a manner directly analogous to that used to analyze constraints in the three-dimensional theory (see sec. 3.8). For example, the usual incompressibility constraint requires

$$\mathrm{J} = \det \mathbf{F} = 1 \ , \tag{4.9.1}$$

which can be differentiated to deduce that

$$\dot{\mathrm{J}} = \mathrm{J}\, \mathbf{F}^{-T} \bullet \dot{\mathbf{F}} = 0 \ , \quad \mathbf{I} \bullet \mathbf{D} = 0 \ , \tag{4.9.2}$$

where the expressions (4.6.8) and (4.6.11) have been used to obtain the form (4.9.2). In this regard, it is noted that the notion of incompressibility depends on the specific constitutive equations that are used for the shell. In particular, it will be shown later in section 4.25 that a modified incompressibility constraint is more convenient than the condition (4.9.1) for the constitutive equations developed later in this chapter.

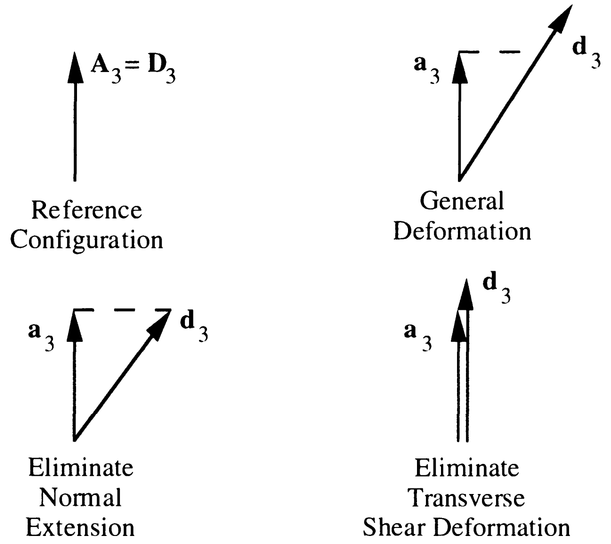

Fig. 4.9.1 Graphical representation of some constraints.

In general, the Cosserat theory allows the director $\mathbf{d}_3$ to be an arbitrary vector that is not tangent to the shell surface $(4.2.4)_2$. To interpret the physical meaning of different possible deformations of $\mathbf{d}_3$, it is most convenient to identify $\mathbf{d}_3$ with a material fiber that was normal to the shell surface in its reference configuration. Then, in the present configuration the shell is said to have experienced *normal extensional deformation* if the component of $\mathbf{d}_3$ in the direction $\mathbf{a}_3$ normal to the shell extends or contracts. Also, the shell is said to have experienced *transverse shear deformation* if at least one of the components of $\mathbf{d}_3$ in the directions $\mathbf{d}_\alpha$ tangent to the shell surface are nonzero.

It then follows that normal extensional deformation will be eliminated if the shell deformation is constrained so that

$$\mathbf{d}_3 \bullet \mathbf{a}_3 = \text{constant} \ . \tag{4.9.3}$$

Recalling that

$$\mathbf{a}_3 = \frac{\mathbf{d}^3}{|\mathbf{d}^3|} \ , \quad \mathbf{d}_3 \bullet \mathbf{d}^3 = 1 \ , \tag{4.9.4}$$

the constraint (4.9.3) is equivalent to the alternative constraint

$$\mathbf{d}^3 \bullet \mathbf{d}^3 = \text{constant} \ . \tag{4.9.5}$$

Moreover, using the facts that

$$\mathbf{d}^3 = \mathbf{F}^{-T}\,\mathbf{D}^3 \ , \quad \overline{\dot{\mathbf{F}^{-T}}} = -\,\mathbf{L}^T\,\mathbf{F}^{-T} \ , \quad \overline{\dot{\mathbf{d}^3}} = -\,\mathbf{L}^T\mathbf{d}^3 \ , \tag{4.9.6}$$

it can be shown that the rate form of (4.9.5) becomes

$$(\mathbf{d}^3\otimes\mathbf{d}^3) \bullet \mathbf{D} = 0 \quad \text{to eliminate normal extensional deformation.} \tag{4.9.7}$$

Similarly, it can be shown that transverse shear deformation will be eliminated if

$$\mathbf{d}_3 \bullet \mathbf{d}_\alpha = 0 \ . \tag{4.9.8}$$

This yields two constraints in rate forms

$$(\mathbf{d}_\alpha\otimes\mathbf{d}_3 + \mathbf{d}_3\otimes\mathbf{d}_\alpha) \bullet \mathbf{D} = 0 \quad \text{to eliminate transverse shear deformation.} \tag{4.9.9}$$

Another possible constraint is to require the material fiber in the $\mathbf{d}_3$ direction to have constant length so that

$$\mathbf{d}_3 \bullet \mathbf{d}_3 = \text{constant} \ . \tag{4.9.10}$$

This then yields the rate form

$$(\mathbf{d}_3\otimes\mathbf{d}_3) \bullet \mathbf{D} = 0 \ . \tag{4.9.11}$$

Some of these constraints are shown graphically in Fig. 4.9.1 by indicating the relative magnitude and orientation of the unit normal $\mathbf{a}_3$ and the director $\mathbf{d}_3$.

Each of these constraints $(4.9.2)_2$, (4.9.7), (4.9.9) and (4.9.11) is a special case of a class of general constraints which require

$$\boldsymbol{\gamma} \bullet \mathbf{D} + \boldsymbol{\gamma}^\alpha \bullet \dot{\boldsymbol{\beta}}_\alpha = 0 \ , \quad \boldsymbol{\gamma}^T = \boldsymbol{\gamma} \ , \tag{4.9.12}$$

where $\boldsymbol{\gamma}$ is a symmetric second order tensor and $\boldsymbol{\gamma}^\alpha$ are vectors. Moreover, it is assumed that under SRBM $\boldsymbol{\gamma}$ and $\boldsymbol{\gamma}^\alpha$ transform by

$$\boldsymbol{\gamma}^+ = \mathbf{Q}\,\boldsymbol{\gamma}\,\mathbf{Q}^T \ , \quad \boldsymbol{\gamma}^{\alpha+} = \boldsymbol{\gamma}^\alpha \ , \tag{4.9.13}$$

so that the constraint equation $(4.9.12)_1$ remains properly invariant under SRBM.

Motivated by the three-dimensional developments in section 3.8, it is possible to develop a constitutive theory for Cosserat shells in the presence of mechanical constraints by making the following five assumptions:

(i) The kinetic quantities $\mathbf{T}$, $\mathbf{t}^i$ and $\mathbf{m}^\alpha$ separate additively into two parts

$$\mathbf{T} = \hat{\mathbf{T}} + \bar{\mathbf{T}} \ , \quad \mathbf{t}^i = \hat{\mathbf{t}}^i + \bar{\mathbf{t}}^i \ , \quad \mathbf{m}^\alpha = \hat{\mathbf{m}}^\alpha + \bar{\mathbf{m}}^\alpha \ ,$$
$$\hat{\mathbf{T}} = a^{-1/2}\,[\,\hat{\mathbf{t}}^i\otimes\mathbf{d}_i + \hat{\mathbf{m}}^\alpha\otimes\mathbf{d}_{3,\alpha}\,] \ , \quad \bar{\mathbf{T}} = a^{-1/2}\,[\,\bar{\mathbf{t}}^i\otimes\mathbf{d}_i + \bar{\mathbf{m}}^\alpha\otimes\mathbf{d}_{3,\alpha}\,] \ , \tag{4.9.14}$$

where $\hat{\mathbf{T}}$, $\hat{\mathbf{t}}^i$ and $\hat{\mathbf{m}}^\alpha$ are determined by constitutive equations that characterize the particular unconstrained shell under consideration, and $\bar{\mathbf{T}}$, $\bar{\mathbf{t}}^i$ and $\bar{\mathbf{m}}^\alpha$ are constraint responses.

(ii) The constraint responses $\bar{\mathbf{T}}$, $\bar{\mathbf{t}}^i$ and $\bar{\mathbf{m}}^\alpha$ are functions of θ^α and t, which are workless in the sense that the mechanical power (4.6.17) vanishes

$$\bar{\mathbf{T}} \bullet \mathbf{D} + a^{-1/2}\,(\mathbf{F}^T\bar{\mathbf{m}}^\alpha) \bullet \dot{\boldsymbol{\beta}}_\alpha = 0 \ , \tag{4.9.15}$$

for all possible motions of the constrained material.

(iii) Both parts $\hat{\mathbf{T}}$ and $\bar{\mathbf{T}}$ of the kinetic quantity $\mathbf{T}$ are symmetric tensors

$$\hat{\mathbf{T}}^T = \hat{\mathbf{T}} \ , \quad \bar{\mathbf{T}}^T = \bar{\mathbf{T}} \ , \tag{4.9.16}$$

so that each of them satisfies the local form (4.4.33) of the balance of angular momentum.

(iv) Both parts $\hat{\mathbf{T}}$ and $\bar{\mathbf{T}}$ of the kinetic quantity $\mathbf{T}$, $\hat{\mathbf{t}}^i$, and $\bar{\mathbf{t}}^i$ of the kinetic quantity $\mathbf{t}^i$, and $\hat{\mathbf{m}}^\alpha$ and $\bar{\mathbf{m}}^\alpha$ of the kinetic quantity $\mathbf{m}^\alpha$ transform under SRBM by

$$\hat{\mathbf{T}}^+ = \mathbf{Q}\,\hat{\mathbf{T}}\,\mathbf{Q}^T \;,\quad \bar{\mathbf{T}}^T = \mathbf{Q}\,\bar{\mathbf{T}}\,\mathbf{Q}^T \;,\quad \hat{\mathbf{t}}^{i+} = \mathbf{Q}\,\hat{\mathbf{t}}^i \;,\quad \bar{\mathbf{t}}^{i+} = \mathbf{Q}\,\bar{\mathbf{t}}^i \;,$$
$$\hat{\mathbf{m}}^{\alpha+} = \mathbf{Q}\,\hat{\mathbf{m}}^\alpha \;,\quad \bar{\mathbf{m}}^{\alpha+} = \mathbf{Q}\,\bar{\mathbf{m}}^\alpha \;, \tag{4.9.17}$$

so that the kinetic quantities $\mathbf{T}$, $\mathbf{t}^i$ and $\mathbf{m}^\alpha$ transform by (4.5.5), and the expression (4.9.15) is properly invariant under SRBM.

(v) The tensors $\bar{\mathbf{T}}$, $\bar{\mathbf{m}}^\alpha$, $\boldsymbol{\gamma}$ and $\boldsymbol{\gamma}^\alpha$ are independent of the rates $\mathbf{L}$ and $\dot{\boldsymbol{\beta}}_\alpha$.

Using a Lagrange multiplier $\gamma(\theta^\alpha,t)$ which is an arbitrary function of (θ^α,t), the equation (4.9.15), subject to the constraint (4.9.12), can be rewritten in the form

$$(\,\bar{\mathbf{T}} - a^{-1/2}\,\gamma\boldsymbol{\gamma}\,) \bullet \mathbf{D} + a^{-1/2}\left[\mathbf{F}^T\bar{\mathbf{m}}^\alpha - \gamma\,\boldsymbol{\gamma}^\alpha\right] \bullet \dot{\boldsymbol{\beta}}_\alpha = 0 \;. \tag{4.9.18}$$

Since at least one of the components of $\boldsymbol{\gamma}$ or $\boldsymbol{\gamma}^\alpha$ in (4.9.12) [say $\boldsymbol{\gamma} \bullet (\mathbf{d}_3\otimes\mathbf{d}_3)$] is nonzero, the value of γ can be specified so that

$$\gamma = \frac{\bar{\mathbf{T}} \bullet (\mathbf{d}_3\otimes\mathbf{d}_3)}{a^{-1/2}\,\boldsymbol{\gamma} \bullet (\mathbf{d}_3\otimes\mathbf{d}_3)} \;. \tag{4.9.19}$$

It then follows that the coefficient of the component $[\mathbf{D} \bullet (\mathbf{d}_3\otimes\mathbf{d}_3)]$ in (4.9.18) vanishes. Consequently, this component of $\mathbf{D}$ can be chosen to satisfy the constraint (4.9.12) for arbitrary values of the other components of $\mathbf{D}$ and $\dot{\boldsymbol{\beta}}_\alpha$. Moreover, since the coefficients of the rates $\{\mathbf{D}\,,\,\dot{\boldsymbol{\beta}}_\alpha\}$ in (4.9.18) are independent of these rates, and the coefficient of $\mathbf{D}$ is a symmetric tensor, it follows that the constraint responses $\bar{\mathbf{T}}$ and $\bar{\mathbf{m}}^\alpha$ must take the forms

$$\bar{\mathbf{T}} = a^{-1/2}\,\gamma\boldsymbol{\gamma} \;,\quad \bar{\mathbf{t}}^i = \gamma\left[\boldsymbol{\gamma}\,\mathbf{d}^i - \mathbf{F}^{-T}\boldsymbol{\gamma}^\alpha(\mathbf{d}_{3,\alpha} \bullet \mathbf{d}^i)\right] ,\quad \bar{\mathbf{m}}^\alpha = \gamma\,\mathbf{F}^{-T}\boldsymbol{\gamma}^\alpha \;, \tag{4.9.20}$$

where γ remains an arbitrary function of (θ^α,t) that is determined by the equations of motion and the boundary conditions.

If more than one constraint is imposed on the shell, then the constraint responses $\bar{\mathbf{T}}$ and $\bar{\mathbf{m}}^\alpha$ can be represented as the sum of a Lagrange multiplier times each of the constraint tensors $\boldsymbol{\gamma}$ and $\boldsymbol{\gamma}^\alpha$. For example, the case of a nonlinear Kirchhoff-Love shell is one of the simplest nonlinear shell theories because it eliminates both normal extensional deformation and transverse shear deformation. For this case, the three constraints (4.9.7) and (4.9.9) are imposed simultaneously. However, since transverse shear deformation is eliminated, it also follows that the constraints (4.9.7) and (4.9.11) become identical so that for the Kirchhoff-Love shell the constraints reduce to

$$(\mathbf{d}_\alpha\otimes\mathbf{d}_3 + \mathbf{d}_3\otimes\mathbf{d}_\alpha) \bullet \mathbf{D} = 0 \;,\quad (\mathbf{d}_3\otimes\mathbf{d}_3) \bullet \mathbf{D} = 0 \;, \tag{4.9.21}$$

which are independent of $\dot{\boldsymbol{\beta}}_\alpha$. It then follows that the constraint responses $\bar{\mathbf{T}}$ and $\bar{\mathbf{m}}^\alpha$ can be represented in the forms

$$\bar{\mathbf{T}} = a^{-1/2}\,\gamma^{3\alpha}\,(\mathbf{d}_\alpha\otimes\mathbf{d}_3 + \mathbf{d}_3\otimes\mathbf{d}_\alpha) + a^{-1/2}\,\gamma^{33}\,(\mathbf{d}_3\otimes\mathbf{d}_3) \;,\quad \bar{\mathbf{m}}^\alpha = 0 \;, \tag{4.9.22}$$

where $\gamma^{3\alpha}$ and γ^{33} are Lagrange multipliers that are functions of (θ^α,t). Thus, with the help of $(4.9.14)_2$ it can be shown that

$$\bar{\mathbf{t}}^\alpha = \gamma^{3\alpha}\,\mathbf{d}_3 \;,\quad \bar{\mathbf{t}}^3 = \gamma^{3\alpha}\,\mathbf{d}_\alpha + \gamma^{33}\,\mathbf{d}_3 \;. \tag{4.9.23}$$

Notice that since all three components of $\bar{\mathbf{t}}^3$ are arbitrary functions of (θ^α,t), the intrinsic director couple $\mathbf{t}^3$ also becomes an arbitrary function of (θ^α,t). This means that the director momentum equation $(4.4.35)_4$ can be satisfied for arbitrary admissible motions of the constrained shell by using $(4.9.14)_3$ and $(4.9.23)_2$ to determine the functions γ^{3i}

$$\gamma^{3i} = \left[\, m\,\mathbf{b}^3 - \hat{\mathbf{t}}^3 + \hat{\mathbf{m}}^\sigma{}_{,\sigma} - m\,(y^3\,\dot{\mathbf{v}} + y^{33}\,\dot{\mathbf{w}}_3)\right] \cdot \mathbf{d}^i \,. \tag{4.9.24}$$

Also, in view of the constraint response $(4.9.23)_1$, the equation of linear momentum $(4.4.35)_3$ includes the effect of $\gamma^{3\alpha}$ and becomes

$$m\,(\dot{\mathbf{v}} + y^3\,\dot{\mathbf{w}}_3) = m\,\mathbf{b} + (\hat{\mathbf{t}}^\alpha + \gamma^{3\alpha}\,\mathbf{d}_3)_{,\alpha} \,, \tag{4.9.25}$$

where $\gamma^{3\alpha}$ is determined by equation (4.9.24).

This nonlinear Kirchhoff-Love shell theory is simpler than the complete Cosserat theory because the constraints (4.9.21) totally determine the director $\mathbf{d}_3$ in terms of deformations of the shell surface. The director momentum equation is satisfied by the Lagrange multipliers $\gamma^{3\alpha}$ and γ^{33}, and the balance laws of the theory reduce to the conservation of mass $(4.4.35)_1$ and the balance of linear momentum (4.9.25).

Moreover, as another example, the constraint responses (4.9.20) associated with the incompressibility constraint (4.9.2) become

$$\bar{\mathbf{T}} = a^{-1/2}\,\gamma\,\mathbf{I} \;\;,\;\; \bar{\mathbf{t}}^i = \gamma\,\mathbf{d}^i \;\;,\;\; \bar{\mathbf{m}}^\alpha = 0 \;. \tag{4.9.26}$$

In general, the constraint responses $\bar{\mathbf{T}}$ and $\bar{\mathbf{m}}^\alpha$ influence the equations of linear and director momentum but not the conservation of mass or the balance of angular momentum equations (since $\bar{\mathbf{T}}$ is a symmetric tensor). Therefore, if more than six independent kinematic constraints are imposed, then $\bar{\mathbf{T}}$ and $\bar{\mathbf{m}}^\alpha$ will have more than six independent components which are arbitrary functions of position and time. This means that even when appropriate boundary conditions are specified, it will not be possible to uniquely determine all components of $\bar{\mathbf{T}}$ and $\bar{\mathbf{m}}^\alpha$. However, it is reasonable to expect that the arbitrariness in $\bar{\mathbf{T}}$ and $\bar{\mathbf{m}}^\alpha$ that remains after the equations of motion and boundary conditions are satisfied will not influence the overall motion of the constrained shell.

Motivated by the discussion of constraints in section 3.8 within the context of the three-dimensional theory, it might seem adequate to eliminate the terms $\dot{\boldsymbol{\beta}}_\alpha$ in (4.9.12) and only consider constraints on $\mathbf{D}$. However, it will presently be shown that the condition that $\mathbf{D}$ vanishes is not sufficient to ensure that the shell moves as a rigid body. Specifically, since $\mathbf{D}$ is defined only on the reference surface of the shell and not pointwise through the thickness of the shell, it does not contain complete information about homogeneity of the three-dimensional deformation of the shell structure. The additional information required is contained in $\dot{\boldsymbol{\beta}}_\alpha$. Consequently, the constraints required to ensure that the shell moves as a rigid body demand that both $\mathbf{D}$ and $\dot{\boldsymbol{\beta}}_\alpha$ vanish pointwise on the reference surface of the shell

$$\mathbf{D} = 0 \;\;,\;\; \dot{\boldsymbol{\beta}}_\alpha = 0 \;. \tag{4.9.27}$$

Moreover, it is convenient to use (4.6.7), (4.6.8) and (4.6.15) to derive the expression

$$\dot{\boldsymbol{\beta}}_\alpha = \mathbf{F}^{-1}\mathbf{L}_{,\alpha}\,\mathbf{d}_3 \;, \tag{4.9.28}$$

which is a measure of the $\mathbf{d}_3$ component of the gradient of $\mathbf{L}$.

In order to be more clear, it is desirable to consider an example where $\mathbf{D}$ vanishes pointwise but the shell is not rigid and $\dot{\boldsymbol{\beta}}_\alpha$ is nonzero. To this end, consider a sector of a right circular cylindrical shell of uniform thickness and radius R in its reference configuration which is deformed into a sector of a right circular cylindrical shell of uniform thickness and radius r(t) in its present configuration (see Fig. 4.9.2). The kinematics of the reference configuration are specified in terms of cylindrical polar base vectors by

$$\mathbf{X} = R\,\mathbf{e}_r\left(\frac{\theta^1}{R}\right) + \theta^2\,\mathbf{e}_z \;,\quad \mathbf{D}_1 = \mathbf{e}_\theta\left(\frac{\theta^1}{R}\right) \;,\quad \mathbf{D}_2 = \mathbf{e}_z \;,\quad \mathbf{D}_3 = \mathbf{e}_r\left(\frac{\theta^1}{R}\right) \;. \tag{4.9.29}$$

Similarly, the kinematics of the present configuration are specified by

$$\mathbf{x} = r\,\mathbf{e}_r\left(\frac{\theta^1}{r}\right) + \theta^2\,\mathbf{e}_z \;,\quad \mathbf{d}_1 = \mathbf{e}_\theta\left(\frac{\theta^1}{r}\right) \;,\quad \mathbf{d}_2 = \mathbf{e}_z \;,\quad \mathbf{d}_3 = \mathbf{e}_r\left(\frac{\theta^1}{r}\right) \;, \tag{4.9.30}$$

where it is important to note that the orientation of the base vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$ in (4.9.30) are different from those in (4.9.29) because the angles θ^1/r and θ^1/R associated with the same material point (θ^1=constant) change with time. It then follows from the definitions (4.2.9), (4.6.7), (4.6.11) and (4.6.15) that

$$\mathbf{F} = \mathbf{e}_\theta\left(\frac{\theta^1}{r}\right)\otimes\mathbf{e}_\theta\left(\frac{\theta^1}{R}\right) + \mathbf{e}_z\otimes\mathbf{e}_z + \mathbf{e}_r\left(\frac{\theta^1}{r}\right)\otimes\mathbf{e}_r\left(\frac{\theta^1}{R}\right) \;,$$

$$\mathbf{D} = 0 \;,\quad \mathbf{L} = \mathbf{W} = -\frac{\dot{r}\theta^1}{r^2}\left[\mathbf{d}_1\otimes\mathbf{d}_3 - \mathbf{d}_3\otimes\mathbf{d}_1\right] \;,$$

$$\boldsymbol{\beta}_1 = \left[\frac{1}{r} - \frac{1}{R}\right]\mathbf{D}_1 \;,\quad \boldsymbol{\beta}_2 = 0 \;,\quad \dot{\boldsymbol{\beta}}_1 = -\frac{\dot{r}}{r^2}\mathbf{D}_1 \;,\quad \dot{\boldsymbol{\beta}}_2 = 0 \;. \tag{4.9.31}$$

Thus, although $\mathbf{D}$ vanishes pointwise the shell is not rigid because it is flattened as the radius r increases.

Before closing this section, it should be mentioned that it is often necessary to modify the functional form of the strain energy Σ when constraints are imposed on the shell. For example, consider a rectangular plate whose sides are normal to the rectangular Cartesian base vectors $\mathbf{e}_\alpha$ and whose major surfaces are normal to the vector $\mathbf{e}_3$ which is in the thickness direction. If the plate is pulled in uniaxial tension in the $\mathbf{e}_1$ direction, then the Poisson effect usually causes the length of the plate in the $\mathbf{e}_2$ direction and its thickness to decrease. Now, if for simplicity, the plate is constrained so that its thickness cannot change (i.e. the director $\mathbf{d}_3$ remains of constant length), then the predicted response to uniaxial tension in the $\mathbf{e}_1$ direction will be too stiff if the same functional form for the strain energy Σ is used in the constrained theory. Thus, from a constitutive point of view, the constrained theory can be more complicated than the general theory (Simo et al, 1990).

To avoid this problem, it is possible to modify the constitutive equation for Σ so that the response of the constrained theory will simulate the softer response of the unconstrained plate. This procedure is similar to that used in modifying the elastic moduli for generalized plane stress problems relative to plane strain problems [see sec. 67 in (Sokolnikoff, 1956)]. Moreover, it will be shown later that the director inertia coefficients y^3 and y^{33} model not only the mass distribution through the thickness of the shell but also information about mode shapes of vibration. Therefore, the appropriate values of y^3 and y^{33} for a deformable shell can be different from those for a constrained shell (e.g. a rigid shell).

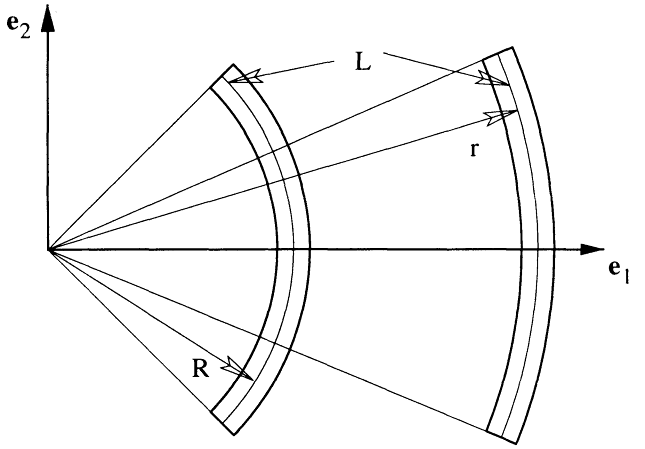

Fig. 4.9.2 Deformation of a sector of a right circular cylindrical shell.

It is also important to note that constraints can influence the nature of boundary conditions since not all components of the velocities $\mathbf{v}$ and $\mathbf{w}_3$ remain independent. For example, the discussion by Naghdi (1972, sec. 15) of a restricted theory of shells is similar to one that could be developed for a theory of shells in which the director is constrained to be the unit normal to the shell surface. Also, it is of interest to note that some constraints are singular in nature in the sense that the number of boundary conditions for the linearized theory can be different from that for the nonlinear theory (Rubin, 1987d).

For the special case of a constrained elastic shell, the parts $\hat{\mathbf{T}}$, $\hat{\mathbf{t}}^i$, $\hat{\mathbf{m}}^\alpha$ of the kinetic quantities associated with the constitutive equations satisfy the condition (4.8.1) that the mechanical power due to these parts is equal to the rate of change of the strain energy function. Then, $\hat{\mathbf{T}}$, $\hat{\mathbf{t}}^i$, $\hat{\mathbf{m}}^\alpha$ are determined in terms of derivatives of the strain energy function by formulas of the type (4.8.12).

4.10 Initial and boundary conditions

The local forms of conservation of mass $(4.4.35)_1$, the balance of linear momentum $(4.4.35)_2$ and the balance of director momentum $(4.4.35)_3$ are partial differential equations which require both initial and boundary conditions. Specifically, the conservation of mass $(4.4.35)_1$ is first order in time with respect to density ρ so it is necessary to specify the initial value of density at each point of the shell

$$\rho(\theta^\alpha,0) = \bar{\rho}(\theta^\alpha) \quad \text{on P for } t = 0 \,. \tag{4.10.1}$$

Also, the balance of linear momentum $(4.4.35)_2$ and the balance of director momentum $(4.4.35)_3$ are second order in time with respect to position $\mathbf{x}$ and the director $\mathbf{d}_3$ so that it is necessary to specify the initial values of $\mathbf{x}$ and $\mathbf{d}_3$, as well as the initial values of the velocities $\mathbf{v}$ and $\mathbf{w}_3$ at each point of the shell

$$\hat{\mathbf{x}}(\theta^\alpha,0) = \bar{\mathbf{x}}(\theta^\alpha) \;,\quad \hat{\mathbf{v}}(\theta^\alpha,0) = \bar{\mathbf{v}}(\theta^\alpha) \;,$$
$$\hat{\mathbf{d}}_3(\theta^\alpha,0) = \bar{\mathbf{d}}_3(\theta^\alpha) \;,\quad \hat{\mathbf{w}}_3(\theta^\alpha,0) = \bar{\mathbf{w}}_3(\theta^\alpha) \quad \text{on P for } t = 0 \,. \tag{4.10.2}$$

Guidance for determining the appropriate forms of boundary conditions is usually obtained by considering the rate of work done by the resultant forces and moments applied to the boundary ∂P of the shell. Consequently, from (4.6.1) it is observed that $\mathbf{t} \cdot \mathbf{v}$ is the rate of work of the resultant force and $\mathbf{m}^3 \cdot \mathbf{w}_3$ is the rate of work of the director couple, both measured per unit length of the curve ∂P. At each point of the curve ∂P it is possible to define a right-handed orthogonal coordinate system with base vectors $\{\mathbf{n}\,,\boldsymbol{\tau}\,,\mathbf{a}_3\}$, where $\mathbf{n}$ is the unit outward normal to ∂P in the plane P, $\boldsymbol{\tau}$ is tangent to the curve ∂P in the plane P, and $\mathbf{a}_3$ is normal to the plane P. Then, with reference to these base vectors the expressions $\mathbf{t} \cdot \mathbf{v}$ and $\mathbf{m}^3 \cdot \mathbf{w}_3$ can be written in the forms

$$\mathbf{t} \cdot \mathbf{v} = (\mathbf{t} \cdot \mathbf{n})\,(\mathbf{v} \cdot \mathbf{n}) + (\mathbf{t} \cdot \boldsymbol{\tau})\,(\mathbf{v} \cdot \boldsymbol{\tau}) + (\mathbf{t} \cdot \mathbf{a}_3)\,(\mathbf{v} \cdot \mathbf{a}_3) \;,$$
$$\mathbf{m}^3 \cdot \mathbf{w}_3 = (\mathbf{m}^3 \cdot \mathbf{n})\,(\mathbf{w}_3 \cdot \mathbf{n}) + (\mathbf{m}^3 \cdot \boldsymbol{\tau})\,(\mathbf{w}_3 \cdot \boldsymbol{\tau}) + (\mathbf{m}^3 \cdot \mathbf{a}_3)\,(\mathbf{w}_3 \cdot \mathbf{a}_3) \;, \quad \text{on } \partial\text{P} \tag{4.10.3}$$

Using these representations, it is possible to define four types of boundary conditions

<u>Kinematic</u>: All three components of the velocities are specified

$$(\mathbf{v} \cdot \mathbf{n}) \;,\; (\mathbf{v} \cdot \boldsymbol{\tau}) \;,\; (\mathbf{v} \cdot \mathbf{a}_3) \;,\; (\mathbf{w}_3 \cdot \mathbf{n}) \;,\; (\mathbf{w}_3 \cdot \boldsymbol{\tau}) \;,\; (\mathbf{w}_3 \cdot \mathbf{a}_3) \quad \text{specified on } \partial\text{P for all } t \geq 0 \;, \tag{4.10.4}$$

<u>Kinetic</u>: All three components of the resultant force and director couple are specified

$$(\mathbf{t} \cdot \mathbf{n}) \;,\; (\mathbf{t} \cdot \boldsymbol{\tau}) \;,\; (\mathbf{t} \cdot \mathbf{a}_3) \;,\; (\mathbf{m}^3 \cdot \mathbf{n}) \;,\; (\mathbf{m}^3 \cdot \boldsymbol{\tau}) \;,\; (\mathbf{m}^3 \cdot \mathbf{a}_3) \;, \quad \text{specified on } \partial\text{P for all } t \geq 0 \;, \tag{4.10.5}$$

<u>Mixed</u>: Kinematic boundary conditions are specified on part of the boundary ∂P and kinetic boundary conditions are specified on the remaining part of the boundary.

<u>Mixed-Mixed</u>: Conjugate components of both the velocities and the resultant force and director couple are specified

$$\{(\mathbf{t} \cdot \mathbf{n}) \text{ or } (\mathbf{v} \cdot \mathbf{n})\} \;,\; \{(\mathbf{t} \cdot \boldsymbol{\tau}) \text{ or } (\mathbf{v} \cdot \boldsymbol{\tau})\} \;,\; \{(\mathbf{t} \cdot \mathbf{a}_3) \text{ or } (\mathbf{v} \cdot \mathbf{a}_3)\} \;,$$
$$\{(\mathbf{m}^3 \cdot \mathbf{n}) \text{ or } (\mathbf{w}_3 \cdot \mathbf{n})\} \;,\; \{(\mathbf{m}^3 \cdot \boldsymbol{\tau}) \text{ or } (\mathbf{w}_3 \cdot \boldsymbol{\tau})\} \;,\; \{(\mathbf{m}^3 \cdot \mathbf{a}_3) \text{ or } (\mathbf{w}_3 \cdot \mathbf{a}_3)\} \;, \quad \text{specified on } \partial\text{P for all } t \geq 0 \;, \tag{4.10.6}$$

Essentially, the conjugate components $(\mathbf{t} \cdot \mathbf{n})$, $(\mathbf{t} \cdot \boldsymbol{\tau})$, $(\mathbf{t} \cdot \mathbf{a}_3)$ are the responses to the motions $(\mathbf{v} \cdot \mathbf{n})$, $(\mathbf{v} \cdot \boldsymbol{\tau})$, $(\mathbf{v} \cdot \mathbf{a}_3)$, respectively, and the components $(\mathbf{m}^3 \cdot \mathbf{n})$, $(\mathbf{m}^3 \cdot \boldsymbol{\tau})$, $(\mathbf{m}^3 \cdot \mathbf{a}_3)$ are the responses to the motions $(\mathbf{w}_3 \cdot \mathbf{n})$, $(\mathbf{w}_3 \cdot \boldsymbol{\tau})$, $(\mathbf{w}_3 \cdot \mathbf{a}_3)$, respectively. Therefore, it is important to emphasize that, for example, both $(\mathbf{v} \cdot \mathbf{n})$ and $(\mathbf{t} \cdot \mathbf{n})$ cannot be specified at the same point of ∂P because this would mean that both the motion and the response can be specified independently of the material properties and geometry of the body. Notice also, that since the initial position of points on the boundary ∂P are specified by the initial condition $(4.10.2)_1$, the velocity boundary conditions (4.10.4) can be used to determine the position of the boundary for all time. This means that the kinematic boundary conditions (4.10.4) could also be characterized by specifying the position of points on the boundary for all time. Similar comments apply to the initial condition $(4.10.2)_3$ and velocity boundary conditions (4.10.4) associated with the director $\mathbf{d}_3$. Furthermore, for static problems, the position vector and the director will include a measure of arbitrariness if insufficient kinematic boundary conditions are supplied to specify the three translational and three rotational rigid-body degrees of freedom.

With regard to the boundary conditions for the moment applied to the shell, it is convenient to consider the balance of angular momentum (4.4.7) and define the moment $\mathbf{m}$ (about the position $\mathbf{x}$), which is measured per unit arclength ds and is applied to the boundary ∂P, by the equation

$$\mathbf{m} = \mathbf{d}_3 \times \mathbf{m}^3 \ . \tag{4.10.7}$$

Here, it is important to distinguish between this moment $\mathbf{m}$ and the director couple $\mathbf{m}^3$. Specifically, it is noted that in general $\mathbf{m}^3$ can have a component in the $\mathbf{d}_3$ direction which resists stretching of the director $\mathbf{d}_3$ and makes no contribution to the moment $\mathbf{m}$.

4.11 Further restrictions on constitutive equations for shells constructed from homogeneous anisotropic nonlinear elastic materials

From the developments in section 4.3 it can be seen that the force $\mathbf{t}$ (4.3.25) and the couple $\mathbf{m}^3$ (4.3.39) which are applied to the boundary ∂P of the shell, are related to weighted integrals of the stress vector through the thickness of the shell structure. This means that the constitutive equations (4.8.14) for the kinetic quantities $\mathbf{t}^i$ and $\mathbf{m}^\alpha$ necessarily couple the effects of geometric properties of the shell with the constitutive properties of the material from which the shell is constructed. Even if the shell is constructed using a homogeneous material, curvature and variable thickness of the shell in its reference configuration can significantly influence the constitutive equations for the Cosserat shell model.

Recently, Naghdi and Rubin (1995) have developed restrictions on constitutive equations for shells constructed from homogeneous anisotropic nonlinear elastic materials. These restrictions which ensure that solutions of the Cosserat theory can reproduce exactly the complete class of homogeneous solutions of the three-dimensional theory. The main results of this work are summarized below.

First, it is recalled from (3.2.34), (4.2.7), (4.2.8), (4.3.2), (4.3.24), (4.3.34), (4.3.38), (4.4.10) and (4.4.32), that

$$a^{1/2}\,\mathbf{T} = \mathbf{t}^i \otimes \mathbf{d}_i + \mathbf{m}^\alpha \otimes \mathbf{d}_{3,\alpha}$$
$$= \int_{\xi_1}^{\xi_2} g^{1/2}\,\mathbf{T}^* \left[\, \mathbf{g}^i \otimes \mathbf{d}_i + \mathbf{g}^\alpha \otimes \theta^3\,\mathbf{d}_{3,\alpha} \,\right] d\theta^3 \ ,$$
$$= \int_{\xi_1}^{\xi_2} g^{1/2}\,\mathbf{T}^* \left[\, \mathbf{g}^\alpha \otimes (\mathbf{d}_\alpha + \theta^3\,\mathbf{d}_{3,\alpha}) + \mathbf{g}^3 \otimes \mathbf{d}_3 \,\right] d\theta^3 \ ,$$
$$= \int_{\xi_1}^{\xi_2} g^{1/2}\,\mathbf{T}^* \left[\, \mathbf{g}^i \otimes \mathbf{g}_i \,\right] d\theta^3 \ . \tag{4.11.1}$$

However, since $\mathbf{g}^i \otimes \mathbf{g}_i = \mathbf{I}$, it follows that

$$a^{1/2}\,\mathbf{T} = \int_{\xi_1}^{\xi_2} g^{1/2}\,\mathbf{T}^*\, d\theta^3 \ . \tag{4.11.2}$$

Next, with the help of (3.2.28), (3.7.15), $(4.4.35)_1$ and $(4.8.12)_1$, it can be shown that

$$a^{1/2}\,\mathbf{T} = 2\,m\,\mathbf{F}\frac{\partial \Sigma}{\partial \mathbf{C}}\mathbf{F}^T \ , \ \ g^{1/2}\,\mathbf{T}^* = 2\,m^*\,\mathbf{F}^*\frac{\partial \Sigma^*}{\partial \mathbf{C}^*}\mathbf{F}^{*T} \ , \tag{4.11.3}$$

so that equation (4.11.2) can be rewritten in the form

$$a^{1/2}\,\mathbf{T} = 2\,m\,\mathbf{F}\frac{\partial \Sigma}{\partial \mathbf{C}}\mathbf{F}^T = 2\int_{\xi_1}^{\xi_2} m^*\,\mathbf{F}^*\frac{\partial \Sigma^*}{\partial \mathbf{C}^*}\mathbf{F}^{*T}\, d\theta^3 \ ,$$
$$m\frac{\partial \Sigma}{\partial \mathbf{C}} = \int_{\xi_1}^{\xi_2} m^*\,\mathbf{F}^{-1}\,\mathbf{F}^*\frac{\partial \Sigma^*}{\partial \mathbf{C}^*}\mathbf{F}^{*T}\,\mathbf{F}^{-T} d\theta^3 \ . \tag{4.11.4}$$

Moreover, (4.3.38), (4.8.12) and (4.11.3) can be used to determine that

$$m\,\mathbf{F}^{-T}\frac{\partial \Sigma}{\partial \boldsymbol{\beta}_\alpha} = \mathbf{m}^\alpha = \int_{\xi_1}^{\xi_2} \theta^3\,g^{1/2}\,\mathbf{T}^*\,\mathbf{g}^\alpha\ d\theta^3 \ ,$$
$$m\frac{\partial \Sigma}{\partial \boldsymbol{\beta}_\alpha} = \int_{\xi_1}^{\xi_2} \theta^3\,2\,m^*\,\mathbf{F}^T\,\mathbf{F}^*\frac{\partial \Sigma^*}{\partial \mathbf{C}^*}\mathbf{F}^{*T}\,\mathbf{g}^\alpha\ d\theta^3 \ ,$$
$$m\frac{\partial \Sigma}{\partial \boldsymbol{\beta}_\alpha} = \int_{\xi_1}^{\xi_2} \theta^3\,2\,m^*\,\mathbf{F}^T\,\mathbf{F}^*\frac{\partial \Sigma^*}{\partial \mathbf{C}^*}\mathbf{G}^\alpha\ d\theta^3 \ . \tag{4.11.5}$$

Now, if the material from which the shell is constructed is homogeneous and uniform, then ρ_0^* and Σ^* are explicitly independent of the coordinates θ^i

$$\rho_0^* = \text{constant} \ , \ \ \Sigma^* = \hat{\Sigma}^*(\mathbf{C}^*) \ . \tag{4.11.6}$$

Thus, with the help of (3.2.28) and (4.3.5) it can be shown that

$$m = \int_{\xi_1}^{\xi_2} m^*\, d\theta^3 = \rho_0^*\,A^{1/2}\,\bar{H} \ , \ \ A^{1/2}\,\bar{H}(\theta^\alpha) = \int_{\xi_1}^{\xi_2} G^{1/2}\, d\theta^3 \ , \tag{4.11.7}$$

where $\bar{H}$ is a scalar function of θ^α only. Next, attention is restricted to three-dimensionally homogeneous deformations [(4.2.14)-(4.2.16)] for which

$$\mathbf{F}^* = \mathbf{F} = \mathbf{F}(t) \ , \ \ \mathbf{C}^* = \mathbf{C} = \mathbf{F}^T\,\mathbf{F} \ , \ \ \boldsymbol{\beta}_\alpha = 0 \ , \tag{4.11.8}$$

so that (4.11.4) and (4.11.5) yield the restrictions

$$\left.\frac{\partial \Sigma}{\partial \mathbf{C}}\right|_{\boldsymbol{\beta}_\beta=0} = \frac{\partial \hat{\Sigma}^*(\mathbf{C})}{\partial \mathbf{C}} \ , \quad \left.\frac{\partial \Sigma}{\partial \boldsymbol{\beta}_\alpha}\right|_{\boldsymbol{\beta}_\beta=0} = 2\,\mathbf{C}\,\frac{\partial \hat{\Sigma}^*(\mathbf{C})}{\partial \mathbf{C}}\,\mathbf{H}^\alpha \ , \tag{4.11.9}$$

where $\mathbf{H}^\alpha(\theta^\beta)$ are tensor functions of θ^β defined by

$$A^{1/2}\,\bar{H}\,\mathbf{H}^\alpha(\theta^\beta) = \int_{\xi_1}^{\xi_2} \theta^3\, G^{1/2}\, \mathbf{G}^\alpha \; d\theta^3 \ . \tag{4.11.10}$$

For comparison purposes, it is important to note that the quantities $\{\mathbf{S}$ and $\mathrm{H}\}$ defined in (Naghdi and Rubin, 1995) equal the quantities $\{A^{1/2}\,\mathbf{S}$ and $A^{1/2}\,\bar{H}\}$ defined here. Also, here use has been made of the result (4.11.7) to simplify (4.11.9).

The expressions (4.11.9) place necessary restrictions on the constitutive equations for shells which ensure consistency with exact solutions for all homogeneous deformations. Specifically, it will presently be shown that for homogeneous deformations the restrictions (4.11.9) also cause

$$m\,\mathbf{b}_c + \mathbf{t}^\alpha{}_{,\alpha} = 0 \ , \quad m\,\mathbf{b}_c^3 - \mathbf{t}^3 + \mathbf{m}^\alpha{}_{,\alpha} = 0 \ , \tag{4.11.11}$$

where the external force $\mathbf{b}_c$ and couple $\mathbf{b}_c^3$ due to tractions on the major surfaces of the shell are given by the expressions (4.3.15) and (4.3.36). Thus, the balances of linear momentum and director momentum (4.4.35) reduce to the equations

$$m\,(\dot{\mathbf{v}} + y^3\,\dot{\mathbf{w}}_3) = m\,\mathbf{b}_b \ , \quad m\,(y^3\,\dot{\mathbf{v}} + y^{33}\,\dot{\mathbf{w}}_3) = m\,\mathbf{b}_b^3 \ , \tag{4.11.12}$$

which express the familiar result that for homogeneous deformations the accelerations are balanced by terms associated with body forces only.

To prove equations (4.11.11), it is first observed from (4.8.14) and (4.11.9) that for homogeneous deformations the constitutive equations for $\mathbf{t}^i$ and $\mathbf{m}^\alpha$ can be written in the forms

$$\mathbf{t}^i = 2\,m\,\mathbf{F}\,\frac{\partial \hat{\Sigma}^*(\mathbf{C})}{\partial \mathbf{C}}\,\mathbf{F}^T\left[\mathbf{d}^i - \mathbf{F}^{-T}\,\mathbf{H}^\alpha\,(\mathbf{d}_{3,\alpha}\bullet\mathbf{d}^i)\right] \ ,$$

$$\mathbf{m}^\alpha = 2\,m\,\mathbf{F}\,\frac{\partial \hat{\Sigma}^*(\mathbf{C})}{\partial \mathbf{C}}\,\mathbf{F}^T\left[\mathbf{F}^{-T}\,\mathbf{H}^\alpha\right] \ . \tag{4.11.13}$$

Moreover, with the help of (3.2.7), $(3.2.33)_1$ and (3.7.15), it follows that for homogeneous deformations

$$J^*\,\mathbf{T}^* = \left[2\,\rho_0^*\,\mathbf{F}\,\frac{\partial \hat{\Sigma}^*(\mathbf{C})}{\partial \mathbf{C}}\,\mathbf{F}^T\right] \ . \tag{4.11.14}$$

Thus, for homogeneous deformations $\mathbf{T}^*$ is independent of the coordinates θ^i so that (4.11.7), (4.11.10), (4.11.13) and (4.11.14) can be used to deduce that

$$\mathbf{t}^i = \mathbf{T}^*\int_{\xi_1}^{\xi_2} g^{1/2}\left[\mathbf{d}^i - \theta^3\,\mathbf{g}^\alpha\,(\mathbf{d}_{3,\alpha}\bullet\mathbf{d}^i)\right] d\theta^3 = \mathbf{T}^*\int_{\xi_1}^{\xi_2} g^{1/2}\,\mathbf{g}^i\, d\theta^3 \ ,$$

$$\mathbf{m}^\alpha = \mathbf{T}^*\int_{\xi_1}^{\xi_2} \theta^3\, g^{1/2}\,\mathbf{g}^\alpha\, d\theta^3 \ , \tag{4.11.15}$$

where use has been made of (4.1.12) and (4.2.15) to derive the expressions

$$\mathbf{d}_{3,\alpha}\bullet\mathbf{d}^i = (\mathbf{F}\,\mathbf{D}_{3,\alpha})\bullet(\mathbf{F}^{-T}\,\mathbf{D}^i) = \mathbf{D}_{3,\alpha}\bullet\mathbf{D}^i \ ,$$

$$\mathbf{d}^i = \mathbf{F}^{-T}\,\mathbf{D}^i = (\mathbf{g}^j \otimes \mathbf{G}_j)\,\mathbf{D}^i = \mathbf{g}^i + \theta^3\,\mathbf{g}^\beta\,(\mathbf{D}_{3,\beta} \bullet \mathbf{D}^i) \ , \tag{4.11.16}$$

for homogeneous deformations. Obviously, the results (4.11.15) are compatible with the expressions $(4.3.24)_1$, $(4.3.34)_1$ and $(4.3.38)_1$. To complete the proof, it follows from $(2.3.15)_2$ that

$$\begin{aligned}(g^{1/2}\,\mathbf{g}^\alpha)_{,\alpha} &= -\,(g^{1/2}\,\mathbf{g}^3)_{,3} \ , \\ (\theta^3\,g^{1/2}\,\mathbf{g}^\alpha)_{,\alpha} &= g^{1/2}\,\mathbf{g}^3 - (\theta^3\,g^{1/2}\,\mathbf{g}^3)_{,3} \ .\end{aligned} \tag{4.11.17}$$

Consequently, using the fact that for homogeneous deformations $\mathbf{T}^*$ is constant, differentiation of (4.11.15) and use of the expressions (4.3.15) and (4.3.36), yields

$$\begin{aligned}
\mathbf{t}^\alpha{}_{,\alpha} &= \mathbf{T}^*\,\Big[\{g^{1/2}\xi_{2,\alpha}\mathbf{g}^\alpha\}_{\theta^3=\xi_2} - \{g^{1/2}\xi_{1,\alpha}\mathbf{g}^\alpha\}_{\theta^3=\xi_1} \\
&\qquad + \int_{\xi_1}^{\xi_2}\{g^{1/2}\,\mathbf{g}^\alpha\}_{,\alpha}\,d\theta^3\Big] \ , \\
\mathbf{t}^\alpha{}_{,\alpha} &= -\,\mathbf{T}^*\,\Big[\{g^{1/2}(\mathbf{g}^3 - \xi_{2,\alpha}\mathbf{g}^\alpha)\}_{\theta^3=\xi_2} - \{g^{1/2}(\mathbf{g}^3 - \xi_{1,\alpha}\mathbf{g}^\alpha)\}_{\theta^3=\xi_1}\Big] \ , \\
\mathbf{t}^\alpha{}_{,\alpha} &= -\,m\,\mathbf{b}_c \ , \\
\mathbf{m}^\alpha{}_{,\alpha} &= \mathbf{T}^*\,\Big[\{g^{1/2}\xi_2\xi_{2,\alpha}\mathbf{g}^\alpha\}_{\theta^3=\xi_2} - \{g^{1/2}\xi_1\xi_{1,\alpha}\mathbf{g}^\alpha\}_{\theta^3=\xi_1} \\
&\qquad + \int_{\xi_1}^{\xi_2}\{\theta^3 g^{1/2}\,\mathbf{g}^\alpha\}_{,\alpha}\,d\theta^3\Big] \ , \\
\mathbf{m}^\alpha{}_{,\alpha} &= -\,\mathbf{T}^*\,\Big[\{g^{1/2}\xi_2(\mathbf{g}^3 - \xi_{2,\alpha}\mathbf{g}^\alpha)\}_{\theta^3=\xi_2} - \{g^{1/2}\xi_1(\mathbf{g}^3 - \xi_{1,\alpha}\mathbf{g}^\alpha)\}_{\theta^3=\xi_1}\Big] \\
&\qquad + \int_{\xi_1}^{\xi_2}\{g^{1/2}\,\mathbf{g}^3\}\,d\theta^3\Big] \ , \\
\mathbf{m}^\alpha{}_{,\alpha} &= -\,m\,\mathbf{b}_c^3 + \mathbf{t}^3 \ ,
\end{aligned} \tag{4.11.18}$$

which completes the proof.

It was shown in (Naghdi and Rubin, 1995) that the restrictions (4.11.9) can easily be satisfied by defining an alternative deformation measure $\bar{\mathbf{C}}$ through the expression

$$\bar{\mathbf{C}} = \bar{\mathbf{C}}(\mathbf{C},\boldsymbol{\beta}_\alpha,\mathbf{H}^\beta) = [\mathbf{I} + \boldsymbol{\beta}_\alpha \otimes \mathbf{H}^\alpha]^T\,\mathbf{C}\,[\mathbf{I} + \boldsymbol{\beta}_\beta \otimes \mathbf{H}^\beta] = \bar{\mathbf{C}}^T \ , \tag{4.11.19}$$

which has the property that for homogeneous deformations

$$\bar{\mathbf{C}}(\mathbf{C},0,\mathbf{H}^\beta) = \mathbf{C} \ . \tag{4.11.20}$$

Now, in general, the strain energy Σ can be specified in terms of the three-dimensional strain energy function Σ^* by the form

$$\Sigma = \Sigma^*(\bar{\mathbf{C}}) + \Psi(\mathbf{C},\boldsymbol{\beta}_\alpha,\mathcal{V}) \ , \quad \mathcal{V} = \{\boldsymbol{\Lambda}\,,\,\mathbf{D}_i\,,\,\bar{\mathrm{H}}\,,\,\mathbf{H}^\alpha\,,\,\theta^\beta\} \ , \tag{4.11.21}$$

where Ψ is an additive part of the strain energy function due to inhomogeneous deformations. Also, for generality in modeling the response of an inhomogeneous material and general geometry in the reference configuration, Ψ is allowed to depend on the reference geometry as well as on the coordinates θ^β through the parameters $\mathcal{V}$.

Next, using the expression (4.11.19), it can be shown that

$$\frac{\partial \Sigma}{\partial \mathbf{C}} = [\mathbf{I} + \boldsymbol{\beta}_\alpha \otimes \mathbf{H}^\alpha] \frac{\partial \Sigma^*(\bar{\mathbf{C}})}{\partial \bar{\mathbf{C}}} [\mathbf{I} + \boldsymbol{\beta}_\beta \otimes \mathbf{H}^\beta]^T + \frac{\partial \Psi}{\partial \mathbf{C}} \ ,$$

$$\frac{\partial \Sigma}{\partial \boldsymbol{\beta}_\alpha} = 2\,\mathbf{C}\,[\mathbf{I} + \boldsymbol{\beta}_\beta \otimes \mathbf{H}^\beta] \frac{\partial \Sigma^*(\bar{\mathbf{C}})}{\partial \bar{\mathbf{C}}} \mathbf{H}^\alpha + \frac{\partial \Psi}{\partial \boldsymbol{\beta}_\alpha} \ . \tag{4.11.22}$$

It then follows that the functional form (4.11.21) will satisfy the restrictions (4.11.9), provided that Ψ satisfies the restrictions

$$\left.\frac{\partial \Psi}{\partial \mathbf{C}}\right|_{\boldsymbol{\beta}_\beta = 0} = 0 \ , \quad \left.\frac{\partial \Psi}{\partial \boldsymbol{\beta}_\alpha}\right|_{\boldsymbol{\beta}_\beta = 0} = 0 \ . \tag{4.11.23}$$

At present, it is not known how to determine an expression for Ψ which will depend explicitly on the form of the three dimensional strain energy Σ^*. The restrictions (4.11.23) indicate that Ψ cannot be a linear function of $\boldsymbol{\beta}_\alpha$. Consequently, for simplicity it is assumed that Ψ is a quadratic function of $\boldsymbol{\beta}_\alpha$ which is independent of $\mathbf{C}$ and takes the form

$$\rho_0 \Psi = \frac{1}{2} \bar{H}\, |\mathbf{D}_3|^{-2}\, \mathbf{K} \bullet \big[(\boldsymbol{\beta}_\alpha \otimes \mathbf{D}^\alpha) \otimes (\boldsymbol{\beta}_\beta \otimes \mathbf{D}^\beta)\big] = \frac{1}{2} \bar{H}\, |\mathbf{D}_3|^{-2}\, \mathbf{K}^{\alpha\beta} \bullet (\boldsymbol{\beta}_\alpha \otimes \boldsymbol{\beta}_\beta), \tag{4.11.24}$$

where the fourth order tensor $\mathbf{K}$ and the second order tensors $\mathbf{K}^{\alpha\beta}$ are related to each other and have the properties that

$$\mathbf{K}^{T(2)} = \mathbf{K} \ , \quad \mathbf{D}^3\, {}^{LT}\mathbf{K} = 0 \ , \quad \mathbf{K}\mathbf{D}^3 = 0 \ ,$$

$$\mathbf{K} = {}^{LT}\big[\mathbf{D}_\alpha \otimes \mathbf{K}^{\alpha\beta} \otimes \mathbf{D}_\beta\big] \ , \quad {}^{LT}\mathbf{K}^{\beta\alpha} = \mathbf{K}^{\alpha\beta} \ , \quad \mathbf{K}^{\alpha\beta} = \mathbf{D}^\alpha\, {}^{LT}\mathbf{K}\mathbf{D}^\beta \ . \tag{4.11.25}$$

In view of these symmetries, it can be shown that the tensors $\mathbf{K}$ and $\mathbf{K}^{\alpha\beta}$ each have twenty-one independent components. These tensors are independent of time but they can depend on the reference geometry through the parameters $\mathcal{V}$. Also, it is noted that the term $|\mathbf{D}_3|$ and the second order tensor $(\boldsymbol{\beta}_\alpha \otimes \mathbf{D}^\alpha)$ are used in the definition (4.11.24) to remove dependence of $\mathbf{K}$ on the explicit choice of the units of the coordinates defining $\mathbf{D}_i$ and $\boldsymbol{\beta}_\alpha$. However, as can be seen from $(4.11.25)_5$, the tensors $\mathbf{K}^{\alpha\beta}$ depend explicitly on this choice of coordinates. Next, differentiation of (4.11.24) yields the results

$$\rho_0 \frac{\partial \Psi}{\partial \mathbf{C}} = 0 \ , \quad \rho_0 \frac{\partial \Psi}{\partial \boldsymbol{\beta}_\alpha} = \bar{H}\, |\mathbf{D}_3|^{-2}\, \mathbf{K}^{\alpha\beta}\, \boldsymbol{\beta}_\beta \ , \tag{4.11.26}$$

which satisfy the restrictions (4.11.23).

In summary, it is assumed that for a general shell the strain energy function is specified so that

$$\bar{\mathbf{C}} = [\mathbf{I} + \boldsymbol{\beta}_\alpha \otimes \mathbf{H}^\alpha]^T\, \mathbf{C}\, [\mathbf{I} + \boldsymbol{\beta}_\beta \otimes \mathbf{H}^\beta] \ ,$$

$$m\,\Sigma = m\,\Sigma^*(\bar{\mathbf{C}}) + \frac{1}{2} A^{1/2} \bar{H}\, |\mathbf{D}_3|^{-2}\, \mathbf{K}^{\alpha\beta} \bullet (\boldsymbol{\beta}_\alpha \otimes \boldsymbol{\beta}_\beta) \ ,$$

$$m \frac{\partial \Sigma}{\partial \mathbf{C}} = m\, [\mathbf{I} + \boldsymbol{\beta}_\alpha \otimes \mathbf{H}^\alpha] \frac{\partial \Sigma^*(\bar{\mathbf{C}})}{\partial \bar{\mathbf{C}}} [\mathbf{I} + \boldsymbol{\beta}_\beta \otimes \mathbf{H}^\beta]^T \ ,$$

$$m \frac{\partial \Sigma}{\partial \boldsymbol{\beta}_\alpha} = 2\,m\,\mathbf{C}\,[\mathbf{I} + \boldsymbol{\beta}_\beta \otimes \mathbf{H}^\beta] \frac{\partial \Sigma^*(\bar{\mathbf{C}})}{\partial \bar{\mathbf{C}}} \mathbf{H}^\alpha + A^{1/2} \bar{H}\, |\mathbf{D}_3|^{-2}\, \mathbf{K}^{\alpha\beta}\, \boldsymbol{\beta}_\beta \ , \tag{4.11.27}$$

where the conservation of mass $(4.4.35)_1$ has been used. Moreover, the constitutive equations (4.8.12) and (4.8.14) for $\mathbf{T}$, $\mathbf{m}^\alpha$ and $\mathbf{t}^i$ associated with (4.11.27) can be represented in the forms

$$a^{1/2}\,\mathbf{T} = 2\,m\,\mathbf{F}\,[\mathbf{I} + \boldsymbol{\beta}_\alpha \otimes \mathbf{H}^\alpha]\,\frac{\partial \Sigma^*(\bar{\mathbf{C}})}{\partial \bar{\mathbf{C}}}\,[\mathbf{I} + \boldsymbol{\beta}_\beta \otimes \mathbf{H}^\beta]^T\,\mathbf{F}^T \ ,$$

$$\mathbf{m}^\alpha = 2\,m\,\mathbf{F}\,[\mathbf{I} + \boldsymbol{\beta}_\beta \otimes \mathbf{H}^\beta]\,\frac{\partial \Sigma^*(\bar{\mathbf{C}})}{\partial \bar{\mathbf{C}}}\,\mathbf{H}^\alpha + A^{1/2}\,\bar{H}\,|\mathbf{D}_3|^{-2}\,\mathbf{F}^{-T}\,\mathbf{K}^{\alpha\beta}\,\boldsymbol{\beta}_\beta \ ,$$

$$\mathbf{t}^i = 2\,m\,\mathbf{F}\,[\mathbf{I} + \boldsymbol{\beta}_\alpha \otimes \mathbf{H}^\alpha]\,\frac{\partial \Sigma^*(\bar{\mathbf{C}})}{\partial \bar{\mathbf{C}}}\,[\mathbf{I} + \boldsymbol{\beta}_\beta \otimes \mathbf{H}^\beta]^T\,\mathbf{D}^i - \mathbf{m}^\alpha\,(\mathbf{d}_{3,\alpha} \bullet \mathbf{d}^i) \ . \tag{4.11.28}$$

The functional forms (4.11.27) and (4.11.28) of the constitutive equations depend explicitly on the three-dimensional strain energy function and they ensure consistency with exact solutions for all nonlinear homogeneous deformations. However, it is still necessary to determine functional forms for $\mathbf{K}^{\alpha\beta}$ which depend on the geometrical and material properties of the shell structure. Usually, this is accomplished by first determining values for $\mathbf{K}$ by matching solutions of the linearized theory of plates for pure bending and torsion. Then, the values of $\mathbf{K}^{\alpha\beta}$ are determined by equations (4.11.25). Specific forms for $\mathbf{K}$ and $\mathbf{K}^{\alpha\beta}$ will be developed later for orthotropic shells using this procedure.

4.12 A small strain theory

In order to better understand the nature of the constitutive assumption (4.11.27), it is of interest to consider the simpler case of a small strain theory where the three-dimensional strain energy function is given by the form (3.12.2) which is a quadratic function of the Lagrangian strain. Moreover, with the help of the conservation of mass $(4.4.35)_1$ and the definition (4.11.7), it can be shown that when the three-dimensional mass density is independent of the thickness coordinate θ^3 of the shell it follows that

$$\rho_0 = \rho_0^*\,\bar{H} \ . \tag{4.12.1}$$

Then, in view of the assumption (4.11.27), the strain energy function is expressed in the form

$$\rho_0\,\Sigma = \frac{1}{2}\,\bar{H}\,\mathbf{K}^* \bullet (\bar{\mathbf{E}} \otimes \bar{\mathbf{E}}) + \frac{1}{2}\bar{H}\,|\mathbf{D}_3|^{-2}\,\mathbf{K}^{\alpha\beta} \bullet (\boldsymbol{\beta}_\alpha \otimes \boldsymbol{\beta}_\beta) \ , \tag{4.12.2}$$

where it is convenient to define the strains $\mathbf{E}$ and $\bar{\mathbf{E}}$ by

$$\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{I}) \ , \quad \bar{\mathbf{E}} = \frac{1}{2}(\bar{\mathbf{C}} - \mathbf{I}) \ . \tag{4.12.3}$$

Also, $\mathbf{K}^*$ is the value associated with the three-dimensional material which is evaluated on the reference surface (θ^3=0).

Next, using the definitions (4.11.27) and (4.12.3), it can be shown that

$$\bar{\mathbf{E}} = (\mathbf{I}+\boldsymbol{\beta}_\alpha\otimes\mathbf{H}^\alpha)^T\mathbf{E}\,(\mathbf{I}+\boldsymbol{\beta}_\beta\otimes\mathbf{H}^\beta) + \frac{1}{2}(\boldsymbol{\beta}_\alpha\otimes\mathbf{H}^\alpha+\mathbf{H}^\alpha\otimes\boldsymbol{\beta}_\alpha)$$
$$+\frac{1}{2}(\boldsymbol{\beta}_\alpha\otimes\mathbf{H}^\alpha)^T(\boldsymbol{\beta}_\beta\otimes\mathbf{H}^\beta)\ ,$$
$$\mathrm{m}\,\Sigma = \frac{1}{2}A^{1/2}\,\bar{H}\,\mathbf{K}^*\bullet(\bar{\mathbf{E}}\otimes\bar{\mathbf{E}}) + \frac{1}{2}A^{1/2}\,\bar{H}\,|\mathbf{D}_3|^{-2}\,\mathbf{K}^{\alpha\beta}\bullet(\boldsymbol{\beta}_\alpha\otimes\boldsymbol{\beta}_\beta)\ ,$$
$$2\mathrm{m}\frac{\partial\Sigma}{\partial\mathbf{C}} = (\mathbf{I}+\boldsymbol{\beta}_\alpha\otimes\mathbf{H}^\alpha)\big[A^{1/2}\,\bar{H}\,\mathbf{K}^*\bullet\bar{\mathbf{E}}\big](\mathbf{I}+\boldsymbol{\beta}_\beta\otimes\mathbf{H}^\beta)^T\ ,$$
$$\mathrm{m}\frac{\partial\Sigma}{\partial\boldsymbol{\beta}_\alpha} = \mathbf{C}(\mathbf{I}+\boldsymbol{\beta}_\beta\otimes\mathbf{H}^\beta)\big[A^{1/2}\,\bar{H}\,\mathbf{K}^*\bullet\bar{\mathbf{E}}\big]\mathbf{H}^\alpha + A^{1/2}\,\bar{H}\,|\mathbf{D}_3|^{-2}\,\mathbf{K}^{\alpha\beta}\bullet\boldsymbol{\beta}_\beta\ ,\tag{4.12.4}$$

where use has been made of the conservation of mass $(4.4.35)_1$. Moreover, with the help of (4.8.12) and (4.8.14), the constitutive equations for $\mathbf{T}$, $\mathbf{m}^\alpha$ and $\mathbf{t}^i$ associated with (4.12.4) can be written in the forms

$$a^{1/2}\,\mathbf{T} = \mathbf{F}(\mathbf{I}+\boldsymbol{\beta}_\alpha\otimes\mathbf{H}^\alpha)\big[A^{1/2}\,\bar{H}\,\mathbf{K}^*\bullet\bar{\mathbf{E}}\big](\mathbf{I}+\boldsymbol{\beta}_\beta\otimes\mathbf{H}^\beta)^T\mathbf{F}^T\ ,$$
$$\mathbf{m}^\alpha = \mathbf{F}(\mathbf{I}+\boldsymbol{\beta}_\beta\otimes\mathbf{H}^\beta)\big[A^{1/2}\,\bar{H}\,\mathbf{K}^*\bullet\bar{\mathbf{E}}\big]\mathbf{H}^\alpha + \mathbf{F}^{-T}\big[A^{1/2}\,\bar{H}\,|\mathbf{D}_3|^{-2}\,\mathbf{K}^{\alpha\beta}\,\boldsymbol{\beta}_\beta\big]\ ,$$
$$\mathbf{t}^i = \mathbf{F}(\mathbf{I}+\boldsymbol{\beta}_\alpha\otimes\mathbf{H}^\alpha)\big[A^{1/2}\,\bar{H}\,\mathbf{K}^*\bullet\bar{\mathbf{E}}\big](\mathbf{I}+\boldsymbol{\beta}_\beta\otimes\mathbf{H}^\beta)^T\mathbf{D}^i - \mathbf{m}^\alpha\,(\mathbf{d}_{3,\alpha}\bullet\mathbf{d}^i)\ .\tag{4.12.5}$$

Since these equations satisfy the restrictions (4.11.9), they ensure consistency with exact solutions for all homogeneous deformations when the strain energy function of the three-dimensional material is given by (3.12.2).

It is observed that $\bar{\mathbf{E}}$ in (4.12.4) is a nonlinear function of the strains $\mathbf{E}$ and $\boldsymbol{\beta}_\alpha$. However, for the small strain theory, quadratic and higher order terms in these strains are neglected. It then follows that equations (4.12.4) are approximated by

$$\bar{\mathbf{E}} = \mathbf{E} + \frac{1}{2}(\boldsymbol{\beta}_\alpha\otimes\mathbf{H}^\alpha+\mathbf{H}^\alpha\otimes\boldsymbol{\beta}_\alpha)\ ,$$
$$\mathrm{m}\,\Sigma = \frac{1}{2}A^{1/2}\,\bar{H}\,\mathbf{K}^*\bullet(\bar{\mathbf{E}}\otimes\bar{\mathbf{E}}) + \frac{1}{2}A^{1/2}\,\bar{H}\,|\mathbf{D}_3|^{-2}\,\mathbf{K}^{\alpha\beta}\bullet(\boldsymbol{\beta}_\alpha\otimes\boldsymbol{\beta}_\beta)\ ,$$
$$2\mathrm{m}\frac{\partial\Sigma}{\partial\mathbf{C}} = A^{1/2}\,\bar{H}\,\mathbf{K}^*\bullet\bar{\mathbf{E}}\ ,$$
$$\mathrm{m}\frac{\partial\Sigma}{\partial\boldsymbol{\beta}_\alpha} = \big[A^{1/2}\,\bar{H}\,\mathbf{K}^*\bullet\bar{\mathbf{E}}\big]\,\mathbf{H}^\alpha + A^{1/2}\,\bar{H}\,|\mathbf{D}_3|^{-2}\,\mathbf{K}^{\alpha\beta}\,\boldsymbol{\beta}_\beta\ .\tag{4.12.6}$$

Also, the constitutive equations for $\mathbf{T}$, $\mathbf{m}^\alpha$ and $\mathbf{t}^i$ associated with (4.12.6) can be represented in the forms

$$a^{1/2}\,\mathbf{T} = \mathbf{F}\big[A^{1/2}\,\bar{H}\,\mathbf{K}^*\bullet\bar{\mathbf{E}}\big]\mathbf{F}^T\ ,$$
$$\mathbf{m}^\alpha = A^{1/2}\,\bar{H}\,\mathbf{F}^{-T}\big[\,\{\mathbf{K}^*\bullet\bar{\mathbf{E}}\}\mathbf{H}^\alpha + |\mathbf{D}_3|^{-2}\,\mathbf{K}^{\alpha\beta}\,\boldsymbol{\beta}_\beta\,\big]\ ,$$
$$\mathbf{t}^i = \mathbf{F}\big[A^{1/2}\,\bar{H}\,\mathbf{K}^*\bullet\bar{\mathbf{E}}\big]\,\mathbf{D}^i - \mathbf{m}^\alpha\,(\mathbf{d}_{3,\alpha}\bullet\mathbf{d}^i)\ .\tag{4.12.7}$$

Due to the approximate nature of the strain energy function (4.12.6), it does not satisfy the restrictions (4.11.9) exactly. Consequently, in general (4.12.6) will not be consistent with exact solutions for all homogeneous deformations. Nevertheless, (4.12.6) satisfies the restrictions (4.11.9) when quadratic and higher order terms in the strains $\mathbf{E}$ and $\boldsymbol{\beta}_\alpha$ have been neglected. Moreover, the strain energy function (4.12.6) and the constitutive equations (4.12.7) remain properly invariant under superposed rigid body motions. This means that the strain energy function (4.12.6) can be viewed as a special simple constitutive assumption that is valid for large deformations and large rotations of the shell but small strains. Also, it is noted that when $\mathbf{H}^\alpha$ vanishes (which can happen when the shell is a flat plate in its reference configuration), the expressions (4.12.4) and (4.12.5) are identical to (4.12.6) and (4.12.7), respectively.

For later reference, it is convenient to introduce a right-handed orthonormal set of vectors $\mathbf{M}_i$ defined by

$$\mathbf{M}'_1 = \mathbf{M}'_2 \times \mathbf{M}'_3 \;, \quad \mathbf{M}'_2 = \frac{\mathbf{M}'_3 \times \mathbf{D}_1}{|\mathbf{M}'_3 \times \mathbf{D}_1|} \;, \quad \mathbf{M}'_3 = \frac{\mathbf{D}_3}{|\mathbf{D}_3|} \;,$$

$$\mathbf{M}_1 = \cos\phi\, \mathbf{M}'_1 + \sin\phi\, \mathbf{M}'_2 \;, \quad \mathbf{M}_2 = -\sin\phi\, \mathbf{M}'_1 + \cos\phi\, \mathbf{M}'_2 \;,$$
$$\mathbf{M}_3 = \mathbf{M}'_3 \;. \tag{4.12.8}$$

This specification causes $\mathbf{M}_1$ and $\mathbf{M}_2$ to lie in the plane that is normal to the director $\mathbf{D}_3$. Also, it allows the axes $\mathbf{M}_\alpha$ of anisotropy to be rotated about the $\mathbf{D}_3$ axis by the angle ϕ relative to the $\mathbf{M}'_1$ axis such as occurs in the layers of many laminated shells. Using these definitions, the components of $\bar{\mathbf{E}}$, $\boldsymbol{\beta}_\alpha$, $\mathbf{K}^*$, $\mathbf{K}$ and $\mathbf{K}^{\alpha\beta}$ relative $\mathbf{M}_i$ are defined such that

$$\bar{\mathbf{E}} = \bar{E}_{ij}\,(\mathbf{M}_i \otimes \mathbf{M}_j) \;, \quad \bar{E}_{ij} = \bar{\mathbf{E}} \bullet (\mathbf{M}_i \otimes \mathbf{M}_j) \;,$$
$$\boldsymbol{\beta}_\alpha = \beta_{i\alpha}\, \mathbf{M}_i \;, \quad \beta_{i\alpha} = \boldsymbol{\beta}_\alpha \bullet \mathbf{M}_i \;,$$
$$\mathbf{K}^* = K^*_{ijkl}(\mathbf{M}_i \otimes \mathbf{M}_j \otimes \mathbf{M}_k \otimes \mathbf{M}_l) \;, \quad K^*_{ijkl} = \mathbf{K}^* \bullet (\mathbf{M}_i \otimes \mathbf{M}_j \otimes \mathbf{M}_k \otimes \mathbf{M}_l) \;,$$
$$\mathbf{K} = K_{i\alpha j\beta}\,(\mathbf{M}_i \otimes \mathbf{M}_\alpha \otimes \mathbf{M}_j \otimes \mathbf{M}_\beta) \;, \quad K_{i\alpha j\beta} = \mathbf{K} \bullet (\mathbf{M}_i \otimes \mathbf{M}_\alpha \otimes \mathbf{M}_j \otimes \mathbf{M}_\beta) \;,$$
$$\mathbf{K}^{\alpha\beta} = K^{\alpha\beta}_{ij}\,(\mathbf{M}_i \otimes \mathbf{M}_j) \;, \quad K^{\alpha\beta}_{ij} = \mathbf{K}^{\alpha\beta} \bullet (\mathbf{M}_i \otimes \mathbf{M}_j) \;,$$
$$K_{j\delta i\gamma} = K_{i\gamma j\delta} \;, \quad K^{\beta\alpha}_{ij} = K^{\alpha\beta}_{ji} \;, \tag{4.12.9}$$

where by definition, $\mathbf{K}$ automatically satisfies the restrictions $(4.11.25)_{2,3}$. Moreover, using $(4.11.25)_6$, it can be shown that

$$K^{\alpha\beta}_{ij} = (\mathbf{D}^\alpha \bullet \mathbf{M}_\gamma)\,(\mathbf{D}^\beta \bullet \mathbf{M}_\delta)\, K_{i\gamma j\delta} \;. \tag{4.12.10}$$

For the case of an orthotropic material relative to the basis $\mathbf{M}_i$ the nine nontrivial components of K^*_{ijkl} are given by (3.12.12) and it is reasonable to assume that the eight nontrivial components of $K_{i\gamma j\delta}$ are

$$\begin{pmatrix} K_{1111} & K_{1122} & K_{1212} & K_{1221} \\ K_{2121} & K_{2222} & K_{3131} & K_{3232} \end{pmatrix} . \tag{4.12.11}$$

Furthermore, for the case of an isotropic material the values of K^*_{ijkl} are given by (3.12.13) in terms of the two moduli K^* and μ^* and it is assumed that

$$K_{\alpha\beta\gamma\delta} = K_1\,(\delta_{\alpha\beta}\,\delta_{\gamma\delta}) + K_2\,(\delta_{\alpha\gamma}\,\delta_{\beta\delta}) + K_3\,(\delta_{\alpha\delta}\,\delta_{\beta\gamma}) \;,$$
$$K_{1111} = K_{2222} = K_1 + K_2 + K_3 \;, \quad K_{1122} = K_1 \;,$$
$$K_{1212} = K_{2121} = K_2 \;, \quad K_{1221} = K_3 \;, \quad K_{3131} = K_{3232} = K_4 \;, \tag{4.12.12}$$

where $\delta_{\alpha\beta}$ is the two-dimensional Kronecker delta. For either case, the values of $K_{i\gamma j\delta}$ are independent of time but can be functions of the reference geometry.

Moreover, for the simple case when the directors $\mathbf{D}_i$ are orthonormal vectors that are equal to $\mathbf{M}_i$

$$\mathbf{D}_i = \mathbf{M}_i \ , \tag{4.12.13}$$

then (4.12.9) reduces to

$$K_{ij}^{\alpha\beta} = K_{i\alpha j\beta} \ , \tag{4.12.14}$$

and the expanded form of the strain energy (4.12.2) for an isotropic material becomes

$$\begin{aligned}
2\rho_0\Sigma = \bar{H}\left[(K^* - \tfrac{2}{3}\mu^*)\,(\delta_{\alpha\beta}\,\delta_{\gamma\delta}) + \mu^*\,(\delta_{\alpha\gamma}\,\delta_{\beta\delta} + \delta_{\alpha\delta}\,\delta_{\beta\gamma})\right]\bar{E}_{\alpha\beta}\,\bar{E}_{\gamma\delta} \\
+ 2\,\bar{H}\left[K^* - \tfrac{2}{3}\mu^*\right]\bar{E}_{\alpha\alpha}\,\bar{E}_{33} + \bar{H}\left[K^* + \tfrac{4}{3}\mu^*\right]\bar{E}_{33}\,\bar{E}_{33} \\
+ \bar{H}\left[\mu^*\right](\bar{E}_{\alpha 3} + \bar{E}_{3\alpha})\,(\bar{E}_{\alpha 3} + \bar{E}_{3\alpha}) \\
+ \bar{H}\left[K_1\,(\delta_{\alpha\beta}\,\delta_{\gamma\delta}) + K_2\,(\delta_{\alpha\gamma}\,\delta_{\beta\delta}) + K_3\,(\delta_{\alpha\delta}\,\delta_{\beta\gamma})\right]\beta_{\alpha\beta}\,\beta_{\gamma\delta} \\
+ \bar{H}\left[K_4\right]\beta_{3\alpha}\,\beta_{3\alpha}\,.
\end{aligned} \tag{4.12.15}$$

In order to compare this expression with the one given by (Naghdi, 1972, p.557), it is noted that the quantities $\{\psi\ ,\ A^{\alpha\beta}\ ,\ e_{\alpha\beta}\ ,\ \gamma_\alpha\ ,\ \gamma_3\ ,\ \kappa_{i\alpha}\}$ there should be replaced by the quantities $\{\Sigma\ ,\ \delta_{\alpha\beta}\ ,\ E_{\alpha\beta}\ ,\ 2E_{\alpha 3}\ ,\ E_{33}\ ,\ \beta_{i\alpha}\}$, respectively. Then, (4.12.15) gives expressions for the coefficients α_i in (Naghdi, 1972, p.557) which are

$$\begin{aligned}
\alpha_1 = \alpha_9 = \bar{H}\,(K^* - \tfrac{2}{3}\mu^*) = \frac{2\bar{H}\mu^*\nu^*}{(1-2\nu^*)}\ , \\
\alpha_2 = \alpha_3 = \bar{H}\,\mu^*\ ,\quad \alpha_4 = \bar{H}\,(K^* + \tfrac{4}{3}\mu^*) = \frac{2\bar{H}\mu^*(1-\nu^*)}{(1-2\nu^*)}\ , \\
\alpha_5 = \bar{H}\,K_1\ ,\quad \alpha_6 = \bar{H}\,K_2\ ,\quad \alpha_7 = \bar{H}\,K_3\ ,\quad \alpha_8 = \bar{H}\,K_4\ , \\
\alpha_{10} = \alpha_{11} = \alpha_{12} = \alpha_{13} = 0\ ,
\end{aligned} \tag{4.12.16}$$

where the Table 3.12.1 has been used to express the results in terms of μ^* and ν^*. The values of $\{\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_9\}$ in (4.12.16) have been obtained here by using the restrictions of the previous section which ensure consistency with exact solutions for all nonlinear homogeneous deformations. Therefore, it is not surprising that these values (apart from α_3) are the same as those obtained by Naghdi (1972, sec. 24) who considered specific small deformation homogeneous solutions of the linearized theory of plates. Difficulties with determining the values of α_3 and α_8 have been discussed by Naghdi (1972, sec. 24), and the relationship of the value for α_3 obtained here with those proposed by Reissner (1945) and Mindlin (1951) has been discussed by Naghdi and Rubin (1995).

Values of the constitutive coefficients (4.12.11) associated with inhomogeneous deformations will be developed in later sections by comparing solutions of bending and torsion of an orthotropic plate with exact solutions of the linearized three-dimensional theory.

4.13 Small deformations superimposed on a large deformation

In general, the equations of motion of an elastic shell are nonlinear partial differential equations for which only a few exact analytical solutions are known. The notion of small deformations superimposed on a large deformation is used to develop approximate equations that are linear functions of the superimposed small deformations and therefore are simpler to solve (e.g. Green and Naghdi, 1971). Such equations can be used to analyze vibrations of prestressed or rotating structures such as space satellites, and buckling of structures. Moreover, if the large deformation represents an actual solution of the equations of motion, then the small deformation equations can be used to analyze linear stability of the large deformation solution.

To develop these small deformation equations, the position vector $\mathbf{x}(\theta^\alpha,t)$ is represented as an additive function of the large deformation $\hat{\mathbf{x}}(\theta^\alpha,t)$ and the small displacement vector $\mathbf{u}(\theta^\alpha,t)$. Also, the directors $\mathbf{d}_i(\theta^\alpha,t)$ are represented as additive functions of the large deformations $\hat{\mathbf{d}}_i(\theta^\alpha,t)$ and the small director displacements $\boldsymbol{\delta}_i(\theta^\alpha,t)$ such that

$$\mathbf{x}(\theta^\alpha,t) = \hat{\mathbf{x}}(\theta^\alpha,t) + \mathbf{u}(\theta^\alpha,t) \ ,$$

$$\mathbf{d}_i(\theta^\alpha,t) = \hat{\mathbf{d}}_i(\theta^\alpha,t) + \boldsymbol{\delta}_i(\theta^\alpha,t) \ , \quad \boldsymbol{\delta}_\alpha = \mathbf{u}_{,\alpha} \ . \tag{4.13.1}$$

The displacement vector $\mathbf{u}$ and the director displacements $\boldsymbol{\delta}_i$ are considered to be small in the sense that their magnitudes and the magnitudes of their space and time derivatives are small enough that quadratic and higher order terms in these quantities can be neglected. Thus, for example

$$|\mathbf{u}|^2 << |\mathbf{u}| \ , \quad |\boldsymbol{\delta}_i|^2 << |\boldsymbol{\delta}_i| \ . \tag{4.13.2}$$

Of course, the values of $\mathbf{u}$, $\boldsymbol{\delta}_i$, and their space and time derivatives must be appropriately normalized in order to express these inequalities in unitless forms.

Quantities other than the displacement vector $\mathbf{u}$ and the director displacements $\boldsymbol{\delta}_i$ are separated additively into a part associated with the large deformation which is denoted by placing a hat ($\hat{\ }$) over the symbol, and a part associated with the small deformation which is denoted by placing a tilde ($\tilde{\ }$) over the same symbol. For example, in general the external force associated with the large deformation to be analyzed can be nonzero. Thus, the external force $\mathbf{b}$ is represented in the form

$$\mathbf{b} = \hat{\mathbf{b}} + \tilde{\mathbf{b}} \ . \tag{4.13.3}$$

In order to develop the equations of motion of the small deformation associated with (4.4.35), it is necessary to substitute (4.13.1) into the constitutive equations (4.8.14) and to expand the resulting quantities in a Taylor series to develop expressions for the vectors $\mathbf{t}^i$ and the couples $\mathbf{m}^\alpha$

$$\mathbf{t}^i = \hat{\mathbf{t}}^i + \tilde{\mathbf{t}}^i \ , \quad \mathbf{m}^\alpha = \hat{\mathbf{m}}^\alpha + \tilde{\mathbf{m}}^\alpha \ . \tag{4.13.4}$$

To this end, it is noted that

$$\mathbf{F} = \hat{\mathbf{F}} + \tilde{\mathbf{F}} \ , \quad \hat{\mathbf{F}} = \hat{\mathbf{d}}_i \otimes \mathbf{D}^i \ , \quad \tilde{\mathbf{F}} = \boldsymbol{\delta}_i \otimes \mathbf{D}^i \ ,$$

$$\mathbf{F}^{-1} = \hat{\mathbf{F}}^{-1} + \tilde{\mathbf{F}}^{-1} \;,\quad \hat{\mathbf{F}}^{-1} = \mathbf{D}_i \otimes \hat{\mathbf{d}}^i \;,\quad \tilde{\mathbf{F}}^{-1} = -\,\hat{\mathbf{F}}^{-1}\boldsymbol{\delta}_i \otimes \hat{\mathbf{d}}^i \;,$$
$$\tilde{\mathbf{F}}\,\hat{\mathbf{F}}^{-1} = \boldsymbol{\delta}_i \otimes \hat{\mathbf{d}}^i \;,\quad \tilde{\mathbf{F}}^{-T}\,\hat{\mathbf{F}}^{T} = -\,\hat{\mathbf{d}}^i \otimes \boldsymbol{\delta}_i \;,$$
$$\mathbf{C} = \hat{\mathbf{C}} + \tilde{\mathbf{C}} \;,\quad \hat{\mathbf{C}} = \hat{\mathbf{F}}^T\hat{\mathbf{F}} \;,\quad \tilde{\mathbf{C}} = \hat{\mathbf{F}}^T(\boldsymbol{\delta}_i \otimes \mathbf{D}^i) + (\mathbf{D}^i \otimes \boldsymbol{\delta}_i)\hat{\mathbf{F}} \;,$$
$$\mathbf{E} = \hat{\mathbf{E}} + \tilde{\mathbf{E}} \;,\quad \hat{\mathbf{E}} = \frac{1}{2}(\hat{\mathbf{F}}^T\hat{\mathbf{F}} - \mathbf{I}) \;,\quad \tilde{\mathbf{E}} = \frac{1}{2}\,\tilde{\mathbf{C}} \;,$$
$$\boldsymbol{\beta}_\alpha = \hat{\boldsymbol{\beta}}_\alpha + \tilde{\boldsymbol{\beta}}_\alpha \;,\quad \hat{\boldsymbol{\beta}}_\alpha = \hat{\mathbf{F}}^{-1}\hat{\mathbf{d}}_{3,\alpha} - \mathbf{D}_{3,\alpha} \;,\quad \tilde{\boldsymbol{\beta}}_\alpha = \hat{\mathbf{F}}^{-1}\left[\,\boldsymbol{\delta}_{3,\alpha} - (\hat{\mathbf{d}}^i \bullet \hat{\mathbf{d}}_{3,\alpha})\,\boldsymbol{\delta}_i\,\right] \;,$$
$$\hat{d}^{1/2} = \hat{\mathbf{d}}_1 \times \hat{\mathbf{d}}_2 \bullet \hat{\mathbf{d}}_3 \;,$$
$$\hat{\mathbf{d}}^1 = \hat{d}^{-1/2}\left[\hat{\mathbf{d}}_2 \times \hat{\mathbf{d}}_3\right] ,\quad \hat{\mathbf{d}}^2 = \hat{d}^{-1/2}\left[\hat{\mathbf{d}}_3 \times \hat{\mathbf{d}}_1\right] ,\quad \hat{\mathbf{d}}^3 = \hat{d}^{-1/2}\,\hat{\mathbf{d}}_1 \times \hat{\mathbf{d}}_2 \;,$$
$$\mathbf{d}^i = \hat{\mathbf{d}}^i + \tilde{\mathbf{d}}^i \;,\quad \tilde{\mathbf{d}}^i = -\left[\hat{\mathbf{d}}^i \bullet \boldsymbol{\delta}_j\right]\hat{\mathbf{d}}^j \;,\quad \mathbf{a}_i = \hat{\mathbf{a}}_i + \tilde{\mathbf{a}}_i \;,\quad \hat{\mathbf{a}}_\alpha = \hat{\mathbf{d}}_\alpha \;,$$
$$\hat{a}^{1/2} = |\,\hat{\mathbf{a}}_1 \times \hat{\mathbf{a}}_2\,| \;,\quad \hat{\mathbf{a}}_3 = \hat{a}^{-1/2}\,\hat{\mathbf{a}}_1 \times \hat{\mathbf{a}}_2 \;,$$
$$\hat{\mathbf{a}}^1 = \hat{a}^{-1/2}\left[\hat{\mathbf{a}}_2 \times \hat{\mathbf{a}}_3\right] ,\quad \hat{\mathbf{a}}^2 = \hat{a}^{-1/2}\left[\hat{\mathbf{a}}_3 \times \hat{\mathbf{a}}_1\right] ,\quad \hat{\mathbf{a}}^3 = \hat{\mathbf{a}}_3 \;,$$
$$a^{1/2} = \hat{a}^{1/2} + \tilde{a}^{1/2} \;,\quad \tilde{a}^{1/2} = \hat{a}^{1/2}\left[\,\hat{\mathbf{a}}^\alpha \bullet \boldsymbol{\delta}_\alpha\right] ,\quad \tilde{\mathbf{a}}_\alpha = \boldsymbol{\delta}_\alpha \;,\quad \tilde{\mathbf{a}}_3 = -\left[\hat{\mathbf{a}}_3 \bullet \boldsymbol{\delta}_\alpha\right]\hat{\mathbf{a}}^\alpha \;,$$
$$\mathbf{a}^i = \hat{\mathbf{a}}^i + \tilde{\mathbf{a}}^i \;,\quad \tilde{\mathbf{a}}^i = -\left[\hat{\mathbf{a}}^i \bullet \tilde{\mathbf{a}}_j\right]\hat{\mathbf{a}}^j \;, \tag{4.13.5}$$

where the symbol $\tilde{\mathbf{F}}^{-1}$ does not denote the inverse of $\tilde{\mathbf{F}}$. Next, the conservation of mass $(4.4.35)_1$ can be written in the form

$$m = \rho_0\,A^{1/2} = \rho\,a^{1/2} = \hat{\rho}\,\hat{a}^{1/2} \;,\quad \rho = \hat{\rho}\left[1 - \hat{\mathbf{a}}^\alpha \bullet \boldsymbol{\delta}_\alpha\right] . \tag{4.13.6}$$

Then, expanding Σ in a Taylor series and neglecting quadratic terms in the small deformation quantities yields

$$\frac{\partial\Sigma}{\partial\mathbf{C}} = \frac{\partial\Sigma}{\partial\hat{\mathbf{C}}} + \frac{\partial\Sigma}{\partial\tilde{\mathbf{C}}} \;,\quad \frac{\partial\Sigma}{\partial\boldsymbol{\beta}_\alpha} = \frac{\partial\Sigma}{\partial\hat{\boldsymbol{\beta}}_\alpha} + \frac{\partial\Sigma}{\partial\tilde{\boldsymbol{\beta}}_\alpha} \;, \tag{4.13.7}$$

where the first terms on the right-hand sides are evaluated taking $\mathbf{C}=\hat{\mathbf{C}}$ and $\boldsymbol{\beta}_\alpha=\hat{\boldsymbol{\beta}}_\alpha$, and the second terms are first order in the small deformation quantities. Specifically, for the functional form (4.11.27)

$$\bar{\mathbf{C}} = \hat{\bar{\mathbf{C}}} + \tilde{\bar{\mathbf{C}}} \;,\quad \hat{\bar{\mathbf{C}}} = (\mathbf{I}+\hat{\boldsymbol{\beta}}_\alpha \otimes \mathbf{H}^\alpha)^T\,\hat{\mathbf{C}}\,(\mathbf{I}+\hat{\boldsymbol{\beta}}_\beta \otimes \mathbf{H}^\beta) \;,$$
$$\tilde{\bar{\mathbf{C}}} = (\mathbf{I}+\hat{\boldsymbol{\beta}}_\alpha \otimes \mathbf{H}^\alpha)^T\,\tilde{\mathbf{C}}\,(\mathbf{I}+\hat{\boldsymbol{\beta}}_\beta \otimes \mathbf{H}^\beta) + (\tilde{\boldsymbol{\beta}}_\alpha \otimes \mathbf{H}^\alpha)^T\,\hat{\mathbf{C}}\,(\mathbf{I}+\hat{\boldsymbol{\beta}}_\beta \otimes \mathbf{H}^\beta)$$
$$+\,(\mathbf{I}+\hat{\boldsymbol{\beta}}_\alpha \otimes \mathbf{H}^\alpha)^T\,\hat{\mathbf{C}}\,(\mathbf{I}+\hat{\boldsymbol{\beta}}_\beta \otimes \mathbf{H}^\beta) \;,$$
$$m\,\frac{\partial\Sigma}{\partial\hat{\mathbf{C}}} = (\mathbf{I}+\hat{\boldsymbol{\beta}}_\alpha \otimes \mathbf{H}^\alpha)\left[m\,\frac{\partial\Sigma^*(\hat{\bar{\mathbf{C}}})}{\partial\hat{\bar{\mathbf{C}}}}\right](\mathbf{I}+\hat{\boldsymbol{\beta}}_\beta \otimes \mathbf{H}^\beta)^T \;,$$
$$m\,\frac{\partial\Sigma}{\partial\tilde{\mathbf{C}}} = (\mathbf{I}+\hat{\boldsymbol{\beta}}_\alpha \otimes \mathbf{H}^\alpha)\left[m\,\frac{\partial^2\Sigma^*(\hat{\bar{\mathbf{C}}})}{\partial\hat{\bar{\mathbf{C}}} \otimes \partial\hat{\bar{\mathbf{C}}}} \bullet \tilde{\bar{\mathbf{C}}}\right](\mathbf{I}+\hat{\boldsymbol{\beta}}_\beta \otimes \mathbf{H}^\beta)^T$$
$$+\,(\tilde{\boldsymbol{\beta}}_\alpha \otimes \mathbf{H}^\alpha)\left[m\,\frac{\partial\Sigma^*(\hat{\bar{\mathbf{C}}})}{\partial\hat{\bar{\mathbf{C}}}}\right](\mathbf{I}+\hat{\boldsymbol{\beta}}_\beta \otimes \mathbf{H}^\beta)^T$$

$$+ (\mathbf{I}+\hat{\boldsymbol{\beta}}_\alpha \otimes \mathbf{H}^\alpha) \left[m \frac{\partial \Sigma^*(\hat{\bar{\mathbf{C}}})}{\partial \hat{\bar{\mathbf{C}}}}\right] (\tilde{\boldsymbol{\beta}}_\beta \otimes \mathbf{H}^\beta)^T ,$$

$$m \frac{\partial \Sigma}{\partial \hat{\boldsymbol{\beta}}_\alpha} = 2\, \hat{\mathbf{C}}\, (\mathbf{I}+\hat{\boldsymbol{\beta}}_\beta \otimes \mathbf{H}^\beta) \left[m \frac{\partial \Sigma^*(\hat{\bar{\mathbf{C}}})}{\partial \hat{\bar{\mathbf{C}}}}\right] \mathbf{H}^\alpha + A^{1/2}\, \bar{H}\, |\mathbf{D}_3|^{-2}\, \mathbf{K}^{\alpha\beta}\, \hat{\boldsymbol{\beta}}_\beta ,$$

$$m \frac{\partial \Sigma}{\partial \tilde{\boldsymbol{\beta}}_\alpha} = 2\, \hat{\mathbf{C}}\, (\mathbf{I}+\hat{\boldsymbol{\beta}}_\beta \otimes \mathbf{H}^\beta) \left[m \frac{\partial^2 \Sigma^*(\hat{\bar{\mathbf{C}}})}{\partial \hat{\bar{\mathbf{C}}} \otimes \partial \hat{\bar{\mathbf{C}}}} \bullet \tilde{\bar{\mathbf{C}}}\right] \mathbf{H}^\alpha$$

$$+ 2\, \tilde{\mathbf{C}}\, (\mathbf{I}+\hat{\boldsymbol{\beta}}_\beta \otimes \mathbf{H}^\beta) \left[m \frac{\partial \Sigma^*(\hat{\bar{\mathbf{C}}})}{\partial \hat{\bar{\mathbf{C}}}}\right] \mathbf{H}^\alpha$$

$$+ 2\, \hat{\mathbf{C}}\, (\mathbf{I}+\hat{\boldsymbol{\beta}}_\beta \otimes \mathbf{H}^\beta) \left[m \frac{\partial \Sigma^*(\hat{\bar{\mathbf{C}}})}{\partial \hat{\bar{\mathbf{C}}}}\right] \mathbf{H}^\alpha + A^{1/2}\, \bar{H}\, |\mathbf{D}_3|^{-2}\, \mathbf{K}^{\alpha\beta}\, \tilde{\boldsymbol{\beta}}_\beta , \qquad (4.13.8)$$

where the derivatives of Σ^* are evaluated with $\bar{\mathbf{C}}=\hat{\bar{\mathbf{C}}}$. If the three-dimensional strain energy function Σ^* is a quadratic function of strain (3.12.2), then these equations simplify somewhat with

$$\hat{\bar{\mathbf{E}}} = \frac{1}{2}\,(\hat{\bar{\mathbf{C}}} - \mathbf{I}) \ , \quad \tilde{\bar{\mathbf{E}}} = \frac{1}{2}\, \tilde{\bar{\mathbf{C}}} \ ,$$

$$m \frac{\partial \Sigma^*(\hat{\bar{\mathbf{C}}})}{\partial \hat{\bar{\mathbf{C}}}} = \frac{1}{2} A^{1/2}\, \bar{H}\, \mathbf{K}^* \bullet \hat{\bar{\mathbf{E}}} \ , \quad m \frac{\partial^2 \Sigma^*(\hat{\bar{\mathbf{C}}})}{\partial \hat{\bar{\mathbf{C}}} \otimes \partial \hat{\bar{\mathbf{C}}}} = \frac{1}{4} A^{1/2}\, \bar{H}\, \mathbf{K}^* . \qquad (4.13.9)$$

Moreover, if quadratic and higher order terms in the strains $\mathbf{E}$ and $\boldsymbol{\beta}_\alpha$ are neglected, then equations (4.12.6) yield

$$\bar{\mathbf{E}} = \hat{\bar{\mathbf{E}}} + \tilde{\bar{\mathbf{E}}} \ ,$$

$$\hat{\bar{\mathbf{E}}} = \hat{\mathbf{E}} + \frac{1}{2}(\hat{\boldsymbol{\beta}}_\alpha \otimes \mathbf{H}^\alpha + \mathbf{H}^\alpha \otimes \hat{\boldsymbol{\beta}}_\alpha) \ , \quad \tilde{\bar{\mathbf{E}}} = \tilde{\mathbf{E}} + \frac{1}{2}(\tilde{\boldsymbol{\beta}}_\alpha \otimes \mathbf{H}^\alpha + \mathbf{H}^\alpha \otimes \tilde{\boldsymbol{\beta}}_\alpha) \ ,$$

$$m \frac{\partial \Sigma}{\partial \hat{\mathbf{C}}} = \frac{1}{2} A^{1/2}\, \bar{H}\, \mathbf{K}^* \bullet \hat{\bar{\mathbf{E}}} \ , \quad m \frac{\partial \Sigma}{\partial \tilde{\mathbf{C}}} = \frac{1}{2} A^{1/2}\, \bar{H}\, \mathbf{K}^* \bullet \tilde{\bar{\mathbf{E}}} \ ,$$

$$m \frac{\partial \Sigma}{\partial \hat{\boldsymbol{\beta}}_\alpha} = \left[A^{1/2}\, \bar{H}\, \mathbf{K}^* \bullet \hat{\bar{\mathbf{E}}}\right] \mathbf{H}^\alpha + A^{1/2}\, \bar{H}\, |\mathbf{D}_3|^{-2}\, \mathbf{K}^{\alpha\beta}\, \hat{\boldsymbol{\beta}}_\beta \ ,$$

$$m \frac{\partial \Sigma}{\partial \tilde{\boldsymbol{\beta}}_\alpha} = \left[A^{1/2}\, \bar{H}\, \mathbf{K}^* \bullet \tilde{\bar{\mathbf{E}}}\right] \mathbf{H}^\alpha + A^{1/2}\, \bar{H}\, |\mathbf{D}_3|^{-2}\, \mathbf{K}^{\alpha\beta}\, \tilde{\boldsymbol{\beta}}_\beta \ . \qquad (4.13.10)$$

For either of these cases, (4.13.7) can be used to expand the constitutive equations (4.8.14) to deduce that

$$\hat{\mathbf{m}}^\alpha = m\, \hat{\mathbf{F}}^{-T} \frac{\partial \Sigma}{\partial \hat{\boldsymbol{\beta}}_\alpha} \ , \quad \tilde{\mathbf{m}}^\alpha = m\, \hat{\mathbf{F}}^{-T} \frac{\partial \Sigma}{\partial \tilde{\boldsymbol{\beta}}_\alpha} - (\hat{\mathbf{d}}^j \otimes \boldsymbol{\delta}_j)\, \hat{\mathbf{m}}^\alpha \ ,$$

$$\hat{\mathbf{t}}^i = 2\,\mathrm{m}\,\hat{\mathbf{F}}\,\frac{\partial\Sigma}{\partial\hat{\mathbf{C}}}\mathbf{D}^i - \hat{\mathbf{m}}^\alpha\,(\hat{\mathbf{d}}_{3,\alpha}\bullet\hat{\mathbf{d}}^i)\ ,$$

$$\tilde{\mathbf{t}}^i = 2\,\mathrm{m}\,\hat{\mathbf{F}}\,\frac{\partial\Sigma}{\partial\tilde{\mathbf{C}}}\mathbf{D}^i + (\boldsymbol{\delta}_j\otimes\hat{\mathbf{d}}^j)\,\hat{\mathbf{t}}^i + \left[(\boldsymbol{\delta}_j\otimes\hat{\mathbf{d}}^j)\,\hat{\mathbf{m}}^\alpha - \tilde{\mathbf{m}}^\alpha\right](\hat{\mathbf{d}}_{3,\alpha}\bullet\hat{\mathbf{d}}^I)$$
$$-\hat{\mathbf{m}}^\alpha\left[\boldsymbol{\delta}_{3,\alpha}\bullet\hat{\mathbf{d}}^i - (\hat{\mathbf{d}}_{3,\alpha}\bullet\hat{\mathbf{d}}^j)(\hat{\mathbf{d}}^i\bullet\boldsymbol{\delta}_j)\right]\ . \qquad (4.13.11)$$

Also, the expression (4.4.32) expands to give

$$\mathbf{T} = \hat{\mathbf{T}} + \tilde{\mathbf{T}}\ ,\quad \hat{\mathbf{T}} = \hat{\mathrm{a}}^{-1/2}\left[\hat{\mathbf{t}}^i\otimes\hat{\mathbf{d}}_i + \hat{\mathbf{m}}^\alpha\otimes\hat{\mathbf{d}}_{3,\alpha}\right]\ ,$$

$$\tilde{\mathbf{T}} = \hat{\mathrm{a}}^{-1/2}\left[\hat{\mathbf{t}}^i\otimes\boldsymbol{\delta}_i + \tilde{\mathbf{t}}^i\otimes\hat{\mathbf{d}}_i + \hat{\mathbf{m}}^\alpha\otimes\boldsymbol{\delta}_{3,\alpha} + \tilde{\mathbf{m}}^\alpha\otimes\hat{\mathbf{d}}_{3,\alpha}\right] - (\hat{\mathbf{a}}^\alpha\bullet\boldsymbol{\delta}_\alpha)\,\hat{\mathbf{T}}\ . \qquad (4.13.12)$$

Next, substitution of (4.13.11) into (4.13.12) yields

$$\hat{\mathbf{T}} = 2\,\hat{\rho}\,\hat{\mathbf{F}}\,\frac{\partial\Sigma}{\partial\hat{\mathbf{C}}}\,\hat{\mathbf{F}}^T\ ,$$

$$\tilde{\mathbf{T}} = 2\,\hat{\rho}\,\hat{\mathbf{F}}\,\frac{\partial\Sigma}{\partial\tilde{\mathbf{C}}}\,\hat{\mathbf{F}}^T + (\boldsymbol{\delta}_j\otimes\hat{\mathbf{d}}^j)\,\hat{\mathbf{T}} + \hat{\mathbf{T}}\,(\hat{\mathbf{d}}^j\otimes\boldsymbol{\delta}_j) - (\hat{\mathbf{a}}^\alpha\bullet\boldsymbol{\delta}_\alpha)\,\hat{\mathbf{T}}\ , \qquad (4.13.13)$$

which can be seen to be equivalent to a direct expansion of the constitutive equation $(4.8.12)_1$. Moreover, since both the terms $\hat{\mathbf{T}}$ and $\tilde{\mathbf{T}}$ are symmetric tensors, the reduced form (4.4.33) of the balance of angular momentum is satisfied by the small deformation terms.

Using these expressions, the balances of linear momentum and director momentum (4.4.35) become

$$\mathrm{m}\,(\ddot{\mathbf{u}} + \mathrm{y}^3\,\ddot{\boldsymbol{\delta}}_3) - \mathrm{m}\,\tilde{\mathbf{b}} - \tilde{\mathbf{t}}^\alpha{}_{,\alpha} = -\left[\,\mathrm{m}\,(\ddot{\hat{\mathbf{x}}} + \mathrm{y}^3\,\ddot{\hat{\mathbf{d}}}_3) - \mathrm{m}\,\hat{\mathbf{b}} - \hat{\mathbf{t}}^\alpha{}_{,\alpha}\,\right]\ ,$$

$$\mathrm{m}\,(\mathrm{y}^3\,\ddot{\mathbf{u}} + \mathrm{y}^{33}\,\ddot{\boldsymbol{\delta}}_3) - \mathrm{m}\,\tilde{\mathbf{b}}^3 + \tilde{\mathbf{t}}^3 - \tilde{\mathbf{m}}^\alpha{}_{,\alpha} = -\left[\,\mathrm{m}\,(\mathrm{y}^3\,\ddot{\hat{\mathbf{x}}} + \mathrm{y}^{33}\,\ddot{\hat{\mathbf{d}}}_3)\right.$$
$$\left. - \mathrm{m}\,\hat{\mathbf{b}}^3 + \hat{\mathbf{t}}^3 - \hat{\mathbf{m}}^\alpha{}_{,\alpha}\,\right]\ . \qquad (4.13.14)$$

With $\hat{\mathbf{x}}$, $\hat{\mathbf{d}}_3$, $\hat{\mathbf{b}}$, $\hat{\mathbf{b}}^3$, $\tilde{\mathbf{b}}$ and $\tilde{\mathbf{b}}^3$ specified, these equations become linear equations to determine the displacement fields $\mathbf{u}$ and $\boldsymbol{\delta}_3$. Moreover, the equations must be supplemented by initial and boundary conditions. Also, when $\hat{\mathbf{x}}$ and $\hat{\mathbf{d}}_3$ are the solutions of the nonlinear linear momentum and director momentum equations, then the right hand sides of (4.13.14) vanish.

As a special case, consider a homogeneous material and let $\hat{\mathbf{x}}$ and $\hat{\mathbf{d}}_3$ be associated with a static homogeneous solution with vanishing body force such that

$$\hat{\mathbf{x}} = \hat{\mathbf{F}}\,\mathbf{X}\ ,\quad \hat{\mathbf{d}}_3 = \hat{\mathbf{F}}\,\mathbf{D}_3\ ,\quad \dot{\hat{\mathbf{F}}} = 0\ ,\quad \hat{\mathbf{b}} = \hat{\mathbf{b}}_c\ ,\quad \hat{\mathbf{b}}^3 = \hat{\mathbf{b}}_c^3\ , \qquad (4.13.15)$$

where $\hat{\mathbf{b}}_c$ and $\hat{\mathbf{b}}_c^3$ are specified by the forms (4.3.15) and (4.3.36), respectively, with $\mathbf{F}^*=\hat{\mathbf{F}}$. It then follows from the results (4.11.11) that the right-hand sides of (4.13.14) vanish so the linear momentum and director momentum equations reduce to

$$\mathrm{m}\,(\ddot{\mathbf{u}} + \mathrm{y}^3\,\ddot{\boldsymbol{\delta}}_3) = \mathrm{m}\,\tilde{\mathbf{b}} + \tilde{\mathbf{t}}^\alpha{}_{,\alpha}\ ,$$

$$\mathrm{m}\,(\mathrm{y}^3\,\ddot{\mathbf{u}} + \mathrm{y}^{33}\,\ddot{\boldsymbol{\delta}}_3) = \mathrm{m}\,\tilde{\mathbf{b}}^3 - \tilde{\mathbf{t}}^3 + \tilde{\mathbf{m}}^\alpha{}_{,\alpha}\ . \qquad (4.13.16)$$

In particular, it is noted that $\tilde{\mathbf{t}}^i$ and $\tilde{\mathbf{m}}^\alpha$ retain a dependence on the values $\hat{\mathbf{t}}^i$ and $\hat{\mathbf{m}}^\alpha$ associated with the large deformation.

Moreover, for the fully linearized theory, $\hat{\mathbf{F}} = \mathbf{I}$ and $\hat{\boldsymbol{\beta}}_\alpha = 0$ so that the quantities $\hat{\mathbf{t}}^i$ and $\hat{\mathbf{m}}^\alpha$ vanish. Then, the motion is determined by the equations (4.13.16) with

$$\tilde{\mathbf{E}} = \frac{1}{2}(\boldsymbol{\delta}_i \otimes \mathbf{D}^i + \mathbf{D}^i \otimes \boldsymbol{\delta}_i) \ , \quad \boldsymbol{\delta}_\alpha = \mathbf{u}_{,\alpha} \ , \quad \tilde{\boldsymbol{\beta}}_\alpha = \boldsymbol{\delta}_{3,\alpha} - (\mathbf{D}_{3,\alpha} \bullet \mathbf{D}^i)\,\boldsymbol{\delta}_i \ ,$$

$$\tilde{\tilde{\mathbf{E}}} = \tilde{\mathbf{E}} + \frac{1}{2}(\tilde{\boldsymbol{\beta}}_\alpha \otimes \mathbf{H}^\alpha + \mathbf{H}^\alpha \otimes \tilde{\boldsymbol{\beta}}_\alpha) \ ,$$

$$\tilde{\mathbf{m}}^\alpha = \left[A^{1/2}\,\bar{H}\,\mathbf{K}^* \bullet \tilde{\tilde{\mathbf{E}}}\right]\mathbf{H}^\alpha + A^{1/2}\,\bar{H}\,|\mathbf{D}_3|^{-2}\,\mathbf{K}^{\alpha\beta}\,\tilde{\boldsymbol{\beta}}_\beta \ ,$$

$$\tilde{\mathbf{t}}^i = \left[A^{1/2}\,\bar{H}\,\mathbf{K}^* \bullet \tilde{\tilde{\mathbf{E}}}\right]\mathbf{D}^i - \tilde{\mathbf{m}}^\alpha\,(\mathbf{D}_{3,\alpha} \bullet \mathbf{D}^i) \ . \qquad (4.13.17)$$

Also, for the linearized theory, quadratic terms in the displacements are neglected in the expression for the material derivative.

4.14 Pure bending of an orthotropic rectangular plate

The restrictions on the constitutive equations developed in section 4.11 ensure consistency with exact solutions for all nonlinear homogeneous deformations, even when the shell has an arbitrary shape in its reference configuration. It was further seen in section 4.12 that these restrictions determined all of the constitutive coefficients associated with homogeneous deformations in terms of the three-dimensional material and geometric properties of the shell. However, these restrictions do not provide any guidance for determining values of the constitutive coefficients $K_{i\gamma j\delta}$ in (4.12.10) associated with inhomogeneous deformations. The objective of the present section is to determine some of the coefficients $K_{i\gamma j\delta}$ by comparing solutions of the linear Cosserat theory for an orthotropic rectangular plate with exact bending solutions of the three-dimensional theory given in section 3.14. Comparison will also be made with the values given in (Green and Naghdi, 1982).

The kinematic expression (4.2.7) requires the three-dimensional position vector $\mathbf{x}^*$ to remain a linear function of the thickness coordinate θ^3. Although it was shown at the end of section 4.2 that the expression (4.2.7) is exact for homogeneous deformations, it is quite clear that for inhomogeneous deformations, $\mathbf{x}^*$ can be a nonlinear function of θ^3. In particular, the bending solutions developed in section 3.14 for the linearized three-dimensional equations indicate that the displacement vector is a quadratic function of θ^3. This means that, even if an exact three-dimensional solution for $\mathbf{x}^*$ is known, there is no unique way of identifying values of $\mathbf{x}$ and $\mathbf{d}_3$ associated with the Cosserat theory when $\mathbf{x}^*$ is a nonlinear function of θ^3. A discussion of this point related to the determination of the shear deformation coefficient in beam theory has recently been given by Rubin (1996). In that work, it was suggested that the kinematics of the Cosserat theory can be related to those of the three-dimensional theory such that the structure of the constitutive equation for stress is preserved. The discussion that follows shows how this can be done for the linearized equations of plates.

By way of background, it is first noted that a shell structure is considered to be a flat plate if its reference surface is flat in its reference configuration. This means that the unit normal vector $\mathbf{A}_3$ to the reference surface is constant

$$\mathbf{A}_{3,\alpha} = 0 \ . \tag{4.14.1}$$

The formulas developed in the previous sections are valid for an arbitrary specification of the director $\mathbf{D}_3$ which satisfies the restriction $(4.1.3)_2$. However, from a physical point of view, it often convenient to take

$$\mathbf{D}_3 = \mathbf{A}_3 \ , \tag{4.14.2}$$

so that for elastic shells the director $\mathbf{d}_3$ can be identified with a material fiber that was normal to the reference surface of the shell in its reference configuration. With this specification, the quantities $\{\mathbf{G}_i, \mathbf{G}^i \ , G^{1/2}\}$ are independent of the coordinate θ^3 and become

$$\mathbf{G}_i = \mathbf{D}^i \ , \quad \mathbf{G}^i = \mathbf{D}^i \ , \quad G^{1/2} = D^{1/2} = A^{1/2} \ . \tag{4.14.3}$$

Furthermore, confining attention to a plate with constant thickness H and taking the reference surface to be the middle surface, it follows from $(4.11.7)_2$ and (4.11.10) that

$$\xi_1 = -\frac{H}{2} \ , \quad \xi_2 = \frac{H}{2} \ , \quad \bar{H} = H \ , \quad \mathbf{H}^\alpha = 0 \ . \tag{4.14.4}$$

Under these conditions, the kinematic expression (4.2.7) causes the three-dimensional displacement vector of the linearized theory to be a linear function of the coordinate θ^3 [see (3.13.1) with $\hat{\mathbf{x}}^*=\mathbf{X}^*$ and (4.13.1) with $\hat{\mathbf{x}}=\mathbf{X}$ and $\hat{\mathbf{d}}_i=\mathbf{D}_i$]

$$\mathbf{u}^*(\theta^i,t) = \mathbf{u}(\theta^\alpha,t) + \theta^3 \, \boldsymbol{\delta}_3(\theta^\alpha,t) \ . \tag{4.14.5}$$

It therefore follows from (3.2.34), (3.13.18) and (4.14.3) that the linearized three-dimensional strain $\tilde{\mathbf{E}}^*$ and stress $\tilde{\mathbf{T}}^*$ associated with a plate become

$$\tilde{\mathbf{E}}^* = \frac{1}{2}\left[\mathbf{u}^*{}_{,\alpha}\otimes\mathbf{D}^\alpha + \mathbf{D}^\alpha\otimes\mathbf{u}^*{}_{,\alpha} + \mathbf{u}^*{}_{,3}\otimes\mathbf{D}^3 + \mathbf{D}^3\otimes\mathbf{u}^*{}_{,3}\right] \ ,$$

$$\tilde{\mathbf{T}}^* = \mathbf{K}^* \bullet \tilde{\mathbf{E}}^* \ . \tag{4.14.6}$$

Next, it is observed from the linearized form of (4.11.2), that the quantity $\tilde{\mathbf{T}}$ associated with the Cosserat theory can be expressed in the form

$$A^{1/2}\,\tilde{\mathbf{T}} = \int_{-H/2}^{H/2} G^{1/2}\,\tilde{\mathbf{T}}^*\, d\theta^3 \ . \tag{4.14.7}$$

Now, substitution of (4.14.6) into (4.14.7) and use of $(4.11.7)_2$ and (4.14.3), yields the equations

$$\tilde{\mathbf{T}} = H\,\mathbf{K}^* \bullet \tilde{\mathbf{E}} \ , \quad \tilde{\mathbf{E}} = \frac{1}{2}\left[\boldsymbol{\delta}_i\otimes\mathbf{D}^i + \mathbf{D}^i\otimes\boldsymbol{\delta}_i\right] \ ,$$

$$\mathbf{u}(\theta^\alpha,t) = \frac{1}{H}\int_{-H/2}^{H/2} \mathbf{u}^*\, d\theta^3 \ , \quad \boldsymbol{\delta}_\alpha = \mathbf{u}_{,\alpha} \ ,$$

$$\boldsymbol{\delta}_3(\theta^\alpha,t) = \frac{1}{H}\int_{-H/2}^{H/2} \mathbf{u}^*{}_{,3}\, d\theta^3 = \frac{1}{H}\left[\mathbf{u}^*(\theta^\alpha,H/2,t) - \mathbf{u}^*(\theta^\alpha,-H/2,t)\right] \ , \tag{4.14.8}$$

which relate the Cosserat displacements $\mathbf{u}$ and $\boldsymbol{\delta}_3$ to the three-dimensional displacement $\mathbf{u}^*$. At this point it should be emphasized that the expressions (4.14.8) for $\tilde{\mathbf{T}}$, $\mathbf{u}$, and $\boldsymbol{\delta}_i$ are not necessarily equal to similar quantities that are obtained by solving the equations for a Cosserat plate. Instead, these quantities are used to compare the predictions of the three-dimensional theory with those of the Cosserat theory.

Now, with the help of (4.14.4), the expressions (4.13.17) for the linearized theory of a plate and the linearized form of (4.4.32) yield

$$\tilde{\mathbf{E}} = \frac{1}{2}(\boldsymbol{\delta}_i \otimes \mathbf{D}^i + \mathbf{D}^i \otimes \boldsymbol{\delta}_i), \quad \boldsymbol{\delta}_\alpha = \mathbf{u}_{,\alpha} \ , \quad \tilde{\boldsymbol{\beta}}_\alpha = \boldsymbol{\delta}_{3,\alpha} \ ,$$

$$\tilde{\mathbf{m}}^\alpha = A^{1/2}\, H\, \mathbf{K}^{\alpha\beta}\, \tilde{\boldsymbol{\beta}}_\beta \ , \quad \tilde{\mathbf{t}}^i = A^{1/2}\, H \left[\mathbf{K}^* \bullet \tilde{\mathbf{E}}\right] \mathbf{D}^i \ , \quad \tilde{\mathbf{T}} = H\, \mathbf{K}^* \bullet \tilde{\mathbf{E}} \ . \tag{4.14.9}$$

Also, the equations of equilibrium associated with (4.13.16) reduce to

$$m\, \tilde{\mathbf{b}} + \tilde{\mathbf{t}}^\alpha{}_{,\alpha} = 0 \ , \quad m\, \tilde{\mathbf{b}}^3 - \tilde{\mathbf{t}}^3 + \tilde{\mathbf{m}}^\alpha{}_{,\alpha} = 0 \ . \tag{4.14.10}$$

In particular, it is observed that the result $(4.14.8)_1$ is consistent with the expression $(4.14.9)_6$. This indicates that the interpretations of $\mathbf{u}$ and $\boldsymbol{\delta}_3$ proposed in (4.14.8) preserve compatibility of the three-dimensional constitutive equations with those of the Cosserat theory for homogeneous deformations.

To be more specific, let the plate occupy the region of space defined by (3.14.1) and take the position vector $\mathbf{X}^*$ in the form (3.14.2) so that the base vectors are related to the constant rectangular Cartesian base vectors $\mathbf{e}_i$ by the equations

$$\mathbf{G}_i = \mathbf{G}^i = \mathbf{D}_i = \mathbf{D}^i = \mathbf{e}_i \ . \tag{4.14.11}$$

For the Cosserat theory this means that

$$\mathbf{X} = \theta^\alpha\, \mathbf{e}_\alpha \ , \quad \mathbf{D}_3 = \mathbf{e}_3 \ . \tag{4.14.12}$$

In order to compare the Cosserat solutions with the corresponding three-dimensional solutions of section 3.14, it is necessary to apply the same loading to the plate. This is accomplished by using the linearized forms of $(4.3.24)_1$, $(4.3.38)_1$, (4.3.15), and (4.3.36) to deduce that

$$\tilde{\mathbf{t}}^\alpha = \int_{-H/2}^{H/2} \tilde{\mathbf{T}}^*\, \mathbf{e}_\alpha\, d\theta^3 \ , \quad \tilde{\mathbf{m}}^\alpha = \int_{-H/2}^{H/2} \theta^3\, \tilde{\mathbf{T}}^*\, \mathbf{e}_\alpha\, d\theta^3 \ ,$$

$$m\, \tilde{\mathbf{b}}_c = \left[\tilde{\mathbf{T}}^*(\theta^\alpha, H/2) - \tilde{\mathbf{T}}^*(\theta^\alpha, -H/2)\right] \mathbf{e}_3 \ ,$$

$$m\, \tilde{\mathbf{b}}_c^3 = \frac{H}{2}\left[\tilde{\mathbf{T}}^*(\theta^\alpha, H/2) + \tilde{\mathbf{T}}^*(\theta^\alpha, -H/2)\right] \mathbf{e}_3 \ . \tag{4.14.13}$$

Moreover, in the absence of three-dimensional body force the linearized version of the expressions (4.3.10) and (4.3.32) yield

$$\tilde{\mathbf{b}}_b = 0 \ , \quad \tilde{\mathbf{b}} = \tilde{\mathbf{b}}_c \ , \quad \tilde{\mathbf{b}}_b^3 = 0 \ , \quad \tilde{\mathbf{b}}^3 = \tilde{\mathbf{b}}_c^3 \ . \tag{4.14.14}$$

Thus, substitution of the three-dimensional stresses (3.14.8) into (4.14.13) yields

$$\tilde{\mathbf{t}}^1 = \left[\frac{12 M_{21}}{W^3}\right] \theta^2\, \mathbf{e}_1 \ , \quad \tilde{\mathbf{t}}^2 = \left[\frac{12 M_{12}}{L^3}\right] \theta^1\, \mathbf{e}_2 \ ,$$

$$\tilde{\mathbf{m}}^1 = \left[\frac{M_{31}}{W}\right] \mathbf{e}_1 \ , \quad \tilde{\mathbf{m}}^2 = \left[\frac{M_{32}}{L}\right] \mathbf{e}_2 \ ,$$

$$m\, \tilde{\mathbf{b}} = 0 \ , \quad m\, \tilde{\mathbf{b}}^3 = H\left[\left\{\frac{12 M_{13}}{L^3 W}\right\} \theta^1 + \left\{\frac{12 M_{23}}{L W^3}\right\} \theta^2\right] \mathbf{e}_3 \ , \tag{4.14.15}$$

where the values of $\tilde{\mathbf{t}}^1$ and $\tilde{\mathbf{m}}^1$ are applied only on the boundaries $\theta^1 = \pm L/2$, and the values of $\tilde{\mathbf{t}}^2$ and $\tilde{\mathbf{m}}^2$ are applied only on the boundaries $\theta^2 = \pm W/2$.

In making the specification (4.14.15) the axes of orthotropy of the plate have tacitly been specified so that $\phi=0$ in (4.12.8) with

$$\mathbf{M}_i = \mathbf{e}_i \ . \tag{4.14.16}$$

Consequently, referring all tensor quantities to the base vectors $\mathbf{e}_i$

$$u_i = \mathbf{e}_i \cdot \mathbf{u} \ , \quad \tilde{\delta}_{ij} = \mathbf{e}_i \cdot \boldsymbol{\delta}_j \ , \quad \beta_{i\alpha} = \mathbf{e}_i \cdot \tilde{\boldsymbol{\beta}}_\alpha = \mathbf{e}_i \cdot \boldsymbol{\delta}_{3,\alpha} \ ,$$

$$E_{ij} = (\mathbf{e}_i \otimes \mathbf{e}_j) \cdot \tilde{\mathbf{E}} \ , \quad T_{ij} = (\mathbf{e}_i \otimes \mathbf{e}_j) \cdot \tilde{\mathbf{T}} = \mathbf{e}_i \cdot \tilde{\mathbf{t}}^j \ , \quad m^{i\alpha} = \mathbf{e}_i \cdot \tilde{\mathbf{m}}^\alpha \ , \tag{4.14.17}$$

it follows with the help of (3.12.12), (4.12.9), (4.12.11) and (4.12.14), that for an orthotropic plate

$$E_{\alpha\beta} = \frac{1}{2}\left[u_{\alpha,\beta} + u_{\beta,\alpha}\right] \ , \quad E_{\alpha 3} = \frac{1}{2}\left[u_{3,\alpha} + \tilde{\delta}_{\alpha 3}\right] \ ,$$

$$E_{33} = \tilde{\delta}_{33} \ , \quad \beta_{i\alpha} = \tilde{\delta}_{i3,\alpha} \ ,$$

$$T_{11} = H\left[K^*_{1111} E_{11} + K^*_{1122} E_{22} + K^*_{1133} E_{33}\right] \ ,$$

$$T_{22} = H\left[K^*_{1122} E_{11} + K^*_{2222} E_{22} + K^*_{2233} E_{33}\right] \ ,$$

$$T_{33} = H\left[K^*_{1133} E_{11} + K^*_{2233} E_{22} + K^*_{3333} E_{33}\right] \ ,$$

$$T_{12} = 2\,H\,K^*_{1212} E_{12} \ , \quad T_{13} = 2\,HK^*_{1313} E_{13} \ , \quad T_{23} = 2\,HK^*_{2323} E_{23} \ ,$$

$$m^{11} = HK_{1111}\beta_{11} + HK_{1122}\beta_{22} \ , \quad m^{12} = HK_{1212}\beta_{12} + HK_{1221}\beta_{21} \ ,$$

$$m^{21} = HK_{1221}\beta_{12} + HK_{2121}\beta_{21} \ , \quad m^{22} = HK_{1122}\beta_{11} + HK_{2222}\beta_{22} \ ,$$

$$m^{31} = HK_{3131}\beta_{31} \ , \quad m^{32} = HK_{3232}\beta_{32} \ , \tag{4.14.18}$$

where the symbol tilde ($\tilde{}$) has been used over the components of $\boldsymbol{\delta}_i$ to distinguish them from the Kronecker delta symbol δ_{ij}.

Now, substitution of the three-dimensional displacements (3.14.12) into the expressions (4.14.8) would suggest that the displacements of the Cosserat theory are given by

$$\begin{aligned}
u_1 = {} & C^*_{1111}\left[\left\{\frac{12M_{21}}{W^3H}\right\}\theta^1\theta^2\right] + C^*_{1122}\left[\left\{\frac{12M_{12}}{L^3H}\right\}\frac{(\theta^1)^2}{2}\right] \\
& + C^*_{1133}\left[\left\{\frac{12M_{13}}{L^3W}\right\}\frac{(\theta^1)^2}{2} + \left\{\frac{12M_{23}}{LW^3}\right\}\theta^1\theta^2\right] \\
& - \left[\frac{12M_{12}}{L^3H}\right]\left[C^*_{2222}\,\frac{(\theta^2)^2}{2}\right] - \left[\frac{12M_{13}}{L^3W}\right]\left[C^*_{2233}\,\frac{(\theta^2)^2}{2}\right] \\
& + \left[\alpha^*_{12}\,\theta^2 + c_1\right] \ , \\
u_2 = {} & C^*_{1122}\left[\left\{\frac{12M_{21}}{W^3H}\right\}\frac{(\theta^2)^2}{2}\right] + C^*_{2222}\left[\left\{\frac{12M_{12}}{L^3H}\right\}\theta^1\theta^2\right] \\
& + C^*_{2233}\left[\left\{\frac{12M_{13}}{L^3W}\right\}\theta^1\theta^2 + \left\{\frac{12M_{23}}{LW^3}\right\}\frac{(\theta^2)^2}{2}\right] \\
& - C^*_{1111}\left[\left\{\frac{12M_{21}}{W^3H}\right\}\frac{(\theta^1)^2}{2}\right] - C^*_{1133}\left[\left\{\frac{12M_{23}}{LW^3}\right\}\frac{(\theta^1)^2}{2}\right]
\end{aligned}$$

$$+ [-\alpha^*_{12}\,\theta^1 + c_2] \ ,$$

$$u_3 = -\ [\frac{12M_{31}}{WH^3}]\,[C^*_{1111}\,\frac{(\theta^1)^2}{2} + C^*_{1122}\frac{(\theta^2)^2}{2}]$$

$$-[\frac{12M_{32}}{LH^3}]\,[C^*_{1122}\frac{(\theta^1)^2}{2} + C^*_{2222}\,\frac{(\theta^2)^2}{2}]$$

$$+ [-\alpha^*_{13}\,\theta^1 - \alpha^*_{23}\,\theta^2 + c_3] \ ,$$

$$\tilde{\delta}_{13} = C^*_{1111}\,[\{\frac{12M_{31}}{WH^3}\}\theta^1] + C^*_{1122}\,[\{\frac{12M_{32}}{LH^3}\}\theta^1] + \alpha^*_{13} \ ,$$

$$\tilde{\delta}_{23} = C^*_{1122}\,[\{\frac{12M_{31}}{WH^3}\}\theta^2] + C^*_{2222}\,[\{\frac{12M_{32}}{LH^3}\}\theta^2] + \alpha^*_{23} \ ,$$

$$\tilde{\delta}_{33} = C^*_{1133}\,[\{\frac{12M_{21}}{W^3H}\}\theta^2] + C^*_{2233}\,[\{\frac{12M_{12}}{L^3H}\}\theta^1]$$

$$+ C^*_{3333}\,[\{\frac{12M_{13}}{L^3W}\}\theta^1 + \{\frac{12M_{23}}{LW^3}\}\theta^2] \ , \tag{4.14.19}$$

where the constants c_i are defined by

$$c_1 = -C^*_{2233}\,[\frac{HM_{12}}{2L^3}] - C^*_{3333}\,[\frac{H^2M_{13}}{2L^3W}] + c^*_1 \ ,$$

$$c_2 = -C^*_{1133}\,[\frac{HM_{21}}{2W^3}] - C^*_{3333}\,[\frac{H^2M_{23}}{2LW^3}] + c^*_2 \ ,$$

$$c_3 = C^*_{1133}\,[\frac{M_{31}}{2WH}] + C^*_{2233}\,[\frac{M_{32}}{2LH}] + c^*_3 \ . \tag{4.14.20}$$

Obviously, since c^*_i are arbitrary constants, they can absorb the remaining terms in (4.14.20) causing c_i to be arbitrary constants associated with rigid body translations. Next, the strains associated with (4.14.19) are calculated to be

$$E_{11} = C^*_{1111}\,[\{\frac{12M_{21}}{W^3H}\}\theta^2] + C^*_{1122}\,[\{\frac{12M_{12}}{L^3H}\}\theta^1]$$

$$+ C^*_{1133}\,[\{\frac{12M_{13}}{L^3W}\}\theta^1 + \{\frac{12M_{23}}{LW^3}\}\theta^2] \ ,$$

$$E_{22} = C^*_{1122}\,[\{\frac{12M_{21}}{W^3H}\}\theta^2] + C^*_{2222}\,[\{\frac{12M_{12}}{L^3H}\}\theta^1]$$

$$+ C^*_{2233}\,[\{\frac{12M_{13}}{L^3W}\}\theta^1 + \{\frac{12M_{23}}{LW^3}\}\theta^2] \ ,$$

$$E_{33} = \tilde{\delta}_{33} \ , \quad E_{12} = E_{13} = E_{23} = 0 \ ,$$

$$\beta_{11} = C^*_{1111}\,[\frac{12M_{31}}{WH^3}] + C^*_{1122}\,[\frac{12M_{32}}{LH^3}] \ , \quad \beta_{21} = 0 \ ,$$

$$\beta_{31} = C^*_{2233}\,[\{\frac{12M_{12}}{L^3H}\}] + C^*_{3333}\,[\{\frac{12M_{13}}{L^3W}\}] \ , \ \beta_{12} = 0 \ ,$$

$$\beta_{22} = C^*_{1122}\,[\frac{12M_{31}}{WH^3}] + C^*_{2222}\,[\frac{12M_{32}}{LH^3}] \ ,$$
$$\beta_{32} = C^*_{1133}\,[\{\frac{12M_{21}}{W^3H}\}] + C^*_{3333}\,[\{\frac{12M_{23}}{LW^3}\}] \ . \tag{4.14.21}$$

At this point it is of interest to recall the physical interpretation of the result that $E_{\alpha 3}$ vanishes. In particular, it follows from (4.14.9) and (4.14.12) that for the linearized theory of a plate

$$\mathbf{d}_\alpha \bullet \mathbf{d}_3 = \boldsymbol{\delta}_\alpha \bullet \mathbf{e}_3 + \mathbf{e}_\alpha \bullet \boldsymbol{\delta}_3 = 2\,E_{\alpha 3} = 0 \ . \tag{4.14.22}$$

This indicates that the director remains normal to the plate's reference surface. Physically, this means that the material fiber that was normal to the plate's reference surface in its reference configuration remains normal to the reference surface in the present configuration, which is a known result for pure bending.

With the help of the strains (4.14.21), the constitutive equations (4.14.18) and the expressions (3.14.11), it can easily be shown that

$$T_{11} = H\,[\{\frac{12M_{21}}{W^3H}\}\theta^2] \ , \quad T_{22} = H\,[\{\frac{12M_{12}}{L^3H}\}\theta^1] \ ,$$
$$T_{33} = H\,[\{\frac{12M_{13}}{L^3W}\}\theta^1 + \{\frac{12M_{23}}{LW^3}\}\theta^2] \ ,$$
$$T_{12} = T_{13} = T_{23} = 0 \ , \quad \tilde{\mathbf{t}}^3 = m\,\tilde{\mathbf{b}}^3 \ , \tag{4.14.23}$$

and that $\tilde{\mathbf{t}}^\alpha$ are given by $(4.14.15)_{1,2}$ for all values of θ^α, not just at the boundaries of the plate. Moreover, it follows that $\tilde{\mathbf{m}}^\alpha$ are constants so the equations of equilibrium (4.14.10) are satisfied. Now, the conditions that $\tilde{\mathbf{m}}^\alpha$ equal their boundary values yield the four equations

$$K_{1111}\,C^*_{1111} + K_{1122}\,C^*_{1122} = \frac{H^2}{12} \ , \quad K_{1111}\,C^*_{1122} + K_{1122}\,C^*_{2222} = 0 \ ,$$
$$K_{1122}\,C^*_{1122} + K_{2222}\,C^*_{2222} = \frac{H^2}{12} \ , \quad K_{1122}\,C^*_{1111} + K_{2222}\,C^*_{1122} = 0 \ , \tag{4.14.24}$$

and the results

$$K_{3131} = K_{3232} = 0 \ . \tag{4.14.25}$$

Although (4.14.23) are four equations for three unknown constitutive coefficients, they are dependent equations that can be solved to deduce that

$$K_{1111} = \frac{H^2}{12}\,[\frac{C^*_{2222}}{C^*_{1111}C^*_{2222} - C^*_{1122}C^*_{1122}}] \ ,$$
$$K_{2222} = \frac{H^2}{12}\,[\frac{C^*_{1111}}{C^*_{1111}C^*_{2222} - C^*_{1122}C^*_{1122}}] \ ,$$
$$K_{1122} = -\frac{H^2}{12}\,[\frac{C^*_{1122}}{C^*_{1111}C^*_{2222} - C^*_{1122}C^*_{1122}}] \ . \tag{4.14.26}$$

However, with the help of (3.14.11), it can be shown that

$$C^*_{1111}\,C^*_{2222} - C^*_{1122}\,C^*_{1122} = \frac{K^*_{3333}}{C^*} \ . \tag{4.14.27}$$

Thus, (4.14.26) can be written in the alternative forms

$$K_{1111} = \frac{H^2}{12}\Big[\ K^*_{1111} - \frac{K^*_{1133}K^*_{1133}}{K^*_{3333}}\Big] \ ,$$

$$K_{2222} = \frac{H^2}{12}\Big[\ K^*_{2222} - \frac{K^*_{2233}K^*_{2233}}{K^*_{3333}}\Big] \ ,$$

$$K_{1122} = \frac{H^2}{12}\Big[\ K^*_{1122} - \frac{K^*_{1133}K^*_{2233}}{K^*_{3333}}\Big] \ , \tag{4.14.28}$$

which are identical to those given by Green and Naghdi (1982).

Since the results (4.14.25) appear to be new, it is of interest to discuss their physical origin. To this end, it suffices to consider the responses to the moments M_{12} and M_{21}. Specifically, it can be seen from (4.14.19) that due to the Poisson effect these moments cause the thickness of the plate to vary linearly with the coordinates θ^α. Moreover, it is observed from (3.14.8) that the three-dimensional stresses associated with these moments are independent of the thickness coordinate θ^3. Consequently, the couples $\tilde{\mathbf{m}}^\alpha$ must vanish even though the thickness of the plate does not remain constant.

It is also of interest to note that the results (4.14.25) influence the discussion of boundary conditions for the plate. In particular, it can be seen from (4.14.18) and (4.14.25) that the couples $m^{3\alpha}$ vanish. This means that there is no work done by changes in the normal component of the director $\mathbf{d}_3$ so that this normal component cannot be specified by boundary conditions.

At this point, it is important to comment on the relationship of the Cosserat theory with a standard Galerkin approximate solution of the three-dimensional equations. Most of the developments in section 4.3 are consistent with the Galerkin approach which uses the kinematic assumption (4.2.7) and the two weighting functions $\{1 \text{ and } \theta^3\}$. However, the direct approach of the Cosserat theory in the remaining sections in chapter 4 does not rely explicitly on this kinematic assumption. In particular, the constitutive equations in section 4.8 are developed in a manner that directly parallels that of the three-dimensional theory, with the kinetic quantities being derivable from a strain energy function. The procedure for determining the functional form of this strain energy is identical to that of the three-dimensional theory. Specifically, a functional form is proposed and the constitutive coefficients are determined to best match experimental data. In this section, the constitutive constants for a plate were determined by matching exact solutions of the three-dimensional equations (which themselves are presumed to accurately predict relevant experiments).

In contrast, within the context of the standard Galerkin approach the constitutive coefficients are determined by direct integration of the three-dimensional constitutive equations that are obtained using the kinematic assumption (4.2.7). In particular, for linear plate theory it can be shown that the components E^*_{ij} of the three-dimensional strain associated with the kinematic assumption (4.14.5) can be expressed in terms of the Cosserat strains E_{ij} and $\beta_{i\alpha}$ by the equations

$$E^*_{\alpha\beta} = E_{\alpha\beta} + \frac{\theta^3}{2}(\beta_{\alpha\beta} + \beta_{\beta\alpha}) \ , \quad E^*_{3\alpha} = E_{3\alpha} + \frac{\theta^3}{2}\beta_{3\alpha} \ , \quad E^*_{33} = E_{33} \ . \tag{4.14.29}$$

It therefore, follows from $(4.14.6)_2$, $(4.14.13)_2$ and (4.14.17) that for these strains direct integration for a plate produces the Galerkin form of $m^{i\alpha}$

$$m^{i\alpha} = \frac{H^3}{12} K^*_{i\alpha j\beta}\, \beta_{j\beta} \ , \tag{4.14.30}$$

which can be compared with the associated Cosserat form

$$m^{i\alpha} = HK_{i\alpha j\beta}\, \beta_{j\beta} \ , \tag{4.14.31}$$

to deduce that the constitutive coefficients $K_{i\alpha j\beta}$ should be specified by

$$K_{i\alpha j\beta} = \frac{H^2}{12} K^*_{i\alpha j\beta} \ . \tag{4.14.32}$$

Obviously, the values (4.14.32) are not compatible with the results (4.14.24) and (4.14.26) which were obtained by forcing the Cosserat solution to match the three-dimensional solutions [comments to this effect were mentioned in (Green and Naghdi, 1996)]. This is mainly due to the fact that the kinematic approximation (4.14.5) causes the strain E^*_{33} in (4.14.29) to be independent of the thickness coordinate θ^3. For this reason, the associated three-dimensional stress distribution is not compatible with the exact distribution (3.14.8). Consequently, this is a specific example where the Cosserat approach produces values of the constitutive coefficients that are more accurate than those obtained using standard Galerkin methods.

For the special case of an isotropic material the expressions (3.14.13) apply and the results (4.14.26) reduce to

$$K_{1111} = K_{2222} = \frac{H^2E^*}{12(1-\nu^{*2})} = \frac{H^2\mu^*}{6(1-\nu^*)} \ ,$$
$$K_{1122} = \frac{H^2E^*\nu^*}{12(1-\nu^{*2})} = \frac{H^2\mu^*\nu^*}{6(1-\nu^*)} \ , \tag{4.14.33}$$

where use has been made of Table 3.12.1. It also follows from (4.12.12) and (4.14.25) that

$$K_1 = \frac{H^2\mu^*\nu^*}{6(1-\nu^*)} \ , \quad K_2 + K_3 = \frac{H^2}{6} \ , \quad K_4 = 0 \ . \tag{4.14.34}$$

Moreover, for a plate with $\bar{H}$=H, equations (4.12.16) and (4.14.34) yield

$$\alpha_5 = \frac{H^3\mu^*\nu^*}{6(1-\nu^*)} \ , \quad \alpha_8 = 0 \ . \tag{4.14.35}$$

This value of α_5 is that same as that determined by Naghdi (1972, sec. 24) and the value of α_8 is consistent with the discussion in (Naghdi, 1972, sec. 24).

The work of Green and Naghdi (1982) on laminated composite plates considers a finite number of Cosserat plates that are bonded together on their major surfaces. This bonding is accomplished by demanding continuity of the displacements and appropriate stresses at the major surfaces. Such bonding was also used by Rubin (1987b) to formulate the numerical solution of spherically symmetric problems using the Cosserat theory for spherical shells. In view of these developments, it is of interest to examine how accurately the Cosserat theory can predict the displacements of the major surfaces of the plate. To this end, the kinematic assumption (4.14.5) is used to define the displacements $\mathbf{u}_1$ and $\mathbf{u}_2$ of these major surfaces through the expressions

$$\mathbf{u}_1 = \mathbf{u} - \frac{H}{2}\boldsymbol{\delta}_3 \ , \quad \mathbf{u}_2 = \mathbf{u} + \frac{H}{2}\boldsymbol{\delta}_3 \ . \tag{4.14.36}$$

Next, with the help of the exact displacements (3.14.12) and the Cosserat displacements (4.14.19), it can be shown that

$$\begin{aligned}
\mathbf{u}_1 - \mathbf{u}^*(\theta^\alpha,-H/2) = {} & \left[c_1 - c_1^* + C^*_{2233}\left\{\frac{3M_{12}H}{2L^3}\right\} + C^*_{3333}\left\{\frac{3M_{13}H^2}{2L^3W}\right\}\right]\mathbf{e}_1 \\
& + \left[c_2 - c_2^* + C^*_{1133}\left\{\frac{3M_{21}H^2}{2LW^3}\right\} + C^*_{3333}\left\{\frac{3M_{23}H^2}{2LW^3}\right\}\right]\mathbf{e}_2 \\
& + \left[c_3 - c_3^* - C^*_{1133}\left\{\frac{3M_{31}}{2WH}\right\} - C^*_{2233}\left\{\frac{3M_{32}}{2LH}\right\}\right]\mathbf{e}_3 \ , \\
\mathbf{u}_2 - \mathbf{u}^*(\theta^\alpha,H/2) = {} & \mathbf{u}_1 - \mathbf{u}^*(\theta^\alpha,-H/2) \ .
\end{aligned} \tag{4.14.37}$$

Thus, the differences between the exact values of the displacements and the Cosserat predictions at the major surfaces are equal to a constant vector that can be eliminated by the values c_i associated with superposed constant translation. In this regard, it is of interest to note that even though the identification $(4.14.8)_3$ does not connect $\mathbf{u}$ with the exact displacement of the middle surface of the plate, the predictions (4.14.36) are quite accurate for this problem.

4.15 Torsion of an orthotropic rectangular plate

The analysis of the previous section produced expressions for five of the eight constitutive coefficients $K_{i\gamma j\delta}$ for a orthotropic plate by considering bending solutions. In the present section the remaining coefficients K_{1212}, K_{1221} and K_{2121} are determined by comparing solutions of the linear Cosserat theory for an orthotropic rectangular plate with exact torsion solutions of the three-dimensional theory given in section 3.15. Also, the formulation of the plate equations is the same as that described in section 4.14.

It can be seen that the kinematic assumption (4.14.5) is compatible with the exact displacement field (3.15.1) associated with the twist ω_3 and the moment M_3. Therefore, when only the moment M_3 is applied (i.e. M_1 and M_2 vanish), it is expected that the solution of the Cosserat theory will be an exact three-dimensional solution. For this case, the displacement fields u_i and $\tilde{\delta}_{i3}$ of the Cosserat theory are given by

$$\begin{gathered}
u_1 = 0 \ , \quad u_2 = 0 \ , \quad u_3 = \omega_3\phi_3(\theta^1,\theta^2) \ , \\
\tilde{\delta}_{13} = -\,\omega_3\theta^2 \ , \quad \tilde{\delta}_{23} = \omega_3\theta^1 \ , \quad \tilde{\delta}_{33} = 0 \ .
\end{gathered} \tag{4.15.1}$$

Thus, with the help of (4.14.17) and (4.14.18) it follows that

$$E_{\alpha\beta}=0 \ , \ \ E_{13}=\frac{\omega_3}{2}\left[\phi_{3,1}-\theta^2\right] \ , \ \ E_{23}=\frac{\omega_3}{2}\left[\phi_{3,2}+\theta^1\right] \ , \ E_{33}=0 \ ,$$
$$\beta_{11}=0 \ , \ \ \beta_{21}=\omega_3 \ , \ \ \beta_{31}=0 \ , \beta_{12}=-\,\omega_3 \ , \ \ \beta_{22}=0 \ , \ \ \beta_{32}=0 \ ,$$
$$T_{11}=T_{22}=T_{33}=T_{12}=0 \ ,$$
$$T_{13}=\omega_3 H K^*_{1313}\left[\phi_{3,1}-\theta^2\right] \ , \ \ T_{23}=\omega_3 H K^*_{2323}\left[\phi_{3,2}+\theta^1\right] ,$$
$$\tilde{\mathbf{t}}^1=\omega_3 H K^*_{1313}\left[\phi_{3,1}-\theta^2\right]\mathbf{e}_3 \ , \ \ \tilde{\mathbf{t}}^2=\omega_3 H K^*_{2323}\left[\phi_{3,2}+\theta^1\right]\mathbf{e}_3$$
$$\tilde{\mathbf{t}}^3=\omega_3 H K^*_{1313}\left[\phi_{3,1}-\theta^2\right]\mathbf{e}_1+\omega_3 H K^*_{2323}\left[\phi_{3,2}+\theta^1\right]\mathbf{e}_2 \ ,$$
$$\tilde{\mathbf{m}}^1=\omega_3 H\left[K_{2121}-K_{1221}\right]\mathbf{e}_2 \ , \ \ \tilde{\mathbf{m}}^2=\omega_3 H\left[K_{1221}-K_{1212}\right]\mathbf{e}_1 \ . \qquad (4.15.2)$$

Moreover, in the absence of three-dimensional body force the assigned fields associated with (4.14.13) and (4.14.14) become

$$m\,\tilde{\mathbf{b}}=0 \ ,$$
$$m\,\tilde{\mathbf{b}}^3=\omega_3 H\left[K^*_{1313}\left\{\phi_{3,1}-\theta^2\right\}\mathbf{e}_1+K^*_{2323}\left\{\phi_{3,2}+\theta^1\right\}\mathbf{e}_2\right] \ . \qquad (4.15.3)$$

It then follows that the equations of equilibrium (4.14.10) demand that

$$K^*_{1313}\,\phi_{3,11}+K^*_{2323}\,\phi_{3,22}=0 \ . \qquad (4.15.4)$$

Also, the boundary conditions that the edges of the plate are free, can be expressed in the forms

$$\tilde{\mathbf{t}}^1=0 \ , \ \ \tilde{\mathbf{m}}^1=0 \ , \ \ \text{for} \ \ \theta^1=\pm L/2 \ ,$$
$$\tilde{\mathbf{t}}^2=0 \ , \ \ \tilde{\mathbf{m}}^2=0 \ , \ \ \text{for} \ \ \theta^2=\pm W/2 \ . \qquad (4.15.5)$$

Next, using (4.15.2), these boundary conditions require

$$\left[\phi_{3,1}\ -\theta^2\right]_{\theta^1=\pm L/2}=0 \ , \ \ \left[\phi_{3,2}\ +\theta^1\right]_{\theta^2=\pm W/2}=0 \ . \qquad (4.15.6)$$

as well as the restrictions

$$K_{1212}=K_{1221}=K_{2121} \ . \qquad (4.15.7)$$

Noticing that (4.15.4) and (4.15.6) are identical to $(3.15.3)_3$ and $(3.15.4)_{5,6}$, respectively, it follows that the solution for ϕ_3 is identical to the three-dimensional solution (3.15.5). This means that the Cosserat solution for torsion about the $\mathbf{e}_3$ axis will be exact provided that the constitutive coefficients satisfy (4.15.7).

In view of the restriction (4.15.7), it follows that there is only one remaining value K_{1212} to determine for an orthotropic plate. Here, it will be shown that by proper choice of this value, the values of the torsional rigidities for both of the solutions for torsion about the $\mathbf{e}_1$ and $\mathbf{e}_2$ axes become quite close to the three-dimensional values. To this end, it is first noted that with the help of (3.15.1) and (3.15.5) with M_3=0, the equations (4.14.8) suggest that the displacements for the Cosserat theory associated with torsion about the $\mathbf{e}_1$ and $\mathbf{e}_2$ axes should be given by

$$u_1=0 \ , \ \ u_2=0 \ , \ \ u_3=\omega_1\theta^1\theta^2-\omega_2\theta^1\theta^2 \ , \qquad (4.15.8)$$

and that the director displacements are close to the values

$$\tilde{\delta}_{13} = \omega_1\Big[-\theta^2 + \sum_{n=1}^{\infty}\Big[\frac{16\, a_2\, a_3^2}{H\, k_{1n}^3 \cosh\{\frac{k_{1n}W}{2a_2}\}}\Big]\sinh\{\frac{k_{1n}\theta^2}{a_2}\}\Big] + \omega_2\theta^2 ,$$

$$\tilde{\delta}_{23} = -\,\omega_1\theta^1 + \omega_2\Big[\theta^1 - \sum_{n=1}^{\infty}\Big[\frac{16\, b_1\, b_3^2}{H\, k_{2n}^3 \cosh\{\frac{k_{2n}L}{2b_1}\}}\Big]\sinh\{\frac{k_{2n}\theta^1}{b_1}\}\Big] ,$$

$$\tilde{\delta}_{33} = 0 \ . \tag{4.15.9}$$

It will presently be shown that the solution of the Cosserat equations is consistent with the displacements in (4.15.8), but that the director displacements are given by

$$\tilde{\delta}_{13} = \omega_1\big[-\theta^2 + f_1(\theta^2)\big] + \omega_2\theta^2 , \quad \tilde{\delta}_{23} = -\,\omega_1\theta^1 + \omega_2\big[\theta^1 - f_2(\theta^1)\big] ,$$

$$\tilde{\delta}_{33} = 0 \ , \tag{4.15.10}$$

where $f_1(\theta^2)$ and $f_2(\theta^1)$ are functions that need to be determined.

Using (4.14.17), (4.14.18), the restriction (4.15.7), (4.15.8) and (4.15.10), it follows that

$$E_{\alpha\beta} = 0 \ , \quad E_{13} = \frac{\omega_1}{2} f_1(\theta^2) , \quad E_{23} = -\frac{\omega_2}{2} f_2(\theta^1) \ , \quad E_{33} = 0 \ ,$$

$$\beta_{11} = 0 \ , \quad \beta_{21} = -\,\omega_1 + \omega_2\big[1 - f_{2,1}\big] , \quad \beta_{31} = 0 \ ,$$

$$\beta_{12} = \omega_1\big[-1 + f_{1,2}\big] + \omega_2 \ , \quad \beta_{22} = 0 \ , \quad \beta_{32} = 0 \ ,$$

$$T_{11} = T_{22} = T_{33} = T_{12} = 0 \ ,$$

$$T_{13} = \omega_1 H K^*_{1313}\, f_1 \ , \quad T_{23} = -\,\omega_2 H K^*_{2323}\, f_2 \ ,$$

$$\tilde{\mathbf{t}}^1 = \big[\omega_1 H K^*_{1313}\, f_1\big]\, \mathbf{e}_3 \ , \quad \tilde{\mathbf{t}}^2 = -\big[\omega_2 H K^*_{2323}\, f_2\big]\, \mathbf{e}_3$$

$$\tilde{\mathbf{t}}^3 = \big[\omega_1 H K^*_{1313}\, f_1\big]\, \mathbf{e}_1 - \big[\omega_2 H K^*_{2323}\, f_2\big]\, \mathbf{e}_2 \ ,$$

$$\tilde{\mathbf{m}}^1 = H K_{1212}\big[\omega_1\{-2 + f_{1,2}\} + \omega_2\{2 - f_{2,1}\}\big]\, \mathbf{e}_2 \ ,$$

$$\tilde{\mathbf{m}}^2 = H K_{1212}\big[\omega_1\{-2 + f_{1,2}\} + \omega_2\{2 - f_{2,1}\}\big]\, \mathbf{e}_1 \ . \tag{4.15.11}$$

Moreover, in the absence of three-dimensional body force the assigned fields associated with (4.14.13) and (4.14.14) vanish

$$\tilde{\mathbf{b}} = 0 \ , \quad \tilde{\mathbf{b}}^3 = 0 \ . \tag{4.15.12}$$

Thus, the equations of equilibrium (4.14.10) demand that

$$K_{1212}\, f_{1,22} - K^*_{1313}\, f_1 = 0 \ , \quad K_{1212}\, f_{2,11} - K^*_{2323}\, f_2 = 0 \ . \tag{4.15.13}$$

Also, the boundary conditions associated with the free edges ($\theta^2 = \pm W/2$ for ω_1; and $\theta^1 = \pm L/2$ for ω_2) become

$$\big[f_{1,2} - 2\big]_{\theta^1 = \pm L/2} = 0 \ , \quad \big[f_{2,1} - 2\big]_{\theta^2 = \pm W/2} = 0 \ . \tag{4.15.14}$$

Therefore, the solutions can be written in the forms

$$f_1 = \frac{2\sinh\{\lambda_1\theta^2/2\}}{\lambda_1\cosh\{\lambda_1 L/2\}} \ , \quad \lambda_1^2 = \frac{K^*_{1313}}{K_{1212}} \ ,$$

$$f_2 = \frac{2\sinh\{\lambda_2\theta^2/2\}}{\lambda_2\cosh\{\lambda_2 L/2\}} \ , \quad \lambda_2^2 = \frac{K^*_{2323}}{K_{1212}} \ . \qquad (4.15.15)$$

In order to compare these results with those of the three-dimensional solution, it is convenient to calculate the total resultant moment M_1 applied to the edge θ^1=L/2, and the moment M_2 applied to the edge θ^2=W/2. This is accomplished by linearizing the last term in the balance of angular momentum (4.4.7) and writing

$$M_1 = \mathbf{e}_1 \bullet \int_{-W/2}^{W/2} [\mathbf{X} \times \tilde{\mathbf{t}}^1 + \mathbf{e}_3 \times \tilde{\mathbf{m}}^1]_{\theta^1 = L/2} \, d\theta^2 \ ,$$

$$M_2 = \mathbf{e}_2 \bullet \int_{-L/2}^{L/2} [\mathbf{X} \times \tilde{\mathbf{t}}^2 + \mathbf{e}_3 \times \tilde{\mathbf{m}}^2]_{\theta^2 = W/2} \, d\theta^1 \ . \qquad (4.15.16)$$

Using these expressions, it can be shown that

$$M_1 = B_1\,\omega_1 \ , \quad M_2 = B_2\,\omega_2 \ , \qquad (4.15.17)$$

where the torsional rigidities B_1 and B_2 are given by

$$B_1 = 4WHK_{1212}\,[1 - \frac{2}{\lambda_1 W}\tanh\{\frac{\lambda_1 W}{2}\}] \ ,$$

$$B_2 = 4LHK_{1212}\,[1 - \frac{2}{\lambda_2 L}\tanh\{\frac{\lambda_2 L}{2}\}] \ . \qquad (4.15.18)$$

Next, in order for these expressions to reproduce the exact limiting values (3.15.15) for a thin plate, it is necessary to specify the constitutive coefficients in the forms

$$K_{1212} = K_{1221} = K_{2121} = \frac{H^2}{12} K^*_{1212} \ , \qquad (4.15.19)$$

which can be seen to be identical to the values given by Green and Naghdi (1982). Moreover, using these values, the functions B_1 and B_2 can be rewritten in the forms

$$B_1 = \frac{W^2H^2}{3}\,[K^*_{1212}\,K^*_{1313}]^{1/2}\; b(\eta_1) \ , \quad \eta_1 = \frac{W}{H}\,[\frac{K^*_{1313}}{K^*_{1212}}]^{1/2} \ ,$$

$$B_2 = \frac{L^2H^2}{3}\,[K^*_{1212}\,K^*_{2323}]^{1/2}\; b(\eta_2) \ , \quad \eta_2 = \frac{L}{H}\,[\frac{K^*_{2323}}{K^*_{1212}}]^{1/2} \ ,$$

$$b(\eta) = \frac{1}{\eta}\,[1 - \frac{1}{\eta\sqrt{3}}\tanh\{\eta\sqrt{3}\}] \ . \qquad (4.15.20)$$

Figure 4.15.1 shows that the Cosserat torsional rigidity function $b(\eta)$ is quite close to the exact function $b^*(\eta)$ for values of η greater than about 2. Since the values of η depend on both the geometry and material properties of the plate, the value η=2 can correspond to either a square or rectangular cross-section. However, for a fixed cross-sectional area and fixed material properties, the function $b(\eta)$ shows the decrease in the torsional rigidity as the thickness of the plate decreases.

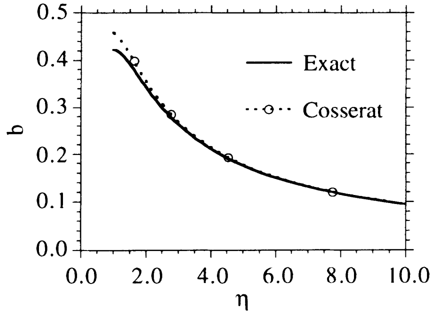

Fig. 4.15.1 Torsional rigidity functions: b^* (Exact) and b (Cosserat).

For the special case of an isotropic material the expressions (3.12.13) apply, and the results (4.15.19) reduce to

$$K_{1212} = K_{1221} = K_{2121} = \frac{H^2\mu^*}{12} \ , \qquad (4.15.21)$$

It also follows from (4.12.12) that

$$K_2 = K_3 = \frac{H^2\mu^*}{12} \ , \qquad (4.15.22)$$

which is compatible with $(4.14.34)_2$. Moreover, for a plate with $\bar{H}$=H, equations (4.12.16) and (4.15.22) yield

$$\alpha_6 = \alpha_7 = \frac{H^3\mu^*}{12} \ , \qquad (4.15.23)$$

which are the same values determined by Naghdi (1972, sec. 24).

Finally, with the help of (4.12.9)-(4.12.11), (4.14.25) and (4.15.7), it follows that the constitutive tensors $\mathbf{K}^{\alpha\beta}$ for an orthotropic plate can be written in the forms

$$\begin{aligned}
\mathbf{K}^{\alpha\beta} = {} & (\mathbf{D}^\alpha \bullet \mathbf{M}_1)(\mathbf{D}^\beta \bullet \mathbf{M}_1)[K_{1111}\, \mathbf{M}_1\otimes\mathbf{M}_1 + K_{1212}\, \mathbf{M}_2\otimes\mathbf{M}_2] \\
& + (\mathbf{D}^\alpha \bullet \mathbf{M}_1)(\mathbf{D}^\beta \bullet \mathbf{M}_2)[K_{1122}\, \mathbf{M}_1\otimes\mathbf{M}_2 + K_{1212}\, \mathbf{M}_2\otimes\mathbf{M}_1] \\
& + (\mathbf{D}^\alpha \bullet \mathbf{M}_2)(\mathbf{D}^\beta \bullet \mathbf{M}_1)[K_{1212}\, \mathbf{M}_1\otimes\mathbf{M}_2 + K_{1122}\, \mathbf{M}_2\otimes\mathbf{M}_1] \\
& + (\mathbf{D}^\alpha \bullet \mathbf{M}_2)(\mathbf{D}^\beta \bullet \mathbf{M}_2)[K_{1212}\, \mathbf{M}_1\otimes\mathbf{M}_1 + K_{2222}\, \mathbf{M}_2\otimes\mathbf{M}_2] \ , \qquad (4.15.24)
\end{aligned}$$

where K_{1111}, K_{1122}, K_{2222} are given by (4.14.28), and K_{1212} is given by (4.15.19). Also, using (4.12.12), (4.14.33), (4.14.34), (4.15.21) and (4.15.22), these expressions become

$$\begin{aligned}
\mathbf{K}^{\alpha\beta} = {} & (\mathbf{D}^\alpha \bullet \mathbf{M}_1)(\mathbf{D}^\beta \bullet \mathbf{M}_1)\frac{H^2\mu^*}{12}[\{\frac{2}{1-\nu^*}\}\mathbf{M}_1\otimes\mathbf{M}_1 + \mathbf{M}_2\otimes\mathbf{M}_2] \\
& + (\mathbf{D}^\alpha \bullet \mathbf{M}_1)(\mathbf{D}^\beta \bullet \mathbf{M}_2)\frac{H^2\mu^*}{12}[\{\frac{2\nu^*}{1-\nu^*}\}\mathbf{M}_1\otimes\mathbf{M}_2 + \mathbf{M}_2\otimes\mathbf{M}_1] \\
& + (\mathbf{D}^\alpha \bullet \mathbf{M}_2)(\mathbf{D}^\beta \bullet \mathbf{M}_1)\frac{H^2\mu^*}{12}[\mathbf{M}_1\otimes\mathbf{M}_2 + \{\frac{2\nu^*}{1-\nu^*}\}\mathbf{M}_2\otimes\mathbf{M}_1]
\end{aligned}$$

$$+ (\mathbf{D}^\alpha \bullet \mathbf{M}_2)(\mathbf{D}^\beta \bullet \mathbf{M}_2)\frac{H^2\mu^*}{12}\left[\mathbf{M}_1\otimes\mathbf{M}_1 + \left\{\frac{2}{1-\nu^*}\right\}\mathbf{M}_2\otimes\mathbf{M}_2\right] \ . \quad (4.15.25)$$

for an isotropic plate.

4.16 Forced shearing vibrations of an orthotropic rectangular plate

In the previous sections the constitutive coefficients for the static response of an orthotropic plate have been determined for both homogeneous and inhomogeneous deformations. This section proposes values for the inertia properties of the plate by comparing predictions of the Cosserat theory with the exact results of section 3.16 for forced shearing vibrations of an orthotropic rectangular parallelepiped.

The formulation of the plate equations is the same as that described in section 4.14. In particular, it is noted that in view of (3.2.28) and the condition (4.14.3), it follows that

$$m^* = \rho_0^* A^{1/2} \ . \quad (4.16.1)$$

Moreover, since the reference surface of the plate is taken to be the middle surface (4.14.4) and the mass density ρ_0^* is presumed to be constant, direct integration of (4.3.5) and (4.3.8) yields

$$m = \rho_0^* H A^{1/2} \ , \quad y^3 = 0 \ , \quad (4.16.2)$$

Also, direct integration of (4.3.30) yields the value

$$y^{33} = \frac{H^2}{12} \ . \quad (4.16.3)$$

It will presently be shown that the Cosserat theory can predict more accurate values for vibrational frequencies if the value of the director inertia coefficient y^{33} is taken to be different from (4.16.3). In this regard, it should be mentioned that within the context of the Cosserat theory, the quantity m and the director inertia coefficients y^3 and y^{33} are independent of time and therefore can be determined in the reference configuration. Moreover, within the context of the direct approach these quantities require constitutive equations which need not be determined by the integrals (4.3.5), (4.3.8) and (4.3.30). Nevertheless, since the value $(4.16.2)_1$ for m ensures that the plate will have the same mass per unit area as the associated parallelepiped, it is retained in the Cosserat theory. Also, the value $(4.16.2)_2$ for y^3 is retained for a plate since it indicates that the mass is distributed symmetrically about the middle surface of the plate.

In contrast, the value (4.16.3) for the director inertia coefficient y^{33} will be modified to cause the Cosserat theory to predict vibrational frequencies more accurately. In this regard, it should be mentioned that the value (4.16.3) is the same as the value obtained using the Galerkin method and the kinematic approximation (4.2.7).

To predict a value for y^{33} it is sufficient to consider only one of the vibrational modes discussed in section 3.16. However, to check the consistency of the value obtained, two of these modes of vibration are analyzed. To this end, only the vibrations associated with the amplitudes A_{13}^* and A_{23}^* will be considered. Also, attention is focused on only the first mode of vibration since it is not reasonable to expect a plate theory to accurately predict higher order modes through the thickness. Consequently, the three-dimensional displacements and stresses associated with these modes can be written in the forms

$$u_1^* = A_{13}^* \sin(\omega_{13}^* t) \sin(k_3^* \theta^3) \ , \quad u_2^* = A_{23}^* \sin(\omega_{23}^* t) \sin(k_3^* \theta^3) \ , u_3^* = 0 \ ,$$

$$\tilde{\mathbf{t}}^{*1} = K_{1313}^* \left[A_{13}^* k_3^* \sin(\omega_{13}^* t) \cos(k_3^* \theta^3)\right] \mathbf{e}_3 \ ,$$

$$\tilde{\mathbf{t}}^{*2} = K_{2323}^* \left[A_{23}^* k_3^* \sin(\omega_{23}^* t) \cos(k_3^* \theta^3)\right] \mathbf{e}_3 \ ,$$

$$\tilde{\mathbf{t}}^{*3} = K_{1313}^* \left[A_{13}^* k_3^* \sin(\omega_{13}^* t) \cos(k_3^* \theta^3)\right] \mathbf{e}_1 + K_{2323}^* \left[A_{23}^* k_3^* \sin(\omega_{23}^* t) \cos(k_3^* \theta^3)\right] \mathbf{e}_2 \ , \tag{4.16.4}$$

where the frequencies and the wave number are given by

$$\omega_{13}^* = \left[\frac{K_{1313}^*}{\rho_0^*}\right]^{1/2} \frac{\pi}{H} \ , \quad \omega_{23}^* = \left[\frac{K_{2323}^*}{\rho_0^*}\right]^{1/2} \frac{\pi}{H} \ , \quad k_3^* = \frac{\pi}{H} \ . \tag{4.16.5}$$

In order for the loading of the plate to be the same as that of the parallelepiped, it is necessary to specify the assigned fields by

$$\tilde{\mathbf{b}} = 0 \ , \quad \tilde{\mathbf{b}}^3 = 0 \ , \tag{4.16.6}$$

and the boundary conditions by

$$\tilde{\mathbf{t}}^1(\pm L/2, \theta^2, t) = 2K_{1313}^* \left[A_{13}^* \sin(\omega_{13}^* t)\right] \mathbf{e}_3 \ , \quad \tilde{\mathbf{m}}^1(\pm L/2, \theta^2, t) = 0 \ ,$$

$$\tilde{\mathbf{t}}^2(\theta^1, \pm W/2, t) = 2K_{2323}^* \left[A_{23}^* \sin(\omega_{23}^* t)\ \right] \mathbf{e}_3 \ , \quad \tilde{\mathbf{m}}^2(\theta^1, \pm W/2, t) = 0 \ , \tag{4.16.7}$$

where use has been made of the linearized forms of (4.3.10), (4.3.15), (4.3.24), (4.3.28), (4.3.32), (4.3.36), (4.3.38) and (4.3.42). Next, the expressions (4.14.8) suggest that the Cosserat displacements associated with (4.16.4) are given by

$$\mathbf{u} = 0 \ , \quad \boldsymbol{\delta}_3 = \frac{2}{H} \left[A_{13}^* \sin(\omega_{13}^* t)\, \mathbf{e}_1 + A_{23}^* \sin(\omega_{23}^* t)\, \mathbf{e}_2\right] . \tag{4.16.8}$$

Using these displacements the constitutive equations (4.14.10) for an orthotropic plate yield the expressions

$$\tilde{\mathbf{E}} = \frac{1}{H} \left[A_{13}^* \sin(\omega_{13}^* t)\, (\mathbf{e}_1 \otimes \mathbf{e}_3 + \mathbf{e}_3 \otimes \mathbf{e}_1) + A_{23}^* \sin(\omega_{23}^* t)\, (\mathbf{e}_2 \otimes \mathbf{e}_3 + \mathbf{e}_3 \otimes \mathbf{e}_2)\right] \ ,$$

$$\tilde{\mathbf{t}}^1 = \left[2K_{1313}^* A_{13}^* \sin(\omega_{13}^* t)\right] \mathbf{e}_3 \ , \quad \tilde{\mathbf{t}}^2 = \left[2K_{2323}^* A_{23}^* \sin(\omega_{23}^* t)\right] \mathbf{e}_3 \ ,$$

$$\tilde{\mathbf{t}}^3 = \left[2K_{1313}^* A_{13}^* \sin(\omega_{13}^* t)\right] \mathbf{e}_1 + \left[2K_{2323}^* A_{23}^* \sin(\omega_{23}^* t)\right] \mathbf{e}_2 \ ,$$

$$\tilde{\boldsymbol{\beta}}_\alpha = 0 \ , \quad \tilde{\mathbf{m}}^\alpha = 0 \ , \tag{4.16.9}$$

which automatically satisfy the boundary conditions (4.16.7). Also, with the help of (4.14.12), it follows that $A^{1/2}=1$ so that

$$m = \rho_0^* H \ . \tag{4.16.10}$$

Now, the linearized equations of motion (4.13.6) reduce to the following two scalar equations

$$\rho_0^* y^{33} \left\{\omega_{13}^*\right\}^2 = K_{1313}^* \ , \quad \rho_0^* y^{33} \left\{\omega_{23}^*\right\}^2 = K_{2323}^* \ . \tag{4.16.11}$$

However, since the frequencies are given by (4.16.5), it can be seen that both of these equations can be satisfied if the director inertia coefficient y^{33} is specified by

$$y^{33} = \frac{H^2}{\pi^2} \ , \tag{4.16.12}$$

instead of the Galerkin value (4.16.3). In this regard, it is noted that the director inertia coefficients model both the mass distribution in the plate as well as the distribution of acceleration in potential vibrational modes. This is similar to the stiffness coefficients of the plate which depend on both the material and geometric properties of the structure.

4.17 Free isochoric vibrations of an isotropic cube

To further examine the validity of the specification $(4.14.34)_3$ of the constitutive constant K_4 and the specification (4.16.12) of the director inertia coefficient y^{33}, it is of interest to use the Cosserat plate equations to predict free isochoric vibrations of an isotropic cube discussed in section 3.17. The formulation of the plate equations is the same as that described in sections 4.14 and 4.16, except here the dimensions of the plate [see (3.14.1)] are taken to be equal

$$L = W = H \ . \tag{4.17.1}$$

Also, the material is assumed to be isotropic so that $(4.14.9)_5$ becomes

$$\tilde{\mathbf{T}} = H \left[(K^* - \frac{2}{3}\mu^*) (\tilde{\mathbf{E}} \bullet \mathbf{I}) \mathbf{I} + 2\mu^* \tilde{\mathbf{E}} \right] \ . \tag{4.17.2}$$

For simplicity, attention is focused on only two of the three modes of vibration described by (3.17.2) so that the three-dimensional displacement field is given by

$$\begin{aligned} u_1^* &= A_{13}^* \sin(\omega^* t) \sin(k^* \theta^1) \cos(k^* \theta^3) \ , \\ u_2^* &= A_{23}^* \sin(\omega^* t) \sin(k^* \theta^2) \cos(k^* \theta^3) \ , \\ u_3^* &= - A_{13}^* \sin(\omega^* t) \cos(k^* \theta^1) \sin(k^* \theta^3) \\ &\quad - A_{23}^* \sin(\omega^* t) \cos(k^* \theta^2) \sin(k^* \theta^3) \ . \end{aligned} \tag{4.17.3}$$

Also, attention is focused on only the first mode of vibration so the frequency and wave number are given by

$$\omega^* = \left[\frac{2\mu^*}{\rho_0^*}\right]^{1/2} \frac{\pi}{H} \ , \quad k^* = \frac{\pi}{H} \ . \tag{4.17.4}$$

In order for the loading of the plate to be the same as that of the cube, it is necessary to specify the assigned fields by

$$\tilde{\mathbf{b}} = 0 \ , \quad \tilde{\mathbf{b}}^3 = 0 \ , \tag{4.17.5}$$

and the boundary conditions by

$$\begin{aligned} \tilde{\mathbf{t}}^1(\pm H/2, \theta^2, t) &= 0 \ , \quad \tilde{\mathbf{m}}^1(\pm H/2, \theta^2, t) = 0 \ , \\ \tilde{\mathbf{t}}^2(\theta^1, \pm H/2, t) &= 0 \ , \quad \tilde{\mathbf{m}}^2(\theta^1, \pm H/2, t) = 0 \ . \end{aligned} \tag{4.17.6}$$

Next, the expressions (4.14.8) suggest that the Cosserat displacements associated with (4.17.3) are given by

$$\mathbf{u} = \left[\frac{2}{\pi} A_{13}^* \sin(\omega^* t) \sin(k^* \theta^1)\right] \mathbf{e}_1 + \left[\frac{2}{\pi} A_{23}^* \sin(\omega^* t) \sin(k^* \theta^2)\right] \mathbf{e}_2 \ ,$$

$$\boldsymbol{\delta}_3 = -\left[\frac{2}{H}A_{13}^* \sin(\omega^* t)\cos(k^*\theta^1) + \frac{2}{H}A_{23}^* \sin(\omega^* t)\cos(k^*\theta^2)\right]\mathbf{e}_3 \ . \quad (4.17.7)$$

Now, with the help of (4.14.9), the strains associated with these displacements become

$$\tilde{\mathbf{E}} = \left[\frac{2}{H}A_{13}^* \sin(\omega^* t)\cos(k^*\theta^1)\right](\mathbf{e}_1\otimes\mathbf{e}_1 - \mathbf{e}_3\otimes\mathbf{e}_3)$$
$$+ \left[\frac{2}{H}A_{23}^* \sin(\omega^* t)\cos(k^*\theta^2)\right](\mathbf{e}_2\otimes\mathbf{e}_2 - \mathbf{e}_3\otimes\mathbf{e}_3) \ ,$$
$$\tilde{\boldsymbol{\beta}}_1 = \left[\frac{2\pi}{H^2}A_{13}^* \sin(\omega^* t)\sin(k^*\theta^1)\right]\mathbf{e}_3 \ ,$$
$$\tilde{\boldsymbol{\beta}}_2 = \left[\frac{2\pi}{H^2}A_{23}^* \sin(\omega^* t)\sin(k^*\theta^2)\right]\mathbf{e}_3 \ . \quad (4.17.8)$$

It can easily be seen that the strain $\tilde{\mathbf{E}}$ is isochoric since

$$\tilde{\mathbf{E}} \bullet \mathbf{I} = 0 \ . \quad (4.17.9)$$

Moreover, for an isotropic material the kinetic quantities in (4.14.9) associated with (4.17.8) become

$$\tilde{\mathbf{t}}^1 = 4\mu^*\left[A_{13}^* \sin(\omega^* t)\cos(k^*\theta^1)\right]\mathbf{e}_1 \ , \quad \tilde{\mathbf{t}}^2 = 4\mu^*\left[A_{23}^* \sin(\omega^* t)\cos(k^*\theta^2)\right]\mathbf{e}_2 \ ,$$
$$\tilde{\mathbf{t}}^3 = 4\mu^*\left[\{A_{13}^* \cos(k^*\theta^1)\}\mathbf{e}_1 + \{A_{23}^* \cos(k^*\theta^2)\}\mathbf{e}_2\right]\sin(\omega^* t) \ ,$$
$$\tilde{\mathbf{m}}^\alpha = 0 \ , \quad (4.17.10)$$

which automatically satisfy the boundary conditions (4.17.6). In this regard, it should be mentioned that this result provides additional justification for the specification $(4.14.34)_2$, which causes $\tilde{\mathbf{m}}^\alpha$ to vanish even though $\tilde{\boldsymbol{\beta}}_\alpha$ do not vanish.

In view of the values $(4.16.2)_2$, (4.16.10) and (4.16.12), the linearized equations of motion (4.13.16) reduce to

$$\rho_0^* H \ddot{\mathbf{u}} = \tilde{\mathbf{t}}^\alpha{}_{,\alpha} \ , \quad \rho_0^* H\left\{\frac{H^2}{\pi^2}\right\}\ddot{\boldsymbol{\delta}}_3 = -\tilde{\mathbf{t}}^3 \ . \quad (4.17.11)$$

Now, substitution of (4.17.7) and (4.17.10) into these equations yields two scalar equations

$$\rho_0^* \frac{H}{\pi}\{\omega^*\}^2 = 2\mu^* k^* \ , \quad \rho_0^*\left\{\frac{H^2}{\pi^2}\right\}\{\omega^*\}^2 = 2\mu^* \ , \quad (4.17.12)$$

which are both satisfied by the values (4.17.4) of the frequency and the wave number. Consequently, this result again justifies the specification (4.16.12) of the director inertia coefficient y^{33}.

4.18 An orthotropic rectangular plate loaded by its own weight

In the previous sections the stiffness and inertia coefficients for a plate have been determined by comparison with exact three-dimensional solutions. Within the context of the Cosserat theory, it is also necessary to specify values for the assigned fields $\mathbf{b}$ and $\mathbf{b}^3$. In determining the restrictions in section 4.11 associated with homogeneous deformations, it was necessary to use the expressions (4.3.15) and (4.3.36) for the parts

$\mathbf{b}_c$ and $\mathbf{b}_c^3$ of these assigned fields that are due to loading on the major surfaces of the shell. For this reason, the expressions (4.3.15) and (4.3.36) are retained. The main objective of this section is to examine the validity of the expressions (4.3.10) and (4.3.32) for the parts $\mathbf{b}_b$ and $\mathbf{b}_b^3$ of the assigned fields associated with body force. This will be explored by comparing the Cosserat solutions of an orthotropic rectangular plate loaded by its own weight with the three-dimensional solutions recorded in section 3.18.

In particular, it will be assumed that the reference values of the three-dimensional mass density ρ_0^* and body force $\mathbf{b}^*$ are constants so that with the help of the conservation of mass $(3.2.35)_1$, the expressions (4.3.10) and (4.3.32) yield

$$m\,\mathbf{b}_b = \rho_0^*\,\mathbf{b}^* \int_{\xi_1}^{\xi_2} G^{1/2}\, d\theta^3 \quad , \quad m\,\mathbf{b}_b^3 = \rho_0^*\,\mathbf{b}^* \int_{\xi_1}^{\xi_2} G^{1/2}\,\theta^3\, d\theta^3 \quad . \tag{4.18.1}$$

In particular, for a plate, the expressions (3.18.1), (4.14.3) and (4.14.4) can be used to deduce that

$$m\,\mathbf{b}_b = \rho_0^*\, H\, \mathbf{b}^* \quad , \quad m\,\mathbf{b}_b^3 = 0 \quad , \tag{4.18.2}$$

so that there is no distinction between the linearized values of $\tilde{\mathbf{b}}_c$ and $\tilde{\mathbf{b}}_c^3$ and the corresponding nonlinear values.

Moreover, for the solutions of section 3.18 the specific body force $\mathbf{b}^*$ is specified by (3.18.1) so that

$$m\,\mathbf{b}_b = -\,\rho_0^*\, H\, g_i^*\, \mathbf{e}_i \quad . \tag{4.18.3}$$

Also, with the help of the linearized forms of (4.3.15) and (4.3.36) and the expressions (3.18.2) for the stresses, the assigned fields due to surface tractions applied to the major surfaces of the plate are given by

$$m\,\tilde{\mathbf{b}}_c = \rho_0^*\, H\, g_3^*\, \mathbf{e}_3 \quad , \quad m\,\tilde{\mathbf{b}}_c^3 = -\frac{H}{2}\,\rho_0^*\, H\, g_3^*\, \mathbf{e}_3 \quad , \tag{4.18.4}$$

so that the total assigned fields can be expressed in the forms

$$m\,\tilde{\mathbf{b}} = -\,\rho_0^*\, H\,\left[g_1^*\,\mathbf{e}_1 + g_2^*\,\mathbf{e}_2\right] \quad , \quad m\,\tilde{\mathbf{b}}^3 = -\frac{H}{2}\,\rho_0^*\, H\, g_3^*\, \mathbf{e}_3 \quad . \tag{4.18.5}$$

Furthermore, since three of the six surfaces of the plate are stress free and the other three are loaded with constant compressive stresses, the boundary conditions become

$$\begin{aligned}
\tilde{\mathbf{t}}^1 &= \rho_0^*\, g_1^*\, L\, \mathbf{e}_1 \ , \quad \tilde{\mathbf{m}}^1 = 0 \ , \quad \text{for } \theta^1 = -L/2 \ , \\
\tilde{\mathbf{t}}^1 &= 0 \ , \quad \tilde{\mathbf{m}}^1 = 0 \ , \quad \text{for } \theta^1 = L/2 \ , \\
\tilde{\mathbf{t}}^2 &= \rho_0^*\, g_2^*\, W\, \mathbf{e}_2 \ , \quad \tilde{\mathbf{m}}^2 = 0 \ , \quad \text{for } \theta^2 = -W/2 \ , \\
\tilde{\mathbf{t}}^2 &= 0 \ , \quad \tilde{\mathbf{m}}^2 = 0 \ , \quad \text{for } \theta^2 = W/2 \ .
\end{aligned} \tag{4.18.6}$$

Next, the displacements (3.18.4) and the expressions (4.14.8) suggest that the Cosserat displacements are given by

$$\begin{aligned}
u_1 = \rho_0^*\, &\Big[\tfrac{1}{2} C_{1111}^*\, g_1^*\, (\theta^1 - \tfrac{L}{2})^2 + C_{1122}^*\, g_2^*\, \theta^1 (\theta^2 - \tfrac{W}{2}) - \tfrac{1}{2} C_{1133}^*\, g_3^*\, \theta^1 H \\
&- \tfrac{1}{2} C_{1122}^*\, g_1^*\, (\theta^2)^2\Big] + \left[\alpha_{12}^*\,\theta^2 + c_1\right] \ , \\
u_2 = \rho_0^*\, &\Big[C_{1122}^*\, g_1^*\, (\theta^1 - \tfrac{L}{2})\,\theta^2 + \tfrac{1}{2} C_{2222}^*\, g_2^*\, (\theta^2 - \tfrac{W}{2})^2 - \tfrac{1}{2} C_{2233}^*\, g_3^*\, \theta^2 H
\end{aligned}$$

$$-\frac{1}{2}C^*_{1122}\,g^*_2\,(\theta^1)^2\big] + \big[-\alpha^*_{12}\,\theta^1 + c_2\,\big]\,,$$

$$u_3 = \rho^*_0\,\big[\,\frac{1}{6}C^*_{3333}\,g^*_3\,H^2 - \frac{1}{2}C^*_{1133}\,g^*_3\,(\theta^1)^2 - \frac{1}{2}C^*_{2233}\,g^*_3\,(\theta^2)^2\big]$$

$$+ \big[-\alpha^*_{13}\,\theta^1 - \alpha^*_{23}\,\theta^2 + c_3\,\big]\,,$$

$$\tilde{\delta}_{13} = \rho^*_0\,\big[C^*_{1133}\,g^*_3\,\theta^1\big] + \alpha^*_{13}\ ,\quad \tilde{\delta}_{23} = \rho^*_0\,\big[C^*_{2233}\,g^*_3\,\theta^2\big] + \alpha^*_{23}\ ,$$

$$\tilde{\delta}_{33} = \rho^*_0\,\big[C^*_{1133}\,g^*_1\,(\theta^1 - \frac{L}{2}) + C^*_{2233}\,g^*_2\,(\theta^2 - \frac{W}{2}) - \frac{1}{2}C^*_{3333}\,g^*_3\,H\big]\ ,\tag{4.18.7}$$

where the constants c_i are expressed in terms of the arbitrary constants c^*_i by

$$c_1 = -\frac{1}{24}\rho^*_0\,C^*_{1133}\,g^*_1\,H^2 + c^*_1\ ,$$

$$c_2 = -\frac{1}{24}\rho^*_0\,C^*_{2233}\,g^*_2\,H^2 + c^*_2\ ,\quad c_3 = c^*_3\ .\tag{4.18.8}$$

Now, with the help of (4.14.17) and (4.18.17), it follows that

$$E_{11} = \rho^*_0\,\big[C^*_{1111}\,g^*_1\,(\theta^1 - \frac{L}{2}) + C^*_{1122}\,g^*_2\,(\theta^2 - \frac{W}{2}) - \frac{1}{2}C^*_{1133}\,g^*_3\,H\big]\ ,$$

$$E_{22} = \rho^*_0\,\big[C^*_{1122}\,g^*_1\,(\theta^1 - \frac{L}{2}) + C^*_{2222}\,g^*_2\,(\theta^2 - \frac{W}{2}) - \frac{1}{2}C^*_{2233}\,g^*_3\,H\big]\ ,$$

$$E_{33} = \rho^*_0\,\big[C^*_{1133}\,g^*_1\,(\theta^1 - \frac{L}{2}) + C^*_{2233}\,g^*_2\,(\theta^2 - \frac{W}{2}) - \frac{1}{2}C^*_{3333}\,g^*_3\,H\big]\ ,$$

$$E_{12} = E_{13} = E_{23} = 0\ ,$$

$$\beta_{11} = \rho^*_0\,\big[C^*_{1133}g^*_3\big]\ ,\quad \beta_{21} = 0\,,\quad \beta_{31} = \rho^*_0\,\big[C^*_{1133}\,g^*_1\big]\,,$$

$$\beta_{12} = 0\,,\quad \beta_{22} = \rho^*_0\,\big[C^*_{2233}\,g^*_3\big]\ ,\quad \beta_{32} = \rho^*_0\,\big[C^*_{2233}\,g^*_2\big]\ ,$$

$$\tilde{\mathbf{t}}^1 = \rho^*_0\,g^*_1\,H\,(\theta^1 - \frac{L}{2})\,\mathbf{e}_1\ ,\quad \tilde{\mathbf{t}}^2 = \rho^*_0\,g^*_2\,H\,(\theta^2 - \frac{W}{2})\,\mathbf{e}_2\ ,\quad \tilde{\mathbf{t}}^3 = -\frac{1}{2}\rho^*_0\,g^*_3\,H^2\,\mathbf{e}_3\,,$$

$$\tilde{\mathbf{m}}^1 = \rho^*_0\,g^*_3\,H\,\big[K^*_{1111}\,C^*_{1133} + K^*_{1122}\,C^*_{2233}\big]\,\mathbf{e}_1\ ,$$

$$\tilde{\mathbf{m}}^2 = \rho^*_0\,g^*_3\,H\,\big[K^*_{1122}\,C^*_{1133} + K^*_{2222}\,C^*_{2233}\big]\,\mathbf{e}_2\ .\tag{4.18.9}$$

It can easily be seen that these results satisfy the linearized equilibrium equations (4.14.10), and that the boundary conditions (4.18.6) on $\tilde{\mathbf{t}}^\alpha$ are satisfied, but not those on $\tilde{\mathbf{m}}^\alpha$. Moreover, it can be seen that the values (4.18.9) for $\tilde{\mathbf{m}}^\alpha$ are constants that are associated only with the solution for g^*_3. This means that the Cosserat solutions for gravity acting in both the negative $\mathbf{e}_1$ and $\mathbf{e}_2$ directions (associated with g^*_1 and g^*_2) are consistent with the three-dimensional solutions. In this regard, since the thickness of the plate does not remain constant for these solutions, yet the associated $\tilde{\mathbf{m}}^\alpha$ still vanish, these solutions again justify the validity of the constitutive result (4.14.25).

On the other hand, when gravity acts in the negative $\mathbf{e}_3$ direction (associated with g^*_3), then the kinematics (4.18.7) which are consistent with the three-dimensional solutions, are not compatible with the boundary conditions on the director couples $\tilde{\mathbf{m}}^\alpha$. Of course, since these couples are constants, it is possible to superimpose solutions for pure bending like those discussed in section 4.14 to satisfy the equations of equilibrium and the boundary conditions. However, the resulting predictions of the unit normals to the lateral surfaces ($\theta^1=\pm L/2$ and $\theta^2=\pm W/2$) and the curvature of the reference surface of the plate will not be compatible with the three-dimensional solution. This indicates a limitation of the Cosserat plate theory which is due to the fact that the kinematics of the Cosserat theory are not compatible with the exact quadratic variation of the displacements through the thickness of the plate when gravity acts in the thickness direction. It is interesting to contrast this result with that for pure bending (section 4.14), which also corresponds to a quadratic variation of the displacements through the thickness, but which can be modeled by specification of the constitutive coefficients (4.14.26).

4.19 Elastic shells

The constitutive restrictions (4.11.9) developed in section 4.11 are valid for elastic shells that have general reference geometry. However, the shell is presumed to be constructed using a homogeneous material. Therefore, in this section and in the remainder of this chapter attention will be limited to such shells for which the three-dimensional mass density ρ_0^* and the strain energy function Σ^* are explicitly independent of position in the reference configuration

$$\rho_0^* = \text{constant} \ , \quad \Sigma^* = \hat{\Sigma}^*(\mathbf{C}^*) \ . \tag{4.19.1}$$

The restrictions (4.11.9) ensure that solutions of the Cosserat theory can reproduce exactly the complete class of nonlinear homogeneous solutions of the three-dimensional theory. Consequently, they provide important information about the influence of the geometry of the shell on the strain energy function which describes its response. However, they do not provide information about inhomogeneous deformations like bending and torsion. Nevertheless, these restrictions were used to propose a relatively simple form (4.11.27) for the strain energy function for shells Σ in terms of the three-dimensional strain energy function Σ^*

$$m\,\Sigma = m\,\Sigma^*(\bar{\mathbf{C}}) + \frac{1}{2} A^{1/2}\,\bar{H}\,|\mathbf{D}_3|^{-2}\,\mathbf{K}^{\alpha\beta} \bullet (\boldsymbol{\beta}_\alpha \otimes \boldsymbol{\beta}_\beta) \ . \tag{4.19.2}$$

The derivatives of this function and the associated constitutive equations have been recorded in (4.11.27) and (4.11.28).

For the simpler case when Σ^* is a quadratic function of three-dimensional strain, the expression for Σ becomes

$$m\,\Sigma = \frac{1}{2} A^{1/2}\,\bar{H}\,\mathbf{K}^* \bullet (\bar{\mathbf{E}} \otimes \bar{\mathbf{E}}) + \frac{1}{2} A^{1/2}\,\bar{H}\,|\mathbf{D}_3|^{-2}\,\mathbf{K}^{\alpha\beta} \bullet (\boldsymbol{\beta}_\alpha \otimes \boldsymbol{\beta}_\beta) \ , \tag{4.19.3}$$

with the associated constitutive equations being given by (4.12.4) and (4.12.5). These equations have the properties that the stiffness $\mathbf{K}^*$ for the linearized three-dimensional material can be used directly, and that they satisfy the restrictions (4.11.9) exactly. However, the resulting constitutive equations for the shell remain nonlinear functions of the strains $\mathbf{E}$ and $\boldsymbol{\beta}_\alpha$ associated with the shell.

For the even simpler case when quadratic terms in the strains $\mathbf{E}$ and $\boldsymbol{\beta}_\alpha$ are neglected, the constitutive equations further reduce to (4.12.6) and (4.12.7). The resulting equations are much simpler but they only satisfy the restrictions (4.11.9) to within the small strain limit. Furthermore, it is noted that in each of these three cases the constitutive equations remain properly invariant under SRBM.

Sections 4.14 and 4.15 focused attention on determining values for the constitutive tensors $\mathbf{K}^{\alpha\beta}$ associated with a flat plate with constant thickness H. Consequently, the part of the strain energy function corresponding to inhomogeneous deformation ($\boldsymbol{\beta}_\alpha \neq 0$) represents a generalization for curved shells. In subsequent sections it will be shown that this generalization seems to work rather well for circular cylindrical and spherical shells with constant thickness. Consequently, these generalized constitutive equations are assumed to be valid for curved shells with variable thickness.

The constitutive restrictions (4.11.9) depend on the definitions $(4.11.7)_2$ and (4.11.10) of $\bar{\mathrm{H}}$ and $\mathbf{H}^\alpha$. In order to present explicit forms for these quantities, it is desirable to be specific about the specification of the director $\mathbf{D}_3$ in the reference configuration. Therefore, here and throughout the rest of this chapter the director $\mathbf{D}_3$ will be taken to be equal to the unit normal $\mathbf{A}_3$ to the shell's reference surface so that

$$\mathbf{D}_3 = \mathbf{A}_3 \ . \tag{4.19.4}$$

It then follows from the definitions (4.1.2) and (4.1.3) that

$$\mathbf{D}_i = \mathbf{A}_i \ , \quad \mathrm{D}^{1/2} = \mathrm{A}^{1/2} \ , \quad \mathbf{D}^i = \mathbf{A}^i \ . \tag{4.19.5}$$

Also, the reference surface of the shell will be taken to be the middle surface so that the major surfaces of the shell are characterize by

$$\xi_1 = -\frac{\mathrm{H}}{2} \ , \quad \xi_2 = \frac{\mathrm{H}}{2} \ , \tag{4.19.6}$$

where H is the shell's thickness.

Next, with the help of (2.3.3) and the kinematic assumption (4.1.4), the quantities $\mathrm{G}^{1/2}$ and $\mathrm{G}^{1/2}\,\mathbf{G}^\alpha$ can be written in the forms

$$\begin{aligned}
\mathrm{G}^{1/2} &= \mathrm{D}^{1/2}\left[1 + \theta^3 (\mathbf{D}_{3,\alpha} \bullet \mathbf{D}^\alpha) + (\theta^3)^2\, \mathrm{D}^{-1/2} (\mathbf{D}_{3,1} \times \mathbf{D}_{3,2} \bullet \mathbf{D}_3)\right] \ , \\
\mathrm{G}^{1/2}\,\mathbf{G}^1 &= \mathrm{D}^{1/2}\left[\mathbf{D}^1 + \theta^3\, \mathrm{D}^{-1/2} (\mathbf{D}_{3,2} \times \mathbf{D}_3)\right] \ , \\
\mathrm{G}^{1/2}\,\mathbf{G}^2 &= \mathrm{D}^{1/2}\left[\mathbf{D}^2 + \theta^3\, \mathrm{D}^{-1/2} (\mathbf{D}_3 \times \mathbf{D}_{3,1})\right] \ .
\end{aligned} \tag{4.19.7}$$

Moreover, in view of the specification (4.19.4), it can be seen that $\mathrm{G}^{1/2}$ is a function of the mean curvature and the Gaussian curvature (Naghdi, 1972, sec. A.2) of the reference surface of the shell in its reference configuration, which are defined by

$$\begin{aligned}
\text{Mean curvature} &= -\frac{1}{2} (\mathbf{A}_{3,\alpha} \bullet \mathbf{A}^\alpha) = -\frac{1}{2} (\mathbf{D}_{3,\alpha} \bullet \mathbf{D}^\alpha) \ , \\
\text{Gaussian curvature} &= \mathrm{A}^{-1/2} (\mathbf{A}_{3,1} \times \mathbf{A}_{3,2} \bullet \mathbf{A}_3) = \mathrm{D}^{-1/2} (\mathbf{D}_{3,1} \times \mathbf{D}_{3,2} \bullet \mathbf{D}_3) \ .
\end{aligned} \tag{4.19.8}$$

The sign convention in the definition $(4.19.8)_1$ is such that the mean curvature is positive for a cylindrical surface when the unit normal vector points inward towards the axis of the cylinder. Now, using (4.19.7), the expressions $(4.11.7)_2$ and (4.11.10) yield

$$\begin{aligned}
\mathrm{A}^{1/2}\,\bar{\mathrm{H}} &= \mathrm{D}^{1/2}\,\mathrm{H}\left[1 + \frac{\mathrm{H}^2}{12} \mathrm{D}^{-1/2} (\mathbf{D}_{3,1} \times \mathbf{D}_{3,2} \bullet \mathbf{D}_3)\right] \ , \\
\mathrm{A}^{1/2}\,\bar{\mathrm{H}}\,\mathbf{H}^1 &= \mathrm{D}^{1/2}\,\mathrm{H}\left[\frac{\mathrm{H}^2}{12} \mathrm{D}^{-1/2} (\mathbf{D}_{3,2} \times \mathbf{D}_3)\right] \ , \\
\mathrm{A}^{1/2}\,\bar{\mathrm{H}}\,\mathbf{H}^2 &= \mathrm{D}^{1/2}\,\mathrm{H}\left[\frac{\mathrm{H}^2}{12} \mathrm{D}^{-1/2} (\mathbf{D}_3 \times \mathbf{D}_{3,1})\right] \ .
\end{aligned} \tag{4.19.9}$$

Within the context of the Cosserat theory it is also necessary to specify constitutive equations for the inertia quantities m, y^3 and y^{33}. The form (4.11.7) for m

$$m = \rho_0^* A^{1/2} \bar{H} \ , \tag{4.19.10}$$

is retained for shells because it ensures that the Cosserat model of the shell has the same mass as the actual shell. Next, using the conservation of mass (3.2.28) and the fact that ρ_0^* is constant, the expressions (4.3.8) and (4.3.30) become

$$m\, y^3 = \rho_0^* \int_{-H/2}^{H/2} \theta^3\, G^{1/2}\, d\theta^3 \ , \qquad m\, y^{33} = \rho_0^* \int_{-H/2}^{H/2} \theta^3\theta^3\, G^{1/2}\, d\theta^3 \ . \tag{4.19.11}$$

Thus, with the help of $(4.19.7)_1$, direct integration yields the Galerkin values

$$m\, y^3 = (\rho_0^* D^{1/2} H) \left[\frac{H^2}{12} (\mathbf{D}_{3,\alpha} \bullet \mathbf{D}^\alpha)\right] \ ,$$

$$m\, y^{33} = (\rho_0^* D^{1/2} H) \left[\frac{H^2}{12} + \frac{H^4}{80} D^{-1/2} (\mathbf{D}_{3,1} \times \mathbf{D}_{3,2} \bullet \mathbf{D}_3)\right] \ . \tag{4.19.12}$$

However, it was seen in sections 4.16 and 4.17 that the Galerkin value (4.16.3) for y^{33} that is consistent with $(4.19.12)_2$ is not the best value to represent vibrations of the plate. Consequently, for the Cosserat theory of shells it is assumed that m is given by the form (4.19.10) with use of $(4.19.9)_1$, but that the director inertia coefficients y^3 and y^{33} are given by

$$m\, y^3 = (\rho_0^* D^{1/2} H)\, H \left[\gamma_1\, H\, (\mathbf{D}_{3,\alpha} \bullet \mathbf{D}^\alpha) + \gamma_2\, H^2\, D^{-1/2} (\mathbf{D}_{3,1} \times \mathbf{D}_{3,2} \bullet \mathbf{D}_3)\right] \ ,$$

$$m\, y^{33} = (\rho_0^* D^{1/2} H) \frac{H^2}{\pi^2} \left[1 + \gamma_3\, H\, (\mathbf{D}_{3,\alpha} \bullet \mathbf{D}^\alpha)\right.$$

$$\left. + \gamma_4\, H^2\, D^{-1/2} (\mathbf{D}_{3,1} \times \mathbf{D}_{3,2} \bullet \mathbf{D}_3)\right] \ , \tag{4.19.13}$$

where γ_1–γ_4 are material constants that need to be determined. Specifically, it will be shown in the following sections that these material constants can be determined by considering vibrations of cylinders and spheres. Moreover, the forms for these expressions were restricted so that they reproduce the values (4.16.2) and (4.16.3) for a plate.

As mentioned in section 4.18, it is also necessary to specify values for the assigned fields $\mathbf{b}$ and $\mathbf{b}^3$. The expressions (4.3.15) and (4.3.36) for the parts $\mathbf{b}_c$ and $\mathbf{b}_c^3$ of these assigned fields that are due to loading on the major surfaces of the shell are retained for shells because they were used in developing the restrictions (4.11.9) for shells of arbitrary shape. In order to determine values of the parts $\mathbf{b}_b$ and $\mathbf{b}_b^3$ of the assigned fields associated with body force, it will be assumed that the three-dimensional specific body force $\mathbf{b}^*$ is constant. Thus, with the help of the conservation of mass $(3.2.35)_1$ and $(4.19.7)_1$, the expressions (4.3.10) and (4.3.32) yield

$$m\, \mathbf{b}_b = m\, \mathbf{b}^* \ , \quad m\, \mathbf{b}_b^3 = (\rho_0^* D^{1/2} H)\, \mathbf{b}^* \left[\frac{H^2}{12} (\mathbf{D}_{3,\alpha} \bullet \mathbf{D}^\alpha)\right] \ . \tag{4.19.14}$$

Section 4.18 showed that these values are good for a plate. It will be shown in a subsequent section that they are also good for a circular cylindrical shell. Consequently, they are proposed for all shells.

4.20 Plane strain expansion of an isotropic circular cylindrical shell

The constitutive equations discussed in section 4.19 generalize equations for plates to curved shells. As a first test of the validity of these equations in the presence of curvature it is reasonable to consider the linearized theory for static plane strain expansion of an isotropic circular cylindrical shell. To this end, the shell is presumed to have internal radius R_1, external radius R_2 and length W in its reference configuration. Also, the inner surface of the shell is loaded by the pressure p_1 and the outer surface is loaded by the pressure p_2.

For the Cosserat model, the reference surface of the shell in its reference configuration is defined by the position vector $\mathbf{X}$ in terms of cylindrical polar base vectors $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ (see appendix B.1) by

$$\mathbf{X} = R\,\mathbf{e}_r(\theta) + \theta^2\,\mathbf{e}_z \;,\quad \theta^1 = \theta \;,\quad |\theta^1| \le \pi \;,\quad |\theta^2| \le \frac{W}{2} \;, \tag{4.20.1}$$

and the radius R of the middle surface and the thickness H of the shell are given by

$$R = \frac{1}{2}(R_1 + R_2) \;,\quad H = R_2 - R_1 \;. \tag{4.20.2}$$

Next, using the definitions (4.1.3) and (4.9.4), it follows that

$$\mathbf{D}_1 = R\,\mathbf{e}_\theta \;,\quad \mathbf{D}^1 = \frac{1}{R}\mathbf{e}_\theta \;,$$

$$\mathbf{D}_2 = \mathbf{e}_z = \mathbf{D}^2 \;,\quad \mathbf{D}_3 = \mathbf{e}_r = \mathbf{D}^3 \;,\quad D^{1/2} = A^{1/2} = R \;, \tag{4.20.3}$$

so the expressions (4.19.9), (4.19.10), (4.19.13) and (4.19.14) yield

$$\bar{H} = H \;,\quad \mathbf{H}^1 = 0 \;,\quad \mathbf{H}^2 = \frac{H^2}{12R}\mathbf{e}_z \;,$$

$$m = \rho_0^*\, R\, H \;,\quad y^3 = H\left[\gamma_1 \frac{H}{R}\right] \;,\quad y^{33} = \frac{H^2}{\pi^2}\left[1 + \gamma_3 \frac{H}{R}\right] \;,$$

$$m\,\mathbf{b}_b = m\,\mathbf{b}^* \;,\quad m\,\mathbf{b}_b^3 = m\,\mathbf{b}^*\left[\frac{H^2}{12R}\right] \;. \tag{4.20.4}$$

Also, with the help of (4.1.4) and the linearized forms of (4.3.12), (4.3.13), (4.3.15) and (4.3.36), it can be shown that

$$\mathbf{X}^* = (R + \theta^3)\,\mathbf{e}_r + \theta^2\,\mathbf{e}_z \;,\quad G^{1/2} = (R + \theta^3) \;,\quad \mathbf{G}^3 = \mathbf{e}_r \;,$$

$$\mathbf{n}^* = \mathbf{e}_r \;,\quad \alpha(H/2) = 1 \;,\quad \mathbf{t}^* = -\,p_2\,\mathbf{e}_r \;,\quad \text{for } \theta^3 = \frac{H}{2} \;,$$

$$\mathbf{n}^* = -\,\mathbf{e}_r \;,\quad \alpha(H/2) = 1 \;,\quad \mathbf{t}^* = p_1\,\mathbf{e}_r \;,\quad \text{for } \theta^3 = -\frac{H}{2} \;, \tag{4.20.5}$$

so that the parts $\tilde{\mathbf{b}}_c$ and $\tilde{\mathbf{b}}_c^3$ of the assigned fields due to surface tractions can be written as

$$m\,\tilde{\mathbf{b}}_c = (R_1 p_1 - R_2 p_2)\,\mathbf{e}_r \;,\quad m\,\tilde{\mathbf{b}}_c^3 = -\frac{H}{2}(R_1 p_1 + R_2 p_2)\,\mathbf{e}_r \;, \tag{4.20.6}$$

where use has been made of the specifications (4.19.6). Thus, the total assigned fields for the linearized theory become

$$m\,\tilde{\mathbf{b}} = m\,\mathbf{b}^* + (R_1 p_1 - R_2 p_2)\,\mathbf{e}_r \;,$$

$$m\,\tilde{\mathbf{b}}^3 = m\,\mathbf{b}^*\left[\frac{H^2}{12R}\right] - \frac{H}{2}(R_1 p_1 + R_2 p_2)\,\mathbf{e}_r \;. \tag{4.20.7}$$

Now, for the axisymmetric plane strain deformation of interest, the displacements can be written in the forms

$$\mathbf{u} = \mathrm{u}\,\mathbf{e}_r \ , \quad \boldsymbol{\delta}_3 = \delta\,\mathbf{e}_r \ , \tag{4.20.8}$$

where u and δ are functions of time only to be determined. Moreover, (4.13.17) can be used to determine the strains associated with these displacements

$$\tilde{\bar{\mathbf{E}}} = \tilde{\mathbf{E}} = \frac{\mathrm{u}}{\mathrm{R}}\mathbf{e}_\theta\otimes\mathbf{e}_\theta + \delta\,\mathbf{e}_r\otimes\mathbf{e}_r \ , \quad \tilde{\boldsymbol{\beta}}_1 = (\delta - \frac{\mathrm{u}}{\mathrm{R}})\,\mathbf{e}_\theta \ , \quad \tilde{\boldsymbol{\beta}}_2 = 0 \ . \tag{4.20.9}$$

Next, taking ϕ=0 in (4.12.8) gives

$$\mathbf{M}_1 = \mathbf{e}_\theta \ , \quad \mathbf{M}_2 = \mathbf{e}_z \ , \quad \mathbf{M}_3 = \mathbf{e}_r \ , \tag{4.20.10}$$

so that with the help of (3.12.13), (3.12.14), Table 3.12.1, (4.12.9) and (4.15.25), the constitutive equations become

$$\tilde{\mathbf{m}}^1 = \frac{\mu^* \mathrm{H}^3}{6\mathrm{R}}\left\{\frac{1}{1-\nu^*}\right\}\left\{\delta - \frac{\mathrm{u}}{\mathrm{R}}\right\}\mathbf{e}_\theta \ ,$$

$$\tilde{\mathbf{m}}^2 = \frac{\mu^* \mathrm{H}^3}{6}\left[\left\{\frac{\nu^*}{1-2\nu^*}\right\}\left\{\frac{\mathrm{u}}{\mathrm{R}} + \delta\right\} + \left\{\frac{\nu^*}{1-\nu^*}\right\}\left\{\delta - \frac{\mathrm{u}}{\mathrm{R}}\right\}\right]\mathbf{e}_z \ ,$$

$$\tilde{\mathbf{t}}^1 = \mu^* \mathrm{H}\left[\left\{\frac{2\nu^*}{1-2\nu^*}\right\}\left\{\frac{\mathrm{u}}{\mathrm{R}} + \delta\right\} + \frac{\mathrm{u}}{\mathrm{R}} - \frac{\mathrm{H}^2}{6\mathrm{R}^2}\left\{\frac{1}{1-\nu^*}\right\}\left\{\delta - \frac{\mathrm{u}}{\mathrm{R}}\right\}\right]\mathbf{e}_\theta \ ,$$

$$\tilde{\mathbf{t}}^2 = \mathrm{R}\mu^* \mathrm{H}\left[\left\{\frac{2\nu^*}{1-2\nu^*}\right\}\left\{\frac{\mathrm{u}}{\mathrm{R}} + \delta\right\}\right]\mathbf{e}_z \ ,$$

$$\tilde{\mathbf{t}}^3 = \mathrm{R}\mu^* \mathrm{H}\left[\left\{\frac{2\nu^*}{1-2\nu^*}\right\}\left\{\frac{\mathrm{u}}{\mathrm{R}} + \delta\right\} + \delta\right]\mathbf{e}_r \ . \tag{4.20.11}$$

Thus, in the absence of body force ($\mathbf{b}^*$=0) the equations of motion (4.13.16) reduce to

$$\begin{aligned}
\rho_0^* \mathrm{RH}\,(\ddot{\mathrm{u}} + \mathrm{y}^3\,\ddot{\delta}) &= (\mathrm{R}_1\mathrm{p}_1 - \mathrm{R}_2\mathrm{p}_2) - 2\mu^*\mathrm{H}\left[\left\{\frac{\nu^*}{1-2\nu^*}\right\}\left\{\frac{\mathrm{u}}{\mathrm{R}} + \delta\right\} + \frac{\mathrm{u}}{\mathrm{R}}\right. \\
&\qquad \left. - \frac{\mathrm{H}^2}{12\mathrm{R}^2}\left\{\frac{1}{1-\nu^*}\right\}\left\{\delta - \frac{\mathrm{u}}{\mathrm{R}}\right\}\right] \ , \\
\rho_0^* \mathrm{RH}\,(\mathrm{y}^3\,\ddot{\mathrm{u}} + \mathrm{y}^{33}\,\ddot{\delta}) &= -\frac{\mathrm{H}}{2}(\mathrm{R}_1\mathrm{p}_1 + \mathrm{R}_2\mathrm{p}_2) - 2\mu^*\mathrm{RH}\left[\left\{\frac{\nu^*}{1-2\nu^*}\right\}\left\{\frac{\mathrm{u}}{\mathrm{R}} + \delta\right\} + \delta\right. \\
&\qquad \left. + \frac{\mathrm{H}^2}{12\mathrm{R}^2}\left\{\frac{1}{1-\nu^*}\right\}\left\{\delta - \frac{\mathrm{u}}{\mathrm{R}}\right\}\right] \ .
\end{aligned} \tag{4.20.12}$$

However, these equations can be written in an alternative form by multiplying $(4.20.12)_1$ by R and adding the result to and subtracting it from $(4.20.12)_2$ to obtain

$$\begin{aligned}
&\rho_0^* \mathrm{RH}\,(\mathrm{y}^3\,\ddot{\mathrm{u}} + \mathrm{y}^{33}\,\ddot{\delta}) + \rho_0^* \mathrm{R}^2\mathrm{H}\,(\ddot{\mathrm{u}} + \mathrm{y}^3\,\ddot{\delta}) = (\mathrm{R}_1^2\mathrm{p}_1 - \mathrm{R}_2^2\mathrm{p}_2) - \frac{2\mu^*\mathrm{RH}}{(1-2\nu^*)}\left\{\frac{\mathrm{u}}{\mathrm{R}} + \delta\right\} \ , \\
&\rho_0^* \mathrm{RH}\,(\mathrm{y}^3\,\ddot{\mathrm{u}} + \mathrm{y}^{33}\,\ddot{\delta}) - \rho_0^* \mathrm{R}^2\mathrm{H}\,(\ddot{\mathrm{u}} + \mathrm{y}^3\,\ddot{\delta}) = \mathrm{R}_1\mathrm{R}_2(\mathrm{p}_2 - \mathrm{p}_1) \\
&\qquad - 2\mu^*\mathrm{RH}\left[1 + \frac{\mathrm{H}^2}{6\mathrm{R}^2}\left\{\frac{1}{1-2\nu^*}\right\}\right]\left\{\delta - \frac{\mathrm{u}}{\mathrm{R}}\right\} \ .
\end{aligned} \tag{4.20.13}$$

In particular, for static deformation it can be seen from $(4.20.13)_2$ that when the pressures are equal ($p_1=p_2$), then the deformation is homogeneous (since $\boldsymbol{\beta}_\alpha=0$) with $\delta=u/R$.

In order to compare with the exact solution, it is convenient to use the kinematic assumption (4.14.5) to express u and δ in terms of the radial displacements u_1 of the inner surface and u_2 of the outer surface by the expressions

$$u = \frac{u_1+u_2}{2} \ , \quad \delta = \frac{u_2 - u_1}{H} \ ,$$

$$\frac{u}{R} + \delta = \frac{R_2 u_2 - R_1 u_1}{RH} \ , \quad \delta - \frac{u}{R} = \frac{R_1 u_2 - R_2 u_1}{RH} \ . \tag{4.20.14}$$

Then, in the absence of inertia the equations (4.20.13) can be solved for u_1 and u_2 to deduce that

$$\frac{u_1}{H} = \left[C_1 + \frac{C_2}{C_3}\right]\frac{R_1}{H} \ , \quad \frac{u_2}{H} = \left[C_1 + \frac{C_2}{C_3}\left\{\frac{R_1^2}{R_2^2}\right\}\right]\frac{R_2}{H} \ ,$$

$$C_1 = \frac{(R_1^2 p_1 - R_2^2 p_2)(1-2\nu^*)}{2\mu^*(R_2^2-R_1^2)} \ , \quad C_2 = \frac{R_2^2\,(p_1-p_2)}{2\mu^*(R_2^2-R_1^2)} \ ,$$

$$C_3 = 1 + \frac{H^2}{6R^2}\left\{\frac{1}{1-\nu^*}\right\} \ . \tag{4.20.15}$$

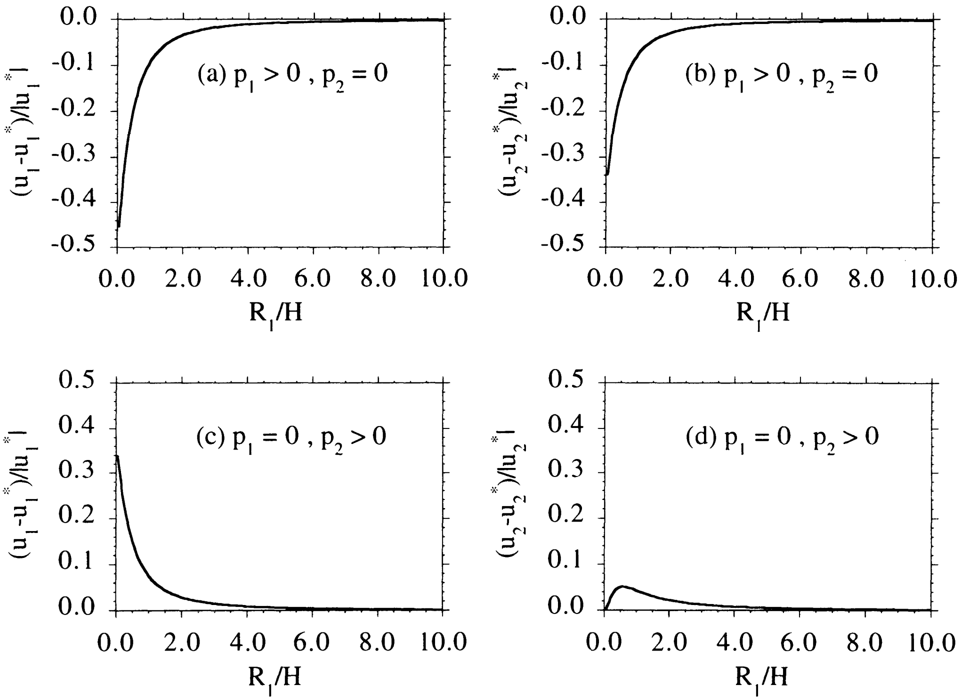

Fig. 4.20.1 Plane strain expansion of a cylindrical shell.

It can easily be shown using the stresses given by Timoshenko and Goodier (1951, sec. 26) that the exact displacements u_1^* and u_2^* associated with the plane strain solution can be written in the forms

$$\frac{u_1^*}{H} = [C_1 + C_2]\frac{R_1}{H} \ , \ \frac{u_2^*}{H} = [C_1 + C_2\{\frac{R_1^2}{R_2^2}\}]\frac{R_2}{H} \ . \tag{4.20.16}$$

Thus, the only difference between the Cosserat solution (4.20.15) and the exact solution (4.20.16) is the presence of C_3. This difference vanishes when the deformation is homogeneous with C_2=0, which occurs for a hollow cylinder (R_1>0) when the pressures are equal (p_1=p_2) or occurs when the cylinder is solid (R_1=0). This difference also vanishes when the shell becomes thin with H/R approaching zero.

To quantitatively examine the Cosserat solution, it is convenient to display the results of two inhomogeneous problems: one with internal pressure only (p_1>0, p_2=0) and the other with external pressure only (p_1=0, p_2>0). Also, the value of ν^* is taken to be 1/3. These results indicate that the relative errors become quite significant as the shell becomes thick with the value of R_1/H being less than about 2. However, for external pressure the value of the external displacement is reasonably accurate even for very thick shells.

At this point it is of interest to examine the importance of the restrictions (4.11.9) on the constitutive equations for shells. In this regard, it is mentioned that the constitutive equations for the linearized theory of shells are usually specified by ignoring the effect of $\mathbf{H}^\alpha$ in (4.12.6) and (4.13.17) to obtain

$$m\,\Sigma = \frac{1}{2}\,A^{1/2}\,\bar{H}\,\mathbf{K}^* \bullet (\mathbf{E}\otimes\mathbf{E}) + \frac{1}{2}A^{1/2}\,\bar{H}\,\mathbf{K}^{\alpha\beta} \bullet (\boldsymbol{\beta}_\alpha\otimes\boldsymbol{\beta}_\beta) \ ,$$

$$\tilde{\bar{\mathbf{E}}} = \tilde{\mathbf{E}} \ , \ \tilde{\mathbf{m}}^\alpha = A^{1/2}\,\bar{H}\,\mathbf{K}^{\alpha\beta}\,\tilde{\boldsymbol{\beta}}_\beta \ ,$$

$$\tilde{\mathbf{t}}^i = [A^{1/2}\,\bar{H}\,\mathbf{K}^* \bullet \tilde{\bar{\mathbf{E}}}]\,\mathbf{D}^i - \tilde{\mathbf{m}}^\alpha\,(\mathbf{D}_{3,\alpha} \bullet \mathbf{D}^i) \ . \tag{4.20.17}$$

For the particular axisymmetric problem considered here these approximate constitutive equations yield the same results as (4.20.11), except that $\tilde{\mathbf{m}}^2$ there is replaced by the expression

$$\tilde{\mathbf{m}}^2 = \frac{\mu^* H^3}{6}[\{\frac{\nu^*}{1-\nu^*}\}\{\delta - \frac{u}{R}\}]\,\mathbf{e}_z \ . \tag{4.20.18}$$

Moreover, the equilibrium equations remain unchanged so that the solution (4.20.16) is still valid. By solving the equilibrium forms of equations (4.20.13) for the strains in terms of the pressures, the more general Cosserat expression $(4.20.11)_2$ becomes

$$\tilde{\mathbf{m}}^2 = [\frac{\nu^* H^2(R_1^2 p_1 - R_2^2 p_2)}{12R} + \frac{\nu^* H^2 R_1 R_2 (p_2 - p_1)}{12R(1-\nu^*)C_3}]\,\mathbf{e}_z \ . \tag{4.20.19}$$

whereas the approximate expression (4.20.18) takes the form

$$\tilde{\mathbf{m}}^2 = [\frac{\nu^* H^2 R_1 R_2 (p_2 - p_1)}{12R(1-\nu^*)C_3}]\,\mathbf{e}_z \ . \tag{4.20.20}$$

However, using the linearized form of $(4.3.38)_1$ and the solution for the stresses in (Timoshenko and Goodier, 1951, sec.26), it can be shown that the exact value is given by

$$\tilde{\mathbf{m}}^2 = \left[\frac{\nu^* H^2 (R_1^2 p_1 - R_2^2 p_2)}{12R} \right] \mathbf{e}_z \; . \qquad (4.20.21)$$

Inspection of these results indicates that although both of the expressions (4.20.19) and (4.20.20) incorrectly depend on the term $(p_2 - p_1)$ corresponding to inhomogeneous deformations, the first term in (4.20.19) is the correct value. Moreover, for the case of a solid cylinder ($R_1=0$) the Cosserat expression (4.20.19) becomes exact. This gives some small indication of the importance of the restrictions (4.11.9). A more significant difference will be presented later with reference to the expansion of a spherical shell.

4.21 Plane strain free vibrations of an isotropic solid circular cylinder

It will be shown presently that values of the quantities γ_1 and γ_3 in (4.19.13) can be determined by considering plane strain free vibration of an isotropic solid circular cylinder. Even though this is an extreme case where the shell is as thick as possible, it will be seen that the Cosserat theory can predict accurate results. To this end, it is first noted that for a solid cylinder it follows from (4.20.2), (4.20.4) and (4.20.14) that

$$R_1 = 0 \;,\quad R = \frac{H}{2} \;,\quad H = R_2 \;,$$
$$m = \frac{1}{2}\rho_0^* H^2 \;,\quad y^3 = 2H\gamma_1 \;,\quad y^{33} = \frac{H^2}{\pi^2}[1 + 2\gamma_3] \;,$$
$$u_1 = 0 \;,\quad u = \frac{u_2}{2} \;,\quad \delta = \frac{u}{R} = \frac{u_2}{H} \; . \qquad (4.21.1)$$

Then, equations (4.20.13) reduce to

$$\frac{1}{2}\rho_0^* H^3 \left[2\gamma_1 + \frac{1}{\pi^2}\{1 + 2\gamma_3\} + \frac{1}{4}\right] \ddot{u}_2 = - H^2 p_2 - \left[\frac{2\mu^* H}{(1-2\nu^*)}\right] u_2 \;,$$
$$\frac{1}{2}\rho_0^* H^3 \left[\frac{1}{\pi^2}\{1 + 2\gamma_3\} - \frac{1}{4}\right] \ddot{u}_2 = 0 \; . \qquad (4.21.2)$$

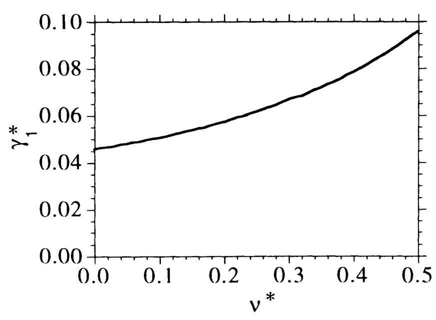

Fig. 4.21.1 Values of γ_1^* which match the exact first frequency of vibration.

These equations will have a nontrivial solution only if the coefficient in the second equation vanishes with γ_3 being given by

$$\gamma_3 = \frac{\pi^2 - 4}{8} , \tag{4.21.3}$$

so that $(4.21.2)_1$ reduces to

$$\frac{1}{4}\rho_0^* H^3 [4\gamma_1 + 1] \ddot{u}_2 = - H^2 p_2 - [\frac{2\mu^* H}{(1-2v^*)}] u_2 . \tag{4.21.4}$$

Now, for free vibrations the pressure p_2 on the outer surface vanishes

$$p_2 = 0 , \tag{4.21.5}$$

and the natural frequency ω can be written in the form

$$\omega = [\frac{2\mu^*}{\rho_0^*}]^{1/2} \frac{\Omega}{H} , \quad \Omega = [\frac{4}{(1-2v^*)(4\gamma_1+1)}]^{1/2} . \tag{4.21.6}$$

The exact solution for this vibration was recorded in section 3.20. Comparison of the Cosserat and exact solutions indicates that these solutions will be identical if the parameter γ_1 is given by the value γ_1^* which equates $(4.21.6)_2$ and $(3.20.5)_2$

$$\gamma_1^* = \frac{1}{(1-v^*)\beta^2} - \frac{1}{4} . \tag{4.21.7}$$

Values of this quantity associated with the first frequency of vibration [i.e. the first root of (3.20.6)] are plotted in Fig. 4.21.1 for a range of Poisson's ratio v^*.

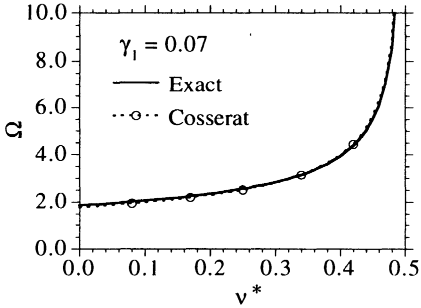

Fig. 4.21.2 Normalized frequencies Ω^* (Exact) and Ω (Cosserat) for $\gamma_1 = 0.07$.

Although this value γ_1^* of γ_1 varies somewhat, the influence of γ_1 on the natural frequency is not too great so it is possible to take γ_1 to be equal to the constant value

$$\gamma_1 = 0.07 , \tag{4.21.8}$$

which is near the value of γ_1^* for $v^* = 0.33$. To examine the accuracy of this specification, the normalized frequency Ω in $(4.21.6)_2$ for the Cosserat theory and the exact value Ω^* in $(3.20.5)_2$ are plotted in Fig. 4.21.2. Since the predictions of the Cosserat theory are so close to the exact results for this wide range of Poisson's ratio, the value (4.21.8) is considered to be quite reasonable. However, this value is not unique since the frequency is rather insensitive to small changes in γ_1.

Before closing this section, it is important to recall that the director inertia coefficients y^3 and y^{33} are required to satisfy the restriction (4.3.49). In particular, the expressions (4.20.4) for a general circular cylinder can be used to deduce that

$$y^{33} - y^3 y^3 = H^2 \left[\frac{1}{\pi^2} \left\{ 1 + \gamma_3 \frac{H}{R} \right\} - \gamma_1^2 \left\{ \frac{H}{R} \right\}^2 \right] . \tag{4.21.9}$$

Next, using (4.21.3) for γ_3 and (4.21.8) for γ_1 it can be shown that this expression satisfies the restriction (4.3.49) that it remains positive for the full range of H/R (≤ 2).

4.22 Expansion of an isotropic spherical shell

The results of section 4.20 showed that the generalized constitutive equations for shells discussed in section 4.19, accurately predict the inner and outer displacements of a circular cylindrical shell in plane strain with internal and external pressures. For a circular cylinder the mean curvature is nonzero but the Gaussian curvature vanishes. In contrast, a spherical shell has both nonzero mean and Gaussian curvatures. Therefore, it is of interest here to consider the problem of expansion of an isotropic spherical shell to explore whether the constitutive equations for shells predict accurate results in the presence of nonzero Gaussian curvature. To this end, the spherical shell is presumed to have internal radius R_1 and external radius R_2 in its reference configuration. Also, the inner surface of the shell is loaded by the pressure p_1 and the outer surface is loaded by the pressure p_2.

For the Cosserat model the reference surface of the shell in its reference configuration is defined by the position vector $\mathbf{X}$ in terms of spherical polar base vectors $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi\}$ (see appendix B.2) by

$$\mathbf{X} = R\, \mathbf{e}_r(\theta,\phi) \ , \quad \theta^1 = \theta \ , \quad \theta^2 = \phi \ , \tag{4.22.1}$$

and the radius R of the middle surface and the thickness H of the shell are given by

$$R = \frac{1}{2}(R_1 + R_2) \ , \quad H = R_2 - R_1 \ . \tag{4.22.2}$$

Next, using the definitions (4.1.3) and (4.9.4), it follows that

$$\mathbf{D}_1 = R\, \mathbf{e}_\theta \ , \quad \mathbf{D}^1 = \frac{1}{R}\, \mathbf{e}_\theta \ , \mathbf{D}_2 = R \sin\theta\, \mathbf{e}_\phi \ , \quad \mathbf{D}^2 = \frac{1}{R \sin\theta}\, \mathbf{e}_\phi \ ,$$

$$\mathbf{D}_3 = \mathbf{e}_r = \mathbf{D}^3 \ , \quad D^{1/2} = A^{1/2} = R^2 \sin\theta \ . \tag{4.22.3}$$

Consequently, in the absence of body force ($\mathbf{b}^*$=0) the expressions (4.19.9), (4.19.10), (4.19.13) and (4.19.14) yield

$$A^{1/2}\, \bar{H} = \left\{ \frac{R_2^3 - R_1^3}{3} \right\} \sin\theta \ , \quad \mathbf{H}^1 = \frac{H^3}{4(R_2^3 - R_1^3)}\, \mathbf{e}_\theta \ , \mathbf{H}^2 = \frac{H^3}{4(R_2^3 - R_1^3)\sin\theta}\, \mathbf{e}_\phi \ ,$$

$$A^{1/2}\, \bar{H}\, \mathbf{H}^1 = \frac{H^3}{12} \sin\theta\, \mathbf{e}_\theta \ , \quad A^{1/2}\, \bar{H}\, \mathbf{H}^2 = \frac{H^3}{12}\, \mathbf{e}_\phi \ ,$$

$$m = \rho_0^* \left\{ \frac{R_2^3 - R_1^3}{3} \right\} \sin\theta \ , \quad y^3 = \left\{ \frac{3R^2 H}{R_2^3 - R_1^3} \right\} H \left[2\gamma_1 \left\{ \frac{H}{R} \right\} + \gamma_2 \left\{ \frac{H}{R} \right\}^2 \right] \ ,$$

$$y^{33} = \left\{\frac{3R^2H}{R_2^3-R_1^3}\right\} \frac{H^2}{\pi^2}\left[1 + 2\gamma_3\left\{\frac{H}{R}\right\} + \gamma_4\left\{\frac{H}{R}\right\}^2\right] ,$$

$$m\,\mathbf{b}_b = 0 \ , \ m\,\mathbf{b}_b^3 = 0 \ . \tag{4.22.4}$$

Also, with the help of (4.1.4) and the linearized forms of (4.3.12), (4.3.13), (4.3.15) and (4.3.36), it can be shown that

$$\mathbf{X}^* = (R + \theta^3)\,\mathbf{e}_r \ , \ G^{1/2} = (R + \theta^3)^2 \sin\theta \ , \ \mathbf{G}^3 = \mathbf{e}_r \ ,$$

$$\mathbf{n}^* = \mathbf{e}_r \ , \ \alpha(H/2) = 1 \ , \ \mathbf{t}^* = -\,p_2\,\mathbf{e}_r \ , \ \text{for } \theta^3 = \frac{H}{2} \ ,$$

$$\mathbf{n}^* = -\,\mathbf{e}_r \ , \ \alpha(H/2) = 1 \ , \ \mathbf{t}^* = p_1\,\mathbf{e}_r \ , \ \text{for } \theta^3 = -\frac{H}{2} \ , \tag{4.22.5}$$

so that the parts $\tilde{\mathbf{b}}_c$ and $\tilde{\mathbf{b}}_c^3$ of the assigned fields due to surface tractions can be written as

$$m\,\tilde{\mathbf{b}}_c = (R_1^2p_1 - R_2^2p_2)\sin\theta\,\mathbf{e}_r \ , \ m\,\tilde{\mathbf{b}}_c^3 = -\frac{H}{2}(R_1^2p_1 + R_2^2p_2)\sin\theta\,\mathbf{e}_r \ , \tag{4.22.6}$$

where use has been made of the specifications (4.19.6). Thus, the total assigned fields for the linearized theory become

$$m\,\tilde{\mathbf{b}} = m\,\tilde{\mathbf{b}}_c = (R_1^2p_1 - R_2^2p_2)\sin\theta\,\mathbf{e}_r \ ,$$

$$m\,\tilde{\mathbf{b}}^3 = m\,\tilde{\mathbf{b}}_c^3 = -\frac{H}{2}(R_1^2p_1 + R_2^2p_2)\sin\theta\,\mathbf{e}_r \ . \tag{4.22.7}$$

Now, for spherically symmetric deformation, the displacements can be written in the forms

$$\mathbf{u} = u\,\mathbf{e}_r \ , \ \boldsymbol{\delta}_3 = \delta\,\mathbf{e}_r \ , \tag{4.22.8}$$

where u and δ are functions of time only to be determined. Moreover, (4.13.17) can be used to determine the strains associated with these displacements

$$\tilde{\mathbf{E}} = \frac{u}{R}(\mathbf{e}_\theta\otimes\mathbf{e}_\theta + \mathbf{e}_\phi\otimes\mathbf{e}_\phi) + \delta\,\mathbf{e}_r\otimes\mathbf{e}_r \ , \ \tilde{\boldsymbol{\beta}}_1 = (\delta - \frac{u}{R})\,\mathbf{e}_\theta \ ,$$

$$\tilde{\boldsymbol{\beta}}_2 = (\delta - \frac{u}{R})\sin\theta\,\mathbf{e}_\phi \ , \ \tilde{\boldsymbol{\beta}}_\alpha\otimes\mathbf{H}^\alpha = \frac{H^3}{4(R_2^3-R_1^3)}(\delta - \frac{u}{R})(\mathbf{e}_\theta\otimes\mathbf{e}_\theta + \mathbf{e}_\phi\otimes\mathbf{e}_\phi)$$

$$\tilde{\tilde{\mathbf{E}}} = \left[\left\{\delta - \frac{3R^2H}{(R_2^3-R_1^3)}(\delta - \frac{u}{R})\right\}(\mathbf{e}_\theta\otimes\mathbf{e}_\theta + \mathbf{e}_\phi\otimes\mathbf{e}_\phi) + \delta\,\mathbf{e}_r\otimes\mathbf{e}_r\right] , \tag{4.22.9}$$

where use has been made of the identities

$$\frac{H^3}{4(R_2^3-R_1^3)} = 1 - \frac{3R^2H}{(R_2^3-R_1^3)} \ ,$$

$$\frac{u}{R} + \frac{H^3}{4(R_2^3-R_1^3)}(\delta - \frac{u}{R}) = \delta - \frac{3R^2H}{(R_2^3-R_1^3)}(\delta - \frac{u}{R}) \ . \tag{4.22.10}$$

Next, taking ϕ=0 in (4.12.8) gives

$$\mathbf{M}_1 = \mathbf{e}_\theta \ , \ \mathbf{M}_2 = \mathbf{e}_\phi \ , \ \mathbf{M}_3 = \mathbf{e}_r \ , \tag{4.22.11}$$

so that with the help of (3.12.13), (3.12.14), Table 3.12.1, (4.12.9) and (4.15.25), the constitutive equations become

$$\mathbf{K}^* \bullet \tilde{\tilde{\mathbf{E}}} = 2\mu^*\left[\left\{\frac{3\nu^*}{1-2\nu^*}\right\}\left\{\delta - \frac{2R^2H}{(R_2^3-R_1^3)}(\delta - \frac{u}{R})\right\}\mathbf{I} + \tilde{\tilde{\mathbf{E}}}\right] ,$$

$$\tilde{\mathbf{m}}^1 = M \sin\theta\, \mathbf{e}_\theta \ , \quad \tilde{\mathbf{m}}^2 = M\, \mathbf{e}_\phi \ ,$$

$$M = \frac{\mu^* H^3(1+\nu^*)}{6(1-2\nu^*)}\Big[\delta - \frac{3R^2H}{(1+\nu^*)(R_2^3-R_1^3)}(\delta - \frac{u}{R})\Big] + \frac{\mu^* H^2(R_2^3-R_1^3)(1+\nu^*)}{18R^2(1-\nu^*)}(\delta - \frac{u}{R}) \ ,$$

$$\tilde{\mathbf{t}}^1 = T \sin\theta\, \mathbf{e}_\theta \ , \quad \tilde{\mathbf{t}}^2 = T\, \mathbf{e}_\phi \ , \quad \tilde{\mathbf{t}}^3 = \bar{T} \sin\theta\, \mathbf{e}_r \ ,$$

$$T = \frac{2\mu^*(R_2^3-R_1^3)(1+\nu^*)}{3R(1-2\nu^*)}\Big[\delta - \frac{3R^2H}{(1+\nu^*)(R_2^3-R_1^3)}(\delta - \frac{u}{R})\Big] - \frac{M}{R} \ ,$$

$$\bar{T} = \frac{2\mu^*(R_2^3-R_1^3)(1+\nu^*)}{3(1-2\nu^*)}\Big[\delta - \frac{6\nu^* R^2H}{(1+\nu^*)(R_2^3-R_1^3)}(\delta - \frac{u}{R})\Big] \ . \tag{4.22.12}$$

Thus, it can be shown that

$$\tilde{\mathbf{t}}^\alpha{}_{,\alpha} = -\, 2T \sin\theta\, \mathbf{e}_r \ , \quad \tilde{\mathbf{m}}^\alpha{}_{,\alpha} = -\, 2M \sin\theta\, \mathbf{e}_r \ , \tag{4.22.13}$$

so the equations of motion (4.13.16) reduce to

$$\rho_0^*\Big\{\frac{R_2{}^3-R_1^3}{3}\Big\}(\ddot{u} + y^3\, \ddot{\delta}) = (R_1^2 p_1 - R_2^2 p_2) - 2\,T \ ,$$

$$\rho_0^*\Big\{\frac{R_2{}^3-R_1^3}{3}\Big\}(y^3\, \ddot{u} + y^{33}\, \ddot{\delta}) = -\frac{H}{2}(R_1^2 p_1 + R_2^2 p_2) - \bar{T} - 2M \ . \tag{4.22.14}$$

However, these equations can be written in an alternative form. First, $(4.22.14)_1$ is multiplied by R and the result is added to $(4.22.14)_2$ to obtain

$$\rho_0^*\Big\{\frac{R_2{}^3-R_1^3}{3}\Big\}\Big[(y^3\, \ddot{u} + y^{33}\, \ddot{\delta}) + R(\ddot{u} + y^3\, \ddot{\delta})\Big]$$

$$= (R_1^3 p_1 - R_2^3 p_2) - \bar{T} - 2M - 2RT \ . \tag{4.22.15}$$

Then $(4.22.14)_1$ is multiplied by $(R_1^2 + R_2^2)/2$ and the result is subtracted from 2R times $(4.22.14)_2$ to deduce that

$$\rho_0^*\ \Big\{\frac{R_2{}^3-R_1^3}{3}\Big\}\Big[2R(y^3\, \ddot{u} + y^{33}\, \ddot{\delta}) - \Big\{\frac{R_1^2+R_2^2}{2}\Big\}(\ddot{u} + y^3\, \ddot{\delta})\Big]$$

$$= R_1^2\, R_2^2(p_2-p_1) - 2R(\bar{T} + 2M) + (R_1^2+R_2^2)T \ . \tag{4.22.16}$$

Moreover, using (4.22.12) it can be shown that

$$\bar{T} + 2M + 2RT = \frac{2\mu^*(R_2^3-R_1^3)(1+\nu^*)}{(1-2\nu^*)}\Big[\delta - \frac{2R^2H}{(R_2^3-R_1^3)}(\delta - \frac{u}{R})\Big] \ ,$$

$$2R(\bar{T} + 2M) - (R_1^2+R_2^2)T \ = \ 4\mu^* R^3 H\Big[1 + \frac{(R_2^3-R_1^3)^2(1+\nu^*)}{36R^6(1-\nu^*)}\Big](\delta - \frac{u}{R}) \ , \tag{4.22.17}$$

and that in the absence of accelerations the equilibrium equations (4.22.15) and (4.22.16) become

$$\frac{2\mu^*(R_2^3-R_1^3)(1+\nu^*)}{(1-2\nu^*)}\Big[\delta - \frac{2R^2H}{(R_2^3-R_1^3)}(\delta - \frac{u}{R})\Big] = (R_1^3 p_1 - R_2^3 p_2) \ ,$$

$$4\mu^* R^3 H \left[1 + \frac{(R_2^3 - R_1^3)^2 (1+\nu^*)}{36R^6(1-\nu^*)}\right] (\delta - \frac{u}{R}) = - R_1^2 R_2^2 (p_1 - p_2) \ . \tag{4.22.18}$$

In particular, it is observed that when the pressures are equal ($p_1 = p_2$), then these equations predict that the deformation field is homogeneous with $\delta = u/R$ and $\boldsymbol{\beta}_\alpha = 0$. Consequently, for this case the solution is exact because the constitutive equations satisfy the restrictions (4.11.9).

In order to compare with the exact solution, it is convenient to use the kinematic assumption (4.14.5) to express u and δ in terms of the radial displacements u_1 of the inner surface and u_2 of the outer surface by the expressions (4.20.14). Then, equations (4.22.18) can be solved to obtain

$$\frac{u_1}{H} = \left[C_1 + \frac{C_2}{C_3}\left\{\frac{R_1 R_2}{R^2}\right\}\right]\frac{R_1}{H} \ , \quad \frac{u_2}{H} = \left[C_1 + \frac{C_2}{C_3}\left\{\frac{R_1 R_2}{R^2}\right\}\left\{\frac{R_1^3}{R_2^3}\right\}\right]\frac{R_2}{H} \ , \tag{4.22.19}$$

where the constants C_1, C_2 and C_3 are defined by

$$C_1 = \frac{(1-2\nu^*)(R_1^3 p_1 - R_2^3 p_2)}{2\mu^*(1+\nu^*)(R_2^3 - R_1^3)} \ , \quad C_2 = \frac{R_2^3(p_1 - p_2)}{4\mu^*(R_2^3 - R_1^3)} \ ,$$

$$C_3 = 1 + \frac{(R_2^3 - R_1^3)^2(1+\nu^*)}{36R^6(1-\nu^*)} \ . \tag{4.22.20}$$

Moreover, the displacements u_1^* and u_2^* associated with the exact solution (Sokolnikoff, 1956, sec. 94) can be written in the forms

$$\frac{u_1^*}{H} = \left[C_1 + C_2\right]\frac{R_1}{H} \ , \quad \frac{u_2^*}{H} = \left[C_1 + C_2\left\{\frac{R_1^3}{R_2^3}\right\}\right]\frac{R_2}{H} \ . \tag{4.22.21}$$

From these expressions, it can be seen that the difference between the Cosserat solution (4.22.19) and the exact solution (4.22.21) vanishes when the deformation is homogeneous, which occurs for a hollow sphere ($R_1 > 0$) when the pressures are equal ($p_1 = p_2$). This difference also vanishes when the shell becomes thin with H/R approaching zero.

The solution for expansion of a spherical shell was considered previously by Naghdi and Rubin (1995) using a different approach where the constitutive coefficients for the Cosserat theory were specified to reproduce the exact linearized solution. In contrast, here the constitutive equations of section 4.19 were used without modification. Moreover, in that paper it was shown that the influence of the restrictions (4.11.9) on the spherical shell geometry is quite significant. To examine this latter point further, the approximate equations (4.20.17) yield constitutive equations of the forms (4.22.12) for $\tilde{\mathbf{m}}^\alpha$ and $\tilde{\mathbf{t}}^\alpha$, with M, T and $\bar{T}$ replaced by

$$M = \frac{\mu^* H^2 (R_2^3 - R_1^3)(1+\nu^*)}{18R^2(1-\nu^*)} (\delta - \frac{u}{R}) \ ,$$

$$T = \frac{2\mu^*(R_2^3 - R_1^3)(1+\nu^*)}{3R(1-2\nu^*)}\left[\delta - \left\{\frac{1}{1+\nu^*}\right\}(\delta - \frac{u}{R})\right] - \frac{M}{R} \ ,$$

$$\bar{T} = \frac{2\mu^*(R_2^3-R_1^3)(1+\nu^*)}{3(1-2\nu^*)}\left[\delta - \left\{\frac{2\nu^*}{1+\nu^*}\right\}\left(\delta - \frac{u}{R}\right)\right] . \tag{4.22.22}$$

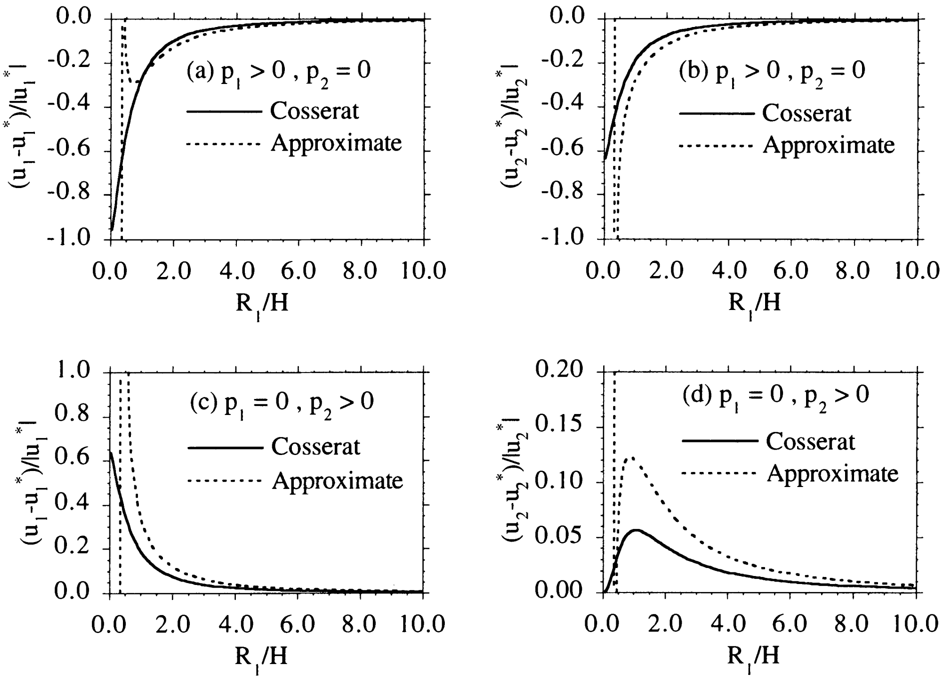

Fig. 4.22.1 Expansion of a spherical shell. Errors of the Cosserat solution (4.22.19) and the approximate solution (4.22.25) relative to the exact solution (4.22.21).

It then follows that

$$\bar{T} + 2M + 2RT = \frac{2\mu^*(R_2^3-R_1^3)(1+\nu^*)}{(1-2\nu^*)}\left[\delta - \frac{2}{3}\left(\delta - \frac{u}{R}\right)\right] ,$$

$$2R(\bar{T} + 2M) - (R_1^2+R_2^2)T = -\frac{\mu^* H^2(R_2^3-R_1^3)(1+\nu^*)}{3R(1-2\nu^*)}\,[\delta]$$

$$+ \frac{4\mu^* R(R_2^3-R_1^3)}{3}\left[1 + \frac{H^2}{4R^2(1-2\nu^*)} + \frac{H(R_2^3-R_1^3)(1+\nu^*)}{12R^4(1-\nu^*)}\right]\left(\delta - \frac{u}{R}\right) , \tag{4.22.23}$$

so that the equilibrium equations (4.22.15) and (4.22.16) become

$$\frac{2\mu^*(R_2^3-R_1^3)(1+\nu^*)}{(1-2\nu^*)}\left[\delta - \frac{2}{3}\left(\delta - \frac{u}{R}\right)\right] = (R_1^3 p_1 - R_2^3 p_2) ,$$

$$\frac{4\mu^* R(R_2^3-R_1^3)}{3}\left[1 + \frac{H^2}{4R^2(1-2\nu^*)} + \frac{H(R_2^3-R_1^3)(1+\nu^*)}{12R^4(1-\nu^*)}\right]\left(\delta - \frac{u}{R}\right)$$

$$-\frac{\mu^* H^2(R_2^3-R_1^3)(1+\nu^*)}{3R(1-2\nu^*)}[\delta] = -R_1^2R_2^2(p_1-p_2) \ . \qquad (4.22.24)$$

In contrast with equations (4.22.18), the approximate equations (4.22.24) predict that the deformation field is not homogeneous when the pressures are equal ($p_1=p_2$) since $\delta \neq u/R$. This observation was made previously by Naghdi and Rubin (1995). However, here the quantitative implications of this error are examined. To this end, the expressions (4.20.14) are used in equations (4.22.24) to obtain the displacements

$$\frac{u_1}{H} = \frac{C_1}{B_2}\left[\frac{R_1}{H} - B_1\right] + \frac{C_2}{B_2B_3}\left[\left\{3 - \frac{2R_1}{R}\right\}\frac{R_1^2}{HR_2}\right] \ ,$$

$$\frac{u_2}{H} = \frac{C_1}{B_2}\left[\frac{R_2}{H} - B_1\right] + \frac{C_2}{B_2B_3}\left[\left\{3 - \frac{2R_2}{R}\right\}\frac{R_1^2}{HR_2}\right] \ , \qquad (4.22.25)$$

where C_1 and C_2 are given by (4.22.20) and the remaining constants are defined by

$$B_1 = \frac{H(1+\nu^*)}{4R(1-2\nu^*)B_3} \ , \quad B_2 = 1 - \frac{2HB_1}{3R} \ ,$$

$$B_3 = 1 + \frac{H^2}{4R^2(1-2\nu^*)} + \frac{H(R_2^3-R_1^3)(1+\nu^*)}{12R^4(1-\nu^*)} \ . \qquad (4.22.26)$$

To quantitatively examine the accuracy of the solution (4.22.19) of the Cosserat theory which satisfies the restrictions (4.11.9), and the solution (4.22.25) of the approximate constitutive equations which do not satisfy the restrictions (4.11.9), it is desirable to consider the error relative to the exact solution (4.22.21). Figure 4.22.1 shows the results of two problems with inhomogeneous deformation: one with internal pressure only ($p_1>0$, $p_2=0$) and the other with external pressure only ($p_1=0$, $p_2>0$). Also, the value of ν^* is taken to be 1/3. These results indicate that the relative errors become quite significant as the shell becomes thick with the value R_1/H being less than about 2. These results also show that the effects of the restrictions (4.11.9) are quite significant since the Cosserat solution is better than the approximate solution for all values of R_1/H greater than about 1. Moreover, the vertical dotted lines in Fig. 4.22.1 correspond to the locations where the approximate solution becomes infinite and changes sign. In contrast, the more general Cosserat solution remains well behaved (although in error) even when the inner radius approaches zero.

Next, consider a related problem in which a hollow ($R_1>0$) spherical shell is compressed ($p_2>0$) onto a rigid spherical ball of radius R_1. For this problem the rigid core prevents displacement of the inner surface of the shell so that

$$u_1 = 0 \quad \text{for } R_1 > 0 \ , \qquad (4.22.27)$$

and a contact stress p_1 develops at the interface of the rigid core and the shell.

Using the condition (4.22.27), the exact equations (4.22.21) can be solved for the exact values u_2^* of the outer displacement and p_1^* of the contact stress to obtain

$$\frac{u_2^*}{H} = -\frac{1}{C_4}\left[\frac{(1-2\nu^*)}{2(1+\nu^*)}\left\{1 - \frac{R_1^3}{R_2^3}\right\}\right]\frac{R_2}{H}\left\{\frac{p_2}{\mu^*}\right\} \ ,$$

$$\frac{p_1^*}{\mu^*} = \frac{1}{C_4}\left[1 + \frac{2(1-2\nu^*)}{(1+\nu^*)}\right]\left\{\frac{p_2}{\mu^*}\right\} \ , \quad C_4 = 1 + \frac{2(1-2\nu^*)}{(1+\nu^*)}\left\{\frac{R_1^3}{R_2^3}\right\} , \tag{4.22.28}$$

and the Cosserat equations (4.22.19) can be solved for u_2 and p_1 to deduce that

$$\frac{u_2}{H} = -\frac{1}{C_6}\left[\frac{(1-2\nu^*)}{2(1+\nu^*)}\left\{1 - \frac{R_1^3}{R_2^3}\right\}\right]\frac{R_2}{H}\left\{\frac{p_2}{\mu^*}\right\} \ ,$$

$$\frac{p_1}{\mu^*} = \frac{1}{C_6}\left[\frac{R_1}{R_2} + \frac{2(1-2\nu^*)}{(1+\nu^*)}C_5\right]\frac{R_2}{R_1}\left\{\frac{p_2}{\mu^*}\right\} \ ,$$

$$C_5 = \left[1 + \frac{(R_2^3-R_1^3)^2(1+\nu^*)}{36R^6(1-\nu^*)}\right]\frac{R^2}{R_2^2} \ , \quad C_6 = 1 + \frac{2(1-2\nu^*)}{(1+\nu^*)}C_5\left\{\frac{R_1^2}{R_2^2}\right\} . \tag{4.22.29}$$

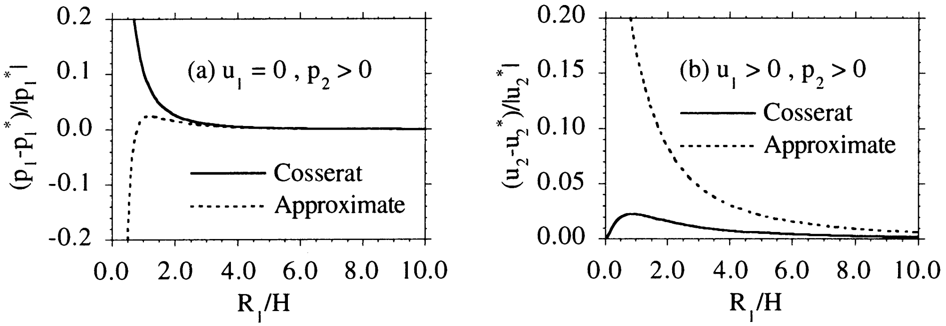

Fig. 4.22.2 Compression of a spherical shell onto a rigid spherical ball. Errors of the Cosserat solution (4.22.29) and the approximate solution (4.22.30) relative to the exact solution (4.22.28).

Also, the approximate equations (4.22.25) can be solved to obtain

$$\frac{u_2}{H} = -\frac{1}{B_5}\left[\frac{(1-2\nu^*)}{2(1+\nu^*)}\left\{3 - \frac{2B_1H}{R}\right\}\right]\left\{\frac{p_2}{\mu^*}\right\} \ ,$$

$$\frac{p_1}{\mu^*} = \frac{1}{B_5}\left[\left\{3 - \frac{2R_1}{R}\right\}\frac{R_1^2}{R_2^2} + \frac{2(1-2\nu^*)}{(1+\nu^*)}B_4\right]\frac{R_2^2}{R_1^2}\left\{\frac{p_2}{\mu^*}\right\} \ ,$$

$$B_4 = \left[\frac{R_1}{R_2} - \frac{B_1H}{R_2}\right]B_3 \ , \quad B_5 = \left\{3 - \frac{2R_1}{R}\right\} + \frac{2(1-2\nu^*)}{(1+\nu^*)}B_4\left\{\frac{R_1}{R_2}\right\} , \tag{4.22.30}$$

where the constants B_1 and B_3 were defined in (4.22.26). It can easily be seen that in the limit of a thin shell these solutions converge to the exact solution. However, in the limit of a nearly solid sphere ($R_1 \to 0$), the displacement of the Cosserat solution (4.22.29) is exact but the displacement of the approximate solution (4.22.30) is not. Also, in this limit both the Cosserat and the approximate solutions predict that the contact pressure becomes infinite, but with different characters of the singularities. The errors of these two solutions are compared quantitatively in Fig. 4.22.2 for $p_2>0$ and $\nu^*=1/3$. In particular, notice that the displacement of the Cosserat solution is more accurate than that of the approximate solution and that the contact pressure of the approximate solution becomes negative as R_1 decreases, whereas the Cosserat contact pressure remains positive.

In summary, the constitutive equations for shells discussed in section 4.19 predict reasonably accurate results for expansion problems of both cylindrical (sec. 4.20) and spherical shells even when the shells become relatively thick. Therefore, those constitutive equations will be presumed valid for general shell structures.

4.23 Free vibrations of an isotropic solid sphere

It will be shown presently that values of the quantities γ_2 and γ_4 in (4.19.13) can be determined by considering free vibration of an isotropic solid sphere. Even though this is an extreme case where the shell is as thick as possible, it will be seen that the Cosserat theory can predict accurate results. To this end, it is first noted that for a solid sphere it follows from (4.22.2), (4.22.4) and (4.20.14) that

$$R_1 = 0 \ , \quad R = \frac{H}{2} \ , \quad H = R_2 \ , \quad m = \frac{1}{3}\rho_0^* H^3 \ , \quad y^3 = 3H(\gamma_1+\gamma_2) \ ,$$
$$y^{33} = \frac{3H^2}{4\pi^2}[1 + 4(\gamma_3+\gamma_4)] \ , \quad u_1 = 0 \ , \quad u = \frac{u_2}{2} \ , \quad \delta = \frac{u}{R} = \frac{u_2}{H} \ . \tag{4.23.1}$$

Then, with the help of (4.22.17), equations (4.22.15) and (4.22.16) reduce to

$$\rho_0^* H^2 \left[(\gamma_1+\gamma_2) + \frac{1}{\pi^2}\left\{\frac{1}{4} + \gamma_3+\gamma_4\right\} + \frac{1}{12}\right] \ddot{u}_2 = -Hp_2 - \left[\frac{2\mu^* H(1+\nu^*)}{(1-2\nu^*)}\right] u_2 \ ,$$
$$\rho_0^* \left[\frac{1}{\pi^2}\left\{\frac{1}{4} + \gamma_3+\gamma_4\right\} - \frac{1}{12}\right] \ddot{u}_2 = 0 \ . \tag{4.23.2}$$

These equations will have a nontrivial solution only if the coefficient in the second equation vanishes with

$$\gamma_3 + \gamma_4 = \frac{\pi^2 - 3}{12} \ , \tag{4.23.3}$$

which with the help of the value (4.21.3) requires γ_4 to be specified by

$$\gamma_4 = -\frac{\pi^2 - 6}{24} \ . \tag{4.23.4}$$

Moreover, with these specifications equation $(4.23.2)_1$ becomes

$$\rho_0^* H^2 \left[(\gamma_1+\gamma_2) + \frac{1}{6}\right] \ddot{u}_2 = -Hp_2 - \left[\frac{2\mu^* H(1+\nu^*)}{(1-2\nu^*)}\right] u_2 \ . \tag{4.23.5}$$

Now, for free vibrations the pressure p_2 on the outer surface vanishes

$$p_2 = 0 \ , \tag{4.23.6}$$

and the natural frequency ω can be written in the form

$$\omega = \left[\frac{2\mu^*}{\rho_0^*}\right]^{1/2} \frac{\Omega}{H} \ , \quad \Omega = \left[\frac{(1+\nu^*)}{(1-2\nu^*)(\gamma_1+\gamma_2+1/6)}\right]^{1/2} \ . \tag{4.23.7}$$

Next, using the solution for free vibration of a solid sphere recorded in (Love, 1944, sec. 196), the exact natural frequency ω^* can be written in the form

$$\omega^* = \left[\frac{2\mu^*}{\rho_0^*}\right]^{1/2} \frac{\Omega^*}{H} \ , \quad \Omega^* = \left[\frac{(1-\nu^*)}{(1-2\nu^*)}\right]^{1/2} \beta \ , \tag{4.23.8}$$

where β are the roots of the equation

$$\frac{\tan \beta}{\beta} - \frac{1}{1 - \dfrac{(1-\nu^*)}{2(1-2\nu^*)}\beta^2} = 0 \ . \tag{4.23.9}$$

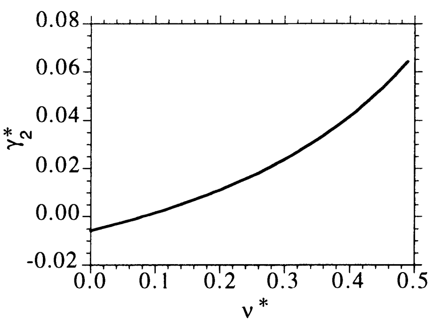

Fig. 4.23.1 Values of γ_2^* which match the exact first frequency of vibration.

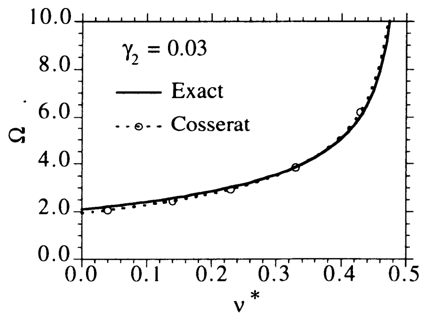

Fig. 4.23.2 Normalized frequencies Ω^* (Exact) and Ω (Cosserat) for $\gamma_2 = 0.03$.

Comparison of the Cosserat and exact solutions indicates that these solutions will be identical if the parameter γ_2 is given by the value γ_2^* which equates $(4.23.7)_2$ and $(4.23.81)_2$

$$\gamma_2^* = \frac{(1+\nu^*)}{(1-\nu^*)\beta^2} - \gamma_1 - \frac{1}{6} \ , \tag{4.23.10}$$

Values of this quantity associated with the first frequency of vibration [i.e. the first root of (4.23.9)], and the value (4.21.8) of γ_1=0.07 are plotted in Fig. 4.23.1 for a range of Poisson's ratio ν^*.

Although this value γ_2^* of γ_2 varies somewhat, the influence of γ_2 on the natural frequency is not too great so it is possible to take γ_2 to be equal to the constant value

$$\gamma_2 = 0.03 \quad , \tag{4.23.11}$$

which is near the value of γ_2^* for $\nu^* = 0.34$. To examine the accuracy of this specification, the normalized frequency Ω in $(4.23.7)_2$ for the Cosserat theory and the exact value Ω^* in $(4.23.8)_2$ are plotted in Fig. 4.23.2. Since the predictions of the Cosserat theory are so close to the exact results for this wide range of Poisson's ratio, the value (4.23.11) is considered to be quite reasonable. However, this value is not unique since the frequency is rather insensitive to small changes in γ_2.

Before closing this section, it is noted that with the specifications (4.21.14) for γ_1, (4.23.11) for γ_2, (4.21.3) for γ_3, and (4.23.4) for γ_4, it can be shown that the expressions (4.22.4) for y^3 and y^{33} satisfy the restriction (4.3.49) for the full range of spherical shells ($H/R \le 2$). By design, the expressions for y^3 and y^{33} satisfy the restriction (4.3.49) for general thin shells. However, at present it is not known how to prove that (4.3.49) is satisfied for general thick shells.

4.24 An isotropic circular cylindrical shell loaded by its own weight

Section 4.19 presented various expressions which attempt to generalize results for plates to curved shells. The objective of the present section is to examine the validity of the generalization (4.19.14) of the assigned director couple due to body force applied to a shell. Specifically, the problem of an isotropic circular cylindrical shell subjected to gravity acting in the axial direction is considered. In its reference configuration, the shell has the same geometry as the one discussed in section 4.20 so the results (4.20.1)-(4.20.4) remain valid with the three-dimensional specific body force $\mathbf{b}^*$ being specified in terms of the constant force of gravity per unit mass g^* by

$$\mathbf{b}^* = -\, g^*\, \mathbf{e}_z \ . \tag{4.24.1}$$

Moreover, the inner and outer surfaces of the shell are presumed to be stress free so that the linearized assigned fields (4.19.14) are given by

$$\tilde{\mathbf{b}}_c = 0 \ , \quad m\,\tilde{\mathbf{b}} = m\,\tilde{\mathbf{b}}_b = -\,\rho_0^*\, g^*\, HR\, \mathbf{e}_z \ ,$$

$$\tilde{\mathbf{b}}_c^3 = 0 \ , \quad m\,\tilde{\mathbf{b}}^3 = m\,\tilde{\mathbf{b}}_b^3 = -\,\rho_0^*\, g^* \left[\frac{H^3}{12}\right] \mathbf{e}_z \ . \tag{4.24.2}$$

Also, the bottom edge ($\theta^2 = -W/2$) is presumed to support the weight of the shell and the top edge ($\theta^2 = W/2$) is presumed to be free. Thus, with the help of (4.3.24), (4.3.38) and the exact solution in section 3.19, the boundary conditions can be written in the forms

$$\tilde{\mathbf{t}}^2 = -\,\rho_0^*\, g^*\, W \left[\int_{-H/2}^{H/2} (R+\theta^3)\, d\theta^3\right] \mathbf{e}_z = -\,\rho_0^*\, g^*\, WRH\, \mathbf{e}_z \ ,$$

$$\tilde{\mathbf{m}}^2 = -\rho_0^* g^* W \left[\int_{-H/2}^{H/2} (R+\theta^3)\,\theta^3\, d\theta^3\right] \mathbf{e}_z = -\rho_0^* g^* W\left\{\frac{H^3}{12}\right\}\mathbf{e}_z \ , \ \text{for } \theta^2 = -W/2 \ ,$$

$$\tilde{\mathbf{t}}^2 = 0 \ , \ \tilde{\mathbf{m}}^2 = 0 \ , \ \text{for } \theta^2 = W/2 \ . \qquad (4.24.3)$$

Since the deformed shell remains axisymmetric, the displacement fields are given by

$$\mathbf{u} = u_r(\theta^2)\,\mathbf{e}_r + u_z(\theta^2)\,\mathbf{e}_z \ , \ \boldsymbol{\delta}_1 = u_r\,\mathbf{e}_\theta \ ,$$

$$\boldsymbol{\delta}_2 = \frac{du_r}{d\theta^2}\,\mathbf{e}_r + \frac{du_z}{d\theta^2}\,\mathbf{e}_z \ , \ \boldsymbol{\delta}_3 = \delta_r(\theta^2)\,\mathbf{e}_r + \delta_z(\theta^2)\,\mathbf{e}_z \ . \qquad (4.24.4)$$

Thus, (4.13.7) can be used to determine the strains associated with these displacements

$$\tilde{\mathbf{E}} = \left[\frac{u_r}{R}\right](\mathbf{e}_\theta\otimes\mathbf{e}_\theta) + \left[\frac{du_z}{d\theta^2} + \frac{H^2}{12R^2}\frac{d\delta_z}{d\theta^2}\right](\mathbf{e}_z\otimes\mathbf{e}_z) + \left[\delta_r\right](\mathbf{e}_r\otimes\mathbf{e}_r)$$

$$+ \left[\frac{\delta_z}{2} + \frac{1}{2}\left\{\frac{du_r}{d\theta^2} + \frac{H^2}{12R}\frac{d\delta_r}{d\theta^2}\right\}\right](\mathbf{e}_r\otimes\mathbf{e}_z + \mathbf{e}_z\otimes\mathbf{e}_r) \ ,$$

$$\tilde{\boldsymbol{\beta}}_1 = \left[\delta_r - \frac{u_r}{R}\right]\mathbf{e}_\theta \ , \ \tilde{\boldsymbol{\beta}}_2 = \frac{d\delta_r}{d\theta^2}\,\mathbf{e}_r + \frac{d\delta_z}{d\theta^2}\,\mathbf{e}_z \ . \qquad (4.24.5)$$

Next, taking $\phi=0$ in (4.12.8), leads to $\mathbf{M}_i$ being given by (4.20.10). Also, for convenience, the exact solution (3.19.10) is written as

$$\mathbf{u}^*(\theta^1,\theta^2,\theta^3) = u_r^*\,\mathbf{e}_r + u_z^*\,\mathbf{e}_z \ , \ u_r^* = -\frac{\nu^*\rho_0^* g^*}{E^*}(R+\theta^3)(\theta^2 - \frac{W}{2}) \ ,$$

$$u_z^* = \frac{\rho_0^* g^*}{2E^*}\left[\left\{(\theta^2 - \frac{W}{2})^2 - W^2\right\} + \nu^*\left\{(R+\theta^3)^2 - R^2\right\}\right] \ . \qquad (4.24.6)$$

Now, the linearized form of the kinematic assumption (4.2.7) suggests that $\mathbf{u}^*$ is approximated by

$$\mathbf{u} + \theta^3\,\boldsymbol{\delta}_3 = \left[u_r + \theta^3\,\delta_r\right]\mathbf{e}_r + \left[u_z + \theta^3\,\delta_z\right]\mathbf{e}_z \ . \qquad (4.24.7)$$

Thus, comparison of these expressions reveals that the displacement field (4.24.7) is not exact since u_z and δ_z are independent of θ^3. However, it will be a close approximation of the exact field if

$$u_r = -\frac{\nu^*\rho_0^* g^* R}{E^*}(\theta^2 - \frac{W}{2}) \ , \ u_z = \frac{\rho_0^* g^*}{2E^*}\left\{(\theta^2 - \frac{W}{2})^2 - W^2\right\} \ ,$$

$$\delta_r = -\frac{\nu^*\rho_0^* g^*}{E^*}(\theta^2 - \frac{W}{2}) \ , \ \delta_z = \text{constant} \ , \qquad (4.24.8)$$

where the constants associated with rigid body motion have been chosen to be consistent with (4.24.6). Moreover, it will be shown that this displacement field is consistent with the constitutive equations, equilibrium equations and boundary conditions of the shell.

Next, with the help of (3.12.14), Table 3.12.1, (4.12.9), (4.13.17), (4.15.25) and (4.24.8), the constitutive equations become

$$\tilde{\mathbf{m}}^1 = 0 \ , \ \tilde{\mathbf{m}}^2 = \frac{\mu^* H^3}{12}\left[\left\{\delta_z - \frac{\nu^*\rho_0^* g^* R}{E^*}\left[1 + \frac{H^2}{12R^2}\right]\right\}\mathbf{e}_r + \left\{\frac{\rho_0^* g^*}{\mu^*}(\theta^2 - \frac{W}{2})\right\}\mathbf{e}_z\right] \ ,$$

$$\tilde{\mathbf{t}}^1 = 0 \ , \quad \tilde{\mathbf{t}}^2 = \mu^* HR \left[\left\{ \delta_z - \frac{\nu^* \rho_0^* g^* R}{E^*} \left[1 + \frac{H^2}{12R^2} \right] \right\} \mathbf{e}_r + \left\{ \frac{\rho_0^* g^*}{\mu^*} (\theta^2 - \frac{W}{2}) \right\} \mathbf{e}_z \right] ,$$

$$\tilde{\mathbf{t}}^3 = \mu^* HR \left[\delta_z - \frac{\nu^* \rho_0^* g^* R}{E^*} \left\{ 1 + \frac{H^2}{12R^2} \right\} \right] \mathbf{e}_z \ . \tag{4.24.9}$$

It then follows that the boundary conditions (4.24.3) will be satisfied if δ_z is given by

$$\delta_z = \frac{\nu^* \rho_0^* g^* R}{E^*} \left\{ 1 + \frac{H^2}{12R^2} \right\} \ , \tag{4.24.10}$$

so that (4.24.9) reduce to

$$\tilde{\mathbf{m}}^1 = 0 \ , \quad \tilde{\mathbf{m}}^2 = \left[\frac{\rho_0^* g^* H^3}{12} (\theta^2 - \frac{W}{2}) \right] \mathbf{e}_z \ ,$$

$$\tilde{\mathbf{t}}^1 = 0 \ , \quad \tilde{\mathbf{t}}^2 = \left[\rho_0^* \, g^* HR \, (\theta^2 - \frac{W}{2}) \right] \mathbf{e}_z \ , \quad \tilde{\mathbf{t}}^3 = 0 \ . \tag{4.24.11}$$

Thus, using (4.24.2) and (4.24.11), it can be seen that the equations of equilibrium (4.13.16) are satisfied.

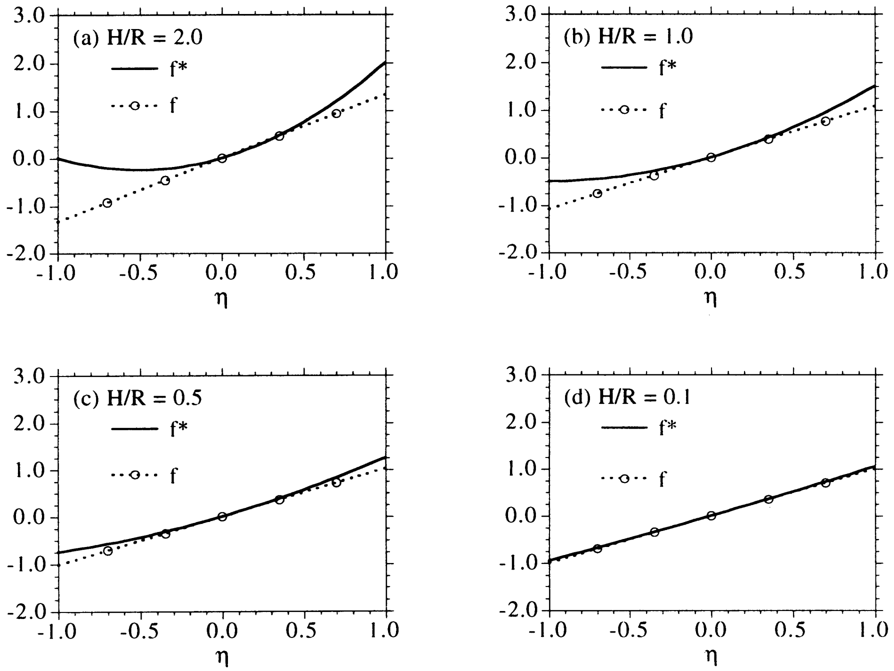

Fig. 4.24.1 Plots of the functions f* (Exact) and f (Cosserat) versus the normalized thickness coordinate η for different thickness (H) to radius (R) ratios.

Furthermore, comparison of the displacement fields (4.24.6) and (4.24.7) reveals that the radial components are identical

$$u_r^* = u_r + \theta^3\, \delta_r \ , \tag{4.24.12}$$

but that the axial components are not. Specifically, the axial component of these displacements at the bottom edge ($\theta^2 = -W/2$) can be rewritten in the forms

$$u_z^*(\theta^1, W/2, \theta^3) = \left[\frac{\nu^* \rho_0^* g^* HR}{2E^*}\right] f^*(\eta) \ ,$$

$$u_z(W/2) + \theta^3\, \delta_z = \left[\frac{\nu^* \rho_0^* g^* HR}{2E^*}\right] f^*(\eta) \ , \tag{4.24.13}$$

where the normalized thickness coordinate η and the functions $f^*(\eta)$ and $f(\eta)$ are defined by

$$\eta = \frac{2\theta^3}{H} \ , \quad -1 \le \eta \le 1 \ , \quad f^*(\eta) = \eta \left[1 + \frac{H\eta}{4R}\right] \ , \quad f(\eta) = \eta \left[1 + \frac{H^2}{12R^2}\right] . \tag{4.24.14}$$

Figure 4.24.1 plots $f^*(\eta)$ associated with the exact solution, and $f(\eta)$ associated with the Cosserat solution, for shells of different thicknesses. Notice that the Cosserat solution produces an average value for the slope of the edge of the shell which is reasonable even for a solid cylinder with H/R=2, and which becomes exact as the cylinder becomes thin (H/R<<1). This indicates that the generalization (4.19.14) of the assigned director couple due to body force applied to a shell is quite reasonable.

4.25 Isotropic nonlinear elastic shells

The objective of this section is to use the three-dimensional constitutive equations of section 3.11 for isotropic nonlinear elastic materials, together with the restrictions (4.11.9), to exhibit explicit constitutive equations for isotropic nonlinear elastic shells. Motivated by the definitions (3.1.7), (3.7.8), (3.11.9), (4.11.19), it is convenient to define the kinematic quantities $\bar{\mathbf{F}}$, $\bar{\mathbf{B}}$, $\bar{J}$, $\bar{\alpha}_1$ and $\bar{\alpha}_2$ by the expressions

$$\bar{\mathbf{F}} = \mathbf{F}\,(\mathbf{I} + \boldsymbol{\beta}_\alpha \otimes \mathbf{H}^\alpha) \ , \quad \bar{\mathbf{C}} = \bar{\mathbf{F}}^T \bar{\mathbf{F}} \ , \quad \bar{\mathbf{B}} = \bar{\mathbf{F}} \bar{\mathbf{F}}^T \ ,$$

$$\bar{J} = \det \bar{\mathbf{F}} \ , \quad \bar{\alpha}_1 = \bar{J}^{-2/3}\, \bar{\mathbf{C}} \bullet \mathbf{I} = \bar{J}^{-2/3}\, \bar{\mathbf{B}} \bullet \mathbf{I} \ , \quad \bar{\alpha}_2 = \bar{J}^{-4/3}\, \bar{\mathbf{C}} \bullet \bar{\mathbf{C}} = \bar{J}^{-4/3}\, \bar{\mathbf{B}} \bullet \bar{\mathbf{B}} . \tag{4.25.1}$$

Then, when the strain energy function Σ^* in (3.11.10) for an isotropic material is expressed as a function of $\bar{\mathbf{C}}$ instead of $\mathbf{C}^*$, it follows from (3.11.12) that

$$\Sigma^*(\bar{\mathbf{C}}) = \hat{\Sigma}^*(\bar{\alpha}_1, \bar{\alpha}_2, \bar{J}) \ ,$$

$$\frac{\partial \Sigma^*}{\partial \bar{\mathbf{C}}} = \frac{1}{2} \bar{J}\, \frac{\partial \hat{\Sigma}^*}{\partial \bar{J}}\, \bar{\mathbf{C}}^{-1} + \bar{J}^{-2/3} \frac{\partial \hat{\Sigma}^*}{\partial \bar{\alpha}_1} \left[\, \mathbf{I} - \frac{1}{3}(\bar{\mathbf{C}} \bullet \mathbf{I})\, \bar{\mathbf{C}}^{-1} \right]$$

$$+\, 2\bar{J}^{-4/3} \frac{\partial \hat{\Sigma}^*}{\partial \bar{\alpha}_2} \left[\, \bar{\mathbf{C}} - \frac{1}{3}(\bar{\mathbf{C}}^2 \bullet \mathbf{I})\, \bar{\mathbf{C}}^{-1} \right] \ . \tag{4.25.2}$$

Moreover, with the help of (4.11.28) it can be shown that $\mathbf{T}$ is given by

$$\mathbf{T} = -\bar{p}\,\mathbf{I} + \mathbf{T}' \ , \quad \mathbf{T}' \bullet \mathbf{I} = 0 \ , \quad a^{1/2}\,\mathbf{T} = 2\,m\,\bar{\mathbf{F}}\,\frac{\partial \hat{\Sigma}^*}{\partial \bar{\mathbf{C}}}\,\bar{\mathbf{F}}^T \ , \quad a^{1/2}\,\bar{p} = -\,m\,\bar{J}\,\frac{\partial \hat{\Sigma}^*}{\partial \bar{J}} \ ,$$

$$a^{1/2}\,\mathbf{T}' = 2m\,\bar{J}^{-2/3}\,\frac{\partial \hat{\Sigma}^*}{\partial \bar{\alpha}_1}\big[\bar{\mathbf{B}} - \tfrac{1}{3}(\bar{\mathbf{B}} \bullet \mathbf{I})\,\mathbf{I}\big] + 4m\,\bar{J}^{-4/3}\,\frac{\partial \hat{\Sigma}^*}{\partial \bar{\alpha}_2}\big[\,\bar{\mathbf{B}}^2 - \tfrac{1}{3}(\bar{\mathbf{B}}^2 \bullet \mathbf{I})\,\mathbf{I}\big] . \quad (4.25.3)$$

Comparison of (4.25.3) with (3.11.6) suggests that $\bar{p}$ and $\mathbf{T}'$ can be interpreted as an integrated pressure and deviatoric stress, respectively.

Next, when the strain energy function for the shell is given by (4.19.2) with the specification (4.25.2), it can be shown with the help of (4.4.32), that the constitutive equations (4.11.28) for $\mathbf{m}^\alpha$ and $\mathbf{t}^i$ can be written in the forms

$$\mathbf{m}^\alpha = 2m\,\bar{\mathbf{F}}\,\frac{\partial \Sigma^*}{\partial \bar{\mathbf{C}}}\,\mathbf{H}^\alpha + A^{1/2}\bar{H}\,\mathbf{F}^{-T}\mathbf{K}^{\alpha\beta}\,\boldsymbol{\beta}_\beta \ ,$$

$$\mathbf{t}^i = a^{1/2}\big[-\bar{p}\,\mathbf{I} + \mathbf{T}'\big]\,\mathbf{d}^i - \mathbf{m}^\alpha\,(\mathbf{d}_{3,\alpha} \bullet \mathbf{d}^i) \ , \quad (4.25.4)$$

where for isotropic materials $\mathbf{K}^{\alpha\beta}$ is determined by (4.15.25).

In the above constitutive equations, it is natural to use the quantity $\bar{J} = \det(\bar{\mathbf{F}})$ as a measure of dilatation instead of the quantity $J = \det(\mathbf{F})$. For most isotropic constitutive equations the limit of nearly incompressible response causes the bulk modulus to approach infinity. Consequently, the measure of dilatation becomes nearly unity. This means that the natural incompressibility constraint for the constitutive equations of this section which satisfy the restrictions of section 4.11, is

$$\bar{J} = 1 \ , \quad (4.25.5)$$

instead of the simpler condition (4.9.1).

Now, in order to develop the constraint responses associated with (4.25.5), this condition is first written in the rate form

$$\dot{\bar{J}} = \bar{J}\,\bar{\mathbf{F}}^{-T} \bullet \dot{\bar{\mathbf{F}}} = 0 \ , \quad (4.25.6)$$

which with the help of (4.25.1) can be rewritten as

$$\mathbf{D} \bullet \mathbf{I} + [\mathbf{F}^T\,\bar{\mathbf{F}}^{-T}\mathbf{H}^\alpha] \bullet \dot{\boldsymbol{\beta}}_\alpha = 0 \ . \quad (4.25.7)$$

It then follows from (4.9.12) and (4.9.20) that the constraint responses associated with this constraint are

$$a^{1/2}\bar{\mathbf{T}} = \gamma\,\mathbf{I} \ , \quad \bar{\mathbf{t}}^i = \gamma\,\big[\mathbf{d}^i - \{\bar{\mathbf{F}}^{-T}\mathbf{H}^\alpha\}(\mathbf{d}_{3,\alpha} \bullet \mathbf{d}^i)\big] \ , \quad \bar{\mathbf{m}}^\alpha = \gamma\,\bar{\mathbf{F}}^{-T}\mathbf{H}^\alpha \ , \quad (4.25.8)$$

where γ is an arbitrary function of (θ^α, t). Moreover, using (4.9.14) the quantities $\mathbf{T}$, $\mathbf{t}^i$ and $\mathbf{m}^\alpha$ are modified by these constraint responses so that

$$a^{1/2}\,\mathbf{T} = \gamma\,\mathbf{I} + a^{1/2}\,\hat{\mathbf{T}} \ ,$$

$$\mathbf{t}^i = \gamma\,\big[\mathbf{d}^i - \{\bar{\mathbf{F}}^{-T}\mathbf{H}^\alpha\}(\mathbf{d}_{3,\alpha} \bullet \mathbf{d}^i)\big] + \big[a^{1/2}\,\hat{\mathbf{T}}\,\mathbf{d}^i - \hat{\mathbf{m}}^\alpha\,(\mathbf{d}_{3,\alpha} \bullet \mathbf{d}^i)\big] \ ,$$

$$\mathbf{m}^\alpha = \gamma\,\bar{\mathbf{F}}^{-T}\mathbf{H}^\alpha + \hat{\mathbf{m}}^\alpha \ , \quad a^{1/2}\,\hat{\mathbf{T}} = -\,a^{1/2}\,\bar{p}\,\mathbf{I} + a^{1/2}\mathbf{T}' \ ,$$

$$\hat{\mathbf{m}}^\alpha = 2m\,\bar{\mathbf{F}}\,\frac{\partial \Sigma^*}{\partial \bar{\mathbf{C}}}\,\mathbf{H}^\alpha + A^{1/2}\bar{H}\,\mathbf{F}^{-T}\mathbf{K}^{\alpha\beta}\,\boldsymbol{\beta}_\beta \ . \quad (4.25.9)$$

Now, for the special case when Σ^* is given by (3.11.16) with (3.11.18) and (3.11.19), it follows that

$$m\,\hat{\Sigma}^*(\bar{\alpha}_1,\bar{\alpha}_2,\bar{J}) = \frac{1}{2} A^{1/2}\,\bar{H}\,\mu_0^*\left[(1-4C_2)(\bar{\alpha}_1-3) + C_2(\bar{\alpha}_2-3)\right]$$
$$+ A^{1/2}\,\bar{H}\,K_0^*\left[(\bar{J}-1) - \ln(\bar{J})\right] \;, \tag{4.25.10}$$

where C_2 is a material constant controlling nonlinear elastic effects. For this constitutive equation $\bar{p}$ is a function of $\bar{J}$ only so that the constraint (4.25.5) will cause $\bar{p}$ to vanish. Moreover, in general the arbitrariness of the Lagrange multiplier γ can absorb any dependence of $\bar{p}$ on $\bar{J}$. Consequently, for an incompressible shell it is possible to consider a restricted theory for which the explicit dependence of the strain energy Σ^* on J^* is removed so that $\Sigma^*(\bar{\mathbf{C}})$ is specified by

$$\Sigma^*(\bar{\mathbf{C}}) = \hat{\Sigma}^*(\bar{\alpha}_1,\bar{\alpha}_2,1) = \tilde{\Sigma}^*(\bar{\alpha}_1,\bar{\alpha}_2) \;, \quad \frac{\partial\tilde{\Sigma}^*}{\partial\bar{J}} = 0 \;. \tag{4.25.11}$$

Then, (4.25.3), (4.25.4) and (4.25.6) hold with $\hat{\Sigma}^*=\tilde{\Sigma}^*$ and $\bar{p}=0$. Also, for an incompressible shell, it is necessary to specify Poison's ratio ν^* in (4.15.25) by

$$\nu^* = \frac{1}{2} \;. \tag{4.25.12}$$

4.26 A simple derivation of the local equations for shells

Section 4.3 showed how the global balance laws for shells can be developed using the kinematic assumption (4.2.7) and the balance laws of the three-dimensional theory. In that derivation certain complicating features related to the boundaries of the three-dimensional shell-like structure were considered as they naturally occurred. The objective of the present section is to provide a simple derivation of the local equations for shells. To this end, it is convenient to recall the local forms (3.2.35) of the balance laws of the three-dimensional theory

$$\dot{m}^* = 0 \;, \quad m^*\,\dot{\mathbf{v}}^* = m^*\,\mathbf{b}^* + \mathbf{t}^{*i}{}_{,i} \;. \tag{4.26.1}$$

Moreover, with the help of (3.2.34) the reduced form of the balance of angular momentum $(3.2.35)_4$ can be rewritten as

$$g^{1/2}\,\mathbf{T}^* = \mathbf{t}^{*i}\otimes\mathbf{g}_i = \mathbf{g}_i\otimes\mathbf{t}^{*i} = g^{1/2}\,\mathbf{T}^{*T} \;. \tag{4.26.2}$$

Furthermore, the kinematic assumption (4.2.7) and the definitions (2.1.5) and (4.2.4)-(4.2.6) yield

$$\mathbf{x}^*(\theta^i,t) = \mathbf{x}(\theta^\alpha,t) + \theta^3\,\mathbf{d}_3(\theta^\alpha,t) \;, \quad \mathbf{g}_\alpha = \mathbf{d}_\alpha + \theta^3\,\mathbf{d}_{3,\alpha} \;, \quad \mathbf{g}_3 = \mathbf{d}_3 \;,$$
$$\mathbf{v}^*(\theta^i,t) = \mathbf{v}(\theta^\alpha,t) + \theta^3\,\mathbf{w}_3(\theta^\alpha,t) \;,$$
$$\dot{\mathbf{v}}^*(\theta^i,t) = \dot{\mathbf{v}}(\theta^\alpha,t) + \theta^3\,\dot{\mathbf{w}}_3(\theta^\alpha,t) \;, \quad \mathbf{w}_\alpha = \mathbf{v}_{,\alpha} \;. \tag{4.26.3}$$

Also, for simplicity the reference surface will be specified to be the middle surface of the shell in its reference configuration so that (4.19.6) hold with

$$-\frac{H}{2} \le \theta^3 \le \frac{H}{2} \;, \quad H = H(\theta^\alpha) \;, \tag{4.26.4}$$

where H is the thickness of the shell in its reference configuration along the normal to the shell surface.

Section 4.3 introduced a number of equations which connect quantities defined for the Cosserat shell with expressions associated with integration through the thickness of related three-dimensional quantities. In most cases, these equations will be referred to, but not be repeated here.

The balance laws for conservation of mass and balance of linear momentum for the shell can be obtained by merely integrating (4.26.1) over the thickness of the shell to get equations for average quantities of the forms

$$\frac{d}{dt}\int_{-H/2}^{H/2} m^* \, d\theta^3 = 0 \ ,$$

$$\Big[\int_{-H/2}^{H/2} m^* \, d\theta^3\Big] \dot{\mathbf{v}} + \Big[\int_{-H/2}^{H/2} m^* \, \theta^3 \, d\theta^3\Big] \dot{\mathbf{w}}_3$$

$$= \int_{-H/2}^{H/2} \left\{ m^* \, \mathbf{b}^* + \mathbf{t}^{*\alpha}{}_{,\alpha} + \mathbf{t}^{*3}{}_{,3} \right\} d\theta^3 \ , \tag{4.26.5}$$

where use has been made of the fact that θ^i are convected coordinates and that H is independent of time. However,

$$\int_{-H/2}^{H/2} \mathbf{t}^{*\alpha}{}_{,\alpha} \, d\theta^3 = \Big[\int_{-H/2}^{H/2} \mathbf{t}^{*\alpha} \, d\theta^3 \,\Big]_{,\alpha} - \frac{1}{2} \, H_{,\alpha} \left\{ \mathbf{t}^{*\alpha}(\theta^\beta,H/2) + \mathbf{t}^{*\alpha}(\theta^\beta,-H/2) \right\} \ ,$$

$$\int_{-H/2}^{H/2} \mathbf{t}^{*3}{}_{,3} \, d\theta^3 = \mathbf{t}^{*3}(\theta^\alpha,H/2) - \mathbf{t}^{*3}(\theta^\alpha,-H/2) \ , \tag{4.26.6}$$

so that $(4.26.5)_2$ can be rewritten in the form

$$\Big[\int_{-H/2}^{H/2} m^* \, d\theta^3\Big] \dot{\mathbf{v}} + \Big[\int_{-H/2}^{H/2} m^* \, \theta^3 \, d\theta^3\Big] \dot{\mathbf{w}}_3$$

$$= \Big[\int_{-H/2}^{H/2} m^* \, \mathbf{b}^* \, d\theta^3 + \left\{ \mathbf{t}^{*3}(\theta^\beta,H/2) - \frac{1}{2} H_{,\alpha} \, \mathbf{t}^{*\alpha}(\theta^\beta,H/2) \right\}$$

$$- \left\{ \mathbf{t}^{*3}(\theta^\beta,-H/2) + \frac{1}{2} H_{,\alpha} \, \mathbf{t}^{*\alpha}(\theta^\beta,-H/2) \right\} \Big] + \Big[\int_{-H/2}^{H/2} \mathbf{t}^{*\alpha} \, d\theta^3 \,\Big]_{,\alpha} \ . \tag{4.26.7}$$

Thus, with the help of the definitions (4.3.5), (4.3.8), (4.3.10), (4.3.15), (4.3.24) and (4.3.28), equations $(4.26.5)_1$ and (4.26.7) yield the local forms of the conservation of mass and the balance of linear momentum equations for the shell

$$\dot{m} = 0 \ , \quad m\,(\dot{\mathbf{v}} + y^3 \, \dot{\mathbf{w}}_3) = m \, \mathbf{b} + \mathbf{t}^{\alpha}{}_{,\alpha} \ . \tag{4.26.8}$$

As has been seen, these equations represent a zeroth order moment (or mere integration) through the thickness of the shell of the three-dimensional conservation of mass and linear momentum equations.

It will presently be shown that the balance of director momentum equation represents a first order moment through the thickness of the shell of the three-dimensional linear momentum equation. Specifically, $(4.26.1)_2$ is multiplied by θ^3 and then integrated through the thickness of the shell to obtain

$$\Big[\int_{-H/2}^{H/2} m^* \, \theta^3 \, d\theta^3\Big] \dot{\mathbf{v}} + \Big[\int_{-H/2}^{H/2} m^* \, \theta^3\theta^3 \, d\theta^3\Big] \dot{\mathbf{w}}_3$$

$$= \int_{-H/2}^{H/2} \left\{ m^* \, \mathbf{b}^* + \mathbf{t}^{*\alpha}{}_{,\alpha} + \mathbf{t}^{*3}{}_{,3} \right\} \theta^3 \, d\theta^3 \ . \tag{4.26.9}$$

However, it can be shown that

$$\int_{-H/2}^{H/2} \mathbf{t}^{*\alpha}{}_{,\alpha}\, \theta^3\, d\theta^3 = \Big[\int_{-H/2}^{H/2} \mathbf{t}^{*\alpha}\, \theta^3\, d\theta^3\, \Big]_{,\alpha} - \frac{H}{4} H_{,\alpha} \{ \mathbf{t}^{*\alpha}(\theta^\beta, H/2) - \mathbf{t}^{*\alpha}(\theta^\beta, -H/2) \} \;,$$

$$\int_{-H/2}^{H/2} \mathbf{t}^{*3}{}_{,3}\, \theta^3\, d\theta^3 = \frac{H}{2} \{ \mathbf{t}^{*3}(\theta^\beta, H/2) + \mathbf{t}^{*3}(\theta^\beta, -H/2) \} - \int_{-H/2}^{H/2} \mathbf{t}^{*3}\, d\theta^3 \;, \tag{4.26.10}$$

so that (4.26.9) can be rewritten in the form

$$\begin{aligned}
&\Big[\int_{-H/2}^{H/2} m^*\, \theta^3\, d\theta^3\Big]\, \dot{\mathbf{v}} + \Big[\int_{-H/2}^{H/2} m^*\, \theta^3\theta^3\, d\theta^3\Big]\, \dot{\mathbf{w}}_3 \\
&\quad = \Big[\int_{-H/2}^{H/2} \{ m^*\, \mathbf{b}^*\, \theta^3 \} d\theta^3 + \frac{H}{2} \{ \mathbf{t}^{*3}(\theta^\beta, H/2) - \frac{1}{2} H_{,\alpha}\, \mathbf{t}^{*\alpha}(\theta^\beta, H/2) \} \\
&\qquad + \frac{H}{2} \{ \mathbf{t}^{*3}(\theta^\beta, -H/2) + \frac{1}{2} H_{,\alpha}\, \mathbf{t}^{*\alpha}(\theta^\beta, -H/2) \} \Big] \\
&\qquad - \Big[\int_{-H/2}^{H/2} \mathbf{t}^{*3}\, d\theta^3\Big] + \Big[\int_{-H/2}^{H/2} \mathbf{t}^{*\alpha}\, \theta^3\, d\theta^3\, \Big]_{,\alpha} \;.
\end{aligned} \tag{4.26.11}$$

Thus, with the help of the definitions (4.3.8), (4.3.30), (4.3.32), (4.3.34), (4.3.36) (4.3.38) and (4.3.42), equation (4.26.11) yields the local form of the balance of director momentum for the shell

$$m\, (y^3\, \dot{\mathbf{v}} + y^{33}\, \dot{\mathbf{w}}_3) = m\, \mathbf{b}^3 - \mathbf{t}^3 + \mathbf{m}^\alpha{}_{,\alpha} \;. \tag{4.26.12}$$

Finally, the reduced form of the balance of angular momentum for the shell is obtained by substituting $(4.26.3)_{2,3}$ into (4.26.2) and integrating through the thickness to obtain

$$\begin{aligned}
&\int_{-H/2}^{H/2} g^{1/2}\, \mathbf{T}^*\, d\theta^3 = \Big[\int_{-H/2}^{H/2} \mathbf{t}^{*\alpha}\, d\theta^3\, \Big]\otimes\mathbf{d}_\alpha + \Big[\int_{-H/2}^{H/2} \mathbf{t}^{*3}\, d\theta^3\, \Big]\otimes\mathbf{d}_3 \\
&\qquad + \Big[\int_{-H/2}^{H/2} \mathbf{t}^{*\alpha}\, \theta^3\, d\theta^3\, \Big]\otimes\mathbf{d}_{3,\alpha} = \int_{-H/2}^{H/2} g^{1/2}\, \mathbf{T}^{*T}\, d\theta^3 \;.
\end{aligned} \tag{4.26.13}$$

Thus, with the help of the definitions (4.3.24), (4.3.34), (4.3.38) and (4.11.2), equation (4.26.13) yields

$$a^{1/2}\, \mathbf{T} = \mathbf{t}^i\otimes\mathbf{d}_i + \mathbf{m}^\alpha\otimes\mathbf{d}_{3,\alpha} = a^{1/2}\, \mathbf{T}^T \;. \tag{4.26.14}$$

4.27 A brief summary of the equations for shells

This section summarizes some of the main equations of the Cosserat theory of shells. For simplicity, attention will be confined to shells which are made of homogeneous materials (ρ_0^* = constant) and which have normal thickness H. The shell's reference surface will be taken to be the middle surface and the director $\mathbf{D}_3$ will be taken in the direction of the normal to this surface in the reference configuration. Also, the body force $\mathbf{b}^*$ per unit mass will be assumed to be a constant vector. In principle, the equations can be used for shells of variable thickness with $H(\theta^\alpha)$. However, their validity has only been examined for specific problems of plates and shells of uniform thickness.

An attempt has been made to record a complete set of equations that can be used to determine the response of general elastic shells. Also, for convenient reference, equation numbers are recorded below to indicate the locations in the previous sections where the quantities or related quantities have been explained in more detail.

KINEMATICS

$$\mathbf{X}(\theta^\alpha) \;,\; \mathbf{D}_\alpha = \mathbf{X},_\alpha \;,\; \mathbf{D}_3(\theta^\alpha) = \mathbf{A}_3 = \frac{\mathbf{D}_1\times\mathbf{D}_2}{|\mathbf{D}_1\times\mathbf{D}_2|} \;, \qquad (4.1.1)\text{-}(4.1.3)$$

$$D^{1/2}\,\mathbf{D}^1 = \mathbf{D}_2\times\mathbf{D}_3 \;,\; D^{1/2}\,\mathbf{D}^2 = \mathbf{D}_3\times\mathbf{D}_1 \;,\; D^{1/2}\,\mathbf{D}^3 = \mathbf{D}_1\times\mathbf{D}_2 \;, \qquad (4.1.14)$$

$$D^{1/2} = A^{1/2} = \mathbf{D}_1\times\mathbf{D}_2 \cdot \mathbf{D}_3, \;\; \mathbf{A}_i = \mathbf{D}_i, \qquad (4.1.14),(4.19.5)$$

$$\mathbf{x}(\theta^\alpha,t) \;,\; \mathbf{d}_\alpha = \mathbf{x},_\alpha \;,\; \mathbf{d}_3(\theta^\alpha,t) \;, \qquad (4.2.1),(4.2.4),(4.2.3)$$

$$d^{1/2}\,\mathbf{d}^1 = \mathbf{d}_2\times\mathbf{d}_3 \;,\; d^{1/2}\,\mathbf{d}^2 = \mathbf{d}_3\times\mathbf{d}_1 \;,\; d^{1/2}\,\mathbf{d}^3 = \mathbf{d}_1\times\mathbf{d}_2 \;, \qquad (4.2.8)$$

$$d^{1/2} = \mathbf{d}_1\times\mathbf{d}_2 \cdot \mathbf{d}_3 \;, \qquad (4.2.4)$$

$$\mathbf{a}_3 = \frac{\mathbf{d}_1\times\mathbf{d}_2}{|\mathbf{d}_1\times\mathbf{d}_2|} \;,\; a^{1/2} = \mathbf{d}_1\times\mathbf{d}_2 \cdot \mathbf{a}_3 \;, \qquad (4.2.2)$$

$$\mathbf{v} = \dot{\mathbf{x}} \;,\; \mathbf{w}_i = \dot{\mathbf{d}}_i \;, \qquad (4.2.5)$$

$$\mathbf{F} = \mathbf{d}_i\otimes\mathbf{D}^i \;,\; \mathbf{C} = \mathbf{F}^T\mathbf{F} \;,\; \mathbf{E} = \frac{1}{2}(\mathbf{C}-\mathbf{I}) \;, \qquad (4.2.9),(4.8.8),(4.12.3)$$

$$\boldsymbol{\beta}_\alpha = \mathbf{F}^{-1}\mathbf{d}_3,_\alpha - \mathbf{D}_3,_\alpha \;, \qquad (4.6.15)$$

$$\xi_1(\theta^\alpha) = -\frac{H}{2} \;,\; \xi_2(\theta^\alpha) = \frac{H}{2} \;, \qquad (4.1.19),(4.19.6)$$

$$A^{1/2}\,\bar{H} = D^{1/2}\,H\left[1 + \frac{H^2}{12}D^{-1/2}\,(\mathbf{D}_3,_1\times\mathbf{D}_3,_2 \cdot \mathbf{D}_3)\right] \;, \qquad (4.19.9)$$

$$A^{1/2}\,\bar{H}\,\mathbf{H}^1 = D^{1/2}\,H\left[\frac{H^2}{12}D^{-1/2}\,(\mathbf{D}_3,_2\times\mathbf{D}_3)\right] \;, \qquad (4.19.9)$$

$$A^{1/2}\,\bar{H}\,\mathbf{H}^2 = D^{1/2}\,H\left[\frac{H^2}{12}D^{-1/2}\,(\mathbf{D}_3\times\mathbf{D}_3,_1)\right] \;, \qquad (4.19.9)$$

$$\bar{\mathbf{F}} = \mathbf{F}\,[\mathbf{I} + \boldsymbol{\beta}_\alpha\otimes\mathbf{H}^\alpha] \;,\; \bar{J} = \det\bar{\mathbf{F}} \;,\; \bar{\mathbf{B}} = \bar{\mathbf{F}}\,\bar{\mathbf{F}}^T \;, \qquad (4.25.1)$$

$$\bar{\alpha}_1 = \bar{J}^{-2/3}\,\bar{\mathbf{C}}\cdot\mathbf{I} = \bar{J}^{-2/3}\,\bar{\mathbf{B}}\cdot\mathbf{I} \;,\; \bar{\alpha}_2 = \bar{J}^{-4/3}\,\bar{\mathbf{C}}\cdot\bar{\mathbf{C}} = \bar{J}^{-4/3}\,\bar{\mathbf{B}}\cdot\bar{\mathbf{B}} \;, \qquad (4.25.1)$$

$$\bar{\mathbf{C}} = \bar{\mathbf{F}}^T\,\bar{\mathbf{F}} \;,\; \bar{\mathbf{E}} = \frac{1}{2}(\bar{\mathbf{C}}-\mathbf{I}) \;, \qquad (4.11.19),(4.12.3)$$

$$\mathbf{M}'_1 = \mathbf{M}'_2\times\mathbf{M}'_3 \;,\; \mathbf{M}'_2 = \frac{\mathbf{M}'_3\times\mathbf{D}_1}{|\mathbf{M}'_3\times\mathbf{D}_1|} \;,\; \mathbf{M}'_3 = \frac{\mathbf{D}_3}{|\mathbf{D}_3|} \;, \qquad (4.12.8)$$

$$\mathbf{M}_1 = \cos\phi\,\mathbf{M}'_1 + \sin\phi\,\mathbf{M}'_2 \;,\; \mathbf{M}_2 = -\sin\phi\,\mathbf{M}'_1 + \cos\phi\,\mathbf{M}'_2 \;,\; \mathbf{M}_3 = \mathbf{M}'_3 \;, \qquad (4.12.8)$$

BALANCE LAWS

$$m = \rho\,a^{1/2} = \rho_0\,A^{1/2} = m(\theta^\alpha) \;,\; \dot{m} = 0 \;, \qquad (4.4.17),(4.4.35)$$

$$m\,(\dot{\mathbf{v}} + y^3\,\dot{\mathbf{w}}_3) = m\,\mathbf{b} + \mathbf{t}^\alpha,_\alpha \;, \qquad (4.4.35)$$

$$m\,(y^3\,\dot{\mathbf{v}} + y^{33}\,\dot{\mathbf{w}}_3) = m\,\mathbf{b}^3 - \mathbf{t}^3 + \mathbf{m}^\alpha,_\alpha \;, \qquad (4.4.35)$$

$$\mathbf{T} = a^{-1/2}\,(\mathbf{t}^i\otimes\mathbf{d}_i + \mathbf{m}^\alpha\otimes\mathbf{d}_3,_\alpha) = \mathbf{T}^T \;, \qquad (4.4.32),(4.4.33)$$

ASSIGNED FIELDS

$$\mathbf{b}_b = \mathbf{b}^* \;,\; m\,\mathbf{b}_b^3 = (\rho_0^*\,D^{1/2}\,H)\,\mathbf{b}^*\left[\frac{H^2}{12}(\mathbf{D}_3,_\alpha\cdot\mathbf{D}^\alpha)\right] \;, \qquad (4.19.14)$$

$$g^{1/2}\alpha(\tfrac{H}{2})\ \mathbf{n}^* = \left[\mathbf{x} + \tfrac{H}{2}\mathbf{d}_3\right]_{,1} \times \left[\mathbf{x} + \tfrac{H}{2}\mathbf{d}_3\right]_{,2} \quad \text{on } \theta^3 = \tfrac{H}{2} \ , \qquad (4.3.12)$$

$$g^{1/2}\alpha(-\tfrac{H}{2})\ \mathbf{n}^* = -\left[\mathbf{x} - \tfrac{H}{2}\mathbf{d}_3\right]_{,1} \times \left[\mathbf{x} - \tfrac{H}{2}\mathbf{d}_3\right]_{,2} \quad \text{on } \theta^3 = -\tfrac{H}{2} \ , \qquad (4.3.13)$$

$$\mathbf{n}^* \bullet \mathbf{n}^* = 1 \ , \qquad (4.3.12)$$

$$m\,\mathbf{b}_c = \left[g^{1/2}\alpha(\tfrac{H}{2})\ \mathbf{t}^*\right]\Big|_{\theta^3 = H/2} + \left[g^{1/2}\alpha(-\tfrac{H}{2})\ \mathbf{t}^*\right]\Big|_{\theta^3 = -H/2} \ , \qquad (4.3.15)$$

$$m\,\mathbf{b}_c^3 = \tfrac{H}{2}\left[g^{1/2}\alpha(\tfrac{H}{2})\ \mathbf{t}^*\right]\Big|_{\theta^3 = H/2} - \tfrac{H}{2}\left[g^{1/2}\alpha(-\tfrac{H}{2})\ \mathbf{t}^*\right]\Big|_{\theta^3 = -H/2} \ , \qquad (4.3.36)$$

$$\mathbf{b} = \mathbf{b}_b + \mathbf{b}_c \ , \quad \mathbf{b}^3 = \mathbf{b}_b^3 + \mathbf{b}_c^3 \ , \qquad (4.3.28),(4.3.42)$$

INERTIA QUANTITIES

$$m = \rho_0^* A^{1/2}\,\bar{H}(\theta^\alpha) \ , \quad \rho_0 = \rho_0^*\ \bar{H} \ , \qquad (4.4.35),(4.19.10)$$

$$m\,y^3 = (\rho_0^* D^{1/2} H)\, H\left[\gamma_1 H (\mathbf{D}_{3,\alpha} \bullet \mathbf{D}^\alpha) + \gamma_2 H^2 D^{-1/2} (\mathbf{D}_{3,1}\times\mathbf{D}_{3,2} \bullet \mathbf{D}_3)\right] \ , \qquad (4.19.13)$$

$$m\,y^{33} = (\rho_0^* D^{1/2} H)\frac{H^2}{\pi^2}\left[1 + \gamma_3 H (\mathbf{D}_{3,\alpha} \bullet \mathbf{D}^\alpha)\right.$$

$$\left. + \gamma_4 H^2 D^{-1/2} (\mathbf{D}_{3,1}\times\mathbf{D}_{3,2} \bullet \mathbf{D}_3)\right] \ , \qquad (4.19.13)$$

$$\gamma_1 = 0.07 \ , \quad \gamma_2 = 0.03 \ , \qquad (4.21.8),(4.23.11)$$

$$\gamma_3 = \frac{\pi^2 - 4}{8} \ , \quad \gamma_4 = -\frac{\pi^2 - 6}{24} \ , \qquad (4.21.3),(4.23.4)$$

BOUNDARY VALUES OF FORCE, COUPLE AND MOMENT

$$\mathbf{n}\,ds = \mathbf{d}_\alpha \frac{\partial\tilde{\theta}^\alpha}{\partial s}\,ds \times \mathbf{a}_3 \ , \quad \mathbf{n} \bullet \mathbf{n} = 1 \ , \qquad (4.3.23)$$

$$\mathbf{t} = a^{-1/2}\,(\mathbf{t}^\alpha \otimes \mathbf{d}_\alpha)\,\mathbf{n} \ , \qquad (4.3.24),(4.3.26)$$

$$\mathbf{m}^3 = a^{-1/2}\,(\mathbf{m}^\alpha \otimes \mathbf{d}_\alpha)\,\mathbf{n} \ , \qquad (4.3.38),(4.3.40)$$

$$\mathbf{m} = \mathbf{d}_3\times\mathbf{m}^3 \ , \qquad (4.10.7)$$

GENERAL CONSTITUTIVE EQUATIONS

$$m\,\dot{\Sigma} = a^{1/2}\,\mathcal{P} = a^{1/2}\,\mathbf{T} \bullet \mathbf{D} + (\mathbf{F}^T\mathbf{m}^\alpha) \bullet \dot{\boldsymbol{\beta}}_\alpha \ . \qquad (4.4.17),(4.8.1)$$

$$m\,\Sigma = m\,\Sigma^*(\bar{\mathbf{C}}) + \tfrac{1}{2}A^{1/2}\,\bar{H}\,\mathbf{K}^{\alpha\beta} \bullet (\boldsymbol{\beta}_\alpha\otimes\boldsymbol{\beta}_\beta) \ , \qquad (4.11.27)$$

$$\mathbf{m}^\alpha = 2\,m\,\bar{\mathbf{F}}\,\frac{\partial\Sigma^*(\bar{\mathbf{C}})}{\partial\bar{\mathbf{C}}}\,\mathbf{H}^\alpha + A^{1/2}\,\bar{H}\,\mathbf{F}^{-T}\,\mathbf{K}^{\alpha\beta}\,\boldsymbol{\beta}_\beta \ , \qquad (4.11.28)$$

$$\mathbf{t}^i = 2\,m\,\bar{\mathbf{F}}\,\frac{\partial\Sigma^*(\bar{\mathbf{C}})}{\partial\bar{\mathbf{C}}}\,\bar{\mathbf{F}}^T\,\mathbf{d}^i - \mathbf{m}^\alpha\,(\mathbf{d}_{3,\alpha} \bullet \mathbf{d}^i) \ , \qquad (4.11.28)$$

$$\mathbf{K}^{\alpha\beta} = \left[(\mathbf{D}^\alpha \bullet \mathbf{M}_\gamma)\,(\mathbf{D}^\beta \bullet \mathbf{M}_\delta)\,K_{i\gamma j\delta}\right](\mathbf{M}_i\otimes\mathbf{M}_j) \ , \qquad (4.12.9),(4.12.10)$$

CONSTRAINTS

$$\mathbf{t}^i = \hat{\mathbf{t}}^i + \bar{\mathbf{t}}^i \ , \quad \mathbf{m}^\alpha = \hat{\mathbf{m}}^\alpha + \bar{\mathbf{m}}^\alpha \ , \qquad (4.9.14)$$

Incompressibility

$$\bar{J} = 1 \ , \quad \bar{\mathbf{t}}^i = \gamma\left[\mathbf{d}^i - \{\bar{\mathbf{F}}^{-T}\mathbf{H}^\alpha\}(\mathbf{d}_{3,\alpha} \bullet \mathbf{d}^i)\right] \ , \qquad (4.25.5),(4.25.8)$$

$$\bar{\mathbf{m}}^\alpha = \gamma\, \bar{\mathbf{F}}^{-T}\mathbf{H}^\alpha \;,\quad \nu^* = \frac{1}{2} \;, \qquad (4.25.8),(4.25.12)$$

Eliminate normal extension only

$$\mathbf{d}_3 \bullet \mathbf{a}_3 = 1 \;,\quad \bar{\mathbf{t}}^i = \gamma^{33}\,(\mathbf{d}^3 \bullet \mathbf{d}^i)\,\mathbf{d}^3 \;,\quad \bar{\mathbf{m}}^\alpha = 0 \;, \quad (4.9.3),(4.9.7),(4.9.20)$$

Eliminate director extension only

$$\mathbf{d}_3 \bullet \mathbf{d}_3 = 1 \;,\quad \bar{\mathbf{t}}^\alpha = 0 \;,\quad \bar{\mathbf{t}}^3 = \gamma^{33}\,\mathbf{d}_3 \;,\quad \bar{\mathbf{m}}^\alpha = 0 \;, \quad (4.9.10),(4.9.11),(4.9.20)$$

Eliminate transverse shear deformation only

$$\mathbf{d}_3 \bullet \mathbf{d}_\alpha = 0 \;,\quad \bar{\mathbf{t}}^\alpha = \gamma^{3\alpha}\,\mathbf{d}_3 \;,\quad \bar{\mathbf{t}}^3 = \gamma^{3\beta}\,\mathbf{d}_\beta \;,\quad \bar{\mathbf{m}}^\alpha = 0 \;, \quad (4.9.8),(4.9.9),(4.9.20)$$

Eliminate normal extension and transverse shear deformation

$$\mathbf{d}_3 = \mathbf{a}_3 \;,\quad \bar{\mathbf{t}}^\alpha = \gamma^{3\alpha}\,\mathbf{d}_3 \;,\quad \bar{\mathbf{t}}^3 = \gamma^{3\beta}\,\mathbf{d}_\beta + \gamma^{33}\,\mathbf{d}_3 \;,\quad \bar{\mathbf{m}}^\alpha = 0 \;, \quad (4.9.21),(4.9.23)$$

ORTHOTROPIC SHELLS - SMALL STRAINS (LARGE DISPLACEMENTS)

$$\bar{\mathbf{E}} = \mathbf{E} + \frac{1}{2}(\boldsymbol{\beta}_\alpha \otimes \mathbf{H}^\alpha + \mathbf{H}^\alpha \otimes \boldsymbol{\beta}_\alpha) \;, \qquad (4.12.6)$$

$$m\,\Sigma = \frac{1}{2} A^{1/2}\,\bar{H}\,\mathbf{K}^* \bullet (\bar{\mathbf{E}} \otimes \bar{\mathbf{E}}) + \frac{1}{2} A^{1/2}\,\bar{H}\,\mathbf{K}^{\alpha\beta} \bullet (\boldsymbol{\beta}_\alpha \otimes \boldsymbol{\beta}_\beta) \;, \qquad (4.12.6)$$

$$\mathbf{m}^\alpha = A^{1/2}\,\bar{H}\,\mathbf{F}^{-T}\big[\{\mathbf{K}^* \bullet \bar{\mathbf{E}}\}\mathbf{H}^\alpha + \mathbf{K}^{\alpha\beta}\,\boldsymbol{\beta}_\beta\,\big] \;, \qquad (4.12.7)$$

$$\mathbf{t}^i = \mathbf{F}\big[A^{1/2}\,\bar{H}\,\mathbf{K}^* \bullet \bar{\mathbf{E}}\big]\,\mathbf{D}^i - \mathbf{m}^\alpha\,(\mathbf{d}_{3,\alpha} \bullet \mathbf{d}^i) \;, \qquad (4.12.7)$$

$$\bar{E}_{ij} = \bar{\mathbf{E}} \bullet (\mathbf{M}_i \otimes \mathbf{M}_j) \;,\quad K^*_{ijkl} = \mathbf{K}^* \bullet (\mathbf{M}_i \otimes \mathbf{M}_j \otimes \mathbf{M}_k \otimes \mathbf{M}_l) \;, \qquad (4.12.9)$$

$$\begin{aligned}
\mathbf{K}^* \bullet \bar{\mathbf{E}} = {} & \big[K^*_{1111}\,\bar{E}_{11} + K^*_{1122}\,\bar{E}_{22} + K^*_{1133}\,\bar{E}_{33}\big]\,(\mathbf{M}_1 \otimes \mathbf{M}_1) \\
& + \big[K^*_{1122}\,\bar{E}_{11} + K^*_{2222}\,\bar{E}_{22} + K^*_{2233}\,\bar{E}_{33}\big]\,(\mathbf{M}_2 \otimes \mathbf{M}_2) \\
& + \big[K^*_{1133}\,\bar{E}_{11} + K^*_{2233}\,\bar{E}_{22} + K^*_{3333}\,\bar{E}_{33}\big]\,(\mathbf{M}_3 \otimes \mathbf{M}_3) \\
& + \big[K^*_{1212}\,(\bar{E}_{12} + \bar{E}_{21})\big]\,(\mathbf{M}_1 \otimes \mathbf{M}_2 + \mathbf{M}_2 \otimes \mathbf{M}_1) \\
& + \big[K^*_{1313}\,(\bar{E}_{13} + \bar{E}_{31})\big]\,(\mathbf{M}_1 \otimes \mathbf{M}_3 + \mathbf{M}_3 \otimes \mathbf{M}_1) \\
& + \big[K^*_{2323}\,(\bar{E}_{23} + \bar{E}_{32})\big]\,(\mathbf{M}_2 \otimes \mathbf{M}_3 + \mathbf{M}_3 \otimes \mathbf{M}_2) \;, \qquad (3.12.12)
\end{aligned}$$

$$\begin{aligned}
\mathbf{K}^{\alpha\beta} = {} & (\mathbf{D}^\alpha \bullet \mathbf{M}_1)(\mathbf{D}^\beta \bullet \mathbf{M}_1)\big[K_{1111}\,\mathbf{M}_1 \otimes \mathbf{M}_1 + K_{1212}\,\mathbf{M}_2 \otimes \mathbf{M}_2\big] \\
& + (\mathbf{D}^\alpha \bullet \mathbf{M}_1)(\mathbf{D}^\beta \bullet \mathbf{M}_2)\big[K_{1122}\,\mathbf{M}_1 \otimes \mathbf{M}_2 + K_{1212}\,\mathbf{M}_2 \otimes \mathbf{M}_1\big] \\
& + (\mathbf{D}^\alpha \bullet \mathbf{M}_2)(\mathbf{D}^\beta \bullet \mathbf{M}_1)\big[K_{1212}\,\mathbf{M}_1 \otimes \mathbf{M}_2 + K_{1122}\,\mathbf{M}_2 \otimes \mathbf{M}_1\big] \\
& + (\mathbf{D}^\alpha \bullet \mathbf{M}_2)(\mathbf{D}^\beta \bullet \mathbf{M}_2)\big[K_{1212}\,\mathbf{M}_1 \otimes \mathbf{M}_1 + K_{2222}\,\mathbf{M}_2 \otimes \mathbf{M}_2\big] \;, \qquad (4.15.24)
\end{aligned}$$

$$K_{1111} = \frac{H^2}{12}\Big[\;K^*_{1111} - \frac{K^*_{1133}K^*_{1133}}{K^*_{3333}}\Big] \;, \qquad (4.14.28)$$

$$K_{2222} = \frac{H^2}{12}\Big[\;K^*_{2222} - \frac{K^*_{2233}K^*_{2233}}{K^*_{3333}}\Big] \;, \qquad (4.14.28)$$

$$K_{1122} = \frac{H^2}{12}\Big[\;K^*_{1122} - \frac{K^*_{1133}K^*_{2233}}{K^*_{3333}}\Big] \;, \qquad (4.14.28)$$

$$K_{1212} = \frac{H^2}{12} K^*_{1212} \ , \qquad (4.15.19)$$

ISOTROPIC SHELLS - SMALL STRAINS (LARGE DISPLACEMENTS)

Use the equations for orthotropic shells - small strains (large displacements) with

$$\bar{\mathbf{E}} = \mathbf{E} + \frac{1}{2}(\boldsymbol{\beta}_\alpha \otimes \mathbf{H}^\alpha + \mathbf{H}^\alpha \otimes \boldsymbol{\beta}_\alpha) \ , \qquad (4.12.6)$$

$$\mathbf{K}^* \bullet \bar{\mathbf{E}} = 2\mu^* \left[\left\{\frac{\nu^*}{1-2\nu^*}\right\}(\bar{\mathbf{E}} \bullet \mathbf{I})\,\mathbf{I} + \bar{\mathbf{E}}\right] \ , \quad \text{Table 3.12.1,(3.12.15)}$$

$$K^*_{1111} = K^*_{2222} = K^*_{3333} = 2\mu^* \left[\frac{1-\nu^*}{1-2\nu^*}\right] \ , \quad \text{Table 3.12.1,(3.12.13)}$$

$$K^*_{1122} = K^*_{1133} = K^*_{2233} = 2\mu^* \left[\frac{\nu^*}{1-2\nu^*}\right] \ , \quad \text{Table 3.12.1,(3.12.13)}$$

$$K^*_{1212} = K^*_{1313} = K^*_{2323} = \mu^* \ , \qquad (3.12.13)$$

$$K_{1111} = K_{2222} = \frac{H^2\mu^*}{6(1-\nu^*)} \ , \quad K_{1122} = \frac{H^2\mu^*\nu^*}{6(1-\nu^*)} \ , \qquad (4.14.33)$$

$$K_{1212} = \frac{H^2\mu^*}{12} \ , \qquad (4.15.21)$$

$$\begin{aligned}
\mathbf{K}^{\alpha\beta} = {} & (\mathbf{D}^\alpha \bullet \mathbf{M}_1)(\mathbf{D}^\beta \bullet \mathbf{M}_1)\frac{H^2\mu^*}{12}\left[\left\{\frac{2}{1-\nu^*}\right\}\mathbf{M}_1\otimes\mathbf{M}_1 + \mathbf{M}_2\otimes\mathbf{M}_2\right] \\
& + (\mathbf{D}^\alpha \bullet \mathbf{M}_1)(\mathbf{D}^\beta \bullet \mathbf{M}_2)\frac{H^2\mu^*}{12}\left[\left\{\frac{2\nu^*}{1-\nu^*}\right\}\mathbf{M}_1\otimes\mathbf{M}_2 + \mathbf{M}_2\otimes\mathbf{M}_1\right] \\
& + (\mathbf{D}^\alpha \bullet \mathbf{M}_2)(\mathbf{D}^\beta \bullet \mathbf{M}_1)\frac{H^2\mu^*}{12}\left[\mathbf{M}_1\otimes\mathbf{M}_2 + \left\{\frac{2\nu^*}{1-\nu^*}\right\}\mathbf{M}_2\otimes\mathbf{M}_1\right] \\
& + (\mathbf{D}^\alpha \bullet \mathbf{M}_2)(\mathbf{D}^\beta \bullet \mathbf{M}_2)\frac{H^2\mu^*}{12}\left[\mathbf{M}_1\otimes\mathbf{M}_1 + \left\{\frac{2}{1-\nu^*}\right\}\mathbf{M}_2\otimes\mathbf{M}_2\right] \ , \qquad (4.15.25)
\end{aligned}$$

NONLINEAR ISOTROPIC SHELL

Use the general constitutive equations, with $\mathbf{K}^{\alpha\beta}$ given by the expression for isotropic shells - small strains (large displacements) and with

$$\Sigma^*(\bar{\mathbf{C}}) = \hat{\Sigma}^*(\bar{\alpha}_1, \bar{\alpha}_2, \bar{J}) \ , \qquad (4.25.2)$$

$$\begin{aligned}
m\,\hat{\Sigma}^*(\bar{\alpha}_1, \bar{\alpha}_2, \bar{J}) = {} & \frac{1}{2} A^{1/2}\, \bar{H}\, \mu_0^* \left[(1-4C_2)(\bar{\alpha}_1 - 3) + C_2(\bar{\alpha}_2 - 3)\right] \\
& + A^{1/2}\, \bar{H}\, K_0^* \left[(\bar{J}-1) - \ln(\bar{J})\right] \ , \qquad (4.25.9)
\end{aligned}$$

$$\mathbf{T} = -\bar{p}\,\mathbf{I} + \mathbf{T}' \ , \quad \mathbf{T}' \bullet \mathbf{I} = 0 \ , \qquad (4.25.3)$$

$$a^{1/2}\,\mathbf{T} = 2\,m\,\bar{\mathbf{F}}\frac{\partial \Sigma^*}{\partial \bar{\mathbf{C}}}\bar{\mathbf{F}}^T \ , \quad a^{1/2}\,\bar{p} = \bar{J}\,A^{1/2}\,\bar{H}\,K_0^*\left[\frac{1}{\bar{J}} - 1\right] \ , \qquad (4.25.3)$$

$$\begin{aligned}
a^{1/2}\,\mathbf{T}' = {} & \bar{J}^{-2/3} A^{1/2}\,\bar{H}\,\mu_0^*\left[(1-4C_2)\left\{\bar{\mathbf{B}} - \frac{1}{3}(\bar{\mathbf{B}} \bullet \mathbf{I})\,\mathbf{I}\right\}\right. \\
& \left. + 2\,C_2\,\bar{J}^{-2/3}\left\{\bar{\mathbf{B}}^2 - \frac{1}{3}(\bar{\mathbf{B}}^2 \bullet \mathbf{I})\,\mathbf{I}\right\}\right] \ , \qquad (4.25.3)
\end{aligned}$$

$$\mathbf{m}^\alpha = a^{1/2}\left[-\bar{p}\,\mathbf{I} + \mathbf{T}'\right]\bar{\mathbf{F}}^{-T}\mathbf{H}^\alpha + A^{1/2}\bar{H}\,\mathbf{F}^{-T}\mathbf{K}^{\alpha\beta}\,\boldsymbol{\beta}_\beta \quad ,(4.25.3),(4.25.4)$$

$$\mathbf{t}^i = a^{1/2}\left[-\bar{p}\,\mathbf{I} + \mathbf{T}'\right]\mathbf{d}^i - \mathbf{m}^\alpha(\mathbf{d}_{3,\alpha}\bullet\mathbf{d}^i)\ , \qquad (4.25.4)$$

LINEARIZED EQUATIONS

Use the general values for m, y^3, y^{33}

$$\mathbf{x} = \mathbf{X} + \mathbf{u}\ ,\quad \mathbf{d}_i = \mathbf{D}_i + \boldsymbol{\delta}_i\ ,\quad \boldsymbol{\delta}_\alpha = \mathbf{u},_\alpha\ , \qquad (4.13.1)$$

$$\mathbf{a}_\alpha = \mathbf{D}_\alpha + \boldsymbol{\delta}_\alpha\ ,\quad \mathbf{a}_3 = \mathbf{A}_3 - \left[\mathbf{A}_3\bullet\boldsymbol{\delta}_\alpha\right]\mathbf{A}^\alpha\ , \qquad (4.13.5)$$

$$\tilde{\mathbf{E}} = \frac{1}{2}\left[(\boldsymbol{\delta}_i\otimes\mathbf{D}^i) + (\mathbf{D}^i\otimes\boldsymbol{\delta}_i)\right]\ ,\quad \tilde{\boldsymbol{\beta}}_\alpha = \boldsymbol{\delta}_{3,\alpha} - (\mathbf{D}^i\bullet\mathbf{D}_{3,\alpha})\,\boldsymbol{\delta}_i\ , \qquad (4.13.5)$$

$$\tilde{\tilde{\mathbf{E}}} = \tilde{\mathbf{E}} + \frac{1}{2}(\tilde{\boldsymbol{\beta}}_\alpha\otimes\mathbf{H}^\alpha + \mathbf{H}^\alpha\otimes\tilde{\boldsymbol{\beta}}_\alpha)\ , \qquad (4.13.17)$$

$$\tilde{\mathbf{b}}_b = \tilde{\mathbf{b}}^*\ ,\quad m\,\tilde{b}_b^3 = (\rho_0^*\,D^{1/2}\,H)\,\tilde{\mathbf{b}}^*\left[\frac{H^2}{12}(\mathbf{D}_{3,\alpha}\bullet\mathbf{D}^\alpha)\right]\ , \qquad (4.19.14)$$

$$G^{1/2}\alpha(\tfrac{H}{2})\,\mathbf{N}^* = \left[\mathbf{X} + \tfrac{H}{2}\mathbf{D}_3\right],_1\times\left[\mathbf{X} + \tfrac{H}{2}\mathbf{D}_3\right],_2 \quad \text{on } \theta^3 = \frac{H}{2}\ , \qquad (4.3.12)$$

$$G^{1/2}\alpha(-\tfrac{H}{2})\,\mathbf{N}^* = -\left[\mathbf{X} - \tfrac{H}{2}\mathbf{D}_3\right],_1\times\left[\mathbf{X} - \tfrac{H}{2}\mathbf{D}_3\right],_2 \quad \text{on } \theta^3 = -\frac{H}{2}\ , \qquad (4.3.13)$$

$$\mathbf{N}^*\bullet\mathbf{N}^* = 1\ , \qquad (4.3.12)$$

$$m\,\tilde{\mathbf{b}}_c = \left[G^{1/2}\alpha(\tfrac{H}{2})\,\tilde{\mathbf{t}}^*\right]\Big|_{\theta^3 = H/2} + \left[G^{1/2}\alpha(-\tfrac{H}{2})\,\tilde{\mathbf{t}}^*\right]\Big|_{\theta^3 = -H/2}\ , \qquad (4.3.15)$$

$$m\,\tilde{\mathbf{b}}_c^3 = \frac{H}{2}\left[G^{1/2}\alpha(\tfrac{H}{2})\,\tilde{\mathbf{t}}^*\right]\Big|_{\theta^3 = H/2} - \frac{H}{2}\left[G^{1/2}\alpha(-\tfrac{H}{2})\,\tilde{\mathbf{t}}^*\right]\Big|_{\theta^3 = -H/2}\ , \qquad (4.3.36)$$

$$\mathbf{b} = \tilde{\mathbf{b}} = \tilde{\mathbf{b}}_b + \tilde{\mathbf{b}}_c\ ,\quad \mathbf{b}^3 = \tilde{\mathbf{b}}^3 = \tilde{\mathbf{b}}_b^3 + \tilde{\mathbf{b}}_c^3\ , \quad (4.3.28),(4.3.42),(4.13.3)$$

$$\mathbf{t}^i = \tilde{\mathbf{t}}^i\ ,\quad \mathbf{m}^\alpha = \tilde{\mathbf{m}}^\alpha\ , \qquad (4.13.4)$$

$$\rho = \rho_0\left[1 - \mathbf{A}^\alpha\bullet\boldsymbol{\delta}_\alpha\right]\ , \qquad (4.13.6)$$

$$m(\ddot{\mathbf{u}} + y^3\,\ddot{\boldsymbol{\delta}}_3) = m\,\tilde{\mathbf{b}} + \tilde{\mathbf{t}}^\alpha,_\alpha\ , \qquad (4.13.16)$$

$$m(y^3\,\ddot{\mathbf{u}} + y^{33}\,\ddot{\boldsymbol{\delta}}_3) = m\,\tilde{\mathbf{b}}^3 - \tilde{\mathbf{t}}^3 + \tilde{\mathbf{m}}^\alpha,_\alpha\ , \qquad (4.13.16)$$

$$\tilde{\mathbf{m}}^\alpha = \left[A^{1/2}\bar{H}\ \mathbf{K}^*\bullet\tilde{\tilde{\mathbf{E}}}\ \right]\mathbf{H}^\alpha + A^{1/2}\bar{H}\ \mathbf{K}^{\alpha\beta}\,\tilde{\boldsymbol{\beta}}_\beta\ , \qquad (4.13.17)$$

$$\tilde{\mathbf{t}}^i = \left[A^{1/2}\bar{H}\ \mathbf{K}^*\bullet\tilde{\tilde{\mathbf{E}}}\ \right]\mathbf{D}^i - \tilde{\mathbf{m}}^\alpha(\mathbf{D}_{3,\alpha}\bullet\mathbf{D}^i)\ , \qquad (4.13.17)$$

4.28 Generalized membranes and membrane-like shells

A shell-like structure can be modeled as a membrane if the influence of the bending moments are negligible relative to the influence of the resultant forces developed in the structure when it is loaded. Within the context of the theory of a Cosserat surface this means that the influence of $\mathbf{m}^\alpha$ is negligible relative to that of $\mathbf{t}^i$ in the equations of motion (4.4.35). From a constitutive point of view this suggests that the strain energy function is independent of $\boldsymbol{\beta}_\alpha$. Thus, Σ can be specified by the three-dimensional strain energy function Σ^* of the material used to construct the shell by the expression

$$\Sigma = \Sigma^*(\mathbf{C}) \ . \tag{4.28.1}$$

It then follows from (4.8.12) and (4.8.14) that

$$a^{1/2}\,\mathbf{T} = 2\,m\,\mathbf{F}\frac{\partial\Sigma^*}{\partial\mathbf{C}}\mathbf{F}^T \ , \ \mathbf{t}^i = a^{1/2}\mathbf{T}\,\mathbf{d}^i = 2\,m\,\mathbf{F}\frac{\partial\Sigma^*}{\partial\mathbf{C}}\mathbf{D}^i \ , \ \mathbf{m}^\alpha = 0 \ . \tag{4.28.2}$$

Moreover, the equation of linear momentum $(4.4.35)_3$ remains unchanged but the equation of director momentum $(4.4.35)_4$ is simplified such that

$$m\,(\dot{\mathbf{v}} + y^3\,\dot{\mathbf{w}}_3) = m\,\mathbf{b} + \mathbf{t}^\alpha{}_{,\alpha} \ , \ m\,(y^3\,\dot{\mathbf{v}} + y^{33}\,\dot{\mathbf{w}}_3) = m\,\mathbf{b}^3 - \mathbf{t}^3 \ . \tag{4.28.3}$$

Also, for this theory, the couple $\mathbf{m}^3$ in (4.4.8) and (4.6.1) vanishes so the rate of work of loads applied to the boundary ∂P of the shell reduces to

$$\int_{\partial P} \mathbf{t} \bullet \mathbf{v}\, ds \ , \ \mathbf{t} = a^{-1/2}\,(\mathbf{t}^\alpha \otimes \mathbf{d}_\alpha)\,\mathbf{n} \ . \tag{4.28.4}$$

Consequently, the boundary conditions for this membrane require specification of the components the velocity $\mathbf{v}$ or of the resultant force $\mathbf{t}$, as described in (4.10.4)-(4.10.6).

This membrane is called a *generalized membrane* because the equations of motion (4.28.3) require the determination of both the position vector $\mathbf{x}$ and the director $\mathbf{d}_3$. In particular, this theory includes the effects of transverse shear deformation and normal extension. However, if the membrane is loaded so that it remains a plate and so that the director remains in the direction $\mathbf{a}_3$ (which is orthogonal to the plane of this plate), then these equations can be related to the two-dimensional theory of generalized plane stress (Sokolnikoff, 1956). A simpler membrane theory that omits the director $\mathbf{d}_3$ will be discussed in the next section.

Since the strain energy function (4.28.1) depends only on the deformation tensor $\mathbf{C}$, it follows that it does not satisfy the restriction $(4.11.9)_2$ when the shell has a general geometry in its reference configuration ($\mathbf{H}^\alpha \neq 0$). Consequently, it is of interest to consider what will be called a *membrane-like shell* which is characterized by the strain energy function

$$\Sigma = \Sigma^*(\bar{\mathbf{C}}) \ , \tag{4.28.5}$$

where $\bar{\mathbf{C}}$ is given by (4.11.19). It then follows from (4.25.1) and (4.11.28) that

$$\bar{\mathbf{F}} = \mathbf{F}\,(\mathbf{I} + \boldsymbol{\beta}_\alpha \otimes \mathbf{H}^\alpha) \ , \ \bar{\mathbf{C}} = \bar{\mathbf{F}}^T\bar{\mathbf{F}} \ ,$$

$$a^{1/2}\,\mathbf{T} = 2\,m\,\bar{\mathbf{F}}\,\frac{\partial\Sigma^*(\bar{\mathbf{C}})}{\partial\bar{\mathbf{C}}}\,\bar{\mathbf{F}}^T \ , \ \mathbf{m}^\alpha = 2\,m\,\bar{\mathbf{F}}\,\frac{\partial\Sigma^*(\bar{\mathbf{C}})}{\partial\bar{\mathbf{C}}}\,\mathbf{H}^\alpha \ ,$$

$$\mathbf{t}^i = 2\,m\,\bar{\mathbf{F}}\,\frac{\partial\Sigma^*(\bar{\mathbf{C}})}{\partial\bar{\mathbf{C}}}\,\bar{\mathbf{F}}^T\,\mathbf{d}^i - \mathbf{m}^\alpha\,(\mathbf{d}_{3,\alpha} \bullet \mathbf{d}^i) \ , \tag{4.28.6}$$

and that the general equations of motion (4.4.35) apply. Consequently, this membrane-like shell has the advantage over the generalized membrane discussed previously that the equations are consistent with exact solutions for all homogeneous deformations. Moreover, it follows that the director couple $\mathbf{m}^3$ does not vanish when $\mathbf{H}^\alpha$ is nonzero. This means that boundary conditions associated with this membrane-like shell will depend on the specific value of $\mathbf{H}^\alpha$ and on the type of loading.

4.29 Simple membranes

In contrast with the theory of a generalized membrane discussed in section 4.28, the theory of a simple membrane characterizes a surface that has no thickness. From the point of view of the Cosserat theory the kinematics of such a surface are determined by the position vector $\mathbf{x}$ in (4.2.1) and there is no need to introduce the director vector $\mathbf{d}_3$ as an independent kinematic quantity. In particular, the tangent vectors $\mathbf{d}_\alpha$ are defined by (4.2.4)

$$\mathbf{d}_\alpha = \mathbf{x},_\alpha \ , \tag{4.29.1}$$

and $\mathbf{d}_3$ is taken to be the unit normal vector $\mathbf{a}_3$ which is defined in terms of the position vector $\mathbf{x}$ of points on the surface such that

$$\mathbf{d}_3 = \mathbf{a}_3 = \frac{\mathbf{d}_1 \times \mathbf{d}_2}{|\mathbf{d}_1 \times \mathbf{d}_2|} \ , \quad d^{1/2} = a^{1/2} = \mathbf{d}_1 \times \mathbf{d}_2 \bullet \mathbf{d}_3 \ . \tag{4.29.2}$$

For such a theory the global forms of the conservation of mass and the balance of linear momentum can be written as

$$\frac{d}{dt}\int_P \rho \, d\sigma = 0 \ , \quad \frac{d}{dt}\int_P \rho \, \mathbf{v} \, d\sigma = \int_P \rho \, \mathbf{b} \, d\sigma + \int_{\partial P} \mathbf{t} \, ds \ , \tag{4.29.3}$$

and the balance of angular momentum is given by

$$\frac{d}{dt}\int_P \rho \, (\mathbf{x} \times \mathbf{v}) \, d\sigma = \int_P (\mathbf{x} \times \rho \, \mathbf{b}) \, d\sigma + \int_{\partial P} (\mathbf{x} \times \mathbf{t}) \, ds \ , \tag{4.29.4}$$

where the quantities $\{d\sigma \ , \ ds\}$ and $\{\rho, \mathbf{b} \ , \ \mathbf{t}\}$ have the same meanings as for the more general Cosserat theory. Also, there is no need to introduce a director momentum equation, and the integral form of the mechanical power $\mathcal{P}$ is given by

$$\int_P \mathcal{P} \, d\sigma = \mathcal{W} - \dot{\mathcal{K}} \ , \quad \mathcal{W} = \int_P \rho \, (\mathbf{b} \bullet \mathbf{v}) \, d\sigma + \int_{\partial P} (\mathbf{t} \bullet \mathbf{v}) \, ds \ ,$$

$$\mathcal{K} = \int_P \frac{1}{2} \rho \, (\mathbf{v} \bullet \mathbf{v}) \, d\sigma \ , \tag{4.29.5}$$

where $\mathcal{W}$ is the rate of work of the assigned field $\mathbf{b}$ and the resultant contact force $\mathbf{t}$, and $\mathcal{K}$ is the kinetic energy.

These equations are consistent with those that would be obtained by neglecting all terms associated with the director in the balance laws (4.4.6), (4.4.7) and in the definition (4.6.1). Moreover, the kinematic and kinetic quantities transform under SRBM by expressions similar to those discussed in section 4.5, with all terms associated with the director being neglected.

Using standard arguments in continuum mechanics it can be shown that the contact force $\mathbf{t}$ is a linear function of the unit outward normal $\mathbf{n}$ to the boundary curve ∂P so that $(4.4.8)_1$, $(4.4.9)_1$ and $(4.4.10)_{1,2,3}$ hold and

$$\mathbf{t} = \mathbf{N} \, \mathbf{n} \ , \quad \mathbf{N} = a^{-1/2} \, \mathbf{t}^\alpha \otimes \mathbf{d}_\alpha \ . \tag{4.29.6}$$

Next, with the help of the transport theorem (4.4.16) and the divergence theorem (4.4.25), and with the additional result (4.4.34), it can be shown that the local forms of the balance laws (4.29.3) become

$$m = \rho\, a^{1/2} = \rho_0\, A^{1/2} = m(\theta^\alpha) \text{ or } \dot{\rho} + \rho\, \mathbf{v}_{,\alpha} \bullet \mathbf{a}^\alpha = 0 \ ,$$

$$m\, \dot{\mathbf{v}} = m\, \mathbf{b} + \mathbf{t}^\alpha{}_{,\alpha} \ , \tag{4.29.7}$$

the local form of the balance of angular momentum (4.29.4) becomes

$$\mathbf{T} = a^{-1/2}\, \mathbf{t}^\alpha \otimes \mathbf{d}_\alpha = \mathbf{T}^T \ , \tag{4.29.8}$$

and the local form of (4.29.5) yields

$$\mathcal{P} = a^{-1/2}\, \mathbf{t}^\alpha \bullet \mathbf{w}_\alpha = \mathbf{T} \bullet \mathbf{D} \ . \tag{4.29.9}$$

Also, for this theory the deformation quantities $\{\mathbf{F}, \mathbf{L}, \mathbf{D}, \mathbf{W}, \mathbf{C}\}$ are given by (4.2.9), (4.6.7), (4.6.11) and (4.8.8), and the reference value of $\mathbf{D}_3$ is taken to be the unit normal $\mathbf{A}_3$ to the reference surface so that

$$\mathbf{D}_3 = \mathbf{A}_3 \ , \quad \mathbf{D}^3 = \mathbf{A}_3 \ . \tag{4.29.10}$$

It then follows that for a nonlinear elastic anisotropic membrane the strain energy function is taken in the form

$$\Sigma = \hat{\Sigma}\, (\mathbf{C}; \theta^\alpha) \ , \tag{4.29.11}$$

and the assumptions of section 4.8, which include

$$\rho\, \dot{\Sigma} = \mathcal{P} = \mathbf{T} \bullet \mathbf{D} \ , \tag{4.29.12}$$

yield the results that

$$a^{1/2}\, \mathbf{T} = 2\, m\, \mathbf{F} \frac{\partial \Sigma}{\partial \mathbf{C}} \mathbf{F}^T \ , \quad \mathbf{t}^\alpha = 2\, m\, \mathbf{F} \frac{\partial \Sigma}{\partial \mathbf{C}} \mathbf{D}^\alpha \ . \tag{4.29.13}$$

However, the components of $\mathbf{C}$ relative to the basis $\mathbf{D}_i$ are given by

$$\mathbf{C} \bullet (\mathbf{D}_i \otimes \mathbf{D}_j) = d_{ij} = \mathbf{d}_i \bullet \mathbf{d}_j \ ,$$

$$d_{\alpha\beta} = \mathbf{d}_\alpha \bullet \mathbf{d}_\beta \ , \quad d_{\alpha 3} = 0 \ , \quad d_{33} = 1 \ , \tag{4.29.14}$$

so that the strain energy Σ can be written in the alternative form

$$\Sigma = \tilde{\Sigma}\, (d_{\alpha\beta}\, ; \theta^\alpha\,) \ , \tag{4.29.15}$$

which can be used to deduce the results that

$$\frac{\partial \hat{\Sigma}}{\partial \mathbf{C}} = \frac{\partial \tilde{\Sigma}}{\partial d_{\alpha\beta}} (\mathbf{D}_\alpha \otimes \mathbf{D}_\beta) \ , \quad a^{1/2}\, \mathbf{T} = 2\, m\, \frac{\partial \tilde{\Sigma}}{\partial d_{\alpha\beta}} (\mathbf{d}_\alpha \otimes \mathbf{d}_\beta) \ . \tag{4.29.16}$$

Consequently, the constitutive equation for $\mathbf{t}^\alpha$ is given by

$$\mathbf{t}^\alpha = 2\, m \frac{\partial \tilde{\Sigma}}{\partial d_{\alpha\beta}}\, \mathbf{d}_\beta \ . \tag{4.29.17}$$

Since the strain energy in (4.29.15) is influenced only by the strain of the surface, the above theory has features of a plane strain theory. However, it is perhaps more accurate to consider it to be a generalized plane strain theory because the membrane surface is not restricted to be planar. Further in this regard, it is important to emphasize that the theory is merely a model of a real structure that has small, but finite thickness, and has negligible resistance to bending. It follows that the actual thickness of this real structure can change when it is subjected only to resultant forces in its plane. This means that if surface tractions are not applied to the major surfaces of the membrane which maintain its thickness, then the in-plane response of the membrane will not be accurately characterized by plane strain constitutive equations. In fact, the response may be more accurately modeled by generalized plane-stress-type constitutive equations.

For example, ignoring the influence of bending, Table 3.12.1 and equations (3.12.15) and (4.12.6) can be used to express the small-strain strain energy of an isotropic elastic plate in the form

$$m\,\Sigma = A^{1/2}\,\bar{H}\,\mu^*\,\Big[\Big\{\frac{\nu^*}{1-2\nu^*}\Big\}(\mathbf{E}\bullet\mathbf{I})^2 + \mathbf{E}\bullet\mathbf{E}\Big] \ ,$$

$$m\frac{\partial\Sigma}{\partial\mathbf{C}} = A^{1/2}\,\bar{H}\,\mu^*\,\Big[\Big\{\frac{\nu^*}{1-2\nu^*}\Big\}(\mathbf{E}\bullet\mathbf{I})\,\mathbf{I} + \mathbf{E}\Big] \ . \tag{4.29.18}$$

Now, one of the conditions of generalized plane stress requires

$$m\frac{\partial\Sigma}{\partial\mathbf{C}}\bullet(\mathbf{D}_3\otimes\mathbf{D}_3) = A^{1/2}\,\bar{H}\,\mu^*\,\Big[\Big\{\frac{\nu^*}{1-2\nu^*}\Big\}(\mathbf{E}\bullet\mathbf{I}) + \mathbf{E}\bullet(\mathbf{D}_3\otimes\mathbf{D}_3)\Big] = 0 \ . \tag{4.29.19}$$

Thus, by using the restriction (4.29.10) and taking

$$\mathbf{I} = \mathbf{D}_i\otimes\mathbf{D}^i = \mathbf{D}_\alpha\otimes\mathbf{D}^\alpha + \mathbf{D}_3\otimes\mathbf{D}_3 \ , \quad \mathbf{D}_\alpha\otimes\mathbf{D}^\alpha = \mathbf{D}^\alpha\otimes\mathbf{D}_\alpha \ , \tag{4.29.20}$$

it can be shown that (4.29.19) yields the results

$$\mathbf{E}\bullet\mathbf{I} = \mathbf{E}\bullet(\mathbf{D}^\alpha\otimes\mathbf{D}_\alpha) + \mathbf{E}\bullet(\mathbf{D}_3\otimes\mathbf{D}_3) \ ,$$

$$\Big\{\frac{\nu^*}{1-2\nu^*}\Big\}(\mathbf{E}\bullet\mathbf{I}) = \Big\{\frac{\bar{\nu}^*}{1-2\bar{\nu}^*}\Big\}\,\big[\mathbf{E}\bullet(\mathbf{D}^\alpha\otimes\mathbf{D}_\alpha)\big] \ , \tag{4.29.21}$$

where the constant $\bar{\nu}^*$ is defined by

$$\bar{\nu}^* = \frac{\nu^*}{1+\nu^*} \ . \tag{4.29.22}$$

Next, using the expression (4.29.18)-(4.29.22) and the fact that for membranes the strain $\mathbf{E}$ is essentially two-dimensional (4.29.14), it follows that the constitutive equations for plane-strain deformation of a membrane can be written in the forms

$$m\,\Sigma = A^{1/2}\,\bar{H}\,\mu^*\,\Big[\Big\{\frac{\nu^*}{1-2\nu^*}\Big\}\big\{\mathbf{E}\bullet(\mathbf{D}_\alpha\otimes\mathbf{D}^\alpha)\big\}^2$$
$$+ \big\{(\mathbf{D}_\alpha\otimes\mathbf{D}^\alpha)\,\mathbf{E}\,(\mathbf{D}^\beta\otimes\mathbf{D}_\beta)\big\}\bullet\mathbf{E}\Big] \ ,$$

$$m\frac{\partial\Sigma}{\partial\mathbf{C}} = A^{1/2}\,\bar{H}\,\mu^*\,\Big[\Big\{\frac{\nu^*}{1-2\nu^*}\Big\}\big\{\mathbf{E}\bullet(\mathbf{D}_\alpha\otimes\mathbf{D}^\alpha)\big\}(\mathbf{D}_\beta\otimes\mathbf{D}^\beta)$$
$$+ \big\{(\mathbf{D}^\alpha\otimes\mathbf{D}_\alpha)\,\mathbf{E}\,(\mathbf{D}_\beta\otimes\mathbf{D}^\beta)\big\}\Big] \ ,$$

$$\mathbf{t}^\alpha = 2\,A^{1/2}\,\bar{H}\,\mu^*\,\Big[\Big\{\frac{\nu^*}{1-2\nu^*}\Big\}\big\{\mathbf{E}\bullet(\mathbf{D}_\sigma\otimes\mathbf{D}^\sigma)\big\}\big\{\mathbf{D}^\alpha\bullet\mathbf{D}^\beta\big\}\mathbf{d}_\beta$$
$$+ \big\{\mathbf{E}\bullet(\mathbf{D}^\alpha\otimes\mathbf{D}^\beta)\big\}\,\mathbf{d}_\beta\Big] \ , \tag{4.29.23}$$

whereas those for generalized plane-stress are similar to (4.29.23), but with the Poisson's ratio ν^* replaced by the value $\bar{\nu}^*$ in (4.29.22) so that

$$m\,\Sigma = A^{1/2}\,\bar{H}\,\mu^*\,\Big[\Big\{\frac{\bar{\nu}^*}{1-2\bar{\nu}^*}\Big\}\Big\{\mathbf{E}\bullet(\mathbf{D}_\alpha\otimes\mathbf{D}^\alpha)\Big\}^2 + \Big\{(\mathbf{D}_\alpha\otimes\mathbf{D}^\alpha)\,\mathbf{E}\,(\mathbf{D}^\beta\otimes\mathbf{D}_\beta)\Big\}\bullet\mathbf{E}\Big]\ ,$$

$$m\frac{\partial\Sigma}{\partial\mathbf{C}} = A^{1/2}\,\bar{H}\,\mu^*\,\Big[\Big\{\frac{\bar{\nu}^*}{1-2\bar{\nu}^*}\Big\}\Big\{\mathbf{E}\bullet(\mathbf{D}_\alpha\otimes\mathbf{D}^\alpha)\Big\}(\mathbf{D}_\beta\otimes\mathbf{D}^\beta) + \Big\{(\mathbf{D}^\alpha\otimes\mathbf{D}_\alpha)\,\mathbf{E}\,(\mathbf{D}_\beta\otimes\mathbf{D}^\beta)\Big\}\Big]\ ,$$

$$\mathbf{t}^\alpha = 2\,A^{1/2}\,\bar{H}\;\mu^*\,\Big[\Big\{\frac{\bar{\nu}^*}{1-2\bar{\nu}^*}\Big\}\Big\{\mathbf{E}\bullet(\mathbf{D}_\sigma\otimes\mathbf{D}^\sigma)\Big\}\Big\{\mathbf{D}^\alpha\bullet\mathbf{D}^\beta\Big\}\mathbf{d}_\beta + \Big\{\mathbf{E}\bullet(\mathbf{D}^\alpha\otimes\mathbf{D}^\beta)\Big\}\,\mathbf{d}_\beta\,\Big]\ , \tag{4.29.24}$$

Using the constitutive equations (4.19.23) or (4.19.24), the equation of linear momentum $(4.29.7)_3$ determines the position vector $\mathbf{x}$ subject to initial conditions and boundary conditions on either $\mathbf{x}$ or the resultant force $\mathbf{t}$ in $(4.29.6)_1$. Moreover, if the constitutive response cannot be modeled as either generalized plane strain or plane stress, then it may be most convenient to use the theory of a generalized membrane of section 4.28 for which the director $\mathbf{d}_3$ is determined by the director momentum equation $(4.28.3)_2$, so that it is compatible with the specified loading on the major surfaces of the structure.

Before closing this section, it is of interest to note that special attention must also be focused on the constitutive equation for the more general theory that admits the director, but introduces the constraint (4.9.3). In particular, for this theory the director $\mathbf{d}_3$ is forced to have a constant component in the normal direction. Consequently, if generalized plane stress type constitutive response is anticipated for this constrained theory, then it is necessary to modify the constitutive equations along lines similar to those used to obtain the results (4.29.24).

4.30 Expansion of an incompressible isotropic spherical shell

It is well known that a spherical shell can become unstable as the pressure inside it increases and the shell expands. An extensive discussion of related bifurcation and stability problems can be found in (Antman, 1995). Here, it is of interest to reconsider this problem because it provides an example of the nonlinear theory of shells with a constraint. For simplicity, attention is focused on expansion of an incompressible isotropic spherical shell. In its reference configuration the shell has internal radius R_1, external radius R_2 and thickness H. In its present configuration the shell has internal radius r_1, external radius r_2, and it is subjected to an internal pressure p_1 and no external pressure. Also, the influence of body force $\mathbf{b}^*$ is neglected. It then follows that the expressions (4.22.1)-(4.22.4) hold, and that radius r of the middle surface and the thickness h of the shell in its present configuration are given by

$$r = \frac{1}{2}(r_1 + r_2) \ , \quad h = r_2 - r_1 \ . \tag{4.30.1}$$

Moreover, the position vector $\mathbf{x}$, directors $\mathbf{d}_i$ and reciprocal vectors $\mathbf{d}^i$ can be expressed in terms of a spherical polar coordinate system such that

$$\mathbf{x} = r\,\mathbf{e}_r(\theta,\phi) \ , \quad \mathbf{d}_1 = r\,\mathbf{e}_\theta \ , \quad \mathbf{d}^1 = \frac{1}{r}\,\mathbf{e}_\theta \ , \quad \mathbf{d}_2 = r\sin\theta\,\mathbf{e}_\phi \ , \quad \mathbf{d}^2 = \frac{1}{r\sin\theta}\,\mathbf{e}_\phi \ ,$$

$$\mathbf{d}_3 = \frac{h}{H}\,\mathbf{e}_r \ , \quad \mathbf{d}^3 = \frac{H}{h}\,\mathbf{e}_r \ . \tag{4.30.2}$$

Next, using the equations summarized in section 4.27 it can be shown that

$$\bar{\mathbf{F}} = \frac{r}{R}(1+\beta)\,(\mathbf{e}_\theta\otimes\mathbf{e}_\theta + \mathbf{e}_\phi\otimes\mathbf{e}_\phi) + \frac{h}{H}(\mathbf{e}_r\otimes\mathbf{e}_r) \ ,$$

$$\beta = \frac{H^3}{12V}\left[\frac{Rh}{rH} - 1\right] \ , \quad V = \frac{R_2^3 - R_1^3}{3} \ . \tag{4.30.3}$$

However, due to the incompressibility constraint (4.25.5) it follows that

$$\bar{J} = \det \bar{\mathbf{F}} = \left\{\frac{r}{R}(1+\beta)\right\}^2 \frac{h}{H} = 1 \ , \tag{4.30.4}$$

so it is convenient to define the stretch λ by

$$\lambda = \frac{r}{R}(1+\beta) \ , \quad \frac{h}{H} = \frac{1}{\lambda^2} \ . \tag{4.30.5}$$

Then, the expression for $\bar{\mathbf{F}}$ reduces to

$$\bar{\mathbf{F}} = \lambda\,(\mathbf{e}_\theta\otimes\mathbf{e}_\theta + \mathbf{e}_\phi\otimes\mathbf{e}_\phi) + \frac{1}{\lambda^2}\,(\mathbf{e}_r\otimes\mathbf{e}_r) \ , \tag{4.30.6}$$

and $(4.30.3)_2$ can be used to determine that

$$\frac{r}{R} = \frac{\lambda - \dfrac{H^3}{12V\lambda^2}}{1 - \dfrac{H^3}{12V}} \ . \tag{4.30.7}$$

Next, the kinematical expressions summarized in section 4.27 can be used to deduce that

$$\boldsymbol{\beta}_1 = \left[\frac{R}{r\lambda^2} - 1\right]\mathbf{e}_\theta \ , \quad \boldsymbol{\beta}_2 = \left[\frac{R}{r\lambda^2} - 1\right]\sin\theta\,\mathbf{e}_\phi \ ,$$

$$A^{1/2}\bar{H} = V\sin\theta \ , \quad \mathbf{H}^1 = \frac{H^3}{12V}\mathbf{e}_\theta \ , \quad \mathbf{H}^2 = \frac{H^3}{12V\sin\theta}\mathbf{e}_\theta \ ,$$

$$\boldsymbol{\beta}_\alpha\otimes\mathbf{H}^\alpha = \beta\,(\mathbf{e}_\theta\otimes\mathbf{e}_\theta + \mathbf{e}_\phi\otimes\mathbf{e}_\phi) \ , \quad \beta = \frac{H^3}{12V}\left[\frac{R}{r\lambda^2} - 1\right] \ ,$$

$$\bar{\mathbf{B}} - \frac{1}{3}(\bar{\mathbf{B}}\cdot\mathbf{I})\,\mathbf{I} = \frac{1}{3}\left[\lambda^2 - \frac{1}{\lambda^4}\right]\left[(\mathbf{e}_\theta\otimes\mathbf{e}_\theta + \mathbf{e}_\phi\otimes\mathbf{e}_\phi) - 2\,(\mathbf{e}_r\otimes\mathbf{e}_r)\right] \ ,$$

$$\bar{\mathbf{B}}^2 - \frac{1}{3}(\bar{\mathbf{B}}^2\cdot\mathbf{I})\,\mathbf{I} = \frac{1}{3}\left[\lambda^4 - \frac{1}{\lambda^8}\right]\left[(\mathbf{e}_\theta\otimes\mathbf{e}_\theta + \mathbf{e}_\phi\otimes\mathbf{e}_\phi) - 2\,(\mathbf{e}_r\otimes\mathbf{e}_r)\right] \ . \tag{4.30.8}$$

Consequently, the kinematics of the present configuration are determined by the single parameter λ.

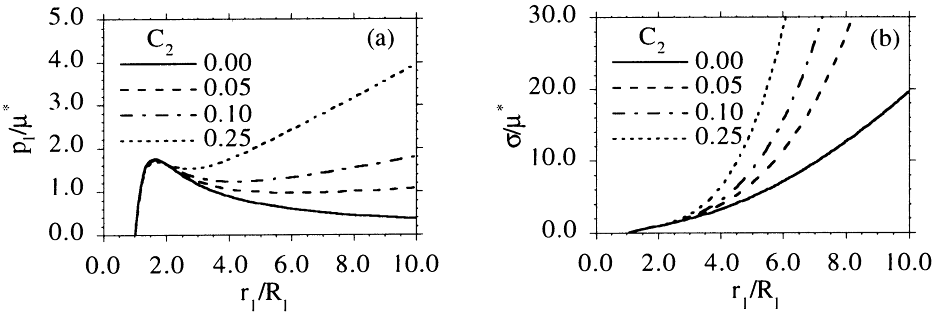

Fig. 4.30.1 Expansion of a spherical shell with R/H = 1.0.

In view of the discussion at the end of section 4.25 related to the incompressibility constraint, a restricted form of the strain energy functions is used in which only its distortional part is specified. In particular, (4.25.10) and (4.25.11) are combined to write

$$m\,\Sigma^*(\bar{\mathbf{C}}) = \frac{1}{2} A^{1/2}\,\bar{H}\,\mu^*\big[(1-4C_2)\,(\bar{\alpha}_1 - 3) + C_2\,(\bar{\alpha}_2 - 3)\big] \ , \tag{4.30.9}$$

where μ^* is the zero-stress shear modulus and C_2 is a material constant. Also, ν^* is set equal to 1/2 in the expressions for $\mathbf{K}^{\alpha\beta}$. It can then be shown that

$$a^{1/2}\,\mathbf{T} = \tilde{\gamma}\sin\theta\,\mathbf{I} + a^{1/2}\,\hat{\mathbf{T}} \ , \quad \bar{\mathbf{m}}^\alpha = \tilde{\gamma}\sin\theta\,\bar{\mathbf{F}}^{-T}\mathbf{H}^\alpha + \hat{\mathbf{m}}^\alpha \ ,$$

$$a^{1/2}\,\hat{\mathbf{T}} = f(\lambda)\,\sin\theta\,\big[(\mathbf{e}_\theta\otimes\mathbf{e}_\theta + \mathbf{e}_\phi\otimes\mathbf{e}_\phi) - 2\,(\mathbf{e}_r\otimes\mathbf{e}_r)\big] \ ,$$

$$f(\lambda) = \frac{\mu^* V}{3}\,\Big[(1-4C_2)\Big\{\lambda^2 - \frac{1}{\lambda^4}\Big\} + 2C_2\,\Big\{\lambda^4 - \frac{1}{\lambda^8}\Big\}\Big] \ ,$$

$$\hat{\mathbf{m}}^\alpha = a^{1/2}\,\hat{\mathbf{T}}\,\bar{\mathbf{F}}^{-T}\mathbf{H}^\alpha + A^{1/2}\bar{H}\,\mathbf{F}^{-T}\mathbf{K}^{\alpha\beta}\,\boldsymbol{\beta}_\beta \ ,$$

$$A^{1/2}\bar{H}\,\mathbf{F}^{-T}\mathbf{K}^{1\beta}\,\boldsymbol{\beta}_\beta = \hat{M}_1\,\sin\theta\,\mathbf{e}_\theta \ , \quad A^{1/2}\bar{H}\,\mathbf{F}^{-T}\mathbf{K}^{2\beta}\,\boldsymbol{\beta}_\beta = \hat{M}_1\,\mathbf{e}_\phi \ ,$$

$$\bar{\mathbf{m}}^1 = \bar{M}\,\sin\theta\,\mathbf{e}_\theta \ , \quad \bar{\mathbf{m}}^1 = \bar{M}\,\mathbf{e}_\phi \ ,$$

$$a^{1/2}\,\hat{\mathbf{T}}\,\bar{\mathbf{F}}^{-T}\mathbf{H}^1 = \hat{M}_2\,\sin\theta\,\mathbf{e}_\theta \ , \quad a^{1/2}\,\hat{\mathbf{T}}\,\bar{\mathbf{F}}^{-T}\mathbf{H}^2 = \hat{M}_2\,\mathbf{e}_\phi \ ,$$

$$\hat{M}_1 = \frac{\mu^* V H^2}{2rR}\Big[\frac{R}{r\lambda^2} - 1\Big] \ , \quad \hat{M}_2 = \frac{H^3 f(\lambda)}{12V\lambda} \ , \quad \bar{M} = \frac{\tilde{\gamma}H^3}{12V\lambda} \ ,$$

$$\mathbf{m}^1 = M\,\sin\theta\,\mathbf{e}_\theta \ , \quad \mathbf{m}^2 = M\,\mathbf{e}_\phi \ , \quad M = \hat{M} + \bar{M} \ , \quad \hat{M} = \hat{M}_1 + \hat{M}_2$$

$$\mathbf{t}^1 = T\,\sin\theta\,\mathbf{e}_\theta \ , \quad \mathbf{t}^2 = T\,\mathbf{e}_\phi \ , \quad \mathbf{t}^3 = T_3\,\sin\theta\,\mathbf{e}_r \ ,$$

$$T = \frac{1}{r}\Big[\tilde{\gamma}\Big\{1 - \frac{H^3}{12V\lambda^3}\Big\} + f(\lambda) - \frac{\hat{M}}{\lambda^2}\Big] , \quad T_3 = \lambda^2\,\big[\tilde{\gamma} - 2\,f(\lambda)\big] \ , \tag{4.30.10}$$

where T is the tension force, and M is the moment (both per unit radian) acting on any edge of the shell which is a great circle. Also, the Lagrange multiplier γ associated with the constraint response in (4.25.5) has been expressed in the form

$$\gamma = \tilde{\gamma}\,\sin\theta \ . \tag{4.30.11}$$

Furthermore, the assigned fields due to the internal pressure become

$$\mathrm{m}\,\mathbf{b} = \mathrm{m}\,\mathbf{b}_c = r^2\, p_l \left[1 - \frac{h}{2r}\right]^2 \sin\theta\, \mathbf{e}_r \ ,$$

$$\mathrm{m}\,\mathbf{b}^3 = \mathrm{m}\,\mathbf{b}_c^3 = -\frac{H}{2}\left[r^2\, p_l\right]\left[1 - \frac{h}{2r}\right]^2 \sin\theta\, \mathbf{e}_r \ . \qquad (4.30.12)$$

Now, the equations of equilibrium yield two scalar equations of the forms

$$r^2\, p_l \left[1 - \frac{h}{2r}\right]^2 - 2\,T = 0 \ , \quad \frac{H}{2}\left[r^2\, p_l\right]\left[1 - \frac{h}{2r}\right]^2 + T_3 + 2\,M = 0 \ . \ (4.30.13)$$

However, since $\tilde{\gamma}$ is an arbitrary Lagrange multiplier, it follows that the tension T is also arbitrary and can be used to solve equation $(4.30.13)_1$ such that

$$T = \frac{r^2 p_l}{2}\left[1 - \frac{h}{2r}\right]^2 \ . \qquad (4.30.14)$$

Thus, defining the average circumferential stress σ in the usual manner, it can be seen that

$$\sigma = \frac{T}{rh} = \left[\frac{r p_l}{2h}\right]\left[1 - \frac{h}{2r}\right]^2 \ , \qquad (4.30.15)$$

where the first term in square brackets is recognized as the strength of materials value for thin shells ($h/r << 1$). Next, with the help of equations (4.30.10), the equilibrium equations (4.30.13) can be solved for the Lagrange multiplier $\tilde{\gamma}$ and the pressure p_l to obtain

$$p_l = \frac{6\left[f(\lambda) - \frac{\hat{M}}{\lambda^2}\right]}{r^3\left[1 - \frac{h}{2r}\right]^2\left[1 + \frac{H}{r\lambda^2} + \frac{H^3}{6V\lambda^3}\left\{1 - \frac{H}{2r\lambda^2}\right\}\right]} \ . \qquad (4.30.16)$$

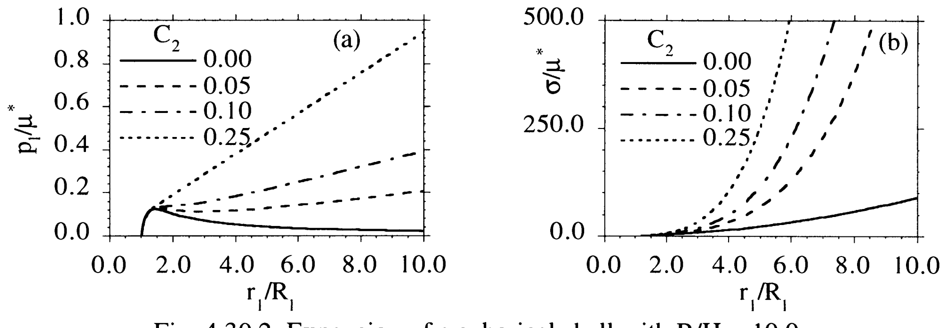

Fig. 4.30.2 Expansion of a spherical shell with R/H = 10.0.

Figures 4.30.1 plot the internal pressure (4.30.16) and circumferential stress (4.30.15) for a thick shell (R/H=1), and Figs. 4.30.2 plot the values of these quantities for a thinner shell (R/H=10). The values of the constant C_2 span the range which causes Σ to be independent of $\bar{\alpha}_2$ (C_2=0) and to be independent of $\bar{\alpha}_1$ (C_2=0.25). Figures 4.30.1a and 4.30.2a indicate that for the thick shell, all values of C_2 exhibit an instability near the value r/R = 1.7. Specifically, if the internal pressure were continually increased, the theory would predict that, for C_2=0, the shell would fail whereas, for C_2=0.10 or 0.25

the shell would expand rapidly to a larger radius and then continue to expand stabily. Moreover, from Fig. 4.30.1a it cannot be determined if, for C_2=0.05 the pressure for large values of r_1/R_1 increases above the peak pressure. Figures 4.30.1b and 4.30.2b also indicate that increasing the value of C_2 causes stiffer elastic response which reduces the tendency for an instability. In particular, notice that the peak pressure at the instability is smaller for the thinner shell and notice from Fig. 4.30.2a that, for C_2=0.10 there does not appear to be any instability. However, for all of these cases the value of the stress σ become so large that failure in tension is expected for most materials.

Finally, in view of the results of section 4.20 it is expected that the actual value of the instability point predicted here for the thick shell may not be very accurate. This point is investigated further in section 7.2 where a more accurate numerical method is described for spherically symmetric problems of both compressible and incompressible shells.

4.31 Bending of an orthotropic plate into a circular cylindrical surface

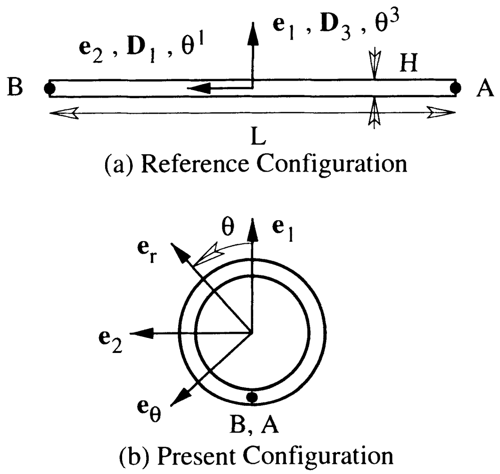

Fig. 4.31 Two dimensional sketch showing (a) the reference configuration of the plate and (b) the deformed present configuration.

This section considers an example where it is essential to use a large deformation theory but not necessarily a large strain theory. Specifically, consider an orthotropic plate which in its reference configuration is stress-free and has length L, width W and thickness H. In its present configuration the plate is bent into a circular cylindrical shell with its ends being bonded together. The orientation of the base vectors $\mathbf{e}_i$ of rectangular Cartesian coordinate system shown in Fig. 4.31a for the plate have been chosen to simplify the description of the shell in Fig. 4.31b in terms of the usual base vectors $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ of a circular cylindrical coordinate system. It will be shown that a solution exists for which the inner and outer surfaces of the cylinder are free from surface tractions, the edges ($\theta^2=\pm W/2$) are free from resultant force, but they require a resultant moment to be applied to maintain the circular cylindrical shape.

For this problem the reference configuration of the plate is characterized by the quantities

$$\mathbf{X} = \theta^1 \mathbf{e}_2 + \theta^2 \mathbf{e}_z \ , \quad \mathbf{e}_3 = \mathbf{e}_z \ , \quad |\theta^1| \le \frac{L}{2} \ , \quad |\theta^2| \le \frac{W}{2} \ ,$$

$$\mathbf{D}_1 = \mathbf{e}_2 \ , \quad \mathbf{D}_2 = \mathbf{e}_z \ , \quad \mathbf{D}_3 = \mathbf{e}_1 \ , \quad \mathbf{D}^i = \mathbf{D}_i \ , \quad \mathbf{D}_{3,\alpha} = 0 \ ,$$

$$A^{1/2} = 1 \ , \quad \bar{H} = H \ , \quad \mathbf{H}^\alpha = 0 \ . \tag{4.31.1}$$

Also, the present configuration is specified by

$$\mathbf{x} = r\,\mathbf{e}_r(\theta) + z\,\mathbf{e}_z \ , \quad \mathbf{d}_3 = c_3\,\mathbf{e}_r \ ,$$

$$r = c_1 R \ , \quad R = \frac{L}{2\pi} \ , \quad \theta = \frac{\theta^1}{R} \ , \quad z = c_2\,\theta^2 \ , \tag{4.31.2}$$

where c_1, c_2 and c_3 are positive constants to be determined and the convected coordinates θ^α should not be confused with powers of the angle θ. Next, using the kinematic expressions summarized in section 4.27, it can be shown that

$$\mathbf{d}_1 = c_1\,\mathbf{e}_\theta \ , \quad \mathbf{d}_2 = c_2\,\mathbf{e}_z \ , \quad \mathbf{d}_3 = c_3\,\mathbf{e}_r \ ,$$

$$\mathbf{d}^1 = \frac{1}{c_1}\,\mathbf{e}_\theta \ , \quad \mathbf{d}^2 = \frac{1}{c_2}\,\mathbf{e}_z \ , \quad \mathbf{d}^3 = \frac{1}{c_3}\,\mathbf{e}_r \ ,$$

$$a^{1/2} = c_1 c_2 \ , \quad \mathbf{a}_3 = \mathbf{e}_r \ ,$$

$$\mathbf{F} = c_1\,(\mathbf{e}_\theta \otimes \mathbf{D}_1) + c_2\,(\mathbf{e}_z \otimes \mathbf{D}_2) + c_3\,(\mathbf{e}_r \otimes \mathbf{D}_3) \ ,$$

$$\bar{\mathbf{E}} = \mathbf{E} = E_{11}\,(\mathbf{D}_1 \otimes \mathbf{D}_1) + E_{22}\,(\mathbf{D}_2 \otimes \mathbf{D}_2) + E_{33}\,(\mathbf{D}_3 \otimes \mathbf{D}_3) \ ,$$

$$E_{11} = \frac{1}{2}(c_1^2 - 1) \ , \quad E_{22} = \frac{1}{2}(c_2^2 - 1) \ , \quad E_{33} = \frac{1}{2}(c_3^2 - 1) \ ,$$

$$\boldsymbol{\beta}_1 = \frac{c_3}{c_1 R}\,\mathbf{D}_1 \ , \quad \boldsymbol{\beta}_2 = 0 \ . \tag{4.31.3}$$

Moreover, the direction of orthotropy is chosen so that ϕ vanishes in (4.12.8) with

$$\mathbf{M}_i = \mathbf{D}_i \ . \tag{4.31.4}$$

Now, for small strains but large displacements, the constitutive equations summarized in section 4.27 yield

$$\mathbf{m}^1 = M_1\,\mathbf{e}_\theta \ , \quad M_1 = HK_{1111}\left\{\frac{c_3}{c_1^2 R}\right\} \ ,$$

$$\mathbf{m}^2 = M_2\,\mathbf{e}_z \ , \quad M_2 = HK_{1122}\left\{\frac{c_3}{c_1 c_2 R}\right\} \ ,$$

$$\mathbf{t}^1 = T_1\,\mathbf{e}_\theta \ , \quad T_1 = H(K^*_{1111}E_{11} + K^*_{1122}E_{22} + K^*_{1133}E_{33})c_1 - \left\{\frac{c_3}{c_1 R}\right\}M_1 \ ,$$

$$\mathbf{t}^2 = T_2\,\mathbf{e}_z \ , \quad T_2 = H(K^*_{1122}E_{11} + K^*_{2222}E_{22} + K^*_{2233}E_{33})c_2 \ ,$$

$$\mathbf{t}^3 = T_3\,\mathbf{e}_r \ , \quad T_3 = H(K^*_{1133}E_{11} + K^*_{2233}E_{22} + K^*_{3333}E_{33})c_3 \ , \tag{4.31.5}$$

where K_{1111} and K_{1122} are given by the expressions (4.14.28).

In the absence of body force and surface tractions on the inner and outer surfaces of the cylindrical shell, the assigned fields vanish

$$\mathbf{b} = 0 \ , \quad \mathbf{b}^3 = 0 \ , \tag{4.31.6}$$

so that the equations of equilibrium reduce to

$$0 = \mathbf{t}^\alpha{}_{,\alpha} \ , \quad 0 = -\,\mathbf{t}^3 + \mathbf{m}^\alpha{}_{,\alpha} \ . \tag{4.31.7}$$

Next, with the help of the expressions (4.31.5), the component forms of the equilibrium equations become

$$\mathrm{T}_1 = 0 \ , \quad \mathrm{T}_2 = \text{constant} \ ,$$
$$\mathrm{T}_3 + \frac{\mathrm{M}_1}{\mathrm{R}} = 0 \ , \quad \mathrm{M}_2 = \text{constant} \ . \tag{4.31.8}$$

Furthermore, the boundary conditions on the force, couple and moment yield the expressions

$$\mathbf{t} = -\frac{\mathrm{T}_2}{\mathrm{c}_1}\,\mathbf{e}_z \ , \quad \mathbf{m}^3 = -\frac{\mathrm{M}_2}{\mathrm{c}_1}\,\mathbf{e}_z \ , \quad \mathbf{m} = \frac{\mathrm{M}_2\mathrm{c}_3}{\mathrm{c}_1}\,\mathbf{e}_\theta \quad \text{for } \theta^2 = -\,\mathrm{W}/2 \ ,$$
$$\mathbf{t} = \frac{\mathrm{T}_2}{\mathrm{c}_1}\,\mathbf{e}_z \ , \quad \mathbf{m}^3 = \frac{\mathrm{M}_2}{\mathrm{c}_1}\,\mathbf{e}_z \ , \quad \mathbf{m} = -\,\frac{\mathrm{M}_2\mathrm{c}_3}{\mathrm{c}_1}\,\mathbf{e}_\theta \quad \text{for } \theta^2 = \mathrm{W}/2 \ . \tag{4.31.9}$$

Since these ends have been assumed to be free of resultant force, it follows that

$$\mathrm{T}_2 = 0 \ . \tag{4.31.10}$$

Now, equations $(4.31.8)_1$, (4.31.10) and $(4.31.8)_3$ yield three equations

$$\mathrm{K}^*_{1111}(\mathrm{c}_1^2 - 1) + \mathrm{K}^*_{1122}(\mathrm{c}_2^2 - 1) + \mathrm{K}^*_{1133}(\mathrm{c}_3^2 - 1) = \frac{2\mathrm{K}_{1111}}{\mathrm{R}^2}\left\{\frac{\mathrm{c}_3^2}{\mathrm{c}_1^4}\right\} \ ,$$
$$\mathrm{K}^*_{1122}(\mathrm{c}_1^2 - 1) + \mathrm{K}^*_{2222}(\mathrm{c}_2^2 - 1) + \mathrm{K}^*_{2233}(\mathrm{c}_3^2 - 1) = 0 \ ,$$
$$\mathrm{K}^*_{1133}(\mathrm{c}_1^2 - 1) + \mathrm{K}^*_{2233}(\mathrm{c}_2^2 - 1) + \mathrm{K}^*_{3333}(\mathrm{c}_3^2 - 1) = -\,\frac{2\mathrm{K}_{1111}}{\mathrm{R}^2}\left\{\frac{1}{\mathrm{c}_1^2}\right\} \ , \tag{4.31.11}$$

to determine the three unknown constants $\{\mathrm{c}_1, \mathrm{c}_2, \mathrm{c}_3\}$. Equations $(4.31.11)_{2,3}$ can be solved for c_2^2 and c_3^2 as functions of c_1^2 to deduce that

$$\mathrm{c}_2^2 = 1 - \frac{\mathrm{K}^*_{1122}(\mathrm{c}_1^2 - 1) + \mathrm{K}^*_{2233}(\mathrm{c}_3^2 - 1)}{\mathrm{K}^*_{2222}} \ ,$$

$$\mathrm{c}_3^2 = 1 - \left[\frac{\mathrm{K}^*_{1133}\mathrm{K}^*_{2222} - \mathrm{K}^*_{1122}\mathrm{K}^*_{2233}}{\mathrm{K}^*_{2222}\mathrm{K}^*_{3333} - \mathrm{K}^*_{2233}\mathrm{K}^*_{2233}}\right](\mathrm{c}_1^2 - 1)$$
$$-\left[\frac{\mathrm{K}^*_{2222}}{\mathrm{K}^*_{2222}\mathrm{K}^*_{3333} - \mathrm{K}^*_{2233}\mathrm{K}^*_{2233}}\right]\left[\frac{2\mathrm{K}_{1111}}{\mathrm{R}^2}\right]\frac{1}{\mathrm{c}_1^2} \ . \tag{4.31.12}$$

Then, these expressions can be substituted into $(4.31.11)_1$ to obtain a fourth order algebraic equation for c_1^2 of the form

$$\mathrm{A}_2\left\{\mathrm{c}_1^8 - \mathrm{c}_1^6\right\} - \mathrm{A}_1\,\mathrm{c}_1^2 + \mathrm{A}_0 = 0 \ ,$$

$$\mathrm{A}_0 = \left[\frac{\mathrm{K}^*_{2222}\mathrm{K}^*_{2222}}{\mathrm{K}^*_{2222}\mathrm{K}^*_{3333} - \mathrm{K}^*_{2233}\mathrm{K}^*_{2233}}\right]\left[\frac{2\mathrm{K}_{1111}}{\mathrm{R}^2}\right]^2 \ ,$$

$$A_1 = K^*_{2222}\left[1 + \frac{K^*_{1133}K^*_{2222} - K^*_{1122}K^*_{2233}}{K^*_{2222}K^*_{3333} - K^*_{2233}K^*_{2233}}\right]\left[\frac{2K_{1111}}{R^2}\right] ,$$

$$A_2 = \left[K^*_{1111}K^*_{2222} - K^*_{1122}K^*_{1122}\right] - \left[\frac{(K^*_{1133}K^*_{2222} - K^*_{1122}K^*_{2233})^2}{K^*_{2222}K^*_{3333} - K^*_{2233}K^*_{2233}}\right] . \quad (4.31.13)$$

This equation can be simplified by using the expression (4.14.28), introducing the definitions

$$\frac{2K_{1111}}{R^2} = \bar{K}_{1111}\left\{\frac{H^2}{6R^2}\right\} , \quad \bar{K}_{1111} = \left[K^*_{1111} - \frac{K^*_{1133}K^*_{1133}}{K^*_{3333}}\right] ,$$

$$\bar{A}_0 = A_0\left\{\frac{6R^2}{H^2}\right\}^2 , \quad \bar{A}_1 = \frac{A_1}{\bar{A}_0}\left\{\frac{6R^2}{H^2}\right\} , \quad \bar{A}_2 = \frac{A_2}{\bar{A}_0} , \quad (4.31.14)$$

and rewriting (4.31.14) in the equivalent form

$$\bar{A}_2\left\{c_1^8 - c_1^6\right\} - \bar{A}_1 c_1^2\left\{\frac{H^2}{6R^2}\right\} + \left\{\frac{H^2}{6R^2}\right\}^2 = 0 . \quad (4.31.15)$$

Once $(4.31.13)_1$ is solved for c_1^2, then (4.31.12) are solved for c_2^2 and c_3^2, and the values $\{c_1, c_2, c_3\}$ become the positive square roots of these quantities. Then, the values of the moments M_1 and M_2 are determined by the expressions (4.31.5). In particular, it is noted from (4.31.9) that it is necessary to apply bending moments to the ends $\theta^2 = \pm W/2$ in order to maintain the cylindrical shape of the shell.

An approximate solution of (4.31.15) can be obtained by assuming that

$$c_1 = 1 + \bar{c}_1\left\{\frac{H^2}{6R^2}\right\} , \quad c_2 = 1 + \bar{c}_2\left\{\frac{H^2}{6R^2}\right\} , \quad c_3 = 1 + \bar{c}_3\left\{\frac{H^2}{6R^2}\right\} , \quad (4.31.16)$$

and neglecting quadratic terms in $\{H^2/6/R^2\}$ in (4.31.12) and (4.31.15) to deduce that

$$\bar{c}_1 = \frac{\bar{A}_1}{2\bar{A}_2} , \quad \bar{c}_2 = -\left[\frac{K^*_{1122}}{K^*_{2222}}\right]\bar{c}_1 - \left[\frac{K^*_{2233}}{K^*_{2222}}\right]\bar{c}_3 ,$$

$$\bar{c}_3 = -\left[\frac{K^*_{1133}K^*_{2222} - K^*_{1122}K^*_{2233}}{K^*_{2222}K^*_{3333} - K^*_{2233}K^*_{2233}}\right]\bar{c}_1 - \frac{1}{2}\left[\frac{\bar{K}_{1111}K^*_{2222}}{K^*_{2222}K^*_{3333} - K^*_{2233}K^*_{2233}}\right] . \quad (4.31.17)$$

It then follows from (4.31.5) and (4.31.16) that the moments M_1 and M_2 can be approximated by

$$M_1 = HK_{1111}\left[1 + (\bar{c}_3 - 2\bar{c}_1)\left\{\frac{H^2}{6R^2}\right\}\right]\frac{1}{R} ,$$

$$M_2 = HK_{1122}\left[1 + (\bar{c}_3 - \bar{c}_1 - \bar{c}_2)\left\{\frac{H^2}{6R^2}\right\}\right]\frac{1}{R} . \quad (4.31.18)$$

These results can be compared with those of linearized theory for bending of a plate in section 4.14. Specifically, the plate can be bent into a circular cylindrical shape by taking $\beta_{\alpha\beta}$ in (4.14.18) to be

$$\beta_{11} = \frac{1}{R} \ , \quad \beta_{12} = \beta_{21} = \beta_{22} = 0 \ , \tag{4.31.19}$$

so that (4.14.15) and (4.14.18) can be used to write the moments in the forms

$$\tilde{\mathbf{m}}^1 = \tilde{M}_1 \, \mathbf{e}_1 \ , \quad \tilde{M}_1 = HK_{1111} \left\{\frac{1}{R}\right\} \ ,$$
$$\tilde{\mathbf{m}}^2 = \tilde{M}_2 \, \mathbf{e}_2 \ , \quad \tilde{M}_2 = HK_{1122} \left\{\frac{1}{R}\right\} \ . \tag{4.31.20}$$

Thus, it can be seen that the magnitudes of the moments (3.31.18) predicted by the nonlinear theory for bending of a plate into a circular cylindrical shell, coincide with the expressions (4.31.20) when the final shell is thin with $\{H^2/6/R^2\}$ being much smaller than unity.

4.32 Linear theory of an isotropic plate

In order to examine the influence of the Kirchhoff-Love constraints (4.9.21) discussed in section 4.9, it is convenient to consider the linear theory of an isotropic plate. Also, for simplicity attention will be confined to a uniform rectangular plate with length L, width W and constant thickness H. Throughout the following analysis use will be made of the equations of the linearized theory which have been summarized in section 4.27.

The reference configuration of the plate is characterized by the quantities

$$\mathbf{X} = \theta^1 \, \mathbf{e}_1 + \theta^2 \, \mathbf{e}_2 \ , \quad |\theta^1| \le \frac{L}{2} \ , \quad |\theta^2| \le \frac{W}{2} \ ,$$
$$\mathbf{D}_1 = \mathbf{e}_1 \ , \quad \mathbf{D}_2 = \mathbf{e}_2 \ , \quad \mathbf{D}_3 = \mathbf{e}_3 \ , \quad \mathbf{D}^i = \mathbf{D}_i \ , \quad \mathbf{D}_{3,\alpha} = 0 \ ,$$
$$D^{1/2} = 1 \ , \quad \bar{H} = H \ , \quad \mathbf{H}^\alpha = 0 \ , \tag{4.32.1}$$

where $\mathbf{e}_i$ are fixed base vectors of a rectangular Cartesian coordinate system. Also, the present configuration is characterized by the displacement fields $\mathbf{u}$ and $\boldsymbol{\delta}_i$ such that

$$\mathbf{x} = \mathbf{X} + \mathbf{u} \ , \quad \mathbf{d}_i = \mathbf{D}_i + \boldsymbol{\delta}_i \ , \quad \boldsymbol{\delta}_\alpha = u_{i,\alpha} \, \mathbf{e}_i \ , \quad \boldsymbol{\delta}_3 = \bar{\delta}_{3i}(\theta^\alpha, t) \, \mathbf{e}_i \ . \tag{4.32.2}$$

Moreover, the linearized strains are given by

$$\bar{\tilde{\mathbf{E}}} = \tilde{\mathbf{E}} = E_{ij} \, (\mathbf{e}_i \otimes \mathbf{e}_j) \ , \quad E_{\alpha\beta} = E_{\beta\alpha} = \frac{1}{2}(u_{\alpha,\beta} + u_{\beta,\alpha}) \ ,$$
$$E_{3\alpha} = E_{\alpha 3} = \frac{1}{2}(u_{3,\alpha} + \bar{\delta}_{3\alpha}) \ , \quad E_{33} = \bar{\delta}_{33} \ , \quad \tilde{\boldsymbol{\beta}}_\alpha = \beta_{\alpha i} \, \mathbf{e}_i \ , \quad \beta_{\alpha i} = \bar{\delta}_{3i,\alpha} \ , \tag{4.32.3}$$

and the constitutive equations become

$$\tilde{\mathbf{m}}^\alpha = m_{\alpha\beta} \, \mathbf{e}_\beta \ , \quad m_{\alpha\beta} = m_{\beta\alpha} \ ,$$
$$m_{11} = \frac{\mu^* H^3}{6(1-\nu^*)} \left[\beta_{11} + \nu^* \beta_{22}\right] \ , \quad m_{22} = \frac{\mu^* H^3}{6(1-\nu^*)} \left[\nu^* \beta_{11} + \beta_{22}\right] \ ,$$
$$m_{12} = m_{21} = \frac{\mu^* H^3}{12} \left[\beta_{12} + \beta_{21}\right] \ , \quad \tilde{\mathbf{t}}^i = t_{ij} \, \mathbf{e}_j \ , \quad t_{ij} = t_{ji} \ ,$$

$$t_{11} = \frac{2\mu^* H}{(1-2\nu^*)} \left[(1-\nu^*)\, E_{11} + \nu^*\, E_{22} + \nu^*\, E_{33} \right] \ ,$$

$$t_{22} = \frac{2\mu^* H}{(1-2\nu^*)} \left[\nu^*\, E_{11} + (1-\nu^*)\, E_{22} + \nu^*\, E_{33} \right] \ ,$$

$$t_{33} = \frac{2\mu^* H}{(1-2\nu^*)} \left[\nu^*\, E_{11} + \nu^*\, E_{22} + (1-\nu^*)\, E_{33} \right] + \gamma_{33} \ ,$$

$$t_{12} = t_{21} = 2\mu^* H\, E_{12} \ , \quad t_{13} = t_{31} = 2\mu^* H\, E_{13} + \gamma_{31} \ ,$$

$$t_{23} = t_{32} = 2\mu^* H\, E_{23} + \gamma_{32} \ , \tag{4.32.4}$$

where μ^* and ν^* are the material constants of the three-dimensional material. Also, the constraint responses $\gamma_{3\alpha}$ and γ_{33} [see (4.9.23)] are associated with the linearized forms of the constraints (4.9.21). In particular, when normal extension is omitted, then γ_{33} becomes an arbitrary function of space and time and the constraint $(4.9.21)_2$ reduces to

$$\mathbf{D}_3 \bullet \bar{\boldsymbol{\delta}}_3 = 0 \ \Rightarrow \ E_{33} = 0 \ . \tag{4.32.5}$$

Similarly, when transverse shear deformation is omitted, then $\gamma_{3\alpha}$ become arbitrary functions of space and time and the constraints $(4.9.21)_1$ reduce to

$$\mathbf{u}_{,\alpha} \bullet \mathbf{D}_3 + \mathbf{D}_\alpha \bullet \bar{\boldsymbol{\delta}}_3 = 0 \ \Rightarrow \ E_{\alpha 3} = 0 \ . \tag{4.32.6}$$

When either one of these constraints is not imposed, then the associated constraint response vanishes.

Next, it is recalled from section 4.27 that the inertia quantities associated with the plate are given by

$$m = \rho_0^* H \ , \quad y^3 = 0 \ , \quad m\, y^{33} = \frac{\rho_0^*\, H^3}{\pi^2} \ , \tag{4.32.7}$$

where ρ_0^* is the mass density of the three-dimensional material. Then, taking g^* to be the force of gravity per unit mass acting in the negative $\mathbf{e}_3$ direction, and assuming that the plate is loaded by a pressure $\bar{p}$ acting on its bottom surface (θ^3=–H/2) in the positive $\mathbf{e}_3$ direction and by a pressure $\hat{p}$ acting on its top surface (θ^3=H/2) in the negative $\mathbf{e}_3$ direction, the assigned fields become

$$m\, \tilde{\mathbf{b}} = \left[-\rho_0^* H\, g^* + \bar{p} - \hat{p} \right] \mathbf{e}_3 \ , \quad m\, \tilde{\mathbf{b}}^3 = -\frac{H}{2} \left[\bar{p} + \hat{p} \right] \mathbf{e}_3 \ . \tag{4.32.8}$$

It is well known (Naghdi, 1972, p. 596) that the balance laws of linear plate theory separate into two distinct sets of equations. One set corresponds to the in-plane components of linear momentum and the normal component of director momentum, and it characterizes extensional deformation by the equations

$$m\, \ddot{u}_\alpha = t_{\alpha\beta,\beta} \ , \quad m\, y^{33}\, \ddot{\bar{\delta}}_{33} = -\frac{H}{2} \left[\bar{p} + \hat{p} \right] - t_{33} \ . \tag{4.32.9}$$

These expressions are used to determine the displacements $\{u_\alpha \ , \ \bar{\delta}_{33}\}$ and the kinetic quantities $\{t_{\alpha\beta} \ , t_{33}\}$. The second set of equations corresponds to the normal component of linear momentum and the in-plane components of director momentum, and it characterizes bending deformation by the equations

$$m\,\ddot{u}_3 = \left[-\rho_0^* H g^* + \bar{p} - \hat{p}\right] + t_{3\alpha,\alpha} \;, \qquad m\,y^{33}\,\ddot{\bar{\delta}}_{3\alpha} = -\,t_{3\alpha} + m_{\alpha\beta,\beta} \;. \quad (4.32.10)$$

These expressions are used to determine the displacements $\{u_3 \,,\ \bar{\delta}_{3\alpha}\}$ and the kinetic quantities $\{\, t_{3\alpha} \,,\, m_{\alpha\beta} \,\}$. Notice from $(4.32.10)_1$ that the bending solution is influenced by the effects of gravity and the difference between the pressures on the top and bottom surfaces. Also, from $(4.32.9)_2$ it is seen that the extensional theory is uninfluenced by gravity and that it depends only on the average pressure applied to the top and bottom surfaces. Moreover, the constraint (4.32.5) on normal extension only influences the extensional theory, and the constraint (4.32.6) on transverse shear deformation only influences the bending theory.

For example, if normal extension is omitted, then (4.32.5) requires E_{33} to vanish and γ_{33} is determined by satisfying $(4.32.9)_2$ so that

$$t_{33} = -\frac{H}{2}\left[\,\bar{p} + \hat{p}\,\right] \;,\; \gamma_{33} = -\frac{H}{2}\left[\,\bar{p} + \hat{p}\,\right] - \frac{2\mu^* H}{(1-2\nu^*)}\left[\,\nu^* E_{11} + \nu^* E_{22}\,\right] . \quad (4.32.11)$$

Also, the constitutive equations for $t_{\alpha\beta}$ reduce to

$$t_{11} = \frac{2\mu^* H}{(1-2\nu^*)}\left[\,(1-\nu^*)\,E_{11} + \nu^* E_{22}\,\right] \;,\; t_{22} = \frac{2\mu^* H}{(1-2\nu^*)}\left[\,\nu^* E_{11} + (1-\nu^*)\,E_{22}\right] \;,$$

$$t_{12} = t_{21} = 2\mu^* H\,E_{12} \;. \quad (4.32.12)$$

These equations characterize the extensional response to plane strain deformations. However, it is also possible to use this simplified constrained theory for the response to generalized plane stress if ν^* in (4.32.12) is replaced by the expression $\bar{\nu}^*$ defined in (4.29.22). In this regard, it should be emphasized that since the bending equations (4.32.10) are uncoupled from the extensional equations (4.32.9), the quantity ν^* in the constitutive equations for $t_{3\alpha}$ and $m_{\alpha\beta}$ should not be replaced by $\bar{\nu}^*$ when generalized plane stress is modeled.

As another example, consider the bending theory when transverse shear deformation is omitted (4.32.6) so that $E_{3\alpha}$ vanishes and $\gamma_{3\alpha}$ are obtained by satisfying $(4.32.10)_2$ with

$$\bar{\delta}_{3\alpha} = -\,u_{3,\alpha} \;,\quad t_{3\alpha} = \gamma_{3\alpha} = m_{\alpha\beta,\beta} + m\,y^{33}\,\ddot{u}_{3,\alpha} \;, \quad (4.32.13)$$

Then, $(4.32.10)_1$ becomes an equation of the form

$$m\,\ddot{u}_3 = \left[-\rho_0^* H g^* + \bar{p} - \hat{p}\right] - \frac{\mu^* H^3}{6(1-\nu^*)}\,u_{3,\alpha\alpha\beta\beta} + m\,y^{33}\,\ddot{u}_{3,\alpha\alpha} \;, \quad (4.32.14)$$

where use has been made of the result that in the absence of shear deformation

$$m_{\alpha\beta,\alpha\beta} = -\frac{\mu^* H^3}{6(1-\nu^*)}\,u_{3,\alpha\alpha\beta\beta} \;. \quad (4.32.15)$$

Thus, (4.32.14) is used to determine the normal component of displacement u_3. Moreover, it is noted that the term associated with director inertia (y^{33}) in equation (4.32.14) can significantly influence natural frequencies when higher modes are considered.

To exhibit the quantitative influence of omitting transverse shear deformation, it is convenient to consider the simple case when the plate is bent into a right cylindrical surface. Specifically, inertia terms are neglected and the load is taken to be constant with

$$p = -\rho_0^* H g^* + \bar{p} - \hat{p} = \text{constant} \ , \tag{4.32.16}$$

so that

$$u_3 = u_3(X) \ , \quad \bar{\delta}_{31} = \bar{\delta}_{31}(X) \ , \quad \bar{\delta}_{32} = 0 \ , \tag{4.32.17}$$

where X is a normalized length coordinate defined by

$$X = \frac{\theta^1}{L} \ , \quad |X| \leq \frac{1}{2} \ . \tag{4.32.18}$$

Moreover, it is assumed that the edges X=±1/2 are built in so that the displacements satisfy the boundary conditions

$$u_3 = 0 \ , \quad \bar{\delta}_{31} = 0 \ , \quad \text{for } X = \pm 1/2 \ . \tag{4.32.19}$$

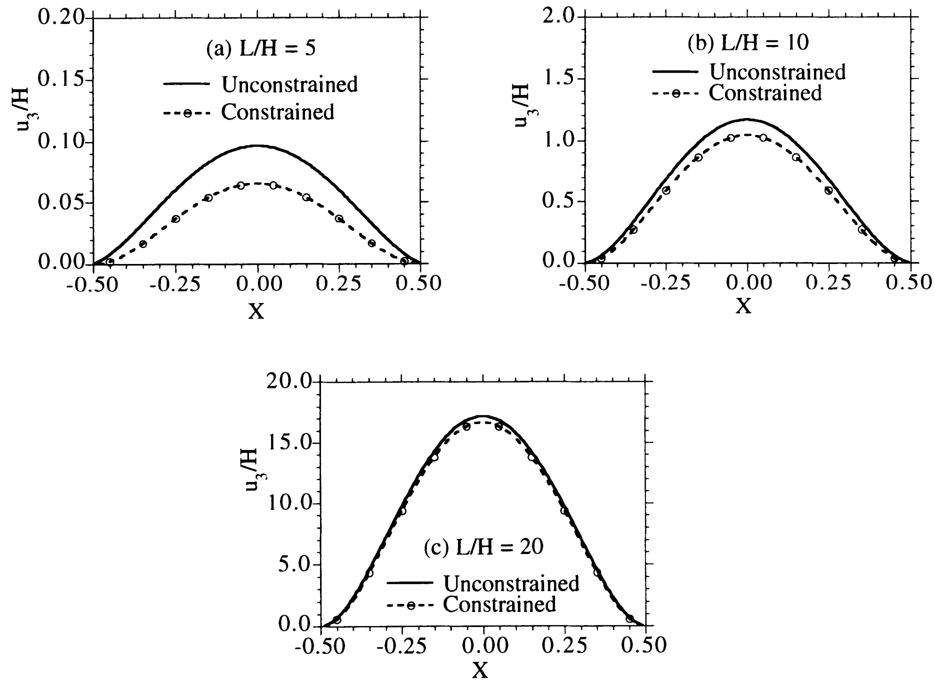

Fig. 4.32.1 Normalized normal displacement (u_3/H) of a plate loaded by the pressure $\bar{p} = 0.01\ \mu^*$, with Poisson's ratio $\nu^* = 1/3$, for different length to thickness ratios.

It can easily be shown that, under these conditions, the bending equations of equilibrium associated with both the unconstrained and the constrained theories become

$$p + t_{31},_1 = 0 \ , \quad -t_{31} + m_{11},_1 = 0 \ . \tag{4.32.20}$$

Moreover, the two displacement equations for the unconstrained theory take the forms

$$p + \mu^* H (u_{3,11} + \bar{\delta}_{31,1}) = 0 \ , \ -\mu^* H (u_{3,1} + \bar{\delta}_{31}) + \frac{\mu^* H^3}{6(1-\nu^*)} \bar{\delta}_{31,11} = 0 \ , \quad (4.32.21)$$

whereas the one displacement equation for the constrained theory becomes

$$p - \frac{\mu^* H^3}{6(1-\nu^*)} u_{3,1111} = 0 \ . \quad (4.32.22)$$

Next, it can be shown that, for both the unconstrained and the constrained theories, the values of $\{\bar{\delta}_{31}, t_{31}, t_{32}, m_{11}, m_{22}, m_{12}\}$ are the same

$$\bar{\delta}_{31} = \frac{p}{\mu^*}(1-\nu^*)\left\{\frac{L}{H}\right\}^3\left[\frac{1}{4} - X^2\right] X \ , \ t_{31} = -pLX \ , \ t_{32} = 0 \ ,$$

$$m_{11} = -\frac{pL^2}{2}\left[X^2 - \frac{1}{12}\right] \ , \ m_{22} = \nu^* m_{11} \ , \ m_{12} = 0 \ . \quad (4.32.23)$$

In particular, notice that in order to maintain the shape of a right cylinder it is necessary to apply moments m_{22} to the edges ($\theta^2 = \pm W/2$). Moreover, it can be shown that the normalized displacement associated with the unconstrained theory is given by

$$\frac{u_3}{H} = \frac{p}{\mu^*}\left[\frac{1}{2}\left\{\frac{L}{H}\right\}^2\left\{\frac{1}{4} - X^2\right\} + \left\{\frac{1-\nu^*}{4}\right\}\left\{\frac{L}{H}\right\}^4\left\{\frac{1}{4} - X^2\right\}^2\right] \ , \quad (4.32.24)$$

and that the displacement associated with the constrained theory is given by

$$\frac{u_3}{H} = \frac{p}{\mu^*}\left\{\frac{1-\nu^*}{4}\right\}\left\{\frac{L}{H}\right\}^4\left\{\frac{1}{4} - X^2\right\}^2 \ . \quad (4.32.25)$$

Examination of these two solutions indicates that for thin plates (L/H>>1) the two solutions coincide.

Figure 4.32.1 shows plots of the normalized normal displacement (4.32.24) for the unconstrained theory and (4.32.5) for the constrained theory, as functions of the normalized length coordinate X, for three different length to thickness ratios (L/H). The quantitative values of these quantities were determined for p and ν^* specified by

$$p = 0.01\,\mu^* \ , \ \nu^* = \frac{1}{3} \ . \quad (4.32.26)$$

Notice from this figure that the unconstrained plate is less stiff than the constrained plate since the centerline displacements are larger in the unconstrained plate. Also, note that the centerline displacements increase by more than a factor of ten as the length is doubled. This is mainly due to the presence of the term $(L/H)^4$ in the solution.

4.33 Dissipation inequality and material damping

The previous sections have limited attention to purely elastic response, which exhibits no dissipation. Consequently, shells made from such materials exhibit the unrealistic feature that free vibrations persist forever. In order to eliminate this unphysical response it is necessary to include a model for material damping. To this end, it is noted that within the context of the purely mechanical theory, it is possible to define the rate of material dissipation $\mathcal{D}$ per unit present area by the formula

$$\int_P \mathcal{D}\, d\sigma = \mathcal{W} - \dot{\mathcal{K}} - \dot{\mathcal{U}} \geq 0 \ , \tag{4.33.1}$$

where $\mathcal{W}$, $\mathcal{K}$ and $\mathcal{U}$ are defined by (4.3.46) and (4.8.4). In words, this equation means that the rate of material dissipation is equal to the rate of work done by the externally applied forces and couples $\mathcal{W}$, minus the rates of change of kinetic energy $\mathcal{K}$ and strain energy $\mathcal{U}$. Moreover, it is assumed that the rate of material dissipation is nonnegative.

Next, with the help of the conservation of mass and the balances of linear, angular and director momentum, it can be shown using (4.6.1), (4.6.17) and (4.8.4) that the local form of equation (4.31.1) becomes

$$\mathcal{D} = \mathbf{T} \bullet \mathbf{D} + a^{-1/2}\, (\mathbf{F}^T\mathbf{m}^\alpha) \bullet \dot{\boldsymbol{\beta}}_\alpha - \rho\, \dot{\Sigma} \ \geq 0 \ . \tag{4.33.2}$$

Moreover, in view of the assumption (4.8.1) it is seen that an elastic shell is an ideal shell since the rate of dissipation $\mathcal{D}$ vanishes. Consequently, the assumption that the rate of material dissipation is nonnegative requires that for a given motion, the work done on a dissipative material is greater than that done on an ideal elastic material. Also, using the transformation relations (4.5.4), (4.5.5), (4.6.15) and (4.8.3), it can be shown that $\mathcal{D}$ remains unaltered by SRBM

$$\mathcal{D}^+ = \mathcal{D} \ . \tag{4.33.3}$$

Now, a model for a shell constructed from a dissipative material can be developed by assuming that $\mathbf{T}$, $\mathbf{t}^i$ and $\mathbf{m}^\alpha$ separate additively into three parts

$$\mathbf{T} = \hat{\mathbf{T}} + \bar{\mathbf{T}} + \check{\mathbf{T}} \ , \quad \mathbf{t}^i = \hat{\mathbf{t}}^i + \bar{\mathbf{t}}^i + \check{\mathbf{t}}^i \ , \quad \mathbf{m}^\alpha = \hat{\mathbf{m}}^\alpha + \bar{\mathbf{m}}^\alpha + \check{\mathbf{m}}^\alpha \ ,$$
$$\hat{\mathbf{T}} = a^{-1/2}\, [\, \hat{\mathbf{t}}^i \otimes \mathbf{d}_i + \hat{\mathbf{m}}^\alpha \otimes \mathbf{d}_{3,\alpha} \,] \ , \quad \bar{\mathbf{T}} = a^{-1/2}\, [\, \bar{\mathbf{t}}^i \otimes \mathbf{d}_i + \bar{\mathbf{m}}^\alpha \otimes \mathbf{d}_{3,\alpha} \,] \ ,$$
$$\check{\mathbf{T}} = a^{-1/2}\, [\check{\mathbf{t}}^i \otimes \mathbf{d}_i + \check{\mathbf{m}}^\alpha \otimes \mathbf{d}_{3,\alpha}\,] \ , \tag{4.33. 4}$$

with $\hat{\mathbf{T}}$, $\hat{\mathbf{t}}^i$ and $\hat{\mathbf{m}}^\alpha$ being the parts associated with elastic deformation [which balance the rate of change in strain energy (4.8.1)]

$$\hat{\mathbf{T}} \bullet \mathbf{D} + a^{-1/2}\, (\mathbf{F}^T\hat{\mathbf{m}}^\alpha) \bullet \dot{\boldsymbol{\beta}}_\alpha = \rho\, \dot{\Sigma} \ , \tag{4.33.5}$$

$\bar{\mathbf{T}}$, $\bar{\mathbf{t}}^i$ and $\bar{\mathbf{m}}^\alpha$ being the constraint responses [which do no work (4.9.15)]

$$\bar{\mathbf{T}} \bullet \mathbf{D} + a^{-1/2}\, (\mathbf{F}^T\bar{\mathbf{m}}^\alpha) \bullet \dot{\boldsymbol{\beta}}_\alpha = 0 \ , \tag{4.33.6}$$

and $\check{\mathbf{T}}$, $\check{\mathbf{t}}^i$ and $\check{\mathbf{m}}^\alpha$ being the parts due to material dissipation. Thus, the restriction (4.33.2) reduces to

$$\mathcal{D} = \check{\mathbf{T}} \bullet \mathbf{D} + a^{-1/2}\, (\mathbf{F}^T\check{\mathbf{m}}^\alpha) \bullet \dot{\boldsymbol{\beta}}_\alpha \geq 0 \ . \tag{4.33.7}$$

For viscous damping, $\check{\mathbf{T}}$ and $\check{\mathbf{m}}^\alpha$ are assumed to be functions of $\mathbf{D}$ and $\dot{\boldsymbol{\beta}}_\beta$

$$\check{\mathbf{T}} = \check{\mathbf{T}}(\mathbf{D}, \dot{\boldsymbol{\beta}}_\beta) \ , \quad \check{\mathbf{m}}^\alpha = \check{\mathbf{m}}^\alpha(\mathbf{D}, \dot{\boldsymbol{\beta}}_\beta) \ . \tag{4.33.8}$$

However, invariance under SRBM requires these functions to satisfy the restrictions

$$\mathbf{Q}\, \check{\mathbf{T}}(\mathbf{D}, \dot{\boldsymbol{\beta}}_\beta)\, \mathbf{Q}^T = \check{\mathbf{T}}(\mathbf{Q}\, \mathbf{D}\, \mathbf{Q}^T, \dot{\boldsymbol{\beta}}_\beta) \ , \quad \mathbf{Q}\, \check{\mathbf{m}}^\alpha(\mathbf{D}, \dot{\boldsymbol{\beta}}_\beta) = \check{\mathbf{m}}^\alpha(\mathbf{Q}\, \mathbf{D}\, \mathbf{Q}^T, \dot{\boldsymbol{\beta}}_\beta) \ , \tag{4.33.9}$$

for all proper orthogonal tensors $\mathbf{Q}$. Consequently, $\check{\mathbf{T}}$ and $\check{\mathbf{m}}^\alpha$ must be isotropic functions of the argument $\mathbf{D}$. Furthermore, as a special simple case it is possible to assume that $\check{\mathbf{T}}$ is a linear function of $\mathbf{D}$, and $\check{\mathbf{m}}^\alpha$ are linear functions of $\dot{\boldsymbol{\beta}}_\alpha$ of the forms

$$a^{1/2} \overset{\vee}{\mathbf{T}} = A^{1/2} \bar{H} \left[\eta_1 (\mathbf{D} \bullet \mathbf{I}) \mathbf{I} + 2\eta_2 \mathbf{D}' \right] ,$$

$$\overset{\vee}{\mathbf{m}}^1 = \eta_3 A^{1/2} \bar{H} (\mathbf{D}^1 \bullet \mathbf{D}^1) \mathbf{F}^{-T} \dot{\boldsymbol{\beta}}_1 \ , \ \overset{\vee}{\mathbf{m}}^2 = \eta_4 A^{1/2} \bar{H} (\mathbf{D}^2 \bullet \mathbf{D}^2) \mathbf{F}^{-T} \dot{\boldsymbol{\beta}}_2 \ , \tag{4.33.10}$$

where $\bar{H}$ is defined in (4.19.9), η_1–η_4 are material constants, and the deviatoric tensor $\mathbf{D}'$ is a pure measure of rate of distortional deformation

$$\mathbf{D}' = \mathbf{D} - \frac{1}{3} (\mathbf{D} \bullet \mathbf{I}) \mathbf{I} \ , \ \mathbf{D}' \bullet \mathbf{I} = 0 \ . \tag{4.33.11}$$

Consequently, η_1 is the viscosity to dilatational deformation rate, η_2 is the viscosity to distortional deformation, and η_3 and η_4 are the viscosities to the inhomogeneous deformation rates $\dot{\boldsymbol{\beta}}_1$ and $\dot{\boldsymbol{\beta}}_2$, respectively. Also, it can be shown that the restriction (4.33.7) is satisfied for all motions provided that η_1–η_4 are all nonnegative

$$\eta_1 \geq 0 \ , \ \eta_2 \geq 0 \ , \ \eta_3 \geq 0 \ , \ \eta_4 \geq 0 \ . \tag{4.33.12}$$

Finally, it is noted that the viscosity constants η_1–η_4 can be determined by attempting to match the rate of damping associated with free vibrations of a structure.

CHAPTER 5

COSSERAT RODS

5.1 Description of a rod structure

A rod-like structure, or rod, is a three-dimensional body that has special geometric features. Most importantly, the rod is a three-dimensional body that is considered to be "thin" in two of its dimensions (see Fig. 5.1.1). In particular, the rod is characterized by its ends and its lateral surface. From another point of view, the rod is considered to be a material curve $\mathcal{C}$ which has some finite thickness bounded by the rod's lateral surface. If this curve $\mathcal{C}$ is a straight line, then the rod-like structure is called a beam, otherwise it is called a rod. Such rod-like structures appear in practice in many applications. For example, the main supporting structures in buildings, and the connecting bars in trusses can be modeled as beams, whereas the curved reinforcement ribs of airplane wings and submarines, and the double helix of DNA molecules can be modeled as rods.

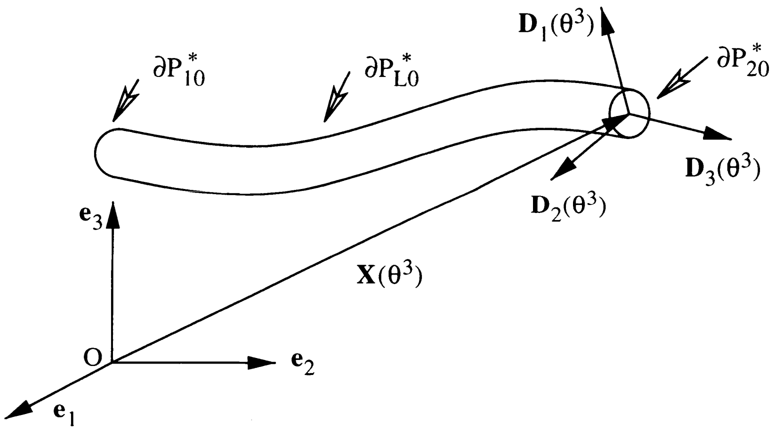

Fig. 5.1.1 A rod-like structure in its reference configuration.

From a mathematical point of view, it is necessary to clearly define in what sense the rod is considered to be "thin". To this end, it is convenient to consider the rod in its reference configuration and to denote the material curve by $\mathcal{C}_0$. Material points in the rod are located relative to a fixed origin O by the three-dimensional position vector $\mathbf{X}^*(\theta^i)$, and material points on the curve $\mathcal{C}_0$ are located by the position vector $\mathbf{X}(\theta^3)$ which corresponds to the curve for which θ^α $(\alpha=1,2)$ vanish

$$\mathbf{X} = \mathbf{X}(\theta^3) = \mathbf{X}^*(0,0,\theta^3) \ . \tag{5.1.1}$$

The tangent vector $\mathbf{D}_3$ in the θ^3 direction and the element of arclength dS to the curve $\mathcal{C}_0$ are given by

$$\mathbf{D}_3(\theta^3) = \frac{d\mathbf{X}}{d\theta^3} = \mathbf{X}_{,3} \ , \quad dS = D_{33}^{1/2}\, d\theta^3 \ , \quad D_{33} = \mathbf{D}_3 \bullet \mathbf{D}_3 \ . \tag{5.1.2}$$

In the following, it is also convenient to introduce a right-handed triad of linearly independent vectors $\mathbf{D}_i$ such that

$$D^{1/2} = \mathbf{D}_1 \times \mathbf{D}_2 \bullet \mathbf{D}_3 > 0 \ . \tag{5.1.3}$$

Since these vectors $\mathbf{D}_i$ are linearly independent, it can be shown that a finite region of three-dimensional space can be characterized by specifying the position vector $\mathbf{X}^*$ in the form

$$\mathbf{X}^*(\theta^i) = \mathbf{X}(\theta^3) + \theta^\alpha\, \mathbf{D}_\alpha(\theta^3) \ . \tag{5.1.4}$$

An arbitrary material part P_0^* of the rod is bounded by the ends ∂P_{10}^* and ∂P_{20}^*, and the smooth lateral surface ∂P_{L0}^* (see Fig. 5.1). In general, these ends can be curved surfaces which are characterized by functions that depend on all three coordinates

$$f_1^*(\theta^i) = f_1^*(\theta^\alpha,\theta^3) = 0 \quad \text{on} \quad \partial P_{10}^* \ ,$$
$$f_2^*(\theta^i) = f_2^*(\theta^\alpha,\theta^3) = 0 \quad \text{on} \quad \partial P_{20}^* \ . \tag{5.1.5}$$

However, for simplicity attention will be confined to rod-like structures which have planar ends. It then follows from (5.1.4) that the directors $\mathbf{D}_\alpha$ can be chosen so that the ends of the rod are specified by

$$\theta^3 = \xi_1 \quad \text{on} \quad \partial P_{10}^* \ , \quad \theta^3 = \xi_2 \quad \text{on} \quad \partial P_{20}^* \ , \tag{5.1.6}$$

where ξ_1 and ξ_2 are constants. The lateral surface ∂P_{L0}^* of the rod is assumed to be a smooth surface defined by the function

$$F^*(\theta^\alpha,\theta^3) = 0 \ , \tag{5.1.7}$$

such that θ^3=constant are closed curves that define the cross-section $\mathcal{A}(\theta^3)$ of the rod. Also, it is assumed that the curve (5.1.4) associated with θ^α=0 pierces this region $\mathcal{A}(\theta^3)$ of the plane defined by $\mathbf{D}_1$ and $\mathbf{D}_2$. Moreover, assuming that $|\partial F/\partial\theta^1|+|\partial F/\partial\theta^2|\neq 0$, the implicit function theorem can be used to determine functions $\hat{\theta}^\alpha(\zeta,\theta^3)$ such that

$$F(\hat{\theta}^\alpha(\zeta,\theta^3),\, \theta^3) = 0 \ , \tag{5.1.8}$$

where the parameter ζ defines points on the boundary $\partial\mathcal{A}(\theta^3)$ of the cross-section. Thus, without loss in generality, the lateral surface can be characterized by

$$\theta^\alpha = \hat{\theta}^\alpha(\zeta,\theta^3) \ , \quad \zeta_1 \le \zeta < \zeta_2 \ , \quad \xi_1 \le \theta^3 \le \xi_2 \ , \tag{5.1.9}$$

where ζ_1 and ζ_2 ($> \zeta_1$) characterize the same point on the boundary $\partial\mathcal{A}(\theta^3)$.

Next, it is of interest to determine the maximum extent of the finite rod-like region that can be described by the position vector (5.1.4). To answer this question, it is necessary to recall a fundamental property of the position vector. Specifically, the position vector is required to provide a one-to-one mapping between the convected coordinates θ^i which define a material point in the rod, and the three-dimensional Euclidean space occupied by the rod. Mathematically, this means that the base vectors $\mathbf{G}_i$ associated with the representation (5.1.4) must be linearly independent

$$\mathbf{G}_i = \mathbf{X}^*{}_{,i} \;,\; \mathbf{G}_\alpha = \mathbf{D}_\alpha \;,\; \mathbf{G}_3 = \mathbf{D}_3 + \theta^\alpha \mathbf{D}_{\alpha,3} \;,\; G^{1/2} = \mathbf{G}_1 \times \mathbf{G}_2 \bullet \mathbf{G}_3 > 0 \;. \quad (5.1.10)$$

In order to write a simplified expression for $G^{1/2}$ it is convenient to introduce the reciprocal vectors $\mathbf{D}^i$ which are defined by formulas of the type (2.1.6) and (2.1.10)

$$\mathbf{D}^1 = D^{-1/2} (\mathbf{D}_2 \times \mathbf{D}_3) \;,\; \mathbf{D}^2 = D^{-1/2} (\mathbf{D}_3 \times \mathbf{D}_1) \;,\; \mathbf{D}^3 = D^{-1/2} (\mathbf{D}_1 \times \mathbf{D}_2) \;,$$
$$D^{1/2} = \mathbf{D}_1 \times \mathbf{D}_2 \bullet \mathbf{D}_3 > 0 \;,\; \mathbf{D}_i \bullet \mathbf{D}^j = \delta_i^j \;. \quad (5.1.11)$$

It then follows that $G^{1/2}$ can be written as a linear function of θ^α in the form

$$G^{1/2} = D^{1/2} \left[\, 1 + \theta^\alpha (\mathbf{D}^3 \bullet \mathbf{D}_{\alpha,3}) \,\right] \;. \quad (5.1.12)$$

Obviously, for $\mathbf{D}^3 \bullet \mathbf{D}_{\alpha,3} = 0$ and for small values of θ^α when $\mathbf{D}^3 \bullet \mathbf{D}_{\alpha,3} \neq 0$, the expression for $G^{1/2}$ will remain positive. To determine the maximum possible range of θ^α for which $G^{1/2}$ is positive, it is necessary to keep the cross-section $\mathcal{A}(\theta^3)$ in the half plane (θ^3=constant) which contains θ^α=0 and is bounded by the line $G^{1/2}$=0.

For later convenience, it is desirable to define two second order tensors $\mathbf{\Lambda}_\alpha$ such that

$$\mathbf{\Lambda}_\alpha = \mathbf{D}_{\alpha,3} \otimes \mathbf{D}^3 \;,\; \mathbf{D}_{\alpha,3} = \mathbf{\Lambda}_\alpha \mathbf{D}_3 \;,\; \mathbf{\Lambda}_\alpha \mathbf{D}_\beta = 0 \;, \quad (5.1.13)$$

and to express the base vectors $\mathbf{G}_i$ in the forms

$$\mathbf{G}_i = (\mathbf{I} + \theta^\alpha \mathbf{\Lambda}_\alpha)\, \mathbf{D}_i \;. \quad (5.1.14)$$

Moreover, with the help of (A.7.16), (A.7.22) and (A.7.23), it can be shown that the reciprocal vectors $\mathbf{G}^i$ and the quantity $G^{1/2}$ are given by

$$\mathbf{G}^i = (\mathbf{I} + \theta^\alpha \mathbf{\Lambda}_\alpha)^{-T}\, \mathbf{D}^i \;,$$
$$G^{1/2} = D^{1/2} \det (\mathbf{I} + \theta^\alpha \mathbf{\Lambda}_\alpha) = D^{1/2} \left[\, 1 + \theta^\alpha \mathbf{\Lambda}_\alpha \bullet \mathbf{I} \,\right] \;. \quad (5.1.15)$$

In summary, a rod-like structure is considered to be "thin" if all material points in it can be mapped by the position vector (5.1.4), and if the magnitude of $(\theta^\alpha \mathbf{\Lambda}_\alpha \bullet \mathbf{I})$ remains small. Moreover, the material part P_0^* of the rod in its reference configuration is defined by the three-dimensional position vector (5.1.4) with θ^α in the cross-section $\mathcal{A}(\theta^3)$, and with θ^3 restricted to the range $[\xi_1,\xi_2]$ associated with the ends ∂P_{10}^* and ∂P_{20}^*

$$\xi_1 \leq \theta^3 \leq \xi_2 \;. \quad (5.1.16)$$

The lateral surface ∂P_{L0}^* of the rod is defined by (5.1.9), and the cross-section $\mathcal{A}(\theta^3)$ is the region contained within the closed curve $\partial\mathcal{A}(\theta^3)$ which is the intersection of the lateral surface with the plane θ^3=constant. Also, the curve $\mathcal{C}_0$ associated with θ^α=0 is called the reference curve of the rod, and it is restricted to lie within the region P_0^* occupied by the rod.

For later convenience, it is noted that the arclength parameter S in $(5.1.2)_2$ of the curve (5.1.1) can be written as a function of the coordinate θ^3 such that

$$S = \hat{S}(\theta^3) \;,\; S_1 = \hat{S}(\xi_1) \;,\; S_2 = \hat{S}(\xi_2) \;, \quad (5.1.17)$$

where S_1 and S_2 also represent the ends of the rod. Then, the region P_0 and the ends ∂P_{10} and ∂P_{20} can be defined by

$$\xi_1 \leq \theta^3 \leq \xi_2 \;,\; S_1 \leq S \leq S_2 \quad \text{in } P_0 \;,\; \theta^3 = \xi_1 \;,\; S = S_1 \quad \text{on } \partial P_{10} \;,$$
$$\theta^3 = \xi_2 \;,\; S = S_2 \quad \text{on } \partial P_{20} \;. \quad (5.1.18)$$

5.2 The Cosserat model of a rod

Within the context of the Cosserat theory, a rod is modeled as a material curve $\mathcal{C}$ with some additional kinematic structure to provide limited information about deformation through the cross-section of the rod. Specifically, with respect to the present configuration at time t, the material curve $\mathcal{C}$ is mapped by the convected coordinate θ^3, and the position vector (from a fixed origin) is denoted by

$$\mathbf{x} = \hat{\mathbf{x}}(\theta^3,t) \ . \tag{5.2.1}$$

The tangent vector $\mathbf{d}_3$ in the θ^3 direction and the element of arc-length ds of the curve $\mathcal{C}$ are given by

$$\mathbf{d}_3(\theta^3,t) = \frac{\partial \mathbf{x}}{\partial \theta^3} = \mathbf{x}_{,3} \ , \quad ds = d_{33}^{1/2}\, d\theta^3 \ , \quad d_{33} = \mathbf{d}_3 \cdot \mathbf{d}_3 \ . \tag{5.2.2}$$

In addition, the Cosserat theory endows the curve $\mathcal{C}$ with two director vectors

$$\mathbf{d}_\alpha = \hat{\mathbf{d}}_\alpha(\theta^3,t) \ , \tag{5.2.3}$$

which are defined at each material point of the curve, are nonparallel, and are nowhere tangent to the curve. This means that the vectors $\mathbf{d}_i$ are linearly independent and form a triad. Moreover, they are ordered so that the triad is right-handed with

$$d^{1/2} = \mathbf{d}_1 \times \mathbf{d}_2 \cdot \mathbf{d}_3 > 0 \ . \tag{5.2.4}$$

In a more general theory, it is possible to introduce a finite set of N director vectors, but here attention will be confined to the simpler theory with only two director vectors.

From a physical point of view, it is desirable to think of the directors $\mathbf{d}_\alpha$ as vectors that describe the deformation of material fibers in the cross-section of the rod (see Fig. 5.2.1). In order to interpret different physical types of deformation of these fibers, it is convenient to define the projections $\bar{\mathbf{d}}_\alpha$ of the vectors $\mathbf{d}_\alpha$ into the plane normal to the curve $\mathcal{C}$ by the expressions

$$\bar{\mathbf{d}}_\alpha = \mathbf{d}_\alpha - d_{33}^{-1}\, (\mathbf{d}_\alpha \cdot \mathbf{d}_3)\, \mathbf{d}_3 \ . \tag{5.2.5}$$

Then, for the general case (Naghdi and Rubin, 1984) the theory allows for: (a) normal cross-sectional extension when the magnitudes of the projections $\bar{\mathbf{d}}_\alpha$ change; (b) tangential shear deformation when the components of $\mathbf{d}_\alpha$ in the direction of $\mathbf{d}_3$ change; and (c) normal cross-sectional shear deformation when the angle between $\bar{\mathbf{d}}_1$ and $\bar{\mathbf{d}}_2$ changes.

The velocity vector $\mathbf{v}$ and the director velocities $\mathbf{w}_i$ derived from the functions (5.2.1), (5.2.2) and (5.2.3) are defined by

$$\mathbf{v} = \mathbf{v}(\theta^3,t) = \dot{\mathbf{x}} \ , \quad \mathbf{w}_i = \mathbf{w}_i(\theta^3,t) = \dot{\mathbf{d}}_i \ , \tag{5.2.6}$$

where the superposed dot $(\dot{\ })$ denotes material time differentiation holding θ^3 fixed. Furthermore, due to the fact that θ^3 is considered to be a convected Lagrangian coordinate, it follows that the director velocity $\mathbf{w}_3$ is related to the derivative of $\mathbf{v}$ by the expression

$$\mathbf{w}_3 = \mathbf{v}_{,3} \ . \tag{5.2.7}$$

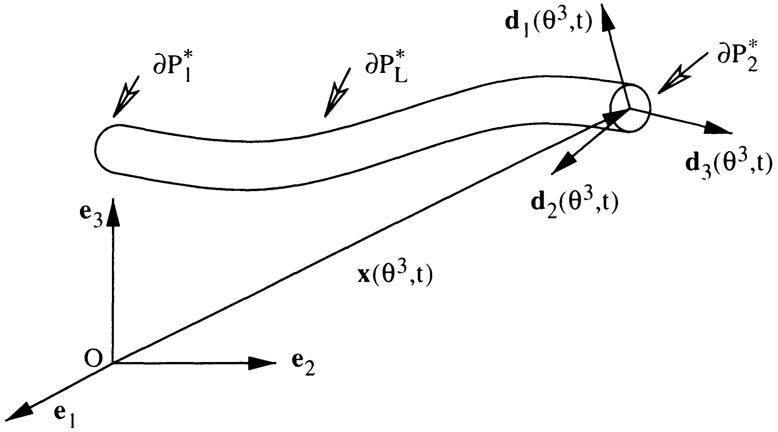

Fig. 5.2.1 The Cosserat model of a rod in its present configuration.

Motivated by the representation (5.1.4), it is natural to consider an associated kinematic assumption that the position vector $\mathbf{x}^*(\theta^i,t)$ of material points in the rod in its present configuration can be represented in the form

$$\mathbf{x}^*(\theta^i,t) = \mathbf{x}(\theta^3,t) + \theta^\alpha\, \mathbf{d}_\alpha(\theta^3,t) \ . \tag{5.2.8}$$

In contrast with the representation (5.1.4), which is always valid for a rod-like structure in its reference configuration, the representation (5.2.8) restricts material line elements in the cross-section (θ^3=constant) of the rod to remain straight. This kinematic assumption (5.2.8) will be used to motivate forms for certain quantities that appear in the balance laws of the Cosserat theory, and it can be used to provide physical interpretation of results of the theory. However, it will be seen that, strictly speaking, the Cosserat theory is used to determine the vectors $\mathbf{x}$ and $\mathbf{d}_\alpha$ which depend on only one space coordinate θ^3 and time. Consequently, within the context of the Cosserat theory there is not necessarily a direct dependence on the assumption (5.2.8). Moreover, it will be seen later that the constitutive equations of the Cosserat theory do not enforce the kinematic assumption (5.2.8) to be valid pointwise through the rod's cross-section.

In order to expand on physical interpretations based on the assumption (5.2.8), it is desirable to consider the base vectors $\mathbf{g}_i$ and the reciprocal vectors $\mathbf{d}^i$ defined by

$$\mathbf{g}_i = \mathbf{x}^*{}_{,i} \ , \quad \mathbf{g}_\alpha = \mathbf{d}_\alpha \ , \quad \mathbf{g}_3 = \mathbf{d}_3 + \theta^\alpha\, \mathbf{d}_{\alpha,3} \ ,$$
$$g^{1/2} = \mathbf{g}_1 \times \mathbf{g}_2 \bullet \mathbf{g}_3 \ , \quad \mathbf{d}^i \bullet \mathbf{d}_j = \delta^i_j \ . \tag{5.2.9}$$

Now, following the work of Naghdi (1982) and Rubin (1996), it is convenient to introduce the nonsingular second order tensor $\mathbf{F}$ such that

$$\mathbf{F} = \mathbf{d}_i \otimes \mathbf{D}^i \ , \quad \mathbf{F}^{-1} = \mathbf{D}_i \otimes \mathbf{d}^i \ , \quad \mathbf{d}_i = \mathbf{F}\,\mathbf{D}_i \ , \quad \det \mathbf{F} = d^{1/2}\, D^{-1/2} \ , \tag{5.2.10}$$

and the two second order tensors $\boldsymbol{\lambda}_\alpha$ defined such that

$$\boldsymbol{\lambda}_\alpha = \mathbf{F}^{-1}(\mathbf{d}_{\alpha,3} \otimes \mathbf{D}^3) \ , \quad \mathbf{d}_{\alpha,3} = \mathbf{F}\boldsymbol{\lambda}_\alpha\mathbf{D}_3 \ , \quad \boldsymbol{\lambda}_\alpha\mathbf{D}_\beta = 0 \ . \tag{5.2.11}$$

Thus, the base vectors $\mathbf{g}_i$ can be written in the alternative forms

$$\mathbf{g}_i = \mathbf{F}(\mathbf{I} + \theta^\alpha \boldsymbol{\lambda}_\alpha)\, \mathbf{D}_i \ . \tag{5.2.12}$$

Also, the quantity $g^{1/2}$ becomes

$$g^{1/2} = \mathbf{d}_1 \times \mathbf{d}_2 \bullet \left[\, \mathbf{d}_3 + \theta^\alpha \mathbf{d}_{\alpha,3} \,\right] = d^{1/2} \left[\, 1 + \theta^\alpha (\mathbf{d}_{\alpha,3} \bullet \mathbf{d}^3) \,\right] \ ,$$
$$g^{1/2} = d^{1/2} \det (\mathbf{I} + \theta^\alpha \boldsymbol{\lambda}_\alpha) = d^{1/2} (1 + \theta^\alpha \boldsymbol{\lambda}_\alpha \bullet \mathbf{I}) \ . \tag{5.2.13}$$

Next, with the help of (5.1.15) and the definition (2.3.8) of the three-dimensional deformation gradient $\mathbf{F}^*$, it follows that

$$\mathbf{F}^*(\theta^i,t) = \mathbf{F}(\mathbf{I} + \theta^\alpha \boldsymbol{\lambda}_\alpha)\, (\mathbf{I} + \theta^\beta \boldsymbol{\Lambda}_\beta)^{-1} \ . \tag{5.2.14}$$

This expression shows that the tensor $\mathbf{F}$ is the value of $\mathbf{F}^*$ on the reference curve (θ^α=0) of the rod. Moreover, it will presently be shown that within the context of the kinematic assumption (5.2.8), the necessary and sufficient condition for the associated three-dimensional deformation to be homogeneous is

$$\mathbf{F}(\theta^3,t) = \bar{\mathbf{F}}(t) \ , \tag{5.2.15}$$

where $\bar{\mathbf{F}}(t)$ is an arbitrary nonsingular tensor function of time only whose determinant is positive. To prove that this is a sufficient condition, use is made of the expressions $(5.1.13)_1$, $(5.2.10)_3$, $(5.2.11)_1$ and (5.2.15) to deduce that

$$\mathbf{d}_{\alpha,3} = (\mathbf{F}\mathbf{D}_\alpha)_{,3} = \mathbf{F}\mathbf{D}_{\alpha,3} \ , \quad \boldsymbol{\lambda}_\alpha = \boldsymbol{\Lambda}_\alpha \ . \tag{5.2.16}$$

Now, the expressions (5.2.15) and $(5.2.16)_2$ can be substituted into (5.2.14) to obtain

$$\mathbf{F}^*(\theta^i,t) = \bar{\mathbf{F}}(t) \ , \tag{5.2.17}$$

which proves that the three-dimensional deformation is homogeneous. On the other hand, if (5.2.17) is presumed, then (5.2.14) becomes

$$\bar{\mathbf{F}}(t) = \mathbf{F}(\mathbf{I} + \theta^\alpha \boldsymbol{\lambda}_\alpha)\, (\mathbf{I} + \theta^\beta \boldsymbol{\Lambda}_\beta)^{-1} \ . \tag{5.2.18}$$

However, since the left-hand side of (5.2.18) is independent of θ^α, it follows by setting θ^α equal to zero that $\mathbf{F}(\theta^3,t)$ is independent of the coordinates θ^3

$$\mathbf{F}(\theta^3,t) = \bar{\mathbf{F}}(t) \ , \tag{5.2.19}$$

which completes the proof. In summary, the associated three-dimensional deformation will be homogeneous if and only if $\mathbf{F}$ is independent of the coordinate θ^3 (5.2.19).

Next, using the definition (3.1.6) of the three-dimensional deformation gradient $\mathbf{F}^*$, it can be shown by integration that for homogeneous deformations

$$\mathbf{x}^* = \mathbf{F}(t)\, \mathbf{X}^* + \mathbf{c}(t) \ , \tag{5.2.20}$$

where $\mathbf{c}(t)$ is an arbitrary vector function of time only representing translation of the rod. Then, substitution of the kinematic expression (5.1.4) for $\mathbf{X}^*$ into (5.2.20), and use of $(5.2.10)_3$ yields

$$\mathbf{x}^* = \left[\mathbf{F}(t)\, \mathbf{X}(\theta^3) + \mathbf{c}(t)\right] + \theta^\alpha\, \mathbf{F}(t)\, \mathbf{D}_\alpha(\theta^3) \ ,$$
$$\mathbf{x}^*(\theta^i,t) = \mathbf{x}(\theta^3,t) + \theta^\alpha\, \mathbf{d}_\alpha(\theta^3,t) \ , \tag{5.2.21}$$

which shows that the kinematic expression (5.2.8) is exact for homogeneous deformations.

5.3 Derivation of the balance laws from the three-dimensional theory

The global forms of the balance laws of the Cosserat theory of rods are similar to those of the three-dimensional theory in the sense that they include the notions of conservation of mass and balances of linear and angular momentum. Moreover, these equations are used to determine the current values of a mass density ρ and the position vector $\mathbf{x}$ of points on the curve C of the rod. Also, the balance of angular momentum will place restrictions on the constitutive equations of the rod theory that are similar in nature to the restrictions (3.2.32) associated with the three-dimensional theory. However, in contrast with the three-dimensional theory, the Cosserat theory of rods introduces the additional kinematic quantities $\mathbf{d}_\alpha$ at each point of the curve C of the rod which also must be determined by balance laws. Consequently, the Cosserat theory of rods requires two additional balance laws called the balances of director momentum.

In this section, it will be shown that the balance laws of the Cosserat theory can be developed by using the kinematic assumption (5.2.8) and the balance laws of the three-dimensional theory. Here, attention is focused on the development of the global forms of the balance laws. However, a simpler derivation of the local equations is provided in section 5.25 for convenience. It is noted that the three-dimensional region P_0^* (with boundary ∂P_0^*) of the rod-like structure in its reference configuration is mapped to the region P^* (with boundary ∂P^*) in the present configuration. Moreover, the ends ∂P_{10}^* and ∂P_{20}^*, and the lateral surface ∂P_{L0}^* are mapped to the surfaces ∂P_1^*, ∂P_2^*, and ∂P_L^*, respectively, in the present configuration. Consequently, the boundary ∂P^* is the union of the boundaries ∂P_1^*, ∂P_2^*, and ∂P_L^*

$$\partial P^* = \partial P_1^* \cup \partial P_2^* \cup \partial P_L^* \ . \tag{5.3.1}$$

Also, since θ^i are convected coordinates, it follows from (5.1.6) and (5.1.9) that

$$\theta^3 = \xi_1 \ \text{ on } \ \partial P_1^* \ , \quad \theta^3 = \xi_2 \ \text{ on } \ \partial P_2^* \ ,$$

$$\theta^\alpha = \hat{\theta}^\alpha(\zeta,\theta^3) \ , \quad \zeta_1 \le \zeta < \zeta_2 \ , \quad \xi_1 \le \theta^3 \le \xi_2 \ \text{ on } \ \partial P_L^* \ , \tag{5.3.2}$$

where for constant θ^3, the functions $\hat{\theta}^\alpha(\zeta,\theta^3)$ in (5.1.9) and $(5.3.2)_3$ define the boundary curve $\partial\mathcal{A}$ of the extent of the cross-section $\mathcal{A}(\theta^3)$ of the rod.

The arclength parameter s of the curve (5.2.1) in the present configuration can be expressed as a function of the reference value S of the arclength by the function

$$s = \hat{s}(S,t) \ , \tag{5.3.3}$$

so that the ends of the rod S_1 and S_2 map to the values s_1 and s_2, respectively, with

$$s_1 = \hat{s}(S_1,t) \ , \quad s_2 = \hat{s}(S_2,t) \ . \tag{5.3.4}$$

Thus, the region P_0 and the ends ∂P_{10} and ∂P_{20} defined by (5.1.18), are mapped to the region P and the ends ∂P_1 and ∂P_2 such that

$$s_1 \le s \le s_2 \ \text{ in } P \ , \quad s = s_1 \ \text{ on } \partial P_1 \ , \quad s = s_2 \ \text{ on } \partial P_2 \ . \tag{5.3.5}$$

Now, with the help of the expression (3.2.5) for the element of volume dv^*, and the representation (5.3.2) of the region P^*, the total mass $(3.2.26)_1$ in P^* can be written as

$$\int_{P^*} \rho^* \, dv^* = \int_{\xi_1}^{\xi_2} [\int_{\mathcal{A}} \rho^* g^{1/2} \, d\theta^1 \, d\theta^2] \, d\theta^3$$

$$= \int_{\xi_1}^{\xi_2} m \, d\theta^3 = \int_{s_1}^{s_2} \rho \, ds = \int_P \rho \, ds \ , \tag{5.3.6}$$

where the mass density ρ (mass per unit arclength ds) has been defined so that it represents the integrated effect of the mass density through the cross-section of the rod

$$m = \rho \, d_{33}^{1/2} = \int_{\mathcal{A}} \rho^* g^{1/2} \, d\theta^1 \, d\theta^2 = \int_{\mathcal{A}} m^* \, d\theta^1 \, d\theta^2 \ . \tag{5.3.7}$$

It then follows from (3.2.1) and (5.3.6), that the global form of conservation of mass of the Cosserat rod becomes

$$\frac{d}{dt} \int_P \rho \, ds = 0 \ . \tag{5.3.8}$$

Also, it is noted that the units of m depend on the specification of the convected coordinate θ^3. However, m will have the units of mass per unit reference length if θ^3 has the units of length.

Next, with the help of the kinematic assumption (5.2.8) and the definitions (5.2.6) for the velocity $\mathbf{v}$ and the director velocities $\mathbf{w}_\alpha$, the three-dimensional balance of linear momentum (3.2.2) applied to the rod-like region P^* becomes

$$\frac{d}{dt} \int_P [\int_{\mathcal{A}} m^* (\mathbf{v} + \theta^\alpha \mathbf{w}_\alpha) \, d\theta^1 \, d\theta^2] \, d\theta^3 = \int_P [\int_{\mathcal{A}} m^* \mathbf{b}^* \, d\theta^1 \, d\theta^2] \, d\theta^3$$

$$+ \int_{\partial P_1^*} \mathbf{t}^* \, da^* + \int_{\partial P_2^*} \mathbf{t}^* \, da^* + \int_{\partial P_L^*} \mathbf{t}^* \, da^* \ . \tag{5.3.9}$$

The main objective here is to rewrite the integrals in (5.3.9) as integrals over the region P and the ends s_1 and s_2 associated with the rod's curve $\mathcal{C}$. To this end, it is noted that $\mathbf{v}$ and $\mathbf{w}_\alpha$ are independent of the coordinates θ^α so that it is convenient to define the director inertia coefficients y^α by the expression

$$m \, y^\alpha = [\int_{\mathcal{A}} m^* \theta^\alpha \, d\theta^1 \, d\theta^2] \ . \tag{5.3.10}$$

Then, the linear momentum in the part P of the rod can be written as

$$\int_{P^*} \rho^* \mathbf{v}^* \, dv^* = \int_P \rho \, (\mathbf{v} + y^\alpha \mathbf{w}_\alpha) \, ds \ . \tag{5.3.11}$$

Also, it is convenient to define the body force $\mathbf{b}_b$ per unit mass by the expression

$$m \, \mathbf{b}_b = [\int_{\mathcal{A}} m^* \mathbf{b}^* \, d\theta^1 \, d\theta^2] \ , \tag{5.3.12}$$

so that the total body force applied to the part P^* of the rod can be rewritten as

$$\int_{P^*} \rho^* \mathbf{b}^* \, dv^* = \int_P \rho \, \mathbf{b}_b \, ds \ . \tag{5.3.13}$$

The integral in (5.3.9) over the lateral surface ∂P_L^* can be expressed as an integral over the region P by using (5.2.8) and (5.3.2) to develop an expression for the unit outward normal $\mathbf{n}^*$ and the element of area da^* on this surface. In particular, on the lateral surface it follows that

$$\text{On } \partial P_L^*: \ \mathbf{x}^* = \mathbf{x}^*(\hat{\theta}^\alpha(\zeta,\theta^3),\theta^3,t) \ ,$$
$$\mathbf{n}^* \, da^* = \frac{\partial \mathbf{x}^*}{\partial \zeta} d\zeta \times \mathbf{x}^*{}_{,3} \, d\theta^3 \ ,$$
$$\mathbf{n}^* \, da^* = (\mathbf{g}_\alpha \frac{\partial \hat{\theta}^\alpha}{\partial \zeta}) \times (\mathbf{g}_\beta \frac{\partial \hat{\theta}^\beta}{\partial \theta^3} + \mathbf{g}_3) \, d\zeta \, d\theta^3 \ ,$$
$$\mathbf{n}^* \, da^* = g^{1/2} [\frac{\partial \hat{\theta}^2}{\partial \zeta} \mathbf{g}^1 - \frac{\partial \hat{\theta}^1}{\partial \zeta} \mathbf{g}^2 + \{ \frac{\partial \hat{\theta}^1}{\partial \zeta} \frac{\partial \hat{\theta}^2}{\partial \theta^3} - \frac{\partial \hat{\theta}^2}{\partial \zeta} \frac{\partial \hat{\theta}^1}{\partial \theta^3} \} \mathbf{g}^3] \, d\zeta \, d\theta^3,$$
$$\mathbf{n}^* = \frac{ [\frac{\partial \hat{\theta}^2}{\partial \zeta} \mathbf{g}^1 - \frac{\partial \hat{\theta}^1}{\partial \zeta} \mathbf{g}^2 + \{ \frac{\partial \hat{\theta}^1}{\partial \zeta} \frac{\partial \hat{\theta}^2}{\partial \theta^3} - \frac{\partial \hat{\theta}^2}{\partial \zeta} \frac{\partial \hat{\theta}^1}{\partial \theta^3} \} \mathbf{g}^3] }{\alpha(\zeta)} \ ,$$
$$\alpha(\zeta,\theta^3) = \left| \frac{\partial \hat{\theta}^2}{\partial \zeta} \mathbf{g}^1 - \frac{\partial \hat{\theta}^1}{\partial \zeta} \mathbf{g}^2 + \{ \frac{\partial \hat{\theta}^1}{\partial \zeta} \frac{\partial \hat{\theta}^2}{\partial \theta^3} - \frac{\partial \hat{\theta}^2}{\partial \zeta} \frac{\partial \hat{\theta}^1}{\partial \theta^3} \} \mathbf{g}^3 \right| \ ,$$
$$da^* = g^{1/2} \, \alpha(\zeta,\theta^3) \, d\zeta \, d\theta^3 \ , \tag{5.3.14}$$

where all quantities in (5.3.14) are evaluated on the lateral surface with $\theta^\alpha = \hat{\theta}^\alpha(\zeta,\theta^3)$. Now, using these expressions the total force applied to the lateral surface can be expressed as

$$\int_{\partial P_L^*} \mathbf{t}^* \, da^* = \int_P \rho \, \mathbf{b}_c \, ds \ , \tag{5.3.15}$$

where $\mathbf{b}_c$ is the contact force per unit mass applied to the lateral surface defined by

$$m \, \mathbf{b}_c = \int_{\partial \mathcal{A}} g^{1/2} \, \mathbf{T}^* \, [\frac{\partial \hat{\theta}^2}{\partial \zeta} \mathbf{g}^1 - \frac{\partial \hat{\theta}^1}{\partial \zeta} \mathbf{g}^2 + \{ \frac{\partial \hat{\theta}^1}{\partial \zeta} \frac{\partial \hat{\theta}^2}{\partial \theta^3} - \frac{\partial \hat{\theta}^2}{\partial \zeta} \frac{\partial \hat{\theta}^1}{\partial \theta^3} \} \mathbf{g}^3] \, d\zeta \ ,$$
$$m \, \mathbf{b}_c = \int_{\partial \mathcal{A}} [\, g^{1/2} \, \alpha(\zeta,\theta^3) \, \mathbf{t}^* \,] \, d\zeta \ . \tag{5.3.16}$$

In the above, the stress vector $\mathbf{t}^*$ on the lateral surface is defined by (3.2.3) in terms of the unit outward normal to this surface.

Similarly, on the end ∂P_2^* it can be shown that

$$\text{On } \partial P_2^*: \ \mathbf{x}^* = \mathbf{x}^*(\theta^\alpha,\xi_2,t) \ ,$$
$$\mathbf{n}^* \, da^* = \mathbf{x}^*{}_{,1} d\theta^1 \times \mathbf{x}^*{}_{,2} d\theta^2 = \mathbf{g}_1 \times \mathbf{g}_2 d\theta^1 d\theta^2$$
$$\mathbf{n}^* \, da^* = g^{1/2} \, \mathbf{g}^3 \, d\theta^1 \, d\theta^2 \ , \quad \mathbf{n}^* = \frac{\mathbf{g}^3}{| \mathbf{g}^3 |} \ ,$$
$$da^* = g^{1/2} \, | \mathbf{g}^3 | \, d\theta^1 \, d\theta^2 \ , \tag{5.3.17}$$

where all quantities in (5.3.17) are evaluated on the end $\xi=\xi_2$. Also, by recognizing that the outward normal to the end ∂P_1^* has a positive component in the negative $\mathbf{g}^3$ direction, it follows that

$$\text{On } \partial P_1^*: \ \mathbf{x}^* = \mathbf{x}^*(\theta^\alpha,\xi_1,t)$$
$$\mathbf{n}^* \, da^* = - \mathbf{x}^*{}_{,1} d\theta^1 \times \mathbf{x}^*{}_{,2} d\theta^2 = - \mathbf{g}_1 \times \mathbf{g}_2 d\theta^1 d\theta^2$$

$$\mathbf{n}^* \, da^* = - g^{1/2} \, \mathbf{g}^3 \, d\theta^1 \, d\theta^2 \ , \quad \mathbf{n}^* = - \frac{\mathbf{g}^3}{| \mathbf{g}^3 |} \ ,$$

$$da^* = g^{1/2} \, | \mathbf{g}^3 | \, d\theta^1 \, d\theta^2 \ . \tag{5.3.18}$$

where all quantities in (5.3.18) are evaluated on the end $\xi=\xi_1$. Next, to rewrite the integrals over the ends of the rod, it is convenient to define $\mathbf{t}^3$ as the total force acting on an end of the rod θ^3=constant which has an outward normal with a positive component in the $\mathbf{g}^3$ direction such that

$$\mathbf{t}^3 = \int_{\mathcal{A}} \mathbf{t}^{*3} \, d\theta^1 \, d\theta^2 = \int_{\mathcal{A}} g^{1/2} \, | \mathbf{g}^3 | \, \mathbf{t}^* \, d\theta^1 \, d\theta^2 \ . \tag{5.3.19}$$

Consequently, the integrals over the ends of the rod can be expressed in the forms

$$\int_{\partial P_1^*} \mathbf{t}^* \, da^* = - \mathbf{t}^3(s_1) \ , \quad \int_{\partial P_2^*} \mathbf{t}^* \, da^* = \mathbf{t}^3(s_2) \ , \tag{5.3.20}$$

where $\mathbf{t}^3(s_1)$ and $\mathbf{t}^3(s_2)$ denote the values of $\mathbf{t}^3$ on the ends $s=s_1$ and $s=s_2$, respectively. Thus, the total force applied to both ends can be written as

$$\int_{\partial P_1^*} \mathbf{t}^* \, da^* + \int_{\partial P_2^*} \mathbf{t}^* \, da^* = [\, \mathbf{t}^3 \,]_1^2 \ , \tag{5.3.21}$$

where the notation

$$[\, f \,]_1^2 = f(s_2) - f(s_1) \ , \tag{5.3.22}$$

has been introduced for convenience.

Now, with the help of (3.2.2) and the expressions (5.3.11), (5.3.13), (5.3.15), (5.3.21), the balance of linear momentum associated with the Cosserat theory can be written in the global form

$$\frac{d}{dt} \int_P \rho \, (\mathbf{v} + y^\alpha \, \mathbf{w}_\alpha) \, ds = \int_P \rho \, \mathbf{b} \, ds + [\, \mathbf{t}^3 \,]_1^2 \ , \tag{5.3.23}$$

where the external force $\mathbf{b}$ per unit mass applied to the rod is defined by

$$\mathbf{b} = \mathbf{b}_b + \mathbf{b}_c \ . \tag{5.3.24}$$

In this regard, it should be emphasized that the external force $\mathbf{b}$ is due to two different physical sources. One part $\mathbf{b}_b$ is due to the three-dimensional body force applied to the rod, and the other part $\mathbf{b}_c$ is due to contact force applied to the lateral surface of the rod.

Before developing the balance of angular momentum, it is convenient to develop the balances of director momentum. Here, the balances of director momentum will be developed as averaged first moments of the balance of linear momentum with respect to the coordinates θ^α in the rod's cross-section. Specifically, the averaged form (3.6.3) of the balance of linear momentum is applied to the rod-like region P^* with the weighting function ϕ taken equal to the coordinates θ^α, so that with the help of the kinematic assumption (5.2.8), the expression (3.6.3) yields

$$\frac{d}{dt} \int_P \Big[\int_{\mathcal{A}} m^* \, (\theta^\alpha \mathbf{v} + \theta^\alpha \theta^\beta \, \mathbf{w}_\beta) \, d\theta^1 \, d\theta^2 \, \Big] \, d\theta^3$$

$$= \int_P \Big[\int_{\mathcal{A}} \{ m^* \, \theta^\alpha \mathbf{b}^* - \mathbf{t}^{*\alpha} \} \, d\theta^1 \, d\theta^2 \, \Big] \, d\theta^3$$

$$+ \int_{\partial P_1^*} \theta^\alpha \mathbf{t}^* \, da^* + \int_{\partial P_2^*} \theta^\alpha \mathbf{t}^* \, da^* + \int_{\partial P_L^*} \theta^\alpha \mathbf{t}^* \, da^* \ . \qquad (5.3.25)$$

Inspection of (5.3.25) indicates that it is convenient to introduce the additional director inertia coefficients $y^{\alpha\beta}$ that are defined in terms of the second moment of the mass density through the rod's cross-section by the equation

$$m \, y^{\alpha\beta} = m \, y^{\beta\alpha} = \int_{\mathcal{A}} m^* \, \theta^\alpha \theta^\beta \, d\theta^1 \, d\theta^2 \ , \qquad (5.3.26)$$

so that the first moments of linear momentum can be expressed as

$$\int_{P^*} \rho^* \, \theta^\alpha \, \mathbf{v}^* \, dv^* = \int_P \rho \, (y^\alpha \, \mathbf{v} + y^{\alpha\beta} \, \mathbf{w}_\beta \,) \, ds \ . \qquad (5.3.27)$$

Also, it is convenient to define the external director couples $\mathbf{b}_b^\alpha$ as the first moments of the body force per unit mass by the expressions

$$m \, \mathbf{b}_b^\alpha = \int_{\mathcal{A}} \{ m^* \, \theta^\alpha \mathbf{b}^* \} \, d\theta^1 \, d\theta^2 \ , \qquad (5.3.28)$$

so that the first moments of the total body force applied to the part P^* of the rod can be rewritten as

$$\int_{P^*} \rho^* \, \theta^\alpha \, \mathbf{b}^* \, dv^* = \int_P \rho \, \mathbf{b}_b^\alpha \, ds \ . \qquad (5.3.29)$$

Furthermore, comparison of (5.3.9) and (5.3.25) indicates the presence of extra terms in (5.3.25) which are related to the integrated effect of the quantities $\mathbf{t}^{*\alpha}$. Consequently, it is convenient to define the intrinsic director couples $\mathbf{t}^\alpha$ such that

$$\mathbf{t}^\alpha = \int_{\mathcal{A}} \mathbf{t}^{*\alpha} \, d\theta^1 \, d\theta^2 \ , \quad \int_P \Big[\int_{\mathcal{A}} \mathbf{t}^{*\alpha} \, d\theta^1 \, d\theta^2 \Big] \, d\theta^3 = \int_P d_{33}^{-1/2} \, \mathbf{t}^\alpha \, ds \ . \qquad (5.3.30)$$

Next, attention is focused on the effect of the traction vector on the lateral surface. In particular, with the help of the expressions (5.3.14) it follows that the first moments of the total force applied to the lateral surface can be written in the form

$$\int_{\partial P_L^*} \theta^\alpha \, \mathbf{t}^* \, da^* = \int_P \rho \, \mathbf{b}_c^\alpha \, ds \ , \qquad (5.3.31)$$

where $\mathbf{b}_c^\alpha$ are the first moments of the contact force per unit mass applied to the lateral surface defined by

$$m \, \mathbf{b}_c^\alpha = \int_{\partial\mathcal{A}} g^{1/2} \, \hat{\theta}^\alpha(\zeta,\theta^3) \, \mathbf{T}^* \Big[\frac{\partial\hat{\theta}^2}{\partial\zeta} \, \mathbf{g}^1 - \frac{\partial\hat{\theta}^1}{\partial\zeta} \, \mathbf{g}^2 + \Big\{ \frac{\partial\hat{\theta}^1}{\partial\zeta} \frac{\partial\hat{\theta}^2}{\partial\theta^3} - \frac{\partial\hat{\theta}^2}{\partial\zeta} \frac{\partial\hat{\theta}^1}{\partial\theta^3} \Big\} \mathbf{g}^3 \Big] \, d\zeta \ ,$$

$$m \, \mathbf{b}_c^\alpha = \int_{\partial\mathcal{A}} \Big[\, g^{1/2} \, \hat{\theta}^\alpha(\zeta,\theta^3) \, \alpha(\zeta,\theta^3) \, \mathbf{t}^* \, \Big] \, d\zeta \ . \qquad (5.3.32)$$

Next, it is convenient to define $\mathbf{m}^\alpha$ as the first moments of the traction vector acting on an end of the rod θ^3=constant which has an outward normal with a positive component in the $\mathbf{g}^3$ direction such that

$$\mathbf{m}^\alpha = \int_{\mathcal{A}} \theta^\alpha \, \mathbf{t}^{*3} \, d\theta^1 \, d\theta^2 = \int_{\mathcal{A}} g^{1/2} \, \theta^\alpha \, | \, \mathbf{g}^3 \, | \, \mathbf{t}^* \, d\theta^1 \, d\theta^2 \ . \qquad (5.3.33)$$

Then, the first moments of the traction vectors on the ends can be expressed in the forms

$$\int_{\partial P_1^*} \theta^\alpha \, \mathbf{t}^* \, da^* = -\,\mathbf{m}^\alpha(s_1) \ , \quad \int_{\partial P_2^*} \theta^\alpha \, \mathbf{t}^* \, da^* = \mathbf{m}^\alpha(s_2) \ , \tag{5.3.34}$$

where $\mathbf{m}^\alpha(s_1)$ and $\mathbf{m}^\alpha(s_2)$ denote the values of $\mathbf{m}^\alpha$ on the ends $s{=}s_1$ and $s{=}s_2$, respectively. Thus, the total of the first moments applied to both ends can be written as

$$\int_{\partial P_1^*} \theta^\alpha \, \mathbf{t}^* \, da^* + \int_{\partial P_2^*} \theta^\alpha \, \mathbf{t}^* \, da^* = \Big[\, \mathbf{m}^\alpha \,\Big]_1^2 \ . \tag{5.3.35}$$

Using the expressions (5.3.27), (5.3.29)-(5.3.31) and (5.3.35), the balances of director momentum (5.3.25) can be written in the simpler forms

$$\frac{d}{dt}\int_P \rho \, (y^\alpha \, \mathbf{v} + y^{\alpha\beta} \, \mathbf{w}_\beta) \, ds = \int_P [\rho \, \mathbf{b}^\alpha - d_{33}^{-1/2} \, \mathbf{t}^\alpha \,] \, ds + \Big[\, \mathbf{m}^\alpha \,\Big]_1^2 \ , \tag{5.3.36}$$

where the external director couples $\mathbf{b}^\alpha$ per unit mass applied to the rod are the sums of the parts $\mathbf{b}_b^\alpha$ due to body force and the parts $\mathbf{b}_c^\alpha$ due to contact force on the lateral surface

$$\mathbf{b}^\alpha = \mathbf{b}_b^\alpha + \mathbf{b}_c^\alpha \ . \tag{5.3.37}$$

Here, it is of interest to note that the theoretical structure of the balances of director momentum (5.3.36) differs from that of the balance of linear momentum (5.3.23) due to the presence of the intrinsic director couples $\mathbf{t}^\alpha$. In this sense, the averaged form (3.6.3) of the three-dimensional balance of linear momentum provided important theoretical guidance for motivating the form (5.3.36) of director momentum.

Returning to the analysis of angular momentum, it follows that when the three-dimensional form (3.2.4) for the balance of angular momentum is applied to the rod-like region P^*, and the kinematic assumption (5.2.8) is used, it can be shown that

$$\begin{aligned} &\frac{d}{dt}\int_{P^*} \rho^* \Big[\mathbf{x} \times (\mathbf{v} + \theta^\alpha \, \mathbf{w}_\alpha) + \mathbf{d}_\alpha \times (\theta^\alpha \, \mathbf{v} + \theta^\alpha \, \theta^\beta \, \mathbf{w}_\beta) \,\Big] \, dv^* \\ &= \int_{P^*} \rho^* \Big[\, (\mathbf{x} \times \mathbf{b}^*) + \, (\mathbf{d}_\alpha \times \theta^\alpha \, \mathbf{b}^*) \,\Big] \, dv^* \\ &+ \int_{\partial P_1^*} \{(\mathbf{x} \times \mathbf{t}^*) + (\mathbf{d}_\alpha \times \theta^\alpha \, \mathbf{t}^*)\} \, da^* + \int_{\partial P_2^*} \{(\mathbf{x} \times \mathbf{t}^*) + (\mathbf{d}_\alpha \times \theta^\alpha \, \mathbf{t}^*)\} \, da^* \\ &\qquad + \int_{\partial P_L^*} \{(\mathbf{x} \times \mathbf{t}^*) + (\mathbf{d}_\alpha \times \theta^\alpha \, \mathbf{t}^*)\} \, da^* \ . \end{aligned} \tag{5.3.38}$$

Moreover, the previous definitions can be used to rewrite the global form of the balance of angular momentum of the Cosserat theory in the simpler form

$$\begin{aligned} &\frac{d}{dt}\int_P \rho \, \Big[\mathbf{x} \times (\mathbf{v} + y^\alpha \, \mathbf{w}_\alpha) + \mathbf{d}_\alpha \times (y^\alpha \, \mathbf{v} + y^{\alpha\beta} \, \mathbf{w}_\beta)\Big] \, ds \\ &= \int_P [\mathbf{x} \times \rho \, \mathbf{b} + \mathbf{d}_\alpha \times \rho \, \mathbf{b}^\alpha \,] \, ds + \Big[\, \mathbf{x} \times \mathbf{t}^3 + \mathbf{d}_\alpha \times \mathbf{m}^\alpha \,\Big]_1^2 \ . \end{aligned} \tag{5.3.39}$$

Inspection of (5.3.39) reveals that the intrinsic director couples $\mathbf{t}^\alpha$ do not contribute to the balance of angular momentum even though they do contribute to the balances of director momentum.

Before closing this section, it is desirable to develop expressions for the rate of work $\mathcal{W}$ done on the rod and the kinetic energy $\mathcal{K}$ of the rod. To this end, the kinematic assumption (5.2.8) is used and the expressions (3.4.1) and (3.4.2) of the three-dimensional theory are evaluated for the rod-like region P^* to obtain

$$\mathcal{W}= \int_{P^*} \rho^* \mathbf{b}^* \bullet (\mathbf{v} + \theta^\alpha \mathbf{w}_\alpha)\, dv^* + \int_{\partial P_1^*} \mathbf{t}^* \bullet (\mathbf{v} + \theta^\alpha \mathbf{w}_\alpha)\, da^*$$

$$+ \int_{\partial P_2^*} \mathbf{t}^* \bullet (\mathbf{v} + \theta^\alpha \mathbf{w}_\alpha)\, da^* + \int_{\partial P_L^*} \mathbf{t}^* \bullet (\mathbf{v} + \theta^\alpha \mathbf{w}_\alpha)\, da^* ,$$

$$\mathcal{K}= \int_{P^*} \frac{1}{2} \rho^* (\mathbf{v} + \theta^\alpha \mathbf{w}_\alpha) \bullet (\mathbf{v} + \theta^\beta \mathbf{w}_\beta)\, dv^* \ . \tag{5.3.40}$$

Now, with the help of the above definitions, these expressions can be written in the simpler forms

$$\mathcal{W}= \int_P \rho\, (\mathbf{b} \bullet \mathbf{v} + \mathbf{b}^\alpha \bullet \mathbf{w}_\alpha)\, ds + \left[\, \mathbf{t}^3 \bullet \mathbf{v} + \mathbf{m}^\alpha \bullet \mathbf{w}_\alpha \,\right]_1^2 \ ,$$

$$\mathcal{K}= \int_P \frac{1}{2} \rho \, (\, \mathbf{v} \bullet \mathbf{v} + 2\, y^\alpha\, \mathbf{v} \bullet \mathbf{w}_\alpha + y^{\alpha\beta}\, \mathbf{w}_\alpha \bullet \mathbf{w}_\alpha \,)\, ds \ . \tag{5.3.41}$$

Also, it is noted that in view of the local three-dimensional form (3.2.28) of the conservation of mass, the director inertia coefficients y^α in (5.3.10) and $y^{\alpha\beta}$ in (5.3.26) are functions of the coordinate θ^3 only and therefore are independent of time

$$y^\alpha = y^\alpha(\theta^3)\ ,\quad \dot{y}^\alpha = 0\ ,\quad y^{\alpha\beta} = y^{\alpha\beta}(\theta^3)\ ,\quad \dot{y}^{\alpha\beta} = 0\ . \tag{5.3.42}$$

Moreover, since the kinetic energy must be a nonnegative function of the velocities, it follows from the expression

$$\mathbf{v} \bullet \mathbf{v} + 2\, y^\alpha\, \mathbf{v} \bullet \mathbf{w}_\alpha + y^{\alpha\beta}\, \mathbf{w}_\alpha \bullet \mathbf{w}_\beta = (\mathbf{v} + y^\alpha\, \mathbf{w}_\alpha) \bullet (\mathbf{v} + y^\beta\, \mathbf{w}_\beta)$$

$$+ (y^{\alpha\beta} - y^\alpha\, y^\beta)\, \mathbf{w}_\alpha \bullet \mathbf{w}_\beta\ , \tag{5.3.43}$$

that y^α and $y^{\alpha\beta}$ are further restricted by the condition that $(y^{\alpha\beta} - y^\alpha y^\beta)$ is positive semi-definite. This means that the eigenvalues of $(y^{\alpha\beta} - y^\alpha y^\beta)$ must be nonnegative. In particular, the characteristic equation for these eigenvalues λ can be written in the form

$$\lambda^2 - I_1\, \lambda + I_2 = 0\ , \tag{5.3.44}$$

where I_1, I_2 are the principle invariants of $(y^{\alpha\beta} - y^\alpha y^\beta)$ which are defined by

$$I_1 = y^{\alpha\alpha} - y^\alpha\, y^\alpha\ ,\quad I_2 = \det\, (y^{\alpha\beta} - y^\alpha\, y^\beta)\ . \tag{5.3.45}$$

Since $(y^{\alpha\beta} - y^\alpha y^\beta)$ is real and symmetric, the roots of (5.3.44) are real. Now the standard solution of (5.3.44) gives the roots

$$\lambda = (I_1/2) \pm \sqrt{(I_1/2)^2 - I_2}\ . \tag{5.3.46}$$

Thus, the necessary and sufficient conditions for these roots to also be nonnegative, can easily be shown to be

$$I_1 \geq 0\ ,\quad I_2 \geq 0\ . \tag{5.3.47}$$

Finally, it is noted that the kinetic energy is required to be positive semi-definite instead of positive definite, because occasionally in the Cosserat theory it is desirable to ignore director inertia (y^α=0; $y^{\alpha\beta}$=0) due to deformation and rotation of the directors, while keeping the effect of inertia ($m \neq 0$) in the balance of linear momentum.

5.4 Balance laws by the direct approach

In the previous section, the global forms of conservation of mass and the balances of linear momentum, director momentum and angular momentum were developed by using the kinematic assumption (5.2.8), together with the balance laws of the three-dimensional theory. From the point of view of the full three-dimensional theory, the Cosserat theory of rods with two directors is an approximate theory whenever line elements in the rod's cross-section do not remain straight.

However, as will be seen in this section, the Cosserat theory of rods can be presented by a direct approach in which the theory is an exact nonlinear theory. From this perspective, the Cosserat theory is considered to be a *model* of a rod-like structure, and the balance laws are postulated without any direct connection with the three-dimensional theory. Specifically, the Cosserat theory models the rod as a curve $\mathcal{C}$ in the present configuration at time t. Material points on this curve are identified by constant values of the convected Lagrangian coordinate θ^3, and the kinematics of the Cosserat theory include the position vector $\mathbf{x}(\theta^3,t)$ (relative to a fixed origin O) and the director vectors $\mathbf{d}_\alpha(\theta^3,t)$ which are specified by the functional forms

$$\mathbf{x} = \hat{\mathbf{x}}(\theta^3,t) \;\;,\;\; \mathbf{d}_\alpha = \hat{\mathbf{d}}_\alpha(\theta^3,t) \;\;. \tag{5.4.1}$$

Also, the velocity $\mathbf{v}$ and the director velocities $\mathbf{w}_i$ are defined by (5.2.6). Moreover, each material point is endowed with a mass density ρ per unit present arclength of $\mathcal{C}$, and director inertia coefficients y^α and $y^{\alpha\beta}$ which can depend on θ^3 but are independent of time

$$\rho = \rho(\theta^3,t) \;\;,\;\; y^\alpha = y^\alpha(\theta^3) \;\;,\;\; y^{\alpha\beta} = y^{\alpha\beta}(\theta^3) = y^{\beta\alpha} \;, \tag{5.4.2}$$

with $y^{\alpha\beta}$ being symmetric.

An arbitrary material part of the Cosserat curve $\mathcal{C}$ is denoted by P and its ends ∂P_1 and ∂P_2 are defined by $\theta^3=\xi_1$ on ∂P_1, and $\theta^3=\xi_2$ on ∂P_2 (with $\xi_2>\xi_1$). Each material point of P is subjected to a specific (per unit mass) external force $\mathbf{b}$ and specific external director couples $\mathbf{b}^\alpha$ which are due to body forces ($\mathbf{b}_b$ and $\mathbf{b}_b^\alpha$) and contact forces ($\mathbf{b}_c$ and $\mathbf{b}_c^\alpha$) applied to the lateral surface of the rod

$$\mathbf{b} = \mathbf{b}(\theta^3,t) = \mathbf{b}_b(\theta^3,t) + \mathbf{b}_c(\theta^3,t) \;\;,\;\; \mathbf{b}^\alpha = \mathbf{b}^\alpha(\theta^3,t) = \mathbf{b}_b^\alpha(\theta^3,t) + \mathbf{b}_c^\alpha(\theta^3,t) \;. \tag{5.4.3}$$

In addition, each material point of P is subjected to intrinsic director couples $\mathbf{t}^\alpha$ per unit length $d\theta^3$ of $\mathcal{C}$

$$\mathbf{t}^\alpha = \mathbf{t}^\alpha(\theta^3,t) \;\;. \tag{5.4.4}$$

Also, an end θ^3=constant of the rod which has an outward normal with a positive component in the $\mathbf{d}_3$ direction, is subjected to a contact force $\mathbf{t}^3$ and contact director couples $\mathbf{m}^\alpha$ such that

$$\mathbf{t}^3 = \mathbf{t}^3(\theta^3,t) \;\;,\;\; \mathbf{m}^\alpha = \mathbf{m}^\alpha(\theta^3,t) \;\;. \tag{5.4.5}$$

Using these definitions, the conservation of mass and the balances of linear momentum and director momentum are postulated in the forms

$$\frac{d}{dt}\int_P \rho \, ds = 0 \ , \ \ \frac{d}{dt}\int_P \rho \, (\mathbf{v} + y^\alpha \mathbf{w}_\alpha) \, ds = \int_P \rho \, \mathbf{b} \, ds + [\ \mathbf{t}^3\ ]_1^2 \ ,$$
$$\frac{d}{dt}\int_P \rho \, (y^\alpha \mathbf{v} + y^{\alpha\beta} \mathbf{w}_\beta) \, ds = \int_P [\rho \, \mathbf{b}^\alpha - d_{33}^{-1/2} \, \mathbf{t}^\alpha\ ] \, ds + [\ \mathbf{m}^\alpha\ ]_1^2 \ , \tag{5.4.6}$$

where ds is the present element of arclength of P, and the notation $[\ \mathbf{t}^3\ ]_1^2$ for the ends of the rod is defined in (5.3.22). Also, the balance of angular momentum about the fixed origin O is postulated in the form

$$\frac{d}{dt}\int_P \rho \, [\mathbf{x} \times (\mathbf{v} + y^\alpha \mathbf{w}_\alpha) + \mathbf{d}_\alpha \times (y^\alpha \mathbf{v} + y^{\alpha\beta} \mathbf{w}_\beta)] \, ds$$
$$= \int_P [\mathbf{x} \times \rho \, \mathbf{b} + \mathbf{d}_\alpha \times \rho \, \mathbf{b}^\alpha\ ] \, ds + [\ \mathbf{x} \times \mathbf{t}^3 + \mathbf{d}_\alpha \times \mathbf{m}^\alpha\ ]_1^2 \ . \tag{5.4.7}$$

In order to develop the local forms of the balance laws (5.4.6) and (5.4.7), it is necessary to develop an expression for the time derivative of an integral over the material region P which changes with time, and it is necessary to convert the effects of the forces and couples applied to the ends of the rod to integrals over the region P.

To this end, it is noted that the material region P in the present configuration which depends on time, can be mapped to the material region P_0 in a fixed reference configuration of the rod which does not depend on time. Specifically, the arclength element ds in the present configuration $(5.2.2)_2$ can be related to the arclength element dS in the reference configuration $(5.1.2)_2$ by the expression

$$ds = (d_{33}^{1/2} \, D_{33}^{-1/2}) \, dS \ . \tag{5.4.8}$$

Thus, an integral over the material region P can be expressed as an integral over the corresponding material region P_0 such that

$$\frac{d}{dt}\int_P (\rho \, \phi) \, ds = \frac{d}{dt}\int_{P_0} (\rho \, \phi \, d_{33}^{1/2}) \, D_{33}^{-1/2} \, dS \ . \tag{5.4.9}$$

However, since P_0, $D_{33}^{1/2}$ and dS are independent of time, the differentiation operation and the integration operation can be interchanged (assuming sufficient continuity) so that (5.4.9) can be rewritten in the form

$$\frac{d}{dt}\int_P (\rho \, \phi) \, ds = \int_{P_0} \overset{\bullet}{\overline{(\rho \ \phi \ d_{33}^{1/2})}} \, D_{33}^{-1/2} \, dS \ . \tag{5.4.10}$$

Now, by taking the material derivative of the expression (5.2.2), it can be shown that

$$\overset{\bullet}{\overline{(d_{33}^{1/2})}} = d_{33}^{1/2} \, \mathrm{div}_c \, \mathbf{v} = d_{33}^{1/2} \, [\ \mathbf{v}_{,3} \bullet d_{33}^{-1} \, \mathbf{d}_3\ ] = d_{33}^{1/2} \, [\frac{\partial \mathbf{v}}{\partial s} \bullet \frac{\partial \mathbf{x}}{\partial s}] \ , \tag{5.4.11}$$

where the divergence operator div_c of an arbitrary tensor function $\mathbf{T}(\theta^3,t)$ with respect to the curve C, is defined by the equation

$$\mathrm{div}_c \, \mathbf{T} = \mathbf{T}_{,3} \bullet d_{33}^{-1} \, \mathbf{d}_3 = \frac{\partial \mathbf{T}}{\partial s} \bullet \frac{\partial \mathbf{x}}{\partial s} \ , \tag{5.4.12}$$

and use has been made of $(5.2.2)_2$ to relate derivatives of an arbitrary function f by

$$f = \hat{f}(\theta^3,t) = \tilde{f}(s,t) \ , \ \ \frac{\partial \tilde{f}}{\partial s} = d_{33}^{1/2} \frac{\partial \hat{f}}{\partial \theta^3} \ . \tag{5.4.13}$$

Also, the vector $\partial\mathbf{x}/\partial s$ in (5.4.11) is a unit vector tangent to the curve C in the present configuration. It then follows that (5.4.10) can be rewritten in the form

$$\frac{d}{dt}\int_P (\rho\,\phi)\;ds = \int_P \left[\,\rho\,\dot{\phi} + \phi\,\{\dot{\rho} + \rho\,\mathrm{div}_c\,\mathbf{v}\}\,\right] ds \;. \tag{5.4.14}$$

Next, taking ϕ equal to unity in (5.4.14), and assuming sufficient continuity, and that the conservation of mass $(5.4.6)_1$ holds for arbitrary material parts P, the local form of conservation of mass becomes

$$\dot{\rho} + \rho\,\mathrm{div}_c\,\mathbf{v} = 0 \;\;,\quad m = \rho\,d_{33}^{1/2} = m(\theta^3) \;\;,\quad \dot{m} = 0 \;. \tag{5.4.15}$$

Also, equation (5.4.14) reduces to

$$\frac{d}{dt}\int_P (\rho\,\phi)\;ds = \int_P \left[\,\rho\,\dot{\phi}\,\right] ds \;. \tag{5.4.16}$$

Thus, the left-hand sides of the balance laws $(5.4.6)_{2,3}$ and (5.4.7) can be expressed as

$$\frac{d}{dt}\int_P \rho\,(\mathbf{v} + y^\alpha\,\mathbf{w}_\alpha)\;ds = \int_P \rho\,(\dot{\mathbf{v}} + y^\alpha\,\dot{\mathbf{w}}_\alpha)\;ds \;\;,$$

$$\frac{d}{dt}\int_P \rho\,(y^\alpha\,\mathbf{v} + y^{\alpha\beta}\,\mathbf{w}_\beta)\;ds = \int_P \rho\,(y^\alpha\,\dot{\mathbf{v}} + y^{\alpha\beta}\,\dot{\mathbf{w}}_\beta)\;ds \;\;,$$

$$\frac{d}{dt}\int_P \rho\,\left[\mathbf{x}\times(\mathbf{v} + y^\alpha\,\mathbf{w}_\alpha) + \mathbf{d}_\alpha\times(y^\alpha\,\mathbf{v} + y^{\alpha\beta}\,\mathbf{w}_\beta)\right] ds$$

$$= \int_P \rho\,\left[\mathbf{x}\times(\dot{\mathbf{v}} + y^\alpha\,\dot{\mathbf{w}}_\alpha) + \mathbf{d}_\alpha\times(y^\alpha\,\dot{\mathbf{v}} + y^{\alpha\beta}\,\dot{\mathbf{w}}_\beta)\right] ds \;. \tag{5.4.17}$$

Moreover, with the help of (5.3.22), it can be shown that

$$\left[\,\mathbf{t}^3\,\right]_1^2 = \int_P \frac{\partial\mathbf{t}^3}{\partial s}\,ds \;\;,\quad \left[\,\mathbf{m}^\alpha\,\right]_1^2 = \int_P \frac{\partial\mathbf{m}^\alpha}{\partial s}\;ds \;\;,$$

$$\left[\,\mathbf{x}\times\mathbf{t}^3 + \mathbf{d}_\alpha\times\mathbf{m}^\alpha\,\right]_1^2 = \int_P \Big[\,\Big\{\mathbf{x}\times\frac{\partial\mathbf{t}^3}{\partial s} + \mathbf{d}_\alpha\times\frac{\partial\mathbf{m}^\alpha}{\partial s}\Big\}$$

$$+\, d_{33}^{-1/2}\,\left\{\mathbf{d}_3\times\mathbf{t}^3 + \mathbf{d}_{\alpha,3}\times\mathbf{m}^\alpha\right\}\Big]\;ds \;. \tag{5.4.18}$$

Using these results and again assuming sufficient continuity, and that the balance laws $(5.4.6)_{2,3}$ and (5.4.7) hold for arbitrary material parts P, the local forms of the balances of linear and director momentum become

$$\rho\,(\dot{\mathbf{v}} + y^\alpha\,\dot{\mathbf{w}}_\alpha) = \rho\,\mathbf{b} + \frac{\partial\mathbf{t}^3}{\partial s} \;\;,\quad \rho\,(y^\alpha\,\dot{\mathbf{v}} + y^{\alpha\beta}\,\dot{\mathbf{w}}_\beta) = \rho\,\mathbf{b}^\alpha - d_{33}^{-1/2}\,\mathbf{t}^\alpha + \frac{\partial\mathbf{m}^\alpha}{\partial s} \;, \tag{5.4.19}$$

and the local form of the balance of angular momentum reduces to

$$\mathbf{t}^i\times\mathbf{d}_i + \mathbf{m}^\alpha\times\mathbf{d}_{\alpha,3} = 0 \;\;, \tag{5.4.20}$$

where the order of the cross product has been inverted for later convenience. Next, recalling from appendix A that the permutation tensor $\boldsymbol{\varepsilon}$ has the property (A.5.15) that for any two vectors $\mathbf{a}$ and $\mathbf{b}$

$$\mathbf{a}\times\mathbf{b} = \boldsymbol{\varepsilon}\bullet(\mathbf{a}\otimes\mathbf{b}) \;\;, \tag{5.4.21}$$

it follows that the balance of angular momentum (5.4.20) can be rewritten as

$$\boldsymbol{\varepsilon}\bullet\left[\mathbf{t}^i\otimes\mathbf{d}_i + \mathbf{m}^\alpha\otimes\mathbf{d}_{\alpha,3}\right] = 0 \;. \tag{5.4.22}$$

Thus, by defining the second order tensor $\mathbf{T}$

$$d_{33}^{1/2}\,\mathbf{T} = \mathbf{t}^i \otimes \mathbf{d}_i + \mathbf{m}^\alpha \otimes \mathbf{d}_{\alpha,3} \ , \tag{5.4.23}$$

the reduced form of the balance of angular momentum (5.4.22) requires $\mathbf{T}$ to be a symmetric tensor

$$\mathbf{T}^T = \mathbf{T} \ , \tag{5.4.24}$$

which is similar to the result (3.2.32) associated with the three-dimensional theory.

Thus, the local forms of the conservation of mass (5.4.15) and the balances of linear momentum and director momentum (5.4.19) can be summarized as

$$m = \rho\, d_{33}^{1/2} = \rho_0\, D_{33}^{1/2} = m(\theta^3) \quad \text{or} \quad \dot{\rho} + \rho\, \mathrm{div}_c\, \mathbf{v} = 0 \ ,$$

$$m\,(\dot{\mathbf{v}} + y^\alpha\, \dot{\mathbf{w}}_\alpha) = m\,\mathbf{b} + \mathbf{t}^3{}_{,3} \ , \quad m\,(y^\alpha\, \dot{\mathbf{v}} + y^{\alpha\beta}\, \dot{\mathbf{w}}_\beta) = m\,\mathbf{b}^\alpha - \mathbf{t}^\alpha + \mathbf{m}^\alpha{}_{,3} \ , \tag{5.4.25}$$

where ρ_0 is the mass density per unit arclength of the reference curve in the reference configuration, and use has been made of (5.4.13).

In general, equations (5.4.24) and (5.4.25) represent a system of nonlinear partial differential equations which require specification of initial and boundary conditions. These balance laws are quite general because they are valid for all materials. However, this system of equations is not complete because it represents a system of thirteen scalar equations to determine twenty-five scalar unknowns $\{\rho\ ,\ \mathbf{x}\ ,\ \mathbf{d}_\alpha\ ,\ \mathbf{t}^i\ ,\ \mathbf{m}^\alpha\}$, once the external loads $\{\mathbf{b}\ ,\ \mathbf{b}^\alpha\}$ have been specified. As in the three-dimensional theory, these equations must be supplemented by constitutive equations for the quantities $\{\mathbf{t}^i\ ,\ \mathbf{m}^\alpha\}$ (here, it is tacitly assumed that the director intertias y^α and $y^{\alpha\beta}$ have been specified).

Furthermore, it is noted that the reduced form of the balance of angular momentum (5.4.24) places three restrictions on the constitutive equations that must be satisfied for all possible motions of the continuum. Therefore, the balance of angular momentum has a different character from the other three balance laws because it is not used to determine the motion or deformation of the continuum. In contrast, the conservation of mass $(5.4.25)_1$ and the balances of linear and director momentum $(5.4.25)_{2,3}$ are used to determine the mass density ρ and the motion of the continuum through the functional forms for $\{\mathbf{x}\ ,\ \mathbf{d}_\alpha\}$.

5.5 Invariance under superposed rigid body motions

Motivated by the relationships developed in section 5.3 and the discussion of superposed rigid body motions (SRBM) in section 3.3 for the three-dimensional theory, it is assumed that under SRBM the Cosserat rod is transformed from its present configuration occupying region P at time t, to its superposed configuration occupying region P^+ at time t^+ such that

$$t^+ = t + a \ , \quad \mathbf{x}^+ = \mathbf{c}(t) + \mathbf{Q}(t)\,\mathbf{x} \ , \quad \mathbf{d}_\alpha^+ = \mathbf{Q}(t)\,\mathbf{d}_\alpha \ , \tag{5.5.1}$$

where a is a constant, $\mathbf{c}(t)$ is an arbitrary vector function of time only, and $\mathbf{Q}(t)$ is a proper orthogonal tensor function of time only

$$\mathbf{Q}^T\mathbf{Q} = \mathbf{I} \ , \quad \det \mathbf{Q} = +1 \ , \tag{5.5.2}$$

which is related to a skew-symmetric tensor $\mathbf{\Omega}(t)$ function of time only through the equations

$$\dot{\mathbf{Q}} = \mathbf{\Omega} \, \mathbf{Q} \ , \quad \mathbf{\Omega}^T = -\mathbf{\Omega} \ . \tag{5.5.3}$$

Throughout the text, quantities associated with the superposed configuration will be denoted using the same symbol as those associated with the present configuration, but with a superposed $(^+)$. Next, using (5.5.1) it follows that since θ^3 is a convected coordinate, it is unaffected by SRBM so that

$$\mathbf{D}_i^+ = \mathbf{D}_i \ , \quad \mathbf{D}^{i+} = \mathbf{D}^i \ , \quad \mathbf{d}_i^+ = \mathbf{Q} \, \mathbf{d}_i \ , \quad \mathbf{d}^{i+} = \mathbf{Q} \, \mathbf{d}^i \ ,$$
$$(d_{33}^{1/2})^+ = d_{33}^{1/2} \ , \quad (ds)^+ = ds \ , \quad \mathbf{F}^+ = \mathbf{Q} \, \mathbf{F} \ , \quad \boldsymbol{\lambda}_\alpha^+ = \boldsymbol{\lambda}_\alpha \ . \tag{5.5.4}$$

Moreover, the various kinetic quantities are assumed to transform to their superposed values by the equations

$$m^+ = m \ , \quad \rho^+ = \rho \ , \quad y^{\alpha+} = y^\alpha \ , \quad y^{\alpha\beta+} = y^{\alpha\beta} \ ,$$
$$\mathbf{t}^{i+} = \mathbf{Q} \, \mathbf{t}^i \ , \quad \mathbf{m}^{\alpha+} = \mathbf{Q} \, \mathbf{m}^\alpha \ , \quad \mathbf{T}^+ = \mathbf{Q} \, \mathbf{T} \, \mathbf{Q}^T \ ,$$
$$\mathbf{b}^+ = (\dot{\mathbf{v}}^+ + y^{\alpha+} \, \dot{\mathbf{w}}_\alpha^+) + \mathbf{Q} \, [\mathbf{b} - (\dot{\mathbf{v}} + y^\alpha \, \dot{\mathbf{w}}_\alpha)] \ ,$$
$$\mathbf{b}_b^+ = (\dot{\mathbf{v}}^+ + y^{\alpha+} \, \dot{\mathbf{w}}_\alpha^+) + \mathbf{Q} \, [\mathbf{b}_b - (\dot{\mathbf{v}} + y^\alpha \, \dot{\mathbf{w}}_\alpha)] \ , \quad \mathbf{b}_c^+ = \mathbf{Q} \, \mathbf{b}_c \ ,$$
$$\mathbf{b}^{\alpha+} = (y^{\alpha+} \, \dot{\mathbf{v}}^+ + y^{\alpha\beta+} \, \dot{\mathbf{w}}_\beta^+) + \mathbf{Q} \, [\mathbf{b}^\alpha - (y^\alpha \, \dot{\mathbf{v}} + y^{\alpha\beta} \, \dot{\mathbf{w}}_\beta)] \ ,$$
$$\mathbf{b}_b^{\alpha+} = (y^{\alpha+} \, \dot{\mathbf{v}}^+ + y^{\alpha\beta+} \, \dot{\mathbf{w}}_\beta^+) + \mathbf{Q} \, [\mathbf{b}_b^\alpha - (y^\alpha \, \dot{\mathbf{v}} + y^{\alpha\beta} \, \dot{\mathbf{w}}_\beta)] \ , \quad \mathbf{b}_c^{\alpha+} = \mathbf{Q} \, \mathbf{b}_c^\alpha \ . \tag{5.5.5}$$

Also, using expressions of the form (3.5.4) and (3.5.6), it follows that

$$\mathrm{div}_c^+ \, \mathbf{v}^+ = \frac{\partial}{\partial s}(\mathbf{Q} \, \mathbf{v} + \dot{\mathbf{c}} + \mathbf{\Omega} \, \mathbf{Q} \, \mathbf{x}) \bullet \frac{\partial}{\partial s}(\mathbf{c} + \mathbf{Q} \, \mathbf{x}) = \mathrm{div}_c \, \mathbf{v} \ ,$$
$$\mathbf{t}^{3+}{}_{,3} = (\mathbf{Q}\mathbf{t}^3)_{,3} = \mathbf{Q} \, \mathbf{t}^3{}_{,3} \ , \quad \mathbf{m}^{\alpha+}{}_{,3} = (\mathbf{Q}\mathbf{m}^\alpha)_{,3} = \mathbf{Q} \, \mathbf{m}^\alpha{}_{,3} \ . \tag{5.5.6}$$

Now, with the help of these results, it can be shown that the balance laws (5.4.24) and (5.4.25) remain form invariant under SRBM. Furthermore, it will be shown in a later section that these conditions place important physical restrictions on constitutive assumptions for the quantities $\{\mathbf{t}^i \, , \mathbf{m}^\alpha\}$.

5.6 Mechanical power

For the purely mechanical theory it is convenient to define the notion of the mechanical power $\mathcal{P}$ due to the kinetic quantities $\{\mathbf{b} \, , \mathbf{b}^\alpha \, , \mathbf{t}^3 \, , \mathbf{m}^\alpha\}$ by the equation

$$\int_P \mathcal{P} \, ds = \mathcal{W} - \dot{\mathcal{K}} = \int_P \rho \, (\mathbf{b} \bullet \mathbf{v} + \mathbf{b}^\alpha \bullet \mathbf{w}_\alpha) \, ds + \Big[\, \mathbf{t}^3 \bullet \mathbf{v} + \mathbf{m}^\alpha \bullet \mathbf{w}_\alpha \, \Big]_1^2$$
$$- \frac{d}{dt} \int_P \frac{1}{2} \rho \, (\, \mathbf{v} \bullet \mathbf{v} + 2 \, y^\alpha \, \mathbf{v} \bullet \mathbf{w}_\alpha + y^{\alpha\beta} \, \mathbf{w}_\alpha \bullet \mathbf{w}_\beta \,) \, ds \ , \tag{5.6.1}$$

where the expressions (5.3.41) have been used for the rate of work $\mathcal{W}$ applied to the rod, and for the kinetic energy $\mathcal{K}$.

Next, assuming sufficient continuity of the functions and using the definitions (5.2.6) and the result that

$$\frac{\partial}{\partial s}(\mathbf{t}^3 \bullet \mathbf{v} + \mathbf{m}^\alpha \bullet \mathbf{w}_\alpha) = \frac{\partial \mathbf{t}^3}{\partial s} \bullet \mathbf{v} + \frac{\partial \mathbf{m}^\alpha}{\partial s} \bullet \mathbf{w}_\alpha + d_{33}^{-1/2} \left\{ \mathbf{t}^3 \bullet \mathbf{w}_3 + \mathbf{m}^\alpha \bullet \mathbf{w}_{\alpha,3} \right\}, \quad (5.6.2)$$

it can be shown that the local form of equation (5.6.1) requires the mechanical power to be given by the expression

$$d_{33}^{1/2}\, \mathcal{P} = \mathbf{t}^i \bullet \mathbf{w}_i + \mathbf{m}^\alpha \bullet \mathbf{w}_{3,\alpha} \ . \quad (5.6.3)$$

Also, it can be shown using the transformation relations of section 5.5, that under SRBM

$$\mathbf{w}_i^+ = \mathbf{Q}\, \mathbf{w}_i + \boldsymbol{\Omega}\, \mathbf{Q}\, \mathbf{d}_i \ , \quad \mathbf{w}_{\alpha,3}^+ = \mathbf{Q}\, \mathbf{w}_{\alpha,3} + \boldsymbol{\Omega}\, \mathbf{Q}\, \mathbf{d}_{\alpha,3} \ , \quad (5.6.4)$$

so that with the help of (3.5.6), the mechanical power in the superposed configuration becomes

$$(d_{33}^{1/2})^+\, \mathcal{P}^+ = d_{33}^{1/2}\, \mathcal{P} + (\boldsymbol{\Omega}\, \mathbf{Q}\, \mathbf{d}_i) \bullet (\mathbf{Q}\, \mathbf{t}^i) + (\boldsymbol{\Omega}\, \mathbf{Q}\, \mathbf{d}_{\alpha,3}) \bullet (\mathbf{Q}\, \mathbf{m}^\alpha) \ ,$$

$$(d_{33}^{1/2})^+\, \mathcal{P}^+ = d_{33}^{1/2}\, \mathcal{P} + (\mathbf{Q}^T \boldsymbol{\omega}) \bullet (\mathbf{d}_i \times \mathbf{t}^i + \mathbf{d}_{\alpha,3} \times \mathbf{m}^\alpha) \ , \quad (5.6.5)$$

where $\boldsymbol{\omega}$ is the axial vector associated with $\boldsymbol{\Omega}$. However, using the local form (5.4.20) of the balance of angular momentum, it follows that the mechanical power is unaltered by SRBM

$$\mathcal{P}^+ = \mathcal{P} \ . \quad (5.6.6)$$

Before closing this section, it is desirable to rewrite the expression for the mechanical power in an alternative form that is more similar to the expression (3.4.5) of the three-dimensional theory. To this end, it is convenient to define the second order tensor $\mathbf{L}$ by the expressions

$$\mathbf{L} = \mathbf{w}_i \otimes \mathbf{d}^i \ , \quad \mathbf{w}_i = \mathbf{L}\, \mathbf{d}_i \ . \quad (5.6.7)$$

It then follows from the definition (5.2.10) of the tensor $\mathbf{F}$ that

$$\dot{\mathbf{F}} = \mathbf{L}\, \mathbf{F} \ . \quad (5.6.8)$$

Consequently, comparison of these expressions with (3.2.12) suggests that $\mathbf{F}$ is similar to the three-dimensional deformation gradient and $\mathbf{L}$ is similar to the three-dimensional velocity gradient. Moreover, using the definition (5.2.11), it can be shown that

$$\mathbf{w}_{\alpha,3} = \mathbf{L}\, \mathbf{d}_{\alpha,3} + \mathbf{F}\, \dot{\boldsymbol{\lambda}}_\alpha\, \mathbf{D}_3 \ . \quad (5.6.9)$$

Thus, using the definition (5.4.23) the mechanical power can be rewritten in the form

$$\mathcal{P} = \mathbf{T} \bullet \mathbf{L} + d_{33}^{-1/2}\, (\mathbf{F}^T \mathbf{m}^\alpha \otimes \mathbf{D}_3) \bullet \dot{\boldsymbol{\lambda}}_\alpha \ . \quad (5.6.10)$$

Moreover, by separating $\mathbf{L}$ into its symmetric part $\mathbf{D}$ and its skew-symmetric part $\mathbf{W}$

$$\mathbf{L} = \mathbf{D} + \mathbf{W} \ , \quad \mathbf{D} = \frac{1}{2}(\mathbf{L} + \mathbf{L}^T) = \mathbf{D}^T \ , \quad \mathbf{W} = \frac{1}{2}(\mathbf{L} - \mathbf{L}^T) = -\mathbf{W}^T \ , \quad (5.6.11)$$

and by using the result (5.4.24) of the balance of angular momentum that $\mathbf{T}$ is a symmetric tensor, it follows that the mechanical power reduces to

$$\mathcal{P} = \mathbf{T} \bullet \mathbf{D} + d_{33}^{-1/2}\, (\mathbf{F}^T \mathbf{m}^\alpha \otimes \mathbf{D}_3) \bullet \dot{\boldsymbol{\lambda}}_\alpha \ . \quad (5.6.12)$$

Next, it is noted that under SRBM the quantities $\mathbf{L}$, $\mathbf{D}$, $\mathbf{W}$ transform by

$$\mathbf{L}^+ = \mathbf{Q}\, \mathbf{L}\, \mathbf{Q}^T + \boldsymbol{\Omega} \ , \quad \mathbf{D}^+ = \mathbf{Q}\, \mathbf{D}\, \mathbf{Q}^T, \quad \mathbf{W}^+ = \mathbf{Q}\, \mathbf{W}\, \mathbf{Q}^T + \boldsymbol{\Omega} \ , \quad (5.6.13)$$

and that $(\mathbf{F}^T\, \mathbf{m}^\alpha \otimes \mathbf{D}_3)$ remains unaltered by SRBM so that the mechanical power is again shown to be unaltered by SRBM.

For future developments, it is convenient to introduce the second order tensors $\boldsymbol{\kappa}_\alpha$ as strain measures defined by

$$\boldsymbol{\kappa}_\alpha = \boldsymbol{\lambda}_\alpha - \boldsymbol{\Lambda}_\alpha \ . \tag{5.6.14}$$

Moreover, in view of the definitions (5.1.13) and (5.2.11), it follows that $\boldsymbol{\kappa}_\alpha \mathbf{D}_\beta$ vanish so that $\boldsymbol{\kappa}_\alpha$ can be characterized by the two vectors $\boldsymbol{\beta}_\alpha$ such that

$$\boldsymbol{\beta}_\alpha = \boldsymbol{\kappa}_\alpha \mathbf{D}_3 = \mathbf{F}^{-1} \mathbf{d}_{\alpha,3} - \mathbf{D}_{\alpha,3} \ , \quad \boldsymbol{\kappa}_\alpha = \boldsymbol{\beta}_\alpha \otimes \mathbf{D}^3 \ . \tag{5.6.15}$$

Then, the expression (5.6.10) for the mechanical power simplifies somewhat to become

$$\mathcal{P} = \mathbf{T} \bullet \mathbf{L} + d_{33}^{-1/2} (\mathbf{F}^T \mathbf{m}^\alpha) \bullet \dot{\boldsymbol{\beta}}_\alpha \ , \tag{5.6.16}$$

which, with the help of the symmetry of $\mathbf{T}$, reduces to

$$\mathcal{P} = \mathbf{T} \bullet \mathbf{D} + d_{33}^{-1/2} (\mathbf{F}^T \mathbf{m}^\alpha) \bullet \dot{\boldsymbol{\beta}}_\alpha \ , \tag{5.6.17}$$

Also, it can be shown that under SRBM the quantities $\boldsymbol{\kappa}_\alpha$ and $\boldsymbol{\beta}_\alpha$ transform by

$$\boldsymbol{\kappa}_\alpha^+ = \boldsymbol{\kappa}_\alpha \ , \quad \boldsymbol{\beta}_\alpha^+ = \boldsymbol{\beta}_\alpha \ . \tag{5.6.18}$$

5.7 An alternative derivation of the balance laws

From the point of view presented previously, the condition that the balance laws remain form invariant under SRBM requires the kinematic and kinetic quantities to satisfy the transformation relations (5.5.1) and (5.5.5), as well a number of other expressions [like (5.5.4)] that can be derived directly from these relations. In this regard, the notion of invariance under SRBM is a fundamental notion that causes intimate interconnections between the balance laws. To demonstrate the fundamental nature of invariance under SRBM, it will be shown that the global forms of the conservation of mass and the balances of linear momentum and angular momentum can be derived by assuming that the global form (5.6.1) of the expression for mechanical power remains form invariant, and that the local forms of the above transformation relations are valid. This means that these balance laws can be derived from a single scalar equation by demanding invariance for the class of all possible SRBM. However, this method does not produce the global or local forms of the balances of director momentum.

To this end, it is first noted that with respect to the superposed configuration, the global equation (5.6.1) can be written in the form

$$\begin{aligned}\int_{P^+} \mathcal{P}^+ \, ds^+ = \int_{P^+} \rho^+ (\mathbf{b}^+ \bullet \mathbf{v}^+ + \mathbf{b}^{\alpha+} \bullet \mathbf{w}_\alpha^+) \, ds^+ + \left[\mathbf{t}^{3+} \bullet \mathbf{v}^+ + \mathbf{m}^{\alpha+} \bullet \mathbf{w}_\alpha^+ \right]_1^2 \\ - \frac{d}{dt^+} \int_{P^+} \frac{1}{2} \rho^+ (\mathbf{v}^+ \bullet \mathbf{v}^+ + 2 \, y^{\alpha+} \mathbf{v}^+ \bullet \mathbf{w}_\alpha^+ + y^{\alpha\beta+} \mathbf{w}_\alpha^+ \bullet \mathbf{w}_\beta^+) \, ds^+ \ . \end{aligned} \tag{5.7.1}$$

Next, consider the special SRBM which is characterized by a superposed constant translational velocity with magnitude u and unit direction $\mathbf{u}$ so that

$$\mathbf{c}(t) = u \, \mathbf{u} \, t \ , \quad \dot{\mathbf{c}} = u \, \mathbf{u} \ , \quad \ddot{\mathbf{c}} = 0 \ , \quad \mathbf{u} \bullet \mathbf{u} = 1 \ , \quad \mathbf{Q} = \mathbf{I} \ , \quad \dot{\mathbf{Q}} = 0 \ , \quad \ddot{\mathbf{Q}} = 0 \ ,$$
$$\mathbf{v}^+ = \mathbf{v} + u \, \mathbf{u} \ , \quad \dot{\mathbf{v}}^+ = \dot{\mathbf{v}} \ , \quad \mathbf{F}^+ = \mathbf{F} \ , \quad (d_{33}^{1/2})^+ = d_{33}^{1/2} \ ,$$

$$\boldsymbol{\beta}_\alpha^+ = \boldsymbol{\beta}_\alpha \ , \quad \mathbf{L}^+ = \mathbf{L} \ , \quad \mathbf{w}_i^+ = \mathbf{w}_i \ , \quad \dot{\mathbf{w}}_\alpha^+ = \dot{\mathbf{w}}_\alpha \ ,$$
$$\rho^+ = \rho \ , \quad y^{\alpha +} = y^\alpha \ , \quad y^{\alpha\beta +} = y^{\alpha\beta} \ , \quad \mathbf{b}^+ = \mathbf{b} \ , \quad \mathbf{b}^{\alpha +} = \mathbf{b}^\alpha \ ,$$
$$\mathbf{t}^{3+} = \mathbf{t}^3 \ , \quad \mathbf{m}^{\alpha +} = \mathbf{m}^\alpha \ , \quad \mathbf{T}^+ = \mathbf{T} \ , \quad \mathcal{P}^+ = \mathcal{P} \ . \tag{5.7.2}$$

Now, substituting (5.7.2) into (5.7.1), transforming the integrals over the superposed region P^+ to the present region P and ∂P, and subtracting the equation (5.6.1) from the result, yields the expression

$$u\,\mathbf{u} \bullet \left[\frac{d}{dt} \int_P \rho\ (\mathbf{v} + y^\alpha\ \mathbf{w}_\alpha)\ ds - \int_P \rho\ \mathbf{b}\ ds - [\ \mathbf{t}^3\ ]_1^2 \ \right]$$
$$+ \frac{1}{2} u^2 \left[\frac{d}{dt} \int_P \rho\ ds \right] = 0 \ , \tag{5.7.3}$$

which must be valid for all values of u and the unit vector $\mathbf{u}$. Moreover, since the coefficients in (5.7.3) are independent of u and $\mathbf{u}$, it follows that each of them must vanish. This procedure yields the global forms of conservation of mass $(5.4.6)_1$ and the balance of linear momentum $(5.4.6)_2$.

To derive the global form of the balance of angular momentum, it is convenient to consider a superposed constant rigid body rotation that is characterized by

$$\mathbf{c} = 0 \ , \quad \dot{\mathbf{c}} = 0 \ , \quad \ddot{\mathbf{c}} = 0 \ , \quad \mathbf{Q} = \mathbf{Q}(t) \ , \quad \dot{\mathbf{Q}} = \boldsymbol{\Omega}\,\mathbf{Q} \ , \quad \dot{\boldsymbol{\Omega}} = 0 \ ,$$
$$\mathbf{v}^+ = \mathbf{Q}\ \mathbf{v} + \boldsymbol{\Omega}\ \mathbf{Q}\ \mathbf{x} \ , \quad \dot{\mathbf{v}}^+ = \mathbf{Q}\ \dot{\mathbf{v}} + 2\ \boldsymbol{\Omega}\ \mathbf{Q}\ \mathbf{v} + \boldsymbol{\Omega}^2\ \mathbf{Q}\ \mathbf{x} \ ,$$
$$\mathbf{F}^+ = \mathbf{Q}\ \mathbf{F} \ , \quad (d_{33}^{1/2})^+ = d_{33}^{1/2} \ , \quad \boldsymbol{\beta}_\alpha^+ = \boldsymbol{\beta}_\alpha \ , \quad \mathbf{L}^+ = \mathbf{Q}\ \mathbf{L}\ \mathbf{Q}^T + \boldsymbol{\Omega} \ ,$$
$$\mathbf{w}_i^+ = \mathbf{Q}\ \mathbf{w}_i + \boldsymbol{\Omega}\ \mathbf{Q}\ \mathbf{d}_i \ , \quad \dot{\mathbf{w}}_\alpha^+ = \mathbf{Q}\ \dot{\mathbf{w}}_\alpha + 2\ \boldsymbol{\Omega}\ \mathbf{Q}\ \mathbf{w}_\alpha + \boldsymbol{\Omega}^2\ \mathbf{Q}\ \mathbf{d}_\alpha \ ,$$
$$\rho^+ = \rho \ , \quad y^{\alpha +} = y^\alpha \ , \quad y^{\alpha\beta +} = y^{\alpha\beta} \ ,$$
$$\mathbf{b}^+ = \mathbf{Q}\ \mathbf{b} + 2\ \boldsymbol{\Omega}\ \mathbf{Q}\ (\mathbf{v} + y^\alpha\ \mathbf{w}_\alpha) + \boldsymbol{\Omega}^2\ \mathbf{Q}\ (\mathbf{x} + y^\alpha\ \mathbf{d}_\alpha) \ ,$$
$$\mathbf{b}^{\alpha +} = \mathbf{Q}\ \mathbf{b}^\alpha + 2\ \boldsymbol{\Omega}\ \mathbf{Q}\ (y^\alpha\ \mathbf{v} + y^{\alpha\beta}\ \mathbf{w}_\beta) + \boldsymbol{\Omega}^2\ \mathbf{Q}\ (y^\alpha\ \mathbf{x} + y^{\alpha\beta}\ \mathbf{d}_\beta) \ ,$$
$$\mathbf{t}^{3+} = \mathbf{Q}\ \mathbf{t}^3 \ , \quad \mathbf{m}^{\alpha +} = \mathbf{Q}\ \mathbf{m}^\alpha \ , \quad \mathbf{T}^+ = \mathbf{Q}\ \mathbf{T}\ \mathbf{Q}^T \ , \quad \mathcal{P}^+ = \mathcal{P} \ . \tag{5.7.4}$$

Thus, with the help of the definition (5.6.16) and the last of (5.7.4), it follows that

$$\mathcal{P}^+ = \mathbf{T}^+ \bullet \mathbf{L}^+ + d_{33}^{-1/2}\ (\mathbf{F}^T\ \mathbf{m}^\alpha) \bullet \dot{\boldsymbol{\beta}}_\alpha = \mathcal{P} + \mathbf{T} \bullet (\mathbf{Q}^T \boldsymbol{\Omega}\ \mathbf{Q}) = \mathcal{P} \ ,$$
$$\mathbf{T} \bullet (\mathbf{Q}^T \boldsymbol{\Omega}\ \mathbf{Q}) = 0 \ . \tag{5.7.5}$$

However, since $\mathbf{T}$ does not depend on $\boldsymbol{\Omega}$ and since $\boldsymbol{\Omega}$ is a skew-symmetric tensor, the quantity $\mathbf{Q}^T\boldsymbol{\Omega}\ \mathbf{Q}$ is also a skew-symmetric tensor so that (5.7.5) requires $\mathbf{T}$ to be a symmetric tensor. This yields the local form (5.4.24) of the balance of angular momentum. Furthermore, using the results (3.5.6) it can be shown that

$$\mathbf{b}^+ \bullet \mathbf{v}^+ + \mathbf{b}^{\alpha +} \bullet \mathbf{w}_\alpha^+ = \mathbf{b} \bullet \mathbf{v} + \mathbf{b}^\alpha \bullet \mathbf{w}_\alpha + (\mathbf{Q}^T \boldsymbol{\omega}) \bullet (\mathbf{x} \times \mathbf{b} + \mathbf{d}_\alpha \times \mathbf{b}^\alpha)$$
$$+ (\boldsymbol{\Omega}\ \mathbf{Q}\mathbf{x}) \bullet (\boldsymbol{\Omega}\ \mathbf{Q}\mathbf{v}) + y^\alpha\ (\boldsymbol{\Omega}\ \mathbf{Q}\mathbf{x}) \bullet (\boldsymbol{\Omega}\ \mathbf{Q}\mathbf{w}_\alpha)$$
$$+ y^\alpha\ (\boldsymbol{\Omega}\ \mathbf{Q}\mathbf{d}_\alpha) \bullet (\boldsymbol{\Omega}\ \mathbf{Q}\mathbf{v}) + y^{\alpha\beta}\ (\boldsymbol{\Omega}\ \mathbf{Q}\mathbf{d}_\alpha) \bullet (\boldsymbol{\Omega}\ \mathbf{Q}\mathbf{w}_\beta) \ ,$$
$$\mathbf{t}^{3+} \bullet \mathbf{v}^+ + \mathbf{m}^{\alpha +} \bullet \mathbf{w}_\alpha^+ = \mathbf{t}^3 \bullet \mathbf{v} + \mathbf{m}^\alpha \bullet \mathbf{w}_\alpha + (\mathbf{Q}^T \boldsymbol{\omega}) \bullet (\mathbf{x} \times \mathbf{t}^3 + \mathbf{d}_\alpha \times \mathbf{m}^\alpha) \ ,$$

$$\mathcal{K}^+ = \mathcal{K} + (\mathbf{Q}^T\boldsymbol{\omega}) \cdot \int_P \left[\mathbf{x} \times \rho\,(\mathbf{v} + y^\alpha\,\mathbf{w}_\alpha) + \mathbf{d}_\alpha \times \rho\,(y^\alpha\,\mathbf{v} + y^{\alpha\beta}\,\mathbf{w}_\beta)\right] ds$$
$$+ \int_P \frac{1}{2}\rho\left[(\boldsymbol{\Omega}\,\mathbf{Q}\mathbf{x}) \cdot (\boldsymbol{\Omega}\,\mathbf{Q}\mathbf{x}) + 2\,y^\alpha\,(\boldsymbol{\Omega}\,\mathbf{Q}\mathbf{x}) \cdot (\boldsymbol{\Omega}\,\mathbf{Q}\mathbf{d}_\alpha)\right.$$
$$\left. + y^{\alpha\beta}\,(\boldsymbol{\Omega}\,\mathbf{Q}\mathbf{d}_\alpha) \cdot (\boldsymbol{\Omega}\,\mathbf{Q}\mathbf{d}_\beta)\right] ds\ , \tag{5.7.6}$$

where the integrals over the superposed region P^+ have been transformed to integrals over the present region P. Also, with the help of (3.5.6), $(3.5.9)_1$ and the conservation of mass equation, it can be shown that

$$\dot{\mathcal{K}}^+ = \dot{\mathcal{K}} + (\mathbf{Q}^T\boldsymbol{\omega}) \cdot \frac{d}{dt}\int_P \left[\mathbf{x} \times \rho\,(\mathbf{v} + y^\alpha\,\mathbf{w}_\alpha) + \mathbf{d}_\alpha \times \rho\,(y^\alpha\,\mathbf{v} + y^{\alpha\beta}\,\mathbf{w}_\beta)\right] ds$$
$$+ \int_P \rho\left[(\boldsymbol{\Omega}\,\mathbf{Q}\mathbf{x}) \cdot (\boldsymbol{\Omega}\,\mathbf{Q}\mathbf{v}) + y^\alpha\,(\boldsymbol{\Omega}\,\mathbf{Q}\mathbf{x}) \cdot (\boldsymbol{\Omega}\,\mathbf{Q}\mathbf{w}_\alpha)\right.$$
$$\left. + y^\alpha\,(\boldsymbol{\Omega}\,\mathbf{Q}\mathbf{d}_\alpha) \cdot (\boldsymbol{\Omega}\,\mathbf{Q}\mathbf{v}) + y^{\alpha\beta}\,(\boldsymbol{\Omega}\,\mathbf{Q}\mathbf{d}_\alpha) \cdot (\boldsymbol{\Omega}\,\mathbf{Q}\mathbf{w}_\beta)\right] ds\ . \tag{5.7.7}$$

Now, substituting (5.7.6) and (5.7.7) into (5.7.1) and subtracting (5.6.1) from the result, yields the expression

$$(\mathbf{Q}^T\boldsymbol{\omega}) \cdot \left[\ \frac{d}{dt}\int_P \left[\mathbf{x} \times \rho\,(\mathbf{v} + y^\alpha\,\mathbf{w}_\alpha) + \mathbf{d}_\alpha \times \rho\,(y^\alpha\,\mathbf{v} + y^{\alpha\beta}\,\mathbf{w}_\beta)\right] ds\right.$$
$$\left. - \int_P (\mathbf{x} \times \rho\,\mathbf{b} + \mathbf{d}_\alpha \times \rho\,\mathbf{b}^\alpha)\,ds - \left[\ \mathbf{x} \times \mathbf{t}^3 + \mathbf{d}_\alpha \times \mathbf{m}^\alpha\ \right]_1^2\ \right] = 0\ . \tag{5.7.8}$$

Furthermore, since (5.7.8) must be valid for all $\boldsymbol{\omega}$, and the coefficient in the square brackets is independent of $\boldsymbol{\omega}$, it follows that (5.7.8) yields the global form of the balance of angular momentum (5.4.7).

In the above it has been shown that the global forms of the conservation of mass and the balances of linear and angular momentum are necessary conditions for the global expression (5.6.1) of mechanical power to remain form invariant under SRBM. However, this procedure does not produce the global or local forms of the balances of director momentum.

5.8 Anisotropic nonlinear elastic rods

In the previous sections kinematical expressions and balance laws were discussed that are valid for rod-like structures which are made from arbitrary materials. Here, constitutive equations will be developed for rods that are composed of general anisotropic nonlinear elastic materials. Such elastic rods are considered to be ideal rods in the same sense that elastic materials are considered to be ideal materials in the three-dimensional theory. For example, the response of an elastic rod is insensitive to the rate of loading. Other fundamental features of elastic rods will be discussed presently.

The constitutive equations of elastic rods can be characterized by the following four assumptions:

Assumption 1: A strain energy Σ per unit mass exists for which

$$\rho \dot{\Sigma} = \mathcal{P} = \mathbf{T} \bullet \mathbf{D} + d_{33}^{-1/2} (\mathbf{F}^T \mathbf{m}^\alpha) \bullet \dot{\boldsymbol{\beta}}_\alpha \quad . \tag{5.8.1}$$

Assumption 2: The strain energy Σ is a function of the deformation tensors $\mathbf{F}$ and $\boldsymbol{\beta}_\alpha$, and the convected coordinate θ^3

$$\Sigma = \tilde{\Sigma} (\mathbf{F}, \boldsymbol{\beta}_\alpha; \theta^3) \ , \tag{5.8.2}$$

where dependence on θ^3 is included to allow for the possibility that the rod can be composed of an inhomogeneous material or can have nonuniform geometrical properties in its reference configuration.

Assumption 3: The strain energy Σ is invariant under SRBM

$$\Sigma^+ = \Sigma \ . \tag{5.8.3}$$

Assumption 4: The kinetic quantities $\mathbf{T}$ and $\mathbf{m}^\alpha$ are independent of the rates of deformations $\mathbf{L}$ and $\dot{\boldsymbol{\beta}}_\alpha$.

In order to explore the physical consequences of the assumption (5.8.1), it is convenient to define the total strain energy $\mathcal{U}$ of the rod

$$\mathcal{U} = \int_P \rho \, \Sigma \, ds \ , \tag{5.8.4}$$

and to use the result (5.4.16) and the assumption (5.8.1) to deduce that

$$\dot{\mathcal{U}} = \int_P \rho \, \dot{\Sigma} \, ds = \int_P \mathcal{P} \, ds \ . \tag{5.8.5}$$

Thus, by substituting (5.8.5) into the mechanical power equation (5.6.1) it is possible to derive the following theorem

$$\mathcal{W} = \dot{\mathcal{K}} + \dot{\mathcal{U}} \ , \tag{5.8.6}$$

which states that for an elastic rod the rate of work done on the rod due to external forces and couples and contact forces and couples, equals the rate of change of kinetic and strain energies. Since the strain energy Σ depends on the present configuration only through the present values of $\mathbf{F}$ and $\boldsymbol{\beta}_\alpha$ [assumption (5.8.2)], the value of the strain energy Σ is independent of the particular loading path which caused $\mathbf{F}$ and $\boldsymbol{\beta}_\alpha$. Consequently, the total work done on the body vanishes for any closed cycle in which the values of velocity $\mathbf{v}$, the director velocities $\mathbf{w}_i$, and the deformation tensors $\mathbf{F}$ and $\boldsymbol{\beta}_\alpha$ are the same at the beginning and end of the cycle. Next, consider a special cycle which is composed of a loading path from one state A to another state B, followed by the reversal of this loading path. Then, in view of assumption 4 the work done on the body from A to B is fully recovered during the reverse loading from B to A. In this sense, the elastic rod is considered to be an ideal rod.

The assumption (5.8.3) places restrictions on the functional form (5.8.2). To develop these restrictions, it is recalled that under SRBM $\mathbf{F}^+ = \mathbf{QF}$ and $\boldsymbol{\beta}_\alpha^+ = \boldsymbol{\beta}_\alpha$ so that (5.8.2) requires

$$\Sigma^+ = \tilde{\Sigma} (\mathbf{F}^+, \boldsymbol{\beta}_\alpha^+; \theta^3) = \tilde{\Sigma} (\mathbf{QF}, \boldsymbol{\beta}_\alpha; \theta^3) = \tilde{\Sigma} (\mathbf{F}, \boldsymbol{\beta}_\alpha; \theta^3) \ , \tag{5.8.7}$$

to hold for arbitrary proper orthogonal $\mathbf{Q}$. However, the polar decomposition theorem states that $\mathbf{F}$ can be separated multiplicatively into a rotation tensor $\mathbf{R}$ and positive definite symmetric stretch tensors $\mathbf{U}$ and $\mathbf{V}$ such that

$$\mathbf{F} = \mathbf{RU} = \mathbf{VR} \;\;,\;\; \mathbf{C} = \mathbf{F}^T\mathbf{F} = \mathbf{C}^T \;,\;\; \mathbf{U}^T = \mathbf{U} = \mathbf{C}^{1/2} \;,$$
$$\mathbf{B} = \mathbf{FF}^T = \mathbf{B}^T \;,\;\; \mathbf{V}^T = \mathbf{V} = \mathbf{B}^{1/2} \;,\;\; \mathbf{R}^T\mathbf{R} = \mathbf{RR}^T = \mathbf{I} \;,\;\; \det \mathbf{R} = 1 \;, \tag{5.8.8}$$

where $\mathbf{C}$ and $\mathbf{B}$ are analogues of the right Cauchy-Green deformation tensor and the left Cauchy-Green deformation tensor in the three-dimensional theory. Following a similar argument to that associated with the three-dimensional theory (sec. 3.7), it can be shown that the strain energy function Σ can depend on $\mathbf{F}$ only through the deformation tensor $\mathbf{C}$ so that Σ necessarily must reduce to the form

$$\tilde{\Sigma}\,(\mathbf{F},\boldsymbol{\beta}_\alpha;\theta^3) = \hat{\Sigma}\,(\mathbf{C},\boldsymbol{\beta}_\alpha;\theta^3) \;. \tag{5.8.9}$$

Now, with the help of (5.8.1) and (5.8.9), it can be shown that

$$\mathbf{T} \bullet \mathbf{D} + d_{33}^{-1/22}\,(\mathbf{F}^T\,\mathbf{m}^\alpha) \bullet \dot{\boldsymbol{\beta}}_\alpha = 2\rho(\mathbf{F}\,\frac{\partial\Sigma}{\partial\mathbf{C}}\,\mathbf{F}^T) \bullet \mathbf{D} + \rho\,\frac{\partial\Sigma}{\partial\boldsymbol{\beta}_\alpha} \bullet \dot{\boldsymbol{\beta}}_\alpha \;\;,$$

$$(\mathbf{T} - 2\rho\,\mathbf{F}\,\frac{\partial\Sigma}{\partial\mathbf{C}}\,\mathbf{F}^T) \bullet \mathbf{D} + (d_{33}^{-1/2}\,\mathbf{F}^T\,\mathbf{m}^\alpha - \rho\,\frac{\partial\Sigma}{\partial\boldsymbol{\beta}_\alpha}) \bullet \dot{\boldsymbol{\beta}}_\alpha \;= 0 \;\;, \tag{5.8.10}$$

where use has been made of the expression (5.6.8) and (5.6.11) to deduce that

$$\dot{\mathbf{C}} = 2\,\mathbf{F}^T\,\mathbf{D}\,\mathbf{F} \;\;. \tag{5.8.11}$$

In order to analyze the consequences of equation (5.8.10), it is noted that the coefficients of $\mathbf{D}$ and $\dot{\boldsymbol{\beta}}_\alpha$ are independent of the rates $(\mathbf{D},\, \dot{\boldsymbol{\beta}}_\alpha)$, and that the coefficient of $\mathbf{D}$ is also symmetric. Thus, for any fixed values of $\mathbf{F}$, $\boldsymbol{\beta}_\alpha$ and θ^3, the coefficients in (5.8.10) are fixed even though the rates $\mathbf{D}$ and $\dot{\boldsymbol{\beta}}_\alpha$ can be chosen arbitrarily. Therefore, the necessary condition that (5.8.10) be valid for arbitrary motions is that the kinetic quantities $\mathbf{T}$ and $\mathbf{m}^\alpha$ are given by derivatives of the strain energy

$$\mathbf{T} = 2\rho\,\mathbf{F}\,\frac{\partial\Sigma}{\partial\mathbf{C}}\,\mathbf{F}^T \;\;,\;\; \mathbf{m}^\alpha = \mathrm{m}\,\mathbf{F}^{-T}\,\frac{\partial\Sigma}{\partial\boldsymbol{\beta}_\alpha} \;\;, \tag{5.8.12}$$

where m is defined by $(5.4.25)_1$.

Next, by using the conservation of mass equation in the form $(5.4.25)_1$, it follows that comparison of $(5.8.12)_1$ with the expressions (3.7.16) and (3.7.17) associated with the three-dimensional theory suggests defining the symmetric tensor $\mathbf{S}$ such that

$$\mathbf{S} = 2\rho_0\,\frac{\partial\Sigma}{\partial\mathbf{C}} \;,\;\; d_{33}^{1/2}\,\mathbf{T} = D_{33}^{1/2}\,\mathbf{F}\,\mathbf{S}\,\mathbf{F}^T \;\;. \tag{5.8.13}$$

This causes $\mathbf{S}$ to be an analogue of the symmetric Piola-Kirchhoff stress in the three-dimensional theory.

Notice that once a form is specified for the strain energy function Σ, then the definition (5.4.23) can be used to obtain

$$\mathbf{m}^\alpha = \mathrm{m}\,\mathbf{F}^{-T}\,\frac{\partial\Sigma}{\partial\boldsymbol{\beta}_\alpha} \;\;,$$

$$\mathbf{t}^i = (d_{33}^{1/2}\,\mathbf{T} - \mathbf{m}^\alpha \otimes \mathbf{d}_{\alpha,3})\,\mathbf{d}^i = 2\,\mathrm{m}\,\mathbf{F}\frac{\partial \Sigma}{\partial \mathbf{C}}\,\mathbf{D}^i - \mathbf{m}^\alpha\,(\mathbf{d}_{\alpha,3} \bullet \mathbf{d}^i)\,. \qquad (5.8.14)$$

These expressions determine the kinetic quantities that appear in the local forms (5.4.25) of the balance laws.

Before closing this section, it is desirable to emphasize an important distinction between constitutive equations for a material in the three-dimensional theory and constitutive equations for a Cosserat rod. Within the context of the three-dimensional theory, constitutive equations characterize the response of a material at each material point and are independent of the shape of the three-dimensional body composed of the material. In contrast, within the context of the Cosserat rod theory, the constitutive equations necessarily couple influences of the geometry of the rod-like structure with those of the response of the three-dimensional material from which the rod is constructed. For example, consider the case of three rods that are constructed from the same homogeneous nonlinear elastic three-dimensional material. With respect to their stress-free reference configurations, let the first rod be a straight beam with uniform cross-sectional area, let the second rod be a straight beam with variable cross-sectional area, and let the third rod be an arbitrary space curve with uniform cross-sectional area. Due to the homogeneity of the geometric properties of the first rod, the strain energy function can be taken to be independent of the coordinate θ^3. Whereas, the variable cross-sectional area of the second rod and the variable curvature of the third rod tend to cause the strain energy functions for these rods to depend on the coordinate θ^3.

In general, the complicated coupling of material and geometrical properties of the rod must be modeled by the Cosserat constitutive equations. In this regard, it will be seen later that under conditions of homogeneous deformations some guidance can be deduced from the three-dimensional theory which helps separate these influences of material and geometric properties.

Furthermore, it is noted that in section 5.3 the theoretical structure of the balance laws for thc Cosserat rod theory was developed from the three-dimensional theory by using the kinematic assumption (5.2.8). From this perspective, constitutive equations for the kinetic quantities (5.8.14) are directly related to integrals through the cross-section of the rod of three-dimensional constitutive equations [see (5.3.19), (5.3.30) and (5.3.33)]. Even for elastic materials, these integrals usually cannot be evaluated exactly when large deformations are included. However, from the perspective of the direct approach of section 5.4, the balance laws of the Cosserat theory are postulated without any specific connection with the three-dimensional theory. Moreover, the mechanical power equation of section 5.6 and the constitutive equations of this section are also developed by a direct approach. This has the advantage that the constitutive equations are necessarily consistent with the balance laws of the Cosserat theory. In particular, the constitutive equations of nonlinear elastic rods retain the fundamental properties associated with the ideal character of an elastic material, as discussed previously. In contrast, if the constitutive equations for elastic rods were obtained by approximate integration of the three-dimensional constitutive equations, then special care would have to be imposed on the integration procedure to ensure that these fundamental characteristics of elastic materials are preserved.

5.9 Constraints

Mechanical constraints in the Cosserat theory of rods can be considered in a manner directly analogous to that used to analyze constraints in the three-dimensional theory (see sec. 3.8). For example, the usual incompressibility constraint requires

$$J = \det \mathbf{F} = 1 \ , \tag{5.9.1}$$

which can be differentiated to deduce that

$$\dot{J} = J\, \mathbf{F}^{-T} \bullet \dot{\mathbf{F}} = 0 \ , \quad \mathbf{I} \bullet \mathbf{D} = 0 \ , \tag{5.9.2}$$

where the expressions (5.6.8) and (5.6.11) have been used to obtain the form (5.9.2). Also, for such an incompressibility constraint, the value of Poisson's ratio in (5.16.13) and (5.16.14) must be specified by

$$\nu^* = \frac{1}{2} \ . \tag{5.9.3}$$

A hierarchy of constrained theories of rods has been developed by Naghdi and Rubin (1984), where more detailed developments can be found. Here, only a brief summary will be presented. In general, the Cosserat theory allows the directors $\mathbf{d}_\alpha$ to be arbitrary vectors that are not tangent to the rod's curve (5.2.4). To interpret the physical meaning of different possible deformations of $\mathbf{d}_\alpha$ it is most convenient to identify $\mathbf{d}_\alpha$ with material fibers that were normal to the rod's curve in its reference configuration. Then, in the present configuration and with reference to the definition (5.2.5) of the projections $\bar{\mathbf{d}}_\alpha$, the rod is said to have experienced: *normal cross-sectional extension* when the magnitudes of $\bar{\mathbf{d}}_\alpha$ change; *tangential shear deformation* when the components of $\mathbf{d}_\alpha$ in the direction of $\mathbf{d}_3$ change; and *normal cross-sectional shear deformation* when the angle between $\bar{\mathbf{d}}_1$ and $\bar{\mathbf{d}}_2$ changes.

It then follows that normal cross-sectional extension will be eliminated if the rod deformation is constrained so that

$$\bar{\mathbf{d}}_1 \bullet \bar{\mathbf{d}}_1 = \mathbf{d}_1 \bullet \mathbf{d}_1 - d_{33}^{-1} (\mathbf{d}_1 \bullet \mathbf{d}_3)^2 = \text{constant} \ ,$$

$$\bar{\mathbf{d}}_2 \bullet \bar{\mathbf{d}}_2 = \mathbf{d}_2 \bullet \mathbf{d}_2 - d_{33}^{-1} (\mathbf{d}_2 \bullet \mathbf{d}_3)^2 = \text{constant} \ . \tag{5.9.4}$$

Taking the material derivative of these expressions and recalling the definition (5.6.7), it can be shown that (5.9.4) can be written in the rate forms

$$\left[(\mathbf{d}_1 \otimes \mathbf{d}_1) + d_{33}^{-2} (\mathbf{d}_1 \bullet \mathbf{d}_3)^2 (\mathbf{d}_3 \otimes \mathbf{d}_3) - d_{33}^{-1} (\mathbf{d}_1 \bullet \mathbf{d}_3) (\mathbf{d}_1 \otimes \mathbf{d}_3 + \mathbf{d}_3 \otimes \mathbf{d}_1)\right] \bullet \mathbf{D} = 0 \ ,$$

$$\left[(\mathbf{d}_2 \otimes \mathbf{d}_2) + d_{33}^{-2} (\mathbf{d}_2 \bullet \mathbf{d}_3)^2 (\mathbf{d}_3 \otimes \mathbf{d}_3) - d_{33}^{-1} (\mathbf{d}_2 \bullet \mathbf{d}_3) (\mathbf{d}_2 \otimes \mathbf{d}_3 + \mathbf{d}_3 \otimes \mathbf{d}_2)\right] \bullet \mathbf{D} = 0 \ ,$$

to eliminate normal cross-sectional extension. (5.9.5)

Similarly, it can be shown that tangential shear deformation will be eliminated if

$$\mathbf{d}_\alpha \bullet \mathbf{d}_3 = 0 \ . \tag{5.9.6}$$

This yields two constraints in the rate forms

$$(\mathbf{d}_\alpha \otimes \mathbf{d}_3 + \mathbf{d}_3 \otimes \mathbf{d}_\alpha) \bullet \mathbf{D} = 0 \text{ to eliminate tangential shear deformation.} \tag{5.9.7}$$

Furthermore, normal cross-sectional shear deformation will be eliminated if

$$\bar{\mathbf{d}}_1 \bullet \bar{\mathbf{d}}_2 = 0 \ . \tag{5.9.8}$$

This yields the rate form

$$\Big[(\mathbf{d}_1 \otimes \mathbf{d}_2 + \mathbf{d}_2 \otimes \mathbf{d}_1) + 2\, d_{33}^{-2}\, (\mathbf{d}_1 \bullet \mathbf{d}_3)\, (\mathbf{d}_2 \bullet \mathbf{d}_3)\, (\mathbf{d}_3 \otimes \mathbf{d}_3)$$
$$- d_{33}^{-1}\, (\mathbf{d}_2 \bullet \mathbf{d}_3)\, (\mathbf{d}_1 \otimes \mathbf{d}_3 + \mathbf{d}_3 \otimes \mathbf{d}_1) - d_{33}^{-1}\, (\mathbf{d}_1 \bullet \mathbf{d}_3)\, (\mathbf{d}_2 \otimes \mathbf{d}_3 + \mathbf{d}_3 \otimes \mathbf{d}_2)\Big] \bullet \mathbf{D} = 0$$
$$\text{to eliminate normal cross-sectional shear deformation.} \tag{5.9.9}$$

Also, tangential extensional deformation will be eliminated if

$$\mathbf{d}_3 \bullet \mathbf{d}_3 = \text{constant} \ , \tag{5.9.10}$$

which yields the rate form

$$\big[\mathbf{d}_3 \otimes \mathbf{d}_3\big] \bullet \mathbf{D} = 0 \ . \tag{5.9.11}$$

Figure 5.9.1 shows a graphical representation of the reference configuration and general deformation of the present configuration, and Fig. 5.9.2 shows a graphical representation of some of these constraints by indicating the relative magnitude and orientation of the tangent vector $\mathbf{d}_3$, the directors $\mathbf{d}_\alpha$ and the projections $\bar{\mathbf{d}}_\alpha$.

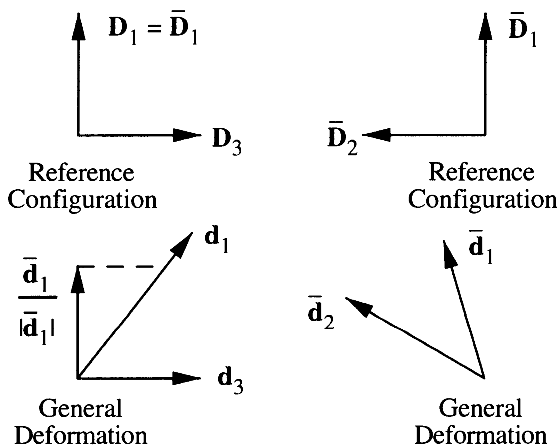

Fig. 5.9.1 Graphical representation of the reference configuration and general deformation of the present configuration.

Each of these constraints $(5.9.2)_2$, (5.9.5), (5.9.7), (5.9.9) and (5.9.10) is a special case of a class of general constraints which require

$$\boldsymbol{\gamma} \bullet \mathbf{D} + \boldsymbol{\gamma}^\alpha \bullet \dot{\boldsymbol{\beta}}_\alpha = 0 \ , \quad \boldsymbol{\gamma}^T = \boldsymbol{\gamma} \ , \tag{5.9.12}$$

where $\boldsymbol{\gamma}$ is a symmetric second order tensor and $\boldsymbol{\gamma}^\alpha$ are vectors. Moreover, it is assumed that under SRBM $\boldsymbol{\gamma}$ and $\boldsymbol{\gamma}^\alpha$ transform by

$$\boldsymbol{\gamma}^+ = \mathbf{Q}\, \boldsymbol{\gamma}\, \mathbf{Q}^T \ , \quad \boldsymbol{\gamma}^{\alpha +} = \boldsymbol{\gamma}^\alpha \ , \tag{5.9.13}$$

so that the constraint equation $(5.9.12)_1$ remains properly invariant under SRBM.

Motivated by the three-dimensional developments in section 3.8, it is possible to develop a constitutive theory for Cosserat rods in the presence of mechanical constraints by making the following five assumptions:

(i) The kinetic quantities $\mathbf{T}$, $\mathbf{t}^i$ and $\mathbf{m}^\alpha$ separate additively into two parts

$$\mathbf{T} = \hat{\mathbf{T}} + \bar{\mathbf{T}} \ , \quad \mathbf{t}^i = \hat{\mathbf{t}}^i + \bar{\mathbf{t}}^i \ , \quad \mathbf{m}^\alpha = \hat{\mathbf{m}}^\alpha + \bar{\mathbf{m}}^\alpha \ ,$$
$$\hat{\mathbf{T}} = d_{33}^{-1/2} \left[\hat{\mathbf{t}}^i \otimes \mathbf{d}_i + \hat{\mathbf{m}}^\alpha \otimes \mathbf{d}_{\alpha,3} \right] \ ,$$
$$\bar{\mathbf{T}} = d_{33}^{-1/2} \left[\bar{\mathbf{t}}^i \otimes \mathbf{d}_i + \bar{\mathbf{m}}^\alpha \otimes \mathbf{d}_{\alpha,3} \right] \ , \tag{5.9.14}$$

where $\hat{\mathbf{T}}$, $\hat{\mathbf{t}}^i$ and $\hat{\mathbf{m}}^\alpha$ are determined by constitutive equations that characterize the particular unconstrained rod under consideration, and $\bar{\mathbf{T}}$, $\bar{\mathbf{t}}^i$ and $\bar{\mathbf{m}}^\alpha$ are constraint responses.

(ii) The constraint responses $\bar{\mathbf{T}}$, $\bar{\mathbf{t}}^i$ and $\bar{\mathbf{m}}^\alpha$ are functions of θ^3 and t which are workless in the sense that

$$\bar{\mathbf{T}} \bullet \mathbf{D} + d_{33}^{-1/2} \left(\mathbf{F}^T \bar{\mathbf{m}}^\alpha \right) \bullet \dot{\boldsymbol{\beta}}_\alpha = 0 \ , \tag{5.9.15}$$

for all possible motions of the constrained material.

(iii) Both parts $\hat{\mathbf{T}}$ and $\bar{\mathbf{T}}$ of the kinetic quantity $\mathbf{T}$ are symmetric tensors

$$\hat{\mathbf{T}}^T = \hat{\mathbf{T}} \ , \quad \bar{\mathbf{T}}^T = \bar{\mathbf{T}} \ , \tag{5.9.16}$$

so that each of them satisfies the local form (5.4.24) of the balance of angular momentum.

(iv) Both parts $\hat{\mathbf{T}}$ and $\bar{\mathbf{T}}$ of the kinetic quantity $\mathbf{T}$, $\hat{\mathbf{t}}^i$ and $\bar{\mathbf{t}}^i$ of the kinetic quantity $\mathbf{t}^i$, and $\hat{\mathbf{m}}^\alpha$ and $\bar{\mathbf{m}}^\alpha$ of the kinetic quantity $\mathbf{m}^\alpha$ transform under SRBM by

$$\hat{\mathbf{T}}^+ = \mathbf{Q}\,\hat{\mathbf{T}}\,\mathbf{Q}^T \ , \quad \bar{\mathbf{T}}^T = \mathbf{Q}\,\bar{\mathbf{T}}\,\mathbf{Q}^T \ , \quad \hat{\mathbf{t}}^{i+} = \mathbf{Q}\,\hat{\mathbf{t}}^i \ , \quad \bar{\mathbf{t}}^{i+} = \mathbf{Q}\,\bar{\mathbf{t}}^i \ ,$$
$$\hat{\mathbf{m}}^{\alpha+} = \mathbf{Q}\,\hat{\mathbf{m}}^\alpha \ , \quad \bar{\mathbf{m}}^{\alpha+} = \mathbf{Q}\,\bar{\mathbf{m}}^\alpha \ , \tag{5.9.17}$$

so that the kinetic quantity $\mathbf{T}$ transforms by (5.5.5) and the expression (5.9.15) is properly invariant under SRBM.

(v) The tensors $\bar{\mathbf{T}}$, $\bar{\mathbf{m}}^\alpha$, $\boldsymbol{\gamma}$ and $\boldsymbol{\gamma}^\alpha$ are independent of the rates $\mathbf{L}$ and $\dot{\boldsymbol{\beta}}_\alpha$.

Using a Lagrange multiplier $\gamma(\theta^3,t)$ which is an arbitrary function of (θ^3,t), the equation (5.9.15), subject to the constraint (5.9.12), can be rewritten in the form

$$\left(\bar{\mathbf{T}} - d_{33}^{-1/2}\, \gamma \boldsymbol{\gamma} \right) \bullet \mathbf{D} + d_{33}^{-1/2} \left[\mathbf{F}^T \bar{\mathbf{m}}^\alpha - \gamma \boldsymbol{\gamma}^\alpha \right] \bullet \dot{\boldsymbol{\beta}}_\alpha = 0 \ . \tag{5.9.18}$$

Since at least one of the components of $\boldsymbol{\gamma}$ or $\boldsymbol{\gamma}^\alpha$ in (5.9.12) [say $\boldsymbol{\gamma} \bullet (\mathbf{d}_3 \otimes \mathbf{d}_3)$] is nonzero, the value of γ can be specified so that

$$\gamma = \frac{\bar{\mathbf{T}} \bullet (\mathbf{d}_3 \otimes \mathbf{d}_3)}{d_{33}^{-1/2}\, \boldsymbol{\gamma} \bullet (\mathbf{d}_3 \otimes \mathbf{d}_3)} \ . \tag{5.9.19}$$

It then follows that the coefficient of the component $[\mathbf{D} \bullet (\mathbf{d}_3 \otimes \mathbf{d}_3)]$ in (5.9.18) vanishes. Consequently, this component of $\mathbf{D}$ can be chosen to satisfy the constraint (5.9.12) for arbitrary values of the other components of $\mathbf{D}$ and $\dot{\boldsymbol{\beta}}_\alpha$. Moreover, since the coefficients of the rates $\{\mathbf{D} \ , \dot{\boldsymbol{\beta}}_\alpha\}$ in (5.9.18) are independent of these rates and the coefficient of $\mathbf{D}$ is a symmetric tensor, it follows that the constraint responses $\bar{\mathbf{T}}$ and $\bar{\mathbf{m}}^\alpha$ must take the forms

$$\bar{\mathbf{T}} = d_{33}^{-1/2}\,\gamma\boldsymbol{\gamma}\ ,\quad \bar{\mathbf{t}}^i = \gamma\left[\boldsymbol{\gamma}\,\mathbf{d}^i - \mathbf{F}^{-T}\boldsymbol{\gamma}^\alpha(\mathbf{d}_{3,\alpha}\bullet\mathbf{d}^i)\right]\ ,\quad \bar{\mathbf{m}}^\alpha = \gamma\,\mathbf{F}^{-T}\boldsymbol{\gamma}^\alpha\ , \tag{5.9.20}$$

where γ remains an arbitrary function of (θ^3,t) that is determined by the equations of motions and the boundary conditions.

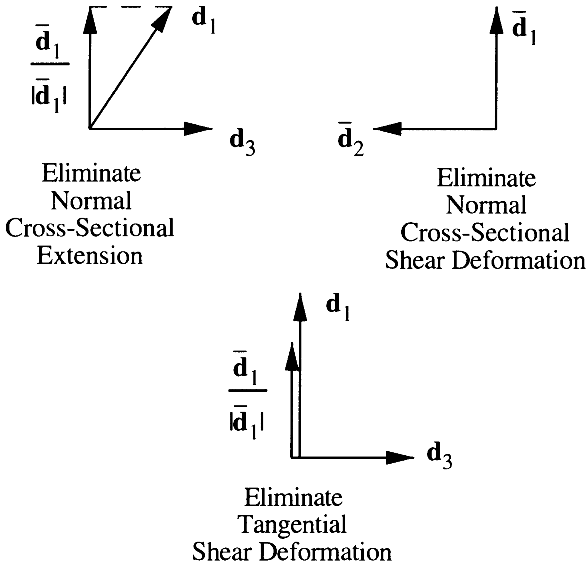

Fig. 5.9.2 Graphical representation of some constraints.

If more than one constraint is imposed on the rod, then the constraint responses $\bar{\mathbf{T}}$ and $\bar{\mathbf{m}}^\alpha$ can be represented as the sum of a Lagrange multiplier times each of the constraint tensors $\boldsymbol{\gamma}$ and $\boldsymbol{\gamma}^\alpha$. For example, the case of a nonlinear Bernoulli-Euler rod is one of the simplest nonlinear rod theories because it eliminates normal cross-sectional extension, tangential shear deformation, and normal cross-sectional shear deformation. For this case the five constraints (5.9.5), (5.9.7) and (5.9.9) are imposed simultaneously. However, since tangential shear deformation is eliminated, these constraints can be written in the simplified forms

$$(\mathbf{d}_1\otimes\mathbf{d}_1)\bullet\mathbf{D} = 0\ ,\quad (\mathbf{d}_2\otimes\mathbf{d}_2)\bullet\mathbf{D} = 0\ ,$$

$$(\mathbf{d}_\alpha\otimes\mathbf{d}_3 + \mathbf{d}_3\otimes\mathbf{d}_\alpha)\bullet\mathbf{D} = 0\ .\quad (\mathbf{d}_1\otimes\mathbf{d}_2 + \mathbf{d}_2\otimes\mathbf{d}_1)\bullet\mathbf{D} = 0\ , \tag{5.9.21}$$

which are independent of $\boldsymbol{\beta}_\alpha$. It then follows that the constraint responses $\bar{\mathbf{T}}$ and $\bar{\mathbf{m}}^\alpha$ can be represented in the forms

$$\bar{\mathbf{T}} = \frac{1}{2}d_{33}^{-1/2}\,\gamma^{\alpha\beta}\,(\mathbf{d}_\alpha\otimes\mathbf{d}_\beta + \mathbf{d}_\beta\otimes\mathbf{d}_\alpha) + d_{33}^{-1/2}\,\gamma^{3\alpha}\,(\mathbf{d}_3\otimes\mathbf{d}_\alpha + \mathbf{d}_\alpha\otimes\mathbf{d}_3)\ ,$$

$$\bar{\mathbf{m}}^\alpha = 0\ ,\quad \gamma^{\alpha\beta} = \gamma^{\beta\alpha}\ , \tag{5.9.22}$$

where $\gamma^{i\alpha}$ are Lagrange multipliers that are functions of (θ^3,t) with $\gamma^{\alpha\beta}$ being symmetric. Thus, with the help of $(5.9.14)_2$ it can be shown that

$$\bar{\mathbf{t}}^\alpha = d_{33}^{1/2}\,\bar{\mathbf{T}}\,\mathbf{d}^\alpha = \gamma^{\alpha\beta}\,\mathbf{d}_\beta + \gamma^{3\alpha}\,\mathbf{d}_3 \ , \quad \bar{\mathbf{t}}^3 = d_{33}^{1/2}\,\bar{\mathbf{T}}\,\mathbf{d}^3 = \gamma^{3\alpha}\,\mathbf{d}_\alpha \ . \tag{5.9.23}$$

Notice that five of the six components of $\bar{\mathbf{t}}^\alpha$ are arbitrary functions of (θ^3,t). This means that all but one of the six components of the director momentum equations $(5.4.25)_3$ can be satisfied for arbitrary admissible motions of the constrained rod by using $(5.9.14)_3$ and $(5.9.23)_1$ to determine the functions $\gamma^{i\alpha}$

$$\gamma^{i\alpha} = \left[\, m\,\mathbf{b}^\alpha - \hat{\mathbf{t}}^\alpha + \mathbf{m}^\alpha{}_{,3} - m\,(y^\alpha\,\dot{\mathbf{v}} + y^{\alpha\beta}\,\dot{\mathbf{w}}_\beta)\right] \bullet \mathbf{d}^i \ . \tag{5.9.24}$$

Moreover, since $\gamma^{\alpha\beta}$ is symmetric, the director momentum equations reduce to a single equation (associated with the condition that $\gamma^{12}=\gamma^{21}$)

$$\left[\, m\,\mathbf{b}^1 - \hat{\mathbf{t}}^1 + \mathbf{m}^1{}_{,3} - m\,(y^1\,\dot{\mathbf{v}} + y^{1\beta}\,\dot{\mathbf{w}}_\beta)\right] \bullet \mathbf{d}^2$$
$$= \left[\, m\,\mathbf{b}^2 - \hat{\mathbf{t}}^2 + \mathbf{m}^2{}_{,3} - m\,(y^2\,\dot{\mathbf{v}} + y^{2\beta}\,\dot{\mathbf{w}}_\beta)\right] \bullet \mathbf{d}^1 \ , \tag{5.9.25}$$

which is used to determine the twist of the rod. Also, in view of the constraint response $(5.9.23)_2$, the equation of linear momentum $(5.4.25)_2$ includes the effect of $\gamma^{3\alpha}$ and becomes

$$m\,(\dot{\mathbf{v}} + y^\alpha\,\dot{\mathbf{w}}_\alpha) = m\,\mathbf{b} + \left[\, \hat{\mathbf{t}}^3 + \gamma^{3\alpha}\,\mathbf{d}_\alpha \,\right]_{,3} \ , \tag{5.9.26}$$

where $\gamma^{3\alpha}$ is determined by equation (5.9.24).

This nonlinear Bernoulli-Euler rod theory is simpler than the complete Cosserat theory because the constraints (5.9.21) determine the directors $\mathbf{d}_\alpha$ to within a twist about the $\mathbf{d}_3$ axis in terms of deformations of the rod's curve. Five of the director momentum equations are satisfied by the Lagrange multipliers $\gamma^{i\alpha}$, and the balance laws of the theory reduce to the conservation of mass $(5.4.25)_1$, the balance of linear momentum (5.9.26), and one component of the balances of director momentum (5.9.25).

Moreover, as another example, the constraint responses (5.9.20) associated with the incompressibility constraint (5.9.2) become

$$\bar{\mathbf{T}} = d_{33}^{-1/2}\,\gamma\,\mathbf{I} \ , \quad \bar{\mathbf{t}}^i = \gamma\,\mathbf{d}^i \ , \quad \bar{\mathbf{m}}^\alpha = 0 \ . \tag{5.9.27}$$

In general, the constraint responses $\bar{\mathbf{T}}$ and $\bar{\mathbf{m}}^\alpha$ influence the equations of linear and director momentum but not the conservation of mass or angular momentum equations (since $\bar{\mathbf{T}}$ is a symmetric tensor). Therefore, if more than nine independent kinematic constraints are imposed, then $\bar{\mathbf{T}}$ and $\bar{\mathbf{m}}^\alpha$ will have more than nine independent components which are arbitrary functions of position and time. This means that even when appropriate boundary conditions are specified, it will not be possible to uniquely determine all components of $\bar{\mathbf{T}}$ and $\bar{\mathbf{m}}^\alpha$. However, it is reasonable to expect that the arbitrariness in $\bar{\mathbf{T}}$ and $\bar{\mathbf{m}}^\alpha$ that remains after the equations of motion and boundary conditions are satisfied will not influence the overall motion of the constrained rod.

Motivated by the discussion of constraints in section 3.8, within the context of the three-dimensional theory it might seem adequate to eliminate the terms $\dot{\boldsymbol{\beta}}_\alpha$ in (5.9.12) and only consider constraints on $\mathbf{D}$. However, it will presently be shown that the condition that $\mathbf{D}$ vanishes is not sufficient to ensure that the rod moves as a rigid body. Specifically, since $\mathbf{D}$ is defined only on the reference curve of the rod and not pointwise through the cross-section of the rod, it does not contain complete information about homogeneity of the three-dimensional deformation of the rod structure. The additional information required is contained in $\boldsymbol{\beta}_\alpha$. Consequently, the constraints required to ensure that the rod moves as a rigid body demand that both $\mathbf{D}$ and $\dot{\boldsymbol{\beta}}_\alpha$ vanish pointwise on the reference curve of the rod

$$\mathbf{D} = 0 \ , \quad \dot{\boldsymbol{\beta}}_\alpha = 0 \ . \tag{5.9.28}$$

Moreover, it is convenient to use (5.6.7), (5.6.8) and (5.6.15) to derive the expression

$$\dot{\boldsymbol{\beta}}_\alpha = \mathbf{F}^{-1}\mathbf{L}_{,3}\, \mathbf{d}_\alpha \ , \tag{5.9.29}$$

which is a measure of the $\mathbf{d}_\alpha$ components of the gradient of the $\mathbf{L}$.

For clarity, it is desirable to consider an example where $\mathbf{D}$ vanishes pointwise but the rod is not rigid and $\dot{\boldsymbol{\beta}}_\alpha$ is nonzero. To this end, consider a right circular cylindrical rod of uniform cross-section in its reference configuration. The rod is deformed by torsion applied along its axis of symmetry so that the radius of the cross-section and the axial length remain constant. The kinematics of the reference configuration are specified in terms of fixed rectangular Cartesian base vectors $\mathbf{e}_i$ by

$$\mathbf{X} = \theta^3\, \mathbf{e}_3 \ , \quad \mathbf{D}_i = \mathbf{e}_i \ . \tag{5.9.30}$$

Similarly, the kinematics of the present configuration are specified by

$$\begin{gathered}\mathbf{x} = \theta^3\, \mathbf{e}_3 \ , \quad \mathbf{d}_1 = \cos\theta\, \mathbf{D}_1 + \sin\theta\, \mathbf{D}_2 = \mathbf{e}_r(\theta) \ , \\ \mathbf{d}_2 = -\sin\theta\, \mathbf{D}_1 + \cos\theta\, \mathbf{D}_2 = \mathbf{e}_\theta(\theta) \ , \quad \mathbf{d}_3 = \mathbf{e}_3 \ , \quad \theta = k(t)\, \theta^3 \ ,\end{gathered} \tag{5.9.31}$$

where k(t) determines the twist per unit length. It then follows from the definitions (5.2.10), (5.6.7), (5.6.11) and (5.6.15), that

$$\begin{gathered}\mathbf{F} = \mathbf{e}_r(\theta)\otimes\mathbf{e}_1 + \mathbf{e}_\theta(\theta)\otimes\mathbf{e}_2 + \mathbf{e}_3\otimes\mathbf{e}_3 \ , \\ \mathbf{D} = 0 \ , \quad \mathbf{L} = \mathbf{W} = -\dot{k}\, \theta^3 \left[\mathbf{d}_1\otimes\mathbf{d}_2 - \mathbf{d}_2\otimes\mathbf{d}_3\right] \ , \\ \boldsymbol{\beta}_1 = k\, \mathbf{D}_2 \ , \quad \boldsymbol{\beta}_2 = -k\, \mathbf{D}_1 \ , \quad \dot{\boldsymbol{\beta}}_1 = \dot{k}\, \mathbf{D}_2 \ , \quad \dot{\boldsymbol{\beta}}_2 = -\dot{k}\, \mathbf{D}_1 \ .\end{gathered} \tag{5.9.32}$$

Thus, although $\mathbf{D}$ vanishes pointwise the rod is not rigid because it is twisted about its axis of symmetry.

Before closing this section, it should be mentioned that it may be desirable to modify the functional form of the strain energy Σ when constraints are imposed on the rod. For example, consider a straight beam with rectangular cross-section whose lateral surfaces are normal to the rectangular Cartesian base vectors $\mathbf{e}_\alpha$ and whose ends are normal to the vector $\mathbf{e}_3$ which is in the tangential direction. If the beam is pulled in uniaxial tension in the $\mathbf{e}_3$ direction, then the Poisson effect usually causes the lengths of the rod in the $\mathbf{e}_1$ and $\mathbf{e}_2$ directions to decrease. Now, if for simplicity the beam is constrained so that normal cross-sectional extension is eliminated, then the predicted response to uniaxial tension in the $\mathbf{e}_3$ direction will be too stiff if the same functional form for the strain energy Σ is used in the constrained theory. To avoid this problem it is possible to modify the constitutive equation for Σ so that the response of the constrained theory will simulate the softer response of the unconstrained beam. Moreover, it will be shown later that the director inertia coefficients y^α and $y^{\alpha\beta}$ model not only the mass distribution through the cross-section of the rod but also model information about mode shapes of vibration. Therefore, the appropriate values of y^α and $y^{\alpha\beta}$ for a deformable rod can be different from those for a constrained rod (e.g. a rigid rod). It is also important to note that constraints can influence the nature of boundary conditions, since not all components of the velocities $\mathbf{v}$ and $\mathbf{w}_\alpha$ are independent.

For the special case of a constrained elastic rod, the parts $\hat{\mathbf{T}}$, $\hat{\mathbf{t}}^i$, $\hat{\mathbf{m}}^\alpha$ of the kinetic quantities associated with constitutive equations satisfy the condition (5.8.1) that the mechanical power due to these parts is equal to the rate of change of the strain energy function. Then, $\hat{\mathbf{T}}$, $\hat{\mathbf{t}}^i$, $\hat{\mathbf{m}}^\alpha$ are determined in terms of derivatives of the strain energy function by formulas of the type (5.8.12).

5.10 Initial and boundary conditions

The local forms of conservation of mass $(5.4.25)_1$, the balance of linear momentum $(5.4.25)_2$ and the balances of director momentum $(5.4.25)_3$ are partial differential equations which require both initial and boundary conditions. Specifically, the conservation of mass $(5.4.25)_1$ is first order in time with respect to density ρ so it is necessary to specify the initial value of density at each point of the rod

$$\rho(\theta^3,0) = \bar{\rho}(\theta^3) \quad \text{on P for } t = 0 \,. \tag{5.10.1}$$

Also, the balance of linear momentum $(5.4.25)_2$ and the balances of director momentum $(5.4.25)_3$ are second order in time with respect to position $\mathbf{x}$ and the directors $\mathbf{d}_\alpha$ so that it is necessary to specify the initial values of $\mathbf{x}$ and $\mathbf{d}_\alpha$, as well as the initial values of the velocities $\mathbf{v}$ and $\mathbf{w}_\alpha$ at each point of the rod

$$\hat{\mathbf{x}}(\theta^3,0) = \bar{\mathbf{x}}(\theta^3) \;\;,\;\; \hat{\mathbf{v}}(\theta^3,0) = \bar{\mathbf{v}}(\theta^3) \;\;,$$
$$\hat{\mathbf{d}}_\alpha(\theta^3,0) = \bar{\mathbf{d}}_\alpha(\theta^3) \;\;,\;\; \hat{\mathbf{w}}_\alpha(\theta^3,0) = \bar{\mathbf{w}}_\alpha(\theta^3) \quad \text{on P for } t = 0 \,, \tag{5.10.2}$$

where an overbar is temporarily used here to denote specified values, and $\bar{\mathbf{d}}_\alpha$ in (5.10.2) should not be confused with the projections defined in (5.2.5).

Guidance for determining the appropriate forms of the boundary conditions is usually obtained by considering the rate of work done by the resultant forces and moments applied to the ends ∂P_1 and ∂P_2 of the rod. Consequently, from (5.6.1) it is observed that $\mathbf{t}^3 \bullet \mathbf{v}$ is the rate of work of the resultant force and $\mathbf{m}^\alpha \bullet \mathbf{w}_\alpha$ is the rate of work of the director couples applied to the ends of the rod. At each point θ^3 of the rod, it is possible to define a right-handed orthogonal coordinate system with base vectors $\{\boldsymbol{\tau}_1 \,,\, \boldsymbol{\tau}_2 \,,\, \mathbf{n}\}$, where $\mathbf{n}$ is tangent to the rod's curve and $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_2$ are vectors in the normal cross-section. Then, with reference to these base vectors, the expressions $\mathbf{t}^3 \bullet \mathbf{v}$ and $\mathbf{m}^\alpha \bullet \mathbf{w}_\alpha$ can be written in the forms

$$\mathbf{t}^3 \bullet \mathbf{v} = (\mathbf{t}^3 \bullet \boldsymbol{\tau}_1)\,(\mathbf{v} \bullet \boldsymbol{\tau}_1) + (\mathbf{t}^3 \bullet \boldsymbol{\tau}_2)\,(\mathbf{v} \bullet \boldsymbol{\tau}_2) + (\mathbf{t}^3 \bullet \mathbf{n})\,(\mathbf{v} \bullet \mathbf{n}) \;\;,$$
$$\mathbf{m}^\alpha \bullet \mathbf{w}_\alpha = (\mathbf{m}^\alpha \bullet \boldsymbol{\tau}_1)\,(\mathbf{w}_\alpha \bullet \boldsymbol{\tau}_1) + (\mathbf{m}^\alpha \bullet \boldsymbol{\tau}_2)\,(\mathbf{w}_\alpha \bullet \boldsymbol{\tau}_2) + (\mathbf{m}^\alpha \bullet \mathbf{n})\,(\mathbf{w}_\alpha \bullet \mathbf{n}) \;\;,$$
$$\text{on } \partial P_1 \text{ and } \partial P_2 \;. \tag{5.10.3}$$

Using these representations it is possible to define four types of boundary conditions

Kinematic: All three components of the velocities are specified

$$(\mathbf{v} \bullet \boldsymbol{\tau}_1) \;,\; (\mathbf{v} \bullet \boldsymbol{\tau}_2) \;,\; (\mathbf{v} \bullet \mathbf{n}) \;,\; (\mathbf{w}_\alpha \bullet \boldsymbol{\tau}_1) \;,\; (\mathbf{w}_\alpha \bullet \boldsymbol{\tau}_2) \;,\; (\mathbf{w}_\alpha \bullet \mathbf{n}) \;,$$
$$\text{specified on } \partial P_1 \text{ and } \partial P_2 \text{ for all } t \geq 0 \;, \tag{5.10.4}$$

Kinetic: All three components of the resultant force and director couples are specified

$$(\mathbf{t}^3 \bullet \boldsymbol{\tau}_1) \;,\; (\mathbf{t}^3 \bullet \boldsymbol{\tau}_2) \;,\; (\mathbf{t}^3 \bullet \mathbf{n}) \;,\; (\mathbf{m}^\alpha \bullet \boldsymbol{\tau}_1) \;,\; (\mathbf{m}^\alpha \bullet \boldsymbol{\tau}_2) \;,\; (\mathbf{m}^\alpha \bullet \mathbf{n}) \;,$$
$$\text{specified on } \partial P_1 \text{ and } \partial P_2 \text{ for all } t \geq 0 \;, \tag{5.10.5}$$

Mixed: Kinematic boundary conditions are specified on one end of the rod and kinetic boundary conditions are specified on the other end of the rod.

Mixed-Mixed: Conjugate components of both the velocities and the resultant force and director couples are specified

$$\{(\mathbf{v} \bullet \boldsymbol{\tau}_1) \text{ or } (\mathbf{t}^3 \bullet \boldsymbol{\tau}_1)\} \;,\; \{(\mathbf{v} \bullet \boldsymbol{\tau}_2) \text{ or } (\mathbf{t}^3 \bullet \boldsymbol{\tau}_2)\} \;,\; \{(\mathbf{v} \bullet \mathbf{n}) \text{ or } (\mathbf{t}^3 \bullet \mathbf{n})\} \;,$$
$$\{(\mathbf{w}_\alpha \bullet \boldsymbol{\tau}_1) \text{ or } (\mathbf{m}^\alpha \bullet \boldsymbol{\tau}_1)\} \;,\; \{(\mathbf{w}_\alpha \bullet \boldsymbol{\tau}_2) \text{ or } (\mathbf{m}^\alpha \bullet \boldsymbol{\tau}_2)\} \;,\; \{(\mathbf{w}_\alpha \bullet \mathbf{n}) \text{ or } (\mathbf{m}^\alpha \bullet \mathbf{n})\} \;,$$
$$\text{specified on } \partial P_1 \text{ and } \partial P_2 \text{ for all } t \geq 0 \;, \tag{5.10.6}$$

Essentially, the conjugate components $(\mathbf{t}^3 \bullet \boldsymbol{\tau}_1)$, $(\mathbf{t}^3 \bullet \boldsymbol{\tau}_2)$, $(\mathbf{t}^3 \bullet \mathbf{n})$ are the responses to the motions $(\mathbf{v} \bullet \boldsymbol{\tau}_1)$, $(\mathbf{v} \bullet \boldsymbol{\tau}_2)$, $(\mathbf{v} \bullet \mathbf{n})$, respectively, and the components $(\mathbf{m}^\alpha \bullet \boldsymbol{\tau}_1)$, $(\mathbf{m}^\alpha \bullet \boldsymbol{\tau}_2)$, $(\mathbf{m}^\alpha \bullet \mathbf{n})$ are the responses to the motions $(\mathbf{w}_\alpha \bullet \boldsymbol{\tau}_1)$, $(\mathbf{w}_\alpha \bullet \boldsymbol{\tau}_2)$, $(\mathbf{w}_\alpha \bullet \mathbf{n})$, respectively. Therefore, it is important to emphasize that for example, both $(\mathbf{v} \bullet \mathbf{n})$ and $(\mathbf{t}^3 \bullet \mathbf{n})$ cannot be specified at the same end of the rod because this would mean that both the motion and the response can be specified independently of the material properties and geometry of the body. Notice also that since the initial position of the ends ∂P_1 and ∂P_2 are specified by the initial condition $(5.10.2)_1$, the velocity boundary conditions (5.10.4) can be used to determine the positions of these ends for all time. This means that the kinematic boundary conditions (5.10.4) could also be characterized by specifying the positions of the ends for all time. Similar comments apply to the initial condition $(5.10.2)_3$ and velocity boundary conditions (5.10.4) associated with the directors $\mathbf{d}_\alpha$. Furthermore, for static problems the position vector and the directors will include a measure of arbitrariness if insufficient kinematic boundary conditions are supplied to specify the three translational and three rotational rigid-body degrees of freedom.

With regard to the boundary conditions for the moment applied to the ends of the rod, it is convenient to consider the balance of angular momentum (5.4.7) and define the moment $\mathbf{m}$ (about the position $\mathbf{x}$) applied to the cross-section by the equation

$$\mathbf{m} = \mathbf{d}_\alpha \times \mathbf{m}^\alpha \;. \tag{5.10.7}$$

Here, it is important to distinguish between this moment $\mathbf{m}$ and the director couples $\mathbf{m}^\alpha$. Specifically, it is noted that in general, $\mathbf{m}^1$ can have a component in the $\mathbf{d}_1$ direction which resists stretching of the director $\mathbf{d}_1$, and $\mathbf{m}^2$ can have a component in the $\mathbf{d}_2$ direction which resists stretching of the director $\mathbf{d}_2$. Moreover, these components of $\mathbf{m}^\alpha$ make no contribution to the moment $\mathbf{m}$.

5.11 Further restrictions on constitutive equations for rods constructed from homogeneous anisotropic nonlinear elastic materials

From the developments in section 5.3 it can be seen that the force $\mathbf{t}^3$ (5.3.19) and the couples $\mathbf{m}^\alpha$ (5.3.33) applied to the ends of the rod, are related to weighted integrals of the stress vector through the cross-section of the rod structure. This means that the constitutive equations (5.8.14) for the kinetic quantities $\mathbf{t}^i$ and $\mathbf{m}^\alpha$ necessarily couple the effects of geometric properties of the rod with the constitutive properties of the material from which the rod is constructed. Even if the rod is constructed using a homogeneous material, curvature and variable cross-section of the rod in its reference configuration can significantly influence the constitutive equations for the Cosserat rod model.

Recently, Rubin (1996) has developed restrictions on constitutive equations for rods constructed from uniform homogeneous anisotropic nonlinear elastic materials which ensure that solutions of the Cosserat theory can reproduce exactly the complete class of homogeneous solutions of the three-dimensional theory. The main results of this work are summarized below.

First, it is recalled from (3.2.34), (5.2.8), (5.2.9), (5.3.2), (5.3.19), (5.3.30), (5.3.33) and (5.4.23), that

$$\begin{aligned} d_{33}^{1/2}\ \mathbf{T} &= (\mathbf{t}^i \otimes \mathbf{d}_i + \mathbf{m}^\alpha \otimes \mathbf{d}_{\alpha,3}) \\ &= \int_{\mathcal{A}} g^{1/2}\, \mathbf{T}^* \left[\, \mathbf{g}^i \otimes \mathbf{d}_i + \mathbf{g}^3 \otimes \theta^\alpha\, \mathbf{d}_{\alpha,3} \,\right] d\theta^1\, d\theta^2 \ , \\ &= \int_{\mathcal{A}} g^{1/2}\, \mathbf{T}^* \left[\, \mathbf{g}^\alpha \otimes \mathbf{d}_\alpha + \mathbf{g}^3 \otimes (\mathbf{d}_3 + \theta^\alpha\, \mathbf{d}_{\alpha,3}) \,\right] d\theta^1\, d\theta^2 \ , \\ &= \int_{\mathcal{A}} g^{1/2}\, \mathbf{T}^* \left[\, \mathbf{g}^i \otimes \mathbf{g}_i \,\right] d\theta^1\, d\theta^2 \ . \end{aligned} \tag{5.11.1}$$

However, since $\mathbf{g}^i \otimes \mathbf{g}_i = \mathbf{I}$ it follows that

$$d_{33}^{1/2}\ \mathbf{T} = \int_{\mathcal{A}} g^{1/2}\, \mathbf{T}^*\, d\theta^1\, d\theta^2 \ . \tag{5.11.2}$$

Next, with the help of (3.2.28), (3.7.15), $(5.4.25)_1$ and $(5.8.13)_1$ it can be shown that

$$d_{33}^{1/2}\ \mathbf{T} = 2\, m\, \mathbf{F} \frac{\partial \Sigma}{\partial \mathbf{C}}\, \mathbf{F}^T \ , \quad g^{1/2}\, \mathbf{T}^* = 2\, m^*\, \mathbf{F}^* \frac{\partial \Sigma^*}{\partial \mathbf{C}^*}\, \mathbf{F}^{*T} \ , \tag{5.11.3}$$

so that equation (5.11.2) can be rewritten in the form

$$\begin{aligned} d_{33}^{1/2}\ \mathbf{T} = 2\, m\, \mathbf{F} \frac{\partial \Sigma}{\partial \mathbf{C}} \mathbf{F}^T &= 2 \int_{\mathcal{A}} m^*\, \mathbf{F}^* \frac{\partial \Sigma^*}{\partial \mathbf{C}^*}\, \mathbf{F}^{*T}\, d\theta^1\, d\theta^2 \ , \\ m \frac{\partial \Sigma}{\partial \mathbf{C}} &= \int_{\mathcal{A}} m^*\, \mathbf{F}^{-1}\, \mathbf{F}^* \frac{\partial \Sigma^*}{\partial \mathbf{C}^*}\, \mathbf{F}^{*T}\, \mathbf{F}^{-T}\, d\theta^1\, d\theta^2 \ . \end{aligned} \tag{5.11.4}$$

Moreover, (3.2.34), (5.3.33), $(5.4.25)_1$, (5.8.12), (5.8.13) and (5.11.3) can be used to determine that

$$\mathbf{m}^\alpha = m\, \mathbf{F}^{-T} \frac{\partial \Sigma}{\partial \boldsymbol{\beta}_\alpha} = \int_{\mathcal{A}} \theta^\alpha\, g^{1/2}\, \mathbf{T}^*\, \mathbf{g}^3\, d\theta^1\, d\theta^2 \ ,$$

$$m \frac{\partial \Sigma}{\partial \boldsymbol{\beta}_\alpha} = \int_{\mathcal{A}} \theta^\alpha \, 2 \, m^* \, \mathbf{F}^T \mathbf{F}^* \frac{\partial \Sigma^*}{\partial \mathbf{C}^*} \mathbf{F}^{*T} \mathbf{g}^3 \, d\theta^1 \, d\theta^2 \ ,$$

$$m \frac{\partial \Sigma}{\partial \boldsymbol{\beta}_\alpha} = \int_{\mathcal{A}} \theta^\alpha \, 2 \, m^* \, \mathbf{F}^T \mathbf{F}^* \frac{\partial \Sigma^*}{\partial \mathbf{C}^*} \mathbf{G}^3 \, d\theta^1 \, d\theta^2 \ . \tag{5.11.5}$$

Now, if the material from which the rod is constructed is uniform and homogeneous, then ρ_0^* and Σ^* are explicitly independent of the coordinates θ^i

$$\rho_0^* = \text{constant} \ , \quad \Sigma^* = \hat{\Sigma}^*(\mathbf{C}^*) \ . \tag{5.11.6}$$

Thus, with the help of (3.2.28) and (5.3.7) it can be shown that

$$m = \int_{\mathcal{A}} m^* \, d\theta^1 \, d\theta^2 = \rho_0^* \, D_{33}^{1/2} \, A \ , \ D_{33}^{1/2} \, A(\theta^3) = \int_{\mathcal{A}} G^{1/2} \, d\theta^1 \, d\theta^2 \ , \tag{5.11.7}$$

where A is scalar function of θ^3 only, related to the cross-sectional area. Next, attention is restricted to three-dimensionally homogeneous deformations [(5.2.15)-(5.2.17)] for which

$$\mathbf{F}^* = \mathbf{F} = \mathbf{F}(t) \ , \quad \mathbf{C}^* = \mathbf{C} = \mathbf{F}^T \mathbf{F} \ , \quad \boldsymbol{\beta}_\alpha = 0 \ , \tag{5.11.8}$$

so that (5.11.4) and (5.11.5) yield the restrictions

$$\left. \frac{\partial \Sigma}{\partial \mathbf{C}} \right|_{\boldsymbol{\beta}_\beta = 0} = \frac{\partial \hat{\Sigma}^*(\mathbf{C})}{\partial \mathbf{C}} \ , \quad \left. \frac{\partial \Sigma}{\partial \boldsymbol{\beta}_\alpha} \right|_{\boldsymbol{\beta}_\beta = 0} = 2 \, A^\alpha \, \mathbf{C} \frac{\partial \hat{\Sigma}^*(\mathbf{C})}{\partial \mathbf{C}} \, \mathbf{D}^3 \ , \tag{5.11.9}$$

where $A^\alpha(\theta^3)$ are functions of θ^3 defined such that

$$D_{33}^{1/2} \, A \, A^\alpha(\theta^3) = D^{1/2} \left[\int_{\mathcal{A}} \theta^\alpha \, d\theta^1 \, d\theta^2 \right] \ ,$$

$$m \, A^\alpha \, \mathbf{D}^3 = \int_{\mathcal{A}} \theta^\alpha \, m^* \, \mathbf{G}^3 \, d\theta^1 \, d\theta^2 \ , \tag{5.11.10}$$

and use has been made of (5.1.10) and (5.1.11) to deduce that

$$G^{1/2} \, \mathbf{G}^3 = D^{1/2} \, \mathbf{D}^3 \ . \tag{5.11.11}$$

Also, here use has been made of the result (5.11.7) to simplify (5.11.9).

The expressions (5.11.9) place necessary restrictions on the constitutive equations for rods which ensure consistency with exact solutions for all homogeneous deformations. Specifically, it will presently be shown that for homogeneous deformations the restrictions (5.11.9) also cause

$$m \, \mathbf{b}_c + \mathbf{t}^3{}_{,3} = 0 \ , \quad m \, \mathbf{b}_c^\alpha - \mathbf{t}^\alpha + \mathbf{m}^\alpha{}_{,3} = 0 \ , \tag{5.11.12}$$

when the external force $\mathbf{b}_c$ and couples $\mathbf{b}_c^\alpha$ due to tractions on the lateral surface of the rod are given by the expressions (5.3.16) and (5.3.32). Thus, the balances of linear momentum and director momentum (5.4.25) reduce to the equations

$$m \, (\dot{\mathbf{v}} + y^\alpha \, \dot{\mathbf{w}}_\alpha) = m \, \mathbf{b}_b \ , \quad m \, (y^\alpha \, \dot{\mathbf{v}} + y^{\alpha\beta} \, \dot{\mathbf{w}}_\beta) = m \, \mathbf{b}_b^\alpha \ , \tag{5.11.13}$$

which express the familiar result that for homogeneous deformations the accelerations are balanced by terms associated with body forces only.

To prove equations (5.11.12), it is first observed from (5.8.14) and (5.11.9) that for homogeneous deformations, the constitutive equations for $\mathbf{t}^i$ and $\mathbf{m}^\alpha$ can be written in the forms

$$\mathbf{t}^i = 2\,m\,\mathbf{F}\,\frac{\partial \hat{\Sigma}^*(\mathbf{C})}{\partial \mathbf{C}}\,\mathbf{F}^T\left[\mathbf{d}^i - A^\alpha\,\mathbf{d}^3\,(\mathbf{d}_{\alpha,3}\bullet \mathbf{d}^i)\right] \ ,$$

$$\mathbf{m}^\alpha = 2\,m\,A^\alpha\,\mathbf{F}\,\frac{\partial \hat{\Sigma}^*(\mathbf{C})}{\partial \mathbf{C}}\,\mathbf{D}^3 \ . \tag{5.11.14}$$

Moreover, with the help of (3.2.7), $(3.2.33)_1$ and (3.7.15) it follows that for homogeneous deformations

$$J^*\,\mathbf{T}^* = \left[2\,\rho_0^*\ \mathbf{F}\,\frac{\partial \hat{\Sigma}^*(\mathbf{C})}{\partial \mathbf{C}}\,\mathbf{F}^T\right] \ . \tag{5.11.15}$$

Thus, for homogeneous deformations $\mathbf{T}^*$ is independent of the coordinates θ^i so that (5.11.7), (5.11.10), (5.11.11), (5.11.14) and (5.11.15) can be used to deduce that

$$\mathbf{t}^i = \mathbf{T}^*\int_{\mathcal{A}} g^{1/2}\left[\mathbf{d}^i - \theta^\alpha\,\mathbf{g}^3\,(\mathbf{d}_{\alpha,3}\bullet \mathbf{d}^i)\right] d\theta^1 d\theta^2 = \mathbf{T}^*\int_{\mathcal{A}} g^{1/2}\,\mathbf{g}^i\,d\theta^1 d\theta^2 \ ,$$

$$\mathbf{m}^\alpha = \mathbf{T}^*\int_{\mathcal{A}} \theta^\alpha\,g^{1/2}\,\mathbf{g}^3\,d\theta^1 d\theta^2 \ , \tag{5.11.16}$$

where use has been made of (5.1.10) and (5.2.16) to derive the expressions

$$\mathbf{d}_{\alpha,3}\bullet \mathbf{d}^i = (\mathbf{F}\,\mathbf{D}_{\alpha,3})\bullet(\mathbf{F}^{-T}\,\mathbf{D}^i) = \mathbf{D}_{\alpha,3}\bullet \mathbf{D}^i \ ,$$

$$\mathbf{d}^i = \mathbf{F}^{-T}\,\mathbf{D}^i = (\mathbf{g}^j\otimes\mathbf{G}_j)\,\mathbf{D}^i = \mathbf{g}^i + \theta^\alpha\,\mathbf{g}^3\,(\mathbf{D}_{\alpha,3}\bullet \mathbf{D}^i) \ , \tag{5.11.17}$$

for homogeneous deformations. Obviously, the results (5.11.16) are compatible with the expressions (5.3.19), $(5.3.30)_1$ and (5.3.33). To complete the proof, it is first recalled from the definition (2.3.16) of the divergence operator that for an arbitrary function

$$\phi = \phi(\theta^\beta) \ , \tag{5.11.18}$$

it can be shown that

$$\phi\,\mathrm{div}^*\,\mathbf{T}^* = \mathrm{div}^*(\phi\,\mathbf{T}^*) - \mathbf{T}^*\bullet\frac{d\phi}{d\theta^\beta}\,\mathbf{g}^\beta \ . \tag{5.11.19}$$

Thus, integrating (5.11.19) over the region P^* bounding an arbitrary portion of the rod characterized by $\xi_1\le\theta^3$ with θ^3 being variable, it follows that

$$\int_{P^*}\phi\,\mathrm{div}^*\,\mathbf{T}^*\,dv^* = -\int_{P^*}\mathbf{T}^*\bullet\frac{d\phi}{d\theta^\beta}\,\mathbf{g}^\beta\,dv^*$$

$$+\int_{\partial P_L^*}\phi\,\mathbf{t}^*\,da^* + \int_{\partial P_2^*}\phi\,\mathbf{t}^*\,da^* + \int_{\partial P_1^*}\phi\,\mathbf{t}^*\,da^* \ , \tag{5.11.20}$$

where the divergence theorem has been used to convert the volume integral of $\mathrm{div}(\phi\,\mathbf{T}^*)$ to a integral over the boundary of the region. Next, using the expressions $dv^*=g^{1/2}d\theta^1 d\theta^2 d\theta^3$, $\mathbf{t}^* = \mathbf{T}^*\mathbf{n}^*$, and the results (5.3.14), (5.3.17) and (5.3.18), the equation (5.11.20) can be rewritten in the alternative form

$$\int_{\xi_1}^{\theta^3}\int_{\mathcal{A}(\theta^3)}\phi\,\mathrm{div}^*\,\mathbf{T}^*\,g^{1/2}d\theta^1 d\theta^2 d\theta^3 = -\int_{\xi_1}^{\theta^3}\int_{\mathcal{A}(\theta^3)}\mathbf{T}^*\bullet\frac{d\phi}{d\theta^\beta}\,\mathbf{g}^\beta\,g^{1/2}d\theta^1 d\theta^2 d\theta^3$$

$$+\int_{\xi_1}^{\theta^3}\int_{\partial\mathcal{A}}\phi\,g^{1/2}\,\alpha(\zeta,\theta^3)\,\mathbf{t}^*\,d\zeta + \int_{\mathcal{A}(\theta^3)}\phi\,\mathbf{T}^*\,g^{1/2}\,\mathbf{g}^3\,d\theta^1 d\theta^2$$

$$- \int_{\mathcal{A}(\xi_1)} \phi \, \mathbf{T}^* \, g^{1/2} \, \mathbf{g}^3 \, d\theta^1 d\theta^2 \ . \tag{5.11.21}$$

Then, differentiation of (5.11.21) with respect to θ^3 yields

$$\int_{\mathcal{A}(\theta^3)} \phi \, \mathrm{div}^* \, \mathbf{T}^* \, g^{1/2} d\theta^1 d\theta^2 = - \int_{\mathcal{A}(\theta^3)} \mathbf{T}^* \bullet \frac{d\phi}{d\theta^\beta} \mathbf{g}^\beta \, g^{1/2} d\theta^1 d\theta^2$$
$$+ \int_{\partial\mathcal{A}} \phi \, g^{1/2} \, \alpha(\zeta,\theta^3) \, \mathbf{t}^* \, d\zeta + \frac{\partial}{\partial\theta^3} \int_{\mathcal{A}(\theta^3)} \phi \, \mathbf{T}^* \, g^{1/2} \, \mathbf{g}^3 \, d\theta^1 d\theta^2 \ . \tag{5.11.22}$$

However, for homogeneous deformations $\mathbf{T}^*$ is constant so the divergence $\mathrm{div}^*\mathbf{T}^*$ vanishes. This means that with the help of (5.11.16), equation (5.11.22) with ϕ=1 reduces to

$$\int_{\partial\mathcal{A}} \phi \, g^{1/2} \, \alpha(\zeta,\theta^3) \, \mathbf{t}^* \, d\zeta + \mathbf{t}^3{}_{,3} = 0 \ , \tag{5.11.23}$$

and equation (5.11.22) with $\phi=\theta^\alpha$ reduces to

$$\int_{\partial\mathcal{A}} \hat{\theta}^{\,\alpha}(\zeta,\theta^3) \, g^{1/2} \, \alpha(\zeta,\theta^3) \, \mathbf{t}^* \, d\zeta - \mathbf{t}^\alpha + \mathbf{m}^\alpha{}_{,3} = 0 \ . \tag{5.11.24}$$

Thus, with the help of the definitions (5.3.16) for $m\mathbf{b}_c$ and (5.3.32) for $m\mathbf{b}_c^\alpha$, it can be seen that (5.11.23) and (5.11.24) prove the validity of (5.11.12) for homogeneous deformations.

The expressions (5.11.9) place necessary restrictions on the constitutive equations for rods which ensure consistency with exact solutions for all homogeneous deformations. It was shown in (Rubin, 1996) that these restrictions can easily be satisfied by defining an alternative deformation measure $\bar{\mathbf{C}}$ by the expression

$$\bar{\mathbf{C}} = \bar{\mathbf{C}}(\mathbf{C},\boldsymbol{\beta}_\alpha,A^\alpha) = [\mathbf{I} + A^\alpha(\boldsymbol{\beta}_\alpha \otimes \mathbf{D}^3)]^T \, \mathbf{C} \, [\mathbf{I} + A^\beta(\boldsymbol{\beta}_\beta \otimes \mathbf{D}^3)] = \bar{\mathbf{C}}^T, \tag{5.11.25}$$

which has the property that for homogeneous deformations

$$\bar{\mathbf{C}}(\mathbf{C},0,A^\alpha) = \mathbf{C} \ . \tag{5.11.26}$$

Now, in general the strain energy Σ can be specified in terms of the three-dimensional strain energy function Σ^* by the form

$$\Sigma = \Sigma^*(\bar{\mathbf{C}}) + \Psi(\mathbf{C},\boldsymbol{\beta}_\alpha,\mathcal{V}) \ , \quad \mathcal{V} = \{\mathbf{D}_i \, , \, A \, , \, A^\beta \, , \, \theta^3\} \ , \tag{5.11.27}$$

where Ψ is an additive part of the strain energy function due to inhomogeneous deformations. Also, for generality in modeling the response of an inhomogeneous material and general geometry in the reference configuration, Ψ is allowed to depend on the reference geometry as well as on the coordinate θ^3 through the parameters $\mathcal{V}$.

Next, using the expression (5.11.25) it can be shown that

$$\frac{\partial\Sigma}{\partial\mathbf{C}} = [\mathbf{I} + A^\alpha(\boldsymbol{\beta}_\alpha \otimes \mathbf{D}^3)] \frac{\partial\Sigma^*(\bar{\mathbf{C}})}{\partial\bar{\mathbf{C}}} [\mathbf{I} + A^\beta(\boldsymbol{\beta}_\beta \otimes \mathbf{D}^3)]^T + \frac{\partial\Psi}{\partial\mathbf{C}} \ ,$$

$$\frac{\partial\Sigma}{\partial\boldsymbol{\beta}_\alpha} = 2 \, A^\alpha \, \mathbf{C} \, [\mathbf{I} + A^\beta(\boldsymbol{\beta}_\beta \otimes \mathbf{D}^3)] \frac{\partial\Sigma^*(\bar{\mathbf{C}})}{\partial\bar{\mathbf{C}}} \, \mathbf{D}^3 + \frac{\partial\Psi}{\partial\boldsymbol{\beta}_\alpha} \ . \tag{5.11.28}$$

It then follows that the functional form (5.11.27) will satisfy the restrictions (5.11.9) provided that Ψ satisfies the restrictions

$$\left.\frac{\partial \Psi}{\partial \mathbf{C}}\right|_{\boldsymbol{\beta}_\beta=0} = 0 \ , \quad \left.\frac{\partial \Psi}{\partial \boldsymbol{\beta}_\alpha}\right|_{\boldsymbol{\beta}_\beta=0} = 0 \ . \tag{5.11.29}$$

At present, it is not known how to determine an expression for Ψ which will depend explicitly on the form of the three dimensional strain energy Σ^*. The restrictions (5.11.29) indicate that Ψ cannot be a linear function of $\boldsymbol{\beta}_\alpha$. Consequently, for simplicity it is assumed that Ψ is a quadratic function of $\boldsymbol{\beta}_\alpha$ which is independent of $\mathbf{C}$ and takes the form

$$\rho_0 \Psi = \frac{1}{2}\, D_{33}^{-1}\, A\, \mathbf{K} \bullet \left[(\boldsymbol{\beta}_\alpha \otimes \mathbf{D}^\alpha) \otimes (\boldsymbol{\beta}_\beta \otimes \mathbf{D}^\beta)\right] = \frac{1}{2}\, D_{33}^{-1}\, A\, \mathbf{K}^{\alpha\beta} \bullet (\boldsymbol{\beta}_\alpha \otimes \boldsymbol{\beta}_\beta) \ , \tag{5.11.30}$$

where the fourth order tensor $\mathbf{K}$ and the second order tensors $\mathbf{K}^{\alpha\beta}$ are related to each other and have the properties that

$$\mathbf{K}^{T(2)} = \mathbf{K} \ , \quad \mathbf{D}^3\, {}^{LT}\mathbf{K} = 0 \ , \quad \mathbf{K}\mathbf{D}^3 = 0 \ , \quad \mathbf{K} = {}^{LT}\!\left[\mathbf{D}_\alpha \otimes \mathbf{K}^{\alpha\beta} \otimes \mathbf{D}_\beta\right] \ ,$$
$$ {}^{LT}\mathbf{K}^{\beta\alpha} = \mathbf{K}^{\alpha\beta} \ , \quad \mathbf{K}^{\alpha\beta} = \mathbf{D}^\alpha\, {}^{LT}\mathbf{K}\mathbf{D}^\beta \ . \tag{5.11.31}$$

In view of these symmetries, it can be shown that the tensors $\mathbf{K}$ and $\mathbf{K}^{\alpha\beta}$ each have twenty-one independent components. These tensors are independent of time but they can depend on the reference geometry through the parameters $\mathcal{V}$. Also, it is noted that the term D_{33} and the second order tensor $(\boldsymbol{\beta}_\alpha \otimes \mathbf{D}^\alpha)$ are used in the definition (5.11.31) to remove dependence of $\mathbf{K}$ on the explicit choice of the units of the coordinates defining $\mathbf{D}_i$ and $\boldsymbol{\beta}_\alpha$. However, as can be seen from $(5.11.31)_6$, the tensors $\mathbf{K}^{\alpha\beta}$ depend explicitly on this choice of coordinates. Next, differentiation of (5.11.30) yields the results

$$\rho_0 \frac{\partial \Psi}{\partial \mathbf{C}} = 0 \ , \quad \rho_0 \frac{\partial \Psi}{\partial \boldsymbol{\beta}_\alpha} = D_{33}^{-1}\, A\, \mathbf{K}^{\alpha\beta}\, \boldsymbol{\beta}_\beta \ , \tag{5.11.32}$$

which satisfy the restrictions (5.11.29).

In summary, it is assumed that for a general rod the strain energy function is specified so that

$$\bar{\mathbf{C}} = [\mathbf{I} + A^\alpha(\boldsymbol{\beta}_\alpha \otimes \mathbf{D}^3)]^T\, \mathbf{C}\, [\mathbf{I} + A^\beta(\boldsymbol{\beta}_\beta \otimes \mathbf{D}^3)] \ ,$$
$$m\,\Sigma = m\,\Sigma^*(\bar{\mathbf{C}}) + \frac{1}{2}\, D_{33}^{-1/2}\, A\, \mathbf{K}^{\alpha\beta} \bullet (\boldsymbol{\beta}_\alpha \otimes \boldsymbol{\beta}_\beta) \ ,$$
$$m \frac{\partial \Sigma}{\partial \mathbf{C}} = m\, [\mathbf{I} + A^\alpha(\boldsymbol{\beta}_\alpha \otimes \mathbf{D}^3)]\, \frac{\partial \Sigma^*(\bar{\mathbf{C}})}{\partial \bar{\mathbf{C}}}\, [\mathbf{I} + A^\beta(\boldsymbol{\beta}_\beta \otimes \mathbf{D}^3)]^T \ ,$$
$$m \frac{\partial \Sigma}{\partial \boldsymbol{\beta}_\alpha} = 2\, m\, \mathbf{C}\, [\mathbf{I} + A^\beta(\boldsymbol{\beta}_\beta \otimes \mathbf{D}^3)]\, \frac{\partial \Sigma^*(\bar{\mathbf{C}})}{\partial \bar{\mathbf{C}}}\, \mathbf{D}^3 + D_{33}^{-1/2}\, A\, \mathbf{K}^{\alpha\beta}\, \boldsymbol{\beta}_\beta \ , \tag{5.11.33}$$

where the conservation of mass $(5.4.25)_1$ has been used. Moreover, the constitutive equations (5.8.12) and (5.8.14) for $\mathbf{T}$, $\mathbf{m}^\alpha$ and $\mathbf{t}^i$ associated with (5.11.33) can be represented in the forms

$$d_{33}^{1/2}\,\mathbf{T} = 2\,m\,\mathbf{F}\,[\mathbf{I} + A^{\alpha}(\boldsymbol{\beta}_{\alpha}\otimes\mathbf{D}^3)]\,\frac{\partial\Sigma^*(\bar{\mathbf{C}})}{\partial\bar{\mathbf{C}}}\,[\mathbf{I} + A^{\beta}(\boldsymbol{\beta}_{\beta}\otimes\mathbf{D}^3)]^T\,\mathbf{F}^T \ ,$$

$$\mathbf{m}^{\alpha} = 2\,m\,\mathbf{F}\,[\mathbf{I} + A^{\beta}(\boldsymbol{\beta}_{\beta}\otimes\mathbf{D}^3)]\,\frac{\partial\Sigma^*(\bar{\mathbf{C}})}{\partial\bar{\mathbf{C}}}\,\mathbf{D}^3 + D_{33}^{-1/2}\,A\,\mathbf{F}^{-T}\,\mathbf{K}^{\alpha\beta}\,\boldsymbol{\beta}_{\beta} \ ,$$

$$\mathbf{t}^{i} = 2\,m\,\mathbf{F}\,[\mathbf{I} + A^{\alpha}(\boldsymbol{\beta}_{\alpha}\otimes\mathbf{D}^3)]\,\frac{\partial\Sigma^*(\bar{\mathbf{C}})}{\partial\bar{\mathbf{C}}}\,[\mathbf{I} + A^{\beta}(\boldsymbol{\beta}_{\beta}\otimes\mathbf{D}^3)]^T\,\mathbf{D}^i$$
$$- \mathbf{m}^{\alpha}\,(\mathbf{d}_{\alpha,3}\bullet\mathbf{d}^i) \ . \tag{5.11.34}$$

The functional forms (5.11.33) and (5.11.34) of the constitutive equations depend explicitly on the three-dimensional strain energy function and they ensure consistency with exact solutions for all nonlinear homogeneous deformations. However, it is still necessary to determine functional forms for $\mathbf{K}^{\alpha\beta}$ which depend on the geometrical and material properties of the rod structure. Usually, this is accomplished by first determining values for $\mathbf{K}$ by matching solutions of the linearized theory of beams for pure bending and torsion. Then, the values of $\mathbf{K}^{\alpha\beta}$ are determined by equations (5.11.31). Specific forms for $\mathbf{K}$ and $\mathbf{K}^{\alpha\beta}$ will be developed later for orthotropic rods using this procedure.

Examination of (5.11.10) indicates that if the reference curve $\theta^{\alpha}=0$ is taken to be the centroid of the cross section determined with respect to the coordinates θ^{α}, then

$$A^{\alpha} = 0 \ , \tag{5.11.35}$$

and the constitutive equations (5.11.33) and (5.11.34) simplify considerably and reduce to

$$\bar{\mathbf{C}} = \mathbf{C} \ , \quad m\,\Sigma = m\,\Sigma^*(\mathbf{C}) + \frac{1}{2}\,D_{33}^{-1/2}\,A\,\mathbf{K}^{\alpha\beta}\bullet(\boldsymbol{\beta}_{\alpha}\otimes\boldsymbol{\beta}_{\beta}) \ ,$$

$$d_{33}^{1/2}\,\mathbf{T} = 2\,m\,\mathbf{F}\,\frac{\partial\Sigma^*(\mathbf{C})}{\partial\mathbf{C}}\,\mathbf{F}^T \ , \quad \mathbf{m}^{\alpha} = D_{33}^{-1/2}\,A\,\mathbf{F}^{-T}\,\mathbf{K}^{\alpha\beta}\,\boldsymbol{\beta}_{\beta} \ ,$$

$$\mathbf{t}^{i} = 2\,m\,\mathbf{F}\,\frac{\partial\Sigma^*(\mathbf{C})}{\partial\mathbf{C}}\,\mathbf{D}^i - \mathbf{m}^{\alpha}\,(\mathbf{d}_{\alpha,3}\bullet\mathbf{d}^i) \ . \tag{5.11.36}$$

In view of these simplified forms the specification (5.11.35) will be used throughout the remainder of the text. However, it should be mentioned that in general, the specification that $A^{\alpha}=0$ is not always compatible with the specification that the director inertia coefficients y^{α} in (5.3.10) vanish.

5.12 A small strain theory

In order to better understand the nature of the constitutive assumption (5.11.36) it is of interest to consider the simpler case of a small strain theory where the three-dimensional strain energy function is given by the form (3.12.2) which is a quadratic function of the Lagrangian strain. Moreover, with the help of the conservation of mass $(5.4.25)_1$ and the definition (5.11.7), it can be shown that when the three-dimensional mass density is independent of the coordinates θ^{α} in the cross-section of the rod, that

$$\rho_0 = \rho_0^* A \ . \qquad (5.12.1)$$

Then, in view of the assumption (5.11.36), the strain energy function is expressed in the form

$$\rho_0 \Sigma = \frac{1}{2} A \mathbf{K}^* \bullet (\mathbf{E} \otimes \mathbf{E}) + \frac{1}{2} D_{33}^{-1} A \mathbf{K}^{\alpha\beta} \bullet (\boldsymbol{\beta}_\alpha \otimes \boldsymbol{\beta}_\beta) \ , \qquad (5.12.2)$$

where it is convenient to define the strain $\mathbf{E}$ by

$$\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{I}) \ . \qquad (5.12.3)$$

Also, $\mathbf{K}^*$ is the value associated with the three-dimensional material which is evaluated on the reference curve (θ^α=0).

Next, using the definitions (5.11.36) and (5.12.3) it can be shown that

$$m \Sigma = \frac{1}{2} D_{33}^{1/2} A \mathbf{K}^* \bullet (\mathbf{E} \otimes \mathbf{E}) + \frac{1}{2} D_{33}^{-1/2} A \mathbf{K}^{\alpha\beta} \bullet (\boldsymbol{\beta}_\alpha \otimes \boldsymbol{\beta}_\beta) \ ,$$

$$2m \frac{\partial \Sigma}{\partial \mathbf{C}} = D_{33}^{1/2} A \mathbf{K}^* \bullet \mathbf{E} \ , \quad m \frac{\partial \Sigma}{\partial \boldsymbol{\beta}_\alpha} = D_{33}^{-1/2} A \mathbf{K}^{\alpha\beta} \bullet \boldsymbol{\beta}_\beta \ ,$$

$$d_{33}^{1/2} \mathbf{T} = D_{33}^{1/2} A \mathbf{F} \left[\mathbf{K}^* \bullet \mathbf{E}\right] \mathbf{F}^T \ , \quad \mathbf{m}^\alpha = D_{33}^{-1/2} A \mathbf{F}^{-T} \mathbf{K}^{\alpha\beta} \boldsymbol{\beta}_\beta \ ,$$

$$\mathbf{t}^i = D_{33}^{1/2} A \mathbf{F} \left[\mathbf{K}^* \bullet \mathbf{E}\right] \mathbf{D}^i - \mathbf{m}^\alpha (\mathbf{d}_{\alpha,3} \bullet \mathbf{d}^i) \ , \qquad (5.12.4)$$

where use has been made of the conservation of mass $(5.4.25)_1$. Since these equations satisfy the restrictions (5.11.9) (with A^α=0), they ensure consistency with exact solutions for all homogeneous deformations when the strain energy function of the three-dimensional material is given by (3.12.2). Moreover, the constitutive equations (5.12.4) remain properly invariant under superposed rigid body motions. This means that the strain energy function (5.12.2) can be viewed as a special simple constitutive assumption that is valid for large deformations and large rotations of the rod, but small strains.

In general, the vectors $\mathbf{D}_i$ are not orthonormal. Consequently, it is convenient to introduce a right-handed orthonormal set of vectors $\mathbf{M}_i$ defined by

$$\mathbf{M}_1 = \mathbf{M}_2 \times \mathbf{M}_3 \ , \quad \mathbf{M}_2 = \frac{\mathbf{M}_3 \times \mathbf{D}_1}{|\mathbf{M}_3 \times \mathbf{D}_1|} \ , \quad \mathbf{M}_3 = \frac{\mathbf{D}_3}{|\mathbf{D}_3|} \ . \qquad (5.12.5)$$

This specification causes $\mathbf{M}_1$ and $\mathbf{M}_2$ to lie in the normal cross-sectional plane of the beam. Then, using these definitions the components of $\mathbf{E}$, $\boldsymbol{\beta}_\alpha$, $\mathbf{K}^*$, $\mathbf{K}$ and $\mathbf{K}^{\alpha\beta}$, relative to $\mathbf{M}_i$, are defined such that

$$\mathbf{E} = E_{ij} (\mathbf{M}_i \otimes \mathbf{M}_j) \ , \quad E_{ij} = \mathbf{E} \bullet (\mathbf{M}_i \otimes \mathbf{M}_j) \ , \quad \boldsymbol{\beta}_\alpha = \beta_{i\alpha} \mathbf{M}_i \ , \quad \beta_{i\alpha} = \boldsymbol{\beta}_\alpha \bullet \mathbf{M}_i \ ,$$

$$\mathbf{K}^* = K_{ijkl}^* (\mathbf{M}_i \otimes \mathbf{M}_j \otimes \mathbf{M}_k \otimes \mathbf{M}_l) \ , \quad K_{ijkl}^* = \mathbf{K}^* \bullet (\mathbf{M}_i \otimes \mathbf{M}_j \otimes \mathbf{M}_k \otimes \mathbf{M}_l) \ ,$$

$$\mathbf{K} = K_{i\alpha j\beta} (\mathbf{M}_i \otimes \mathbf{M}_\alpha \otimes \mathbf{M}_j \otimes \mathbf{M}_\beta) \ , \quad K_{i\alpha j\beta} = \mathbf{K} \bullet (\mathbf{M}_i \otimes \mathbf{M}_\alpha \otimes \mathbf{M}_j \otimes \mathbf{M}_\beta) \ ,$$

$$\mathbf{K}^{\alpha\beta} = K_{ij}^{\alpha\beta} (\mathbf{M}_i \otimes \mathbf{M}_j) \ , \quad K_{ij}^{\alpha\beta} = \mathbf{K}^{\alpha\beta} \bullet (\mathbf{M}_i \otimes \mathbf{M}_j) \ ,$$

$$K_{j\delta i\gamma} = K_{i\gamma j\delta} \ , \quad K_{ij}^{\beta\alpha} = K_{ji}^{\alpha\beta} \ , \qquad (5.12.6)$$

where by definition $\mathbf{K}$ automatically satisfies the restrictions $(5.11.31)_{2,3}$. Moreover, using $(5.11.31)_6$ it can be shown that

$$K_{ij}^{\alpha\beta} = (\mathbf{D}^\alpha \bullet \mathbf{M}_\gamma)(\mathbf{D}^\beta \bullet \mathbf{M}_\delta) K_{i\gamma j\delta} \ , \qquad (5.12.7)$$

where the values of $K_{i\gamma j\delta}$ are independent of time, but can be functions of the reference geometry.

For the case of an orthotropic material relative to the basis $\mathbf{M}_i$ the nine nontrivial components of K^*_{ijkl} are given by (3.12.12), but $K_{i\gamma j\delta}$ remains general because the cross-section of the rod may not be symmetrical. If however, the cross-section is characterized by the plane containing $\mathbf{M}_1$ and $\mathbf{M}_2$ and its shape is symmetric about the planes which intersect the centroid and are normal to $\mathbf{M}_1$ and $\mathbf{M}_2$, then it is reasonable to assume that the eight nontrivial components of $K_{i\gamma j\delta}$ are

$$\begin{pmatrix} K_{1111} & K_{1122} & K_{1212} & K_{1221} \\ K_{2121} & K_{2222} & K_{3131} & K_{3232} \end{pmatrix} . \tag{5.12.8}$$

For the case of an isotropic material the values of K^*_{ijkl} are given by (3.12.13), but again $K_{i\gamma j\delta}$ remains general because the cross-section of the rod may not be symmetrical. If however, the cross-section is characterized by the plane containing $\mathbf{M}_1$ and $\mathbf{M}_2$ and its shape is symmetric about the planes which intersect the centroid and are normal to $\mathbf{M}_1$ and $\mathbf{M}_2$, then it is reasonable to assume that the eight nontrivial components of $K_{i\gamma j\delta}$ are again given by (5.12.8). For example, if the cross-section of the rod is rectangular, then the response to bending will in general be orthotropic even though the material is isotropic.

Moreover, for the simple case when the directors $\mathbf{D}_i$ are orthonormal vectors that are equal to $\mathbf{M}_i$

$$\mathbf{D}_i = \mathbf{M}_i \ , \tag{5.12.9}$$

then (5.12.7) reduces to

$$K^{\alpha\beta}_{ij} = K_{i\alpha j\beta} \ , \tag{5.12.10}$$

and the indicial form of the strain energy (5.12.2) for an isotropic material can be written as

$$2\rho_0\Sigma = A\left[(K^* - \frac{2}{3}\mu^*)\, E_{ii}\, E_{jj} + 2\mu^*\, E_{ij}\, E_{ij}\right] + A\, K_{i\alpha j\beta}\, \beta_{i\alpha}\, \beta_{j\beta} \ . \tag{5.12.11}$$

Furthermore, when the cross-section has the symmetry required for $K_{i\alpha j\beta}$ to take the orthotropic form (5.12.8), then (5.12.11) can be expanded to give

$$\begin{aligned} 2\rho_0\Sigma = A\Big[&(K^* + \frac{4}{3}\mu^*)(E_{11}^2 + E_{22}^2 + E_{33}^2) \\ &+ 2(K^* - \frac{2}{3}\mu^*)(E_{11}E_{22} + E_{11}E_{33} + E_{22}E_{33}) \\ &+ \mu^*\left\{(E_{12}+E_{21})^2 + (E_{13}+E_{31})^2 + (E_{23}+E_{32})^2\right\}\Big] \\ &+ A\left[K_{1111}\, \beta_{11}^2 + K_{2222}\, \beta_{22}^2 + 2\, K_{1122}\, \beta_{11}\, \beta_{22}\right. \\ &+ K_{1212}\, \beta_{12}^2 + K_{2121}\, \beta_{21}^2 + 2\, K_{1221}\, \beta_{12}\, \beta_{21} \\ &\left.+ K_{3131}\, \beta_{31}^2 + K_{3232}\, \beta_{32}^2\right] \ . \end{aligned} \tag{5.12.12}$$

In order to compare this expression with the one given in (Green et. al., 1974b, pp.503-505), it is noted that the quantities $\{\lambda\ ,\ \psi_{F1}+\psi_{F1}+\psi_E+\psi_T\ ,\ \gamma_{ij}\ ,\ \kappa_{\alpha i}\}$ there should be replaced by the quantities $\{\rho_0\ ,\ \Sigma\ ,\ 2E_{ij}\ ,\ \beta_{i\alpha}\}$, respectively. Then, (5.12.12) gives expressions for the coefficients k_i in (Green et. al., 1974b) which are

$$k_1 = k_2 = k_3 = \frac{1}{4}(K^* + \frac{4}{3}\mu^*)A = \frac{1}{2}\left[\frac{1-\nu^*}{1-2\nu^*}\right]\mu^*A \ , \ k_4 = k_5 = k_6 = \mu^*A \ ,$$

$$k_7 = k_8 = k_9 = \frac{1}{2}(K^* - \frac{2}{3}\mu^*)A = \left[\frac{\nu^*}{1-2\nu^*}\right]\mu^*A \ ,$$

$$k_{10} = K_{1111}\,A \ , \ k_{11} = K_{2222}\,A \ , \ k_{12} = K_{2121}\,A \ ,$$

$$k_{13} = K_{1212}\,A \ , \ k_{14} = 2K_{1221}\,A \ , \ k_{15} = K_{3232}\,A \ ,$$

$$k_{16} = K_{3131}\,A \ , \ k_{17} = 2K_{1122}\,A \ , \tag{5.12.13}$$

where the Table 3.12.1 has been used to express the results in terms of μ^* and ν^*. The values of $\{k_1-k_9\}$ in (5.12.13) have been obtained here by using the restrictions of the previous section which ensure consistency with exact solutions for all nonlinear homogeneous deformations. Therefore, it is not surprising that these values (apart from k_4-k_6) are the same as those obtained by Green et. al. (1974b) who considered specific small deformation homogeneous solutions of the linearized theory of straight uniform beams. The relationship of the values of the shear coefficients (k_4-k_6) in (5.12.13) to the more common expressions for transverse shear coefficients has been discussed by Rubin (1996).

Values of the constitutive coefficients in (5.12.8) and ($k_{10}-k_{17}$) in (5.12.13) associated with inhomogeneous deformations will be developed in later sections by comparing solutions of bending and torsion of an orthotropic beam with rectangular cross-section, with exact solutions of the linearized three-dimensional theory.

5.13 Small deformations superimposed on a large deformation

In general, the equations of motion of an elastic rod are nonlinear partial differential equations for which only a few exact analytical solutions are known. The notion of small deformations superimposed on a large deformation is used to develop approximate equations that are linear functions of the superimposed small deformations and therefore are simpler to solve (e.g. Green et al, 1968). Such equations can be used to analyze vibrations of pre-stressed or rotating structures such as space satellites, and buckling of structures. Moreover, if the large deformation represents an actual solution of the equations of motion, then the small deformation equations can be used to analyze linear stability of the large deformation solution.

To develop these small deformation equations, the position vector $\mathbf{x}(\theta^3,t)$ is represented as an additive function of the large deformation $\hat{\mathbf{x}}(\theta^3,t)$ and the small displacement vector $\mathbf{u}(\theta^3,t)$. Also, the directors $\mathbf{d}_i(\theta^3,t)$ are represented as additive functions of the large deformations $\hat{\mathbf{d}}_i(\theta^3,t)$ and the small director displacements $\boldsymbol{\delta}_i(\theta^3,t)$, such that

$$\mathbf{x}(\theta^3,t) = \hat{\mathbf{x}}(\theta^3,t) + \mathbf{u}(\theta^3,t) \ , \ \mathbf{d}_i(\theta^3,t) = \hat{\mathbf{d}}_i(\theta^3,t) + \boldsymbol{\delta}_i(\theta^3,t) \ , \ \boldsymbol{\delta}_3 = \mathbf{u}_{,3} \ . \tag{5.13.1}$$

The displacement vector $\mathbf{u}$ and the director displacements $\boldsymbol{\delta}_i$ are considered to be small, in the sense that their magnitudes and the magnitudes of their space and time derivatives are small enough that quadratic and higher order terms in these quantities can be neglected. Thus, for example

$$|\mathbf{u}|^2 << |\mathbf{u}| \ , \quad |\boldsymbol{\delta}_i|^2 << |\boldsymbol{\delta}_i| \ . \tag{5.13.2}$$

Of course, the values of $\mathbf{u}$, $\boldsymbol{\delta}_i$, and their space and time derivatives must be appropriately normalized in order to express these inequalities in unitless forms.

Quantities other than the displacement vector $\mathbf{u}$ and the director displacements $\boldsymbol{\delta}_i$ are separated additively into a part associated with the large deformation which is denoted by placing a hat ($\hat{}$) over the symbol, and a part associated with the small deformation which is denoted by placing a tilde ($\tilde{}$) over the same symbol. For example, in general the external force associated with the large deformation to be analyzed can be nonzero. Thus, the external force $\mathbf{b}$ is represented in the form

$$\mathbf{b} = \hat{\mathbf{b}} + \tilde{\mathbf{b}} \ . \tag{5.13.3}$$

In order to develop the equations of motion of the small deformation associated with (5.4.25), it is necessary to substitute (5.13.1) into the constitutive equations (5.8.14) and to expand the resulting quantities in a Taylor series to develop expressions for the vectors $\mathbf{t}^i$ and the couples $\mathbf{m}^\alpha$

$$\mathbf{t}^i = \hat{\mathbf{t}}^i + \tilde{\mathbf{t}}^i \ , \quad \mathbf{m}^\alpha = \hat{\mathbf{m}}^\alpha + \tilde{\mathbf{m}}^\alpha \ . \tag{5.13.4}$$

To this end, it is noted that

$$\begin{gathered}
\mathbf{F} = \hat{\mathbf{F}} + \tilde{\mathbf{F}} \ , \quad \hat{\mathbf{F}} = \hat{\mathbf{d}}_i \otimes \mathbf{D}^i \ , \quad \tilde{\mathbf{F}} = \boldsymbol{\delta}_i \otimes \mathbf{D}^i \ , \\
\mathbf{F}^{-1} = \hat{\mathbf{F}}^{-1} + \tilde{\mathbf{F}}^{-1} \ , \quad \hat{\mathbf{F}}^{-1} = \mathbf{D}_i \otimes \hat{\mathbf{d}}^i \ , \quad \tilde{\mathbf{F}}^{-1} = -\hat{\mathbf{F}}^{-1} \boldsymbol{\delta}_i \otimes \hat{\mathbf{d}}^i \ , \\
\tilde{\mathbf{F}} \hat{\mathbf{F}}^{-1} = \boldsymbol{\delta}_i \otimes \hat{\mathbf{d}}^i \ , \quad \tilde{\mathbf{F}}^{-T} \hat{\mathbf{F}}^{T} = -\hat{\mathbf{d}}^i \otimes \boldsymbol{\delta}_i \ , \\
\mathbf{C} = \hat{\mathbf{C}} + \tilde{\mathbf{C}} \ , \quad \hat{\mathbf{C}} = \hat{\mathbf{F}}^T \hat{\mathbf{F}} \ , \quad \tilde{\mathbf{C}} = \hat{\mathbf{F}}^T (\boldsymbol{\delta}_i \otimes \mathbf{D}^i) + (\mathbf{D}^i \otimes \boldsymbol{\delta}_i) \hat{\mathbf{F}} \ , \\
\mathbf{E} = \hat{\mathbf{E}} + \tilde{\mathbf{E}} \ , \quad \hat{\mathbf{E}} = \frac{1}{2}(\hat{\mathbf{F}}^T \hat{\mathbf{F}} - \mathbf{I}) \ , \quad \tilde{\mathbf{E}} = \frac{1}{2} \tilde{\mathbf{C}} \ , \\
\boldsymbol{\beta}_\alpha = \hat{\boldsymbol{\beta}}_\alpha + \tilde{\boldsymbol{\beta}}_\alpha \ , \quad \hat{\boldsymbol{\beta}}_\alpha = \hat{\mathbf{F}}^{-1} \hat{\mathbf{d}}_{3,\alpha} - \mathbf{D}_{3,\alpha} \ , \\
\tilde{\boldsymbol{\beta}}_\alpha = \hat{\mathbf{F}}^{-1} \left[\boldsymbol{\delta}_{\alpha,3} - (\hat{\mathbf{d}}^i \bullet \hat{\mathbf{d}}_{\alpha,3}) \boldsymbol{\delta}_i \right] \ , \quad \hat{d}^{1/2} = \hat{\mathbf{d}}_1 \times \hat{\mathbf{d}}_2 \bullet \hat{\mathbf{d}}_3 \ , \\
\hat{\mathbf{d}}^1 = \hat{d}^{-1/2} \left[\hat{\mathbf{d}}_2 \times \hat{\mathbf{d}}_3 \right] \ , \quad \hat{\mathbf{d}}^2 = \hat{d}^{-1/2} \left[\hat{\mathbf{d}}_3 \times \hat{\mathbf{d}}_1 \right] \ , \quad \hat{\mathbf{d}}^3 = \hat{d}^{-1/2} \hat{\mathbf{d}}_1 \times \hat{\mathbf{d}}_2 \ , \\
\mathbf{d}^i = \hat{\mathbf{d}}^i + \tilde{\mathbf{d}}^i \ , \quad \tilde{\mathbf{d}}^i = -\left[\hat{\mathbf{d}}^i \bullet \boldsymbol{\delta}_j \right] \hat{\mathbf{d}}^j \ , \quad \hat{d}_{33}^{1/2} = (\hat{\mathbf{d}}_3 \bullet \hat{\mathbf{d}}_3)^{1/2} \ , \\
d_{33}^{1/2} = \hat{d}_{33}^{1/2} + \tilde{d}_{33}^{1/2} \ , \quad \tilde{d}_{33}^{1/2} = \hat{d}_{33}^{-1/2} \hat{\mathbf{d}}_3 \bullet \boldsymbol{\delta}_3 \ ,
\end{gathered} \tag{5.13.5}$$

where the symbol $\tilde{\mathbf{F}}^{-1}$ does not denote the inverse of $\tilde{\mathbf{F}}$. Next, the conservation of mass $(5.4.25)_1$ can be written in the form

$$m = \rho_0 D_{33}^{1/2} = \rho \, d_{33}^{1/2} = \hat{\rho} \, \hat{d}_{33}^{1/2} \ , \quad \rho = \hat{\rho} \left[1 - \hat{d}_{33}^{-1} \hat{\mathbf{d}}_3 \bullet \boldsymbol{\delta}_3 \right] \ . \tag{5.13.6}$$

Then, expanding Σ in a Taylor series and neglecting quadratic terms in the small deformation quantities yields

$$\frac{\partial \Sigma}{\partial \mathbf{C}} = \frac{\partial \Sigma}{\partial \hat{\mathbf{C}}} + \frac{\partial \Sigma}{\partial \tilde{\mathbf{C}}} \ , \quad \frac{\partial \Sigma}{\partial \boldsymbol{\beta}_\alpha} = \frac{\partial \Sigma}{\partial \hat{\boldsymbol{\beta}}_\alpha} + \frac{\partial \Sigma}{\partial \tilde{\boldsymbol{\beta}}_\alpha} \ , \tag{5.13.7}$$

where the first terms on the right-hand sides are evaluated taking $\mathbf{C}=\hat{\mathbf{C}}$ and $\boldsymbol{\beta}_\alpha=\hat{\boldsymbol{\beta}}_\alpha$, and the second terms are first order in the small deformation quantities. Specifically, for the functional form (5.11.36)

$$m\frac{\partial\Sigma}{\partial\hat{\mathbf{C}}} = \left[m\frac{\partial\Sigma^*(\hat{\mathbf{C}})}{\partial\hat{\mathbf{C}}}\right] \ , \quad m\frac{\partial\Sigma}{\partial\tilde{\mathbf{C}}} = \left[m\frac{\partial^2\Sigma^*(\hat{\mathbf{C}})}{\partial\hat{\mathbf{C}}\otimes\partial\hat{\mathbf{C}}}\right]\bullet\tilde{\mathbf{C}} \ ,$$

$$m\frac{\partial\Sigma}{\partial\hat{\boldsymbol{\beta}}_\alpha} = D_{33}^{1/2}\, A\, \mathbf{K}^{\alpha\beta}\, \hat{\boldsymbol{\beta}}_\beta \ , \quad m\frac{\partial\Sigma}{\partial\tilde{\boldsymbol{\beta}}_\alpha} = D_{33}^{-1/2}\, A\, \mathbf{K}^{\alpha\beta}\, \tilde{\boldsymbol{\beta}}_\beta \ , \tag{5.13.8}$$

where the derivatives of Σ^* are evaluated with $\mathbf{C}=\hat{\mathbf{C}}$. If the three-dimensional strain energy function Σ^* is a quadratic function of strain (3.12.2), then these equations simplify somewhat with

$$m\frac{\partial\Sigma^*(\hat{\mathbf{C}})}{\partial\hat{\mathbf{C}}} = \frac{1}{2}D_{33}^{1/2}\, A\, \mathbf{K}^*\bullet\hat{\mathbf{E}} \ , \quad m\frac{\partial^2\Sigma^*(\hat{\mathbf{C}})}{\partial\hat{\mathbf{C}}\otimes\partial\hat{\mathbf{C}}} = \frac{1}{4}D_{33}^{1/2}\, A\, \mathbf{K}^* \ . \tag{5.13.9}$$

For either of these cases, (5.13.7) can be used to expand the constitutive equations (5.11.36) to deduce that

$$\hat{\mathbf{m}}^\alpha = m\,\hat{\mathbf{F}}^{-T}\frac{\partial\Sigma}{\partial\hat{\boldsymbol{\beta}}_\alpha} \ , \quad \tilde{\mathbf{m}}^\alpha = m\,\hat{\mathbf{F}}^{-T}\frac{\partial\Sigma}{\partial\tilde{\boldsymbol{\beta}}_\alpha} - (\hat{\mathbf{d}}^j\otimes\boldsymbol{\delta}_j)\,\hat{\mathbf{m}}^\alpha \ ,$$

$$\hat{\mathbf{t}}^i = 2\,m\,\hat{\mathbf{F}}\frac{\partial\Sigma}{\partial\hat{\mathbf{C}}}\mathbf{D}^i - \hat{\mathbf{m}}^\alpha\,(\hat{\mathbf{d}}_{\alpha,3}\bullet\hat{\mathbf{d}}^i) \ ,$$

$$\tilde{\mathbf{t}}^i = 2\,m\,\hat{\mathbf{F}}\frac{\partial\Sigma}{\partial\tilde{\mathbf{C}}}\,\mathbf{D}^i + (\boldsymbol{\delta}_j\otimes\hat{\mathbf{d}}^j)\,\hat{\mathbf{t}}^i + \left[(\boldsymbol{\delta}_j\otimes\hat{\mathbf{d}}^j)\,\hat{\mathbf{m}}^\alpha - \tilde{\mathbf{m}}^\alpha\right](\hat{\mathbf{d}}_{\alpha,3}\bullet\hat{\mathbf{d}}^i)$$
$$- \hat{\mathbf{m}}^\alpha\left[\boldsymbol{\delta}_{\alpha,3}\bullet\hat{\mathbf{d}}^i - (\hat{\mathbf{d}}_{\alpha,3}\bullet\hat{\mathbf{d}}^j)\,(\hat{\mathbf{d}}^i\bullet\boldsymbol{\delta}_j)\right] \ . \tag{5.13.10}$$

Also, the expression (5.4.23) expands to give

$$\mathbf{T} = \hat{\mathbf{T}} + \tilde{\mathbf{T}} \ , \quad \hat{\mathbf{T}} = \hat{d}_{33}^{-1/2}\left[\hat{\mathbf{t}}^i\otimes\hat{\mathbf{d}}_i + \hat{\mathbf{m}}^\alpha\otimes\hat{\mathbf{d}}_{\alpha,3}\right] \ ,$$

$$\tilde{\mathbf{T}} = \hat{d}_{33}^{-1/2}\left[\hat{\mathbf{t}}^i\otimes\boldsymbol{\delta}_i + \tilde{\mathbf{t}}^i\otimes\hat{\mathbf{d}}_i + \hat{\mathbf{m}}^\alpha\otimes\boldsymbol{\delta}_{\alpha,3} + \tilde{\mathbf{m}}^\alpha\otimes\hat{\mathbf{d}}_{\alpha,3}\right] - (\hat{d}_{33}^{-1}\,\hat{\mathbf{d}}_3\bullet\boldsymbol{\delta}_3)\,\hat{\mathbf{T}} \ . \tag{5.13.11}$$

Next, substitution of (5.13.10) into (5.13.11) yields

$$\hat{\mathbf{T}} = 2\,\hat{\rho}\,\hat{\mathbf{F}}\frac{\partial\Sigma}{\partial\hat{\mathbf{C}}}\,\hat{\mathbf{F}}^T \ ,$$

$$\tilde{\mathbf{T}} = 2\,\hat{\rho}\,\hat{\mathbf{F}}\frac{\partial\Sigma}{\partial\tilde{\mathbf{C}}}\,\hat{\mathbf{F}}^T + (\boldsymbol{\delta}_j\otimes\hat{\mathbf{d}}^j)\,\hat{\mathbf{T}} + \hat{\mathbf{T}}\,(\hat{\mathbf{d}}^j\otimes\boldsymbol{\delta}_j) - (\hat{d}_{33}^{-1}\,\hat{\mathbf{d}}_3\bullet\boldsymbol{\delta}_3)\,\hat{\mathbf{T}} \ , \tag{5.13.12}$$

which can be seen to be equivalent to a direct expansion of the constitutive equation $(5.8.12)_1$. Moreover, since both the terms $\hat{\mathbf{T}}$ and $\tilde{\mathbf{T}}$ are symmetric tensors, the reduced form (5.4.24) of the balance of angular momentum is satisfied by the small deformation terms.

Using these expressions the balances of linear momentum and director momentum (5.4.25) become

$$m\,(\ddot{\mathbf{u}} + y^\alpha\,\ddot{\boldsymbol{\delta}}_\alpha) - m\,\tilde{\mathbf{b}} - \tilde{\mathbf{t}}^3{}_{,3} = -\left[\, m\,(\ddot{\hat{\mathbf{x}}} + y^\alpha\,\ddot{\hat{\mathbf{d}}}_\alpha) - m\,\hat{\mathbf{b}} - \hat{\mathbf{t}}^3{}_{,3}\,\right] ,$$

$$m\,(y^{\alpha}\,\ddot{\mathbf{u}} + y^{\alpha\beta}\,\ddot{\boldsymbol{\delta}}_{\beta}) - m\,\tilde{\mathbf{b}}^{\alpha} + \tilde{\mathbf{t}}^{\alpha} - \tilde{\mathbf{m}}^{\alpha}{}_{,3} = -\,[\; m\,(y^{\alpha}\,\ddot{\hat{\mathbf{x}}} + y^{\alpha\beta}\,\ddot{\hat{\mathbf{d}}}_{\beta}) - m\,\hat{\mathbf{b}}^{\alpha} + \hat{\mathbf{t}}^{\alpha} - \hat{\mathbf{m}}^{\alpha}{}_{,3}\;] \;. \quad (5.13.13)$$

With $\hat{\mathbf{x}}$, $\hat{\mathbf{d}}_{\alpha}$, $\hat{\mathbf{b}}$, $\hat{\mathbf{b}}^{\alpha}$, $\tilde{\mathbf{b}}$ and $\tilde{\mathbf{b}}^{\alpha}$ specified, these equations become linear equations to determine the displacement fields $\mathbf{u}$ and $\boldsymbol{\delta}_{\alpha}$. Moreover, the equations must be supplemented by initial and boundary conditions. Also, when $\hat{\mathbf{x}}$ and $\hat{\mathbf{d}}_{\alpha}$ are the solutions of the nonlinear linear momentum and director momentum equations, then the right hand sides of (5.13.13) vanish.

As a special case, consider a homogeneous material and let $\hat{\mathbf{x}}$ and $\hat{\mathbf{d}}_{\alpha}$ be associated with a static homogeneous solution, with vanishing body force such that

$$\hat{\mathbf{x}} = \hat{\mathbf{F}}\,\mathbf{X} \;\;,\;\; \hat{\mathbf{d}}_{\alpha} = \hat{\mathbf{F}}\,\mathbf{D}_{\alpha} \;\;,\;\; \dot{\hat{\mathbf{F}}} = 0 \;\;,\;\; \hat{\mathbf{b}} = \hat{\mathbf{b}}_{c} \;\;,\;\; \hat{\mathbf{b}}^{\alpha} = \hat{\mathbf{b}}^{\alpha}_{c} \;\;, \qquad (5.13.14)$$

where $\hat{\mathbf{b}}_{c}$ and $\hat{\mathbf{b}}^{\alpha}_{c}$ are specified by the forms (5.3.16) and (5.3.32), respectively, with $\mathbf{F}^{*}=\hat{\mathbf{F}}$. It then follows from the results (5.11.12) and (5.11.13) that the right-hand sides of (5.13.13) vanish so the linear momentum and director momentum equations reduce to

$$m\,(\ddot{\mathbf{u}} + y^{\alpha}\,\ddot{\boldsymbol{\delta}}_{\alpha}) = m\,\tilde{\mathbf{b}} + \tilde{\mathbf{t}}^{3}{}_{,3} \;\;,\;\; m\,(y^{\alpha}\,\ddot{\mathbf{u}} + y^{\alpha\beta}\,\ddot{\boldsymbol{\delta}}_{\beta}) = m\,\tilde{\mathbf{b}}^{\alpha} - \tilde{\mathbf{t}}^{\alpha} + \tilde{\mathbf{m}}^{\alpha}{}_{,3} \;\;. \;(5.13.15)$$

In particular, it is noted that $\tilde{\mathbf{t}}^{i}$ and $\tilde{\mathbf{m}}^{\alpha}$ retain a dependence on the values $\hat{\mathbf{t}}^{i}$ and $\hat{\mathbf{m}}^{\alpha}$ associated with the large deformation.

Moreover, for the fully linearized theory $\hat{\mathbf{F}}=\mathbf{I}$ and $\hat{\boldsymbol{\beta}}_{\alpha}=0$ so that the quantities $\hat{\mathbf{t}}^{i}$ and $\hat{\mathbf{m}}^{\alpha}$ vanish. Then, the motion is determined by the equations (5.13.15) with

$$\tilde{\mathbf{E}} = \frac{1}{2}(\boldsymbol{\delta}_{i}\otimes\mathbf{D}^{i} + \mathbf{D}^{i}\otimes\boldsymbol{\delta}_{i}) \;\;,\;\; \boldsymbol{\delta}_{3} = \mathbf{u}_{,3} \;\;,\;\; \tilde{\boldsymbol{\beta}}_{\alpha} = \boldsymbol{\delta}_{\alpha,3} - (\mathbf{D}_{\alpha,3}\bullet\mathbf{D}^{i})\,\boldsymbol{\delta}_{i} \;\;,$$

$$\tilde{\mathbf{m}}^{\alpha} = D_{33}^{-1/2}\,A\,\mathbf{K}^{\alpha\beta}\,\tilde{\boldsymbol{\beta}}_{\beta} \;\;,\;\; \tilde{\mathbf{t}}^{i} = \left[D_{33}^{1/2}\,A\,\mathbf{K}^{*}\bullet\tilde{\mathbf{E}}\right]\mathbf{D}^{i} - \tilde{\mathbf{m}}^{\alpha}\,(\mathbf{D}_{\alpha,3}\bullet\mathbf{D}^{i}) \;\;.(5.13.16)$$

Also, for the linearized theory quadratic terms in the displacements are neglected in the expression for the material derivative.

5.14 Pure bending of an orthotropic beam with rectangular cross-section

The restrictions on the constitutive equations developed in section 5.11 ensure consistency with exact solutions for all nonlinear homogeneous deformations even when the rod has an arbitrary shape in its reference configuration. It was further seen in section 5.12 that these restrictions determined all of the constitutive coefficients associated with homogeneous deformations in terms of the three-dimensional material and geometric properties of the rod. However, these restrictions do not provide any guidance for determining values of the constitutive coefficients $K_{i\gamma j\delta}$ in (5.12.7) associated with inhomogeneous deformations. The objective of the present section is to determine some of the coefficients $K_{i\gamma j\delta}$ by comparing solutions of the linear Cosserat theory for an orthotropic beam with rectangular cross-section, with exact bending solutions of the three-dimensional theory given in section 3.14.

The kinematic expression (5.2.8) requires the three-dimensional position vector $\mathbf{x}^*$ to remain a linear function of the coordinate θ^α through the cross-section of the rod. Although it was shown at the end of section 5.2 that the expression (5.2.8) is exact for homogeneous deformations, it is quite clear that for inhomogeneous deformations $\mathbf{x}^*$ can be a nonlinear function of θ^α. In particular, the bending solutions developed in section 3.14 for the linearized three-dimensional equations indicate that the displacement vector is a quadratic function of θ^α. This means that even if an exact three-dimensional solution for $\mathbf{x}^*$ is known, there is no unique way of identifying values of $\mathbf{x}$ and $\mathbf{d}_\alpha$ associated with the Cosserat theory when $\mathbf{x}^*$ is a nonlinear function of θ^α. A discussion of this point related to the determination of the shear deformation coefficient in beam theory has recently been given by Rubin (1996). In that work, it was suggested that the kinematics of the Cosserat theory can be related to those of the three-dimensional theory such that the structure of the constitutive equation for stress is preserved. The discussion that follows briefly reviews aspects of that work.

By way of background, it is first noted that a rod structure is considered to be a beam if its reference curve is straight in its reference configuration. This means that the tangent vector $\mathbf{D}_3$ to the reference curve has a constant direction

$$\frac{d}{d\theta^3}\left[\frac{\mathbf{D}_3}{D_{33}^{1/2}}\right] = 0 \ . \tag{5.14.1}$$

The formulas developed in the previous sections are valid for an arbitrary specification of the directors $\mathbf{D}_\alpha$ which satisfies the restriction (5.1.3). However, from a physical point of view, it often convenient to take the directors $\mathbf{D}_\alpha$ to be unit vectors which are orthogonal to the reference curve and are parallel to the axes of symmetry of the rectangular cross-section

$$\mathbf{D}_1 \bullet \mathbf{D}_1 = 1 \ , \quad \mathbf{D}_1 \bullet \mathbf{D}_3 = 0 \ , \quad \mathbf{D}_1 \bullet \mathbf{D}_2 = 0 \ , \quad \mathbf{D}_2 \bullet \mathbf{D}_2 = 1 \ , \quad \mathbf{D}_2 \bullet \mathbf{D}_3 = 0 \ . \tag{5.14.2}$$

so that for elastic rods the directors $\mathbf{d}_\alpha$ can be identified with material fibers that were normal to the reference curve of the rod in its reference configuration.

Here, attention is confined to a beam which in its reference configuration has constant height H, constant width W and length L. To be specific, the position vector $\mathbf{X}^*$ in the reference configuration is given by (3.14.2), and the region occupied by the beam is specified by

$$| \theta^1 | \le \frac{H}{2} \ , \quad | \theta^2 | \le \frac{W}{2} \ , \quad | \theta^3 | \le \frac{L}{2} \ . \tag{5.14.3}$$

Also, the reference curve $\mathbf{X}$ is taken to be the line that connects the centroids of the cross-section, and the directors are specified by

$$\mathbf{X} = \theta^3 \mathbf{e}_3 \ , \quad \mathbf{D}_i = \mathbf{e}_i \ . \tag{5.14.4}$$

With these specifications, the quantities $\{\mathbf{G}_i, \mathbf{G}^i, G^{1/2}\}$ are independent of the coordinate θ^3 and become

$$\mathbf{G}_i = \mathbf{G}^i = \mathbf{D}_i = \mathbf{D}^i = \mathbf{e}_i \ , \quad G^{1/2} = D^{1/2} = D_{33}^{1/2} = 1 \ . \tag{5.14.5}$$

Furthermore, it follows from $(5.11.7)_2$ and (5.11.10) that

$$A = HW \ , \quad A^\alpha = 0 \ . \tag{5.14.6}$$

Under these conditions, the kinematic expression (5.2.8) causes the three-dimensional displacement vector of the linearized theory to be a linear function of the coordinates θ^α [see (3.13.1) with $\hat{\mathbf{x}}^*=\mathbf{X}^*$, and (5.13.1) with $\hat{\mathbf{x}}=\mathbf{X}$ and $\hat{\mathbf{d}}_i=\mathbf{D}_i$]

$$\mathbf{u}^*(\theta^i,t) = \mathbf{u}(\theta^3,t) + \theta^\alpha\, \boldsymbol{\delta}_\alpha(\theta^3,t) \ . \tag{5.14.7}$$

It therefore, follows from (3.2.34), (3.13.18) and (5.14.5) that the linearized three-dimensional strain $\tilde{\mathbf{E}}^*$ and stress $\tilde{\mathbf{T}}^*$ associated with a beam become

$$\tilde{\mathbf{E}}^* = \frac{1}{2}\left[\mathbf{u}^*{}_{,\alpha}\otimes\mathbf{D}^\alpha + \mathbf{D}^\alpha\otimes\mathbf{u}^*{}_{,\alpha} + \mathbf{u}^*{}_{,3}\otimes\mathbf{D}^3 + \mathbf{D}^3\otimes\mathbf{u}^*{}_{,3}\right] \ ,$$

$$\tilde{\mathbf{T}}^* = \mathbf{K}^* \bullet \tilde{\mathbf{E}}^* \ . \tag{5.14.8}$$

Next, it is observed from the linearized form of (5.11.2) that the quantity $\tilde{\mathbf{T}}$ associated with the Cosserat theory can be expressed in the form

$$D_{33}^{1/2}\, \tilde{\mathbf{T}} = \int_{-W/2}^{W/2} \int_{-H/2}^{H/2} G^{1/2}\, \tilde{\mathbf{T}}^*\, d\theta^1 d\theta^2 \ . \tag{5.14.9}$$

Now, substitution of (5.14.7) into (5.14.8) and use of $(5.11.7)_2$ and (5.14.5) yields the equations

$$\tilde{\mathbf{T}} = A\, \mathbf{K}^* \bullet \tilde{\mathbf{E}} \ , \quad \tilde{\mathbf{E}} = \frac{1}{2}\left[\boldsymbol{\delta}_i\otimes\mathbf{D}^i + \mathbf{D}^i\otimes\boldsymbol{\delta}_i\right] \ ,$$

$$\mathbf{u}(\theta^3,t) = \frac{1}{A}\int_{-W/2}^{W/2}\int_{-H/2}^{H/2} \mathbf{u}^*\, d\theta^1 d\theta^2 \ , \quad \boldsymbol{\delta}_3 = \mathbf{u}_{,3} \ ,$$

$$\boldsymbol{\delta}_1(\theta^3,t) = \frac{1}{A}\int_{-W/2}^{W/2}\int_{-H/2}^{H/2} \mathbf{u}^*{}_{,1}\, d\theta^1 d\theta^2$$

$$= \frac{1}{A}\int_{-W/2}^{W/2}\left[\mathbf{u}^*(H/2,\theta^2,\theta^3,t) - \mathbf{u}^*(-H/2,\theta^2,\theta^3,t)\right] d\theta^2 \ ,$$

$$\boldsymbol{\delta}_2(\theta^3,t) = \frac{1}{A}\int_{-W/2}^{W/2}\int_{-H/2}^{H/2} \mathbf{u}^*{}_{,2}\, d\theta^1 d\theta^2$$

$$= \frac{1}{A}\int_{-H/2}^{H/2}\left[\mathbf{u}^*(\theta^1,W/2,\theta^3,t) - \mathbf{u}^*(\theta^1,-W/2,\theta^3,t)\right] d\theta^1 \ , \tag{5.14.10}$$

which relate the Cosserat displacements $\mathbf{u}$ and $\boldsymbol{\delta}_\alpha$ to the three-dimensional displacement $\mathbf{u}^*$. At this point it should be emphasized that the expressions (5.14.10) for $\tilde{\mathbf{T}}$, $\mathbf{u}$, and $\boldsymbol{\delta}_i$ are not necessarily equal to similar quantities that are obtained by solving the equations for a Cosserat beam. Instead, these quantities are used to compare the predictions of the three-dimensional theory with those of the Cosserat theory.

Now, with the help of (5.14.5) the expressions (5.13.16) for the linearized theory of a beam and the linearized form of (5.4.23) yield

$$\tilde{\mathbf{E}} = \frac{1}{2}(\boldsymbol{\delta}_i\otimes\mathbf{D}^i + \mathbf{D}^i\otimes\boldsymbol{\delta}_i) \ , \quad \boldsymbol{\delta}_3 = \mathbf{u}_{,3} \ , \quad \tilde{\boldsymbol{\beta}}_\alpha = \boldsymbol{\delta}_{\alpha,3} \ , \quad \tilde{\mathbf{m}}^\alpha = D_{33}^{-1/2}\, A\, \mathbf{K}^{\alpha\beta}\, \tilde{\boldsymbol{\beta}}_\beta \ ,$$

$$\tilde{\mathbf{t}}^i = D_{33}^{1/2}\, A\left[\mathbf{K}^* \bullet \tilde{\mathbf{E}}\right] \mathbf{D}^i \ , \quad \tilde{\mathbf{T}} = A\, \mathbf{K}^* \bullet \tilde{\mathbf{E}} \ . \tag{5.14.11}$$

Also, the equations of equilibrium associated with (5.13.15) reduce to

$$m\, \tilde{\mathbf{b}} + \tilde{\mathbf{t}}^3{}_{,3} = 0 \ , \quad m\, \tilde{\mathbf{b}}^\alpha - \tilde{\mathbf{t}}^\alpha + \tilde{\mathbf{m}}^\alpha{}_{,3} = 0 \ . \tag{5.14.12}$$

In particular, it is observed that the result $(5.14.10)_1$ is consistent with the expression $(5.14.11)_6$. This indicates that the interpretations of $\mathbf{u}$ and $\boldsymbol{\delta}_\alpha$ proposed in (5.14.10) preserve compatibility of the three-dimensional constitutive equations with those of the Cosserat theory for homogeneous deformations.

In order to compare the Cosserat solutions with the corresponding three-dimensional solutions of section 3.14, it is necessary to apply the same loading to the rod. This is accomplished by using the linearized forms of (5.3.19), $(5.3.33)_1$, (5.3.16), and (5.3.32), to deduce that

$$\tilde{\mathbf{t}}^3 = \int_{-W/2}^{W/2}\int_{-H/2}^{H/2} \tilde{\mathbf{T}}^*\mathbf{e}_3\, d\theta^1 d\theta^2 \ , \quad \tilde{\mathbf{m}}^\alpha = \int_{-W/2}^{W/2}\int_{-H/2}^{H/2} \theta^\alpha\, \tilde{\mathbf{T}}^*\mathbf{e}_3\, d\theta^1 d\theta^2 \ ,$$

$$m\,\tilde{\mathbf{b}}_c = \int_{-W/2}^{W/2}\big[\tilde{\mathbf{T}}^*(H/2,\theta^2,\theta^3) - \tilde{\mathbf{T}}^*(-H/2,\theta^2,\theta^3)\big]\,\mathbf{e}_1\, d\theta^2$$
$$+ \int_{-H/2}^{H/2}\big[\tilde{\mathbf{T}}^*(\theta^1,W/2,\theta^3) - \tilde{\mathbf{T}}^*(\theta^1,-W/2,\theta^3)\big]\,\mathbf{e}_2\, d\theta^1 \ ,$$

$$m\,\tilde{\mathbf{b}}_c^1 = \frac{H}{2}\int_{-W/2}^{W/2}\big[\tilde{\mathbf{T}}^*(H/2,\theta^2,\theta^3) + \tilde{\mathbf{T}}^*(-H/2,\theta^2,\theta^3)\big]\,\mathbf{e}_1\, d\theta^2$$
$$+ \int_{-H/2}^{H/2}\big[\tilde{\mathbf{T}}^*(\theta^1,W/2,\theta^3) - \tilde{\mathbf{T}}^*(\theta^1,-W/2,\theta^3)\big]\,\mathbf{e}_2\,\theta^1\, d\theta^1 \ ,$$

$$m\,\tilde{\mathbf{b}}_c^2 = \int_{-W/2}^{W/2}\big[\tilde{\mathbf{T}}^*(H/2,\theta^2,\theta^3) - \tilde{\mathbf{T}}^*(-H/2,\theta^2,\theta^3)\big]\,\mathbf{e}_1\,\theta^2\, d\theta^2$$
$$+ \frac{W}{2}\int_{-H/2}^{H/2}\big[\tilde{\mathbf{T}}^*(\theta^1,W/2,\theta^3) + \tilde{\mathbf{T}}^*(\theta^1,-W/2,\theta^3)\big]\,\mathbf{e}_2\, d\theta^1 \ . \qquad (5.14.13)$$

Moreover, in the absence of three-dimensional body force the linearized versions of the expressions (5.3.12) and (5.3.28) yield

$$\tilde{\mathbf{b}}_b = 0 \ , \quad \tilde{\mathbf{b}} = \tilde{\mathbf{b}}_c \ , \quad \tilde{\mathbf{b}}_b^\alpha = 0 \ , \quad \tilde{\mathbf{b}}^\alpha = \tilde{\mathbf{b}}_c^\alpha \ . \qquad (5.14.14)$$

Comparison of (3.14.1) and (5.14.3) indicates that the variables H and L have been interchanged for the present description of the beam. Also, for the present purposes it is sufficient to omit two of the bending solutions by taking

$$M_{21} = M_{12} = 0 \ . \qquad (5.14.15)$$

Therefore, the relevant three-dimensional stresses are obtained by using (5.14.15) and interchanging the roles of L and H in (3.14.8) to obtain

$$T_{11}^* = \Big[\frac{12M_{31}}{WL^3}\Big]\,\theta^3 \ , \quad T_{22}^* = \Big[\frac{12M_{32}}{HL^3}\Big]\,\theta^3 \ ,$$
$$T_{33}^* = \Big[\frac{12M_{13}}{H^3W}\Big]\,\theta^1 + \Big[\frac{12M_{23}}{HW^3}\Big]\,\theta^2 \ , \quad T_{12}^* = T_{13}^* = T_{23}^* = 0 \ . \qquad (5.14.16)$$

Thus, substitution of these expressions into (5.14.13) yields

$$\tilde{\mathbf{t}}^3 = 0 \ , \quad \tilde{\mathbf{m}}^1 = M_{13}\,\mathbf{e}_3 \ , \quad \tilde{\mathbf{m}}^2 = M_{23}\,\mathbf{e}_3 \ , \quad m\,\tilde{\mathbf{b}} = 0 \ ,$$
$$m\,\tilde{\mathbf{b}}^1 = \Big[\frac{12HM_{31}}{L^3}\Big]\,\theta^3\,\mathbf{e}_1 \ , \quad m\,\tilde{\mathbf{b}}^2 = \Big[\frac{12WM_{32}}{L^3}\Big]\,\theta^3\,\mathbf{e}_2 \ , \qquad (5.14.17)$$

where the values of $\tilde{\mathbf{t}}^3$ and $\tilde{\mathbf{m}}^\alpha$ are applied only on the ends $\theta^3=\pm L/2$.

In (5.14.17), use has been made of (5.14.2) and (5.14.4) which require the axes of orthotropy of the beam to be parallel to $\mathbf{D}_i$, so that (5.12.5) reduces to

$$\mathbf{M}_i = \mathbf{e}_i \ . \qquad (5.14.18)$$

Consequently, referring all tensor quantities to the base vectors $\mathbf{e}_i$

$$u_i = \mathbf{e}_i \bullet \mathbf{u}\ ,\quad \tilde{\delta}_{ij} = \mathbf{e}_i \bullet \boldsymbol{\delta}_j\ ,\quad \beta_{i\alpha} = \mathbf{e}_i \bullet \tilde{\boldsymbol{\beta}}_\alpha = \mathbf{e}_i \bullet \boldsymbol{\delta}_{\alpha,3}\ ,\quad E_{ij} = (\mathbf{e}_i \otimes \mathbf{e}_j) \bullet \tilde{\mathbf{E}}\ ,$$
$$T_{ij} = (\mathbf{e}_i \otimes \mathbf{e}_j) \bullet \tilde{\mathbf{T}} = \mathbf{e}_i \bullet \tilde{\mathbf{t}}^j\ ,\quad m^{i\alpha} = \mathbf{e}_i \bullet \tilde{\mathbf{m}}^\alpha\ , \tag{5.14.19}$$

it follows with the help of (3.12.12), (5.12.6), (5.12.8) and (5.12.10) that for an orthotropic beam

$$E_{\alpha\beta} = \frac{1}{2}\left[\tilde{\delta}_{\alpha\beta} + \tilde{\delta}_{\beta\alpha}\right]\ ,\quad E_{\alpha 3} = \frac{1}{2}\left[u_{\alpha,3} + \tilde{\delta}_{3\alpha}\right]\ ,\quad E_{33} = u_{3,3}\ ,\quad \beta_{i\alpha} = \tilde{\delta}_{i\alpha,3}\ ,$$
$$T_{11} = A\left[K^*_{1111} E_{11} + K^*_{1122} E_{22} + K^*_{1133} E_{33}\right]\ ,$$
$$T_{22} = A\left[K^*_{1122} E_{11} + K^*_{2222} E_{22} + K^*_{2233} E_{33}\right]\ ,$$
$$T_{33} = A\left[K^*_{1133} E_{11} + K^*_{2233} E_{22} + K^*_{3333} E_{33}\right]\ ,$$
$$T_{12} = 2\,A K^*_{1212} E_{12}\ ,\quad T_{13} = 2\,AK^*_{1313} E_{13}\ ,\quad T_{23} = 2\,AK^*_{2323} E_{23}\ ,$$
$$m^{11} = AK_{1111}\beta_{11} + AK_{1122}\beta_{22}\ ,\quad m^{12} = AK_{1212}\beta_{12} + AK_{1221}\beta_{21}\ ,$$
$$m^{21} = AK_{1221}\beta_{12} + AK_{2121}\beta_{21}\ ,\quad m^{22} = AK_{1122}\beta_{11} + AK_{2222}\beta_{22}\ ,$$
$$m^{31} = AK_{3131}\beta_{31}\ ,\quad m^{32} = AK_{3232}\beta_{32}\ , \tag{5.14.20}$$

where the symbol tilde ($\tilde{}$) has been used over the components of $\boldsymbol{\delta}_i$ to distinguish them from the Kronecker delta symbol δ_{ij}.

Now, substitution of the three-dimensional displacements (3.14.12) [with the roles of H and L interchanged and with the specifications (5.14.15)] into the expressions (5.14.10) would suggest that the displacements of the Cosserat theory are given by

$$u_1 = -\,C^*_{3333}\left[\left\{\frac{12M_{13}}{H^3W}\right\}\frac{(\theta^3)^2}{2}\right] + \left[\alpha^*_{13}\,\theta^3 + c_1\right]\ ,$$
$$u_2 = -\,C^*_{3333}\left[\left\{\frac{12M_{23}}{HW^3}\right\}\frac{(\theta^3)^2}{2}\right] + \left[\alpha^*_{23}\,\theta^3 + c_2\right]\ ,$$
$$u_3 = C^*_{1133}\left[\left\{\frac{12M_{31}}{WL^3}\right\}\frac{(\theta^3)^2}{2}\right] + C^*_{2233}\left[\left\{\frac{12M_{32}}{HL^3}\right\}\frac{(\theta^3)^2}{2}\right] + c_3\ ,$$
$$\tilde{\delta}_{11} = C^*_{1111}\left[\left\{\frac{12M_{31}}{WL^3}\right\}\theta^3\right] + C^*_{1122}\left[\left\{\frac{12M_{32}}{HL^3}\right\}\theta^3\right]\ ,$$
$$\tilde{\delta}_{21} = -\,\alpha^*_{12}\ ,\quad \tilde{\delta}_{31} = C^*_{3333}\left[\left\{\frac{12M_{13}}{H^3W}\right\}\theta^3\right] - \alpha^*_{13}\ ,\quad \tilde{\delta}_{12} = \alpha^*_{12}\ ,$$
$$\tilde{\delta}_{22} = C^*_{1122}\left[\left\{\frac{12M_{31}}{WL^3}\right\}\theta^3\right] + C^*_{2222}\left[\left\{\frac{12M_{32}}{HL^3}\right\}\theta^3\right]\ ,$$
$$\tilde{\delta}_{32} = C^*_{3333}\left[\left\{\frac{12M_{23}}{HW^3}\right\}\theta^3\right] - \alpha^*_{23}\ , \tag{5.14.21}$$

where the constants c_i are defined by

$$c_1 = C^*_{1133}\left[\frac{M_{13}}{2HW}\right] - C^*_{2233}\left[\frac{WM_{13}}{2H^3}\right] + c^*_1\ ,$$
$$c_2 = C^*_{2233}\left[\frac{M_{23}}{2HW}\right] - C^*_{1133}\left[\frac{HM_{23}}{2W^3}\right] + c^*_2\ ,$$

$$c_3 = -C^*_{1111}\,[\frac{H^2M_{31}}{2WL^3}] - C^*_{1122}\,[\frac{WM_{31}}{2L^3}]$$
$$- C^*_{1122}\,[\frac{HM_{32}}{2L^3}] - C^*_{2222}\,[\frac{W^2M_{32}}{2HL^3}] \ . \tag{5.14.22}$$

Obviously, since c^*_i are arbitrary constants, they can absorb the remaining terms in (5.14.22) causing c_i to be arbitrary constants associated with rigid body translations. Next, the strains associated with (5.14.21) are calculated to be

$$E_{11} = \tilde{\delta}_{11} \ , \ E_{22} = \tilde{\delta}_{22} \ ,$$
$$E_{33} = C^*_{1133}\,[\{\frac{12M_{31}}{WL^3}\}\theta^3] + C^*_{2233}\,[\{\frac{12M_{32}}{HL^3}\}\theta^3] \ ,$$
$$E_{12} = E_{13} = E_{23} = 0 \ , \ \beta_{11} = C^*_{1111}\,[\frac{12M_{31}}{WL^3}] + C^*_{1122}\,[\frac{12M_{32}}{HL^3}] \ ,$$
$$\beta_{21} = 0 \ , \ \beta_{31} = C^*_{3333}\,[\{\frac{12M_{13}}{H^3W}\}] \ , \ \beta_{12} = 0 \ ,$$
$$\beta_{22} = C^*_{1122}\,[\frac{12M_{31}}{WL^3}] + C^*_{2222}\,[\frac{12M_{32}}{HL^3}] \ , \ \beta_{32} = C^*_{3333}\,[\{\frac{12M_{23}}{HW^3}\}] \ . \tag{5.14.23}$$

At this point, it is of interest to recall the physical interpretation of the result that $E_{\alpha 3}$ vanishes. In particular, it follows from (5.14.4) and (5.14.11) that for the linearized theory of a beam

$$\mathbf{d}_\alpha \bullet \mathbf{d}_3 = \boldsymbol{\delta}_\alpha \bullet \mathbf{e}_3 + \mathbf{e}_\alpha \bullet \boldsymbol{\delta}_3 = 2\,E_{\alpha 3} = 0 \ . \tag{5.14.24}$$

This indicates that the directors remains normal to the rod's reference curve. Physically, this means that the material fiber that was normal to the rod's reference curve in its reference configuration remains normal to the reference curve in the present configuration, which is a known result for pure bending.

With the help of the strains (5.14.23), the constitutive equations (5.14.20) and the expressions (3.14.11), it can easily be shown that

$$T_{11} = A\,[\{\frac{12M_{31}}{WL^3}\}\theta^3] \ , \ T_{22} = A\,[\{\frac{12M_{32}}{HL^3}\}\theta^3] \ ,$$
$$T_{33} = T_{12} = T_{13} = T_{23} = 0 \ , \ \tilde{\mathbf{t}}^\alpha = m\,\tilde{\mathbf{b}}^\alpha \ , \tag{5.14.25}$$

and that $\tilde{\mathbf{t}}^3$ is given by $(5.14.17)_1$ for all values of θ^3, not just at the ends of the beam. Moreover, it follows that $\tilde{\mathbf{m}}^\alpha$ are constants so the equations of equilibrium (5.14.12) are satisfied. Now, the conditions that $\tilde{\mathbf{m}}^\alpha$ equal the values $(5.14.17)_{2,3}$ at the ends of the beam yield the four equations

$$K_{1111}\,C^*_{1111} + K_{1122}\,C^*_{1122} = 0 \ , \ K_{1111}\,C^*_{1122} + K_{1122}\,C^*_{2222} = 0 \ ,$$
$$K_{1122}\,C^*_{1111} + K_{2222}\,C^*_{1122} = 0 \ , \ K_{1122}\,C^*_{1122} + K_{2222}\,C^*_{2222} = 0 \ , \tag{5.14.26}$$

and the results

$$K_{3131} = \frac{H^2}{12}\,[\frac{1}{C^*_{3333}}] \ , \ K_{3232} = \frac{W^2}{12}\,[\frac{1}{C^*_{3333}}] \ . \tag{5.14.27}$$

Although (5.14.23) are four equations for three unknown constitutive coefficients, they are dependent equations that can be solved to deduce that

$$K_{1111} = K_{1122} = K_{2222} = 0 \ . \qquad (5.14.28)$$

Since the results (5.14.28) appear to be new, it is of interest to discuss their physical origin. To this end, it suffices to consider the responses to the moments M_{31} and M_{32}. Specifically, it can be seen from (5.14.21) that due to the Poisson effect these moments cause the thickness of the beam to vary linearly with the coordinate θ^3. Moreover, it is observed from (3.14.8) that the three-dimensional stresses associated with these moments are independent of the cross-sectional coordinates θ^α. Consequently, the couples $\tilde{\mathbf{m}}^\alpha$ must vanish even though the cross-section of the beam does not remain constant.

It is also of interest to note that the results (5.14.28) influence the discussion of boundary conditions for the beam. In particular, it can be seen from (5.14.20) and (5.14.28) that the couples $m^{\alpha\beta}$ vanish. This means that there is no work done by changes in the normal components of the directors $\mathbf{d}_\alpha$ so that these normal components cannot be specified by boundary conditions. For example, the problem considered by Green et. al. (1974b) related to their equations (9.16)-(9.18) cannot be formulated with the specification (5.14.28). However, the specification (5.14.28) is used here because it was determined by comparison with exact three-dimensional solutions.

At this point, it is important to comment on the relationship of the Cosserat theory with a standard Galerkin approximate solution of the three-dimensional equations. Most of the developments in section 5.3 are consistent with the Galerkin approach which uses the kinematic assumption (5.2.8) and the three weighting functions $\{1 \text{ and } \theta^\alpha\}$. However, the direct approach of the Cosserat theory in the remaining sections in chapter 5 does not rely explicitly on this kinematic assumption. In particular, the constitutive equations in section 5.8 are developed in a manner that directly parallels that of the three-dimensional theory, with the kinetic quantities being derivable from a strain energy function. The procedure for determining the functional form of this strain energy is identical to that of the three-dimensional theory. Specifically, a functional form is proposed and the constitutive coefficients are determined to best match experimental data. In this section the constitutive constants for a beam were determined by matching exact solutions of the three-dimensional equations (which themselves are presumed to accurately predict relevant experiments).

In contrast, within the context of the standard Galerkin approach the constitutive coefficients are determined by direct integration of the three-dimensional constitutive equations that are obtained using the kinematic assumption (5.2.8). In particular, for linear beam theory it can be shown that the components E^*_{ij} of the three-dimensional strain associated with the kinematic assumption (5.14.8) can be expressed in terms of the Cosserat strains E_{ij} and $\beta_{i\alpha}$ by the equations

$$E^*_{\alpha\beta} = E_{\alpha\beta} \ , \quad E^*_{3\alpha} = E_{3\alpha} + \frac{\theta^\beta}{2}\beta_{\alpha\beta} \ , \quad E^*_{33} = E_{33} + \frac{\theta^\beta}{2}\beta_{3\beta} \ . \qquad (5.14.29)$$

It therefore follows from $(5.14.6)_1$, $(5.14.8)_2$, $(5.14.13)_2$ and (5.14.19) that for these strains, direct integration for a beam produces the Galerkin form of $m^{i\alpha}$

$$m^{i1} = A\left\{\frac{H^2}{12}\right\}\left[\, K^*_{i33\beta}\, \beta_{\beta 1} + \frac{1}{2} K^*_{i333}\, \beta_{31} \,\right] ,$$

$$m^{i2} = A\left\{\frac{W^2}{12}\right\}\left[\, K^*_{i33\beta}\, \beta_{\beta 2} + \frac{1}{2} K^*_{i333}\, \beta_{32} \,\right] , \tag{5.14.30}$$

which can be compared with the associated Cosserat form

$$m^{i\alpha} = AK_{i\alpha j\beta}\, \beta_{j\beta} , \tag{5.14.31}$$

to deduce that the constitutive coefficients $K_{i\alpha j\beta}$ should be specified by

$$K_{i1\beta 1} = \frac{H^2}{12} K^*_{i33\beta} , \quad K_{i131} = \frac{H^2}{24} K^*_{i333} , \quad K_{i1\beta 2} = 0 , \quad K_{i132} = 0 ,$$

$$K_{i2\beta 2} = \frac{W^2}{12} K^*_{i33\beta} , \quad K_{i232} = \frac{W^2}{24} K^*_{i333} , \quad K_{i2\beta 1} = 0 , \quad K_{i231} = 0 . \tag{5.14.32}$$

In particular, these specifications require

$$K_{1111} = \frac{H^2}{12} K^*_{1331} , \quad K_{1122} = 0 , \quad K_{2222} = \frac{W^2}{12} K^*_{1331} ,$$

$$K_{3131} = \frac{H^2}{24} K^*_{3333} , \quad K_{3232} = \frac{W^2}{24} K^*_{3333} . \tag{5.14.33}$$

Obviously, these values are not compatible with the results (5.14.27) and (5.14.28), which were obtained by forcing the Cosserat solution to match the three-dimensional solutions [comments to this effect were mentioned in (Green and Naghdi, 1996)]. This is mainly due to the fact that the kinematic approximation (5.14.7) causes the strains E^*_{11} and E^*_{22} in (5.14.29) to be independent of the coordinates θ^α. For this reason, the associated three-dimensional stress distribution is not compatible with the exact distribution (3.14.8). Consequently, this is a specific example where the Cosserat approach produces values of the constitutive coefficients that are more accurate than those obtained using standard Galerkin methods.

For the special case of an isotropic material, the expressions (3.14.13) apply and the results (5.14.27) reduce to

$$K_{3131} = \frac{E^*H^2}{12} = \frac{\mu^*(1+\nu^*)H^2}{6} , \quad K_{3232} = \frac{E^*W^2}{12} = \frac{\mu^*(1+\nu^*)W^2}{6} , \tag{5.14.34}$$

where use has been made of Table 3.12.1. Moreover, it can be seen that the expressions (5.14.34) are compatible with those developed by Green et. al. (1974b).

The work of Green and Naghdi (1982) on laminated composite plates considers a finite number of Cosserat plates that are bonded together on their major surfaces. This bonding is accomplished by demanding continuity of the displacements and appropriate stresses at the major surfaces. Such bonding could also be used to formulate equations for a layered composite beam. Another problem where it is important to predict the displacements of the lateral surface of a beam occurs when analyzing the problem of a beam in contact with another surface (Naghdi and Rubin, 1989). In view of these developments, it is of interest to examine how accurately the Cosserat theory can predict the displacements of the lateral surface of the beam. To this end, the kinematic assumption (5.14.7) is used to define the displacements $\bar{\mathbf{u}}_\alpha$ and $\hat{\mathbf{u}}_\alpha$ of the centerlines of the lateral surfaces through the expressions

$$\bar{\mathbf{u}}_1 = \mathbf{u} - \frac{H}{2}\boldsymbol{\delta}_1 \ , \quad \hat{\mathbf{u}}_1 = \mathbf{u} + \frac{H}{2}\boldsymbol{\delta}_1 \ , \quad \bar{\mathbf{u}}_2 = \mathbf{u} - \frac{W}{2}\boldsymbol{\delta}_2 \ , \quad \hat{\mathbf{u}}_2 = \mathbf{u} + \frac{W}{2}\boldsymbol{\delta}_2 \ . \tag{5.14.35}$$

Next, with the help of the exact displacements (3.14.12) [associated with the specification (5.14.15) and with the roles of L and H interchanged] and the Cosserat displacements (5.14.21), it can be shown that

$$\begin{aligned}
\bar{\mathbf{u}}_1 - \mathbf{u}^*(-H/2,0,\theta^3) &= \Big[c_1 - c_1^* - C_{1133}^*\Big\{\frac{3M_{13}}{2HW}\Big\}\Big]\,\mathbf{e}_1 \\
&+ \Big[c_2 - c_2^* + C_{1133}^*\Big\{\frac{3HM_{23}}{2W^3}\Big\}\Big]\,\mathbf{e}_2 \\
&+ \Big[c_3 - c_3^* + C_{1111}^*\Big\{\frac{3H^2M_{31}}{2WL^3}\Big\} + C_{1122}^*\Big\{\frac{3HM_{32}}{2L^3}\Big\}\Big]\,\mathbf{e}_3 \ , \\
\hat{\mathbf{u}}_1 - \mathbf{u}^*(H/2,0,\theta^3) &= \bar{\mathbf{u}}_1 - \mathbf{u}^*(-H/2,0,\theta^3) \ , \\
\bar{\mathbf{u}}_2 - \mathbf{u}^*(0,-W/2,\theta^3) &= \Big[c_1 - c_1^* + C_{2233}^*\Big\{\frac{3WM_{13}}{2H^3}\Big\}\Big]\,\mathbf{e}_1 \\
&+ \Big[c_2 - c_2^* - C_{2233}^*\Big\{\frac{3M_{23}}{2HW}\Big\}\Big]\,\mathbf{e}_2 \\
&+ \Big[c_3 - c_3^* + C_{1122}^*\Big\{\frac{3WM_{31}}{2L^3}\Big\} + C_{2222}^*\Big\{\frac{3W^2M_{32}}{2HL^3}\Big\}\Big]\,\mathbf{e}_3 \ , \\
\hat{\mathbf{u}}_2 - \mathbf{u}^*(0,W/2,\theta^3) &= \bar{\mathbf{u}}_2 - \mathbf{u}^*(0,-W/2,\theta^3) \ ,
\end{aligned} \tag{5.14.36}$$

Thus, the differences between the exact values of the displacements and the Cosserat predictions on the centerlines of the lateral surfaces are equal to constant vectors. However, the values of these constant vectors are different for both sets of the lateral surfaces ($\theta^1=\pm H/2$) and ($\theta^2=\pm W/2$). This means that the values of c_i, associated with superposed constant translation, can be chosen to eliminate these differences on either set of these lateral surfaces, but not on both. Moreover, it is of interest to note that, even though the identification $(5.14.10)_1$ does not connect $\mathbf{u}$ with the exact displacement of the reference curve of the beam, the predictions (5.14.35) are still reasonably accurate for this problem.

5.15 Torsion of an orthotropic beam with rectangular cross-section

The analysis of the previous section produced expressions for five of the eight constitutive coefficients $K_{i\gamma j\delta}$ for an orthotropic beam with rectangular cross-section by considering bending solutions. In the present section two restrictions on the remaining coefficients K_{1212}, K_{1221} and K_{2121} are determined by comparing solutions of the linear Cosserat theory, for an orthotropic beam with rectangular cross-section with an exact torsion solution of the three-dimensional theory given in section 3.15. Also, the formulation of the beam equations is the same as that described in section 5.14 so that the roles of H and L in the three-dimensional solution of section 3.15 must be interchanged.

It can be seen from (3.15.1) that the kinematic assumption (5.14.7) is not capable of modeling torsion about the $\mathbf{e}_1$ and $\mathbf{e}_2$ axes of the beam. However, as will be seen, it is possible to model torsion about the $\mathbf{e}_3$ axes of the beam even though the kinematics of warping of the cross-section are not included in the simplest Cosserat theory with only two directors. Consequently, attention will be focused only on the solution in section 3.15 associated with the twist ω_3 and the moment M_3. Therefore, when M_1 and M_2 vanish and the roles of L and H are interchanged, equations (3.15.1), (3.15.2), (3.15.5), (3.15.7), and (3.15.12)-(3.15.14) yield expressions for the exact solution of the forms

$$u_1^* = -\,\omega_3\theta^2\theta^3 \ , \quad u_2^* = \omega_3\theta^1\theta^3 \ , \quad u_3^* = \omega_3\phi_3(\theta^1,\theta^2) \ ,$$

$$\phi_3 = -\,\theta^1\theta^2 - \sum_{n=1}^{\infty} \Big[\frac{8\,(-1)^n\, c_1\, c_2^2}{W\, k_{3n}^3 \cosh\{\frac{k_{3n}H}{2c_1}\}} \Big] \sinh\{\frac{k_{3n}\theta^1}{c_1}\} \sin\{\frac{k_{3n}\theta^2}{c_2}\} \ ,$$

$$T_{11}^* = T_{22}^* = T_{33}^* = T_{12}^* = 0 \ ,$$

$$T_{13}^* = K_{1313}^*\, \omega_3 \left[-\,\theta^2 + \phi_{3,1} \right] \ , \quad T_{23}^* = K_{2323}^*\, \omega_3 \left[\theta^1 + \phi_{3,2} \right] \ ,$$

$$k_{3n} = \frac{\pi(2n-1)c_2}{W} \ , \quad M_3 = B_3^*\, \omega_3 \ ,$$

$$B_3^* = \frac{H^2W^2}{3} \left[K_{1313}^*\, K_{2323}^* \right]^{1/2} b^*(\eta_3) \ , \quad \eta_3 = \frac{H}{W} \Big[\frac{K_{2323}^*}{K_{1313}^*} \Big]^{1/2} \ ,$$

$$b^*(\eta) = b^*(1/\eta) = \frac{1}{\eta} \Big[1 - \frac{192}{\pi^5\eta} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^5} \tanh\{\frac{\pi(2n-1)\eta}{2}\} \Big] \ . \tag{5.15.1}$$

Next, substitution of the displacement field (5.15.1) into the expressions (5.14.10) suggests that the displacement fields $\mathbf{u}$ and $\boldsymbol{\delta}_i$ of the Cosserat theory are given by

$$\mathbf{u} = 0 \ , \quad \boldsymbol{\delta}_1 = \omega_3\theta^3\, \mathbf{e}_2 \ , \quad \boldsymbol{\delta}_2 = -\,\omega_3\theta^3\, \mathbf{e}_1 \ , \quad \boldsymbol{\delta}_3 = 0 \ . \tag{5.15.2}$$

Thus, with the help of (5.14.19) and (5.14.20) it follows that

$$E_{ij} = 0 \ , \quad \beta_{12} = -\,\omega_3 \ , \quad \beta_{21} = \omega_3 \ , \quad \text{all other } \beta_{i\alpha} = 0 \ , \quad T_{ij} = 0 \ , \quad \tilde{\mathbf{t}}^i = 0 \ ,$$

$$\tilde{\mathbf{m}}^1 = \omega_3 A\left[K_{2121} - K_{1221} \right] \mathbf{e}_2 \ , \quad \tilde{\mathbf{m}}^2 = \omega_3 A\left[K_{1221} - K_{1212} \right] \mathbf{e}_1 \ . \tag{5.15.3}$$

Moreover, in the absence of three-dimensional body force and using the fact that the surface tractions vanish on the lateral surface of the beam, the assigned fields associated with (5.14.13) and (5.14.14) also vanish

$$\tilde{\mathbf{b}} = 0 \ , \quad \tilde{\mathbf{b}}^\alpha = 0 \ . \tag{5.15.4}$$

It then follows that the equations of equilibrium (5.14.12) are automatically satisfied. Also, the boundary conditions at the ends of the beam require the resultant force to vanish and the resultant moment to be a pure torsional moment of magnitude M_3. The resultant force vanishes because $\tilde{\mathbf{t}}^3$ vanishes, and the expression for the torsional moment is obtained by linearizing the last term in the balance of angular momentum (5.4.7) to

$$M_3\,\mathbf{e}_3 = \mathbf{e}_\alpha \times \tilde{\mathbf{m}}^\alpha = A\left[K_{1212} + K_{2121} - 2\,K_{1221}\right]\omega_3\,\mathbf{e}_3\,. \tag{5.15.5}$$

Thus, with the help of (5.15.1) it follows that the Cosserat theory will predict the correct value for the torsional rigidity B_3^* if the constitutive coefficients satisfy the restriction

$$A\left[K_{1212} + K_{2121} - 2\,K_{1221}\right] = B_3^*\,. \tag{5.15.6}$$

Following the discussion in (Green et al, 1974b) it is of interest to substitute the exact stress field (5.15.1) into (5.3.33) to determine approximate expressions for $\tilde{\mathbf{m}}^\alpha$ of the forms

$$\tilde{\mathbf{m}}^1 = \frac{1}{2}\,B_3^*\,\omega_3\,\mathbf{e}_2\ ,\quad \tilde{\mathbf{m}}^2 = -\frac{1}{2}B_3^*\,\omega_3\,\mathbf{e}_1\,. \tag{5.15.7}$$

Thus, comparison of (5.15.3) and (5.15.7) would suggest that K_{1212} and K_{2121} are equal

$$K_{1212} = K_{2121}\,. \tag{5.15.8}$$

The validity of this last result will be discussed in the next section where values of the constitutive coefficients are determined by considering another solution where the Cosserat theory reproduces the exact three-dimensional results.

5.16 Inhomogeneous shear of an orthotropic beam with rectangular cross-section

In the previous section, two restrictions on the values of the constitutive coefficients K_{1212}, K_{1221} and K_{2121} were determined by considering the problem of torsion about the axis of the beam. Here, values for these coefficients will be determined by considering an alternative problem of inhomogeneous shear. Again, in order to compare with the formulation of the beam equations described in section 5.14, the roles of H and L are interchanged in the description of the three-dimensional region of section 3.14.

For the inhomogeneous shear problem under consideration, the displacements u_i^*, strains E_{ij}^* and stresses T_{ij}^* associated with the three-dimensional theory (3.14.5) and (3.14.7) are given by

$$\begin{gathered} u_1^* = \gamma_1\theta^2\theta^3\ ,\quad u_2^* = \gamma_2\theta^1\theta^3\ ,\quad u_3^* = 0\ ,\\ E_{11}^* = E_{22}^* = E_{33}^* = 0\ ,\quad E_{12}^* = \frac{(\gamma_1+\gamma_2)}{2}\theta^3\ ,\quad E_{13}^* = \frac{\gamma_1}{2}\theta^2\ ,\quad E_{23}^* = \frac{\gamma_2}{2}\theta^1\ ,\\ T_{11}^* = T_{22}^* = T_{33}^* = 0\ ,\quad T_{12}^* = K_{1212}^*\,(\gamma_1+\gamma_2)\,\theta^3\ ,\\ T_{13}^* = K_{1313}^*\,\gamma_1\theta^2\ ,\quad T_{23}^* = K_{2323}^*\,\gamma_2\theta^1\ , \end{gathered} \tag{5.16.1}$$

where γ_1 and γ_2 are constants. It can easily be seen that these stresses satisfy the equilibrium equations (3.14.6) in the absence of body forces. Moreover, since the shear strains are functions of the coordinates, the shearing deformation is inhomogeneous.

Next, substitution of the displacement field (5.16.1) into the expressions (5.14.10) suggests that the displacement fields $\mathbf{u}$ and $\boldsymbol{\delta}_i$ of the Cosserat theory are given by

$$\mathbf{u} = 0\ ,\quad \boldsymbol{\delta}_1 = \gamma_2\theta^3\,\mathbf{e}_2\ ,\quad \boldsymbol{\delta}_2 = \gamma_1\theta^3\,\mathbf{e}_1\ ,\quad \boldsymbol{\delta}_3 = 0\,. \tag{5.16.2}$$

Thus, with the help of (5.14.19) and (5.14.20) it follows that

$$E_{12} = \frac{(\gamma_1+\gamma_2)}{2}\theta^3 \ , \quad \text{all other } E_{ij} = 0 \ , \quad \beta_{12} = \gamma_1 \ , \quad \beta_{21} = \gamma_2 \ , \quad \text{all other } \beta_{i\alpha} = 0 \ ,$$
$$T_{12} = AK^*_{1212}\,(\gamma_1+\gamma_2)\,\theta^3 \ , \quad \text{all other } T_{ij} = 0 \ ,$$
$$\tilde{\mathbf{t}}^1 = AK^*_{1212}\,(\gamma_1+\gamma_2)\theta^3\,\mathbf{e}_2 \ , \quad \tilde{\mathbf{t}}^2 = AK^*_{1212}\,(\gamma_1+\gamma_2)\,\theta^3\,\mathbf{e}_1 \ , \quad \tilde{\mathbf{t}}^3 = 0 \ ,$$
$$\tilde{\mathbf{m}}^1 = A\big[K_{1221}\,\gamma_1 + K_{2121}\,\gamma_2\big]\,\mathbf{e}_2 \ , \quad \tilde{\mathbf{m}}^2 = A\big[K_{1212}\,\gamma_1 + K_{1221}\,\gamma_2\big]\,\mathbf{e}_1 \ . \tag{5.16.3}$$

Moreover, in the absence of three-dimensional body force the expressions (5.14.13) and (5.14.14) can be used to deduce that the assigned fields associated with (5.16.1) are given by

$$\tilde{\mathbf{b}} = 0 \ , \quad m\,\tilde{\mathbf{b}}^\alpha = \tilde{\mathbf{t}}^\alpha \ . \tag{5.16.4}$$

It then follows that the equations of equilibrium (5.14.12) are automatically satisfied.

Since the kinematics (5.16.2) of the Cosserat theory, together with the kinematic assumption (5.14.7), reproduce the exact kinematics of the three-dimensional solution (5.16.1), it is expected that the Cosserat theory will give exact results. This also means that the correspondence between the three-dimensional stresses and the resultant quantities $\tilde{\mathbf{t}}^i$ and $\tilde{\mathbf{m}}^i$ will be exact. In particular, it can be shown with the help of (5.14.9) and (5.14.11) that integration of (5.16.1) yields the values for $\tilde{\mathbf{t}}^i$ given in (5.16.3) . Also, with the help of (5.14.13), integration of (5.16.1) yields the expressions

$$\tilde{\mathbf{m}}^1 = \Big[\frac{H^3W}{12}\Big]\,K^*_{2323}\,\gamma_2\,\mathbf{e}_1 \ , \quad \tilde{\mathbf{m}}^2 = \Big[\frac{HW^3}{12}\Big]\,K^*_{1313}\,\gamma_1\,\mathbf{e}_1 \ . \tag{5.16.5}$$

Moreover, since γ_1 and γ_2 are independent quantities, comparison of (5.16.3) and (5.16.5) suggests that the coupling term K_{1221} vanishes

$$K_{1221} = 0 \ , \tag{5.16.6}$$

and that the other constitutive coefficients are given by

$$A\,K_{1212} = \Big[\frac{HW^3}{12}\Big]\,K^*_{1313} \ , \quad A\,K_{2121} = \Big[\frac{H^3W}{12}\Big]\,K^*_{2323} \ . \tag{5.16.7}$$

At this point it is convenient to use the definition (5.15.1) for η_3 to rewrite (5.16.7) in the form

$$A\,K_{1212} = \Big[\frac{H^2W^2}{12}\Big]\,\big[K^*_{1313}\,K^*_{2323}\big]^{1/2}\,\frac{1}{\eta_3} \ ,$$
$$A\,K_{2121} = \Big[\frac{H^2W^2}{12}\Big]\,\big[K^*_{1313}\,K^*_{2323}\big]^{1/2}\,\eta_3 \ . \tag{5.16.8}$$

It then follows that these two coefficients will be equal and will be compatible with the result (5.15.8) only for the special case when η_3 equals unity. Moreover, even when η_3 equals unity the values (5.16.8) will not yield the correct value (5.15.6) for the torsional rigidity of the beam. This means that some form of compromise is required when specifying values for these constitutive coefficients.

On the one hand, it is quite compelling to specify these constitutive coefficients by the values (5.16.6) and (5.16.8) because this solution produces exact results relative to the three-dimensional theory. On the other hand, torsion of a beam is a much more common physical phenomena to be modeled than the inhomogeneous shear considered in this section. Therefore, a reasonable compromise is accept the conclusions (5.15.6) and (5.15.8) of the torsion problem as well as the conclusion (5.16.6) of the present problem and to ignore the additional conclusions (5.16.8). This causes the constitutive coefficients to be specified by

$$A\, K_{1212} = A\, K_{2121} = \frac{1}{2} B_3^* \;,\quad K_{1221} = 0 \;, \tag{5.16.9}$$

where the torsional rigidity B_3^* is given by the exact expression in (5.15.1).

In order to summarize the constitutive equations associated with inhomogeneous deformations of an orthotropic rod with rectangular cross section, it is necessary to recognize that the values that have been developed tacitly depend on the fact that the directions of orthotropy are orthogonal to the planes defining the lateral surface of the beam. This means that it is necessary to specify the directors $\mathbf{D}_i$ in the reference configuration of the beam to be orthogonal vectors which are parallel to the directions of orthotropy. Moreover, for simplicity the vectors $\mathbf{D}_\alpha$ will be taken to be unit vectors so that

$$\mathbf{D}_i \bullet \mathbf{D}_\alpha = \delta_{i\alpha} \;, \tag{5.16.10}$$

and (5.12.5) reduces to

$$\mathbf{M}_\alpha = \mathbf{D}_\alpha \;,\quad \mathbf{M}_3 = \frac{\mathbf{D}_3}{|\mathbf{D}_3|} \;. \tag{5.16.11}$$

It then follows from (5.12.6), (5.12.7), (5.14.28) and (5.16.9) that the tensors $\mathbf{K}^{\alpha\beta}$ are given by

$$\mathbf{K}^{11} = K_{1212}\, \mathbf{M}_2 \otimes \mathbf{M}_2 + K_{3131}\, \mathbf{M}_3 \otimes \mathbf{M}_3 \;,$$
$$\mathbf{K}^{22} = K_{1212}\, \mathbf{M}_1 \otimes \mathbf{M}_1 + K_{3232}\, \mathbf{M}_3 \otimes \mathbf{M}_3 \;,\quad \mathbf{K}^{12} = \mathbf{K}^{21} = 0 \;, \tag{5.16.12}$$

where K_{3131} and K_{3232} are specified by (5.14.27), and K_{1212} is specified by (5.16.9). Also, using (3.12.13), (5.14.34), (5.15.1) and (5.16.9) these expressions become

$$\mathbf{K}^{11} = \frac{\mu^* H^2}{6} \Big[\Big\{ \frac{b^*(\eta)}{\eta} \Big\}\, \mathbf{M}_2 \otimes \mathbf{M}_2 + (1+\nu^*)\, \mathbf{M}_3 \otimes \mathbf{M}_3 \Big] \;,$$

$$\mathbf{K}^{22} = \frac{\mu^* W^2}{6} \Big[\big\{ \eta\, b^*(\eta) \big\}\, \mathbf{M}_1 \otimes \mathbf{M}_1 + (1+\nu^*)\, \mathbf{M}_3 \otimes \mathbf{M}_3 \Big] \;,\quad \mathbf{K}^{12} = \mathbf{K}^{21} = 0 \;, \tag{5.16.13}$$

for an isotropic beam. In the above, $b^*(\eta)$ is the function given by (5.15.1) and the parameter η reduces to

$$\eta = \frac{H}{W} \;. \tag{5.16.14}$$

5.17 Forced shearing vibrations of an orthotropic beam with rectangular cross-section

In the previous sections the constitutive coefficients for the static response of an orthotropic beam with rectangular cross-section have been determined for both homogeneous and inhomogeneous deformations. This section proposes values for the inertia properties of the beam by comparing predictions of the Cosserat theory with the exact results of section 3.16 for forced shearing vibrations of an orthotropic rectangular parallelepiped. Again, in order to compare with the formulation of the beam equations described in section 5.14, the roles of H and L are interchanged in the description of the three-dimensional region of section 3.14.

The formulation of the beam equations is the same as that described in section 5.14. In particular, it is noted that in view (3.2.28) and the condition (5.14.5), it follows that

$$m^* = \rho_0^* \quad . \tag{5.17.1}$$

Moreover, since the reference curve of the beam is taken to be the middle line (5.14.4) and the mass density ρ_0^* is presumed to be constant, direct integration of (5.3.7) and (5.3.10) yields

$$m = \rho_0^* A = \rho_0^* HW \quad , \quad y^\alpha = 0 \quad , \tag{5.17.2}$$

where the cross-sectional area A is defined by (5.14.6). Also, direct integration of (5.3.26) yields the values

$$y^{11} = \frac{H^2}{12} \quad , \quad y^{22} = \frac{W^2}{12} \quad , \quad y^{12} = y^{21} = 0 \quad . \tag{5.17.3}$$

It will presently be shown that the Cosserat theory can predict more accurate values for vibrational frequencies if the values of the director inertia coefficients y^{11} and y^{22} are taken to be different from (5.17.3). In this regard, it should be mentioned that within the context of the Cosserat theory the quantity m and the director inertia coefficients y^α and $y^{\alpha\beta}$ are independent of time and therefore can be determined in the reference configuration. Moreover, within the context of the direct approach these quantities require constitutive equations and, in particular, they need not be determined by integrals (5.3.7), (5.3.10) and (5.3.26). Nevertheless, since the value $(5.17.2)_1$ for m ensures that the beam will have the same mass per unit length as the associated parallelepiped, it is retained in the Cosserat theory. Also, the values $(5.17.2)_2$ for y^α and the value $(5.17.3)_3$ for y^{12} are retained for a beam since they indicate that the mass is distributed symmetrically about the middle line of the beam.

In contrast, the values (5.17.3) for the director inertia coefficients y^{11} and y^{22} will be modified to cause the Cosserat theory to predict vibrational frequencies more accurately. In this regard, it should be mentioned that the values (5.17.3) are the same as the values obtained using the Galerkin method and the kinematic approximation (5.2.8).

To predict values for y^{11} and y^{22} it is sufficient to consider only two of the vibrational modes discussed in section 3.16. To this end, only the vibrations associated with the amplitudes A_{31}^* and A_{32}^* will be considered. Also, attention is focused on only the first mode of vibration since it is not reasonable to expect a beam theory to accurately predict higher order modes through the thickness. Consequently, the three-dimensional displacements and stresses associated with these modes can be written in the forms

$$u_1^* = 0 \;, \quad u_2^* = 0 \quad , \quad u_3^* = A_{31}^* \sin(\omega_{31}^* t) \sin(k_1^* \theta^1) + A_{32}^* \sin(\omega_{32}^* t) \sin(k_2^* \theta^2) \quad ,$$

$$\tilde{\mathbf{t}}^{*1} = K_{1313}^* \left[A_{31}^* k_1^* \sin(\omega_{31}^* t) \cos(k_1^* \theta^1) \right] \mathbf{e}_3 \quad ,$$

$$\tilde{\mathbf{t}}^{*2} = K_{2323}^* \left[A_{32}^* k_2^* \sin(\omega_{32}^* t) \cos(k_2^* \theta^2) \right] \mathbf{e}_3 \quad ,$$

$$\tilde{\mathbf{t}}^{*3} = K^*_{1313}\left[A^*_{31}k^*_1 \sin(\omega^*_{31}t)\cos(k^*_1\theta^1)\right]\mathbf{e}_1 + K^*_{2323}\left[A^*_{32}k^*_2 \sin(\omega^*_{32}t)\cos(k^*_2\theta^2)\right]\mathbf{e}_2 \ , \tag{5.17.4}$$

where the frequencies and the wave numbers are given by

$$\omega^*_{31} = \left[\frac{K^*_{1313}}{\rho^*_0}\right]^{1/2}\frac{\pi}{H} \ , \quad \omega^*_{32} = \left[\frac{K^*_{2323}}{\rho^*_0}\right]^{1/2}\frac{\pi}{W} \ , \quad k^*_1 = \frac{\pi}{H} \ , \quad k^*_2 = \frac{\pi}{W} \ . \tag{5.17.5}$$

In order for the loading of the beam to be the same as that of the parallelepiped, it is necessary to specify the assigned fields by

$$\tilde{\mathbf{b}} = 0 \ , \quad \tilde{\mathbf{b}}^\alpha = 0 \ , \tag{5.17.6}$$

and the boundary conditions by

$$\tilde{\mathbf{t}}^3(\pm L/2,t) = \left[2W\, K^*_{1313}\left\{A^*_{31}\sin(\omega^*_{31}t)\right\}\mathbf{e}_1 + 2H\, K^*_{2323}\left\{A^*_{32}\sin(\omega^*_{32}t)\right\}\mathbf{e}_2\right] \ ,$$
$$\tilde{\mathbf{m}}^1(\pm L/2,t) = 0 \ , \quad \tilde{\mathbf{m}}^2(\pm L/2,t) = 0 \ , \tag{5.17.7}$$

where use has been made of the linearized forms of (5.3.12), (5.3.16), (5.3.19), (5.3.24), (5.3.28), (5.3.30), (5.3.32), (5.3.33) and (5.3.37). Next, the expressions (5.14.10) suggest that the Cosserat displacements associated with (5.17.4) are given by

$$\mathbf{u} = 0 \ , \quad \boldsymbol{\delta}_3 = 0 \ , \quad \boldsymbol{\delta}_1 = \frac{2}{H}\left[A^*_{31}\sin(\omega^*_{31}t)\right]\mathbf{e}_3 \ , \quad \boldsymbol{\delta}_2 = \frac{2}{W}\left[A^*_{32}\sin(\omega^*_{32}t)\right]\mathbf{e}_3 \ . \tag{5.17.8}$$

Using these displacements the constitutive equations (5.14.11), with the specifications (5.16.10)-(5.16.12) for an orthotropic beam, yield the expressions

$$\tilde{\mathbf{E}} = \left[\frac{1}{H}\left\{A^*_{31}\sin(\omega^*_{31}t)\right\}(\mathbf{e}_1\otimes\mathbf{e}_3 + \mathbf{e}_3\otimes\mathbf{e}_1) + \frac{1}{W}\left\{A^*_{32}\sin(\omega^*_{32}t)\right\}(\mathbf{e}_2\otimes\mathbf{e}_3 + \mathbf{e}_3\otimes\mathbf{e}_2)\right] \ ,$$
$$\tilde{\mathbf{t}}^1 = \left[2WK^*_{1313}\, A^*_{31}\sin(\omega^*_{31}t)\right]\mathbf{e}_3 \ , \quad \tilde{\mathbf{t}}^2 = \left[2HK^*_{2323}\, A^*_{32}\sin(\omega^*_{32}t)\right]\mathbf{e}_3 \ ,$$
$$\tilde{\mathbf{t}}^3 = \left[\left\{2WK^*_{1313}\, A^*_{31}\sin(\omega^*_{31}t)\right\}\mathbf{e}_1 + \left\{2HK^*_{2323}\, A^*_{32}\sin(\omega^*_{32}t)\right\}\mathbf{e}_2\right] \ ,$$
$$\tilde{\boldsymbol{\beta}}_\alpha = 0 \ , \quad \tilde{\mathbf{m}}^\alpha = 0 \ , \tag{5.17.9}$$

which automatically satisfy the boundary conditions (5.17.7).

Now, the linearized equations of motion (5.13.15) reduce to the following two scalar equations

$$\rho^*_0\, y^{11}\left\{\omega^*_{31}\right\}^2 = K^*_{1313} \ , \quad \rho^*_0\, y^{22}\left\{\omega^*_{32}\right\}^2 = K^*_{2323} \ . \tag{5.17.10}$$

However, since the frequencies are given by (5.17.5), it can be seen that both of these equations can be satisfied if the director inertia coefficients y^{11} and y^{22} are specified by

$$y^{11} = \frac{H^2}{\pi^2} \ , \quad y^{22} = \frac{W^2}{\pi^2} \ , \tag{5.17.11}$$

instead of the Galerkin values (5.17.3). In this regard, it is noted that the director inertia coefficients model both the mass distribution in the beam as well as the distribution of acceleration in potential vibrational modes. This is similar to the stiffness coefficients of the beam which depend on both the material and geometric properties of the structure.

5.18 Free isochoric vibrations of an isotropic cube

To further examine the validity of the specification (5.14.28) of the constitutive constant K_4 and the specifications (5.17.11) of the director inertia coefficients y^{11} and y^{22}, it is of interest to use the Cosserat beam equations to predict the free isochoric vibrations of an isotropic cube discussed in section 3.17. Again, in order to compare with the formulation of the beam equations described in section 5.14, the roles of H and L are interchanged in the description of the three-dimensional region of section 3.14.

The formulation of the beam equations is the same as that described in sections 5.14 and 5.17, except that here the dimensions of the beam [see (3.14.1)] are taken to be equal

$$L = W = H \ . \tag{5.18.1}$$

Also, the material is assumed to be isotropic so that $(5.14.11)_6$ becomes

$$\tilde{\mathbf{T}} = H^2 \left[(K^* - \tfrac{2}{3}\mu^*) (\tilde{\mathbf{E}} \bullet \mathbf{I}) \, \mathbf{I} + 2\mu^* \, \tilde{\mathbf{E}} \right] \ . \tag{5.18.2}$$

For simplicity attention is focused on only two of the three modes of vibration described by (3.17.2) so that the three-dimensional displacements and stresses associated with these modes can be written in the forms

$$\begin{aligned}
u_1^* &= A_{13}^* \sin(\omega^* t) \sin(k^* \theta^1) \cos(k^* \theta^3) \ , \\
u_2^* &= A_{23}^* \sin(\omega^* t) \sin(k^* \theta^2) \cos(k^* \theta^3) \ , \\
u_3^* &= - A_{13}^* \sin(\omega^* t) \cos(k^* \theta^1) \sin(k^* \theta^3) - A_{23}^* \sin(\omega^* t) \cos(k^* \theta^2) \sin(k^* \theta^3) \ , \\
\tilde{\mathbf{t}}^{*1} &= 2\mu^* k^* \left[A_{13}^* \cos(k^* \theta^3) \right] \sin(\omega^* t) \cos(k^* \theta^1) \, \mathbf{e}_1 \ , \\
\tilde{\mathbf{t}}^{*2} &= 2\mu^* k^* \left[A_{23}^* \cos(k^* \theta^3) \right] \sin(\omega^* t) \cos(k^* \theta^2) \, \mathbf{e}_2 \ , \\
\tilde{\mathbf{t}}^{*3} &= - 2\mu^* k^* \left[A_{13}^* \cos(k^* \theta^1) + A_{23}^* \cos(k^* \theta^2) \right] \sin(\omega^* t) \cos(k^* \theta^3) \, \mathbf{e}_3 \ .
\end{aligned} \tag{5.18.3}$$

Also, attention is focused on only the first mode of vibration so the frequency and wave number are given by

$$\omega^* = \left[\frac{2\mu^*}{\rho_0^*} \right]^{1/2} \frac{\pi}{H} \ , \quad k^* = \frac{\pi}{H} \ . \tag{5.18.4}$$

In order for the loading of the beam to be the same as that of the cube it is necessary to specify the assigned fields by

$$\tilde{\mathbf{b}} = 0 \ , \quad \tilde{\mathbf{b}}^\alpha = 0 \ , \tag{5.18.5}$$

and the boundary conditions by

$$\tilde{\mathbf{t}}^3(\pm L/2, t) = 0 \ , \quad \tilde{\mathbf{m}}^1(\pm L/2, t) = 0 \ , \quad \tilde{\mathbf{m}}^2(\pm L/2, t) = 0 \ , \tag{5.18.6}$$

Next, the expressions (5.14.10) suggest that the Cosserat displacements associated with (5.18.3) are given by

$$\begin{aligned}
\mathbf{u} &= -\frac{2}{\pi} \left[A_{13}^* + A_{23}^* \right] \sin(\omega^* t) \sin(k^* \theta^3) \, \mathbf{e}_3 \ , \\
\boldsymbol{\delta}_1 &= \frac{2}{H} \left[A_{13}^* \right] \sin(\omega^* t) \cos(k^* \theta^3) \, \mathbf{e}_1 \ , \quad \boldsymbol{\delta}_2 = \frac{2}{H} \left[A_{23}^* \right] \sin(\omega^* t) \cos(k^* \theta^3) \, \mathbf{e}_2 \ , \\
\boldsymbol{\delta}_3 &= -\frac{2}{H} \left[A_{13}^* + A_{23}^* \right] \sin(\omega^* t) \cos(k^* \theta^3) \, \mathbf{e}_3 \ .
\end{aligned} \tag{5.18.7}$$

Now, with the help of (5.14.9), the strains associated with these displacements become

$$\tilde{\mathbf{E}} = \frac{2}{H}[A_{13}^* \, \mathbf{e}_1\otimes\mathbf{e}_1 + A_{23}^* \, \mathbf{e}_2\otimes\mathbf{e}_2 - (A_{13}^* + A_{23}^*) \, \mathbf{e}_3\otimes\mathbf{e}_3] \sin(\omega^* t) \cos(k^*\theta^3) \ ,$$

$$\tilde{\boldsymbol{\beta}}_1 = -\frac{2}{\pi}[A_{13}^*] \sin(\omega^* t) \sin(k^*\theta^3) \, \mathbf{e}_1 \ , \ \tilde{\boldsymbol{\beta}}_2 = -\frac{2}{\pi}[A_{23}^*] \sin(\omega^* t) \sin(k^*\theta^3) \, \mathbf{e}_2 \ . \quad (5.18.8)$$

It can easily be seen that the strain $\tilde{\mathbf{E}}$ is isochoric since

$$\tilde{\mathbf{E}} \bullet \mathbf{I} = 0 \ . \quad (5.18.9)$$

Moreover, for an isotropic material the constitutive equations (5.14.11), (5.14.20), (5.14.28) and (5.18.2) associated with (5.18.8) become

$$\tilde{\mathbf{t}}^1 = 4\mu^* H \left[A_{13}^* \sin(\omega^* t) \cos(k^*\theta^3)\right] \mathbf{e}_1 \ , \quad \tilde{\mathbf{t}}^2 = 4\mu^* H \left[A_{23}^* \sin(\omega^* t) \cos(k^*\theta^3)\right] \mathbf{e}_2 \ ,$$

$$\tilde{\mathbf{t}}^3 = -4\mu^* H \left[(A_{13}^* + A_{23}^*) \sin(\omega^* t) \cos(k^*\theta^3)\right] \mathbf{e}_3 \ , \quad \tilde{\mathbf{m}}^\alpha = 0 \ , \quad (5.18.10)$$

which automatically satisfy the boundary conditions (5.17.7). In this regard, it should be mentioned that this result provides additional justification for the specifications (5.14.28), which cause $\tilde{\mathbf{m}}^\alpha$ to vanish even though $\tilde{\boldsymbol{\beta}}_\alpha$ do not vanish.

In view of the values (5.17.2), $(5.17.3)_3$ and (5.17.11), the linearized equations of motion (5.13.15) reduce to

$$\rho_0^* H^2 \ddot{\mathbf{u}} = \tilde{\mathbf{t}}^3{}_{,3} \ , \quad \rho_0^* H^2 \left\{\frac{H^2}{\pi^2}\right\} \ddot{\boldsymbol{\delta}}_\alpha = -\tilde{\mathbf{t}}^\alpha \ . \quad (5.18.11)$$

Now, substitution of (5.18.7) and (5.18.10) into these equations yields two scalar equations

$$\rho_0^* \frac{H^2}{\pi} \left\{\omega^*\right\}^2 = 2\mu^* H k^* \ , \quad \rho_0^* \left\{\frac{H^2}{\pi^2}\right\} \left\{\omega^*\right\}^2 = 2\mu^* \ , \quad (5.18.12)$$

which are both satisfied by the values (5.18.4) of the frequency and the wave number. Consequently, this result again justifies the specifications (5.17.11) of the director inertia coefficients y^{11} and y^{22}.

5.19 An orthotropic beam with rectangular cross-section loaded by its own weight

In the previous sections the stiffness and inertia coefficients for a beam have been determined by comparison with exact three-dimensional solutions. Within the context of the Cosserat theory it is also necessary to specify values for the assigned fields $\mathbf{b}$ and $\mathbf{b}^\alpha$. In determining the restrictions in section 5.11 associated with homogeneous deformations, it was necessary to use the expressions (5.3.16) and (5.3.32) for the parts $\mathbf{b}_c$ and $\mathbf{b}_c^\alpha$ of the assigned fields that are due to loading on the lateral surface of the beam. For this reason, the expressions (5.3.16) and (5.3.32) are retained. The main objective of this section is to examine the validity of the expressions (5.3.12) and (5.3.28) for the parts $\mathbf{b}_b$ and $\mathbf{b}_b^\alpha$ of the assigned fields associated with body force. This will be explored by comparing the Cosserat solution of an orthotropic beam with rectangular cross-section, which is loaded by its own weight with the three-dimensional solution recorded in section

3.18. Again, in order to compare with the formulation of the beam equations described in section 5.14, the roles of H and L are interchanged in the description of the three-dimensional region of section 3.18.

In particular, it will be assumed that the reference values of the three-dimensional mass density ρ_0^* and body force $\mathbf{b}^*$ are constants so that with the help of the conservation of mass $(3.2.35)_1$, the expressions (5.3.12) and (5.3.28) yield

$$m\,\mathbf{b}_b = m\,\mathbf{b}^* \ , \quad m\,\mathbf{b}_b^\alpha = m\,y^\alpha\,\mathbf{b}^* \ , \tag{5.19.1}$$

where use has been made of (5.3.7) and (5.3.10). In particular, for a beam the expressions (5.17.2) can be used to deduce that

$$m\,\mathbf{b}_b = \rho_0^*\,HW\,\mathbf{b}^* \ , \quad m\,\mathbf{b}_b^\alpha = 0 \ , \tag{5.19.2}$$

so that there is no distinction between the linearized values $\tilde{\mathbf{b}}_c$ and $\tilde{\mathbf{b}}_c^\alpha$ and the corresponding nonlinear values.

Moreover, for the solutions of section 3.18 the specific body force $\mathbf{b}^*$ is specified by (3.18.1) so that

$$m\,\mathbf{b}_b = -\,\rho_0^*\,HW\,g_i^*\,\mathbf{e}_i \ . \tag{5.19.3}$$

Also, with the help of the linearized forms of (5.3.16) and (5.3.32), and the expressions (3.18.2) for the stresses, the assigned fields due to surface tractions applied to the lateral surface of the beam are given by

$$m\,\tilde{\mathbf{b}}_c = \rho_0^*\,HW\,[\,g_1^*\,\mathbf{e}_1 + g_2^*\,\mathbf{e}_2\,] \ , \quad m\,\tilde{\mathbf{b}}_c^1 = -\frac{H}{2}\rho_0^*\,HW\,[\,g_1^*\,\mathbf{e}_1 + g_2^*\,\mathbf{e}_2\,] \ ,$$
$$m\,\tilde{\mathbf{b}}_c^2 = -\frac{W}{2}\rho_0^*\,HW\,[\,g_1^*\,\mathbf{e}_1 + g_2^*\,\mathbf{e}_2\,] \ , \tag{5.19.4}$$

so that the total assigned fields can be expressed in the forms

$$m\,\tilde{\mathbf{b}} = -\,\rho_0^*\,HW\,\big[g_3^*\,\mathbf{e}_3\big] \ , \quad m\,\tilde{\mathbf{b}}^1 = -\frac{1}{2}\rho_0^*\,H^2W\,[\,g_1^*\,\mathbf{e}_1 + g_2^*\,\mathbf{e}_2\,] \ ,$$
$$m\,\tilde{\mathbf{b}}^2 = -\frac{1}{2}\rho_0^*\,HW^2\,[\,g_1^*\,\mathbf{e}_1 + g_2^*\,\mathbf{e}_2\,] \ . \tag{5.19.5}$$

Furthermore, since three of the six surfaces of the beam are stress free, and the other three are loaded with constant compressive stresses, the boundary conditions become

$$\tilde{\mathbf{t}}^3 = -\,\rho_0^*\,g_3^*\,HWL\,\mathbf{e}_1 \ , \quad \tilde{\mathbf{m}}^1 = 0 \ , \quad \tilde{\mathbf{m}}^2 = 0 \ , \quad \text{for } \theta^3 = -L/2 \ ,$$
$$\tilde{\mathbf{t}}^3 = 0 \ , \quad \tilde{\mathbf{m}}^1 = 0 \ , \quad \tilde{\mathbf{m}}^2 = 0 \ , \quad \text{for } \theta^3 = L/2 \ . \tag{5.19.6}$$

Next, the displacements (3.18.4) and the expressions (5.14.10) and (5.4.19) suggest that the Cosserat displacements are given by

$$u_1 = -\,\rho_0^*\,\big[\tfrac{1}{2}C_{1133}^*\,g_1^*\,(\theta^3)^2\big] + \big[\alpha_{13}^*\theta^3 + c_1\,\big] \ ,$$
$$u_2 = -\,\rho_0^*\,\big[\tfrac{1}{2}C_{2233}^*\,g_2^*\,(\theta^3)^2\big] + \big[\alpha_{23}^*\theta^3 + c_2\,\big] \ ,$$
$$u_3 = \rho_0^*\,\big[-\tfrac{H}{2}C_{1133}^*\,g_1^*\,\theta^3 - \tfrac{W}{2}C_{2233}^*\,g_2^*\,\theta^3 + \tfrac{1}{2}C_{3333}^*\,g_3^*\,(\theta^3 - \tfrac{L}{2})^2\big] + c_3 \ ,$$
$$\tilde{\delta}_{11} = \rho_0^*\,\big[-\tfrac{H}{2}C_{1111}^*\,g_1^* - \tfrac{W}{2}C_{1122}^*\,g_2^* + C_{1133}^*\,g_3^*\,(\theta^3 - \tfrac{L}{2})\big] \ ,$$
$$\tilde{\delta}_{21} = -\,\alpha_{12}^* \ , \quad \tilde{\delta}_{31} = \rho_0^*\,\big[C_{1133}^*\,g_1^*\,\theta^3\big] - \alpha_{13}^* \ ,$$

$$\tilde{\delta}_{12} = \alpha_{12}^* \ , \quad \tilde{\delta}_{22} = \rho_0^* \left[-\frac{H}{2} C_{1122}^* g_1^* - \frac{W}{2} C_{2222}^* g_2^* + C_{2233}^* g_3^* (\theta^3 - \frac{L}{2}) \right] \ ,$$

$$\tilde{\delta}_{32} = \rho_0^* \left[C_{2233}^* g_2^* \theta^3 \right] - \alpha_{23}^* \ ,$$

$$\tilde{\delta}_{13} = - \rho_0^* \left[C_{1133}^* g_1^* \theta^3 \right] + \alpha_{13}^* \ , \quad \tilde{\delta}_{23} = - \rho_0^* \left[C_{2233}^* g_2^* \theta^3 \right] + \alpha_{23}^* \ ,$$

$$\tilde{\delta}_{33} = \rho_0^* \left[-\frac{H}{2} C_{1133}^* g_1^* - \frac{W}{2} C_{2233}^* g_2^* + C_{3333}^* g_3^* (\theta^3 - \frac{L}{2}) \right] \ , \tag{5.19.7}$$

where the constants c_i are expressed in terms of the arbitrary constants c_i^* by

$$c_1 = \rho_0^* \left[\frac{H^2}{6} C_{1111}^* g_1^* - \frac{W^2}{24} C_{1111}^* g_1^* \right] + c_1^* \ ,$$

$$c_2 = \rho_0^* \left[\frac{W^2}{6} C_{2222}^* g_2^* - \frac{H^2}{24} C_{1122}^* g_2^* \right] + c_2^* \ ,$$

$$c_3 = - \rho_0^* \left[\frac{H^2}{24} C_{1133}^* g_3^* + \frac{W^2}{24} C_{2233}^* g_3^* \right] + c_3^* \ . \tag{5.19.8}$$

Now, with the help of (3.14.11), (5.14.11), (5.14.19), (5.14.20), (5.14.27) and (5.14.28), it follows that

$$E_{11} = \rho_0^* \left[-\frac{H}{2} C_{1111}^* g_1^* - \frac{W}{2} C_{1122}^* g_2^* + C_{1133}^* g_3^* (\theta^3 - \frac{L}{2}) \right] \ ,$$

$$E_{22} = \rho_0^* \left[-\frac{H}{2} C_{1122}^* g_1^* - \frac{W}{2} C_{2222}^* g_2^* + C_{2233}^* g_3^* (\theta^3 - \frac{L}{2}) \right] \ ,$$

$$E_{33} = \rho_0^* \left[-\frac{H}{2} C_{1133}^* g_1^* - \frac{W}{2} C_{2233}^* g_2^* + C_{3333}^* g_3^* (\theta^3 - \frac{L}{2}) \right] \ ,$$

$$E_{12} = E_{13} = E_{23} = 0 \ , \quad \beta_{11} = \rho_0^* \left[C_{1133}^* g_3^* \right] \ , \quad \beta_{21} = 0 \ , \quad \beta_{31} = \rho_0^* \left[C_{1133}^* g_1^* \right] \ ,$$

$$\beta_{12} = 0 \ , \quad \beta_{22} = \rho_0^* \left[C_{2233}^* g_3^* \right] \ , \quad \beta_{32} = \rho_0^* \left[C_{2233}^* g_2^* \right] \ ,$$

$$\tilde{\mathbf{t}}^1 = -\frac{1}{2} \rho_0^* H^2 W g_1^* \mathbf{e}_1 \ , \quad \tilde{\mathbf{t}}^2 = -\frac{1}{2} \rho_0^* H W^2 g_2^* \mathbf{e}_2 \ , \quad \tilde{\mathbf{t}}^3 = \rho_0^* H W g_3^* (\theta^3 - \frac{L}{2}) \mathbf{e}_3 \ ,$$

$$\tilde{\mathbf{m}}^1 = \frac{H^3 W}{12} \left[\frac{C_{1133}^*}{C_{3333}^*} \right] \rho_0^* g_1^* \mathbf{e}_3 \ , \quad \tilde{\mathbf{m}}^2 = \frac{H W^3}{12} \left[\frac{C_{2233}^*}{C_{3333}^*} \right] \rho_0^* g_2^* \mathbf{e}_3 \ . \tag{5.19.9}$$

It can easily be seen that these results satisfy the linearized equilibrium equations (5.14.12), and that the boundary conditions (5.19.6) on $\tilde{\mathbf{t}}^3$ are satisfied, but not those on $\tilde{\mathbf{m}}^\alpha$. Moreover, it can be seen that the values (5.19.9) for $\tilde{\mathbf{m}}^\alpha$ are constants that are associated only with the solutions for g_1^* and g_2^*. This means that the Cosserat solution for gravity acting in the negative $\mathbf{e}_3$ direction (associated with g_3^*) is consistent with the three-dimensional solution. In this regard, since the thickness and width of the cross-section of the beam do not remain constant for this solution, and the associated $\tilde{\mathbf{m}}^\alpha$ still vanish, these solutions again justify the validity of the constitutive results (5.14.28).

On the other hand, when gravity acts in the negative $\mathbf{e}_1$ and $\mathbf{e}_2$ directions (g_1^* and g_2^* are nonzero), then the kinematics (5.19.7) that are consistent with the three-dimensional solutions are not compatible with the boundary conditions on the director couples $\tilde{\mathbf{m}}^\alpha$. Of course, since these couples are constants it is possible to superimpose solutions for pure bending like those discussed in section 5.14 to satisfy the equations of equilibrium and the boundary conditions. However, the resulting predictions of the unit normals to the lateral surfaces ($\theta^1=\pm H/2$ and $\theta^2=\pm W/2$) and the curvature of the reference curve of the beam will not be compatible with the three-dimensional solution. This indicates a limitation of the Cosserat beam theory which is due to the fact that the kinematics of the Cosserat theory are not compatible with the exact quadratic variation of the displacements through the cross-section of the beam when gravity acts through the cross-section. It is interesting to contrast this result with that for pure bending (section 5.14) which also corresponds to a quadratic variation of the displacements through the cross-section, but which can be modeled by specification of the constitutive coefficients (5.14.27).

5.20 Elastic rods

The constitutive restrictions (5.11.9) developed in section 5.11 are valid for elastic rods that have general reference geometry. However, the rod is presumed to be constructed using a homogeneous material. Therefore, in this section and in the remainder of this chapter, attention will be limited to such rods for which the three-dimensional mass density ρ_0^* and the strain energy function Σ^* are explicitly independent of position in the reference configuration

$$\rho_0^* = \text{constant} \; , \quad \Sigma^* = \hat{\Sigma}^*(\mathbf{C}^*) \; . \tag{5.20.1}$$

The restrictions (5.11.9) ensure that solutions of the Cosserat theory can reproduce exactly the complete class of nonlinear homogeneous solutions of the three-dimensional theory. Consequently, they provide important information about the influence of the geometry of the rod on the strain energy function which describes its response. However, they do not provide information about inhomogeneous deformations like bending and torsion. Nevertheless, these restrictions were used to propose a relatively simple form (5.11.27) for the strain energy function for rods Σ in terms of the three-dimensional strain energy function Σ^*

$$\mathrm{m}\,\Sigma = \mathrm{m}\,\Sigma^*(\bar{\mathbf{C}}) + \frac{1}{2}\mathrm{D}_{33}^{-1/2}\,\mathrm{A}\,\mathbf{K}^{\alpha\beta} \bullet (\boldsymbol{\beta}_\alpha \otimes \boldsymbol{\beta}_\beta) \; . \tag{5.20.2}$$

The derivatives of this function and the associated constitutive equations have been recorded in (5.11.33) and (5.11.34).

Since the reference curve ($\theta^\alpha=0$) of the rod can always be specified to be the centroid of the cross section, the functions A^α vanish (5.11.35) and the strain energy function (5.20.2) simplifies by setting $\bar{\mathbf{C}} = \mathbf{C}$ to give

$$\mathrm{m}\,\Sigma = \mathrm{m}\,\Sigma^*(\mathbf{C}) + \frac{1}{2}\mathrm{D}_{33}^{-1/2}\,\mathrm{A}\,\mathbf{K}^{\alpha\beta} \bullet (\boldsymbol{\beta}_\alpha \otimes \boldsymbol{\beta}_\beta) \; . \tag{5.20.3}$$

Moreover, the constitutive equations reduce to the forms (5.11.36).

For the simpler case when Σ^* is a quadratic function of three-dimensional strain, the expression for Σ becomes

$$\mathrm{m}\,\Sigma = \frac{1}{2}\mathrm{D}_{33}^{1/2}\,\mathrm{A}\,\mathbf{K}^* \bullet (\mathbf{E} \otimes \mathbf{E}) + \frac{1}{2}\mathrm{D}_{33}^{-1/2}\,\mathrm{A}\,\mathbf{K}^{\alpha\beta} \bullet (\boldsymbol{\beta}_\alpha \otimes \boldsymbol{\beta}_\beta) \; , \tag{5.20.4}$$

with the associated constitutive equations being given by (5.12.4). These equations have the properties that the stiffness $\mathbf{K}^*$ for the linearized three-dimensional material can be used directly, and that they satisfy the restrictions (5.11.9) exactly. However, the resulting constitutive equations for the rod remain nonlinear functions of the strains $\mathbf{E}$ and $\boldsymbol{\beta}_\alpha$ associated with the rod.

Sections 5.14 and 5.15 focused attention on determining values for the constitutive tensors $\mathbf{K}^{\alpha\beta}$ associated with a straight beam with rectangular cross-section with constant thickness H and width W. Consequently, the part of the strain energy function corresponding to inhomogeneous deformation ($\boldsymbol{\beta}_\alpha \neq 0$) represents a generalization for curved rods. In subsequent sections it will be shown that this generalization seems to work rather well for modeling a circular ring and a circular plate with constant thickness. Consequently, these generalized constitutive equations are assumed to be valid for curved rods with variable cross-section.

Within the context of the Cosserat theory it is necessary to specify constitutive equations for the inertia quantities m, y^α and $y^{\alpha\beta}$ which depend on the shape of the cross-section of the rod. In order to develop explicit expressions for these quantities, it is convenient to confine attention to a rod which has a rectangular cross-section with height H and width W in its reference configuration. Moreover, it is assumed that the directors $\mathbf{D}_\alpha$ are oriented so that this cross-section is defined by

$$|\theta^1| \leq \frac{H}{2} \;, \quad |\theta^2| \leq \frac{W}{2} \;. \tag{5.20.5}$$

Next, using the expressions (5.1.13) and (5.1.15) for $G^{1/2}$

$$G^{1/2} = D^{1/2} \left[1 + \theta^\alpha \mathbf{D}_{\alpha,3} \bullet \mathbf{D}^3 \right] \;, \tag{5.20.6}$$

it can be seen that the specification (5.20.5) is consistent with $\theta^\alpha = 0$ being the centroid of the cross-section (5.11.35). Also, (5.11.7) yields

$$D_{33}^{1/2} \, A = D^{1/2} \, HW \;. \tag{5.20.7}$$

In this regard, it should be noted that the cross-section need not be uniform since both H and W can be functions of θ^3.

Now, the form (5.11.7) for m

$$m = \rho_0^* \, D^{1/2} \, HW \;, \tag{5.20.8}$$

is retained for rods because it ensures that the Cosserat model of the rod has the same mass as the actual rod. Next, using the conservation of mass (3.2.28) and the fact that ρ_0^* is constant, the expressions (5.3.10) and (5.3.26) become

$$m \, y^\alpha = \rho_0^* \int_{-W/2}^{W/2} \int_{-H/2}^{H/2} \theta^\alpha \, G^{1/2} \, d\theta^1 d\theta^2 \;,$$

$$m \, y^{\alpha\beta} = \rho_0^* \int_{-W/2}^{W/2} \int_{-H/2}^{H/2} \theta^\alpha \theta^\beta \, G^{1/2} \, d\theta^1 d\theta^2 \;. \tag{5.20.9}$$

Thus, with the help of (5.20.6), direct integration yields the Galerkin values

$$y^1 = \left[\frac{H^2}{12} (\mathbf{D}_{1,3} \bullet \mathbf{D}^3)\right] \;, \quad y^2 = \left[\frac{W^2}{12} (\mathbf{D}_{2,3} \bullet \mathbf{D}^3)\right] \;,$$

$$y^{11} = \left[\frac{H^2}{12}\right] \;, \quad y^{12} = y^{21} = 0 \;, \quad y^{22} = \left[\frac{W^2}{12}\right] \;, \tag{5.20.10}$$

However, it was seen in sections 5.17 and 5.18 that the Galerkin values (5.13.3) for y^{11} and y^{22} that are consistent with (5.20.12), are not the best values to represent vibrations of a beam. Consequently, for the Cosserat theory of rods it is assumed that m is given by the form (5.20.8) with use of (5.20.7), but that the director inertia coefficients y^α and $y^{\alpha\beta}$ are given by

$$y^1 = \frac{H^2}{|\mathbf{D}_1|^2}\left[\gamma_1 (\mathbf{D}_{1,3} \bullet \mathbf{D}^3)\right] \ , \ y^2 = \frac{W^2}{|\mathbf{D}_2|^2}\left[\gamma_1 (\mathbf{D}_{2,3} \bullet \mathbf{D}^3)\right] \ , \ y^{12} = y^{21} = 0 \ ,$$

$$y^{11} = \frac{H^2}{\pi^2|\mathbf{D}_1|^2}\left[1 + \gamma_2 \frac{H}{|\mathbf{D}_1|}(\mathbf{D}_{1,3} \bullet \mathbf{D}^3)\right] ,$$

$$y^{22} = \frac{W^2}{\pi^2|\mathbf{D}_2|^2}\left[1 + \gamma_2 \frac{W}{|\mathbf{D}_2|}(\mathbf{D}_{2,3} \bullet \mathbf{D}^3)\right] , \tag{5.20.11}$$

where γ_1 and γ_2 are material constants that need to be determined, and the terms $|\mathbf{D}_1|$ and $|\mathbf{D}_2|$ have been introduced to make the definitions insensitive to the units of θ^α. Specifically, it will be shown in the following sections that these material constants can be determined by considering vibrations of a circular ring. Moreover, the forms for these expressions were restricted so that they reproduce the values (5.17.2) and (5.17.11) for a beam when $\mathbf{D}_\alpha$ are constant vectors.

As mentioned in section 5.19, it is also necessary to specify values for the assigned fields $\mathbf{b}$ and $\mathbf{b}^\alpha$. The expressions (5.3.16) and (5.3.32) for the parts $\mathbf{b}_c$ and $\mathbf{b}^\alpha_c$ of these assigned fields that are due to loading on the lateral surface of the rod, are retained for rods because they were used in developing the restrictions (5.11.9) for rods of arbitrary shape. In order to determine values of the parts $\mathbf{b}_b$ and $\mathbf{b}^\alpha_b$ of the assigned fields associated with body force, it will be assumed that the three-dimensional specific body force $\mathbf{b}^*$ is constant. Thus, with the help of the conservation of mass $(3.2.35)_1$ and (5.3.7), the expressions (5.3.12) and (5.3.28) yield

$$\mathbf{b}_b = \mathbf{b}^* \ , \ \mathbf{b}^1_b = \mathbf{b}^* \left[\frac{H^2}{12}(\mathbf{D}_{1,3} \bullet \mathbf{D}^3)\right] \ , \ \mathbf{b}^2_b = \mathbf{b}^* \left[\frac{W^2}{12}(\mathbf{D}_{2,3} \bullet \mathbf{D}^3)\right] \ . \tag{5.20.12}$$

The validity of these expressions for beams has already been demonstrated in section 5.19. Later, in section 5.23 it will be shown that they produce reasonable results for a specific problem of a curved rod.

5.21 Plane strain expansion of an isotropic circular cylindrical shell

The constitutive equations discussed in section 5.20 generalize equations for beams to curved rods. As a first test of the validity of these equations in the presence of curvature, it is reasonable to consider the linearized theory for static plane strain expansion of an isotropic circular cylindrical shell of constant height and thickness. To this end, the rod is presumed to have internal radius R_1, external radius R_2, and height H in its reference configuration. Also, the inner surface of the rod is loaded by the pressure p_1, and the outer surface is loaded by the pressure p_2.

For the Cosserat model, the reference curve of the rod in its reference configuration is defined by the position vector $\mathbf{X}$ in terms of cylindrical polar base vectors $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ (see appendix B.1) by

$$\mathbf{X} = R\,\mathbf{e}_r(\theta) \ , \quad \theta^3 = \theta \ , \quad |\theta^3| \le \pi \ , \tag{5.21.1}$$

and the radius R of the centroid of the cross-section and the thickness W of the cylindrical shell are given by

$$R = \frac{1}{2}(R_1 + R_2) \ , \quad W = R_2 - R_1 \ . \tag{5.21.2}$$

Next, using the definition (5.1.2) the directors are specified by

$$\mathbf{D}_1 = \mathbf{e}_z \ , \quad \mathbf{D}^1 = \mathbf{e}_z \ , \quad \mathbf{D}_2 = \mathbf{e}_r \ , \quad \mathbf{D}^2 = \mathbf{e}_r \ ,$$
$$\mathbf{D}_3 = R\,\mathbf{e}_\theta \ , \quad \mathbf{D}^3 = \frac{1}{R}\mathbf{e}_\theta \ , \quad D^{1/2} = D_{33}^{1/2} = R \ , \tag{5.21.3}$$

so the expressions (5.20.7), (5.20.8), (5.20.11) and (5.20.12) yield

$$A = HW \ , \quad m = \rho_0^* R\,HW \ , \quad y^1 = 0 \ , \quad y^2 = \gamma_1 \left[\frac{W^2}{R}\right] \ , \quad y^{12} = y^{21} = 0 \ ,$$
$$y^{11} = \frac{H^2}{\pi^2} \ , \quad y^{22} = \frac{W^2}{\pi^2}\left[1 + \gamma_2 \frac{W}{R}\right] \ ,$$
$$\mathbf{b}_b = \mathbf{b}^* \ , \quad \mathbf{b}_b^1 = 0 \ , \quad \mathbf{b}_b^2 = \left[\frac{W^2}{12R}\right]\mathbf{b}^* \ . \tag{5.21.4}$$

Also, with the help of (5.1.4) and the linearized forms of (5.3.14), (5.3.16) and (5.3.32), it can be shown that

$$\mathbf{X}^* = (R + \theta^2)\,\mathbf{e}_r + \theta^1\,\mathbf{e}_z \ , \quad G^{1/2} = (R + \theta^2) \ , \quad \mathbf{G}^3 = \mathbf{e}_r \ ,$$
$$\mathbf{n}^* = \mathbf{e}_r \ , \quad G^{1/2}{=}R_2 \ , \quad \alpha{=}1 \ , \quad \mathbf{t}^* = -\,p_2\,\mathbf{e}_r \ \text{ for } \zeta{=}{-}\theta^1 \ , \quad \theta^2 = \frac{W}{2} \ ,$$
$$\mathbf{n}^* = -\,\mathbf{e}_z \ , \quad G^{1/2}{=}R{-}\zeta \ , \quad \alpha = 1 \ , \quad \mathbf{t}^* = -\,\sigma\,\mathbf{e}_z \ \text{ for } \zeta{=}{-}\theta^2 \ , \quad \theta^1 = -\frac{H}{2} \ ,$$
$$\mathbf{n}^* = -\,\mathbf{e}_r \ , \quad G^{1/2}{=}R_1 \ , \quad \alpha = 1 \ , \quad \mathbf{t}^* = p_1\,\mathbf{e}_r \ \text{ for } \zeta{=}\theta^1 \ , \quad \theta^2 = -\frac{W}{2} \ ,$$
$$\mathbf{n}^* = \mathbf{e}_z \ , \quad G^{1/2}{=}R{+}\zeta \ , \quad \alpha = 1 \ , \quad \mathbf{t}^* = \sigma\,\mathbf{e}_z \ \text{ for } \zeta{=}\theta^2 \ , \quad \theta^1 = \frac{H}{2} \ , \tag{5.21.5}$$

where σ is the stress required to ensure plane strain, which is independent of θ^1 and θ^3 but may depend on θ^2. Also, it follows that the parts $\tilde{\mathbf{b}}_c$ and $\tilde{\mathbf{b}}_c^\alpha$ of the assigned fields due to surface tractions can be written as

$$m\,\tilde{\mathbf{b}}_c = H\,(R_1 p_1 - R_2 p_2)\,\mathbf{e}_r \ , \quad m\,\tilde{\mathbf{b}}_c^1 = H\left[\int_{-W/2}^{W/2} \sigma\,(R{+}\theta^2)\,d\theta^2\right]\mathbf{e}_z \ ,$$
$$m\,\tilde{\mathbf{b}}_c^2 = -\frac{HW}{2}(R_1 p_1 + R_2 p_2)\,\mathbf{e}_r \ . \tag{5.21.6}$$

Thus, using (5.21.4) the total assigned fields for the linearized theory become

$$m\,\tilde{\mathbf{b}} = m\,\mathbf{b}^* + H\,(R_1 p_1 - R_2 p_2)\,\mathbf{e}_r \ , \quad m\,\tilde{\mathbf{b}}^1 = H\left[\int_{-W/2}^{W/2} \sigma\,(R{+}\theta^2)\,d\theta^2\right]\mathbf{e}_z ,$$
$$m\,\tilde{\mathbf{b}}^2 = m\left[\frac{W^2}{12R}\right]\mathbf{b}^* - \frac{HW}{2}(R_1 p_1 + R_2 p_2)\,\mathbf{e}_r \ . \tag{5.21.7}$$

Now, for the axisymmetric plane strain deformation of interest, the displacements can be written in the forms

$$\mathbf{u} = u\,\mathbf{e}_r \ , \quad \boldsymbol{\delta}_1 = 0 \ , \quad \boldsymbol{\delta}_2 = \delta\,\mathbf{e}_r \ , \quad \boldsymbol{\delta}_3 = u\,\mathbf{e}_\theta \ , \tag{5.21.8}$$

where u and δ are functions of time only to be determined. Moreover, (5.13.16) can be used to determine the strains associated with these displacements

$$\tilde{\mathbf{E}} = \delta\,\mathbf{e}_r\otimes\mathbf{e}_r + \frac{u}{R}\mathbf{e}_\theta\otimes\mathbf{e}_\theta \ , \quad \tilde{\boldsymbol{\beta}}_1 = 0 \ , \quad \tilde{\boldsymbol{\beta}}_2 = (\delta - \frac{u}{R})\,\mathbf{e}_\theta \ . \tag{5.21.9}$$

Next, using (3.12.14), Table 3.12.1, (5.16.10), (5.16.11), (5.16.13), (5.21.3), (5.21.4) and (5.21.8), the vectors $\mathbf{M}_i$ become

$$\mathbf{M}_1 = \mathbf{e}_z \ , \quad \mathbf{M}_2 = \mathbf{e}_r \ , \quad \mathbf{M}_3 = \mathbf{e}_\theta \ , \tag{5.21.10}$$

and the constitutive equations (5.13.16) yield

$$\tilde{\mathbf{m}}^1 = 0 \ , \quad \tilde{\mathbf{m}}^2 = \frac{\mu^* H W^3(1+v^*)}{6R}\{\delta - \frac{u}{R}\}\mathbf{e}_\theta \ ,$$

$$\tilde{\mathbf{t}}^1 = \frac{2\mu^* RHW}{(1-2v^*)}[\, v^*\{\frac{u}{R} + \delta\}\,]\,\mathbf{e}_z \ , \quad \tilde{\mathbf{t}}^2 = \frac{2\mu^* RHW}{(1-2v^*)}[\,(1-v^*)\,\delta + v^*\frac{u}{R}\,]\,\mathbf{e}_r \ ,$$

$$\tilde{\mathbf{t}}^3 = \frac{2\mu^* HW}{(1-2v^*)}[(1-v^*)\frac{u}{R} + v^*\,\delta - \frac{(1+v^*)(1-2v^*)W^2}{12R^2}\{\delta - \frac{u}{R}\}]\,\mathbf{e}_\theta \ . \tag{5.21.11}$$

Thus, in the absence of body force ($\mathbf{b}^*$=0) the equations of motion (5.13.15) reduce to

$$(\rho_0^* RHW)\,(\ddot{u} + y^2\,\ddot{\delta}) = H\,(R_1p_1 - R_2p_2) - \frac{2\mu^* HW}{(1-2v^*)}[(1-v^*)\frac{u}{R} + v^*\,\delta$$

$$- \frac{(1+v^*)(1-2v^*)W^2}{12R^2}\{\delta - \frac{u}{R}\}] \ ,$$

$$0 = H\,[\int_{-W/2}^{W/2}\sigma\,(R+\theta^2)\,d\theta^2] - \frac{2\mu^* RHW}{1-2v^*}[\, v^*\{\frac{u}{R} + \delta\}\,] \ ,$$

$$\rho_0^* RHW\,(y^2\,\ddot{u} + y^{22}\,\ddot{\delta}) = -\frac{HW}{2}(R_1p_1 + R_2p_2) - \frac{2\mu^* RHW}{(1-2v^*)}[\, v^*\frac{u}{R} + (1-v^*)\,\delta$$

$$+ \frac{(1+v^*)(1-2v^*)W^2}{12R^2}\{\delta - \frac{u}{R}\}\,] \ , \tag{5.21.12}$$

where the second equation merely places a restriction on the stress σ. The remaining equations can be written in alternative forms by multiplying $(5.21.12)_1$ by (R/H), dividing $(5.21.12)_3$ by H, then adding and subtracting the results to obtain

$$\rho_0^* RW\,(y^2\,\ddot{u} + y^{22}\,\ddot{\delta}) + \rho_0^* R^2W\,(\ddot{u} + y^2\,\ddot{\delta}) = (R_1^2p_1 - R_2^2p_2) - \frac{2\mu^* RW}{(1-2v^*)}\{\frac{u}{R} + \delta\} \ ,$$

$$\rho_0^* RW\,(y^2\,\ddot{u} + y^{22}\,\ddot{\delta}) - \rho_0^* R^2W\,(\ddot{u} + y^2\,\ddot{\delta}) = R_1R_2(p_2-p_1)$$

$$- 2\mu^* RW\,[1 + \frac{(1+v^*)W^2}{6R^2}]\{\delta - \frac{u}{R}\} \ . \tag{5.21.13}$$

In particular, for static deformation it can be seen from $(5.20.13)_2$ that when the pressures are equal (p_1=p_2), then the deformation is homogeneous (since $\boldsymbol{\beta}_\alpha$=0) with δ=u/R .

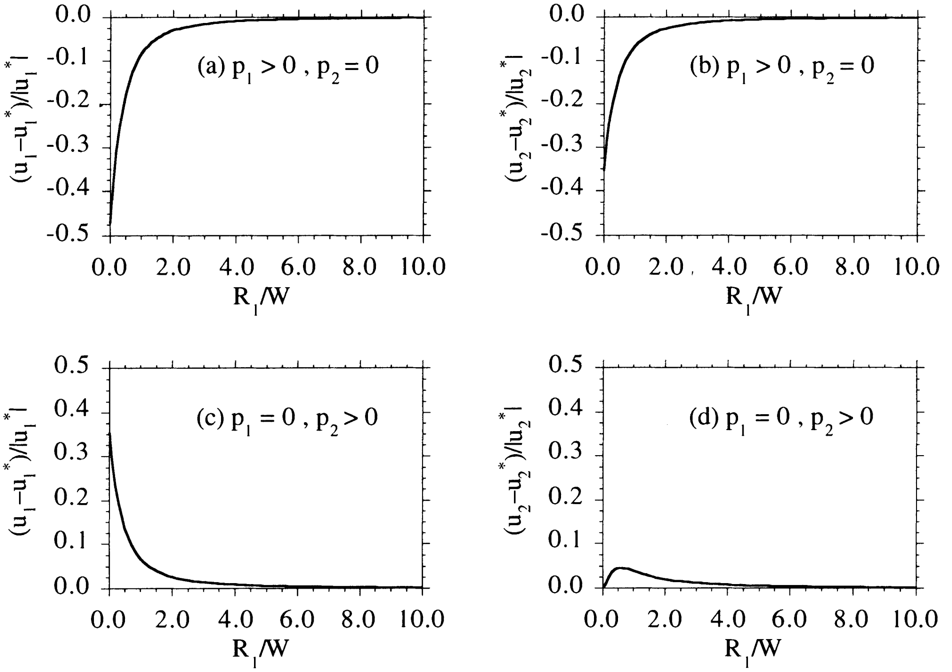

Fig. 5.21.1 Plane strain expansion of a circular cylindrical shell. Errors of the Cosserat solution (5.21.15) relative to the exact solution (5.21.16).

In order to compare with the exact solution, it is convenient to use the kinematic assumption (5.14.7) to express u and δ in terms of the radial displacements u_1 of the inner surface and u_2 of the outer surface, by the expressions

$$u = \frac{u_1+u_2}{2} \ , \quad \delta = \frac{u_2 - u_1}{W} \ ,$$

$$\frac{u}{R} + \delta = \frac{R_2 u_2 - R_1 u_1}{RW} \ , \quad \delta - \frac{u}{R} = \frac{R_1 u_2 - R_2 u_1}{RW} \ . \tag{5.21.14}$$

Then, in the absence of inertia the equations (5.21.13) can be solved for u_1 and u_2 to deduce that

$$\frac{u_1}{W} = [C_1 + \frac{C_2}{C_3}]\frac{R_1}{W} \ , \quad \frac{u_2}{W} = [C_1 + \frac{C_2}{C_3}\{\frac{R_1^2}{R_2^2}\}]\frac{R_2}{W} \ ,$$

$$C_1 = \frac{(R_1^2 p_1 - R_2^2 p_2)(1-2\nu^*)}{2\mu^*(R_2^2 - R_1^2)} \ , \quad C_2 = \frac{R_2^2 (p_1 - p_2)}{2\mu^*(R_2^2 - R_1^2)} \ , \quad C_3 = 1 + \frac{(1+\nu^*)W^2}{6R^2} \ . \tag{5.21.15}$$

It can easily be shown using the stresses given by Timoshenko and Goodier (1951, sec. 26), that the exact displacements u_1^* and u_2^* associated with the plane strain solution can be written in the forms

$$\frac{u_1^*}{W} = [C_1 + C_2]\frac{R_1}{W} \ , \quad \frac{u_2^*}{W} = [C_1 + C_2\{\frac{R_1^2}{R_2^2}\}]\frac{R_2}{W} \ . \tag{5.21.16}$$

Thus, the only difference between the Cosserat solution (5.21.15) and the exact solution (5.21.16) is the presence of C_3. This difference vanishes when the deformation is homogeneous with $C_2=0$ [which occurs for a ring ($R_1>0$) when the pressures are equal ($p_1=p_2$), or occurs when the ring becomes a solid plate ($R_1=0$)]. This difference also vanishes when the ring becomes thin with W/R approaching zero.

To quantitatively examine the Cosserat solution, it is convenient to display the results of two inhomogeneous problems, one with internal pressure only ($p_1>0$, $p_2=0$), and the other with external pressure only ($p_1=0$, $p_2>0$). Also, the value of v^* is taken to be 1/3. Figure 5.21.1 shows that the relative errors of the displacements become quite significant as the thin ring (large values of R_1/W) becomes a solid plate ($R_1/W=0$). However, the prediction of the outer displacement remains accurate for the case of external pressure.

5.22 Plane strain free vibrations of an isotropic solid circular cylinder

It will be shown presently that values of the quantities γ_1 and γ_2 in (5.20.11) can be determined by considering plane strain free vibration of an isotropic solid circular cylinder. Even though this is an extreme case where the rod is as thick as possible, it will be seen that the Cosserat theory can predict accurate results. To this end, it is first noted that for a solid cylinder, it follows from (5.21.2), (5.21.4) and (5.21.14) that

$$R_1 = 0 \ , \quad R = \frac{W}{2} \ , \quad W = R_2 \ , \quad m = \frac{1}{2}\rho_0^* W^2 \ ,$$

$$y^1 = 0 \ , \quad y^2 = 2W\gamma_1 \ , \quad y^{12} = y^{21} = 0 \ , \quad y^{11} = \frac{H^2}{\pi^2} \ , \quad y^{22} = \frac{W^2}{\pi^2}\,[1 + 2\gamma_2\,] \ ,$$

$$u_1 = 0 \ , \quad u = \frac{u_2}{2} \ , \quad \delta = \frac{u}{R} = \frac{u_2}{W} \ . \tag{5.22.1}$$

Then, equations (5.21.13) reduce to

$$\frac{1}{2}\rho_0^* W^3\,[\,2\gamma_1 + \frac{1}{\pi^2}\{1 + 2\gamma_2\} + \frac{1}{4}\,]\,\ddot{u}_2 = -\,W^2 p_2 - [\frac{2\mu^* W}{(1-2v^*)}\,]\,u_2 \ ,$$
$$\frac{1}{2}\rho_0^* W^3\,[\frac{1}{\pi^2}\{1 + 2\gamma_2\} - \frac{1}{4}\,]\,\ddot{u}_2 = 0 \ . \tag{5.22.2}$$

These equations will have a nontrivial solution only if the coefficient in the second equation vanishes, with γ_2 being given by

$$\gamma_2 = \frac{\pi^2 - 4}{8} \ , \tag{5.22.3}$$

so that $(5.22.2)_1$ reduces to

$$\frac{1}{4}\rho_0^* W^3\,[4\gamma_1 + 1]\,\ddot{u}_2 = -\,W^2 p_2 - [\frac{2\mu^* W}{(1-2v^*)}]\,u_2 \ . \tag{5.22.4}$$

Now, for free vibrations the pressure p_2 on the outer surface vanishes

$$p_2 = 0 \ , \tag{5.22.5}$$

and the natural frequency ω can be written in the form

$$\omega = \left[\frac{2\mu^*}{\rho_0^*}\right]^{1/2} \frac{\Omega}{W} \ , \quad \Omega = \left[\frac{4}{(1-2\nu^*)(4\gamma_1+1)}\right]^{1/2} \tag{5.22.6}$$

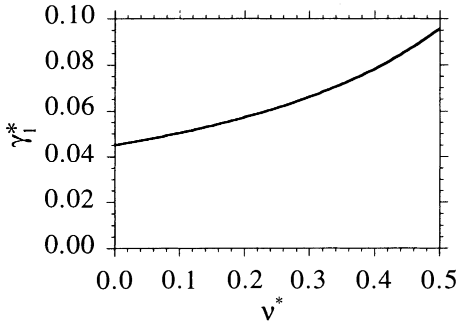

Fig. 5.22.1 Values of γ_1^* which match the exact first frequency of vibration.

The exact solution for this vibration was recorded in section 3.20. Comparison of the Cosserat and exact solutions indicates that these solutions will be identical if the parameter γ_1 is given by the value γ_1^* which equates $(5.22.6)_2$ and $(3.20.5)_2$

$$\gamma_1^* = \frac{1}{(1-\nu^*)\beta^2} - \frac{1}{4} \ . \tag{5.22.7}$$

Values of this quantity associated with the first frequency of vibration [i.e. the first root of (3.20.6)] are plotted in Fig. 5.22.1 for a range of Poisson's ratio ν^*.

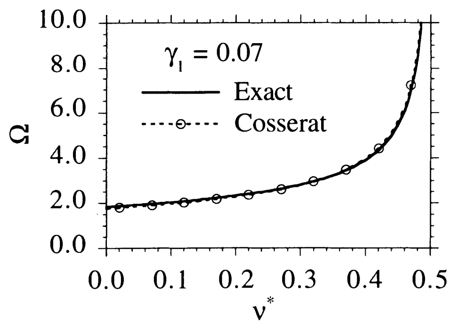

Fig. 5.22.2 Normalized frequencies Ω^* (Exact) and Ω (Cosserat) for $\gamma_1 = 0.07$.

Although this value γ_1^* of γ_1 varies somewhat, the influence of γ_1 on the natural frequency is not too great so it is possible to take γ_1 to be equal to the constant value

$$\gamma_1 = 0.07 \ , \tag{5.22.8}$$

which is near the value of γ_1^* for $\nu^* = 0.33$. To examine the accuracy of this specification, the normalized frequency Ω in $(5.22.6)_2$ for the Cosserat theory and the exact value Ω^* in $(3.20.5)_2$ are plotted in Fig. 5.22.2. Since the predictions of the Cosserat theory are so close to the exact results for this wide range of Poisson's ratio, the value (5.22.8) is considered to be quite reasonable. However, this value is not unique since the frequency is rather insensitive to small changes in γ_1.

Before closing this section, it is important to recall that the director inertia coefficients y^α and $y^{\alpha\beta}$ are required to satisfy the restrictions (5.3.47) which ensure that the kinetic energy is semi-definite. In particular, the expressions (5.21.4) for a general circular solid cylinder can be used to deduce that

$$I_1 = \frac{H^2}{\pi^2} + W^2 \left[\frac{1}{\pi^2} \{ 1+2\gamma_2 \frac{W}{R} \} - \gamma_1^2 \frac{W^2}{R^2} \right] ,$$

$$I_2 = \frac{H^2W^2}{\pi^2} \left[\frac{1}{\pi^2} \{ 1+2\gamma_2 \frac{W}{R} \} - \gamma_1^2 \frac{W^2}{R^2} \right] . \tag{5.22.9}$$

Next, using (5.22.3) for γ_2 and (5.22.8) for γ_1, it can be shown that both I_1 and I_2 remain positive and satisfy the restrictions (5.3.47) for the full range of W/R (≤ 2).

5.23 An isotropic circular cylindrical shell loaded by its own weight

Section 5.20 presented various expressions which attempt to generalize results for beams to curved rods. The objective of the present section is to examine the validity of the generalization (5.20.12) of the assigned director couples due to body force applied to a rod. Specifically, the problem of an isotropic circular cylindrical shell subjected to gravity acting in the axial direction, is considered. In its reference configuration, the shell has the same geometry as the one discussed in section 5.21 so the results (5.21.1)-(5.21.4) remain valid with the three-dimensional specific body force $\mathbf{b}^*$ being specified in terms of the constant force of gravity per unit mass g^* by

$$\mathbf{b}^* = -\, g^* \, \mathbf{e}_z \ . \tag{5.23.1}$$

Moreover, the lateral surface of the rod is assumed to be stress free, except for the surface θ^1=–H/2 which has a constant stress $\rho_0^* \, g^* H$ applied in the axial direction $\mathbf{e}_z$ to support the weight of the cylindrical shell. Thus, equations (5.21.5) reduce to

$$\mathbf{X}^* = (R + \theta^2)\, \mathbf{e}_r + \theta^1 \, \mathbf{e}_z \ , \quad G^{1/2} = (R + \theta^2) \ , \quad \mathbf{G}^3 = \mathbf{e}_r \ ,$$

$$\mathbf{n}^* = \mathbf{e}_r \ , \quad G^{1/2}{=}R_2 \, , \quad \alpha{=}1, \quad \mathbf{t}^* = 0 \quad \text{for } \zeta{=}{-}\theta^1 \ , \quad \theta^2 = \frac{W}{2} \, ,$$

$$\mathbf{n}^* = -\, \mathbf{e}_z \ , \quad G^{1/2}{=}R{+}\theta^2 \, , \quad \alpha = 1 \ , \quad \mathbf{t}^* = \rho_0^* \, g^* H \, \mathbf{e}_z \quad \text{for } \zeta{=}{-}\theta^2 \ , \quad \theta^1 = -\frac{H}{2} \, ,$$

$$\mathbf{n}^* = -\, \mathbf{e}_r \ , \quad G^{1/2}{=}R_1 \, , \quad \alpha = 1 \ , \ \mathbf{t}^* = 0 \quad \text{for } \zeta{=}\theta^1 \ , \quad \theta^2 = -\frac{W}{2} \, ,$$

$$\mathbf{n}^* = \mathbf{e}_z \ , \quad G^{1/2}{=}R{+}\zeta \, , \quad \alpha = 1, \quad \mathbf{t}^* = 0 \quad \text{for } \zeta{=}\theta^2 \ , \quad \theta^1 = \frac{H}{2} \ . \tag{5.23.2}$$

Also, it follows from (5.3.12) and (5.3.28) that the parts $\tilde{\mathbf{b}}_c$ and $\tilde{\mathbf{b}}_c^\alpha$ of the assigned fields due to surface tractions can be written as

$$m \, \tilde{\mathbf{b}}_c = [\rho_0^* g^* RHW] \, \mathbf{e}_z \ , \quad m \, \tilde{\mathbf{b}}_c^1 = -\, [\tfrac{1}{2} \rho_0^* g^* RH^2 W] \, \mathbf{e}_z \ ,$$

$$m\,\tilde{\mathbf{b}}_c^2 = [\frac{\rho_0^* g^* H W^3}{12}]\,\mathbf{e}_z \ , \tag{5.23.3}$$

so that with the help of (5.21.4) and (5.23.3) the assigned director couples become

$$m\,\tilde{\mathbf{b}} = 0 \ , \quad m\,\tilde{\mathbf{b}}^1 = -\,[\,\frac{1}{2}\rho_0^* g^* R H^2 W]\,\mathbf{e}_z \ , \quad m\,\tilde{\mathbf{b}}^2 = 0 \ . \tag{5.23.4}$$

Since the deformed rod remains axisymmetric, the displacement fields are given by

$$\mathbf{u} = u_r\,\mathbf{e}_r + u_z\,\mathbf{e}_z \ , \quad \boldsymbol{\delta}_1 = \delta_{1r}\,\mathbf{e}_r + \delta_{1z}\,\mathbf{e}_z \ , \quad \boldsymbol{\delta}_2 = \delta_{2r}\,\mathbf{e}_r + \delta_{2z}\,\mathbf{e}_z \ , \quad \boldsymbol{\delta}_3 = u_r\,\mathbf{e}_\theta \ , \tag{5.23.5}$$

and (5.13.16) can be used to determine the associated strains

$$\tilde{\mathbf{E}} = [\delta_{2r}](\mathbf{e}_r\otimes\mathbf{e}_r) + [\frac{u_r}{R}]\,(\mathbf{e}_\theta\otimes\mathbf{e}_\theta) + [\delta_{1z}](\mathbf{e}_z\otimes\mathbf{e}_z) + [\frac{1}{2}(\delta_{1r} + \delta_{2z})]\,(\mathbf{e}_r\otimes\mathbf{e}_z + \mathbf{e}_z\otimes\mathbf{e}_r) \ ,$$

$$\tilde{\boldsymbol{\beta}}_1 = [\delta_{1r}]\,\mathbf{e}_\theta \ , \quad \tilde{\boldsymbol{\beta}}_2 = [\delta_{2r} - \frac{u_r}{R}]\,\mathbf{e}_\theta \ . \tag{5.23.6}$$

Next, using (3.12.14), Table 3.12.1, (5.16.10), (5.16.11), (5.16.13), (5.21.3), (5.21.4), (5.21.10) and (5.23.6), the constitutive equations (5.13.16) yield

$$\tilde{\mathbf{m}}^1 = \frac{\mu^* H^3 W(1+\nu^*)}{6R}[\,\delta_{1r}\,]\,\mathbf{e}_\theta \ , \quad \tilde{\mathbf{m}}^2 = \frac{\mu^* H W^3(1+\nu^*)}{6R}[\delta_{2r} - \frac{u_r}{R}]\,\mathbf{e}_\theta \ ,$$

$$\tilde{\mathbf{t}}^1 = \frac{2\mu^* RHW}{(1-2\nu^*)}[\,\frac{1}{2}(1-2\nu^*)\{\delta_{1r}+\delta_{2z}\}\mathbf{e}_r + \{\nu^*(\delta_{2r}+\frac{u_r}{R}) + (1-\nu^*)\delta_{1z}\}\mathbf{e}_z\,] \ ,$$

$$\tilde{\mathbf{t}}^2 = \frac{2\mu^* RHW}{(1-2\nu^*)}[\{(1-\nu^*)\delta_{2r} + \nu^*(\delta_{1z}+\frac{u_r}{R})\}\mathbf{e}_r + \frac{1}{2}(1-2\nu^*)\{\delta_{1r}+\delta_{2z}\}\mathbf{e}_z\,] \ ,$$

$$\tilde{\mathbf{t}}^3 = [\frac{2\mu^* HW}{(1-2\nu^*)}\{(1-\nu^*)\frac{u_r}{R} + \nu^*\,(\delta_{1z} + \delta_{2r})\} - \frac{\mu^*(1+\nu^*)HW^3}{6R^2}\{\delta_{2r} - \frac{u_r}{R}\}\,]\,\mathbf{e}_\theta \ . \tag{5.23.7}$$

Consequently, with the help of (5.23.4) and (5.23.7) the linearized equations of equilibrium (5.13.15) yield

$$-\,[\frac{2\mu^* HW}{(1-2\nu^*)}\{(1-\nu^*)\frac{u_r}{R} + \nu^*\,(\delta_{1z} + \delta_{2r})\} - \frac{\mu^*(1+\nu^*)HW^3}{6R^2}\{\delta_{2r} - \frac{u_r}{R}\}\,]\,\mathbf{e}_r = 0 \ ,$$

$$-\,[\,\frac{1}{2}\rho_0^* g^* R H^2 W]\,\mathbf{e}_z - \frac{2\mu^* RHW}{(1-2\nu^*)}[\,\frac{1}{2}(1-2\nu^*)\{\delta_{1r}+\delta_{2z}\}\mathbf{e}_r$$

$$+ \{\nu^*(\delta_{2r}+\frac{u_r}{R}) + (1-\nu^*)\delta_{1z}\}\mathbf{e}_z\,] - \frac{\mu^* H^3 W(1+\nu^*)}{6R}[\,\delta_{1r}\,]\,\mathbf{e}_r = 0 \ ,$$

$$-\,\frac{2\mu^* RHW}{(1-2\nu^*)}[\{(1-\nu^*)\delta_{2r} + \nu^*(\delta_{1z}+\frac{u_r}{R})\}\mathbf{e}_r + \frac{1}{2}(1-2\nu^*)\{\delta_{1r}+\delta_{2z}\}\mathbf{e}_z\,]$$

$$-\,\frac{\mu^* H W^3(1+\nu^*)}{6R}[\delta_{2r} - \frac{u_r}{R}]\,\mathbf{e}_r = 0 \ , \tag{5.23.8}$$

which can be solved to deduce that

$$\frac{u_r}{R} = \delta_{2r} = -\,\nu^*\,\delta_{1z} \ , \quad \delta_{1z} = -\,\frac{\rho_0^* g^* H}{2E^*} \ , \quad \delta_{1r} = \delta_{2z} = 0 \ ,$$

$$\mathbf{u} = [\frac{\nu^* \rho_0^* g^* R H}{2E^*}]\,\mathbf{e}_r + u_z\,\mathbf{e}_z \ , \quad \boldsymbol{\delta}_1 = -\,[\frac{\rho_0^* g^* H}{2E^*}]\,\mathbf{e}_z \ , \quad \boldsymbol{\delta}_2 = [\frac{\nu^* \rho_0^* g^* H}{2E^*}]\,\mathbf{e}_r \ ,$$

$$\boldsymbol{\delta}_3 = [\frac{v^* \overset{*}{\rho}_0 g^* RH}{2E^*}]\, \mathbf{e}_\theta \,, \quad \tilde{\boldsymbol{\beta}}_1 = 0 \,, \quad \tilde{\boldsymbol{\beta}}_2 = 0 \,, \quad \tilde{\mathbf{m}}^1 = 0 \,, \quad \tilde{\mathbf{m}}^2 = 0 \,,$$

$$\tilde{\mathbf{t}}^1 = -[\frac{\overset{*}{\rho}_0 g^* RH^2 W}{2E^*}]\, \mathbf{e}_z \,, \quad \tilde{\mathbf{t}}^2 = 0 \,, \quad \tilde{\mathbf{t}}^3 = 0 \,, \tag{5.23.9}$$

Moreover, the linearized form of the kinematic assumption (5.2.8) suggests that $\mathbf{u}^*$ is approximated by

$$\mathbf{u} + \theta^\alpha \boldsymbol{\delta}_\alpha = [u_r + \theta^2\, \delta_{2r}]\, \mathbf{e}_r + [\{\frac{H}{2} + \theta^1\}\delta_{1z}]\, \mathbf{e}_z \,,$$

$$\mathbf{u} + \theta^\alpha \boldsymbol{\delta}_\alpha = \frac{\overset{*}{\rho}_0 \overset{*}{g}}{E^*} [\, v^* \frac{H}{2}(R + \theta^2)\, \mathbf{e}_r - \frac{H}{2}\{\frac{H}{2} + \theta^1\}\mathbf{e}_z \,] \,, \tag{5.23.10}$$

where the constant u_z ($=H\delta_{1z}/2$) associated with rigid body displacement has been determined by the condition that the vertical component of displacement ($\mathbf{u} + \theta^\alpha \boldsymbol{\delta}_\alpha$) vanishes for $\theta^1 = -H/2$ and $\theta^2 = 0$.

The solution (5.23.9) and (5.23.10) is consistent with the same physical boundary conditions as have been imposed on the exact solution of section 3.19. However, in order to compare the exact displacement field (3.19.10) with the approximate one (5.23.10), it is necessary to replace $\{\theta^1, \theta^2, \theta^3, H, W\}$ in (3.19.10) with $\{\theta^3, \theta^1, \theta^2, W, H\}$, respectively, to obtain

$$\mathbf{u}^* = \frac{\overset{*}{\rho}_0 \overset{*}{g}}{E^*} [\, v^* (\frac{H}{2} - \theta^1)(R+\theta^2)\, \mathbf{e}_r$$
$$-\frac{1}{2}\{[H^2 - (\theta^1 - \frac{H}{2})^2] - v^*[(R+\theta^2)^2 - R^2]\}\mathbf{e}_z \,] \,. \tag{5.23.11}$$

Comparison of the solutions (5.23.10) and (5.23.11) reveals that the simple approximation of the Cosserat rod theory is not able to reproduce the exact solution. In particular, it cannot reproduce the facts that the exact radial displacement is a function of θ^1, and the exact axial displacement is a function of θ^2. This effect causes the cylindrical shell to become conical in shape with the top free edge having a smaller radius than the bottom loaded edge. In spite of this deficiency, the Cosserat solution is able to reproduce some significant features of the exact solution. For example, the radial displacement of the centerline of the rod, the vertical height of the rod, and the radial thickness of the rod, are exact since

$$\mathbf{u}^*(0,0,\theta^3) \bullet \mathbf{e}_r = u_r = \frac{v^* \overset{*}{\rho}_0 g^* RH}{2E^*} \,,$$

$$[\mathbf{u}^*(H/2,0,\theta^3) - \mathbf{u}^*(-H/2,0,\theta^3)] \bullet \mathbf{e}_z = H\delta_{1z} = -\frac{\overset{*}{\rho}_0 g^* H^2}{2E^*} \,,$$

$$[\mathbf{u}^*(0,W/2,\theta^3) - \mathbf{u}^*(0,-W/2,\theta^3)] \bullet \mathbf{e}_r = W\delta_{2r} = \frac{v^* \overset{*}{\rho}_0 g^* HW}{2E^*} \,. \tag{5.23.12}$$

Of course, it should not be expected that a simple rod theory can capture all of the features of the exact solution for this inhomogeneous deformation problem. However, the results (5.23.12) indicate that the generalization (5.20.12) of the assigned director couples due to body force applied to a rod is reasonable.

5.24 Isotropic nonlinear elastic rods

The objective of this section is to use the three-dimensional constitutive equations of section 3.11 for isotropic nonlinear elastic materials, together with the restrictions (5.11.9), to exhibit explicit constitutive equations for isotropic nonlinear elastic rods. Motivated by the definitions (3.1.7), (3.7.8), (3.11.9), (5.2.10), (5.11.25) and the specification (5.11.35), it is convenient to define the kinematic quantities $\mathbf{C}$, $\mathbf{B}$, J, α_1 and α_2 by the expressions

$$\mathbf{C} = \mathbf{F}^T\mathbf{F} \ , \quad \mathbf{B} = \mathbf{F}\mathbf{F}^T \ , \quad J = \det \mathbf{F} \ ,$$
$$\alpha_1 = J^{-2/3}\, \mathbf{C} \bullet \mathbf{I} = J^{-2/3}\, \mathbf{B} \bullet \mathbf{I} \ , \quad \alpha_2 = J^{-4/3}\, \mathbf{C} \bullet \mathbf{C} = J^{-4/3}\, \mathbf{B} \bullet \mathbf{B} \ . \tag{5.24.1}$$

Then, when the strain energy function Σ^* in (3.11.10) for an isotropic material is expressed as a function of $\mathbf{C}$ instead of $\mathbf{C}^*$, it follows from (3.11.12) that

$$\Sigma^*(\mathbf{C}) = \hat{\Sigma}^*(\alpha_1,\alpha_2,J) \ ,$$
$$\frac{\partial \Sigma^*}{\partial \mathbf{C}} = \frac{1}{2} J \frac{\partial \hat{\Sigma}^*}{\partial J} \mathbf{C}^{-1} + J^{-2/3} \frac{\partial \hat{\Sigma}^*}{\partial \alpha_1} \Big[\mathbf{I} - \frac{1}{3} (\mathbf{C} \bullet \mathbf{I})\, \mathbf{C}^{-1} \Big]$$
$$+ 2J^{-4/3} \frac{\partial \hat{\Sigma}^*}{\partial \alpha_2} \Big[\mathbf{C} - \frac{1}{3} (\mathbf{C}^2 \bullet \mathbf{I})\, \mathbf{C}^{-1} \Big] \ . \tag{5.24.2}$$

Moreover, with the help of (5.11.36) it can be shown that $\mathbf{T}$ is given by

$$\mathbf{T} = -\, p\, \mathbf{I} + \mathbf{T}' \ , \quad \mathbf{T}' \bullet \mathbf{I} = 0 \ ,$$
$$d_{33}^{1/2}\, \mathbf{T} = 2\, m\, \mathbf{F} \frac{\partial \Sigma^*(\mathbf{C})}{\partial \mathbf{C}} \mathbf{F}^T \ , \quad d_{33}^{1/2}\, p = -\, m\, J \frac{\partial \hat{\Sigma}^*}{\partial J} \ ,$$
$$d_{33}^{1/2}\, \mathbf{T}' = 2m\, J^{-2/3} \frac{\partial \hat{\Sigma}^*}{\partial \alpha_1} \Big[\mathbf{B} - \frac{1}{3} (\mathbf{B} \bullet \mathbf{I})\, \mathbf{I} \Big] + 4m\, J^{-4/3} \frac{\partial \hat{\Sigma}^*}{\partial \alpha_2} \Big[\mathbf{B}^2 - \frac{1}{3} (\mathbf{B}^2 \bullet \mathbf{I})\, \mathbf{I} \Big] \ . \tag{5.24.3}$$

Thus, comparison of (5.24.3) with (3.11.6) suggests that p and $\mathbf{T}'$ can be interpreted as an integrated pressure and deviatoric stress, respectively.

Next, when the strain energy function for the rod is given by (5.20.2) with the specification (5.24.2), it can be shown with the help of (5.4.23) that the constitutive equations (5.11.36) for $\mathbf{m}^\alpha$ and $\mathbf{t}^i$ can be written in the forms

$$\mathbf{m}^\alpha = D_{33}^{-1/2}\, A\, \mathbf{F}^{-T}\, \mathbf{K}^{\alpha\beta}\, \boldsymbol{\beta}_\beta \ , \quad \mathbf{t}^i = d_{33}^{1/2} \big[-\, p\, \mathbf{I} + \mathbf{T}' \big]\, \mathbf{d}^i \, - \mathbf{m}^\alpha\, (\mathbf{d}_{\alpha,3} \bullet \mathbf{d}^i) \ , \tag{5.24.4}$$

where for isotropic materials with the specifications (5.16.10) and (5.16.11), the tensor $\mathbf{K}^{\alpha\beta}$ is determined by (5.16.13).

If the rod is modeled as incompressible, then the incompressibility constraint (5.9.1) is imposed

$$J = 1 \ , \tag{5.24.5}$$

and $\mathbf{T}$ and $\mathbf{t}^i$ are modified by the constraint responses (5.9.27) so that

$$d_{33}^{1/2}\,\mathbf{T} = \bar{\gamma}\,\mathbf{I} + d_{33}^{1/2}\,\mathbf{T}' \ , \ \ \mathbf{t}^i = \bar{\gamma}\,\mathbf{d}^i + d_{33}^{1/2}\,\mathbf{T}'\,\mathbf{d}^i - \ \mathbf{m}^\alpha\,(\mathbf{d}_{3,\alpha} \bullet \mathbf{d}^i) \ , \tag{5.24.6}$$

where $\bar{\gamma}$ is given by the expression

$$\bar{\gamma} = \gamma - d_{33}^{1/2}\,p \ . \tag{5.24.7}$$

Here, γ is an arbitrary function of (θ^3,t) which can absorb any dependence of the function of p on its arguments. Moreover, for the special case when Σ^* is given by (3.11.16) with (3.11.18) and (3.11.19), it follows from (5.11.7) that

$$\begin{aligned} m\,\hat{\Sigma}^*(\alpha_1,\alpha_2,J) &= \frac{1}{2} D_{33}^{1/2}\,A\,\mu_0^*\left[(1-4C_2)\,(\alpha_1-3) + C_2\,(\alpha_2-3)\right] \\ &\quad + D_{33}^{1/2}\,A\,K_0^*\left[(J-1) - \ln(J)\right] \ , \end{aligned} \tag{5.24.8}$$

where C_2 is a material constant controlling nonlinear elastic effects.

5.25 A simple derivation of the local equations for rods with rectangular cross-sections

Section 5.3 showed how the global balance laws for rods can be developed using the kinematic assumption (5.2.8) and the balance laws of the three-dimensional theory. In that derivation certain complicating features related to the boundaries of the three-dimensional rod-like structure were considered as they naturally occurred. The objective of the present section is to provide a simple derivation of the local equations for rods with rectangular cross-sections in their reference configurations. Specifically, the coordinates θ^i, the position vector $\mathbf{X}$, and the directors $\mathbf{D}_\alpha$ are specified so that the conditions (5.14.2) hold. Moreover, the geometry of the rod in its reference configuration is specified by the position vectors $\mathbf{X}^*$ (5.1.4), and the limits (5.14.3) on the coordinates θ^i such that

$$\begin{gathered} \mathbf{X}^* = \mathbf{X}(\theta^3) + \theta^\alpha \mathbf{D}_\alpha(\theta^3) \ , \ \ \mathbf{D}_3 = \mathbf{X}_{,3} \ , \\ |\,\theta^1\,| \le \frac{H}{2} \ , \ \ |\,\theta^2\,| \le \frac{W}{2} \ , \ \ |\,\theta^3\,| \le \frac{L}{2} \ . \end{gathered} \tag{5.25.1}$$

where the thickness H and width W can both be functions of θ^3.

Next, it is convenient to recall the local forms (3.2.35) of the balance laws of the three-dimensional theory

$$\dot{m}^* = 0 \ , \ \ m^*\,\dot{\mathbf{v}}^* = m^*\,\mathbf{b}^* + \mathbf{t}^{*i}{}_{,i} \ . \tag{5.25.2}$$

Also, with the help of (3.2.34) the reduced form of the balance of angular momentum $(3.2.35)_4$ can be rewritten as

$$g^{1/2}\,\mathbf{T}^* = \mathbf{t}^{*i} \otimes \mathbf{g}_i = \mathbf{g}_i \otimes \mathbf{t}^{*i} = g^{1/2}\,\mathbf{T}^{*T} \ . \tag{5.25.3}$$

Now, the kinematic assumption (5.2.8) and the definitions (2.1.5) and (5.2.2)-(5.2.7), yield

$$\begin{gathered} \mathbf{x}^*(\theta^i,t) = \mathbf{x}(\theta^3,t) + \theta^\alpha\,\mathbf{d}_\alpha(\theta^3,t) \ , \ \ \mathbf{g}_\alpha = \mathbf{d}_\alpha \ , \ \ \mathbf{g}_3 = \mathbf{d}_3 + \theta^\alpha\,\mathbf{d}_{\alpha,3} \ , \\ \mathbf{v}^*(\theta^i,t) = \mathbf{v}(\theta^3,t) + \theta^\alpha\,\mathbf{w}_\alpha(\theta^3,t) \ , \\ \dot{\mathbf{v}}^*(\theta^i,t) = \dot{\mathbf{v}}(\theta^3,t) + \theta^\alpha\,\dot{\mathbf{w}}_\alpha(\theta^3,t) \ , \ \ \mathbf{w}_3 = \mathbf{v}_{,3} \ . \end{gathered} \tag{5.25.4}$$

Section 5.3 introduced a number of equations which connect quantities defined for the Cosserat rod with expressions associated with integration over the cross-section of related three-dimensional quantities. In most cases, these equations will be referred to, but not be repeated here.

The balance laws for conservation of mass and balance of linear momentum for the rod can be obtained by merely integrating (5.25.2) over the cross-section of the rod to obtain equations for average quantities of the forms

$$\frac{d}{dt}\int_{-W/2}^{W/2}\int_{-H/2}^{H/2} m^*\, d\theta^1 d\theta^2 = 0 \ ,$$

$$[\int_{-W/2}^{W/2}\int_{-H/2}^{H/2} m^*\, d\theta^1 d\theta^2]\,\dot{\mathbf{v}} + [\int_{-W/2}^{W/2}\int_{-H/2}^{H/2} m^*\,\theta^\alpha\, d\theta^1 d\theta^2]\,\dot{\mathbf{w}}_\alpha$$
$$= \int_{-W/2}^{W/2}\int_{-H/2}^{H/2} \{ m^*\,\mathbf{b}^* + \mathbf{t}^{*\alpha}{}_{,\alpha} + \mathbf{t}^{*3}{}_{,3} \}\, d\theta^1 d\theta^2 \ , \qquad (5.25.5)$$

where use has been made of the fact that θ^i are convected coordinates, and that H and W are independent of time. However,

$$\int_{-W/2}^{W/2}\int_{-H/2}^{H/2} \mathbf{t}^{*\alpha}{}_{,\alpha}\, d\theta^1 d\theta^2 = [\int_{-W/2}^{W/2}\{ \mathbf{t}^{*1}(H/2,\theta^2,\theta^3) - \mathbf{t}^{*1}(-H/2,\theta^2,\theta^3)\}\, d\theta^2]$$
$$+ [\int_{-H/2}^{H/2}\{ \mathbf{t}^{*2}(\theta^1,W/2,\theta^3) - \mathbf{t}^{*2}(\theta^1,-W/2,\theta^3)\}\, d\theta^1] \ ,$$

$$\int_{-W/2}^{W/2}\int_{-H/2}^{H/2} \mathbf{t}^{*3}{}_{,3}\, d\theta^1 d\theta^2 = [\int_{-W/2}^{W/2}\int_{-H/2}^{H/2} \mathbf{t}^{*3}\, d\theta^1 d\theta^2]_{,3}$$
$$-\frac{1}{2}H_{,3}\,[\int_{-W/2}^{W/2} \{\mathbf{t}^{*3}(H/2,\theta^2,\theta^3) + \mathbf{t}^{*3}(-H/2,\theta^2,\theta^3)\}\, d\theta^2]$$
$$-\frac{1}{2}W_{,3}\,[\int_{-H/2}^{H/2} \{\mathbf{t}^{*3}(\theta^1,W/2,\theta^3) + \mathbf{t}^{*3}(\theta^1,-W/2,\theta^3)\}\, d\theta^1] \ , \qquad (5.25.6)$$

so that $(5.25.5)_2$ can be rewritten in the form

$$[\int_{-W/2}^{W/2}\int_{-H/2}^{H/2} m^*\, d\theta^1 d\theta^2]\,\dot{\mathbf{v}} + [\int_{-W/2}^{W/2}\int_{-H/2}^{H/2} m^*\,\theta^\alpha\, d\theta^1 d\theta^2]\,\dot{\mathbf{w}}_\alpha$$
$$= [\int_{-W/2}^{W/2}\int_{-H/2}^{H/2} \{ m^*\,\mathbf{b}^* \}\, d\theta^1 d\theta^2$$
$$+ \int_{-W/2}^{W/2}\{ \mathbf{t}^{*1}(H/2,\theta^2,\theta^3) - \frac{1}{2}H_{,3}\,\mathbf{t}^{*3}(H/2,\theta^2,\theta^3)\}\, d\theta^2$$
$$+ \int_{-H/2}^{H/2}\{ \mathbf{t}^{*2}(\theta^1,W/2,\theta^3) - \frac{1}{2}W_{,3}\,\mathbf{t}^{*3}(\theta^1,W/2,\theta^3)\}\, d\theta^1$$
$$- \int_{-W/2}^{W/2}\{ \mathbf{t}^{*1}(-H/2,\theta^2,\theta^3) + \frac{1}{2}H_{,3}\,\mathbf{t}^{*3}(-H/2,\theta^2,\theta^3)\}\, d\theta^2 \ ]$$
$$- \int_{-H/2}^{H/2}\{ \mathbf{t}^{*1}(\theta^1,-W/2,\theta^3) + \frac{1}{2}W_{,3}\,\mathbf{t}^{*3}(\theta^1,-W/2,\theta^3)\}\, d\theta^1$$
$$+ [\int_{-W/2}^{W/2}\int_{-H/2}^{H/2} \mathbf{t}^{*3}\, d\theta^1 d\theta^2]_{,3} \ . \qquad (5.25.7)$$

Next, it is noted that the boundary of the cross-section $\partial\mathcal{A}$ is defined by specifying the functions (5.1.9) on the four edges such that

$$\theta^1 = \frac{H}{2} \ \text{ and } \ \zeta = \theta^2 \quad \text{for } \theta^2 \in [-\frac{W}{2}, \frac{W}{2}] \ ,$$
$$\zeta = -\theta^1 \ \text{ and } \ \theta^2 = \frac{W}{2} \quad \text{for } \theta^1 \in [\frac{H}{2}, -\frac{H}{2}] \ ,$$

$$\theta^1 = -\frac{H}{2} \ \text{ and } \ \zeta = -\theta^2 \quad \text{for } \theta^2 \in [\frac{W}{2}, -\frac{W}{2}] \ ,$$
$$\zeta = \theta^1 \ \text{ and } \ \theta^2 = -\frac{W}{2} \quad \text{for } \theta^1 \in [-\frac{H}{2}, \frac{H}{2}] \ . \tag{5.25.8}$$

Thus, with the help of the definitions (5.3.7), (5.3.10), (5.3.12), (5.3.16), (5.3.19) and (5.3.24), equations $(5.25.5)_1$ and (5.25.7) yield the local forms of the conservation of mass and the balance of linear momentum equations for the rod

$$\dot{m} = 0 \ , \quad m\,(\dot{\mathbf{v}} + y^\alpha\, \dot{\mathbf{w}}_\alpha) = m\,\mathbf{b} + \mathbf{t}^3{}_{,3} \ . \tag{5.25.9}$$

As has been seen, these equations represent a zeroth order moment (or mere integration) over the cross-section of the rod of the three-dimensional conservation of mass and the balance of linear momentum equations.

It will presently be shown that the balances of director momentum equations represent first order moments over the cross-section of the rod of the three-dimensional linear momentum equation. Specifically, $(5.25.2)_2$ is multiplied by θ^α and then integrated over the cross-section of the rod to obtain

$$\Big[\int_{-W/2}^{W/2} \int_{-H/2}^{H/2} m^*\, \theta^\alpha\, d\theta^1 d\theta^2\Big]\, \dot{\mathbf{v}} + \Big[\int_{-W/2}^{W/2} \int_{-H/2}^{H/2} m^*\, \theta^\alpha \theta^\beta\, d\theta^1 d\theta^2\Big]\, \dot{\mathbf{w}}_\beta$$
$$= \int_{-W/2}^{W/2} \int_{-H/2}^{H/2} \big\{ m^*\, \mathbf{b}^* + \mathbf{t}^{*\beta}{}_{,\beta} + \mathbf{t}^{*3}{}_{,3} \big\}\, \theta^\alpha\, d\theta^1 d\theta^2. \tag{5.25.10}$$

However, it can be shown that

$$\int_{-W/2}^{W/2} \int_{-H/2}^{H/2} \big\{\mathbf{t}^{*\beta}{}_{,\beta}\big\}\, \theta^\alpha\, d\theta^1 d\theta^2 = \int_{-W/2}^{W/2} \int_{-H/2}^{H/2} \big\{ (\mathbf{t}^{*\beta}\, \theta^\alpha)_{,\beta} - \mathbf{t}^{*\beta} \delta^\alpha_\beta \big\}\, d\theta^1 d\theta^2$$

$$\int_{-W/2}^{W/2} \int_{-H/2}^{H/2} \big\{\mathbf{t}^{*\beta}{}_{,\beta}\big\}\, \theta^1\, d\theta^1 d\theta^2 = \Big[\frac{H}{2} \int_{-W/2}^{W/2} \big\{ \mathbf{t}^{*1}(H/2,\theta^2,\theta^3) + \mathbf{t}^{*1}(-H/2,\theta^2,\theta^3) \big\}\, d\theta^2\Big]$$
$$+ \Big[\int_{-H/2}^{H/2} \big\{ \mathbf{t}^{*2}(\theta^1,W/2,\theta^3) - \mathbf{t}^{*2}(\theta^1,-W/2,\theta^3) \big\}\, \theta^1 d\theta^1\Big]$$
$$- \Big[\int_{-W/2}^{W/2} \int_{-H/2}^{H/2} \big\{\mathbf{t}^{*1}\big\}\, d\theta^1 d\theta^2\Big] \ ,$$

$$\int_{-W/2}^{W/2} \int_{-H/2}^{H/2} \big\{\mathbf{t}^{*\beta}{}_{,\beta}\big\}\, \theta^2\, d\theta^1 d\theta^2 = \Big[\int_{-W/2}^{W/2} \big\{ \mathbf{t}^{*1}(H/2,\theta^2,\theta^3) - \mathbf{t}^{*1}(-H/2,\theta^2,\theta^3) \big\}\, \theta^2 d\theta^2\Big]$$
$$+ \Big[\frac{W}{2} \int_{-H/2}^{H/2} \big\{ \mathbf{t}^{*2}(\theta^1,W/2,\theta^3) + \mathbf{t}^{*2}(\theta^1,-W/2,\theta^3) \big\}\, d\theta^1\Big]$$
$$- \Big[\int_{-W/2}^{W/2} \int_{-H/2}^{H/2} \big\{\mathbf{t}^{*2}\big\}\, d\theta^1 d\theta^2\Big] \ ,$$

$$\int_{-W/2}^{W/2} \int_{-H/2}^{H/2} \mathbf{t}^{*3}{}_{,3}\, \theta^1\, d\theta^1 d\theta^2 = \Big[\int_{-W/2}^{W/2} \int_{-H/2}^{H/2} \mathbf{t}^{*3}\, \theta^1 d\theta^1 d\theta^2\Big]_{,3}$$
$$- \frac{1}{2} H_{,3} \Big[\frac{H}{2} \int_{-W/2}^{W/2} \big\{ \mathbf{t}^{*3}(H/2,\theta^2,\theta^3) - \mathbf{t}^{*3}(-H/2,\theta^2,\theta^3) \big\}\, d\theta^2\Big]$$
$$- \frac{1}{2} W_{,3} \Big[\int_{-H/2}^{H/2} \big\{ \mathbf{t}^{*3}(\theta^1,W/2,\theta^3) + \mathbf{t}^{*3}(\theta^1,-W/2,\theta^3) \big\}\, \theta^1 d\theta^1\Big] \ ,$$

$$\int_{-W/2}^{W/2} \int_{-H/2}^{H/2} \mathbf{t}^{*3}{}_{,3}\, \theta^2\, d\theta^1 d\theta^2 = \Big[\int_{-W/2}^{W/2} \int_{-H/2}^{H/2} \mathbf{t}^{*3}\, \theta^2 d\theta^1 d\theta^2\Big]_{,3}$$
$$- \frac{1}{2} H_{,3} \Big[\int_{-W/2}^{W/2} \big\{ \mathbf{t}^{*3}(H/2,\theta^2,\theta^3) + \mathbf{t}^{*3}(-H/2,\theta^2,\theta^3) \big\}\, \theta^2 d\theta^2\Big]$$
$$- \frac{1}{2} W_{,3} \Big[\frac{W}{2} \int_{-H/2}^{H/2} \big\{ \mathbf{t}^{*3}(\theta^1,W/2,\theta^3) - \mathbf{t}^{*3}(\theta^1,-W/2,\theta^3) \big\}\, d\theta^1\Big] \ , \tag{5.25.11}$$

so that with the help of (5.25.8) the equations (5.25.10) can be rewritten in the forms

$$
\begin{aligned}
&[\int_{-W/2}^{W/2} \int_{-H/2}^{H/2} m^* \theta^1 \, d\theta^1 d\theta^2] \, \dot{\mathbf{v}} + [\int_{-W/2}^{W/2} \int_{-H/2}^{H/2} m^* \theta^1 \theta^\beta \, d\theta^1 d\theta^2] \, \dot{\mathbf{w}}_\beta \\
&\quad = [\int_{-W/2}^{W/2} \int_{-H/2}^{H/2} \{ m^* \mathbf{b}^* \} \theta^1 \, d\theta^1 d\theta^2 \\
&\quad + \frac{H}{2} \int_{-W/2}^{W/2} \{ \mathbf{t}^{*1}(H/2,\theta^2,\theta^3) - \frac{1}{2} H_{,3} \, [\mathbf{t}^{*3}(H/2,\theta^2,\theta^3) \} \, d\theta^2 \\
&\quad + \int_{-H/2}^{H/2} \{ \mathbf{t}^{*2}(\theta^1,W/2,\theta^3) - \frac{1}{2} W_{,3} \, \mathbf{t}^{*3}(\theta^1,W/2,\theta^3) \} \, \theta^1 d\theta^1 \\
&\quad + \frac{H}{2} \int_{-W/2}^{W/2} \{ \mathbf{t}^{*1}(-H/2,\theta^2,\theta^3)] + \frac{1}{2} H_{,3} \, \mathbf{t}^{*3}(-H/2,\theta^2,\theta^3) \} \, d\theta^2 \\
&\quad - \int_{-H/2}^{H/2} \{ \mathbf{t}^{*2}(\theta^1,-W/2,\theta^3)] + \frac{1}{2} W_{,3} \, \mathbf{t}^{*3}(\theta^1,-W/2,\theta^3) \} \, \theta^1 d\theta^1] \\
&\quad - [\int_{-W/2}^{W/2} \int_{-H/2}^{H/2} \{ \mathbf{t}^{*1} \} d\theta^1 d\theta^2] + [\int_{-W/2}^{W/2} \int_{-H/2}^{H/2} \mathbf{t}^{*3} \, \theta^1 d\theta^1 d\theta^2]_{,3} \\
&[\int_{-W/2}^{W/2} \int_{-H/2}^{H/2} m^* \theta^2 \, d\theta^1 d\theta^2] \, \dot{\mathbf{v}} + [\int_{-W/2}^{W/2} \int_{-H/2}^{H/2} m^* \theta^2 \theta^\beta \, d\theta^1 d\theta^2] \, \dot{\mathbf{w}}_\beta \\
&\quad = [\int_{-W/2}^{W/2} \int_{-H/2}^{H/2} \{ m^* \mathbf{b}^* \} \theta^2 \, d\theta^1 d\theta^2 \\
&\quad + \int_{-W/2}^{W/2} \{ \mathbf{t}^{*1}(H/2,\theta^2,\theta^3) - \frac{1}{2} H_{,3} \, \mathbf{t}^{*3}(H/2,\theta^2,\theta^3) \} \, \theta^2 d\theta^2 \\
&\quad + \frac{W}{2} \int_{-H/2}^{H/2} \{ \mathbf{t}^{*2}(\theta^1,W/2,\theta^3) - \frac{1}{2} W_{,3} \, \mathbf{t}^{*3}(\theta^1,W/2,\theta^3) \} \, d\theta^1 \\
&\quad - \int_{-W/2}^{W/2} \{ \mathbf{t}^{*1}(-H/2,\theta^2,\theta^3) + \frac{1}{2} H_{,3} \, \mathbf{t}^{*3}(-H/2,\theta^2,\theta^3) \} \, \theta^2 d\theta^2 \\
&\quad + \frac{W}{2} \int_{-H/2}^{H/2} \{ \mathbf{t}^{*2}(\theta^1,-W/2,\theta^3) + \frac{1}{2} W_{,3} \, \mathbf{t}^{*3}(\theta^1,-W/2,\theta^3) \} \, d\theta^1] \\
&\quad - [\int_{-W/2}^{W/2} \int_{-H/2}^{H/2} \{ \mathbf{t}^{*2} \} d\theta^1 d\theta^2] + [\int_{-W/2}^{W/2} \int_{-H/2}^{H/2} \mathbf{t}^{*3} \, \theta^2 d\theta^1 d\theta^2]_{,3} \, . \qquad (5.25.12)
\end{aligned}
$$

Thus, with the help of the definitions (5.3.10), (5.3.26), (5.3.28), (5.3.30), (5.3.32), (5.3.33) and (5.3.37), equations (5.25.12) yield the local forms of the balances of director momentum for the rod

$$
m \, (y^\alpha \, \dot{\mathbf{v}} + y^{\alpha\beta} \, \dot{\mathbf{w}}_\beta) = m \, \mathbf{b}^\alpha - \mathbf{t}^\alpha + \mathbf{m}^\alpha{}_{,3} \; . \qquad (5.25.13)
$$

Finally, the reduced form of the balance of angular momentum for the rod is obtained by substituting $(5.25.4)_{2,3}$ into (5.25.3) and integrating over the cross-section to obtain

$$
\begin{aligned}
&[\int_{-W/2}^{W/2} \int_{-H/2}^{H/2} g^{1/2} \, \mathbf{T}^* \, d\theta^1 d\theta^2 = [\int_{-W/2}^{W/2} \int_{-H/2}^{H/2} \mathbf{t}^{*\alpha} \, d\theta^1 d\theta^2 \,] \otimes \mathbf{d}_\alpha \\
&\quad + [\int_{-W/2}^{W/2} \int_{-H/2}^{H/2} \mathbf{t}^{*3} \, d\theta^1 d\theta^2 \,] \otimes \mathbf{d}_3 + [\int_{-W/2}^{W/2} \int_{-H/2}^{H/2} \mathbf{t}^{*3} \, \theta^\alpha \, d\theta^1 d\theta^2 \,] \otimes \mathbf{d}_{\alpha,3} \\
&\quad = \int_{-W/2}^{W/2} \int_{-H/2}^{H/2} g^{1/2} \, \mathbf{T}^{*T} \, d\theta^1 d\theta^2 \; . \qquad (5.25.14)
\end{aligned}
$$

Thus, with the help of the definitions (5.3.19), (5.3.30), (5.3.33) and (5.11.2), equation (5.25.14) yields

$$
d_{33}^{1/2} \, \mathbf{T} = \mathbf{t}^i \otimes \mathbf{d}_i + \mathbf{m}^\alpha \otimes \mathbf{d}_{3,\alpha} = d_{33}^{1/2} \, \mathbf{T}^T \; . \qquad (5.25.15)
$$

5.26 A brief summary of the equations for rods

This section summarizes some of the main equations of the Cosserat theory of rods. For simplicity attention will be confined to rods which are made of homogeneous materials ($\rho_0^* =$ constant) and which have rectangular cross-sections with thickness H and width W. The rod's reference curve will be taken to be the centroid of the cross-section, and the directors $\mathbf{D}_\alpha$ will be taken to be unit orthogonal vectors which are normal to this centroid and are oriented so that θ^1 is the thickness coordinate ($-H/2 \le \theta^1 \le H/2$), and θ^2 is the width coordinate ($-W/2 \le \theta^2 \le W/2$). Also, the body force $\mathbf{b}^*$ per unit mass will be assumed to be a constant vector. In principle, the equations can be used for rods of variable thickness and width with $H(\theta^3)$ and $W(\theta^3)$. However, their validity has only been examined for specific problems of beams and rods of uniform thickness and width.

An attempt has been made to record a complete set of equations that can be used to determine the response of general elastic rods. Also, for convenient reference, equation numbers are recorded below to indicate the locations in the previous sections where the quantities, or related quantities, have been explained in more detail.

KINEMATICS

$$\mathbf{X}(\theta^3)\ ,\quad \mathbf{D}_\alpha(\theta^3)\ ,\quad \mathbf{D}_3(\theta^3) = \mathbf{X}_{,3}\ ,\quad D_{33} = \mathbf{D}_3 \bullet \mathbf{D}_3\ , \qquad (5.1.1)\text{-}(5.1.2)$$

$$D^{1/2}\,\mathbf{D}^1 = \mathbf{D}_2 \times \mathbf{D}_3\ ,\quad D^{1/2}\,\mathbf{D}^2 = \mathbf{D}_3 \times \mathbf{D}_1\ ,\quad D^{1/2}\,\mathbf{D}^3 = \mathbf{D}_1 \times \mathbf{D}_2\ , \qquad (5.1.11)$$

$$D^{1/2} = \mathbf{D}_1 \times \mathbf{D}_2 \bullet \mathbf{D}_3\ , \qquad (5.1.3)$$

$$\mathbf{x}(\theta^3,t)\ ,\quad \mathbf{d}_\alpha(\theta^3,t)\ ,\quad \mathbf{d}_3(\theta^3,t) = \mathbf{x}_{,3}\ ,\quad d_{33} = \mathbf{d}_3 \bullet \mathbf{d}_3\ , \qquad (5.2.1),(5.2.2),(5.2.3)$$

$$d^{1/2}\,\mathbf{d}^1 = \mathbf{d}_2 \times \mathbf{d}_3\ ,\quad d^{1/2}\,\mathbf{d}^2 = \mathbf{d}_3 \times \mathbf{d}_1\ ,\quad d^{1/2}\,\mathbf{d}^3 = \mathbf{d}_1 \times \mathbf{d}_2\ , \qquad (5.2.9)$$

$$d^{1/2} = \mathbf{d}_1 \times \mathbf{d}_2 \bullet \mathbf{d}_3\ , \qquad (5.2.4)$$

$$\mathbf{v} = \dot{\mathbf{x}}\ ,\quad \mathbf{w}_i = \dot{\mathbf{d}}_i\ , \qquad (5.2.6)$$

$$\mathbf{F} = \mathbf{d}_i \otimes \mathbf{D}^i\ ,\quad \mathbf{C} = \mathbf{F}^T\mathbf{F}\ ,\quad \mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{I})\ , \qquad (5.2.10),(5.8.8),(5.12.3)$$

$$\boldsymbol{\beta}_\alpha = \mathbf{F}^{-1}\,\mathbf{d}_{\alpha,3} - \mathbf{D}_{\alpha,3}\ , \qquad (5.6.15)$$

$$\theta^1 = \frac{H}{2} \text{ and } \theta^2 = \zeta \text{ for } \theta^2 \in [-\frac{W}{2}, \frac{W}{2}]\ , \qquad (5.1.8),(5.25.8)$$

$$\theta^1 = -\zeta \text{ and } \theta^2 = \frac{W}{2} \text{ for } \theta^1 \in [\frac{H}{2}, -\frac{H}{2}]\ , \qquad (5.1.8),(5.25.8)$$

$$\theta^1 = -\frac{H}{2} \text{ and } \theta^2 = -\zeta \text{ for } \theta^2 \in [\frac{W}{2}, -\frac{W}{2}]\ , \qquad (5.1.8),(5.25.8)$$

$$\theta^1 = \zeta \text{ and } \theta^2 = -\frac{W}{2} \text{ for } \theta^1 \in [-\frac{H}{2}, \frac{H}{2}]\ , \qquad (5.1.8),(5.25.8)$$

$$D_{33}^{1/2}\ A = D^{1/2}\,HW\ ,\quad A^\alpha = 0\ , \qquad (5.20.7),(5.11.35)$$

$$J = \det \mathbf{F}\ ,\quad \mathbf{B} = \mathbf{F}\mathbf{F}^T\ , \qquad (5.24.1)$$

$$\alpha_1 = J^{-2/3}\,\mathbf{C} \bullet \mathbf{I} = J^{-2/3}\,\mathbf{B} \bullet \mathbf{I}\ ,\quad \alpha_2 = J^{-4/3}\,\mathbf{C} \bullet \mathbf{C} = J^{-4/3}\,\mathbf{B} \bullet \mathbf{B}\ , \qquad (5.24.1)$$

$$\mathbf{M}_1 = \mathbf{M}_2 \times \mathbf{M}_3\ ,\quad \mathbf{M}_2 = \frac{\mathbf{M}_3 \times \mathbf{D}_1}{|\mathbf{M}_3 \times \mathbf{D}_1|}\ ,\quad \mathbf{M}_3 = \frac{\mathbf{D}_3}{|\mathbf{D}_3|}\ , \qquad (5.12.5)$$

BALANCE LAWS

$$m = \rho\, d_{33}^{1/2} = \rho_0\, D_{33}^{1/2} = m(\theta^3) \ , \ \dot{m} = 0 \ , \qquad (5.4.15),(5.4.25)$$

$$m\,(\dot{\mathbf{v}} + y^{\alpha}\, \dot{\mathbf{w}}_{\alpha}) = m\,\mathbf{b} + \mathbf{t}^3{}_{,3} \ , \qquad (5.4.25)$$

$$m\,(y^{\alpha}\, \dot{\mathbf{v}} + y^{\alpha\beta}\, \dot{\mathbf{w}}_{\beta}) = m\,\mathbf{b}^{\alpha} - \mathbf{t}^{\alpha} + \mathbf{m}^{\alpha}{}_{,3} \ , \qquad (5.4.25)$$

$$d_{33}^{1/2}\, \mathbf{T} = \mathbf{t}^i \otimes \mathbf{d}_i + \mathbf{m}^{\alpha} \otimes \mathbf{d}_{\alpha,3} = \mathbf{T}^T \ , \qquad (5.4.23),(5.4.24)$$

ASSIGNED FIELDS

$$\mathbf{b}_b = \mathbf{b}^* \ , \ \mathbf{b}_b^1 = \mathbf{b}^* \left[\frac{H^2}{12}(\mathbf{D}_{1,3} \bullet \mathbf{D}^3)\right] \ , \ \mathbf{b}_b^2 = \mathbf{b}^* \left[\frac{W^2}{12}(\mathbf{D}_{2,3} \bullet \mathbf{D}^3)\right] \ , \ (5.20.12)$$

$$\text{for } \theta^1 = \frac{H}{2},\ g^{1/2}\alpha(\tfrac{H}{2},\theta^2)\mathbf{n}^* = \mathbf{d}_2 \times \left[\mathbf{x} + \frac{H}{2}\mathbf{d}_1 + \theta^2\mathbf{d}_2\right]_{,3} \ , \ (5.2.8),(5.3.14),(5.25.8)$$

$$\text{for } \theta^2 = \frac{W}{2},\ g^{1/2}\alpha(\theta^1,\tfrac{W}{2})\mathbf{n}^* = \left[\mathbf{x} + \theta^1\mathbf{d}_1 + \frac{W}{2}\mathbf{d}_2\right]_{,3} \times \mathbf{d}_1 \ , \ (5.2.8),(5.3.14),(5.25.8)$$

$$\text{for } \theta^1 = -\frac{H}{2},\ g^{1/2}\alpha(-\tfrac{H}{2},\theta^2)\mathbf{n}^* = -\,\mathbf{d}_2 \times \left[\mathbf{x} - \frac{H}{2}\mathbf{d}_1 + \theta^2\mathbf{d}_2\right]_{,3} \ , \ (5.2.8),(5.3.14),(5.25.8)$$

$$\text{for } \theta^2 = -\frac{W}{2},\ g^{1/2}\alpha(\theta^1,-\tfrac{W}{2})\mathbf{n}^* = -\left[\mathbf{x} + \theta^1\mathbf{d}_1 - \frac{W}{2}\mathbf{d}_2\right]_{,3} \times \mathbf{d}_1 \quad (5.2.8),(5.3.14),(5.25.8)$$

$$\mathbf{n}^* \bullet \mathbf{n}^* = 1 \ , \qquad (5.3.14)$$

$$m\,\mathbf{b}_c = \int_{-W/2}^{W/2}\left\{g^{1/2}\alpha(\tfrac{H}{2},\theta^2)\,\mathbf{t}^*(\tfrac{H}{2},\theta^2,\theta^3)\right\}d\theta^2 + \int_{-H/2}^{H/2}\left\{g^{1/2}\alpha(\theta^1,\tfrac{W}{2})\,\mathbf{t}^*(\theta^1,\tfrac{W}{2},\theta^3)\right\}d\theta^1$$
$$+ \int_{-W/2}^{W/2}\left\{g^{1/2}\alpha(-\tfrac{H}{2},\theta^2)\,\mathbf{t}^*(-\tfrac{H}{2},\theta^2,\theta^3)\right\}d\theta^2$$
$$+ \int_{-H/2}^{H/2}\left\{g^{1/2}\alpha(\theta^1,-\tfrac{W}{2})\,\mathbf{t}^*(\theta^1,-\tfrac{W}{2},\theta^3)\right\}d\theta^1 \ , \qquad (5.3.16)$$

$$m\,\mathbf{b}_c^1 = \frac{H}{2}\int_{-W/2}^{W/2}\left\{g^{1/2}\alpha(\tfrac{H}{2},\theta^2)\,\mathbf{t}^*(\tfrac{H}{2},\theta^2,\theta^3)\right\}d\theta^2$$
$$+ \int_{-H/2}^{H/2}\left\{g^{1/2}\alpha(\theta^1,\tfrac{W}{2})\,\mathbf{t}^*(\theta^1,\tfrac{W}{2},\theta^3)\right\}\theta^1 d\theta^1$$
$$- \frac{H}{2}\int_{-W/2}^{W/2}\left\{g^{1/2}\alpha(-\tfrac{H}{2},\theta^2)\,\mathbf{t}^*(-\tfrac{H}{2},\theta^2,\theta^3)\right\}d\theta^2$$
$$+ \int_{-H/2}^{H/2}\left\{g^{1/2}\alpha(\theta^1,-\tfrac{W}{2})\,\mathbf{t}^*(\theta^1,-\tfrac{W}{2},\theta^3)\right\}\theta^1 d\theta^1 \ , \qquad (5.3.32)$$

$$m\,\mathbf{b}_c^2 = \int_{-W/2}^{W/2}\left\{g^{1/2}\alpha(\tfrac{H}{2},\theta^2)\,\mathbf{t}^*(\tfrac{H}{2},\theta^2,\theta^3)\right\}\theta^2 d\theta^2$$
$$+ \frac{W}{2}\int_{-H/2}^{H/2}\left\{g^{1/2}\alpha(\theta^1,\tfrac{W}{2})\,\mathbf{t}^*(\theta^1,\tfrac{W}{2},\theta^3)\right\}d\theta^1$$
$$+ \int_{-W/2}^{W/2}\left\{g^{1/2}\alpha(-\tfrac{H}{2},\theta^2)\,\mathbf{t}^*(-\tfrac{H}{2},\theta^2,\theta^3)\right\}\theta^2 d\theta^2$$
$$- \frac{W}{2}\int_{-H/2}^{H/2}\left\{g^{1/2}\alpha(\theta^1,-\tfrac{W}{2})\,\mathbf{t}^*(\theta^1,-\tfrac{W}{2},\theta^3)\right\}d\theta^1 \ , \qquad (5.3.32)$$

$$\mathbf{b} = \mathbf{b}_b + \mathbf{b}_c \ , \ \mathbf{b}^{\alpha} = \mathbf{b}_b^{\alpha} + \mathbf{b}_c^{\alpha} \ , \qquad (5.3.24),(5.3.37)$$

INERTIA QUANTITIES

$$m = \rho_0^*\, D^{1/2}\, HW \ , \ \rho_0 = \rho_0^*\, A \ , \qquad (5.4.25),(5.20.8)$$

$$y^1 = \frac{H^2}{|\mathbf{D}_1|^2}\left[\gamma_1 (\mathbf{D}_{1,3} \bullet \mathbf{D}^3)\right] \;,\quad y^2 = \frac{W^2}{|\mathbf{D}_2|^2}\left[\gamma_1 (\mathbf{D}_{2,3} \bullet \mathbf{D}^3)\right] \;, \qquad (5.20.11)$$

$$y^{11} = \frac{H^2}{\pi^2|\mathbf{D}_1|^2}\left[1 + \gamma_2 \frac{H}{|\mathbf{D}_1|}(\mathbf{D}_{1,3} \bullet \mathbf{D}^3)\right] \;,\; y^{12} = y^{21} = 0 \;, \qquad (5.20.11)$$

$$y^{22} = \frac{W^2}{\pi^2|\mathbf{D}_2|^2}\left[1 + \gamma_2 \frac{W}{|\mathbf{D}_2|}(\mathbf{D}_{2,3} \bullet \mathbf{D}^3)\right] \;, \qquad (5.20.11)$$

$$\gamma_1 = 0.07 \;,\; \gamma_2 = \frac{\pi^2 - 4}{8} \;, \qquad (5.22.3),(5.22.8)$$

BOUNDARY VALUES OF FORCE, COUPLES AND MOMENT

$$\mathbf{t}^3 \;,\; \mathbf{m}^\alpha \;,\; \mathbf{m} = \mathbf{d}_\alpha \times \mathbf{m}^\alpha \;, \qquad (5.10.3),(5.10.7)$$

GENERAL CONSTITUTIVE EQUATIONS

$$m\,\dot{\Sigma} = d_{33}^{1/2}\,\mathcal{P} = d_{33}^{1/2}\,\mathbf{T} \bullet \mathbf{D} + (\mathbf{F}^T\mathbf{m}^\alpha) \bullet \dot{\boldsymbol{\beta}}_\alpha \;, \qquad (5.6.17),(5.8.1)$$

$$m\,\Sigma = m\,\Sigma^*(\mathbf{C}) + \frac{1}{2}\,D_{33}^{-1/2}\,A\,\mathbf{K}^{\alpha\beta} \bullet (\boldsymbol{\beta}_\alpha \otimes \boldsymbol{\beta}_\beta) \;, \qquad (5.11.36)$$

$$\mathbf{m}^\alpha = D_{33}^{-1/2}\,A\,\mathbf{F}^{-T}\,\mathbf{K}^{\alpha\beta}\,\boldsymbol{\beta}_\beta \;, \qquad (5.11.36)$$

$$\mathbf{t}^i = 2\,m\,\mathbf{F}\frac{\partial \Sigma^*(\mathbf{C})}{\partial \mathbf{C}}\mathbf{D}^i - \mathbf{m}^\alpha\,(\mathbf{d}_{\alpha,3} \bullet \mathbf{d}^i) \;, \qquad (5.11.36)$$

$$\mathbf{K}^{\alpha\beta} = \left[(\mathbf{D}^\alpha \bullet \mathbf{M}_\gamma)\,(\mathbf{D}^\beta \bullet \mathbf{M}_\delta)\,K_{i\gamma j\delta}\right](\mathbf{M}_i \otimes \mathbf{M}_j) \;, \qquad (5.12.6),(5.12.7)$$

CONSTRAINTS

$$\mathbf{t}^i = \hat{\mathbf{t}}^i + \bar{\mathbf{t}}^i \;,\; \mathbf{m}^\alpha = \hat{\mathbf{m}}^\alpha + \bar{\mathbf{m}}^\alpha \;, \qquad (5.9.14)$$

Incompressibility

$$J = 1 \;,\; \bar{\mathbf{t}}^i = \gamma\,\mathbf{d}^i \;,\; \bar{\mathbf{m}}^\alpha = 0 \;,\; \nu^* = \frac{1}{2} \;, \qquad (5.9.1),(5.9.27),(5.16.13),(5.16.14)$$

Eliminate normal cross-sectional extension only

$$\bar{\mathbf{d}}_\alpha = \mathbf{d}_\alpha - d_{33}^{-1}\,(\mathbf{d}_\alpha \bullet \mathbf{d}_3)\,\mathbf{d}_3 \;. \qquad (5.2.5)$$

$$\bar{\mathbf{d}}_1 \bullet \bar{\mathbf{d}}_1 = \mathbf{d}_1 \bullet \mathbf{d}_1 - d_{33}^{-1}\,(\mathbf{d}_1 \bullet \mathbf{d}_3)^2 = \text{constant} \;, \qquad (5.9.4)$$

$$\bar{\mathbf{t}}^i = \gamma^{11}\left[(\mathbf{d}_1 \otimes \mathbf{d}_1) + d_{33}^{-2}\,(\mathbf{d}_1 \bullet \mathbf{d}_3)^2\,(\mathbf{d}_3 \otimes \mathbf{d}_3) - d_{33}^{-1}\,(\mathbf{d}_1 \bullet \mathbf{d}_3)\,(\mathbf{d}_1 \otimes \mathbf{d}_3 + \mathbf{d}_3 \otimes \mathbf{d}_1)\right]\mathbf{d}^i \;,\; \bar{\mathbf{m}}^\alpha = 0 \;, \qquad (5.9.5),(5.9.20)$$

$$\bar{\mathbf{d}}_2 \bullet \bar{\mathbf{d}}_2 = \mathbf{d}_2 \bullet \mathbf{d}_2 - d_{33}^{-1}\,(\mathbf{d}_2 \bullet \mathbf{d}_3)^2 = \text{constant} \;, \qquad (5.9.3)$$

$$\bar{\mathbf{t}}^i = \gamma^{22}\left[(\mathbf{d}_2 \otimes \mathbf{d}_2) + d_{33}^{-2}\,(\mathbf{d}_2 \bullet \mathbf{d}_3)^2\,(\mathbf{d}_3 \otimes \mathbf{d}_3) - d_{33}^{-1}\,(\mathbf{d}_2 \bullet \mathbf{d}_3)\,(\mathbf{d}_2 \otimes \mathbf{d}_3 + \mathbf{d}_3 \otimes \mathbf{d}_2)\right]\mathbf{d}^i \;,\; \bar{\mathbf{m}}^\alpha = 0 \;, \qquad (5.9.5),(5.9.20)$$

Eliminate normal cross-sectional shear deformation only

$$\bar{\mathbf{d}}_1 \bullet \bar{\mathbf{d}}_2 = 0 \;,\; \bar{\mathbf{t}}^i = \gamma^{12}\left[(\mathbf{d}_1 \otimes \mathbf{d}_2 + \mathbf{d}_2 \otimes \mathbf{d}_1) + 2\,d_{33}^{-2}\,(\mathbf{d}_1 \bullet \mathbf{d}_3)\,(\mathbf{d}_2 \bullet \mathbf{d}_3)\,(\mathbf{d}_3 \otimes \mathbf{d}_3) - d_{33}^{-1}\,(\mathbf{d}_2 \bullet \mathbf{d}_3)\,(\mathbf{d}_1 \otimes \mathbf{d}_3 + \mathbf{d}_3 \otimes \mathbf{d}_1) - d_{33}^{-1}\,(\mathbf{d}_1 \bullet \mathbf{d}_3)\,(\mathbf{d}_2 \otimes \mathbf{d}_3 + \mathbf{d}_3 \otimes \mathbf{d}_2)\right]\mathbf{d}^i \;, \qquad (5.9.8),(5.9.9),(5.9.20)$$

$$\gamma^{12} = \gamma^{21} \ , \quad \bar{\mathbf{m}}^{\alpha} = 0 \ , \qquad (5.9.8),(5.9.9),(5.9.20)$$

Eliminate tangential shear deformation only

$$\mathbf{d}_{\alpha} \bullet \mathbf{d}_3 = 0 \ , \quad \bar{\mathbf{t}}^{i} = \gamma^{3\alpha} \left[\mathbf{d}_{\alpha}\otimes\mathbf{d}_3 + \mathbf{d}_3\otimes\mathbf{d}_{\alpha}\right] \mathbf{d}^{i} \ , \quad \bar{\mathbf{m}}^{\alpha} = 0 \ , \quad (5.9.6),(5.9.7),(5.9.20)$$

Eliminate tangential extensional deformation only

$$\mathbf{d}_3 \bullet \mathbf{d}_3 = \text{constant} \ , \quad \bar{\mathbf{t}}^{\alpha} = 0 \ , \quad \bar{\mathbf{t}}^{3} = \gamma^{33}\, \mathbf{d}_3 \ , \quad \bar{\mathbf{m}}^{\alpha} = 0 \ , \quad (5.9.10),(5.9.11),(5.9.20)$$

Eliminate normal cross-sectional extension, normal cross-sectional shear deformation and tangential shear deformation

$$(\mathbf{d}_1\otimes\mathbf{d}_1) \bullet \mathbf{D} = 0 \ , \quad (\mathbf{d}_2\otimes\mathbf{d}_2) \bullet \mathbf{D} = 0 \ , \qquad (5.9.21)$$

$$(\mathbf{d}_{\alpha}\otimes\mathbf{d}_3 + \mathbf{d}_3\otimes\mathbf{d}_{\alpha}) \bullet \mathbf{D} = 0 \ , \quad (\mathbf{d}_1\otimes\mathbf{d}_2 + \mathbf{d}_2\otimes\mathbf{d}_1) \bullet \mathbf{D} = 0 \ , \qquad (5.9.21)$$

$$\bar{\mathbf{t}}^{\alpha} = \gamma^{\alpha\beta}\, \mathbf{d}_{\beta} + \gamma^{3\alpha}\, \mathbf{d}_3 \ , \quad \bar{\mathbf{t}}^{3} = \gamma^{3\alpha}\, \mathbf{d}_{\alpha} \ , \quad \bar{\mathbf{m}}^{\alpha} = 0 \ , \quad (5.9.22),(5.9.23)$$

ORTHOTROPIC RODS - SMALL STRAINS (LARGE DISPLACEMENTS)

$$\mathrm{m}\,\Sigma = \frac{1}{2} D_{33}^{1/2}\, A\, \mathbf{K}^{*} \bullet (\mathbf{E}\otimes\mathbf{E}) + \frac{1}{2} D_{33}^{-1/2}\, A\, \mathbf{K}^{\alpha\beta} \bullet (\boldsymbol{\beta}_{\alpha}\otimes\boldsymbol{\beta}_{\beta}) \ , \qquad (5.12.4)$$

$$\mathbf{m}^{\alpha} = D_{33}^{-1/2}\, A\, \mathbf{F}^{-T}\, \mathbf{K}^{\alpha\beta}\, \boldsymbol{\beta}_{\beta} \ , \qquad (5.12.4)$$

$$\mathbf{t}^{i} = D_{33}^{1/2}\, A\, \mathbf{F} \left[\mathbf{K}^{*} \bullet \mathbf{E}\right] \mathbf{D}^{i} \ - \mathbf{m}^{\alpha} \left(\mathbf{d}_{\alpha,3} \bullet \mathbf{d}^{i}\right) \ , \qquad (5.12.4)$$

$$\mathbf{E} = E_{ij}\, (\mathbf{M}_i \otimes \mathbf{M}_j) \ , \quad K^{*}_{ijkl} = \mathbf{K}^{*} \bullet (\mathbf{M}_i\otimes\mathbf{M}_j\otimes\mathbf{M}_k\otimes\mathbf{M}_l) \ , \qquad (5.12.6)$$

$$\mathbf{M}_{\alpha} = \mathbf{D}_{\alpha} \ , \quad \mathbf{M}_3 = \frac{\mathbf{D}_3}{|\mathbf{D}_3|} \ , \qquad (5.16.11)$$

$$\begin{aligned}
\mathbf{K}^{*} \bullet \mathbf{E} = {} & \left[K^{*}_{1111}\, E_{11} + K^{*}_{1122}\, E_{22} + K^{*}_{1133}\, E_{33}\right] (\mathbf{M}_1\otimes\mathbf{M}_1) \\
& + \left[K^{*}_{1122}\, E_{11} + K^{*}_{2222}\, E_{22} + K^{*}_{2233}\, E_{33}\right] (\mathbf{M}_2\otimes\mathbf{M}_2) \\
& + \left[K^{*}_{1133}\, E_{11} + K^{*}_{2233}\, E_{22} + K^{*}_{3333}\, E_{33}\right] (\mathbf{M}_3\otimes\mathbf{M}_3) \\
& + \left[K^{*}_{1212}\, (E_{12} + E_{21})\right] (\mathbf{M}_1\otimes\mathbf{M}_2 + \mathbf{M}_2\otimes\mathbf{M}_1) \\
& + \left[K^{*}_{1313}\, (E_{13} + E_{31})\right] (\mathbf{M}_1\otimes\mathbf{M}_3 + \mathbf{M}_3\otimes\mathbf{M}_1) \\
& + \left[K^{*}_{2323}\, (E_{23} + E_{32})\right] (\mathbf{M}_2\otimes\mathbf{M}_3 + \mathbf{M}_3\otimes\mathbf{M}_2) \ , \qquad (3.12.12)
\end{aligned}$$

$$\mathbf{K}^{11} = K_{1212}\, \mathbf{M}_2\otimes\mathbf{M}_2 + K_{3131}\, \mathbf{M}_3\otimes\mathbf{M}_3 \ , \quad \mathbf{K}^{12} = \mathbf{K}^{21} = 0 \ , \qquad (5.16.12)$$

$$\mathbf{K}^{22} = K_{1212}\, \mathbf{M}_1\otimes\mathbf{M}_1 + K_{3232}\, \mathbf{M}_3\otimes\mathbf{M}_3 \ , \qquad (5.16.12)$$

$$K_{3131} = \frac{H^2}{12}\left[\frac{1}{C^{*}_{3333}}\right] , \quad K_{3232} = \frac{W^2}{12}\left[\frac{1}{C^{*}_{3333}}\right] , (3.14.11),(5.14.27)$$

$$A\, K_{1212} = A\, K_{2121} = \frac{1}{2} B^{*}_{3} \ , \qquad (5.15.1),(5.16.9)$$

ISOTROPIC RODS - SMALL STRAINS (LARGE DISPLACEMENTS)

Use the equations for orthotropic rods - small strains (large displacements) with

$$\mathbf{K}^{*} \bullet \mathbf{E} = 2\mu^{*} \left[\left\{\frac{\nu^{*}}{1-2\nu^{*}}\right\} (\mathbf{E} \bullet \mathbf{I})\, \mathbf{I} + \mathbf{E}\right] , \qquad \text{Table 3.12.1, (3.12.15)}$$

$$K^{*}_{1111} = K^{*}_{2222} = K^{*}_{3333} = 2\mu^{*} \left[\frac{1-\nu^{*}}{1-2\nu^{*}}\right] , \qquad \text{Table 3.12.1, (3.12.13)}$$

$$K^*_{1122} = K^*_{1133} = K^*_{2233} = 2\mu^* \left[\frac{\nu^*}{1-2\nu^*}\right] , \quad \text{Table 3.12.1, (3.12.13)}$$

$$K^*_{1212} = K^*_{1313} = K^*_{2323} = \mu^* , \quad (3.12.13)$$

$$K_{1212} = \frac{\mu^* H^2}{6}\left\{\frac{b^*(\eta)}{\eta}\right\} = \frac{\mu^* W^2}{6}\left\{\eta\, b^*(\eta)\right\} = \frac{\mu^* HW}{6}\, b^*(\eta) , \quad (5.15.1),(5.16.3)$$

$$K_{3131} = \frac{\mu^*(1+\nu^*)H^2}{6} , \quad K_{3232} = \frac{\mu^*(1+\nu^*)W^2}{6} , \quad (5.14.34)$$

$$\mathbf{K}^{11} = \frac{\mu^* H^2}{6}\left[\left\{\frac{b^*(\eta)}{\eta}\right\} \mathbf{M}_2\otimes\mathbf{M}_2 + (1+\nu^*)\,\mathbf{M}_3\otimes\mathbf{M}_3\right] , \quad \mathbf{K}^{12} = \mathbf{K}^{21} = 0 , \quad (5.16.13)$$

$$\mathbf{K}^{22} = \frac{\mu^* W^2}{6}\left[\left\{\eta\, b^*(\eta)\right\} \mathbf{M}_1\otimes\mathbf{M}_1 + (1+\nu^*)\,\mathbf{M}_3\otimes\mathbf{M}_3\right] , \quad \eta = \frac{H}{W} , \quad (5.16.13),(5.16.14)$$

NONLINEAR ISOTROPIC ROD

Use the general constitutive equations, with $\mathbf{K}^{\alpha\beta}$ given by the expression for isotropic rods - small strains (large displacements) and with

$$\Sigma^*(\mathbf{C}) = \hat{\Sigma}^*(\alpha_1,\alpha_2,J) , \quad (5.24.4)$$

$$m\,\hat{\Sigma}^*(\alpha_1,\alpha_2,J) = D_{33}^{1/2}\, A\, \mu_0^* \left[(1-4C_2)(\alpha_1-3) + C_2(\alpha_2-3)\right]$$
$$+ D_{33}^{1/2}\, A\, K_0^* \left[(J-1) - \ln(J)\right] , \quad (5.24.8)$$

$$\mathbf{T} = -p\,\mathbf{I} + \mathbf{T}' , \quad \mathbf{T}'\bullet\mathbf{I} = 0 , \quad (5.24.3)$$

$$d_{33}^{1/2}\,\mathbf{T} = 2\,m\,\mathbf{F}\frac{\partial\Sigma^*(\mathbf{C})}{\partial\mathbf{C}}\,\mathbf{F}^T , \quad d_{33}^{1/2}\,p = -\,m\,J\frac{\partial\hat{\Sigma}^*}{\partial J} , \quad (5.24.3)$$

$$d_{33}^{1/2}\,\mathbf{T}' = 2m\,J^{-2/3}\frac{\partial\hat{\Sigma}^*}{\partial\alpha_1}\left[\mathbf{B} - \frac{1}{3}(\mathbf{B}\bullet\mathbf{I})\,\mathbf{I}\right]$$
$$+ 4m\,J^{-4/3}\frac{\partial\hat{\Sigma}^*}{\partial\alpha_2}\left[\mathbf{B}^2 - \frac{1}{3}(\mathbf{B}^2\bullet\mathbf{I})\,\mathbf{I}\right] . \quad (5.24.3)$$

$$\mathbf{m}^\alpha = D_{33}^{-1/2}\, A\, \mathbf{F}^{-T}\,\mathbf{K}^{\alpha\beta}\,\boldsymbol{\beta}_\beta , \quad (5.24.4)$$

$$\mathbf{t}^i = d_{33}^{1/2}\left[-p\,\mathbf{I} + \mathbf{T}'\right]\mathbf{d}^i - \mathbf{m}^\alpha\,(\mathbf{d}_{\alpha,3}\bullet\mathbf{d}^i) , \quad (5.24.4)$$

LINEARIZED EQUATIONS

Use the general values for m, y^α, $y^{\alpha\beta}$

$$\mathbf{x} = \mathbf{X} + \mathbf{u} , \quad \mathbf{d}_i = \mathbf{D}_i + \boldsymbol{\delta}_i , \quad \boldsymbol{\delta}_3 = \mathbf{u}_{,3} , \quad (5.13.1)$$

$$\tilde{\mathbf{E}} = \frac{1}{2}(\boldsymbol{\delta}_i\otimes\mathbf{D}^i + \mathbf{D}^i\otimes\boldsymbol{\delta}_i) , \quad \tilde{\boldsymbol{\beta}}_\alpha = \boldsymbol{\delta}_{\alpha,3} - (\mathbf{D}_{\alpha,3}\bullet\mathbf{D}^i)\,\boldsymbol{\delta}_i , \quad (5.13.16)$$

$$\tilde{\mathbf{b}}_b = \tilde{\mathbf{b}}^* , \quad \tilde{\mathbf{b}}_b^1 = \tilde{\mathbf{b}}^*\left[\frac{H^2}{12}(\mathbf{D}_{1,3}\bullet\mathbf{D}^3)\right] , \quad (5.20.12)$$

$$\tilde{\mathbf{b}}_b^2 = \tilde{\mathbf{b}}^*\left[\frac{W^2}{12}(\mathbf{D}_{2,3}\bullet\mathbf{D}^3)\right] , \quad (5.20.12)$$

for $\theta^1 = \frac{H}{2}$, $G^{1/2}\alpha(\frac{H}{2},\theta^2)\mathbf{N}^* = \mathbf{D}_2\times\left[\mathbf{X} + \frac{H}{2}\mathbf{D}_1 + \theta^2\mathbf{D}_2\right]_{,3}$,(5.2.8),(5.3.14),(5.25.8)

for $\theta^2 = \frac{W}{2}$, $G^{1/2}\alpha(\theta^1,\frac{W}{2})\mathbf{N}^* = \left[\mathbf{X} + \theta^1\mathbf{D}_1 + \frac{W}{2}\mathbf{D}_2\right]_{,3}\times\mathbf{D}_1$, (5.2.8),(5.3.14),(5.25.8)

$$\text{for } \theta^1 = -\frac{H}{2},\ G^{1/2}\alpha(-\frac{H}{2},\theta^2)\mathbf{N}^* = -\mathbf{D}_2\times[\mathbf{X} - \frac{H}{2}\mathbf{D}_1 + \theta^2\mathbf{D}_2]_{,3} \quad (5.2.8),(5.3.14),(5.25.8)$$

$$\text{for } \theta^2 = -\frac{W}{2},\ G^{1/2}\alpha(\theta^1,-\frac{W}{2})\mathbf{N}^* = -[\mathbf{X} + \theta^1\mathbf{D}_1 - \frac{W}{2}\mathbf{D}_2]_{,3}\times\mathbf{D}_1 \quad (5.2.8),(5.3.14),(5.25.8)$$

$$\mathbf{N}^* \bullet \mathbf{N}^* = 1 \ , \qquad (5.3.14)$$

$$\begin{aligned} m\,\tilde{\mathbf{b}}_c = &\int_{-W/2}^{W/2}\{G^{1/2}\alpha(\frac{H}{2},\theta^2)\ \tilde{\mathbf{t}}^*(\frac{H}{2},\theta^2,\theta^3)\}\,d\theta^2 \\ &+ \int_{-H/2}^{H/2}\{G^{1/2}\alpha(\theta^1,\frac{W}{2})\ \tilde{\mathbf{t}}^*(\theta^1,\frac{W}{2},\theta^3)\}\,d\theta^1 \\ &+ \int_{-W/2}^{W/2}\{G^{1/2}\alpha(-\frac{H}{2},\theta^2)\ \tilde{\mathbf{t}}^*(-\frac{H}{2},\theta^2,\theta^3)\}\,d\theta^2 \\ &+ \int_{-H/2}^{H/2}\{G^{1/2}\alpha(\theta^1,-\frac{W}{2})\ \tilde{\mathbf{t}}^*(\theta^1,-\frac{W}{2},\theta^3)\}\,d\theta^1 \ , \end{aligned} \qquad (5.3.16)$$

$$\begin{aligned} m\,\tilde{\mathbf{b}}_c^1 = &\frac{H}{2}\int_{-W/2}^{W/2}\{G^{1/2}\alpha(\frac{H}{2},\theta^2)\ \tilde{\mathbf{t}}^*(\frac{H}{2},\theta^2,\theta^3)\}\,d\theta^2 \\ &+ \int_{-H/2}^{H/2}\{G^{1/2}\alpha(\theta^1,\frac{W}{2})\ \tilde{\mathbf{t}}^*(\theta^1,\frac{W}{2},\theta^3)\}\,\theta^1 d\theta^1 \\ &- \frac{H}{2}\int_{-W/2}^{W/2}\{G^{1/2}\alpha(-\frac{H}{2},\theta^2)\ \tilde{\mathbf{t}}^*(-\frac{H}{2},\theta^2,\theta^3)\}\,d\theta^2 \\ &+ \int_{-H/2}^{H/2}\{G^{1/2}\alpha(\theta^1,-\frac{W}{2})\ \tilde{\mathbf{t}}^*(\theta^1,-\frac{W}{2},\theta^3)\}\,\theta^1 d\theta^1 \ , \end{aligned} \qquad (5.3.32)$$

$$\begin{aligned} m\,\tilde{\mathbf{b}}_c^2 = &\int_{-W/2}^{W/2}\{G^{1/2}\alpha(\frac{H}{2},\theta^2)\ \tilde{\mathbf{t}}^*(\frac{H}{2},\theta^2,\theta^3)\}\,\theta^2 d\theta^2 \\ &+ \frac{W}{2}\int_{-H/2}^{H/2}\{G^{1/2}\alpha(\theta^1,\frac{W}{2})\ \tilde{\mathbf{t}}^*(\theta^1,\frac{W}{2},\theta^3)\}\,d\theta^1 \\ &+ \int_{-W/2}^{W/2}\{G^{1/2}\alpha(-\frac{H}{2},\theta^2)\ \tilde{\mathbf{t}}^*(-\frac{H}{2},\theta^2,\theta^3)\}\,\theta^2 d\theta^2 \\ &- \frac{W}{2}\int_{-H/2}^{H/2}\{G^{1/2}\alpha(\theta^1,-\frac{W}{2})\ \tilde{\mathbf{t}}^*(\theta^1,-\frac{W}{2},\theta^3)\}\,d\theta^1 \ , \end{aligned} \qquad (5.3.32)$$

$$\mathbf{b} = \tilde{\mathbf{b}} = \tilde{\mathbf{b}}_b + \tilde{\mathbf{b}}_c \ , \ \mathbf{b}^\alpha = \tilde{\mathbf{b}}^\alpha = \tilde{\mathbf{b}}_b^\alpha + \tilde{\mathbf{b}}_c^\alpha \ , \quad (5.3.24),(5.3.37),(5.13.3)$$

$$\mathbf{t}^i = \tilde{\mathbf{t}}^i \ , \ \mathbf{m}^\alpha = \tilde{\mathbf{m}}^\alpha \ , \qquad (5.13.4)$$

$$\rho = \rho_0\left[1 - D_{33}^{-1}\,\mathbf{D}_3 \bullet \boldsymbol{\delta}_3\right] \ , \qquad (5.13.6)$$

$$m\,(\ddot{\mathbf{u}} + y^\alpha\,\ddot{\boldsymbol{\delta}}_\alpha) = m\,\tilde{\mathbf{b}} + \tilde{\mathbf{t}}^3{}_{,3} \ , \qquad (5.13.15)$$

$$m\,(y^\alpha\,\ddot{\mathbf{u}} + y^{\alpha\beta}\,\ddot{\boldsymbol{\delta}}_\beta) = m\,\tilde{\mathbf{b}}^\alpha - \tilde{\mathbf{t}}^\alpha + \tilde{\mathbf{m}}^\alpha{}_{,3} \ , \qquad (5.13.15)$$

$$\tilde{\mathbf{m}}^\alpha = D_{33}^{-1/2}\,A\,\mathbf{K}^{\alpha\beta}\,\tilde{\boldsymbol{\beta}}_\beta \ , \ \tilde{\mathbf{t}}^i = \left[D_{33}^{1/2}\,A\,\mathbf{K}^* \bullet \tilde{\mathbf{E}}\right]\mathbf{D}^i - \tilde{\mathbf{m}}^\alpha\,(\mathbf{D}_{\alpha,3} \bullet \mathbf{D}^i) \ ,(5.13.16)$$

5.27 Linearized equations for beams with rectangular cross-sections

This section presents linearized equations for isotropic beams with rectangular cross-sections as specified by (5.14.3). Using the equations that are summarized in section 5.26 the kinematics of the beam are characterized by

$$\mathbf{X} = \theta^3\,\mathbf{e}_3 \ , \quad \mathbf{D}_i = \mathbf{e}_i \ , \quad \mathbf{x} = \mathbf{X} + \mathbf{u} \ , \quad \mathbf{d}_i = \mathbf{D}_i + \boldsymbol{\delta}_i \ , \quad \mathbf{u} = u_i\,\mathbf{e}_i \ ,$$

$$\boldsymbol{\delta}_\alpha = \tilde{\delta}_{i\alpha}\,\mathbf{e}_i \;,\quad \boldsymbol{\delta}_3 = (u_{i,3})\,\mathbf{e}_i \;,$$

$$\tilde{\mathbf{E}} = \frac{1}{2}\,[\tilde{\delta}_{i\alpha}\,(\mathbf{e}_i\otimes\mathbf{e}_\alpha + \mathbf{e}_\alpha\otimes\mathbf{e}_i) + u_{i,3}\,(\mathbf{e}_i\otimes\mathbf{e}_3 + \mathbf{e}_3\otimes\mathbf{e}_i)] \;,\quad \tilde{\boldsymbol{\beta}}_\alpha = \tilde{\delta}_{i\alpha,3}\,\mathbf{e}_i \,. \quad (5.27.1)$$

Next, specifying the material directions $\mathbf{M}_i{=}\mathbf{D}_i$, considering an isotropic material, and using the constitutive equations of section 5.26 it can be shown that

$$\tilde{\mathbf{m}}^1 = \frac{\mu^* HW}{6}\,[\{HW\,b^*(\eta)\}(\tilde{\delta}_{21,3})\,\mathbf{e}_2 + H^2(1+\nu^*)(\tilde{\delta}_{31,3})\,\mathbf{e}_3] \;,\quad \eta = \frac{H}{W} \;,$$

$$\tilde{\mathbf{m}}^2 = \frac{\mu^* HW}{6}\,[-\{HW\,b^*(\eta)\}(\tilde{\delta}_{12,3})\,\mathbf{e}_1 + W^2(1+\nu^*)(\tilde{\delta}_{32,3})\,\mathbf{e}_3] \;,$$

$$\tilde{\mathbf{m}} = \mathbf{e}_\alpha \times \tilde{\mathbf{m}}^\alpha = \frac{\mu^* HW}{6}\,[W^2(1+\nu^*)(\tilde{\delta}_{32,3})\,\mathbf{e}_1 - H^2(1+\nu^*)(\tilde{\delta}_{31,3})\,\mathbf{e}_2$$
$$+ \{HW\,b^*(\eta)\}(\tilde{\delta}_{21,3} + \tilde{\delta}_{12,3})\,\mathbf{e}_3] \;,$$

$$\tilde{\mathbf{t}}^1 = 2\mu^* HW\,[\{\frac{1}{1-2\nu^*}\}\{(1-\nu^*)\,\tilde{\delta}_{11} + \nu^*\,\tilde{\delta}_{22} + \nu^*\,u_{3,3})\}\mathbf{e}_1 + \frac{1}{2}(\tilde{\delta}_{12} + \tilde{\delta}_{21})\,\mathbf{e}_2$$
$$+ \frac{1}{2}(u_{1,3} + \tilde{\delta}_{31})\,\mathbf{e}_3\,] \;,$$

$$\tilde{\mathbf{t}}^2 = 2\mu^* HW\,[\,\frac{1}{2}(\tilde{\delta}_{12} + \tilde{\delta}_{21})\,\mathbf{e}_1 + \{\frac{1}{1-2\nu^*}\}\{\nu^*\,\tilde{\delta}_{11} + (1-\nu^*)\,\tilde{\delta}_{22} + \nu^*\,u_{3,3})\}\mathbf{e}_2$$
$$+ \frac{1}{2}(u_{2,3} + \tilde{\delta}_{32})\,\mathbf{e}_3\,] \;,$$

$$\tilde{\mathbf{t}}^3 = 2\mu^* HW\,[\,\frac{1}{2}(u_{1,3} + \tilde{\delta}_{31})\,\mathbf{e}_1 + \frac{1}{2}(u_{2,3} + \tilde{\delta}_{32})\,\mathbf{e}_2$$
$$+ \{\frac{1}{1-2\nu^*}\}\{\nu^*\,\tilde{\delta}_{11} + \nu^*\,\tilde{\delta}_{22} + (1-\nu^*)\,u_{3,3})\}\mathbf{e}_3\,] \;, \qquad (5.27.2)$$

where $b^*(\eta)$ characterizes the torsional rigidity and is given by (5.15.1). Moreover, the assigned fields are specified by

$$\tilde{\mathbf{b}}_b = \tilde{\mathbf{b}}^* \;,\quad \tilde{\mathbf{b}}_b^\alpha = 0 \;,\quad \mathbf{b} = \tilde{\mathbf{b}} = \tilde{\mathbf{b}}_b + \tilde{\mathbf{b}}_c \;,\quad \mathbf{b}^\alpha = \tilde{\mathbf{b}}^\alpha = \tilde{\mathbf{b}}_b^\alpha + \tilde{\mathbf{b}}_c^\alpha \;, \qquad (5.27.3)$$

where the assigned fields $\tilde{\mathbf{b}}_b^\alpha$ and $\tilde{\mathbf{b}}_c^\alpha$ due to surface tractions on the beam are summarized in section 5.26. Also, the inertia quantities are given by

$$m = \rho_0^*\,HW \;,\quad \rho_0 = \rho_0^*\,HW \;,$$

$$y^\alpha = 0 \;,\quad y^{11} = \frac{H^2}{\pi^2} \;,\quad y^{12} = y^{21} = 0 \;,\quad y^{22} = \frac{W^2}{\pi^2} \,. \qquad (5.27.4)$$

Now, using these results the linearized versions of the conservation of mass and the balances of linear momentum and director momentum reduce to

$$\rho = \rho_0\,[1 - u_{3,3}\,] \;,$$

$$m\,\ddot{u}_1 = m\,(\mathbf{e}_1 \bullet \tilde{\mathbf{b}}) + \mu^* HW\,(u_{1,33} + \tilde{\delta}_{31,3}) \;,$$

$$m\,\ddot{u}_2 = m\,(\mathbf{e}_2 \bullet \tilde{\mathbf{b}}) + \mu^* HW\,(u_{2,33} + \tilde{\delta}_{32,3}) \;,$$

$$m\,\ddot{u}_3 = m\,(\mathbf{e}_3 \bullet \tilde{\mathbf{b}}) + \{\frac{2\mu^* HW}{1-2\nu^*}\}\{\nu^*\,\tilde{\delta}_{11,3} + \nu^*\,\tilde{\delta}_{22,3} + (1-\nu^*)\,u_{3,33})\} \;,$$

$$m\, y^{11}\, \ddot{\tilde{\delta}}_{11} = m\,(\mathbf{e}_1 \bullet \tilde{\mathbf{b}}^1) - \{\frac{2\mu^* HW}{1-2\nu^*}\}\{(1-\nu^*)\,\tilde{\delta}_{11} + \nu^*\,\tilde{\delta}_{22} + \nu^*\, u_{3,3})\} \ ,$$

$$m\, y^{11}\, \ddot{\tilde{\delta}}_{21} = m\,(\mathbf{e}_2 \bullet \tilde{\mathbf{b}}^1) - \mu^* HW\,(\tilde{\delta}_{12} + \tilde{\delta}_{21}) + \{\frac{\mu^* H^2 W^2 b^*(\eta)}{6}\}(\tilde{\delta}_{21,33}) \ ,$$

$$m\, y^{11}\, \ddot{\tilde{\delta}}_{31} = m\,(\mathbf{e}_3 \bullet \tilde{\mathbf{b}}^1) - \mu^* HW\,(u_{1,3} + \tilde{\delta}_{31}) + \frac{E^* H^3 W}{12}(\tilde{\delta}_{31,33}) \ ,$$

$$m\, y^{22}\, \ddot{\tilde{\delta}}_{12} = m\,(\mathbf{e}_1 \bullet \tilde{\mathbf{b}}^2) - \mu^* HW\,(u_{2,3} + \tilde{\delta}_{32}) - \{\frac{\mu^* H^2 W^2 b^*(\eta)}{6}\}(\tilde{\delta}_{12,33}) \ ,$$

$$m\, y^{22}\, \ddot{\tilde{\delta}}_{22} = m\,(\mathbf{e}_2 \bullet \tilde{\mathbf{b}}^2) - \{\frac{2\mu^* HW}{1-2\nu^*}\}\{\nu^*\,\tilde{\delta}_{11} + (1-\nu^*)\,\tilde{\delta}_{22} + \nu^*\, u_{3,3})\} \ ,$$

$$m\, y^{22}\, \ddot{\tilde{\delta}}_{32} = m\,(\mathbf{e}_3 \bullet \tilde{\mathbf{b}}^2) - \mu^* HW\,(u_{2,3} + \tilde{\delta}_{32}) + \frac{E^* HW^3}{12}(\tilde{\delta}_{32,33}) \ , \qquad (5.27.5)$$

where the definition of Young's modulus E^* had been used.

In order to discuss the boundary conditions it is first noted that the linearized versions of the velocity $\mathbf{v}$ and the director velocities $\mathbf{w}_\alpha$ become

$$\mathbf{v} = \dot{u}_i\, \mathbf{e}_i \ , \quad \mathbf{w}_\alpha = \dot{\tilde{\delta}}_{i\alpha}\, \mathbf{e}_i \ . \qquad (5.27.6)$$

Thus, the rate of work of contact force and director couples applied to the ends of the beam can be written in the form

$$\tilde{\mathbf{t}}^3 \bullet \dot{\mathbf{u}} + \mathbf{m}^\alpha \bullet \mathbf{w}_\alpha = (\mathbf{e}_i \bullet \tilde{\mathbf{t}}^3)\, \dot{u}_i + (\mathbf{e}_2 \bullet \tilde{\mathbf{m}}^1)\, \dot{\tilde{\delta}}_{21} + (\mathbf{e}_3 \bullet \tilde{\mathbf{m}}^1)\, \dot{\tilde{\delta}}_{31}$$
$$+ (\mathbf{e}_1 \bullet \tilde{\mathbf{m}}^2)\, \dot{\tilde{\delta}}_{12} + (\mathbf{e}_3 \bullet \tilde{\mathbf{m}}^2)\, \dot{\tilde{\delta}}_{32} \ , \qquad (5.27.7)$$

which indicates that kinematic boundary conditions associated with (5.27.7) are characterized by specified values of

$$\{u_1 \,,\, u_2 \,,\, u_3 \,,\, \tilde{\delta}_{21} \,,\, \tilde{\delta}_{31} \,,\, \tilde{\delta}_{12} \,,\, \tilde{\delta}_{32}\} \ . \qquad (5.27.8)$$

In particular, it is noted that since $(\mathbf{e}_1 \bullet \tilde{\mathbf{m}}^1)$ and $(\mathbf{e}_2 \bullet \tilde{\mathbf{m}}^2)$ vanish, there are no boundary conditions specified for $\tilde{\delta}_{11}$ and $\tilde{\delta}_{22}$.

5.28 Bernoulli-Euler rods

The rod theory that has been developed in the previous sections includes the full kinematics of the Cosserat theory, and requires balance laws of linear and director momentum to determine three vectors quantities

$$\{\mathbf{x} \,,\, \mathbf{d}_\alpha\} \ . \qquad (5.28.1)$$

In particular, this general theory includes the effects of normal cross-sectional extension, tangential shear deformation, and normal cross-sectional shear deformation. Consequently, the directors $\mathbf{d}_\alpha$ are independent kinematic quantities. However, for some problems it is possible to obtain sufficiently accurate results by using a simpler constrained theory of rods.

It was shown in section 5.9 that when the effects of normal cross-sectional extension, tangential shear deformation, and normal cross-sectional shear deformation are eliminated, then the directors $\mathbf{d}_\alpha$ are constrained by the equations

$$\mathbf{d}_\alpha \bullet \mathbf{d}_\beta = \mathbf{D}_\alpha \bullet \mathbf{D}_\beta \ , \quad \mathbf{d}_\alpha \bullet \mathbf{d}_3 = 0 \ , \tag{5.28.2}$$

which in rate forms become

$$\mathbf{w}_\alpha \bullet \mathbf{d}_\beta + \mathbf{d}_\alpha \bullet \mathbf{w}_\beta = 0 \ , \quad \mathbf{w}_\alpha \bullet \mathbf{d}_3 + \mathbf{d}_\alpha \bullet \mathbf{w}_3 = 0 \ ,$$
$$[\mathbf{d}_\alpha \otimes \mathbf{d}_\beta + \mathbf{d}_\beta \otimes \mathbf{d}_\alpha] \bullet \mathbf{D} = 0 \ , \quad [\mathbf{d}_\alpha \otimes \mathbf{d}_3 + \mathbf{d}_3 \otimes \mathbf{d}_\alpha] \bullet \mathbf{D} = 0 \ . \tag{5.28.3}$$

Consequently, with the help of (5.9.12), (5.9.22) and (5.9.23) the constitutive equations and constraint responses can be written in the forms

$$\mathbf{T} = \hat{\mathbf{T}} + \bar{\mathbf{T}} \ , \quad \mathbf{t}^i = \hat{\mathbf{t}}^i + \bar{\mathbf{t}}^i \ , \quad \mathbf{m}^\alpha = \hat{\mathbf{m}}^\alpha + \bar{\mathbf{m}}^\alpha \ ,$$
$$\bar{\mathbf{t}}^\alpha = \gamma^{\alpha\beta}\, \mathbf{d}_\beta + \gamma^{3\alpha}\, \mathbf{d}_3 \ , \quad \bar{\mathbf{t}}^3 = \gamma^{3\alpha}\, \mathbf{d}_\alpha \ , \quad \gamma^{\alpha\beta} = \gamma^{\beta\alpha} \ , \quad \bar{\mathbf{m}}^\alpha = 0 \ ,$$
$$\bar{\mathbf{T}} = \frac{1}{2} d_{33}^{-1/2}\, \gamma^{\alpha\beta}\, (\mathbf{d}_\alpha \otimes \mathbf{d}_\beta + \mathbf{d}_\beta \otimes \mathbf{d}_\alpha) + d_{33}^{-1/2}\, \gamma^{3\alpha}\, (\mathbf{d}_3 \otimes \mathbf{d}_\alpha + \mathbf{d}_\alpha \otimes \mathbf{d}_3) \ , \tag{5.28.4}$$

where $\gamma^{i\alpha}$ are arbitrary function of (θ^3,t), with $\gamma^{\alpha\beta}$ being symmetric. This means that five components of the vectors $\mathbf{t}^i$ are arbitrary functions of time. Moreover, this means that all but one of the six components of the director momentum equations $(5.4.25)_3$ can be satisfied for arbitrary admissible motions of the constrained rod by using (5.28.4) and determining the functions $\gamma^{i\alpha}$ by (5.9.24)

$$\gamma^{i\alpha} = \mathbf{d}^i \bullet [\, m\, \mathbf{b}^\alpha - \hat{\mathbf{t}}^\alpha + \mathbf{m}^\alpha{}_{,3} - m\, (y^\alpha\, \dot{\mathbf{v}} + y^{\alpha\beta}\, \dot{\mathbf{w}}_\beta)] \ . \tag{5.28.5}$$

Also, since $\gamma^{\alpha\beta}$ is symmetric, two of these six equations are dependent and yield the condition (5.9.25).

This nonlinear Bernoulli-Euler rod theory is simpler than the complete Cosserat theory because the constraints (5.28.2) determine the directors $\mathbf{d}_\alpha$ to within a twist about the $\mathbf{d}_3$ axis in terms of deformations of the rod's curve. Five of the director momentum equations $(5.4.25)_3$ are satisfied by the Lagrange multipliers $\gamma^{i\alpha}$, and the balance laws of the theory reduce to the conservation of mass $(5.4.25)_1$, the balance of linear momentum (5.9.26), and one component of the balances of director momentum (5.9.25)

$$m = \rho\, d_{33}^{1/2} = \rho_0\, D_{33}^{1/2} \ , \quad m\, (\dot{\mathbf{v}} + y^\alpha\, \dot{\mathbf{w}}_\alpha) = m\, \mathbf{b} + [\, \hat{\mathbf{t}}^3 + \bar{\mathbf{t}}^3 \,]_{,3} \ ,$$
$$[\, m\, \mathbf{b}^1 - \hat{\mathbf{t}}^1 + \mathbf{m}^1{}_{,3} - m\, (y^1\, \dot{\mathbf{v}} + y^{1\beta}\, \dot{\mathbf{w}}_\beta)] \bullet \mathbf{d}^2$$
$$= [\, m\, \mathbf{b}^2 - \hat{\mathbf{t}}^2 + \mathbf{m}^2{}_{,3} - m\, (y^2\, \dot{\mathbf{v}} + y^{2\beta}\, \dot{\mathbf{w}}_\beta)] \bullet \mathbf{d}^1 \ , \tag{5.28.6}$$

Also, the constraint responses $\bar{\mathbf{t}}^i$ are determined by (5.28.4) and (5.28.5), and can be written in the forms

$$\bar{\mathbf{t}}^\alpha = [\mathbf{d}^\alpha \bullet \{ m\, \mathbf{b}^\beta - \hat{\mathbf{t}}^\beta + \mathbf{m}^\beta{}_{,3} - m\, (y^\beta\, \dot{\mathbf{v}} + y^{\beta\gamma}\, \dot{\mathbf{w}}_\gamma) \}]\, \mathbf{d}_\beta$$
$$+ \ [\mathbf{d}^3 \bullet \{ m\, \mathbf{b}^\alpha - \hat{\mathbf{t}}^\alpha + \mathbf{m}^\alpha{}_{,3} - m\, (y^\alpha\, \dot{\mathbf{v}} + y^{\alpha\beta}\, \dot{\mathbf{w}}_\beta) \}]\, \mathbf{d}_3$$
$$\bar{\mathbf{t}}^3 = [\mathbf{d}^3 \bullet \{ m\, \mathbf{b}^\alpha - \hat{\mathbf{t}}^\alpha + \mathbf{m}^\alpha{}_{,3} - m\, (y^\alpha\, \dot{\mathbf{v}} + y^{\alpha\beta}\, \dot{\mathbf{w}}_\beta) \}]\, \mathbf{d}_\alpha \ . \tag{5.28.7}$$

Furthermore, due to the simplifying constraints (5.28.2), it can be shown that

$$\mathbf{C} = \mathbf{D}_\alpha \otimes \mathbf{D}^\beta + d_{33}\, \mathbf{D}^3 \otimes \mathbf{D}^3 \ , \quad \mathbf{E} = \frac{1}{2} [d_{33} - D_{33}]\, \mathbf{D}^3 \otimes \mathbf{D}^3 \ , \tag{5.28.8}$$

which is consistent with uniaxial strain. This means that if (5.28.8) were substituted into the constitutive equations summarized in section 5.26, then the predicted response would correspond to uniaxial strain. However, the response of a thin rod-like structure usually is closer to uniaxial stress than to uniaxial strain. For this reason, the constitutive equations of the Bernoulli-Euler rod theory must be modified relative to those developed for the more general theory.

To this end, first consider a beam with rectangular cross-section that experiences homogeneous deformation for which

$$\mathbf{X} = \theta^3\,\mathbf{e}_3\ ,\quad \mathbf{D}_\alpha = \mathbf{e}_\alpha\ ,\quad \mathbf{x} = \lambda_3\theta^3\,\mathbf{e}_3\ ,\quad \mathbf{d}_1 = \lambda_1\,\mathbf{e}_1\ ,\quad \mathbf{d}_2 = \lambda_2\,\mathbf{e}_2\ , \tag{5.28.9}$$

where λ_i are constants, θ^i have the units of length, and $\mathbf{e}_i$ are constant orthonormal vectors. For this deformation it follows that

$$\mathbf{F} = \lambda_1\,\mathbf{e}_1\otimes\mathbf{e}_1 + \lambda_2\,\mathbf{e}_2\otimes\mathbf{e}_2 + \lambda_3\,\mathbf{e}_3\otimes\mathbf{e}_3\ ,\quad \mathrm{J} = \lambda_1\lambda_2\lambda_3\ ,$$
$$\mathbf{C} = \mathbf{B} = \lambda_1^2\,\mathbf{e}_1\otimes\mathbf{e}_1 + \lambda_2^2\,\mathbf{e}_2\otimes\mathbf{e}_2 + \lambda_3^2\,\mathbf{e}_3\otimes\mathbf{e}_3\ ,$$
$$\mathbf{E} = \mathrm{E}_{11}\,\mathbf{e}_1\otimes\mathbf{e}_1 + \mathrm{E}_{22}\,\mathbf{e}_2\otimes\mathbf{e}_2 + \mathrm{E}_{33}\,\mathbf{e}_3\otimes\mathbf{e}_3\ ,\quad \boldsymbol{\beta}_\alpha = 0$$
$$\mathrm{E}_{11} = \frac{1}{2}(\lambda_1^2 - 1)\ ,\quad \mathrm{E}_{22} = \frac{1}{2}(\lambda_2^2 - 1)\ ,\quad \mathrm{E}_{33} = \frac{1}{2}(\lambda_3^2 - 1)\ . \tag{5.28.10}$$

Moreover, for isotropic response with small strains the equations summarized in section 5.26 can be used to deduce that

$$\mathrm{m}\,\Sigma = \mu^*\,\mathrm{D}_{33}^{1/2}\,\mathrm{A}\,\Big[\Big\{\frac{\nu^*}{1-2\nu^*}\Big\}(\mathbf{E}\bullet\mathbf{I})^2 + \mathbf{E}\bullet\mathbf{E}\Big] + \frac{1}{2}\mathrm{D}_{33}^{-1/2}\,\mathrm{A}\,\mathbf{K}^{\alpha\beta}\bullet(\boldsymbol{\beta}_\alpha\otimes\boldsymbol{\beta}_\beta)\ ,$$
$$\hat{\mathbf{t}}^i = 2\mu^*\,\mathrm{HW}\,\mathbf{F}\,\Big[\Big\{\frac{\nu^*}{1-2\nu^*}\Big\}(\mathbf{E}\bullet\mathbf{I})\,\mathbf{I} + \mathbf{E}\Big]\,\mathbf{D}^i\ ,$$
$$\hat{\mathbf{t}}^1 = 2\mu^*\,\mathrm{HW}\,\lambda_1\,\Big[\Big\{\frac{\nu^*}{1-2\nu^*}\Big\}(\mathrm{E}_{11}+\mathrm{E}_{22}+\mathrm{E}_{33}) + \mathrm{E}_{11}\Big]\,\mathbf{e}_1\ ,$$
$$\hat{\mathbf{t}}^2 = 2\mu^*\,\mathrm{HW}\,\lambda_2\,\Big[\Big\{\frac{\nu^*}{1-2\nu^*}\Big\}(\mathrm{E}_{11}+\mathrm{E}_{22}+\mathrm{E}_{33}) + \mathrm{E}_{22}\Big]\,\mathbf{e}_2\ ,$$
$$\hat{\mathbf{t}}^3 = 2\mu^*\,\mathrm{HW}\,\lambda_3\,\Big[\Big\{\frac{\nu^*}{1-2\nu^*}\Big\}(\mathrm{E}_{11}+\mathrm{E}_{22}+\mathrm{E}_{33}) + \mathrm{E}_{33}\Big]\,\mathbf{e}_3\ ,\quad \hat{\mathbf{m}}^i = 0\ . \tag{5.28.11}$$

Therefore, for uniaxial stress these equations reduce to

$$\hat{\mathbf{t}}^1 = \hat{\mathbf{t}}^2 = 0\ ,\quad \mathrm{E}_{11} = \mathrm{E}_{22} = -\,\nu^*\,\mathrm{E}_{33}\ ,$$
$$\hat{\mathbf{t}}^3 = 2(1+\nu^*)\mu^*\,\mathrm{HW}\,\lambda_3\,\mathrm{E}_{33}\,\mathbf{e}_3 = \mathrm{HW}\,\mathrm{E}^*\,\lambda_3\,\mathrm{E}_{33}\,\mathbf{e}_3\ ,$$
$$\hat{\mathbf{t}}^3 = \frac{1}{2}\mathrm{HW}\,\mathrm{E}^*\,\lambda_3\,(\lambda_3^2 - 1)\,\mathbf{e}_3 = \mathrm{HW}\,\mathrm{E}^*\sqrt{1+2\mathrm{E}_{33}}\,\mathrm{E}_{33}\,\mathbf{e}_3\ , \tag{5.28.12}$$

where use has been made of Table 3.12.1 to express the results in terms of Young's modulus E^*. Consequently, even though the strain energy is a quadratic function of the strain $\mathbf{E}$, the force $\hat{\mathbf{t}}^3$ acting on the end of the rod is a nonlinear function of the strain E_{33}. Moreover, for the Bernoulli-Euler theory it is desirable to develop a more general constitutive equation for $\hat{\mathbf{t}}^3$ which can model the response of uniaxial stress.

In view of the above discussion, it is convenient to define the tangential stretch λ by

$$\lambda = \frac{d_{33}^{1/2}}{D_{33}^{1/2}} = \sqrt{\frac{\mathbf{d}_3 \bullet \mathbf{d}_3}{\mathbf{D}_3 \bullet \mathbf{D}_3}} \ , \quad \lambda^+ = \lambda \ , \tag{5.28.13}$$

which is trivially invariant under SRBM. Now, for a general Bernoulli-Euler theory the strain energy function can be specified in the form

$$\Sigma = \Sigma(\lambda, \boldsymbol{\beta}_\alpha) \ . \tag{5.28.14}$$

Next, using the fact that

$$\dot{\lambda} = \frac{\mathbf{d}_3 \bullet \mathbf{w}_3}{\lambda D_{33}} = \lambda \left[\frac{\mathbf{d}_3 \otimes \mathbf{d}_3}{d_{33}}\right] \bullet \mathbf{D} \ , \tag{5.28.15}$$

and using the assumption (5.8.1) for the quantities $\hat{\mathbf{T}}$ and $\hat{\mathbf{m}}^\alpha$ associated with constitutive equations, it can be shown that

$$[d_{33}^{1/2}\, \hat{\mathbf{T}} - m\frac{\partial \Sigma}{\partial \lambda} \lambda \{\frac{\mathbf{d}_3 \otimes \mathbf{d}_3}{d_{33}}\}] \bullet \mathbf{D} + [\mathbf{F}^T \hat{\mathbf{m}}^\alpha - m\frac{\partial \Sigma}{\partial \boldsymbol{\beta}_\alpha}] \bullet \dot{\boldsymbol{\beta}}_\alpha = 0 \ . \tag{5.28.16}$$

Since this equation must be valid for all motions, it follows that the constitutive equations for the Bernoulli-Euler rod become

$$d_{33}^{1/2}\, \hat{\mathbf{T}} = m\frac{\partial \Sigma}{\partial \lambda} \lambda \{\frac{\mathbf{d}_3 \otimes \mathbf{d}_3}{d_{33}}\} \ , \quad \mathbf{m}^\alpha = \hat{\mathbf{m}}^\alpha = \mathbf{F}^{-T}\, m\frac{\partial \Sigma}{\partial \boldsymbol{\beta}_\alpha} \ ,$$
$$\hat{\mathbf{t}}^i = [d_{33}^{1/2}\, \hat{\mathbf{T}} - \mathbf{m}^\alpha \otimes \mathbf{d}_{\alpha,3}]\, \mathbf{d}^i \ , \quad \hat{\mathbf{t}}^\alpha = -\, \mathbf{m}^\beta\, (\mathbf{d}_{\beta,3} \bullet \mathbf{d}^\alpha) \ ,$$
$$\hat{\mathbf{t}}^3 = m\frac{\partial \Sigma}{\partial \lambda} \lambda\, d_{33}^{-1}\, \mathbf{d}_3 - \mathbf{m}^\beta\, (\mathbf{d}_{\beta,3} \bullet \mathbf{d}^3) \ . \tag{5.28.17}$$

As a special case it is possible to specify the strain energy function in the form

$$m\, \Sigma = E^*\, D_{33}^{1/2}\, A\, f(\lambda) + \frac{1}{2} D_{33}^{-1/2}\, A\, \mathbf{K}^{\alpha\beta} \bullet (\boldsymbol{\beta}_\alpha \otimes \boldsymbol{\beta}_\beta) \ , \tag{5.28.18}$$

where E^* is Young's modulus for small deformation uniaxial stress, the function $f(\lambda)$ satisfies the restrictions that

$$f = 0 \quad \text{and} \quad \frac{df}{d\lambda} = 0 \quad \text{for } \lambda = 1 \ , \tag{5.28.19}$$

and the remaining quantities are summarized in section 5.26. Then, the constitutive equations associated with (5.28.18) become

$$d_{33}^{1/2}\, \hat{\mathbf{T}} = E^*\, d_{33}^{-1/2}\, A\, \frac{df(\lambda)}{d\lambda}\, (\mathbf{d}_3 \otimes \mathbf{d}_3) \ , \quad \mathbf{m}^\alpha = \hat{\mathbf{m}}^\alpha = D_{33}^{-1/2}\, A\, \mathbf{F}^{-T} \mathbf{K}^{\alpha\beta}\, \boldsymbol{\beta}_\beta \ ,$$
$$\hat{\mathbf{t}}^i = [d_{33}^{1/2}\, \hat{\mathbf{T}} - \mathbf{m}^\alpha \otimes \mathbf{d}_{\alpha,3}]\, \mathbf{d}^i \ , \quad \hat{\mathbf{t}}^\alpha = -\, \mathbf{m}^\beta\, (\mathbf{d}_{\beta,3} \bullet \mathbf{d}^\alpha) \ ,$$
$$\hat{\mathbf{t}}^3 = E^*\, d_{33}^{-1/2}\, A\, \frac{df(\lambda)}{d\lambda}\, \mathbf{d}_3 - \mathbf{m}^\beta\, (\mathbf{d}_{\beta,3} \bullet \mathbf{d}^3) \ . \tag{5.28.20}$$

In particular, the constitutive equations (5.28.20) will yield the same response for homogeneous deformation as (5.28.12) provided that the function $f(\lambda)$ is specified by

$$f(\lambda) = \frac{1}{2}\, [E_{33}]^2 \ , \quad E_{33} = \frac{1}{2}\, (\lambda^2 - 1) \ . \tag{5.28.21}$$

Of course, more general dependence on the stretch λ can be obtained by retaining a general functional form for $f(\lambda)$.

In order to discuss the nature of boundary conditions for this constrained theory, it is convenient introduce an orthonormal triad of vectors $\mathbf{a}_i$ defined by

$$\mathbf{a}_\alpha = \mathbf{d}_\alpha \ , \quad \mathbf{a}_3 = \frac{\mathbf{d}_3}{|\mathbf{d}_3|} \ . \tag{5.28.22}$$

It then follows that the time rate of change of this triad is characterized by the equation

$$\dot{\mathbf{a}}_i = \boldsymbol{\omega} \times \mathbf{a}_i \ , \tag{5.28.23}$$

where the angular velocity vector $\boldsymbol{\omega}$ is given by

$$\boldsymbol{\omega} = \left[\frac{\mathbf{a}_3 \times \mathbf{w}_3}{|\mathbf{d}_3|}\right] + \omega_3\, \mathbf{a}_3 = \omega_i\, \mathbf{a}_i \ , \quad \omega_3 = \mathbf{w}_1 \bullet \mathbf{a}_2 = -\, \mathbf{w}_2 \bullet \mathbf{a}_1 \ . \tag{5.28.24}$$

Moreover, the rate of work done by the resultant force $\mathbf{t}^3$ and the resultant director couples $\mathbf{m}^\alpha$ applied to the end of the rod [see (5.6.1)] becomes

$$\mathbf{t}^3 \bullet \mathbf{v} + \mathbf{m}^\alpha \bullet \mathbf{w}_\alpha = \mathbf{t}^3 \bullet \mathbf{v} + \mathbf{m} \bullet \boldsymbol{\omega} \ , \tag{5.28.25}$$

where the resultant moment $\mathbf{m}$ has been defined by (5.10.7)

$$\mathbf{m} = \mathbf{d}_\alpha \times \mathbf{m}^\alpha \ . \tag{5.28.26}$$

In particular, it is noted that the first term on the right-hand-side of $(5.28.24)_1$ influences the bending components of $\mathbf{m}$ and is totally determined by the motion of the reference curve since

$$\mathbf{d}_3 = \mathbf{x},_3 \ , \quad \omega_3 = \dot{\mathbf{d}}_3 = \mathbf{v},_3 \ . \tag{5.28.27}$$

Also, the term ω_3 in (5.28.24) influences the torsional component of the moment $\mathbf{m}$.

Furthermore, it follows that kinematic, kinetic, mixed and mixed-mixed boundary conditions of the type discussed in section 5.10, which are related to components of $\{\mathbf{v} \bullet \mathbf{a}_i$ or $\mathbf{t}^3 \bullet \mathbf{a}_i\}$ and $\{\boldsymbol{\omega} \bullet \mathbf{a}_i$ or $\mathbf{m} \bullet \mathbf{a}_i\}$, can be proposed for the Bernoulli-Euler rod.

Next, consider the linearized theory for a beam with rectangular cross-section as specified by (5.14.3). Then, using the equations that are summarized in section 5.26 the kinematics of the beam are specified by

$$\mathbf{X} = \theta^3\, \mathbf{e}_3 \ , \quad \mathbf{D}_i = \mathbf{e}_i \ , \quad \mathbf{x} = \mathbf{X} + \mathbf{u} \ , \quad \mathbf{d}_i = \mathbf{D}_i + \boldsymbol{\delta}_i \ , \quad \boldsymbol{\delta}_3 = (u_i,_3)\, \mathbf{e}_i \ ,$$
$$\mathbf{u} = u_i\, \mathbf{e}_i \ , \quad \boldsymbol{\delta}_1 = \delta\, \mathbf{e}_2 - (u_1,_3)\, \mathbf{e}_3 \ , \quad \boldsymbol{\delta}_2 = -\, \delta\, \mathbf{e}_1 - (u_2,_3)\, \mathbf{e}_3 \ ,$$
$$\tilde{\boldsymbol{\beta}}_1 = \ \delta,_3\, \mathbf{e}_2 - (u_1,_{33})\, \mathbf{e}_3 \ , \quad \tilde{\boldsymbol{\beta}}_1 = -\, \delta,_3\, \mathbf{e}_1 - (u_2,_{33})\, \mathbf{e}_3 \ , \tag{5.28.28}$$

where the expressions for $\boldsymbol{\delta}_\alpha$ were obtained by satisfying the linearized version of the constraints (5.28.2) which require

$$\mathbf{e}_\alpha \bullet \boldsymbol{\delta}_i + \boldsymbol{\delta}_\alpha \bullet \mathbf{e}_i = \delta_{\alpha i} \ . \tag{5.28.29}$$

The assigned fields are specified by

$$\tilde{\mathbf{b}}_b = \tilde{\mathbf{b}}^* \ , \quad \tilde{\mathbf{b}}^\alpha_b = 0 \ , \quad \mathbf{b} = \tilde{\mathbf{b}} = \tilde{\mathbf{b}}_b + \tilde{\mathbf{b}}_c \ , \quad \mathbf{b}^\alpha = \tilde{\mathbf{b}}^\alpha = \tilde{\mathbf{b}}^\alpha_b + \tilde{\mathbf{b}}^\alpha_c \ , \tag{5.28.30}$$

where the assigned fields $\tilde{\mathbf{b}}^\alpha_b$ and $\tilde{\mathbf{b}}^\alpha_c$, due to surface tractions on the beam are summarized in section 5.26. Also, the inertia quantities are given by

$$m = \rho_0^* HW \ , \quad \rho_0 = \rho_0^* HW \ ,$$
$$y^\alpha = 0 \ , \quad y^{11} = \frac{H^2}{\pi^2} \ , \quad y^{12} = y^{21} = 0 \ , \quad y^{22} = \frac{W^2}{\pi^2} \ . \tag{5.28.31}$$

Next, specifying the material directions $\mathbf{M}_i=\mathbf{D}_i$, considering an isotropic material, using the constitutive equations of section 5.26, linearizing (5.28.20) with the specification (5.28.21), and linearizing the constraint equations (5.28.7), it can be shown that

$$\tilde{\mathbf{m}}^1 = \hat{\mathbf{m}}^1 = \frac{\mu^* HW}{6}\left[\left\{HW\, b^*(\eta)\right\}(\delta,_3)\,\mathbf{e}_2 - H^2(1+\nu^*)(u_1,_{33})\,\mathbf{e}_3\right] ,$$

$$\tilde{\mathbf{m}}^2 = \hat{\mathbf{m}}^2 = \frac{\mu^* HW}{6}\left[-\left\{HW\, b^*(\eta)\right\}(\delta,_3)\,\mathbf{e}_1 - W^2(1+\nu^*)(u_2,_{33})\,\mathbf{e}_3\right] ,$$

$$\tilde{\mathbf{m}} \doteq \mathbf{e}_\alpha \times \tilde{\mathbf{m}}^\alpha = \frac{\mu^* HW}{6}\left[- W^2(1+\nu^*)(u_2,_{33})\,\mathbf{e}_1 + H^2(1+\nu^*)(u_1,_{33})\,\mathbf{e}_2 + \left\{2HW\, b^*(\eta)\right\}(\delta,_3)\,\mathbf{e}_3\right] ,$$

$$\hat{\mathbf{t}}^\alpha = 0\ , \quad \hat{\mathbf{t}}^3 = E^* HW\,(u_3,_3)\,\mathbf{e}_3\ , \quad \eta = \frac{H}{W}\ ,$$

$$\tilde{\mathbf{t}}^1 = \bar{\mathbf{t}}^1 = \left[m\,(\mathbf{e}_1 \bullet \tilde{\mathbf{b}}^1)\right]\mathbf{e}_1 + \left[m\,(\mathbf{e}_1 \bullet \tilde{\mathbf{b}}^2) - \left\{\frac{\mu^* H^2 W^2 b^*(\eta)}{6}\right\}(\delta,_{33}) + m\, y^{22}\, \ddot{\delta}\,\right]\mathbf{e}_2$$
$$+ \left[m\,(\mathbf{e}_3 \bullet \tilde{\mathbf{b}}^1) - \left\{\frac{E^* H^3 W}{12}\right\}(u_1,_{333}) + m\, y^{11}\,(\ddot{u}_1,_3)\right]\mathbf{e}_3\ ,$$

$$\tilde{\mathbf{t}}^2 = \bar{\mathbf{t}}^2 = \left[m\,(\mathbf{e}_2 \bullet \tilde{\mathbf{b}}^1) + \left\{\frac{\mu^* H^2 W^2 b^*(\eta)}{6}\right\}(\delta,_{33}) - m\, y^{11}\, \ddot{\delta}\,\right]\mathbf{e}_1 + \left[m\,(\mathbf{e}_2 \bullet \tilde{\mathbf{b}}^2)\,\right]\mathbf{e}_2$$
$$+ \left[m\,(\mathbf{e}_3 \bullet \tilde{\mathbf{b}}^2) - \left\{\frac{E^* HW^3}{12}\right\}(u_2,_{333}) + m\, y^{22}\,(\ddot{u}_2,_3)\right]\mathbf{e}_3\ ,$$

$$\tilde{\mathbf{t}}^3 = \hat{\mathbf{t}}^3 + \bar{\mathbf{t}}^3 = \left[m\,(\mathbf{e}_3 \bullet \tilde{\mathbf{b}}^1) - \left\{\frac{E^* H^3 W}{12}\right\}(u_1,_{333}) + m\, y^{11}\,(\ddot{u}_1,_3)\right]\mathbf{e}_1$$
$$+ \left[m\,(\mathbf{e}_3 \bullet \tilde{\mathbf{b}}^2) - \left\{\frac{E^* HW^3}{12}\right\}(u_2,_{333}) + m\, y^{22}\,(\ddot{u}_2,_3)\right]\mathbf{e}_2$$
$$+ \left[E^* HW\,(u_3,_3)\right]\mathbf{e}_3\ , \tag{5.28.32}$$

where $b^*(\eta)$ characterizes the torsional rigidity and is given by (5.15.1), and the definition for Young's modulus E^* has been used.

Now, the linearized versions of the conservation of mass and the balances of linear momentum and director momentum (5.28.6) reduce to

$$\rho = \rho_0\left[1 - u_3,_3\,\right]\ ,$$

$$m\,\ddot{u}_1 = m\left[\,(\mathbf{e}_1 \bullet \tilde{\mathbf{b}}) + (\mathbf{e}_3 \bullet \tilde{\mathbf{b}}^1),_3\right] - \left\{\frac{E^* H^3 W}{12}\right\}(u_1,_{3333}) + m\, y^{11}\,(\ddot{u}_1,_{33})\ ,$$

$$m\,\ddot{u}_2 = m\left[\,(\mathbf{e}_2 \bullet \tilde{\mathbf{b}}) + (\mathbf{e}_3 \bullet \tilde{\mathbf{b}}^2),_3\right] - \left\{\frac{E^* HW^3}{12}\right\}(u_2,_{3333}) + m\, y^{22}\,(\ddot{u}_2,_{33})\ ,$$

$$m\,\ddot{u}_3 = m\,(\mathbf{e}_3 \bullet \tilde{\mathbf{b}}) + E^* HW\,(u_3,_{33})\ ,$$

$$m\left\{y^{11} + y^{22}\right\}\ddot{\delta} = \left\{\frac{\mu^* H^2 W^2 b^*(\eta)}{3}\right\}(\delta,_{33}) + m\left\{(\mathbf{e}_2 \bullet \tilde{\mathbf{b}}^1) - (\mathbf{e}_1 \bullet \tilde{\mathbf{b}}^2)\right\}\ . \tag{5.28.33}$$

In particular, notice that the bending equations $(5.28.33)_{2,3}$ include rotary inertia terms (y^{11} and y^{22}) which are known to influence high frequency response (Graff, 1975, sec. 3.4). Furthermore, the linearized versions of the velocity $\dot{\mathbf{u}}$ and of the angular velocity $\boldsymbol{\omega}$ in (5.28.24), become

$$\dot{\mathbf{u}} = \dot{u}_i \, \mathbf{e}_i \ , \quad \tilde{\boldsymbol{\omega}} = \mathbf{e}_3 \times \dot{\mathbf{u}},_3 + \tilde{\omega}_3 \, \mathbf{e}_3 = \tilde{\omega}_i \, \mathbf{e}_i \ ,$$

$$\tilde{\omega}_1 = - \dot{u}_{2,3} \ , \quad \tilde{\omega}_2 = \dot{u}_{1,3} \ , \quad \tilde{\omega}_3 = \dot{\delta} \ , \tag{5.28.34}$$

which indicate that kinematic boundary conditions associated with (5.28.25) are characterized by specified values of

$$\{u_1 \, , u_2 \, , u_3 \, , u_{1,3} \ , u_{2,3} \ , \delta\} \ . \tag{5.28.35}$$

5.29 Timoshenko rods

Timoshenko (1921) recognized that the effect of shear deformation in beams can significantly influence the response to vibrations and wave propagation (also see Graff, 1975, sec. 3.4). In its simplest form the Timoshenko theory of beams allows the cross-section that was normal to the beam in its reference configuration, to shear away from the normal cross-section in the present configuration. Of course, the general Cosserat theory of rods models this phenomena, but it also models other modes of deformation of the cross-section which can be neglected in some applications. Therefore, it is of interest to develop a nonlinear constrained theory rods which includes tangential shear deformation, but is less general that the full Cosserat theory. To this end, it is possible to consider different alternative constraints. One possibility is to require the dimensions of the normal cross-section in the present configuration to remain unchanged. This can be accomplished by eliminating the effects of normal cross-sectional extension and normal cross-sectional shear deformation. A specific nonlinear constrained theory for rods of this type can be developed within the context of the Cosserat theory of rods by imposing the constraints (5.9.4) and (5.9.8).

Another possibility which has been extensively used by Antman (1972, 1995), is to assume that the dimensions of the material cross-section that was normal in the reference configuration remain constant. This can be accomplished by imposing the constraints that the directors $\mathbf{d}_\alpha$ remain orthonormal vectors

$$\mathbf{d}_\alpha \bullet \mathbf{d}_\beta = \delta_{\alpha\beta} \ . \tag{5.29.1}$$

Physically, this is a different set of constraints from (5.9.4) and (5.9.8) because the constraints (5.29.1) allow the dimensions of the normal cross-section in the present configuration to change with time, whereas the other ones do not. Now, taking the material derivative of (5.29.1) and recalling the definition (5.6.7), it can be shown that the rate forms of these constraints become

$$(\mathbf{d}_\alpha \otimes \mathbf{d}_\beta + \mathbf{d}_\beta \otimes \mathbf{d}_\alpha) \bullet \mathbf{D} = 0 \ . \tag{5.29.2}$$

Consequently, with the help of (5.9.12), (5.9.22) and (5.9.23), the constitutive equations and constraint responses can be written in the forms

$$\mathbf{T} = \hat{\mathbf{T}} + \bar{\mathbf{T}} \ , \quad \mathbf{t}^i = \hat{\mathbf{t}}^i + \bar{\mathbf{t}}^i \ , \quad \mathbf{m}^\alpha = \hat{\mathbf{m}}^\alpha + \bar{\mathbf{m}}^\alpha \ ,$$

$$\bar{\mathbf{t}}^\alpha = \gamma^{\alpha\beta} \, \mathbf{d}_\beta \, , \quad \bar{\mathbf{t}}^3 = 0 \, , \quad \gamma^{\alpha\beta} = \gamma^{\beta\alpha} \ , \quad \bar{\mathbf{m}}^\alpha = 0 \ ,$$

$$\bar{\mathbf{T}} = \frac{1}{2} d_{33}^{-1/2}\, \gamma^{\alpha\beta}\, (\mathbf{d}_\alpha \otimes \mathbf{d}_\beta + \mathbf{d}_\beta \otimes \mathbf{d}_\alpha) \ , \tag{5.29.3}$$

where the symmetric tensor $\gamma^{\alpha\beta}$ is an arbitrary function of (θ^3,t). Thus, comparison of (5.29.2) with (5.9.5) and (5.9.9) indicates that these forms of the constraints are much simpler than those associated with the constraints (5.9.4) and (5.9.8).

Since $\gamma^{\alpha\beta}$ is symmetric, it means that three components of the vectors $\mathbf{t}^i$ are arbitrary functions of time. Moreover, this means that three of the six components of the director momentum equations $(5.4.25)_3$ can be satisfied for arbitrary admissible motions of the constrained rod by using (5.29.3) and determining the functions $\gamma^{\alpha\beta}$ by

$$\gamma^{\alpha\beta} = \mathbf{d}^\alpha \cdot \left[\, m\, \mathbf{b}^\beta - \hat{\mathbf{t}}^\beta + \mathbf{m}^\beta{}_{,3} - m\, (y^\beta\, \dot{\mathbf{v}} + y^{\beta\gamma}\, \dot{\mathbf{w}}_\gamma) \right] \ . \tag{5.29.4}$$

Also, since $\gamma^{\alpha\beta}$ is symmetric, two of these four equations are dependent and yield the condition (5.9.25).

This nonlinear Timoshenko rod theory is simpler than the complete Cosserat theory because the constraints (5.29.1) determine the directors $\mathbf{d}_\alpha$ by three angles of rotation. Three of the director momentum equations $(5.4.25)_3$ are satisfied by the Lagrange multipliers $\gamma^{\alpha\beta}$, and the balance laws of the theory reduce to the conservation of mass $(5.4.25)_1$, the balance of linear momentum (5.9.26), and three component of the balances of director momentum (5.9.25)

$$\begin{gathered}
m = \rho\, d_{33}^{1/2} = \rho_0\, D_{33}^{1/2} \ , \quad m\,(\dot{\mathbf{v}} + y^\alpha\, \dot{\mathbf{w}}_\alpha) = m\,\mathbf{b} + \hat{\mathbf{t}}^3{}_{,3} \ , \\
\left[m\, \mathbf{b}^\alpha - \hat{\mathbf{t}}^\alpha + \mathbf{m}^\alpha{}_{,3} - m\,(y^\alpha\, \dot{\mathbf{v}} + y^{\alpha\beta}\, \dot{\mathbf{w}}_\beta) \right] \cdot \mathbf{d}^3 = 0 \\
\left[\, m\, \mathbf{b}^1 - \hat{\mathbf{t}}^1 + \mathbf{m}^1{}_{,3} - m\,(y^1\, \dot{\mathbf{v}} + y^{1\beta}\, \dot{\mathbf{w}}_\beta) \right] \cdot \mathbf{d}^2 \\
= \left[\, m\, \mathbf{b}^2 - \hat{\mathbf{t}}^2 + \mathbf{m}^2{}_{,3} - m\,(y^2\, \dot{\mathbf{v}} + y^{2\beta}\, \dot{\mathbf{w}}_\beta) \right] \cdot \mathbf{d}^1 \ ,
\end{gathered} \tag{5.29.5}$$

Also, the constraint responses $\bar{\mathbf{t}}^i$ are determined by (5.29.3) and (5.29.4), and can be written in the forms

$$\bar{\mathbf{t}}^\alpha = \left[\mathbf{d}^\alpha \cdot \left\{ m\, \mathbf{b}^\beta - \hat{\mathbf{t}}^\beta + \mathbf{m}^\beta{}_{,3} - m\,(y^\beta\, \dot{\mathbf{v}} + y^{\beta\gamma}\, \dot{\mathbf{w}}_\gamma) \right\} \right] \mathbf{d}_\beta \ , \quad \bar{\mathbf{t}}^3 = 0 \ . \tag{5.29.6}$$

Furthermore, due to the simplifying constraints (5.29.1) it can be shown that

$$\begin{gathered}
\mathbf{C} = \delta_{\alpha\beta}\, \mathbf{D}^\alpha \otimes \mathbf{D}^\beta + (\mathbf{d}_\alpha \cdot \mathbf{d}_3)\, (\mathbf{D}^\alpha \otimes \mathbf{D}^3 + \mathbf{D}^3 \otimes \mathbf{D}^\alpha) + d_{33}\, \mathbf{D}^3 \otimes \mathbf{D}^3 \ , \\
\mathbf{E} = E_{\alpha 3}\, (\mathbf{D}^\alpha \otimes \mathbf{D}^3 + \mathbf{D}^3 \otimes \mathbf{D}^\alpha) + E_{33}\, \mathbf{D}^3 \otimes \mathbf{D}^3 \ , \\
E_{\alpha 3} = \frac{1}{2} \left[\mathbf{d}_\alpha \cdot \mathbf{d}_3 - \mathbf{D}_\alpha \cdot \mathbf{D}_3 \right] \ , \quad E_{33} = \frac{1}{2} \left[d_{33} - D_{33} \right] \ ,
\end{gathered} \tag{5.29.7}$$

where use has been made of the specification that in the reference configuration

$$\mathbf{D}_\alpha \cdot \mathbf{D}_\beta = \delta_{\alpha\beta} \ . \tag{5.29.8}$$

This means that, if (5.29.7) were substituted into the constitutive equations summarized in section 5.26, then the predicted response would be reasonable for shear, but would predict the response to uniaxial strain for extensional deformations. However, the response of a thin rod-like structure usually is closer to uniaxial stress than to uniaxial strain for extensional deformations. Consequently, the constitutive equations of the Timoshenko rod theory must be modified relative to those developed for the more general theory.

In view of the discussion in the previous section, the strain energy function for an isotropic material is modified to include the effects of shear deformation, and is modified to simulate the response of uniaxial stress by taking

$$m\,\Sigma = D_{33}^{1/2}\,A\,\big[E^*\,f(\lambda) + \mu^*\,E_{\alpha 3}\,(\mathbf{D}^\alpha\otimes\mathbf{D}^3+\mathbf{D}^3\otimes\mathbf{D}^\alpha)\bullet E_{\beta 3}\,(\mathbf{D}^\beta\otimes\mathbf{D}^3+\mathbf{D}^3\otimes\mathbf{D}^\beta)\big]$$
$$+\ \frac{1}{2}D_{33}^{-1/2}\,A\,\mathbf{K}^{\alpha\beta}\bullet(\boldsymbol{\beta}_\alpha\otimes\boldsymbol{\beta}_\beta)\ , \qquad (5.29.9)$$

where λ is the stretch defined by (5.28.13). Next, using the assumption (5.8.1) for the quantities $\hat{\mathbf{T}}$ and $\hat{\mathbf{m}}^\alpha$ associated with constitutive equations, it can be shown that

$$[d_{33}^{1/2}\,\hat{\mathbf{T}} - E^*\,D_{33}^{1/2}\,A\,\frac{df}{d\lambda}\,\lambda\{\frac{\mathbf{d}_3\otimes\mathbf{d}_3}{d_{33}}\}$$
$$+\,D_{33}^{1/2}\,A\,\mu^*\,E_{\alpha 3}(\mathbf{D}^\alpha\otimes\mathbf{D}^3+\mathbf{D}^3\otimes\mathbf{D}^\alpha)\bullet(\mathbf{D}^\beta\otimes\mathbf{D}^3+\mathbf{D}^3\otimes\mathbf{D}^\beta)\,(\mathbf{d}_\beta\otimes\mathbf{d}_3+\mathbf{d}_3\otimes\mathbf{d}_\beta)]\bullet\mathbf{D}$$
$$+\,[\mathbf{F}^T\hat{\mathbf{m}}^\alpha - D_{33}^{-1/2}\,A\,\mathbf{K}^{\alpha\beta}\,\boldsymbol{\beta}_\beta]\bullet\dot{\boldsymbol{\beta}}_\alpha = 0\ . \qquad (5.29.10)$$

Since this equation must be valid for all motions it follows that the constitutive equations for the Timoshenko rod become

$$d_{33}^{1/2}\,\hat{\mathbf{T}} = E^*\,d_{33}^{-1/2}\,A\,\frac{df}{d\lambda}\,(\mathbf{d}_3\otimes\mathbf{d}_3)$$
$$+\,D_{33}^{1/2}\,A\,\mu^*\,E_{\alpha 3}(\mathbf{D}^\alpha\otimes\mathbf{D}^3+\mathbf{D}^3\otimes\mathbf{D}^\alpha)\bullet(\mathbf{D}^\beta\otimes\mathbf{D}^3+\mathbf{D}^3\otimes\mathbf{D}^\beta)\,(\mathbf{d}_\beta\otimes\mathbf{d}_3+\mathbf{d}_3\otimes\mathbf{d}_\beta)\ ,$$
$$\mathbf{m}^\alpha = \hat{\mathbf{m}}^\alpha = \mathbf{F}^{-T}\,D_{33}^{-1/2}\,A\,\mathbf{K}^{\alpha\beta}\,\boldsymbol{\beta}_\beta\ ,\quad \hat{\mathbf{t}}^i = [d_{33}^{1/2}\,\hat{\mathbf{T}} - \mathbf{m}^\alpha\otimes\mathbf{d}_{\alpha,3}]\,\mathbf{d}^i\ ,$$
$$\hat{\mathbf{t}}^\alpha = D_{33}^{1/2}\,A\mu^*E_{\beta 3}(\mathbf{D}^\alpha\otimes\mathbf{D}^3+\mathbf{D}^3\otimes\mathbf{D}^\alpha)\bullet(\mathbf{D}^\beta\otimes\mathbf{D}^3+\mathbf{D}^3\otimes\mathbf{D}^\beta)\,\mathbf{d}_3 - \mathbf{m}^\beta\,(\mathbf{d}_{\beta,3}\bullet\mathbf{d}^\alpha)\ ,$$
$$\hat{\mathbf{t}}^3 = D_{33}^{1/2}\,A\mu^*\,E_{\alpha 3}(\mathbf{D}^\alpha\otimes\mathbf{D}^3+\mathbf{D}^3\otimes\mathbf{D}^\alpha)\bullet(\mathbf{D}^\beta\otimes\mathbf{D}^3+\mathbf{D}^3\otimes\mathbf{D}^\beta)\,\mathbf{d}_\beta + E^*\,d_{33}^{-1/2}\,A\,\frac{df}{d\lambda}\mathbf{d}_3$$
$$-\,\mathbf{m}^\beta\,(\mathbf{d}_{\beta,3}\bullet\mathbf{d}^3)\ . \qquad (5.29.11)$$

As a special case the function $f(\lambda)$ is given by the quadratic form (5.28.21). Furthermore, it is noted that the coefficient of the shear terms in (5.29.9) was chosen so that the theory can reproduce exact solutions for simple shear through the thickness of the beam. The relationship between this value and the more common values for transverse shear has been discussed in (Rubin, 1996).

In order to discuss the nature of boundary conditions for this constrained theory, it is convenient introduce an orthonormal triad of vectors $\mathbf{a}_i$ defined by

$$\mathbf{a}_\alpha = \mathbf{d}_\alpha\ ,\quad \mathbf{a}_3 = \mathbf{a}_1\times\mathbf{a}_2\ . \qquad (5.29.12)$$

It then follows that the time rate of change of this triad is characterized by the equation

$$\dot{\mathbf{a}}_i = \boldsymbol{\omega}\times\mathbf{a}_i\ , \qquad (5.29.13)$$

where the angular velocity vector $\boldsymbol{\omega}$ is given by

$$\boldsymbol{\omega} = \omega_i\,\mathbf{a}_i\ ,\quad \omega_1 = \mathbf{a}_3\bullet\mathbf{w}_2\ ,\quad \omega_2 = -\,\mathbf{a}_3\bullet\mathbf{w}_1\ ,\quad \omega_3 = \mathbf{a}_2\bullet\mathbf{w}_1 = -\,\mathbf{a}_1\bullet\mathbf{w}_2\ . \qquad (5.29.14)$$

Moreover, the rate of work done by the resultant force $\mathbf{t}^3$ and the resultant director couples $\mathbf{m}^\alpha$ applied to the end of the rod [see (5.6.1)] becomes

$$\mathbf{t}^3\bullet\mathbf{v} + \mathbf{m}^\alpha\bullet\mathbf{w}_\alpha = \mathbf{t}^3\bullet\mathbf{v} + \mathbf{m}\bullet\boldsymbol{\omega}\ , \qquad (5.29.15)$$

where the resultant moment $\mathbf{m}$ has been defined by (5.10.7)

$$\mathbf{m} = \mathbf{d}_\alpha \times \mathbf{m}^\alpha \ . \tag{5.29.16}$$

Also, it follows that kinematic, kinetic, mixed and mixed-mixed boundary conditions of the type discussed in section 5.10, which are related to components of $\{\mathbf{v} \bullet \mathbf{a}_i \text{ or } \mathbf{t}^3 \bullet \mathbf{a}_i\}$ and $\{\boldsymbol{\omega} \bullet \mathbf{a}_i \text{ or } \mathbf{m} \bullet \mathbf{a}_i\}$, can be proposed for the Timoshenko rod.

Next, consider the linearized theory for a beam with rectangular cross-section as specified by (5.14.3). Then, using the equations that are summarized in section 5.26 the kinematics of the beam are specified by

$$\mathbf{X} = \theta^3 \mathbf{e}_3 \ , \ \mathbf{D}_i = \mathbf{e}_i \ , \ \mathbf{x} = \mathbf{X} + \mathbf{u} \ , \ \mathbf{d}_i = \mathbf{D}_i + \boldsymbol{\delta}_i \ , \ \boldsymbol{\delta}_3 = (u_{i,3})\, \mathbf{e}_i \ ,$$
$$\mathbf{u} = u_i \mathbf{e}_i \ , \ \boldsymbol{\delta}_1 = \delta\, \mathbf{e}_2 + \tilde{\delta}_{31} \mathbf{e}_3 \ , \ \boldsymbol{\delta}_2 = -\delta\, \mathbf{e}_1 + \tilde{\delta}_{32} \mathbf{e}_3 \ ,$$
$$\tilde{\boldsymbol{\beta}}_1 = \delta_{,3}\, \mathbf{e}_2 + \tilde{\delta}_{31,3}\, \mathbf{e}_3 \ , \ \tilde{\boldsymbol{\beta}}_1 = -\delta_{,3}\, \mathbf{e}_1 + \tilde{\delta}_{32,3}\, \mathbf{e}_3 \ , \tag{5.29.17}$$

where the expressions for $\boldsymbol{\delta}_\alpha$ were obtained by satisfying the linearized version of the constraints (5.29.1) which require

$$\mathbf{e}_\alpha \bullet \boldsymbol{\delta}_\beta + \boldsymbol{\delta}_\alpha \bullet \mathbf{e}_\beta = \delta_{\alpha\beta} \ . \tag{5.29.18}$$

The assigned fields are specified by

$$\tilde{\mathbf{b}}_b = \tilde{\mathbf{b}}^* \ , \ \tilde{\mathbf{b}}_b^\alpha = 0 \ , \ \mathbf{b} = \tilde{\mathbf{b}} = \tilde{\mathbf{b}}_b + \tilde{\mathbf{b}}_c \ , \ \mathbf{b}^\alpha = \tilde{\mathbf{b}}^\alpha = \tilde{\mathbf{b}}_b^\alpha + \tilde{\mathbf{b}}_c^\alpha \ , \tag{5.29.19}$$

where the assigned fields $\tilde{\mathbf{b}}_b^\alpha$ and $\tilde{\mathbf{b}}_c^\alpha$ due to surface tractions on the beam are summarized in section 5.26. Also, the inertia quantities are given by

$$m = \rho_0^* HW \ , \ \rho_0 = \rho_0^* HW \ ,$$
$$y^\alpha = 0 \ , \ y^{11} = \frac{H^2}{\pi^2} \ , \ y^{12} = y^{21} = 0 \ , \ y^{22} = \frac{W^2}{\pi^2} \ . \tag{5.29.20}$$

Next, specifying the material directions $\mathbf{M}_i = \mathbf{D}_i$, considering an isotropic material, using the constitutive equations of section 5.26, linearizing (5.29.11) with the specification (5.28.21), and linearizing the constraint equations (5.29.6), it can be shown that

$$\tilde{\mathbf{m}}^1 = \hat{\mathbf{m}}^1 = \frac{\mu^* HW}{6} \Big[\big\{ HW\, b^*(\eta) \big\} (\delta_{,3})\, \mathbf{e}_2 + H^2(1+\nu^*)(\tilde{\delta}_{31,3})\, \mathbf{e}_3 \Big] \ ,$$
$$\tilde{\mathbf{m}}^2 = \hat{\mathbf{m}}^2 = \frac{\mu^* HW}{6} \Big[-\big\{ HW\, b^*(\eta) \big\} (\delta_{,3})\, \mathbf{e}_1 + W^2(1+\nu^*)(\tilde{\delta}_{32,3})\, \mathbf{e}_3 \Big] \ ,$$
$$\tilde{\mathbf{m}} = \mathbf{e}_\alpha \times \tilde{\mathbf{m}}^\alpha = \frac{\mu^* HW}{6} \Big[W^2(1+\nu^*)(\tilde{\delta}_{32,3})\, \mathbf{e}_1 - H^2(1+\nu^*)(\tilde{\delta}_{31,3})\, \mathbf{e}_2$$
$$+ \big\{ 2HW\, b^*(\eta) \big\} (\delta_{,3})\, \mathbf{e}_3 \Big] \ ,$$
$$\hat{\mathbf{t}}^\alpha = \mu^* HW\, (u_{\alpha,3} + \tilde{\delta}_{3\alpha})\, \mathbf{e}_3 \ , \ \eta = \frac{H}{W} \ ,$$
$$\tilde{\mathbf{t}}^1 = \hat{\mathbf{t}}^1 + \bar{\mathbf{t}}^1 = \Big[m\, (\mathbf{e}_1 \bullet \tilde{\mathbf{b}}^1) \Big]\, \mathbf{e}_1 + \Big[m\, (\mathbf{e}_1 \bullet \tilde{\mathbf{b}}^2) - \Big\{ \frac{\mu^* H^2 W^2 b^*(\eta)}{6} \Big\} (\delta_{,33}) + m\, y^{22}\, \ddot{\delta} \Big] \mathbf{e}_2$$
$$+ \Big[\mu^* HW\, (u_{1,3} + \tilde{\delta}_{31}) \Big]\, \mathbf{e}_3 \ ,$$
$$\tilde{\mathbf{t}}^2 = \hat{\mathbf{t}}^2 + \bar{\mathbf{t}}^2 = \Big[m\, (\mathbf{e}_2 \bullet \tilde{\mathbf{b}}^1) + \Big\{ \frac{\mu^* H^2 W^2 b^*(\eta)}{6} \Big\} (\delta_{,33}) - m\, y^{11}\, \ddot{\delta} \Big]\, \mathbf{e}_1$$
$$+ \Big[m\, (\mathbf{e}_2 \bullet \tilde{\mathbf{b}}^2) \Big]\, \mathbf{e}_2 + \Big[\mu^* HW\, (u_{2,3} + \tilde{\delta}_{32}) \Big]\, \mathbf{e}_3 \ ,$$

$$\tilde{\mathbf{t}}^3 = \hat{\mathbf{t}}^3 = \mu^* HW\,(u_{\alpha,3} + \tilde{\delta}_{3\alpha})\,\mathbf{e}_\alpha + E^* HW\,(u_{3,3})\,\mathbf{e}_3 \;, \tag{5.29.21}$$

where $b^*(\eta)$ characterizes the torsional rigidity and is given by (5.15.1), and the definition for Young's modulus E^* has been used.

Now, the linearized versions of the conservation of mass and the balances of linear momentum and director momentum (5.29.5) reduce to

$$\begin{aligned}
\rho &= \rho_0\,[1 - u_{3,3}] \;,\\
m\,\ddot{u}_1 &= m\,(\mathbf{e}_1 \bullet \tilde{\mathbf{b}}) + \mu^* HW\,(u_{1,33} + \tilde{\delta}_{31,3}) \;,\\
m\,\ddot{u}_2 &= m\,(\mathbf{e}_2 \bullet \tilde{\mathbf{b}}) + \mu^* HW\,(u_{2,33} + \tilde{\delta}_{32,3}) \;,\\
m\,\ddot{u}_3 &= m\,(\mathbf{e}_3 \bullet \tilde{\mathbf{b}}) + E^* HW\,(u_{3,33}) \;,\\
m\,y^{11}\,\ddot{\tilde{\delta}}_{31} &= m\,(\mathbf{e}_3 \bullet \tilde{\mathbf{b}}^1) - \mu^* HW\,(u_{1,3} + \tilde{\delta}_{31}) + \frac{E^* H^3 W}{12}(\tilde{\delta}_{31,33}) \;,\\
m\,y^{22}\,\ddot{\tilde{\delta}}_{32} &= m\,(\mathbf{e}_3 \bullet \tilde{\mathbf{b}}^2) - \mu^* HW\,(u_{2,3} + \tilde{\delta}_{32}) + \frac{E^* H W^3}{12}(\tilde{\delta}_{32,33}) \;,\\
m\{y^{11} + y^{22}\}\,\ddot{\delta} &= \Big\{\frac{\mu^* H^2 W^2 b^*(\eta)}{3}\Big\}(\delta_{,33}) + m\,\{(\mathbf{e}_2 \bullet \tilde{\mathbf{b}}^1) - (\mathbf{e}_1 \bullet \tilde{\mathbf{b}}^2)\} \;.
\end{aligned} \tag{5.29.22}$$

In particular, notice that the director momentum equations $(5.29.22)_{5,6,7}$ include rotary inertia terms (y^{11} and y^{22}) which are known to influence high frequency response (Graff, 1975, sec. 3.4). Furthermore, the linearized versions of the velocity $\dot{\mathbf{u}}$ and of the angular velocity $\boldsymbol{\omega}$ in (5.29.14), become

$$\dot{\mathbf{u}} = \dot{u}_i\,\mathbf{e}_i \;,\quad \tilde{\boldsymbol{\omega}} = \tilde{\omega}_i\,\mathbf{e}_i \;,\quad \tilde{\omega}_1 = \dot{\tilde{\delta}}_{32} \;,\quad \tilde{\omega}_2 = -\,\dot{\tilde{\delta}}_{31} \;,\quad \tilde{\omega}_3 = \dot{\delta} \;, \tag{5.29.23}$$

which indicate that kinematic boundary conditions associated with (5.29.15) are characterized by specified values of

$$\{u_1\,,\,u_2\,,\,u_3\,,\,\tilde{\delta}_{31}\,,\,\tilde{\delta}_{32}\,,\,\delta\} \;. \tag{5.29.24}$$

5.30 Generalized strings

A rod-like structure can be modeled as a string if the influence of the bending moments are negligible relative to the influence of the resultant forces developed in the structure when it is loaded. Within the context of the theory of a Cosserat rod, this means that the influence of $\mathbf{m}^\alpha$ is negligible relative to that of $\mathbf{t}^i$ in the equations of motion (5.4.25). From a constitutive point of view, this suggests that the strain energy function is independent of $\boldsymbol{\beta}_\alpha$. Thus, Σ can be specified by the three-dimensional strain energy function Σ^* of the material used to construct the rod, by the expression

$$\Sigma = \Sigma^*(\mathbf{C}) \;. \tag{5.30.1}$$

It then follows from (5.8.12) and (5.8.14) that

$$d_{33}^{1/2}\,\mathbf{T} = 2\,m\,\mathbf{F}\,\frac{\partial \Sigma^*}{\partial \mathbf{C}}\,\mathbf{F}^T \;,\quad \mathbf{t}^i = d_{33}^{1/2}\,\mathbf{T}\,\mathbf{d}^i = 2\,m\,\mathbf{F}\,\frac{\partial \Sigma^*}{\partial \mathbf{C}}\,\mathbf{D}^i \;,\quad \mathbf{m}^\alpha = 0 \;. \tag{5.30.2}$$

Moreover, the equation of linear momentum $(5.4.25)_3$ remains unchanged, but the equation of director momentum $(5.4.25)_4$ is simplified such that

$$\mathrm{m}\,(\dot{\mathbf{v}} + \mathrm{y}^\alpha\,\dot{\mathbf{w}}_\alpha) = \mathrm{m}\,\mathbf{b} + \mathbf{t}^3{}_{,3}\ ,\quad \mathrm{m}\,(\mathrm{y}^\alpha\,\dot{\mathbf{v}} + \mathrm{y}^{\alpha\beta}\,\dot{\mathbf{w}}_\beta) = \mathrm{m}\,\mathbf{b}^\alpha - \mathbf{t}^\alpha\ . \tag{5.30.3}$$

Also, since the couples $\mathbf{m}^\alpha$ vanish, it follows from (5.3.41) that the rate of work of loads applied to the boundary ends of the generalized string reduces to

$$\mathbf{t}^3 \bullet \mathbf{v}\ . \tag{5.30.4}$$

Consequently, the boundary conditions for this string require specification of the components the velocity $\mathbf{v}$, or of the resultant force $\mathbf{t}^3$, as described in (5.10.4)-(5.10.6). In particular, it is noted that there are no boundary conditions for the directors $\mathbf{d}_\alpha$.

This string is called a generalized string because the equations of motion (5.30.3) require the determination of both the position vector $\mathbf{x}$ and the directors $\mathbf{d}_\alpha$. In particular, this theory includes the effects of tangential shear deformation, normal cross-sectional extension, and normal cross-sectional shear deformation. A simpler string theory, that omits the directors $\mathbf{d}_\alpha$ will be discussed in the next section. The main advantage of the theory of a generalized string over that of a simple string is that the three-dimensional constitutive equation can be used directly. This means that the response to uniaxial stress in the tangential direction can be obtained by solving equations (5.30.3) without the need for making special constitutive assumptions such as those discussed in sections 5.28 and 5.29 for Bernoulli-Euler and Timoshenko rods, respectively. This is because the equations (5.30.3) also determine the deformations of the directors $\mathbf{d}_\alpha$ which characterize contractions of the cross-section.

5.31 Simple strings

In contrast with the theory of a generalized string discussed in section 5.30, the theory of a simple string characterizes a curve that has no thickness. From the point of view of the Cosserat theory, the kinematics of such a curve are determined by the position vector $\mathbf{x}$ in (5.2.1), and there is no need to introduce the director vectors $\mathbf{d}_\alpha$ as a independent kinematic quantities. In particular, the tangent vector $\mathbf{d}_3$ and the arclength ds of the curve in the present configuration are defined by (5.2.2) such that

$$\mathbf{d}_3 = \mathbf{x}_{,3}\ ,\quad \mathrm{ds} = \mathrm{d}_{33}^{1/2}\,\mathrm{d}\theta^3\ ,\quad \mathrm{d}_{33} = \mathbf{d}_3 \bullet \mathbf{d}_3\ . \tag{5.31.1}$$

Also, the tangential stretch λ of the curve is given by (5.28.13) such that

$$\lambda = \frac{\mathrm{d}_{33}^{1/2}}{\mathrm{D}_{33}^{1/2}}\ , \tag{5.31.2}$$

where D_{33} is the reference value of d_{33}. Next, recalling the definitions of the velocities, and defining rate of deformation quantities $\{\mathbf{L}, \mathbf{D}, \mathbf{W}\}$ in a similar manner to (5.6.7) and (5.6.11), it can be shown that

$$\mathbf{v} = \dot{\mathbf{x}}\ ,\quad \mathbf{w}_3 = \mathbf{v}_{,3} = \dot{\mathbf{d}}_3\ ,\quad \mathbf{L} = \mathbf{w}_3 \otimes \mathbf{d}^3\ ,\quad \mathbf{d}^3 = \frac{\mathbf{d}_3}{\mathrm{d}_{33}}\ ,$$

$$\mathbf{D} = \frac{1}{2}(\mathbf{L} + \mathbf{L}^T) = \mathbf{D}^T \ , \quad \mathbf{W} = \frac{1}{2}(\mathbf{L} - \mathbf{L}^T) = -\,\mathbf{W}^T \ , \tag{5.31.3}$$

so that the material derivative of the stretch λ becomes

$$\dot{\lambda} = \lambda\, \mathbf{w}_3 \bullet \mathbf{d}^3 = \lambda\, [\frac{\mathbf{d}_3 \otimes \mathbf{d}_3}{d_{33}}] \bullet \mathbf{D} \ , \quad \frac{\dot{\lambda}}{\lambda} = \mathbf{D} \bullet \mathbf{I} \ . \tag{5.31.4}$$

For the simple string theory, the global forms of the conservation of mass and the balance of linear momentum can be written as

$$\frac{d}{dt}\int_P \rho\, ds = 0 \ , \quad \frac{d}{dt}\int_P \rho\, \mathbf{v}\, ds = \int_P \rho\, \mathbf{b}\, ds + [\ \mathbf{t}^3\ ]_1^2 \ , \tag{5.31.5}$$

where P is a material part of the curve and the notation $[\ \mathbf{t}^3\ ]_1^2$ for the ends of the string is defined in (5.3.22). Furthermore, the balance of angular momentum about the fixed origin O is postulated in the form

$$\frac{d}{dt}\int_P \rho\, (\mathbf{x} \times \mathbf{v})\, ds = \int_P (\mathbf{x} \times \rho\, \mathbf{b})\, ds + [\ \mathbf{x} \times \mathbf{t}^3]_1^2 \ . \tag{5.31.6}$$

Also, there is no need to introduce a director momentum equation, and the integral form of the mechanical power $\mathcal{P}$ is given by

$$\int_P \mathcal{P}\, ds = \mathcal{W} - \dot{\mathcal{K}} \ , \quad \mathcal{W} = \int_P \rho\, (\mathbf{b} \bullet \mathbf{v})\, ds + [\ \mathbf{t}^3 \bullet \mathbf{v}\ ]_1^2 \ ,$$
$$\mathcal{K} = \int_P \frac{1}{2}\rho\, (\mathbf{v} \bullet \mathbf{v})\, ds \ , \tag{5.31.7}$$

where $\mathcal{W}$ is the rate of work of the assigned field $\mathbf{b}$ and the resultant contact force $\mathbf{t}^3$, and $\mathcal{K}$ is the kinetic energy.

These equations are consistent with those that would be obtained by neglecting all terms associated with the directors in the balance laws (5.4.6), (5.4.7) and in the definition (5.6.1). Moreover, the kinematic and kinetic quantities transform under SRBM by expressions similar to those discussed in section 5.5, with all terms associated with the director being neglected.

Next, using standard arguments in continuum mechanics it can be shown that the local forms of the balance laws (5.31.5) become

$$m = \rho\, d_{33}^{1/2} = \rho_0\, D_{33}^{1/2} = m(\theta^3) \quad \text{or} \quad \dot{\rho} + \rho\, \mathbf{w}_3 \bullet \mathbf{d}^3 = 0 \ ,$$
$$m\, \dot{\mathbf{v}} = m\, \mathbf{b} + \mathbf{t}^3{}_{,3} \ , \tag{5.31.8}$$

the local form of the balance of angular momentum (5.31.6) becomes

$$\mathbf{T} = d_{33}^{-1/2}\, \mathbf{t}^3 \otimes \mathbf{d}_3 = \mathbf{T}^T \ , \tag{5.31.9}$$

and the local form of (5.31.7) yields

$$\mathcal{P} = d_{33}^{-1/2}\, \mathbf{t}^3 \bullet \mathbf{w}_3 = \mathbf{T} \bullet \mathbf{D} \ . \tag{5.31.10}$$

For a nonlinear elastic string the strain energy function is taken in the form

$$\Sigma = \hat{\Sigma}\,(\lambda;\, \theta^3) \ , \tag{5.31.11}$$

and the assumptions of section 5.8 which include

$$\rho\, \dot{\Sigma} = \mathcal{P} = \mathbf{T} \bullet \mathbf{D} \ , \tag{5.31.12}$$

yield the results that

$$d_{33}^{1/2}\,\mathbf{T} = m\,\lambda \frac{\partial \Sigma}{\partial \lambda}\,[\frac{\mathbf{d}_3 \otimes \mathbf{d}_3}{d_{33}}] \ , \quad \mathbf{t}^3 = m\,\lambda \frac{\partial \Sigma}{\partial \lambda}\mathbf{d}^3 = N\,[\frac{\mathbf{d}_3}{d_{33}^{1/2}}] \ ,$$

$$N = \rho_0 \frac{\partial \Sigma}{\partial \lambda} = m\,D_{33}^{-1/2} \frac{\partial \Sigma}{\partial \lambda} \ , \tag{5.31.13}$$

where N is the tension in the string, and $\mathbf{t}^3$ acts in the direction tangent to the string. Also, the form for the strain energy function can motivated in a similar manner to that described in section 5.28. In particular, (5.28.18) and (5.28.21) suggest that Σ be specified in the form

$$m\,\Sigma = \frac{1}{2} E^* D_{33}^{1/2} A\ \left[E_{33}\right]^2 \ , \quad E_{33} = \frac{1}{2}(\lambda^2 - 1) \ , \tag{5.31.14}$$

so that the constitutive equation reduces to

$$\mathbf{t}^3 = N\,[\frac{\mathbf{d}_3}{d_{33}^{1/2}}] \ , \quad N = \frac{1}{2} E^* A\,\lambda\,(\lambda^2 - 1) \ . \tag{5.31.15}$$

Thus, the tension N is a nonlinear function of the stretch λ.

As a special case, specify the reference configuration by

$$\mathbf{X} = \theta^3\,\mathbf{e}_3 \ , \quad \mathbf{D}_3 = \mathbf{e}_3 \ , \quad D_{33}^{1/2} = 1 \ , \tag{5.31.16}$$

and let the present configuration be characterized by

$$\mathbf{x} = x_i(\theta^3,t)\,\mathbf{e}_i \ , \quad \mathbf{d}_3 = x_{i,3}\,\mathbf{e}_i \ , \quad \lambda^2 = (x_{1,3})^2 + (x_{2,3})^2 + (x_{3,3})^2 \ . \tag{5.31.17}$$

It then follows that for the simple constitutive equation (5.31.15), the equations of motion reduce to

$$m\,\ddot{x}_i = m\,(\mathbf{e}_i \bullet \mathbf{b}) + \frac{1}{2} E^* A\ \left[(x_{m,3}x_{m,3} - 1)x_{i,3}\right]_{,3} \ . \tag{5.31.18}$$

5.32 Transverse loading of an isotropic beam with a rectangular cross-section

The objective of this section is to consider a specific problem and compare the solutions of the linearized equations of the general Cosserat theory (G), the Timoshenko theory (T), and the Bernoulli-Euler theory (BE). To this end, consider an isotropic beam with rectangular cross-section which is characterized by the equations (5.14.3). The beam is taken to be clamped at each of its ends ($\theta^3 \pm L/2$), and it is subjected to a uniform stress q acting in the negative $\mathbf{e}_1$ direction applied to its upper surface ($\theta^1 = H/2$). Thus, the traction vector applied to the beam's top surface is given by

$$\mathbf{t}^*(H/2,\theta^2,\theta^3) = -\,q\,\mathbf{e}_1 \ , \tag{5.32.1}$$

and the remaining lateral surfaces are stress-free. Moreover, the effect of body force is neglected so that with the help of the equations in section 5.26 the linearized assigned fields are given by

$$m\,\tilde{\mathbf{b}} = -\,qW\,\mathbf{e}_1 \ , \quad m\,\tilde{\mathbf{b}}^1 = -\frac{qHW}{2}\mathbf{e}_1 \ , \quad m\,\tilde{\mathbf{b}}^2 = 0 \ . \tag{5.32.2}$$

In the following analysis the solution of the equations of equilibrium will be obtained for each of the three beam theories.

General Cosserat Theory (G)

For the general Cosserat theory it can be shown that the equations of equilibrium recorded in section 5.27, can be satisfied provided that the displacement **u** and the director displacements $\boldsymbol{\delta}_\alpha$ are given by

$$\mathbf{u} = u_1\,\mathbf{e}_1 + u_3\,\mathbf{e}_3\ ,\quad \boldsymbol{\delta}_1 = \tilde{\delta}_{11}\,\mathbf{e}_1 + \tilde{\delta}_{31}\,\mathbf{e}_3\ ,\quad \boldsymbol{\delta}_2 = \tilde{\delta}_{22}\,\mathbf{e}_2\ . \tag{5.32.3}$$

Moreover, the boundary conditions for clamped ends require

$$u_1 = u_3 = \tilde{\delta}_{31} = 0 \quad \text{for} \quad \theta^3 = \pm\frac{L}{2}\ . \tag{5.32.4}$$

Then, the solution of the general Cosserat theory can be summarized by

$$\frac{u_1}{H} = -\frac{q(1+\nu^*)L^2}{E^*H^2}\left[\frac{1}{4} - Z^2\right] - \frac{qL^4}{2E^*H^4}\left[\frac{1}{4} - Z^2\right]^2\ ,\quad u_3 = 0\ ,$$

$$\tilde{\delta}_{11} = -\frac{q(1-\nu^*)}{4\mu^*}\ ,\quad \tilde{\delta}_{31} = -\frac{2qL^3}{E^*H^3}\left[\frac{1}{4} - Z^2\right]Z\ ,\quad \tilde{\delta}_{22} = \frac{q\nu^*}{4\mu^*}\ ,$$

$$\tilde{\mathbf{m}}^1 = -\frac{qWL^2}{6}\left[\frac{1}{4} - 3Z^2\right]\mathbf{e}_3\ ,\quad \tilde{\mathbf{m}}^2 = 0\ ,\quad \tilde{\mathbf{m}} = \frac{qWL^2}{6}\left[\frac{1}{4} - 3Z^2\right]\mathbf{e}_2\ ,$$

$$\tilde{\mathbf{t}}^1 = -\left[\frac{qHW}{2}\right]\mathbf{e}_1 + (qWL\,Z)\,\mathbf{e}_3\ ,\quad \tilde{\mathbf{t}}^2 = 0\ ,\quad \tilde{\mathbf{t}}^3 = (qWL\,Z)\,\mathbf{e}_1 - \left[\frac{qHW\nu^*}{2}\right]\mathbf{e}_3\ , \tag{5.32.5}$$

where the normalized axial coordinate Z has been defined by

$$Z = \frac{\theta^3}{L}\ . \tag{5.32.6}$$

Of particular interest is the fact that this theory predicts normal cross-sectional contraction in the $\mathbf{e}_1$ direction ($\tilde{\delta}_{11} < 0$), normal cross-sectional expansion in the $\mathbf{e}_2$ direction ($\tilde{\delta}_{22} > 0$) and axial compression ($\mathbf{e}_3 \cdot \tilde{\mathbf{t}}^3 < 0$).

Timoshenko Theory (T)

The linearized equilibrium equations of the Timoshenko theory which are summarized in section 5.29, can be satisfied provided that the displacement **u** and the director displacements $\boldsymbol{\delta}_\alpha$ are given by

$$\mathbf{u} = u_1\,\mathbf{e}_1 + u_3\,\mathbf{e}_3\ ,\quad \boldsymbol{\delta}_1 = \tilde{\delta}_{31}\,\mathbf{e}_3\ ,\quad \boldsymbol{\delta}_2 = 0\ . \tag{5.32.7}$$

Also, the boundary conditions for this theory are the same as (5.32.4), and solution of the Timoshenko theory can be summarized by

$$\frac{u_1}{H} = -\frac{q(1+\nu^*)L^2}{E^*H^2}\left[\frac{1}{4} - Z^2\right] - \frac{qL^4}{2E^*H^4}\left[\frac{1}{4} - Z^2\right]^2\ ,\quad u_3 = 0\ ,$$

$$\tilde{\delta}_{31} = -\frac{2qL^3}{E^*H^3}\left[\frac{1}{4} - Z^2\right]Z\ ,$$

$$\tilde{\mathbf{m}}^1 = -\frac{qWL^2}{6}\left[\frac{1}{4} - 3Z^2\right]\mathbf{e}_3\ ,\quad \tilde{\mathbf{m}}^2 = 0\ ,\quad \tilde{\mathbf{m}} = \frac{qWL^2}{6}\left[\frac{1}{4} - 3Z^2\right]\mathbf{e}_2\ ,$$

$$\tilde{\mathbf{t}}^1 = -\left[\frac{qHW}{2}\right]\mathbf{e}_1 + (qWL\,Z)\,\mathbf{e}_3\ ,\quad \tilde{\mathbf{t}}^2 = 0\ ,\quad \tilde{\mathbf{t}}^3 = (qWL\,Z)\,\mathbf{e}_1\ . \tag{5.32.8}$$

Comparison of (5.32.5) with (5.32.8) indicates that the theories (G) and (T) yield the same functional form for the displacements $\{u_1, u_3, \tilde{\delta}_{31}\}$, for the couples $\{\tilde{\mathbf{m}}^1, \tilde{\mathbf{m}}^2, \tilde{\mathbf{m}}\}$, and for the shearing component $(\mathbf{e}_1 \bullet \tilde{\mathbf{t}}^3)$ of the resultant force $\tilde{\mathbf{t}}^3$. However, the constrained theory (T) does not correctly predict the normal cross-sectional deformation and the axial compression.

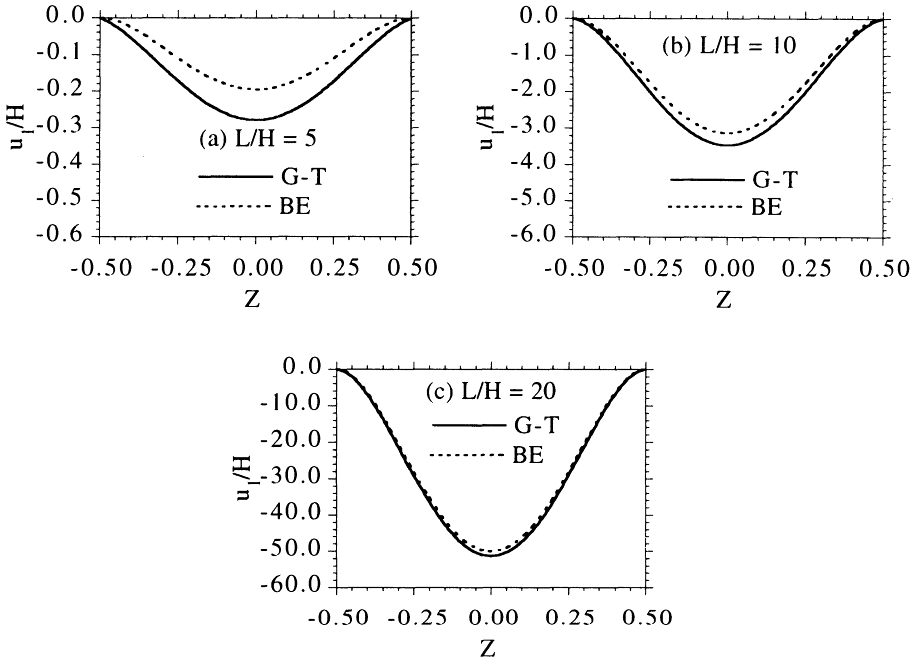

Fig. 5.32 Transverse loading of a clamped-clamped beam, with $q/E^*=0.01$ and $\nu^*=1/3$, for different length to thickness ratios L/H. Comparison of the general Cosserat theory and the Timoshenko theories (G-T) with the Bernoulli-Euler theory (BE).

Bernoulli-Euler (BE)

The linearized equilibrium equations of the Bernoulli-Euler theory which are summarized in section 5.28, can be satisfied provided that the displacement $\mathbf{u}$ and the director displacements $\boldsymbol{\delta}_\alpha$ are given by

$$\mathbf{u} = u_1\,\mathbf{e}_1 + u_3\,\mathbf{e}_3\,,\quad \boldsymbol{\delta}_1 = \tilde{\delta}_{31}\,\mathbf{e}_1 = -\,u_{1,3}\,\mathbf{e}_3\,,\quad \boldsymbol{\delta}_2 = 0\,. \tag{5.32.9}$$

Also, the boundary conditions for this theory are the same as (5.32.4), and solution of the Bernoulli-Euler theory can be summarized by

$$\frac{u_1}{H} = -\frac{qL^4}{2E^*H^4}\left[\frac{1}{4} - Z^2\right]^2\,,\quad u_3 = 0\,,\quad \tilde{\delta}_{31} = -\frac{2qL^3}{E^*H^3}\left[\frac{1}{4} - Z^2\right]Z\,,$$

$$\tilde{\mathbf{m}}^1 = -\frac{qWL^2}{6}\left[\frac{1}{4} - 3\,Z^2\right]\mathbf{e}_3\,,\quad \tilde{\mathbf{m}}^2 = 0\,,\quad \tilde{\mathbf{m}} = \frac{qWL^2}{6}\left[\frac{1}{4} - 3\,Z^2\right]\mathbf{e}_2\,,$$

$$\tilde{\mathbf{t}}^1 = -\left[\frac{qHW}{2}\right]\mathbf{e}_1 + (qWL\,Z)\,\mathbf{e}_3\,,\quad \tilde{\mathbf{t}}^2 = 0\,,\quad \tilde{\mathbf{t}}^3 = (qWL\,Z)\,\mathbf{e}_1\,. \tag{5.32.10}$$

Now, comparison of (5.32.10) with (5.32.8) indicates that the Bernoulli-Euler theory predicts the same results as the Timoshenko theory, except for the displacement field u_1.

Figure 5.32 exhibits the quantitative differences between these displacement fields for three different length to thickness ratios L/H, and for the specifications

$$\frac{q}{E^*} = 0.01 \;\;,\;\; \nu^* = \frac{1}{3} \;. \tag{5.32.11}$$

In particular, note that the inclusion of shear deformation yields a more flexible response with the differences being more significant as the beam becomes thicker.

5.33 Linearized buckling equations

Linearized equations for buckling of beams have been extensively studied and are discussed in (Timoshenko and Gere, 1961). Also, extensive nonlinear analysis of such problems is presented in (Antman, 1995). The objective of this section is to develop linearized equations for buckling of beams within the context of the Cosserat theory. This is accomplished by starting with the nonlinear theory, and linearizing the equations about the finite deformation solution of uniaxial tension of a straight rod (or beam). If the beam is compressed and the tension is negative, then the equations can be used to analyze buckling of the rod. Specific equations will be developed for five theories: a general Cosserat beam, a Timoshenko beam, a Bernoulli-Euler beam, a generalized string, and a simple string.

General Cosserat Beam

To this end, consider an isotropic beam with rectangular cross-section as specified by (5.14.3). Using the equations that are summarized in section 5.26, the kinematics of the reference configuration are specified by

$$\mathbf{X} = \theta^3 \, \mathbf{e}_3 \;\;,\;\; \mathbf{D}_i = \mathbf{e}_i \;\;, \tag{5.33.1}$$

and the kinematics of the deformed present configuration are defined by

$$\mathbf{x} = a\,\theta^3\,\mathbf{e}_3 + u_i\,\mathbf{e}_i \;,\;\; \mathbf{d}_1 = b\,\mathbf{e}_1 + \tilde{\delta}_{i1}\,\mathbf{e}_i \;\;,\;\; \mathbf{d}_2 = b\,\mathbf{e}_2 + \tilde{\delta}_{i2}\,\mathbf{e}_i \;\;,$$
$$\mathbf{d}_3 = a\,\mathbf{e}_3 + u_{i,3}\,\mathbf{e}_3 \;\;,\;\; u_i = u_i(\theta^3,t) \;\;,\;\; \tilde{\delta}_{i\alpha} = \tilde{\delta}_{i\alpha}(\theta^3,t) \;\;. \tag{5.33.2}$$

Here, the constant a describes the finite stretch in the axial direction, and the constant b describes the finite stretch of the cross-section. Also, the displacements u_i and the director displacements $\tilde{\delta}_{i\alpha}$ are measured from this prestretched configuration. Again, using the kinematics summarized in section 5.26 and neglecting quadratic terms in the displacements u_i and $\tilde{\delta}_{i\alpha}$, it can be shown that

$$d_{33}^{1/2} = a + u_{3,3} \;\;,$$
$$\begin{aligned}
\mathbf{F} = &\left[(b + \tilde{\delta}_{11})\,(\mathbf{e}_1\otimes\mathbf{e}_1) + \tilde{\delta}_{21}\,(\mathbf{e}_2\otimes\mathbf{e}_1) + \tilde{\delta}_{31}\,(\mathbf{e}_3\otimes\mathbf{e}_1)\right] \\
&+ \left[\tilde{\delta}_{12}\,(\mathbf{e}_1\otimes\mathbf{e}_2) + (b + \tilde{\delta}_{22})\,(\mathbf{e}_2\otimes\mathbf{e}_2) + \tilde{\delta}_{32}\,(\mathbf{e}_3\otimes\mathbf{e}_2)\right] \\
&+ \left[\,u_{1,3}\,(\mathbf{e}_1\otimes\mathbf{e}_3) + u_{2,3}\,(\mathbf{e}_2\otimes\mathbf{e}_3) + (a + u_{3,3})\,(\mathbf{e}_3\otimes\mathbf{e}_3)\right] \;\;,
\end{aligned}$$
$$\mathbf{E} = \frac{1}{2}(b^2-1)\,(\mathbf{e}_1\otimes\mathbf{e}_1 + \mathbf{e}_2\otimes\mathbf{e}_2) + \frac{1}{2}(a^2-1)\,(\mathbf{e}_3\otimes\mathbf{e}_3)$$

$$+ (b\,\tilde{\delta}_{11})(\mathbf{e}_1\otimes\mathbf{e}_1) + (b\,\tilde{\delta}_{22})(\mathbf{e}_2\otimes\mathbf{e}_2) + (a\,u_{3},_{3})(\mathbf{e}_3\otimes\mathbf{e}_3)$$

$$+ \frac{1}{2} b\,(\tilde{\delta}_{12} + \tilde{\delta}_{21})(\mathbf{e}_1\otimes\mathbf{e}_2 + \mathbf{e}_2\otimes\mathbf{e}_1) + \frac{1}{2}(b\,u_{1},_{3} + a\,\tilde{\delta}_{31})(\mathbf{e}_1\otimes\mathbf{e}_3 + \mathbf{e}_3\otimes\mathbf{e}_1)$$

$$+ \frac{1}{2}(b\,u_{2},_{3} + a\,\tilde{\delta}_{32})(\mathbf{e}_2\otimes\mathbf{e}_3 + \mathbf{e}_3\otimes\mathbf{e}_2) \;,$$

$$\boldsymbol{\beta}_\alpha = \frac{1}{b}\tilde{\delta}_{1\alpha},_{3}\,\mathbf{e}_1 + \frac{1}{b}\tilde{\delta}_{2\alpha},_{3}\,\mathbf{e}_2 + \frac{1}{a}\tilde{\delta}_{3\alpha},_{3}\,\mathbf{e}_3 \;. \tag{5.33.3}$$

Next, specifying the material directions $\mathbf{M}_i=\mathbf{D}_i$, considering an isotropic material, and using section 5.26, the constitutive equations become

$$m\,\Sigma = \frac{1}{2}HW\,\mathbf{K}^* \bullet (\mathbf{E}\otimes\mathbf{E}) + \frac{1}{2}HW\,\mathbf{K}^{\alpha\beta} \bullet (\boldsymbol{\beta}_\alpha\otimes\boldsymbol{\beta}_\beta) \;,$$

$$\mathbf{K}^* \bullet \mathbf{E} = 2\mu^* \left[\left\{ \frac{\nu^*}{1-2\nu^*} \right\} (\mathbf{E}\bullet\mathbf{I})\,\mathbf{I} + \mathbf{E} \right] ,$$

$$\mathbf{K}^{11} = \frac{\mu^* H^2}{6} \left[\left\{ \frac{b^*(\eta)}{\eta} \right\} \mathbf{M}_2\otimes\mathbf{M}_2 + (1+\nu^*)\,\mathbf{M}_3\otimes\mathbf{M}_3 \right] \;,\quad \mathbf{K}^{12} = \mathbf{K}^{21} = 0 \;,$$

$$\mathbf{K}^{22} = \frac{\mu^* W^2}{6} \left[\left\{ \eta\, b^*(\eta) \right\} \mathbf{e}_1\otimes\mathbf{e}_1 + (1+\nu^*)\,\mathbf{e}_3\otimes\mathbf{e}_3 \right] \;,\quad \eta = \frac{H}{W} \;,$$

$$\mathbf{m}^\alpha = HW\,\mathbf{F}^{-T}\,\mathbf{K}^{\alpha\beta}\,\boldsymbol{\beta}_\beta \;,$$

$$\mathbf{m}^1 = \left\{ \frac{\mu^* H^2 W^2 b^*(\eta)}{6b^2} \right\} \tilde{\delta}_{21},_{3}\,\mathbf{e}_2 + \left\{ \frac{E^* H^3 W}{12a^2} \right\} \tilde{\delta}_{31},_{3}\,\mathbf{e}_3 \;,$$

$$\mathbf{m}^2 = \left\{ \frac{\mu^* H^2 W^2 b^*(\eta)}{6b^2} \right\} \tilde{\delta}_{12},_{3}\,\mathbf{e}_1 + \left\{ \frac{E^* H W^3}{12a^2} \right\} \tilde{\delta}_{32},_{3}\,\mathbf{e}_3 \;,$$

$$\mathbf{m} = b\,\mathbf{e}_\alpha \times \mathbf{m}^\alpha = \left\{ \frac{bE^* H W^3}{12a^2} \right\} \tilde{\delta}_{32},_{3}\,\mathbf{e}_1 - \left\{ \frac{bE^* H^3 W}{12a^2} \right\} \tilde{\delta}_{31},_{3}\,\mathbf{e}_2$$

$$+ \left\{ \frac{\mu^* H^2 W^2 b^*(\eta)}{6b} \right\} [\tilde{\delta}_{21},_{3} - \tilde{\delta}_{12},_{3}]\,\mathbf{e}_3 \;,$$

$$\mathbf{t}^i = HW\,\mathbf{F} \left[\mathbf{K}^* \bullet \mathbf{E} \right] \mathbf{e}_i \;,$$

$$\mathbf{K}^* \bullet \mathbf{E} = 2\mu^* \left[\left\{ \frac{\nu^*}{1-2\nu^*} \right\} \left\{ \frac{1}{2}(1-2\nu^*)(a^2-1) + (b\,\tilde{\delta}_{11} + b\,\tilde{\delta}_{22}) + (a\,u_{3},_{3}) \right\} \mathbf{I} + \mathbf{E} \right] ,$$

$$\mathbf{t}^1 = 2\mu^* HW \left[\left\{ \frac{\nu^*}{1-2\nu^*} \right\} \left\{ (b\,\tilde{\delta}_{11} + b\,\tilde{\delta}_{22}) + (a\,u_{3},_{3}) \right\} b\,\mathbf{e}_1 \right.$$

$$\left. + (b^2\,\tilde{\delta}_{11})\,\mathbf{e}_1 + \frac{1}{2}b^2(\tilde{\delta}_{12} + \tilde{\delta}_{21})\,\mathbf{e}_2 + \frac{1}{2}(b\,u_{1},_{3} + a\,\tilde{\delta}_{31})\,a\,\mathbf{e}_3 \right] \;,$$

$$\mathbf{t}^2 = 2\mu^* HW \left[\left\{ \frac{\nu^*}{1-2\nu^*} \right\} \left\{ (b\,\tilde{\delta}_{11} + b\,\tilde{\delta}_{22}) + (a\,u_{3},_{3}) \right\} b\,\mathbf{e}_2 \right.$$

$$\left. + \frac{1}{2}b^2(\tilde{\delta}_{12} + \tilde{\delta}_{21})\,\mathbf{e}_1 + (b^2\,\tilde{\delta}_{22})\,\mathbf{e}_2 + \frac{1}{2}(b\,u_{2},_{3} + a\,\tilde{\delta}_{32})\,a\,\mathbf{e}_3 \right] \;,$$

$$\mathbf{t}^3 = E^* HW \frac{1}{2}(a^2-1) \left[(u_{1},_{3})\,\mathbf{e}_1 + (u_{2},_{3})\,\mathbf{e}_2 + (a + u_{3},_{3})\,\mathbf{e}_3 \right]$$

$$+ 2\mu^* HW \left[\left\{ \frac{\nu^*}{1-2\nu^*} \right\} \left\{ (b\,\tilde{\delta}_{11} + b\,\tilde{\delta}_{22}) + (a\,u_{3},_{3}) \right\} a\,\mathbf{e}_3 \right.$$

$$+\frac{1}{2}(b\, u_{1,3} + a\, \tilde{\delta}_{31})\, b\, \mathbf{e}_1 + \frac{1}{2}(b\, u_{2,3} + a\, \tilde{\delta}_{32})\, b\, \mathbf{e}_2 + (a^2\, u_{3,3})\, \mathbf{e}_3 \Big] \ , \quad (5.33.4)$$

where the expression for Young's modulus E^* has been used, and the value of the stretch b has been determined by the condition that the prestretched configuration is a state of uniaxial tension so that

$$\frac{1}{2}(b^2-1) = -\, v^* \frac{1}{2}(a^2-1) \ , \quad b^2 = 1 - v^*\,(a^2-1) \ . \qquad (5.33.5)$$

In particular, it can be seen from (5.33.4) that for the prestretched configuration (u_i=0, $\tilde{\delta}_{i\alpha}$=0)

$$\mathbf{m}^\alpha = 0 \ , \quad \mathbf{t}^\alpha = 0 \ , \quad \mathbf{t}^3 = N\, \mathbf{e}_3 \ , \quad N = E^* HW \frac{1}{2}(a^2-1)\, a \ , \qquad (5.33.6)$$

where N is the axial tension.

Next, with the help of the expressions (5.27.4) for the inertia quantities, the equations of motion can be written as the conservation of mass

$$\rho = \frac{\rho_0}{a}\Big[1 - \frac{1}{a} u_{3,3}\Big] \ , \qquad (5.33.7)$$

and four other independent sets of equations. One set associated with extensional deformations along the axis of the beam and in its cross-section,

$$m\, \ddot{u}_3 = m\,(\mathbf{e}_3 \bullet \mathbf{b}) + \frac{N}{a}(u_{3,33}) + \Big\{\frac{2\mu^* HW v^*}{1-2v^*}\Big\} a \big\{(b\, \tilde{\delta}_{11,3} + b\, \tilde{\delta}_{22,3}) + (a\, u_{3,33})\big\}$$

$$+ 2\mu^* HW (a^2\, u_{3,33}) \ ,$$

$$m\, y^{11}\, \ddot{\tilde{\delta}}_{11} = m\,(\mathbf{e}_1 \bullet \mathbf{b}^1) - 2\mu^* HW \Big[\Big\{\frac{v^*}{1-2v^*}\Big\} b \big\{(b\, \tilde{\delta}_{11} + b\, \tilde{\delta}_{22}) + (a\, u_{3,3})\big\}$$

$$+ (b^2\, \tilde{\delta}_{11})\Big] \ ,$$

$$m\, y^{22}\, \ddot{\tilde{\delta}}_{22} = m\,(\mathbf{e}_2 \bullet \mathbf{b}^2) - 2\mu^* HW \Big[\Big\{\frac{v^*}{1-2v^*}\Big\} b \big\{(b\, \tilde{\delta}_{11} + b\, \tilde{\delta}_{22}) + (a\, u_{3,3})\big\}$$

$$+ (b^2\, \tilde{\delta}_{22})\Big] \ , \qquad (5.33.8)$$

one set describing bending and shearing in the $\mathbf{e}_1$–$\mathbf{e}_3$ plane,

$$m\, \ddot{u}_1 = m\,(\mathbf{e}_1 \bullet \mathbf{b}) + \frac{N}{a}(u_{1,33}) + \mu^* HW\, b\,(b\, u_{1,33} + a\, \tilde{\delta}_{31,3}) \ ,$$

$$m\, y^{11}\, \ddot{\tilde{\delta}}_{31} = m\,(\mathbf{e}_3 \bullet \mathbf{b}^1) - \mu^* HW\, a\,(b\, u_{1,3} + a\, \tilde{\delta}_{31}) + \Big\{\frac{E^* H^3 W}{12a^2}\Big\} \tilde{\delta}_{31,33} \ , \quad (5.33.9)$$

one set describing bending and shearing in the $\mathbf{e}_2$–$\mathbf{e}_3$ plane,

$$m\, \ddot{u}_2 = m\,(\mathbf{e}_2 \bullet \mathbf{b}) + \frac{N}{a}(u_{2,33}) + \mu^* HW\, b\,(b\, u_{2,33} + a\, \tilde{\delta}_{32,3}) \ ,$$

$$m\, y^{22}\, \ddot{\tilde{\delta}}_{32} = m\,(\mathbf{e}_3 \bullet \mathbf{b}^2) - 2\mu^* HW\, a\,(b\, u_{2,3} + a\, \tilde{\delta}_{32}) + \Big\{\frac{E^* HW^3}{12a^2}\Big\} \tilde{\delta}_{32,33} \ , \quad (5.33.10)$$

and one set describing torsion and shearing of the cross-section,

$$m\, y^{11}\, \ddot{\tilde{\delta}}_{21} = m\,(\mathbf{e}_2 \bullet \mathbf{b}^1) - \mu^* HW\, b^2\,(\tilde{\delta}_{12} + \tilde{\delta}_{21}) + \Big\{\frac{\mu^* H^2 W^2 b^*(\eta)}{6b^2}\Big\} \tilde{\delta}_{21,33} \ ,$$

$$m\, y^{22}\, \ddot{\tilde{\delta}}_{12} = m\,(\mathbf{e}_1 \bullet \mathbf{b}^2) - \mu^* HW\, b^2\, (\tilde{\delta}_{12} + \tilde{\delta}_{21}) + \{\frac{\mu^* H^2 W^2 b^*(\eta)}{6b^2}\}\, \tilde{\delta}_{12,33}\ , \qquad (5.33.11)$$

where the inertia quantities are given by the expressions (5.27.4).

In order to discuss the boundary conditions it is first noted that the velocity $\mathbf{v}$ and the director velocities $\mathbf{w}_\alpha$ become

$$\mathbf{v} = \dot{u}_i\, \mathbf{e}_i\ ,\quad \mathbf{w}_\alpha = \dot{\tilde{\delta}}_{i\alpha}\, \mathbf{e}_i\ . \qquad (5.33.12)$$

Thus, the rate of work of the contact force and the director couples applied to the ends of the beam can be written in the form

$$\mathbf{t}^3 \bullet \dot{\mathbf{u}} + \mathbf{m}^\alpha \bullet \mathbf{w}_\alpha = (\mathbf{e}_i \bullet \mathbf{t}^3)\, \dot{u}_i + (\mathbf{e}_2 \bullet \mathbf{m}^1)\, \dot{\tilde{\delta}}_{21} + (\mathbf{e}_3 \bullet \mathbf{m}^1)\, \dot{\tilde{\delta}}_{31}$$

$$+ (\mathbf{e}_1 \bullet \mathbf{m}^2)\, \dot{\tilde{\delta}}_{12} + (\mathbf{e}_3 \bullet \mathbf{m}^2)\, \dot{\tilde{\delta}}_{32}\ , \qquad (5.33.13)$$

which indicates that kinematic boundary conditions associated with (5.33.13) are characterized by specified values of

$$\{u_1\ , u_2\ , u_3\ , \tilde{\delta}_{21}\ , \tilde{\delta}_{31}\ , \tilde{\delta}_{12}\ , \tilde{\delta}_{32}\}\ . \qquad (5.33.14)$$

In particular, it is noted that since $(\mathbf{e}_1 \bullet \mathbf{m}^1)$ and $(\mathbf{e}_2 \bullet \mathbf{m}^2)$ vanish, there are no boundary conditions specified for $\tilde{\delta}_{11}$ and $\tilde{\delta}_{22}$.

As a special case, consider the simpler problem where the centerline of the beam remains in the $\mathbf{e}_1$–$\mathbf{e}_3$ plane and there is no torsion or shearing of the cross-section so that

$$u_2 = u_3 = 0\ ,\quad \tilde{\delta}_{11} = \tilde{\delta}_{22} = 0\ ,\quad \tilde{\delta}_{12} = \tilde{\delta}_{21} = \tilde{\delta}_{32} = 0\ . \qquad (5.33.15)$$

Then, in the absence of assigned fields $\mathbf{b}$ and $\mathbf{b}^\alpha$ the equations of motion (5.33.8), (5.33.10) and (5.33.11) are satisfied automatically, and (5.33.9) reduce to

$$m\, \ddot{u}_1 = \frac{N}{a}(u_{1,33}) + \mu^* HW\, b\, (b\, u_{1,33} + a\, \tilde{\delta}_{31,3})\ ,$$

$$m\, y^{11}\, \ddot{\tilde{\delta}}_{31} = -\, \mu^* HW\, a\, (b\, u_{1,3} + a\, \tilde{\delta}_{31}) + \{\frac{E^* H^3 W}{12a^2}\}\, \tilde{\delta}_{31,33}\ . \qquad (5.33.16)$$

Now, equation $(5.33.16)_1$ can be solved for $\tilde{\delta}_{31,3}$ to obtain

$$\tilde{\delta}_{31,3} = \frac{1}{\mu^* HWab}\, [m\, \ddot{u}_1 - \frac{N}{a}(u_{1,33}) - \mu^* HW\, (b^2\, u_{1,33})\ ]\ . \qquad (5.33.17)$$

Next, by differentiating $(5.33.16)_2$ with respect to θ^3 and using (5.33.17), it can be shown that the result becomes

$$\{\frac{m y^{11}}{\mu^* HWab}\}[m\, \ddddot{u}_1] - [\{\frac{y^{11} b}{a}\}\{1 + \frac{N}{\mu^* HWab^2}\} + \{\frac{E^* H^2}{12\mu^* a^3 b}\}][m\, \ddot{u}_{1,33}]$$

$$+ \frac{a}{b}[m\, \ddot{u}_1] + \{\frac{E^* H^3 Wb}{12a^3}\}\{1 + \frac{N}{\mu^* HWab^2}\}[u_{1,3333}] - \frac{N}{b}[u_{1,33}] = 0\ . \quad (5.33.18)$$

This equation includes the effects of shear deformation as well as rotary inertia, and it is valid for large pretension (N>0) or precompression (N<0). Also, it is recalled that the value of N appears implicitly through the values of a and b.

Timoshenko Beam

The equations developed above are valid for the general Cosserat theory. Next, it is of interest to develop similar equations for the constrained Timoshenko theory which was discussed in section 5.29. In view of the constraints (5.29.1), the cross-section cannot deform in the prestretched configuration so the kinematics of the present configuration are characterized by (5.33.2) with b set equal to unity

$$\mathbf{x} = a\,\theta^3\,\mathbf{e}_3 + u_i\,\mathbf{e}_i\ , \quad \mathbf{d}_1 = \mathbf{e}_1 + \tilde{\delta}_{i1}\,\mathbf{e}_i\ , \quad \mathbf{d}_2 = \mathbf{e}_2 + \tilde{\delta}_{i2}\,\mathbf{e}_i\ ,$$
$$\mathbf{d}_3 = a\,\mathbf{e}_3 + u_{i,3}\,\mathbf{e}_3\ , \quad u_i = u_i(\theta^3,t)\ , \quad \tilde{\delta}_{i\alpha} = \tilde{\delta}_{i\alpha}(\theta^3,t)\ . \tag{5.33.19}$$

Also, the constraints (5.29.1) yield the conditions that

$$\tilde{\delta}_{11} = \tilde{\delta}_{22} = 0\ , \quad \tilde{\delta}_{21} = -\,\tilde{\delta}_{12} = \delta\ . \tag{5.33.20}$$

Thus, equations (5.33.3) reduce to

$$d_{33}^{1/2} = a + u_{3,3}\ ,$$
$$\mathbf{F} = \big[\,(\mathbf{e}_1\otimes\mathbf{e}_1) + \delta\,(\mathbf{e}_2\otimes\mathbf{e}_1) + \tilde{\delta}_{31}\,(\mathbf{e}_3\otimes\mathbf{e}_1)\big]$$
$$+ \big[-\,\delta\,(\mathbf{e}_1\otimes\mathbf{e}_2) + (\mathbf{e}_2\otimes\mathbf{e}_2) + \tilde{\delta}_{32}\,(\mathbf{e}_3\otimes\mathbf{e}_2)\big]$$
$$+ \big[\,u_{1,3}\,(\mathbf{e}_1\otimes\mathbf{e}_3) + u_{2,3}\,(\mathbf{e}_2\otimes\mathbf{e}_3) + (a + u_{3,3})\,(\mathbf{e}_3\otimes\mathbf{e}_3)\big]\ ,$$
$$\mathbf{E} = \frac{1}{2}(a^2-1)\,(\mathbf{e}_3\otimes\mathbf{e}_3) + (a\,u_{3,3})\,(\mathbf{e}_3\otimes\mathbf{e}_3)$$
$$+ \frac{1}{2}(u_{1,3} + a\,\tilde{\delta}_{31})\,(\mathbf{e}_1\otimes\mathbf{e}_3 + \mathbf{e}_3\otimes\mathbf{e}_1)$$
$$+ \frac{1}{2}(u_{2,3} + a\,\tilde{\delta}_{32})\,(\mathbf{e}_2\otimes\mathbf{e}_3 + \mathbf{e}_3\otimes\mathbf{e}_2)\ ,$$
$$E_{\alpha 3} = \frac{1}{2}(u_{\alpha,3} + a\,\tilde{\delta}_{3\alpha})\ , \quad E_{33} = \frac{1}{2}(a^2-1) + (a\,u_{3,3})\ ,$$
$$\boldsymbol{\beta}_1 = \delta_{,3}\,\mathbf{e}_2 + \frac{1}{a}\tilde{\delta}_{31,3}\,\mathbf{e}_3\ , \quad \boldsymbol{\beta}_2 = -\,\delta_{,3}\,\mathbf{e}_1 + \frac{1}{a}\tilde{\delta}_{32,3}\,\mathbf{e}_3\ . \tag{5.33.21}$$

Now, using (5.33.1) and the quadratic form (5.28.21), the strain energy function for the Timoshenko beam becomes

$$m\,\Sigma = HW\,\Big[\frac{1}{2}E^*\,(E_{33})^2 + 2\mu^*\,E_{\alpha 3}\,E_{\alpha 3}\Big] + \frac{1}{2}\,HW\,\mathbf{K}^{\alpha\beta}\bullet(\boldsymbol{\beta}_\alpha\otimes\boldsymbol{\beta}_\beta)\ , \tag{5.33.22}$$

where $\mathbf{K}^{\alpha\beta}$ are given by (5.33.4). Since there are no constraint responses for the director couples $\mathbf{m}^\alpha$, the constitutive equations (5.29.11) for these quantities are

$$\mathbf{m}^\alpha = HW\,\mathbf{F}^{-T}\,\mathbf{K}^{\alpha\beta}\,\boldsymbol{\beta}_\beta\ , \quad \mathbf{m}^1 = \Big\{\frac{\mu^* H^2 W^2 b^*(\eta)}{6}\Big\}\,\delta_{,3}\,\mathbf{e}_2 + \Big\{\frac{E^* H^3 W}{12a^2}\Big\}\,\tilde{\delta}_{31,3}\,\mathbf{e}_3\ ,$$
$$\mathbf{m}^2 = -\,\Big\{\frac{\mu^* H^2 W^2 b^*(\eta)}{6}\Big\}\,\delta_{,3}\,\mathbf{e}_1 + \Big\{\frac{E^* H W^3}{12a^2}\Big\}\,\tilde{\delta}_{32,3}\,\mathbf{e}_3\ ,$$
$$\mathbf{m} = \mathbf{e}_\alpha\times\mathbf{m}^\alpha = \Big\{\frac{E^* H W^3}{12a^2}\Big\}\,\tilde{\delta}_{32,3}\,\mathbf{e}_1 - \Big\{\frac{E^* H^3 W}{12a^2}\Big\}\,\tilde{\delta}_{31,3}\,\mathbf{e}_2$$
$$+ \Big\{\frac{\mu^* H^2 W^2 b^*(\eta)}{3}\Big\}\,\delta_{,3}\,\mathbf{e}_2\ . \tag{5.33.23}$$

Also, the parts $\hat{\mathbf{t}}^i$ of $\mathbf{t}^i$ which are related to the strain energy function (5.33.22), are given by (5.29.11)

$$\hat{\mathbf{t}}^1 = \mu^*HW\, a\, (u_{1,3} + a\, \tilde{\delta}_{31})\, \mathbf{e}_3 \quad , \quad \hat{\mathbf{t}}^2 = \mu^*HW\, a\, (u_{2,3} + a\, \tilde{\delta}_{32})\, \mathbf{e}_3 \ ,$$

$$\hat{\mathbf{t}}^3 = \mu^*HW\, (u_{1,3} + a\, \tilde{\delta}_{31})\, \mathbf{e}_1 + \mu^*HW\, (u_{2,3} + a\, \tilde{\delta}_{32})\, \mathbf{e}_2 + E^*HW\, (a^2\, u_{3,3})\, \mathbf{e}_3 + \frac{N}{a}\left[(u_{1,3})\, \mathbf{e}_1 + (u_{2,3})\, \mathbf{e}_2 + (a + u_{3,3})\, \mathbf{e}_3\right] \ , \tag{5.33.24}$$

and the constraint responses (5.29.6) reduce to

$$\bar{\mathbf{t}}^1 = m\, (\mathbf{e}_1 \bullet \mathbf{b}^1)\, \mathbf{e}_1 + m\, (\mathbf{e}_2 \bullet \mathbf{b}^1)\, \mathbf{e}_2 \ ,$$

$$\bar{\mathbf{t}}^2 = m\, (\mathbf{e}_1 \bullet \mathbf{b}^2)\, \mathbf{e}_1 + m\, (\mathbf{e}_2 \bullet \mathbf{b}^2)\, \mathbf{e}_2 \ , \quad \bar{\mathbf{t}}^3 = 0 \ , \tag{5.33.25}$$

where use has been made of the definition (5.33.6) for the tension N. Also, in view of $(5.33.25)_3$ it follows that the resultant force is given by $\mathbf{t}^3 = \hat{\mathbf{t}}^3$.

Using these results the equations of motion (5.29.5) reduce to the conservation of mass (5.33.7) and four other independent sets of equations. One equation associated with extensional deformations along the axis of the beam,

$$m\, \ddot{u}_3 = m\, (\mathbf{e}_3 \bullet \mathbf{b}) + \frac{N}{a} (u_{3,33}) + E^*HW\, (a^2\, u_{3,33}) \ , \tag{5.33.26}$$

one set describing bending and shearing in the $\mathbf{e}_1$–$\mathbf{e}_3$ plane,

$$m\, \ddot{u}_1 = m\, (\mathbf{e}_1 \bullet \mathbf{b}) + \frac{N}{a} (u_{1,33}) + \mu^*HW\, (u_{1,33} + a\, \tilde{\delta}_{31,3}) \ ,$$

$$m\, y^{11}\, \ddot{\tilde{\delta}}_{31} = m\, (\mathbf{e}_3 \bullet \mathbf{b}^1) - \mu^*HW\, a\, (u_{1,3} + a\, \tilde{\delta}_{31}) + \left\{\frac{E^*H^3W}{12a^2}\right\} \tilde{\delta}_{31,33} \ , \tag{5.33.27}$$

one set describing bending and shearing in the $\mathbf{e}_2$–$\mathbf{e}_3$ plane,

$$m\, \ddot{u}_2 = m\, (\mathbf{e}_2 \bullet \mathbf{b}) + \frac{N}{a} (u_{2,33}) + \mu^*HW\, (u_{2,33} + a\, \tilde{\delta}_{32,3}) \ ,$$

$$m\, y^{22}\, \ddot{\tilde{\delta}}_{32} = m\, (\mathbf{e}_3 \bullet \mathbf{b}^2) - 2\mu^*HW\, a\, (u_{2,3} + a\, \tilde{\delta}_{32}) + \left\{\frac{E^*HW^3}{12a^2}\right\} \tilde{\delta}_{32,33} \ , \tag{5.33.28}$$

and one equation describing torsion of the cross-section,

$$m\, (y^{11} + y^{22})\, \ddot{\delta} = m\, (\mathbf{e}_2 \bullet \mathbf{b}^1 - \mathbf{e}_1 \bullet \mathbf{b}^2) + \left\{\frac{\mu^*H^2W^2b^*(\eta)}{3}\right\} \delta_{,33} \ . \tag{5.33.29}$$

For this theory the rate of work expression (5.29.15) reduces to

$$\mathbf{t}^3 \bullet \mathbf{v} + \mathbf{m} \bullet \boldsymbol{\omega} = (\mathbf{e}_i \bullet \mathbf{t}^3)\, \dot{u}_i + (\mathbf{e}_i \bullet \mathbf{m})\, \omega_i \ , \tag{5.33.30}$$

where the components ω_i of $\boldsymbol{\omega}$ are given by

$$\omega_1 = \dot{\tilde{\delta}}_{32} \ , \quad \omega_2 = -\dot{\tilde{\delta}}_{31} \ , \quad \omega_3 = \dot{\delta} \ . \tag{5.33.31}$$

This indicates that kinematic boundary conditions associated with (5.33.30) are characterized by specified values of

$$\{u_1 \, , u_2 \, , u_3 \, , \tilde{\delta}_{31} \, , \tilde{\delta}_{32} \, , \delta\} \ . \tag{5.33.32}$$

Moreover, it is noted that equations (5.33.27) and (5.33.28) are the same as (5.33.9) and (5.33.10) with b set equal to unity. Thus, the analysis of the equations for this theory for buckling would lead to an equation of the form (5.33.18) with b equal to unity

$$\{\frac{m y^{11}}{\mu^* HWa}\}[m \ddddot{u}_1] - [\{\frac{y^{11}}{a}\}\{1+\frac{N}{\mu^* HWa}\} + \{\frac{E^* H^2}{12\mu^* a^3}\}][m \ddot{u}_{1,33}]$$
$$+ \; a\,[m \ddot{u}_1] + \{\frac{E^* H^3 W}{12 a^3}\}\{1+\frac{N}{\mu^* HWa}\}[u_{1,3333}] - N\,[u_{1,33}] = 0 \; . \quad (5.33.33)$$

Again, it can be seen that the pretension N appears implicitly in this equation through the value of a.

<u>Bernoulli-Euler Beam</u>

For the Bernoulli-Euler beam which has been discussed in section 5.28, the directors are constrained even more than those in the Timoshenko theory because tangential shear deformation is eliminated. In view of the constraints (5.28.2), the cross-section cannot deform in the prestretched configuration so the kinematics of the present configuration are characterized by (5.33.19) with the constraints (5.28.2) yielding the conditions

$$\tilde{\delta}_{11} = \tilde{\delta}_{22} = 0 \;\; , \;\; \tilde{\delta}_{21} = -\,\tilde{\delta}_{12} = \delta \;\; , \;\; a\,\tilde{\delta}_{3\alpha} = -\,u_{\alpha,3} \;\; . \qquad (5.33.34)$$

Thus, equations (5.33.21) reduce to

$$d_{33}^{1/2} = a + u_{3,3} \;\; ,$$
$$\mathbf{F} = [\,(\mathbf{e}_1\otimes\mathbf{e}_1) + \delta\,(\mathbf{e}_2\otimes\mathbf{e}_1) - \frac{1}{a}u_{1,3}\,(\mathbf{e}_3\otimes\mathbf{e}_1)]$$
$$+ [-\,\delta\,(\mathbf{e}_1\otimes\mathbf{e}_2) + (\mathbf{e}_2\otimes\mathbf{e}_2) \; - \frac{1}{a}u_{2,3}\,(\mathbf{e}_3\otimes\mathbf{e}_2)]$$
$$+ [\,u_{1,3}\,(\mathbf{e}_1\otimes\mathbf{e}_3) + u_{2,3}\,(\mathbf{e}_2\otimes\mathbf{e}_3) + (a + u_{3,3})\,(\mathbf{e}_3\otimes\mathbf{e}_3)] \;\; ,$$
$$\mathbf{E} = \frac{1}{2}(a^2-1)\,(\mathbf{e}_3\otimes\mathbf{e}_3) + (a\,u_{3,3})\,(\mathbf{e}_3\otimes\mathbf{e}_3) \;\; , \;\; E_{33} = \frac{1}{2}(a^2-1) + (a\,u_{3,3}) \;\; ,$$
$$\boldsymbol{\beta}_1 = \delta_{,3}\,\mathbf{e}_2 - \frac{1}{a^2}u_{1,33}\,\mathbf{e}_3 \;\; , \;\; \boldsymbol{\beta}_2 = -\,\delta_{,3}\,\mathbf{e}_1 - \frac{1}{a^2}u_{2,33}\,\mathbf{e}_3 \;\; . \qquad (5.33.35)$$

Now, using (5.33.1) and the quadratic form (5.28.21), the strain energy function for the Bernoulli-Euler beam becomes

$$m\,\Sigma = \frac{1}{2}E^*\,HW\,(E_{33})^2 + \frac{1}{2}\,HW\,\mathbf{K}^{\alpha\beta}\bullet(\boldsymbol{\beta}_\alpha\otimes\boldsymbol{\beta}_\beta) \; , \qquad (5.33.36)$$

where $\mathbf{K}^{\alpha\beta}$ are given by (5.33.4). Since there are no constraint responses for the director couples $\mathbf{m}^\alpha$ the constitutive equations (5.28.20) for these quantities are

$$\mathbf{m}^\alpha = \; HW\,\mathbf{F}^{-T}\,\mathbf{K}^{\alpha\beta}\,\boldsymbol{\beta}_\beta \;\; , \;\; \mathbf{m}^1 = \{\frac{\mu^* H^2 W^2 b^*(\eta)}{6}\}\;\delta_{,3}\,\mathbf{e}_2 - \;\{\frac{E^* H^3 W}{12 a^3}\}\;u_{1,33}\,\mathbf{e}_3 \;\; ,$$
$$\mathbf{m}^2 = -\;\{\frac{\mu^* H^2 W^2 b^*(\eta)}{6}\}\;\delta_{,3}\,\mathbf{e}_1 - \{\frac{E^* H W^3}{12 a^3}\}\;u_{2,33}\,\mathbf{e}_3 \;\; ,$$
$$\mathbf{m} = \mathbf{e}_\alpha\times\mathbf{m}^\alpha = -\{\frac{E^* H W^3}{12 a^3}\}\;u_{2,33}\,\mathbf{e}_1 + \{\frac{E^* H^3 W}{12 a^3}\}\;u_{1,33}\,\mathbf{e}_2$$
$$+ \;\{\frac{\mu^* H^2 W^2 b^*(\eta)}{3}\}\;\delta_{,3}\,\mathbf{e}_2 \;\; . \qquad (5.33.37)$$

Also, the parts $\hat{\mathbf{t}}^i$ of $\mathbf{t}^i$ which are related to the strain energy function (5.33.36), are given by (5.28.20)

$$\hat{\mathbf{t}}^\alpha = 0 \;\; ,$$

$$\hat{\mathbf{t}}^3 = E^*HW\,(a^2\,u_{3,3})\,\mathbf{e}_3 + \frac{N}{a}\left[(u_{1,3})\,\mathbf{e}_1 + (u_{2,3})\,\mathbf{e}_2 + (a + u_{3,3})\,\mathbf{e}_3\right] \ , \qquad (5.33.38)$$

and the constraint responses (5.28.7) reduce to

$$\bar{\mathbf{t}}^1 = [m\,(\mathbf{e}_1 \bullet \mathbf{b}^1)]\,\mathbf{e}_1 + [m\,(\mathbf{e}_1 \bullet \mathbf{b}^2) - \{\frac{\mu^* H^2 W^2 b^*(\eta)}{6}\}\,\delta_{,33} - m y^{22}\,\ddot{\delta}\,]\,\mathbf{e}_2$$
$$+ [m\,(\mathbf{e}_3 \bullet \mathbf{b}^1) - \{\frac{E^* H^3 W}{12a^3}\}\,u_{1,333} + \ m y^{11}\,\ddot{u}_{1,3}\,]\,\mathbf{e}_3 \ ,$$
$$\bar{\mathbf{t}}^2 = [m\,(\mathbf{e}_2 \bullet \mathbf{b}^1) + \{\frac{\mu^* H^2 W^2 b^*(\eta)}{6}\}\,\delta_{,33} - m y^{11}\,\ddot{\delta}\,]\,\mathbf{e}_1$$
$$+ [m\,(\mathbf{e}_2 \bullet \mathbf{b}^2) + \frac{1}{a} m y^{22}\,\ddot{u}_{2,3}]\,\mathbf{e}_2$$
$$+ [m\,(\mathbf{e}_3 \bullet \mathbf{b}^2) - \{\frac{E^* H W^3}{12a^3}\}\,u_{2,333} + m y^{22}\,\ddot{u}_{2,3}\,]\,\mathbf{e}_3 \ ,$$
$$\bar{\mathbf{t}}^3 = \frac{1}{a}\,[m\,(\mathbf{e}_3 \bullet \mathbf{b}^1) - \{\frac{E^* H^3 W}{12a^3}\}\,u_{1,333} + \ m y^{11}\,\ddot{u}_{1,3}\,]\,\mathbf{e}_1$$
$$+ \frac{1}{a}\,[m\,(\mathbf{e}_3 \bullet \mathbf{b}^2) - \{\frac{E^* H W^3}{12a^3}\}\,u_{2,333} + m y^{22}\,\ddot{u}_{2,3}\,]\,\mathbf{e}_2 \ , \qquad (5.33.39)$$

where use has been made of the definition (5.33.6) for the tension N. Also, with the help of (5.33.38) and (5.33.39), it follows that the resultant force becomes

$$\mathbf{t}^3 = \frac{1}{a}\,[m\,(\mathbf{e}_3 \bullet \mathbf{b}^1) - \{\frac{E^* H^3 W}{12a^3}\}\,u_{1,333} + \ m y^{11}\,\ddot{u}_{1,3} + N\,(u_{1,3})]\,\mathbf{e}_1$$
$$+ \frac{1}{a}\,[m\,(\mathbf{e}_3 \bullet \mathbf{b}^2) - \{\frac{E^* H W^3}{12a^3}\}\,u_{2,333} + m y^{22}\,\ddot{u}_{2,3} + N\,(u_{2,3})]\,\mathbf{e}_2$$
$$+ [E^* HW\,(a^2\,u_{3,3}) + \frac{N}{a}\,(a + u_{3,3})]\,\mathbf{e}_3 \, . \qquad (5.33.40)$$

Using these results the equations of motion (5.28.6) reduce to the conservation of mass (5.33.7) and four other independent equations. One equation associated with extensional deformations along the axis of the beam which is the same as (5.33.26), one equation describing bending in the $\mathbf{e}_1$–$\mathbf{e}_3$ plane,

$$m\,\ddot{u}_1 = m\,(\mathbf{e}_1 \bullet \mathbf{b}) + \frac{1}{a}\,[m\,(\mathbf{e}_3 \bullet \mathbf{b}^1)_{,3} - \{\frac{E^* H^3 W}{12a^3}\}\,u_{1,3333}$$
$$+ \ m y^{11}\,\ddot{u}_{1,33} + N\,(u_{1,33})] \ , \qquad (5.33.41)$$

one equation describing bending in the $\mathbf{e}_2$–$\mathbf{e}_3$ plane,

$$m\,\ddot{u}_2 = m\,(\mathbf{e}_2 \bullet \mathbf{b}) + \frac{1}{a}\,[m\,(\mathbf{e}_3 \bullet \mathbf{b}^2)_{,3} - \{\frac{E^* H W^3}{12a^3}\}\,u_{2,3333}$$
$$+ m y^{22}\,\ddot{u}_{2,33} + N\,(u_{2,33})] \ , \qquad (5.33.42)$$

and one equation describing torsion of the cross-section which is the same as (5.33.29). For this theory, the rate of work expression (5.28.25) reduces to (5.33.30) where the components ω_i of $\boldsymbol{\omega}$ are given by

$$\omega_1 = -\frac{1}{a}\,\dot{u}_{1,3} \ , \quad \omega_2 = \frac{1}{a}\,\dot{u}_{2,3} \ , \quad \omega_3 = \dot{\delta} \ . \tag{5.33.43}$$

This indicates that kinematic boundary conditions associated with the rate of work expression are characterized by specified values of

$$\{u_1 \, , u_2 \, , u_3 \, , u_{1,3} \, , u_{2,3} \, , \delta\} \ . \tag{5.33.44}$$

Generalized String

For the generalized string discussed in section 5.30 the kinematics are given by (5.33.1)-(5.33.3). However, the strain energy of bending is eliminated so that the strain energy function becomes

$$m\,\Sigma = \frac{1}{2}HW\,\mathbf{K}^* \bullet (\mathbf{E}\otimes\mathbf{E}) \ , \quad \mathbf{K}^* \bullet \mathbf{E} = 2\mu^* \left[\{\frac{\nu^*}{1-2\nu^*}\} (\mathbf{E} \bullet \mathbf{I})\,\mathbf{I} + \mathbf{E} \right] . \tag{5.33.45}$$

It then follows that the vectors $\mathbf{t}^i$ are given by (5.33.4) with the stretch b being determined by (5.33.5) so that (5.33.6) describes the prestretched configuration.

Next, with the help of the expressions (5.27.4) for the inertia quantities, the equations of motion can be written as the conservation of mass (5.33.7) and four other independent sets of equations. One set associated with extensional deformations along the axis of the beam and in its cross-section which is the same as (5.33.8), one set describing transverse motion and shearing in the $\mathbf{e}_1$–$\mathbf{e}_3$ plane,

$$m\,\ddot{u}_1 = m\,(\mathbf{e}_1 \bullet \mathbf{b}) + \frac{N}{a}(u_{1,33}) + \mu^* HW\,b\,(b\,u_{1,33} + a\,\tilde{\delta}_{31,3}) \ ,$$

$$m\,y^{11}\,\ddot{\tilde{\delta}}_{31} = m\,(\mathbf{e}_3 \bullet \mathbf{b}^1) - \mu^* HW\,a\,(b\,u_{1,3} + a\,\tilde{\delta}_{31}) \ , \tag{5.33.46}$$

one set describing transverse motion and shearing in the $\mathbf{e}_2$–$\mathbf{e}_3$ plane,

$$m\,\ddot{u}_2 = m\,(\mathbf{e}_2 \bullet \mathbf{b}) + \frac{N}{a}(u_{2,33}) + \mu^* HW\,b\,(b\,u_{2,33} + a\,\tilde{\delta}_{32,3}) \ ,$$

$$m\,y^{22}\,\ddot{\tilde{\delta}}_{32} = m\,(\mathbf{e}_3 \bullet \mathbf{b}^2) - 2\mu^* HW\,a\,(b\,u_{2,3} + a\,\tilde{\delta}_{32}) \ , \tag{5.33.47}$$

and one set describing torsion and shearing of the cross-section,

$$m\,y^{11}\,\ddot{\tilde{\delta}}_{21} = m\,(\mathbf{e}_2 \bullet \mathbf{b}^1) - \mu^* HW\,b^2\,(\tilde{\delta}_{12} + \tilde{\delta}_{21}) \ ,$$

$$m\,y^{22}\,\ddot{\tilde{\delta}}_{12} = m\,(\mathbf{e}_1 \bullet \mathbf{b}^2) - \mu^* HW\,b^2\,(\tilde{\delta}_{12} + \tilde{\delta}_{21}) \ . \tag{5.33.48}$$

In order to discuss the boundary conditions for this theory it is recalled that the rate of work of the contact force, applied to the ends of the string can be written in the form

$$\mathbf{t}^3 \bullet \dot{\mathbf{u}} = (\mathbf{e}_i \bullet \mathbf{t}^3)\,\dot{u}_i \ , \tag{5.33.49}$$

which indicates that kinematic boundary conditions are characterized by specified values of

$$\{u_1 \, , u_2 \, , u_3\} \ . \tag{5.33.50}$$

In particular, it is noted that since the director couples vanish there are no boundary conditions specified for $\tilde{\delta}_{i\alpha}$.

For the special case when the kinematics are specified by (5.33.15) and the assigned fields $\mathbf{b}$ and $\mathbf{b}^\alpha$ vanish, the equations of motion (5.33.8), (5.33.47) and (5.33.48) are satisfied automatically, and (5.33.46) reduce to

$$m\,\ddot{u}_1 = \frac{N}{a}(u_{1,33}) + \mu^* HW\, b\,(b\,u_{1,33} + a\,\tilde{\delta}_{31,3}) \ ,$$

$$m\,y^{11}\,\ddot{\tilde{\delta}}_{31} = -\,\mu^* HW\, a\,(b\,u_{1,3} + a\,\tilde{\delta}_{31}) \ . \tag{5.33.51}$$

Thus, equation $(5.33.51)_1$ can be solved for $\tilde{\delta}_{31,3}$ to obtain (5.33.7). Next, by differentiating $(5.33.51)_2$ with respect to θ^3 and using (5.33.17), it can be shown that the result becomes

$$\{\frac{m y^{11}}{\mu^* HWab}\}[m\,\ddddot{u}_1] - [\{\frac{y^{11}b}{a}\}\{1 + \frac{N}{\mu^* HWab^2}\}][m\,\ddot{u}_{1,33}]$$
$$+\ \frac{a}{b}[m\,\ddot{u}_1] - \frac{N}{b}[u_{1,33}] = 0 \ . \tag{5.33.52}$$

<u>Simple String</u>

For the theory of a simple string discussed in section 5.31 the kinematics of the reference configuration are given by

$$\mathbf{X} = \theta^3\,\mathbf{e}_3 \ , \quad \mathbf{D}_3 = \mathbf{e}_3 \ , \quad D_{33}^{1/2} = 1 \ , \tag{5.33.53}$$

and the kinematics of the present configuration are given by

$$\mathbf{x} = a\,\mathbf{e}_3 + u_i\,\mathbf{e}_i \ , \quad \mathbf{d}_3 = a\,\mathbf{e}_3 + u_{i,3}\,\mathbf{e}_i \ ,$$
$$\lambda = a + u_{3,3} \ , \quad E_{33} = \frac{1}{2}[a^2 - 1 + 2\,a\,u_{3,3}] \ . \tag{5.33.54}$$

Also, the strain energy function is given by

$$m\,\Sigma = \frac{1}{2}E^*\,HW\ [E_{33}]^2 \ , \tag{5.33.55}$$

so the constitutive equation for the resultant force $\mathbf{t}^3$ becomes

$$\mathbf{t}^3 = [\frac{N}{a}(u_{1,3})]\,\mathbf{e}_1 + [\frac{N}{a}(u_{2,3})]\,\mathbf{e}_2 + [E^* HW\,(a^2\,u_{3,3}) + \frac{N}{a}(a + u_{3,3})]\,\mathbf{e}_3 \ , \tag{5.33.56}$$

where use has been made of the definition (5.33.6) for the tension N.

Using these results the equations of motion (5.31.8) reduce to the conservation of mass (5.33.7) and three other independent equations. One equation associated with extensional deformations along the axis of the beam which is the same as (5.33.26), one equation describing transverse motion in the $\mathbf{e}_1$–$\mathbf{e}_3$ plane,

$$m\,\ddot{u}_1 = m\,(\mathbf{e}_1 \bullet \mathbf{b}) + [\frac{N}{a}(u_{1,33})] \ , \tag{5.33.57}$$

and one equation describing transverse motion in the $\mathbf{e}_2$–$\mathbf{e}_3$ plane,

$$m\,\ddot{u}_2 = m\,(\mathbf{e}_2 \bullet \mathbf{b}) + [\frac{N}{a}(u_{2,33})] \ . \tag{5.33.58}$$

For this theory, the rate of work of the contact force applied to the ends of the string can be written in the form (5.33.49), and the kinematic boundary conditions are characterized by specified values of (5.33.50).

Discussion

The procedure used above, which linearizes the equations of motion about a finite prestretched configuration, causes the values a and b of the stretches to appear in various coefficients of the equations. However, if the tension N is small relative to Young's modulus E^*, then the values of a and b will be very close to unity. In particular, using $(5.33.6)_4$, the value of a can be approximated by

$$a \approx 1 + \frac{N}{E^* HW} \quad . \tag{5.33.59}$$

In order to explore the effect of this approximation it is noted that the equation of equilibrium associated with the Bernoulli-Euler theory (5.33.41) becomes

$$\{\frac{E^* H^3 W}{12a^3}\} [u_{1},_{3333}] - N [u_{1},_{33}] = 0 \ , \tag{5.33.60}$$

whereas that associated with the Timoshenko theory (5.33.33) becomes

$$\{\frac{E^* H^3 W}{12a^3}\}\{1 + \frac{N}{\mu^* HWa}\}[u_{1},_{3333}] - N [u_{1},_{33}] = 0 \ . \tag{5.33.61}$$

These equations have the same forms as those recorded in (Timoshenko and Gere, 1961), except for the presence of the prestretch a. This means that the corrections due the present of the stretch a in these equations are of the same order as the correction due to shear deformation that is retained in equation (5.33.61). Specifically, if the first order corrections due to a are included, then equation (5.33.60) is approximated by

$$\{\frac{E^* H^3 W}{12}\}\{1 - \frac{3N}{E^* HW}\}[u_{1},_{3333}] - N [u_{1},_{33}] = 0 \ , \tag{5.33.62}$$

and equation (5.33.61) is approximated by

$$\{\frac{E^* H^3 W}{12}\}\{1 - \frac{3N}{E^* HW} + \frac{N}{\mu^* HW}\}[u_{1},_{3333}] - N [u_{1},_{33}] = 0 \ . \tag{5.33.63}$$

It then can be shown that the effect of the prestretch a tends to increase the buckling load (with $N<0$), and the effect of shear deformation tends to decrease the buckling load.

Furthermore, it is noted that the dynamical equations (5.33.41) for the Bernoulli-Euler theory, and (5.33.33) for the Timoshenko theory, have the same structure as those recorded in (Graff, 1975) when the effects of rotary inertia are included. In particular, Fig. 3.13 of (Graff, 1975, p. 187) shows the significance of both rotary inertia and shear deformation on the correct prediction of bending wave propagation in rods when the wave length becomes small.

5.34 An intrinsic formulation of Bernoulli-Euler rods with symmetric cross-sections

Love (1944) discusses the early works of Kirchhoff and Euler who analyzed large deformations of elastic rods in equilibrium. This formulation uses assumptions consistent with the Bernoulli-Euler theory, and it also assumes that the reference curve is inextensible. Consequently, the directors $\mathbf{d}_i$ of the constrained inextensible Cosserat theory form an orthonormal triad which can be described by Euler angles. Recently, Lu and Perkins (1994) have considered rods with symmetric cross-sections, and have reformulated these equations of elastica in terms of intrinsic variables. The objective of this section is to briefly record aspects of a generalization of this intrinsic formulation that was developed in (Rubin, 1997). This generalization includes nonlinear dependence of the strain energy function on extension, curvature and twist of the reference curve. By way of background, it is first recalled that the Serret-Frenet triad associated with the current position $\mathbf{x}$ of the reference curve of the rod is characterized by the unit tangent vector $\mathbf{e}_t$, the unit normal vector $\mathbf{e}_n$, and the unit binormal vector $\mathbf{e}_\tau$, such that

$$\mathbf{e}_t = \frac{\partial \mathbf{x}}{\partial s} = d_{33}^{-1/2}\,\mathbf{d}_3 \ , \quad \frac{\partial \mathbf{e}_t}{\partial s} = d_{33}^{-1/2}\,\mathbf{e}_{t,3} = \kappa\,\mathbf{e}_n \ ,$$
$$\frac{\partial \mathbf{e}_n}{\partial s} = d_{33}^{-1/2}\,\mathbf{e}_{n,3} = -\kappa\,\mathbf{e}_t + \tau\,\mathbf{e}_\tau \ , \quad \frac{\partial \mathbf{e}_\tau}{\partial s} = d_{33}^{-1/2}\,\mathbf{e}_{\tau,3} = -\tau\,\mathbf{e}_n \ ,$$
$$\mathbf{e}_\tau = \mathbf{e}_t \times \mathbf{e}_n \ , \tag{5.34.1}$$

where the current arc-length parameter s is related to the Lagrangian coordinate θ^3 by the formula

$$\mathbf{d}_3 = \mathbf{x}_{,3} \ , \quad \frac{\partial s}{\partial \theta^3} = d_{33}^{1/2} = |\,\mathbf{d}_3\,| \ . \tag{5.34.2}$$

Also, κ is the curvature and τ is the geometric torsion of the reference curve in its present configuration.

Within the context of the Bernoulli-Euler theory discussed in section 5.28, the directors $\mathbf{d}_\alpha$ which describe the cross-section of the rod, remain unit vectors and are orthogonal to the tangent vector $\mathbf{d}_3$ and themselves such that

$$\mathbf{d}_\alpha \bullet \mathbf{d}_i = \delta_{\alpha i} \ . \tag{5.34.3}$$

Moreover, the reciprocal vectors $\mathbf{d}^i$ for this theory become

$$\mathbf{d}^\alpha = \mathbf{d}_\alpha \ , \quad \mathbf{d}^3 = d_{33}^{-1}\,\mathbf{d}_3 \ . \tag{5.34.4}$$

The constitutive equations developed in (Rubin, 1997) assumed that the strain energy function depended on the stretch of the reference curve, its curvature, and a measure of twist associated with torsion. For the present purposes the stretch of the reference curve is defined by

$$\lambda = \frac{\partial s}{\partial S} = \frac{d_{33}^{1/2}}{D_{33}^{1/2}} \ , \tag{5.34.5}$$

where $D_{33}^{1/2}$ is related to the arclength parameter S of the reference curve in its reference configuration, by the formula

$$\mathbf{D}_3 = \mathbf{X}_{,3} \ , \quad \frac{\partial S}{\partial \theta^3} = D_{33}^{1/2} = |\,\mathbf{D}_3\,| \ . \tag{5.34.6}$$

It was shown in (Rubin, 1997) that the constitutive equations simplify if the strain energy function depends on the measure of curvature α defined by

$$\alpha = \lambda\,\kappa = \frac{\partial s}{\partial S}\,\kappa \ . \tag{5.34.7}$$

Moreover, the measure of twist δ associated with torsion is specified by

$$\delta = \frac{1}{2}\, D_{33}^{-1/2}\, (\mathbf{d}_{1},_{3} \bullet \mathbf{d}_2 - \mathbf{d}_{2},_{3} \bullet \mathbf{d}_1) = \frac{1}{2}\Big[\frac{\partial \mathbf{d}_1}{\partial S} \bullet \mathbf{d}_2 - \frac{\partial \mathbf{d}_2}{\partial S} \bullet \mathbf{d}_1\Big] \,. \tag{5.34.8}$$

These formulas indicate that α is the curvature of the reference curve in the present configuration, but measured per unit reference arclength, and δ is the twist measured per unit reference arclength.

Now, using these kinematic variables the strain energy function is taken in the form

$$m\,\Sigma = \mu^*\, D_{33}^{1/2}\, A\, f(\lambda,\alpha,\delta) \;, \tag{5.34.9}$$

where μ^* is the shear modulus, A is the cross-sectional area of the rod, and f is an arbitrary function of its arguments. Furthermore, it should be emphasized that the twist δ which is related to torsion of the cross-section of the rod is physically different from the geometric torsion τ of the reference curve.

In order to develop the constitutive equations associated with the form (5.34.9), it is convenient to first express α in terms of derivatives of the directors. To this end, it is noted from (5.34.1) that

$$\alpha\, \mathbf{e}_n = \lambda\, d_{33}^{-1}\, [\mathbf{d}_{3},_{3} - (\mathbf{e}_t \bullet \mathbf{d}_{3},_{3})\, \mathbf{e}_t] = \lambda^{-1}\, D_{33}^{-1}\, (\mathbf{d}_{3},_{3} \bullet \mathbf{d}_\alpha)\, \mathbf{d}_\alpha \;,$$

$$\mathbf{e}_n = -\,\alpha^{-1}\, \lambda^{-1}\, D_{33}^{-1}\, (\mathbf{d}_{\alpha},_{3} \bullet \mathbf{d}_3)\, \mathbf{d}_\alpha \;, \tag{5.34.10}$$

where use has been made of the constraint (5.34.3). Also, for later reference, this equation is used to determine an expression for the binormal $\mathbf{e}_\tau$ in the form

$$\mathbf{e}_\tau = \mathbf{e}_t \times \mathbf{e}_n = \alpha^{-1}\, \lambda^{-1}\, D_{33}^{-1}\, [(\mathbf{d}_{2},_{3} \bullet \mathbf{d}_3)\, \mathbf{d}_1 - (\mathbf{d}_{1},_{3} \bullet \mathbf{d}_3)\, \mathbf{d}_2\,] \,. \tag{5.34.11}$$

Now, using (5.34.10) the curvature α can be expressed in the alternative form

$$\alpha^2 = \lambda^{-2}\, D_{33}^{-2}\, [(\mathbf{d}_{1},_{3} \bullet \mathbf{d}_3)^2 + (\mathbf{d}_{2},_{3} \bullet \mathbf{d}_3)^2] \,. \tag{5.34.12}$$

Thus, with the help of (5.28.15) it can be shown that

$$\dot{\lambda} = \lambda\, d_{33}^{-1}\, (\mathbf{d}_3 \otimes \mathbf{d}_3) \bullet \mathbf{D} \;,$$

$$\dot{\alpha} = -\,\alpha\, \frac{\dot{\lambda}}{\lambda} + \alpha^{-1}\, \lambda^{-2}\, D_{33}^{-2}\, [(\mathbf{d}_{1},_{3} \bullet \mathbf{d}_3)(\mathbf{w}_{1},_{3} \bullet \mathbf{d}_3 + \mathbf{d}_{1},_{3} \bullet \mathbf{w}_3)$$
$$+\, (\mathbf{d}_{2},_{3} \bullet \mathbf{d}_3)(\mathbf{w}_{2},_{3} \bullet \mathbf{d}_3 + \mathbf{d}_{2},_{3} \bullet \mathbf{w}_3)] \;,$$

$$\dot{\delta} = \frac{1}{2}\, D_{33}^{-1/2}[(\mathbf{w}_{1},_{3} \bullet \mathbf{d}_2 + \mathbf{d}_{1},_{3} \bullet \mathbf{w}_2) - (\mathbf{w}_{2},_{3} \bullet \mathbf{d}_1 + \mathbf{d}_{2},_{3} \bullet \mathbf{w}_1)] \,. \tag{5.34.13}$$

However, equations (5.6.7), (5.6.8) and (5.6.15) can be used to deduce that

$$\mathbf{w}_{\alpha},_{3} = \mathbf{F}\dot{\boldsymbol{\beta}}_\alpha + \mathbf{L}\, \mathbf{d}_{\alpha},_{3} \;,$$

$$\mathbf{w}_{\alpha},_{3} \bullet \mathbf{d}_3 + \mathbf{d}_{\alpha},_{3} \bullet \mathbf{w}_3 = d_{33}\, \mathbf{D}^3 \bullet \dot{\boldsymbol{\beta}}_\alpha + (\mathbf{d}_{\alpha},_{3} \otimes \mathbf{d}_3 + \mathbf{d}_3 \otimes \mathbf{d}_{\alpha},_{3}) \bullet \mathbf{D} \;,$$

$$\mathbf{w}_{\alpha},_{3} \bullet \mathbf{d}_\beta + \mathbf{d}_{\alpha},_{3} \bullet \mathbf{w}_\beta = \mathbf{D}^\beta \bullet \dot{\boldsymbol{\beta}}_\alpha + (\mathbf{d}_{\alpha},_{3} \otimes \mathbf{d}_\beta + \mathbf{d}_\beta \otimes \mathbf{d}_{\alpha},_{3}) \bullet \mathbf{D} \,. \tag{5.34.14}$$

Consequently, the material derivative of the strain energy function (5.34.9) can be written in the form

$$m\,\dot{\Sigma} = \mu^*\, D_{33}^{1/2}\, A\, [d_{33}^{-1}\{\lambda \frac{\partial f}{\partial \lambda} - \alpha \frac{\partial f}{\partial \alpha}\}\, (\mathbf{d}_3 \otimes \mathbf{d}_3)$$

$$+ \alpha^{-1}\,\lambda^{-2}\,D_{33}^{-2}\,\frac{\partial f}{\partial \alpha}\,\{(\mathbf{d}_{\alpha},_3 \bullet \mathbf{d}_3)\,(\mathbf{d}_{\alpha},_3 \otimes \mathbf{d}_3 + \mathbf{d}_3 \otimes \mathbf{d}_{\alpha},_3)\}$$

$$+ \frac{1}{2}\,D_{33}^{-1/2}\,\frac{\partial f}{\partial \delta}\,\{(\mathbf{d}_1,_3 \otimes \mathbf{d}_2 + \mathbf{d}_2 \otimes \mathbf{d}_1,_3) - (\mathbf{d}_2,_3 \otimes \mathbf{d}_1 + \mathbf{d}_1 \otimes \mathbf{d}_2,_3)\}] \bullet \mathbf{D}$$

$$+ \mu^*\,A\,[\alpha^{-1}\,D_{33}^{-1/2}\,\frac{\partial f}{\partial \alpha}\,\{(\mathbf{d}_1,_3 \bullet \mathbf{d}_3)\,\mathbf{D}^3\} + \frac{1}{2}\,\frac{\partial f}{\partial \delta}\mathbf{D}^2] \bullet \dot{\boldsymbol{\beta}}_1$$

$$+ \mu^*\,A\,[\alpha^{-1}\,D_{33}^{-1/2}\,\frac{\partial f}{\partial \alpha}\,\{(\mathbf{d}_2,_3 \bullet \mathbf{d}_3)\,\mathbf{D}^3\} - \frac{1}{2}\,\frac{\partial f}{\partial \delta}\mathbf{D}^1] \bullet \dot{\boldsymbol{\beta}}_2\ . \qquad (5.34.15)$$

For an elastic rod the parts $\hat{\mathbf{T}}$, $\hat{\mathbf{t}}^i$, $\hat{\mathbf{m}}^\alpha$ of the kinetic quantities associated with constitutive equations, satisfy the condition (5.8.1) that the mechanical power due to these parts is equal to the rate of change of the strain energy function. It then follows that

$$d_{33}^{1/2}\,\hat{\mathbf{T}} = [\,\hat{\mathbf{t}}^i \otimes \mathbf{d}_i + \mathbf{m}^\alpha \otimes \mathbf{d}_{\alpha},_3\,] = \mu^*\,D_{33}^{1/2}\,A\,[d_{33}^{-1}\{\lambda\,\frac{\partial f}{\partial \lambda} - \alpha\,\frac{\partial f}{\partial \alpha}\}\,(\mathbf{d}_3 \otimes \mathbf{d}_3)$$

$$+ \alpha^{-1}\,\lambda^{-2}\,D_{33}^{-2}\,\frac{\partial f}{\partial \alpha}\,\{(\mathbf{d}_{\alpha},_3 \bullet \mathbf{d}_3)\,(\mathbf{d}_{\alpha},_3 \otimes \mathbf{d}_3 + \mathbf{d}_3 \otimes \mathbf{d}_{\alpha},_3)\}$$

$$+ \frac{1}{2}\,D_{33}^{-1/2}\,\frac{\partial f}{\partial \delta}\,\{(\mathbf{d}_1,_3 \otimes \mathbf{d}_2 + \mathbf{d}_2 \otimes \mathbf{d}_1,_3) - (\mathbf{d}_2,_3 \otimes \mathbf{d}_1 + \mathbf{d}_1 \otimes \mathbf{d}_2,_3)\}]\ ,$$

$$\mathbf{m}^1 = \hat{\mathbf{m}}^1 = \mu^*\,A\,[\alpha^{-1}\,\lambda^{-2}\,D_{33}^{-3/2}\,\frac{\partial f}{\partial \alpha}\,\{(\mathbf{d}_1,_3 \bullet \mathbf{d}_3)\,\mathbf{d}_3\} + \frac{1}{2}\,\frac{\partial f}{\partial \delta}\mathbf{d}_2]\ ,$$

$$\mathbf{m}^2 = \hat{\mathbf{m}}^2 = \mu^*\,A\,[\alpha^{-1}\,\lambda^{-2}\,D_{33}^{-3/2}\,\frac{\partial f}{\partial \alpha}\,\{(\mathbf{d}_2,_3 \bullet \mathbf{d}_3)\,\mathbf{d}_3\} - \frac{1}{2}\,\frac{\partial f}{\partial \delta}\mathbf{d}_1]\ ,$$

$$\hat{\mathbf{t}}^1 = -\frac{1}{2}\mu^*\,A\,[\frac{\partial f}{\partial \delta}\,\mathbf{d}_2,_3\,]\ ,\quad \hat{\mathbf{t}}^2 = \frac{1}{2}\mu^*\,A\,[\frac{\partial f}{\partial \delta}\,\mathbf{d}_1,_3]\ ,$$

$$\hat{\mathbf{t}}^3 = \mu^*\,A\,[\{\frac{\partial f}{\partial \lambda} - \alpha\,\lambda^{-1}\,\frac{\partial f}{\partial \alpha}\}\mathbf{e}_t + \alpha^{-1}\,\lambda^{-2}\,D_{33}^{-3/2}\,\frac{\partial f}{\partial \alpha}\,(\mathbf{d}_{\alpha},_3 \bullet \mathbf{d}_3)\,\mathbf{d}_{\alpha},_3]\ , \qquad (5.34.16)$$

where use has been made of the condition that for a Bernoulli-Euler rod, the director couples $\mathbf{m}^\alpha$ are totally determined by constitutive equations (i.e. the constraint responses vanish, $\bar{\mathbf{m}}^\alpha$=0).

Moreover, the resultant moment $\mathbf{m}$ applied to the end of the rod becomes

$$\mathbf{m} = \mathbf{d}_\alpha \times \mathbf{m}^\alpha = \mu^*\,A\,[\alpha^{-1}\,\lambda^{-1}\,D_{33}^{-1}\,\frac{\partial f}{\partial \alpha}\,\{\,(\mathbf{d}_2,_3 \bullet \mathbf{d}_3)\,\mathbf{d}_1 - (\mathbf{d}_1,_3 \bullet \mathbf{d}_3)\,\mathbf{d}_2\}$$

$$+ \frac{\partial f}{\partial \delta}\,\mathbf{e}_t]\ . \qquad (5.34.17)$$

Thus, with the help of the expression (5.34.12) it can be seen that $\mathbf{m}$ reduces to the simpler form

$$\mathbf{m} = M\,\mathbf{e}_\tau + T\,\mathbf{e}_t\ ,\quad M = \mu^* A\,\frac{\partial f}{\partial \alpha}\ ,\quad T = \mu^* A\,\frac{\partial f}{\partial \delta}\ , \qquad (5.34.18)$$

where M is the bending moment, and T is the torsional moment. Physically, this means that when the strain energy function is assumed to depend on the curvature of the rod only through the curvature α of the reference curve, then the bending moment M acts in the direction of the binormal $\mathbf{e}_\tau$. This, of course, is physically reasonable only for rods whose cross-sections are symmetric in the sense that the resistance to bending is the same for bending in any plane that contains the tangent vector $\mathbf{e}_t$.

Next, attention is focused on developing alternative expressions for the constraint responses of this Bernoulli-Euler theory. To this end, use is made of the definitions (5.34.18) and the constitutive equations (5.34.16) to obtain

$$\mathbf{m}^1 = [\alpha^{-1}\,\lambda^{-1}\,D_{33}^{-1}\,M\,(\mathbf{d}_{1,3}\bullet\mathbf{d}_3)]\,\mathbf{e}_t + \frac{1}{2}\,T\,\mathbf{d}_2 \ ,$$
$$\mathbf{m}^2 = [\alpha^{-1}\,\lambda^{-1}\,D_{33}^{-1}\,M\,(\mathbf{d}_{2,3}\bullet\mathbf{d}_3)]\,\mathbf{e}_t - \frac{1}{2}\,T\,\mathbf{d}_1 \ ,$$
$$\hat{\mathbf{t}}^1 = -\frac{1}{2}T\,\mathbf{d}_{2,3}\ ,\quad \hat{\mathbf{t}}^2 = \frac{1}{2}T\,\mathbf{d}_{1,3}\ ,$$
$$\hat{\mathbf{t}}^3 = [\alpha^{-1}\,\lambda^{-2}\,D_{33}^{-3/2}\,M\,(\mathbf{d}_{\alpha,3}\bullet\mathbf{d}_3)]\,\mathbf{d}_{\alpha,3} + [N - \alpha\,\lambda^{-1}\,M]\,\mathbf{e}_t\ , \tag{5.34.19}$$

where N is the tangential force acting on the end of the rod

$$N = \mu^*\,A\,\frac{\partial f}{\partial \lambda}\ . \tag{5.34.20}$$

Also, since the reference curve is taken to be the centroid of the cross-section, and since the cross-section is symmetric, the director inertia coefficients y^α and $y^{\alpha\beta}$ satisfy the restrictions that

$$y^\alpha = 0\ ,\quad y^{12} = y^{21} = 0\ . \tag{5.34.21}$$

Then, the constraint responses (5.28.7) reduce to

$$\bar{\mathbf{t}}^\alpha = [\mathbf{d}^\alpha\bullet\{m\,\mathbf{b}^\beta - \hat{\mathbf{t}}^\beta + \mathbf{m}^\beta{}_{,3} - m\,y^{\beta\gamma}\,\dot{\mathbf{w}}_\gamma\}]\,\mathbf{d}_\beta$$
$$+\ [\mathbf{d}^3\bullet\{m\,\mathbf{b}^\alpha - \hat{\mathbf{t}}^\alpha + \mathbf{m}^\alpha{}_{,3} - m\,y^{\alpha\beta}\,\dot{\mathbf{w}}_\beta\}]\,\mathbf{d}_3$$
$$\bar{\mathbf{t}}^3 = [\mathbf{d}^3\bullet\{m\,\mathbf{b}^\alpha - \hat{\mathbf{t}}^\alpha + \mathbf{m}^\alpha{}_{,3} - m\,y^{\alpha\beta}\,\dot{\mathbf{w}}_\beta\}]\,\mathbf{d}_\alpha\ . \tag{5.34.22}$$

However, using the expressions (5.34.19) it can be shown that

$$\mathbf{d}^3\bullet[-\hat{\mathbf{t}}^1 + \mathbf{m}^1{}_{,3}] = d_{33}^{-1}\,T\,(\mathbf{d}_{2,3}\bullet\mathbf{d}_3) + d_{33}^{-1/2}\,[\alpha^{-1}\,\lambda^{-1}\,D_{33}^{-1}\,M\,(\mathbf{d}_{1,3}\bullet\mathbf{d}_3)]_{,3}$$
$$\mathbf{d}^3\bullet[-\hat{\mathbf{t}}^2 + \mathbf{m}^2{}_{,3}] = -\,d_{33}^{-1}\,T\,(\mathbf{d}_{1,3}\bullet\mathbf{d}_3) + d_{33}^{-1/2}\,[\alpha^{-1}\,\lambda^{-1}\,D_{33}^{-1}\,M\,(\mathbf{d}_{2,3}\bullet\mathbf{d}_3)]_{,3}\ . \tag{5.34.23}$$

Thus, the constraint response $\bar{\mathbf{t}}^3$ can be rewritten in the form

$$\bar{\mathbf{t}}^3 = [\mathbf{d}^3\bullet m\{\mathbf{b}^\alpha - y^{\alpha\beta}\,\dot{\mathbf{w}}_\beta\}]\,\mathbf{d}_\alpha + d_{33}^{-1}\,T\,[(\mathbf{d}_{2,3}\bullet\mathbf{d}_3)\,\mathbf{d}_1 - (\mathbf{d}_{1,3}\bullet\mathbf{d}_3)\,\mathbf{d}_2]$$
$$+\ d_{33}^{-1/2}\,[\alpha^{-1}\,\lambda^{-1}\,D_{33}^{-1}\,M\,(\mathbf{d}_{\alpha,3}\bullet\mathbf{d}_3)\,\mathbf{d}_\alpha]_{,3}$$
$$-\ d_{33}^{-1/2}\,[\alpha^{-1}\,\lambda^{-1}\,D_{33}^{-1}\,M\,(\mathbf{d}_{\alpha,3}\bullet\mathbf{d}_3)]\,\mathbf{d}_{\alpha,3}\ . \tag{5.34.24}$$

Next, with the help of (5.34.10), (5.34.11) and (5.34.19), the resultant force $\mathbf{t}^3$ becomes

$$\mathbf{t}^3 = \hat{\mathbf{t}}^3 + \bar{\mathbf{t}}^3 = [N - \alpha\,\lambda^{-1}\,M]\mathbf{e}_t + [\mathbf{d}^3\bullet m\{\mathbf{b}^\alpha - y^{\alpha\beta}\,\dot{\mathbf{w}}_\beta\}]\,\mathbf{d}_\alpha$$
$$+\ \alpha\,\lambda^{-1}\,T\,\mathbf{e}_\tau - d_{33}^{-1/2}\,[M\,\mathbf{e}_n]_{,3}\ . \tag{5.34.25}$$

Thus, with the help of (5.34.1) and (5.34.7) this expression reduces to

$$\mathbf{t}^3 = N\,\mathbf{e}_t + V\,\mathbf{e}_n + W\,\mathbf{e}_\tau + [\mathbf{d}^3\bullet m\{\mathbf{b}^\alpha - y^{\alpha\beta}\,\dot{\mathbf{w}}_\beta\}]\,\mathbf{d}_\alpha\ ,$$
$$V = -\,d_{33}^{-1/2}\,M_{,3}\ ,\quad W = \alpha\,\lambda^{-1}\,T - \tau\,M\ , \tag{5.34.26}$$

where V and W are parts of the shear forces in the normal and binormal directions, respectively.

In order to summarize the equations of motion, it is first noted from (5.28.6) that the resulting scalar equation of director momentum reduces to

$$[\{-\hat{\mathbf{t}}^1+\mathbf{m}^1{}_{,3}\}\bullet\mathbf{d}_2-\{-\hat{\mathbf{t}}^2+\mathbf{m}^2{}_{,3}\}\bullet\mathbf{d}_1$$
$$+\,m\,\{\mathbf{b}^1-y^{11}\,\dot{\mathbf{w}}_1\}\bullet\mathbf{d}_2-m\,\{\mathbf{b}^2-y^{22}\,\dot{\mathbf{w}}_2\}\bullet\mathbf{d}_1]=0\ . \tag{5.34.27}$$

Next, using (5.34.1), (5.34.7), (5.34.19), (5.34.21) and (5.34.26), the equations of motion (5.28.6) become

$$m=\rho\, d_{33}^{1/2}=\rho_0\, D_{33}^{1/2}\ ,$$
$$m\,\dot{\mathbf{v}}=m\,\mathbf{b}+[N_{,3}-\alpha V]\,\mathbf{e}_t+[V_{,3}+\alpha N-d_{33}^{1/2}\,\tau W]\,\mathbf{e}_n$$
$$+\,[W_{,3}+d_{33}^{1/2}\,\tau W\,]\,\mathbf{e}_\tau+[\mathbf{d}^3\bullet m\{\mathbf{b}^\alpha-y^{\alpha\beta}\,\dot{\mathbf{w}}_\beta\}\mathbf{d}_\alpha]_{,3}\ ,$$
$$T_{,3}+[m\,\{\mathbf{b}^1-y^{11}\,\dot{\mathbf{w}}_1\}\bullet\mathbf{d}_2-m\,\{\mathbf{b}^2-y^{22}\,\dot{\mathbf{w}}_2\}\bullet\mathbf{d}_1]=0\ . \tag{5.34.28}$$

In particular, notice that in the absence of assigned fields $\mathbf{b}$ and $\mathbf{b}^\alpha$ and inertia, the equilibrium equations reduce to

$$N_{,3}-\alpha V=0\ ,\quad V_{,3}+\alpha N-d_{33}^{1/2}\,\tau W=0\ ,$$
$$W_{,3}+d_{33}^{1/2}\,\tau W=0\ ,\quad T_{,3}=0\ . \tag{5.34.29}$$

These equations are called intrinsic because they depend only on the intrinsic variables $\{d_{33}^{1/2},\lambda,\alpha,\tau\}$ of the reference curve. Moreover, it is noted that exact integrals of these equations, and exact integrals of more general equations of rods in equilibrium have been considered in (Ericksen, 1970; Antman, 1972; Whitman and DeSilva, 1974; Antman and Jordan, 1975; Rubin, 1997).

When the assigned fields $\mathbf{b}$ and $\mathbf{b}^\alpha$ do not vanish, or inertia is included in the analysis, the equations of motion (5.34.28) are no longer fully intrinsic because it is necessary to calculate the directors $\mathbf{d}_\alpha$ in addition to the position $\mathbf{x}$ of the reference curve. For this case, the expressions (5.34.10) and (5.34.11) can be used to express the vectors $\{\mathbf{e}_t, \mathbf{e}_n, \mathbf{e}_\tau\}$ in terms of the directors $\mathbf{d}_i$.

In the above equations the quantities $\{N, V, W, T, M\}$ are determined by the strain energy function (5.34.9) and the equations (5.34.18), (5.34.20) and (5.34.26). In particular, the equations of an extensible elastica with a square cross-section of thickness H, are obtained when the strain energy function is a quadratic function of strains of the form

$$m\,\Sigma=\mu^*\,D_{33}^{1/2}\,H^2\,f(\lambda,\alpha,\delta)\ ,$$
$$f=\frac{1}{2}\Big[\frac{E^*}{\mu^*}(\lambda-1)^2+\frac{E^*H^2}{12\mu^*}\,\alpha^2+\frac{H^2b^*(1)}{3}\,\delta^2\Big]\ , \tag{5.34.30}$$

where E^* is Young's modulus, and the function $b^*(1)$ is given by (5.15.1). Then, the constitutive equations become

$$N=E^*H^2(\lambda-1)\ ,\quad V=-\,d_{33}^{-1/2}\,M_{,3}\ ,\quad W=\alpha\,\lambda^{-1}\,T-\tau\,M\ ,$$
$$T=\frac{1}{3}\mu^*H^4b^*(1)\,\delta\ ,\quad M=\frac{1}{12}E^*H^4\,\alpha\ . \tag{5.34.31}$$

5.35 Dissipation inequality and material damping

The previous sections have limited attention to purely elastic response which exhibits no dissipation. Consequently, rods made from such materials exhibit the unrealistic feature that free vibrations persist forever. In order to eliminate this unphysical response, it is necessary to include a model for material damping. To this end, it is noted that within the context of the purely mechanical theory it is possible to define the rate of material dissipation $\mathcal{D}$ per unit present arclength by the formula

$$\int_P \mathcal{D}\,ds = \mathcal{W} - \dot{\mathcal{K}} - \dot{\mathcal{U}} \geq 0 \ , \tag{5.35.1}$$

where $\mathcal{W}$, $\mathcal{K}$ and $\mathcal{U}$ are defined by (5.3.41) and (5.8.4). In words, this equation means that the rate of material dissipation is equal to the rate of work $\mathcal{W}$ done by the externally applied forces and couples and by the contact forces and couples at the ends of the rod, minus the rates of change of kinetic energy $\mathcal{K}$ and strain energy $\mathcal{U}$. Moreover, it is assumed that the rate of material dissipation is nonnegative.

Next, with the help of the conservation of mass and the balances of linear, angular and director momentums, it can be shown using (5.6.1), (5.6.17) and (5.8.4) that the local form of equation (5.35.1) becomes

$$\mathcal{D} = \mathbf{T} \bullet \mathbf{D} + d_{33}^{-1/2}\,(\mathbf{F}^T\mathbf{m}^\alpha) \bullet \dot{\boldsymbol{\beta}}_\alpha - \rho\,\dot{\Sigma} \geq 0 \ . \tag{5.35.2}$$

Moreover, in view of the assumption (5.8.1) it is seen that an elastic rod is an ideal rod since the rate of dissipation $\mathcal{D}$ vanishes. Consequently, the assumption that the rate of material dissipation is nonnegative requires that for a given motion, the work done on a dissipative material is greater than that done on an ideal elastic material. Also, using the transformation relations (5.5.4), (5.5.5), (5.6.15) and (5.8.3), it can be shown that $\mathcal{D}$ remains unaltered by SRBM

$$\mathcal{D}^+ = \mathcal{D} \ . \tag{5.35.3}$$

Now, a model for a rod constructed from a dissipative material can be developed by assuming that $\mathbf{T}$, $\mathbf{t}^i$ and $\mathbf{m}^\alpha$ separate additively into three parts

$$\mathbf{T} = \hat{\mathbf{T}} + \bar{\mathbf{T}} + \check{\mathbf{T}} \ , \quad \mathbf{t}^i = \hat{\mathbf{t}}^i + \bar{\mathbf{t}}^i + \check{\mathbf{t}}^i \ , \quad \mathbf{m}^\alpha = \hat{\mathbf{m}}^\alpha + \bar{\mathbf{m}}^\alpha + \check{\mathbf{m}}^\alpha \ ,$$
$$\hat{\mathbf{T}} = d_{33}^{-1/2}\,[\,\hat{\mathbf{t}}^i \otimes \mathbf{d}_i + \hat{\mathbf{m}}^\alpha \otimes \mathbf{d}_{\alpha,3}\,] \ , \quad \bar{\mathbf{T}} = d_{33}^{-1/2}\,[\,\bar{\mathbf{t}}^i \otimes \mathbf{d}_i + \bar{\mathbf{m}}^\alpha \otimes \mathbf{d}_{\alpha,3}\,] \ ,$$
$$\check{\mathbf{T}} = d_{33}^{-1/2}\,[\check{\mathbf{t}}^i \otimes \mathbf{d}_i + \check{\mathbf{m}}^\alpha \otimes \mathbf{d}_{\alpha,3}\,] \ , \tag{5.35. 4}$$

with $\hat{\mathbf{T}}$, $\hat{\mathbf{t}}^i$ and $\hat{\mathbf{m}}^\alpha$ being the parts associated with elastic deformation [which balance the rate of change in strain energy (5.8.1)]

$$\hat{\mathbf{T}} \bullet \mathbf{D} + d_{33}^{-1/2}\,(\mathbf{F}^T\hat{\mathbf{m}}^\alpha) \bullet \dot{\boldsymbol{\beta}}_\alpha = \rho\,\dot{\Sigma} \ , \tag{5.35.5}$$

$\bar{\mathbf{T}}$, $\bar{\mathbf{t}}^i$ and $\bar{\mathbf{m}}^\alpha$ being the constraint responses [which do no work (5.9.15)]

$$\bar{\mathbf{T}} \bullet \mathbf{D} + d_{33}^{-1/2}\,(\mathbf{F}^T\bar{\mathbf{m}}^\alpha) \bullet \dot{\boldsymbol{\beta}}_\alpha = 0 \ , \tag{5.35.6}$$

and $\check{\mathbf{T}}$, $\check{\mathbf{t}}^i$ and $\check{\mathbf{m}}^\alpha$ being the parts due to material dissipation. Thus, the restriction (5.35.2) reduces to

$$\mathcal{D} = \check{\mathbf{T}} \bullet \mathbf{D} + d_{33}^{-1/2}\,(\mathbf{F}^T\check{\mathbf{m}}^\alpha) \bullet \dot{\boldsymbol{\beta}}_\alpha \geq 0 \ . \tag{5.35.7}$$

For viscous damping $\check{\mathbf{T}}$ and $\check{\mathbf{m}}^\alpha$ are assumed to be functions of $\mathbf{D}$ and $\dot{\boldsymbol{\beta}}_\beta$

$$\check{\mathbf{T}} = \check{\mathbf{T}}(\mathbf{D},\dot{\boldsymbol{\beta}}_\beta) \ , \quad \check{\mathbf{m}}^\alpha = \check{\mathbf{m}}^\alpha(\mathbf{D},\dot{\boldsymbol{\beta}}_\beta) \ . \tag{5.35.8}$$

However, invariance under SRBM requires these functions to satisfy the restrictions

$$\mathbf{Q}\,\check{\mathbf{T}}(\mathbf{D},\dot{\boldsymbol{\beta}}_\beta)\,\mathbf{Q}^T = \check{\mathbf{T}}(\mathbf{Q}\,\mathbf{D}\,\mathbf{Q}^T,\dot{\boldsymbol{\beta}}_\beta) \ , \quad \mathbf{Q}\,\check{\mathbf{m}}^\alpha(\mathbf{D},\dot{\boldsymbol{\beta}}_\beta) = \check{\mathbf{m}}^\alpha(\mathbf{Q}\,\mathbf{D}\,\mathbf{Q}^T,\dot{\boldsymbol{\beta}}_\beta) \ , \tag{5.35.9}$$

for all proper orthogonal tensors $\mathbf{Q}$. Consequently, $\check{\mathbf{T}}$ and $\check{\mathbf{m}}^\alpha$ must be isotropic functions of the argument $\mathbf{D}$. Furthermore, as a special simple case it is possible to assume that $\check{\mathbf{T}}$ is a linear function of $\mathbf{D}$ and that $\check{\mathbf{m}}^\alpha$ are linear functions of $\dot{\boldsymbol{\beta}}_\alpha$ of the forms

$$d_{33}^{1/2}\,\check{\mathbf{T}} = D_{33}^{1/2}\,A\,[\eta_1\,(\mathbf{D} \bullet \mathbf{I})\,\mathbf{I} + 2\eta_2\,\mathbf{D}'] \ ,$$

$$\check{\mathbf{m}}^1 = \eta_3\,D_{33}^{1/2}\,A\,(\mathbf{D}^1 \bullet \mathbf{D}^1)\,\mathbf{F}^{-T}\dot{\boldsymbol{\beta}}_1 \ , \quad \check{\mathbf{m}}^2 = \eta_4\,D_{33}^{1/2}\,A\,(\mathbf{D}^2 \bullet \mathbf{D}^2)\,\mathbf{F}^{-T}\dot{\boldsymbol{\beta}}_2 \ , \tag{5.35.10}$$

where A is defined by (5.20.7), η_1–η_4 are material constants, and $\mathbf{D}'$ is a pure measure of rate of distortional deformation

$$\mathbf{D}' = \mathbf{D} - \frac{1}{3}(\mathbf{D} \bullet \mathbf{I})\,\mathbf{I} \ , \quad \mathbf{D}' \bullet \mathbf{I} = 0 \ . \tag{5.35.11}$$

Consequently, η_1 is the viscosity to dilatational deformation rate, η_2 is the viscosity to distortional deformation, and η_3 and η_4 are the viscosities to the inhomogeneous deformation rates $\dot{\boldsymbol{\beta}}_1$ and $\dot{\boldsymbol{\beta}}_2$, respectively. Also, it can be shown that the restriction (5.35.7) is satisfied for all motions provided that η_1–η_4 are all nonnegative

$$\eta_1 \geq 0 \ , \quad \eta_2 \geq 0 \ , \quad \eta_3 \geq 0 \ , \quad \eta_4 \geq 0 \ . \tag{5.35.12}$$

Moreover, it is noted that the viscosity constants η_1–η_4 can be determined by attempting to match the rate of damping associated with free vibrations of a structure.

Before closing this section, it is important to note that for a simple string, the definition $(5.31.3)_5$ of $\mathbf{D}$ does not satisfy the usual invariance properties under SRBM because, with the help of (5.5.3) and (5.5.4) it can be shown that

$$\mathbf{D} = \frac{1}{2d_{33}}(\mathbf{w}_3 \otimes \mathbf{d}_3 + \mathbf{d}_3 \otimes \mathbf{w}_3) \ , \quad \mathbf{w}_3^+ = \mathbf{Q}\mathbf{d}_3 + \boldsymbol{\Omega}\,\mathbf{d}_3^+$$

$$\mathbf{D}^+ = \mathbf{Q}\mathbf{D}\mathbf{Q}^T + \frac{1}{2d_{33}}[\boldsymbol{\Omega}\,(\mathbf{d}_3^+ \otimes \mathbf{d}_3^+) - (\mathbf{d}_3^+ \otimes \mathbf{d}_3^+)\boldsymbol{\Omega}] \ , \quad \mathbf{D}^+ \bullet \mathbf{I} = \mathbf{D} \bullet \mathbf{I} \ . \tag{5.35.13}$$

Consequently, for strings the expression for $\check{\mathbf{T}}$

$$\check{\mathbf{T}} = d_{33}^{-1/2}\,[\check{\mathbf{t}}^3 \otimes \mathbf{d}_3\,] \ , \tag{5.35.14}$$

cannot take the general form (5.35.10) with nonzero η_2 since it would not be properly invariant under SRBM. However, in view of the result $(5.35.13)_4$ it follows that $\check{\mathbf{T}}$ can be specified in the form

$$d_{33}^{1/2}\,\check{\mathbf{T}} = D_{33}^{1/2}\,A\,[\eta_1\,(\mathbf{D} \bullet \mathbf{I})\,\mathbf{I}] \ , \tag{5.35.15}$$

with η_1 being nonnegative. Moreover, using (5.31.4) it can be shown that

$$d_{33}^{1/2}\,\check{\mathbf{T}} = D_{33}^{1/2}\,A\,[\eta_1\{\frac{\dot{\lambda}}{\lambda}\}\mathbf{I}] \ , \tag{5.35.16}$$

which indicates that for strings η_1 represents the viscosity to stretching.

CHAPTER 6

COSSERAT POINTS

6.1 Description of a point-like structure

A point-like structure, or Cosserat point, is a three-dimensional body that has special geometric features. Most importantly, the Cosserat point is a three-dimensional body that is considered to be "thin" in all three of its dimensions. Therefore, it is essentially a zero-dimensional point surrounded by some finite but small region of material. Sometimes this region P^* is bounded by a smooth closed surface ∂P^*, while other times it is a polyhedron bounded by S planar surfaces ∂P_J^* (J=1,2,...,S) (see Fig. 6.1.1). In all cases, the Cosserat point theory provides a simple continuum description of the motion and deformation of this small point-like structure.

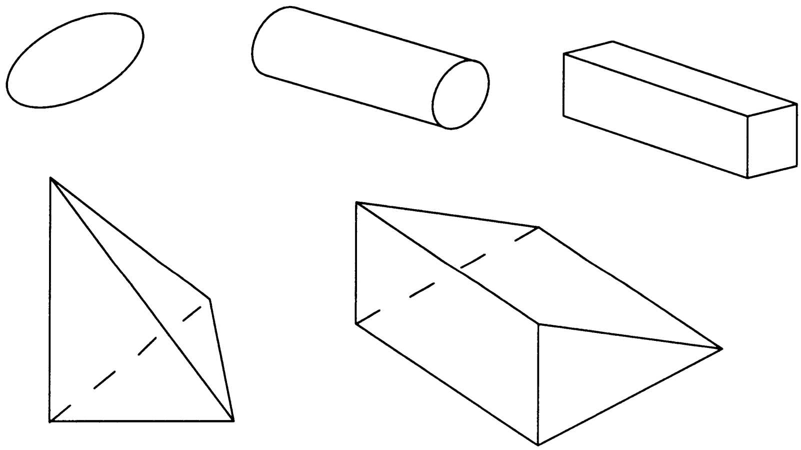

Fig. 6.1.1 Various point-like structures

The work of Slawianowski (1974, 1975, 1982) seems to be the first to analyze the homogeneous deformation of zero-dimensional bodies. Later, Cohen (1981), Muncaster (1984a,b), and Cohen and Muncaster (1984, 1988) developed the theory of pseudo-rigid bodies. This approach was motivated by a desire to generalize the theory of rigid-body dynamics from the usual six degrees of freedom to twelve degrees of freedom associated with homogeneous deformations of the body. A number of solutions to interesting problems are recorded in (Cohen and Muncaster, 1988) and references to more recent work can be found in (Cohen and Sun, 1991; and Cohen and Mac Sithigh, 1993, 1994, 1996).

In contrast, the theory of a Cosserat point attempts to simplify the three-dimensional continuum theory which has infinite degrees of freedom, to a theory with only a finite number of degrees of freedom. For the simplest Cosserat point this theory reduces to twelve degrees of freedom associated with homogeneous deformations. Consequently, the equations of the theory of a Cosserat point have similarity with the theories of zero-dimensional bodies (Slawianowski, 1974, 1975, 1982) and pseudo-rigid bodies (Cohen and Muncaster, 1988), but they are formulated to appear more similar to the theory of a three-dimensional continuum than to the theory of a rigid body.

It will be seen later that the theory of a Cosserat point is a nonlinear continuum theory that requires a discussion of the basic balance laws and constitutive equations which is similar to that of the full three-dimensional continuum theory. In particular, the balance laws are inherently nonlinear and are valid for arbitrary materials. For these reasons, it is expected that the theory of a Cosserat point can be used as a theoretical foundation for developing nonlinear finite elements that are used to obtain numerical solutions of problems in continuum mechanics. In this context, the Cosserat point theory provides a nonlinear continuum model for the response of each element in a specific discretization of a given structure. Then, by describing how each element interacts with its nearest neighbors, it is possible to obtain a finite set of ordinary differential equations that are functions of time only and which describe the dynamic (or static) response of the whole structure. Some work had been done (Rubin, 1985a,b, 1986, 1987a,b 1995) with this objective in mind, but more needs to be done before this expectation if fully realized.

From a mathematical point of view it is necessary to clearly define in what sense the point-like structure is considered to be "thin". To this end, it is convenient to let P_0 denote the material region occupied by the point-like structure in its reference configuration. Material points in P_0 are located relative to a fixed origin O by the three-dimensional position vector $\mathbf{X}^*(\theta^i)$. A single point in P_0 is identified as the reference point and is located by the position vector $\mathbf{X}$ which corresponds to the point for which θ^i vanish

$$\mathbf{X} = \mathbf{X}^*(0,0,0) \ . \tag{6.1.1}$$

In addition, a right-handed triad of linearly independent constant vectors $\mathbf{D}_i$ is defined such that

$$D^{1/2} = \mathbf{D}_1 \times \mathbf{D}_2 \bullet \mathbf{D}_3 > 0 \ . \tag{6.1.2}$$

Then, the position vector $\mathbf{X}^*$ of an arbitrary material point in the region P_0 is specified in the form (see Fig. 6.1.2)

$$\mathbf{X}^*(\theta^i) = \mathbf{X} + \theta^i \, \mathbf{D}_i \ . \tag{6.1.3}$$

Next, it is of interest to determine the maximum extent of the finite point-like region that can be described by the position vector (6.1.3). To answer this question it is necessary to recall a fundamental property of the position vector. Specifically, the position vector is required to provide a one-to-one mapping between the convected coordinates θ^i which define a material point in the point-like structure, and the three-dimensional Euclidean space occupied by it. Mathematically, this means that the base vectors $\mathbf{G}_i$ associated with the representation (6.1.3) must be linearly independent

$$\mathbf{G}_i = \mathbf{X}^*{}_{,i} = \mathbf{D}_i \ , \quad G^{1/2} = \mathbf{G}_1 \times \mathbf{G}_2 \bullet \mathbf{G}_3 > 0 \ . \tag{6.1.4}$$

Obviously, since $\mathbf{D}_i$ are linearly independent it follows that $\mathbf{G}_i$ are also linearly independent for all values of θ^i. This means that the complete three-dimensional space can be mapped by the position vector (6.1.3) so that there is no restriction on the validity of that representation.

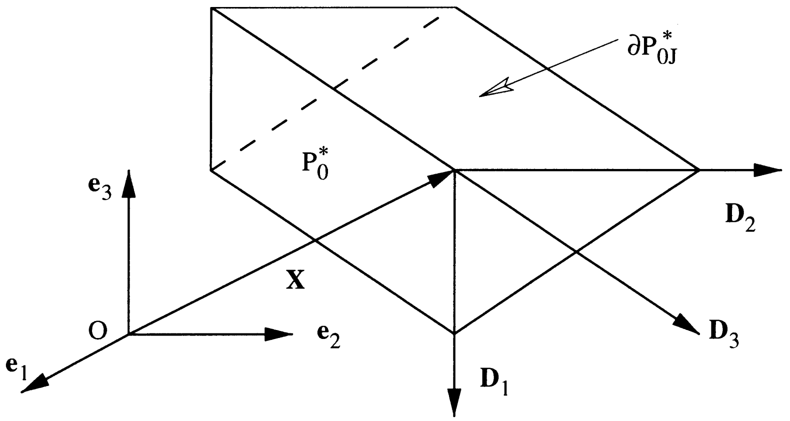

Fig. 6.1.2 A point-like structure in its reference configuration.

Moreover, it is convenient to introduce the reciprocal vectors $\mathbf{D}^i$ which are defined by formulas of the type (2.1.6) and (2.1.10)

$$\mathbf{D}^1 = D^{-1/2}\,(\mathbf{D}_2 \times \mathbf{D}_3) \ , \quad \mathbf{D}^2 = D^{-1/2}\,(\mathbf{D}_3 \times \mathbf{D}_1) \ ,$$

$$\mathbf{D}^3 = D^{-1/2}\,(\mathbf{D}_1 \times \mathbf{D}_2) \ , \quad D^{1/2} = \mathbf{D}_1 \times \mathbf{D}_2 \bullet \mathbf{D}_3 > 0 \ , \quad \mathbf{D}_i \bullet \mathbf{D}^j = \delta_i^j \ . \tag{6.1.5}$$

6.2 The Cosserat point model

Within the context of the Cosserat theory a point-like structure is modeled as a material point P with some additional kinematic structure to provide limited information about deformation through the thickness of the point-like structure. Specifically, with respect to the present configuration at time t, the position vector (from a fixed origin) of the material point P is denoted by

$$\mathbf{x} = \hat{\mathbf{x}}(t) \ . \tag{6.2.1}$$

In addition, the Cosserat theory endows the point P with three director vectors

$$\mathbf{d}_i = \hat{\mathbf{d}}_i(t) \ , \tag{6.2.2}$$

which are linearly independent vectors that form a right-handed triad with

$$d^{1/2} = \mathbf{d}_1 \times \mathbf{d}_2 \bullet \mathbf{d}_3 > 0 \ . \tag{6.2.3}$$

In a more general theory (e.g. Green and Naghdi, 1991) it is possible to introduce a finite set of N director vectors. However, here attention will be confined to the simpler theory with only three director vectors.

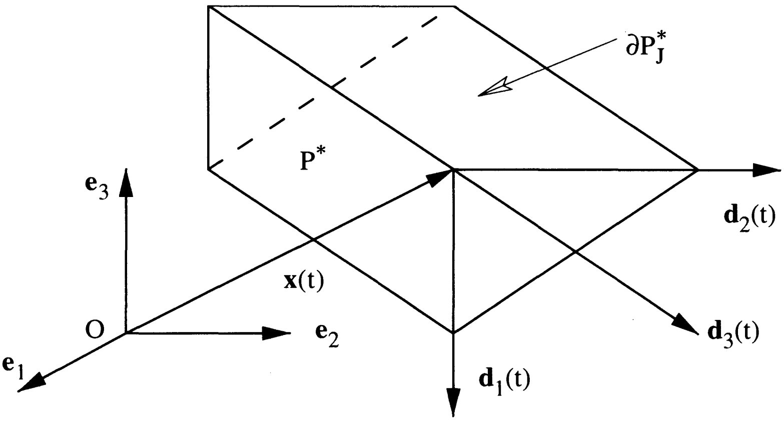

Fig. 6.2.1 The model of a Cosserat point in its present configuration.

From a physical point of view it is desirable to think of the directors $\mathbf{d}_i$ as vectors that describe the deformation of material fibers through the thickness of the point-like structure (see Fig. 6.2.1). In general, each of these fibers is allowed to change its length and its absolute orientation in space. Therefore, the position vector $\mathbf{x}$ contains three degrees of freedom associated with linear translation of the Cosserat point, and $\mathbf{d}_i$ contain nine degrees of freedom: three associated with rotation, and six associated with deformation or straining of the point-like structure. This deformation includes changes in the lengths of each fiber as well as changes in the angle between any two fibers.

The velocity vector $\mathbf{v}$ and the director velocities $\mathbf{w}_i$ derived from the functions (6.2.1) and (6.2.2), are defined by

$$\mathbf{v} = \mathbf{v}(t) = \dot{\mathbf{x}} \;,\quad \mathbf{w}_i = \mathbf{w}_i(t) = \dot{\mathbf{d}}_i \;, \tag{6.2.4}$$

where the superposed dot $(\dot{\ })$ denotes material time differentiation. In this regard, it is noted that since the vectors $\mathbf{x}$ and $\mathbf{d}_i$ are functions of time only, the notion of material time differentiation is indistinguishable from ordinary time differentiation.

Motivated by the representation (6.1.3), it is natural to consider an associated kinematic assumption that the position vector $\mathbf{x}^*(\theta^i,t)$ of material points in the point-like structure in its present configuration can be represented in the form

$$\mathbf{x}^*(\theta^i,t) = \mathbf{x}(t) + \theta^i\, \mathbf{d}_i(t) \;. \tag{6.2.5}$$

In contrast with the representation (6.1.3), which is always valid for a point-like structure in its reference configuration, the representation (6.2.5) restricts material line elements in the point-like structure to remain straight. This kinematic assumption (6.2.5) will be used to motivate forms for certain quantities that appear in the balance laws of the

Cosserat theory, and it can be used to provide physical interpretation of results of the theory. However, it will be seen that, strictly speaking, the Cosserat theory is used to determine the vectors $\mathbf{x}$ and $\mathbf{d}_i$ which depend only on time. Consequently, within the context of the Cosserat theory there is not necessarily a direct dependence on the assumption (6.2.5).

In order to expand on physical interpretations based on the assumption (6.2.5), it is desirable to consider the base vectors $\mathbf{g}_i$ and the reciprocal vectors $\mathbf{d}^i$ defined by

$$\mathbf{g}_i = \mathbf{x}^*{}_{,i} = \mathbf{d}_i \ , \quad g^{1/2} = \mathbf{g}_1 \times \mathbf{g}_2 \bullet \mathbf{g}_3 \ , \quad \mathbf{d}^i \bullet \mathbf{d}_j = \delta^i{}_j \ . \tag{6.2.6}$$

Now, following the work of Naghdi (1982) it is convenient to introduce the nonsingular second order tensor $\mathbf{F}$ such that

$$\mathbf{F} = \mathbf{d}_i \otimes \mathbf{D}^i \ , \quad \mathbf{F}^{-1} = \mathbf{D}_i \otimes \mathbf{d}^i \ , \quad \mathbf{d}_i = \mathbf{F}\,\mathbf{D}_i \ , \quad \det \mathbf{F} = d^{1/2}\,D^{-1/2} \ . \tag{6.2.7}$$

Then, with the help of (6.1.4) and the definition (2.3.8) of the three-dimensional deformation gradient $\mathbf{F}^*$ it follows that

$$\mathbf{F}^*(\theta^i,t) = \mathbf{F}(t) \ . \tag{6.2.8}$$

This expression shows that the associated three-dimensional deformation gradient $\mathbf{F}^*$ equals $\mathbf{F}$ which is independent of the coordinates θ^i and depends only on time. Consequently, the kinematic assumption (6.2.5) causes the associated three-dimensional deformation of the point-like structure to be homogeneous. It also follows that the representation (6.2.5) is exact for all homogeneous deformations of the point-like structure.

6.3 Derivation of the balance laws from the three-dimensional theory

The balance laws of the theory of a Cosserat point are similar to those of the three-dimensional theory in the sense that they include the notions of conservation of mass and balances of linear and angular momentum. Moreover, these equations are used to determine the current values of the mass m and position vector $\mathbf{x}$ of the point P. Also, the balance of angular momentum will place restrictions on the constitutive equations of the Cosserat point theory that are similar in nature to the restriction (3.2.32) associated with the three-dimensional theory. However, in contrast with the three-dimensional theory, the Cosserat point theory introduces the additional kinematic quantities $\mathbf{d}_i$ at P which also must be determined by balance laws. Consequently, the Cosserat point theory requires three additional balance laws called the balances of director momentum.

In this section it will be shown that the balance laws of the Cosserat point theory can be developed by using the kinematic assumption (6.2.5) and the balance laws of the three-dimensional theory. For later convenience, the equations will be developed for a point-like structure that is a polyhedron with S boundary surfaces. However, the case of a region bounded by a smooth closed surface can be handled in a similar manner. To this end, the three-dimensional region P_0^* (with boundaries ∂P_{0J}^*, J=1,2,...S) occupied by the point-like structure in its reference configuration, is mapped to the region P^* (with boundaries ∂P_J^*) in the present configuration.

Now, with the help of the expression (3.2.5) for the element of volume dv^* and the expression $(3.2.28)_1$ for m^*, the total mass m $(3.2.26)_1$ in P^* can be written as

$$m = \rho\, d^{1/2} = \int_{P^*} \rho^*\, dv^* = \int_{P^*} m^*\, d\theta^1\, d\theta^2\, d\theta^3 \quad , \tag{6.3.1}$$

where ρ is the mass density (mass per unit present volume). It then follows from (3.2.1) and (6.3.1) that the conservation of mass of the Cosserat point becomes

$$\dot{m} = 0 \ . \tag{6.3.2}$$

Next, with the help of the kinematic assumption (6.2.5) and the definitions (6.2.4) for the velocity $\mathbf{v}$ and the director velocities $\mathbf{w}_i$, the three-dimensional balance of linear momentum (3.2.2) applied to the point-like region P^* becomes

$$\frac{d}{dt}\int_{P^*} m^*\,(\mathbf{v} + \theta^i\,\mathbf{w}_i)\, d\theta^1\, d\theta^2\, d\theta^3$$

$$= \int_{P^*} m^*\, \mathbf{b}^*\, d\theta^1\, d\theta^2\, d\theta^3 + \sum_{J=1}^{S} \int_{\partial P^*_J} \mathbf{t}^*\, da^* \ . \tag{6.3.3}$$

The main objective here is to rewrite the integrals in (6.3.3) by defining simpler alternative quantities that are functions of time only. To this end, it is noted that $\mathbf{v}$ and $\mathbf{w}_i$ are independent of the coordinates θ^i so that it is convenient to define the director inertia coefficients y^i by the expressions

$$m\, y^i = \int_{P^*} m^*\, \theta^i\, d\theta^1\, d\theta^2 \ . \tag{6.3.4}$$

Then, the linear momentum of the Cosserat point can be written as

$$\int_{P^*} m^*\,(\mathbf{v} + \theta^i\,\mathbf{w}_i)\, d\theta^1\, d\theta^2\, d\theta^3 = m\,(\mathbf{v} + y^i\,\mathbf{w}_i) \ . \tag{6.3.5}$$

Also, it is convenient to define the specific body force $\mathbf{b}_b$ by the expression

$$m\,\mathbf{b}_b = \int_{P^*} m^*\, \mathbf{b}^*\, d\theta^1\, d\theta^2\, d\theta^3 \quad , \tag{6.3.6}$$

and to define the forces $\mathbf{m}^0_J$ applied to each of the surfaces ∂P^*_J by

$$\mathbf{m}^0_J = \int_{\partial P^*_J} \mathbf{t}^*\, da^* \ . \tag{6.3.7}$$

Then, the total specific force $\mathbf{b}_c$ due to surface tractions is given by

$$m\,\mathbf{b}_c = \sum_{J=1}^{S} \mathbf{m}^0_J \ . \tag{6.3.8}$$

Now, with the help of (6.3.3) and the expressions (6.3.5), (6.3.6) and (6.3.8), the balance of linear momentum associated with the Cosserat point can be written in the form

$$\frac{d}{dt}\left[m\,(\mathbf{v} + y^i\,\mathbf{w}_i)\right] = m\,\mathbf{b} \quad , \tag{6.3.9}$$

where the total specific external force $\mathbf{b}$ mass applied to the Cosserat point is defined by

$$\mathbf{b} = \mathbf{b}_b + \mathbf{b}_c \ . \tag{6.3.10}$$

In this regard, it should be emphasized that the external force $\mathbf{b}$ is due to two different physical sources. One part $\mathbf{b}_b$ is due to the three-dimensional body force applied to the

Cosserat point, and the other part $\mathbf{b}_c$ is due to contact forces applied to the boundary surfaces of the Cosserat point .

Before developing the balance of angular momentum, it is convenient to develop the balances of director momentum. Here, the balances of director momentum will be developed as averaged first moments of the balance of linear momentum with respect to the coordinates θ^i in region P^* of the Cosserat point. Specifically, the averaged form (3.6.3) of the balance of linear momentum is applied to the point-like region P^*, with the weighting function ϕ taken equal to the coordinates θ^i so that with the help of the kinematic assumption (6.2.5), the expression (3.6.3) yields

$$\frac{d}{dt}\int_{P^*} m^* (\theta^i \mathbf{v} + \theta^i \theta^j \mathbf{w}_j)\, d\theta^1\, d\theta^2\, d\theta^3$$

$$= \int_{P^*} \{m^* \theta^i \mathbf{b}^* - \mathbf{t}^{*i}\}\, d\theta^1\, d\theta^2\, d\theta^3 + \sum_{J=1}^{S} \int_{\partial P_J^*} \theta^i \mathbf{t}^*\, da^* \quad . \qquad (6.3.11)$$

Inspection of (6.3.11) indicates that it is convenient to introduce the additional director inertia coefficients y^{ij} that are defined in terms of the second moment of the mass density through the region P^* by the equations

$$m\, y^{ij} = m\, y^{ji} = \int_{P^*} m^* \theta^i \theta^j\, d\theta^1\, d\theta^2\, d\theta^3 \ , \qquad (6.3.12)$$

so that the first moments of linear momentum can be expressed as

$$\int_{P^*} \rho^* \theta^i \mathbf{v}^*\, dv^* = m\,(y^i \mathbf{v} + y^{ij} \mathbf{w}_j) \ . \qquad (6.3.13)$$

Also, it is convenient to define the specific external director couples $\mathbf{b}_b^i$ as the first moments of the body force by the expression

$$m\, \mathbf{b}_b^i = \int_{P^*} m^* \theta^i \mathbf{b}^*\, d\theta^1\, d\theta^2\, d\theta^3 \ , \qquad (6.3.14)$$

and to define the couples $\mathbf{m}_J^i$ applied to each of the surfaces ∂P_J^* by

$$\mathbf{m}_J^i = \int_{\partial P_J^*} \theta^i \mathbf{t}^*\, da^* \ . \qquad (6.3.15)$$

Then, the total specific external couples $\mathbf{b}_c^i$ due to surface tractions are given by

$$m\, \mathbf{b}_c^i = \sum_{J=1}^{S} \mathbf{m}_J^i \ . \qquad (6.3.16)$$

Furthermore, comparison of (6.3.3) and (6.3.11) indicates the presence of extra terms in (6.3.11) which are related to the integrated effect of the quantities $\mathbf{t}^{*i}$. Consequently, it is convenient to define the intrinsic director couples $\mathbf{t}^i$ such that

$$\mathbf{t}^i = \int_{P^*} \mathbf{t}^{*i}\, d\theta^1\, d\theta^2\, d\theta^3 \ . \qquad (6.3.17)$$

Using the expressions (6.3.13), (6.3.14), (6.3.16) and (6.3.17), the balances of director momentum (6.3.11) can be written in the simpler forms

$$\frac{d}{dt}\left[m\,(y^i \mathbf{v} + y^{ij} \mathbf{w}_j)\right] = m\, \mathbf{b}^i - \mathbf{t}^i \ , \qquad (6.3.18)$$

where the specific external director couples $\mathbf{b}^i$ applied to the Cosserat point are the sums of the parts $\mathbf{b}^i_b$ due to body force, and the parts $\mathbf{b}^i_c$ due to contact forces applied to the boundary surfaces of the Cosserat point

$$\mathbf{b}^i = \mathbf{b}^i_b + \mathbf{b}^i_c \ . \tag{6.3.19}$$

Here, it is of interest to note that the theoretical structure of the balances of director momentum (6.3.18) differs from that of the balance of linear momentum (6.3.9) due to the presence of the intrinsic director couples $\mathbf{t}^i$. In this sense, the averaged form (3.6.3) of the three-dimensional balance of linear momentum provided important theoretical guidance for motivating the form (6.3.18) of director momentum.

Returning to the analysis of angular momentum, it follows that when the three-dimensional form (3.2.4) for the balance of angular momentum is applied to the point-like region P^* and the kinematic assumption (6.2.5) is used, then it can be shown that

$$\begin{aligned}\frac{d}{dt}\int_{P^*} m^* \left[\mathbf{x} \times (\mathbf{v} + \theta^i \mathbf{w}_i) + \mathbf{d}_i \times (\theta^i \mathbf{v} + \theta^i \theta^j \mathbf{w}_j)\right] d\theta^1 d\theta^2 d\theta^3 \\ = \int_{P^*} m^* \left[(\mathbf{x} \times \mathbf{b}^*) + (\mathbf{d}_i \times \theta^i \mathbf{b}^*) \right] d\theta^1 d\theta^2 d\theta^3 \\ + \sum_{J=1}^{S} \int_{\partial P^*_J} \{(\mathbf{x} \times \mathbf{t}^*) + (\mathbf{d}_i \times \theta^i \mathbf{t}^*)\}\, da^* \ .\end{aligned} \tag{6.3.20}$$

Moreover, the previous definitions can be used to rewrite the balance of angular momentum of the Cosserat theory in the simpler form

$$\frac{d}{dt}\left[\mathbf{x} \times m\,(\mathbf{v} + y^i \mathbf{w}_i) + \mathbf{d}_i \times m\,(y^i \mathbf{v} + y^{ij} \mathbf{w}_j)\right] = m\left[\mathbf{x} \times \mathbf{b} + \mathbf{d}_i \times \mathbf{b}^i\right] . \tag{6.3.21}$$

Inspection of (6.3.21) reveals that the intrinsic director couples $\mathbf{t}^i$ do not contribute to the balance of angular momentum even though they do contribute to the balances of director momentum.

Before closing this section, it is desirable to develop expressions for the rate of work $\mathcal{W}$ done on the Cosserat point and the kinetic energy $\mathcal{K}$ of the Cosserat point. To this end, the kinematic assumption (6.2.5) is used, and the expressions (3.4.1) and (3.4.2) of the three-dimensional theory are evaluated for the point-like region P^* to obtain

$$\begin{aligned}\mathcal{W} = \int_{P^*} \rho^* \mathbf{b}^* \bullet (\mathbf{v} + \theta^i \mathbf{w}_i)\, dv^* + \sum_{J=1}^{S} \int_{\partial P^*_J} \mathbf{t}^* \bullet (\mathbf{v} + \theta^i \mathbf{w}_i)\, da^* \ , \\ \mathcal{K} = \int_{P^*} \frac{1}{2} \rho^* (\mathbf{v} + \theta^i \mathbf{w}_i) \bullet (\mathbf{v} + \theta^j \mathbf{w}_j)\, dv^* \ .\end{aligned} \tag{6.3.22}$$

Now, with the help of the previous definitions these expressions can be written in the simpler forms

$$\mathcal{W} = m\,(\mathbf{b} \bullet \mathbf{v} + \mathbf{b}^i \bullet \mathbf{w}_i) \ , \quad \mathcal{K} = \frac{1}{2} m\,(\mathbf{v} \bullet \mathbf{v} + 2\, y^i\, \mathbf{v} \bullet \mathbf{w}_i + y^{ij}\, \mathbf{w}_i \bullet \mathbf{w}_j) \ . \tag{6.3.23}$$

Also, it is noted that in view of the local three-dimensional form (3.2.28) of the conservation of mass and the fact that θ^i are convected coordinates, the director inertia coefficients y^i in (6.3.4) and y^{ij} in (6.3.12) are independent of time

$$\dot{y}^i = 0 \ , \quad \dot{y}^{ij} = 0 \ . \tag{6.3.24}$$

Moreover, since the kinetic energy must be a nonnegative function of the velocities, it follows from the expression

$$\mathbf{v} \bullet \mathbf{v} + 2\, y^i\, \mathbf{v} \bullet \mathbf{w}_i + y^{ij}\, \mathbf{w}_i \bullet \mathbf{w}_j = (\mathbf{v} + y^i\, \mathbf{w}_i) \bullet (\mathbf{v} + y^j\, \mathbf{w}_j)$$
$$+ (y^{ij} - y^i\, y^j)\, \mathbf{w}_i \bullet \mathbf{w}_j\ , \qquad (6.3.25)$$

that y^i and y^{ij} are further restricted by the condition that $(y^{ij} - y^i\, y^j)$ is positive semi-definite. This means that the eigenvalues of $(y^{ij} - y^i\, y^j)$ must be nonnegative. In particular, the characteristic equation for these eigenvalues λ can be written in the form

$$-\lambda^3 + I_1\, \lambda^2 - I_2\, \lambda + I_3 = 0\ , \qquad (6.3.26)$$

where I_1, I_2, I_3 are the principle invariants of $(y^{ij} - y^i\, y^j)$ which are defined by

$$I_1 = y^{ii} - y^i\, y^i\ ,\quad I_2 = \frac{1}{2}\left[\, I_1^2 - (y^{ij} - y^i\, y^j)\, (y^{ij} - y^i\, y^j)\right]\ ,$$
$$I_3 = \det\, (y^{ij} - y^i\, y^j)\ . \qquad (6.3.27)$$

Since $(y^{ij} - y^i\, y^j)$ is real and symmetric, the roots of (6.3.26) are real. Furthermore, the Routh-Hurwitz criteria (Watkins, 1969) can be used to deduce the necessary and sufficient conditions

$$I_1 \ge 0\ ,\quad I_1\, I_2 - I_3 \ge 0\ ,\quad I_3 \ge 0\ , \qquad (6.3.28)$$

that these roots are also nonnegative. Finally, it is noted that the kinetic energy is required to be positive semi-definite instead of positive definite because, occasionally in the Cosserat theory, it is desirable to ignore director inertia (y^i=0; y^{ij}=0) due to deformation and rotation of the Cosserat point, while keeping the effect of inertia ($m \neq 0$) in the balance of linear momentum.

6.4 Balance laws by the direct approach

In the previous section the forms of conservation of mass and the balances of linear momentum, director momentum and angular momentum were developed by using the kinematic assumption (6.2.5) together with the balance laws of the three-dimensional theory. From the point of view of the full three-dimensional theory, the Cosserat point theory with three directors is an approximate theory whenever line elements through the thickness of the point-like structure do not remain straight.

However, as will be seen in this section, the Cosserat point theory can be presented by a direct approach in which the theory is an exact nonlinear theory. From this perspective, the Cosserat theory is considered to be a *model* of a point-like structure and the balance laws are postulated without any direct connection with the three-dimensional theory. Specifically, the Cosserat theory models the point-like structure as a point P in the present configuration at time t. The kinematics of the Cosserat theory include the position vector $\mathbf{x}(t)$ (relative to a fixed origin O) and the director vectors $\mathbf{d}_i(t)$ which are specified by the functional forms

$$\mathbf{x} = \hat{\mathbf{x}}(t)\ ,\quad \mathbf{d}_i = \hat{\mathbf{d}}_i(t)\ . \qquad (6.4.1)$$

Also, the velocity $\mathbf{v}$ and the director velocities $\mathbf{w}_i$ are defined by (6.2.4). Moreover, the Cosserat point is endowed with a total mass m and director inertia coefficients y^i and y^{ij} which are independent of time

$$m = \text{constant} \ , \quad y^i = \text{constant} \ , \quad y^{ij} = y^{ji} = \text{constant} \ , \tag{6.4.2}$$

with y^{ij} being symmetric.

The Cosserat point P is subjected to a specific external force $\mathbf{b}$ and specific external director couples $\mathbf{b}^i$, which are due to body forces ($\mathbf{b}_b$ and $\mathbf{b}_b^i$) and contact forces ($\mathbf{b}_c$ and $\mathbf{b}_c^i$) applied to the surfaces of the point-like structure

$$\mathbf{b} = \mathbf{b}(t) = \mathbf{b}_b(t) + \mathbf{b}_c(t) \ , \quad \mathbf{b}^i = \mathbf{b}^i(t) = \mathbf{b}_b^i(t) + \mathbf{b}_c^i(t) \ . \tag{6.4.3}$$

In addition, the Cosserat point P is subjected to intrinsic director couples $\mathbf{t}^i$

$$\mathbf{t}^i = \mathbf{t}^i(t) \ . \tag{6.4.4}$$

Using these definitions, the conservation of mass and the balances of linear momentum and director momentum are postulated in the forms

$$\dot{m} = 0 \ , \quad \frac{d}{dt}\left[m\,(\mathbf{v} + y^i\,\mathbf{w}_i\,)\right] = m\,\mathbf{b} \ , \quad \frac{d}{dt}\left[m\,(y^i\,\mathbf{v} + y^{ij}\,\mathbf{w}_j\,)\right] = m\,\mathbf{b}^i - \mathbf{t}^i \ , \tag{6.4.5}$$

Also, the balance of angular momentum about the fixed origin O is postulated in the form

$$\frac{d}{dt}\left[\,\mathbf{x}\times m\,(\mathbf{v} + y^i\,\mathbf{w}_i\,) + \mathbf{d}_i\times m\,(y^i\,\mathbf{v} + y^{ij}\,\mathbf{w}_j\,)\right] = m\,[\mathbf{x}\times\mathbf{b} + \mathbf{d}_i\times\mathbf{b}^i\,] \ . \tag{6.4.6}$$

In order to develop the reduced form of the balance of angular momentum, it is noted that since the inertia quantities m, y^i, y^{ij} are constants, it follows that

$$\frac{d}{dt}\left[m\,(\mathbf{v} + y^i\,\mathbf{w}_i\,)\right] = m\,(\dot{\mathbf{v}} + y^i\,\dot{\mathbf{w}}_i\,) \ , \quad \frac{d}{dt}\left[m\,(y^i\,\mathbf{v} + y^{ij}\,\mathbf{w}_j\,)\right] = m\,(y^i\,\dot{\mathbf{v}} + y^{ij}\,\dot{\mathbf{w}}_j\,) \ ,$$

$$\frac{d}{dt}\left[\,\mathbf{x}\times m\,(\mathbf{v} + y^i\,\mathbf{w}_i\,) + \mathbf{d}_i\times m\,(y^i\,\mathbf{v} + y^{ij}\,\mathbf{w}_j\,)\right]$$
$$= \mathbf{x}\times m\,(\dot{\mathbf{v}} + y^i\,\dot{\mathbf{w}}_i\,) + \mathbf{d}_i\times m\,(y^i\,\dot{\mathbf{v}} + y^{ij}\,\dot{\mathbf{w}}_j\,) \ . \tag{6.4.7}$$

Then, the equations (6.4.5)-(6.4.7) can be used to deduce the reduced form of the balance of angular momentum

$$\mathbf{t}^i\times\mathbf{d}_i = 0 \ , \tag{6.4.8}$$

where the order of the cross product has been inverted for later convenience. Next, recalling from appendix A that the permutation tensor $\boldsymbol{\varepsilon}$ has the property (A.5.15) that for any two vectors $\mathbf{a}$ and $\mathbf{b}$

$$\mathbf{a}\times\mathbf{b} = \boldsymbol{\varepsilon}\bullet(\mathbf{a}\otimes\mathbf{b}) \ , \tag{6.4.9}$$

it follows that the balance of angular momentum (6.4.8) can be rewritten as

$$\boldsymbol{\varepsilon}\bullet\left[\mathbf{t}^i\otimes\mathbf{d}_i\,\right] = 0 \ . \tag{6.4.10}$$

Consequently, by defining the second order tensor $\mathbf{T}$

$$d^{1/2}\,\mathbf{T} = \mathbf{t}^i\otimes\mathbf{d}_i \ , \tag{6.4.11}$$

the reduced form of the balance of angular momentum (6.4.10) requires $\mathbf{T}$ to be a symmetric tensor

$$\mathbf{T}^T = \mathbf{T} \ , \tag{6.4.12}$$

which is similar to the result (3.2.32) associated with the three-dimensional theory.

Thus, with the help of (6.4.7), the conservation of mass and the balances of linear momentum and director momentum (6.4.5) can be summarized as

$$m = \rho\, d^{1/2} = \rho_0\, D^{1/2} = \text{constant} \quad \text{or} \quad \dot{m} = 0 \ ,$$

$$m\,(\dot{\mathbf{v}} + y^i\, \dot{\mathbf{w}}_i\,) = m\,\mathbf{b} \ , \quad m\,(y^i\, \dot{\mathbf{v}} + y^{ij}\, \dot{\mathbf{w}}_j\,) = m\,\mathbf{b}^i - \mathbf{t}^i \ , \tag{6.4.13}$$

where ρ_0 is the mass density in the reference configuration.

In general, equations (6.4.12) and (6.4.13) represent a system of nonlinear ordinary differential equations which require specification of initial conditions. These balance laws are quite general because they are valid for all materials. However, this system of equations is not complete because it represents a system of sixteen scalar equations to determine twenty-two scalar unknowns $\{m\,,\, \mathbf{x}\,,\, \mathbf{d}_i\,,\, \mathbf{t}^i\}$, once the external loads $\{\mathbf{b},\, \mathbf{b}^i\}$ have been specified. As in the three-dimensional theory, these equations must be supplemented by constitutive equations for the quantities $\mathbf{t}^i$ (here it is tacitly assumed that the director inertias y^i and y^{ij} have been specified).

Furthermore, it is noted that the reduced form of the balance of angular momentum (6.4.12) places three restrictions on the constitutive equations that must be satisfied for all possible motions of the continuum. Therefore, the balance of angular momentum has a different character from the other three balance laws because it is not used to determine the motion or deformation of the continuum. In contrast, the conservation of mass $(6.4.13)_1$ is used to determine the mass density ρ, and the balances of linear and director momentum $(6.4.13)_{2,3}$ are used to determine the motion of the continuum through the functional forms for $\{\mathbf{x}\,,\, \mathbf{d}_i\}$.

6.5 Invariance under superposed rigid body motions

Motivated by the relationships developed in section 6.3 and the discussion of superposed rigid body motions (SRBM) in section 3.3 for the three-dimensional theory, it is assumed that under SRBM the Cosserat point is transformed from its present configuration P to its superposed configuration P^+ at time t^+ such that

$$t^+ = t + a \ , \quad \mathbf{x}^+ = \mathbf{c}(t) + \mathbf{Q}(t)\, \mathbf{x} \ , \quad \mathbf{d}_i^+ = \mathbf{Q}(t)\, \mathbf{d}_i \ , \tag{6.5.1}$$

where a is a constant, $\mathbf{c}(t)$ is an arbitrary vector function of time only, and $\mathbf{Q}(t)$ is a proper orthogonal tensor function of time only

$$\mathbf{Q}^T\, \mathbf{Q} = \mathbf{I} \ , \quad \det \mathbf{Q} = +\,1 \ , \tag{6.5.2}$$

which is related to a skew-symmetric tensor $\mathbf{\Omega}(t)$ function of time only through the equations

$$\dot{\mathbf{Q}} = \mathbf{\Omega}\, \mathbf{Q} \ , \quad \mathbf{\Omega}^T = -\,\mathbf{\Omega} \ . \tag{6.5.3}$$

Throughout the text, quantities associated with the superposed configuration will be denoted using the same symbol as those associated with the present configuration, but with a superposed ($^+$). Next, using (6.5.1) it follows that under SRBM

$$\mathbf{D}_i^+ = \mathbf{D}_i \ , \quad \mathbf{D}^{i+} = \mathbf{D}^i \ , \quad \mathbf{d}_i^+ = \mathbf{Q}\, \mathbf{d}_i \ , \quad \mathbf{d}^{i+} = \mathbf{Q}\, \mathbf{d}^i \ , \quad \mathbf{F}^+ = \mathbf{Q}\, \mathbf{F} \ . \tag{6.5.4}$$

Moreover, the various kinetic quantities are assumed to transform to their superposed values by the equations

$$\mathrm{m}^+ = \mathrm{m} \ , \quad \rho^+ = \rho \ , \quad \mathrm{y}^{i+} = \mathrm{y}^i \ , \quad \mathrm{y}^{ij+} = \mathrm{y}^{ij} \ ,$$

$$\mathbf{t}^{i+} = \mathbf{Q}\,\mathbf{t}^i \ , \quad \mathbf{T}^+ = \mathbf{Q}\,\mathbf{T}\,\mathbf{Q}^T \ , \quad \mathbf{b}^+ = (\dot{\mathbf{v}}^+ + \mathrm{y}^{i+}\,\dot{\mathbf{w}}_i^+) + \mathbf{Q}\,[\mathbf{b} - (\dot{\mathbf{v}} + \mathrm{y}^i\,\dot{\mathbf{w}}_i)] \ ,$$

$$\mathbf{b}_b^+ = (\dot{\mathbf{v}}^+ + \mathrm{y}^{i+}\,\dot{\mathbf{w}}_i^+) + \mathbf{Q}\,[\mathbf{b}_b - (\dot{\mathbf{v}} + \mathrm{y}^i\,\dot{\mathbf{w}}_i)] \ , \quad \mathbf{b}_c^+ = \mathbf{Q}\,\mathbf{b}_c \ ,$$

$$\mathbf{b}^{i+} = (\mathrm{y}^{i+}\,\dot{\mathbf{v}}^+ + \mathrm{y}^{ij+}\,\dot{\mathbf{w}}_j^+) + \mathbf{Q}\,[\mathbf{b}^i - (\mathrm{y}^i\,\dot{\mathbf{v}} + \mathrm{y}^{ij}\,\dot{\mathbf{w}}_j)] \ ,$$

$$\mathbf{b}_b^{i+} = (\mathrm{y}^{i+}\,\dot{\mathbf{v}}^+ + \mathrm{y}^{ij+}\,\dot{\mathbf{w}}_j^+) + \mathbf{Q}\,[\mathbf{b}_b^i - (\mathrm{y}^i\,\dot{\mathbf{v}} + \mathrm{y}^{ij}\,\dot{\mathbf{w}}_j)] \ , \quad \mathbf{b}_c^{i+} = \mathbf{Q}\,\mathbf{b}_c^i \ . \tag{6.5.5}$$

Now, with the help of these results it can be shown that the balance laws (6.4.12) and (6.4.13) remain form invariant under SRBM. Furthermore, it will be shown in a later section that these conditions place important physical restrictions on constitutive assumptions for the quantities $\mathbf{t}^i$.

6.6 Mechanical power

For the purely mechanical theory it is convenient to define the notion of the mechanical power $\mathcal{P}$ due to the kinetic quantities $\{\mathbf{b}\ ,\ \mathbf{b}^i\}$ by the equation

$$\begin{aligned} \mathrm{d}^{1/2}\,\mathcal{P} = \mathcal{W} - \dot{\mathcal{K}} &= \mathrm{m}\,(\mathbf{b}\cdot\mathbf{v} + \mathbf{b}^i\cdot\mathbf{w}_i) \\ &\quad - \frac{\mathrm{d}}{\mathrm{dt}}\Big[\frac{1}{2}\mathrm{m}\,(\,\mathbf{v}\cdot\mathbf{v} + 2\,\mathrm{y}^i\,\mathbf{v}\cdot\mathbf{w}_i + \mathrm{y}^{ij}\,\mathbf{w}_i\cdot\mathbf{w}_j\,)\Big] \ , \end{aligned} \tag{6.6.1}$$

where the expressions (6.3.23) have been used for the rate of work $\mathcal{W}$ applied to the Cosserat point and for the kinetic energy $\mathcal{K}$.

Next, using the fact that m, y^i and y^{ij} are constants (6.3.24), and using the balance laws (6.4.13), it can be shown that the expression for the mechanical power can be reduced to

$$\mathrm{d}^{1/2}\,\mathcal{P} = \mathbf{t}^i\cdot\mathbf{w}_i \ . \tag{6.6.2}$$

Also, it can be shown using the transformation relations of section 6.5 that under SRBM

$$\mathbf{w}_i^+ = \mathbf{Q}\,\mathbf{w}_i + \mathbf{\Omega}\,\mathbf{Q}\,\mathbf{d}_i \ , \tag{6.6.3}$$

so that with the help of (3.5.6), the mechanical power in the superposed configuration becomes

$$(\mathrm{d}^{1/2})^+\,\mathcal{P}^+ = \mathrm{d}^{1/2}\,\mathcal{P} + (\mathbf{\Omega}\,\mathbf{Q}\,\mathbf{d}_i)\cdot(\mathbf{Q}\,\mathbf{t}^i) = \mathrm{d}^{1/2}\,\mathcal{P} + (\mathbf{Q}^T\boldsymbol{\omega})\cdot(\mathbf{d}_i\times\mathbf{t}^i) \ , \tag{6.6.4}$$

where $\boldsymbol{\omega}$ is the axial vector associated with $\mathbf{\Omega}$. However, using the form (6.4.8) of the balance of angular momentum, it follows that the mechanical power is unaltered by SRBM

$$\mathcal{P}^+ = \mathcal{P} \ . \tag{6.6.5}$$

Before closing this section, it is desirable to rewrite the expression for the mechanical power in an alternative form that is more similar to the expression (3.4.5) of the three-dimensional theory. To this end, it is convenient to define the second order tensor $\mathbf{L}$ by the expressions

$$\mathbf{L} = \mathbf{w}_i \otimes \mathbf{d}^i \ , \quad \mathbf{w}_i = \mathbf{L}\,\mathbf{d}_i \ . \tag{6.6.6}$$

It then follows from the definition (6.2.7) of the tensor $\mathbf{F}$ that

$$\dot{\mathbf{F}} = \mathbf{L}\,\mathbf{F} \ . \tag{6.6.7}$$

Consequently, comparison of these expressions with (3.2.12) suggests that $\mathbf{F}$ is similar to the three-dimensional deformation gradient and $\mathbf{L}$ is similar to the three-dimensional velocity gradient. Thus, using the definition (6.4.11), the mechanical power can be rewritten in the form

$$\mathcal{P} = \mathbf{T} \bullet \mathbf{L} \ . \tag{6.6.8}$$

Moreover, by separating $\mathbf{L}$ into its symmetric part $\mathbf{D}$ and its skew-symmetric part $\mathbf{W}$

$$\mathbf{L} = \mathbf{D} + \mathbf{W} \ , \quad \mathbf{D} = \frac{1}{2}(\mathbf{L} + \mathbf{L}^T) = \mathbf{D}^T \ , \quad \mathbf{W} = \frac{1}{2}(\mathbf{L} - \mathbf{L}^T) = -\mathbf{W}^T \ , \tag{6.6.9}$$

and by using the result (6.4.12) of the balance of angular momentum that $\mathbf{T}$ is a symmetric tensor, it follows that the mechanical power reduces to

$$\mathcal{P} = \mathbf{T} \bullet \mathbf{D} \ . \tag{6.6.10}$$

Next, it is noted that under SRBM the quantities $\mathbf{L}$, $\mathbf{D}$, $\mathbf{W}$ transform by

$$\mathbf{L}^+ = \mathbf{Q}\,\mathbf{L}\,\mathbf{Q}^T + \boldsymbol{\Omega} \ , \quad \mathbf{D}^+ = \mathbf{Q}\,\mathbf{D}\,\mathbf{Q}^T, \quad \mathbf{W}^+ = \mathbf{Q}\,\mathbf{W}\,\mathbf{Q}^T + \boldsymbol{\Omega} \ , \tag{6.6.11}$$

so that the mechanical power is again shown to be unaltered by SRBM.

6.7 An alternative derivation of the balance laws

From the point of view presented previously, the condition that the balance laws remain form invariant under SRBM requires the kinematic and kinetic quantities to satisfy the transformation relations (6.5.1) and (6.5.5), as well a number of other expressions [like (6.5.4)] that can be derived directly from these relations. In this regard, the notion of invariance under SRBM is a fundamental notion that causes intimate interconnections between the balance laws. To demonstrate the fundamental nature of invariance under SRBM, it will be shown that the conservation of mass $(6.4.5)_1$ and the balances of linear momentum $(6.4.5)_2$ and angular momentum (6.4.6) can be derived by assuming that the form (6.6.1) of the expression for mechanical power remains form invariant, and that the above transformation relations are valid. This means that these balance laws can be derived from a single scalar equation by demanding invariance for the class of all possible SRBM. However, this method does not produce the balances of director momentum $(6.4.5)_3$.

To this end, it is first noted that with respect to the superposed configuration, the equation (6.6.1) can be written in the form

$$\begin{aligned}(d^{1/2})^+ \mathcal{P}^+ = {} & m^+ (\mathbf{b}^+ \bullet \mathbf{v}^+ + \mathbf{b}^{i+} \bullet \mathbf{w}_i^+) \\ & - \frac{d}{dt^+}\Big[\frac{1}{2} m^+ (\mathbf{v}^+ \bullet \mathbf{v}^+ + 2\, y^{i+}\, \mathbf{v}^+ \bullet \mathbf{w}_i^+ + y^{ij+}\, \mathbf{w}_i^+ \bullet \mathbf{w}_j^+)\Big] \ .\end{aligned} \tag{6.7.1}$$

Next, consider the special SRBM which is characterized by a superposed constant translational velocity with magnitude u and unit direction $\mathbf{u}$ so that

$$\mathbf{c}(t) = u\,\mathbf{u}\,t \ , \quad \dot{\mathbf{c}} = u\,\mathbf{u} \ , \quad \ddot{\mathbf{c}} = 0 \ , \quad \mathbf{u} \bullet \mathbf{u} = 1 \ , \quad \mathbf{Q} = \mathbf{I} \ , \quad \dot{\mathbf{Q}} = 0 \ , \quad \ddot{\mathbf{Q}} = 0 \ ,$$

$$\mathbf{v}^+ = \mathbf{v} + \mathrm{u}\,\mathbf{u} \ , \quad \dot{\mathbf{v}}^+ = \dot{\mathbf{v}} \ , \quad \mathbf{F}^+ = \mathbf{F} \ , \quad (\mathrm{d}^{1/2})^+ = \mathrm{d}^{1/2} \ , \quad \mathbf{L}^+ = \mathbf{L} \ ,$$
$$\mathbf{w}_i^+ = \mathbf{w}_i \ , \quad \dot{\mathbf{w}}_i^+ = \dot{\mathbf{w}}_i \ , \quad \mathrm{m}^+ = \mathrm{m} \ , \quad \rho^+ = \rho \ , \quad \mathrm{y}^{i+} = \mathrm{y}^i \ , \quad \mathrm{y}^{ij+} = \mathrm{y}^{ij} \ ,$$
$$\mathbf{b}^+ = \mathbf{b} \ , \quad \mathbf{b}^{i+} = \mathbf{b}^i \ , \quad \mathbf{t}^{i+} = \mathbf{t}^i \ , \quad \mathbf{T}^+ = \mathbf{T} \ , \quad \mathcal{P}^+ = \mathcal{P} \ . \tag{6.7.2}$$

Now, substituting (6.7.2) into (6.7.1), and subtracting the equation (6.6.1) from the result, yields the expression

$$\mathrm{u}\,\mathbf{u} \bullet \Big[\frac{\mathrm{d}}{\mathrm{dt}} \{ \mathrm{m}\,(\mathbf{v} + \mathrm{y}^i\,\mathbf{w}_i\,) \} - \mathrm{m}\,\mathbf{b} \Big] + \frac{1}{2}\mathrm{u}^2 \big[\,\dot{\mathrm{m}}\, \big] = 0 \ , \tag{6.7.3}$$

which must be valid for all values of u and the unit vector **u**. Moreover, since the coefficients in (6.7.3) are independent of u and **u**, it follows that each of them must vanish. This procedure yields the conservation of mass $(6.4.5)_1$ and the balance of linear momentum $(6.4.5)_2$.

To derive the form (6.4.6) of the balance of angular momentum, it is convenient to consider a superposed constant rigid body rotation that is characterized by

$$\mathbf{c} = 0 \ , \quad \dot{\mathbf{c}} = 0 \ , \quad \ddot{\mathbf{c}} = 0 \ , \quad \mathbf{Q} = \mathbf{Q}(t) \ , \quad \dot{\mathbf{Q}} = \mathbf{\Omega}\,\mathbf{Q} \ , \quad \dot{\mathbf{\Omega}} = 0 \ ,$$
$$\mathbf{v}^+ = \mathbf{Q}\,\mathbf{v} + \mathbf{\Omega}\,\mathbf{Q}\,\mathbf{x} \ , \quad \dot{\mathbf{v}}^+ = \mathbf{Q}\,\dot{\mathbf{v}} + 2\,\mathbf{\Omega}\,\mathbf{Q}\,\mathbf{v} + \mathbf{\Omega}^2\,\mathbf{Q}\,\mathbf{x} \ ,$$
$$\mathbf{F}^+ = \mathbf{Q}\,\mathbf{F} \ , \quad (\mathrm{d}^{1/2})^+ = \mathrm{d}^{1/2} \ , \quad \mathbf{L}^+ = \mathbf{Q}\,\mathbf{L}\,\mathbf{Q}^T + \mathbf{\Omega} \ ,$$
$$\mathbf{w}_i^+ = \mathbf{Q}\,\mathbf{w}_i + \mathbf{\Omega}\,\mathbf{Q}\,\mathbf{d}_i \ , \quad \dot{\mathbf{w}}_i^+ = \mathbf{Q}\,\dot{\mathbf{w}}_i + 2\,\mathbf{\Omega}\,\mathbf{Q}\,\mathbf{w}_i + \mathbf{\Omega}^2\,\mathbf{Q}\,\mathbf{d}_i \ ,$$
$$\mathrm{m}^+ = \mathrm{m} \ , \quad \rho^+ = \rho \ , \quad \mathrm{y}^{i+} = \mathrm{y}^i \ , \quad \mathrm{y}^{ij+} = \mathrm{y}^{ij} \ ,$$
$$\mathbf{b}^+ = \mathbf{Q}\,\mathbf{b} + 2\,\mathbf{\Omega}\,\mathbf{Q}\,(\mathbf{v} + \mathrm{y}^i\,\mathbf{w}_i\,) + \mathbf{\Omega}^2\,\mathbf{Q}\,(\mathbf{x} + \mathrm{y}^i\,\mathbf{d}_i) \ ,$$
$$\mathbf{b}^{i+} = \mathbf{Q}\,\mathbf{b}^i + 2\,\mathbf{\Omega}\,\mathbf{Q}\,(\mathrm{y}^i\,\mathbf{v} + \mathrm{y}^{ij}\,\mathbf{w}_j\,) + \mathbf{\Omega}^2\,\mathbf{Q}\,(\mathrm{y}^i\,\mathbf{x} + \mathrm{y}^{ij}\,\mathbf{d}_j\,) \ ,$$
$$\mathbf{t}^{i+} = \mathbf{Q}\,\mathbf{t}^i \ , \quad \mathbf{T}^+ = \mathbf{Q}\,\mathbf{T}\,\mathbf{Q}^T \ , \quad \mathcal{P}^+ = \mathcal{P} \ . \tag{6.7.4}$$

Thus, with the help of the definition (6.6.8) and the last of (6.7.4), it follows that

$$\mathcal{P}^+ = \mathbf{T}^+ \bullet \mathbf{L}^+ = \mathcal{P} + \mathbf{T} \bullet (\mathbf{Q}^T\mathbf{\Omega}\,\mathbf{Q}) = \mathcal{P}, \quad \mathbf{T} \bullet (\mathbf{Q}^T\mathbf{\Omega}\,\mathbf{Q}) = 0 \ . \tag{6.7.5}$$

However, since **T** does not depend on **Ω**, and since **Ω** is a skew-symmetric tensor, the quantity $\mathbf{Q}^T\mathbf{\Omega}\,\mathbf{Q}$ is also a skew-symmetric tensor so that (6.7.5) requires **T** to be a symmetric tensor, which yields the form (6.4.12) of the balance of angular momentum. Furthermore, using the results (3.5.6) it can be shown that

$$\begin{aligned}
\mathbf{b}^+ \bullet \mathbf{v}^+ + \mathbf{b}^{i+} \bullet \mathbf{w}_i^+ &= \mathbf{b} \bullet \mathbf{v} + \mathbf{b}^i \bullet \mathbf{w}_i + (\mathbf{Q}^T\boldsymbol{\omega}) \bullet (\mathbf{x} \times \mathbf{b} + \mathbf{d}_i \times \mathbf{b}^i) \\
&+ (\mathbf{\Omega}\,\mathbf{Q}\mathbf{x}) \bullet (\mathbf{\Omega}\,\mathbf{Q}\mathbf{v}) + \mathrm{y}^i\,(\mathbf{\Omega}\,\mathbf{Q}\mathbf{x}) \bullet (\mathbf{\Omega}\,\mathbf{Q}\mathbf{w}_i) \\
&+ \mathrm{y}^i\,(\mathbf{\Omega}\,\mathbf{Q}\mathbf{d}_i) \bullet (\mathbf{\Omega}\,\mathbf{Q}\mathbf{v}) + \mathrm{y}^{ij}\,(\mathbf{\Omega}\,\mathbf{Q}\mathbf{d}_i) \bullet (\mathbf{\Omega}\,\mathbf{Q}\mathbf{w}_j) \ , \\
\mathcal{K}^+ = \mathcal{K} &+ (\mathbf{Q}^T\boldsymbol{\omega}) \bullet \big[\mathbf{x} \times \mathrm{m}\,(\mathbf{v} + \mathrm{y}^i\,\mathbf{w}_i) + \mathbf{d}_i \times \mathrm{m}\,(\mathrm{y}^i\,\mathbf{v} + \mathrm{y}^{ij}\,\mathbf{w}_j\,)\big] \\
&+ \frac{1}{2}\,\mathrm{m}\,\big[(\mathbf{\Omega}\,\mathbf{Q}\mathbf{x}) \bullet (\mathbf{\Omega}\,\mathbf{Q}\mathbf{x}) + 2\,\mathrm{y}^i\,(\mathbf{\Omega}\,\mathbf{Q}\mathbf{x}) \bullet (\mathbf{\Omega}\,\mathbf{Q}\mathbf{d}_i) \\
&+ \mathrm{y}^{ij}\,(\mathbf{\Omega}\,\mathbf{Q}\mathbf{d}_i) \bullet (\mathbf{\Omega}\,\mathbf{Q}\mathbf{d}_j)\big] \ .
\end{aligned} \tag{6.7.6}$$

Also, with the help of (3.5.6), $(3.5.9)_1$ and the conservation of mass equation, it can be shown that

$$\dot{\mathcal{K}}^+ = \dot{\mathcal{K}} + (\mathbf{Q}^T\boldsymbol{\omega}) \bullet \frac{\mathrm{d}}{\mathrm{dt}}\big[\,\mathbf{x} \times \mathrm{m}\,(\mathbf{v} + \mathrm{y}^i\,\mathbf{w}_i\,) + \mathbf{d}_i \times \mathrm{m}\,(\mathrm{y}^i\,\mathbf{v} + \mathrm{y}^{ij}\,\mathbf{w}_j\,)\big]$$

$$+ \mathrm{m} \left[(\boldsymbol{\Omega}\, \mathbf{Q x}) \bullet (\boldsymbol{\Omega}\, \mathbf{Q v}) + y^i\, (\boldsymbol{\Omega}\, \mathbf{Q x}) \bullet (\boldsymbol{\Omega}\, \mathbf{Q w}_i) \right.$$
$$\left. + y^i\, (\boldsymbol{\Omega}\, \mathbf{Q d}_i) \bullet (\boldsymbol{\Omega}\, \mathbf{Q v}) + y^{ij}\, (\boldsymbol{\Omega}\, \mathbf{Q d}_i) \bullet (\boldsymbol{\Omega}\, \mathbf{Q w}_j) \right] \; . \tag{6.7.7}$$

Now, substituting (6.7.6) and (6.7.7) into (6.7.1) and subtracting (6.6.1) from the result, yields the expression

$$(\mathbf{Q}^T \boldsymbol{\omega}) \bullet \; \left[\frac{d}{dt} \{ \mathbf{x} \times \mathrm{m}\, (\mathbf{v} + y^i\, \mathbf{w}_i) + \mathbf{d}_i \times \mathrm{m}\, (y^i\, \mathbf{v} + y^{ij}\, \mathbf{w}_j) \}) \right.$$
$$\left. - \; \mathrm{m}\, (\mathbf{x} \times \mathbf{b} + \mathbf{d}_i \times \mathbf{b}^i) \right] \; . \tag{6.7.8}$$

Furthermore, since (6.7.8) must be valid for all $\boldsymbol{\omega}$, and the coefficient in the square brackets is independent of $\boldsymbol{\omega}$, it follows that (6.7.8) yields the form (6.4.6) of the balance of angular momentum.

In the above it has been shown that the conservation of mass and the balances of linear and angular momentum are necessary conditions for the expression (6.6.1) of mechanical power to remain form invariant under SRBM. However, this procedure does not produce the balances of director momentum.

6.8 Anisotropic nonlinear elastic Cosserat points

In the previous sections, kinematical expressions and balance laws were discussed that are valid for point-like structures that are made from arbitrary materials. Here, constitutive equations will be developed for Cosserat points that are composed of general anisotropic nonlinear elastic materials. Such elastic Cosserat points are considered to be ideal in the same sense that elastic materials are considered to be ideal materials in the three-dimensional theory. For example, the response of an elastic Cosserat point is insensitive to the rate of loading. Other fundamental features of elastic Cosserat points will be discussed presently.

The constitutive equations of elastic Cosserat points can be characterized by the following four assumptions:

Assumption 1: A strain energy Σ per unit mass exists for which

$$\rho\, \dot{\Sigma} = \mathcal{P} = \mathbf{T} \bullet \mathbf{D} \quad . \tag{6.8.1}$$

Assumption 2: The strain energy Σ is a function of the deformation tensor $\mathbf{F}$ only

$$\Sigma = \tilde{\Sigma}\, (\mathbf{F}) \; . \tag{6.8.2}$$

Assumption 3: The strain energy Σ is invariant under SRBM

$$\Sigma^+ = \Sigma \; . \tag{6.8.3}$$

Assumption 4: The kinetic quantity $\mathbf{T}$ is independent of the rate of deformation $\mathbf{L}$.

In order to explore the physical consequences of the assumption (6.8.1), it is convenient to define the total strain energy $\mathcal{U}$ of the Cosserat point

$$\mathcal{U} = \mathrm{m}\, \Sigma \; , \tag{6.8.4}$$

and to use the conservation of mass $(6.4.5)_1$ and the assumption (6.8.1) to deduce that

$$\dot{\mathcal{U}} = \mathrm{m}\, \dot{\Sigma} = d^{1/2}\, \mathcal{P} \quad . \tag{6.8.5}$$

Thus, by substituting (6.8.5) into the mechanical power equation (6.6.1), it is possible to derive the following theorem

$$\mathcal{W} = \dot{\mathcal{K}} + \dot{\mathcal{U}} \ , \tag{6.8.6}$$

which states that for an elastic Cosserat point the rate of work done on the point due to external forces and couples, equals the rate of change of kinetic and strain energies. Since the strain energy Σ depends on the present configuration only through the present value of $\mathbf{F}$ [assumption (6.8.2)], the value of the strain energy Σ is independent of the particular loading path which caused $\mathbf{F}$. Consequently, the total work done on the body vanishes for any closed cycle in which the values of the velocity $\mathbf{v}$, the director velocities $\mathbf{w}_i$, and the deformation tensor $\mathbf{F}$ are the same at the beginning and end of the cycle. Next, consider a special cycle which is composed of a loading path from one state A to another state B, followed by the reversal of this loading path. Then, in view of assumption 4, the work done on the body from A to B is fully recovered during the reverse loading from B to A. In this sense, the elastic Cosserat point is considered to be ideal.

The assumption (6.8.3) places restrictions on the functional form (6.8.2). To develop these restrictions, it is recalled that under SRBM $\mathbf{F}^+=\mathbf{QF}$ so that (6.8.2) requires

$$\Sigma^+ = \tilde{\Sigma}(\mathbf{F}^+) = \tilde{\Sigma}(\mathbf{QF}) = \tilde{\Sigma}(\mathbf{F}) \ , \tag{6.8.7}$$

to hold for arbitrary proper orthogonal $\mathbf{Q}$. However, the polar decomposition theorem states that $\mathbf{F}$ can be separated multiplicatively into a rotation tensor $\mathbf{R}$, and positive definite symmetric stretch tensors $\mathbf{U}$ and $\mathbf{V}$ such that

$$\mathbf{F} = \mathbf{RU} = \mathbf{VR} \ , \quad \mathbf{C} = \mathbf{F}^T\mathbf{F} = \mathbf{C}^T \ , \quad \mathbf{U}^T = \mathbf{U} = \mathbf{C}^{1/2} \ ,$$
$$\mathbf{B} = \mathbf{FF}^T = \mathbf{B}^T \ , \quad \mathbf{V}^T = \mathbf{V} = \mathbf{B}^{1/2} \ , \quad \mathbf{R}^T\mathbf{R} = \mathbf{RR}^T = \mathbf{I} \ , \quad \det \mathbf{R} = 1 \ , \tag{6.8.8}$$

where $\mathbf{C}$ and $\mathbf{B}$ are analogues of the right Cauchy-Green deformation tensor and the left Cauchy-Green deformation tensor in the three-dimensional theory. Following a similar argument to that associated with the three-dimensional theory (sec. 3.7), it can be shown that the strain energy function Σ can depend on $\mathbf{F}$ only through the deformation tensor $\mathbf{C}$ so that Σ necessarily must reduce to the form

$$\tilde{\Sigma}(\mathbf{F}) = \hat{\Sigma}(\mathbf{C}) \ . \tag{6.8.9}$$

Now, with the help of (6.8.1) and (6.8.9) it can be shown that

$$(\mathbf{T} - 2\,\rho\,\mathbf{F}\frac{\partial\Sigma}{\partial\mathbf{C}}\mathbf{F}^T) \bullet \mathbf{D} = 0 \ , \tag{6.8.10}$$

where use has been made of the expression (6.6.7) and (6.6.9) to deduce that

$$\dot{\mathbf{C}} = 2\,\mathbf{F}^T\,\mathbf{D}\,\mathbf{F} \ . \tag{6.8.11}$$

In order to analyze the consequences of equation (6.8.10), it is noted that the coefficient of $\mathbf{D}$ is symmetric and independent of the rate $\mathbf{D}$. Thus, for any fixed value of $\mathbf{F}$, the coefficient in (6.8.10) is fixed even though the rate $\mathbf{D}$ can be chosen arbitrarily. Therefore, the necessary condition that (6.8.10) be valid for arbitrary motions is that the kinetic quantity $\mathbf{T}$ be given by a derivative of the strain energy

$$\mathbf{T} = 2\,\rho\,\mathbf{F}\frac{\partial \Sigma}{\partial \mathbf{C}}\mathbf{F}^T \ . \tag{6.8.12}$$

Next, it follows that comparison of (6.8.12) with the expressions (3.7.16) and (3.7.17) associated with the three-dimensional theory, suggests defining the symmetric tensor **S** such that

$$\mathbf{S} = 2\,\rho_0\frac{\partial \Sigma}{\partial \mathbf{C}} \ , \quad d^{1/2}\,\mathbf{T} = D^{1/2}\,\mathbf{F}\,\mathbf{S}\,\mathbf{F}^T \ . \tag{6.8.13}$$

This causes **S** to be an analogue of the symmetric Piola-Kirchhoff stress in the three-dimensional theory.

Notice that once a form is specified for the strain energy function Σ, then the definition (6.4.11) can be used to obtain

$$\mathbf{t}^i = d^{1/2}\,\mathbf{T}\,\mathbf{d}^i \ , \tag{6.8.14}$$

which determines the kinetic quantities that appear in the form $(6.4.13)_3$ of the balances of director momentum.

Before closing this section, it is desirable to emphasize an important distinction between constitutive equations for a material in the three-dimensional theory, and constitutive equations for a Cosserat point. Within the context of the three-dimensional theory, constitutive equations characterize the response of a material at each material point and are independent of the shape of the three-dimensional body composed of the material. In contrast, within the context of the Cosserat point theory, the constitutive equations necessarily couple influences of the geometry of the point-like structure with those of the response of the three-dimensional material from which the Cosserat point is constructed. For example, consider the case of four point-like structures that are constructed from the same homogeneous nonlinear elastic three-dimensional material. With respect to their stress-free reference configurations, let the first structure be a rectangular parallelepiped, the second structure be a cylinder with uniform triangular cross-sectional area, the third structure be a tetrahedron, and let the fourth structure be an ellipsoid. In general, the coupling of material and geometrical properties of the point-like structure must be modeled by the Cosserat constitutive equations. Therefore, the constitutive equations for these four point-like structures are in general different. However, it will be seen later that, under conditions of homogeneous deformations and homogeneous uniform materials, it is possible to separate these influences of material and geometric properties and develop reasonably simple constitutive equations for the Cosserat point.

Furthermore, it is noted that in section 6.3 the theoretical structure of the balance laws for the Cosserat point theory was developed from the three-dimensional theory by using the kinematic assumption (6.2.5), which allows three-dimensional homogeneous deformation only. From this perspective, constitutive equations for the kinetic quantities (6.8.14) are directly related to integrals through the thickness of the point-like structure of three-dimensional constitutive equations [see (6.3.17)]. However, from the perspective of the direct approach of section 6.4, the balance laws of the Cosserat theory are postulated without any specific connection with the three-dimensional theory. Moreover, the mechanical power equation of section 6.6 and the constitutive equations of this section are also developed by a direct approach. This has the advantage that the constitutive equations are necessarily consistent with the balance laws of the Cosserat theory. In particular, the constitutive equations of nonlinear elastic Cosserat points retain the fundamental properties associated with the ideal character of an elastic material, as discussed previously. Also, since the equations of the Cosserat point theory do not demand that the kinematic assumption (6.2.5) holds pointwise in the point-like structure, the Cosserat theory can be used to model inhomogeneous deformations like free-vibrations of a parallelepiped (Rubin, 1986).

6.9 Constraints

Mechanical constraints in the Cosserat point theory can be considered in a manner directly analogous to that used to analyze constraints in the three-dimensional theory (see sec. 3.8). For example, the usual incompressibility constraint requires

$$\mathrm{J} = \det \mathbf{F} = 1 \ , \tag{6.9.1}$$

which can be differentiated to deduce that

$$\dot{\mathrm{J}} = \mathrm{J}\, \mathbf{F}^{-\mathrm{T}} \bullet \dot{\mathbf{F}} = 0 \ , \quad \mathbf{I} \bullet \mathbf{D} = 0 \ , \tag{6.9.2}$$

where the expressions (6.6.7) and (6.6.9) have been used to obtain the form (6.9.2).

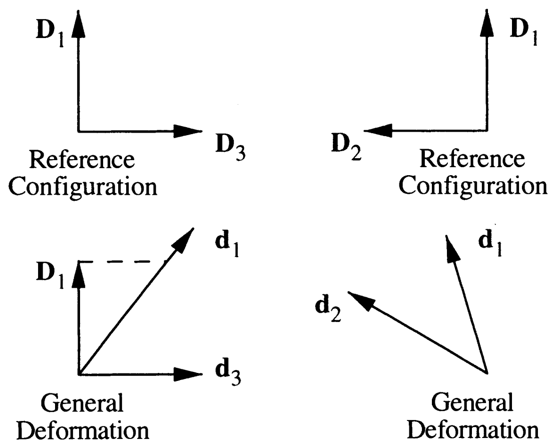

Fig. 6.9.1 Graphical representation of the reference configuration and general deformation of the present configuration.

In general, the Cosserat theory allows the directors $\mathbf{d}_i$ to be arbitrary vectors that are linearly independent (6.2.3). To interpret the physical meaning of different possible deformations of $\mathbf{d}_i$, it is most convenient to identify $\mathbf{d}_i$ with material fibers that were orthogonal to each other in the Cosserat point's reference configuration. Then, in the present configuration the Cosserat point is said to have experienced: *extension* when the magnitudes of $\mathbf{d}_i$ change; and *shear deformation* when the angles between any two $\mathbf{d}_i$ change.

It then follows that extension will be eliminated if the Cosserat point deformation is constrained so that

$$\mathbf{d}_1 \bullet \mathbf{d}_1 = \text{constant} \ , \quad \mathbf{d}_2 \bullet \mathbf{d}_2 = \text{constant} \ , \quad \mathbf{d}_3 \bullet \mathbf{d}_3 = \text{constant} \ . \tag{6.9.3}$$

Taking the material derivative of these expressions and recalling the definition (6.6.6), it can be shown that (6.9.3) can be written in the rate forms

$$(\mathbf{d}_1 \otimes \mathbf{d}_1) \bullet \mathbf{D} = 0 \ , \quad (\mathbf{d}_2 \otimes \mathbf{d}_2) \bullet \mathbf{D} = 0 \ , \quad (\mathbf{d}_3 \otimes \mathbf{d}_3) \bullet \mathbf{D} = 0 \ , \quad \text{to eliminate extension.} \tag{6.9.4}$$

Similarly, it can be shown that shear deformation will be eliminated if

$$\mathbf{d}_1 \bullet \mathbf{d}_2 = \text{constant} \ , \quad \mathbf{d}_1 \bullet \mathbf{d}_3 = \text{constant} \ , \quad \mathbf{d}_2 \bullet \mathbf{d}_3 = \text{constant} \ . \tag{6.9.5}$$

This yields three constraints in rate forms

$$(\mathbf{d}_1 \otimes \mathbf{d}_2 + \mathbf{d}_2 \otimes \mathbf{d}_1) \bullet \mathbf{D} = 0 \ , \quad (\mathbf{d}_1 \otimes \mathbf{d}_3 + \mathbf{d}_3 \otimes \mathbf{d}_1) \bullet \mathbf{D} = 0 \ ,$$
$$(\mathbf{d}_2 \otimes \mathbf{d}_3 + \mathbf{d}_3 \otimes \mathbf{d}_2) \bullet \mathbf{D} = 0 \ , \quad \text{to eliminate shear deformation.} \tag{6.9.6}$$

Figure 6.9.1 shows a graphical representation of the reference configuration and general deformation of the present configuration. Also, Fig. 6.9.2 shows a graphical representation of some of these constraints by indicating the relative magnitude and orientation of the directors $\mathbf{d}_i$.

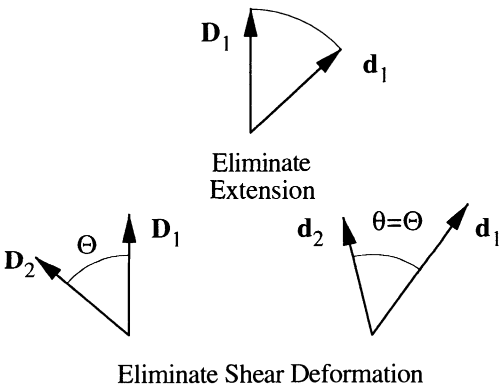

Fig. 6.9.2 Graphical representation of some constraints.

Each of these constraints $(6.9.2)_2$, (6.9.4) and (6.9.6) is a special case of a class of general constraints which require

$$\boldsymbol{\gamma} \bullet \mathbf{D} = 0 \ , \quad \boldsymbol{\gamma}^T = \boldsymbol{\gamma} \ , \tag{6.9.7}$$

where $\boldsymbol{\gamma}$ is a symmetric second order tensor. Moreover, it is assumed that under SRBM, $\boldsymbol{\gamma}$ transforms by

$$\boldsymbol{\gamma}^+ = \mathbf{Q}\, \boldsymbol{\gamma}\, \mathbf{Q}^T \ , \tag{6.9.8}$$

so that the constraint equation $(6.9.7)_1$ remains properly invariant under SRBM.

Motivated by the three-dimensional developments in section 3.8, it is possible to develop a constitutive theory for Cosserat points in the presence of mechanical constraints by making the following five assumptions:

(i) The kinetic quantities $\mathbf{T}$ and $\mathbf{t}^i$ separate additively into two parts

$$\mathbf{T} = \hat{\mathbf{T}} + \bar{\mathbf{T}} \ , \quad \mathbf{t}^i = \hat{\mathbf{t}}^i + \bar{\mathbf{t}}^i \ , \quad \hat{\mathbf{T}} = \mathrm{d}^{-1/2}\, \hat{\mathbf{t}}^i \otimes \mathbf{d}_i \ , \quad \bar{\mathbf{T}} = \mathrm{d}^{-1/2}\, \bar{\mathbf{t}}^i \otimes \mathbf{d}_i \ , \tag{6.9.9}$$

where $\hat{\mathbf{T}}$ and $\hat{\mathbf{t}}^i$ are determined by constitutive equations that characterize the particular unconstrained Cosserat point under consideration, and $\bar{\mathbf{T}}$ and $\bar{\mathbf{t}}^i$ are constraint responses.

(ii) The constraint responses $\bar{\mathbf{T}}$ and $\bar{\mathbf{t}}^i$ are functions of time t which are workless in the sense that

$$\bar{\mathbf{T}} \bullet \mathbf{D} = 0 \ , \tag{6.9.10}$$

for all possible motions of the constrained material.

(iii) Both parts $\hat{\mathbf{T}}$ and $\bar{\mathbf{T}}$ of the kinetic quantity $\mathbf{T}$ are symmetric tensors

$$\hat{\mathbf{T}}^T = \hat{\mathbf{T}} \ , \quad \bar{\mathbf{T}}^T = \bar{\mathbf{T}} \ , \tag{6.9.11}$$

so that each of them satisfies the form (6.4.12) of the balance of angular momentum.

(iv) Both parts $\hat{\mathbf{T}}$ and $\bar{\mathbf{T}}$ of the kinetic quantity $\mathbf{T}$, and $\hat{\mathbf{t}}^i$ and $\bar{\mathbf{t}}^i$ of the kinetic quantity $\mathbf{t}^i$ transform under SRBM by

$$\hat{\mathbf{T}}^+ = \mathbf{Q}\,\hat{\mathbf{T}}\,\mathbf{Q}^T \ , \quad \bar{\mathbf{T}}^+ = \mathbf{Q}\,\bar{\mathbf{T}}\,\mathbf{Q}^T \ , \quad \hat{\mathbf{t}}^{i+} = \mathbf{Q}\,\hat{\mathbf{t}}^i \ , \quad \bar{\mathbf{t}}^{i+} = \mathbf{Q}\,\bar{\mathbf{t}}^i \ , \tag{6.9.12}$$

so that the kinetic quantity $\mathbf{T}$ transforms by (6.5.5), and the expression (6.9.10) is properly invariant under SRBM.

(v) The tensors $\bar{\mathbf{T}}$ and $\boldsymbol{\gamma}$ are independent of the rate $\mathbf{L}$.

Using a Lagrange multiplier $\gamma(t)$ which is an arbitrary function of t, the equation (6.9.10), subject to the constraint (6.9.7), can be rewritten in the form

$$(\bar{\mathbf{T}} - \mathrm{d}^{-1/2}\,\gamma\boldsymbol{\gamma}) \bullet \mathbf{D} = 0 \ . \tag{6.9.13}$$

Since at least one of the components of $\boldsymbol{\gamma}$ in (6.9.7) [say $\boldsymbol{\gamma} \bullet (\mathbf{d}_3 \otimes \mathbf{d}_3)$] is nonzero, the value of γ can be specified so that

$$\gamma = \frac{\bar{\mathbf{T}} \bullet (\mathbf{d}_3 \otimes \mathbf{d}_3)}{\mathrm{d}^{-1/2}\, \boldsymbol{\gamma} \bullet (\mathbf{d}_3 \otimes \mathbf{d}_3)} \ . \tag{6.9.14}$$

It then follows that the coefficient of the component $[\mathbf{D} \bullet (\mathbf{d}_3 \otimes \mathbf{d}_3)]$ in (6.9.13) vanishes. Consequently, this component of $\mathbf{D}$ can be chosen to satisfy the constraint (6.9.7) for arbitrary values of the other components of $\mathbf{D}$. Moreover, since the coefficient of the rate $\mathbf{D}$ in (6.9.13) is symmetric and independent of the rate $\mathbf{D}$, it follows that the constraint response $\bar{\mathbf{T}}$ must take the form

$$\bar{\mathbf{T}} = \mathrm{d}^{-1/2}\,\gamma\boldsymbol{\gamma} \ , \quad \bar{\mathbf{t}}^i = \gamma\,\boldsymbol{\gamma}\,\mathbf{d}^i \ , \tag{6.9.15}$$

where γ remains an arbitrary function of t that is determined by the equations of motions.

If more than one constraint is imposed on the Cosserat point, then the constraint response $\bar{\mathbf{T}}$ can be represented as the sum of a Lagrange multiplier times each of the constraint tensors $\boldsymbol{\gamma}$. For example, the case of a rigid Cosserat point can be obtained by considering all six of the constraints (6.9.4) and (6.9.6) such that

$$(\mathbf{d}_i \otimes \mathbf{d}_j + \mathbf{d}_j \otimes \mathbf{d}_i) \bullet \mathbf{D} = 0 \ . \tag{6.9.16}$$

It then follows that the constraint response $\bar{\mathbf{T}}$ can be represented in the form

$$\bar{\mathbf{T}} = \frac{1}{2} d^{-1/2} \gamma^{ij} (\mathbf{d}_i \otimes \mathbf{d}_j + \mathbf{d}_j \otimes \mathbf{d}_i) \ , \quad \gamma^{ij} = \gamma^{ji} \ , \tag{6.9.17}$$

where γ^{ij} are Lagrange multipliers that are functions of t, with γ^{ij} being symmetric. Thus, with the help of $(5.9.11)_2$ it can be shown that

$$\bar{\mathbf{t}}^i = d^{1/2} \bar{\mathbf{T}} \mathbf{d}^i = \gamma^{ij} \mathbf{d}_j \ . \tag{6.9.18}$$

Also, in view of the constraints (6.9.16), the directors $\mathbf{d}_i$ form a rigid triad which is rotating with angular velocity $\boldsymbol{\omega}$ such that

$$\dot{\mathbf{d}}_i = \boldsymbol{\omega} \times \mathbf{d}_i \ . \tag{6.9.19}$$

Notice that six of the nine components of $\bar{\mathbf{t}}^i$ are arbitrary functions of t. This means that all but three of the nine components of the director momentum equations $(6.4.13)_3$ can be satisfied for arbitrary admissible motions of the constrained Cosserat point by using $(6.9.9)_3$ and (6.9.18) to determine the functions γ^{ij}

$$\gamma^{ij} = \left[m \mathbf{b}^i - \hat{\mathbf{t}}^i - m (y^i \dot{\mathbf{v}} + y^{ik} \dot{\mathbf{w}}_k) \right] \bullet \mathbf{d}^j \ . \tag{6.9.20}$$

Moreover, since γ^{ij} is symmetric, the director momentum equations reduce to three equations (associated with the conditions that $\gamma^{12}=\gamma^{21}$, $\gamma^{13}=\gamma^{31}$, $\gamma^{23}=\gamma^{32}$)

$$\left[m \mathbf{b}^1 - \hat{\mathbf{t}}^1 - m (y^1 \dot{\mathbf{v}} + y^{1j} \dot{\mathbf{w}}_j) \right] \bullet \mathbf{d}^2 = \left[m \mathbf{b}^2 - \hat{\mathbf{t}}^2 - m (y^2 \dot{\mathbf{v}} + y^{2j} \dot{\mathbf{w}}_j) \right] \bullet \mathbf{d}^1 \ ,$$
$$\left[m \mathbf{b}^1 - \hat{\mathbf{t}}^1 - m (y^1 \dot{\mathbf{v}} + y^{1j} \dot{\mathbf{w}}_j) \right] \bullet \mathbf{d}^3 = \left[m \mathbf{b}^3 - \hat{\mathbf{t}}^3 - m (y^3 \dot{\mathbf{v}} + y^{3j} \dot{\mathbf{w}}_j) \right] \bullet \mathbf{d}^1 \ ,$$
$$\left[m \mathbf{b}^2 - \hat{\mathbf{t}}^2 - m (y^2 \dot{\mathbf{v}} + y^{2j} \dot{\mathbf{w}}_j) \right] \bullet \mathbf{d}^3 = \left[m \mathbf{b}^3 - \hat{\mathbf{t}}^3 - m (y^3 \dot{\mathbf{v}} + y^{3j} \dot{\mathbf{w}}_j) \right] \bullet \mathbf{d}^2 \ . \tag{6.9.21}$$

In particular, these equations are used to determine the angular velocity $\boldsymbol{\omega}$ of the rigid Cosserat point. Furthermore, in view of (6.9.9) and (6.9.11) it can be shown that

$$d^{1/2} \hat{\mathbf{T}} \bullet (\mathbf{d}^i \otimes \mathbf{d}^j) = \mathbf{d}^i \bullet \hat{\mathbf{t}}^j = \mathbf{d}^j \bullet \hat{\mathbf{t}}^i = d^{1/2} \hat{\mathbf{T}} \bullet (\mathbf{d}^j \otimes \mathbf{d}^i) \ , \tag{6.9.22}$$

so that (6.9.21) reduces to

$$\left[m \mathbf{b}^1 - m (y^1 \dot{\mathbf{v}} + y^{1j} \dot{\mathbf{w}}_j) \right] \bullet \mathbf{d}^2 = \left[m \mathbf{b}^2 - m (y^2 \dot{\mathbf{v}} + y^{2j} \dot{\mathbf{w}}_j) \right] \bullet \mathbf{d}^1 \ ,$$
$$\left[m \mathbf{b}^1 - m (y^1 \dot{\mathbf{v}} + y^{1j} \dot{\mathbf{w}}_j) \right] \bullet \mathbf{d}^3 = \left[m \mathbf{b}^3 - m (y^3 \dot{\mathbf{v}} + y^{3j} \dot{\mathbf{w}}_j) \right] \bullet \mathbf{d}^1 \ ,$$
$$\left[m \mathbf{b}^2 - m (y^2 \dot{\mathbf{v}} + y^{2j} \dot{\mathbf{w}}_j) \right] \bullet \mathbf{d}^3 = \left[m \mathbf{b}^3 - m (y^3 \dot{\mathbf{v}} + y^{3j} \dot{\mathbf{w}}_j) \right] \bullet \mathbf{d}^2 \ , \tag{6.9.23}$$

which are independent of the constitutive equations for $\hat{\mathbf{t}}^i$, as they should be.

Next, to see that (6.9.23) are equivalent to the usual equations of angular momentum of rigid body dynamics, use is made of (6.4.9) to show that (6.9.23) can be written in the alternative forms

$$\boldsymbol{\varepsilon} \bullet \left[\mathbf{d}_i \otimes \left\{ m \mathbf{b}^i - m (y^i \dot{\mathbf{v}} + y^{ij} \dot{\mathbf{w}}_j) \right\} \right] = 0 \ ,$$
$$\mathbf{d}_i \times \left[m \mathbf{b}^i - m (y^i \dot{\mathbf{v}} + y^{ij} \dot{\mathbf{w}}_j) \right] = 0 \ . \tag{6.9.24}$$

Now, letting $\mathbf{H}_o$ be the angular momentum about the fixed origin, and $\mathbf{M}_o$ be the total moment applied to the point-like structure about that origin, it follows from (6.3.20) and (6.3.21) that the balance of angular momentum can be written in the form

$$\dot{\mathbf{H}}_o = \mathbf{M}_o \ , \tag{6.9.25}$$

where $\mathbf{H}_o$ and $\mathbf{M}_o$ are defined by

$$\mathbf{H}_o = \mathbf{x} \times \mathrm{m}\,(\mathbf{v} + y^i\,\mathbf{w}_i) + \mathbf{d}_i \times \mathrm{m}\,(y^i\,\mathbf{v} + y^{ij}\,\mathbf{w}_j) \ ,$$
$$\mathbf{M}_o = \mathbf{x} \times \mathrm{m}\,\mathbf{b} + \mathbf{d}_i \times \mathrm{m}\,\mathbf{b}^i \ . \tag{6.9.26}$$

Moreover, for notational convenience, let A be the point located by the vector $\mathbf{x}$ and take

$$\mathbf{x}_A = \mathbf{x} \ , \quad \mathbf{v}_A = \mathbf{v} \ . \tag{6.9.27}$$

Also, let $\bar{\mathbf{x}}^*$ be the location, and $\bar{\mathbf{v}}^*$ be the velocity of center of mass of the point-like structure such that

$$\bar{\mathbf{x}}^* - \mathbf{x}_A = y^i\,\mathbf{d}_i \ , \quad \bar{\mathbf{v}}^* = \dot{\bar{\mathbf{x}}}^* \ . \tag{6.9.28}$$

Then, the angular momentum $\mathbf{H}_A$ about the point A, and the total moment $\mathbf{M}_A$ applied to the point-like structure about A become

$$\mathbf{H}_A = \mathbf{d}_i \times \mathrm{m}\,y^{ij}\,\mathbf{w}_j \ , \quad \mathbf{M}_A = \mathbf{d}_i \times \mathrm{m}\,\mathbf{b}^i \ , \tag{6.9.29}$$

so that the expressions (6.9.26) can be rewritten as

$$\mathbf{H}_o = \mathbf{x} \times \mathrm{m}\,(\mathbf{v} + y^i\,\mathbf{w}_i) + (y^i\,\mathbf{d}_i) \times \mathrm{m}\,\mathbf{v} + \mathbf{H}_A \ ,$$
$$\mathbf{M}_o = \mathbf{x} \times \mathrm{m}\,\mathbf{b} + \mathbf{M}_A \ . \tag{6.9.30}$$

Thus, substitution of (6.9.30) into the balance of angular momentum (6.9.25), and use of the conservation of mass $(6.4.13)_1$ and the balance of linear momentum $(6.4.13)_2$, yields the equation

$$(\bar{\mathbf{x}}^* - \mathbf{x}_A) \times \mathrm{m}\,\dot{\mathbf{v}}_A + \dot{\mathbf{H}}_A = \mathbf{M}_A \ , \tag{6.9.31}$$

which can be recognized as the balance of angular momentum of a rigid body about an arbitrary material point A in the body. Moreover, use of the definitions $(6.9.27)_2$, $(6.9.28)_1$ and (6.9.29) reveals that (6.9.31) is equivalent to equation (6.9.24).

It is now of interest to note that once the linear momentum equation $(6.4.13)_2$ and the director momentum equations (6.9.23) have been solved for specified values of $\mathbf{b}$ and $\mathbf{b}^i$, then the motion of the rigid Cosserat point is determined and the functions $\mathbf{x}$ and $\mathbf{d}_i$ are known. Thus, equations (6.9.20) can be used to determine the six unknowns γ^{ij} which determine the constraint response $\bar{\mathbf{T}}$. In particular, for the case of an elastic Cosserat point which is constrained to be rigid, the constitutive equations cause $\hat{\mathbf{T}}$ and $\hat{\mathbf{t}}^i$ to vanish (assuming that they vanish in the unstressed reference configuration) so that $\mathbf{T}$ and $\mathbf{t}^i$ are totally determined for the rigid Cosserat point.

Moreover, as another example, the constraint responses (6.9.15) associated with the incompressibility constraint (6.9.2) become

$$\bar{\mathbf{T}} = \mathrm{d}^{-1/2}\,\gamma\,\mathbf{I} \ , \quad \bar{\mathbf{t}}^i = \gamma\,\mathbf{d}^i \ . \tag{6.9.32}$$

Before closing this section, it should be mentioned that it may be desirable to modify the functional form of the strain energy Σ when constraints are imposed on the Cosserat point. For example, consider a rectangular parallelepiped whose surfaces are normal to the rectangular Cartesian base vectors $\mathbf{e}_i$. Moreover, let $\mathbf{d}_i$ be the directors associated with the line elements which in the reference configuration were oriented in the $\mathbf{D}_i = \mathbf{e}_i$ directions. If the parallelepiped is pulled in uniaxial tension in the $\mathbf{e}_3$ direction, then the Poisson effect usually causes its lengths in the $\mathbf{e}_1$ and $\mathbf{e}_2$ directions to decrease. Now, if for simplicity the parallelepiped is constrained so that the directors $\mathbf{d}_\alpha$ are not allowed to extend, then the predicted response to uniaxial tension in the $\mathbf{e}_3$ direction will be too stiff if the same functional form for the strain energy Σ is used in the constrained theory. To avoid this problem, it is possible to modify the constitutive equation for Σ so that the response of the constrained theory will simulate the softer response of the unconstrained Cosserat point. Moreover, it will be shown later that the director inertia coefficients y^i and y^{ij} model not only the mass distribution through the thickness of the Cosserat point, but also information about mode shapes of vibration. Therefore, the appropriate values of y^i and y^{ij} for a deformable Cosserat point can be different from those for a constrained Cosserat point (e.g. a rigid Cosserat point).

For the special case of a constrained elastic Cosserat point, the parts $\hat{\mathbf{T}}$ and $\hat{\mathbf{t}}^i$ of the kinetic quantities associated with constitutive equations satisfy the condition (6.8.1) that the mechanical power due to these parts is equal to the rate of change of the strain energy function. Then, $\hat{\mathbf{T}}$ and $\hat{\mathbf{t}}^i$ are determined in terms of derivatives of the strain energy function by formulas of the type (6.8.12).

6.10 Initial conditions

In general, the balance laws of the three-dimensional theory are partial differential equations that require both initial and boundary conditions. However, due to the simplified nature of the Cosserat point theory, the resulting balance laws become functions of time only. This is because the effects of the surface tractions applied to the boundary of the point-like structure are absorbed into the definitions of the external force $\mathbf{b}_c$ (6.3.8) and external couples $\mathbf{b}_c^i$ (6.3.16) which are assumed to be prescribed. Therefore, the conservation of mass $(6.4.13)_1$, the balance of linear momentum $(6.4.13)_2$ and the balances of director momentum $(6.4.13)_3$ are ordinary differential equations which require initial conditions only.

Specifically, the conservation of mass $(6.4.13)_1$ is first order in time so it is necessary to specify the initial value of the mass m (or the mass density ρ) of the Cosserat point

$$m(0) = \bar{m} \ , \quad \rho(0) = \bar{\rho} \ , \quad \text{for } t = 0 \ . \tag{6.10.1}$$

Also, the balance of linear momentum $(6.4.13)_2$ and the balances of director momentum $(6.4.13)_3$ are second order in time with respect to position $\mathbf{x}$ and the directors $\mathbf{d}_i$ so that it is necessary to specify the initial values of $\mathbf{x}$ and $\mathbf{d}_i$ as well as the initial values of the velocities $\mathbf{v}$ and $\mathbf{w}_i$

$$\hat{\mathbf{x}}(0) = \bar{\mathbf{x}} \ , \quad \hat{\mathbf{v}}(0) = \bar{\mathbf{v}} \ , \quad \hat{\mathbf{d}}_i(0) = \bar{\mathbf{d}}_i \ , \quad \hat{\mathbf{w}}_i(0) = \bar{\mathbf{w}}_i \ , \quad \text{for } t = 0 \ , \tag{6.10.2}$$

where an overbar is temporarily used here to denote specified initial values of the functions. Moreover, it is noted that since the conservation of mass equation requires the mass to be constant for all time, specification of the initial value of mass is the same as specifying the mass at anytime.

6.11 Further restrictions on constitutive equations for Cosserat points constructed from homogeneous anisotropic nonlinear elastic materials

From the developments in section 6.3 it can be seen that the intrinsic director couples $\mathbf{t}^i$ (6.3.17) are related to weighted integrals of the stress vector through the thickness of the point-like structure. This means that the constitutive equations (6.8.12) and (6.8.14) for $\mathbf{t}^i$ necessarily couple the effects of geometric properties of the point-like structure with the constitutive properties of the material from which the point-like structure is constructed. However, due to the simplified nature of the kinematic assumption (6.2.5), it will be seen that these two effects can be separated out for the case of point-like structures constructed from uniform homogeneous anisotropic nonlinear elastic materials. In particular, following the work of Naghdi and Rubin (1995) for shells and Rubin (1996) for rods, restrictions on constitutive equations for such Cosserat points will be developed here which ensure that solutions of the Cosserat point theory can reproduce exactly the complete class of homogeneous solutions of the three-dimensional theory.

First, it is recalled from (3.2.34), (6.2.6) and (6.3.17) that

$$\begin{aligned} d^{1/2}\,\mathbf{T} = \mathbf{t}^i \otimes \mathbf{d}_i &= \int_{P^*} g^{1/2}\,\mathbf{T}^* \left[\, \mathbf{g}^i \otimes \mathbf{d}_i \,\right] d\theta^1\, d\theta^2\, d\theta^3 \\ &= \int_{P^*} g^{1/2}\,\mathbf{T}^* \left[\, \mathbf{g}^i \otimes \mathbf{g}_i \,\right] d\theta^1\, d\theta^2\, d\theta^3 \ . \end{aligned} \tag{6.11.1}$$

However, since $\mathbf{g}^i \otimes \mathbf{g}_i = \mathbf{I}$, it follows that

$$d^{1/2}\,\mathbf{T} = \int_{P^*} g^{1/2}\,\mathbf{T}^*\, d\theta^1\, d\theta^2\, d\theta^3 \ . \tag{6.11.2}$$

Next, with the help of (3.2.28), (3.7.15) and (6.8.12) it can be shown that

$$d^{1/2}\,\mathbf{T} = 2\,m\,\mathbf{F}\frac{\partial \Sigma}{\partial \mathbf{C}}\mathbf{F}^T \ , \quad g^{1/2}\,\mathbf{T}^* = 2\,m^*\,\mathbf{F}^*\frac{\partial \Sigma^*}{\partial \mathbf{C}^*}\mathbf{F}^{*T} \ , \tag{6.11.3}$$

so that equation (6.11.2) can be rewritten in the form

$$\begin{aligned} d^{1/2}\,\mathbf{T} = 2\,m\,\mathbf{F}\frac{\partial \Sigma}{\partial \mathbf{C}}\mathbf{F}^T &= 2\int_{P^*} m^*\,\mathbf{F}^*\frac{\partial \Sigma^*}{\partial \mathbf{C}^*}\mathbf{F}^{*T}\, d\theta^1\, d\theta^2\, d\theta^3 \ , \\ m\frac{\partial \Sigma}{\partial \mathbf{C}} &= \int_{P^*} m^*\,\mathbf{F}^{-1}\,\mathbf{F}^*\frac{\partial \Sigma^*}{\partial \mathbf{C}^*}\mathbf{F}^{*T}\,\mathbf{F}^{-T}\, d\theta^1\, d\theta^2\, d\theta^3 \ . \end{aligned} \tag{6.11.4}$$

Now, if the material from which the point-like structure is constructed is uniform and homogeneous, then ρ_0^* and Σ^* are explicitly independent of the coordinates θ^i

$$\rho_0^* = \text{constant} \ , \quad \Sigma^* = \hat{\Sigma}^*(\mathbf{C}^*) \ . \tag{6.11.5}$$

Thus, with the help of (3.2.28) and (6.3.1) it can be shown that

$$m = \int_{P^*} m^* \, d\theta^1 \, d\theta^2 \, d\theta^3 = \rho_0^* V \ , \quad D^{1/2} V = \int_{P^*} G^{1/2} \, d\theta^1 \, d\theta^2 \, d\theta^3 \ , \tag{6.11.6}$$

where V is related to the reference volume of the Cosserat point. Next, attention is restricted to three-dimensionally homogeneous deformations (6.2.8) for which

$$\mathbf{F}^* = \mathbf{F} = \mathbf{F}(t) \ , \quad \mathbf{C}^* = \mathbf{C} = \mathbf{F}^T \mathbf{F} \ , \tag{6.11.7}$$

so that (6.11.4) yields the restriction

$$\frac{\partial \Sigma}{\partial \mathbf{C}} = \frac{\partial \hat{\Sigma}^*(\mathbf{C})}{\partial \mathbf{C}} \ . \tag{6.11.8}$$

The expression (6.11.8) places a necessary restriction on the strain energy function for Cosserat points which ensures consistency with exact solutions for all homogeneous deformations. Specifically, it will presently be shown that for homogeneous deformations the restriction (6.11.8) also causes

$$m \, \mathbf{b}_c = 0 \ , \quad m \, \mathbf{b}_c^i - \mathbf{t}^i = 0 \ , \tag{6.11.9}$$

where the external force $\mathbf{b}_c$ and couples $\mathbf{b}_c^i$ due to tractions on the surface of the point-like structure are given by the expressions (6.3.8) and (6.3.16). Thus, the balances of linear momentum and director momentum (6.4.13) reduce to the equations

$$m \, (\dot{\mathbf{v}} + y^i \, \dot{\mathbf{w}}_i) = m \, \mathbf{b}_b \ , \quad m \, (y^i \, \dot{\mathbf{v}} + y^{ij} \, \dot{\mathbf{w}}_j) = m \, \mathbf{b}_b^i \ , \tag{6.11.10}$$

which express the familiar result that for homogeneous deformations the accelerations are balanced by terms associated with body forces only.

To prove equations (6.11.9), it is first observed using (3.2.7), $(3.2.33)_1$ and (3.7.15), that for homogeneous deformations

$$J^* \, \mathbf{T}^* = \left[2 \, \rho_0^* \, \mathbf{F} \, \frac{\partial \hat{\Sigma}^*(\mathbf{C})}{\partial \mathbf{C}} \, \mathbf{F}^T \right] \ . \tag{6.11.11}$$

Thus, for homogeneous deformations $\mathbf{T}^*$ is independent of the coordinates θ^i so that (6.2.6), (6.8.14) and (6.11.2) can be used to deduce that

$$\mathbf{t}^i = d^{1/2} \, \mathbf{T} \, \mathbf{d}^i = \int_{P^*} g^{1/2} \, \mathbf{T}^* \, \mathbf{d}^i \, d\theta^1 \, d\theta^2 \, d\theta^3 = \int_{P^*} g^{1/2} \, \mathbf{T}^* \, \mathbf{g}^i \, d\theta^1 \, d\theta^2 \, d\theta^3 \ , \tag{6.11.12}$$

which with the help of $(3.2.34)_1$, can be seen to be consistent with the expression (6.3.17). To complete the proof, it is recalled from the definition (2.3.16) of the divergence operator that for an arbitrary function

$$\phi = \phi(\theta^j) \ , \tag{6.11.13}$$

it can be shown that

$$\phi \, \mathrm{div}^* \, \mathbf{T}^* = \mathrm{div}^*(\phi \, \mathbf{T}^*) - \mathbf{T}^* \bullet \frac{d\phi}{d\theta^j} \, \mathbf{g}^j \ . \tag{6.11.14}$$

Thus, integrating (6.11.14) over the polyhedron P^* characterizing the point-like structure, it follows that

$$\int_{P^*} \phi \, \mathrm{div}^* \, \mathbf{T}^* \, dv^* = - \int_{P^*} \mathbf{T}^* \bullet \frac{d\phi}{d\theta^j} \, \mathbf{g}^j \, dv^* + \sum_{J=1}^{S} \int_{\partial P_J^*} \phi \, \mathbf{t}^* \, da^* \ , \tag{6.11.15}$$

where the divergence theorem has been used to convert the volume integral of $\mathrm{div}^*(\phi\, \mathbf{T}^*)$ to a integral over the boundary of the region. However, for homogeneous deformations $\mathrm{div}^*\, \mathbf{T}^*$ vanishes and (6.11.15) yields the equation

$$\sum_{J=1}^{S} \int_{\partial P_J^*} \mathbf{t}^*\, da^* = 0 \ , \tag{6.11.16}$$

for $\phi = 1$, and the equation

$$\sum_{J=1}^{S} \int_{\partial P_J^*} \theta^i\, \mathbf{t}^*\, da^* - \int_{P^*} \mathbf{T}^*\, \mathbf{g}^i\, dv^* = 0 \ . \tag{6.11.17}$$

for $\phi = \theta^i$. Thus, with the help of the definitions (6.3.7), (6.3.8) for $m\mathbf{b}_c$, (6.3.15), and (6.3.16) for $m\mathbf{b}_c^i$, it can be seen that (6.11.12), (6.11.16) and (6.11.17) prove the validity of (6.11.9) for homogeneous deformations.

In summary, it is noted that the constitutive restriction (6.11.8) can easily be satisfied by expressing the strain energy Σ of the Cosserat point as a function of the three-dimensional strain energy Σ^* such that

$$\Sigma = \hat{\Sigma}^*(\mathbf{C}) \ . \tag{6.11.18}$$

Then, the constitutive equations (6.8.12) and (6.8.14) for $\mathbf{T}$ and $\mathbf{t}^i$ become

$$\mathbf{T} = 2\,\rho\, \mathbf{F} \frac{\partial \Sigma^*(\mathbf{C})}{\partial \mathbf{C}} \mathbf{F}^T \ , \quad \mathbf{t}^i = 2\, m\, \mathbf{F} \frac{\partial \Sigma^*(\mathbf{C})}{\partial \mathbf{C}} \mathbf{D}^i \ . \tag{6.11.19}$$

6.12 A small strain theory

In order to better understand the nature of the constitutive assumption (6.11.19), it is of interest to consider the simpler case of a small strain theory where the three-dimensional strain energy function is given by the form (3.12.2) which is a quadratic function of the Lagrangian strain. Moreover, with the help of the conservation of mass $(6.4.13)_1$ and the definition (6.11.6), it can be shown that when the three-dimensional mass density is independent of the coordinates θ^i in the region of the Cosserat point that

$$m = \rho_0^*\, V\, D^{1/2} \ . \tag{6.12.1}$$

Then, in view of the assumption (6.11.18), the strain energy function is expressed in the form

$$m\,\Sigma = \frac{1}{2} V\, D^{1/2}\, \mathbf{K}^* \bullet (\mathbf{E} \otimes \mathbf{E}) \ , \tag{6.12.2}$$

where it is convenient to define the strain $\mathbf{E}$ by

$$\mathbf{E} = \frac{1}{2} (\mathbf{C} - \mathbf{I}) \ . \tag{6.12.3}$$

Also, $\mathbf{K}^*$ is the value associated with the three-dimensional material which is evaluated at the point $(\theta^i{=}0)$.

Next, using the definitions (6.11.19) and (6.12.3) it can be shown that

$$m\,\Sigma = \frac{1}{2} V\, D^{1/2}\, \mathbf{K}^* \bullet (\mathbf{E} \otimes \mathbf{E}) \ , \quad 2m \frac{\partial \Sigma}{\partial \mathbf{C}} = V\, D^{1/2}\, \mathbf{K}^* \bullet \mathbf{E} \ ,$$

$$d^{1/2}\, \mathbf{T} = V\, D^{1/2}\, \mathbf{F} \left[\mathbf{K}^* \bullet \mathbf{E}\right] \mathbf{F}^T \ , \quad \mathbf{t}^i = V\, D^{1/2}\, \mathbf{F} \left[\mathbf{K}^* \bullet \mathbf{E}\right] \mathbf{D}^i \ . \tag{6.12.4}$$

Since these equations satisfy the restrictions (6.11.8), they ensure consistency with exact solutions for all homogeneous deformations when the strain energy function of the three-dimensional material is given by (3.12.2). Moreover, the constitutive equations (6.12.4) remain properly invariant under superposed rigid body motions. This means that the strain energy function (6.12.2) can be viewed as a special simple constitutive assumption that is valid for large deformations and large rotations of the Cosserat point, but small strains.

In general, the vectors $\mathbf{D}_i$ are not orthonormal. Consequently, it is convenient to introduce a right-handed orthonormal set of vectors $\mathbf{M}_i$ defined by

$$\mathbf{M}_1 = \mathbf{M}_2 \times \mathbf{M}_3 \ , \quad \mathbf{M}_2 = \frac{\mathbf{M}_3 \times \mathbf{D}_1}{|\mathbf{M}_3 \times \mathbf{D}_1|} \ , \quad \mathbf{M}_3 = \frac{\mathbf{D}_3}{|\mathbf{D}_3|} \ . \tag{6.12.5}$$

This specification causes $\mathbf{M}_1$ and $\mathbf{M}_2$ to lie in the plane normal to $\mathbf{D}_3$, with $\mathbf{M}_2$ being normal to the director $\mathbf{D}_1$. Then, using these definitions the components of $\mathbf{E}$ and $\mathbf{K}^*$ relative $\mathbf{M}_i$ are defined by

$$\mathbf{E} = E_{ij} \, (\mathbf{M}_i \otimes \mathbf{M}_j) \ , \quad E_{ij} = \mathbf{E} \bullet (\mathbf{M}_i \otimes \mathbf{M}_j) \ ,$$

$$\mathbf{K}^* = K^*_{ijkl}(\mathbf{M}_i \otimes \mathbf{M}_j \otimes \mathbf{M}_k \otimes \mathbf{M}_l) \ , \quad K^*_{ijkl} = \mathbf{K}^* \bullet (\mathbf{M}_i \otimes \mathbf{M}_j \otimes \mathbf{M}_k \otimes \mathbf{M}_l) \ . \tag{6.12.6}$$

For the case of an orthotropic material relative to the basis $\mathbf{M}_i$, the nine nontrivial components of K^*_{ijkl} are given by (3.12.12). Whereas, for the case of an isotropic material, the values of K^*_{ijkl} are given by (3.12.13), and the indicial form of the strain energy function (6.12.2) can be written as

$$2m\Sigma = V \, D^{1/2} \left[(K^* - \tfrac{2}{3}\mu^*) \, E_{ii} \, E_{jj} + 2\mu^* \, E_{ij} \, E_{ij} \right] . \tag{6.12.7}$$

6.13 Small deformations superimposed on a large deformation

In general, the equations of motion of an elastic Cosserat point are nonlinear ordinary differential equations which can be quite difficult to solve analytically. The notion of small deformations superimposed on a large deformation is used to develop approximate equations that are linear functions of the superimposed small deformations, and therefore are simpler to solve (e.g. Green et al, 1968). Such equations can be used to analyze vibrations of pre-stressed or rotating structures. Moreover, if the large deformation represents an actual solution of the equations of motion, then the small deformation equations can be used to analyze linear stability of the large deformation solution.

To develop these small deformation equations, the position vector $\mathbf{x}(t)$ is represented as an additive function of the large deformation $\hat{\mathbf{x}}(t)$ and the small displacement vector $\mathbf{u}(t)$. Also, the directors $\mathbf{d}_i(t)$ are represented as additive functions of the large deformations $\hat{\mathbf{d}}_i(t)$ and the small director displacements $\boldsymbol{\delta}_i(t)$ such that

$$\mathbf{x}(t) = \hat{\mathbf{x}}(t) + \mathbf{u}(t) \ , \quad \mathbf{d}_i(t) = \hat{\mathbf{d}}_i(t) + \boldsymbol{\delta}_i(t) \ . \tag{6.13.1}$$

The displacement vector $\mathbf{u}$ and the director displacements $\boldsymbol{\delta}_i$ are considered to be small in the sense that their magnitudes and the magnitudes of their time derivatives are small enough that quadratic and higher order terms in these quantities can be neglected. Thus, for example

$$|\mathbf{u}|^2 << |\mathbf{u}| \ , \quad |\boldsymbol{\delta}_i|^2 << |\boldsymbol{\delta}_i| \ . \tag{6.13.2}$$

Of course, the values of $\mathbf{u}$, $\boldsymbol{\delta}_i$, and their time derivatives must be appropriately normalized in order to express these inequalities in unitless forms.

Quantities other than the displacement vector $\mathbf{u}$ and the director displacements $\boldsymbol{\delta}_i$ are separated additively into a part associated with the large deformation which is denoted by placing a hat ($\hat{}$) over the symbol, and a part associated with the small deformation which is denoted by placing a tilde ($\tilde{}$) over the same symbol. For example, in general the external force associated with the large deformation to be analyzed can be nonzero. Thus, the external force $\mathbf{b}$ is represented in the form

$$\mathbf{b} = \hat{\mathbf{b}} + \tilde{\mathbf{b}} \ . \tag{6.13.3}$$

In order to develop the equations of motion of the small deformation associated with (6.4.13), it is necessary to substitute (6.13.1) into the constitutive equations (6.8.14) and to expand the resulting quantities in a Taylor series to develop expressions for the vectors $\mathbf{t}^i$

$$\mathbf{t}^i = \hat{\mathbf{t}}^i + \tilde{\mathbf{t}}^i \ . \tag{6.13.4}$$

To this end, it is noted that

$$\begin{gathered}
\mathbf{F} = \hat{\mathbf{F}} + \tilde{\mathbf{F}} \ , \quad \hat{\mathbf{F}} = \hat{\mathbf{d}}_i \otimes \mathbf{D}^i \ , \quad \tilde{\mathbf{F}} = \boldsymbol{\delta}_i \otimes \mathbf{D}^i \ , \\
\mathbf{F}^{-1} = \hat{\mathbf{F}}^{-1} + \tilde{\mathbf{F}}^{-1} \ , \quad \hat{\mathbf{F}}^{-1} = \mathbf{D}_i \otimes \hat{\mathbf{d}}^i \ , \quad \tilde{\mathbf{F}}^{-1} = -\hat{\mathbf{F}}^{-1}\boldsymbol{\delta}_i \otimes \hat{\mathbf{d}}^i \ , \\
\tilde{\mathbf{F}}\,\hat{\mathbf{F}}^{-1} = \boldsymbol{\delta}_i \otimes \hat{\mathbf{d}}^i \ , \quad \tilde{\mathbf{F}}^{-T}\,\hat{\mathbf{F}}^{T} = -\hat{\mathbf{d}}^i \otimes \boldsymbol{\delta}_i \ , \quad \mathbf{C} = \hat{\mathbf{C}} + \tilde{\mathbf{C}} \ , \quad \hat{\mathbf{C}} = \hat{\mathbf{F}}^T\hat{\mathbf{F}} \ , \\
\tilde{\mathbf{C}} = \hat{\mathbf{F}}^T(\boldsymbol{\delta}_i \otimes \mathbf{D}^i) + (\mathbf{D}^i \otimes \boldsymbol{\delta}_i)\hat{\mathbf{F}} \ , \quad \mathbf{E} = \hat{\mathbf{E}} + \tilde{\mathbf{E}} \ , \quad \hat{\mathbf{E}} = \frac{1}{2}(\hat{\mathbf{F}}^T\hat{\mathbf{F}} - \mathbf{I}) \ , \quad \tilde{\mathbf{E}} = \frac{1}{2}\tilde{\mathbf{C}} \ , \\
\hat{d}^{1/2} = \hat{\mathbf{d}}_1 \times \hat{\mathbf{d}}_2 \bullet \hat{\mathbf{d}}_3 \ , \quad \hat{\mathbf{d}}^1 = \hat{d}^{-1/2}\,[\hat{\mathbf{d}}_2 \times \hat{\mathbf{d}}_3] \ , \quad \hat{\mathbf{d}}^2 = \hat{d}^{-1/2}\,[\hat{\mathbf{d}}_3 \times \hat{\mathbf{d}}_1] \ , \quad \hat{\mathbf{d}}^3 = \hat{d}^{-1/2}\,\hat{\mathbf{d}}_1 \times \hat{\mathbf{d}}_2 \ , \\
\mathbf{d}^i = \hat{\mathbf{d}}^i + \tilde{\mathbf{d}}^i \ , \quad \tilde{\mathbf{d}}^i = -\,[\hat{\mathbf{d}}^i \bullet \boldsymbol{\delta}_j\,]\,\hat{\mathbf{d}}^j \ , \quad d^{1/2} = \hat{d}^{1/2} + \tilde{d}^{1/2} \ , \quad \tilde{d}^{1/2} = \hat{d}^{1/2}\,\hat{\mathbf{d}}^i \bullet \boldsymbol{\delta}_i \ ,
\end{gathered} \tag{6.13.5}$$

where the symbol $\tilde{\mathbf{F}}^{-1}$ does not denote the inverse of $\tilde{\mathbf{F}}$. Next, the conservation of mass $(6.4.13)_1$ can be written in the form

$$m = \rho_0\, D^{1/2} = \rho\, d^{1/2} = \hat{\rho}\,\hat{d}^{1/2} \ , \quad \rho = \hat{\rho}\,[1 - \hat{\mathbf{d}}^i \bullet \boldsymbol{\delta}_i\,] \ . \tag{6.13.6}$$

Then, expanding Σ in a Taylor series and neglecting quadratic terms in the small deformation quantities yields

$$\frac{\partial \Sigma}{\partial \mathbf{C}} = \frac{\partial \Sigma}{\partial \hat{\mathbf{C}}} + \frac{\partial \Sigma}{\partial \tilde{\mathbf{C}}} \ , \tag{6.13.7}$$

where the first term on the right-hand side is evaluated taking $\mathbf{C}=\hat{\mathbf{C}}$, and the second term is first order in the small deformation quantities. Specifically, for the functional form (6.11.18)

$$m\frac{\partial \Sigma}{\partial \hat{\mathbf{C}}} = \Big[m\frac{\partial \Sigma^*(\hat{\mathbf{C}})}{\partial \hat{\mathbf{C}}}\Big] \ , \quad m\frac{\partial \Sigma}{\partial \tilde{\mathbf{C}}} = \Big[m\frac{\partial^2 \Sigma^*(\hat{\mathbf{C}})}{\partial \hat{\mathbf{C}} \otimes \partial \hat{\mathbf{C}}}\Big] \bullet \tilde{\mathbf{C}} \ , \tag{6.13.8}$$

where the derivatives of Σ^* are evaluated taking $\mathbf{C}=\hat{\mathbf{C}}$. If the three-dimensional strain energy function Σ^* is a quadratic function of strain (3.12.2), then these equations simplify somewhat with

$$m\frac{\partial\Sigma^*(\hat{\mathbf{C}})}{\partial\hat{\mathbf{C}}} = \frac{1}{2}V\,D^{1/2}\,\mathbf{K}^*\bullet\hat{\mathbf{E}}\ ,\quad m\frac{\partial^2\Sigma^*(\hat{\mathbf{C}})}{\partial\hat{\mathbf{C}}\otimes\partial\hat{\mathbf{C}}} = \frac{1}{4}V\,D^{1/2}\,\mathbf{K}^*\ . \tag{6.13.9}$$

For either of these cases, (6.13.8) can be used to expand the constitutive equations (6.11.19) to deduce that

$$\hat{\mathbf{t}}^i = 2\,m\,\hat{\mathbf{F}}\frac{\partial\Sigma}{\partial\hat{\mathbf{C}}}\mathbf{D}^i\ ,\quad \tilde{\mathbf{t}}^i = 2\,m\,\hat{\mathbf{F}}\frac{\partial\Sigma}{\partial\tilde{\mathbf{C}}}\,\mathbf{D}^i + (\boldsymbol{\delta}_j\otimes\hat{\mathbf{d}}^j)\,\hat{\mathbf{t}}^i\ . \tag{6.13.10}$$

Also, the expression (6.4.11) expands to give

$$\mathbf{T} = \hat{\mathbf{T}} + \tilde{\mathbf{T}}\ ,\quad \hat{\mathbf{T}} = \hat{d}^{-1/2}\,\hat{\mathbf{t}}^i\otimes\hat{\mathbf{d}}_i\ ,$$

$$\tilde{\mathbf{T}} = \hat{d}^{-1/2}\,[\,\hat{\mathbf{t}}^i\otimes\boldsymbol{\delta}_i + \tilde{\mathbf{t}}^i\otimes\hat{\mathbf{d}}_i\,] - \hat{d}^{-1/2}\,(\hat{\mathbf{d}}^j\bullet\boldsymbol{\delta}_j)\,\hat{\mathbf{t}}^i\otimes\hat{\mathbf{d}}_i\ . \tag{6.13.11}$$

Next, substitution of (6.13.10) into (6.13.11) yields

$$\hat{\mathbf{T}} = 2\,\hat{\rho}\,\hat{\mathbf{F}}\frac{\partial\Sigma}{\partial\hat{\mathbf{C}}}\,\hat{\mathbf{F}}^T\ ,$$

$$\tilde{\mathbf{T}} = 2\,\hat{\rho}\,\hat{\mathbf{F}}\frac{\partial\Sigma}{\partial\tilde{\mathbf{C}}}\,\hat{\mathbf{F}}^T + (\boldsymbol{\delta}_j\otimes\hat{\mathbf{d}}^j)\,\hat{\mathbf{T}} + \hat{\mathbf{T}}\,(\hat{\mathbf{d}}^j\otimes\boldsymbol{\delta}_j) - (\hat{\mathbf{d}}^j\bullet\boldsymbol{\delta}_j)\,\hat{\mathbf{T}}\ , \tag{6.13.12}$$

which can be seen to be equivalent to a direct expansion of the constitutive equation (6.8.12). Moreover, since both the terms $\hat{\mathbf{T}}$ and $\tilde{\mathbf{T}}$ are symmetric tensors, the reduced form (6.4.12) of the balance of angular momentum is satisfied by the small deformation terms.

Using these expressions, the balances of linear momentum and director momentum (6.4.13) become

$$m\,(\ddot{\mathbf{u}} + y^i\,\ddot{\boldsymbol{\delta}}_i) - m\,\tilde{\mathbf{b}} = -\,[\,m\,(\ddot{\hat{\mathbf{x}}} + y^i\,\ddot{\hat{\mathbf{d}}}_i) - m\,\hat{\mathbf{b}}\,]\ ,$$

$$m\,(y^i\,\ddot{\mathbf{u}} + y^{ij}\,\ddot{\boldsymbol{\delta}}_j) - m\,\tilde{\mathbf{b}}^i + \tilde{\mathbf{t}}^i = -\,[\,m\,(y^i\,\ddot{\hat{\mathbf{x}}} + y^{ij}\,\ddot{\hat{\mathbf{d}}}_j) - m\,\hat{\mathbf{b}}^i + \hat{\mathbf{t}}^i\,]\ . \tag{6.13.13}$$

With $\hat{\mathbf{x}}$, $\hat{\mathbf{d}}_i$, $\hat{\mathbf{b}}$, $\hat{\mathbf{b}}^i$, $\tilde{\mathbf{b}}$ and $\tilde{\mathbf{b}}^i$ specified these equations become linear equations to determine the displacement fields $\mathbf{u}$ and $\boldsymbol{\delta}_i$. Moreover, the equations must be supplemented by initial conditions. Also, when $\hat{\mathbf{x}}$ and $\hat{\mathbf{d}}_i$ are the solutions of the nonlinear linear momentum and director momentum equations, then the right hand sides of (6.13.13) vanish.

As a special case consider a homogeneous material and let $\hat{\mathbf{x}}$ and $\hat{\mathbf{d}}_i$ be associated with a static homogeneous solution with vanishing body force such that

$$\hat{\mathbf{x}} = \hat{\mathbf{F}}\,\mathbf{X}\ ,\quad \hat{\mathbf{d}}_i = \hat{\mathbf{F}}\,\mathbf{D}_i\ ,\quad \dot{\hat{\mathbf{F}}} = 0\ ,\quad \hat{\mathbf{b}} = \hat{\mathbf{b}}_c\ ,\quad \hat{\mathbf{b}}^i = \hat{\mathbf{b}}^i_c\ , \tag{6.13.14}$$

where $\hat{\mathbf{b}}_c$ and $\hat{\mathbf{b}}^i_c$ are specified by the forms (6.3.8) and (6.3.16), respectively, with $\mathbf{F}^*=\hat{\mathbf{F}}$. It then follows from the results (6.11.9) and (6.11.10) that the right-hand sides of (6.13.13) vanish so the linear momentum and director momentum equations reduce to

$$m\,(\ddot{\mathbf{u}} + y^i\,\ddot{\boldsymbol{\delta}}_i) = m\,\tilde{\mathbf{b}} \ , \quad m\,(y^i\,\ddot{\mathbf{u}} + y^{ij}\,\ddot{\boldsymbol{\delta}}_j) = m\,\tilde{\mathbf{b}}^i - \tilde{\mathbf{t}}^i \ . \tag{6.13.15}$$

In particular, it is noted that $\tilde{\mathbf{t}}^i$ retain a dependence on the values $\hat{\mathbf{t}}^i$ associated with the large deformation.

Moreover, for the fully linearized theory, $\hat{\mathbf{F}}=\mathbf{I}$ so that the quantities $\hat{\mathbf{t}}^i$ vanish. Then, the motion is determined by the equations (6.13.15) with

$$\tilde{\mathbf{E}} = \frac{1}{2}(\boldsymbol{\delta}_i\otimes\mathbf{D}^i + \mathbf{D}^i\otimes\boldsymbol{\delta}_i) \ , \quad \tilde{\mathbf{t}}^i = \left[V\,\mathbf{K}^*\bullet\tilde{\mathbf{E}}\right]\mathbf{D}^i \ . \tag{6.13.16}$$

6.14 Forced shearing vibrations of an orthotropic rectangular parallelepiped

In the previous sections the constitutive equations for the response of an orthotropic Cosserat point have been determined which ensure that solutions of the Cosserat point theory can reproduce exactly the complete class of homogeneous solutions of the three-dimensional theory. This section proposes values for the inertia properties of a Cosserat point model of an orthotropic rectangular parallelepiped by comparing predictions of the Cosserat theory with the exact results of section 3.16 for forced shearing vibrations.

To this end, attention is confined to a rectangular parallelepiped which in its reference configuration has constant length L, width W and height H. To be specific, the position vector $\mathbf{X}^*$ in the reference configuration is given by (3.14.2), and the region occupied by the parallelepiped is specified by

$$|\,\theta^1\,| \le \frac{L}{2} \ , \quad |\,\theta^2\,| \le \frac{W}{2} \ , \quad |\,\theta^3\,| \le \frac{H}{2} \ . \tag{6.14.1}$$

Also, the reference point $\mathbf{X}$ is taken to be the centroid of this region and the directors are specified by

$$\mathbf{X} = 0 \ , \quad \mathbf{D}_i = \mathbf{e}_i \ . \tag{6.14.2}$$

With these specifications, the quantities $\{\mathbf{G}_i,\ \mathbf{G}^i,\ G^{1/2}\}$ are independent of the coordinates θ^i and become

$$\mathbf{G}_i = \mathbf{G}^i = \mathbf{D}_i = \mathbf{D}^i = \mathbf{e}_i \ , \quad G^{1/2} = D^{1/2} = 1 \ . \tag{6.14.3}$$

Furthermore, it follows from (6.11.6) that

$$V = LWH \ . \tag{6.14.4}$$

Under these conditions, the kinematic expression (6.2.5) causes the three-dimensional displacement vector of the linearized theory to be a linear function of the coordinates θ^i [see (3.13.1) with $\hat{\mathbf{x}}^*=\mathbf{X}^*$ and (6.13.1) with $\hat{\mathbf{x}}=\mathbf{X}$ and $\hat{\mathbf{d}}_i=\mathbf{D}_i$]

$$\mathbf{u}^*(\theta^i,t) = \mathbf{u}(t) + \theta^i\,\boldsymbol{\delta}_i(t) \ . \tag{6.14.5}$$

It therefore, follows from (3.2.34), (3.13.18) and (6.14.3), that the linearized three-dimensional strain $\tilde{\mathbf{E}}^*$ and stress $\tilde{\mathbf{T}}^*$ associated with the parallelepiped become

$$\tilde{\mathbf{E}}^* = \frac{1}{2}\left[\mathbf{u}^*_{,i}\otimes\mathbf{e}_i + \mathbf{e}_i\otimes\mathbf{u}^*_{,i}\right] \ , \quad \tilde{\mathbf{T}}^* = \mathbf{K}^*\bullet\tilde{\mathbf{E}}^* \ . \tag{6.14.6}$$

Next, it is observed from the linearized form of (6.11.2), that the quantity $\tilde{\mathbf{T}}$ associated with the Cosserat theory can be expressed in the form

$$D^{1/2}\,\tilde{\mathbf{T}} = \int_{-H/2}^{H/2}\int_{-W/2}^{W/2}\int_{-L/2}^{L/2} G^{1/2}\,\tilde{\mathbf{T}}^*\, d\theta^1 d\theta^2 d\theta^3 \ . \tag{6.14.7}$$

Now, substitution of (6.14.6) into (6.14.7) and use of (6.11.6) and (6.14.3), yields the equations

$$\tilde{\mathbf{T}} = V\,\mathbf{K}^* \bullet \tilde{\mathbf{E}} \ , \quad \tilde{\mathbf{E}} = \frac{1}{2}\left[\boldsymbol{\delta}_i\otimes\mathbf{e}_i + \mathbf{e}_i\otimes\boldsymbol{\delta}_i\right] \ , \mathbf{u}(t) = \frac{1}{V}\int_{-H/2}^{H/2}\int_{-W/2}^{W/2}\int_{-L/2}^{L/2} \mathbf{u}^*\, d\theta^1 d\theta^2 d\theta^3 \ ,$$

$$\boldsymbol{\delta}_i(t) = \frac{1}{V}\int_{-H/2}^{H/2}\int_{-W/2}^{W/2}\int_{-L/2}^{L/2} \mathbf{u}^*{}_{,i}\, d\theta^1 d\theta^2 d\theta^3 \ ,$$

$$\boldsymbol{\delta}_1(t) = \frac{1}{V}\int_{-H/2}^{H/2}\int_{-W/2}^{W/2} \left[\mathbf{u}^*(L/2,\theta^2,\theta^3,t) - \mathbf{u}^*(-L/2,\theta^2,\theta^3,t)\right] d\theta^2 d\theta^3 \ ,$$

$$\boldsymbol{\delta}_2(t) = \frac{1}{V}\int_{-H/2}^{H/2}\int_{-L/2}^{L/2} \left[\mathbf{u}^*(\theta^1,W/2,\theta^3,t) - \mathbf{u}^*(\theta^1,-W/2,\theta^3,t)\right] d\theta^1 d\theta^3 \ ,$$

$$\boldsymbol{\delta}_3(t) = \frac{1}{V}\int_{-W/2}^{W/2}\int_{-L/2}^{L/2} \left[\mathbf{u}^*(\theta^1,\theta^2,H/2,t) - \mathbf{u}^*(\theta^1,\theta^2,-H/2,t)\right] d\theta^1 d\theta^2 \ , \tag{6.14.8}$$

which relate the Cosserat displacements $\mathbf{u}$ and $\boldsymbol{\delta}_i$ to the three-dimensional displacement $\mathbf{u}^*$. At this point it should be emphasized that the expressions (6.14.8) for $\tilde{\mathbf{T}}$, $\mathbf{u}$ and $\boldsymbol{\delta}_i$ are not necessarily equal to similar quantities that are obtained by solving the equations for the associated Cosserat point. Instead, these quantities are used to compare the predictions of the three-dimensional theory with those of the Cosserat theory.

Now, with the help of (6.14.3), the expressions (6.13.16) for the linearized theory of the Cosserat point and the linearized form of (6.4.11) yield

$$\tilde{\mathbf{E}} = \frac{1}{2}(\boldsymbol{\delta}_i\otimes\mathbf{e}_i + \mathbf{e}_i\otimes\boldsymbol{\delta}_i) \ , \quad \tilde{\mathbf{t}}^i = \left[V\,\mathbf{K}^* \bullet \tilde{\mathbf{E}}\right]\mathbf{e}_i \ , \quad \tilde{\mathbf{T}} = V\,\mathbf{K}^* \bullet \tilde{\mathbf{E}} \ . \tag{6.14.9}$$

In particular, it is observed that the result $(6.14.8)_1$ is consistent with the expression $(6.14.9)_3$. This indicates that the interpretations of $\mathbf{u}$ and $\boldsymbol{\delta}_i$ proposed in (6.14.8), preserve compatibility of the three-dimensional constitutive equations with those of the Cosserat theory for homogeneous deformations.

Next, attention is focussed on determination of the inertia properties of the parallelepiped. To this end, it is noted that in view (3.2.28) and the condition (6.14.3), it follows that

$$m^* = \rho_0^* \ . \tag{6.14.10}$$

Moreover, since the mass density ρ_0^* is presumed to be constant, direct integration of (6.3.1) and (6.3.4) yields

$$m = \rho_0^*\, V = \rho_0^*\, LWH \ , \quad y^i = 0 \ , \tag{6.14.11}$$

where the volume V is defined by (6.14.4). Also, direct integration of (6.3.12) yields the values

$$y^{11} = \frac{L^2}{12} \ , \quad y^{22} = \frac{W^2}{12} \ , \quad y^{33} = \frac{H^2}{12} \ , \quad \text{all other } y^{ij} = 0 \ . \tag{6.14.12}$$

It will presently be shown that the Cosserat theory can predict more accurate values for vibrational frequencies if the values of the director inertia coefficients y^{11}, y^{22} and y^{33} are taken to be different from (6.14.12). In this regard, it should be mentioned that within the context of the Cosserat theory, the quantity m and the director inertia coefficients y^i and y^{ij} are independent of time and therefore can be determined in the reference configuration. Moreover, within the context of the direct approach, these quantities require constitutive equations and, in particular, they need not be determined by integrals (6.3.1), (6.3.4) and (6.3.12). Nevertheless, since the value $(6.14.11)_1$ for m ensures that the Cosserat point will have the same mass as the parallelepiped, it is retained in the Cosserat theory. Also, the values $(6.14.11)_2$ for y^i and the values $(6.14.12)_4$ for the off-diagonal terms of y^{ij} are retained for the Cosserat model of the parallelepiped since they indicate that the mass is distributed symmetrically about its centroid.

In contrast, the values (6.14.12) for the director inertia coefficients y^{11}, y^{22} and y^{33} will be modified to cause the Cosserat theory to predict vibrational frequencies more accurately. In this regard, it should be mentioned that the values (6.14.12) are the same as the values obtained using the Galerkin method and the kinematic approximation (6.2.5).

To predict values for y^{11}, y^{22} and y^{33}, all three types of vibrational modes discussed in section 3.16 are considered. However, attention is focused on only the first mode of vibration of each type since it is not reasonable to expect the Cosserat point theory to accurately predict higher order modes through the thickness of the parallelepiped. Consequently, the three-dimensional displacements and stresses associated with these modes can be written in the forms (3.16.1) and (3.16.2)

$$
\begin{gathered}
u_1^* = A_{12}^* \sin(\omega_{12}^* t) \sin(k_2^* \theta^2) + A_{13}^* \sin(\omega_{13}^* t) \sin(k_3^* \theta^3) \ , \\
u_2^* = A_{21}^* \sin(\omega_{21}^* t) \sin(k_1^* \theta^1) + A_{23}^* \sin(\omega_{23}^* t) \sin(k_3^* \theta^3) \ , \\
u_3^* = A_{31}^* \sin(\omega_{31}^* t) \sin(k_1^* \theta^1) + A_{32}^* \sin(\omega_{32}^* t) \sin(k_2^* \theta^2) \ , \\
T_{11}^* = T_{22}^* = T_{33}^* = 0 \ , \\
T_{12}^* = K_{1212}^* \left[A_{12}^* k_2^* \sin(\omega_{12}^* t) \cos(k_2^* \theta^2) + A_{21}^* k_1^* \sin(\omega_{21}^* t) \cos(k_1^* \theta^1)\right] \ , \\
T_{13}^* = K_{1313}^* \left[A_{13}^* k_3^* \sin(\omega_{13}^* t) \cos(k_3^* \theta^3) + A_{31}^* k_1^* \sin(\omega_{31}^* t) \cos(k_1^* \theta^1)\right] \ , \\
T_{23}^* = K_{2323}^* \left[A_{23}^* k_3^* \sin(\omega_{23}^* t) \cos(k_3^* \theta^3) + A_{32}^* k_2^* \sin(\omega_{32}^* t) \cos(k_2^* \theta^2)\right] \ , \\
\tilde{\mathbf{t}}^{*1} = \tilde{\mathbf{T}}^* \mathbf{e}_1 = T_{12}^* \mathbf{e}_2 + T_{13}^* \mathbf{e}_3 \ , \quad \tilde{\mathbf{t}}^{*2} = \tilde{\mathbf{T}}^* \mathbf{e}_2 = T_{12}^* \mathbf{e}_1 + T_{23}^* \mathbf{e}_3 \ , \\
\tilde{\mathbf{t}}^{*3} = \tilde{\mathbf{T}}^* \mathbf{e}_3 = T_{13}^* \mathbf{e}_1 + T_{23}^* \mathbf{e}_2 \ ,
\end{gathered}
\tag{6.14.13}
$$

where the frequencies and the wave numbers are given by (3.16.3) and (3.16.4)

$$
\begin{gathered}
\omega_{12}^* = \left[\frac{K_{1212}^*}{\rho_0^*}\right]^{1/2} \frac{\pi}{W} \ , \quad \omega_{13}^* = \left[\frac{K_{1313}^*}{\rho_0^*}\right]^{1/2} \frac{\pi}{H} \ , \quad \omega_{21}^* = \left[\frac{K_{1212}^*}{\rho_0^*}\right]^{1/2} \frac{\pi}{L} \ , \\
\omega_{23}^* = \left[\frac{K_{2323}^*}{\rho_0^*}\right]^{1/2} \frac{\pi}{H} \ , \quad \omega_{31}^* = \left[\frac{K_{1313}^*}{\rho_0^*}\right]^{1/2} \frac{\pi}{L} \ , \quad \omega_{32}^* = \left[\frac{K_{2323}^*}{\rho_0^*}\right]^{1/2} \frac{\pi}{W} \ , \\
k_1^* = \frac{\pi}{L} \ , \quad k_2^* = \frac{\pi}{W} \ , \quad k_3^* = \frac{\pi}{H} \ ,
\end{gathered}
\tag{6.14.14}
$$

and where the directions of orthotropy are aligned with the base vectors such that

$$
\mathbf{D}_i = \mathbf{M}_i = \mathbf{M}_i^* = \mathbf{e}_i \ . \tag{6.14.15}
$$

It then follows that

$$\tilde{\mathbf{t}}^{*1}(\pm L/2,\theta^2,\theta^3,t) = K^*_{1212}\left[A^*_{12}k^*_2 \sin(\omega^*_{12}t)\cos(k^*_2\theta^2)\right]\mathbf{e}_2$$
$$+ K^*_{1313}\left[A^*_{13}k^*_3 \sin(\omega^*_{13}t)\cos(k^*_3\theta^3)\right]\mathbf{e}_3 \ ,$$
$$\tilde{\mathbf{t}}^{*2}(\theta^1,\pm W/2,\theta^3,t) = K^*_{1212}\left[A^*_{21}k^*_1 \sin(\omega^*_{21}t)\cos(k^*_1\theta^1)\right]\mathbf{e}_1$$
$$+ K^*_{2323}\left[A^*_{23}k^*_3 \sin(\omega^*_{23}t)\cos(k^*_3\theta^3)\right]\mathbf{e}_3 \ ,$$
$$\tilde{\mathbf{t}}^{*3}(\theta^1,\theta^2,\pm H/2) = K^*_{1313}\left[A^*_{31}k^*_1 \sin(\omega^*_{31}t)\cos(k^*_1\theta^1)\right]\mathbf{e}_1$$
$$+ K^*_{2323}\left[A^*_{32}k^*_2 \sin(\omega^*_{32}t)\cos(k^*_2\theta^2)\right]\mathbf{e}_2 \ . \tag{6.14.16}$$

In order to compare the Cosserat solutions with the corresponding three-dimensional solutions of section 3.14, it is necessary to apply the same loading to the parallelepiped. This is accomplished by using the linearized forms of (6.3.7), (6.3.8), (6.3.15) and (6.3.16) to deduce that

$$m\,\tilde{\mathbf{b}}_c = \int_{-H/2}^{H/2}\int_{-W/2}^{W/2} \left[\tilde{\mathbf{T}}^*(L/2,\theta^2,\theta^3) - \tilde{\mathbf{T}}^*(-L/2,\theta^2,\theta^3)\right]\mathbf{e}_1\, d\theta^2 d\theta^3$$
$$+ \int_{-H/2}^{H/2}\int_{-L/2}^{L/2} \left[\tilde{\mathbf{T}}^*(\theta^1,W/2,\theta^3) - \tilde{\mathbf{T}}^*(\theta^1,-W/2,\theta^3)\right]\mathbf{e}_2\, d\theta^1 d\theta^3$$
$$+ \int_{-W/2}^{W/2}\int_{-L/2}^{L/2} \left[\tilde{\mathbf{T}}^*(\theta^1,\theta^2,H/2) - \tilde{\mathbf{T}}^*(\theta^1,\theta^2,-H/2)\right]\mathbf{e}_3\, d\theta^1 d\theta^2 \ ,$$
$$m\,\tilde{\mathbf{b}}^1_c = \frac{L}{2}\int_{-H/2}^{H/2}\int_{-W/2}^{W/2} \left[\tilde{\mathbf{T}}^*(L/2,\theta^2,\theta^3) + \tilde{\mathbf{T}}^*(-L/2,\theta^2,\theta^3)\right]\mathbf{e}_1\, d\theta^2 d\theta^3$$
$$+ \int_{-H/2}^{H/2}\int_{-L/2}^{L/2} \theta^1\left[\tilde{\mathbf{T}}^*(\theta^1,W/2,\theta^3) - \tilde{\mathbf{T}}^*(\theta^1,-W/2,\theta^3)\right]\mathbf{e}_2\, d\theta^1 d\theta^3$$
$$+ \int_{-W/2}^{W/2}\int_{-L/2}^{L/2} \theta^1\left[\tilde{\mathbf{T}}^*(\theta^1,\theta^2,H/2) - \tilde{\mathbf{T}}^*(\theta^1,\theta^2,-H/2)\right]\mathbf{e}_3\, d\theta^1 d\theta^2 \ ,$$
$$m\,\tilde{\mathbf{b}}^2_c = \int_{-H/2}^{H/2}\int_{-W/2}^{W/2} \theta^2\left[\tilde{\mathbf{T}}^*(L/2,\theta^2,\theta^3) - \tilde{\mathbf{T}}^*(-L/2,\theta^2,\theta^3)\right]\mathbf{e}_1\, d\theta^2 d\theta^3$$
$$+ \frac{W}{2}\int_{-H/2}^{H/2}\int_{-L/2}^{L/2} \left[\tilde{\mathbf{T}}^*(\theta^1,W/2,\theta^3) + \tilde{\mathbf{T}}^*(\theta^1,-W/2,\theta^3)\right]\mathbf{e}_2\, d\theta^1 d\theta^3$$
$$+ \int_{-W/2}^{W/2}\int_{-L/2}^{L/2} \theta^2\left[\tilde{\mathbf{T}}^*(\theta^1,\theta^2,H/2) - \tilde{\mathbf{T}}^*(\theta^1,\theta^2,-H/2)\right]\mathbf{e}_3\, d\theta^1 d\theta^2 \ ,$$
$$m\,\tilde{\mathbf{b}}^3_c = \int_{-H/2}^{H/2}\int_{-W/2}^{W/2} \theta^3\left[\tilde{\mathbf{T}}^*(L/2,\theta^2,\theta^3) - \tilde{\mathbf{T}}^*(-L/2,\theta^2,\theta^3)\right]\mathbf{e}_1\, d\theta^2 d\theta^3$$
$$+ \int_{-H/2}^{H/2}\int_{-L/2}^{L/2} \theta^3\left[\tilde{\mathbf{T}}^*(\theta^1,W/2,\theta^3) - \tilde{\mathbf{T}}^*(\theta^1,-W/2,\theta^3)\right]\mathbf{e}_2\, d\theta^1 d\theta^3$$
$$+ \frac{H}{2}\int_{-W/2}^{W/2}\int_{-L/2}^{L/2} \left[\tilde{\mathbf{T}}^*(\theta^1,\theta^2,H/2) + \tilde{\mathbf{T}}^*(\theta^1,\theta^2,-H/2)\right]\mathbf{e}_3\, d\theta^1 d\theta^2 \ . \tag{6.14.17}$$

Moreover, in the absence of three-dimensional body force, the linearized versions of the expressions (6.3.10) and (6.3.19) yield

$$\tilde{\mathbf{b}} = \tilde{\mathbf{b}}_c \ , \quad \tilde{\mathbf{b}}^i = \tilde{\mathbf{b}}^i_c \ . \tag{6.14.18}$$

Thus, by substituting (6.14.16) into (6.14.17), these expressions yield

$$\tilde{\mathbf{b}} = 0 \ ,$$
$$m\,\tilde{\mathbf{b}}^1 = 2L\left[H\,K^*_{1212}\,A^*_{12}\sin(\omega^*_{12}t)\,\mathbf{e}_2 + W\,K^*_{1313}\,A^*_{13}\sin(\omega^*_{13}t)\,\mathbf{e}_3\ \right] ,$$
$$m\,\tilde{\mathbf{b}}^2 = 2W\left[H\,K^*_{1212}\,A^*_{21}\sin(\omega^*_{21}t)\,\mathbf{e}_1 + L\,K^*_{2323}\,A^*_{23}\sin(\omega^*_{23}t)\,\mathbf{e}_3\right] \ ,$$

$$m\,\tilde{\mathbf{b}}^3 = 2H\left[W\,K^*_{1313}\,A^*_{31}\sin(\omega^*_{31}t)\,\mathbf{e}_1 + L\,K^*_{2323}\,A^*_{32}\sin(\omega^*_{32}t)\,\mathbf{e}_2\right]. \quad (6.14.19)$$

Next, the expressions (6.14.8) suggest that the Cosserat displacements associated with (6.14.13) are given by

$$\mathbf{u} = 0\ ,\quad \boldsymbol{\delta}_1 = \frac{2}{L}\left[\{A^*_{21}\sin(\omega^*_{21}t)\}\,\mathbf{e}_2 + \{A^*_{31}\sin(\omega^*_{31}t)\}\,\mathbf{e}_3\right],$$
$$\boldsymbol{\delta}_2 = \frac{2}{W}\left[\{A^*_{12}\sin(\omega^*_{12}t)\}\,\mathbf{e}_1 + \{A^*_{32}\sin(\omega^*_{32}t)\}\,\mathbf{e}_3\right],$$
$$\boldsymbol{\delta}_3 = \frac{2}{H}\left[\{A^*_{13}\sin(\omega^*_{13}t)\}\,\mathbf{e}_1 + \{A^*_{23}\sin(\omega^*_{23}t)\}\,\mathbf{e}_2\right]. \quad (6.14.20)$$

Using these displacements, the constitutive equations (6.14.9) with the specifications (3.12.12) and (6.14.15) for an orthotropic material, yield the expressions

$$\tilde{\mathbf{E}} = \left[\frac{1}{L}\{A^*_{21}\sin(\omega^*_{21}t)\} + \frac{1}{W}\{A^*_{12}\sin(\omega^*_{12}t)\}\right](\mathbf{e}_1\otimes\mathbf{e}_2 + \mathbf{e}_2\otimes\mathbf{e}_1)$$
$$+ \left[\frac{1}{L}\{A^*_{31}\sin(\omega^*_{31}t)\} + \frac{1}{H}\{A^*_{13}\sin(\omega^*_{13}t)\}\right](\mathbf{e}_1\otimes\mathbf{e}_3 + \mathbf{e}_3\otimes\mathbf{e}_1)$$
$$+ \left[\frac{1}{W}\{A^*_{32}\sin(\omega^*_{32}t)\} + \frac{1}{H}\{A^*_{23}\sin(\omega^*_{23}t)\}\right](\mathbf{e}_2\otimes\mathbf{e}_3 + \mathbf{e}_3\otimes\mathbf{e}_2)\ ,$$
$$\tilde{\mathbf{t}}^1 = 2H\,K^*_{1212}\left[W\{A^*_{21}\sin(\omega^*_{21}t)\} + L\{A^*_{12}\sin(\omega^*_{12}t)\}\right]\mathbf{e}_2$$
$$+ 2W\,K^*_{1313}\left[W\{A^*_{31}\sin(\omega^*_{31}t)\} + L\{A^*_{13}\sin(\omega^*_{13}t)\}\right]\mathbf{e}_3\ ,$$
$$\tilde{\mathbf{t}}^2 = 2H\,K^*_{1212}\left[W\{A^*_{21}\sin(\omega^*_{21}t)\} + L\{A^*_{12}\sin(\omega^*_{12}t)\}\right]\mathbf{e}_1$$
$$+ 2L\,K^*_{2323}\left[H\{A^*_{32}\sin(\omega^*_{32}t)\} + W\{A^*_{23}\sin(\omega^*_{23}t)\}\right]\mathbf{e}_3\ ,$$
$$\tilde{\mathbf{t}}^3 = 2W\,K^*_{1313}\left[H\{A^*_{31}\sin(\omega^*_{31}t)\} + L\{A^*_{13}\sin(\omega^*_{13}t)\}\right]\mathbf{e}_1$$
$$+ 2L\,K^*_{2323}\left[H\{A^*_{32}\sin(\omega^*_{32}t)\} + W\{A^*_{23}\sin(\omega^*_{23}t)\}\right]\mathbf{e}_2\ . \quad (6.14.21)$$

Now, with the help of (6.14.11) and $(6.14.12)_4$ the linearized equations of motion (6.13.15) reduce to

$$m\,\ddot{\mathbf{u}} = m\,\tilde{\mathbf{b}}\ ,\quad m\,y^{11}\,\ddot{\boldsymbol{\delta}}_1 = m\,\tilde{\mathbf{b}}^1 - \tilde{\mathbf{t}}^1\ ,$$
$$m\,y^{22}\,\ddot{\boldsymbol{\delta}}_1 = m\,\tilde{\mathbf{b}}^2 - \tilde{\mathbf{t}}^2\ ,\quad m\,y^{33}\,\ddot{\boldsymbol{\delta}}_3 = m\,\tilde{\mathbf{b}}^3 - \tilde{\mathbf{t}}^3\ . \quad (6.14.22)$$

Next, it follows from the specifications (6.14.19) and (6.14.20) and the results (6.14.21), that equation $(6.14.22)_1$ is automatically satisfied and the remaining equations (6.14.22) reduce to six scalar equations

$$\rho^*_0\,y^{11}\{\omega^*_{21}\}^2 = K^*_{1212}\ ,\quad \rho^*_0\,y^{11}\{\omega^*_{31}\}^2 = K^*_{1313}\ ,$$
$$\rho^*_0\,y^{22}\{\omega^*_{12}\}^2 = K^*_{1212}\ ,\quad \rho^*_0\,y^{22}\{\omega^*_{32}\}^2 = K^*_{2323}\ ,$$
$$\rho^*_0\,y^{33}\{\omega^*_{13}\}^2 = K^*_{1313}\ ,\quad \rho^*_0\,y^{33}\{\omega^*_{23}\}^2 = K^*_{2323}\ . \quad (6.14.23)$$

However, since the frequencies are given by (6.14.14), it can be seen that all of these equations can be satisfied if the director inertia coefficients y^{11}, y^{22} and y^{33} are specified by

$$y^{11} = \frac{L^2}{\pi^2}\ ,\quad y^{22} = \frac{W^2}{\pi^2}\ ,\quad y^{33} = \frac{H^2}{\pi^2}\ , \quad (6.14.24)$$

instead of the Galerkin values (6.14.12). In this regard, it is noted that the director inertia coefficients model both the mass distribution in the Cosserat point as well as the distribution of acceleration in potential vibrational modes. This is similar to the stiffness coefficients of the Cosserat point which depend on both the material and geometric properties of the structure.

6.15 Free isochoric vibrations of an isotropic cube

Free vibrations of an isotropic parallelepiped have been considered using the theory of a Cosserat point by Rubin (1986). Here, the simpler case of free isochoric vibrations of an isotropic cube which was discussed in section 3.17, is reconsidered to further examine the validity of the specifications (6.14.24) of the director inertia coefficients y^{11}, y^{22} and y^{33}.

The formulation of the Cosserat point equations is the same as that described in section 6.14, except that here the dimensions of the parallelepiped [see (3.14.1)] are taken to be equal

$$L = W = H \ . \tag{6.15.1}$$

Also, the material is assumed to be isotropic so that $(6.14.8)_1$ becomes

$$\tilde{\mathbf{T}} = H^3 \left[(K^* - \tfrac{2}{3}\mu^*) (\tilde{\mathbf{E}} \bullet \mathbf{I}) \mathbf{I} + 2\mu^* \tilde{\mathbf{E}}\right] \ . \tag{6.15.2}$$

Specifically, using (3.17.2) the three-dimensional displacements associated with three types of isochoric modes can be written in the forms

$$\begin{aligned} u_1^* &= A_{12}^* \sin(\omega^* t) \sin(k^*\theta^1) \cos(k^*\theta^2) + A_{13}^* \sin(\omega^* t) \sin(k^*\theta^1) \cos(k^*\theta^3) \ , \\ u_2^* &= -A_{12}^* \sin(\omega^* t) \cos(k^*\theta^1) \sin(k^*\theta^2) + A_{23}^* \sin(\omega^* t) \sin(k^*\theta^2) \cos(k^*\theta^3) \ , \\ u_3^* &= -A_{13}^* \sin(\omega^* t) \cos(k^*\theta^1) \sin(k^*\theta^3) - A_{23}^* \sin(\omega^* t) \cos(k^*\theta^2) \sin(k^*\theta^3) \ . \end{aligned} \tag{6.15.3}$$

Also, attention is focused on only the first mode of vibration of each of these types so the frequency and wave number are given by

$$\omega^* = \left[\frac{2\mu^*}{\rho_0^*}\right]^{1/2} \frac{\pi}{H} \ , \quad k^* = \frac{\pi}{H} \ . \tag{6.15.4}$$

In order for the loading of the Cosserat point to be the same as that of the cube, it is necessary to specify the assigned fields by

$$\tilde{\mathbf{b}} = 0 \ , \quad \tilde{\mathbf{b}}^i = 0 \ . \tag{6.15.5}$$

Next, the expressions (6.14.8) suggest that the Cosserat displacements associated with (6.15.3) are given by

$$\begin{aligned} \mathbf{u} = 0 \ , \quad \boldsymbol{\delta}_1 &= \frac{4}{H^2 k^*} \left[(A_{12}^* + A_{13}^*) \sin(\omega^* t)\right] \mathbf{e}_1 \ , \\ \boldsymbol{\delta}_2 &= \frac{4}{H^2 k^*} \left[(-A_{12}^* + A_{23}^*) \sin(\omega^* t)\right] \mathbf{e}_2 \ , \\ \boldsymbol{\delta}_3 &= -\frac{4}{H^2 k^*} \left[(A_{13}^* + A_{23}^*) \sin(\omega^* t)\right] \mathbf{e}_3 \ . \end{aligned} \tag{6.15.6}$$

Now, with the help of (6.14.8) the strain associated with these displacements becomes

$$\tilde{\mathbf{E}} = \frac{4}{H^2k^*}\Big[(A_{12}^* + A_{13}^*)\,\mathbf{e}_1\otimes\mathbf{e}_1 + (-A_{12}^* + A_{23}^*)\,\mathbf{e}_2\otimes\mathbf{e}_2$$

$$- (A_{13}^* + A_{23}^*)\,\mathbf{e}_3\otimes\mathbf{e}_3\Big]\sin(\omega^* t)\ , \qquad (6.15.7)$$

which can easily be seen to be isochoric since

$$\tilde{\mathbf{E}}\cdot\mathbf{I} = 0\ . \qquad (6.15.8)$$

Moreover, for an isotropic material the constitutive equations (6.14.9) and (6.15.2) associated with (6.15.7) become

$$\tilde{\mathbf{t}}^1 = \frac{8\mu^*H}{k^*}\big[(A_{12}^* + A_{13}^*)\sin(\omega^* t)\big]\,\mathbf{e}_1\ ,\quad \tilde{\mathbf{t}}^2 = \frac{8\mu^*H}{k^*}\big[(-A_{12}^* + A_{23}^*)\sin(\omega^* t)\big]\,\mathbf{e}_2\ ,$$

$$\tilde{\mathbf{t}}^3 = -\frac{8\mu^*H}{k^*}\big[(A_{13}^* + A_{23}^*)\sin(\omega^* t)\big]\,\mathbf{e}_3\ . \qquad (6.15.9)$$

In view of the values (6.14.11), $(6.14.12)_4$ and (6.14.24), the linearized equations of motion (6.13.14) with the specifications (6.15.5), reduce to

$$\rho_0^* H^3\,\ddot{\mathbf{u}} = 0\ ,\quad \rho_0^* H^3\Big\{\frac{H^2}{\pi^2}\Big\}\ddot{\boldsymbol{\delta}}_i = -\tilde{\mathbf{t}}^i\ . \qquad (6.15.10)$$

Now, with the help of (6.15.6) and (6.15.9) it is seen that $(6.15.10)_1$ is automatically satisfied and that the three equations $(6.15.10)_2$ reduce to the single scalar equation

$$\rho_0^*\Big\{\frac{H^2}{\pi^2}\Big\}\big\{\omega^*\big\}^2 = 2\mu^*\ , \qquad (6.15.11)$$

which is satisfied by the value $(6.15.4)_1$ of the frequency. Consequently, this result again justifies the specifications (6.14.24) of the director inertia coefficients y^{11}, y^{22} and y^{33}.

6.16 Isotropic nonlinear elastic Cosserat points

The objective of this section is to use the three-dimensional constitutive equations of section 3.11 for isotropic nonlinear elastic materials, together with the restriction (6.11.8), to exhibit explicit constitutive equations for isotropic nonlinear elastic Cosserat points. Motivated by the definitions (3.1.7), (3.7.8), (3.11.9), (6.2.7) and (6.8.8), it is convenient to define the kinematic quantities $\mathbf{C}$, $\mathbf{B}$, J, α_1 and α_2 by the expressions

$$\mathbf{C} = \mathbf{F}^T\mathbf{F}\ ,\quad \mathbf{B} = \mathbf{F}\mathbf{F}^T\ ,\quad J = \det\mathbf{F}\ ,$$

$$\alpha_1 = J^{-2/3}\,\mathbf{C}\cdot\mathbf{I} = J^{-2/3}\,\mathbf{B}\cdot\mathbf{I}\ ,\quad \alpha_2 = J^{-4/3}\,\mathbf{C}\cdot\mathbf{C} = J^{-4/3}\,\mathbf{B}\cdot\mathbf{B}\ . \qquad (6.16.1)$$

Then, when the strain energy function Σ^* in (3.11.10) for an isotropic material is expressed as a function of $\mathbf{C}$ instead of $\mathbf{C}^*$, it follows from (3.11.12) that

$$\Sigma^*(\mathbf{C}) = \hat{\Sigma}^*(\alpha_1,\alpha_2,J)\ ,$$

$$\frac{\partial\Sigma^*}{\partial\mathbf{C}} = \frac{1}{2}J\frac{\partial\hat{\Sigma}^*}{\partial J}\mathbf{C}^{-1} + J^{-2/3}\frac{\partial\hat{\Sigma}^*}{\partial\alpha_1}\Big[\mathbf{I} - \frac{1}{3}(\mathbf{C}\cdot\mathbf{I})\,\mathbf{C}^{-1}\Big]$$

$$+ 2J^{-4/3}\frac{\partial\hat{\Sigma}^*}{\partial\alpha_2}\left[\mathbf{C}-\frac{1}{3}(\mathbf{C}^2\bullet\mathbf{I})\,\mathbf{C}^{-1}\right] \ . \tag{6.16.2}$$

Moreover, with the help of (6.11.19) it can be shown that **T** is given by

$$\mathbf{T} = -\,p\,\mathbf{I} + \mathbf{T}' \ , \quad \mathbf{T}'\bullet\mathbf{I} = 0 \ , \quad \mathbf{T} = 2\,\rho\,\mathbf{F}\frac{\partial\Sigma^*(\mathbf{C})}{\partial\mathbf{C}}\,\mathbf{F}^T \ , \quad p = -\,\rho\,J\frac{\partial\hat{\Sigma}^*}{\partial J} \ ,$$

$$\mathbf{T}' = 2\,\rho\,J^{-2/3}\frac{\partial\hat{\Sigma}^*}{\partial\alpha_1}\left[\mathbf{B}-\frac{1}{3}(\mathbf{B}\bullet\mathbf{I})\,\mathbf{I}\right] + 4\,\rho\,J^{-4/3}\frac{\partial\hat{\Sigma}^*}{\partial\alpha_2}\left[\mathbf{B}^2-\frac{1}{3}(\mathbf{B}^2\bullet\mathbf{I})\,\mathbf{I}\right] . \tag{6.16.3}$$

Thus, comparison of (6.16.3) with (3.11.6) suggests that p and **T**' can be interpreted as an integrated pressure and deviatoric stress, respectively. Moreover, using the definition (6.4.11) it follows that

$$\mathbf{t}^i = d^{1/2}\left[-\,p\,\mathbf{I} + \mathbf{T}'\right]\mathbf{d}^i \ . \tag{6.16.4}$$

If the Cosserat point is modeled as incompressible, then the incompressibility constraint (6.9.1) is imposed

$$J = \frac{d^{1/2}}{D^{1/2}} = 1 \ , \tag{6.16.5}$$

and **T** and $\mathbf{t}^i$ are modified by the constraint responses (6.9.32) so that

$$\mathbf{T} = d^{-1/2}\,\bar{\gamma}\,\mathbf{I} + \mathbf{T}' \ , \quad \mathbf{t}^i = \bar{\gamma}\,\mathbf{d}^i + d^{1/2}\,\mathbf{T}'\,\mathbf{d}^i \ , \tag{6.16.6}$$

where $\bar{\gamma}$ is given by the expression

$$\bar{\gamma} = \gamma - d^{1/2}\,p \ , \tag{6.16.7}$$

and γ is an arbitrary function of time which can absorb any dependence of the function of p on time. Moreover, for the special case when Σ^* is given by (3.11.16) with (3.11.18) and (3.11.19), it follows from (6.11.6) that

$$m\,\hat{\Sigma}^*(\alpha_1,\alpha_2,J) = \frac{1}{2}V\,D^{1/2}\,\mu_0^*\left[(1-4C_2)\,(\alpha_1-3) + C_2\,(\alpha_2-3)\right]$$
$$+ V\,D^{1/2}\,K_0^*\left[(J-1)-\ln(J)\right] \ , \tag{6.16.8}$$

where C_2 is a material constant controlling nonlinear elastic effects.

6.17 A brief summary of the equations for Cosserat points

This section summarizes some of the main equations for Cosserat points. For simplicity attention will be confined to point-like structures which are made from uniform homogeneous materials (ρ_0^* = constant) and which are rectangular parallelepipeds with length L, width W and height H. The reference point will be taken to be the centroid of the parallelepiped and the directors $\mathbf{D}_i$ will be taken to be unit orthogonal vectors which are oriented so that θ^1 is the length coordinate ($-L/2 \le \theta^1 \le L/2$), θ^2 is the width coordinate ($-W/2 \le \theta^2 \le W/2$), and θ^3 is the height coordinate ($-H/2 \le \theta^3 \le H/2$). Also, the specific body force $\mathbf{b}^*$ will be assumed to be a constant vector.

An attempt has been made to record a complete set of equations that can be used to determine general elastic response of a Cosserat point model of a rectangular parallelepiped. Also, for convenient reference, equation numbers are recorded below to indicate the locations in the previous sections where the quantities, or related quantities, have been explained in more detail.

KINEMATICS

$$\mathbf{X}\ ,\quad \mathbf{D}_i = \mathbf{e}_i\ ,\quad D^{1/2} = \mathbf{D}_1\times\mathbf{D}_2 \bullet \mathbf{D}_3\ , \quad (6.1.1)\text{-}(6.1.2),(6.14.3)$$

$$D^{1/2}\,\mathbf{D}^1 = \mathbf{D}_2\times\mathbf{D}_3\ ,\quad D^{1/2}\,\mathbf{D}^2 = \mathbf{D}_3\times\mathbf{D}_1\ ,\quad D^{1/2}\,\mathbf{D}^3 = \mathbf{D}_1\times\mathbf{D}_2\ , \qquad (6.1.5)$$

$$\mathbf{x}(t)\ ,\quad \mathbf{d}_i(t)\ ,\quad d^{1/2} = \mathbf{d}_1\times\mathbf{d}_2 \bullet \mathbf{d}_3\ , \qquad (6.2.1)\text{-}(6.2.3)$$

$$d^{1/2}\,\mathbf{d}^1 = \mathbf{d}_2\times\mathbf{d}_3\ ,\quad d^{1/2}\,\mathbf{d}^2 = \mathbf{d}_3\times\mathbf{d}_1\ ,\quad d^{1/2}\,\mathbf{d}^3 = \mathbf{d}_1\times\mathbf{d}_2\ , \qquad (6.2.6)$$

$$\mathbf{v} = \dot{\mathbf{x}}\ ,\quad \mathbf{w}_i = \dot{\mathbf{d}}_i\ , \qquad (6.2.4)$$

$$\mathbf{F} = \mathbf{d}_i\otimes\mathbf{D}^i\ ,\quad \mathbf{C} = \mathbf{F}^T\mathbf{F}\ ,\quad \mathbf{E} = \frac{1}{2}(\mathbf{C}-\mathbf{I})\ ,(6.2.7),(6.8.8),(6.12.3)$$

$$|\,\theta^1\,| \le \frac{L}{2}\ ,\quad |\,\theta^2\,| \le \frac{W}{2}\ ,\quad |\,\theta^3\,| \le \frac{H}{2}\ , \qquad (6.14.1)$$

$$V = LWH\ , \qquad (6.14.4)$$

$$J = \det \mathbf{F}\ ,\quad \mathbf{B} = \mathbf{F}\mathbf{F}^T\ , \qquad (6.8.8),(6.16.1)$$

$$\alpha_1 = J^{-2/3}\,\mathbf{C}\bullet\mathbf{I} = J^{-2/3}\,\mathbf{B}\bullet\mathbf{I}\ ,\ \alpha_2 = J^{-4/3}\,\mathbf{C}\bullet\mathbf{C} = J^{-4/3}\,\mathbf{B}\bullet\mathbf{B}\ , \qquad (6.16.1)$$

$$\mathbf{M}_1 = \mathbf{M}_2\times\mathbf{M}_3\ ,\quad \mathbf{M}_2 = \frac{\mathbf{M}_3\times\mathbf{D}_1}{|\mathbf{M}_3\times\mathbf{D}_1|}\ ,\quad \mathbf{M}_3 = \frac{\mathbf{D}_3}{|\mathbf{D}_3|}\ , \qquad (6.12.5)$$

BALANCE LAWS

$$\dot{m} = 0\ ,\quad m\,(\dot{\mathbf{v}} + y^i\,\dot{\mathbf{w}}_i) = m\,\mathbf{b}\ ,\quad m\,(y^i\,\dot{\mathbf{v}} + y^{ij}\,\dot{\mathbf{w}}_j) = m\,\mathbf{b}^i - \mathbf{t}^i\ , \qquad (6.4.5)$$

$$\mathbf{T} = d^{-1/2}\,\mathbf{t}^i\otimes\mathbf{d}_i = \mathbf{T}^T\ , \qquad (6.4.11),(6.4.12)$$

ASSIGNED FIELDS

$$\mathbf{b}_b = \mathbf{b}^*\ ,\quad \mathbf{b}_b^i = 0\ , \qquad (6.3.6)(6.3.14)$$

$$\text{for }\theta^1 = \frac{L}{2}\ ,\ \mathbf{n}^*da^* = \mathbf{d}_2\times\mathbf{d}_3\ d\theta^2 d\theta^3 = d^{1/2}\,\mathbf{d}^1\ d\theta^2 d\theta^3\ ,\ (6.2.5),(6.2.6)$$

$$\text{for }\theta^2 = \frac{W}{2}\ ,\ \mathbf{n}^*da^* = \mathbf{d}_3\times\mathbf{d}_1\ d\theta^1 d\theta^3 = d^{1/2}\,\mathbf{d}^2\ d\theta^1 d\theta^3\ ,(6.2.5),(6.2.6)$$

$$\text{for }\theta^3 = \frac{H}{2},\ \mathbf{n}^*da^* = \mathbf{d}_1\times\mathbf{d}_2\ d\theta^1 d\theta^2 = d^{1/2}\,\mathbf{d}^3\ d\theta^1 d\theta^2\ ,\ (6.2.5),(6.2.6)$$

$$\text{for }\theta^1 = -\frac{L}{2}\ ,\ \mathbf{n}^*da^* = -\,\mathbf{d}_2\times\mathbf{d}_3\ d\theta^2 d\theta^3 = -\,d^{1/2}\,\mathbf{d}^1\ d\theta^2 d\theta^3\ ,\ (6.2.5),(6.2.6)$$

$$\text{for }\theta^2 = -\frac{W}{2}\ ,\ \mathbf{n}^*da^* = -\,\mathbf{d}_3\times\mathbf{d}_1\ d\theta^1 d\theta^3 = -\,d^{1/2}\,\mathbf{d}^2\ d\theta^1 d\theta^3\ ,\ (6.2.5),(6.2.6)$$

$$\text{for }\theta^3 = -\frac{H}{2}\ ,\ \mathbf{n}^*da^* = -\,\mathbf{d}_1\times\mathbf{d}_2\ d\theta^1 d\theta^2 = -\,d^{1/2}\,\mathbf{d}^3\ d\theta^1 d\theta^2\ ,(6.2.5),(6.2.6)$$

$$\mathbf{n}^*\bullet\mathbf{n}^* = 1\ ,$$

$$\begin{aligned} m\,\mathbf{b}_c = &\int_{-H/2}^{H/2}\!\int_{-W/2}^{W/2} d^{1/2}\,|\,\mathbf{d}^1\,|\,\Big\{\mathbf{t}^*(\tfrac{L}{2},\theta^2,\theta^3) + \mathbf{t}^*(-\tfrac{L}{2},\theta^2,\theta^3)\Big\}\,d\theta^2 d\theta^3 \\ &+ \int_{-H/2}^{H/2}\!\int_{-L/2}^{L/2} d^{1/2}\,|\,\mathbf{d}^2\,|\,\Big\{\mathbf{t}^*(\theta^1,\tfrac{W}{2},\theta^3) + \mathbf{t}^*(\theta^1,-\tfrac{W}{2},\theta^3)\Big\}\,d\theta^1 d\theta^3 \\ &+ \int_{-W/2}^{W/2}\!\int_{-L/2}^{L/2} d^{1/2}\,|\,\mathbf{d}^3\,|\,\Big\{\mathbf{t}^*(\theta^1,\theta^2,\tfrac{H}{2}) + \mathbf{t}^*(\theta^1,\theta^2,-\tfrac{H}{2})\Big\}\,d\theta^1 d\theta^2\ , \end{aligned}$$

(6.3.7)-(6.3.8)

$$m\,\mathbf{b}_c^1 = \int_{-H/2}^{H/2}\int_{-W/2}^{W/2} d^{1/2}\,|\,\mathbf{d}^1\,|\;\frac{L}{2}\{\mathbf{t}^*(\frac{L}{2},\theta^2,\theta^3) - \mathbf{t}^*(-\frac{L}{2},\theta^2,\theta^3)\}\,d\theta^2 d\theta^3$$
$$+\int_{-H/2}^{H/2}\int_{-L/2}^{L/2} d^{1/2}\,|\,\mathbf{d}^2\,|\,\theta^1\{\mathbf{t}^*(\theta^1,\frac{W}{2},\theta^3) + \mathbf{t}^*(\theta^1,-\frac{W}{2},\theta^3)\}\,d\theta^1 d\theta^3$$
$$+\int_{-W/2}^{W/2}\int_{-L/2}^{L/2} d^{1/2}\,|\,\mathbf{d}^3\,|\,\theta^1\{\mathbf{t}^*(\theta^1,\theta^2,\frac{H}{2}) + \mathbf{t}^*(\theta^1,\theta^2,-\frac{H}{2})\}\,d\theta^1 d\theta^2\,, \tag{6.3.14)-(6.3.16}$$

$$m\,\mathbf{b}_c^2 = \int_{-H/2}^{H/2}\int_{-W/2}^{W/2} d^{1/2}\,|\,\mathbf{d}^1\,|\;\theta^2\{\mathbf{t}^*(\frac{L}{2},\theta^2,\theta^3) + \mathbf{t}^*(-\frac{L}{2},\theta^2,\theta^3)\}\,d\theta^2 d\theta^3$$
$$+\int_{-H/2}^{H/2}\int_{-L/2}^{L/2} d^{1/2}\,|\,\mathbf{d}^2\,|\,\frac{W}{2}\{\mathbf{t}^*(\theta^1,\frac{W}{2},\theta^3) - \mathbf{t}^*(\theta^1,-\frac{W}{2},\theta^3)\}\,d\theta^1 d\theta^3$$
$$+\int_{-W/2}^{W/2}\int_{-L/2}^{L/2} d^{1/2}\,|\,\mathbf{d}^3\,|\,\theta^2\{\mathbf{t}^*(\theta^1,\theta^2,\frac{H}{2}) + \mathbf{t}^*(\theta^1,\theta^2,-\frac{H}{2})\}\,d\theta^1 d\theta^2\,, \tag{6.3.14)-(6.3.16}$$

$$m\,\mathbf{b}_c^3 = \int_{-H/2}^{H/2}\int_{-W/2}^{W/2} d^{1/2}\,|\,\mathbf{d}^1\,|\;\theta^3\{\mathbf{t}^*(\frac{L}{2},\theta^2,\theta^3) + \mathbf{t}^*(-\frac{L}{2},\theta^2,\theta^3)\}\,d\theta^2 d\theta^3$$
$$+\int_{-H/2}^{H/2}\int_{-L/2}^{L/2} d^{1/2}\,|\,\mathbf{d}^2\,|\,\theta^3\{\mathbf{t}^*(\theta^1,\frac{W}{2},\theta^3) + \mathbf{t}^*(\theta^1,-\frac{W}{2},\theta^3)\}\,d\theta^1 d\theta^3$$
$$+\int_{-W/2}^{W/2}\int_{-L/2}^{L/2} d^{1/2}\,|\,\mathbf{d}^3\,|\,\frac{H}{2}\{\mathbf{t}^*(\theta^1,\theta^2,\frac{H}{2}) - \mathbf{t}^*(\theta^1,\theta^2,-\frac{H}{2})\}\,d\theta^1 d\theta^2\,, \tag{6.3.14)-(6.3.16}$$

$$\mathbf{b} = \mathbf{b}_b + \mathbf{b}_c\ ,\quad \mathbf{b}^i = \mathbf{b}_b^i + \mathbf{b}_c^i\ , \tag{6.3.10),(6.3.19}$$

INERTIA QUANTITIES

$$m = \rho_0^* V = \rho_0^* LWH\ ,\quad y^i = 0\ , \tag{6.4.13),(6.14.11}$$

$$y^{12} = y^{21} = 0\ ,\quad y^{13} = y^{31} = 0\ ,\quad y^{23} = y^{32} = 0\ , \tag{6.14.12}$$

$$y^{11} = \frac{L^2}{\pi^2}\ ,\quad y^{22} = \frac{W^2}{\pi^2}\ ,\quad y^{33} = \frac{H^2}{\pi^2}\ , \tag{6.14.24}$$

GENERAL CONSTITUTIVE EQUATIONS

$$\rho\,\dot{\Sigma} = \mathcal{P} = \mathbf{T}\bullet\mathbf{D}\ ,\quad \Sigma = \Sigma^*(\mathbf{C})\ , \tag{6.8.1),(6.11.18}$$

$$\mathbf{T} = 2\,\rho\,\mathbf{F}\frac{\partial\Sigma^*(\mathbf{C})}{\partial\mathbf{C}}\mathbf{F}^T\ ,\quad \mathbf{t}^i = 2\,m\,\mathbf{F}\frac{\partial\Sigma^*(\mathbf{C})}{\partial\mathbf{C}}\mathbf{D}^i\ , \tag{6.11.19}$$

CONSTRAINTS

$$\mathbf{T} = \hat{\mathbf{T}} + \bar{\mathbf{T}}\ ,\quad \mathbf{t}^i = \hat{\mathbf{t}}^i + \bar{\mathbf{t}}^i\ , \tag{6.9.9}$$

Incompressibility

$$J = 1\ ,\quad \bar{\mathbf{T}} = d^{-1/2}\gamma\,\mathbf{I}\ ,\quad \bar{\mathbf{t}}^i = \gamma\,\mathbf{d}^i\ , \tag{6.9.1),(6.9.32}$$

Rigid body

$$\mathbf{d}_1\bullet\mathbf{d}_1 = \text{constant}\ ,\quad \mathbf{d}_2\bullet\mathbf{d}_2 = \text{constant}\ ,\quad \mathbf{d}_3\bullet\mathbf{d}_3 = \text{constant}\ , \tag{6.9.3}$$

$$\mathbf{d}_1\bullet\mathbf{d}_2 = 0\ ,\quad \mathbf{d}_1\bullet\mathbf{d}_3 = 0\ ,\quad \mathbf{d}_2\bullet\mathbf{d}_3 = 0\ , \tag{6.9.5}$$

$$\bar{\mathbf{T}} = \frac{1}{2}d^{-1/2}\gamma^{ij}(\mathbf{d}_i\otimes\mathbf{d}_j + \mathbf{d}_j\otimes\mathbf{d}_i)\ ,\ \gamma^{ij} = \gamma^{ji}\ ,\quad \bar{\mathbf{t}}^i = d^{1/2}\,\bar{\mathbf{T}}\,\mathbf{d}^i = \gamma^{ij}\,\mathbf{d}_j\ , \tag{6.9.17),(6.9.18}$$

ORTHOTROPIC POINTS - SMALL STRAINS (LARGE DISPLACEMENTS)

$$m\,\Sigma = \frac{1}{2}V\,\mathbf{K}^*\bullet(\mathbf{E}\otimes\mathbf{E})\ ,\quad \mathbf{t}^i = V\,\mathbf{F}\left[\mathbf{K}^*\bullet\mathbf{E}\right]\mathbf{D}^i\ , \tag{6.12.4}$$

$$\mathbf{E} = E_{ij}\,(\mathbf{M}_i\otimes\mathbf{M}_j)\ ,\quad K^*_{ijkl} = \mathbf{K}^*\bullet(\mathbf{M}_i\otimes\mathbf{M}_j\otimes\mathbf{M}_k\otimes\mathbf{M}_l)\ , \tag{6.12.6}$$

$$\mathbf{K}^* \bullet \mathbf{E} = [K^*_{1111}\, E_{11} + K^*_{1122}\, E_{22} + K^*_{1133}\, E_{33}]\, (\mathbf{M}_1 \otimes \mathbf{M}_1)$$
$$+ [K^*_{1122}\, E_{11} + K^*_{2222}\, E_{22} + K^*_{2233}\, E_{33}]\, (\mathbf{M}_2 \otimes \mathbf{M}_2)$$
$$+ [K^*_{1133}\, E_{11} + K^*_{2233}\, E_{22} + K^*_{3333}\, E_{33}]\, (\mathbf{M}_3 \otimes \mathbf{M}_3)$$
$$+ [K^*_{1212}\, (E_{12} + E_{21})]\, (\mathbf{M}_1 \otimes \mathbf{M}_2 + \mathbf{M}_2 \otimes \mathbf{M}_1)$$
$$+ [K^*_{1313}\, (E_{13} + E_{31})]\, (\mathbf{M}_1 \otimes \mathbf{M}_3 + \mathbf{M}_3 \otimes \mathbf{M}_1)$$
$$+ [K^*_{2323}\, (E_{23} + E_{32})]\, (\mathbf{M}_2 \otimes \mathbf{M}_3 + \mathbf{M}_3 \otimes \mathbf{M}_2)\ , \qquad (3.12.12)$$

ISOTROPIC POINTS - SMALL STRAINS (LARGE DISPLACEMENTS)

Use the equations for orthotropic points - small strains (large displacements) with

$$\mathbf{K}^* \bullet \mathbf{E} = 2\mu^* \left[\left\{\frac{\nu^*}{1-2\nu^*}\right\} (\mathbf{E} \bullet \mathbf{I})\, \mathbf{I} + \mathbf{E} \right]\ , \quad \text{Table 3.12.1, (3.12.15)}$$

$$K^*_{1111} = K^*_{2222} = K^*_{3333} = 2\mu^* \left[\frac{1-\nu^*}{1-2\nu^*}\right]\ , \quad \text{Table 3.12.1, (3.12.13)}$$

$$K^*_{1122} = K^*_{1133} = K^*_{2233} = 2\mu^* \left[\frac{\nu^*}{1-2\nu^*}\right]\ , \quad \text{Table 3.12.1, (3.12.13)}$$

$$K^*_{1212} = K^*_{1313} = K^*_{2323} = \mu^*\ , \qquad (3.12.13)$$

NONLINEAR ISOTROPIC POINT

Use the general constitutive equations with

$$\Sigma^*(\mathbf{C}) = \hat{\Sigma}^*(\alpha_1, \alpha_2, J)\ , \qquad (6.16.2)$$

$$m\, \hat{\Sigma}^*(\alpha_1, \alpha_2, J) = \frac{1}{2} V\, \mu_0^* \left[(1 - 4C_2)\, (\alpha_1 - 3) + C_2\, (\alpha_2 - 3)\right]$$
$$+ V\, K_0^* \left[(J - 1) - \ln(J)\right]\ , \qquad (6.16.8)$$

$$\mathbf{T} = -\, p\, \mathbf{I} + \mathbf{T}'\ , \quad \mathbf{T}' \bullet \mathbf{I} = 0\ , \quad \mathbf{T} = 2\, \rho\, \mathbf{F}\, \frac{\partial \Sigma^*(\mathbf{C})}{\partial \mathbf{C}}\, \mathbf{F}^T\ , \quad p = -\, \rho\, J\, \frac{\partial \hat{\Sigma}^*}{\partial J}\ , \quad (6.16.3)$$

$$\mathbf{T}' = 2\, \rho\, J^{-2/3}\, \frac{\partial \hat{\Sigma}^*}{\partial \alpha_1} \left[\mathbf{B} - \frac{1}{3} (\mathbf{B} \bullet \mathbf{I})\, \mathbf{I}\right] + 4\, \rho\, J^{-4/3}\, \frac{\partial \hat{\Sigma}^*}{\partial \alpha_2} \left[\mathbf{B}^2 - \frac{1}{3} (\mathbf{B}^2 \bullet \mathbf{I})\, \mathbf{I}\right]\ . \quad (6.16.3)$$

$$\mathbf{t}^i = d^{1/2} \left[-\, p\, \mathbf{I} + \mathbf{T}'\right] \mathbf{d}^i\ , \qquad (6.16.4)$$

LINEARIZED EQUATIONS

Use the general values for m, y^α, $y^{\alpha\beta}$

$$\mathbf{x} = \mathbf{X} + \mathbf{u}\ , \quad \mathbf{d}_i = \mathbf{D}_i + \boldsymbol{\delta}_i\ , \quad \tilde{\mathbf{E}} = \frac{1}{2} (\boldsymbol{\delta}_i \otimes \mathbf{D}^i + \mathbf{D}^i \otimes \boldsymbol{\delta}_i)\ , \quad (6.13.1),(6.13.16)$$

$$\tilde{\mathbf{b}}_b = \tilde{\mathbf{b}}^*\ , \quad \tilde{\mathbf{b}}_b^i = 0\ , \qquad (6.3.6)(6.3.14)$$

$$\text{for } \theta^1 = \frac{L}{2}\ , \quad \mathbf{N}^* da^* = \mathbf{D}_2 \times \mathbf{D}_3\, d\theta^2 d\theta^3 = \mathbf{e}_1\, d\theta^2 d\theta^3\ , \quad (6.2.5),(6.2.6)$$

$$\text{for } \theta^2 = \frac{W}{2}\ , \quad \mathbf{N}^* da^* = \mathbf{D}_3 \times \mathbf{D}_1\, d\theta^1 d\theta^3 = \mathbf{e}_2\, d\theta^1 d\theta^3\ , \quad (6.2.5),(6.2.6)$$

$$\text{for } \theta^3 = \frac{H}{2}\ , \quad \mathbf{N}^* da^* = \mathbf{D}_1 \times \mathbf{D}_2\, d\theta^1 d\theta^2 = \mathbf{e}_3\, d\theta^1 d\theta^2\ , \quad (6.2.5),(6.2.6)$$

$$\text{for } \theta^1 = -\frac{L}{2}\ , \quad \mathbf{N}^* da^* = -\, \mathbf{D}_2 \times \mathbf{D}_3\, d\theta^2 d\theta^3 = -\, \mathbf{e}_1\, d\theta^2 d\theta^3\ , \quad (6.2.5),(6.2.6)$$

$$\text{for } \theta^2 = -\frac{W}{2} \ , \quad \mathbf{N}^* da^* = -\mathbf{D}_3 \times \mathbf{D}_1 \, d\theta^1 d\theta^3 = -\mathbf{e}_2 \, d\theta^1 d\theta^3 \ , \quad (6.2.5),(6.2.6)$$

$$\text{for } \theta^3 = -\frac{H}{2} \ , \quad \mathbf{N}^* da^* = -\mathbf{D}_1 \times \mathbf{D}_2 \, d\theta^1 d\theta^2 = -\mathbf{e}_3 \, d\theta^1 d\theta^2 \ , \quad (6.2.5),(6.2.6)$$

$$\mathbf{N}^* \bullet \mathbf{N}^* = 1 \ ,$$

$$m\,\tilde{\mathbf{b}}_c = \int_{-H/2}^{H/2}\int_{-W/2}^{W/2} \{ \tilde{\mathbf{t}}^*(\tfrac{L}{2},\theta^2,\theta^3) + \tilde{\mathbf{t}}^*(-\tfrac{L}{2},\theta^2,\theta^3) \} d\theta^2 d\theta^3$$
$$+ \int_{-H/2}^{H/2}\int_{-L/2}^{L/2} \{ \tilde{\mathbf{t}}^*(\theta^1,\tfrac{W}{2},\theta^3) + \tilde{\mathbf{t}}^*(\theta^1,-\tfrac{W}{2},\theta^3) \} d\theta^1 d\theta^3$$
$$+ \int_{-W/2}^{W/2}\int_{-L/2}^{L/2} \{ \tilde{\mathbf{t}}^*(\theta^1,\theta^2,\tfrac{H}{2}) + \tilde{\mathbf{t}}^*(\theta^1,\theta^2,-\tfrac{H}{2}) \} d\theta^1 d\theta^2 \ , \quad (6.3.7)\text{-}(6.3.8)$$

$$m\,\tilde{\mathbf{b}}_c^1 = \int_{-H/2}^{H/2}\int_{-W/2}^{W/2} \frac{L}{2}\{ \tilde{\mathbf{t}}^*(\tfrac{L}{2},\theta^2,\theta^3) - \tilde{\mathbf{t}}^*(-\tfrac{L}{2},\theta^2,\theta^3) \} d\theta^2 d\theta^3$$
$$+ \int_{-H/2}^{H/2}\int_{-L/2}^{L/2} \theta^1\{ \tilde{\mathbf{t}}^*(\theta^1,\tfrac{W}{2},\theta^3) + \tilde{\mathbf{t}}^*(\theta^1,-\tfrac{W}{2},\theta^3) \} d\theta^1 d\theta^3$$
$$+ \int_{-W/2}^{W/2}\int_{-L/2}^{L/2} \theta^1\{ \tilde{\mathbf{t}}^*(\theta^1,\theta^2,\tfrac{H}{2}) + \tilde{\mathbf{t}}^*(\theta^1,\theta^2,-\tfrac{H}{2}) \} d\theta^1 d\theta^2 \ , \quad (6.3.14)\text{-}(6.3.16)$$

$$m\,\tilde{\mathbf{b}}_c^2 = \int_{-H/2}^{H/2}\int_{-W/2}^{W/2} \theta^2\{ \tilde{\mathbf{t}}^*(\tfrac{L}{2},\theta^2,\theta^3) + \tilde{\mathbf{t}}^*(-\tfrac{L}{2},\theta^2,\theta^3) \} d\theta^2 d\theta^3$$
$$+ \int_{-H/2}^{H/2}\int_{-L/2}^{L/2} \frac{W}{2}\{ \tilde{\mathbf{t}}^*(\theta^1,\tfrac{W}{2},\theta^3) - \tilde{\mathbf{t}}^*(\theta^1,-\tfrac{W}{2},\theta^3) \} d\theta^1 d\theta^3$$
$$+ \int_{-W/2}^{W/2}\int_{-L/2}^{L/2} \theta^2\{ \tilde{\mathbf{t}}^*(\theta^1,\theta^2,\tfrac{H}{2}) + \tilde{\mathbf{t}}^*(\theta^1,\theta^2,-\tfrac{H}{2}) \} d\theta^1 d\theta^2 \ , \quad (6.3.14)\text{-}(6.3.16)$$

$$m\,\tilde{\mathbf{b}}_c^3 = \int_{-H/2}^{H/2}\int_{-W/2}^{W/2} \theta^3\{ \tilde{\mathbf{t}}^*(\tfrac{L}{2},\theta^2,\theta^3) + \tilde{\mathbf{t}}^*(-\tfrac{L}{2},\theta^2,\theta^3) \} d\theta^2 d\theta^3$$
$$+ \int_{-H/2}^{H/2}\int_{-L/2}^{L/2} \theta^3\{ \tilde{\mathbf{t}}^*(\theta^1,\tfrac{W}{2},\theta^3) + \tilde{\mathbf{t}}^*(\theta^1,-\tfrac{W}{2},\theta^3) \} d\theta^1 d\theta^3$$
$$+ \int_{-W/2}^{W/2}\int_{-L/2}^{L/2} \frac{H}{2}\{ \tilde{\mathbf{t}}^*(\theta^1,\theta^2,\tfrac{H}{2}) - \tilde{\mathbf{t}}^*(\theta^1,\theta^2,-\tfrac{H}{2}) \} d\theta^1 d\theta^2 \ , \quad (6.3.14)\text{-}(6.3.16)$$

$$\mathbf{b} = \tilde{\mathbf{b}} = \tilde{\mathbf{b}}_b + \tilde{\mathbf{b}}_c \ , \quad \mathbf{b}^i = \tilde{\mathbf{b}}^i = \tilde{\mathbf{b}}_b^i + \tilde{\mathbf{b}}_c^i \ , \qquad (6.3.10),(6.3.19)$$

$$\mathbf{t}^i = \tilde{\mathbf{t}}^i \ , \quad \tilde{\mathbf{t}}^i = [V\,\mathbf{K}^* \bullet \tilde{\mathbf{E}}]\,\mathbf{D}^i \qquad (6.13.16)$$

$$m\,\ddot{\mathbf{u}} = m\,\tilde{\mathbf{b}} \ , \quad m\,y^{ij}\,\ddot{\boldsymbol{\delta}}_j = m\,\tilde{\mathbf{b}}^i - \tilde{\mathbf{t}}^i \ , \qquad (6.13.15)$$

6.18 Dissipation inequality and material damping

The previous sections have limited attention to purely elastic response which exhibits no dissipation. Consequently, Cosserat points made from such materials exhibit the unrealistic feature that free vibrations persist forever. In order to eliminate this unphysical response, it is necessary to include a model for material damping. To this end, it is noted that within the context of the purely mechanical theory it is possible to define the rate of material dissipation $\mathcal{D}$ by the formula

$$d^{1/2}\,\mathcal{D} = \mathcal{W} - \dot{\mathcal{K}} - \dot{\mathcal{U}} \geq 0 \ , \qquad (6.18.1)$$

where $\mathcal{W}$, $\mathcal{K}$ and $\mathcal{U}$ are defined by (6.3.23) and (6.8.4). In words, this equation means that the rate of material dissipation is equal to the rate of work $\mathcal{W}$ done by the externally applied forces and couples, minus the rates of change of kinetic energy $\mathcal{K}$ and strain energy $\mathcal{U}$. Moreover, it is assumed that the rate of material dissipation is nonnegative.

Next, with the help of the conservation of mass and the balances of linear, angular and director momentums, it can be shown using (6.6.1), (6.6.10) and (6.8.4) that the local form of equation (6.18.1) becomes

$$\mathcal{D} = \mathbf{T} \bullet \mathbf{D} - \rho \dot{\Sigma} \geq 0 \ . \tag{6.18.2}$$

Moreover, in view of the assumption (6.8.1) it is seen that an elastic Cosserat point is an ideal Cosserat point since the rate of dissipation $\mathcal{D}$ vanishes. Consequently, the assumption that the rate of material dissipation is nonnegative requires that for a given motion, the work done on a dissipative material is greater than that done on an ideal elastic material. Also, using the transformation relations (6.5.5), (6.6.11) and (6.8.3), it can be shown that $\mathcal{D}$ remains unaltered by SRBM

$$\mathcal{D}^+ = \mathcal{D} \ . \tag{6.18.3}$$

Now, a model for a Cosserat point constructed from a dissipative material can be developed by assuming that $\mathbf{T}$ and $\mathbf{t}^i$ separate additively into three parts

$$\mathbf{T} = \hat{\mathbf{T}} + \bar{\mathbf{T}} + \check{\mathbf{T}} \ , \quad \mathbf{t}^i = \hat{\mathbf{t}}^i + \bar{\mathbf{t}}^i + \check{\mathbf{t}}^i \ ,$$
$$\hat{\mathbf{T}} = d^{-1/2}\, \hat{\mathbf{t}}^i \otimes \mathbf{d}_i \ , \quad \bar{\mathbf{T}} = d^{-1/2}\, \bar{\mathbf{t}}^i \otimes \mathbf{d}_i \ , \quad \check{\mathbf{T}} = d^{-1/2}\, \check{\mathbf{t}}^i \otimes \mathbf{d}_i \ , \tag{6.18. 4}$$

with $\hat{\mathbf{T}}$ and $\hat{\mathbf{t}}^i$ being the parts associated with elastic deformation [which balance the rate of change in strain energy (6.8.1)]

$$\hat{\mathbf{T}} \bullet \mathbf{D} = \rho \dot{\Sigma} \ , \tag{6.18.5}$$

$\bar{\mathbf{T}}$ and $\bar{\mathbf{t}}^i$ being the constraint responses [which do no work (6.9.10)]

$$\bar{\mathbf{T}} \bullet \mathbf{D} = 0 \ , \tag{6.18.6}$$

and $\check{\mathbf{T}}$ and $\check{\mathbf{t}}^i$ being the parts due to material dissipation. Thus, the restriction (6.18.2) reduces to

$$\mathcal{D} = \check{\mathbf{T}} \bullet \mathbf{D} \geq 0 \ . \tag{6.18.7}$$

For viscous damping $\check{\mathbf{T}}$ is assumed to be a function of $\mathbf{D}$

$$\check{\mathbf{T}} = \check{\mathbf{T}}(\mathbf{D}) \ . \tag{6.18.8}$$

However, invariance under SRBM requires this function to satisfy the restrictions

$$\mathbf{Q}\, \check{\mathbf{T}}(\mathbf{D})\, \mathbf{Q}^T = \check{\mathbf{T}}(\mathbf{Q}\, \mathbf{D}\, \mathbf{Q}^T) \ , \tag{6.18.9}$$

for all proper orthogonal tensors $\mathbf{Q}$. Consequently, $\check{\mathbf{T}}$ must be an isotropic function of the argument $\mathbf{D}$. Furthermore, as a special simple case it is possible to assume that $\check{\mathbf{T}}$ is a linear function of $\mathbf{D}$ of the form

$$d^{1/2}\, \check{\mathbf{T}} = D^{1/2}\, V \left[\eta_1\, (\mathbf{D} \bullet \mathbf{I})\, \mathbf{I} + 2\eta_2\, \mathbf{D}'\right] \ , \tag{6.18.10}$$

where V is defined by (6.14.4), η_1 and η_2 are material constants and $\mathbf{D}'$ is a pure measure of rate of distortional deformation

$$\mathbf{D}' = \mathbf{D} - \frac{1}{3}(\mathbf{D} \bullet \mathbf{I})\, \mathbf{I} \ , \quad \mathbf{D}' \bullet \mathbf{I} = 0 \ . \tag{6.18.11}$$

Consequently, η_1 is the viscosity to dilatational deformation rate and η_2 is the viscosity to distortional deformation, respectively. Also, it can be shown that the restriction (6.18.7) is satisfied for all motions provided that η_1 and η_2 are all nonnegative

$$\eta_1 \geq 0 \ , \quad \eta_2 \geq 0 \ . \tag{6.18.12}$$

Finally, it is noted that the viscosity constants η_1 and η_2 can be determined by attempting to match the rate of damping associated with free vibrations of the Cosserat point.

CHAPTER 7

NUMERICAL SOLUTIONS USING COSSERAT THEORIES

7.1 The Cosserat approach to numerical solution procedures for problems in continuum mechanics

Numerical solution procedures for problems in continuum mechanics using finite element methods are very well developed and are used in many commercial computer codes for engineering design and analysis. A number of books on finite elements describe both the engineering and mathematical aspects of the finite element method (e.g. Strang and Fix, 1973; Huebner, 1975; Desai, 1979; Becker et al, 1981; Bathe, 1982; Carey and Oden, 1983 and 1984; Reddy, 1985; Hughes, 1987; and Zienkiewicz and Taylor, 1989 and 1991). In its simplest form the finite element method determines an approximate solution of a set of partial differential equations by first representing these equations in a weak form. This weak form can be developed by the method of weighted residuals or can be expressed in a functional representation when one exits. Then, the independent variables are expressed in terms of shape functions (over space and/or time), and a set of ordinary differential equations in time, or algebraic equations, are obtained for the coefficients of these shape functions. Much research focuses on developing efficient methods for solving the resulting large numbers of nonlinear algebraic equations and on determining special shape functions and element formulations which have numerically desirable properties.

For the simple displacement approach to the solution of elasticity problems, it is common to first express the displacements in terms of shape functions. These expressions are substituted into the constitutive equations (strain-displacement relations and the stress-strain relations) and then the balance laws of conservation of mass and momentum are rewritten in weak functional forms which depend on the coefficients of these shape functions. If the equations are nonlinear, then it often is not possible to exactly evaluate the integrals over the region occupied by the body. Consequently, additional approximations are made in evaluating these integrals.

The Cosserat approach to numerical solution procedures for problems in continuum mechanics is more similar to the direct approach to finite elements where the response of each element is determined by a set of constitutive equations. Then, the equations expressing the interactions of neighboring elements are used to determine the response of the complete structure. Moreover, in some situations the Cosserat approach is similar to mixed or hybrid finite element methods which present equations for both the contact forces and the displacements at the nodes.

Specifically, the Cosserat approach starts with a continuum model of a structural element (like a shell, rod, or point) which includes fundamental balance laws of conservation of mass, and balances of linear momentum, director momentum and angular momentum. The response of the overall structure is determined by connecting neighboring structural elements. Most importantly, the continuum model for each element has been developed so that it possesses the same fundamental properties as those exhibited by the full three-dimensional theory. Specifically, the Cosserat model of the element is valid for large deformations, it is properly invariant under superposed rigid body motions, and the constitutive equations for elastic structures are determined directly from a strain energy function in such a way that they are compatible with the kinematics of the structural element. Consequently, fundamental notions such as path independence of the work done on an elastic structure are automatically satisfied in the Cosserat approach. In contrast, approximate integration of nonlinear elastic three-dimensional constitutive equations will not automatically yield response functions for the structural element that are suitably restricted for a strain energy function to exist. In this regard, the Cosserat approach can compliment standard finite element procedures by providing additional ways of examining fundamental features of the resulting algebraic equations.

The initial attempts to use the Cosserat approach for the numerical solution of problems in continuum mechanics (Rubin, 1985a,b; 1987a,b) were limited to essentially one dimensional problems where the topology of neighboring elements was trivial. For example, spherically symmetric problems were modeled in (Rubin, 1987b) using concentric spherical shells. Also, nonlinear string problems were modeled in (Rubin, 1987a) as a chain of vectors which represented connected Cosserat points. In each of these problems, it was possible to connect the elements by demanding continuity of position and stress at each interface of two neighboring elements. In (Rubin, 1995) it was shown that this notion of demanding continuity of both position and stress at the interfaces of neighboring elements does not generalize to two- and three-dimensional problems. Instead, for these more general topologically complex problems, continuity was only satisfied at common nodes of neighboring elements.

The objective of the following sections is to briefly describe the Cosserat approach for a few simple cases. Obviously, the Cosserat approach is still in its infancy and there is much more research that needs to be done to explore the relationship of this approach to standard finite elements. It is presently known that for elements that are only allowed to experience homogeneous strain, the Cosserat approach and the standard Galerkin approach can be placed into a one-to-one correspondence [except that the Cosserat approach allows for a more general mass matrix since the director inertia coefficients are specified by constitutive equations (Rubin and Gottlieb, 1996)]. However, it is anticipated that the Cosserat approach can be particularly useful in modeling nonlinear elastic inhomogeneous deformations and inelastic deformations.

7.2 Formulation of the numerical solution of spherically symmetric problems using the theory of a Cosserat shell

In (Rubin, 1987b) it was shown that the numerical solution of spherically problems in continuum mechanics can be formulated using the theory of a Cosserat shell. The main idea is to divide the spherical region into N concentric spherical shells that are in contact at their common interfaces and are subjected to spherically symmetric motions. The response of the I'th spherical shell is modeled using the theory of a Cosserat shell. Moreover, since the theory of a Cosserat shell allows for general surface tractions to be applied to its major surfaces, it is possible to develop both kinematic and kinetic coupling conditions at these interfaces.

In formulating this problem it is convenient to first write the equations for a single spherical shell and then to write the equations for the I'th shell of a collection of N connected spherical shells that model a spherical region. To this end, consider a shell made of a homogeneous material with constant reference mass density ρ_0^*, inner radius R_1^* and outer radius R_2^*. Moreover, the material that is used to construct the shell is taken to be isotropic and nonlinear elastic with material viscous dissipation.

Specifically, using the spherical polar coordinates described in appendix B, the position vector $\mathbf{X}^*$ of points in the three-dimensional spherical region associated with the stress-free reference configuration is characterized by

$$\mathbf{X}^* = R^*\,\mathbf{e}_r(\theta,\phi) \ , \quad R_1^* \le R^* \le R_2^* \ , \quad \theta^i = (\theta,\phi,R^*) \ , \tag{7.2.1}$$

and the mean radius R and thickness H associated with this spherical region are

$$R = \frac{1}{2}(R_1^* + R_2^*) \ , \quad H = R_2^* - R_1^* \ . \tag{7.2.2}$$

Moreover, for the Cosserat model of this shell the position vector of the reference surface $\mathbf{X}$ and the director $\mathbf{D}_3$ associated with the reference configuration are taken to be

$$\mathbf{X} = R\,\mathbf{e}_r(\theta,\phi) \ , \quad \mathbf{D}_3 = \mathbf{e}_r(\theta,\phi) \ . \tag{7.2.3}$$

It then follows from the results (4.22.3) and (4.22.4), that the reference configuration of the shell is characterized by

$$\mathbf{D}_1 = R\,\mathbf{e}_\theta \ , \quad \mathbf{D}^1 = \frac{1}{R}\,\mathbf{e}_\theta \ , \quad \mathbf{D}_2 = R\sin\theta\,\mathbf{e}_\phi \ , \quad \mathbf{D}^2 = \frac{1}{R\sin\theta}\,\mathbf{e}_\phi \ ,$$

$$\mathbf{D}_3 = \mathbf{e}_r = \mathbf{D}^3 \ , \quad D^{1/2} = A^{1/2} = R^2\sin\theta \ , \quad A^{1/2}\,\bar{H} = V\sin\theta \ , \quad V = \frac{R_2^{*3} - R_1^{*3}}{3} \ ,$$

$$\mathbf{H}^1 = \frac{H^3}{12V}\,\mathbf{e}_\theta \ , \quad \mathbf{H}^2 = \frac{H^3}{12V\sin\theta}\,\mathbf{e}_\phi \ ,$$

$$m = \rho_0^*\,V\sin\theta \ , \quad y^3 = \left\{\frac{RH^3}{V}\right\}\left[2\gamma_1 + \gamma_2\left\{\frac{H}{R}\right\}\right] \ ,$$

$$y^{33} = \left\{\frac{R^2H^3}{\pi^2 V}\right\}\left[1 + 2\gamma_3\left\{\frac{H}{R}\right\} + \gamma_4\left\{\frac{H}{R}\right\}^2\right] \ , \tag{7.2.4}$$

where the quantity V has been introduced for convenience and the constants $\gamma_1, \gamma_2, \gamma_3$ and γ_4 are given by (4.21.8), (4.23.11), (4.21.3) and (4.23.4),

$$\gamma_1 = 0.07 \ , \ \ \gamma_2 = 0.03 \ , \ \ \gamma_3 = \frac{\pi^2 - 4}{8} \ , \ \ \gamma_4 = -\frac{\pi^2 - 6}{24} \ . \tag{7.2.5}$$

In the present configuration the shell region is characterized by its inner radius $r_1^*(t)$, its outer radius $r_2^*(t)$, its means radius $r(t)$ and its thickness $h(t)$ such that

$$r(t) = \frac{1}{2}(r_1^* + r_2^*) \ , \ \ h = r_2^* - r_1^* \ . \tag{7.2.6}$$

Also, the present configuration of the Cosserat shell is characterized by the position vector $\mathbf{x}$ to the reference surface of the shell and the director $\mathbf{d}_3$

$$\mathbf{x} = r(t)\,\mathbf{e}_r(\theta,\phi) \ , \ \ \mathbf{d}_3 = \frac{h}{H}\,\mathbf{e}_r(\theta,\phi) \ . \tag{7.2.7}$$

Then, using the kinematic expressions summarized in section 4.27 it can be shown that

$$\mathbf{d}_1 = r\,\mathbf{e}_\theta \ , \ \ \mathbf{d}_2 = r\sin\theta\,\mathbf{e}_\phi \ , \ \ a^{1/2} = r^2\sin\theta \ ,$$

$$\mathbf{d}^1 = \frac{1}{r}\mathbf{e}_\theta \ , \ \ \mathbf{d}^2 = \frac{1}{r\sin\theta}\mathbf{e}_\phi \ , \ \ \mathbf{d}^3 = \frac{H}{h}\mathbf{e}_r \ ,$$

$$d^{1/2} = \frac{r^2 h}{H}\sin\theta \ , \ \ \mathbf{d}_{3,1} = \frac{h}{H}\mathbf{e}_\theta \ , \ \ \mathbf{d}_{3,2} = \frac{h}{H}\sin\theta\,\mathbf{e}_\phi \ ,$$

$$\mathbf{F} = \frac{h}{H}(\mathbf{e}_r\otimes\mathbf{e}_r) + \frac{r}{R}(\mathbf{e}_\theta\otimes\mathbf{e}_\theta + \mathbf{e}_\phi\otimes\mathbf{e}_\phi) \ , \ \ \mathbf{F}^{-1} = \frac{H}{h}(\mathbf{e}_r\otimes\mathbf{e}_r) + \frac{R}{r}(\mathbf{e}_\theta\otimes\mathbf{e}_\theta + \mathbf{e}_\phi\otimes\mathbf{e}_\phi) \ ,$$

$$\boldsymbol{\beta}_1 = [\frac{Rh}{rH} - 1]\,\mathbf{e}_\theta \ , \ \ \boldsymbol{\beta}_2 = [\frac{Rh}{rH} - 1]\sin\theta\,\mathbf{e}_\phi \ ,$$

$$\boldsymbol{\beta}_\alpha\otimes\mathbf{H}^\alpha = \beta\,(\mathbf{e}_\theta\otimes\mathbf{e}_\theta + \mathbf{e}_\phi\otimes\mathbf{e}_\phi) \ , \ \ \beta = \frac{H^3}{12V}[\frac{Rh}{rH} - 1] \ ,$$

$$\bar{\mathbf{F}} = \frac{h}{H}(\mathbf{e}_r\otimes\mathbf{e}_r) + \frac{r}{R}(1+\beta)(\mathbf{e}_\theta\otimes\mathbf{e}_\theta + \mathbf{e}_\phi\otimes\mathbf{e}_\phi) \ , \ \ \bar{J} = \frac{h}{H}\{\frac{r}{R}(1+\beta)\}^2 \ ,$$

$$\bar{\mathbf{B}} - \frac{1}{3}(\bar{\mathbf{B}}\bullet\mathbf{I})\,\mathbf{I} = \frac{1}{3}[\{\frac{h}{H}\}^2 - \{\frac{r}{R}(1+\beta)\}^2]\,[2\,(\mathbf{e}_r\otimes\mathbf{e}_r) - (\mathbf{e}_\theta\otimes\mathbf{e}_\theta + \mathbf{e}_\phi\otimes\mathbf{e}_\phi)] \ ,$$

$$\bar{\mathbf{B}}^2 - \frac{1}{3}(\bar{\mathbf{B}}^2\bullet\mathbf{I})\,\mathbf{I} = \frac{1}{3}[\{\frac{h}{H}\}^4 - \{\frac{r}{R}(1+\beta)\}^4]\,[2\,(\mathbf{e}_r\otimes\mathbf{e}_r) - (\mathbf{e}_\theta\otimes\mathbf{e}_\theta + \mathbf{e}_\phi\otimes\mathbf{e}_\phi)] \ ,$$

$$\mathbf{D} = \frac{\dot{h}}{h}(\mathbf{e}_r\otimes\mathbf{e}_r) + \frac{\dot{r}}{r}(\mathbf{e}_\theta\otimes\mathbf{e}_\theta + \mathbf{e}_\phi\otimes\mathbf{e}_\phi) \ ,$$

$$\mathbf{D}' = \frac{1}{3}[\frac{\dot{h}}{h} - \frac{\dot{r}}{r}]\,[2\,(\mathbf{e}_r\otimes\mathbf{e}_r) - (\mathbf{e}_\theta\otimes\mathbf{e}_\theta + \mathbf{e}_\phi\otimes\mathbf{e}_\phi)] \ ,$$

$$\dot{\boldsymbol{\beta}}_1 = \frac{Rh}{rH}[\frac{\dot{h}}{h} - \frac{\dot{r}}{r}]\,\mathbf{e}_\theta \ , \ \ \dot{\boldsymbol{\beta}}_2 = \frac{Rh}{rH}[\frac{\dot{h}}{h} - \frac{\dot{r}}{r}]\sin\theta\,\mathbf{e}_\phi \ . \tag{7.2.8}$$

In this section it is of interest to model nonlinear elastic response with viscous dissipation so using (4.33.4) the constitutive equations are separated into two parts

$$\mathbf{T} = \hat{\mathbf{T}} + \check{\mathbf{T}} \ , \ \ \mathbf{t}^i = \hat{\mathbf{t}}^i + \check{\mathbf{t}}^i \ , \ \ \mathbf{m}^\alpha = \hat{\mathbf{m}}^\alpha + \check{\mathbf{m}}^\alpha \ , \tag{7.2.9}$$

where $\{\hat{\mathbf{T}}, \hat{\mathbf{t}}^i, \hat{\mathbf{m}}^\alpha\}$ characterize the elastic parts and $\{\check{\mathbf{T}}, \check{\mathbf{t}}^i, \check{\mathbf{m}}^\alpha\}$ characterize the viscous parts. Specifically, the strain energy function for the isotropic nonlinear elastic response of the shell is given by (4.11.27), (4.25.9), (4.15.25)

$$m\,\Sigma = m\,\Sigma^* + \frac{1}{2}\,A^{1/2}\,\bar{H}\,\mathbf{K}^{\alpha\beta}\bullet(\boldsymbol{\beta}_\alpha\otimes\boldsymbol{\beta}_\beta) \ ,$$

$$m\,\Sigma^* = \frac{1}{2}A^{1/2}\,\bar{H}\,\mu^*\,\left[(1 - 4C_2)\,(\bar{\alpha}_1 - 3) + C_2\,(\bar{\alpha}_2 - 3)\right]$$

$$+ A^{1/2}\bar{H}\,K^*\big[(\bar{J}-1)-\ln(\bar{J})\big] \ ,$$

$$\mathbf{K}^{11} = \frac{H^2\mu^*}{12R^2}\Big[\Big\{\frac{2}{1-\nu^*}\Big\}\mathbf{e}_\theta\otimes\mathbf{e}_\theta + \mathbf{e}_\phi\otimes\mathbf{e}_\phi\Big] \ ,$$

$$\mathbf{K}^{22} = \frac{H^2\mu^*}{12R^2\sin^2\theta}\Big[\mathbf{e}_\theta\otimes\mathbf{e}_\theta + \Big\{\frac{2}{1-\nu^*}\Big\}\ \mathbf{e}_\phi\otimes\mathbf{e}_\phi\Big] \ ,$$

$$\mathbf{K}^{12} = \frac{H^2\mu^*}{12R^2\sin\theta}\Big[\Big\{\frac{2\nu^*}{1-\nu^*}\Big\}\mathbf{e}_\theta\otimes\mathbf{e}_\phi + \mathbf{e}_\phi\otimes\mathbf{e}_\theta\Big] \ , \quad \mathbf{K}^{21} = (\mathbf{K}^{12})^T \ , \tag{7.2.10}$$

where the material constants μ^*, K^* and ν^* associated with small deformations are related by Table 3.12.1, C_2 is a material constant associated with nonlinear elastic response, and the vectors $\mathbf{M}_i$ have been specified by (4.12.8) so that

$$\mathbf{M}_1 = \mathbf{e}_\theta \ , \quad \mathbf{M}_2 = \mathbf{e}_\phi \ , \quad \mathbf{M}_3 = \mathbf{e}_r \ . \tag{7.2.11}$$

Also, the elastic constitutive equations associated with the forms (7.2.10) are given by (4.25.3) and (4.25.4)

$$\hat{\mathbf{T}} = -\hat{p}\,\mathbf{I} + \mathbf{T}' \ , \quad \hat{\mathbf{T}}'\bullet\mathbf{I} = 0 \ , \quad a^{1/2}\hat{p} = \bar{J}\,A^{1/2}\bar{H}\,K^*\Big[\frac{1}{\bar{J}} - 1\Big] \ ,$$

$$a^{1/2}\hat{\mathbf{T}}' = \bar{J}^{-2/3}A^{1/2}\bar{H}\,\mu^*\Big[(1-4C_2)\Big\{\bar{\mathbf{B}} - \frac{1}{3}(\bar{\mathbf{B}}\bullet\mathbf{I})\,\mathbf{I}\Big\}$$

$$+ 2\,C_2\,\bar{J}^{-2/3}\Big\{\bar{\mathbf{B}}^2 - \frac{1}{3}(\bar{\mathbf{B}}^2\bullet\mathbf{I})\,\mathbf{I}\Big\}\Big] \ ,$$

$$\hat{\mathbf{m}}^\alpha = a^{1/2}\hat{\mathbf{T}}\,\bar{\mathbf{F}}^{-T}\mathbf{H}^\alpha + A^{1/2}\bar{H}\,\mathbf{F}^{-T}\mathbf{K}^{\alpha\beta}\,\boldsymbol{\beta}_\beta \ , \quad \hat{\mathbf{t}}^i = a^{1/2}\hat{\mathbf{T}}\,\mathbf{d}^i - \hat{\mathbf{m}}^\alpha\,(\mathbf{d}_{3,\alpha}\bullet\mathbf{d}^i) \ . \tag{7.2.12}$$

Moreover, using the expressions (7.2.8) it can be shown that

$$A^{1/2}\bar{H}\,\mathbf{F}^{-T}\mathbf{K}^{1\beta}\,\boldsymbol{\beta}_\beta = \hat{M}_1\sin\theta\,\mathbf{e}_\theta \ , \quad A^{1/2}\bar{H}\,\mathbf{F}^{-T}\mathbf{K}^{2\beta}\,\boldsymbol{\beta}_\beta = \hat{M}_1\,\mathbf{e}_\phi \ ,$$

$$\hat{M}_1 = \Big[\frac{VH^2\mu^*}{6Rr}\Big]\Big[\frac{1+\nu^*}{1-\nu^*}\Big]\Big[\frac{Rh}{rH} - 1\Big] \ ,$$

$$a^{1/2}\hat{\mathbf{T}} = \frac{\mu^*V}{3}\Big[\frac{3K^*}{\mu^*}(\bar{J}-1)\,\mathbf{I} + \Big\{\bar{J}^{-2/3}(1-4C_2)\Big(\Big\{\frac{h}{H}\Big\}^2 - \Big\{\frac{r}{R}(1+\beta)\Big\}^2\Big)$$

$$+ 2C_2\bar{J}^{-4/3}\Big(\Big\{\frac{h}{H}\Big\}^4 - \Big\{\frac{r}{R}(1+\beta)\Big\}^4\Big)\Big\}\big\{2\,(\mathbf{e}_r\otimes\mathbf{e}_r) - (\mathbf{e}_\theta\otimes\mathbf{e}_\theta + \mathbf{e}_\phi\otimes\mathbf{e}_\phi)\big\}\Big]\sin\theta \ ,$$

$$a^{1/2}\hat{\mathbf{T}}\,\mathbf{d}^1 = \hat{T}_1\sin\theta\,\mathbf{e}_\theta \ , \quad a^{1/2}\hat{\mathbf{T}}\,\mathbf{d}^2 = \hat{T}_1\,\mathbf{e}_\phi \ , \quad a^{1/2}\hat{\mathbf{T}}\,\mathbf{d}^3 = \hat{T}_3\sin\theta\,\mathbf{e}_r \ ,$$

$$\hat{T}_1 = \frac{\mu^*V}{3r}\Big[\frac{3K^*}{\mu^*}(\bar{J}-1) - \bar{J}^{-2/3}(1-4C_2)\Big(\Big\{\frac{h}{H}\Big\}^2 - \Big\{\frac{r}{R}(1+\beta)\Big\}^2\Big)$$

$$- 2C_2\bar{J}^{-4/3}\Big(\Big\{\frac{h}{H}\Big\}^4 - \Big\{\frac{r}{R}(1+\beta)\Big\}^4\Big)\Big] \ ,$$

$$\hat{T}_3 = \frac{\mu^*VH}{3h}\Big[\frac{3K^*}{\mu^*}(\bar{J}-1) + 2\,\bar{J}^{-2/3}(1-4C_2)\Big(\Big\{\frac{h}{H}\Big\}^2 - \Big\{\frac{r}{R}(1+\beta)\Big\}^2\Big)$$

$$+ 4C_2\bar{J}^{-4/3}\Big(\Big\{\frac{h}{H}\Big\}^4 - \Big\{\frac{r}{R}(1+\beta)\Big\}^4\Big)\Big] \ ,$$

$$a^{1/2}\hat{\mathbf{T}}\,\bar{\mathbf{F}}^{-T}\mathbf{H}^1 = \hat{M}_2\sin\theta\,\mathbf{e}_\theta \ , \quad a^{1/2}\hat{\mathbf{T}}\,\bar{\mathbf{F}}^{-T}\mathbf{H}^2 = \hat{M}_2\,\mathbf{e}_\phi \ , \quad \hat{M}_2 = \frac{H^3R}{12V(1+\beta)}\,[\hat{T}_1] \ ,$$

$$\hat{\mathbf{m}}^1 = \hat{M} \sin\theta\, \mathbf{e}_\theta \ , \quad \hat{\mathbf{m}}^2 = \hat{M}\, \mathbf{e}_\phi \ , \quad \hat{M} = \hat{M}_1 + \hat{M}_2 \ ,$$
$$\hat{\mathbf{t}}^1 = [\hat{T}_1 - \frac{h}{Hr}\hat{M}] \sin\theta\, \mathbf{e}_\theta \ , \quad \hat{\mathbf{t}}^2 = [\hat{T}_1 - \frac{h}{Hr}\hat{M}]\, \mathbf{e}_\phi \ , \quad \hat{\mathbf{t}}^3 = \hat{T}_3 \sin\theta\, \mathbf{e}_r \ . \quad (7.2.13)$$

Also, the viscous parts are given by (4.33.10)

$$a^{1/2} \check{\mathbf{T}} = A^{1/2} \bar{H}\, [\eta_1\, (\mathbf{D} \bullet \mathbf{I})\, \mathbf{I} + 2\eta_2\, \mathbf{D}'] \ ,$$
$$\check{\mathbf{m}}^1 = \eta_3\, A^{1/2} \bar{H}\, (\mathbf{D}^1 \bullet \mathbf{D}^1)\, \mathbf{F}^{-T} \dot{\boldsymbol{\beta}}_1 \ , \quad \check{\mathbf{m}}^2 = \eta_4\, A^{1/2} \bar{H}\, (\mathbf{D}^2 \bullet \mathbf{D}^2)\, \mathbf{F}^{-T} \dot{\boldsymbol{\beta}}_2 \ ,$$
$$\eta_4 = \eta_3 \ , \quad \check{\mathbf{t}}^i = a^{1/2}\, \check{\mathbf{T}}\, \mathbf{d}^i - \check{\mathbf{m}}^\alpha\, (\mathbf{d}_{3,\alpha} \bullet \mathbf{d}^i) \ , \quad (7.2.14)$$

where η_4 has been taken to be equal to η_3 due to spherical symmetry. Then, with the help of (7.2.8) these equations can be rewritten in the forms

$$\check{\mathbf{m}}^1 = \check{M} \sin\theta\, \mathbf{e}_\theta \ , \quad \check{\mathbf{m}}^2 = \check{M}\, \mathbf{e}_\phi \ , \quad \check{M} = \eta_3\, \bar{H}\, [\frac{R^2 h}{r^2 H}][\frac{\dot{h}}{h} - \frac{\dot{r}}{r}] \ ,$$
$$a^{1/2}\, \check{\mathbf{T}}\, \mathbf{d}^1 = \check{T}_1 \sin\theta\, \mathbf{e}_\theta \ , \quad a^{1/2}\, \check{\mathbf{T}}\, \mathbf{d}^2 = \check{T}_1\, \mathbf{e}_\phi \ , \quad a^{1/2}\, \check{\mathbf{T}}\, \mathbf{d}^3 = \check{T}_3 \sin\theta\, \mathbf{e}_r \ ,$$
$$\check{T}_1 = \frac{R^2}{r} \bar{H}\, [\eta_1\, \{\frac{\dot{h}}{h} + 2\frac{\dot{r}}{r}\} - \frac{2}{3}\eta_2 \{\frac{\dot{h}}{h} - \frac{\dot{r}}{r}\}] \ ,$$
$$\check{T}_3 = \frac{R^2 H}{h} \bar{H}\, [\eta_1\, \{\frac{\dot{h}}{h} + 2\frac{\dot{r}}{r}\} + \frac{4}{3}\eta_2 \{\frac{\dot{h}}{h} - \frac{\dot{r}}{r}\}] \ ,$$
$$\check{\mathbf{t}}^1 = [\check{T}_1 - \frac{h}{H}\check{M}\,] \sin\theta\, \mathbf{e}_\theta \ , \quad \check{\mathbf{t}}^2 = [\check{T}_1 - \frac{h}{H}\check{M}\,]\, \mathbf{e}_\phi \ , \quad \check{\mathbf{t}}^3 = \check{T}_3 \sin\theta\, \mathbf{e}_r \ . \quad (7.2.15)$$

Next, collecting these results, the constitutive equations (7.2.9) become

$$\mathbf{m}^1 = M \sin\theta\, \mathbf{e}_\theta \ , \quad \mathbf{m}^2 = M\, \mathbf{e}_\phi \ , \quad M = \hat{M}_1 + \hat{M}_2 + \check{M} \ , \quad \mathbf{m}^\alpha{}_{,\alpha} = -\,2\, M \sin\theta\, \mathbf{e}_r \ ,$$
$$\mathbf{t}^1 = T_1 \sin\theta\, \mathbf{e}_\theta \ , \quad \mathbf{t}^2 = T_1\, \mathbf{e}_\phi \ , \quad \mathbf{t}^3 = T_3 \sin\theta\, \mathbf{e}_r \ ,$$
$$T_1 = [\hat{T}_1 + \check{T}_1] - \frac{h}{Hr}[M] \ , \quad T_3 = \hat{T}_3 + \check{T}_3 \ , \quad \mathbf{t}^\alpha{}_{,\alpha} = -\,2\, T_1 \sin\theta\, \mathbf{e}_r \ . \quad (7.2.16)$$

For the spherically symmetric problems under consideration, the external body force $\mathbf{b}^*$ is neglected so from (4.19.14) the assigned fields associated with this body force vanish

$$\mathbf{b}_b = 0 \ , \quad \mathbf{b}_b^3 = 0 \ , \quad (7.2.17)$$

and the assigned fields (4.3.28) and (4.3.42) are determined solely by their parts associated with contact forces on the major surfaces of the shell

$$\mathbf{b} = \mathbf{b}_c \ , \quad \mathbf{b}^3 = \mathbf{b}_c^3 \ . \quad (7.2.18)$$

Next, using the equations summarized in section 4.27, and taking p_1 to be the pressure applied to the inner surface of the shell, and p_2 to be the pressure applied to the outer surface of the shell, it can be shown that

$$m\, \mathbf{b} = [r_1^{*2}\, p_1 - r_2^{*2}\, p_2] \sin\theta\, \mathbf{e}_r \ , \quad m\, \mathbf{b}^3 = -\frac{H}{2}[r_1^{*2}\, p_1 + r_2^{*2}\, p_2] \sin\theta\, \mathbf{e}_r \ . \quad (7.2.19)$$

Now, with the help of (7.2.4), (7.2.7), (7.2.16) and (7.2.19), the balances of linear momentum and director momentum (4.4.35) for this shell reduce to two scalar equations of the forms

$$\rho_0^* V [\ddot{r} + y^3 \frac{\ddot{h}}{H}] = [r_1^{*2} p_1 - r_2^{*2} p_2] - 2 T_1 \ ,$$

$$\rho_0^* V [y^3 \ddot{r} + y^{33} \frac{\ddot{h}}{H}] = - \frac{H}{2} [r_1^{*2} p_1 + r_2^{*2} p_2] - T_3 - 2 M \ , \tag{7.2.20}$$

where it is recalled that the quantities $\{T_1, T_3, M\}$ include parts due to viscous dissipation which are linear in the rates $\{\dot{h}, \dot{r}\}$. Moreover, these equations can be solved for the pressures p_1 and p_2 to obtain

$$r_1^{*2} p_1 = T_1 - \frac{1}{H} [T_3 + 2 M] + \frac{1}{2} \rho_0^* V [\ddot{r} + y^3 \frac{\ddot{h}}{H}] - \frac{1}{H} \rho_0^* V [y^3 \ddot{r} + y^{33} \frac{\ddot{h}}{H}] \ ,$$

$$r_2^{*2} p_2 = - T_1 - \frac{1}{H} [T_3 + 2 M] - \frac{1}{2} \rho_0^* V [\ddot{r} + y^3 \frac{\ddot{h}}{H}] - \frac{1}{H} \rho_0^* V [y^3 \ddot{r} + y^{33} \frac{\ddot{h}}{H}] \ . \tag{7.2.21}$$

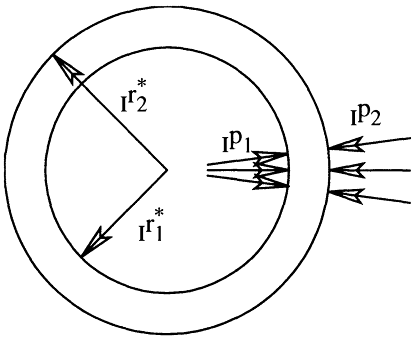

Fig. 7.2.1 Present configuration of the I'th spherical Cosserat shell showing the geometry and the contact pressures.

Next, consider a spherical region in its reference configuration which is divided into N concentric spherical shells which are in contact. The inner radius of the I'th spherical shell is denoted by R_I^*, its outer radius is denoted by R_{I+1}^*, its mean radius is denoted by R_I and its thickness is denoted by H_I such that

$$R_I = \frac{1}{2} (R_I^* + R_{I+1}^*) \ , \quad H_I = R_{I+1}^* - R_I^* \qquad \text{for I=1,2,\ldots,N} \ . \tag{7.2.22}$$

Also, in its present configuration the inner radius of the I'th spherical shell is denoted by $_I r_1^*$, its outer radius is denoted by $_I r_2^*$, its mean radius is denoted by r_I, and its thickness is denoted by h_I (see Fig. 7.2.1) such that

$$r_I(t) = \frac{1}{2} ({}_I r_1^* + {}_I r_2^*) \ , \quad h_I(t) = {}_I r_2^* - {}_I r_1^* \qquad \text{for I=1,2,\ldots,N} \ . \tag{7.2.23}$$

Moreover, the equations of motion for this I'th shell can be written in forms similar to (7.2.21)

$${}_I r_1^{*2} \, {}_I p_1 = {}_I T_1 - \frac{1}{H_I} [{}_I T_3 + 2 \, {}_I M] + \frac{1}{2} \rho_0^* V_I [\ddot{r}_I + {}_I y^3 \frac{\ddot{h}_I}{H_I}]$$

$$-\frac{1}{H_I}\rho_0^* V_I\,[\,{}_Iy^3\,\ddot{r}_I + {}_Iy^{33}\frac{\ddot{h}_I}{H_I}]\ ,$$

$$ {}_Ir_2^{*2}\,{}_Ip_2 = -\,{}_IT_1 - \frac{1}{H_I}[{}_IT_3 + 2\,{}_IM] - \frac{1}{2}\rho_0^* V_I\,[\,\ddot{r}_I + {}_Iy^3\frac{\ddot{h}_I}{H_I}]$$

$$-\frac{1}{H_I}\rho_0^* V_I\,[\,{}_Iy^3\,\ddot{r}_I + {}_Iy^{33}\frac{\ddot{h}_I}{H_I}] \quad \text{for } I=1,2,\ldots,N\ , \tag{7.2.24}$$

where the subscript I identifies terms associated with the I'th shell and no sum is indicated over repeated upper cased letters. Thus, for example, ${}_Ip_1$ is the pressure applied to the inner surface of the I'th shell and ${}_Ip_2$ is the pressure applied to its outer surface. Also, the kinematics and constitutive equations for the I'th shell are summarized by the following equations

$$V_I = \frac{R_{I+1}^{*\,3} - R_I^{*3}}{3}\ ,\quad {}_Iy^3 = \{\frac{R_IH_I^3}{V_I}\}\ [2\gamma_1 + \gamma_2\{\frac{H_I}{R_I}\}]\ ,$$

$$ {}_Iy^{33} = \{\frac{R_I^2H_I^3}{\pi^2 V_I}\}\ [1 + 2\gamma_3\{\frac{H_I}{R_I}\} + \gamma_4\{\frac{H_I}{R_I}\}^2]\ ,$$

$$\beta_I = \frac{H_I^3}{12V_I}[\frac{R_Ih_I}{r_IH_I} - 1]\ ,\quad \bar{J}_I = \frac{h_I}{H_I}\{\frac{r_I}{R_I}(1+\beta_I)\}^2\ ,\quad {}_I\hat{M}_1 = [\frac{V_IH_I^2\mu^*}{6R_Ir_I}][\frac{1+v^*}{1-v^*}][\frac{R_Ih_I}{r_IH_I} - 1]\ ,$$

$$ {}_I\hat{M}_2 = \frac{H_I^3R_I}{12V_I(1+\beta_I)}\ [{}_I\hat{T}_1]\ ,\quad {}_I\check{M} = \eta_3\,[\frac{R_I^2h_I}{r_I^2H_I}][\frac{\dot{h}_I}{h_I} - \frac{\dot{r}_I}{r_I}]\ ,\quad {}_IM = {}_I\hat{M}_1 + {}_I\hat{M}_2 + {}_I\check{M}\ ,$$

$$ {}_I\hat{T}_1 = \frac{\mu^*V_I}{3r_I}\,[\,\frac{3K^*}{\mu^*}(\bar{J}_I - 1) - \bar{J}_I^{-2/3}\,(1-4C_2)(\{\frac{h_I}{H_I}\}^2 - \{\frac{r_I}{R_I}(1+\beta_I)\}^2)$$

$$ - 2C_2\bar{J}_I^{-4/3}\,(\{\frac{h_I}{H_I}\}^4 - \{\frac{r_I}{R_I}(1+\beta_I)\}^4)]\ ,$$

$$ {}_I\hat{T}_3 = \frac{\mu^*V_IH_I}{3h_I}\,[\,\frac{3K^*}{\mu^*}(\bar{J}_I - 1) + 2\,\bar{J}_I^{-2/3}\,(1-4C_2)(\{\frac{h_I}{H_I}\}^2 - \{\frac{r_I}{R_I}(1+\beta_I)\}^2)$$

$$ + 4C_2\bar{J}_I^{-4/3}\,(\{\frac{h_I}{H_I}\}^4 - \{\frac{r_I}{R_I}(1+\beta_I)\}^4)]\ ,$$

$$ {}_I\check{T}_1 = \frac{R_I^2}{r_I}[\eta_1\,\{\frac{\dot{h}_I}{h_I} + 2\frac{\dot{r}_I}{r_I}\} - \frac{2}{3}\eta_2\{\frac{\dot{h}_I}{h_I} - \frac{\dot{r}_I}{r_I}\}]\ ,$$

$$ {}_I\check{T}_3 = \frac{R_I^2H_I}{h_I}[\eta_1\,\{\frac{\dot{h}_I}{h_I} + 2\frac{\dot{r}_I}{r_I}\} + \frac{4}{3}\eta_2\{\frac{\dot{h}_I}{h_I} - \frac{\dot{r}_I}{r_I}\}]\ ,$$

$$ {}_IT_1 = [{}_I\hat{T}_1 + {}_I\check{T}_1] - \frac{h_I}{H_Ir_I}[{}_IM]\ ,\quad {}_IT_3 = {}_I\hat{T}_3 + {}_I\check{T}_3\ . \tag{7.2.25}$$

The equations (7.2.24) model the motion of the I'th spherical shell. In order to solve for the motion of the entire spherical region it is necessary to couple the equations of motion of neighboring spherical shells at their common interfaces. This is done by specifying kinematic coupling conditions which require

$$ {}_{I-1}r_2^* = {}_I r_1^* = r_I^*(t) \quad \text{for } I=2,3,\ldots,N \ , \tag{7.2.26}$$

where r_I^* is the radius of the common interface of the I'th and the (I+1)'th spherical shells. Specifically, the coupling equation (7.2.26) requires the outer surface of the (I−1)'th shell to be located at the same radius as the inner surface of the I'th shell. Moreover, it is necessary to specify kinetic coupling equations which require

$$ {}_{I-1}p_2 = {}_I p_1 = p_I^*(t) \quad \text{for } I=2,3,\ldots,N \ , \tag{7.2.27}$$

where $p_I^*(t)$ is the contact pressure acting on the common interface of the I'th and the (I+1)'th spherical shells. Specifically, the coupling equation (7.2.27) requires the pressure acting on the outer surface of the (I−1)'th shell to be the same as that acting on the inner surface of the I'th shell.

The kinetic coupling equations (7.2.27) represent (N−1) equations to determine N+1 unknown locations r_I^* (I=1,2,...,N+1). The remaining two equations are specified by boundary conditions at the inner surface r_1^*, and the outer surface r_{N+1}^* of the entire spherical region. Specifically, these boundary conditions require specification of either the kinematics or the kinetics and can be summarized in the forms

$$\begin{gathered} \text{Specify } \{r_1^* \text{ or } p_1^*\} \text{ on the inner surface} \ , \\ \text{Specify } \{r_{N+1}^* \text{ or } p_{N+1}^*\} \text{ on the outer surface} \ . \end{gathered} \tag{7.2.28}$$

It then follows with the help of (7.2.24) that the pressure p_I^* is given by

$$\begin{gathered}
r_1^{*2}\, p_1^* = {}_1T_1 - \frac{1}{H_1}[{}_1T_3 + 2\,{}_1M] + \frac{1}{2}\rho_0^* V_1 \,[\, \ddot{r}_1 + {}_1y^3 \frac{\ddot{h}_1}{H_1}] \\
- \frac{1}{H_1}\rho_0^* V_1 \,[\, {}_1y^3\, \ddot{r}_1 + {}_1y^{33} \frac{\ddot{h}_1}{H_1}] \ , \\
r_I^{*2}\, p_I^* = {}_IT_1 - \frac{1}{H_I}[{}_IT_3 + 2\,{}_IM] + \frac{1}{2}\rho_0^* V_I \,[\, \ddot{r}_I + {}_Iy^3 \frac{\ddot{h}_I}{H_I}] \\
- \frac{1}{H_I}\rho_0^* V_I \,[\, {}_Iy^3\, \ddot{r}_I + {}_Iy^{33} \frac{\ddot{h}_I}{H_I}] \quad \text{for } I=2,3,\ldots,N \ , \\
r_{N+1}^{*\ 2}\, p_{N+1}^* = -\,{}_NT_1 - \frac{1}{H_N}[{}_NT_3 + 2\,{}_NM] - \frac{1}{2}\rho_0^* V_N \,[\, \ddot{r}_N + {}_Ny^3 \frac{\ddot{h}_N}{H_N}] \\
- \frac{1}{H_N}\rho_0^* V_N \,[\, {}_Ny^3\, \ddot{r}_N + {}_Ny^{33} \frac{\ddot{h}_N}{H_N}] \ .
\end{gathered} \tag{7.2.29}$$

Also, the coupling equations (7.2.27) can be rewritten in the forms

$$\begin{gathered}
-\,{}_{I-1}T_1 - \frac{1}{H_{I-1}}[{}_{I-1}T_3 + 2\,{}_{I-1}M] - \frac{1}{2}\rho_0^* V_{I-1} \,[\, \ddot{r}_{I-1} + {}_{I-1}y^3 \frac{\ddot{h}_{I-1}}{H_{I-1}}] \\
- \frac{1}{H_{I-1}}\rho_0^* V_{I-1} \,[\, {}_{I-1}y^3\, \ddot{r}_{I-1} + {}_{I-1}y^{33} \frac{\ddot{h}_{I-1}}{H_{I-1}}]
\end{gathered}$$

$$= {}_I T_1 - \frac{1}{H_I}[{}_I T_3 + 2\,{}_I M] + \frac{1}{2}\rho_0^* V_I\,[\,\ddot{r}_I + {}_I y^3 \frac{\ddot{h}_I}{H_I}]$$

$$- \frac{1}{H_I}\rho_0^* V_I\,[\,{}_I y^3\,\ddot{r}_I + {}_I y^{33}\frac{\ddot{h}_I}{H_I}] \;. \tag{7.2.30}$$

Thus, the boundary conditions (7.2.28) and the coupling conditions (7.2.30) represent N+1 equations to determine the N+1 unknowns

$$\{r_1^* \text{ or } p_1^*\}\;,\;\{r_I^* \text{ for I=2,3,...,N}\}\;,\;\{r_{N+1}^* \text{ or } p_{N+1}^*\}\;. \tag{7.2.31}$$

This Cosserat formulation is valid for spherically symmetric problems that include dynamic vibration and wave propagation. However, if attention is confined to static problems, then this formulation can be compared with well known linear elastic solutions, and numerical integration of the full three-dimensional equations. To this end, it is first recalled from (Sokolnikoff, 1956, sec. 94) that the linear elastic solution for static spherically symmetric problems can be written in the form

$$T_{rr}^* = B_1 - \frac{B_2}{R^{*2}}\;,\quad T_{\theta\theta}^* = B_1 + \frac{B_2}{2R^{*2}}\;,\quad u_r^* = r^* - R^* = C_1 R^* + \frac{C_2}{R^{*2}}\;,$$

$$B_1 = \frac{p_1^*\,R_1^{*3} - p_{N+1}^*\,R_{N+1}^{*\;3}}{R_{N+1}^{*\;3} - R_1^{*3}}\;,\quad B_2 = \frac{R_1^{*3}\,R_{N+1}^{*\;3}\,(p_1^* - p_{N+1}^*)}{R_{N+1}^{*\;3} - R_1^{*3}}\;,$$

$$C_1 = \frac{(1-2\nu^*)B_1}{2\mu^*(1+\nu^*)}\;,\quad C_2 = \frac{B_2}{4\mu^*}\;, \tag{7.2.32}$$

where R^* is the reference radius, r^* is the present radius, u^* is the displacement of a material point, T_{rr}^* is the radial stress, and $T_{\theta\theta}^*$ is the circumferential stress. This form of the solution is convenient when the contact pressures p_1^* and p_{N+1}^* at the inner and outer boundaries are specified. However, in the examples considered below it is of interest to consider an additional problem where the displacement $u_1^* = u^*(R_1^*)$ of the inner boundary and the contact pressure p_{N+1}^* on the outer boundary are specified. For this case, the constants B_1 and B_2 are given by

$$B_1 = \frac{-\,p_{N+1}^* + \dfrac{4\mu^* R_1^{*2} u_1^*}{R_{N+1}^{*\;3}}}{1 + \dfrac{2(1-2\nu^*)R_1^{*3}}{(1+\nu^*)R_{N+1}^{*\;3}}}\;,\quad B_2 = 4\mu^* R_1^{*2} u_1^* - \frac{2(1-2\nu^*)R_1^{*3}B_1}{(1+\nu^*)}\;. \tag{7.2.33}$$

For the nonlinear three-dimensional theory, the position vectors $\mathbf{X}^*$ and $\mathbf{x}^*$ in the reference and present configurations are specified using spherical polar coordinates by

$$\mathbf{X}^* = R^*\,\mathbf{e}_r(\theta,\phi)\;,\quad \mathbf{x}^* = r^*(R^*)\,\mathbf{e}_r(\theta,\phi)\;. \tag{7.2.34}$$

Next, using the definitions in sections 2.3 and 3.1 and the expressions in appendix B it can be shown that the main kinematical quantities become

$$\theta^i = (R^*, \theta, \phi)\;,\quad G^{1/2} = R^{*2}\sin\theta\;,\;\mathbf{G}_1 = \mathbf{e}_r\;,\;\;\mathbf{G}_2 = R^*\,\mathbf{e}_\theta\;,\;\;\mathbf{G}_3 = R^*\sin\theta\,\mathbf{e}_\phi\;,$$

$$\mathbf{g}_1 = \frac{dr^*}{dR^*}\mathbf{e}_r \ , \quad \mathbf{g}_2 = r^* \, \mathbf{e}_\theta \ , \quad \mathbf{g}_3 = r^* \sin\theta \, \mathbf{e}_\phi \ ,$$

$$\mathbf{B}^* - \frac{1}{3}(\mathbf{B}^* \bullet \mathbf{I})\,\mathbf{I} = \frac{1}{3}\Big[\Big\{\frac{dr^*}{dR^*}\Big\}^2 - \Big\{\frac{r^*}{R^*}\Big\}^2\Big]\,[2\,(\mathbf{e}_r \otimes \mathbf{e}_r) - (\mathbf{e}_\theta \otimes \mathbf{e}_\theta + \mathbf{e}_\phi \otimes \mathbf{e}_\phi)] \ ,$$

$$J^* = \Big[\frac{r^*}{R^*}\Big]^2 \frac{dr^*}{dR^*} \ . \tag{7.2.35}$$

Moreover, with the help of the results in section 3.11 when C_2 vanishes, the three-dimensional strain energy function Σ^* associated with the form given in (7.2.10) becomes

$$\rho_0^* \Sigma^* = \frac{1}{2}\mu^* \, (\alpha_1^* - 3) + K^* \, [(J^* - 1) - \ln(J^*)] \ , \ \alpha_1^* = J^{*-2/3}\, \mathbf{B}^* \bullet \mathbf{I} \ , \tag{7.2.36}$$

and the three-dimensional stresses can be written in the forms

$$T_{rr}^* = K^* \, [1 - \frac{1}{J^*}] + \frac{2\mu^*}{3J^{*5/3}} \, \Big[\Big\{\frac{dr^*}{dR^*}\Big\}^2 - \Big\{\frac{r^*}{R^*}\Big\}^2\Big] \ ,$$

$$T_{\theta\theta}^* = K^* \, [1 - \frac{1}{J^*}] - \frac{\mu^*}{3J^{*5/3}} \, \Big[\Big\{\frac{dr^*}{dR^*}\Big\}^2 - \Big\{\frac{r^*}{R^*}\Big\}^2\Big] \ . \tag{7.2.37}$$

Also, with the help of (B.2.8) it can be shown that the equations of equilibrium reduce to a single equation of the form

$$\frac{dT_{rr}^*}{dR^*} + \Big[\frac{2(T_{rr}^* - T_{\theta\theta}^*)}{r^*}\Big] \frac{dr^*}{dR^*} = 0 \ , \tag{7.2.38}$$

which yields a second order differential equation for r^* of the form

$$\frac{d^2 r^*}{dR^{*2}} = \frac{b_1}{b_2} \ ,$$

$$b_1 = -\Big[\frac{K^*}{J^{*2}} - \frac{10\mu^*}{9J^{*8/3}}\Big(\Big\{\frac{dr^*}{dR^*}\Big\}^2 - \Big\{\frac{r^*}{R^*}\Big\}^2\Big)\Big]\Big[\frac{2r^*}{R^{*2}}\frac{dr^*}{dR^*}\Big]\Big[\frac{dr^*}{dR^*} - \frac{r^*}{R^*}\Big]$$

$$+ \frac{2\mu^*}{3J^{*5/3}}\Big[\frac{2r^*}{R^{*2}}\Big]\Big[\frac{dr^*}{dR^*} - \frac{r^*}{R^*}\Big] - \Big[\frac{2(T_{rr}^* - T_{\theta\theta}^*)}{r^*}\Big]\frac{dr^*}{dR^*} \ ,$$

$$b_2 = \Big[\frac{K^*}{J^{*2}} - \frac{10\mu^*}{9J^{*8/3}}\Big(\Big\{\frac{dr^*}{dR^*}\Big\}^2 - \Big\{\frac{r^*}{R^*}\Big\}^2\Big)\Big]\Big\{\frac{r^*}{R^*}\Big\}^2 + \Big[\frac{4\mu^*}{3J^{*5/3}}\Big]\frac{dr^*}{dR^*} \ . \tag{7.2.39}$$

These equations can be integrated numerically using the shooting method which specifies guesses for the values of

$$r^* \ \text{and} \ \frac{dr^*}{dR^*} \ \text{for} \ R^* = R_1^* \ , \tag{7.2.40}$$

and iterates on these guesses to satisfy boundary conditions on the inner and outer surfaces of the spherical region. For example, when the inner radius is specified to be r_1^* and the contact pressure on the outer surface is specified to be p_{N+1}^*, then

$$r^* = r_1^* \ \ \text{for} \ R^* = R_1^* \ \ \text{and} \ \ T_{rr}^* = -\,p_{N+1}^* \ \ \text{for} \ R^* = R_{N+1}^* \ . \tag{7.2.41}$$

For these specifications, iteration is performed on the guess for the derivative dr^*/dR^* in (7.2.40) until the stress condition $(7.2.41)_2$ is satisfied.

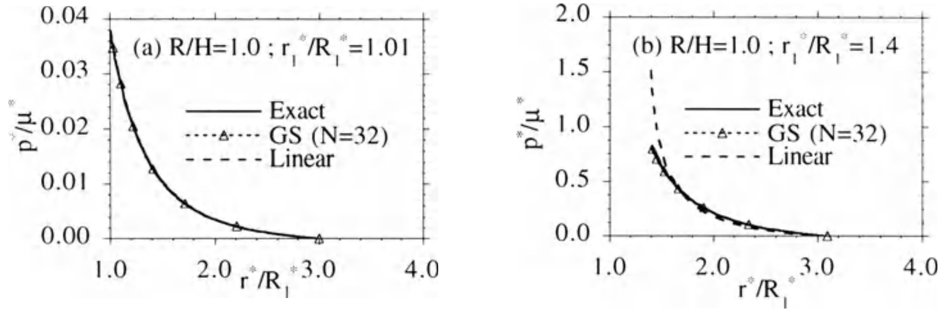

Fig. 7.2.2 Comparison of the exact, general shell (GS), and linear solutions for: (a) small expansion ($r_I^*/R_I^* = 1.01$); and (b) moderate expansion ($r_I^*/R_I^* = 1.4$) of a thick spherical shell (R/H=1).

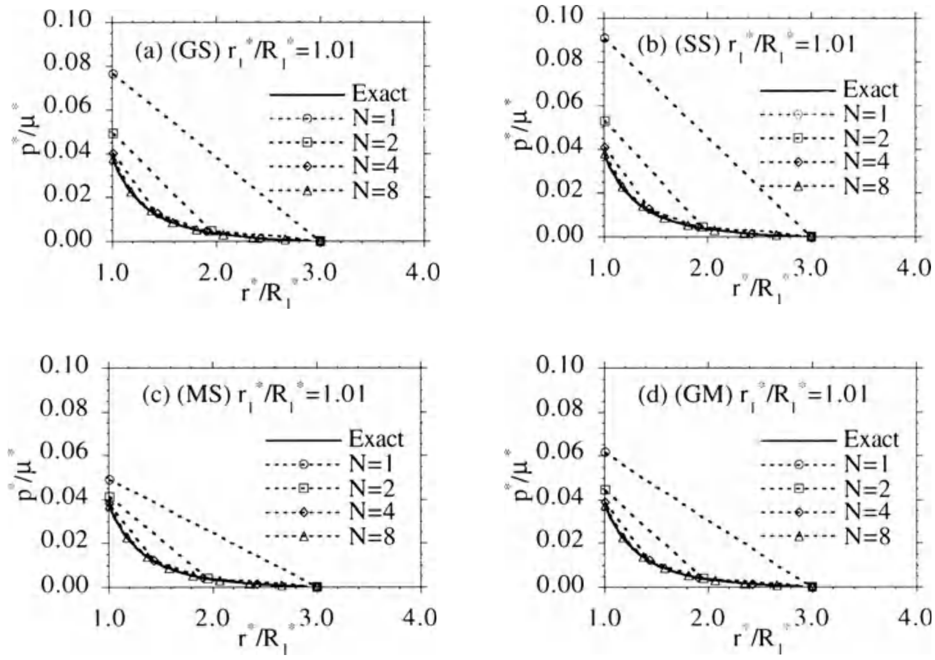

Fig. 7.2.3 Convergence of the Cosserat solutions for: (a) a general shell (GS); (b) a simple shell (SS); (c) a membrane-like shell (MS); and (d) a generalized membrane (GM); with the exact solution for small expansion ($r_I^*/R_I^* = 1.01$) of a thick spherical shell (R/H=1).

In the following, a number of example problems are considered to explore various aspects of the Cosserat solution. Specifically, it is of interest to examine the effect of different constitutive assumptions for the Cosserat shell model which are characterized by different strain energy functions.

GENERAL SHELL (GS)

For the general shell model, the constitutive equations satisfy the restrictions of section 4.11 which ensure that the Cosserat theory reproduces exact solutions for all homogeneous deformations. This model is denoted by the symbol (GS) and the strain energy function is specified in the form

$$m\,\Sigma = m\,\Sigma^*(\bar{\mathbf{C}}) + \frac{1}{2}\, A^{1/2}\,\bar{H}\,\mathbf{K}^{\alpha\beta} \bullet (\boldsymbol{\beta}_\alpha \otimes \boldsymbol{\beta}_\beta) \ . \tag{7.2.42}$$

Moreover, the constitutive equations of the I'th Cosserat shell take the forms given by (7.2.25) with

$${}_{I}M = {}_{I}\hat{M}_1 + {}_{I}\hat{M}_2 + {}_{I}\check{M} \ , \ {}_{I}T_1 = \left[{}_{I}\hat{T}_1 + {}_{I}\check{T}_1\right] - \frac{h_I}{H_I r_I}\left[{}_{I}M\right] \ , \ {}_{I}T_3 = {}_{I}\hat{T}_3 + {}_{I}\check{T}_3 \ . \tag{7.2.43}$$

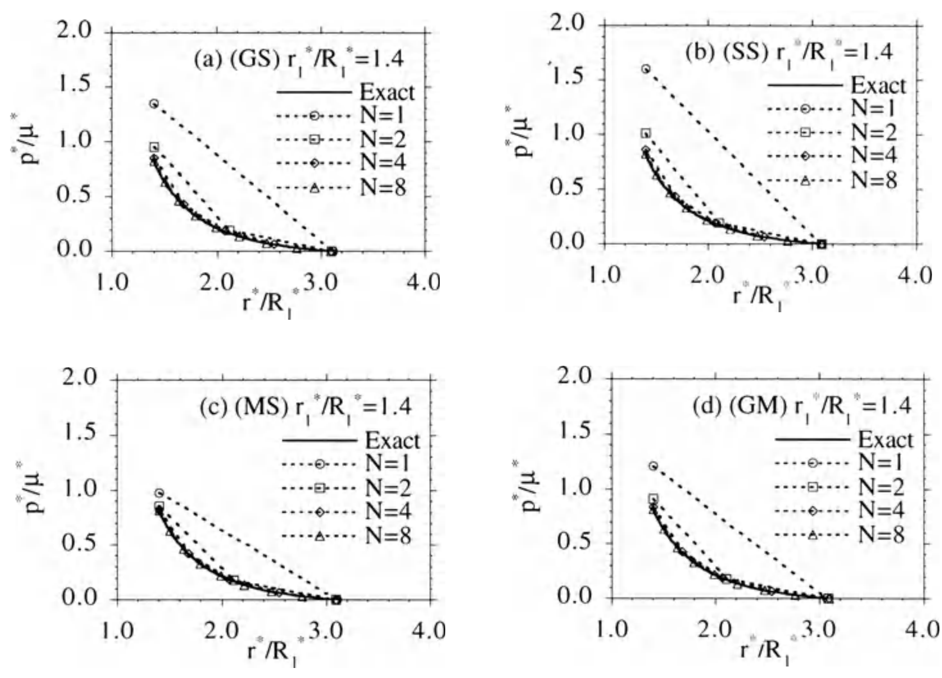

Fig. 7.2.4 Convergence of the Cosserat solutions for: (a) a general shell (GS); (b) a simple shell (SS); (c) a membrane-like shell (MS); and (d) a generalized membrane (GM); with the exact solution for moderate expansion (r_I^*/R_I^* =1.4) of a thick spherical shell (R/H=1).

SIMPLE SHELL (SS)

For a simple shell model, the constitutive equations do not satisfy the restrictions of section 4.11. This model is denoted by the symbol (SS) and the strain energy function is specified in the form

$$m\,\Sigma = m\,\Sigma^*(\mathbf{C}) + \frac{1}{2}\,A^{1/2}\,\bar{H}\,\mathbf{K}^{\alpha\beta}\bullet(\boldsymbol{\beta}_\alpha\otimes\boldsymbol{\beta}_\beta)\ , \tag{7.2.44}$$

where $\bar{\mathbf{C}}$ has been replaced by $\mathbf{C}$. Moreover, the constitutive equations of the I'th Cosserat surface take the forms given by (7.2.43) with

$$_{I}\hat{M}_2 = 0\ ,\quad {}_{I}M = {}_{I}\hat{M}_1 + {}_{I}\check{M}\ ,\quad \beta_I = 0 \text{ in the expressions for } \bar{J}_I,\ {}_{I}\hat{T}_1 \text{ and } {}_{I}\hat{T}_3\ ,$$
$$_{I}T_1 = [{}_{I}\hat{T}_1 + {}_{I}\check{T}_1] - \frac{h_I}{H_I r_I}[{}_{I}M]\ ,\quad {}_{I}T_3 = {}_{I}\hat{T}_3 + {}_{I}\check{T}_3\ . \tag{7.2.45}$$

Here, it is noted that although β_I is set equal to zero, the values of r_I and h_I remain independent and are not determined by the condition that β_I vanishes.

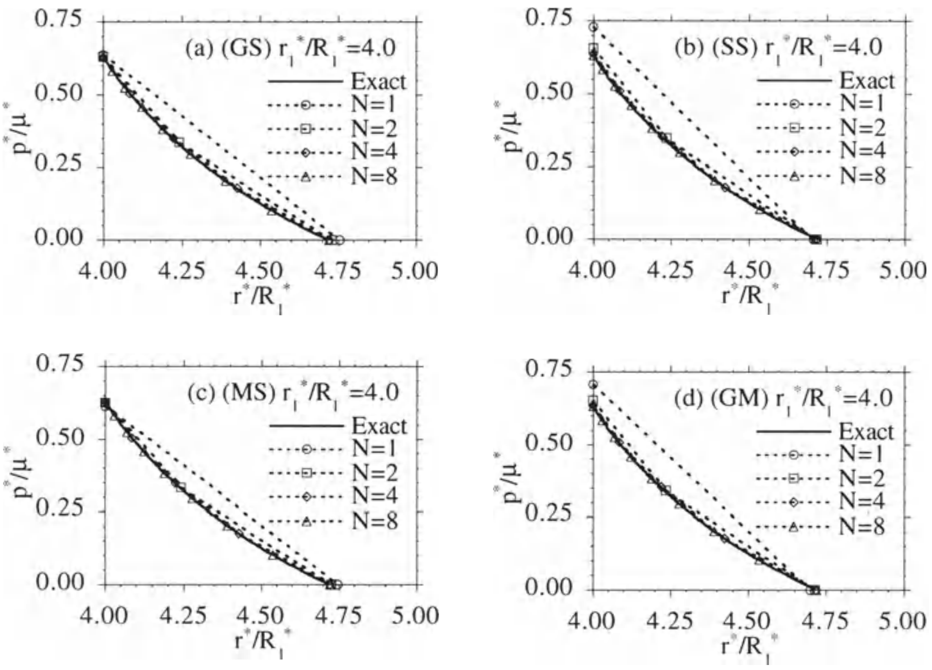

Fig. 7.2.5 Convergence of the Cosserat solutions for: (a) a general shell (GS); (b) a simple shell (SS); (c) a membrane-like shell (MS); and (d) a generalized membrane (GM); with the exact solution for large expansion (r_I^*/R_I^* =4.0) of a thick spherical shell (R/H=1).

MEMBRANE-LIKE SHELL (MS)

For the membrane-like shell of section 4.28, the constitutive equations satisfy the restrictions of section 4.11, but the strain energy of bending is neglected. This model is denoted by the symbol (MS) and the strain energy function is specified in the form

$$m\,\Sigma = m\,\Sigma^*(\bar{\mathbf{C}})\ . \tag{7.2.46}$$

Moreover, the constitutive equations of the I'th Cosserat shell take the forms given by (7.2.43) with

$$ {}_I\hat{M}_1 = 0 \ , \ {}_I\check{M} = 0 \ , \ {}_IM = {}_I\hat{M}_2 \ . \tag{7.2.47}$$

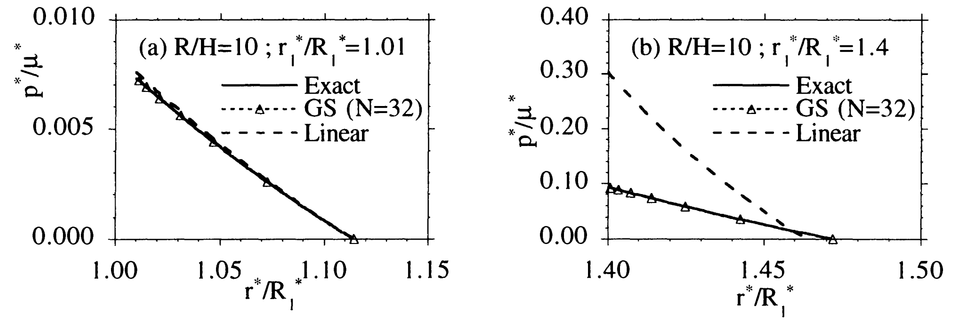

Fig. 7.2.6 Comparison of the exact, general shell (GS), and linear solution for: (a) small expansion (r_1^*/R_1^* =1.01); and (b) moderate expansion (r_1^*/R_1^* =1.4) of a thinner spherical shell (R/H=10).

GENERALIZED MEMBRANE (GM)

For the generalized membrane of section 4.28 the constitutive equations do not satisfy the restrictions of section 4.11 and the strain energy of bending is neglected. This model is denoted by the symbol (GM) and the strain energy function is specified in the form

$$ m\,\Sigma = m\,\Sigma^*(\mathbf{C}) \ , \tag{7.2.48}$$

where $\bar{\mathbf{C}}$ is replaced by $\mathbf{C}$. Moreover, the constitutive equations of the I'th Cosserat shell take the forms given by (7.2.43) with

$$ {}_IM = 0 \ , \ \ \beta_I = 0 \ \text{ in the expressions for } \bar{J}_I,\ {}_I\hat{T}_1 \text{ and } {}_I\hat{T}_3 \ . \tag{7.2.49}$$

A shell is considered to be thin if its thickness is small relative to its minimum radius of curvature. Therefore, if the spherical region is divided into shells with equal thicknesses, the inner most shells will have a larger ratio of the thickness to minimum radius of curvature than the outer shells. Consequently, it is convenient to divide the spherical region into N shells with thicknesses which increase with increasing radius. Specifically, for the examples considered here, the thickness of the I'th shell is specified by

$$ H_I = s^{I-1}\,H_1 \quad \text{for I=1,2,...,N} \ , \ H_1 = \Big[\frac{s-1}{s^N-1}\Big](R_{N+1}^* - R_1^*) \ , \ \ s \ge 1 \ , \tag{7.2.50}$$

where for most of the problems the value of s is taken to be

$$ s = 1.1 \ . \tag{7.2.51}$$

Also, for simplicity the stresses are normalized by the shear modulus μ^* and the elastic constants are specified by

$$ \nu^* = \frac{1}{3} \ , \ \frac{K^*}{\mu^*} = \frac{2(1+\nu^*)}{3(1-2\nu^*)} = \frac{8}{3} \ , \ \ C_2 = 0 \ . \tag{7.2.52}$$

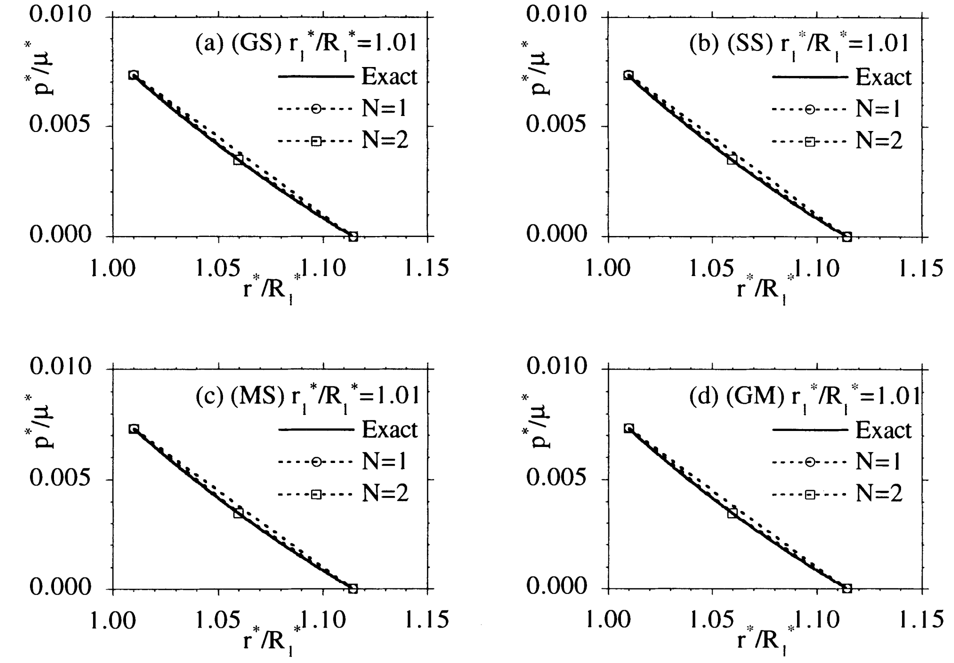

Fig. 7.2.7 Convergence of the Cosserat solutions for: (a) a general shell (GS); (b) a simple shell (SS); (c) a membrane-like shell (MS); and (d) a generalized membrane (GM); with the exact solution for small expansion (r_1^*/R_1^* =1.01) of a thinner spherical shell (R/H=10).

EXPANSION

The first example considers static expansion of a thick shell for which

$$R_1^* = 5 \text{ mm} \ , \quad R_{N+1}^* = 15 \text{ mm} \ , \quad \frac{R}{H} = \frac{R_1^* + R_{N+1}^*}{2(R_{N+1}^* - R_1^*)} = 1.0 \ , \tag{7.2.53}$$

where R/H characterizes the ratio of the mean radius to the thickness of the spherical region. For this example the inner radius r_1^* is specified and the outer pressure p_{N+1}^* vanishes

$$r_1^* = \text{specified} \ , \quad p_{N+1}^* = 0 \ . \tag{7.2.54}$$

Figures 7.2.2a,b compare the numerical solution of the exact three-dimensional nonlinear equation (7.2.39), the numerical solution of the equations (7.2.30) and (7.2.31) for a general shell (GS) (with N=32 elements), and the linear solution (7.2.32) and (7.2.33), for small expansion (r_1^*/R_1^* =1.01) and moderate expansion (r_1^*/R_1^* =1.4), respectively. These figures show the distribution of the normalized contact pressure

$$\frac{p^*}{\mu^*} = -\frac{T_{rr}^*}{\mu^*} \ . \tag{7.2.55}$$

as a function of the normalized current radial position (r^*/R_I^*). Specifically, for the linear solution, the current position is determined by the formula

$$r^* = R^* + u_r^*(R^*) \quad . \tag{7.2.56}$$

The results in these figures indicate that for small expansion the exact and linear solutions coincide, whereas for moderate expansion the linear solution significantly over estimates the stress concentration at the inner boundary. Also, these figures indicated that the Cosserat solution converges to the exact solution.

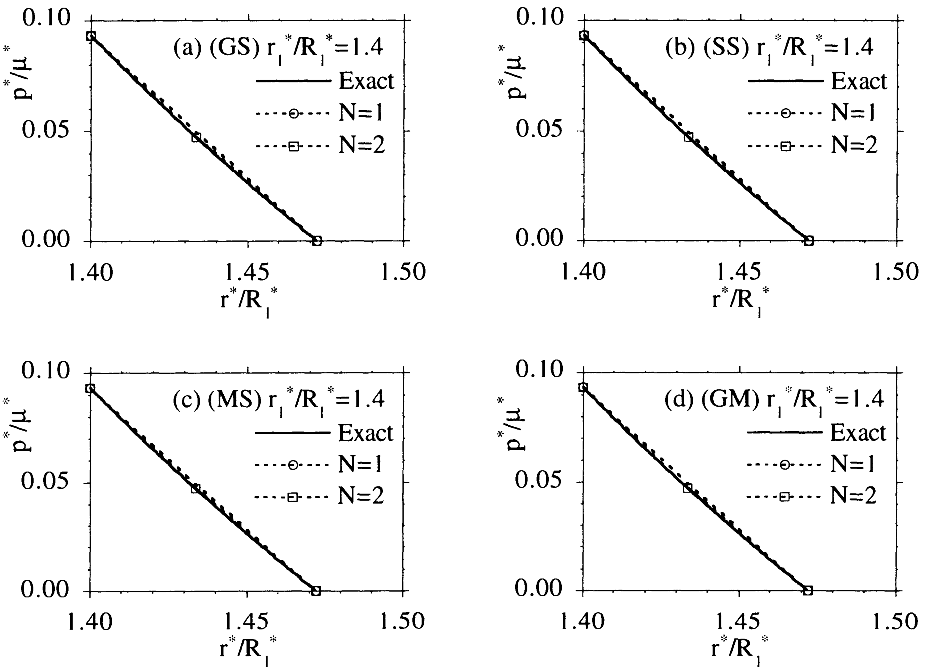

Fig. 7.2.8 Convergence of the Cosserat solutions for: (a) a general shell (GS); (b) a simple shell (SS); (c) a membrane-like shell (MS); and (d) a generalized membrane (GM); with the exact solution for moderate expansion (r_I^*/R_I^* =1.4) of a thinner spherical shell (R/H=10).

Next, the convergence of the Cosserat solutions for each of the four models (GS, SS, MS, GM) is examined by comparing the exact solution with the Cosserat solution for different numbers N of concentric shells. Specifically, Fig. 7.2.3, Fig. 7.2.4 and Fig. 7.2.5 show the contact pressure distributions for small expansion (r_I^*/R_I^* =1.01), moderate expansion (r_I^*/R_I^* =1.4) and large expansion (r_I^*/R_I^* =4.0) of this thick spherical shell. The results shown in these figures indicate that for thick shells the effect of the restriction of section 4.11 is significant since the convergence rate of the membrane-like shell (MS) is similar to that of the general shell (GS). Again, it is noted from Fig. 7.2.5 that the Cosserat solution converges to the exact solution.

Attention is now focused on static expansion of a thinner shell for which

$$R_1^* = 95 \text{ mm} \ , \quad R_{N+1}^* = 105 \text{ mm} \ , \quad \frac{R}{H} = \frac{R_1^* + R_{N+1}^*}{2(R_{N+1}^* - R_1^*)} = 10.0 \ . \qquad (7.2.57)$$

Figure 7.2.6a,b compare the numerical solution of the exact three-dimensional nonlinear equation (7.2.39), the numerical solution of the equations (7.2.30) and (7.2.31) for a general shell (GS) (with N=32 elements), and the linear solution (7.2.32) and (7.2.33), for small expansion (r_1^*/R_1^* =1.01) and moderate expansion (r_1^*/R_1^* =1.4), respectively. Comparison of Fig. 7.2.2b and Fig. 7.2.6b indicates that the differences between the linear and nonlinear solutions remains significant for moderate expansion even when the shell becomes thinner.

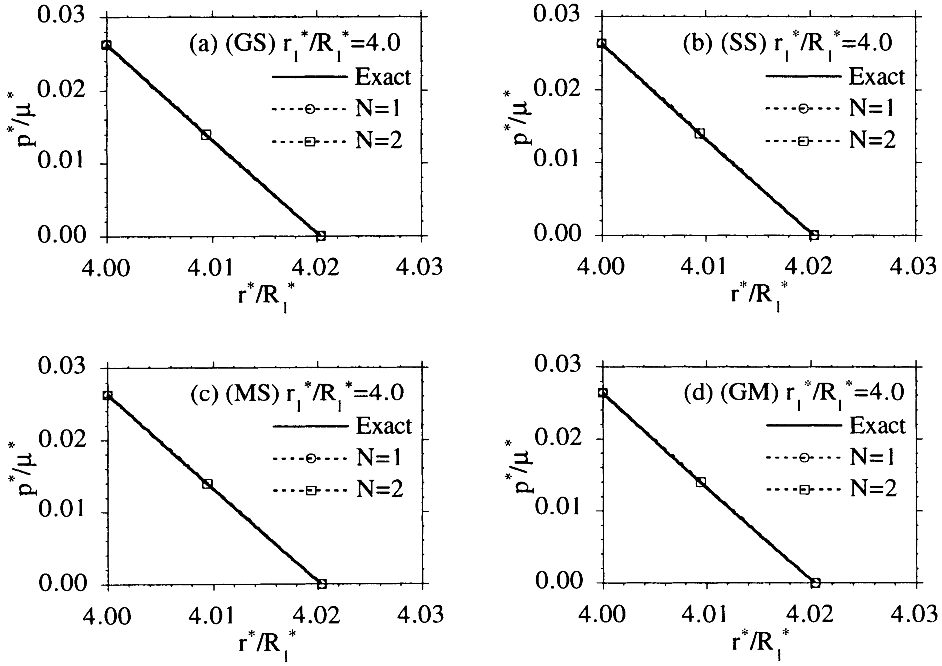

Fig. 7.2.9 Convergence of the Cosserat solutions for: (a) a general shell (GS); (b) a simple shell (SS); (c) a membrane-like shell (MS); and (d) a generalized membrane (GM); with the exact solution for large expansion (r_1^*/R_1^* =4.0) of a thinner spherical shell (R/H=10).

Figure 7.2.7, Fig. 7.2.8 and Fig. 7.2.9 show the contact pressure distributions for small expansion (r_1^*/R_1^* =1.01), moderate expansion (r_1^*/R_1^* =1.4) and large expansion (r_1^*/R_1^* =4.0) of this thinner spherical shell. The results shown in these figures indicate that that the effect of the restriction of section 4.11 is relatively insignificant since the convergence rates for all models are similar.

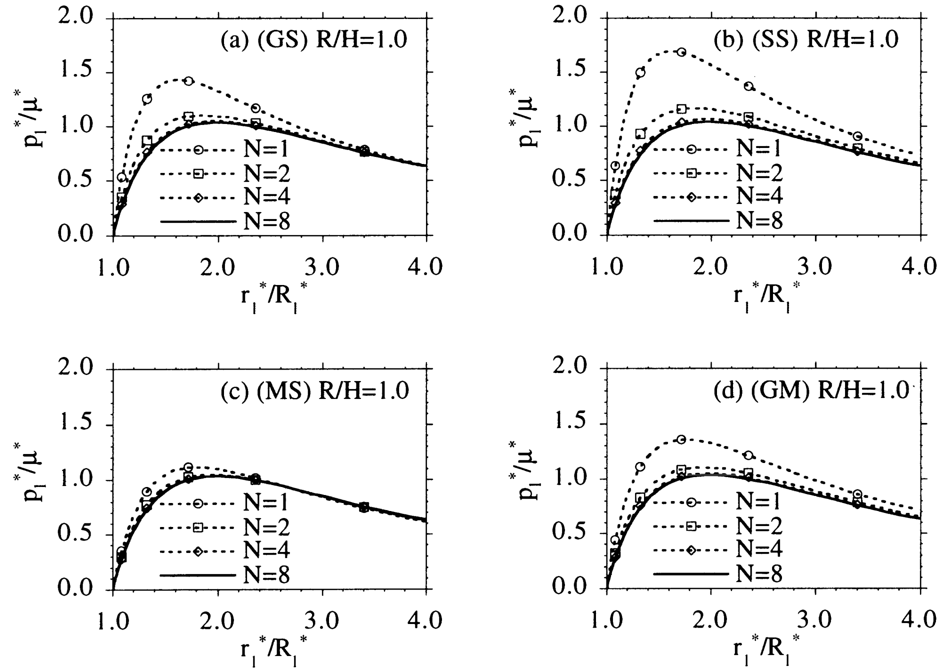

Fig. 7.2.10 Convergence of the Cosserat solutions for expansion of a thick spherical shell (R/H=1) using the models of: (a) a general shell (GS); (b) a simple shell (SS); (c) a membrane-like shell (MS); and (d) a generalized membrane (GM).

The expansion of a spherical shell can be viewed from a different perspective by plotting the internal pressure p_1^* required to maintain equilibrium as a function of the internal radius. Figures 7.2.10 and 7.2.11 examine the convergence of the four Cosserat models for a thick shell (R/H=1.0) and a thinner shell (R/H=10). These results indicate that pressure controlled expansion would be unstable since the pressure exhibits a maximum as the shell expands. Moreover, these figures show that the spherical region must be divided into a number of elements in order to accurately predict the peak pressure. However, even for the thick shell the pressure curve past the peak pressure is predicted fairly accurately using only a single element. This is because the shell effectively becomes thinner as it expands.

Next, the effect of compressibility is examined. By way of background it is recalled that as the value of Poisson's ratio ν^* approaches 0.5, the material becomes incompressible since the bulk modulus approaches infinity. Within the context of the general shell model (GS) this causes the value of $\bar{J}$ to remain near unity

$$\bar{J} \approx 1 \ , \tag{7.2.58}$$

whereas within the context of the simple shell model (SS) this causes the value of J to remain near unity

$$J \approx 1 \ . \tag{7.2.59}$$

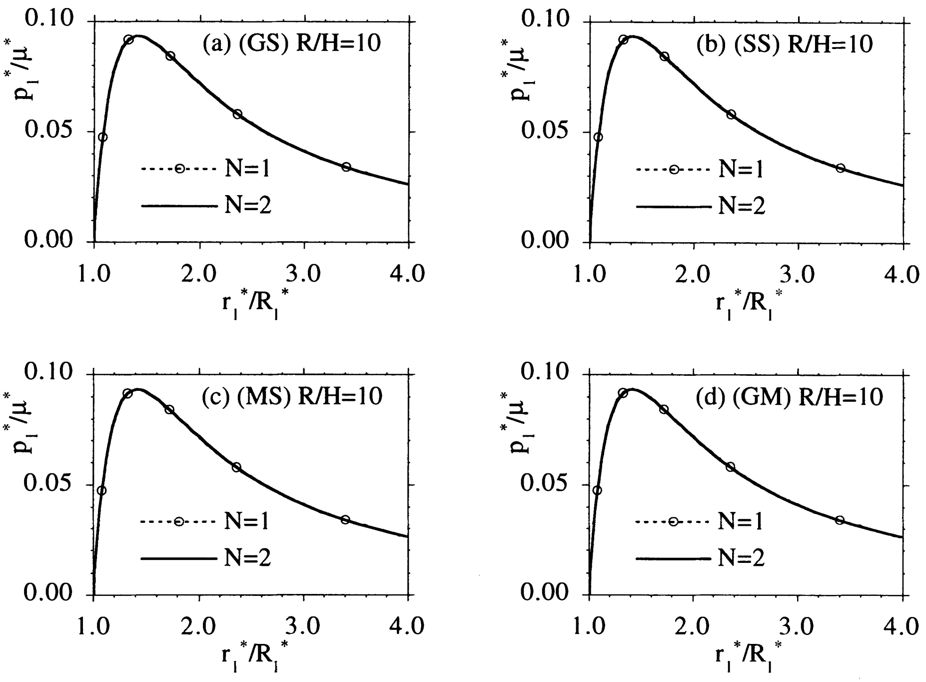

Fig. 7.2.11 Convergence of the Cosserat solutions for expansion of a thinner spherical shell (R/H=10) using the models of: (a) a general shell (GS); (b) a simple shell (SS); (c) a membrane-like shell (MS); and (d) a generalized membrane (GM).

Therefore, in general the response of an incompressible general shell (GS) will not be the same as that for an incompressible simple shell (SS). In order to quantitatively analyze the difference between these two models, it is convenient to consider the exact incompressibility condition which requires J^* in (7.2.35) to remain unity. Integration of this condition yields the well known result that

$$r^{*3} = R^{*3} + r_1^{*3} - R_1^{*3} \ , \tag{7.2.60}$$

where the constant of integration has been determined so that the present location of the inner boundary ($R^*=R_1^*$) is given by ($r^*=r_1^*$). Considering a single shell element (N=1), it then follows that the outer boundary ($R^*=R_2^*$, $r^*=r_2^*$) becomes

$$r_2^{*3} = R_2^{*3} + r_1^{*3} - R_1^{*3} \ , \tag{7.2.61}$$

so that the mean radius r and thickness h are given by

$$r = \frac{1}{2}(r_1^* + r_2^*) \ , \quad h = r_2^* - r_1^* \ . \tag{7.2.62}$$

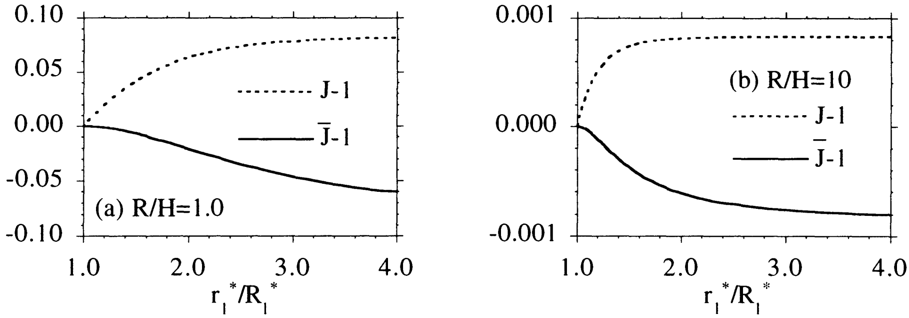

Fig. 7.2.12 Values of J–1 and $\bar{J}$–1 associated with the exact incompressible condition for a thick (R/H=1.0) and a thinner (R/H=10) shell.

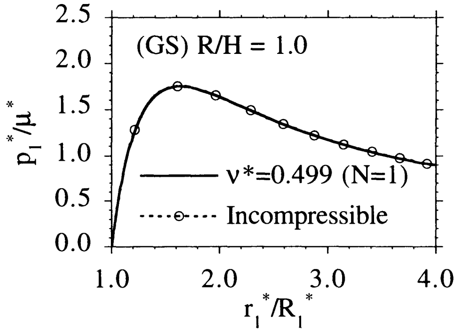

Fig. 7.2.13 Comparison of the incompressible limit (ν^*=0.499) of the compressible solution (GS) with the analytical incompressible solution of section 4.30 for a thick shell (R/H=1.0).

Using these expressions and the formulas in (7.2.28) it is possible to determine values of J and $\bar{J}$ associated with the exact incompressibility condition. Figure 7.2.12 plots these values for both a thick shell (R/H=1.0) and a thinner shell (R/H=10). For the thick shell the difference between (J–1) and ($\bar{J}$–1) can be as much as 20% whereas for the thinner shell this difference is less than 2%. This means that differences between the models (GS) and (SS) also become negligible in the incompressible limit. Consequently, attention will be mainly confined to the thick shell case (7.2.53). Specifically, the limit of incompressibility will be approached by obtaining solutions for the compressible equations with Poisson's ratio given by

$$\nu^* = 0.499 \ . \tag{7.2.63}$$

Figure 7.2.13 shows that the compressible solution for the general shell model (GS) of this section [with the specification (7.2.63) and a single element, N=1] produces the same results as the incompressible solution of section 4.30.

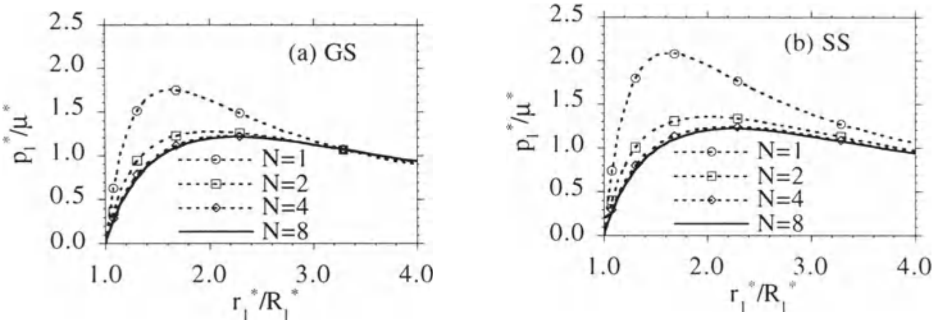

Fig. 7.2.14 Convergence of the general shell (GS) and simple shell (SS) solutions for the incompressible limit (ν^*=0.499) of a thick shell (R/H=1.0).

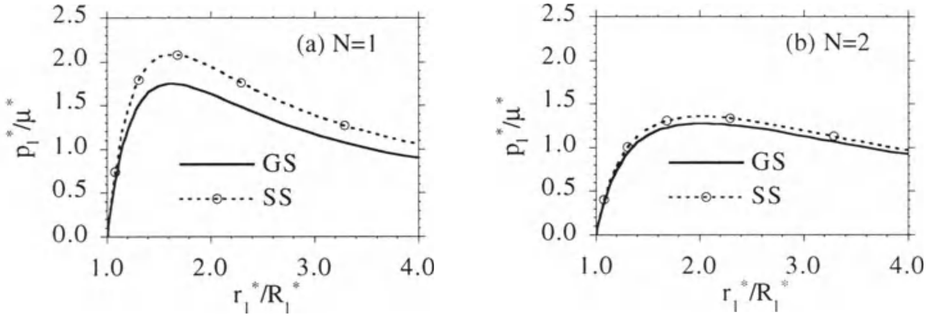

Fig. 7.2.15 Comparison of the general shell (GS) and simple shell (SS) solutions for the incompressible limit (ν^*=0.499) of a thick shell (R/H=1.0) for two discretizations.

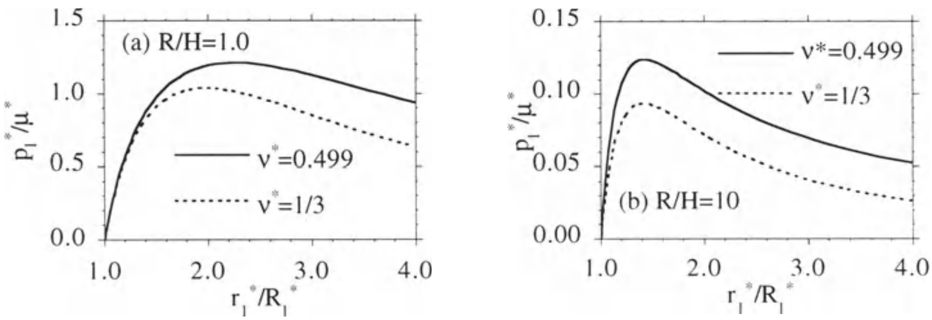

Fig. 7.2.16 Effect of compressibility using the general shell model (GS): (a) a thick shell with N=8 elements and (b) a thinner shell with N=1 element.

Figure 7.2.14 shows convergence of the general shell (GS) and the simple shell (SS) solutions for the incompressible limit (ν^*=0.499) of a thick shell (R/H=1.0). In particular, it is noted that significant errors in the prediction of the internal contact pressure can occur when only a single element (N=1) is used to model a thick shell. This means that the quantitative results for the incompressible shell in section 4.30 are not be accurate in the thick shell limit. Also, Fig. 7.2.15 compares the general shell (GS) and the simple shell (SS) solutions for two discretizations. Specifically, it can be seen that the solution (GS) converges faster than the solution (SS) since it lies below the solution (SS). This again indicates the importance of the restrictions of section 4.11.

Figure 7.2.16 compares the incompressible limit (ν^*=0.499) with the compressible solution (ν^*=1/3) for a thick shell (R/H=1.0) and a thinner shell (R/H=10). In both cases it is seen that the internal contact pressure p_1^* for expansion is greater for the incompressible shell than for the compressible shell.

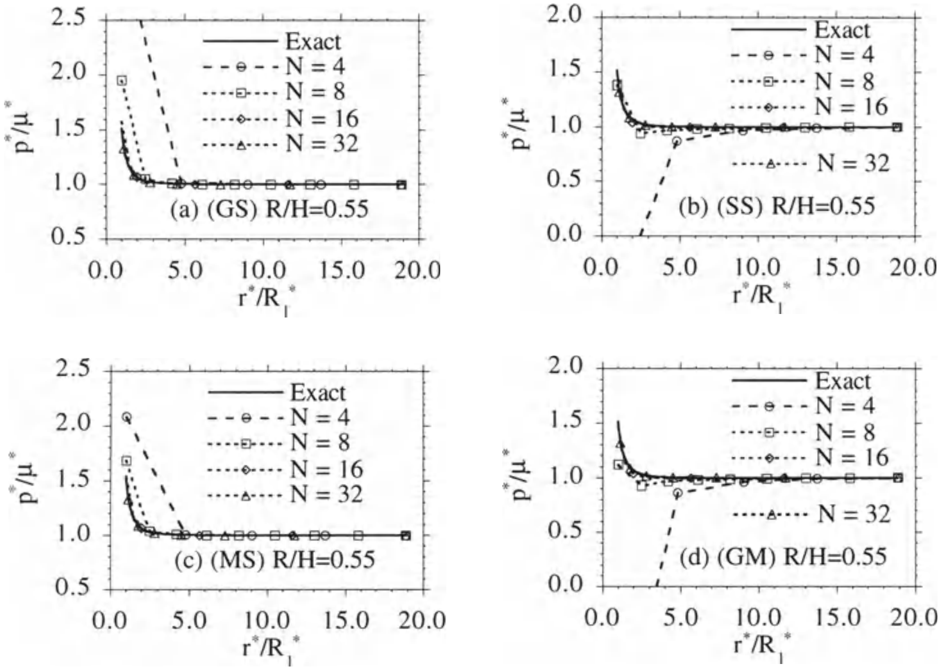

Fig. 7.2.17 Convergence of the Cosserat solutions for: (a) a general shell (GS); (b) a simple shell (SS); (c) a membrane-like shell (MS); and (d) a generalized membrane (GM); with the exact solution for compression of a thick spherical shell (R/H=0.55).

COMPRESSION

Finally, to further emphasize the influence of the restrictions of section 4.11, consider the problem of compression of a very thick shell onto a rigid ball. For this problem the shell geometry is specified by

$$R_1^* = 1 \text{ mm} \ , \quad R_{N+1}^* = 21 \text{ mm} \ , \quad \frac{R}{H} = \frac{R_1^* + R_{N+1}^*}{2(R_{N+1}^* - R_1^*)} = 0.55 \ , \qquad (7.2.64)$$

and the boundary conditions are taken to be

$$r_1^* = R_1^* \ , \quad p_{N+1}^* = \mu^* \ . \tag{7.2.65}$$

Figure 7.2.17 shows the convergence of the Cosserat solutions for the four models relative to the exact solution (7.2.39), with s given by (7.2.51) and with ν^* given by (7.2.52) for the compressible case. Notice that the solutions for the general shell (GS) and the membrane-like shell (MS) which satisfy the restrictions of section 4.11, both converge to the exact solution from above. In contrast, the solutions for the simple shell (SS) and the generalized membrane (GM) which do not satisfy the restrictions of section 4.11, both converge to the exact solution from below. Specifically, these latter solutions predict the unphysical result that the pressure does not monotonically increase towards the center of the sphere and that it can even become negative for too coarse of a discretization.

Examination of the solutions in Fig. 7.2.17 indicates that the contact pressure p^* is essentially uniform for most of the shell region and it exhibits a stress concentration of about 1.6 near the inner boundary. This means that the outer region of the shell experiences nearly homogeneous deformation. Consequently, it is expected that only a single general shell (GS) element can be used to describe this entire outer region. Figure 7.2.18 shows the effect of changing the scale factor s in (7.2.50). In particular, for s=5 it is possible to obtain a reasonably accurate solution with only N=3 elements.

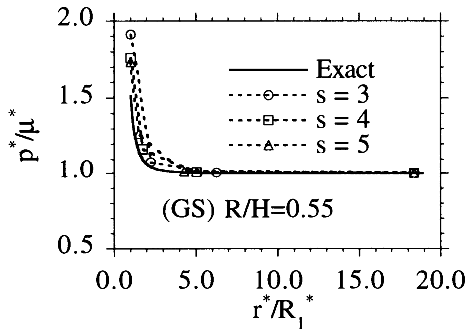

Fig. 7.2.18 Convergence of the Cosserat solution for a general shell (GS) with the exact solution for compression of a thick spherical shell (R/H=0.55). The number of elements is N=3 and s is scale factor in (7.2.50).

7.3 Formulation of the numerical solution of string problems using the theory of a Cosserat point

In section 5.31 the equations of motion of a simple string were presented in both global and local forms. These equations are partial differential equations which are functions of one space variable θ^3 and time t. Since these nonlinear partial differential equations are quite difficult to solve analytically, it is necessary to resort to some numerical method which is applicable to general problems of strings. One approach is the standard Galerkin method which proposes a spatial discretization and an approximate integration procedure to deduce a system of ordinary differential equations in time only, which themselves can be solved using standard methods. Alternatively, it is possible to divide the string into finite elements and to use the theory of a Cosserat point to model the response of each of these elements (Rubin, 1987a). Then, the equations for each element are coupled to the equations for neighboring elements by using kinematic and kinetic coupling conditions at the common boundaries of the elements (which is similar to the direct approach for finite elements). This approach produces a system of coupled nonlinear ordinary differential equations which are functions of time only, as in the Galerkin procedure. An example of this approach can be found in (Rubin and Gottlieb, 1996) where the nonlinear whirling motion of a string was considered. Here, this system of equations is developed using a notation that is consistent with that of this book, but which differs from that in the previous references (Rubin, 1987a; Rubin and Gottlieb, 1996).

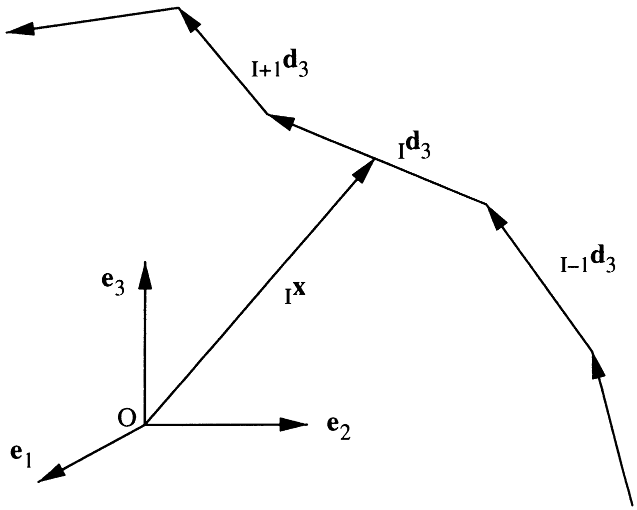

Fig. 7.3.1 Sketch of a string modeled as a chain of director vectors.

In this section the string is modeled as a collection of N (I=1,2,...,N) Cosserat points. The I'th Cosserat point is modeled by its location ${}_{I}\mathbf{x}(t)$ relative to a fixed origin and its director vector ${}_{I}\mathbf{d}_3(t)$ so that the kinematics of the I'th point are characterized by the two vectors

$$\{{}_{I}\mathbf{x}(t)\,,\,{}_{I}\mathbf{d}_3(t)\}\ , \tag{7.3.1}$$

which are functions of time only. Also, the velocity ${}_{I}\mathbf{v}$, the director velocity ${}_{I}\mathbf{w}_3$, and additional kinematic variables are defined by

$$\begin{gathered}
{}_{I}\mathbf{v} = {}_{I}\dot{\mathbf{x}}\ ,\quad {}_{I}\mathbf{w}_3 = {}_{I}\dot{\mathbf{d}}_3\ ,\quad {}_{I}d_{33} = {}_{I}\mathbf{d}_3 \bullet {}_{I}\mathbf{d}_3\ ,\quad {}_{I}\mathbf{d}^3 = {}_{I}d_{33}^{-1}\,{}_{I}\mathbf{d}_3\ ,\quad {}_{I}\mathbf{d}_3 \bullet {}_{I}\mathbf{d}^3 = 1\ ,\\
{}_{I}\mathbf{L} = {}_{I}\mathbf{w}_3 \otimes {}_{I}\mathbf{d}^3 = {}_{I}\mathbf{D} + {}_{I}\mathbf{W}\ ,\\
{}_{I}\mathbf{D} = \frac{1}{2}({}_{I}\mathbf{L} + {}_{I}\mathbf{L}^T) = {}_{I}\mathbf{D}^T,\quad {}_{I}\mathbf{W} = \frac{1}{2}({}_{I}\mathbf{L} - {}_{I}\mathbf{L}^T) = -\,{}_{I}\mathbf{W}^T\ ,
\end{gathered} \tag{7.3.2}$$

where the subscript I is used to denote quantities related to the I'th Cosserat point, and it is recalled that there is no sum on repeated upper case indices. Moreover, the complete string is modeled as a chain of N director vectors ${}_I\mathbf{d}_3$ which are connected head to tail (see Fig. 7.3.1).

For clarity, the theory of a Cosserat point for strings will be developed by two different approaches. In the first approach, the equations of motion of a string recorded in section 5.31 will be used to motivate the form for the balance laws of the theory of a Cosserat point. In the second approach, the theory of a Cosserat point will be developed by the direct method where the balance laws are postulated. For both approaches the constitutive equations are developed using the direct approach in terms of a strain energy function and a dissipation inequality.

To this end, recall the global forms of the balance laws of a simple string, as recorded in section 5.31. Here, the same symbol is used to denote similar quantities in the theory of a simple string and in the associated theory of a Cosserat point. However, to distinguish between the two theories, a superposed (**) is used to indicate that the quantity is related to the string theory. Thus, for example the symbol $\mathbf{x}^{**}$ denotes the position vector of a material point in the string theory of section 5.31, whereas the symbol ${}_I\mathbf{x}$ denotes the position vector of the I'th Cosserat point. In particular, the position vector $\mathbf{x}^{**}$ and the velocity $\mathbf{v}^{**}$ of a material point on the string are given by

$$\mathbf{x}^{**} = \mathbf{x}^{**}(\theta^3,t) \ , \quad \mathbf{v}^{**} = \mathbf{v}^{**}(\theta^3,t) = \dot{\mathbf{x}}^{**} \quad , \quad 0 \le \theta^3 \le L_T \ , \tag{7.3.3}$$

where θ^3 is a convected Lagrangian coordinate, a superposed dot denotes material time differentiation holding θ^3 fixed, and L_T is the total stress-free reference length of the string.

Then, using the definitions (5.31.1) and $(5.31.8)_1$, the conservation of mass and the balance of linear momentum can be written in the alternative forms

$$\frac{d}{dt}\int_{\xi_I}^{\xi_{I+1}} m^{**}\, d\theta^3 = 0 \ ,$$

$$\frac{d}{dt}\int_{\xi_I}^{\xi_{I+1}} m^{**}\, \mathbf{v}^{**}\, d\theta^3 = \int_{\xi_I}^{\xi_{I+1}} m^{**}\, \mathbf{b}^{**}\, d\theta^3 + \left[\, \mathbf{t}^{3**} \,\right]_I^{I+1} \ , \tag{7.3.4}$$

where $\xi_I \le \theta^3 \le \xi_{I+1}$ denotes the I'th material region of the string and the notation $\left[\, \mathbf{t}^{3**} \,\right]_I^{I+1}$ is defined in a manner similar to (5.3.22) such that

$$\left[\, \mathbf{t}^{3**} \,\right]_I^{I+1} = \mathbf{t}^{3**}(\xi_{I+1},t) - \mathbf{t}^{3**}(\xi_I,t) \ . \tag{7.3.5}$$

Also, the balance of angular momentum (5.31.6) of this material region of the string can be written in the alternative form

$$\frac{d}{dt}\int_{\xi_I}^{\xi_{I+1}} m^{**}\, \mathbf{x}^{**} \times \mathbf{v}^{**}\, d\theta^3 = \int_{\xi_I}^{\xi_{I+1}} m^{**}\, \mathbf{x}^{**} \times \mathbf{b}^{**}\, d\theta^3 + \left[\, \mathbf{x}^{**} \times \mathbf{t}^{3**} \,\right]_I^{I+1} \ . \tag{7.3.6}$$

Furthermore, to develop the form for the balance of director momentum of the theory of a Cosserat point, it is necessary to develop an averaged form of the balance of linear momentum of the string which is similar to (3.6.3). To this end, the local form of the balance of linear momentum $(5.31.8)_3$ is recorded as

$$m^{**}\,\dot{\mathbf{v}}^{**} = m^{**}\,\mathbf{b}^{**} + \mathbf{t}^{3**}{}_{,3}\ . \tag{7.3.7}$$

Next, multiplying this equation by the weighting function $\phi(\theta^3)$ and integrating over the material region it can be shown that

$$\frac{d}{dt}\int_{\xi_I}^{\xi_{I+1}} \phi\, m^{**}\,\mathbf{v}^{**}\,d\theta^3 = \int_{\xi_I}^{\xi_{I+1}} [\,\phi\, m^{**}\,\mathbf{b}^{**} - \phi_{,3}\,\mathbf{t}^{3**}\,]\,d\theta^3 + \Big[\,\phi\,\mathbf{t}^{3**}\,\Big]_I^{I+1}. \tag{7.3.8}$$

In order to motivate the balance laws of the theory of a Cosserat point for strings, it is convenient to introduce the following definitions

$$\begin{gathered}
{}_Im = \int_{\xi_I}^{\xi_{I+1}} m^{**}\,d\theta^3\ ,\quad {}_Im\,{}_Iy^3 = \int_{\xi_I}^{\xi_{I+1}} {}_I\bar{\theta}^3\,m^{**}\,d\theta^3\ ,\\
{}_Im\,{}_Iy^{33} = \int_{\xi_I}^{\xi_{I+1}} {}_I\bar{\theta}^3\,{}_I\bar{\theta}^3\,m^{**}\,d\theta^3\ ,\\
{}_Im\,{}_I\mathbf{B}_b = \int_{\xi_I}^{\xi_{I+1}} m^{**}\,\mathbf{b}_b^{**}\,d\theta^3\ ,\quad {}_Im\,{}_I\mathbf{B}_c = \int_{\xi_I}^{\xi_{I+1}} m^{**}\,\mathbf{b}_c^{**}\,d\theta^3\ ,\\
{}_Im\,{}_I\mathbf{B}_b^3 = \int_{\xi_I}^{\xi_{I+1}} {}_I\bar{\theta}^3\,m^{**}\,\mathbf{b}_b^{**}\,d\theta^3\ ,\quad {}_Im\,{}_I\mathbf{B}_c^3 = \int_{\xi_I}^{\xi_{I+1}} {}_I\bar{\theta}^3\,m^{**}\,\mathbf{b}_c^{**}\,d\theta^3\ ,\\
{}_I\mathbf{m}_1^0 = -\,\mathbf{t}^{3**}(\xi_I,t)\ ,\quad {}_I\mathbf{m}_2^0 = \mathbf{t}^{3**}(\xi_{I+1},t)\ ,\quad {}_I\mathbf{m}_1^3 = \frac{{}_IL}{2}\mathbf{t}^{3**}(\xi_I,t)\ ,\\
{}_I\mathbf{m}_2^3 = \frac{{}_IL}{2}\mathbf{t}^{3**}(\xi_{I+1},t)\ ,\quad {}_I\mathbf{t}^3 = \int_{\xi_I}^{\xi_{I+1}} \mathbf{t}^{**}\,d\theta^3\ ,
\end{gathered} \tag{7.3.9}$$

where ${}_I\bar{\theta}^3$ is a convected Lagrangian coordinate defined in the I'th region by

$${}_I\bar{\theta}^3 = \frac{2\theta^3 - \xi_I - \xi_{I+1}}{2}\ ,\quad -\frac{{}_IL}{2} \le {}_I\bar{\theta}^3 \le \frac{{}_IL}{2}\ ,\quad {}_IL = \xi_{I+1} - \xi_I\ , \tag{7.3.10}$$

and ${}_IL$ is the reference length.

Now, the theory of a Cosserat point for strings can be developed by introducing the kinematic assumption that the position vector $\mathbf{x}^{**}$ in the I'th region of the string is represented in the form

$$\mathbf{x}^{**} = {}_I\mathbf{x}(t) + {}_I\bar{\theta}^3\,{}_I\mathbf{d}_3(t)\ \ , \tag{7.3.11}$$

Then, using the definitions (7.3.2), (7.3.9), and substituting (7.3.11) into the balance laws (7.3.4) and (7.3.8) (with $\phi = {}_I\bar{\theta}^3$), it is possible to develop the conservation of mass and the balances of linear and director momentum, respectively, in the forms

$$\begin{gathered}
{}_I\dot{m} = 0\ ,\quad \frac{d}{dt}[\,{}_Im\,({}_I\mathbf{v} + {}_Iy^3\,{}_I\mathbf{w}_3)\,] = {}_Im\,{}_I\mathbf{b}\ ,\\
\frac{d}{dt}[\,{}_Im\,({}_Iy^3\,{}_I\mathbf{v} + {}_Iy^{33}\,{}_I\mathbf{w}_3)\,] = {}_Im\,{}_I\mathbf{b}^3 - {}_I\mathbf{t}^3\,.
\end{gathered} \tag{7.3.12}$$

Also, substituting (7.3.11) into (7.3.6), the balance of angular momentum becomes

$$\frac{d}{dt}[\,{}_I\mathbf{x}\times{}_Im\,({}_I\mathbf{v} + {}_Iy^3\,{}_I\mathbf{w}_3) + {}_I\mathbf{d}_3\times{}_Im\,({}_Iy^3\,{}_I\mathbf{v} + {}_Iy^{33}\,{}_I\mathbf{w}_3)\,]$$

$$= {}_I\mathbf{x} \times {}_I\mathrm{m}\, {}_I\mathbf{b} + {}_I\mathbf{d}_3 \times {}_I\mathrm{m}\, {}_I\mathbf{b}^3 \ . \tag{7.3.13}$$

In these equations, ${}_I\mathrm{m}$ is the mass, ${}_I\mathrm{y}^3$ and ${}_I\mathrm{y}^{33}$ are the director inertia coefficients, ${}_I\mathbf{b}$ is the specific (per unit mass) external assigned force, ${}_I\mathbf{b}^3$ is the specific external assigned director couple, and ${}_I\mathbf{t}^3$ is the intrinsic director couple. Moreover, the director inertia coefficients are constants

$${}_I\dot{\mathrm{y}}^3 = 0 \ , \ {}_I\dot{\mathrm{y}}^{33} = 0 \ , \tag{7.3.14}$$

and the assigned fields ${}_I\mathbf{b}$ and ${}_I\mathbf{b}^3$ can be expressed in the forms

$${}_I\mathrm{m}\, {}_I\mathbf{b} = {}_I\mathrm{m}\, {}_I\mathbf{B} + {}_I\mathbf{m}_1^0 + {}_I\mathbf{m}_2^0 \ , \quad {}_I\mathbf{B} = {}_I\mathbf{B}_b + {}_I\mathbf{B}_c \ ,$$

$${}_I\mathrm{m}\, {}_I\mathbf{b}^3 = {}_I\mathrm{m}\, {}_I\mathbf{B}^3 + {}_I\mathbf{m}_1^3 + {}_I\mathbf{m}_2^3 \ , \quad {}_I\mathbf{B}^3 = {}_I\mathbf{B}_b^3 + {}_I\mathbf{B}_c^3 \ . \tag{7.3.15}$$

Here, the terms ${}_I\mathbf{B}_b$ and ${}_I\mathbf{B}_b^3$ are associated with the external body force; the terms ${}_I\mathbf{B}_c$ and ${}_I\mathbf{B}_c^3$ are associated with the tractions applied to the lateral surface of the string section; ${}_I\mathbf{m}_1^0$ and ${}_I\mathbf{m}_2^0$ are the forces applied to the ends ξ_I and ξ_{I+1}, respectively, of the string section; and ${}_I\mathbf{m}_1^3$ and ${}_I\mathbf{m}_2^3$ are the director couples applied to the ends ξ_I and ξ_{I+1}, respectively, of the string section.

The equations of motion (7.3.12) and (7.3.13) have been developed by integration of the equations of motion of a string. This approach is used to motivate the structure of the equations of motion of the Cosserat point. However, within the context of the direct approach these same equations of motion are postulated as the balance laws for the theory of a Cosserat point. The remaining equations describing the theory of a Cosserat point are developed within the context of the direct approach.

Specifically, using the equations of motion (7.3.12) the balance of angular momentum (7.3.13) can be reduced to the form

$${}_I\mathbf{d}_3 \times {}_I\mathbf{t}^3 = 0 \ . \tag{7.3.16}$$

Moreover, by introducing the definition

$${}_I\mathrm{d}_{33}^{1/2}\, {}_I\mathbf{T} = {}_I\mathbf{t}^3 \otimes {}_I\mathbf{d}_3 \ , \tag{7.3.17}$$

it follows that the reduced form of the balance of angular momentum can be written as

$${}_I\mathbf{T}^T = {}_I\mathbf{T} \ , \tag{7.3.18}$$

which is similar to the expression (3.2.32) associated with the three-dimensional theory.

For elastic Cosserat points, it is convenient to introduce the stretch ${}_I\lambda$ of the I'th element by the formula

$${}_I\lambda = \frac{{}_I\mathrm{d}_{33}^{1/2}}{{}_I\mathrm{D}_{33}^{1/2}} \ , \quad {}_I\mathrm{D}_{33} = {}_I\mathbf{D}_3 \bullet {}_I\mathbf{D}_3 \ , \tag{7.3.19}$$

where ${}_I\mathbf{D}_3$ is the value of the director ${}_I\mathbf{d}_3$ in the stress-free reference configuration. Consequently, it can be shown using the definitions (7.3.2) that

$${}_I\dot{\lambda} = {}_I\lambda \left[\frac{{}_I\mathbf{d}_3 \otimes {}_I\mathbf{d}_3}{{}_I\mathrm{d}_{33}}\right] \bullet {}_I\mathbf{D} \ , \quad \frac{{}_I\dot{\lambda}}{{}_I\lambda} = {}_I\mathbf{D} \bullet \mathbf{I} \ . \tag{7.3.20}$$

Also, for an elastic Cosserat point the specific strain energy ${}_I\Sigma$ is a function of the stretch only

$$ {}_I\Sigma = {}_I\hat{\Sigma}({}_I\lambda) \; . \tag{7.3.21} $$

For the purely mechanical theory it is convenient to define: the rate of dissipation ${}_I\mathcal{D}$; the mechanical power ${}_I\mathcal{P}$; the rate of work ${}_I\mathcal{W}$ of the assigned fields $\{{}_I\mathbf{b}\, , {}_I\mathbf{b}^3\}$; the kinetic energy ${}_I\mathcal{K}$; and the total internal energy ${}_I\mathcal{U}$ by the formulas

$$ {}_I d_{33}^{1/2}\, {}_I\mathcal{D} = {}_I\mathcal{W} - {}_I\dot{\mathcal{K}} - {}_I\dot{\mathcal{U}} \, , \quad {}_I\mathcal{P} = {}_I\mathcal{W} - {}_I\dot{\mathcal{K}} \, , \quad {}_I\mathcal{W} = {}_I m \, ({}_I\mathbf{b} \bullet {}_I\mathbf{v} + {}_I\mathbf{b}^3 \bullet {}_I\mathbf{w}_3) \; , $$

$$ {}_I\mathcal{K} = \frac{1}{2}\, {}_I m \, ({}_I\mathbf{v} \bullet {}_I\mathbf{v} + 2\, {}_I y^3 \, {}_I\mathbf{v} \bullet {}_I\mathbf{w}_3 + {}_I y^{33}\, {}_I\mathbf{w}_3 \bullet {}_I\mathbf{w}_3) \, , \quad {}_I\mathcal{U} = {}_I m \, {}_I\Sigma \; . \tag{7.3.22} $$

Next, with the help of the equations of motion (7.3.12), the definition (7.3.17), and the result (7.3.18), it can be shown that the rate of dissipation reduces to

$$ {}_I\mathcal{D} = {}_I\mathbf{T} \bullet {}_I\mathbf{D} - {}_I\rho \, {}_I\dot{\Sigma} \ \geq 0 \; , \tag{7.3.23} $$

which is required to be nonnegative. In this equation use has been made of the Lagrangian form of conservation of mass

$$ {}_I m = {}_I\rho \, {}_I d_{33}^{1/2} = {}_I\rho_0 \, {}_I D_{33}^{1/2} \; , \tag{7.3.24} $$

where ${}_I\rho$ is the mass density in the present configuration and ${}_I\rho_0$ is its reference value. Moreover, using the result (7.3.20) it follows that

$$ {}_I\mathcal{D} = [\, {}_I\mathbf{T} - {}_I\rho \, {}_I\lambda \frac{\partial_I \hat{\Sigma}}{\partial_I \lambda} \{ \frac{{}_I\mathbf{d}_3 \otimes {}_I\mathbf{d}_3}{{}_I d_{33}} \}] \bullet {}_I\mathbf{D} \geq 0 \; , \tag{7.3.25} $$

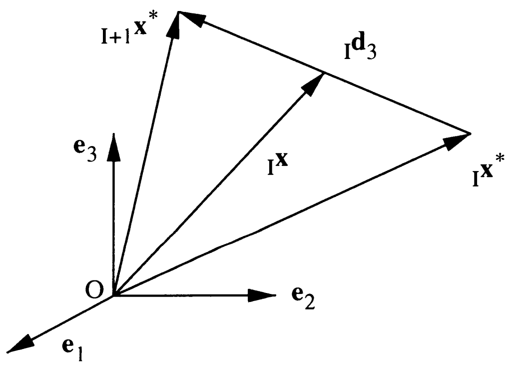

Fig. 7.3.2 Sketch of the I'th Cosserat point showing the definition of the endpoints ${}_I\mathbf{x}^*$ and ${}_{I+1}\mathbf{x}^*$.

For an elastic Cosserat point, assumptions similar to those described in section 6.8 are made. In particular, the dissipation ${}_I\mathcal{D}$ vanishes and ${}_I\mathbf{T}$ is independent of the rate ${}_I\mathbf{L}$,

$$ {}_I\mathbf{T} = {}_I\hat{\mathbf{T}}({}_I\mathbf{d}_3) \; , \tag{7.3.26} $$

Consequently, it can be shown using (7.3.23) that

$$ {}_I\mathbf{T} = {}_I\hat{\mathbf{T}} = {}_I\rho \, {}_I\lambda \frac{\partial_I \hat{\Sigma}}{\partial_I \lambda} [\frac{{}_I\mathbf{d}_3 \otimes {}_I\mathbf{d}_3}{{}_I d_{33}}] \; , \tag{7.3.27} $$

which automatically satisfies the balance of angular momentum (7.3.18). Once a constitutive equation for ${}_I\mathbf{T}$ has been specified, the value of ${}_I\mathbf{t}^3$ which appears in the equations of motion (7.3.12) and in the definition (7.3.17) can be determined by the expression

$$ {}_I\mathbf{t}^3 = {}_I d_{33}^{1/2}\, {}_I\mathbf{T}\, {}_I\mathbf{d}^3 , \tag{7.3.28} $$

so that

$$ {}_I\mathbf{t}^3 = {}_I\hat{\mathbf{t}}^3 = {}_I N\, {}_I L\, [\frac{{}_I\mathbf{d}_3}{{}_I d_{33}^{1/2}}] \ , \quad {}_I N = \frac{1}{{}_I L}\, {}_I\rho_0 \frac{\partial_I \hat{\Sigma}}{\partial_I \lambda} = \frac{1}{{}_I L}\, {}_I m\, {}_I D_{33}^{-1/2} \frac{\partial_I \hat{\Sigma}}{\partial_I \lambda} \ . \tag{7.3.29} $$

where ${}_I N$ is a measure of the tension in the I'th element.

Now, a simple model for a Cosserat point constructed from a dissipative material can be developed by assuming that ${}_I\mathbf{T}$ and ${}_I\mathbf{t}^3$ separate additively into two parts

$$ {}_I\mathbf{T} = {}_I\hat{\mathbf{T}} + {}_I\check{\mathbf{T}} , \quad {}_I\mathbf{t}^3 = {}_I\hat{\mathbf{t}}^3 + {}_I\check{\mathbf{t}}^3 , $$
$$ {}_I\hat{\mathbf{T}} = {}_I d_{33}^{-1/2}\, {}_I\hat{\mathbf{t}}^3 \otimes {}_I\mathbf{d}_3 = {}_I\hat{\mathbf{T}}^T , \quad {}_I\check{\mathbf{T}} = {}_I d_{33}^{-1/2}\, {}_I\check{\mathbf{t}}^i \otimes {}_I\mathbf{d}_3 = {}_I\check{\mathbf{T}}^T , \tag{7.3.30} $$

with ${}_I\hat{\mathbf{T}}$ and ${}_I\hat{\mathbf{t}}^3$ being the parts associated with elastic deformation [which balance the rate of change in strain energy (7.3.21)]

$$ {}_I\hat{\mathbf{T}} \bullet {}_I\mathbf{D} = {}_I\rho\, {}_I\dot{\Sigma} \ , \tag{7.3.31} $$

and ${}_I\check{\mathbf{T}}$ and ${}_I\check{\mathbf{t}}^3$ being the parts due to material dissipation. Thus, the restriction (7.3.23) reduces to

$$ {}_I\mathcal{D} = {}_I\check{\mathbf{T}} \bullet {}_I\mathbf{D} \geq 0 \ . \tag{7.3.32} $$

As a simple case it is possible to assume that ${}_I\check{\mathbf{T}}$ and ${}_I\check{\mathbf{t}}^3$ are linear functions of ${}_I\mathbf{D}$ such that

$$ {}_I d_{33}^{1/2}\, {}_I\check{\mathbf{T}} = {}_I D_{33}^{1/2}\, {}_I A\, {}_I L\, [{}_I\eta_1\, ({}_I\mathbf{D} \bullet \mathbf{I})\, \mathbf{I}] \ , $$
$$ {}_I\check{\mathbf{t}}^3 = {}_I d_{33}^{1/2}\, {}_I\check{\mathbf{T}}\, {}_I\mathbf{d}^3 = {}_I A\, {}_I L \frac{{}_I\eta_1}{{}_I\lambda}\, ({}_I\mathbf{D} \bullet \mathbf{I})\, [\frac{{}_I\mathbf{d}_3}{{}_I d_{33}^{1/2}}] \ , \tag{7.3.33} $$

where ${}_I A$ is the reference cross-sectional area of the string and ${}_I\eta_1$ is a material constant that controls the viscosity to stretching of the Cosserat point [see $(7.3.20)_2$ and the discussion at the end of section 5.35 related to the form (5.35.15)]. Also, it can be shown that the restriction (7.3.32) is satisfied for all motions provided that ${}_I\eta_1$ is nonnegative

$$ {}_I\eta_1 \geq 0 \ . \tag{7.3.34} $$

For the numerical solution of string problems using the theory of a Cosserat point, it is necessary to satisfy the balance laws (7.3.12) and the constitutive equations (7.3.30) for each Cosserat point. Moreover, it is necessary to couple the equations for each Cosserat point to those of its nearest neighbors using kinematic and kinetic coupling conditions. Specifically, the kinematic assumption (7.3.11) can be used, together with the definitions

$$ {}_I\mathbf{x}^* = {}_I\mathbf{x}^*(t) = \mathbf{x}^{**}(\xi_I, t) \ , \tag{7.3.35} $$

of the positions of the end points of each Cosserat point, to deduce that ${}_I\mathbf{x}$ and ${}_I\mathbf{d}_3$ are given by (see Fig. 7.3.2)

$$ {}_I\mathbf{x} = \frac{1}{2}[\,{}_I\mathbf{x}^* + {}_{I+1}\mathbf{x}^*\,] \;,\quad {}_I\mathbf{d}_3 = \frac{1}{{}_IL}[\,{}_{I+1}\mathbf{x}^* - {}_I\mathbf{x}^*\,] \;. \tag{7.3.36}$$

These kinematic coupling equations reduce the number of degrees of freedom from 2N vectors $\{{}_I\mathbf{x}, {}_I\mathbf{d}_3\}$ to only N+1 vectors $\{{}_I\mathbf{x}^*\}$.

Next, using the definitions (7.3.9) it follows that the director couples ${}_I\mathbf{m}_1^3$ and ${}_I\mathbf{m}_2^3$ are related to the forces ${}_I\mathbf{m}_1^0$ and ${}_I\mathbf{m}_2^0$ by the equations

$$ {}_I\mathbf{m}_1^3 = -\frac{{}_IL}{2}\,{}_I\mathbf{m}_1^0 \;,\quad {}_I\mathbf{m}_2^3 = \frac{{}_IL}{2}\,{}_I\mathbf{m}_2^0 \;. \tag{7.3.37}$$

Consequently, with the help of the expressions (7.3.15) and (7.3.36), the equations of motion can be solved for the two unknowns ${}_I\mathbf{m}_1^0$ and ${}_I\mathbf{m}_2^0$ to deduce that

$$ {}_I\mathbf{m}_1^0 = -\frac{1}{2}\,{}_Im\,{}_I\mathbf{B} + \frac{1}{{}_IL}\,{}_Im\,{}_I\mathbf{B}^3 - \frac{1}{{}_IL}\,{}_I\mathbf{t}^3 $$
$$ + {}_Im\,[\{\frac{1}{4} - \frac{{}_Iy^3}{{}_IL} + \frac{{}_Iy^{33}}{{}_IL^2}\}\,{}_I\ddot{\mathbf{x}}^* + \{\frac{1}{4} - \frac{{}_Iy^{33}}{{}_IL^2}\}\,{}_{I+1}\ddot{\mathbf{x}}^*\,] \;, $$
$$ {}_I\mathbf{m}_2^0 = -\frac{1}{2}\,{}_Im\,{}_I\mathbf{B} - \frac{1}{{}_IL}\,{}_Im\,{}_I\mathbf{B}^3 + \frac{1}{{}_IL}\,{}_I\mathbf{t}^3 $$
$$ + {}_Im\,[\{\frac{1}{4} - \frac{{}_Iy^{33}}{{}_IL^2}\}\,{}_I\ddot{\mathbf{x}}^* + \{\frac{1}{4} + \frac{{}_Iy^3}{{}_IL} + \frac{{}_Iy^{33}}{{}_IL^2}\}\,{}_{I+1}\ddot{\mathbf{x}}^*\,] \;. \tag{7.3.38}$$

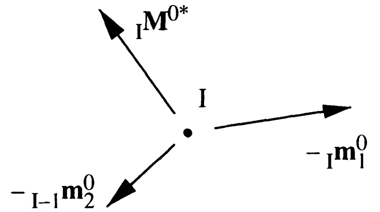

Fig. 7.3.3 Force balance of the I'th interior node.

For generality, it is assumed that external concentrated forces ${}_I\mathbf{M}^{0*}$ are applied to the interior nodes (I=2,3,...,N). Then, considering the force balance of the I'th node (Fig. 7.3.3), the kinetic coupling equations become

$$ {}_{I-1}\mathbf{m}_2^0 + {}_I\mathbf{m}_1^0 = {}_I\mathbf{M}^{0*} \;,\quad \text{for I=2,3,...,N} \;. \tag{7.3.39}$$

Moreover, with the help of (7.3.38) the coupling equations require

$$ [{}_{I-1}m\{\frac{1}{4} - \frac{{}_{I-1}y^{33}}{{}_{I-1}L^2}\}]\,{}_{I-1}\ddot{\mathbf{x}}^* + [{}_{I-1}m\{\frac{1}{4} + \frac{{}_{I-1}y^3}{{}_{I-1}L} + \frac{{}_{I-1}y^{33}}{{}_{I-1}L^2}\} + {}_Im\{\frac{1}{4} - \frac{{}_Iy^3}{{}_IL} + \frac{{}_Iy^{33}}{{}_IL^2}\}]\,{}_I\ddot{\mathbf{x}}^* $$
$$ + [{}_Im\,\{\frac{1}{4} - \frac{{}_Iy^{33}}{{}_IL^2}\}]\,{}_{I+1}\ddot{\mathbf{x}}^* = \frac{1}{2}\,{}_{I-1}m\,{}_{I-1}\mathbf{B} + \frac{1}{{}_{I-1}L}\,{}_{I-1}m\,{}_{I-1}\mathbf{B}^3 - \frac{1}{{}_{I-1}L}\,{}_{I-1}\mathbf{t}^3 $$
$$ + \frac{1}{2}\,{}_Im\,{}_I\mathbf{B} - \frac{1}{{}_IL}\,{}_Im\,{}_I\mathbf{B}^3 + \frac{1}{{}_IL}\,{}_I\mathbf{t}^3 + {}_I\mathbf{M}^{0*} \;,\quad \text{for I=2,3,...,N} \;. \tag{7.3.40}$$

These expressions represent (N−1) vector equations for the (N+1) position vectors ${}_I\mathbf{x}^*$. The remaining two vector equations are determined by boundary conditions. To explore the nature of these boundary conditions, it is convenient to use (7.3.15), the kinematic conditions (7.3.36), and the results (7.3.37), to rewrite the rate of work ${}_I\mathcal{W}$ in (7.3.22) in the form

$$
{}_I\mathcal{W} = {}_I\mathrm{m}\,({}_I\mathbf{B} \bullet {}_I\mathbf{v} + {}_I\mathbf{B}^3 \bullet {}_I\mathbf{w}_3) + {}_I\mathbf{m}_1^0 \bullet {}_I\dot{\mathbf{x}}^* + {}_I\mathbf{m}_2^0 \bullet {}_{I+1}\dot{\mathbf{x}}^* \quad . \tag{7.3.41}
$$

Moreover, it follows by using the conditions (7.3.39) and summing ${}_I\mathcal{W}$ over all N elements, that the total rate of external work $\mathcal{W}$ applied to the entire string can be expressed as

$$
\mathcal{W} = \sum_{I=1}^{N} {}_I\mathcal{W} = \sum_{I=1}^{N} {}_I\mathrm{m}\,({}_I\mathbf{B} \bullet {}_I\mathbf{v} + {}_I\mathbf{B}^3 \bullet {}_I\mathbf{w}_3) + \sum_{I=2}^{N} {}_I\mathbf{M}^{0*} \bullet {}_I\dot{\mathbf{x}}^*
$$

$$
+ {}_1\mathbf{m}_1^0 \bullet {}_1\dot{\mathbf{x}}^* + {}_N\mathbf{m}_2^0 \bullet {}_{N+1}\dot{\mathbf{x}}^* \quad . \tag{7.3.42}
$$

Using this expression it can be seen that the external forces ${}_I\mathbf{M}^{0*}$ do work at the interior nodes, the force ${}_1\mathbf{m}_1^0$ does work on the end I=1, and the force ${}_{N+1}\mathbf{m}_2^0$ does work on the end I=N+1. Consequently, the boundary conditions on these ends require specification of

$$
\{\,{}_1\mathbf{x}^*(t) \text{ or } {}_1\mathbf{m}_1^0(t)\} \quad \text{and} \quad \{{}_{N+1}\mathbf{x}^*(t) \text{ or } {}_N\mathbf{m}_2^0(t)\} \quad , \tag{7.3.43}
$$

where ${}_1\mathbf{m}_1^0$ and ${}_N\mathbf{m}_2^0$ are determined by the equations (7.3.38). Moreover, the solution of the (N+1) vector equations (7.3.40) and (7.3.43) requires specification of the initial conditions for the position and velocities of the nodes

$$
{}_I\mathbf{x}^* = {}_I\mathbf{x}^*(0) \quad \text{and} \quad {}_I\dot{\mathbf{x}}^* = {}_I\dot{\mathbf{x}}^*(0) \quad . \tag{7.3.44}
$$

Also, it can be shown that these equations are properly invariant under SRBM so they are valid for large deformations and rotations of the string.

As a specific example, consider a string of stress-free total reference length L_T which is made from a homogeneous material with uniform cross-sectional area A, and constant three-dimensional density ρ_0^*. Moreover, the string is divided into N equal parts of length L so that the reference values ${}_I\mathbf{X}^*$ of the position vectors ${}_I\mathbf{x}^*$ are specified by

$$
{}_I\mathbf{X}^* = \Big[-\frac{L_T}{2} + (I-1)\,L\Big]\,\mathbf{e}_3 \quad \text{for } I=1,2,\ldots,N+1 \quad , \quad L = \frac{L_T}{N} \quad . \tag{7.3.45}
$$

Also, the strain energy function for the I'th Cosserat point is specified by a form similar to (5.31.14) such that

$$
{}_I\mathrm{m}\,{}_I\Sigma = \frac{1}{2}E^*\,{}_I D_{33}^{1/2}\,AL\,\big[{}_I E_{33}\big]^2 \; , \quad {}_I E_{33} = \frac{1}{2}({}_I\lambda^2 - 1) \; ,
$$

$$
{}_I N = \frac{1}{2}E^*\,A\,{}_I\lambda\,({}_I\lambda^2 - 1) \quad , \tag{7.3.46}
$$

where E^* is Young's modulus of the three-dimensional material. Furthermore, with the help of (7.3.9), and the assumptions that the specific body force $\mathbf{b}_b^*$ is constant and that there are no surface tractions applied to the lateral surface of the string, it follows that

$$
{}_I\mathbf{D}_3 = \mathbf{e}_3 \;\; , \;\; {}_I D_{33} = 1 \;\; , \;\; {}_I\mathrm{m} = \mathrm{m} = \rho_0^* AL \;\; , \;\; {}_I y^3 = 0 \;\; , \;\; {}_I y^{33} = y^{33} \;\; ,
$$

$$
{}_I\mathbf{B}_b = \mathbf{b}_b^* \;\; , \;\; {}_I\mathbf{B}_c = 0 \;\; , \;\; {}_I\mathbf{B}_b^3 = 0 \;\; , \;\; {}_I\mathbf{B}_c^3 = 0 \;\; , \tag{7.3.47}
$$

where the value y^{33} needs to be specified by a constitutive equation. Also, using the constitutive equations (7.3.29), (7.3.30) and (7.3.33) it can be shown that

$$ {}_I\mathbf{t}^3 = L\,[{}_IN + {}_I\eta_1\,\frac{{}_I\dot{\lambda}}{{}_I\lambda^2}]\,[\frac{{}_I\mathbf{d}_3}{{}_Id_{33}^{1/2}}] \ . \tag{7.3.48} $$

Now, with the help of these results the equations (7.3.38) reduce to

$$ {}_I\mathbf{m}_1^0 = -\frac{1}{2}m\,\mathbf{b}_b^* - \frac{1}{L}\,{}_I\mathbf{t}^3 + m\,[\{\frac{1}{4}+\frac{y^{33}}{L^2}\}\,{}_I\ddot{\mathbf{x}}^* + \{\frac{1}{4}-\frac{y^{33}}{L^2}\}\,{}_{I+1}\ddot{\mathbf{x}}^*\,] \ , $$

$$ {}_I\mathbf{m}_2^0 = -\frac{1}{2}m\,\mathbf{b}_b^* + \frac{1}{L}\,{}_I\mathbf{t}^3 + m\,[\{\frac{1}{4}-\frac{y^{33}}{L^2}\}\,{}_I\ddot{\mathbf{x}}^* + \{\frac{1}{4}+\frac{y^{33}}{L^2}\}\,{}_{I+1}\ddot{\mathbf{x}}^*\,] \ , \tag{7.3.49} $$

and the equations of motion (7.3.40) become

$$ m\,[\{\frac{1}{4}-\frac{y^{33}}{L^2}\}\,{}_{I-1}\ddot{\mathbf{x}}^* + 2m\{\frac{1}{4}+\frac{y^{33}}{L^2}\}\,{}_I\ddot{\mathbf{x}}^* + \{\frac{1}{4}-\frac{y^{33}}{L^2}\}\,{}_{I+1}\ddot{\mathbf{x}}^*\,] $$

$$ = m\,\mathbf{b}_b^* - \frac{1}{L}\,{}_{I-1}\mathbf{t}^3 + \frac{1}{L}\,{}_I\mathbf{t}^3 + {}_I\mathbf{M}^{0*} \quad , \quad \text{for } I=2,3,\dots,N \ . \tag{7.3.50} $$

Furthermore, for a uniform string it can be shown that direct integration of $(7.3.9)_3$ yields

$$ y^{33} = \frac{L^2}{12} \ , \tag{7.3.51} $$

which would be associated with the Galerkin approach. Also, it can be seen from (7.3.49) and (7.3.50) that the value

$$ y^{33} = \frac{L^2}{4} \ , \tag{7.3.52} $$

will cause the mass matrix to be diagonal. However, a more physical value for y^{33} can be obtained by matching the first frequency for the small deformation free axial vibrations of a string with that predicted by a string which is modeled by a single Cosserat point (N=1). More specifically, the equations (7.3.49) will produce the exact first frequency (Graff, 1975, p. 87) for a single element provided that

$$ y^{33} = \frac{L^2}{\pi^2} \ , \tag{7.3.53} $$

which is similar to the result (6.14.24) obtained for free vibrations of a Cosserat point. Also, it was shown in (Rubin and Gottlieb, 1996) that a formula for y^{33} can be developed which causes the first frequency of the small deformation lateral vibration of a taut string to be exact for any level of discretization (N). For large values of N, that formula again yields the result (7.3.53). Consequently, the value (7.3.53) will be used in the examples that follow.

As a specific example, consider the equilibrium of a string that is subjected to gravity acting in the negative $\mathbf{e}_1$ direction

$$ \mathbf{b}_b^* = -\,g\,\mathbf{e}_1 \ , \tag{7.3.54} $$

where g is the specific force of gravity. Thus, in the absence of concentrated forces the equations (7.3.49) and (7.3.50) yield

$$ {}_I\mathbf{m}_1^0 = \frac{1}{2}m\,g\,\mathbf{e}_1 - \frac{1}{L}\,{}_I\mathbf{t}^3 \quad , \quad {}_I\mathbf{m}_2^0 = \frac{1}{2}m\,g\,\mathbf{e}_1 + \frac{1}{L}\,{}_I\mathbf{t}^3 \quad \text{for } I=1,2,\dots,N+1 \ , $$

$$ -\,m\,g\,\mathbf{e}_1 - \frac{1}{L}\,{}_{I-1}\mathbf{t}^3 + \frac{1}{L}\,{}_I\mathbf{t}^3 = 0 \quad , \quad \text{for } I=2,3,\dots,N \ . \tag{7.3.55} $$

Assuming that the string is made from rubber, the material constants are given by (Kolsky, 1963, p. 201)

$$\rho_0^* = 0.93 \text{ Mg/m}^3 \ , \quad E^* = 2.0 \text{ MPa} \ . \tag{7.3.56}$$

Also, the geometry of the string is taken to be

$$L_T = 1.0 \text{ m} \ , \quad A = 1.0 \times 10^{-6} \text{ m}^2 \ , \tag{7.3.57}$$

and the present value of ${}_I\mathbf{x}^*$ is specified by

$${}_I\mathbf{x}^* = {}_I x_1 \, \mathbf{e}_1 + {}_I x_3 \, \mathbf{e}_3 \ . \tag{7.3.58}$$

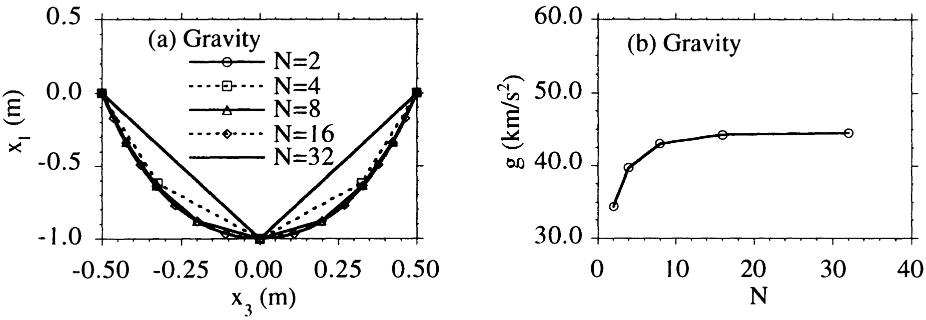

Fig. 7.3.4 Convergence of (a) the deformed shapes, and (b) the values of g, for loading due to gravity.

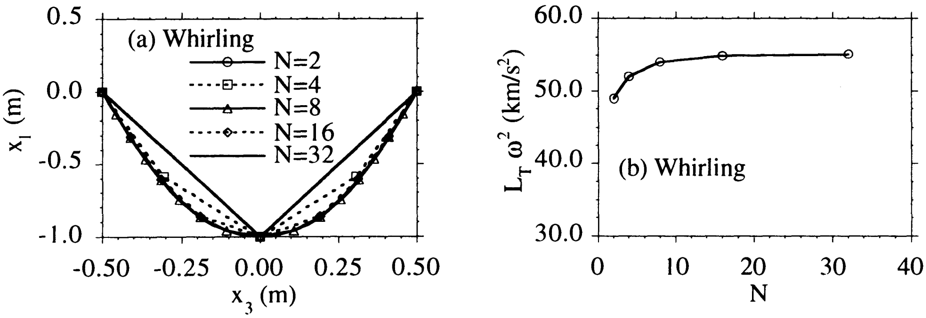

Fig. 7.3.5 Convergence of (a) the deformed shapes, and (b) the values of $L_T\omega^2$, for whirling with $\theta=\pi$.

A computer program was developed using MATLAB 5.2.0 (The MathWorks Inc., 1996) to iteratively solve the equations (7.3.55) for the values $\{{}_I x_1 \ , \ {}_I x_3 \ , \ g\}$ together with the boundary conditions

$${}_1 x_1 = 0 \, , \quad {}_1 x_3 = -\frac{L_T}{2} \ , \quad {}_{N+1} x_1 = 0 \ , \quad {}_{N+1} x_3 = \frac{L_T}{2} \ , \tag{7.3.59}$$

and the constraint that

$${}_{(N/2+1)} x_1 = -L_T \quad \text{for even values of N} \ . \tag{7.3.60}$$

This latter constraint was imposed in order to compare the present solution with the one developed next for whirling of a string.

Figure 7.3.4a shows the convergence of the deformed shapes for loading due to gravity by comparing the solutions for different values of N. It can be seen that even for N=4 the nodes of the string model are very close to those of the more refined solution with N=32. Also, Fig. 7.3.4b shows the convergence of the value g for gravity. In particular, it can be seen that in order to obtain these large deformations the value of g has to be about 4,500 times the usual value on earth.

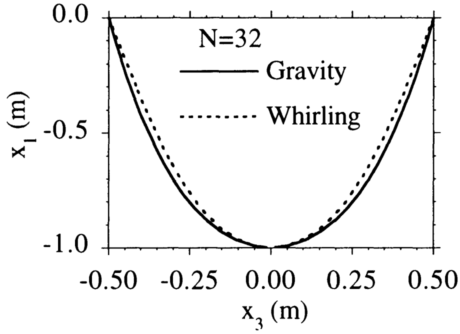

Fig. 7.3.6 Comparison of the deformed shapes for gravity loading and whirling with $\theta=\pi$.

As a second problem, consider the steady whirling of a string about the $\mathbf{e}_3$ axis. For this case, the position vectors ${}_I\mathbf{x}^*$ are expressed in terms of cylindrical polar coordinates, such that

$$ {}_I\mathbf{x}^* = {}_Ix_1\,\mathbf{e}_r(\theta) + {}_Ix_3\,\mathbf{e}_3 \ , \quad \theta = \omega\, t \ , \tag{7.3.61}$$

where ω is the constant angular velocity of whirling. Now, in the absence of gravity (g=0) and concentrated forces, and with the use of the specifications (7.3.47), (7.3.53), (7.3.56) and (7.3.57), the equations (7.3.49) and (7.3.50) become

$$ {}_I\mathbf{m}_1^0 = -\frac{1}{L}\,{}_I\mathbf{t}^3 - m\omega^2\,[\{\frac{1}{4}+\frac{y^{33}}{L^2}\}\,{}_Ix_1 + \{\frac{1}{4}-\frac{y^{33}}{L^2}\}\,{}_{I+1}x_1\,]\,\mathbf{e}_r \ ,$$

$$ {}_I\mathbf{m}_2^0 = \frac{1}{L}\,{}_I\mathbf{t}^3 - m\omega^2\,[\{\frac{1}{4}-\frac{y^{33}}{L^2}\}\,{}_Ix_1 + \{\frac{1}{4}+\frac{y^{33}}{L^2}\}\,{}_{I+1}x_1\,]\,\mathbf{e}_r \ ,$$

$$ m\omega^2\,[\{\frac{1}{4}-\frac{y^{33}}{L^2}\}\,{}_{I-1}x_1 + 2\{\frac{1}{4}+\frac{y^{33}}{L^2}\}\,{}_Ix_1 + \{\frac{1}{4}-\frac{y^{33}}{L^2}\}]\,{}_{I+1}x_1\,]\,\mathbf{e}_r$$
$$ -\frac{1}{L}\,{}_{I-1}\mathbf{t}^3 + \frac{1}{L}\,{}_I\mathbf{t}^3 = 0 \quad , \quad \text{for } I=2,3,\ldots,N \ . \tag{7.3.62}$$

A computer program was again developed using MATLAB 5.2.0 (The MathWorks Inc., 1996) to iteratively solve the equations (7.3.62) for the values $\{{}_Ix_1 \ , \ {}_Ix_3 \ , \ \omega\}$ together with the boundary conditions

$$ {}_1x_1 = 0 \ , \ {}_1x_3 = -\frac{L_T}{2} \ , \ {}_{N+1}x_1 = 0 \ , \ {}_{N+1}x_3 = \frac{L_T}{2} \ , \tag{7.3.63}$$

and the constraint that

$$ {}_{(N/2+1)}x_1 = L_T \quad \text{for even values of N} \ . \tag{7.3.64}$$

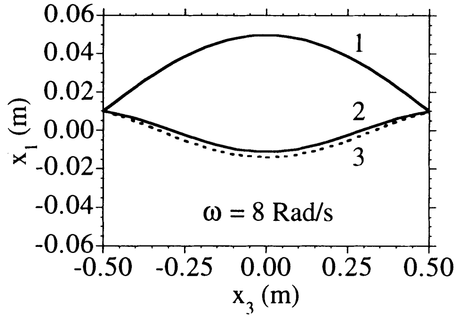

Fig. 7.3.7 Steady state whirling for ω = 8 rad/s, showing Modes 1,2 and 3.

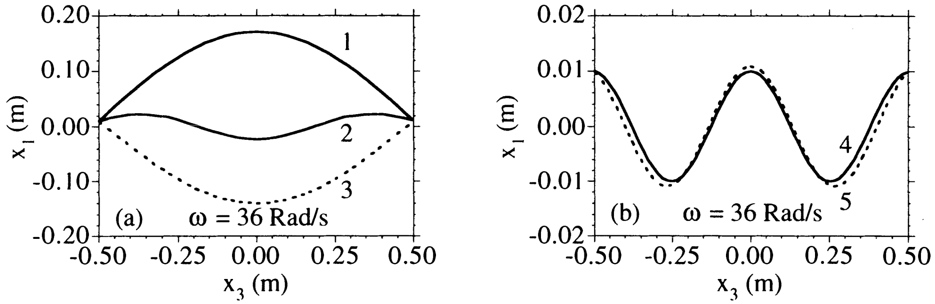

Fig. 7.3.8 Steady state whirling with ω = 36 rad/s, showing
(a) Modes 1,2 and 3; and (b) Modes 4 and 5.

Figure 7.3.5a shows the convergence of the deformed shapes due to whirling by comparing the solutions for different values of N. It can be seen that even for N=4, the nodes of the string model are very close to those of the more refined solution with N=32. Also, Fig. 7.3.5b shows the convergence of the value for $L_T\omega^2$. In particular, it is observed that the value of $L_T\omega^2$ is greater than the value of g shown in Fig. 7.3.4b. This is consistent with the fact that the centripetal acceleration for whirling is a linear function of the radius, whereas the effect of gravity associated with Fig. 7.3.4b is uniform over the string. Moreover, the difference between these two loads causes the shapes of the string to be slightly different as is shown in Fig. 7.3.6.

As a third example, consider a string whose ends are attached to the edges of two wheels which are rotating with constant angular velocity ω about the same axis $\mathbf{e}_3$. The wheels are located a constant distance L_T apart so that before the rotation is started, the string is taut but not in tension. Moreover, since the string is attached to the edges of the wheels, its center of mass has an eccentricity relative to the axis of rotation which is equal to the radius e of each wheel.

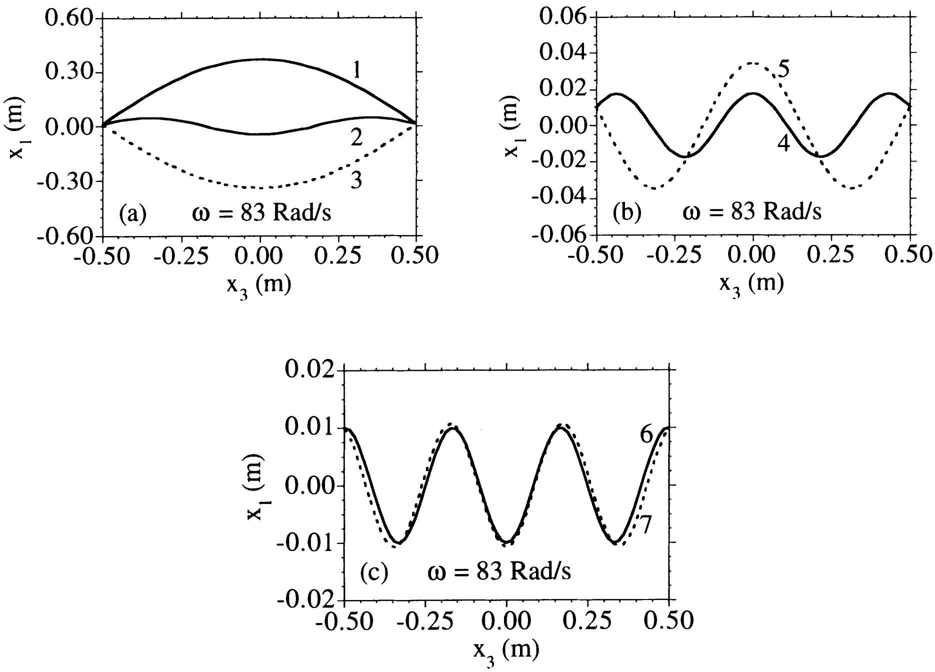

Fig. 7.3.9 Steady state whirling with ω = 83 rad/s, showing (a) Modes 1,2 and 3; (b) Modes 4 and 5; and (c) Modes 6 and 7.

For the general dynamical case, the position vectors ${}_I\mathbf{x}^*$ are expressed in terms of cylindrical polar coordinates such that

$$ {}_I\mathbf{x}^* = {}_Ix_1\,\mathbf{e}_r(\theta) + {}_Ix_2\,\mathbf{e}_\theta(\theta) + {}_Ix_3\,\mathbf{e}_3 \ , \quad \theta = \omega t \ . \tag{7.3.65} $$

In the absence of gravity (g=0) and concentrated forces, and with the use of the specifications (7.3.47), (7.3.53), (7.3.56) and (7.3.57), the equations (7.3.49) and (7.3.50) become

$$ {}_I\mathbf{m}_1^0 = -\frac{1}{L}\,{}_I\mathbf{t}^3 + m\,[\{\tfrac{1}{4}+\frac{y^{33}}{L^2}\}\,{}_I\ddot{\mathbf{x}}^* + \{\tfrac{1}{4}-\frac{y^{33}}{L^2}\}\,{}_{I+1}\ddot{\mathbf{x}}^*\,] \ , $$

$$ {}_I\mathbf{m}_2^0 = \frac{1}{L}\,{}_I\mathbf{t}^3 + m\,[\{\tfrac{1}{4}-\frac{y^{33}}{L^2}\}\,{}_I\ddot{\mathbf{x}}^* + \{\tfrac{1}{4}+\frac{y^{33}}{L^2}\}\,{}_{I+1}\ddot{\mathbf{x}}^*\,] \ , $$

$$ \begin{aligned} m\,[\{\tfrac{1}{4}-\frac{y^{33}}{L^2}\}\,{}_{I-1}\ddot{\mathbf{x}}^* &+ 2m\{\tfrac{1}{4}+\frac{y^{33}}{L^2}\}\,{}_I\ddot{\mathbf{x}}^* + \{\tfrac{1}{4}-\frac{y^{33}}{L^2}\}\,{}_{I+1}\ddot{\mathbf{x}}^*\,] \\ &= -\frac{1}{L}\,{}_{I-1}\mathbf{t}^3 + \frac{1}{L}\,{}_I\mathbf{t}^3 \ , \quad \text{for } I=2,3,\ldots,N \ , \end{aligned} \tag{7.3.66} $$

where the accelerations ${}_I\ddot{\mathbf{x}}^*$ are given by

$$ {}_I\ddot{\mathbf{x}}^* = [{}_I\ddot{x}_1 - 2\,\omega\,{}_I\dot{x}_2 - \omega^2\,{}_Ix_1]\,\mathbf{e}_r + [{}_I\ddot{x}_2 + 2\,\omega\,{}_I\dot{x}_1 - \omega^2\,{}_Ix_2]\,\mathbf{e}_\theta + [{}_I\ddot{x}_3]\,\mathbf{e}_3 \ . \tag{7.3.67} $$

Moreover, the boundary conditions are specified by

$$ {}_1x_1 = e \ , \quad {}_1x_2 = 0 \ , \quad {}_1x_3 = -\frac{L_T}{2} \ , \quad {}_{N+1}x_1 = e \ , \quad {}_{N+1}x_2 = 0 \ , \quad {}_{N+1}x_3 = \frac{L_T}{2} \ . \tag{7.3.68} $$

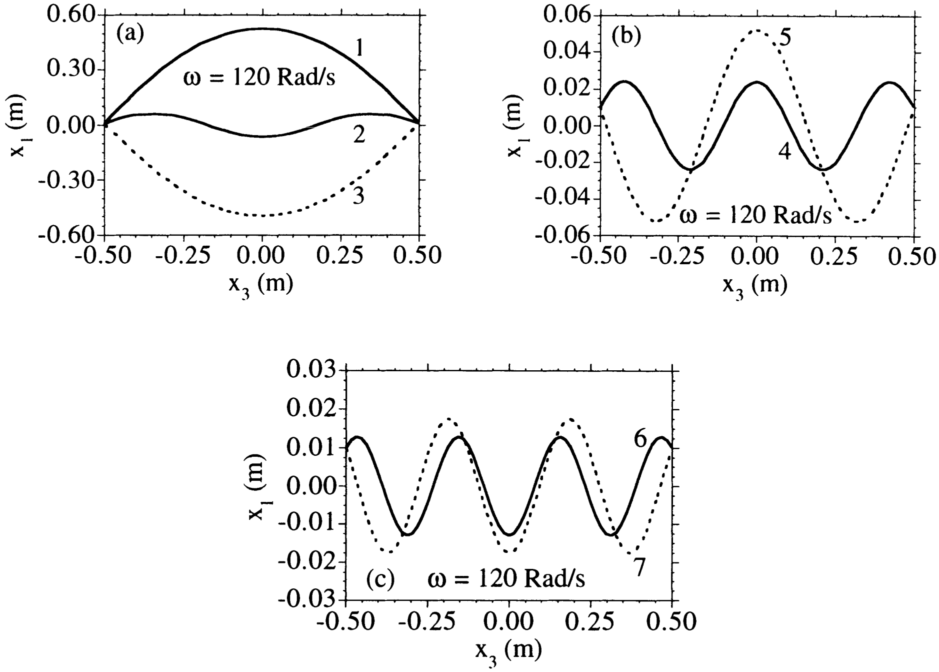

Fig. 7.3.10 Steady state whirling with ω = 120 rad/s, showing (a) Modes 1,2 and 3; (b) Modes 4 and 5; and (c) Modes 6 and 7.

For general initial conditions the solution of equations (7.3.66) and the boundary conditions (7.3.68) yield complicated dynamics with the components $\{{}_Ix_1, {}_Ix_2, {}_Ix_3\}$ being time dependent. However, for simple steady state solutions these components are independent of time and the string remains in $\mathbf{e}_r$–$\mathbf{e}_3$ plane such that

$$ {}_Ix_2 = 0 \ , \quad {}_I\ddot{\mathbf{x}}^* = -[\omega^2 \, {}_Ix_1] \, \mathbf{e}_r \ . \tag{7.3.69} $$

A computer program was again developed using MATLAB 5.2.0 (The MathWorks Inc., 1996) to iteratively solve the equations (7.3.66) for the values $\{{}_Ix_1 \ , \ {}_Ix_3\}$ together with the boundary conditions (7.3.68). Moreover, for the examples considered here the eccentricity e was specified by

$$ e = 0.01 \text{ m} \ . \tag{7.3.70} $$

The steady state solutions of these equations indicate that for ω below a critical value of about 8 rad/s the string attains the typical shape of a whirling jump rope, which is called Mode 1. However, at this critical speed two additional modes appear (Modes 2 and 3) which are shown in Fig. 7.3.7. In particular, it can be observed form this figure that the locations of the center of the string for these two new modes are on the opposite sides of the axis of rotation relative to that of Mode 1.

At a critical speed of about ω = 36 rad/s another two modes (Modes 4 and 5) appear, and at a critical speed of about ω = 83 rad/s still another two modes (Modes 6 and 7) appear. Although, additional calculations have not been performed, it is expected that additional pairs of modes will appear at higher and higher critical speeds. Figures 7.3.8, 7.3.9 and 7.3.10 show these mode shapes at different rotational speeds.

From Figs. 7.3.7, 7.3.8a, 7.3.9a and 7.3.10a it can be observed that Modes 2 and 3 separate with increasing speed. Mode 2 remains centered around the axis of rotation and Mode 3 appears more like the reflection of Mode 1. Also, Modes 1 and 3 have nearly one half wavelengths whereas Mode 2 has nearly two wavelengths. Figures 7.3.8b, 7.3.9b and 7.3.10b indicate that Modes 4 and 5 also separate with increasing speed. Mode 4 has about two and one half wavelengths whereas Mode 5 has about two wavelengths. Similarly, Figs. 7.3.9c and 7.3.10c indicate that Modes 6 and 7 separate with increasing speed and Mode 6 has about three and one half wavelengths and Mode 7 has about two and one half wavelengths.

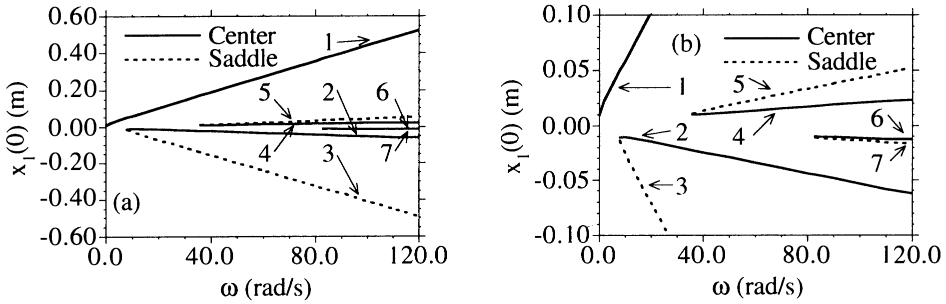

Fig. 7.3.11 Center locations for the steady state whirling solutions.

Figure 7.3.11 plots the location of the center of the string for each of the modes as a function of the rotational speed ω. From this figure it can clearly be observed that the center of the string drops drastically towards the axis of rotation at each critical speed when two new solutions appear. These new solutions tend to keep the string more evenly distributed about the axis of rotation. In particular, the even modes 2, 4 and 6 cause the string to remain close to the axes of rotation. Figure 7.3.12 also shows that the total energy (kinetic energy + elastic strain energy) of the new solutions is much lower than the previous solutions. Thus, from an energetic point of view the newly appearing solutions are preferable to the existing ones.

Linearized stability analysis was also performed by analytically linearizing the equations (7.3.66) and then numerically evaluating the Jacobian matrix and its eigenvalues using the numerical values $\{{}_I x_1$ and ${}_I x_3\}$ associated with the steady state solutions for each of the modes. Also, full dynamical motions in all three directions were allowed for each node. This analysis indicates that Modes 3, 5 and 7 are saddles with positive real eigenvalues. Modes 1 and 2 are centers with nearly pure imaginary eigenvalues and Modes 4 and 6 are either centers or potentially weak sources depending on the speed of rotation. In addition, some full dynamical simulations were performed for the simpler system with only two (N=2) Cosserat points and material damping of the form (7.3.33) with ${}_{I}\eta_1=10^3$ Pa/s. This simple model exhibits only three modes which are similar in character to Modes 1,2 and 3 of the more complete system. These simulations indicate that Mode 1 is a sink (with a negative real part of at least one eigenvalue), Mode 2 is a weak source (with a small positive real part of at least one eigenvalue) and Mode 3 is a strong source (with a large positive real part of at least one eigenvalue).

Also, it is noted that the response of this dynamical system seems to have some similarity with the well known self centering response of a rotor with eccentric mass. Specifically, the linear equations of rotor dynamics predict that above a critical speed the center of mass asymptotically approaches the axis of rotation (Biezeno and Grammel, 1954, p. 184; Darlow, 1989, Figs. 210 and 2.14). For the discrete model of the continuous system described here, higher wavelength solutions like Modes 2, 4 and 6 come into existence as the speed increases. These modes also tend to cause the center of mass of the string to approach the axis of rotation.

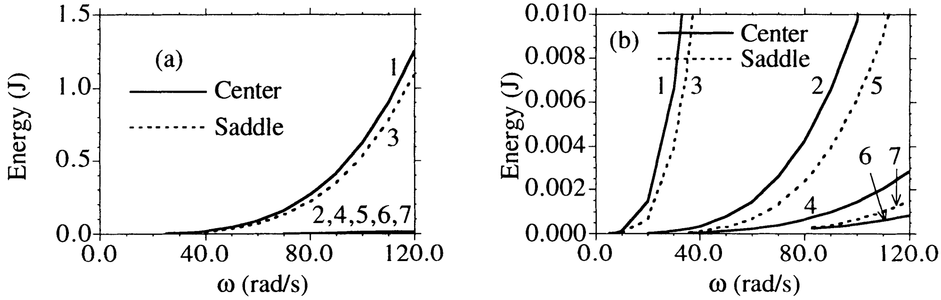

Fig. 7.3.12 Total energy of the steady state whirling solutions.

Finally, it is emphasized that none of these three example problems can be formulated using linearized equations for the string since the string initially is tension-free in its unloaded or unrotated initial state.

7.4 Formulation of the numerical solution of rod problems using the theory of a Cosserat point

In section 7.3 it was shown that the theory of a Cosserat point can be used to formulate the numerical solution of the dynamic motion of nonlinear strings. In this section it will be shown that the theory of a Cosserat point can also be used to formulate the numerical solution of the dynamic motion of nonlinear rods. Such a formulation has recently been presented by Rubin (2000) where a complete set of constitutive equations for elastic beams with rectangular cross-sections was described, and a number of nonlinear example problems were examined.

For the numerical solution procedure the rod is divided into elements, each of which is modeled using balance laws of the theory of a Cosserat point. Then, the equations for each element are coupled to the equations for neighboring elements by using kinematic and kinetic coupling conditions at the common boundaries of the elements (which is similar to the direct approach for finite elements). This approach produces a system of coupled nonlinear ordinary differential equations which are functions of time only, as in the standard Galerkin procedure.

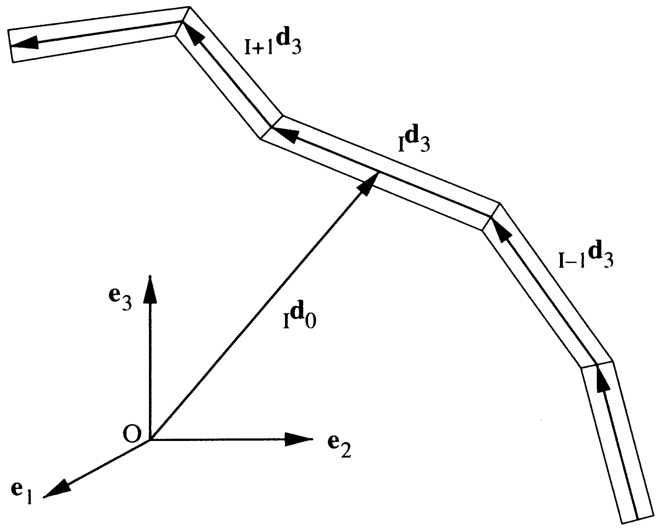

Fig. 7.4.1 Sketch of a rod modeled as a chain of Cosserat points.

More specifically, the rod is modeled as a chain of N (I=1,2,...,N) Cosserat points (see Fig. 7.4.1). The kinematics of the I'th Cosserat point are characterized by six director vectors

$$ {}_I\mathbf{d}_i(t) \quad \text{for } i=0,1,2,...,5 \ , \tag{7.4.1}$$

where for convenience, the location ${}_I\mathbf{x}(t)$ of the Cosserat point relative to a fixed origin is denoted by ${}_I\mathbf{d}_0(t)$

$$ {}_I\mathbf{x}(t) = {}_I\mathbf{d}_0(t) \ , \tag{7.4.2}$$

and the remaining director vectors are used to model deformations of the Cosserat point. Also, the director velocities ${}_I\mathbf{w}_i$ are defined by

$$ {}_I\mathbf{w}_i = {}_I\dot{\mathbf{d}}_i \ , \tag{7.4.3}$$

In the stress-free reference configuration, the directors ${}_I\mathbf{d}_i$ attain the values ${}_I\mathbf{D}_i$. Also, the three vectors $\{{}_I\mathbf{d}_1, {}_I\mathbf{d}_2, {}_I\mathbf{d}_3\}$ are assumed to remain linearly independent such that

$$ {}_I D^{1/2} = {}_I\mathbf{D}_1 \times {}_I\mathbf{D}_2 \bullet {}_I\mathbf{D}_3 > 0 \ , \quad {}_I d^{1/2} = {}_I\mathbf{d}_1 \times {}_I\mathbf{d}_2 \bullet {}_I\mathbf{d}_3 > 0 \ . \tag{7.4.4}$$

Here, the subscript I is used to denote quantities related to the I'th Cosserat point and it is recalled that there is no sum on repeated upper case indices. Moreover, the reciprocal vectors ${}_I\mathbf{d}^i$ and ${}_I\mathbf{D}^i$ (i=1,2,3) are defined so that

$$ {}_I\mathbf{D}_i \bullet {}_I\mathbf{D}^j = \delta_i^j \ , \quad {}_I\mathbf{d}_i \bullet {}_I\mathbf{d}^j = \delta_i^j \ , \quad \text{for } i,j=1,2,3 \ . \tag{7.4.5} $$

Also, a number of kinematic variables can be defined by the expressions

$$ {}_I\mathbf{F} = \sum_{i=1}^{3} {}_I\mathbf{d}_i \otimes {}_I\mathbf{D}^i \ , \quad {}_IJ = \det({}_I\mathbf{F}) = \frac{{}_Id^{1/2}}{{}_ID^{1/2}} \ , \quad {}_I\mathbf{C} = {}_I\mathbf{F}^T {}_I\mathbf{F} \ , \quad {}_I\mathbf{B} = {}_I\mathbf{F}\,{}_I\mathbf{F}^T \ , $$

$$ {}_I\boldsymbol{\beta}_1 = {}_I\mathbf{F}^{-1}\,{}_I\mathbf{d}_4 - {}_I\mathbf{D}_4 \ , \quad {}_I\boldsymbol{\beta}_2 = {}_I\mathbf{F}^{-1}\,{}_I\mathbf{d}_5 - {}_I\mathbf{D}_5 \ , $$

$$ {}_I\dot{\mathbf{F}} = {}_I\mathbf{L}\,{}_I\mathbf{F} \ , \quad {}_I\mathbf{L} = \sum_{i=1}^{3} {}_I\mathbf{w}_i \otimes {}_I\mathbf{d}^i = {}_I\mathbf{D} + {}_I\mathbf{W} \ , $$

$$ {}_I\mathbf{D} = \frac{1}{2}({}_I\mathbf{L} + {}_I\mathbf{L}^T) = {}_I\mathbf{D}^T \ , \quad {}_I\mathbf{W} = \frac{1}{2}({}_I\mathbf{L} - {}_I\mathbf{L}^T) = -\,{}_I\mathbf{W}^T \ . \tag{7.4.6} $$

In these expressions, $\{{}_I\mathbf{F}, {}_I\mathbf{C}, {}_I\mathbf{B}\}$ are related to homogeneous deformations, $\{{}_I\mathbf{L}, {}_I\mathbf{D}, {}_I\mathbf{W}\}$ are related to homogeneous deformation rates, and ${}_I\boldsymbol{\beta}_\alpha$ $(\alpha=1,2)$ are related to inhomogeneous deformations (see Rubin, 2000).

For clarity, the theory of a Cosserat point for rods will be developed by two different approaches. In the first approach, the three-dimensional equations of motion recorded in section 3.2 will be used to motivate the form for the balance laws of the theory of a Cosserat point. In the second approach, the theory of a Cosserat point will be developed by the direct method where the balance laws are postulated. For both approaches, the constitutive equations are developed using the direct approach in terms of a strain energy function and a dissipation inequality.

To this end, it is noted that in its stress-free reference configuration, material points in the rod-like structure are located by the position vector $\mathbf{X}^*(\theta^i)$ in terms of convected Lagrangian coordinates θ^i (i=1,2,3). The cross-section of the rod $\mathcal{A}(\theta^3)$ limits the values of the cross-sectional coordinates θ^α $(\alpha=1,2)$ and is allowed to vary along the rod's axial coordinate θ^3. For definiteness, the total length of the reference curve $\mathbf{X}^*(0,0,\theta^3)$ is taken to be L_T such that

$$ 0 \le \theta^3 \le L_T \ . \tag{7.4.7} $$

Moreover, the three-dimensional region ${}_IP^*$ occupied by the I'th Cosserat point in the present configuration has boundary ${}_I\partial P^*$ which is divided into the cross-section ${}_I\partial P_1^*$ associated with the end $\theta^3={}_I\xi$, the cross-section ${}_I\partial P_2^*$ associated with the end $\theta^3={}_{I+1}\xi$, and the lateral surface ${}_I\partial P_L^*$ such that

$$ {}_I\partial P^* = {}_I\partial P_1^* \cup {}_I\partial P_2^* \cup {}_I\partial P_L^* \ . \tag{7.4.8} $$

Consequently, the conservation of mass (3.2.1) and the balance of linear momentum (3.2.2) can be rewritten in the forms

$$ \frac{d}{dt}\int_{{}_IP^*} \rho^*\, dv^* = 0 \ , $$

$$ \frac{d}{dt}\int_{{}_IP^*} \rho^*\,\mathbf{v}^*\, dv^* = \int_{{}_IP^*} \rho^*\,\mathbf{b}^*\, dv^* + \int_{{}_I\partial P_1^*} \mathbf{t}^*\, da^* + \int_{{}_I\partial P_2^*} \mathbf{t}^*\, da^* + \int_{{}_I\partial P_L^*} \mathbf{t}^*\, da^* \ , \tag{7.4.9} $$

and the balance of angular momentum becomes

$$\frac{d}{dt}\int_{{}_IP^*} \mathbf{x}^* \times \rho^* \mathbf{v}^* \, dv^* = \int_{{}_IP^*} \mathbf{x}^* \times \rho^* \mathbf{b}^* \, dv^* + \int_{{}_I\partial P_1^*} \mathbf{x}^* \times \mathbf{t}^* \, da^*$$
$$+ \int_{{}_I\partial P_2^*} \mathbf{x}^* \times \mathbf{t}^* \, da^* + \int_{{}_I\partial P_L^*} \mathbf{x}^* \times \mathbf{t}^* \, da^* \,, \tag{7.4.10}$$

Furthermore, the averaged form (3.6.3) of the balance of linear momentum is rewritten as

$$\frac{d}{dt}\int_{{}_IP^*} \phi \, \rho^* \mathbf{v}^* \, dv^* = \int_{{}_IP^*} [\phi \, \rho^* \mathbf{b}^* - g^{-1/2} \, \mathbf{t}^{*j} \, \phi_{,j}] \, dv^* + \int_{{}_I\partial P_1^*} \phi \, \mathbf{t}^* \, da^*$$
$$+ \int_{{}_I\partial P_2^*} \phi \, \mathbf{t}^* \, da^* + \int_{{}_I\partial P_L^*} \phi \, \mathbf{t}^* \, da^* \,, \tag{7.4.11}$$

where $\phi(\theta^i)$ is a weighting function of the convected coordinates θ^i only.

In order to motivate the balance laws of the theory of a Cosserat point for rods, it is convenient to introduce the following definitions

$$ {}_Im = \int_{{}_IP^*} \rho^* \, dv^* \;, \quad {}_Iy^{00} = 1 \;, \quad {}_Im \, {}_Iy^{0\alpha} = \int_{{}_IP^*} \theta^\alpha \rho^* \, dv^* \;,$$

$$ {}_Im \, {}_Iy^{03} = \int_{{}_IP^*} {}_I\bar{\theta}^3 \rho^* \, dv^* \;, \quad {}_Im \, {}_Iy^{04} = \int_{{}_IP^*} \theta^1 \, {}_I\bar{\theta}^3 \rho^* \, dv^* \;,$$

$$ {}_Im \, {}_Iy^{05} = \int_{{}_IP^*} \theta^2 \, {}_I\bar{\theta}^3 \rho^* \, dv^* \;, \quad {}_Im \, {}_Iy^{\alpha\beta} = \int_{{}_IP^*} \theta^\alpha \theta^\beta \rho^* \, dv^* \;,$$

$$ {}_Im \, {}_Iy^{\alpha 3} = \int_{{}_IP^*} \theta^\alpha \, {}_I\bar{\theta}^3 \rho^* \, dv^* \;, \quad {}_Im \, {}_Iy^{\alpha 4} = \int_{{}_IP^*} \theta^\alpha \theta^1 \, {}_I\bar{\theta}^3 \rho^* \, dv^* \;,$$

$$ {}_Im \, {}_Iy^{\alpha 5} = \int_{{}_IP^*} \theta^\alpha \theta^2 \, {}_I\bar{\theta}^3 \rho^* \, dv^* \;, \quad {}_Im \, {}_Iy^{33} = \int_{{}_IP^*} {}_I\bar{\theta}^3 \, {}_I\bar{\theta}^3 \rho^* \, dv^* \;,$$

$$ {}_Im \, {}_Iy^{34} = \int_{{}_IP^*} \theta^1 \, {}_I\bar{\theta}^3 \, {}_I\bar{\theta}^3 \rho^* \, dv^* \;, \quad {}_Im \, {}_Iy^{35} = \int_{{}_IP^*} \theta^2 \, {}_I\bar{\theta}^3 \, {}_I\bar{\theta}^3 \rho^* \, dv^* \;,$$

$$ {}_Im \, {}_Iy^{44} = \int_{{}_IP^*} \theta^1 \theta^1 \, {}_I\bar{\theta}^3 \, {}_I\bar{\theta}^3 \rho^* \, dv^* \,, \quad {}_Im \, {}_Iy^{45} = \int_{{}_IP^*} \theta^1 \theta^2 \, {}_I\bar{\theta}^3 \, {}_I\bar{\theta}^3 \rho^* \, dv^* \,,$$

$$ {}_Im \, {}_Iy^{55} = \int_{{}_IP^*} \theta^2 \theta^2 \, {}_I\bar{\theta}^3 \, {}_I\bar{\theta}^3 \rho^* \, dv^* \,, \quad {}_Iy^{ij} = {}_Iy^{ji} \quad \text{for i,j=0,1,2,...,5} \;,$$

$$ {}_Im \, {}_I\mathbf{B}_b^0 = \int_{{}_IP^*} \rho^* \mathbf{b}^* \, dv^* \;, \quad {}_Im \, {}_I\mathbf{B}_c^0 = \int_{{}_I\partial P_L^*} \mathbf{t}^* \, da^* \,,$$

$$ {}_Im \, {}_I\mathbf{B}_b^\alpha = \int_{{}_IP^*} \theta^\alpha \rho^* \mathbf{b}^* \, dv^* \;, \quad {}_Im \, {}_I\mathbf{B}_c^\alpha = \int_{{}_I\partial P_L^*} \theta^\alpha \, \mathbf{t}^* \, da^* \,,$$

$$ {}_Im \, {}_I\mathbf{B}_b^3 = \int_{{}_IP^*} {}_I\bar{\theta}^3 \rho^* \mathbf{b}^* \, dv^* \;, \quad {}_Im \, {}_I\mathbf{B}_c^3 = \int_{{}_I\partial P_L^*} {}_I\bar{\theta}^3 \, \mathbf{t}^* \, da^* \,,$$

$$ {}_Im \, {}_I\mathbf{B}_b^4 = \int_{{}_IP^*} \theta^1 \, {}_I\bar{\theta}^3 \rho^* \mathbf{b}^* \, dv^* \;, \quad {}_Im \, {}_I\mathbf{B}_c^4 = \int_{{}_I\partial P_L^*} \theta^1 \, {}_I\bar{\theta}^3 \, \mathbf{t}^* \, da^* \,,$$

$$ {}_Im \, {}_I\mathbf{B}_b^5 = \int_{{}_IP^*} \theta^2 \, {}_I\bar{\theta}^3 \rho^* \mathbf{b}^* \, dv^* \;, \quad {}_Im \, {}_I\mathbf{B}_c^5 = \int_{{}_I\partial P_L^*} \theta^1 \, {}_I\bar{\theta}^3 \, \mathbf{t}^* \, da^* \,,$$

$$
{}_I\mathbf{m}^0_\alpha = \int_{{}_I\partial P^*_\alpha} \mathbf{t}^*\, da^* \ , \quad {}_I\mathbf{m}^\beta_\alpha = \int_{{}_I\partial P^*_\alpha} \theta^\beta\, \mathbf{t}^*\, da^* \ , \quad {}_I\mathbf{m}^3_\alpha = \int_{{}_I\partial P^*_\alpha} {}_I\bar{\theta}^3\, \mathbf{t}^*\, da^* \ ,
$$

$$
{}_I\mathbf{m}^4_\alpha = \int_{{}_I\partial P^*_\alpha} \theta^1\, {}_I\bar{\theta}^3\, \mathbf{t}^*\, da^* \ , \quad {}_I\mathbf{m}^5_\alpha = \int_{{}_I\partial P^*_\alpha} \theta^2\, {}_I\bar{\theta}^3\, \mathbf{t}^*\, da^* \ ,
$$

$$
{}_I\mathbf{t}^0 = 0 \ , \quad {}_I\mathbf{t}^\alpha = \int_{{}_IP^*} g^{-1/2}\, \mathbf{t}^{*\alpha}\, dv^* \ , \quad {}_I\mathbf{t}^3 = \int_{{}_IP^*} g^{-1/2}\, \mathbf{t}^{*3}\, dv^* \ ,
$$

$$
{}_I\mathbf{t}^4 = \int_{{}_IP^*} g^{-1/2}\, [{}_I\bar{\theta}^3\, \mathbf{t}^{*1} + \theta^1\, \mathbf{t}^{*3}\,]\, dv^*, {}_I\mathbf{t}^5 = \int_{{}_IP^*} g^{-1/2}\, [{}_I\bar{\theta}^3\, \mathbf{t}^{*2} + \theta^2\, \mathbf{t}^{*3}\,]\, dv^*, \tag{7.4.12}
$$

where ${}_I\bar{\theta}^3$ is a convected Lagrangian coordinate defined in the I'th region by

$$
{}_I\bar{\theta}^3 = \frac{2\theta^3 - \xi_I - \xi_{I+1}}{2} \ , \quad -\frac{{}_IL}{2} \le {}_I\bar{\theta}^3 \le \frac{{}_IL}{2} \ , \quad {}_IL = \xi_{I+1} - \xi_I \ , \tag{7.4.13}
$$

and ${}_IL$ is the reference length of this region.

Now, the theory of a Cosserat point for rods can be developed by introducing the kinematic assumption that the three-dimensional position vector $\mathbf{x}^*$ in the I'th region of the rod is represented in the form

$$
\mathbf{x}^*(\theta^i,t) = {}_I\mathbf{d}_0(t) + \theta^1\, [{}_I\mathbf{d}_1(t) + {}_I\bar{\theta}^3{}_I\mathbf{d}_4(t)] + \theta^2\, [{}_I\mathbf{d}_2(t) + {}_I\bar{\theta}^3{}_I\mathbf{d}_5(t)] + {}_I\bar{\theta}^3{}_I\mathbf{d}_3(t) \ , \tag{7.4.14}
$$

Then, using the definitions (7.4.3) and substituting (7.4.12) into the balance laws (7.4.9) and (7.4.11) (with $\phi = \{\theta^1, \theta^2, {}_I\bar{\theta}^3, \theta^1{}_I\bar{\theta}^3, \theta^2{}_I\bar{\theta}^3\}$), it is possible to develop the conservation of mass and the balances of linear and director momentum, respectively, in the forms

$$
{}_I\dot{m} = 0 \ , \quad \frac{d}{dt}[\sum_{j=0}^{5} {}_Im\, {}_Iy^{ij}\, {}_I\mathbf{w}_j\,] = {}_Im\, {}_I\mathbf{b}^i - {}_I\mathbf{t}^i \quad \text{for i=0,1,2,...,5 , with } {}_I\mathbf{t}^0 = 0 \ . \tag{7.4.15}
$$

Also, substituting (7.4.12) into (7.4.10), the balance of angular momentum becomes

$$
\frac{d}{dt} \sum_{i=0}^{5} \sum_{j=0}^{5} [{}_I\mathbf{d}_i \times {}_Im\, {}_Iy^{ij}\, {}_I\mathbf{w}_j] = \sum_{i=0}^{5} {}_I\mathbf{d}_i \times {}_Im\, {}_I\mathbf{b}^i \ . \tag{7.4.16}
$$

In these equations, ${}_Im$ is the mass, ${}_Iy^{ij}$ are the director inertia coefficients, ${}_I\mathbf{b}^i$ are the specific (per unit mass) external assigned director couples, and ${}_I\mathbf{t}^i$ are the intrinsic director couples. Moreover, the director inertia coefficients are constants

$$
{}_I\dot{y}^{ij} = 0 \ , \tag{7.4.17}
$$

and the assigned fields ${}_I\mathbf{b}^i$ can be expressed in the forms

$$
{}_Im\, {}_I\mathbf{b}^i = {}_Im\, {}_I\mathbf{B}^i + {}_I\mathbf{m}^i_1 + {}_I\mathbf{m}^i_2 \ , \quad {}_I\mathbf{B}^i = {}_I\mathbf{B}^i_b + {}_I\mathbf{B}^i_c \ . \tag{7.4.18}
$$

Here, the terms ${}_I\mathbf{B}^i_b$ are associated with the external body force; the terms ${}_I\mathbf{B}^i_c$ are associated with the tractions applied to the lateral surface of the rod section; ${}_I\mathbf{m}^0_1$ and ${}_I\mathbf{m}^0_2$ are the forces applied to the ends ξ_I and ξ_{I+1}, respectively, of the rod section; and ${}_I\mathbf{m}^i_1$ and ${}_I\mathbf{m}^i_2$ (i=1,2,...,5) are the director couples applied to the ends ξ_I and ξ_{I+1}, respectively, of the rod section.

The equations of motion (7.4.15) and (7.4.16) have been developed by integration of the equations of motion of a three-dimensional continuum. This approach is used to motivate the structure of the equations of motion of the Cosserat point. However, within the context of the direct approach these same equations of motion are postulated as the balance laws for the theory of a Cosserat point. The remaining equations describing the theory of a Cosserat point are developed within the context of the direct approach.

Specifically, using the equations of motion (7.4.15) the balance of angular momentum (7.4.16) can be reduced to the form

$$\sum_{i=1}^{5} {}_I\mathbf{d}_i \times {}_I\mathbf{t}^i = 0 \ . \tag{7.4.19}$$

Moreover, by introducing the definition

$${}_I\mathbf{T} = {}_I d^{-1/2} \sum_{i=1}^{5} {}_I\mathbf{t}^i \otimes {}_I\mathbf{d}_i \ , \tag{7.4.20}$$

it follows that the reduced form of the balance of angular momentum can be written as

$${}_I\mathbf{T}^T = {}_I\mathbf{T} \ , \tag{7.4.21}$$

which is similar to the expression (3.2.32) associated with the three-dimensional theory.

For an elastic Cosserat point the specific strain energy ${}_I\Sigma$ is a function of the deformation measures ${}_I\mathbf{C}$ and ${}_I\boldsymbol{\beta}_\alpha$ only

$${}_I\Sigma = {}_I\hat{\Sigma}({}_I\mathbf{C}, {}_I\boldsymbol{\beta}_\alpha) \ . \tag{7.4.22}$$

Moreover, for the purely mechanical theory it is convenient to define: the rate of dissipation ${}_I\mathcal{D}$; the mechanical power ${}_I\mathcal{P}$; the rate of work ${}_I\mathcal{W}$ of the assigned fields ${}_I\mathbf{b}^i$; the kinetic energy ${}_I\mathcal{K}$; and the total internal energy ${}_I\mathcal{U}$ by the formulas

$${}_I d^{1/2}\, {}_I\mathcal{D} = {}_I\mathcal{W} - {}_I\dot{\mathcal{K}} - {}_I\dot{\mathcal{U}} \ , \quad {}_I d^{1/2}\, {}_I\mathcal{P} = {}_I\mathcal{W} - {}_I\dot{\mathcal{K}} \ ,$$

$${}_I\mathcal{W} = \sum_{i=0}^{5} {}_I m\, {}_I\mathbf{b}^i \cdot {}_I\mathbf{w}_i \ , \quad {}_I\mathcal{K} = \sum_{i=0}^{5} \frac{1}{2}\, {}_I m\, {}_I y^{ij}\, {}_I\mathbf{w}_i \cdot {}_I\mathbf{w}_j \ , \quad {}_I\mathcal{U} = {}_I m\, {}_I\Sigma \ . \tag{7.4.23}$$

Next, with the help of the definitions (7.4.6) it can be shown that

$$\overline{{}_I\mathbf{F}^{-1}}^{\bullet} = -\, {}_I\mathbf{F}^{-1}\, {}_I\mathbf{L} \ , \quad {}_I\mathbf{w}_i = {}_I\mathbf{L}\, {}_I\mathbf{d}_i \quad \text{for i=1,2,3} \ ,$$

$${}_I\mathbf{w}_4 = {}_I\mathbf{F}\, {}_I\dot{\boldsymbol{\beta}}_1 + {}_I\mathbf{L}\, {}_I\mathbf{d}_4 \ , \quad {}_I\mathbf{w}_5 = {}_I\mathbf{F}\, {}_I\dot{\boldsymbol{\beta}}_2 + {}_I\mathbf{L}\, {}_I\mathbf{d}_5 \ . \tag{7.4.24}$$

Thus, using the equations of motion (7.4.15), the definition (7.4.20), and the result (7.4.21), it follows that the rate of dissipation reduces to

$${}_I d^{1/2}\, {}_I\mathcal{D} = {}_I d^{1/2}\, {}_I\mathbf{T} \cdot {}_I\mathbf{D} + ({}_I\mathbf{F}^T {}_I\mathbf{t}^4) \cdot {}_I\dot{\boldsymbol{\beta}}_1 + ({}_I\mathbf{F}^T {}_I\mathbf{t}^5) \cdot {}_I\dot{\boldsymbol{\beta}}_2 - {}_I m\, {}_I\dot{\Sigma} \geq 0 \ , \tag{7.4.25}$$

which is required to be nonnegative.

For an elastic Cosserat point, assumptions similar to those described in section 6.8 are made. In particular, the dissipation ${}_I\mathcal{D}$ vanishes and ${}_I\mathbf{T}$, ${}_I\mathbf{t}^4$ and ${}_I\mathbf{t}^5$ are independent of the rates $\{{}_I\mathbf{L} , {}_I\dot{\boldsymbol{\beta}}_\alpha\}$

$${}_I\mathbf{T} = {}_I\hat{\mathbf{T}}({}_I\mathbf{d}_i, {}_I\boldsymbol{\beta}_\alpha) \ , \quad {}_I\mathbf{t}^4 = {}_I\hat{\mathbf{t}}^4({}_I\mathbf{d}_i, {}_I\boldsymbol{\beta}_\alpha) \ , \quad {}_I\mathbf{t}^5 = {}_I\hat{\mathbf{t}}^5({}_I\mathbf{d}_i, {}_I\boldsymbol{\beta}_\alpha) \ , \ \text{for i=1,2,...,5} \ , \tag{7.4.26}$$

Consequently, it can be shown using (7.4.25) that

$${}_I\mathbf{T} = {}_I\hat{\mathbf{T}} = 2\, {}_I\rho\, {}_I\mathbf{F} \frac{\partial_I\hat{\Sigma}}{\partial_I\mathbf{C}}\, {}_I\mathbf{F}^T \ , \quad {}_I\mathbf{t}^4 = {}_I\hat{\mathbf{t}}^4 = {}_I\mathbf{F}^{-T}\, {}_I m \frac{\partial_I\hat{\Sigma}}{\partial_I\boldsymbol{\beta}_1} \ , \quad {}_I\mathbf{t}^5 = {}_I\hat{\mathbf{t}}^5 = {}_I\mathbf{F}^{-T}\, {}_I m \frac{\partial_I\hat{\Sigma}}{\partial_I\boldsymbol{\beta}_2} \ , \tag{7.4.27}$$

which automatically satisfy the balance of angular momentum (7.4.21). Also, in this equation use has been made of the Lagrangian form of conservation of mass

$$ {}_I m = {}_I\rho \, {}_I d^{1/2} = {}_I\rho_0 \, {}_I D^{1/2} \ , \tag{7.4.28} $$

where ${}_I\rho$ is the mass density in the present configuration and ${}_I\rho_0$ is its reference value. Once constitutive equations for ${}_I\mathbf{T}$, ${}_I\mathbf{t}^4$ and ${}_I\mathbf{t}^5$ are specified, the values of ${}_I\mathbf{t}^i$ (i=1,2,3) which appear in the equations of motion (7.4.15) and in the definition (7.4.20), can be determined by the expression

$$ {}_I\mathbf{t}^i = [{}_I d^{1/2} \, {}_I\mathbf{T} - {}_I\mathbf{t}^4 \otimes {}_I\mathbf{d}_4 - {}_I\mathbf{t}^5 \otimes {}_I\mathbf{d}_5 \,] {}_I\mathbf{d}^i \quad \text{for } i=1,2,3 \ . \tag{7.4.29} $$

Now, a simple model for a Cosserat point constructed from a dissipative material can be developed by assuming that ${}_I\mathbf{T}$ and ${}_I\mathbf{t}^i$ separate additively into two parts

$$ {}_I\mathbf{T} = {}_I\hat{\mathbf{T}} + {}_I\check{\mathbf{T}} \ , \quad {}_I\mathbf{t}^i = {}_I\hat{\mathbf{t}}^i + {}_I\check{\mathbf{t}}^i \ , \quad {}_I\hat{\mathbf{t}}^0 = 0 \ , \quad {}_I\check{\mathbf{t}}^0 = 0 \ , $$

$$ {}_I\hat{\mathbf{T}} = {}_I d^{-1/2} \sum_{i=1}^{5} {}_I\hat{\mathbf{t}}^i \otimes {}_I\mathbf{d}_i = {}_I\hat{\mathbf{T}}^T \ , \quad {}_I\check{\mathbf{T}} = {}_I d^{-1/2} \sum_{i=1}^{5} {}_I\check{\mathbf{t}}^i \otimes {}_I\mathbf{d}_i = {}_I\check{\mathbf{T}}^T \ , \tag{7.4.30} $$

with ${}_I\hat{\mathbf{T}}$ and ${}_I\hat{\mathbf{t}}^i$ being the parts associated with elastic deformation [which balance the rate of change in strain energy (7.4.22)]

$$ {}_I d^{1/2} \, {}_I\hat{\mathbf{T}} \bullet {}_I\mathbf{D} + ({}_I\mathbf{F}^T {}_I\hat{\mathbf{t}}^4) \bullet {}_I\dot{\boldsymbol{\beta}}_1 + ({}_I\mathbf{F}^T {}_I\hat{\mathbf{t}}^5) \bullet {}_I\dot{\boldsymbol{\beta}}_2 = {}_I m \, {}_I\dot{\Sigma} \ , \tag{7.4.31} $$

and ${}_I\check{\mathbf{T}}$ and ${}_I\check{\mathbf{t}}^i$ being the parts due to material dissipation. Thus, the restriction (7.4.25) reduces to

$$ {}_I d^{1/2} \, {}_I\mathcal{D} = {}_I d^{1/2} \, {}_I\check{\mathbf{T}} \bullet {}_I\mathbf{D} + ({}_I\mathbf{F}^T {}_I\check{\mathbf{t}}^4) \bullet {}_I\dot{\boldsymbol{\beta}}_1 + ({}_I\mathbf{F}^T {}_I\check{\mathbf{t}}^5) \bullet {}_I\dot{\boldsymbol{\beta}}_2 \geq 0 \ . \tag{7.4.32} $$

As a simple case it is possible to assume that ${}_I\check{\mathbf{T}}$, ${}_I\check{\mathbf{t}}^4$ and ${}_I\check{\mathbf{t}}^5$ are linear functions of rates of the forms

$$ {}_I d^{1/2} \, {}_I\check{\mathbf{T}} = {}_I D^{1/2} \, {}_I V \, [{}_I\eta_1 \, ({}_I\mathbf{D} \bullet \mathbf{I}) \, \mathbf{I} + 2 \, {}_I\eta_2 \, {}_I\mathbf{D}' \,] \ , $$

$$ {}_I\check{\mathbf{t}}^4 = {}_I\eta_3 \, {}_I D^{1/2} \, {}_I V \, ({}_I\mathbf{D}^1 \bullet {}_I\mathbf{D}^1) \, {}_I\mathbf{F}^{-T} {}_I\dot{\boldsymbol{\beta}}_1 \ , \ {}_I\check{\mathbf{t}}^5 = {}_I\eta_4 \, {}_I D^{1/2} \, {}_I V \, ({}_I\mathbf{D}^2 \bullet {}_I\mathbf{D}^2) \, {}_I\mathbf{F}^{-T} {}_I\dot{\boldsymbol{\beta}}_2 \ , \tag{7.4.33} $$

where ${}_I V$ is related to the volume of the element [see (7.4.50)], ${}_I\eta_1$ – ${}_I\eta_4$ are material constants and ${}_I\mathbf{D}'$ is a pure measure of the rate of distortional deformation

$$ {}_I\mathbf{D}' = {}_I\mathbf{D} - \frac{1}{3} ({}_I\mathbf{D} \bullet \mathbf{I}) \, \mathbf{I} \ , \quad {}_I\mathbf{D}' \bullet \mathbf{I} = 0 \ . \tag{7.4.34} $$

Consequently, ${}_I\eta_1$ is the viscosity to dilatational deformation rate, ${}_I\eta_2$ is the viscosity to distortional deformation rate, and ${}_I\eta_3$ and ${}_I\eta_4$ are the viscosities to the inhomogeneous deformation rates ${}_I\dot{\boldsymbol{\beta}}_1$ and ${}_I\dot{\boldsymbol{\beta}}_2$, respectively. Also, it can be shown that the restriction (7.4.32) is satisfied for all motions provided that ${}_I\eta_1 - {}_I\eta_4$ are all nonnegative

$$ {}_I\eta_1 \geq 0 \ , \quad {}_I\eta_2 \geq 0 \ , \quad {}_I\eta_3 \geq 0 \ , \quad {}_I\eta_4 \geq 0 \ . \tag{7.4.35} $$

Moreover, it is noted that the viscosity constants ${}_I\eta_1$ – ${}_I\eta_4$ can be determined by attempting to match the rate of damping associated with free vibrations of the structure.

For the numerical solution of rod problems using the theory of a Cosserat point, it is necessary to satisfy the balance laws (7.4.15) and the constitutive equations (7.4.27), (7.4.30) and (7.4.33) for each Cosserat point. Moreover, it is necessary to couple the equations for each Cosserat point to those of its nearest neighbors using kinematic and kinetic coupling conditions. Specifically, the kinematic assumption (7.4.14) can be used together with the definitions

$$
\begin{gathered}
{}_{\mathrm{I}}\mathbf{d}_0^*(t) = \mathbf{x}^*(0,0,\xi_{\mathrm{I}},t) = {}_{\mathrm{I}}\mathbf{d}_0(t) - \frac{{}_{\mathrm{I}}L}{2}\,{}_{\mathrm{I}}\mathbf{d}_3(t) \;, \\
{}_{\mathrm{I+1}}\mathbf{d}_0^*(t) = \mathbf{x}^*(0,0,\xi_{\mathrm{I+1}},t) = {}_{\mathrm{I}}\mathbf{d}_0(t) + \frac{{}_{\mathrm{I}}L}{2}\,{}_{\mathrm{I}}\mathbf{d}_3(t) \;, \\
{}_{\mathrm{I}}\mathbf{d}_1^*(t) = {}_{\mathrm{I}}\mathbf{d}_1(t) - \frac{{}_{\mathrm{I}}L}{2}\,{}_{\mathrm{I}}\mathbf{d}_4(t) \;\;,\;\; {}_{\mathrm{I+1}}\mathbf{d}_1^*(t) = {}_{\mathrm{I}}\mathbf{d}_1(t) + \frac{{}_{\mathrm{I}}L}{2}\,{}_{\mathrm{I}}\mathbf{d}_4(t) \;\;, \\
{}_{\mathrm{I}}\mathbf{d}_2^*(t) = {}_{\mathrm{I}}\mathbf{d}_2(t) - \frac{{}_{\mathrm{I}}L}{2}\,{}_{\mathrm{I}}\mathbf{d}_5(t) \;\;,\;\; {}_{\mathrm{I+1}}\mathbf{d}_2^*(t) = {}_{\mathrm{I}}\mathbf{d}_2(t) + \frac{{}_{\mathrm{I}}L}{2}\,{}_{\mathrm{I}}\mathbf{d}_5(t) \;\;,
\end{gathered}
\tag{7.4.36}
$$

of the positions of the centroids ${}_{\mathrm{I}}\mathbf{d}_0^*$ and the directors ${}_{\mathrm{I}}\mathbf{d}_\alpha^*$ of the end points of each Cosserat point, to deduce that ${}_{\mathrm{I}}\mathbf{d}_i$ are given by (see Fig. 7.4.2)

$$
\begin{gathered}
{}_{\mathrm{I}}\mathbf{d}_0 = \frac{1}{2}\left[{}_{\mathrm{I}}\mathbf{d}_0^* + {}_{\mathrm{I+1}}\mathbf{d}_0^*\right] \;,\; {}_{\mathrm{I}}\mathbf{d}_1 = \frac{1}{2}\left[{}_{\mathrm{I}}\mathbf{d}_1^* + {}_{\mathrm{I+1}}\mathbf{d}_1^*\right] \;,\; {}_{\mathrm{I}}\mathbf{d}_2 = \frac{1}{2}\left[{}_{\mathrm{I}}\mathbf{d}_2^* + {}_{\mathrm{I+1}}\mathbf{d}_2^*\right] \;, \\
{}_{\mathrm{I}}\mathbf{d}_3 = \frac{1}{{}_{\mathrm{I}}L}\left[{}_{\mathrm{I+1}}\mathbf{d}_0^* - {}_{\mathrm{I}}\mathbf{d}_0^*\right] \;,\; {}_{\mathrm{I}}\mathbf{d}_4 = \frac{1}{{}_{\mathrm{I}}L}\left[{}_{\mathrm{I+1}}\mathbf{d}_1^* - {}_{\mathrm{I}}\mathbf{d}_1^*\right] \;,\; {}_{\mathrm{I}}\mathbf{d}_5 = \frac{1}{{}_{\mathrm{I}}L}\left[{}_{\mathrm{I+1}}\mathbf{d}_2^* - {}_{\mathrm{I}}\mathbf{d}_2^*\right] \;.
\end{gathered}
\tag{7.4.37}
$$

These kinematic coupling equations reduce the number of degrees of freedom from 6N vectors $\{{}_{\mathrm{I}}\mathbf{d}_i$ for i=0,1,...,5 and I=1,2,...,N$\}$ to only 3(N+1) vectors $\{{}_{\mathrm{I}}\mathbf{d}_i^*$ for i=0,1,2 and I=1,2,...,N+1$\}$. Also, it is noted that the reference values ${}_{\mathrm{I}}\mathbf{D}_i$ of ${}_{\mathrm{I}}\mathbf{d}_i$ are related to the reference values ${}_{\mathrm{I}}\mathbf{D}_i^*$ of ${}_{\mathrm{I}}\mathbf{d}_i^*$ by formulas similar to (7.4.36) and (7.4.37).

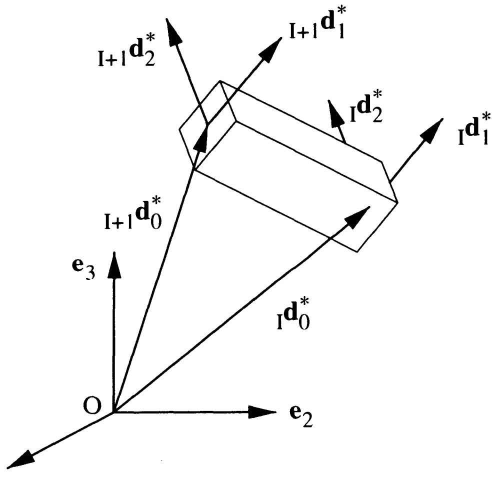

Fig. 7.4.2 Sketch of the I'th Cosserat point showing the I'th and (I+1)'th cross-sections.

Next, using the definitions (7.4.12) it follows that the director couples $\{{}_{\mathrm{I}}\mathbf{m}_\alpha^3,\ {}_{\mathrm{I}}\mathbf{m}_\alpha^4,\ {}_{\mathrm{I}}\mathbf{m}_\alpha^5\}$ are related to the forces and couples $\{{}_{\mathrm{I}}\mathbf{m}_\alpha^0,\ {}_{\mathrm{I}}\mathbf{m}_\alpha^\beta\}$ by the equations

$$
\begin{gathered}
{}_{\mathrm{I}}\mathbf{m}_1^3 = -\,\frac{{}_{\mathrm{I}}L}{2}\,{}_{\mathrm{I}}\mathbf{m}_1^0 \;,\; {}_{\mathrm{I}}\mathbf{m}_2^3 = \frac{{}_{\mathrm{I}}L}{2}\,{}_{\mathrm{I}}\mathbf{m}_2^0 \;,\; {}_{\mathrm{I}}\mathbf{m}_1^4 = -\,\frac{{}_{\mathrm{I}}L}{2}\,{}_{\mathrm{I}}\mathbf{m}_1^1 \;, \\
{}_{\mathrm{I}}\mathbf{m}_2^4 = \frac{{}_{\mathrm{I}}L}{2}\,{}_{\mathrm{I}}\mathbf{m}_2^1 \;,\; {}_{\mathrm{I}}\mathbf{m}_1^5 = -\,\frac{{}_{\mathrm{I}}L}{2}\,{}_{\mathrm{I}}\mathbf{m}_1^2 \;,\; {}_{\mathrm{I}}\mathbf{m}_2^5 = \frac{{}_{\mathrm{I}}L}{2}\,{}_{\mathrm{I}}\mathbf{m}_2^2 \;.
\end{gathered}
\tag{7.4.38}
$$

Consequently, with the help of the expressions (7.4.18) and (7.4.38), the equations of motion (7.4.15) can be solved for the six unknowns $\{{}_I\mathbf{m}_\alpha^0,\ {}_I\mathbf{m}_\alpha^\beta\}$ to deduce that

$$
\begin{aligned}
{}_I\mathbf{m}_1^0 &= -\frac{1}{{}_I L}\,{}_I\mathbf{t}^3 - \frac{1}{2}\,{}_I m\,{}_I\mathbf{B}^0 + \frac{1}{{}_I L}\,{}_I m\,{}_I\mathbf{B}^3 + \sum_{j=0}^{5} {}_I m\left\{\frac{{}_I y^{0j}}{2} - \frac{{}_I y^{3j}}{{}_I L}\right\}\,{}_I\dot{\mathbf{w}}_j\ ,\\
{}_I\mathbf{m}_2^0 &= \frac{1}{{}_I L}\,{}_I\mathbf{t}^3 - \frac{1}{2}\,{}_I m\,{}_I\mathbf{B}^0 - \frac{1}{{}_I L}\,{}_I m\,{}_I\mathbf{B}^3 + \sum_{j=0}^{5} {}_I m\left\{\frac{{}_I y^{0j}}{2} + \frac{{}_I y^{3j}}{{}_I L}\right\}\,{}_I\dot{\mathbf{w}}_j\ ,\\
{}_I\mathbf{m}_1^1 &= \frac{1}{2}\,{}_I\mathbf{t}^1 - \frac{1}{{}_I L}\,{}_I\mathbf{t}^4 - \frac{1}{2}\,{}_I m\,{}_I\mathbf{B}^1 + \frac{1}{{}_I L}\,{}_I m\,{}_I\mathbf{B}^4 + \sum_{j=0}^{5} {}_I m\left\{\frac{{}_I y^{1j}}{2} - \frac{{}_I y^{4j}}{{}_I L}\right\}\,{}_I\dot{\mathbf{w}}_j\ ,\\
{}_I\mathbf{m}_2^1 &= \frac{1}{2}\,{}_I\mathbf{t}^1 + \frac{1}{{}_I L}\,{}_I\mathbf{t}^4 - \frac{1}{2}\,{}_I m\,{}_I\mathbf{B}^1 - \frac{1}{{}_I L}\,{}_I m\,{}_I\mathbf{B}^4 + \sum_{j=0}^{5} {}_I m\left\{\frac{{}_I y^{1j}}{2} + \frac{{}_I y^{4j}}{{}_I L}\right\}\,{}_I\dot{\mathbf{w}}_j\ ,\\
{}_I\mathbf{m}_1^2 &= \frac{1}{2}\,{}_I\mathbf{t}^2 - \frac{1}{{}_I L}\,{}_I\mathbf{t}^5 - \frac{1}{2}\,{}_I m\,{}_I\mathbf{B}^2 + \frac{1}{{}_I L}\,{}_I m\,{}_I\mathbf{B}^5 + \sum_{j=0}^{5} {}_I m\left\{\frac{{}_I y^{2j}}{2} - \frac{{}_I y^{5j}}{{}_I L}\right\}\,{}_I\dot{\mathbf{w}}_j\ ,\\
{}_I\mathbf{m}_2^2 &= \frac{1}{2}\,{}_I\mathbf{t}^2 + \frac{1}{{}_I L}\,{}_I\mathbf{t}^5 - \frac{1}{2}\,{}_I m\,{}_I\mathbf{B}^2 - \frac{1}{{}_I L}\,{}_I m\,{}_I\mathbf{B}^5 + \sum_{j=0}^{5} {}_I m\left\{\frac{{}_I y^{2j}}{2} + \frac{{}_I y^{5j}}{{}_I L}\right\}\,{}_I\dot{\mathbf{w}}_j\ . \qquad (7.4.39)
\end{aligned}
$$

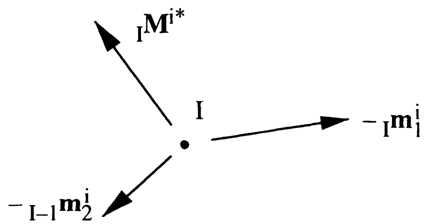

Fig. 7.4.3 Balance of the couples at the I'th interior node for (i=0,1,2).

For generality, it is assumed that external concentrated couples ${}_I\mathbf{M}^{i*}$ (i=0,1,2) are applied to the interior nodes (I=2,3,...,N). Then, considering the balance of couples of this I'th node (Fig. 7.4.3), the kinetic coupling equations become

$$
{}_{I-1}\mathbf{m}_2^i + {}_I\mathbf{m}_1^i = {}_I\mathbf{M}^{i*} \quad \text{for I=2,3,...,N and i=0,1,2}\ . \qquad (7.4.40)
$$

Moreover, with the help of (7.4.39) these coupling equations require

$$
\begin{aligned}
&\frac{1}{{}_{I-1}L}\,{}_{I-1}\mathbf{t}^3 - \frac{1}{2}\,{}_{I-1}m\,{}_{I-1}\mathbf{B}^0 - \frac{1}{{}_{I-1}L}\,{}_{I-1}m\,{}_{I-1}\mathbf{B}^3\\
&\quad + \sum_{j=0}^{5} {}_{I-1}m\left\{\frac{{}_{I-1}y^{0j}}{2} + \frac{{}_{I-1}y^{3j}}{{}_{I-1}L}\right\}\,{}_{I-1}\dot{\mathbf{w}}_j - \frac{1}{{}_I L}\,{}_I\mathbf{t}^3 - \frac{1}{2}\,{}_I m\,{}_I\mathbf{B}^0 + \frac{1}{{}_I L}\,{}_I m\,{}_I\mathbf{B}^3\\
&\qquad + \sum_{j=0}^{5} {}_I m\left\{\frac{{}_I y^{0j}}{2} - \frac{{}_I y^{3j}}{{}_I L}\right\}\,{}_I\dot{\mathbf{w}}_j = {}_I\mathbf{M}^{0*}\ ,
\end{aligned}
$$

$$
\frac{1}{2}\,{}_{I-1}\mathbf{t}^1 + \frac{1}{{}_{I-1}L}\,{}_{I-1}\mathbf{t}^4 - \frac{1}{2}\,{}_{I-1}m\,{}_{I-1}\mathbf{B}^1 - \frac{1}{{}_{I-1}L}\,{}_{I-1}m\,{}_{I-1}\mathbf{B}^4
$$

$$+ \sum_{j=0}^{5} {}_{I-1}m\{\frac{{}_{I-1}y^{1j}}{2} + \frac{{}_{I-1}y^{4j}}{{}_{I-1}L}\} \, {}_{I-1}\dot{\mathbf{w}}_j + \frac{1}{2}{}_I\mathbf{t}^1 - \frac{1}{{}_IL}\,{}_I\mathbf{t}^4 - \frac{1}{2}{}_Im\,{}_I\mathbf{B}^1 + \frac{1}{{}_IL}\,{}_Im\,{}_I\mathbf{B}^4$$

$$+ \sum_{j=0}^{5} {}_Im\{\frac{{}_Iy^{1j}}{2} - \frac{{}_Iy^{4j}}{{}_IL}\} \, {}_I\dot{\mathbf{w}}_j = {}_I\mathbf{M}^{1*} \ ,$$

$$\frac{1}{2}{}_{I-1}\mathbf{t}^2 + \frac{1}{{}_{I-1}L}\,{}_{I-1}\mathbf{t}^5 - \frac{1}{2}{}_{I-1}m\,{}_{I-1}\mathbf{B}^2 - \frac{1}{{}_{I-1}L}\,{}_{I-1}m\,{}_{I-1}\mathbf{B}^5$$

$$+ \sum_{j=0}^{5} {}_{I-1}m\{\frac{{}_{I-1}y^{2j}}{2} + \frac{{}_{I-1}y^{5j}}{{}_{I-1}L}\} \, {}_{I-1}\dot{\mathbf{w}}_j + \frac{1}{2}{}_I\mathbf{t}^2 - \frac{1}{{}_IL}\,{}_I\mathbf{t}^5 - \frac{1}{2}{}_Im\,{}_I\mathbf{B}^2 + \frac{1}{{}_IL}\,{}_Im\,{}_I\mathbf{B}^5$$

$$+ \sum_{j=0}^{5} {}_Im\{\frac{{}_Iy^{2j}}{2} - \frac{{}_Iy^{5j}}{{}_IL}\} \, {}_I\dot{\mathbf{w}}_j = {}_I\mathbf{M}^{2*} \ . \quad \text{for } I=2,3,...,N \ . \tag{7.4.41}$$

In view of the kinematic conditions (7.4.37) these expressions represent 3(N−1) vector equations for the 3(N+1) vectors $\{{}_I\mathbf{d}_0^*, {}_I\mathbf{d}_1^*, {}_I\mathbf{d}_2^*\}$. The remaining six vector equations are determined by boundary conditions. To explore the nature of these boundary conditions it is convenient to use (7.4.18), the kinematic conditions (7.4.37), and the results (7.4.38) to rewrite the rate of work ${}_I\mathcal{W}$ in (7.4.23) in the form

$${}_I\mathcal{W} = \sum_{i=0}^{5} {}_Im\,{}_I\mathbf{B}^i \bullet {}_I\mathbf{w}_i + \sum_{i=0}^{2} {}_I\mathbf{m}_1^i \bullet {}_I\dot{\mathbf{d}}_i^* + \sum_{i=0}^{2} {}_I\mathbf{m}_2^i \bullet {}_{I+1}\dot{\mathbf{d}}_i^* \ . \tag{7.4.42}$$

Moreover, it follows by using the conditions (7.4.40) and summing ${}_I\mathcal{W}$ over all N elements, that the total rate of external work $\mathcal{W}$ applied to the entire rod can be expressed as

$$\mathcal{W} = \sum_{I=1}^{N} {}_I\mathcal{W} = \sum_{I=1}^{N} \sum_{i=0}^{5} {}_Im\,{}_I\mathbf{B}^i \bullet {}_I\mathbf{w}_i + \sum_{I=2}^{N} \sum_{i=0}^{2} {}_I\mathbf{M}^{i*} \bullet {}_I\dot{\mathbf{d}}_i^*$$

$$+ \sum_{i=0}^{2} {}_1\mathbf{m}_1^i \bullet {}_1\dot{\mathbf{d}}_i^* + \sum_{i=0}^{2} {}_N\mathbf{m}_2^i \bullet {}_{I+1}\dot{\mathbf{d}}_i^* \ . \tag{7.4.43}$$

Using this expression it can be seen that the external couples ${}_I\mathbf{M}^{i*}$ (i=0,1,2) do work at the interior nodes, the director couples ${}_1\mathbf{m}_1^i$ (i=0,1,2) do work on the end I=1, and the director couples ${}_N\mathbf{m}_2^i$ (i=0,1,2) do work on the end I=N+1. Consequently, the boundary conditions on these ends require specification of

$$\{{}_1\mathbf{d}_0^*(t) \text{ or } {}_1\mathbf{m}_1^0(t)\}, \ \{{}_1\mathbf{d}_1^*(t) \text{ or } {}_1\mathbf{m}_1^1(t)\}, \ \{{}_1\mathbf{d}_2^*(t) \text{ or } {}_1\mathbf{m}_1^2(t)\},$$

$$\{{}_{N+1}\mathbf{d}_0^*(t) \text{ or } {}_N\mathbf{m}_2^0(t)\}, \ \{{}_{N+1}\mathbf{d}_1^*(t) \text{ or } {}_N\mathbf{m}_2^1(t)\}, \ \{{}_{N+1}\mathbf{d}_2^*(t) \text{ or } {}_N\mathbf{m}_2^2(t)\} \ , \tag{7.4.44}$$

where ${}_1\mathbf{m}_1^i(t)$ and ${}_N\mathbf{m}_2^i$ (i=0,1,2) are determined by the equations (7.4.39). Moreover, the solution of the 3(N+1) vector equations (7.4.41) and (7.4.44) requires specification of the initial conditions for the directors and their velocities at the nodes

$${}_I\mathbf{d}_i^* = {}_I\mathbf{d}_i^*(0) \ \text{ and } \ {}_I\dot{\mathbf{d}}_i^* = {}_I\dot{\mathbf{d}}_i^*(0) \quad \text{for } I=1,2,...,N+1 \text{ and } i=0,1,2 \ . \tag{7.4.45}$$

Furthermore, with regard to boundary conditions it is observed from (7.4.11), the definitions (7.4.12), the kinematic assumption (7.4.14), and the results (7.4.37) and (7.4.38), that the total resultant moments applied to the ends ${}_I\partial P_\alpha^*$ of the rod are given by

$$\int_{{}_{\mathrm{I}}\partial P_1^*} \mathbf{x}^* \times \mathbf{t}^* \, da^* = {}_{\mathrm{I}}\mathbf{d}_0^* \times {}_{\mathrm{I}}\mathbf{m}_1^0 + {}_{\mathrm{I}}\mathbf{m}_1 \ ,$$

$$\int_{{}_{\mathrm{I}}\partial P_2^*} \mathbf{x}^* \times \mathbf{t}^* \, da^* = {}_{\mathrm{I+1}}\mathbf{d}_0^* \times {}_{\mathrm{I}}\mathbf{m}_2^0 + {}_{\mathrm{I}}\mathbf{m}_2 \ , \tag{7.4.46}$$

where the moments ${}_{\mathrm{I}}\mathbf{m}_\alpha$ applied to the ends ${}_{\mathrm{I}}\partial P_\alpha^*$, respectively, about the centroids (θ^α=0), are defined by

$${}_{\mathrm{I}}\mathbf{m}_1 = {}_{\mathrm{I}}\mathbf{d}_1^* \times {}_{\mathrm{I}}\mathbf{m}_1^1 + {}_{\mathrm{I}}\mathbf{d}_2^* \times {}_{\mathrm{I}}\mathbf{m}_1^2 \ , \quad {}_{\mathrm{I}}\mathbf{m}_2 = {}_{\mathrm{I+1}}\mathbf{d}_1^* \times {}_{\mathrm{I}}\mathbf{m}_2^1 + {}_{\mathrm{I+1}}\mathbf{d}_2^* \times {}_{\mathrm{I}}\mathbf{m}_2^2 \ . \tag{7.4.47}$$

Thus, it can be seen that director couples $\{{}_{\mathrm{I}}\mathbf{m}_1^1 \ , \ {}_{\mathrm{I}}\mathbf{m}_1^2\}$ and $\{{}_{\mathrm{I}}\mathbf{m}_2^1 \ , \ {}_{\mathrm{I}}\mathbf{m}_2^2\}$ contain more information than the moments $\{{}_{\mathrm{I}}\mathbf{m}_1 \ , {}_{\mathrm{I}}\mathbf{m}_2\}$. Also, it can be shown (Rubin, 2000) that the equations of the theory of a Cosserat point are properly invariant under SRBM so they are valid for large deformations and rotations of the rod.

In (Rubin, 2000) it was shown that if the rod is made from a uniform homogeneous elastic material, then restrictions on the functional form for the strain energy Σ can be developed to ensure that the equations for the Cosserat point will reproduce exact three-dimensional solutions for all homogeneous motions. Specifically, it was shown there that the strain energy function for the Cosserat point which satisfies these restrictions, can be written in terms of the three-dimensional strain energy $\Sigma^*(\mathbf{C}^*)$ in the form

$${}_{\mathrm{I}}\Sigma = \Sigma^*({}_{\mathrm{I}}\bar{\mathbf{C}}^*) + {}_{\mathrm{I}}\Psi({}_{\mathrm{I}}\mathbf{C}, {}_{\mathrm{I}}\boldsymbol{\beta}_\alpha) \ , \tag{7.4.48}$$

where the function ${}_{\mathrm{I}}\Psi$ characterizes the strain energy due to inhomogeneous deformations and it satisfies the restrictions

$$\frac{\partial_{\mathrm{I}}\Psi}{\partial_{\mathrm{I}}\mathbf{C}} = 0 \ , \quad \frac{\partial_{\mathrm{I}}\Psi}{\partial_{\mathrm{I}}\boldsymbol{\beta}_\alpha} = 0 \quad \text{for } {}_{\mathrm{I}}\boldsymbol{\beta}_\alpha = 0 \ . \tag{7.4.49}$$

Also, in developing these restrictions (Rubin, 2000) it was convenient to introduce the quantities ${}_{\mathrm{I}}V$, ${}_{\mathrm{I}}V^\alpha$ defined by the geometry of the reference configuration of the rod by the equations

$$\mathbf{X}^*(\theta^i,t) = {}_{\mathrm{I}}\mathbf{D}_0(t) + \theta^1 \left[{}_{\mathrm{I}}\mathbf{D}_1(t) + {}_{\mathrm{I}}\bar{\theta}^3{}_{\mathrm{I}}\mathbf{D}_4(t)\right] + \theta^2 \left[{}_{\mathrm{I}}\mathbf{D}_2(t) + {}_{\mathrm{I}}\bar{\theta}^3{}_{\mathrm{I}}\mathbf{D}_5(t)\right] + {}_{\mathrm{I}}\bar{\theta}^3{}_{\mathrm{I}}\mathbf{D}_3(t) \ ,$$

$${}_{\mathrm{I}}\boldsymbol{\Lambda}_1 = {}_{\mathrm{I}}\mathbf{D}_4 \otimes {}_{\mathrm{I}}\mathbf{D}^3 \ , \quad {}_{\mathrm{I}}\boldsymbol{\Lambda}_2 = {}_{\mathrm{I}}\mathbf{D}_5 \otimes {}_{\mathrm{I}}\mathbf{D}^3 \ , \quad {}_{\mathrm{I}}\boldsymbol{\Lambda}_3 = {}_{\mathrm{I}}\mathbf{D}_4 \otimes {}_{\mathrm{I}}\mathbf{D}^1 + {}_{\mathrm{I}}\mathbf{D}_5 \otimes {}_{\mathrm{I}}\mathbf{D}^2 \ ,$$

$${}_{\mathrm{I}}\mathbf{G}_i = (\mathbf{I} + \theta^\alpha{}_{\mathrm{I}}\boldsymbol{\Lambda}_\alpha + {}_{\mathrm{I}}\bar{\theta}^3{}_{\mathrm{I}}\boldsymbol{\Lambda}_3)\, {}_{\mathrm{I}}\mathbf{D}_i \ , \quad {}_{\mathrm{I}}\mathbf{G}^i = (\mathbf{I} + \theta^\alpha{}_{\mathrm{I}}\boldsymbol{\Lambda}_\alpha + {}_{\mathrm{I}}\bar{\theta}^3{}_{\mathrm{I}}\boldsymbol{\Lambda}_3)^{-T}\, {}_{\mathrm{I}}\mathbf{D}^i \ ,$$

$${}_{\mathrm{I}}D^{1/2}\, {}_{\mathrm{I}}V = \int_{{}_{\mathrm{I}}P_0^*} dV^* \ , \quad {}_{\mathrm{I}}D^{1/2}\, {}_{\mathrm{I}}V\, {}_{\mathrm{I}}V^1 = \int_{{}_{\mathrm{I}}P_0^*} \left[{}_{\mathrm{I}}G^1{}_{\mathrm{I}}\bar{\theta}^3 + {}_{\mathrm{I}}G^3\, \theta^1\right] dV^* \ ,$$

$${}_{\mathrm{I}}D^{1/2}\, {}_{\mathrm{I}}V\, {}_{\mathrm{I}}V^2 = \int_{{}_{\mathrm{I}}P_0^*} \left[{}_{\mathrm{I}}G^2{}_{\mathrm{I}}\bar{\theta}^3 + {}_{\mathrm{I}}G^3\, \theta^2\right] dV^* \ . \tag{7.4.50}$$

In these equations, ${}_{\mathrm{I}}P_0^*$ is the region in the reference configuration associated with the material region ${}_{\mathrm{I}}P^*$ in the present configuration, dV^* is the element of volume in the reference configuration and ${}_{\mathrm{I}}V$ is related to the volume of the I'th Cosserat point. Moreover, the mass ${}_{\mathrm{I}}m$ can be expressed in terms of the constant mass density ρ_0^* by

$${}_{\mathrm{I}}m = \rho_0^*\, {}_{\mathrm{I}}V\, {}_{\mathrm{I}}D^{1/2} \ , \tag{7.4.51}$$

and the tensors ${}_{\mathrm{I}}\bar{\mathbf{F}}$ and ${}_{\mathrm{I}}\bar{\mathbf{C}}$ are defined by

$$ {}_I\bar{\mathbf{F}} = {}_I\mathbf{F}\,[\mathbf{I} + {}_I\boldsymbol{\beta}_\alpha \otimes {}_I\mathbf{V}^\alpha] \ , \quad {}_I\bar{\mathbf{C}} = {}_I\bar{\mathbf{F}}^T {}_I\bar{\mathbf{F}} \ , \tag{7.4.52} $$

with the assumption that ${}_I\bar{\mathbf{F}}$ remains nonsingular.

In general, it is possible to define normalized measures ${}_I\kappa^i_\alpha$ of inhomogeneous strains by the expressions

$$ {}_I\kappa^i_\alpha = {}_IL\,{}_I\mathbf{D}^i \bullet {}_I\boldsymbol{\beta}_\alpha \quad \text{for i=1,2,3} \ , \tag{7.4.53} $$

so that ${}_I\Psi$ becomes

$$ {}_I\Psi = {}_I\Psi({}_I\mathbf{C}, {}_I\kappa^i_\alpha) \ . \tag{7.4.54} $$

It then follows from (7.4.27),(7.4.48),(7.4.52)-(7.4.54), that the elastic part of the constitutive equations are given by

$$ {}_Id^{1/2}\,{}_I\hat{\mathbf{T}} = 2\,{}_Im\,[{}_I\bar{\mathbf{F}} \frac{\partial\Sigma^*({}_I\bar{\mathbf{C}})}{\partial_I\bar{\mathbf{C}}} {}_I\bar{\mathbf{F}}^T + {}_I\mathbf{F} \frac{\partial_I\Psi}{\partial_I\mathbf{C}} {}_I\mathbf{F}^T] \ , $$

$$ {}_I\hat{\mathbf{t}}^4 = {}_Im\,[2\,{}_I\bar{\mathbf{F}} \frac{\partial\Sigma^*({}_I\bar{\mathbf{C}})}{\partial_I\bar{\mathbf{C}}} {}_I\mathbf{V}^1 + \sum_{i=1}^{3} {}_IL \frac{\partial_I\Psi}{\partial_I\kappa^i_1} {}_I\mathbf{d}^i\,] \ , $$

$$ {}_I\hat{\mathbf{t}}^5 = {}_Im\,[2\,{}_I\bar{\mathbf{F}} \frac{\partial\Sigma^*({}_I\bar{\mathbf{C}})}{\partial_I\bar{\mathbf{C}}} {}_I\mathbf{V}^2 + \sum_{i=1}^{3} {}_IL \frac{\partial_I\Psi}{\partial_I\kappa^i_2} {}_I\mathbf{d}^i\,] \ . \tag{7.4.55} $$

For the examples considered below attention will be focused on a compressible Mooney-Rivlin isotropic elastic material which is a special case of (3.11.16) and (3.11.19) such that

$$ \rho_0^*\,\Sigma^*({}_I\bar{\mathbf{C}}) = K^*\left[\,{}_I\bar{J} - 1 - \ln({}_I\bar{J})\right] + \mu^*\,(\bar{\alpha}_1 - 3) \ , \quad {}_I\bar{J} = \det({}_I\bar{\mathbf{F}}) \ , $$

$$ {}_I\bar{\mathbf{F}}' = {}_I\bar{J}^{-1/3}\,{}_I\bar{\mathbf{F}} \ , \ \det({}_I\bar{\mathbf{F}}') = 1 \ , \quad {}_I\bar{\mathbf{B}}' = {}_I\bar{\mathbf{F}}'\,{}_I\bar{\mathbf{F}}'^T \ , \quad \bar{\alpha}_1 = {}_I\bar{\mathbf{B}}' \bullet \mathbf{I} \ , \tag{7.4.56} $$

where K^* and μ^* are the constant three-dimensional bulk modulus and shear modulus, respectively. Also, the functional form for ${}_I\Psi$ is taken in the form

$$ \begin{aligned} 2\,{}_Im\,{}_I\Psi = {}_IV\,{}_ID^{1/2}\ & [{}_Ik_1\,({}_I\kappa^3_1)^2 + {}_Ik_2\,({}_I\kappa^3_2)^2 + {}_Ik_3\,({}_I\omega_1)^2 \\ & + {}_Ik_4\,({}_I\kappa^1_1)^2 + {}_Ik_5\,({}_I\kappa^3_2)^2 + {}_Ik_6\,({}_I\omega_2)^2] \ , \\ {}_I\omega_1 = \tfrac{1}{2}({}_I\kappa^2_1 - {}_I\kappa^1_2) \ , & \quad {}_I\omega_2 = \tfrac{1}{2}({}_I\kappa^2_1 + {}_I\kappa^1_2) \ , \end{aligned} \tag{7.4.57} $$

where ${}_Ik_i$ are constants and the variables ${}_I\omega_\alpha$ have been introduced for convenience. Moreover, in (Rubin, 2000) it was shown that $\{{}_Ik_1, {}_Ik_2\}$ control bending, ${}_Ik_3$ controls torsion, $\{{}_Ik_4, {}_Ik_5\}$ control hour glassing due to extension of the cross-section, and ${}_Ik_6$ controls hour glassing due to shearing of the cross-section of the rod element. Thus, for these constitutive assumptions the expressions (7.4.55) reduce to

$$ {}_Id^{1/2}\,{}_I\hat{\mathbf{T}} = {}_IV\,{}_ID^{1/2}\ K^*\,[{}_I\bar{J} - 1]\,\mathbf{I} + {}_IV\,{}_ID^{1/2}\,\mu^*\,[{}_I\bar{\mathbf{B}}' - \tfrac{1}{3}({}_I\bar{\mathbf{B}}' \bullet \mathbf{I})\,\mathbf{I}] \ , $$

$$ {}_I\hat{\mathbf{t}}^4 = {}_IV\,{}_ID^{1/2}\,K^*\,[{}_I\bar{J} - 1]\,{}_I\bar{\mathbf{F}}^{-T}\,{}_I\mathbf{V}^1 + {}_IV\,{}_ID^{1/2}\,\mu^*\,{}_I\bar{J}^{-2/3}\,[{}_I\bar{\mathbf{F}} - \tfrac{1}{3}({}_I\bar{\mathbf{B}} \bullet \mathbf{I})\,{}_I\bar{\mathbf{F}}^{-T}]\,{}_I\mathbf{V}^1 $$

$$ + {}_IV\,{}_ID^{1/2}\,{}_IL[{}_Ik_4\,{}_I\kappa^1_1]\,{}_I\mathbf{d}^1 + \frac{{}_IV_IL_ID^{1/2}}{2}[{}_Ik_3\,{}_I\omega_1 + {}_Ik_6\,{}_I\omega_2]\,{}_I\mathbf{d}^2 $$

$$+ {}_I V \, {}_I D^{1/2} \, {}_I L [{}_I k_1 \, {}_I \kappa_1^3] \, {}_I \mathbf{d}^3 \ ,$$

$${}_I \hat{\mathbf{t}}^5 = {}_I V \, {}_I D^{1/2} \, K^* \, [{}_I \bar{J} - 1] \, {}_I \bar{\mathbf{F}}^{-T} \, {}_I V^2 + {}_I V \, {}_I D^{1/2} \, \mu^* \, {}_I \bar{J}^{-2/3} \, [{}_I \bar{\mathbf{F}} - \frac{1}{3} ({}_I \bar{\mathbf{B}} \bullet \mathbf{I}) \, {}_I \bar{\mathbf{F}}^{-T}] \, {}_I \mathbf{V}^2$$

$$+ \frac{{}_I V_I L_I D^{1/2}}{2} [- {}_I k_3 \, {}_I \omega_1 + {}_I k_6 \, {}_I \omega_2] \, {}_I \mathbf{d}^1 + {}_I V \, {}_I D^{1/2} \, {}_I L [{}_I k_5 \, {}_I \kappa_2^2] \, {}_I \mathbf{d}^2$$

$$+ {}_I V \, {}_I D^{1/2} \, {}_I L [{}_I k_2 \, {}_I \kappa_2^3] \, {}_I \mathbf{d}^3 \ . \tag{7.4.58}$$

Within the context of the direct approach to the development of constitutive equations, the constants $\{{}_I k_1, {}_I k_2\}$ and ${}_I k_3$ are determined by matching experiments or analytical solutions for bending and torsion, respectively. In this regard, it should be mentioned that since inhomogeneous deformations are modeled by these constitutive equations, the values of the constitutive coefficients that are be obtained by the standard Galerkin procedure may not be the same values as those obtained by the direct approach. This is because the Galerkin approach assumes that the kinematic assumption (7.4.14) is valid pointwise, and the constitutive coefficients are obtained by using the three-dimensional constitutive equations and performing direct integration of the expressions (7.4.12). Also, the constants $\{{}_I k_4, {}_I k_5\}$ and ${}_I k_6$ are specified to be small enough that they don't significantly change the stiffness to bending and torsion, and they are specified to be large enough that they control undesirable hour glassing of the elements due to otherwise uncontrolled deformation modes.

In particular, these coefficients were specified in (Rubin, 2000) for a structure that in its reference configuration is a uniform beam with total length L_T, and rectangular cross-sectional with height H and width W such that

$$|\theta^1| \le \frac{H}{2} \ , \quad |\theta^2| \le \frac{W}{2} \ , \quad 0 \le \theta^3 \le L_T \ . \tag{7.4.59}$$

If the beam is divided into N elements of equal length L, then it follows that

$${}_I L = L = \frac{L_T}{N} \ , \ \xi_I = (I-1) \, L \ , \ {}_I m = m = \rho_0^* \, V \ , \ {}_I V = V = HWL \ . \tag{7.4.60}$$

Also, in (Rubin, 2000) it was shown that for this beam the only nonzero director inertia coefficients are

$${}_I y^{00} = 1 \ , \quad {}_I y^{11} = y^{11} = \frac{H^2}{\pi^2} \ , \quad {}_I y^{22} = y^{22} = \frac{W^2}{\pi^2} \ ,$$

$${}_I y^{33} = y^{33} = \frac{L^2}{\pi^2} \ , \quad {}_I y^{44} = y^{44} = {}_I y^{55} = y^{55} = [\frac{2L}{3\pi}]^2 \ , \tag{7.4.61}$$

the bending coefficients are given by

$${}_I k_1 = k_1 = \frac{E^* H^2}{12 L^2} \ , \quad {}_I k_2 = k_2 = \frac{E^* W^2}{12 L^2} \ , \tag{7.4.62}$$

where E^* is Young's modulus, and the torsional coefficient is given by

$${}_I k_3 = k_3 = \frac{\mu^* HW}{3L^2} b^*(\eta) \ , \quad \eta = \frac{H}{W} \ , \tag{7.4.63}$$

where $b^*(\eta)$ is the function defined in (3.15.13). Moreover, the hour glassing coefficients were specified by

$$ {}_Ik_4 = k_4 = k\,\mu^* \left[\frac{H^2}{L^2}\right] , \quad {}_Ik_5 = k_5 = k\,\mu^* \left[\frac{W^2}{L^2}\right] , \quad {}_Ik_6 = k_6 = k\,\mu^* \left[\frac{HW}{L^2}\right] , \tag{7.4.64} $$

where k is determined later.

As a specific example, consider a beam which is rotating in the $\mathbf{e}_r$–$\mathbf{e}_3$ plane (of a cylindrical polar coordinate system) with constant angular velocity Ω about the $\mathbf{e}_3$ axis, which coincides with the centroid of the beam's rectangular cross-section. For this problem, the reference values of the directors are given by

$$ {}_I\mathbf{D}_0^* = \left[-\frac{{}_IL}{2} + \xi_I \right] \mathbf{e}_3 \ , \quad {}_I\mathbf{D}_1^* = \mathbf{e}_1 \ , \quad {}_I\mathbf{D}_2^* = \mathbf{e}_2 \ . \tag{7.4.65} $$

In the present configuration the director vectors are taken to be

$$ {}_I\mathbf{d}_0^* = {}_Iz\,\mathbf{e}_3 \ , \quad {}_I\mathbf{d}_1^* = {}_I\phi\,\mathbf{e}_r \ , \quad {}_I\mathbf{d}_2^* = {}_I\phi\,\mathbf{e}_\theta \ , \quad \theta = \Omega t \ , \tag{7.4.66} $$

where use has been made of the axisymmetric nature of the problem. For steady motion relative to the rotating coordinate system $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_3\}$, the components $\{{}_Iz, {}_I\phi\}$ are constants that need to be determined by the equations of motions. Moreover, for this case the accelerations ${}_I\ddot{\mathbf{d}}_i^*$ (i=0,1,2) become

$$ {}_I\ddot{\mathbf{d}}_0^* = 0 \ , \quad {}_I\ddot{\mathbf{d}}_1^* = -\Omega^2\,{}_I\phi\,\mathbf{e}_r \ , \quad {}_I\ddot{\mathbf{d}}_2^* = -\Omega^2\,{}_I\phi\,\mathbf{e}_\theta \ . \tag{7.4.67} $$

Also, the boundary conditions are specified so that the ends of the rod are attached to rigid plates that rotate about the $\mathbf{e}_3$ axis

$$ {}_1z = -\frac{{}_IL}{2} \ , \quad {}_1\phi = 1 \ , \quad {}_{N+1}z = \frac{{}_IL}{2} \ , \quad {}_{N+1}\phi = 1 \ . \tag{7.4.68} $$

In particular, note that these ends remain rigid, but the interior sections of the rod are allowed to deform due to the rotation. For this problem, body force is neglected, there are no loads on the lateral surface of the rod, and the external couples ${}_I\mathbf{M}^{i*}$ vanish. Moreover, the dissipation terms vanish so with the help of (7.3.37) the equations for the director couples (7.4.39) reduce to

$$ {}_I\mathbf{m}_1^0 = -\frac{1}{L}\,{}_I\mathbf{t}^3 \ , \quad {}_I\mathbf{m}_2^0 = \frac{1}{L}\,{}_I\mathbf{t}^3 \ , $$

$$ {}_I\mathbf{m}_1^1 = \frac{1}{2}\,{}_I\mathbf{t}^1 - \frac{1}{L}\,{}_I\mathbf{t}^4 - m\,\Omega^2 \left[\left\{\frac{y^{11}}{4} + \frac{y^{44}}{L^2}\right\} {}_I\phi + \left\{\frac{y^{11}}{4} - \frac{y^{44}}{L^2}\right\} {}_{I+1}\phi \right] \mathbf{e}_r \ , $$

$$ {}_I\mathbf{m}_2^1 = \frac{1}{2}\,{}_I\mathbf{t}^1 + \frac{1}{L}\,{}_I\mathbf{t}^4 - m\,\Omega^2 \left[\left\{\frac{y^{11}}{4} - \frac{y^{44}}{L^2}\right\} {}_I\phi + \left\{\frac{y^{11}}{4} + \frac{y^{44}}{L^2}\right\} {}_{I+1}\phi \right] \mathbf{e}_r \ , $$

$$ {}_I\mathbf{m}_1^2 = \frac{1}{2}\,{}_I\mathbf{t}^2 - \frac{1}{L}\,{}_I\mathbf{t}^5 - m\,\Omega^2 \left[\left\{\frac{y^{22}}{4} + \frac{y^{55}}{L^2}\right\} {}_I\phi + \left\{\frac{y^{22}}{4} - \frac{y^{55}}{L^2}\right\} {}_{I+1}\phi \right] \mathbf{e}_\theta \ , $$

$$ {}_I\mathbf{m}_2^2 = \frac{1}{2}\,{}_I\mathbf{t}^2 + \frac{1}{{}_IL}\,{}_I\mathbf{t}^5 - m\,\Omega^2 \left[\left\{\frac{y^{22}}{4} - \frac{y^{55}}{L^2}\right\} {}_I\phi + \left\{\frac{y^{22}}{4} + \frac{y^{55}}{L^2}\right\} {}_{I+1}\phi \right] \mathbf{e}_\theta \ , \tag{7.4.69} $$

and the equations of motion (7.4.41) become

$$ \frac{1}{L}\,{}_{I-1}\mathbf{t}^3 - \frac{1}{L}\,{}_I\mathbf{t}^3 = 0 \ , $$

$$ -m\,\Omega^2 \left[\left\{\frac{y^{11}}{4} - \frac{y^{44}}{L^2}\right\} {}_{I-1}\phi + 2\left\{\frac{y^{11}}{4} + \frac{y^{44}}{L^2}\right\} {}_I\phi + \left\{\frac{y^{11}}{4} - \frac{y^{44}}{L^2}\right\} {}_{I+1}\phi \right] \mathbf{e}_r $$

$$+\frac{1}{2}\,{}_{I-1}t^1 + \frac{1}{L}\,{}_{I-1}t^4 + \frac{1}{2}\,{}_{I}t^1 - \frac{1}{L}\,{}_{I}t^4 = 0 \quad \text{for I=2,3,...,N} \ , \tag{7.4.70}$$

with the other director momentum equation

$$-\,m\,\Omega^2\,[\{\frac{y^{22}}{4} - \frac{y^{55}}{L^2}\}\,{}_{I-1}\phi + 2\{\frac{y^{22}}{4} + \frac{y^{55}}{L^2}\}\,{}_{I}\phi + \{\frac{y^{22}}{4} - \frac{y^{55}}{L^2}\}\,{}_{I+1}\phi\,]\,\mathbf{e}_\theta$$

$$+\frac{1}{2}\,{}_{I-1}t^2 + \frac{1}{L}\,{}_{I-1}t^5 + \frac{1}{2}\,{}_{I}t^2 - \frac{1}{L}\,{}_{I}t^5 = 0 \quad \text{for I=2,3,...,N} \ , \tag{7.4.71}$$

being satisfied by the solution of $(7.4.41)_2$.

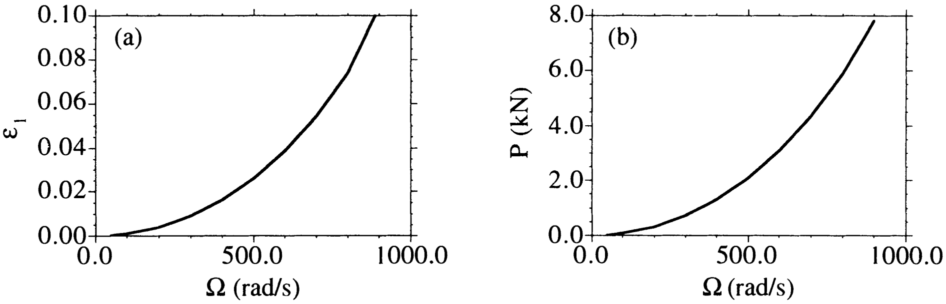

Fig. 7.4.4 Values of the radial strain ε_1 at the middle of the beam and the axial force P for N=16 elements and k=0.5.

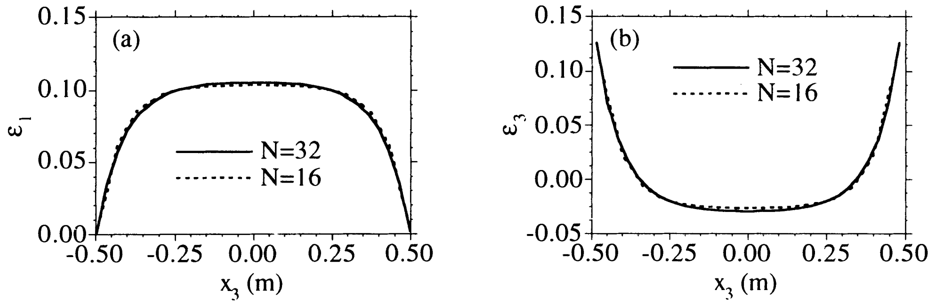

Fig. 7.4.5 Convergence of the values of radial strain ε_1 and axial strain ε_3 of the beam for Ω=900 rad/s and k=0.5.

Here, the beam is assumed to be made of rubber and the material constants associated with (7.4.58) are specified by (Kolsky, 1963, p. 201)

$$\rho_0^* = 0.93 \text{ Mg/m}^3 \ , \quad K^* = 4.7 \text{ MPa} \ , \quad \mu^* = 0.7 \text{ MPa} \ . \tag{7.4.72}$$

Also, for most of the calculations the constant k in (7.4.64) is taken to be

$$k = 0.5 \ . \tag{7.4.73}$$

However, calculations using different values of k are also considered. Furthermore, to emphasize the effect of rotation, the beam is taken to be thick with its reference geometry being specified by

$$L_T = 1.0 \text{ m} \ , \quad H = W = 0.1 \text{ m} \ , \tag{7.4.74}$$

A computer program was developed using MATLAB 5.2.0 (The MathWorks I· c., 1996]) to iteratively solve the equations (7.4.70) for the values $\{{}_I z, {}_I\phi\}$ together with the boundary conditions (7.4.68).

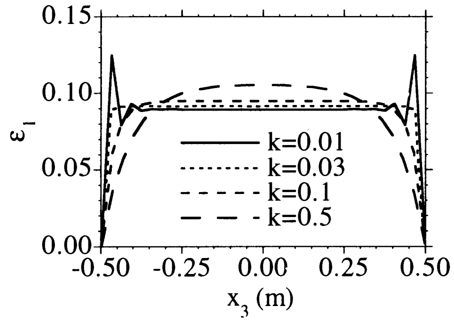

Fig. 7.4.6 The influence of the value of k on the value of radial strain ε_1 of the beam for Ω=900 rad/s and N=32 elements.

Since the rotation causes the radial acceleration to be a linear function of the radial coordinate in the beam's cross-section, the actual displacement field is expected to be significantly different from linear through the cross-section. Consequently, this example may actually be outside of the range of applicability of the Cosserat model. Nevertheless, it will be seen that the Cosserat model predicts (at least qualitatively) the main physical effects which include barreling and axial tension. In order to present these results, it is convenient to define the engineering radial strain ${}_I\varepsilon_1$ and axial strain ${}_I\varepsilon_3$, and the axial force ${}_I P$ by the formulas

$$ {}_I\varepsilon_1 = {}_I\phi - 1 \ , \ {}_I\varepsilon_3 = \frac{{}_{I+1}z - {}_I z}{L} \ , \ {}_I P = {}_I \mathbf{m}_2^0 \bullet \mathbf{e}_3 \ , \quad \text{for } I=1,2,\ldots,N \ . \quad (7.4.75) $$

Also, the axial force [which by $(7.4.70)_1$ is uniform] is denoted by P.

Figures 7.4.4 show the values of the radial strain ε_1 at the middle of the beam and the axial force P for N=16 elements and k=0.5. These predictions are physically reasonable since the cross-section of the beam is expected to expand due to rotation and the axial force is expected to be in tension due the Poisson effect. Moreover, it is noted that neither of these quantities can be predicted by usual beam theory which neglects the effect of normal cross-sectional expansion. Figures 7.4.5 show the values of the radial strain ε_1 and axial strain ε_3 of the beam for Ω=900 rad/s and k=0.5. Figure 7.4.5a shows the barreling effect and Fig. 7.4.5b shows the axial extension near the ends of the beam which is caused by the fact that the cross-section is not allowed to expand there. Finally, Fig. 7.4.6 shows the influence of the value of k on the value of radial strain ε_1 for Ω=900 rad/s and N=32 elements. It is seen that the value of k, which controls hour glassing of the elements, has a significant effect on the character of the solution near the constrained ends of the beam. If the value of k is too small, then unphysical oscillations occur, whereas if the value of k is two large then the barreling effect is dispersed over the entire length of the beam. Future research is needed in order to find a problem which has an exact solution that can be used to determine a physical value of k.

7.5 Formulation of the numerical solution of three-dimensional problems using the theory of a Cosserat point

In this section a numerical procedure is presented for the solution of three-dimensional problems in continuum mechanics. Specifically, attention is focused on a three-dimensional body that in its present configuration occupies the region R^* with boundary ∂R^*. Typical numerical solutions procedures, like the Galerkin method and the finite element method, divide the region into M connected subregions ${}_IP^*$ with boundaries ${}_I\partial P^*$ (I=1,2,...,M). The dynamic response of the continua is approximated by a finite number of degrees of freedom which characterize the motion of the continua. These degrees of freedom can be distributed either in a few elements, each with higher order shape functions, or in a larger number of elements, each with lower order shape functions. In (Rubin, 1995) it was shown that the theory of a Cosserat point can be used to formulate the solution of three-dimensional thermomechanical problems in continuum mechanics. Here, attention will be confined to the purely mechanical theory of nonlinear elastic materials and the notation (used in Rubin, 1989) is modified to be compatible with the development in this book.

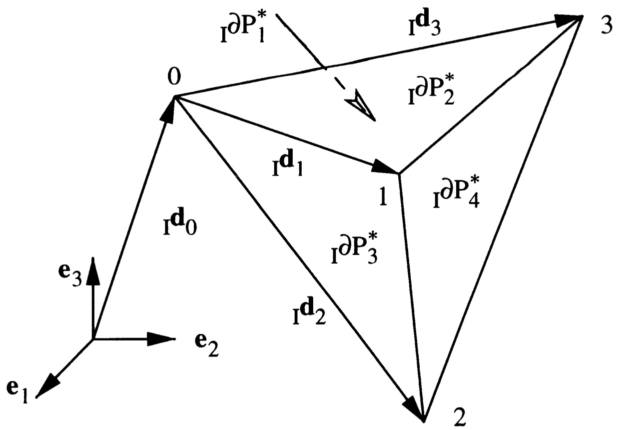

Fig. 7.5.1 The I'th Cosserat point for three-dimensional problems.

For the numerical solution procedure, each of the elements is modeled using balance laws of the theory of a Cosserat point which were developed in sections 6.3 and 6.4. Then, the equations for each element are coupled to the equations of neighboring elements by using kinematic and kinetic coupling conditions at the common nodes of the elements (which is similar to the finite element method). This approach produces a system of coupled nonlinear ordinary differential equations which are functions of time only, as in the standard Galerkin procedure. For the simplest case, which is considered here, the M regions ${}_{\mathrm{I}}P^*$ are modeled as tetrahedrons with a total of N nodal points. Moreover, the deformation in each tetrahedron is limited to be homogeneous deformation.

More specifically, the body is modeled as a collection of M (I=1,2,...,M) Cosserat points, each of which is a tetrahedron with four nodes (see Fig. 7.5.1). Consequently, the kinematics of the I'th Cosserat point can be characterized by only four director vectors

$$ {}_{\mathrm{I}}\mathbf{d}_i(t) \quad \text{for } i=0,1,2,3 \ . \tag{7.5.1}$$

For convenience, the location ${}_{\mathrm{I}}\mathbf{x}(t)$ of the Cosserat point relative to a fixed origin is denoted by ${}_{\mathrm{I}}\mathbf{d}_0(t)$

$$ {}_{\mathrm{I}}\mathbf{x}(t) = {}_{\mathrm{I}}\mathbf{d}_0(t) \ , \tag{7.5.2}$$

and the remaining director vectors are used to model the homogeneous deformations of the Cosserat point. Also, the director velocities ${}_{\mathrm{I}}\mathbf{w}_i$ are defined by

$$ {}_{\mathrm{I}}\mathbf{w}_i = {}_{\mathrm{I}}\dot{\mathbf{d}}_i \ , \tag{7.5.3}$$

In the stress-free reference configuration, the directors ${}_{\mathrm{I}}\mathbf{d}_i$ attain the constant values ${}_{\mathrm{I}}\mathbf{D}_i$. Also, the three vectors $\{{}_{\mathrm{I}}\mathbf{d}_1, {}_{\mathrm{I}}\mathbf{d}_2, {}_{\mathrm{I}}\mathbf{d}_3\}$ are assumed to remain linearly independent such that

$$ {}_{\mathrm{I}}D^{1/2} = {}_{\mathrm{I}}\mathbf{D}_1 \times {}_{\mathrm{I}}\mathbf{D}_2 \bullet {}_{\mathrm{I}}\mathbf{D}_3 > 0 \ , \quad {}_{\mathrm{I}}d^{1/2} = {}_{\mathrm{I}}\mathbf{d}_1 \times {}_{\mathrm{I}}\mathbf{d}_2 \bullet {}_{\mathrm{I}}\mathbf{d}_3 > 0 \ . \tag{7.5.4}$$

Here, the subscript I is used to denote quantities related to the I'th Cosserat point and it is recalled that there is no sum on repeated upper case indices unless indicated explicitly. Moreover, the reciprocal vectors ${}_{\mathrm{I}}\mathbf{d}^i$ and ${}_{\mathrm{I}}\mathbf{D}^i$ (i=1,2,3) are defined so that

$$ {}_{\mathrm{I}}\mathbf{D}_i \bullet {}_{\mathrm{I}}\mathbf{D}^j = \delta_i^j \ , \quad {}_{\mathrm{I}}\mathbf{d}_i \bullet {}_{\mathrm{I}}\mathbf{d}^j = \delta_i^j \ , \quad \text{for } i,j=1,2,3 \ . \tag{7.5.5}$$

Also, a number of useful kinematic variables are defined by the expressions

$$ {}_{\mathrm{I}}\mathbf{F} = \sum_{i=1}^{3} {}_{\mathrm{I}}\mathbf{d}_i \otimes {}_{\mathrm{I}}\mathbf{D}^i \ , \quad {}_{\mathrm{I}}J = \det({}_{\mathrm{I}}\mathbf{F}) = \frac{{}_{\mathrm{I}}d^{1/2}}{{}_{\mathrm{I}}D^{1/2}} \ , \quad {}_{\mathrm{I}}\mathbf{C} = {}_{\mathrm{I}}\mathbf{F}^T {}_{\mathrm{I}}\mathbf{F} \ , \quad {}_{\mathrm{I}}\mathbf{B} = {}_{\mathrm{I}}\mathbf{F} {}_{\mathrm{I}}\mathbf{F}^T \ ,$$

$$ {}_{\mathrm{I}}\dot{\mathbf{F}} = {}_{\mathrm{I}}\mathbf{L} {}_{\mathrm{I}}\mathbf{F} \ , \quad {}_{\mathrm{I}}\mathbf{L} = \sum_{i=1}^{3} {}_{\mathrm{I}}\mathbf{w}_i \otimes {}_{\mathrm{I}}\mathbf{d}^i = {}_{\mathrm{I}}\mathbf{D} + {}_{\mathrm{I}}\mathbf{W} \ ,$$

$$ {}_{\mathrm{I}}\mathbf{D} = \frac{1}{2}({}_{\mathrm{I}}\mathbf{L} + {}_{\mathrm{I}}\mathbf{L}^T) = {}_{\mathrm{I}}\mathbf{D}^T \ , \quad {}_{\mathrm{I}}\mathbf{W} = \frac{1}{2}({}_{\mathrm{I}}\mathbf{L} - {}_{\mathrm{I}}\mathbf{L}^T) = -\,{}_{\mathrm{I}}\mathbf{W}^T \ . \tag{7.5.6}$$

In these expressions, $\{{}_{\mathrm{I}}\mathbf{F}, {}_{\mathrm{I}}\mathbf{C}, {}_{\mathrm{I}}\mathbf{B}\}$ are measures of deformation and $\{{}_{\mathrm{I}}\mathbf{L}, {}_{\mathrm{I}}\mathbf{D}, {}_{\mathrm{I}}\mathbf{W}\}$ are measures of deformation rate.

For clarity, the theory of a Cosserat point for three-dimensional problems will be developed by using features of the derivation from the three-dimensional equations presented in section 6.3 and the direct approach presented in section 6.4. Also, the constitutive equations are developed using the direct approach in terms of a strain energy function and a dissipation inequality.

To this end, it is noted that in its stress-free reference configuration, material points in the I'th Cosserat point are located by the position vector $\mathbf{X}^*({}_{\mathrm{I}}\theta^i)$ in terms of convected Lagrangian coordinates θ^i (i=1,2,3) which are functions of the convected coordinates ${}_{\mathrm{I}}\theta^i$ (i=1,2,3). More specifically, the kinematic assumption (6.1.3) associated with the reference configuration is used to write

$$\mathbf{X}^*({}_I\theta^i) = {}_I\mathbf{D}_0 + {}_I\theta^i\, {}_I\mathbf{D}_i \ , \tag{7.5.7}$$

and the kinematic assumption (6.2.5) associated with the present configuration is used to write

$$\mathbf{x}^*({}_I\theta^i,t) = {}_I\mathbf{d}_0(t) + {}_I\theta^i\, {}_I\mathbf{d}_i(t) \ . \tag{7.5.8}$$

Furthermore, the three-dimensional region ${}_IP^*$ occupied by the I'th Cosserat point in the present configuration has boundary ${}_I\partial P^*$ which is divided into the four surfaces ${}_I\partial P_J^*$ (J=1,2,3,4) of the tetrahedron such that

$$\begin{gathered}
{}_I\partial P^* = {}_I\partial P_1^* \cup {}_I\partial P_2^* \cup {}_I\partial P_3^* \cup {}_I\partial P_4^* \ , \\
{}_I\partial P_1^*: \quad {}_I\theta^1 = 0 \ , \quad 0 \le {}_I\theta^2 \le L \ , \quad 0 \le {}_I\theta^3 \le L \ , \\
{}_I\partial P_2^*: \quad 0 \le {}_I\theta^1 \le L \ , \quad {}_I\theta^2 = 0 \ , \quad 0 \le {}_I\theta^3 \le L \ , \\
{}_I\partial P_3^*: \quad 0 \le {}_I\theta^1 \le L \ , \quad 0 \le {}_I\theta^2 \le L \ , \quad {}_I\theta^3 = 0 \ , \\
{}_I\partial P_4^*: \quad 0 \le {}_I\theta^1 \le L \ , \quad 0 \le {}_I\theta^2 \le L \ , \quad {}_I\theta^3 = L - {}_I\theta^1 - {}_I\theta^2 \ ,
\end{gathered} \tag{7.5.9}$$

where L is a constant having the dimensions of length and the vectors ${}_I\mathbf{d}_i$ have been normalized appropriately. Specifically, it follows that the four nodes of the I'th Cosserat point (see Fig. 7.5.1) are characterized by the vectors ${}_I\mathbf{X}_i^*$ (0,1,2,3) in the reference configuration such that

$$\begin{gathered}
{}_I\mathbf{X}_0^* = {}_I\mathbf{D}_0 \ , \quad {}_I\mathbf{X}_1^* = {}_I\mathbf{D}_0 + L\, {}_I\mathbf{D}_1 \ , \quad {}_I\mathbf{X}_2^* = {}_I\mathbf{D}_0 + L\, {}_I\mathbf{D}_2 \ , \quad {}_I\mathbf{X}_3^* = {}_I\mathbf{D}_0 + L\, {}_I\mathbf{D}_3 \ , \\
{}_I\mathbf{D}_0 = {}_I\mathbf{X}_0^* \ , \quad {}_I\mathbf{D}_1 = \frac{1}{L}\left[{}_I\mathbf{X}_1^* - {}_I\mathbf{X}_0^*\right] \ , \\
{}_I\mathbf{D}_2 = \frac{1}{L}\left[{}_I\mathbf{X}_2^* - {}_I\mathbf{X}_0^*\right] \ , \quad {}_I\mathbf{D}_3 = \frac{1}{L}\left[{}_I\mathbf{X}_3^* - {}_I\mathbf{X}_0^*\right] \ ;
\end{gathered} \tag{7.5.10}$$

and by the vectors ${}_I\mathbf{x}_i^*(t)$ in the present configuration such that

$$\begin{gathered}
{}_I\mathbf{x}_0^* = {}_I\mathbf{d}_0 \ , \quad {}_I\mathbf{x}_1^* = {}_I\mathbf{d}_0 + L\, {}_I\mathbf{d}_1 \ , \quad {}_I\mathbf{x}_2^* = {}_I\mathbf{d}_0 + L\, {}_I\mathbf{d}_2 \ , \quad {}_I\mathbf{x}_3^* = {}_I\mathbf{d}_0 + L\, {}_I\mathbf{d}_3 \ , \\
{}_I\mathbf{d}_0 = {}_I\mathbf{x}_0^* \ , \quad {}_I\mathbf{d}_1 = \frac{1}{L}\left[{}_I\mathbf{x}_1^* - {}_I\mathbf{x}_0^*\right] \ , \\
{}_I\mathbf{d}_2 = \frac{1}{L}\left[{}_I\mathbf{x}_2^* - {}_I\mathbf{x}_0^*\right] \ , \quad {}_I\mathbf{d}_3 = \frac{1}{L}\left[{}_I\mathbf{x}_3^* - {}_I\mathbf{x}_0^*\right] \ .
\end{gathered} \tag{7.5.11}$$

Next, it is recalled that the conservation of mass (3.2.1) and the balance of linear momentum (3.2.2) can be rewritten in the forms

$$\begin{gathered}
\frac{d}{dt}\int_{{}_IP^*} \rho^*\, dv^* = 0 \ , \\
\frac{d}{dt}\int_{{}_IP^*} \rho^*\, \mathbf{v}^*\, dv^* = \int_{{}_IP^*} \rho^*\, \mathbf{b}^*\, dv^* + \sum_{J=1}^{4} \int_{{}_I\partial P_J^*} \mathbf{t}^*\, da^* \ ,
\end{gathered} \tag{7.5.12}$$

and the balance of angular momentum becomes

$$\frac{d}{dt}\int_{{}_IP^*} \mathbf{x}^* \times \rho^*\, \mathbf{v}^*\, dv^* = \int_{{}_IP^*} \mathbf{x}^* \times \rho^*\, \mathbf{b}^*\, dv^* + \sum_{J=1}^{4} \int_{{}_I\partial P_J^*} \mathbf{x}^* \times \mathbf{t}^*\, da^* \ . \tag{7.5.13}$$

Furthermore, the averaged form (3.6.3) of the balance of linear momentum is rewritten as

$$\frac{d}{dt}\int_{{}_IP^*} \phi\, \rho^*\, \mathbf{v}^*\, dv^* = \int_{{}_IP^*} \left[\phi\, \rho^*\, \mathbf{b}^* - g^{-1/2}\, \mathbf{t}^{*j}\, \phi_{,j}\right] dv^* + \sum_{J=1}^{4} \int_{{}_I\partial P_J^*} \phi\, \mathbf{t}^*\, da^* \ , \tag{7.5.14}$$

where $\phi({}_I\theta^i)$ is a weighting function of the convected coordinates ${}_I\theta^i$ only.

In order to motivate the balance laws of the theory of a Cosserat point, it is convenient to introduce the following definitions

$$
\begin{gathered}
{}_I m = \int_{{}_I P^*} \rho^* \, dv^* \ , \quad {}_I y^{00} = 1 \ , \quad {}_I m \, {}_I y^{0j} = \int_{{}_I P^*} {}_I\theta^j \rho^* \, dv^* \ , \\
{}_I m \, {}_I y^{ij} = \int_{{}_I P^*} {}_I\theta^i \, {}_I\theta^j \rho^* \, dv^* \quad \text{for } i,j=1,2,3 \ , \\
{}_I y^{ij} = {}_I y^{ji} \quad \text{for } i,j=0,1,2,3 \ , \\
{}_I m \, {}_I \mathbf{b}_b^0 = \int_{{}_I P^*} \rho^* \, \mathbf{b}^* \, dv^* \ , \quad {}_I \mathbf{m}_J^0 = \int_{{}_I \partial P_J^*} \mathbf{t}^* \, da^* \ , \\
{}_I m \, {}_I \mathbf{b}_b^i = \int_{{}_I P^*} {}_I\theta^i \rho^* \, \mathbf{b}^* \, dv^* \ , \quad {}_I \mathbf{m}_J^i = \int_{{}_I \partial P_J^*} {}_I\theta^i \, \mathbf{t}^* \, da^* \ , \\
{}_I \mathbf{t}^0 = 0 \ , \quad {}_I \mathbf{t}^i = \int_{{}_I P^*} g^{-1/2} \, \mathbf{t}^{*i} \, dv^* \ , \quad \text{for } i=1,2,3 \ . \qquad (7.5.15)
\end{gathered}
$$

It then follows using the kinematic assumption (7.5.11), that for the I'th Cosserat point, the conservation of mass, balance of linear momentum, and balances of director momentum become

$$
\begin{gathered}
{}_I \dot{m} = 0 \ , \quad \frac{d}{dt}\Big[\sum_{j=0}^{3} {}_I m \, {}_I y^{ij} \, {}_I \mathbf{w}_j \Big] = {}_I m \, {}_I \mathbf{b}^i - {}_I \mathbf{t}^i \ , \\
\text{for } i=0,1,2,3 \ \text{ with } \ {}_I \mathbf{t}^0 = 0 \ , \qquad (7.5.16)
\end{gathered}
$$

and the balance of angular momentum becomes

$$
\frac{d}{dt} \sum_{i=0}^{3} \sum_{j=0}^{3} \big[{}_I \mathbf{d}_i \times {}_I m \, {}_I y^{ij} \, {}_I \mathbf{w}_j \big] = \sum_{i=0}^{3} {}_I \mathbf{d}_i \times {}_I m \, {}_I \mathbf{b}^i \ . \qquad (7.5.17)
$$

In these equations, ${}_I m$ is the mass, ${}_I y^{ij}$ are the director inertia coefficients, ${}_I \mathbf{b}^i$ are the specific (per unit mass) external assigned director couples, and ${}_I \mathbf{t}^i$ are the intrinsic director couples. Moreover, the director inertia coefficients are constants

$$
{}_I \dot{y}^{ij} = 0 \ , \qquad (7.5.18)
$$

and the assigned fields ${}_I \mathbf{b}^i$ can be expressed in the forms

$$
{}_I m \, {}_I \mathbf{b}^i = {}_I m \, {}_I \mathbf{b}_b^i + \sum_{J=1}^{4} {}_I \mathbf{m}_J^i \quad \text{for } i=0,1,2,3 \ . \qquad (7.5.19)
$$

Here, the terms ${}_I \mathbf{b}_b^i$ are associated with the external body force; the terms ${}_I \mathbf{m}_J^0$ are the forces applied to the surfaces ${}_I \partial P_J^*$; and ${}_I \mathbf{m}_J^i$ (i=1,2,3) are the director couples applied to the surfaces ${}_I \partial P_J^*$ of the tetrahedron.

The equations of motion (7.5.16) and (7.5.17) have been developed by integration of the equations of motion of a three-dimensional continuum. This approach is used to motivate the structure of the equations of motion of the Cosserat point. However, within the context of the direct approach these same equations of motion are postulated as the balance laws for the theory of a Cosserat point. The remaining equations describing the theory of a Cosserat point are developed within the context of the direct approach.

Specifically, using the equations of motion (7.5.16) the balance of angular momentum (7.5.17) can be reduced to the form

$$\sum_{i=1}^{3} {}_{I}\mathbf{d}_i \times {}_{I}\mathbf{t}^i = 0 \ . \tag{7.5.20}$$

Moreover, by introducing the definition

$${}_{I}\mathbf{T} = {}_{I}d^{-1/2} \sum_{i=1}^{3} {}_{I}\mathbf{t}^i \otimes {}_{I}\mathbf{d}_i \ , \tag{7.5.21}$$

it follows that the reduced form of the balance of angular momentum can be written as

$${}_{I}\mathbf{T}^T = {}_{I}\mathbf{T} \ , \tag{7.5.22}$$

which is similar to the expression (3.2.32) associated with the three-dimensional theory.

For an elastic Cosserat point the specific strain energy ${}_{I}\Sigma$ is a function of the deformation measure ${}_{I}\mathbf{C}$ only

$${}_{I}\Sigma = {}_{I}\hat{\Sigma}({}_{I}\mathbf{C}) \ . \tag{7.5.23}$$

Moreover, for the purely mechanical theory it is convenient to define: the rate of dissipation ${}_{I}\mathcal{D}$; the mechanical power ${}_{I}\mathcal{P}$; the rate of work ${}_{I}\mathcal{W}$ of the assigned fields ${}_{I}\mathbf{b}^i$; the kinetic energy ${}_{I}\mathcal{K}$; and the total internal energy ${}_{I}\mathcal{U}$ by the formulas

$${}_{I}d^{1/2}\,{}_{I}\mathcal{D} = {}_{I}\mathcal{W} - {}_{I}\dot{\mathcal{K}} - {}_{I}\dot{\mathcal{U}} \ , \quad {}_{I}d^{1/2}\,{}_{I}\mathcal{P} = {}_{I}\mathcal{W} - {}_{I}\dot{\mathcal{K}} \ ,$$
$${}_{I}\mathcal{W} = \sum_{i=0}^{3} {}_{I}m\,{}_{I}\mathbf{b}^i \bullet {}_{I}\mathbf{w}_i \ , \quad {}_{I}\mathcal{K} = \sum_{i=0}^{3} \frac{1}{2}\,{}_{I}m\,{}_{I}y^{ij}\,{}_{I}\mathbf{w}_i \bullet {}_{I}\mathbf{w}_j \ ,$$
$${}_{I}\mathcal{U} = {}_{I}m\,{}_{I}\Sigma \ . \tag{7.5.24}$$

Next, with the help of the definitions (7.5.6) it can be shown that

$$\overline{{}_{I}\mathbf{F}^{-1}}^{\bullet} = -\,{}_{I}\mathbf{F}^{-1}\,{}_{I}\mathbf{L} \ , \quad {}_{I}\mathbf{w}_i = {}_{I}\mathbf{L}\,{}_{I}\mathbf{d}_i \quad \text{for } i=1,2,3 \ , \tag{7.5.25}$$

Thus, using the equations of motion (7.5.16), the definition (7.5.21), and the result (7.5.25), it follows that the rate of dissipation reduces to

$${}_{I}d^{1/2}\,{}_{I}\mathcal{D} = {}_{I}d^{1/2}\,{}_{I}\mathbf{T} \bullet {}_{I}\mathbf{D} - {}_{I}m\,{}_{I}\dot{\Sigma} \geq 0 \ , \tag{7.5.26}$$

which is required to be nonnegative.

For an elastic Cosserat point, assumptions similar to those described in section 6.8 are made. In particular, the dissipation ${}_{I}\mathcal{D}$ vanishes and ${}_{I}\mathbf{T}$ is independent of the rate ${}_{I}\mathbf{L}$

$${}_{I}\mathbf{T} = {}_{I}\hat{\mathbf{T}}({}_{I}\mathbf{d}_i) \quad \text{for } i=1,2,3 \ , \tag{7.5.27}$$

Consequently, it can be shown using (7.5.26) that

$${}_{I}\mathbf{T} = {}_{I}\hat{\mathbf{T}} = 2\,{}_{I}\rho\,{}_{I}\mathbf{F}\,\frac{\partial_{I}\hat{\Sigma}}{\partial_{I}\mathbf{C}}\,{}_{I}\mathbf{F}^T \ , \tag{7.5.28}$$

which automatically satisfies the balance of angular momentum (7.5.22). Also, in this equation use has been made of the Lagrangian form of conservation of mass

$$ {}_I m = {}_I\rho \, {}_I d^{1/2} = {}_I\rho_0 \, {}_I D^{1/2} \ , \tag{7.5.29} $$

where ${}_I\rho$ is the mass density in the present configuration and ${}_I\rho_0$ is its reference value. Once the constitutive equation for ${}_I\mathbf{T}$ is specified, the values of ${}_I\mathbf{t}^i$ (i=1,2,3) which appear in the equations of motion (7.5.16) and in the definition (7.5.21), can be determined by the expression

$$ {}_I\mathbf{t}^i = [{}_I d^{1/2} \, {}_I\mathbf{T} \,]{}_I\mathbf{d}^i \quad \text{for } i=1,2,3 \ . \tag{7.5.30} $$

Now, a simple model for a Cosserat point constructed from a dissipative material can be developed by assuming that ${}_I\mathbf{T}$ and ${}_I\mathbf{t}^i$ separate additively into two parts

$$ {}_I\mathbf{T} = {}_I\hat{\mathbf{T}} + {}_I\check{\mathbf{T}} \ , \quad {}_I\mathbf{t}^i = {}_I\hat{\mathbf{t}}^i + {}_I\check{\mathbf{t}}^i \ , \quad {}_I\hat{\mathbf{t}}^0 = 0 \ , \quad {}_I\check{\mathbf{t}}^0 = 0 \ , $$

$$ {}_I\hat{\mathbf{T}} = {}_I d^{-1/2} \sum_{i=1}^{3} {}_I\hat{\mathbf{t}}^i \otimes {}_I\mathbf{d}_i = {}_I\hat{\mathbf{T}}^T \ , \quad {}_I\check{\mathbf{T}} = {}_I d^{-1/2} \sum_{i=1}^{3} {}_I\check{\mathbf{t}}^i \otimes {}_I\mathbf{d}_i = {}_I\check{\mathbf{T}}^T \ , \tag{7.5.31} $$

with ${}_I\hat{\mathbf{T}}$ and ${}_I\hat{\mathbf{t}}^i$ being the parts associated with elastic deformation [which balance the rate of change in strain energy (7.5.23)]

$$ {}_I d^{1/2} \, {}_I\hat{\mathbf{T}} \bullet {}_I\mathbf{D} = {}_I m \, {}_I\dot{\Sigma} \ , \tag{7.5.32} $$

and ${}_I\check{\mathbf{T}}$ and ${}_I\check{\mathbf{t}}^i$ being the parts due to material dissipation. Thus, the restriction (7.5.26) reduces to

$$ {}_I d^{1/2} \, {}_I\mathcal{D} = {}_I d^{1/2} \, {}_I\check{\mathbf{T}} \bullet {}_I\mathbf{D} \geq 0 \ . \tag{7.5.33} $$

As a simple case it is possible to assume that ${}_I\check{\mathbf{T}}$ is a linear function of rate of the form

$$ {}_I d^{1/2} \, {}_I\check{\mathbf{T}} = {}_I D^{1/2} \, [{}_I\eta_1 \, ({}_I\mathbf{D} \bullet \mathbf{I}) \, \mathbf{I} + 2 \, {}_I\eta_2 \, {}_I\mathbf{D}' \,] \ , \tag{7.5.34} $$

where ${}_I\eta_1$ and ${}_I\eta_2$ are material constants and ${}_I\mathbf{D}'$ is a pure measure of the rate of distortional deformation

$$ {}_I\mathbf{D}' = {}_I\mathbf{D} - \frac{1}{3}({}_I\mathbf{D} \bullet \mathbf{I}) \, \mathbf{I} \ , \quad {}_I\mathbf{D}' \bullet \mathbf{I} = 0 \ . \tag{7.5.35} $$

Consequently, ${}_I\eta_1$ is the viscosity of dilatational deformation rate, and ${}_I\eta_2$ is the viscosity of distortional deformation rate, respectively. Also, it can be shown that the restriction (7.5.33) is satisfied for all motions provided that ${}_I\eta_1$ and ${}_I\eta_2$ are nonnegative

$$ {}_I\eta_1 \geq 0 \ , \quad {}_I\eta_2 \geq 0 \ . \tag{7.5.36} $$

Moreover, it is noted that the viscosity constants ${}_I\eta_1$ and ${}_I\eta_4$ can be determined by attempting to match the rate of damping associated with free vibrations of the Cosserat point.

The deformation field associated with the kinematic assumption (7.5.8) is homogeneous. Consequently, for three-dimensionally uniform homogeneous materials, it is expected that the constitutive equation for the strain energy of the Cosserat point can be restricted so that the theory of a Cosserat point will be consistent with exact three-dimensional solutions for all homogeneous deformations. This restriction requires that the strain energy function ${}_I\hat{\Sigma}({}_I\mathbf{C})$ in (7.5.23) have the same form as the three-dimensional strain energy function $\hat{\Sigma}^*(\mathbf{C}^*)$ so that

$$ {}_I\hat{\Sigma}({}_I\mathbf{C}) = \hat{\Sigma}^*({}_I\mathbf{C}) \ . \tag{7.5.37} $$

In order to determine the appropriate forms of the forces that one element applies to its nearest neighbors, it is convenient to record an explicit expression for the rate of work ${}_{\mathrm{I}}\mathcal{W}_c$ supplied to the I'th Cosserat point through its boundary surfaces ${}_{\mathrm{I}}\partial P_J^*$

$$ {}_{\mathrm{I}}\mathcal{W}_c = \sum_{i=0}^{3} \sum_{J=1}^{4} {}_{\mathrm{I}}\mathbf{m}_J^i \bullet {}_{\mathrm{I}}\mathbf{w}_i \ . \tag{7.5.38} $$

Then, using the results (7.5.11) it can be shown that

$$ {}_{\mathrm{I}}\mathcal{W}_c = \sum_{i=0}^{3} {}_{\mathrm{I}}\mathbf{m}^i \bullet {}_{\mathrm{I}}\dot{\mathbf{x}}_i^* \ , \tag{7.5.39} $$

where the nodal forces ${}_{\mathrm{I}}\mathbf{m}^i$ (i=0,1,2,3) of the I'th element are defined by

$$ {}_{\mathrm{I}}\mathbf{m}^0 = \sum_{J=1}^{4} [{}_{\mathrm{I}}\mathbf{m}_J^0 - \frac{1}{L}\{{}_{\mathrm{I}}\mathbf{m}_J^1 + {}_{\mathrm{I}}\mathbf{m}_J^2 + {}_{\mathrm{I}}\mathbf{m}_J^3\}] \ , $$

$$ {}_{\mathrm{I}}\mathbf{m}^i = \frac{1}{L} \sum_{J=1}^{4} {}_{\mathrm{I}}\mathbf{m}_J^i \quad \text{for } i=1,2,3 \ . \tag{7.5.40} $$

These forces represent the contributions of the external forces applied to the I'th element through the boundaries ${}_{\mathrm{I}}\partial P_J^*$. Next, using the expressions (7.5.11), (7.5.19) and (7.5.40), the equations of motion (7.5.16) can be solved to deduce that

$$ \begin{aligned} {}_{\mathrm{I}}\mathbf{m}^0 = & -{}_{\mathrm{I}}m\,[{}_{\mathrm{I}}\mathbf{b}_b^0 - \frac{1}{L}\{\,{}_{\mathrm{I}}\mathbf{b}_b^1 + {}_{\mathrm{I}}\mathbf{b}_b^2 + {}_{\mathrm{I}}\mathbf{b}_b^3\,\}] + \frac{1}{L}[\,{}_{\mathrm{I}}\mathbf{t}^1 + {}_{\mathrm{I}}\mathbf{t}^2 + {}_{\mathrm{I}}\mathbf{t}^3] \\ & + {}_{\mathrm{I}}m\,[1 - \frac{1}{L}\{2\,{}_{\mathrm{I}}y^{01} + 2\,{}_{\mathrm{I}}y^{02} + 2\,{}_{\mathrm{I}}y^{03}\} \\ & + \frac{1}{L^2}\{\,{}_{\mathrm{I}}y^{11} + {}_{\mathrm{I}}y^{22} + {}_{\mathrm{I}}y^{33} + 2\,{}_{\mathrm{I}}y^{12} + 2\,{}_{\mathrm{I}}y^{13} + 2\,{}_{\mathrm{I}}y^{23}\}]\,{}_{\mathrm{I}}\ddot{\mathbf{x}}_0^* \\ & + \frac{1}{L}\,{}_{\mathrm{I}}m\,[{}_{\mathrm{I}}y^{01} - \frac{1}{L}\{\,{}_{\mathrm{I}}y^{11} + {}_{\mathrm{I}}y^{12} + {}_{\mathrm{I}}y^{13}\}]\,{}_{\mathrm{I}}\ddot{\mathbf{x}}_1^* \\ & + \frac{1}{L}\,{}_{\mathrm{I}}m\,[{}_{\mathrm{I}}y^{02} - \frac{1}{L}\{\,{}_{\mathrm{I}}y^{12} + {}_{\mathrm{I}}y^{22} + {}_{\mathrm{I}}y^{23}\}]\,{}_{\mathrm{I}}\ddot{\mathbf{x}}_2^* \\ & + \frac{1}{L}\,{}_{\mathrm{I}}m\,[{}_{\mathrm{I}}y^{03} - \frac{1}{L}\{\,{}_{\mathrm{I}}y^{13} + {}_{\mathrm{I}}y^{23} + {}_{\mathrm{I}}y^{33}\}]\,{}_{\mathrm{I}}\ddot{\mathbf{x}}_3^* \ , \\ {}_{\mathrm{I}}\mathbf{m}^1 = & \frac{1}{L}[-{}_{\mathrm{I}}m\,{}_{\mathrm{I}}\mathbf{b}_b^1 + {}_{\mathrm{I}}\mathbf{t}^1] + {}_{\mathrm{I}}m\,[\frac{1}{L}{}_{\mathrm{I}}y^{01} - \frac{1}{L^2}\{{}_{\mathrm{I}}y^{11} + {}_{\mathrm{I}}y^{12} + {}_{\mathrm{I}}y^{13}\}\,]\,{}_{\mathrm{I}}\ddot{\mathbf{x}}_0^* \\ & + \frac{1}{L^2}{}_{\mathrm{I}}m\,[\,{}_{\mathrm{I}}y^{11}\,{}_{\mathrm{I}}\ddot{\mathbf{x}}_1^* + {}_{\mathrm{I}}y^{12}\,{}_{\mathrm{I}}\ddot{\mathbf{x}}_2^* + {}_{\mathrm{I}}y^{13}\,{}_{\mathrm{I}}\ddot{\mathbf{x}}_3^*\,] \ , \\ {}_{\mathrm{I}}\mathbf{m}^2 = & \frac{1}{L}[-{}_{\mathrm{I}}m\,{}_{\mathrm{I}}\mathbf{b}_b^2 + {}_{\mathrm{I}}\mathbf{t}^2] + {}_{\mathrm{I}}m\,[\frac{1}{L}{}_{\mathrm{I}}y^{02} - \frac{1}{L^2}\{{}_{\mathrm{I}}y^{12} + {}_{\mathrm{I}}y^{22} + {}_{\mathrm{I}}y^{23}\}\,]\,{}_{\mathrm{I}}\ddot{\mathbf{x}}_0^* \\ & + \frac{1}{L^2}{}_{\mathrm{I}}m\,[\,{}_{\mathrm{I}}y^{12}\,{}_{\mathrm{I}}\ddot{\mathbf{x}}_1^* + {}_{\mathrm{I}}y^{22}\,{}_{\mathrm{I}}\ddot{\mathbf{x}}_2^* + {}_{\mathrm{I}}y^{23}\,{}_{\mathrm{I}}\ddot{\mathbf{x}}_3^*\,] \\ {}_{\mathrm{I}}\mathbf{m}^3 = & \frac{1}{L}[-{}_{\mathrm{I}}m\,{}_{\mathrm{I}}\mathbf{b}_b^3 + {}_{\mathrm{I}}\mathbf{t}^3] + {}_{\mathrm{I}}m\,[\frac{1}{L}{}_{\mathrm{I}}y^{03} - \frac{1}{L^2}\{{}_{\mathrm{I}}y^{13} + {}_{\mathrm{I}}y^{23} + {}_{\mathrm{I}}y^{33}\}\,]\,{}_{\mathrm{I}}\ddot{\mathbf{x}}_0^* \\ & + \frac{1}{L^2}{}_{\mathrm{I}}m\,[\,{}_{\mathrm{I}}y^{13}\,{}_{\mathrm{I}}\ddot{\mathbf{x}}_1^* + {}_{\mathrm{I}}y^{23}\,{}_{\mathrm{I}}\ddot{\mathbf{x}}_2^* + {}_{\mathrm{I}}y^{33}\,{}_{\mathrm{I}}\ddot{\mathbf{x}}_3^*\,] \ . \end{aligned} \tag{7.5.41} $$

In view of the results (7.5.11), it follows that the deformation of the I'th Cosserat point is completely determined by the four vectors ${}_I\mathbf{x}_i^*(t)$. Moreover, once the inertia quantities

$${}_I m \ , \ {}_I y^{ij} \quad \text{for } i,j=0,1,2,3,4 \ , \tag{7.5.42}$$

the assigned fields

$${}_I \mathbf{b}_b^i \quad \text{for } i=0,1,2,3,4 \ , \tag{7.5.43}$$

the strain energy function (7.5.37), and the viscosity coefficients ${}_I\eta_1$ and ${}_I\eta_2$ in (7.5.34) have been specified, the constitutive equations for the intrinsic director couples ${}_I\mathbf{t}^i$ in (7.5.30) are determined. Then, the nodal forces ${}_I\mathbf{m}^i$ are also determined by the equations (7.5.41).

Since the body R^* has been divided into M tetrahedral Cosserat points, there are $4\times M$ vectors ${}_I\mathbf{x}_i^*$ associated with the four nodes of each of these Cosserat points. However, since there are only N nodes associated with these M Cosserat points, the deformation of the body should be totally determined by the N vectors $\mathbf{x}_K^*(t)$ (K=1,2,...,N) which characterize the motion of these nodes. Consequently, it is necessary to specify kinematic coupling conditions that are determined by the topology of the specific discretization of the region R^*. These kinematic coupling conditions can be summarized as

$$\{ {}_I\mathbf{x}_0^* \text{ or } {}_I\mathbf{x}_1^* \text{ or } {}_I\mathbf{x}_2^* \text{ or } {}_I\mathbf{x}_3^* \} = \mathbf{x}_K^* \quad \text{for } K=1,2,...,N \ . \tag{7.5.44}$$

Next, summing the contributions to the rate of work (7.5.39) of all M elements, it is convenient to define the external nodal force $\mathbf{m}_K^*$ which is applied to the K'th node by the formula

$$\mathbf{m}_K^* = \sum_{(I,i:K)} {}_I\mathbf{m}^i \ . \tag{7.5.45}$$

In this equation, the special summation symbol indicates that the summation is performed over all forces i (i=0,1,2,3) and all elements I (I=1,2,...,M) which have nodes that coincident with the K'th node of the region R^*. If the K'th node is an interior node of the region R^* then $\mathbf{m}_K^*$ vanishes

$$\mathbf{m}_K^* = 0 \quad \text{for interior nodes of } R^* \ , \tag{7.5.46}$$

because the traction vector that one Cosserat point applies on its neighbor through their common boundary surface is equal and opposite to the traction vector that the same neighbor applies to the same Cosserat point. On the other hand, if the K'th node is an exterior node lying on the boundary ∂R^*, then $\mathbf{m}_K^*$ is either specified by boundary conditions associated with surface tractions or it is determined by the solution as the reaction to specified kinematic conditions on the boundary.

More specifically, the boundary conditions associated with ∂R^* can be kinematic-type conditions for which

$$\mathbf{x}_K^* = \text{specified} \ , \ \ \mathbf{m}_K^* = \text{determined by solution}; \tag{7.5.47}$$

they can be kinetic-type conditions for which

$$\mathbf{m}_K^* = \text{specified} \ , \ \ \mathbf{x}_K^* = \text{determined by solution}; \tag{7.5.48}$$

or they can be mixed-type or mixed-mixed-type conditions where different components of $\mathbf{x}_K^*$ and $\mathbf{m}_K^*$ are specified

$$\mathbf{x}_K^* \bullet \mathbf{b}_i = \text{specified} \;\text{ and }\; \mathbf{m}_K^* \bullet \mathbf{b}_i = \text{determined by solution} \;,$$
$$\text{or}$$
$$\mathbf{x}_K^* \bullet \mathbf{b}_i = \text{determined by solution} \;\text{ and }\; \mathbf{m}_K^* \bullet \mathbf{b}_i = \text{specified} \;. \tag{7.5.49}$$

In these conditions $\mathbf{b}_i$ (i=1,2,3) are an arbitrary set of orthonormal vectors.

Whenever the K'th node is an exterior node, the equations (7.5.9), (7.5.15) and (7.5.40) are used to rewrite $\mathbf{m}_K^*$ in terms of integrals of the traction vector $\mathbf{t}^*$ over the exterior boundaries associated with node K. For example, let the node ${}_3\mathbf{x}_1^*$ associated with the 3rd Cosserat point, the node ${}_5\mathbf{x}_0^*$ associated with the 5th Cosserat point, and the node ${}_9\mathbf{x}_3^*$ associated with the 9th Cosserat point, coincide with the exterior node K, and let the boundaries ${}_3\partial P_3^*$, ${}_5\partial P_1^*$ and ${}_9\partial P_4^*$ be the only exterior boundaries associated with this exterior node. Then, $\mathbf{m}_K^*$ in (7.5.45) is specified by

$$\begin{aligned}\mathbf{m}_K^* &= {}_3\mathbf{m}^1 + {}_5\mathbf{m}^0 + {}_9\mathbf{m}^3 = \frac{1}{L}\,{}_3\mathbf{m}_3^1 + \left[{}_5\mathbf{m}_1^0 - \frac{1}{L}\left\{{}_5\mathbf{m}_1^1 + {}_5\mathbf{m}_1^2 + {}_5\mathbf{m}_1^3\right\}\right] + \frac{1}{L}\,{}_9\mathbf{m}_4^3\\ &= \int_{{}_3\partial P_3^*} \frac{1}{L}\left\{{}_3\theta^1\right\} \mathbf{t}^*\, da^* + \int_{{}_5\partial P_1^*} \left[1 - \frac{1}{L}\left\{{}_5\theta^2 + {}_5\theta^3\right\}\right] \mathbf{t}^*\, da^*\\ &\quad + \int_{{}_9\partial P_4^*} \left[1 - \frac{1}{L}\left\{{}_9\theta^1 - {}_9\theta^2\right\}\right] \mathbf{t}^*\, da^* \;.\end{aligned} \tag{7.5.50}$$

In general, the equations (7.5.46) associated with interior nodes, and the equations (7.5.47) or (7.5.48) associated with boundary nodes, yield N vector equations to determine the N vectors (either $\mathbf{x}_K^*$ or $\mathbf{m}_K^*$) that contain second order time derivatives of $\mathbf{x}_K^*$ (K=1,2,...,N). Consequently, in addition to the boundary conditions described above, it is necessary to specify initial values for

$$\{\mathbf{x}_K^* \,,\, \dot{\mathbf{x}}_K^*\} \;. \tag{7.5.51}$$

Once the values of $\mathbf{x}_K^*$ are determined, the nodal forces ${}_I\mathbf{m}^i$ (i=0,1,2,3) of the I'th Cosserat point are determined by the equations (7.5.41), and the nodal forces $\mathbf{m}_K^*$ are determined by the equations (7.5.45).

7.6 Formulation of the numerical solution of two-dimensional problems using the theory of a Cosserat point

In this section a numerical procedure is presented for the solution of two-dimensional problems in continuum mechanics. Specifically, attention is focused on a three-dimensional body that in its present configuration occupies the region R^* with boundary ∂R^*. For the two-dimensional problems under consideration here, R^* is a right cylindrical region with parallel planar ends ∂R_4^* and ∂R_5^*. Because of these planar ends, the development in this section is slightly more complicated than the one for three-dimensional problems presented in the last section.

Typical numerical solutions procedures, like the Galerkin method and the finite element method, divide the region into M connected subregions ${}_I P^*$ with boundaries ${}_I\partial P^*$

(I=1,2,...,M). The dynamic response of the continua is approximated by a finite number of degrees of freedom which characterize the motion of the continua. These degrees of freedom can be distributed either in a few elements, each with higher order shape functions, or in a larger number of elements, each with lower order shape functions. In (Rubin, 1995) it was shown that the theory of a Cosserat point can be used to formulate the solution of two-dimensional thermomechanical problems in continuum mechanics. Here, attention will be confined to the purely mechanical theory of nonlinear elastic materials, and the notation (used in Rubin, 1989) is modified to be compatible with the development in this book.

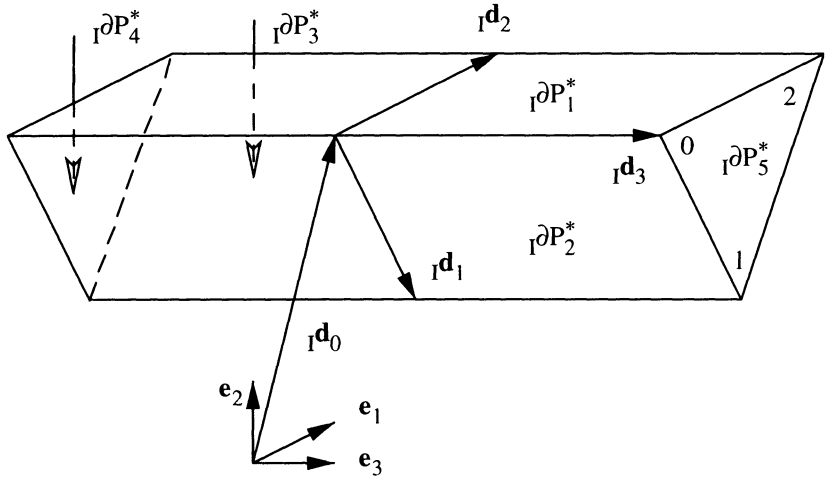

Fig. 7.6.1 Three-dimensional view of the I'th Cosserat point for two-dimensional problems.

For the numerical solution procedure, each of the elements is modeled using balance laws of the theory of a Cosserat point which were developed in sections 6.3 and 6.4. Then, the equations for each element are coupled to the equations of neighboring elements by using kinematic and kinetic coupling conditions at the common nodes of the elements (which is similar to the finite element method). This approach produces a system of coupled nonlinear ordinary differential equations which are functions of time only, as in the standard Galerkin procedure. For the simplest case, which is considered here, the M regions $_IP^*$ are modeled as right cylinders with triangular cross-sections, such that the triangles in the cross-section of the three-dimensional cylindrical region have a total of N nodal points. Moreover, the deformation in each element is limited to be homogeneous deformation.

More specifically, the body is modeled as a collection of M (I=1,2,...,M) Cosserat points, each of which is a right cylinder with a triangular cross-section with three nodes (see Fig. 7.6.1). Consequently, the kinematics of the I'th Cosserat point can be characterized by only four director vectors

$$ {}_I\mathbf{d}_i(t) \quad \text{for } i=0,1,2,3 \ . \tag{7.6.1}$$

For convenience, the location ${}_I\mathbf{x}(t)$ of the Cosserat point relative to a fixed origin is denoted by ${}_I\mathbf{d}_0(t)$

$$ {}_I\mathbf{x}(t) = {}_I\mathbf{d}_0(t) \ , \tag{7.6.2}$$

and the remaining director vectors are used to model the homogeneous deformations of the Cosserat point. Also, the director velocities ${}_I\mathbf{w}_i$ are defined by

$$ {}_I\mathbf{w}_i = {}_I\dot{\mathbf{d}}_i \ . \tag{7.6.3}$$

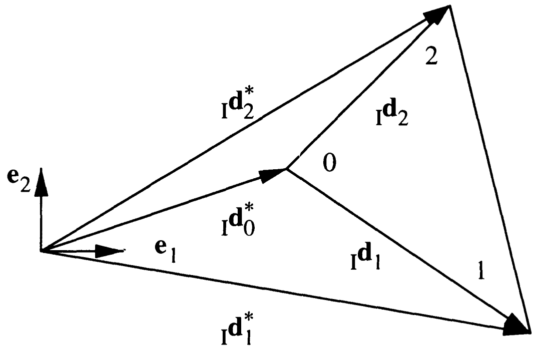

Fig. 7.6.2 Two-dimensional view of the I'th Cosserat point for two-dimensional problems.

In the stress-free reference configuration, the directors ${}_I\mathbf{d}_i$ attain the constant values ${}_I\mathbf{D}_i$. Also, the three vectors $\{{}_I\mathbf{d}_1, {}_I\mathbf{d}_2, {}_I\mathbf{d}_3\}$ are assumed to remain linearly independent such that

$$ {}_ID^{1/2} = {}_I\mathbf{D}_1 \times {}_I\mathbf{D}_2 \bullet {}_I\mathbf{D}_3 > 0 \ , \quad {}_Id^{1/2} = {}_I\mathbf{d}_1 \times {}_I\mathbf{d}_2 \bullet {}_I\mathbf{d}_3 > 0 \ . \tag{7.6.4}$$

Here, the subscript I is used to denote quantities related to the I'th Cosserat point and it is recalled that there is no sum on repeated upper case indices unless indicated explicitly. Moreover, the reciprocal vectors ${}_I\mathbf{d}^i$ and ${}_I\mathbf{D}^i$ (i=1,2,3) are defined so that

$$ {}_I\mathbf{D}_i \bullet {}_I\mathbf{D}^j = \delta_i^j \ , \quad {}_I\mathbf{d}_i \bullet {}_I\mathbf{d}^j = \delta_i^j \ , \quad \text{for } i,j=1,2,3 \ . \tag{7.6.5}$$

Also, a number of useful kinematic variables are defined by the expressions

$$ {}_I\mathbf{F} = \sum_{i=1}^{3} {}_I\mathbf{d}_i \otimes {}_I\mathbf{D}^i \ , \quad {}_IJ = \det({}_I\mathbf{F}) = \frac{{}_Id^{1/2}}{{}_ID^{1/2}} \ , \quad {}_I\mathbf{C} = {}_I\mathbf{F}^T{}_I\mathbf{F} \ , \quad {}_I\mathbf{B} = {}_I\mathbf{F}{}_I\mathbf{F}^T \ ,$$

$$ {}_I\dot{\mathbf{F}} = {}_I\mathbf{L}{}_I\mathbf{F} \ , \quad {}_I\mathbf{L} = \sum_{i=1}^{3} {}_I\mathbf{w}_i \otimes {}_I\mathbf{d}^i = {}_I\mathbf{D} + {}_I\mathbf{W} \ ,$$

$$ {}_I\mathbf{D} = \frac{1}{2}({}_I\mathbf{L} + {}_I\mathbf{L}^T) = {}_I\mathbf{D}^T \ , \quad {}_I\mathbf{W} = \frac{1}{2}({}_I\mathbf{L} - {}_I\mathbf{L}^T) = -{}_I\mathbf{W}^T \ . \tag{7.6.6}$$

In these expressions, $\{{}_I\mathbf{F}, {}_I\mathbf{C}, {}_I\mathbf{B}\}$ are measures of deformation and $\{{}_I\mathbf{L}, {}_I\mathbf{D}, {}_I\mathbf{W}\}$ are measures of deformation rate.

For clarity, the theory of a Cosserat point for two-dimensional problems will be developed by using features of the derivation from the three-dimensional equations presented in section 6.3 and the direct approach presented in section 6.4. Also, the constitutive equations are developed using the direct approach in terms of a strain energy function and a dissipation inequality.

To this end, it is noted that in its stress-free reference configuration, material points in the I'th Cosserat point are located by the position vector $\mathbf{X}^*({}_I\theta^i)$ in terms of convected Lagrangian coordinates θ^i (i=1,2,3) which are functions of the convected coordinates ${}_I\theta^i$ (i=1,2,3). More specifically, the kinematic assumption (6.1.3) associated with the reference configuration is used to write

$$\mathbf{X}^*({}_I\theta^i) = {}_I\mathbf{D}_0 + {}_I\theta^i\, {}_I\mathbf{D}_i \ , \tag{7.6.7}$$

and the kinematic assumption (6.2.5) associated with the present configuration is used to write

$$\mathbf{x}^*({}_I\theta^i,t) = {}_I\mathbf{d}_0(t) + {}_I\theta^i\, {}_I\mathbf{d}_i(t) \ . \tag{7.6.8}$$

Furthermore, the three-dimensional region ${}_IP^*$ occupied by the I'th Cosserat point in the present configuration has boundary ${}_I\partial P^*$ which is divided into the five surfaces ${}_I\partial P^*_J$ (J=1,2,3,4,5) of the right cylindrical triangle (see Fig. 7.6.1) such that

$$\begin{gathered}
{}_I\partial P^* = {}_I\partial P^*_1 \cup {}_I\partial P^*_2 \cup {}_I\partial P^*_3 \cup {}_I\partial P^*_4 \cup {}_I\partial P^*_5 \ , \\
{}_I\partial P^*_1: \quad {}_I\theta^1 = 0 \ , \quad 0 \le {}_I\theta^2 \le L \ , \quad -\frac{L}{2} \le {}_I\theta^3 \le -\frac{L}{2} \ , \\
{}_I\partial P^*_2: \quad 0 \le {}_I\theta^1 \le L \ , \quad {}_I\theta^2 = 0 \ , \quad -\frac{L}{2} \le {}_I\theta^3 \le -\frac{L}{2} \ , \\
{}_I\partial P^*_3: \quad 0 \le {}_I\theta^1 \le L \ , \quad 0 \le {}_I\theta^2 = L - {}_I\theta^1 \ , \quad -\frac{L}{2} \le {}_I\theta^3 \le -\frac{L}{2} \ , \\
{}_I\partial P^*_4: \quad 0 \le {}_I\theta^1 \le L \ , \quad 0 \le {}_I\theta^2 \le L - {}_I\theta^1 \ , \quad {}_I\theta^3 = -\frac{L}{2} \ , \\
{}_I\partial P^*_5: \quad 0 \le {}_I\theta^1 \le L \ , \quad 0 \le {}_I\theta^2 \le L - {}_I\theta^1 \ , \quad {}_I\theta^3 = \frac{L}{2} \ ,
\end{gathered} \tag{7.6.9}$$

where L is a constant having the dimensions of length, and the vectors ${}_I\mathbf{d}_i$ have been normalized appropriately. Specifically, it follows that the three nodes of the I'th Cosserat point (see Fig. 7.6.2) are characterized by the vectors ${}_I\mathbf{X}^*_i$ (i=0,1,2) in the reference configuration such that

$$\begin{gathered}
{}_I\mathbf{X}^*_0 = {}_I\mathbf{D}_0 \ , \quad {}_I\mathbf{X}^*_1 = {}_I\mathbf{D}_0 + L\, {}_I\mathbf{D}_1 \ , \quad {}_I\mathbf{X}^*_2 = {}_I\mathbf{D}_0 + L\, {}_I\mathbf{D}_2 \ , \\
{}_I\mathbf{D}_0 = {}_I\mathbf{X}^*_0 \ , \quad {}_I\mathbf{D}_1 = \frac{1}{L}[{}_I\mathbf{X}^*_1 - {}_I\mathbf{X}^*_0] \ , \quad {}_I\mathbf{D}_2 = \frac{1}{L}[{}_I\mathbf{X}^*_2 - {}_I\mathbf{X}^*_0] \ ,
\end{gathered} \tag{7.6.10}$$

and by the vectors ${}_I\mathbf{x}^*_i(t)$ in the present configuration such that

$$\begin{gathered}
{}_I\mathbf{x}^*_0 = {}_I\mathbf{d}_0 \ , \quad {}_I\mathbf{x}^*_1 = {}_I\mathbf{d}_0 + L\, {}_I\mathbf{d}_1 \ , \quad {}_I\mathbf{x}^*_2 = {}_I\mathbf{d}_0 + L\, {}_I\mathbf{d}_2 \ , \\
{}_I\mathbf{d}_0 = {}_I\mathbf{x}^*_0 \ , \quad {}_I\mathbf{d}_1 = \frac{1}{L}[{}_I\mathbf{x}^*_1 - {}_I\mathbf{x}^*_0] \ , \quad {}_I\mathbf{d}_2 = \frac{1}{L}[{}_I\mathbf{x}^*_2 - {}_I\mathbf{x}^*_0] \ .
\end{gathered} \tag{7.6.11}$$

Next, it is recalled that the conservation of mass (3.2.1) and the balance of linear momentum (3.2.2) can be rewritten in the forms

$$\frac{d}{dt}\int_{{}_IP^*} \rho^*\, dv^* = 0 \ ,$$

$$\frac{d}{dt}\int_{{}_I P^*} \rho^* \mathbf{v}^* dv^* = \int_{{}_I P^*} \rho^* \mathbf{b}^* dv^* + \sum_{J=1}^{5} \int_{{}_I\partial P_J^*} \mathbf{t}^* da^* \ , \qquad (7.6.12)$$

and the balance of angular momentum becomes

$$\frac{d}{dt}\int_{{}_I P^*} \mathbf{x}^* \times \rho^* \mathbf{v}^* dv^* = \int_{{}_I P^*} \mathbf{x}^* \times \rho^* \mathbf{b}^* dv^* + \sum_{J=1}^{5} \int_{{}_I\partial P_J^*} \mathbf{x}^* \times \mathbf{t}^* da^* \ . \ (7.6.13)$$

Furthermore, the averaged form (3.6.3) of the balance of linear momentum is rewritten as

$$\frac{d}{dt}\int_{{}_I P^*} \phi\, \rho^* \mathbf{v}^* dv^* = \int_{{}_I P^*} [\phi\, \rho^* \mathbf{b}^* - g^{-1/2}\, \mathbf{t}^{*j}\, \phi_{,j}\,]\, dv^* + \sum_{J=1}^{5} \int_{{}_I\partial P_J^*} \phi\, \mathbf{t}^* da^*, \quad (7.6.14)$$

where $\phi({}_I\theta^i)$ is a weighting function of the convected coordinates ${}_I\theta^i$ only.

In order to motivate the balance laws of the theory of a Cosserat point, it is convenient to introduce the following definitions

$$
\begin{gathered}
{}_I m = \int_{{}_I P^*} \rho^* dv^* \ , \ \ {}_I y^{00} = 1 \ , \ \ {}_I m\, {}_I y^{0j} = \int_{{}_I P^*} {}_I\theta^j \rho^* dv^* \ , \\
{}_I m\, {}_I y^{ij} = \int_{{}_I P^*} {}_I\theta^i\, {}_I\theta^j \rho^* dv^* \quad \text{for } i,j=1,2,3 \ , \\
{}_I y^{ij} = {}_I y^{ji} \quad \text{for } i,j=0,1,2,3 \ , \\
{}_I m\, {}_I \mathbf{b}_b^0 = \int_{{}_I P^*} \rho^* \mathbf{b}^* dv^* \ , \ \ {}_I \mathbf{m}_J^0 = \int_{{}_I\partial P_J^*} \mathbf{t}^* da^* \ , \\
{}_I m\, {}_I \mathbf{b}_b^i = \int_{{}_I P^*} {}_I\theta^i \rho^* \mathbf{b}^* dv^* \ , \ \ {}_I \mathbf{m}_J^i = \int_{{}_I\partial P_J^*} {}_I\theta^i\, \mathbf{t}^* da^* \ , \\
{}_I \mathbf{t}^0 = 0 \ , \ \ {}_I \mathbf{t}^i = \int_{{}_I P^*} g^{-1/2}\, \mathbf{t}^{*i} dv^* \ , \quad \text{for } i=1,2,3 \ . \qquad (7.6.15)
\end{gathered}
$$

It then follows using the kinematic assumption (7.6.11), that for the I'th Cosserat point, the conservation of mass, the balance of linear momentum, and the balances of director momentum become

$$
\begin{gathered}
{}_I \dot{m} = 0 \ , \ \ \frac{d}{dt}[\sum_{j=0}^{3} {}_I m\, {}_I y^{ij}\, {}_I \mathbf{w}_j\,] = {}_I m\, {}_I \mathbf{b}^i - {}_I \mathbf{t}^i \ , \\
\text{for } i=0,1,2,3 \ \text{ with } \ {}_I \mathbf{t}^0 = 0 \ , \qquad (7.6.16)
\end{gathered}
$$

and the balance of angular momentum becomes

$$\frac{d}{dt} \sum_{i=0}^{3} \sum_{j=0}^{3} [{}_I \mathbf{d}_i \times {}_I m\, {}_I y^{ij}\, {}_I \mathbf{w}_j] = \sum_{i=0}^{3} {}_I \mathbf{d}_i \times {}_I m\, {}_I \mathbf{b}^i \ . \qquad (7.6.17)$$

In these equations ${}_I m$ is the mass, ${}_I y^{ij}$ are the director inertia coefficients, ${}_I \mathbf{b}^i$ are the specific (per unit mass) external assigned director couples, and ${}_I \mathbf{t}^i$ are the intrinsic director couples. Moreover, the director inertia coefficients are constants

$${}_I \dot{y}^{ij} = 0 \ , \qquad (7.6.18)$$

and the assigned fields ${}_I \mathbf{b}^i$ can be expressed in the forms

$$ {}_{I}m\ {}_{I}\mathbf{b}^i = {}_{I}m\ {}_{I}\mathbf{b}^i_b + \sum_{J=1}^{5} {}_{I}\mathbf{m}^i_J \qquad \text{for } i=0,1,2,3 \ . \tag{7.6.19} $$

Here, the terms ${}_{I}\mathbf{b}^i_b$ are associated with the external body force; the terms ${}_{I}\mathbf{m}^0_J$ are the forces applied to the boundary surfaces ${}_{I}\partial P^*_J$; and ${}_{I}\mathbf{m}^i_J$ (i=1,2,3) are the director couples applied to the boundary surfaces ${}_{I}\partial P^*_J$.

The equations of motion (7.6.16) and (7.6.17) have been developed by integration of the equations of motion of a three-dimensional continuum. This approach is used to motivate the structure of the equations of motion of the Cosserat point. However, within the context of the direct approach, these same equations of motion are postulated as the balance laws for the theory of a Cosserat point. The remaining equations describing the theory of a Cosserat point are developed within the context of the direct approach.

Specifically, using the equations of motion (7.6.16) the balance of angular momentum (7.6.17) can be reduced to the form

$$ \sum_{i=1}^{3} {}_{I}\mathbf{d}_i \times {}_{I}\mathbf{t}^i = 0 \ . \tag{7.6.20} $$

Moreover, by introducing the definition

$$ {}_{I}\mathbf{T} = {}_{I}d^{-1/2} \sum_{i=1}^{3} {}_{I}\mathbf{t}^i \otimes {}_{I}\mathbf{d}_i \ , \tag{7.6.21} $$

it follows that the reduced form of the balance of angular momentum can be written as

$$ {}_{I}\mathbf{T}^T = {}_{I}\mathbf{T} \ , \tag{7.6.22} $$

which is similar to the expression (3.2.32) associated with the three-dimensional theory.

For an elastic Cosserat point the specific strain energy ${}_{I}\Sigma$ is a function of the deformation measure ${}_{I}\mathbf{C}$ only

$$ {}_{I}\Sigma = {}_{I}\hat{\Sigma}({}_{I}\mathbf{C}) \ . \tag{7.6.23} $$

Moreover, for the purely mechanical theory it is convenient to define: the rate of dissipation ${}_{I}\mathcal{D}$; the mechanical power ${}_{I}\mathcal{P}$; the rate of work ${}_{I}\mathcal{W}$ of the assigned fields ${}_{I}\mathbf{b}^i$; the kinetic energy ${}_{I}\mathcal{K}$; and the total internal energy ${}_{I}\mathcal{U}$ by the formulas

$$ {}_{I}d^{1/2}\ {}_{I}\mathcal{D} = {}_{I}\mathcal{W} - {}_{I}\dot{\mathcal{K}} - {}_{I}\dot{\mathcal{U}} \ , \quad {}_{I}d^{1/2}\ {}_{I}\mathcal{P} = {}_{I}\mathcal{W} - {}_{I}\dot{\mathcal{K}} \ , $$
$$ {}_{I}\mathcal{W} = \sum_{i=0}^{3} {}_{I}m\ {}_{I}\mathbf{b}^i \cdot {}_{I}\mathbf{w}_i \ , \quad {}_{I}\mathcal{K} = \sum_{i=0}^{3} \frac{1}{2}\ {}_{I}m\ {}_{I}y^{ij}\ {}_{I}\mathbf{w}_i \cdot {}_{I}\mathbf{w}_j \ , $$
$$ {}_{I}\mathcal{U} = {}_{I}m\ {}_{I}\Sigma \ . \tag{7.6.24} $$

Next, with the help of the definitions (7.6.6) it can be shown that

$$ \dot{\overline{{}_{I}\mathbf{F}^{-1}}} = -\ {}_{I}\mathbf{F}^{-1}\ {}_{I}\mathbf{L} \ , \quad {}_{I}\mathbf{w}_i = {}_{I}\mathbf{L}\ {}_{I}\mathbf{d}_i \quad \text{for } i=1,2,3 \ , \tag{7.6.25} $$

Thus, using the equations of motion (7.6.16), the definition (7.6.21), and the result (7.6.25), it follows that the rate of dissipation reduces to

$$ {}_{I}d^{1/2}\ {}_{I}\mathcal{D} = {}_{I}d^{1/2}\ {}_{I}\mathbf{T} \cdot {}_{I}\mathbf{D} - {}_{I}m\ {}_{I}\dot{\Sigma} \geq 0 \ , \tag{7.6.26} $$

which is required to be nonnegative.

For an elastic Cosserat point, assumptions similar to those described in section 6.8 are made. In particular, the dissipation ${}_I\mathcal{D}$ vanishes and ${}_I\mathbf{T}$ is independent of the rate ${}_I\mathbf{L}$

$$ {}_I\mathbf{T} = {}_I\hat{\mathbf{T}}({}_I\mathbf{d}_i) \quad \text{for } i=1,2,3 \ . \tag{7.6.27}$$

Consequently, it can be shown using (7.6.26) that

$$ {}_I\mathbf{T} = {}_I\hat{\mathbf{T}} = 2\ {}_I\rho\ {}_I\mathbf{F}\ \frac{\partial_I\hat{\Sigma}}{\partial_I\mathbf{C}}\ {}_I\mathbf{F}^T \ , \tag{7.6.28}$$

which automatically satisfies the balance of angular momentum (7.6.22). Also, in this equation, use has been made of the Lagrangian form of conservation of mass

$$ {}_Im = {}_I\rho\ {}_Id^{1/2} = {}_I\rho_0\ {}_ID^{1/2} \ , \tag{7.6.29}$$

where ${}_I\rho$ is the mass density in the present configuration and ${}_I\rho_0$ is its reference value. Once the constitutive equation for ${}_I\mathbf{T}$ is specified, the values of ${}_I\mathbf{t}^i$ (i=1,2,3) which appear in the equations of motion (7.6.16) and in the definition (7.6.21), can be determined by the expression

$$ {}_I\mathbf{t}^i = [{}_Id^{1/2}\ {}_I\mathbf{T}\ ]{}_I\mathbf{d}^i \quad \text{for } i=1,2,3 \ . \tag{7.6.30}$$

Now, a simple model for a Cosserat point constructed from a dissipative material can be developed by assuming that ${}_I\mathbf{T}$ and ${}_I\mathbf{t}^i$ separate additively into two parts

$$ {}_I\mathbf{T} = {}_I\hat{\mathbf{T}} + {}_I\check{\mathbf{T}} \ , \quad {}_I\mathbf{t}^i = {}_I\hat{\mathbf{t}}^i + {}_I\check{\mathbf{t}}^i \ , \quad {}_I\hat{\mathbf{t}}^0 = 0 \ , \quad {}_I\check{\mathbf{t}}^0 = 0 \ ,$$

$$ {}_I\hat{\mathbf{T}} = {}_Id^{-1/2} \sum_{i=1}^{3} {}_I\hat{\mathbf{t}}^i \otimes {}_I\mathbf{d}_i = {}_I\hat{\mathbf{T}}^T \ , \quad {}_I\check{\mathbf{T}} = {}_Id^{-1/2} \sum_{i=1}^{3} {}_I\check{\mathbf{t}}^i \otimes {}_I\mathbf{d}_i = {}_I\check{\mathbf{T}}^T \ , \tag{7.6.31}$$

with ${}_I\hat{\mathbf{T}}$ and ${}_I\hat{\mathbf{t}}^i$ being the parts associated with elastic deformation [which balance the rate of change in strain energy (7.6.23)]

$$ {}_Id^{1/2}\ {}_I\hat{\mathbf{T}} \bullet {}_I\mathbf{D} = {}_Im\ {}_I\dot{\Sigma} \ , \tag{7.6.32}$$

and ${}_I\check{\mathbf{T}}$ and ${}_I\check{\mathbf{t}}^i$ being the parts due to material dissipation. Thus, the restriction (7.6.26) reduces to

$$ {}_Id^{1/2}\ {}_I\mathcal{D} = {}_Id^{1/2}\ {}_I\check{\mathbf{T}} \bullet {}_I\mathbf{D} \geq 0 \ . \tag{7.6.33}$$

As a simple case it is possible to assume that ${}_I\check{\mathbf{T}}$ is a linear function of rate of the form

$$ {}_Id^{1/2}\ {}_I\check{\mathbf{T}} = {}_ID^{1/2}\ [{}_I\eta_1\ ({}_I\mathbf{D} \bullet \mathbf{I})\ \mathbf{I} + 2\ {}_I\eta_2\ {}_I\mathbf{D}'\ ] \ , \tag{7.6.34}$$

where ${}_I\eta_1$ and ${}_I\eta_2$ are material constants and ${}_I\mathbf{D}'$ is a pure measure of the rate of distortional deformation

$$ {}_I\mathbf{D}' = {}_I\mathbf{D} - \frac{1}{3}({}_I\mathbf{D} \bullet \mathbf{I})\ \mathbf{I} \ , \quad {}_I\mathbf{D}' \bullet \mathbf{I} = 0 \ . \tag{7.6.35}$$

Consequently, ${}_I\eta_1$ is the viscosity of dilatational deformation rate, and ${}_I\eta_2$ is the viscosity of distortional deformation rate, respectively. Also, it can be shown that the restriction (7.6.33) is satisfied for all motions provided that ${}_I\eta_1$ and ${}_I\eta_2$ are nonnegative

$$ {}_I\eta_1 \geq 0 \ , \quad {}_I\eta_2 \geq 0 \ . \tag{7.6.36}$$

Moreover, it is noted that the viscosity constants ${}_I\eta_1$ and ${}_I\eta_4$ can be determined by attempting to match the rate of damping associated with free vibrations of the Cosserat point.

The deformation field associated with the kinematic assumption (7.6.8) is homogeneous. Consequently, for three-dimensionally uniform homogeneous materials, it is expected that the constitutive equation for the strain energy of the Cosserat point can be restricted so that the theory of a Cosserat point will be consistent with exact three-dimensional solutions for all homogeneous deformations. This restriction requires that the strain energy function ${}_I\hat{\Sigma}({}_I\mathbf{C})$ in (7.6.23) have the same form as the three-dimensional strain energy function $\hat{\Sigma}^*(\mathbf{C}^*)$ so that

$$ {}_I\hat{\Sigma}({}_I\mathbf{C}) = \hat{\Sigma}^*({}_I\mathbf{C}) \ . \tag{7.6.37}$$

In order to determine the appropriate forms of the forces that one element applies to its nearest neighbors, it is convenient to record an explicit expression for the rate of work ${}_I\mathcal{W}_c$ supplied to the I'th Cosserat point through its boundary surfaces ${}_I\partial P_J^*$

$$ {}_I\mathcal{W}_c = \sum_{i=0}^{3} \sum_{J=1}^{5} {}_I\mathbf{m}_J^i \bullet {}_I\mathbf{w}_i \ . \tag{7.6.38}$$

Then, using the results (7.6.11), it can be shown that

$$ {}_I\mathcal{W}_c = \sum_{i=0}^{2} {}_I\mathbf{m}^i \bullet {}_I\dot{\mathbf{x}}_i^* + {}_I\mathbf{m}^3 \bullet {}_I\mathbf{w}_3, \tag{7.6.39}$$

where the nodal forces ${}_I\mathbf{m}^i$ (i=0,1,2,3) if the I'th element are defined by

$$ {}_I\mathbf{m}^i = {}_I\bar{\mathbf{m}}^i + {}_I\hat{\mathbf{m}}^i \quad \text{for i=0,1,2,3}$$

$$ {}_I\bar{\mathbf{m}}^0 = \sum_{J=1}^{3} [{}_I\mathbf{m}_J^0 - \frac{1}{L}\{{}_I\mathbf{m}_J^1 + {}_I\mathbf{m}_J^2\}] \ , \ {}_I\hat{\mathbf{m}}^0 = \sum_{J=4}^{5} [{}_I\mathbf{m}_J^0 - \frac{1}{L}\{{}_I\mathbf{m}_J^1 + {}_I\mathbf{m}_J^2\}] \ ,$$

$$ {}_I\bar{\mathbf{m}}^i = \frac{1}{L}\sum_{J=1}^{3} {}_I\mathbf{m}_J^i \ , \quad {}_I\hat{\mathbf{m}}^i = \frac{1}{L}\sum_{J=4}^{5} {}_I\mathbf{m}_J^i \quad \text{for i=1,2} \ ,$$

$$ {}_I\bar{\mathbf{m}}^3 = \sum_{J=1}^{3} {}_I\mathbf{m}_J^3 \ , \quad {}_I\hat{\mathbf{m}}^3 = \sum_{J=4}^{5} {}_I\mathbf{m}_J^3 \ . \tag{7.6.40}$$

The forces ${}_I\bar{\mathbf{m}}^i$ represent the contributions of the external forces applied to the I'th element through the lateral boundaries ${}_I\partial P_J^*$ (J=1,2,3) and the forces ${}_I\hat{\mathbf{m}}^i$ represent the contributions of the external forces applied through the two ends ${}_I\partial P_4^*$ and ${}_I\partial P_5^*$. Furthermore, for two-dimensional problems it is assumed that the tractions on the lateral surfaces and the mass are symmetrically distributed about the middle plane ${}_I\theta^3=0$ so that

$$ {}_I\bar{\mathbf{m}}^3 = 0 \ , \quad {}_Iy^{03} = {}_Iy^{13} = {}_Iy^{23} = 0 \ . \tag{7.6.41}$$

Next, using the expressions (7.6.11), (7.6.19), (7.6.40) and (7.6.41), the equations of motion (7.6.16) can be solved to deduce that

$$ {}_I\bar{\mathbf{m}}^0 = - \ {}_I\hat{\mathbf{m}}^0 - {}_Im\,{}_I\mathbf{b}_b^0 + \frac{1}{L}[\,{}_Im\,{}_I\mathbf{b}_b^1 - {}_I\mathbf{t}^1] + \frac{1}{L}[{}_Im\,{}_I\mathbf{b}_b^2 - {}_I\mathbf{t}^2]$$

$$ + {}_Im\,[1 - \frac{1}{L}\{2{}_Iy^{01} + 2{}_Iy^{02}\} + \frac{1}{L^2}\{{}_Iy^{11} + 2{}_Iy^{12} + {}_Iy^{22}\}]\,\ddot{\mathbf{x}}_0^*$$

$$ + {}_Im\,[\frac{1}{L}{}_Iy^{01} - \frac{1}{L^2}\{{}_Iy^{11} + {}_Iy^{12}\}]\,\ddot{\mathbf{x}}_1^* + {}_Im\,[\frac{1}{L}{}_Iy^{02} - \frac{1}{L^2}\{{}_Iy^{12} + {}_Iy^{22}\}]\,\ddot{\mathbf{x}}_2^* \ ,$$

$$
{}_I\bar{\mathbf{m}}^1 = -\,{}_I\hat{\mathbf{m}}^1 - \frac{1}{L}\left[\,{}_I m\,{}_I\mathbf{b}_b^1 - {}_I\mathbf{t}^1\right] + {}_I m\left[\frac{1}{L}\,{}_I y^{01} - \frac{1}{L^2}\{{}_I y^{11} + {}_I y^{12}\}\right]\ddot{\mathbf{x}}_0^*
$$
$$
+ {}_I m\left[\frac{1}{L^2}\,{}_I y^{11}\right]\ddot{\mathbf{x}}_1^* + {}_I m\left[\frac{1}{L^2}\,{}_I y^{12}\right]\ddot{\mathbf{x}}_2^*
$$
$$
{}_I\bar{\mathbf{m}}^2 = -\,{}_I\hat{\mathbf{m}}^2 - \frac{1}{L}\left[{}_I m\,{}_I\mathbf{b}_b^2 - {}_I\mathbf{t}^2\right] + {}_I m\left[\frac{1}{L}\,{}_I y^{02} - \frac{1}{L^2}\{{}_I y^{12} + {}_I y^{22}\}\right]\ddot{\mathbf{x}}_0^*
$$
$$
+ {}_I m\left[\frac{1}{L^2}\,{}_I y^{12}\right]\ddot{\mathbf{x}}_1^* + {}_I m\left[\frac{1}{L^2}\,{}_I y^{22}\right]\ddot{\mathbf{x}}_2^*
$$
$$
{}_I\hat{\mathbf{m}}^3 = -\left[{}_I m\,{}_I\mathbf{b}_b^3 - {}_I\mathbf{t}^3\right] + {}_I m\left[{}_I y^{33}\,\ddot{\mathbf{d}}_3\right]\ . \tag{7.6.42}
$$

In view of the results (7.6.11), it follows that the deformation of the I'th Cosserat point is completely determined by the four vectors ${}_I\mathbf{x}_i^*(t)$ (i=0,1,2) and $\mathbf{d}_3(t)$. Moreover, once the inertia quantities

$$
{}_I m\ ,\ {}_I y^{ij} \quad \text{for i,j=0,1,2,3}\ , \tag{7.6.43}
$$

the assigned fields

$$
{}_I\mathbf{b}_b^i \quad \text{for i=0,1,2,3}\ ,\quad {}_I\hat{\mathbf{m}}^i \quad \text{for i=0,1,2} \tag{7.6.44}
$$

the strain energy function (7.6.37), and the viscosity coefficients ${}_I\eta_1$ and ${}_I\eta_2$ in (7.6.34) have been specified, the constitutive equations for the intrinsic director couples ${}_I\mathbf{t}^i$ in (7.6.30) are determined. Then, the nodal forces ${}_I\bar{\mathbf{m}}^i$ (i=0,1,2) and the forces ${}_I\hat{\mathbf{m}}^3$ are also determined by the equations (7.6.42).

Since the body R^* has been divided into M Cosserat points, there are 3×M nodal vectors ${}_I\mathbf{x}_i^*$ and M director vectors ${}_I\mathbf{d}_3$ associated with these Cosserat points. However, since there are only N nodes associated with these M Cosserat points, the deformation of the body should be totally determined by the N vectors $\mathbf{x}_K^*(t)$ (K=1,2,...,N), which characterize the motion of these nodes, and the M director vectors ${}_I\mathbf{d}_3$. Consequently, it is necessary to specify kinematic coupling conditions that are determined by the topology of the specific discretization of the region R^*. These kinematic coupling conditions can be summarized as

$$
\{{}_I\mathbf{x}_0^* \text{ or } {}_I\mathbf{x}_1^* \text{ or } {}_I\mathbf{x}_2^*\} = \mathbf{x}_K^* \quad \text{for K=1,2,...,N}\ . \tag{7.6.45}
$$

Next, summing the contributions to the rate of work (7.6.39) of all M elements, it is convenient to define the external nodal force $\mathbf{m}_K^*$ which is applied to the K'th node by the formula

$$
\mathbf{m}_K^* = \sum_{(I,i:K)} {}_I\bar{\mathbf{m}}^i\ . \tag{7.6.46}
$$

In this equation, the special summation symbol indicates that the summation is performed over all forces i (i=0,1,2), and all elements I (I=1,2,...,M) which have nodes that coincident with the K'th node of the region R^*. If the K'th node is an interior node of the region R^*, then $\mathbf{m}_K^*$ vanishes

$$
\mathbf{m}_K^* = 0 \quad \text{for interior nodes of } R^*\ , \tag{7.6.47}
$$

because the traction vector that one Cosserat point applies on its neighbor through their common boundary surface is equal and opposite to the traction vector that the same neighbor applies to the same Cosserat point. On the other hand, if the K'th node is an exterior node lying on the boundary ∂R^*, then $\mathbf{m}_K^*$ is either specified by boundary conditions associated with surface tractions, or it is determined by the solution as the reaction to specified kinematic conditions on the boundary. In addition to the N position vectors $\mathbf{x}_K^*(t)$, it is necessary to determine the M directors ${}_I\mathbf{d}_3$ associated with the I'th Cosserat point. These latter quantities are determined by the M equations $(7.6.42)_3$.

More specifically, the boundary conditions associated with ∂R^* can be kinematic-type conditions for which

$$\mathbf{x}_K^* = \text{specified} \ , \ \ \mathbf{m}_K^* = \text{determined by solution}; \tag{7.6.48}$$

they can be kinetic-type conditions for which

$$\mathbf{m}_K^* = \text{specified} \ , \ \ \mathbf{x}_K^* = \text{determined by solution}; \tag{7.6.49}$$

or they can be mixed-type or mixed-mixed-type conditions where different components of $\mathbf{x}_K^*$ and $\mathbf{m}_K^*$ are specified

$$\mathbf{x}_K^* \bullet \mathbf{b}_1 = \text{specified and } \ \mathbf{m}_K^* \bullet \mathbf{b}_2 = \text{determined by solution} \ ,$$
$$\text{or}$$
$$\mathbf{x}_K^* \bullet \mathbf{b}_2 = \text{determined by solution and } \ \mathbf{m}_K^* \bullet \mathbf{b}_1 = \text{specified} \, . \tag{7.6.50}$$

In these conditions $\mathbf{b}_\alpha$ (α=1,2) are two orthogonal vectors which are parallel to the planar ends ∂R_4^* and ∂R_5^*. Also, the boundary conditions associated with the direction normal to the ends ∂R_4^* and ∂R_5^* can be kinematic-type conditions for which

$${}_I\mathbf{d}_3 = \text{specified and } {}_I\hat{\mathbf{m}}^3 = \text{determined by solution} \ , \tag{7.6.51}$$

or they can be kinetic-type conditions for which

$${}_I\mathbf{d}_3 = \text{determined by solution and } {}_I\hat{\mathbf{m}}^3 = \text{ specified} \ . \tag{7.6.52}$$

Obviously, there are many combinations of these conditions which need not be discussed in detail.

Whenever the K'th node is an exterior node, the equations (7.6.9), (7.6.15) and (7.6.40) are used to rewrite $\mathbf{m}_K^*$ in terms of integrals of the traction vector $\mathbf{t}^*$ over the exterior boundaries associated with node K. For example, let the node ${}_5\mathbf{x}_1^*$ associated with the 5'th Cosserat point and the node ${}_9\mathbf{x}_0^*$ associated with the 9'th Cosserat point, coincide with the exterior node K, and let the boundaries ${}_5\partial P_2^*$ and ${}_9\partial P_1^*$ be the only exterior boundaries associated with this exterior node. Then, $\mathbf{m}_K^*$ in (7.6.46) is specified by

$$\begin{aligned}\mathbf{m}_K^* &= {}_5\bar{\mathbf{m}}^1 + {}_9\bar{\mathbf{m}}^0 = \frac{1}{L}\,{}_5\mathbf{m}_2^1 + \left[{}_9\mathbf{m}_1^0 - \frac{1}{L}\left\{{}_9\mathbf{m}_1^1 + {}_9\mathbf{m}_1^2\right\}\right] \\ &= \int_{{}_5\partial P_2^*} \frac{1}{L}\left\{{}_5\theta^1\right\} \mathbf{t}^*\, da^* + \int_{{}_9\partial P_1^*} \left[1 - \frac{1}{L}\left\{ {}_9\theta^2\right\}\right] \mathbf{t}^*\, da^* \ .\end{aligned} \tag{7.6.53}$$

In general, the equations (7.6.47) associated with interior nodes, and the equations (7.6.48) or (7.6.49) associated with boundary nodes, yield N vector equations to determine the N vectors (either $\mathbf{x}_K^*$ or $\mathbf{m}_K^*$), that contain second order time derivatives of $\mathbf{x}_K^*$ (K=1,2,...,N). Consequently, in addition to the boundary conditions described above, it is necessary to specify initial values for

$$\{\mathbf{x}_K^* \,,\, \dot{\mathbf{x}}_K^*\} \ . \tag{7.6.54}$$

Furthermore, the equations $(7.6.42)_3$ yield M vector equations to determine the M directors ${}_I\mathbf{d}_3$, that contain second order time derivatives of ${}_I\mathbf{d}_3$. Thus, if the forces ${}_I\hat{\mathbf{m}}^3$ are specified and $(7.6.42)_3$ are used to determine ${}_I\mathbf{d}_3$, then it is also necessary to specify initial values of

$$\{{}_I\mathbf{d}_3 \,,\, {}_I\dot{\mathbf{d}}_3\} \ . \tag{7.6.55}$$

Once the values of $\mathbf{x}_K^*$ are determined, the nodal forces ${}_I\bar{\mathbf{m}}^i$ (I=0,1,2) of the I'th Cosserat point are determined by the equations (7.6.42), and the nodal forces $\hat{\mathbf{m}}_K^*$ are determined by the equations (7.6.46). Also, if ${}_I\mathbf{d}_3$ are specified, then the forces ${}_I\hat{\mathbf{m}}^3$ are determined by $(7.6.42)_3$.

APPENDIX A

TENSORS, TENSOR PRODUCTS AND TENSOR OPERATIONS IN THREE-DIMENSIONS

A.1 Vectors and vector operations

The purpose of this section is to summarize some basic properties of vectors and vector operations. Complete descriptions of linear vector spaces can be found in standard texts on linear algebra (Noble, 1969) and a more complete summary of the use of vectors in continuum mechanics can be found in (Sokolnikoff, 1964). For the present purpose it is sufficient to recall that a vector in three-dimensional space is usually identified with an arrow connecting two points in space. This arrow has a magnitude and a specific direction.

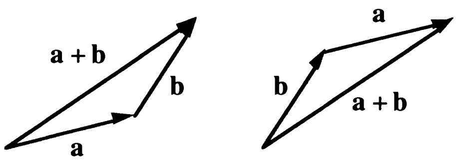

Fig. A.1.1 Parallelogram rule of vector addition.

Some basic vector operations can be summarized as follows. If $\mathbf{a},\mathbf{b}$ are two vectors, then the quantity

$$\mathbf{c} = \mathbf{a} + \mathbf{b} \ , \tag{A.1.1}$$

is also a vector which is defined by the parallelogram law of addition (see Fig. A.1.1). Furthermore, the operations

$$\begin{aligned}
\mathbf{a} + \mathbf{b} &= \mathbf{b} + \mathbf{a} && \text{(commutative law)} \ , \\
(\mathbf{a} + \mathbf{b}) + \mathbf{c} &= \mathbf{a} + (\mathbf{b} + \mathbf{c}) && \text{(associative law)} \ , \\
\alpha\,\mathbf{a} &= \mathbf{a}\,\alpha && \text{(multiplication by a real number)} \ , \\
\mathbf{a} \bullet \mathbf{b} &= \mathbf{b} \bullet \mathbf{a} && \text{(commutative law)} \ , \\
\mathbf{a} \bullet (\mathbf{b} + \mathbf{c}) &= \mathbf{a} \bullet \mathbf{b} + \mathbf{a} \bullet \mathbf{c} && \text{(distributive law)} \ , \\
\alpha\,(\mathbf{a} \bullet \mathbf{b}) &= (\alpha\,\mathbf{a}) \bullet \mathbf{b} && \text{(associative law)} \ , \\
\mathbf{a} \times \mathbf{b} &= -\,\mathbf{b} \times \mathbf{a} && \text{(lack of commutativity)} \ , \\
\mathbf{a} \times (\mathbf{b} + \mathbf{c}) &= \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c} && \text{(distributive law)} \ , \\
\alpha\,(\mathbf{a} \times \mathbf{b}) &= (\alpha\,\mathbf{a}) \times \mathbf{b} && \text{(associative law)} \ ,
\end{aligned} \tag{A.1.2}$$

are satisfied for all vectors $\mathbf{a},\mathbf{b},\mathbf{c}$ and all real numbers α, where $\mathbf{a} \bullet \mathbf{b}$ denotes the dot product (or scalar product), and $\mathbf{a} \times \mathbf{b}$ denotes the cross product (or vector product) between the vectors $\mathbf{a}$ and $\mathbf{b}$.

A.2 Tensors as linear operators

Scalars (or real numbers) are referred to as tensors of order zero, and vectors are referred to as tensors of order one. Here, higher order tensors are defined inductively starting with the notion of a vector.

Tensor of Order M: The quantity $\mathbf{T}$ is called a tensor of order two (or a second order tensor) if it is a linear operator whose domain is the space of all vectors $\mathbf{v}$ and whose range $\mathbf{Tv}$ or $\mathbf{vT}$ is a vector. Similarly, the quantity $\mathbf{T}$ is called a tensor of order three if it is a linear operator whose domain is the space of all vectors $\mathbf{v}$ and whose range $\mathbf{Tv}$ or $\mathbf{vT}$ is a tensor of order two. Consequently, by induction, the quantity $\mathbf{T}$ is called a tensor of order M (M≥1) if it is a linear operator whose domain is the space of all vectors $\mathbf{v}$ and whose range $\mathbf{Tv}$ or $\mathbf{vT}$ is a tensor of order (M–1). Since $\mathbf{T}$ is a linear operator, it satisfies the following rules

$$\mathbf{T}(\mathbf{v}+\mathbf{w}) = \mathbf{Tv}+\mathbf{Tw} \ , \quad \alpha(\mathbf{Tv}) = (\alpha\mathbf{T})\mathbf{v} = \mathbf{T}(\alpha\mathbf{v}) \ ,$$
$$(\mathbf{v}+\mathbf{w})\mathbf{T} = \mathbf{vT}+\mathbf{wT} \ , \quad \alpha(\mathbf{vT}) = (\alpha\mathbf{v})\mathbf{T} = (\mathbf{vT})\alpha \ , \tag{A.2.1}$$

where $\mathbf{v},\mathbf{w}$ are arbitrary vectors and α is an arbitrary scalar. Notice that the tensor $\mathbf{T}$ can be operated upon on its right [e.g. $(A.2.1)_{1,2}$] or on its left [e.g. $(A.2.1)_{3,4}$] and that in general, operation on the right and on the left is not commutative

$$\mathbf{Tv} \neq \mathbf{vT} \quad \text{(Lack of commutativity in general)} \ . \tag{A.2.2}$$

Zero Tensor of Order M: The zero tensor of order M is denoted by $\mathbf{0}(M)$ and is a linear operator whose domain is the space of all vectors $\mathbf{v}$ and whose range $\mathbf{0}(M-1)$ is the zero tensor of order M–1.

$$\mathbf{0}(M)\ \mathbf{v} = \mathbf{v}\ \mathbf{0}(M) = \mathbf{0}(M-1) \ . \tag{A.2.3}$$

Notice that these tensors are defined inductively starting with the known properties of the real number 0, which is the zero tensor $\mathbf{0}(0)$ of order 0. Often, for simplicity in writing a tensor equation, the zero tensor of any order is denoted by the symbol 0.

Addition and Subtraction: The usual rules of addition and subtraction of two tensors $\mathbf{A}$ and $\mathbf{B}$ apply *only* when the two tensors have the *same* order. It should be emphasized that tensors of different orders cannot be added or subtracted.

A.3 Tensor products (special case)

In order to define the operations of tensor product, dot product, and juxtaposition for general tensors, it is convenient to first consider the definitions of these properties for special tensors. Also, the operations of left and right transpose of the tensor product of a string of vectors will be defined. It will be seen later that the operation of dot product is defined to be consistent with the usual notion of the dot product as an inner product because it is a positive definite operation. Consequently, the dot product of a tensor with itself yields the square of the magnitude of the tensor. Also, the operation of juxtaposition is defined to be consistent with the usual procedures for matrix multiplication when two second order tensors are juxtaposed.

Tensor Product (Special Case): The tensor product operation is denoted by the symbol $\otimes$ and it is defined so that for an arbitrary vector $\mathbf{v}$, the quantity $(\mathbf{a}_1 \otimes \mathbf{a}_2)$ is a second order tensor satisfying the relations

$$(\mathbf{a}_1 \otimes \mathbf{a}_2)\, \mathbf{v} = \mathbf{a}_1\, (\mathbf{a}_2 \bullet \mathbf{v}) \ , \quad \mathbf{v}\, (\mathbf{a}_1 \otimes \mathbf{a}_2) = (\mathbf{v} \bullet \mathbf{a}_1)\, \mathbf{a}_2 \ . \tag{A.3.1}$$

Similarly, the quantity $(\mathbf{a}_1 \otimes \mathbf{a}_2 \otimes \mathbf{a}_3)$ is a third order tensor satisfying the relations

$$(\mathbf{a}_1 \otimes \mathbf{a}_2 \otimes \mathbf{a}_3)\, \mathbf{v} = (\mathbf{a}_1 \otimes \mathbf{a}_2)\, (\mathbf{a}_3 \bullet \mathbf{v}) \ , \ \mathbf{v}\, (\mathbf{a}_1 \otimes \mathbf{a}_2 \otimes \mathbf{a}_3) = (\mathbf{v} \bullet \mathbf{a}_1)\, (\mathbf{a}_2 \otimes \mathbf{a}_3) \ . \tag{A.3.2}$$

For convenience in generalizing these ideas, let $\mathbf{A}$ be a special tensor of order M which is formed by the tensor product of a string of M (M≥2) vectors $(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \ldots, \mathbf{a}_M)$, and let $\mathbf{B}$ be a special tensor of order N which is formed by the tensor product of a string of N (N≥2) vectors $(\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \ldots, \mathbf{b}_N)$ so that

$$\mathbf{A} = (\mathbf{a}_1 \otimes \mathbf{a}_2 \otimes \mathbf{a}_3 \otimes \ldots \otimes \mathbf{a}_M) \ , \quad \mathbf{B} = (\mathbf{b}_1 \otimes \mathbf{b}_2 \otimes \mathbf{b}_3 \otimes \ldots \otimes \mathbf{b}_N) \ , \tag{A.3.3}$$

and for definiteness take

$$M \leq N \ . \tag{A.3.4}$$

It then follows that $\mathbf{A}$ satisfies the relations

$$\mathbf{A}\mathbf{v} = (\mathbf{a}_1 \otimes \mathbf{a}_2 \otimes \ldots \otimes \mathbf{a}_M)\, \mathbf{v} = (\mathbf{a}_1 \otimes \mathbf{a}_2 \otimes \ldots \otimes \mathbf{a}_{M-1})\, (\mathbf{a}_M \bullet \mathbf{v}) \ ,$$
$$\mathbf{v}\mathbf{A} = \mathbf{v}\, (\mathbf{a}_1 \otimes \mathbf{a}_2 \otimes \ldots \otimes \mathbf{a}_M) = (\mathbf{v} \bullet \mathbf{a}_1)\, (\mathbf{a}_2 \otimes \mathbf{a}_3 \otimes \ldots \otimes \mathbf{a}_M) \ . \tag{A.3.5}$$

Notice that when $\mathbf{v}$ operates on either the right or left side of $\mathbf{A}$, it has the effect of forming the scalar product with the vector in the string closest to it, and it causes the result $\mathbf{A}\mathbf{v}$ or $\mathbf{v}\mathbf{A}$ to have order one less than the order of $\mathbf{A}$ since one of the tensor products is removed. The remaining vectors in the string of tensor products are unaltered.

Dot Product (Special Case): The dot product operation between two vectors can be generalized to an operation between two tensors of any orders. For example, the dot product between two second order tensors can be written as

$$(\mathbf{a}_1 \otimes \mathbf{a}_2) \bullet (\mathbf{b}_1 \otimes \mathbf{b}_2) = (\mathbf{a}_1 \bullet \mathbf{b}_1)\, (\mathbf{a}_2 \bullet \mathbf{b}_2) \ ,$$
$$(\mathbf{b}_1 \otimes \mathbf{b}_2) \bullet (\mathbf{a}_1 \otimes \mathbf{a}_2) = (\mathbf{b}_1 \bullet \mathbf{a}_1)\, (\mathbf{b}_2 \bullet \mathbf{a}_2) \ , \tag{A.3.6}$$

and the dot product between a second order tensor and a third order tensor can be written as

$$(\mathbf{a}_1 \otimes \mathbf{a}_2) \bullet (\mathbf{b}_1 \otimes \mathbf{b}_2 \otimes \mathbf{b}_3) = (\mathbf{a}_1 \bullet \mathbf{b}_1)\, (\mathbf{a}_2 \bullet \mathbf{b}_2)\, \mathbf{b}_3 \ ,$$
$$(\mathbf{b}_1 \otimes \mathbf{b}_2 \otimes \mathbf{b}_3) \bullet (\mathbf{a}_1 \otimes \mathbf{a}_2) = \mathbf{b}_1\, (\mathbf{b}_2 \bullet \mathbf{a}_1)\, (\mathbf{b}_3 \bullet \mathbf{a}_2) \ . \tag{A.3.7}$$

Also, the dot product between two third order tensors can be written as

$$(\mathbf{a}_1 \otimes \mathbf{a}_2 \otimes \mathbf{a}_3) \bullet (\mathbf{b}_1 \otimes \mathbf{b}_2 \otimes \mathbf{b}_3) = (\mathbf{a}_1 \bullet \mathbf{b}_1)\, (\mathbf{a}_2 \bullet \mathbf{b}_2)\, (\mathbf{a}_3 \bullet \mathbf{b}_3) \ ,$$
$$(\mathbf{b}_1 \otimes \mathbf{b}_2 \otimes \mathbf{b}_3) \bullet (\mathbf{a}_1 \otimes \mathbf{a}_2 \otimes \mathbf{a}_3) = (\mathbf{b}_1 \bullet \mathbf{a}_1)\, (\mathbf{b}_2 \bullet \mathbf{a}_2)\, (\mathbf{b}_3 \bullet \mathbf{a}_3) \ , \tag{A.3.8}$$

and the dot product between a second order tensor and a fourth order tensor can be written as

$$(\mathbf{a}_1 \otimes \mathbf{a}_2) \bullet (\mathbf{b}_1 \otimes \mathbf{b}_2 \otimes \mathbf{b}_3 \otimes \mathbf{b}_4) = (\mathbf{a}_1 \bullet \mathbf{b}_1)\, (\mathbf{a}_2 \bullet \mathbf{b}_2)\, (\mathbf{b}_3 \otimes \mathbf{b}_4) \ ,$$
$$(\mathbf{b}_1 \otimes \mathbf{b}_2 \otimes \mathbf{b}_3 \otimes \mathbf{b}_4) \bullet (\mathbf{a}_1 \otimes \mathbf{a}_2) = (\mathbf{b}_1 \otimes \mathbf{b}_2)\, (\mathbf{b}_3 \bullet \mathbf{a}_1)\, (\mathbf{b}_4 \bullet \mathbf{a}_2) \ . \tag{A.3.9}$$

This dot product operation can be generalized for special tensors of any orders like $\mathbf{A}$ and $\mathbf{B}$ by carefully examining examples (A.3.7) and (A.3.9). These examples indicate that the dot product between tensors of different orders does not necessarily commute, whereas the dot product of two tensors of the same order does commute.

$$\mathbf{A} \bullet \mathbf{B} \neq \mathbf{B} \bullet \mathbf{A} \quad \text{for M<N} \quad , \quad \mathbf{A} \bullet \mathbf{B} = \mathbf{B} \bullet \mathbf{A} \quad \text{for M=N} \quad . \tag{A.3.10}$$

Moreover, it can be seen that the tensor of smallest order controls the outcome of the dot product operation. Specifically, in $(A.3.7)_1$ the second order tensor $(\mathbf{a}_1 \otimes \mathbf{a}_2)$ appears on the left of the third order tensor $(\mathbf{b}_1 \otimes \mathbf{b}_2 \otimes \mathbf{b}_3)$. Since $(\mathbf{a}_1 \otimes \mathbf{a}_2)$ is a second order tensor this causes only the first two vectors on the left of $(\mathbf{b}_1 \otimes \mathbf{b}_2 \otimes \mathbf{b}_3)$ to form inner products with $\mathbf{a}_1$ and $\mathbf{a}_2$. Similarly, since $(\mathbf{a}_1 \otimes \mathbf{a}_2)$ appears on the right side of $(\mathbf{b}_1 \otimes \mathbf{b}_2 \otimes \mathbf{b}_3)$ in $(A.3.7)_2$, this causes only the first two vectors on the right of $(\mathbf{b}_1 \otimes \mathbf{b}_2 \otimes \mathbf{b}_3)$ to form inner products with $\mathbf{a}_1$ and $\mathbf{a}_2$.

In general, the dot product $\mathbf{A} \bullet \mathbf{B}$ of the special tensors defined in (A.3.3) is a tensor of order $|M-N|$. Furthermore, in view of the restriction (A.3.4), only the first M vectors of $\mathbf{B}$ closest to $\mathbf{A}$ will form inner products with the vectors of $\mathbf{A}$. It is also important to note that the order of the strings of vectors in the inner products remains the same as that in the tensors. Specifically, for the dot product $\mathbf{A} \bullet \mathbf{B}$ the first vector on the left side of $\mathbf{A}$ forms an inner product with the first vector on the left side of $\mathbf{B}$ and so forth. Whereas, for the dot product $\mathbf{B} \bullet \mathbf{A}$ the last vector on the right side of $\mathbf{A}$ forms an inner product with the last vector on the right side of $\mathbf{B}$.

Cross Product (Special Case): The cross product operation between two vectors can be generalized to an operation between two tensors of any orders. For example, the cross product between two second order tensors can be written as

$$(\mathbf{a}_1 \otimes \mathbf{a}_2) \times (\mathbf{b}_1 \otimes \mathbf{b}_2) = (\mathbf{a}_1 \times \mathbf{b}_1) \otimes (\mathbf{a}_2 \times \mathbf{b}_2) \quad ,$$

$$(\mathbf{b}_1 \otimes \mathbf{b}_2) \times (\mathbf{a}_1 \otimes \mathbf{a}_2) = (\mathbf{b}_1 \times \mathbf{a}_1) \otimes (\mathbf{b}_2 \times \mathbf{a}_2) \ , \tag{A.3.11}$$

and the cross product between a second order tensor and a third order tensor can be written as

$$(\mathbf{a}_1 \otimes \mathbf{a}_2) \times (\mathbf{b}_1 \otimes \mathbf{b}_2 \otimes \mathbf{b}_3) = (\mathbf{a}_1 \times \mathbf{b}_1) \otimes (\mathbf{a}_2 \times \mathbf{b}_2) \otimes \mathbf{b}_3 \ ,$$

$$(\mathbf{b}_1 \otimes \mathbf{b}_2 \otimes \mathbf{b}_3) \times (\mathbf{a}_1 \otimes \mathbf{a}_2) = \mathbf{b}_1 \otimes (\mathbf{b}_2 \times \mathbf{a}_1) \otimes (\mathbf{b}_3 \times \mathbf{a}_2) \ . \tag{A.3.12}$$

Also, the cross product between two third order tensors can be written as

$$(\mathbf{a}_1 \otimes \mathbf{a}_2 \otimes \mathbf{a}_3) \times (\mathbf{b}_1 \otimes \mathbf{b}_2 \otimes \mathbf{b}_3) = (\mathbf{a}_1 \times \mathbf{b}_1) \otimes (\mathbf{a}_2 \times \mathbf{b}_2) \otimes (\mathbf{a}_3 \times \mathbf{b}_3) \ ,$$

$$(\mathbf{b}_1 \otimes \mathbf{b}_2 \otimes \mathbf{b}_3) \times (\mathbf{a}_1 \otimes \mathbf{a}_2 \otimes \mathbf{a}_3) = (\mathbf{b}_1 \times \mathbf{a}_1) \otimes (\mathbf{b}_2 \times \mathbf{a}_2) \otimes (\mathbf{b}_3 \times \mathbf{a}_3) \ , \tag{A.3.13}$$

and the cross product between a second order tensor and a fourth order tensor can be written as

$$(\mathbf{a}_1 \otimes \mathbf{a}_2) \times (\mathbf{b}_1 \otimes \mathbf{b}_2 \otimes \mathbf{b}_3 \otimes \mathbf{b}_4) = (\mathbf{a}_1 \times \mathbf{b}_1) \otimes (\mathbf{a}_2 \times \mathbf{b}_2) \otimes (\mathbf{b}_3 \otimes \mathbf{b}_4) \quad ,$$

$$(\mathbf{b}_1 \otimes \mathbf{b}_2 \otimes \mathbf{b}_3 \otimes \mathbf{b}_4) \times (\mathbf{a}_1 \otimes \mathbf{a}_2) = (\mathbf{b}_1 \otimes \mathbf{b}_2) \otimes (\mathbf{b}_3 \times \mathbf{a}_1) \otimes (\mathbf{b}_4 \times \mathbf{a}_2) \ . \tag{A.3.14}$$

This cross product operation can be generalized for special tensors of any orders like $\mathbf{A}$ and $\mathbf{B}$ by carefully examining examples (A.3.12) and (A.3.14). These examples indicate that the cross product between tensors of different orders does not necessarily commute, whereas the cross product of two tensors of the same order will commute if the order is even, and it will be changed in sign if the order is odd.

$$\mathbf{A} \times \mathbf{B} \neq \mathbf{B} \times \mathbf{A} \qquad \text{for M<N} \ ,$$

$$\mathbf{A} \times \mathbf{B} = \mathbf{B} \times \mathbf{A} \quad \text{for M=N (even order)} \ ,$$
$$\mathbf{A} \times \mathbf{B} = -\,\mathbf{B} \times \mathbf{A} \quad \text{for M=N (odd order)} \ . \tag{A.3.15}$$

Moreover, it can be seen that the tensor of smallest order controls the outcome of the result of the cross product operation. Also, since the cross product of two vectors is a vector, it is necessary to retain the tensor product operators in the examples (A.3.11)-(A.3.14).

In general, the cross product $\mathbf{A} \times \mathbf{B}$ of the special tensors defined in (A.3.3) is a tensor of order N which is equal to the order of the highest order tensor $\mathbf{B}$. Furthermore, in view of the restriction (A.3.4), only the first M vectors of $\mathbf{B}$ closest to $\mathbf{A}$ will form cross products with the vectors of $\mathbf{A}$. It is also important to note that the order of the strings of vectors in these cross products remains the same as that in the tensors. Specifically, for the cross product $\mathbf{A} \times \mathbf{B}$, the first vector on the left side of $\mathbf{A}$ forms a cross product with the first vector on the left side of $\mathbf{B}$ and so forth. Whereas, for the cross product $\mathbf{B} \times \mathbf{A}$, the last vector on the right side of $\mathbf{A}$ forms a cross product with the last vector on the right side of $\mathbf{B}$.

Juxtaposition (Special Case): The operation of juxtaposition is indicated when two special tensors are placed next to each other without an operator between them. For examples

$$(\mathbf{a}_1 \otimes \mathbf{a}_2)(\mathbf{b}_1 \otimes \mathbf{b}_2) = (\mathbf{a}_2 \bullet \mathbf{b}_1)\,(\mathbf{a}_1 \otimes \mathbf{b}_2) \ ,$$
$$(\mathbf{a}_1 \otimes \mathbf{a}_2)(\mathbf{b}_1 \otimes \mathbf{b}_2 \otimes \mathbf{b}_3) = (\mathbf{a}_2 \bullet \mathbf{b}_1)\,(\mathbf{a}_1) \otimes (\mathbf{b}_2 \otimes \mathbf{b}_3) \ ,$$
$$(\mathbf{b}_1 \otimes \mathbf{b}_2 \otimes \mathbf{b}_3)(\mathbf{a}_1 \otimes \mathbf{a}_2) = (\mathbf{b}_3 \bullet \mathbf{a}_1)\,(\mathbf{b}_1 \otimes \mathbf{b}_2) \otimes (\mathbf{a}_2) \ ,$$
$$(\mathbf{a}_1 \otimes \mathbf{a}_2 \otimes \mathbf{a}_3)(\mathbf{b}_1 \otimes \mathbf{b}_2 \otimes \mathbf{b}_3) = (\mathbf{a}_3 \bullet \mathbf{b}_1)\,(\mathbf{a}_1 \otimes \mathbf{a}_2) \otimes (\mathbf{b}_2 \otimes \mathbf{b}_3) \ . \tag{A.3.16}$$

It can be seen that the juxtaposition operation causes only one of the vectors in each tensor (the vectors closest to each other) to be connected by the inner product. The remaining vectors form a string of vectors connected by tensor products. The order of these vectors in the string is the same as their order in the juxtaposition operation. The order of the resulting tensor is the sum of the orders of the two tensors placed in juxtaposition minus two.

In general, the juxtaposition $\mathbf{AB}$ of the special tensors defined in (A8) is a tensor of order (M+N−2) which is given by

$$\begin{aligned} \mathbf{AB} &= (\mathbf{a}_1 \otimes \mathbf{a}_2 \otimes \mathbf{a}_3 \otimes \ldots \otimes \mathbf{a}_M)(\mathbf{b}_1 \otimes \mathbf{b}_2 \otimes \mathbf{b}_3 \otimes \ldots \otimes \mathbf{b}_N) \\ &= (\mathbf{a}_M \bullet \mathbf{b}_1)\,(\mathbf{a}_1 \otimes \mathbf{a}_2 \otimes \mathbf{a}_3 \otimes \ldots \otimes \mathbf{a}_{M-1}) \otimes (\mathbf{b}_2 \otimes \mathbf{b}_3 \otimes \ldots \otimes \mathbf{b}_N) \ . \end{aligned} \tag{A.3.17}$$

Thus, the juxtaposition operation is not commutative if any one of the tensors $\mathbf{A}$ or $\mathbf{B}$ is of higher order than one

$$\mathbf{AB} \neq \mathbf{BA} \ . \tag{A.3.18}$$

Since only the closest vectors are connected by the inner product, it follows that the juxtaposition of any order tensor $\mathbf{A}$ with a vector $\mathbf{v}$, is the same as the dot product of the two tensors

$$\mathbf{Av} = \mathbf{A} \bullet \mathbf{v} \ , \quad \mathbf{vA} = \mathbf{v} \bullet \mathbf{A} \ . \tag{A.3.19}$$

Consequently, the juxtaposition of two vectors is the same as the dot product between them so that this is a special case when the juxtaposition operation commutes

$$\mathbf{a}_1\mathbf{b}_1 = \mathbf{a}_1 \bullet \mathbf{b}_1 = \mathbf{b}_1\mathbf{a}_1 \ . \tag{A.3.20}$$

In spite of the validity of this result, the dot product between two vectors will usually be expressed explicitly instead of through the juxtaposition operation.

Transpose (Special Case): The transpose of the second order tensor $(\mathbf{a}_1 \otimes \mathbf{a}_2)$ is defined by

$$(\mathbf{a}_1 \otimes \mathbf{a}_2)^T = (\mathbf{a}_2 \otimes \mathbf{a}_1) \ . \tag{A.3.21}$$

Note that the effect of the transpose operation is merely to change the order of the vectors in the tensor product. This transpose operation can be generalized for higher order tensors, for higher order transpose operations, and for left and right transpose operations. Once the left transpose operation is admitted, it is easy to see that a quantity like $\mathbf{A}^T\mathbf{B}$ cannot be interpreted uniquely if the left and right transpose operations are both denoted by a superposed $(^T)$. In particular, it would not be clear if the transpose operation were applied to the right of $\mathbf{A}$ or the left of $\mathbf{B}$. For this reason, the left transpose operation will be denoted by a superposed $(^{LT})$ on the left of the tensor. Thus, for example

$$^{LT}(\mathbf{a}_1 \otimes \mathbf{a}_2) = (\mathbf{a}_2 \otimes \mathbf{a}_1) = (\mathbf{a}_1 \otimes \mathbf{a}_2)^T \ , \tag{A.3.22}$$

which indicates that for second order tensors there is no distinction between the left and right transposes.

These transpose operations can be applied to higher order tensors by generalizing the following examples

$$^{LT}(\mathbf{a}_1 \otimes \mathbf{a}_2 \otimes \mathbf{a}_3) = (\mathbf{a}_2 \otimes \mathbf{a}_1) \otimes \mathbf{a}_3 \ , \quad ^{LT}(\mathbf{a}_1 \otimes \mathbf{a}_2 \otimes \mathbf{a}_3 \otimes \mathbf{a}_4) = (\mathbf{a}_2 \otimes \mathbf{a}_1) \otimes (\mathbf{a}_3 \otimes \mathbf{a}_4) \ ,$$
$$(\mathbf{a}_1 \otimes \mathbf{a}_2 \otimes \mathbf{a}_3)^T = \mathbf{a}_1 \otimes (\mathbf{a}_3 \otimes \mathbf{a}_2) \ , \quad (\mathbf{a}_1 \otimes \mathbf{a}_2 \otimes \mathbf{a}_3 \otimes \mathbf{a}_4)^T = (\mathbf{a}_1 \otimes \mathbf{a}_2) \otimes (\mathbf{a}_4 \otimes \mathbf{a}_3) \ . \tag{A.3.23}$$

Here, the parentheses are included to emphasize that these transpose operators influence only the order of the *two* vectors in the string closest to the operator. In this regard, it is possible to define transpose operators of order $M \geq 2$ by LT(M) and T(M) such that they interchange the order of two sets of M vectors in the tensor product $\mathbf{B}$ of a string of N vectors. Since the special tensor $\mathbf{B}$ has to contain at least two sets of M vectors, it follows that the transpose operator of order M can be applied only to tensors of order $N \geq 2M$. For examples

$$^{LT(2)}(\mathbf{a}_1 \otimes \mathbf{a}_2 \otimes \mathbf{a}_3 \otimes \mathbf{a}_4) = (\mathbf{a}_3 \otimes \mathbf{a}_4) \otimes (\mathbf{a}_1 \otimes \mathbf{a}_2) \ ,$$
$$^{LT(2)}(\mathbf{a}_1 \otimes \mathbf{a}_2 \otimes \mathbf{a}_3 \otimes \mathbf{a}_4 \otimes \mathbf{a}_5) = (\mathbf{a}_3 \otimes \mathbf{a}_4) \otimes (\mathbf{a}_1 \otimes \mathbf{a}_2) \otimes \mathbf{a}_5 \ ,$$
$$(\mathbf{a}_1 \otimes \mathbf{a}_2 \otimes \mathbf{a}_3 \otimes \mathbf{a}_4)^{T(2)} = (\mathbf{a}_3 \otimes \mathbf{a}_4) \otimes (\mathbf{a}_1 \otimes \mathbf{a}_2) \ ,$$
$$(\mathbf{a}_1 \otimes \mathbf{a}_2 \otimes \mathbf{a}_3 \otimes \mathbf{a}_4 \otimes \mathbf{a}_5)^{T(2)} = \mathbf{a}_1 \otimes (\mathbf{a}_4 \otimes \mathbf{a}_5) \otimes (\mathbf{a}_2 \otimes \mathbf{a}_3) \ . \tag{A.3.24}$$

Notice that for these second order transpose operators, only the four vectors closest to the operators are influenced and that the order of the vectors in each of the two sets of two vectors are not changed.

A.4 Indicial notation

Quantities written in indicial notation will have a finite number of indices attached to them. Since the number of indices can be zero, a scalar with no index can also be considered to be written in index notation. The language of index notation is quite simple because only two types of indices can appear in any term. Either the index is a free index or it is a repeated index. Also, a simple summation convention will be defined which applies only to repeated indices. These two types of indices and the summation convention are defined as follows.

Free Indices: Indices that appear only once in a given term are known as free indices. When the free index is a Latin letter, the index will take the values (1,2,3), whereas when it is a Greek letter, it will take only the values (1,2). For example, i and α are free indices in the following expressions

$$\theta^i = (\theta^1, \theta^2, \theta^3) \ , \quad \mathbf{g}_i = (\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3) \ , \quad \mathbf{g}^i = (\mathbf{g}^1, \mathbf{g}^2, \mathbf{g}^3) \quad (i=1,2,3) \ ,$$
$$\theta^\alpha = (\theta^1, \theta^2) \ , \quad \mathbf{a}_\alpha = (\mathbf{a}_1, \mathbf{a}_2) \ , \quad \mathbf{a}^\alpha = (\mathbf{a}^1, \mathbf{a}^2) \quad (\alpha=1,2) \ , \qquad (A.4.1)$$

For general curvilinear coordinates it is necessary to distinguish between indices used as subscripts for covariant quantities, and indices used as superscripts for contravariant quantities. Whereas, for rectangular Cartesian coordinates the base vectors $\mathbf{e}_i$ are orthonormal constant vectors and the distinction between covariant and contravariant quantities disappears so that all indices can be written as subscripts.

Repeated Indices: Indices that *appear twice* in a given term are known as repeated indices. For example, i and j are free indices and m and n are repeated indices in the following expressions

$$a_i \, b_j \, c^m \, T_{mn} \, d^n \ , \quad A_{imn} \, B^{jmn} \ , \quad A_{im}{}^n \, B_j{}^m{}_n \ . \qquad (A.4.2)$$

It is important to emphasize that in the language of indicial notation, an index *can never appear more than twice* in any term. Also, for curvilinear coordinates one of the repeated indices is a subscript and the other is a superscript.

Einstein Summation Convention: When an index appears as a repeated index in a term, that index is understood to take on the values (1,2,3) for Latin indices or (1,2) for Greek indices and the resulting terms are summed. Thus, for example,

$$v^i \, \mathbf{g}_i = v^1 \, \mathbf{g}_1 + v^2 \, \mathbf{g}_2 + v^3 \, \mathbf{g}_3 \ , \quad v_i \, \mathbf{g}^i = v_1 \, \mathbf{g}^1 + v_2 \, \mathbf{g}^2 + v_3 \, \mathbf{g}^3 \ ,$$
$$v^\alpha \, \mathbf{a}_\alpha = v^1 \, \mathbf{a}_1 + v^2 \, \mathbf{a}_2 \ , \quad v_\alpha \, \mathbf{a}^\alpha = v_1 \, \mathbf{a}^1 + v_2 \, \mathbf{a}^2 \ ,$$
$$\mathbf{g}_i \otimes \mathbf{g}^i = \mathbf{g}_1 \otimes \mathbf{g}^1 + \mathbf{g}_2 \otimes \mathbf{g}^2 + \mathbf{g}_3 \otimes \mathbf{g}^3 \ . \qquad (A.4.3)$$

Because of this summation convention, repeated indices are also known as dummy indices since their replacement by any other letter (of the same type, Latin or Greek) not appearing as a free index and also not appearing as another repeated index, does not change the meaning of the term in which they occur. For examples,

$$v^i \, \mathbf{g}_i = v^j \, \mathbf{g}_j \ , \quad a_i \, b_\alpha c^\alpha = a_i \, b_\beta \, c^\beta \ . \qquad (A.4.4)$$

It is important to emphasize that the same free indices must appear in each term in an equation so that for example, the free index i in $(A.4.4)_2$ must appear on each side of the equality.

Kronecker Delta: Using the fact that the contravariant base vectors $\mathbf{g}^i$ defined by (2.1.10) are reciprocal vectors of the covariant base vectors $\mathbf{g}_i$ defined by (2.1.5), it follows that the Kronecker delta symbols δ_i^j and δ^i_j are defined by

$$\delta_i^j = \mathbf{g}_i \bullet \mathbf{g}^j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \quad , \quad \delta^i_j = \mathbf{g}^i \bullet \mathbf{g}_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \quad . \tag{A.4.5}$$

Since the Kronecker deltas vanish unless i=j, they exhibit the following exchange properties

$$\delta_i^j \, v_j = (\delta_1^j \, v_j \, , \delta_2^j \, v_j \, , \delta_3^j \, v_j) = (v_1 \, , v_2 \, , v_3) = v_i \quad ,$$
$$\delta^i_j \, v^j = (\delta^1_j \, v^j \, , \delta^2_j \, v^j \, , \delta^3_j \, v^j) = (v^1 \, , v^2 \, , v^3) = v^i \quad , \tag{A.4.6}$$

Notice that the Kronecker symbol can be removed by replacing the repeated index j in (A.4.6) by the free index i.

In view of the definitions of base tensors and components of tensors discussed in section 2.2, it follows that the Kronecker delta symbols are used in calculating the dot product between two vectors **a** and **b**, since

$$\mathbf{a} = a_i \, \mathbf{g}^i = a^i \, \mathbf{g}_i \quad , \quad \mathbf{b} = b_i \, \mathbf{g}^i = b^i \, \mathbf{g}_i \quad ,$$
$$\mathbf{a} \bullet \mathbf{b} = a_i \, \mathbf{g}^i \bullet b^j \, \mathbf{g}_j = a_i \, (\mathbf{g}^i \bullet \mathbf{g}_j) \, b^j = a_i \, \delta^i_j \, b^j = a_i \, b^i \quad ,$$
$$\mathbf{a} \bullet \mathbf{b} = a^i \, \mathbf{g}_i \bullet b_j \, \mathbf{g}^j = a^i \, (\mathbf{g}_i \bullet \mathbf{g}^j) \, b_j = a^i \, \delta_i^j \, b_j = a^i \, b_i \quad . \tag{A.4.7}$$

Permutation symbol: The permutation symbols ε_{ijk} and ε^{ijk} are defined by

$$\varepsilon_{ijk} = \mathbf{g}_i \times \mathbf{g}_j \bullet \mathbf{g}_k = g^{1/2} \, e_{ijk} \quad , \quad \varepsilon^{ijk} = \mathbf{g}^i \times \mathbf{g}^j \bullet \mathbf{g}^k = g^{-1/2} \, e^{ijk} \quad , \tag{A.4.8}$$

where the scalar $g^{1/2}$ is defined by (2.1.6) and the alternating symbols e_{ijk} and e^{ijk} are defined in terms of the right-handed orthonormal base vectors $\mathbf{e}_i$ of a rectangular Cartesian coordinate system by the equations

$$e_{ijk} = e^{ijk} = \mathbf{e}_i \times \mathbf{e}_j \bullet \mathbf{e}_k \quad ,$$

$$e_{ijk} = \begin{cases} 1 & \text{if (i,j,k) are an even permutation of (1,2,3)} \\ -1 & \text{if (i,j,k) are an odd permutation of (1,2,3)} \\ 0 & \text{if at least two of (i,j,k) have the same value} \end{cases} \tag{A.4.9}$$

From the definition (A.4.9), it appears that the permutation symbols can be used in calculating the vector product between two vectors. To this end, it is necessary to prove that

$$\mathbf{g}_i \times \mathbf{g}_j = \varepsilon_{ijk} \, \mathbf{g}^k \quad , \quad \mathbf{g}^i \times \mathbf{g}^j = \varepsilon^{ijk} \, \mathbf{g}_k \quad . \tag{A.4.10}$$

Proof: Since $\mathbf{g}_i \times \mathbf{g}_j$ is a vector in Euclidean 3-Space for each choice of the values of i and j, and since $\mathbf{g}^k$ forms a complete basis for that space, it follows that $\mathbf{g}_i \times \mathbf{g}_j$ can be represented as a linear combination of the base vectors $\mathbf{g}^k$ such that

$$\mathbf{g}_i \times \mathbf{g}_j = A_{ijk} \, \mathbf{g}^k \quad , \tag{A.4.11}$$

where the components A_{ijk} need to be determined. In particular, by taking the dot product of (A.4.11) with $\mathbf{g}_k$ and using the definition (A.4.8) it can be shown that

$$\varepsilon_{ijk} = \mathbf{g}_i \times \mathbf{g}_j \bullet \mathbf{g}_k = A_{ijm}\, \mathbf{g}^m \bullet \mathbf{g}_k = A_{ijm}\, \delta^m{}_k = A_{ijk} \ , \qquad \text{(A.4.12)}$$

which proves the result $(A.4.10)_1$. A similar proof can be provided for the result $(A.4.10)_2$. Now, using (A.4.10) it follows that the cross product between the vectors **a** and **b** can be represented in the forms

$$\mathbf{a} \times \mathbf{b} = (a^i\, \mathbf{g}_i) \times (b^j\, \mathbf{g}_j) = (\mathbf{g}_i \times \mathbf{g}_j)\, a^i\, b^j = \varepsilon_{ijk}\, a^i\, b^j\, \mathbf{g}^k \ ,$$
$$\mathbf{a} \times \mathbf{b} = (a_i\, \mathbf{g}^i) \times (b_j\, \mathbf{g}^j) = (\mathbf{g}^i \times \mathbf{g}^j)\, a_i\, b_j = \varepsilon^{ijk}\, a_i\, b_j\, \mathbf{g}_k \ . \qquad \text{(A.4.13)}$$

Contraction: Contraction is the process of setting two free indices in a given expression equal to the same repeated index, together with the implied summation convention. For example, the free indices i,j in $\delta_i{}^j$ and $\delta^i{}_j$ can be contracted upon to obtain

$$\delta_i{}^i = \delta_1{}^1 + \delta_2{}^2 + \delta_3{}^3 = 3 \ , \quad \delta^j{}_j = \delta^1{}_1 + \delta^2{}_2 + \delta^3{}_3 = 3 \ . \qquad \text{(A.4.14)}$$

Note that contraction on the set of $9=3^2$ quantities $T_i{}^j$ or $T^i{}_j$ can be performed by multiplying them by the Kronecker delta to obtain

$$T_i{}^j\, \delta^i{}_j = T_i{}^i \ , \quad T^i{}_j\, \delta_i{}^j = T^i{}_i \ . \qquad \text{(A.4.15)}$$

A.5 Tensors products (general case)

Base Tensors and Tensors: Section 2.2 shows how base tensors of any order can be defined using the tensor products of the covariant base vectors $\mathbf{g}_i$ and the contravariant base vectors $\mathbf{g}^i$. Section 2.2 also shows how the covariant, contravariant, and mixed components of a tensor can be defined. For example, let **A** be a second order tensor and **B** be a third order tensor defined by

$$\mathbf{A} = A_{ij}\, (\mathbf{g}^i \otimes \mathbf{g}^j) = A^{ij}\, (\mathbf{g}_i \otimes \mathbf{g}_j) = A_i{}^j\, (\mathbf{g}^i \otimes \mathbf{g}_j) = A^i{}_j\, (\mathbf{g}_i \otimes \mathbf{g}^j) \ ,$$
$$\mathbf{B} = B_{ijk}\, (\mathbf{g}^i \otimes \mathbf{g}^j \otimes \mathbf{g}^k) = B_{ij}{}^k\, (\mathbf{g}^i \otimes \mathbf{g}^j \otimes \mathbf{g}_k) = B_i{}^{jk}\, (\mathbf{g}^i \otimes \mathbf{g}_j \otimes \mathbf{g}_k)$$
$$= B^{ijk}\, (\mathbf{g}_i \otimes \mathbf{g}_j \otimes \mathbf{g}_k) = B^{ij}{}_k\, (\mathbf{g}_i \otimes \mathbf{g}_j \otimes \mathbf{g}^k) = B^i{}_{jk}\, (\mathbf{g}_i \otimes \mathbf{g}^j \otimes \mathbf{g}^k)$$
$$= B^i{}_j{}^k\, (\mathbf{g}_i \otimes \mathbf{g}^j \otimes \mathbf{g}_k) = B_i{}^j{}_k\, (\mathbf{g}^i \otimes \mathbf{g}_j \otimes \mathbf{g}^k) \ . \qquad \text{(A.5.1)}$$

Higher order tensors can be defined in an obvious manner.

Since the components of these tensors can be arbitrary real numbers, these expressions represent general tensors of order two and three. In particular, it should be recognized that a general second order tensor **A** cannot necessarily be expressed as the tensor product of two vectors **a** and **b**, as was done in discussing the special cases of the tensor operations.

Using the fact that components of a tensor are unaffected by the various tensor operations, it is rather straight forward to generalize the tensor operations defined in section A.3 for general tensors. For simplicity, the tensors **A** and **B** defined in (A.5.1) will be used for specific examples. However, since each tensor can be represented in terms of its covariant, contravariant, and mixed components, only a few possible representations of these examples will be exhibited.

Tensor Product (General Case): The tensor product **A**⊗**B** of **A** and **B** can be written as

$$\mathbf{A}\otimes\mathbf{B} = A_{ij}\, B_{qrs}\, (\mathbf{g}^i\otimes\mathbf{g}^j)\otimes(\mathbf{g}^q\otimes\mathbf{g}^r\otimes\mathbf{g}^s) = A_i{}^j\, B_q{}^{rs}\, (\mathbf{g}^i\otimes\mathbf{g}_j)\otimes(\mathbf{g}^q\otimes\mathbf{g}_r\otimes\mathbf{g}_s)\ . \quad (A.5.2)$$

Dot Product (General Case): The dot products $\mathbf{A} \bullet \mathbf{B}$ and $\mathbf{B} \bullet \mathbf{A}$ of $\mathbf{A}$ and $\mathbf{B}$ can be written as

$$\begin{aligned}\mathbf{A} \bullet \mathbf{B} &= A_{ij}\, B_{qrs}\, (\mathbf{g}^i\otimes\mathbf{g}^j) \bullet (\mathbf{g}^q\otimes\mathbf{g}^r\otimes\mathbf{g}^s) = A_{ij}\, B_{qrs}\, g^{iq}\, g^{jr}\, \mathbf{g}^s \\ &= A_{ij}\, B^{ij}{}_s\, \mathbf{g}^s = A^{qr}\, B_{qrs}\, \mathbf{g}^s\ , \\ \mathbf{B} \bullet \mathbf{A} &= B_{qrs}\, A_{ij}\, (\mathbf{g}^q\otimes\mathbf{g}^r\otimes\mathbf{g}^s) \bullet (\mathbf{g}^i\otimes\mathbf{g}^j) = B_{qrs}\, A_{ij}\, (\mathbf{g}^q)\, g^{ri}\, g^{sj} \\ &= B_{qrs}\, A^{rs}\, \mathbf{g}^q = B_q{}^{ij}\, A_{ij}\, \mathbf{g}^q\ . \end{aligned} \quad (A.5.3)$$

Cross Product (General Case): The cross products $\mathbf{A} \times \mathbf{B}$ and $\mathbf{B} \times \mathbf{A}$ of $\mathbf{A}$ and $\mathbf{B}$ can be written as

$$\begin{aligned}\mathbf{A} \times \mathbf{B} &= A_{ij}\, B_{qrs}\, (\mathbf{g}^i\otimes\mathbf{g}^j) \times (\mathbf{g}^q\otimes\mathbf{g}^r\otimes\mathbf{g}^s) = A_{ij}\, B_{qrs}\, (\mathbf{g}^i\times\mathbf{g}^q)\otimes(\mathbf{g}^j\times\mathbf{g}^r)\otimes\mathbf{g}^s \\ &= A_{ij}\, B_{qrs}\, \varepsilon^{iqm}\, \varepsilon^{jrn}\, (\mathbf{g}_m\otimes\mathbf{g}_n\otimes\mathbf{g}^s)\ , \\ \mathbf{B} \times \mathbf{A} &= B_{qrs}\, A_{ij}\, (\mathbf{g}^q\otimes\mathbf{g}^r\otimes\mathbf{g}^s) \times (\mathbf{g}^i\otimes\mathbf{g}^j) = B_{qrs}\, A_{ij}\, \mathbf{g}^q\otimes(\mathbf{g}^r\times\mathbf{g}^i)\otimes(\mathbf{g}^s\times\mathbf{g}^j) \\ &= B_{qrs}\, A_{ij}\, \varepsilon^{rim}\, \varepsilon^{sjn}\, (\mathbf{g}^q\otimes\mathbf{g}_m\otimes\mathbf{g}_n)\ . \end{aligned} \quad (A.5.4)$$

Juxtaposition (General Case): The juxtapositions $\mathbf{AB}$ and $\mathbf{BA}$ of $\mathbf{A}$ and $\mathbf{B}$ can be written as

$$\begin{aligned}\mathbf{AB} &= A_{ij}\, B_{qrs}\, (\mathbf{g}^i\otimes\mathbf{g}^j)(\mathbf{g}^q\otimes\mathbf{g}^r\otimes\mathbf{g}^s) = A_{ij}\, B_{qrs}\, g^{jq}\, \mathbf{g}^i\otimes(\mathbf{g}^r\otimes\mathbf{g}^s) \\ &= A_{ij}\, B^j{}_{rs}\, \mathbf{g}^i\otimes(\mathbf{g}^r\otimes\mathbf{g}^s) = A_i{}^q\, B_{qrs}\, \mathbf{g}^i\otimes(\mathbf{g}^r\otimes\mathbf{g}^s)\ , \\ \mathbf{BA} &= B_{qrs}\, A_{ij}\, (\mathbf{g}^q\otimes\mathbf{g}^r\otimes\mathbf{g}^s)(\mathbf{g}^i\otimes\mathbf{g}^j) = B_{qrs}\, A_{ij}\, g^{si}\, (\mathbf{g}^q\otimes\mathbf{g}^r)\otimes\mathbf{g}^j \\ &= B_{qrs}\, A^s{}_j\, (\mathbf{g}^q\otimes\mathbf{g}^r)\otimes\mathbf{g}^j = B_{qr}{}^i\, A_{ij}\, (\mathbf{g}^q\otimes\mathbf{g}^r)\otimes\mathbf{g}^j\ . \end{aligned} \quad (A.5.5)$$

Transpose of a Tensor: The transpose operations $\mathbf{A}^T$, ${}^{LT}\mathbf{A}$, $\mathbf{B}^T$, and ${}^{LT}\mathbf{B}$ can be written as

$$\begin{aligned}\mathbf{A}^T &= (A_{ij}\, \mathbf{g}^i\otimes\mathbf{g}^j)^T = A_{ij}\, (\mathbf{g}^i\otimes\mathbf{g}^j)^T = A_{ij}\, (\mathbf{g}^j\otimes\mathbf{g}^i) = {}^{LT}\mathbf{A}\ , \\ \mathbf{A}^T &= (A_i{}^j\, \mathbf{g}^i\otimes\mathbf{g}_j)^T = A_i{}^j\, (\mathbf{g}^i\otimes\mathbf{g}_j)^T = A_i{}^j\, (\mathbf{g}_j\otimes\mathbf{g}^i) = {}^{LT}\mathbf{A}\ , \\ \mathbf{B}^T &= (B_{qrs}\, \mathbf{g}^q\otimes\mathbf{g}^r\otimes\mathbf{g}^s)^T = B_{qrs}\, (\mathbf{g}^q\otimes\mathbf{g}^r\otimes\mathbf{g}^s)^T = B_{qrs}\, \mathbf{g}^q\otimes(\mathbf{g}^s\otimes\mathbf{g}^r)\ , \\ \mathbf{B}^T &= (B_{qr}{}^s\, \mathbf{g}^q\otimes\mathbf{g}^r\otimes\mathbf{g}_s)^T = B_{qr}{}^s\, (\mathbf{g}^q\otimes\mathbf{g}^r\otimes\mathbf{g}_s)^T = B_{qr}{}^s\, \mathbf{g}^q\otimes(\mathbf{g}_s\otimes\mathbf{g}^r)\ , \\ {}^{LT}\mathbf{B} &= {}^{LT}(B_{qrs}\, \mathbf{g}^q\otimes\mathbf{g}^r\otimes\mathbf{g}^s) = B_{qrs}\, {}^{LT}(\mathbf{g}^q\otimes\mathbf{g}^r\otimes\mathbf{g}^s) = B_{qrs}\, (\mathbf{g}^r\otimes\mathbf{g}^q)\otimes\mathbf{g}^s\ , \\ {}^{LT}\mathbf{B} &= {}^{LT}(B^q{}_{rs}\, \mathbf{g}_q\otimes\mathbf{g}^r\otimes\mathbf{g}^s) = B^q{}_{rs}\, {}^{LT}(\mathbf{g}_q\otimes\mathbf{g}^r\otimes\mathbf{g}^s) = B^q{}_{rs}\, (\mathbf{g}^r\otimes\mathbf{g}_q)\otimes\mathbf{g}^s\ . \end{aligned} \quad (A5.6)$$

In particular, notice that the transpose operation does not change the order of the indices of the components of the tensor, but merely changes the order of the base vectors. Using these results, it can be shown for a second order tensor $\mathbf{A}$ and an arbitrary vector $\mathbf{v}$ that

$$\mathbf{A}\mathbf{v} = \mathbf{v}\mathbf{A}^T\ , \quad \mathbf{v}\mathbf{A} = \mathbf{A}^T\mathbf{v}\ . \quad (A.5.7)$$

Recalling that the components of a tensor are unaffected by the transpose operation, higher order transpose operations can be expressed as natural generalizations of results like (A.3.24). For example, let $\mathbf{T}$ be a fifth order tensor defined by

$$\mathbf{T} = T_{ijkmn}\, (\mathbf{g}^i\otimes\mathbf{g}^j\otimes\mathbf{g}^k\otimes\mathbf{g}^m\otimes\mathbf{g}^n) = T_{ijk}{}^{mn}\, (\mathbf{g}^i\otimes\mathbf{g}^j\otimes\mathbf{g}^k\otimes\mathbf{g}_m\otimes\mathbf{g}_n)\ . \quad (A.5.8)$$

Then, the second order transpose operations T(2) and LT(2) applied to $\mathbf{T}$ can be written as

$$\mathbf{T}^{T(2)} = (T_{ijkmn}\, \mathbf{g}^i\otimes\mathbf{g}^j\otimes\mathbf{g}^k\otimes\mathbf{g}^m\otimes\mathbf{g}^n)^{T(2)} = T_{ijkmn}\, (\mathbf{g}^i\otimes\mathbf{g}^j\otimes\mathbf{g}^k\otimes\mathbf{g}^m\otimes\mathbf{g}^n)^{T(2)}$$
$$= T_{ijkmn}\, \mathbf{g}^i\otimes(\mathbf{g}^m\otimes\mathbf{g}^n)\otimes(\mathbf{g}^j\otimes\mathbf{g}^k) \ ,$$
$$\mathbf{T}^{T(2)} = (T_{ijk}{}^{mn}\, \mathbf{g}^i\otimes\mathbf{g}^j\otimes\mathbf{g}^k\otimes\mathbf{g}_m\otimes\mathbf{g}_n)^{T(2)} = T_{ijk}{}^{mn}\, (\mathbf{g}^i\otimes\mathbf{g}^j\otimes\mathbf{g}^k\otimes\mathbf{g}_m\otimes\mathbf{g}_n)^{T(2)}$$
$$= T_{ijk}{}^{mn}\, \mathbf{g}^i\otimes(\mathbf{g}_m\otimes\mathbf{g}_n)\otimes(\mathbf{g}^j\otimes\mathbf{g}^k) \ ,$$
$${}^{LT(2)}\mathbf{T} = {}^{LT(2)}(T_{ijkmn}\, \mathbf{g}^i\otimes\mathbf{g}^j\otimes\mathbf{g}^k\otimes\mathbf{g}^m\otimes\mathbf{g}^n) = T_{ijkmn}\, {}^{LT(2)}(\mathbf{g}^i\otimes\mathbf{g}^j\otimes\mathbf{g}^k\otimes\mathbf{g}^m\otimes\mathbf{g}^n)$$
$$= T_{ijkmn}\, (\mathbf{g}^k\otimes\mathbf{g}^m)\otimes(\mathbf{g}^i\otimes\mathbf{g}^j)\otimes\mathbf{g}^n \ ,$$
$${}^{LT(2)}\mathbf{T} = {}^{LT(2)}(T_{ijk}{}^{mn}\, \mathbf{g}^i\otimes\mathbf{g}^j\otimes\mathbf{g}^k\otimes\mathbf{g}_m\otimes\mathbf{g}_n) = T_{ijk}{}^{mn}\, {}^{LT(2)}(\mathbf{g}^i\otimes\mathbf{g}^j\otimes\mathbf{g}^k\otimes\mathbf{g}_m\otimes\mathbf{g}_n)$$
$$= T_{ijk}{}^{mn}\, (\mathbf{g}^k\otimes\mathbf{g}_m)\otimes(\mathbf{g}^i\otimes\mathbf{g}^j)\otimes\mathbf{g}_n \ . \qquad (A.5.9)$$

<u>Identity Tensor of Order 2M</u>: The identity tensor of order 2M (M≥1) is denoted by $\mathbf{I}$(2M) and is a tensor that has the property that the dot product of $\mathbf{I}$(2M) with an arbitrary tensor $\mathbf{A}$ of order M yields the result $\mathbf{A}$ such that

$$\mathbf{I}(2M) \bullet \mathbf{A} = \mathbf{A} \bullet \mathbf{I}(2M) = \mathbf{A} \ . \qquad (A.5.10)$$

Letting (i,j...s,t) be a string of M indices, it follows that $\mathbf{I}$(2M) admits a number of representations which include

$$\mathbf{I}(2M) = (\mathbf{g}_i\otimes\mathbf{g}_j\otimes...\otimes\mathbf{g}_s\otimes\mathbf{g}_t)\otimes(\mathbf{g}^i\otimes\mathbf{g}^j\otimes...\otimes\mathbf{g}^s\otimes\mathbf{g}^t) \ ,$$
$$\mathbf{I}(2M) = (\mathbf{g}^i\otimes\mathbf{g}^j\otimes...\otimes\mathbf{g}^s\otimes\mathbf{g}^t)\otimes(\mathbf{g}_i\otimes\mathbf{g}_j\otimes...\otimes\mathbf{g}_s\otimes\mathbf{g}_t) \ ,$$
$$\mathbf{I}(2M) = (\mathbf{g}_i\otimes\mathbf{g}_j\otimes...\otimes\mathbf{g}^s\otimes\mathbf{g}^t)\otimes(\mathbf{g}^i\otimes\mathbf{g}^j\otimes...\otimes\mathbf{g}_s\otimes\mathbf{g}_t) \ ,$$
$$\mathbf{I}(2M) = (\mathbf{g}^i\otimes\mathbf{g}^j\otimes...\otimes\mathbf{g}_s\otimes\mathbf{g}_t)\otimes(\mathbf{g}_i\otimes\mathbf{g}_j\otimes...\otimes\mathbf{g}^s\otimes\mathbf{g}^t) \ , \qquad (A.5.11)$$

where summation over repeated indices is implied. Since the second order identity tensor appears often in continuum mechanics, it is convenient to denote it by $\mathbf{I}$ instead of $\mathbf{I}$(2). In view of (A.5.11), it follows that the second order identity $\mathbf{I}$ can be represented by

$$\mathbf{I} = \mathbf{g}_i\otimes\mathbf{g}^i = \mathbf{g}^i\otimes\mathbf{g}_i \ . \qquad (A.5.12)$$

Also, the various components of $\mathbf{I}$ can be written as

$$g^{ij} = \mathbf{I} \bullet (\mathbf{g}^i\otimes\mathbf{g}^j) \ , \quad g_{ij} = \mathbf{I} \bullet (\mathbf{g}_i\otimes\mathbf{g}_j) \ , \quad \delta_i^j = \mathbf{I} \bullet (\mathbf{g}_i\otimes\mathbf{g}^j) \ , \quad \delta^i{}_j = \mathbf{I} \bullet (\mathbf{g}^i\otimes\mathbf{g}_j) \ . \quad (A.5.13)$$

<u>Zero Tensor of Order M</u>: Since all components of the zero tensor of order M are 0, and since the order of the tensors in a given equation will usually be obvious from the context, the symbol 0 is used to denote the zero tensor of any order.

<u>Lack of Commutativity</u>: Note that in general, the operations of tensor product, dot product, cross product, and juxtaposition are not commutative so the order of these operations must be preserved. Specifically, it follows that

$$\mathbf{A}\otimes\mathbf{B} \neq \mathbf{B}\otimes\mathbf{A} \ , \quad \mathbf{A}\bullet\mathbf{B} \neq \mathbf{B}\bullet\mathbf{A} \ , \quad \mathbf{A}\times\mathbf{B} \neq \mathbf{B}\times\mathbf{A} \ , \quad \mathbf{AB} \neq \mathbf{BA} \ . \qquad (A.5.14)$$

<u>Permutation Tensor</u>: The permutation tensor $\boldsymbol{\varepsilon}$ is a third order tensor that can be defined such that for any two vectors $\mathbf{a}$ and $\mathbf{b}$

$$(\mathbf{a}\otimes\mathbf{b}) \bullet \boldsymbol{\varepsilon} = \boldsymbol{\varepsilon} \bullet (\mathbf{a}\otimes\mathbf{b}) = \mathbf{a}\times\mathbf{b} \ . \qquad (A.5.15)$$

Moreover, it can be shown that for any vector $\mathbf{c}$

$$\boldsymbol{\varepsilon} \bullet (\mathbf{a}\otimes\mathbf{b}\otimes\mathbf{c}) = \mathbf{a}\times\mathbf{b}\bullet\mathbf{c} \qquad (A.5.16)$$

Thus, with the help of the definitions (A.4.8) it follows that ε_{ijk} are the covariant components and ε^{ijk} are the contravariant components of $\boldsymbol{\varepsilon}$ so that

$$\varepsilon_{ijk} = \boldsymbol{\varepsilon} \bullet (\mathbf{g}_i \otimes \mathbf{g}_j \otimes \mathbf{g}_k) \;,\quad \boldsymbol{\varepsilon} = \varepsilon_{ijk} \, (\mathbf{g}^i \otimes \mathbf{g}^j \otimes \mathbf{g}^k) \;,$$
$$\varepsilon^{ijk} = \boldsymbol{\varepsilon} \bullet (\mathbf{g}^i \otimes \mathbf{g}^j \otimes \mathbf{g}^k) \;,\quad \boldsymbol{\varepsilon} = \varepsilon^{ijk} \, (\mathbf{g}_i \otimes \mathbf{g}_j \otimes \mathbf{g}_k) \;. \tag{A.5.17}$$

Hierarchy of Tensor Operations: To simplify the notation and reduce the need for using parentheses to clarify mathematical equations, it is convenient to define the hierarchy of the tensor operations according to Table A.5.1, with level 1 operations being performed before level 2 operations and so forth. Also, as is usual, the order in which operations in the same level are performed is determined by which operation appears in the most left-hand position in the equation.

Level	Tensor Operation
1	Left Transpose (LT) and Right Transpose (T)
2	Juxtaposition and Tensor product (⊗)
3	Cross product (×)
4	Dot product (•)
5	Addition and Subtraction

Table A.5.1 Hierarchy of tensor operations

A.6 Tensor transformation relations

From a physical point of view it is obvious that a reasonable mathematical representation of any physical law cannot depend on arbitrary mathematical choices. In particular, it cannot depend on the choice of the specific coordinate system with respect to which this law is expressed. For this reason, it is essential to formulate physical laws using mathematical quantities which themselves are automatically independent of the particular choice of coordinate system. Tensors have this mathematical property and thus are essential in continuum mechanics.

Although the tensor **T** is a mathematical quantity that is independent of the particular choice of coordinate system, it is important to emphasize that the components of the tensor **T** depend explicitly on the choice of the coordinate system. Moreover, it is clear from equations like (2.2.4) and (2.2.5), that the covariant, contravariant and mixed components of **T** also depend on the particular choice of the base tensors used to determine them.

To make this dependence on the choice of the coordinate system more clear, let θ'^i be another set of coordinates which are related to θ^i by a one-to-one invertible mapping such that

$$\theta'^i = \theta'^i \, (\theta^j, t) \;,\quad \theta^i = \theta^i \, (\theta'^j, t) \;. \tag{A.6.1}$$

Now, the covariant base vectors $\mathbf{g}'_i$ and contravariant base vectors $\mathbf{g}'^i$ associated with the new coordinates θ'^i are defined by

$$\mathbf{g}'_i = \frac{\partial \mathbf{x}^*}{\partial \theta'^i} \ , \quad \mathbf{g}'^i \bullet \mathbf{g}'_j = \delta^i_j \ . \tag{A.6.2}$$

Consequently, with the help of the chain rule of differentiation it can be shown that

$$\mathbf{g}'_i = \frac{\partial \theta^j}{\partial \theta'^i} \mathbf{g}_j \ , \quad \mathbf{g}'^i = \frac{\partial \theta'^i}{\partial \theta^j} \mathbf{g}^j \ , \quad \mathbf{g}_i = \frac{\partial \theta'^j}{\partial \theta^i} \mathbf{g}'_j \ , \quad \mathbf{g}^i = \frac{\partial \theta^i}{\partial \theta'^j} \mathbf{g}'^j \ . \tag{A.6.3}$$

In particular, notice that the index in the numerator of the partial derivative is a contravariant index and the index in the denominator of the partial derivative is a covariant index. Also, notice that if the primed coordinate is a free index, then it has the same character (covariant or contravariant) as the primed base vector $(A.6.3)_{1,2}$. Moreover, notice that if the primed index is a repeated index, then it has the opposite character from the primed base vector $(A.6.3)_{3,4}$. Using these relations, it can also be shown that

$$\frac{\partial \theta'^i}{\partial \theta^j} = \mathbf{g}'^i \bullet \mathbf{g}_j \ , \quad \frac{\partial \theta^i}{\partial \theta'^j} = \mathbf{g}^i \bullet \mathbf{g}'_j \ . \tag{A.6.4}$$

By definition [(2.2.1) and (2.2.2)], the components v_i or v^i of the vector $\mathbf{v}$ are determined by taking the dot product of $\mathbf{v}$ with the base vectors $\mathbf{g}_i$ or $\mathbf{g}^i$

$$v_i = \mathbf{v} \bullet \mathbf{g}_i \ , \quad v^i = \mathbf{v} \bullet \mathbf{g}^i \ , \quad \mathbf{v} = v_i \, \mathbf{g}^i = v^i \, \mathbf{g}_i \ . \tag{A.6.5}$$

Similarly, the components v'_i or v'^i of $\mathbf{v}$ with respect to the new base vectors $\mathbf{g}'_i$ or $\mathbf{g}'^i$ are defined such that

$$v'_i = \mathbf{v} \bullet \mathbf{g}'_i \ , \quad v'^i = \mathbf{v} \bullet \mathbf{g}'^i \ , \quad \mathbf{v} = v'_i \, \mathbf{g}'^i = v'^i \, \mathbf{g}'_i \ . \tag{A.6.6}$$

However, since $\mathbf{v}$ can be expressed in terms of either the primed or the unprimed base vectors, it follows that the primed and unprimed components must be related. In particular, using the expressions (A.6.3) it can be shown that

$$v'_i = \frac{\partial \theta^j}{\partial \theta'^i} v_j \ , \quad v'^i = \frac{\partial \theta'^i}{\partial \theta^j} v^j \ , \quad v_i = \frac{\partial \theta'^j}{\partial \theta^i} v'_j \ , \quad v^i = \frac{\partial \theta^i}{\partial \theta'^j} v'^j \ . \tag{A.6.7}$$

Similarly, the components of a general tensor $\mathbf{T}$ are defined by the dot product of $\mathbf{T}$ with the base tensors so that for example

$$\begin{aligned}
T_{ijk...}{}^{rst} &= \mathbf{T} \bullet \mathbf{g}_i \otimes \mathbf{g}_j \otimes \mathbf{g}_k \otimes ... \otimes \mathbf{g}^r \otimes \mathbf{g}^s \otimes \mathbf{g}^t \ , \\
\mathbf{T} &= T_{ijk...}{}^{rst} \, (\mathbf{g}^i \otimes \mathbf{g}^j \otimes \mathbf{g}^k \otimes ... \otimes \mathbf{g}_r \otimes \mathbf{g}_s \otimes \mathbf{g}_t) \ , \\
T'_{ijk...}{}^{rst} &= \mathbf{T} \bullet \mathbf{g}'_i \otimes \mathbf{g}'_j \otimes \mathbf{g}'_k \otimes ... \otimes \mathbf{g}'^r \otimes \mathbf{g}'^s \otimes \mathbf{g}'^t \ , \\
\mathbf{T} &= T'_{ijk...}{}^{rst} \, (\mathbf{g}'^i \otimes \mathbf{g}'^j \otimes \mathbf{g}'^k \otimes ... \otimes \mathbf{g}'_r \otimes \mathbf{g}'_s \otimes \mathbf{g}'_t) \ .
\end{aligned} \tag{A.6.8}$$

Again, since $\mathbf{T}$ can be expressed in terms of either the primed or the unprimed base vectors, it follows that the primed and unprimed components must be related so that with the help of (A.6.3), it can be shown that

$$\begin{aligned}
T'_{ijk...}{}^{rst} &= \frac{\partial \theta^a}{\partial \theta'^i} \frac{\partial \theta^b}{\partial \theta'^j} \frac{\partial \theta^c}{\partial \theta'^i} \cdots \frac{\partial \theta'^r}{\partial \theta^d} \frac{\partial \theta'^s}{\partial \theta^e} \frac{\partial \theta'^t}{\partial \theta^f} \, T_{abc...}{}^{def} \ , \\
T_{ijk...}{}^{rst} &= \frac{\partial \theta'^a}{\partial \theta^i} \frac{\partial \theta'^b}{\partial \theta^j} \frac{\partial \theta'^c}{\partial \theta^k} \cdots \frac{\partial \theta^r}{\partial \theta'^d} \frac{\partial \theta^s}{\partial \theta'^e} \frac{\partial \theta^t}{\partial \theta'^f} \, T'_{abc...}{}^{def} \ .
\end{aligned} \tag{A.6.9}$$

A.7 Additional definitions and results

Properties of the dot product: For later convenience it is useful to discuss some properties of the dot product between second order tensors. To this end, let $\mathbf{a}$, $\mathbf{b}$ be general vectors and $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$ be general second order tensors. It then can be shown that

$$\mathbf{a} \bullet \mathbf{B}\mathbf{b} = \mathbf{B}^T\mathbf{a} \bullet \mathbf{b} = \mathbf{B} \bullet \mathbf{a}\otimes\mathbf{b} \ ,$$
$$\mathbf{A}\,\mathbf{a} \bullet \mathbf{B}\mathbf{b} = \mathbf{a} \bullet \mathbf{A}^T\mathbf{B}\mathbf{b} = \mathbf{B}^T\mathbf{A}\,\mathbf{a} \bullet \mathbf{b} = \mathbf{A}^T\mathbf{B} \bullet \mathbf{a}\otimes\mathbf{b} \ , \quad \mathbf{A} \bullet \mathbf{I} = \mathbf{A}^T \bullet \mathbf{I} \ ,$$
$$\mathbf{A} \bullet \mathbf{B} = \mathbf{A}\mathbf{B}^T \bullet \mathbf{I} = \mathbf{B}^T\mathbf{A} \bullet \mathbf{I} = \mathbf{B}^T \bullet \mathbf{A}^T = \mathbf{A}^T \bullet \mathbf{B}^T \ ,$$
$$\mathbf{A} \bullet \mathbf{B}\mathbf{C}\mathbf{D} = \mathbf{B}^T\mathbf{A} \bullet \mathbf{C}\mathbf{D} \ , \quad \mathbf{A} \bullet \mathbf{B}\mathbf{C}\mathbf{D} = \mathbf{A}\mathbf{D}^T \bullet \mathbf{B}\mathbf{C} \ , \quad \mathbf{A} \bullet \mathbf{B}\mathbf{C}\mathbf{D} = \mathbf{B}^T\mathbf{A}\mathbf{D}^T \bullet \mathbf{C} \ , \tag{A.7.1}$$

Symmetric: A second order tensor $\mathbf{A}$ is said to be symmetric if

$$\mathbf{A}^T = \mathbf{A} \ . \tag{A.7.2}$$

It then follows that the components of $\mathbf{A}$ are related to those of $\mathbf{A}^T$ by the following expressions

$$\mathbf{A} = A_{ij}\,(\mathbf{g}^i\otimes\mathbf{g}^j) = A^{ij}\,(\mathbf{g}_i\otimes\mathbf{g}_j) = A_i{}^j\,(\mathbf{g}^i\otimes\mathbf{g}_j) = A^i{}_j\,(\mathbf{g}_i\otimes\mathbf{g}^j) \ ,$$
$$\mathbf{A}^T = A_{ij}\,(\mathbf{g}^j\otimes\mathbf{g}^i) = A^{ij}\,(\mathbf{g}_j\otimes\mathbf{g}_i) = A_i{}^j\,(\mathbf{g}_j\otimes\mathbf{g}^i) = A^i{}_j\,(\mathbf{g}^j\otimes\mathbf{g}_i) \ ,$$
$$A^T{}_{ij} = \mathbf{A}^T \bullet (\mathbf{g}_i\otimes\mathbf{g}_j) = A_{ji} \ , \quad A^{Tij} = \mathbf{A}^T \bullet (\mathbf{g}^i\otimes\mathbf{g}^j) = A^{ji} \ ,$$
$$A^T{}_i{}^j = \mathbf{A}^T \bullet (\mathbf{g}_i\otimes\mathbf{g}^j) = A^j{}_i \ , \quad A^{Ti}{}_j = \mathbf{A}^T \bullet (\mathbf{g}^i\otimes\mathbf{g}_j) = A_j{}^i \ . \tag{A.7.3}$$

Skew-Symmetric: A second order tensor $\mathbf{A}$ is said to be skew-symmetric if

$$\mathbf{A}^T = -\,\mathbf{A} \ . \tag{A.7.4}$$

It then follows that the components of $\mathbf{A}$ are related to those of $\mathbf{A}^T$ by the following expressions

$$A^T{}_{ij} = A_{ji} = -\,A_{ij} \ , \quad A_{11} = A_{22} = A_{33} = 0 \ ,$$
$$A^{Tij} = A^{ji} = -\,A^{ij} \ , \quad A^{11} = A^{22} = A^{33} = 0 \ . \tag{A.7.5}$$

It can also be shown that since a skew-symmetric tensor $\mathbf{A}$ has only three independent components, it is possible to define an axial vector $\boldsymbol{\omega}$ such that for any vector $\mathbf{v}$

$$\boldsymbol{\omega} = -\frac{1}{2}\,\boldsymbol{\varepsilon} \bullet \mathbf{A} \ , \quad \mathbf{A} = -\,\boldsymbol{\varepsilon}\,\boldsymbol{\omega} \ , \quad \mathbf{A}\,\mathbf{v} = \boldsymbol{\omega} \times \mathbf{v} \ . \tag{A.7.6}$$

Symmetric and skew-symmetric parts of a tensor: An arbitrary second order tensor $\mathbf{A}$ can be uniquely separated into the sum of its symmetric part $\mathbf{A}_{sym}$ and its skew-symmetric part $\mathbf{A}_{skew}$ such that

$$\mathbf{A} = \mathbf{A}_{sym} + \mathbf{A}_{skew} \ ,$$
$$\mathbf{A}_{sym} = \frac{1}{2}(\mathbf{A} + \mathbf{A}^T) = \mathbf{A}_{sym}^T \ , \quad \mathbf{A}_{skew} = \frac{1}{2}(\mathbf{A} - \mathbf{A}^T) = -\,\mathbf{A}_{skew}^T \ . \tag{A.7.7}$$

Using the properties of the dot product (A.7.1) it can be shown that the symmetric and skew-symmetric parts of $\mathbf{A}$ are orthogonal to each other since

$$\mathbf{A}_{sym} \bullet \mathbf{A}_{skew} = \mathbf{A}_{sym}^T \bullet \mathbf{A}_{skew}^T = -\,\mathbf{A}_{sym} \bullet \mathbf{A}_{skew} = 0 \ . \tag{A.7.8}$$

This means that $(A.7.7)_1$ separates $\mathbf{A}$ into two orthogonal tensors.

Trace: The trace of a second order tensor $\mathbf{A}$ is defined as the dot product of $\mathbf{A}$ with the second order identity tensor $\mathbf{I}$

$$\mathrm{tr}\,\mathbf{A} = \mathbf{A} \bullet \mathbf{I} \ . \tag{A.7.9}$$

Using the representations (A.7.3) it can be shown that

$$\mathbf{A} \bullet \mathbf{I} = A_{ij}\, g^{ij} = A^{ij}\, g_{ij} = A_i^{\ i} = A^i_{\ i} \ . \tag{A.7.10}$$

Deviatoric tensor: The second order tensor $\mathbf{A}$ is said to be deviatoric if its trace vanishes

$$\mathbf{A} \bullet \mathbf{I} = 0 \ . \tag{A.7.11}$$

Spherical and deviatoric parts of a tensor: An arbitrary second order tensor $\mathbf{A}$ can also be uniquely separated into the sum of its spherical part (A $\mathbf{I}$) and its deviatoric part $\mathbf{A}'$ such that

$$\mathbf{A} = A\, \mathbf{I} + \mathbf{A}' \ , \quad A = \frac{1}{3}(\mathbf{A} \bullet \mathbf{I}) \quad , \quad \mathbf{A}' \bullet \mathbf{I} = 0 \ . \tag{A.7.12}$$

Moreover, it is obvious from (A.7.12) that the spherical and deviatoric parts of $\mathbf{A}$ are orthogonal to each other since

$$(A\, \mathbf{I}) \bullet \mathbf{A}' = 0 \ . \tag{A.7.13}$$

This means that $(A.7.12)_1$ separates $\mathbf{A}$ into two orthogonal tensors.

Polar decomposition theorem: An arbitrary nonsingular second order tensor $\mathbf{F}$ (det $\mathbf{F}$≠0) can be uniquely decomposed into the juxtaposition of an orthogonal rotation tensor $\mathbf{R}$ and two positive definite symmetric tensors $\mathbf{M}$ and $\mathbf{N}$ such that

$$\begin{gathered} \mathbf{F} = \mathbf{RM} = \mathbf{NR} \ , \quad \mathbf{R}^T\mathbf{R} = \mathbf{RR}^T = \mathbf{I} \ , \\ \mathbf{M}^T = \mathbf{M} = (\mathbf{F}^T\mathbf{F})^{1/2} \ , \quad \mathbf{M} \bullet (\mathbf{v} \otimes \mathbf{v}) > 0 \ \text{ for } \mathbf{v} \neq 0 \ , \\ \mathbf{N}^T = \mathbf{N} = (\mathbf{F}\mathbf{F}^T)^{1/2} \ , \quad \mathbf{N} \bullet (\mathbf{v} \otimes \mathbf{v}) > 0 \ \text{ for } \mathbf{v} \neq 0 \ , \end{gathered} \tag{A.7.14}$$

where $\mathbf{v}$ is an arbitrary nonzero vector. Moreover, if the determinant of $\mathbf{F}$ is positive, then $\mathbf{R}$ is a proper orthogonal tensor with

$$\det \mathbf{F} > 0 \ \Rightarrow \ \det \mathbf{R} = +1 \ . \tag{A.7.15}$$

The proof of this theorem can be found in (Malvern, 1969).

An additional property of nonsingular tensors: It can be shown that for an arbitrary nonsingular second order tensor $\mathbf{F}$ and arbitrary vectors $\mathbf{a}$ and $\mathbf{b}$

$$\mathbf{Fa} \times \mathbf{Fb} = (\det \mathbf{F})\ \mathbf{F}^{-T}(\mathbf{a} \times \mathbf{b}) \ . \tag{A.7.16}$$

This result can be proved by first noting that the vector $\mathbf{F}^{-T}(\mathbf{a} \times \mathbf{b})$ is orthogonal to both the vectors $\mathbf{Fa}$ and $\mathbf{Fb}$

$$\begin{gathered} \mathbf{F}^{-T}(\mathbf{a} \times \mathbf{b}) \bullet \mathbf{Fb} = (\mathbf{a} \times \mathbf{b}) \bullet \mathbf{F}^{-1}\mathbf{Fb} = (\mathbf{a} \times \mathbf{b}) \bullet \mathbf{b} = 0 \ , \\ \mathbf{F}^{-T}(\mathbf{a} \times \mathbf{b}) \bullet \mathbf{Fa} = (\mathbf{a} \times \mathbf{b}) \bullet \mathbf{F}^{-1}\mathbf{Fa} = (\mathbf{a} \times \mathbf{b}) \bullet \mathbf{a} = 0 \ , \end{gathered} \tag{A.7.17}$$

so that the vector ($\mathbf{Fa} \times \mathbf{Fb}$) must be parallel to the vector $\mathbf{F}^{-T}(\mathbf{a} \times \mathbf{b})$

$$\mathbf{Fa} \times \mathbf{Fb} = \alpha\ \mathbf{F}^{-T}(\mathbf{a} \times \mathbf{b}) \ . \tag{A.7.18}$$

Moreover, since the left-hand side of (A.7.18) is a linear function of both $\mathbf{a}$ and $\mathbf{b}$, the scalar α cannot depend on either $\mathbf{a}$ or $\mathbf{b}$. It also follows that for an arbitrary vector $\mathbf{c}$

$$\alpha\ (\mathbf{a} \times \mathbf{b} \bullet \mathbf{c}) = \mathbf{Fa} \times \mathbf{Fb} \bullet \mathbf{Fc} \ . \tag{A.7.19}$$

Now, the proof can be completed by using the rectangular Cartesian base vectors $\mathbf{e}_i$ and by taking $\mathbf{a}=\mathbf{e}_1$, $\mathbf{b}=\mathbf{e}_2$, and $\mathbf{c}=\mathbf{e}_3$ to shown that

$$\alpha = \det \mathbf{F} = \mathbf{Fe}_1 \times \mathbf{Fe}_2 \bullet \mathbf{Fe}_3 \ . \tag{A.7.20}$$

<u>Eigenvalues, eigenvectors and principal invariants of a real second order tensor:</u> The vector $\mathbf{v}$ is said to be an eigenvector of a real second order tensor $\mathbf{T}$ with the associated eigenvalue σ if

$$\mathbf{T}\ \mathbf{v} = \sigma\ \mathbf{v}\ . \tag{A.7.21}$$

It follows that the characteristic equation for determining the three values of the eigenvalue σ is given by

$$\det(\mathbf{T} - \sigma\,\mathbf{I}) = -\sigma^3 + I_1(\mathbf{T})\,\sigma^2 - I_2(\mathbf{T})\,\sigma + I_3(\mathbf{T}) = 0\ , \tag{A.7.22}$$

where the principal invariants I_1, I_2, I_3 of $\mathbf{T}$ are determined by the expressions

$$I_1(\mathbf{T}) = \mathbf{T} \bullet \mathbf{I}\ ,\quad I_2(\mathbf{T}) = \frac{1}{2}\left[(\mathbf{T} \bullet \mathbf{I})^2 - \mathbf{T} \bullet \mathbf{T}^T\right]\ ,\quad I_3(\mathbf{T}) = \det \mathbf{T}\ . \tag{A.7.23}$$

For a general nonsymmetric tensor $\mathbf{T}$, the eigenvalues and eigenvectors can have complex values. However, if $\mathbf{T}$ is symmetric, then the three roots $\{\sigma_1,\sigma_2,\sigma_3\}$ of the cubic equation (A.7.22) for σ are real and can be ordered so that

$$\sigma_1 \geq \sigma_2 \geq \sigma_3\ . \tag{A.7.24}$$

Also, the associated eigenvectors $\{\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3\}$ are real linearly independent vectors which can be normalized to form a right-handed orthonormal set. Moreover, $\mathbf{T}$ can be represented in the spectral form

$$\mathbf{T} = \sigma_1\ \mathbf{p}_1 \otimes \mathbf{p}_1 + \sigma_2\ \mathbf{p}_2 \otimes \mathbf{p}_2 + \sigma_3\ \mathbf{p}_3 \otimes \mathbf{p}_3\ . \tag{A.7.25}$$

In the remainder of this section attention will be confined to this symmetric case.

In order to determine explicit expressions for the eigenvalues, it is noted that the characteristic equation of a deviatoric tensor is a cubic equation in standard form since the first invariant I_1 of a deviatoric tensor vanishes. Moreover, using the fact that $\mathbf{T}$ can be separated into its spherical part $T\,\mathbf{I}$ and its deviatoric part $\mathbf{T}'$ such that

$$\mathbf{T} = T\,\mathbf{I} + \mathbf{T}'\ ,\quad T = \frac{1}{3}(\mathbf{T} \bullet \mathbf{I})\ ,\quad \mathbf{T}' \bullet \mathbf{I} = 0\ , \tag{A.7.26}$$

it follows that when $\mathbf{v}$ is an eigenvector of $\mathbf{T}$ it is also an eigenvector of $\mathbf{T}'$

$$\mathbf{T}'\ \mathbf{v} = (\mathbf{T} - T\,\mathbf{I})\ \mathbf{v} = (\sigma - T)\ \mathbf{v} = \sigma'\ \mathbf{v}\ , \tag{A.7.27}$$

with the associated eigenvalue σ' related to σ by

$$\sigma = \sigma' + T\ . \tag{A.7.28}$$

Thus, the values of σ can be deduced by solving the simpler problem for the values of σ' using the characteristic equation

$$\det(\mathbf{T}' - \sigma'\mathbf{I}) = -(\sigma')^3 + \sigma'\left(\frac{\sigma_e^2}{3}\right) + J_3 = 0\ , \tag{A.7.29}$$

where σ_e and J_3 are invariants of $\mathbf{T}'$ defined by

$$\sigma_e^2 = \frac{3}{2}\ \mathbf{T}' \bullet \mathbf{T}' = -3\ I_2(\mathbf{T}')\ ,\quad J_3 = \det \mathbf{T}' = I_3(\mathbf{T}')\ . \tag{A.7.30}$$

Note that if σ_e vanishes, then $\mathbf{T}'$ vanishes so that from (A.7.29), σ' vanishes and (A.7.28) indicates that there is only one distinct eigenvalue

$$\sigma = T\ . \tag{A.7.31}$$

On the other hand, if σ_e does not vanish, (A.7.29) can be divided by $(\sigma_e/3)^3$ to obtain

$$\left(\frac{3\sigma'}{\sigma_e}\right)^3 - 3\left(\frac{3\sigma'}{\sigma_e}\right) - 2\hat{J}_3 = 0 \ , \tag{A.7.32}$$

where the invariant $\hat{J}_3$ is defined by

$$\hat{J}_3 = \frac{27J_3}{2\sigma_e^3} \ . \tag{A.7.33}$$

Now, the solution of (A.7.32) can easily be obtained using the trigonometric form

$$\sin 3\beta = -\hat{J}_3 \ , \quad -\frac{\pi}{6} \le \beta \le \frac{\pi}{6} \ ,$$

$$\sigma'_1 = \frac{2\sigma_e}{3}\cos\left(\frac{\pi}{6} + \beta\right) \ , \quad \sigma'_2 = \frac{2\sigma_e}{3}\sin(\beta) \ , \quad \sigma'_3 = -\frac{2\sigma_e}{3}\cos\left(\frac{\pi}{6} - \beta\right) \ , \tag{A.7.34}$$

where the eigenvalues σ'_1, σ'_2, σ'_3 are automatically ordered so that

$$\sigma'_1 \ge \sigma'_2 \ge \sigma'_3 \ . \tag{A.7.35}$$

Then, the three values of σ are be calculated using (A.7.28).

Furthermore, if **T** is the stress tensor, then β is related to the Lode angle (Hill, 1971, p. 18; Vyalov, 1986, sec. 3.4), and the values of β can be used to identify three states of deviatoric stress denoted by: triaxial compression (TXC); torsion (TOR); and triaxial extension (TXE); and defined by

$$\beta = \frac{\pi}{6} \ \text{for (TXC)} \ , \quad \beta = 0 \ \text{for (TOR)} \ , \quad \beta = -\frac{\pi}{6} \ \text{for (TXE)} \ . \tag{A.7.36}$$

APPENDIX B

SUMMARY OF TENSOR OPERATIONS IN SPECIFIC COORDINATE SYSTEMS

B.1 Cylindrical polar coordinates

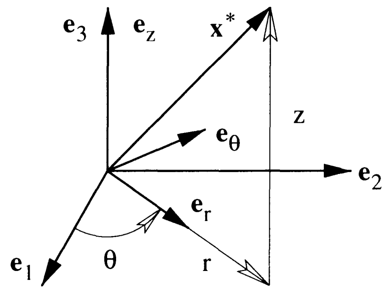

Fig. B.1.1 Definition of cylindrical polar coordinates.

The right-handed orthonormal base vectors $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ associated with the cylindrical polar coordinates $\{r, \theta, z\}$ are defined in terms of the fixed base vectors $\mathbf{e}_i$ of a rectangular Cartesian coordinate system by the equations

$$\mathbf{e}_r = \mathbf{e}_r(\theta) = \cos\theta\,\mathbf{e}_1 + \sin\theta\,\mathbf{e}_2 \ , \ \mathbf{e}_\theta = \mathbf{e}_\theta(\theta) = -\sin\theta\,\mathbf{e}_1 + \cos\theta\,\mathbf{e}_2 \ ,$$
$$\mathbf{e}_z = \mathbf{e}_3 \ , \tag{B.1.1}$$

where $\mathbf{e}_r$ and $\mathbf{e}_\theta$ lie in the $\mathbf{e}_1$–$\mathbf{e}_2$ plane and the angle θ is measured counterclockwise from the $\mathbf{e}_1$ direction (see Fig. B.1.1). Also, the position vector $\mathbf{x}^*$ of a point in the three-dimensional space can be represented in the form

$$\mathbf{x}^* = r\,\mathbf{e}_r(\theta) + z\,\mathbf{e}_z \ . \tag{B.1.2}$$

Since the objective of this appendix is to present formulas for derivatives of tensors expressed in terms of these coordinates, it is necessary to first record the derivatives of the base vectors

$$\frac{d\mathbf{e}_r}{d\theta} = \mathbf{e}_\theta \ , \ \frac{d\mathbf{e}_\theta}{d\theta} = -\,\mathbf{e}_r \ , \ \frac{d\mathbf{e}_z}{d\theta} = 0 \ . \tag{B.1.3}$$

Also, by taking θ^i to be general curvilinear coordinates (not necessarily convected coordinates) and setting

$$\theta^i = \{r \, , \theta \, , z\} \ , \tag{B.1.4}$$

it follows that the covariant base vectors $\mathbf{g}_i$, the contravariant vectors $\mathbf{g}^i$ and the scalar $g^{1/2}$ are given by

$$\mathbf{g}_1 = \frac{\partial \mathbf{x}^*}{\partial r} = \mathbf{e}_r \ , \ \mathbf{g}^1 = \mathbf{e}_r \ , \ \mathbf{g}_2 = \frac{\partial \mathbf{x}^*}{\partial \theta} = r\,\mathbf{e}_\theta \ , \ \mathbf{g}^2 = \frac{1}{r}\,\mathbf{e}_\theta \ ,$$

$$\mathbf{g}_3 = \frac{\partial \mathbf{x}^*}{\partial z} = \mathbf{e}_z \ , \quad \mathbf{g}^3 = \mathbf{e}_z \ , \quad g^{1/2} = r \ . \tag{B.1.5}$$

Next, let f, **v** and **T** be, respectively, scalar, vector and second order tensor functions of $\{r,\theta,z\}$. Furthermore, let **v** and **T** be expressed in terms of their physical components by

$$\begin{aligned}
\mathbf{v} &= v_r\,\mathbf{e}_r + v_\theta\,\mathbf{e}_\theta + v_z\,\mathbf{e}_z \ , \\
\mathbf{T} &= T_{rr}\,(\mathbf{e}_r\otimes\mathbf{e}_r) + T_{r\theta}\,(\mathbf{e}_r\otimes\mathbf{e}_\theta) + T_{rz}\,(\mathbf{e}_r\otimes\mathbf{e}_z) \\
&\quad + T_{\theta r}\,(\mathbf{e}_\theta\otimes\mathbf{e}_r) + T_{\theta\theta}\,(\mathbf{e}_\theta\otimes\mathbf{e}_\theta) + T_{\theta z}\,(\mathbf{e}_\theta\otimes\mathbf{e}_z) \\
&\quad + T_{zr}\,(\mathbf{e}_z\otimes\mathbf{e}_r) + T_{z\theta}\,(\mathbf{e}_z\otimes\mathbf{e}_\theta) + T_{zz}\,(\mathbf{e}_z\otimes\mathbf{e}_z) \ .
\end{aligned} \tag{B.1.6}$$

Then, the gradient operator can be expressed as

$$\begin{aligned}
\mathrm{grad}^* f &= \frac{\partial f}{\partial r}\,\mathbf{e}_r + \frac{1}{r}\,\frac{\partial f}{\partial \theta}\,\mathbf{e}_\theta + \frac{\partial f}{\partial z}\,\mathbf{e}_z \ , \\
\mathrm{grad}^* \mathbf{v} &= \frac{\partial v_r}{\partial r}(\mathbf{e}_r\otimes\mathbf{e}_r) + \frac{1}{r}\left[\frac{\partial v_r}{\partial \theta} - v_\theta\right](\mathbf{e}_r\otimes\mathbf{e}_\theta) + \frac{\partial v_r}{\partial z}(\mathbf{e}_r\otimes\mathbf{e}_z) \\
&\quad + \frac{\partial v_\theta}{\partial r}(\mathbf{e}_\theta\otimes\mathbf{e}_r) + \frac{1}{r}\left[\frac{\partial v_\theta}{\partial \theta} + v_r\right](\mathbf{e}_\theta\otimes\mathbf{e}_\theta) + \frac{\partial v_\theta}{\partial z}(\mathbf{e}_\theta\otimes\mathbf{e}_z) \\
&\quad + \frac{\partial v_z}{\partial r}(\mathbf{e}_z\otimes\mathbf{e}_r) + \frac{1}{r}\,\frac{\partial v_z}{\partial \theta}(\mathbf{e}_z\otimes\mathbf{e}_\theta) + \frac{\partial v_z}{\partial z}(\mathbf{e}_z\otimes\mathbf{e}_z) \ ,
\end{aligned} \tag{B.1.7}$$

the divergence operator can be expressed as

$$\begin{aligned}
\mathrm{div}^* \mathbf{v} &= \frac{\partial v_r}{\partial r} + \frac{1}{r}\left[\frac{\partial v_\theta}{\partial \theta} + v_r\right] + \frac{\partial v_z}{\partial z} \ , \\
\mathrm{div}^* \mathbf{T} &= \left[\frac{\partial T_{rr}}{\partial r} + \frac{1}{r}\frac{\partial T_{r\theta}}{\partial \theta} + \frac{\partial T_{rz}}{\partial z} + \frac{T_{rr} - T_{\theta\theta}}{r}\right]\mathbf{e}_r \\
&\quad + \left[\frac{\partial T_{\theta r}}{\partial r} + \frac{1}{r}\frac{\partial T_{\theta\theta}}{\partial \theta} + \frac{\partial T_{\theta z}}{\partial z} + \frac{T_{\theta r} + T_{r\theta}}{r}\right]\mathbf{e}_\theta \\
&\quad + \left[\frac{\partial T_{zr}}{\partial r} + \frac{1}{r}\frac{\partial T_{z\theta}}{\partial \theta} + \frac{\partial T_{zz}}{\partial z} + \frac{T_{zr}}{r}\right]\mathbf{e}_z \ ,
\end{aligned} \tag{B.1.8}$$

the curl operator can be expressed as

$$\begin{aligned}
\mathrm{curl}^* \mathbf{v} &= \left[\frac{1}{r}\,\frac{\partial v_z}{\partial \theta} - \frac{\partial v_\theta}{\partial z}\right]\mathbf{e}_r + \left[\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}\right]\mathbf{e}_\theta \\
&\quad + \left[\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} - \frac{1}{r}\,\frac{\partial v_r}{\partial \theta}\right]\mathbf{e}_z \ .
\end{aligned} \tag{B.1.9}$$

and the Laplacian operator can be expressed as

$$\nabla^{*2} f = \mathrm{div}^*\,(\mathrm{grad}^*\,f) = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\,\frac{\partial f}{\partial r} + \frac{1}{r^2}\,\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2} \ . \tag{B.1.10}$$

B.2 Spherical polar coordinates

The right-handed orthonormal base vectors $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi\}$ associated with the spherical polar coordinates $\{r, \theta, \phi\}$ are defined in terms of the fixed base vectors $\mathbf{e}_i$ of a rectangular Cartesian coordinate system by the equations

$$\mathbf{e}_r = \mathbf{e}_r(\theta,\phi) = \sin\theta\,(\cos\phi\,\mathbf{e}_1 + \sin\phi\,\mathbf{e}_2) + \cos\theta\,\mathbf{e}_3 \ ,$$
$$\mathbf{e}_\theta = \mathbf{e}_\theta(\theta,\phi) = \cos\theta\,(\cos\phi\,\mathbf{e}_1 + \sin\phi\,\mathbf{e}_2) - \sin\theta\,\mathbf{e}_3 \ ,$$
$$\mathbf{e}_\phi = \mathbf{e}_\phi(\phi) = -\sin\phi\,\mathbf{e}_1 + \cos\phi\,\mathbf{e}_2 \ , \qquad \text{(B.2.1)}$$

where the angle ϕ is measured in the horizontal plane counterclockwise from the $\mathbf{e}_1$ direction to the vertical plane which includes the position vector (see Fig. B.2.1), and θ is the acute angle measured from the vertical direction $\mathbf{e}_3$ to the position vector. Also, the position vector $\mathbf{x}^*$ of a point in the three-dimensional space can be represented in the form

$$\mathbf{x}^* = r\,\mathbf{e}_r(\theta,\phi) \ . \qquad \text{(B.2.2)}$$

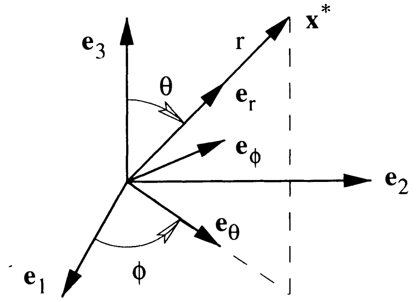

Fig. B.2.1 Definition of spherical polar coordinates.

Since the objective of this appendix is to present formulas for derivatives of tensors expressed in terms of these coordinates, it is necessary to first record the derivatives of the base vectors

$$\frac{\partial \mathbf{e}_r}{\partial \theta} = \mathbf{e}_\theta \ , \quad \frac{\partial \mathbf{e}_r}{\partial \phi} = \sin\theta\,\mathbf{e}_\phi \ , \quad \frac{\partial \mathbf{e}_\theta}{\partial \theta} = -\mathbf{e}_r \ , \quad \frac{\partial \mathbf{e}_\theta}{\partial \phi} = \cos\theta\,\mathbf{e}_\phi \ ,$$
$$\frac{\partial \mathbf{e}_\phi}{\partial \theta} = 0 \ , \quad \frac{\partial \mathbf{e}_\phi}{\partial \phi} = -\sin\theta\,\mathbf{e}_r - \cos\theta\,\mathbf{e}_\theta \ . \qquad \text{(B.2.3)}$$

Also, by taking θ^i to be general curvilinear coordinates (not necessarily convected coordinates) and setting

$$\theta^i = \{r, \theta, \phi\} \ , \qquad \text{(B.2.4)}$$

it follows that the covariant base vectors $\mathbf{g}_i$, the contravariant vectors $\mathbf{g}^i$ and the scalar $g^{1/2}$ are given by

$$\mathbf{g}_1 = \frac{\partial \mathbf{x}^*}{\partial r} = \mathbf{e}_r \ , \quad \mathbf{g}^1 = \mathbf{e}_r \ , \quad \mathbf{g}_2 = \frac{\partial \mathbf{x}^*}{\partial \theta} = r\,\mathbf{e}_\theta \ , \quad \mathbf{g}^2 = \frac{1}{r}\,\mathbf{e}_\theta \ ,$$

$$\mathbf{g}_3 = \frac{\partial \mathbf{x}^*}{\partial \phi} = \mathrm{r}\sin\theta\, \mathbf{e}_\phi \ , \quad \mathbf{g}^3 = \frac{1}{\mathrm{r}\sin\theta}\, \mathbf{e}_\phi \ , \quad \mathrm{g}^{1/2} = \mathrm{r}^2\sin\theta \ . \tag{B.2.5}$$

Next, let f, **v** and **T** be, respectively, scalar, vector and second order tensor functions of $\{r,\theta,\phi\}$. Furthermore, let **v** and **T** be expressed in terms of their physical components by

$$\begin{aligned}
\mathbf{v} &= \mathrm{v_r}\, \mathbf{e}_r + \mathrm{v}_\theta\, \mathbf{e}_\theta + \mathrm{v}_\phi\, \mathbf{e}_\phi \ , \\
\mathbf{T} &= \mathrm{T_{rr}}\, (\mathbf{e}_r\otimes\mathbf{e}_r) + \mathrm{T_{r\theta}}\, (\mathbf{e}_r\otimes\mathbf{e}_\theta) + \mathrm{T_{r\phi}}\, (\mathbf{e}_r\otimes\mathbf{e}_\phi) \\
&+ \mathrm{T_{\theta r}}\, (\mathbf{e}_\theta\otimes\mathbf{e}_r) + \mathrm{T_{\theta\theta}}\, (\mathbf{e}_\theta\otimes\mathbf{e}_\theta) + \mathrm{T_{\theta\phi}}\, (\mathbf{e}_\theta\otimes\mathbf{e}_\phi) \\
&+ \mathrm{T_{\phi r}}\, (\mathbf{e}_\phi\otimes\mathbf{e}_r) + \mathrm{T_{\phi\theta}}\, (\mathbf{e}_\phi\otimes\mathbf{e}_\theta) + \mathrm{T_{\phi\phi}}\, (\mathbf{e}_\phi\otimes\mathbf{e}_\phi) \ .
\end{aligned} \tag{B.2.6}$$

Then, the gradient operator can be expressed as

$$\mathrm{grad}^*\mathrm{f} = \frac{\partial \mathrm{f}}{\partial \mathrm{r}}\, \mathbf{e}_r + \frac{1}{\mathrm{r}}\frac{\partial \mathrm{f}}{\partial \theta}\, \mathbf{e}_\theta + \frac{1}{\mathrm{r}\sin\theta}\frac{\partial \mathrm{f}}{\partial \phi}\, \mathbf{e}_\phi \ ,$$

$$\begin{aligned}
\mathrm{grad}^*\mathbf{v} &= \frac{\partial \mathrm{v_r}}{\partial \mathrm{r}}(\mathbf{e}_r\otimes\mathbf{e}_r) + \frac{1}{\mathrm{r}}\Big[\frac{\partial \mathrm{v_r}}{\partial \theta} - \mathrm{v}_\theta\Big](\mathbf{e}_r\otimes\mathbf{e}_\theta) + \frac{1}{\mathrm{r}\sin\theta}\Big[\frac{\partial \mathrm{v_r}}{\partial \phi} - \mathrm{v}_\phi\sin\theta\Big](\mathbf{e}_r\otimes\mathbf{e}_\phi) \\
&+ \frac{\partial \mathrm{v}_\theta}{\partial \mathrm{r}}(\mathbf{e}_\theta\otimes\mathbf{e}_r) + \frac{1}{\mathrm{r}}\Big[\frac{\partial \mathrm{v}_\theta}{\partial \theta} + \mathrm{v_r}\Big](\mathbf{e}_\theta\otimes\mathbf{e}_\theta) + \frac{1}{\mathrm{r}\sin\theta}\Big[\frac{\partial \mathrm{v}_\theta}{\partial \phi} - \mathrm{v}_\phi\cos\theta\Big](\mathbf{e}_\theta\otimes\mathbf{e}_\phi) \\
&+ \frac{\partial \mathrm{v}_\phi}{\partial \mathrm{r}}(\mathbf{e}_\phi\otimes\mathbf{e}_r) + \frac{1}{\mathrm{r}}\frac{\partial \mathrm{v}_\phi}{\partial \theta}(\mathbf{e}_\phi\otimes\mathbf{e}_\theta) + \frac{1}{\mathrm{r}\sin\theta}\Big[\frac{\partial \mathrm{v}_\phi}{\partial \phi} + \mathrm{v_r}\sin\theta + \mathrm{v}_\theta\cos\theta\Big](\mathbf{e}_\phi\otimes\mathbf{e}_\phi)
\end{aligned} \tag{B.2.7}$$

the divergence operator can be expressed as

$$\mathrm{div}^*\mathbf{v} = \frac{\partial \mathrm{v_r}}{\partial \mathrm{r}} + \frac{2\mathrm{v_r}}{\mathrm{r}} + \frac{1}{\mathrm{r}}\frac{\partial \mathrm{v}_\theta}{\partial \theta} + \frac{\mathrm{v}_\theta\cot\theta}{\mathrm{r}} + \frac{1}{\mathrm{r}\sin\theta}\frac{\partial \mathrm{v}_\phi}{\partial \phi} \ ,$$

$$\begin{aligned}
\mathrm{div}^*\mathbf{T} &= \Big[\frac{\partial \mathrm{T_{rr}}}{\partial \mathrm{r}} + \frac{1}{\mathrm{r}}\frac{\partial \mathrm{T_{r\theta}}}{\partial \theta} + \frac{1}{\mathrm{r}\sin\theta}\frac{\partial \mathrm{T_{r\phi}}}{\partial \phi} + \frac{2\mathrm{T_{rr}} - \mathrm{T_{\theta\theta}} - \mathrm{T_{\phi\phi}} + \mathrm{T_{r\theta}}\cot\theta}{\mathrm{r}}\Big]\, \mathbf{e}_r \\
&+ \Big[\frac{\partial \mathrm{T_{\theta r}}}{\partial \mathrm{r}} + \frac{1}{\mathrm{r}}\frac{\partial \mathrm{T_{\theta\theta}}}{\partial \theta} + \frac{1}{\mathrm{r}\sin\theta}\frac{\partial \mathrm{T_{\theta\phi}}}{\partial \phi} + \frac{2\mathrm{T_{\theta r}} + \mathrm{T_{r\theta}} + (\mathrm{T_{\theta\theta}} - \mathrm{T_{\phi\phi}})\cot\theta}{\mathrm{r}}\Big]\, \mathbf{e}_\theta \\
&+ \Big[\frac{\partial \mathrm{T_{\phi r}}}{\partial \mathrm{r}} + \frac{1}{\mathrm{r}}\frac{\partial \mathrm{T_{\phi\theta}}}{\partial \theta} + \frac{1}{\mathrm{r}\sin\theta}\frac{\partial \mathrm{T_{\phi\phi}}}{\partial \phi} + \frac{2\mathrm{T_{\phi r}} + \mathrm{T_{r\phi}} + (\mathrm{T_{\phi\theta}} + \mathrm{T_{\theta\phi}})\cot\theta}{\mathrm{r}}\Big]\, \mathbf{e}_\phi \ ,
\end{aligned} \tag{B.2.8}$$

the curl operator can be expressed as

$$\begin{aligned}
\mathrm{curl}^*\mathbf{v} &= \Big[\frac{1}{\mathrm{r}}\frac{\partial \mathrm{v}_\phi}{\partial \theta} + \frac{\mathrm{v}_\phi\cot\theta}{\mathrm{r}} - \frac{1}{\mathrm{r}\sin\theta}\frac{\partial \mathrm{v}_\theta}{\partial \phi}\Big]\, \mathbf{e}_r \\
&+ \Big[\frac{1}{\mathrm{r}\sin\theta}\frac{\partial \mathrm{v_r}}{\partial \phi} - \frac{\partial \mathrm{v}_\phi}{\partial \mathrm{r}} - \frac{\mathrm{v}_\phi}{\mathrm{r}}\Big]\, \mathbf{e}_\theta + \Big[\frac{\partial \mathrm{v}_\theta}{\partial \mathrm{r}} + \frac{\mathrm{v}_\theta}{\mathrm{r}} - \frac{1}{\mathrm{r}}\frac{\partial \mathrm{v_r}}{\partial \theta}\Big]\, \mathbf{e}_\phi \ .
\end{aligned} \tag{B.2.9}$$

and the Laplacian operator can be expressed as

$$\nabla^{*2}\mathrm{f} = \mathrm{div}^*(\mathrm{grad}^*\, \mathrm{f}) = \frac{\partial^2 \mathrm{f}}{\partial \mathrm{r}^2} + \frac{2}{\mathrm{r}}\frac{\partial \mathrm{f}}{\partial \mathrm{r}} + \frac{1}{\mathrm{r}^2}\frac{\partial^2 \mathrm{f}}{\partial \theta^2} + \frac{\cot\theta}{\mathrm{r}^2}\frac{\partial \mathrm{f}}{\partial \theta} + \frac{1}{\mathrm{r}^2\sin^2\theta}\frac{\partial^2 \mathrm{f}}{\partial \phi^2} \ . \tag{B.2.10}$$

EXERCISES

In the following, the number of a typical exercise problem is denoted by E2.3-2, which indicates that it is the second exercise related to section 2.3.

Exercises for Chapter 2

Section 2.1

E2.1-1 Show that the expressions (2.1.16) satisfy the equations (2.1.11).

Section 2.2

E2.2-1 Derive the expressions (B1.5) for the covariant vectors $\mathbf{g}_i$, the contravariant vectors $\mathbf{g}^i$, and the scalar $g^{1/2}$ associated with cylindrical polar coordinates.

E2.2-2 Using the representation $(B1.6)_1$ for the vector $\mathbf{v}$ in terms of its physical components in a cylindrical polar coordinate system, derive expressions for its covariant components v_i and its contravariant components v^i.

E2.2-3 Using the representation $(B1.6)_2$ for the tensor $\mathbf{T}$ in terms of its physical components in the cylindrical polar coordinate system, derive expressions for its covariant components T_{ij}, its contravariant components T^{ij}, and its mixed components $T_i{}^j$ and $T^i{}_j$.

E2.2-4 Derive the expressions (B2.5) for the covariant vectors $\mathbf{g}_i$, the contravariant vectors $\mathbf{g}^i$, and the scalar $g^{1/2}$ associated with spherical polar coordinates.

E2.2-5 Using equations (A.7.19) and (A.7.20) it can be shown that the determinant of a second order tensor $\mathbf{T}$ satisfies the equation

$$(\mathbf{a} \times \mathbf{b} \bullet \mathbf{c}) \det \mathbf{T} = \mathbf{Ta} \times \mathbf{Tb} \bullet \mathbf{Tc} \ ,$$

for any vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$. With the help of this result, prove that

$$g^{1/2} \det \mathbf{T} = \varepsilon^{ijk}\, T_{i1}\, T_{j2}\, T_{k3} = g^{-1/2} \det (T_{ij}) \ , \ \det (T_{ij}) = e^{ijk}\, T_{i1}\, T_{j2}\, T_{k3} \ ,$$

$$g^{1/2} \det \mathbf{T} = \varepsilon_{ijk}\, T^{i1}\, T^{j2}\, T^{k3} = g^{1/2} \det (T^{ij}) \ , \ \det (T^{ij}) = e_{ijk}\, T^{i1}\, T^{j2}\, T^{k3} \ ,$$

E2.2-6 Show that the results in exercise E2.2-5 can be rewritten in the alternative forms

$$\det \mathbf{T} = \det (T_{ij})\, (\mathbf{g}^1 \times \mathbf{g}^2 \bullet \mathbf{g}^3)\, (\mathbf{g}^1 \times \mathbf{g}^2 \bullet \mathbf{g}^3) = \det (T^{ij})\, (\mathbf{g}_1 \times \mathbf{g}_2 \bullet \mathbf{g}_3)\, (\mathbf{g}_1 \times \mathbf{g}_2 \bullet \mathbf{g}_3) \ ,$$

$$\det \mathbf{T} = \det (T^i{}_j)\, (\mathbf{g}_1 \times \mathbf{g}_2 \bullet \mathbf{g}_3)\, (\mathbf{g}^1 \times \mathbf{g}^2 \bullet \mathbf{g}^3) = \det (T_i{}^j)\, (\mathbf{g}^1 \times \mathbf{g}^2 \bullet \mathbf{g}^3)\, (\mathbf{g}_1 \times \mathbf{g}_2 \bullet \mathbf{g}_3) \ .$$

E2.2-7 Using the representation $\mathbf{I} = g_{ij}\, \mathbf{g}^i \otimes \mathbf{g}^j = g^{ij}\, \mathbf{g}_i \otimes \mathbf{g}_j$ for the second order identity tensor and using the formulas of exercise E.2.2-5, prove that

$$\det (g_{ij}) = g \ , \ \det (g^{ij}) = g^{-1} \ .$$

E2.2-8 Consider the simple shearing deformation characterized by

$$\mathbf{x}^* = (X + \gamma\, Y)\, \mathbf{e}_1 + Y\, \mathbf{e}_2 + Z\, \mathbf{e}_3 \ , \ \theta^i = (X,Y,Z) \ ,$$

where γ is a constant. Show that

$$\mathbf{g}_1 = \mathbf{e}_1 \ , \ \ \mathbf{g}_2 = \gamma\, \mathbf{e}_1 + \mathbf{e}_2 \ , \ \ \mathbf{g}_3 = \mathbf{e}_3 \ , \ \ g^{1/2} = 1 \ ,$$

$$\mathbf{g}^1\ \mathbf{e}_1 - \gamma\, \mathbf{e}_2 \ , \ \ \mathbf{g}^2 = \mathbf{e}_2 \ , \ \ \mathbf{g}^3 = \mathbf{e}_3 \ ,$$

and sketch the vectors $\mathbf{g}_1$, $\mathbf{g}_2$, $\mathbf{g}^1$, $\mathbf{g}^2$ in the $\mathbf{e}_1$–$\mathbf{e}_2$ plane.

Section 2.3

E2.3-1 Using the representation $(B2.6)_1$ for the vector $\mathbf{v}$, derive the expressions $(B2.7)_2$ for grad* $\mathbf{v}$ and $(B2.8)_1$ for div* $\mathbf{v}$ in spherical polar coordinates.

E2.3-2 Derive the expression (B2.10) for the Laplacian of a scalar f in spherical polar coordinates.

E2.3-3 Prove the validity of the equation

$$(g^{1/2}\, \mathbf{g}^j)_{,j} = 0 \ .$$

E2.3-4 Using the representation $(B2.6)_2$ for the second order tensor $\mathbf{T}$ and the expression (2.3.16), derive the expression $(B2.8)_2$ for div* $\mathbf{T}$ in spherical polar coordinates.

E2.3-5 Consider the axial shearing deformation in cylindrical polar coordinates which is characterized by the coordinates and position vectors

$$\theta^i = \{ R , \Theta , Z \} \ , \ \ r = R \ , \ \ \theta = \Theta \ , \ \ z = Z + \gamma R \ ,$$

$$\mathbf{X}^* = R\, \mathbf{e}_r(\Theta) + Z\, \mathbf{e}_3 \ , \ \ \mathbf{x}^* = R\, \mathbf{e}_r(\theta) + [Z + \gamma R]\, \mathbf{e}_3 \ ,$$

where γ is a constant. Using the expression (2.3.8), show that the deformation gradient $\mathbf{F}^*$ can be represented in the form

$$\mathbf{F}^* = \mathbf{I} + \gamma\, (\mathbf{e}_3 \otimes \mathbf{e}_r) \ .$$

Also, show that the right Cauchy-Green deformation tensor $\mathbf{C}^* = \mathbf{F}^{*T}\mathbf{F}^*$ is given by

$$\mathbf{C}^* = \mathbf{I} + \gamma^2\, \mathbf{e}_r \otimes \mathbf{e}_r + \gamma\, (\mathbf{e}_r \otimes \mathbf{e}_3 + \mathbf{e}_3 \otimes \mathbf{e}_r) \ .$$

In particular, notice that even though the coordinates $\{r,\theta,z\}$ are linear functions of the reference coordinates $\{R,\Theta,Z\}$, the deformation is inhomogeneous since $\mathbf{F}^*$ and $\mathbf{C}^*$ depend on θ through the base vector $\mathbf{e}_r$.

E2.3-6 Consider simple torsion of a cylindrical bar which is characterized by the coordinates and position vectors

$$\theta^i = \{ R , \Theta , Z \} \ , \ \ r = R \ , \ \ \theta = \Theta + \gamma Z \ , \ \ z = Z \ ,$$

$$\mathbf{X}^* = R\, \mathbf{e}_r(\Theta) + Z\, \mathbf{e}_3 \ , \ \ \mathbf{x}^* = R\, \mathbf{e}_r(\theta) + Z\, \mathbf{e}_3 \ ,$$

where γ is a constant. Using the expression (2.3.8), show that the deformation gradient $\mathbf{F}^*$ can be represented in the form

$$\mathbf{F}^* = \mathbf{e}_r(\theta) \otimes \mathbf{e}_r(\Theta) + \mathbf{e}_\theta(\theta) \otimes \mathbf{e}_\theta(\Theta) + \mathbf{e}_3 \otimes \mathbf{e}_3 + \gamma\, R\, \mathbf{e}_\theta(\theta) \otimes \mathbf{e}_3 \ .$$

Also, show that the right Cauchy-Green deformation tensor $\mathbf{C}^* = \mathbf{F}^{*T}\mathbf{F}^*$ is given by

$$\mathbf{C}^* = \mathbf{I} + (\gamma\, R)^2\, \mathbf{e}_3 \otimes \mathbf{e}_3 + \gamma\, R\, [\mathbf{e}_\theta(\Theta) \otimes \mathbf{e}_3 + \mathbf{e}_3 \otimes \mathbf{e}_\theta(\Theta)] \ .$$

In particular, note that since $\theta \neq \Theta$, the directions $\mathbf{e}_r(\theta)$ and $\mathbf{e}_\theta(\theta)$ are different from $\mathbf{e}_r(\Theta)$ and $\mathbf{e}_\theta(\theta)$. Also, note that the deformation is not homogeneous since $\mathbf{F}^*$ and $\mathbf{C}^*$ depend on the coordinates.

E2.3-7 Let f be a scalar function of the convected coordinates θ^i. The Laplacian of f with respect to the reference configuration is denoted by the symbol $\nabla^2 f$ and is defined by

$$\nabla^2 f = \mathrm{Div}^* \, (\mathrm{Grad}^* \, f) \ .$$

Show that $\nabla^2 f$ can be written in the form

$$\nabla^2 f = G^{-1/2} \, [G^{1/2} \, f_{,i} \, G^{ij}]_{,j} \ .$$

E2.3-8 Show that for rectangular Cartesian coordinates with $\mathbf{G}_i = \mathbf{e}_i$, the Laplacian of f in E2.3.7 can be rewritten in the form

$$\nabla^2 f = f_{,jj} \ .$$

Section 2.4

E2.4-1 Using the expressions (2.4.9), prove that the covariant derivatives of the metrics g^{ij} and g_{ij} vanish

$$g^{ij}{}_{|m} = 0 \ , \quad g_{ij|m} = 0 \ .$$

E2.4-2 Using the result of problem E2.4-1 and the equation

$$g_{ik|j} + g_{kj|i} - g_{ij|k} = 0 \ ,$$

prove that the Christoffel symbol $\Gamma_{ij.}^{\ \ m}$ can be expressed in the form

$$\Gamma_{ij.}^{\ \ m} = \frac{1}{2} [\, g_{ik,j} + g_{kj,i} - g_{ij,k} \,] \, g^{km} \ ,$$

which is a function of the metric only. This means that the notion of covariant differentiation (2.4.9) is valid in a general Riemannian space where the metric g_{ij} is specified as a function of the coordinates, but a positioan vector $\mathbf{x}^*$ leading to the base vectors $\mathbf{g}_i = \mathbf{x}^*_{,i}$ does not necessarily exist.

Exercises for Chapter 3

Section 3.2

E3.2-1 Prove the validity of equation (3.2.18).

E3.2-2 Prove the validity of equation (3.2.22).

Section 3.4

E3.4-1 Prove the validity of the expression (3.4.7).

Section 3.5

E3.5-1 Prove the validity of the expressions (3.5.7).

E3.5-2 Prove the validity of the expressions (3.5.8).

E3.5-3 Prove the validity of the expressions (3.5.9).

Section 3.6

E3.6-1 Derive the equation (3.6.3).

Section 3.8

E3.8-1 Derive an expression for the constraint response $\bar{\mathbf{T}}^*$ associated with the constraint (3.8.2).

Section 3.10

E3.10-1 Prove that $\mathbf{H}^*$ in (3.11.4) is an orthogonal tensor.

Section 3.11

E3.11-1 Derive an expression for the material derivative of $\mathbf{B}^{*\prime}$.

E3.11-2 Prove the validity of the restriction (3.11.18).

Section 3.12

E3.12-1 Explain why the result (3.12.9) depends on the symmetry of K^*_{ijkl}.

E3.12-2 Starting with (3.12.13), prove the validity of (3.12.14).

Section 3.13

E3.13-1 Rederive the equation for $g^{1/2}$ in (3.13.5).

E3.13-2 Rederive the equation for ρ in (3.13.6).

Section 3.14

E3.14-1 Show that the stresses (3.14.8) satisfy the equations of equilibrium.

E3.14-2 Show that the constants α^*_{12}, α^*_{13}, α^*_{23} associated with rigid body rotations do not influence the strains.

Section 3.15

E3.15-1 Starting with the displacements (3.15.1), derive the expressions (3.15.2) for the stresses.

E3.15-2 Rederive the boundary conditions (3.15.4).

E3.15-3 Rederive the expressions (3.15.11) for the moments.

Section 3.16

E3.16-1 Starting with the displacements (3.16.1), derive the expressions (3.16.2) for the stresses.

E3.16-2 Prove the validity of the expressions (3.16.3) for the vibrational frequencies.

Section 3.17

E3.17-1 Starting with the displacements (3.17.2), derive the expressions (3.17.4) for the stresses.

E3.17-2 Prove the validity of the expression (3.17.5) for the natural frequency.

Section 3.18

E3.18-1 Show that the stresses (3.18.2) satisfy the equations of equilibrium.

E3.18-2 Starting with the displacements (3.18.4), derive the expressions (3.18.2) for the stresses.

Section 3.19

E3.19-1 Show that the stresses (3.19.5) satisfies the equations of equilibrium.

E3.19-2 Starting with the displacements (3.19.10), derive the expression (3.19.5) for the stress.

Section 3.20

E3.20-1 Rederive the equation of motion (3.20.3).

Exercises for Chapter 4

Section 4.1

E4.1-1 Show that the reciprocal vectors in (4.1.5) satisfy the conditions $(2.3.3)_3$.

E4.1-2 Rederive the expression for $G^{1/2}$ in (4.1.16).

Section 4.2

E4.2-1 Prove the validity of equation (4.2.11).

E4.2-2 The second fundamental form $b_{\alpha\beta}$ which characterizes the curvature of the surface $\mathbf{x}(\theta^\alpha,t)$, is defined in terms of the tangent vectors $\mathbf{a}_\alpha$ and the unit normal vector $\mathbf{a}_3$ by the expression

$$b_{\alpha\beta} = \mathbf{a}_3 \bullet \mathbf{a}_{\alpha,\beta} \ .$$

Prove that $b_{\alpha\beta}$ is symmetric ($b_{\alpha\beta} = b_{\beta\alpha}$) and that $b_{\alpha\beta}$ can also be determined by the equations

$$b_{\alpha\beta} = -\,\mathbf{a}_{3,\alpha} \bullet \mathbf{a}_\beta \ .$$

Furthermore, it is recalled that the Gaussian curvature K and the mean curvature H of a surface are defined by

$$K = \frac{b_{11}b_{22} - b_{12}b_{12}}{a_{11}a_{22} - a_{12}a_{12}} \ , \quad H = \frac{1}{2} a^{\alpha\beta}\, b_{\alpha\beta} \ ,$$

where $a^{\alpha\beta}$ is the metric defined by

$$a^{\alpha\beta} = \mathbf{a}^\alpha \bullet \mathbf{a}^\beta \ .$$

E4.2-3 Consider a right circular cylindrical surface characterized by

$$\mathbf{x}(\theta^\alpha) = r\,\mathbf{e}_r(\theta) + z\,\mathbf{e}_3 \ , \quad \theta^\alpha = (\theta \ , z) \ ,$$

where r is a constant. Show that

$$\mathbf{a}_1 = r\,\mathbf{e}_\theta \ , \quad \mathbf{a}_2 = \mathbf{e}_3 \ , \quad \mathbf{a}_3 = \mathbf{e}_r \ , \quad a^{1/2} = r \ ,$$

$$b_{11} = -r \ , \quad b_{12} = b_{22} = 0 \ , \quad K = 0 \ , \quad H = -\frac{1}{2r} \ ,$$

where K and H are defined in exercise E4.2-2.

E4.2-4 Consider axisymmetric deformation of a circular cylindrical membrane characterized by

$$\mathbf{x} = r(z)\,\mathbf{e}_r(\theta) + z(\theta^2)\,\mathbf{e}_3 \ , \quad \theta^1 = \theta \ .$$

Show that

$$\mathbf{a}_1 = r\,\mathbf{e}_\theta \ , \quad \mathbf{a}_2 = \left[\frac{dr}{dz}\mathbf{e}_r + \mathbf{e}_3\right]\frac{dz}{d\theta^2} \ ,$$

$$\mathbf{a}_3 = \frac{\mathbf{e}_r - \dfrac{dr}{dz}\mathbf{e}_3}{\left[1+\left\{\dfrac{dr}{dz}\right\}^2\right]^{1/2}} \ , \quad a^{1/2} = \left[1+\left\{\frac{dr}{dz}\right\}^2\right]^{1/2} r\,\frac{dz}{d\theta^2} \ ,$$

$$b_{11} = -\frac{r}{\left[1+\left\{\dfrac{dr}{dz}\right\}^2\right]^{1/2}} \ , \quad b_{12} = 0 \ , \quad b_{22} = \frac{\left\{\dfrac{dz}{d\theta^2}\right\}^2 \dfrac{d^2r}{dz^2}}{\left[1+\left\{\dfrac{dr}{dz}\right\}^2\right]^{1/2}} \ ,$$

$$H = -\frac{1}{2r\left[1+\left\{\dfrac{dr}{dz}\right\}^2\right]^{1/2}} + \frac{\dfrac{d^2r}{dz^2}}{2\left[1+\left\{\dfrac{dr}{dz}\right\}^2\right]^{3/2}} \ , \quad K = -\frac{\dfrac{d^2r}{dz^2}}{r\left[1+\left\{\dfrac{dr}{dz}\right\}^2\right]^2} \ ,$$

where K and H are defined in exercise E4.2-2.

E4.2-5 Consider axisymmetric deformation of a circular membrane characterized by

$$\mathbf{x} = r(\theta^1)\,\mathbf{e}_r(\theta) + z(r)\,\mathbf{e}_3 \ , \quad \theta^2 = \theta \ .$$

Show that

$$\mathbf{a}_1 = [\, \mathbf{e}_r + \frac{dz}{dr}\mathbf{e}_3] \frac{dr}{d\theta^1} \;\;,\;\; \mathbf{a}_2 = r\,\mathbf{e}_\theta \;\;,$$

$$\mathbf{a}_3 = \frac{-\frac{dz}{dr}\mathbf{e}_r + \mathbf{e}_3}{[1+\{\frac{dz}{dr}\}^2]^{1/2}} \;\;,\;\; a^{1/2} = [1+\{\frac{dz}{dr}\}^2]^{1/2}\, r\frac{dr}{d\theta^1} \;\;,$$

$$b_{11} = \frac{\{\frac{dr}{d\theta^1}\}^2 \frac{d^2z}{dr^2}}{[1+\{\frac{dz}{dr}\}^2]^{1/2}} \;\;,\;\; b_{12} = 0 \;\;,\;\; b_{22} = \frac{r\frac{dz}{dr}}{[1+\{\frac{dz}{dr}\}^2]^{1/2}} \;\;,$$

$$H = \frac{\frac{d^2z}{dr^2}}{2[1+\{\frac{dz}{dr}\}^2]^{3/2}} + \frac{\frac{dz}{dr}}{2r[1+\{\frac{dz}{dr}\}^2]^{1/2}} \;\;,\;\; K = \frac{\{\frac{dz}{dr}\}\frac{d^2z}{dr^2}}{[1+\{\frac{dz}{dr}\}^2]^2} \;\;,$$

where K and H are defined in exercise E4.2-2.

E4.2-6 Using the fact that $\mathbf{a}^\sigma$ are biorthogonal to the vectors $\mathbf{a}_\beta$ and $\mathbf{a}_3$, prove that

$$\mathbf{a}^\sigma{}_{,\alpha} = -\,(\mathbf{a}_{\beta,\alpha} \bullet \mathbf{a}^\sigma)\,\mathbf{a}^\beta - (\mathbf{a}_{3,\alpha} \bullet \mathbf{a}^\sigma)\,\mathbf{a}_3 \;\;.$$

Section 4.3

E4.3-1 Rederive the expressions in (4.3.12).

E4.3-2 Rederive the expressions in (4.3.15).

E4.3-3 Rederive the expression for $\mathbf{n}^*$ in (4.3.21).

E4.3-4 Prove the validity of equation (4.3.25).

E4.3-5 Prove the validity of equation (4.3.39).

Section 4.4

E4.4-1 Prove the validity of equation (4.4.14).

E4.4-2 Prove the validity of equation $(4.4.19)_3$.

E4.4-3 Prove the validity of equation (4.4.22).

E4.4-4 Prove the validity of equation $(4.4.26)_1$.

E4.4-5 Prove the validity of equation (4.4.28).

E4.4-6 Prove the validity of equation $(4.4.34)_1$.

E4.4-7 Consider an isotropic plate of constant reference thickness H which is subjected to uniform uniaxial tension T on its top and bottom surfaces. For this case, the position vector $\mathbf{x}^*$ and the three-dimensional stress tensor $\mathbf{T}^*$ are given by

$$\mathbf{x}^* = b\,\theta^1\,\mathbf{e}_1 + b\,\theta^2\,\mathbf{e}_2 + c\,\theta^3\,\mathbf{e}_3 \;\;,\;\; \mathbf{T}^* = T\,\mathbf{e}_3\otimes\mathbf{e}_3 \;\;,$$

where b and c are constants determining the stretches of line elements in the plane of the plate and in the thickness directions, respectively. Next, neglect body force, assume equilibrium, and use the definitions (3.2.34), (4.3.24), (4.3.34), (4.3.38), (4.3.15) and (4.3.36) to show that

$$\mathbf{t}^{*\alpha} = 0 \;\;,\;\; \mathbf{t}^{*3} = b^2T\,\mathbf{e}_3 \;\;,\;\; \mathbf{t}^\alpha = 0 \;\;,\;\; \mathbf{t}^3 = b^2HT\,\mathbf{e}_3 \;\;,$$

$$\mathbf{m}^\alpha = 0 \;\;,\;\; \mathbf{b}_c = 0 \;\;,\;\; m\,\mathbf{b}_c^3 = b^2HT\,\mathbf{e}_3 \;\;,$$

and that the equations of equilibrium (4.4.35) are satisfied. In particular, notice the importance of the presence of $\mathbf{t}^3$ in the balance of director momentum.

Section 4.5

E4.5-1 Prove the validity of equation $(4.5.6)_2$.

Section 4.6

E4.6-1 Prove the validity of equation (4.6.2).

E4.6-2 Prove the validity of equation (4.6.3).

E4.6-3 Prove the validity of equation (4.6.5).

E4.6-4 Prove the validity of equation (4.6.12).

E4.6-5 Prove the validity of equation (4.6.17).

Section 4.7

E4.7-1 Prove the validity of (4.7.3).

E4.7-2 Prove the validity of (4.7.7).

Section 4.8

E4.8-1 Prove the validity of (4.8.10).

Section 4.9

E4.9-1 Starting with $\mathbf{F}\mathbf{F}^{-1}=\mathbf{I}$, prove the validity of $(4.9.6)_2$.

E4.9-2 Prove the validity of (4.9.24).

E4.9-3 Prove the validity of (4.9.28).

E4.9-4 Rederive the expressions (4.9.31).

Section 4.11

E4.11-1 Prove the validity of $(4.11.5)_3$.

E4.11-2 Prove the validity of (4.11.16).

E4.11-3 Prove the validity of $(4.11.22)_3$.

E4.11-4 Prove the validity of the expression for $\mathbf{t}^i$ in $(4.11.28)_3$.

Section 4.12

E4.12-1 Rederive the expanded form of the strain energy (4.12.15).

Section 4.13

E4.13-1 Starting with $\mathbf{F}\mathbf{F}^{-1}=\mathbf{I}$, prove the validity of the expression for $\tilde{\mathbf{F}}^{-1}$ in (4.13.5).

E4.13-2 Prove the validity of the expression for ρ in (4.13.6).

Section 4.14

E4.14-1 Show that the expressions (4.14.15) and (4.14.23) satisfy the equilibrium equations (4.14.10).

E4.14-2 Use the displacements (4.14.19) to derive the strains (4.14.21).

E4.14-3 Rederive the expressions (4.14.30).

Section 4.15

E4.15-1 Prove the validity of (4.15.4).

E4.15-2 Prove the validity of (4.15.6).

E4.15-3 Prove the validity of (4.15.20).

Section 4.16

E4.16-1 Prove the validity of (4.16.7).

E4.16-2 Rederive the expressions (4.16.9).

E4.16-3 Prove the validity of (4.16.12).

Section 4.17

E4.17-1 Rederive the expressions (4.17.10).

E4.17-2 Prove the validity of the frequencies (4.17.12).

Section 4.18

E4.18-1 Rederive the expressions (4.18.5).

E4.18-2 Prove the validity of (4.18.6).

Section 4.20

E4.20-1 Rederive the expressions (4.20.7).

E4.20-2 Show that (4.20.12) can be rearranged into the forms (4.20.13).

Section 4.21

E4.21-1 Prove the validity of (4.21.7).

Section 4.22

E4.22-1 Rederive the expressions (4.22.4).

E4.22-2 Rederive the expressions (4.22.9).

E4.22-3 Rederive the expressions (4.22.12).

E4.22-4 Show that (4.22.18) predict the correct displacement field for homogeneous deformations.

Section 4.23

E4.23-1 Rederive the equations of motion in (4.23.2).

Section 4.24

E4.24-1 Rederive the expressions (4.24.3).

E4.24-2 Use the displacements (4.24.4) to derive the strains (4.24.5).

E4.24-3 Use the displacements (4.24.8) to derive the constitutive responses (4.24.9).

Section 4.25

E4.25-1 Calculate the value of $\bar{p}$ associated with the strain energy function (4.25.10).

E4.25-2 Show that if J=1, then the value of $\bar{p}$ in E4.25-1 does not vanish and that it is dependent on the inhomogeneous strains $\boldsymbol{\beta}_\alpha$.

E4.25-3 Explain the motivation for specifying ν^* in (4.15.25) by the value (4.25.12) for an incompressible shell.

Section 4.26

E4.26-1 Prove the validity of equations (4.26.7) and (4.26.8).

E4.26-2 Prove the validity of equation (4.26.11) and (4.26.12).

E4.26-3 Prove the validity of equations (4.26.13) and (4.26.14).

Section 4.28

E4.28-1 Compare the order of the equations of motion (4.28.3) for a generalized membrane with the order of those of a general shell.

Section 4.29

E4.29-1 Describe the deformation modes that are included in the generalized membrane associated with (4.28.1) which are not modeled by a simple membrane.

E4.29-2 Starting with the definition $a^{1/2} = \mathbf{a}_1 \times \mathbf{a}_2 \bullet \mathbf{a}_3$ and using the fact that $\mathbf{a}_3$ is a unit vector, prove that

$$\dot{a}^{1/2} = a^{1/2}\, \mathbf{v}_{,\alpha} \bullet \mathbf{a}^\alpha \ .$$

E4.29-3 Starting with the equation $\mathbf{d}^\alpha \bullet \mathbf{d}_i = \delta^\alpha{}_i$, prove that

$$\mathbf{d}^\alpha{}_{,\alpha} = -\,(\mathbf{d}^\alpha \bullet \mathbf{d}_{m,\alpha})\, \mathbf{d}^m \ .$$

E4.29-4 Using the condition that $\mathbf{d}_3 = \mathbf{a}_3$, prove that

$$(a^{1/2})_{,\alpha} = a^{1/2}\, (\mathbf{d}^\sigma \bullet \mathbf{d}_{\sigma,\alpha}) \ , \quad (a^{1/2}\, \mathbf{d}^\alpha)_{,\alpha} = -\, a^{1/2}\, (\mathbf{d}^\alpha \bullet \mathbf{a}_{3,\alpha})\, \mathbf{a}_3 \ .$$

E4.29-5 Show that the rate form of the constraint $a^{1/2} = A^{1/2}$ is given by

$$(\mathbf{d}^\alpha \otimes \mathbf{d}_\alpha) \bullet \mathbf{D} = 0 \ .$$

E4.29-6 Consider a constrained simple membrane for which the strain energy function vanishes and $a^{1/2} = A^{1/2}$. For this case, the responses $\mathbf{T}$ and $\mathbf{t}^\alpha$ are totally determined by the constraint responses $\bar{\mathbf{T}}$ and $\bar{\mathbf{t}}^\alpha$. Prove that these constraint responses can be written in the forms

$$a^{1/2}\, \bar{\mathbf{T}} = a^{1/2}\, \gamma\, \mathbf{d}^\alpha \otimes \mathbf{d}_\alpha \ , \quad \bar{\mathbf{t}}^\alpha = a^{1/2}\, \gamma\, \mathbf{d}^\alpha \ ,$$

where γ is a Lagrange multiplier which is an arbitrary function of θ^α and t.

E4.29-7 In order to model the pressure difference due to surface tension, it is possible to use the equations of equilibrium for the constrained membrane of problem E4.29-6 and take the assigned field to be

$$m\, \mathbf{b} = a^{1/2}\, (\bar{p} - \hat{p})\, \mathbf{a}_3 \ ,$$

where $\bar{p}$ and $\hat{p}$ are the pressures applied to opposite sides of the membrane. Using this expression, show that the equilibrium equations (4.29.7) are satisfied provided that

$$\bar{p} - \hat{p} = \gamma\, (\mathbf{d}^\alpha \bullet \mathbf{a}_{3,\alpha}) \ ,$$

where γ is the surface tension per unit area.

E4.29-8 Using the condition $\mathbf{d}_3 = \mathbf{d}_\alpha$, it follows that $\mathbf{d}^\alpha = \mathbf{a}^\alpha$. With the help of the definition of the second fundamental form $b_{\alpha\beta}$ given in E4.2-2, show that for this case,

$$\mathbf{d}^\alpha \bullet \mathbf{a}_{3,\alpha} = -\, a^{\alpha\beta}\, b_{\alpha\beta} \ ,$$

so that the pressure difference in E4.29-7 is related to the second fundamental form of the membrane surface.

Section 4.30

E4.30-1 Use the kinematics (4.30.2) to derive the expressions (4.30.3) and (4.30.4).

E4.30-2 Rederive the expressions (4.30.10).

Section 4.31

E4.31-1 Use the kinematics (4.31.3) to derive the expressions (4.31.4).

E4.31-2 Rederive the expressions (4.31.5).

E4.31-3 Rederive the boundary conditions (4.31.9).

Section 4.32

E4.32-1 Rederive the expressions (4.32.3) for the linearized strains.

E4.32-2 Prove the validity of (4.32.6).

E4.32-3 Rederive the equations of motion (4.32.9) and (4.32.10).

E4.32-4 Rederive the constitutive equations (4.32.12) for plane strain.

E4.32-5 Rederive the equations (4.32.14) for a Kirchhoff-Love plate.

E4.32-6 Let $f(\theta^\alpha)$ be a function of the rectangular Cartesian coordinates θ^i which is independent of θ^3. Use the results of problem E2.3-8 to show that the Laplacian $\nabla^2 f$ of f is given by

$$\nabla^2 f = f_{,\alpha\alpha} \ .$$

It then follows that the linearized equation for a plate (4.32.14) can be written in the alternative coordinate-free form

$$m \ddot{u}_3 = [-\rho_0^* H g^* + \bar{p} - \hat{p}] - \frac{\mu^* H^3}{6(1-\nu^*)} \nabla^4 u_3 + m y^{33} \nabla^2 \ddot{u}_3 \ ,$$

where $\nabla^4 u_3$ is the biharmonic operator defined by

$$\nabla^4 u_3 = \nabla^2(\nabla^2 u_3) \ .$$

E4.32-7 Using the results of problems E2.3-7 and E2.3-8, consider a set of convected coordinates θ^i for which

$$\mathbf{G}_\alpha = \mathbf{X}^*_{,\alpha} = \mathbf{A}_\alpha \ , \quad \mathbf{G}_3 = \mathbf{X}^*_{,3} = \mathbf{A}_3 \ .$$

Also, let $f(\theta^\alpha)$ be a function which is independent of θ^3 and show that the Laplacian $\nabla^2 f$ of f is given by

$$\nabla^2 f = A^{-1/2} [A^{1/2} f_{,\alpha} A^{\alpha\beta}]_{,\beta} \ , \quad A^{\alpha\beta} = \mathbf{A}^\alpha \bullet \mathbf{A}^\beta \ .$$

E4.32-8 Consider a planar surface referred to polar coordinates such that

$$\theta^1 = R \ , \quad \theta^2 = \Theta \ , \quad \mathbf{X} = R\, \mathbf{e}_r(\Theta) \ .$$

Show that

$$\mathbf{A}_1 = \mathbf{e}_r(\Theta) \ , \quad \mathbf{A}_2 = R\, \mathbf{e}_\theta(\Theta) \ , \quad \mathbf{A}_3 = \mathbf{e}_3 \ ,$$

$$A^{1/2} = R \ , \quad A^{11} = 1 \ , \quad A^{22} = \frac{1}{R^2} \ , \quad A^{12} = A^{21} = 0 \ ,$$

and that

$$\nabla^2 f = \frac{1}{R}\Big[\frac{\partial}{\partial R}\Big\{R \frac{\partial f}{\partial R}\Big\} + \frac{\partial}{\partial \Theta}\Big\{\frac{1}{R}\frac{\partial f}{\partial \Theta}\Big\}\Big] = \frac{\partial^2 f}{\partial R^2} + \frac{1}{R}\frac{\partial f}{\partial R} + \frac{1}{R^2}\frac{\partial^2 f}{\partial \Theta^2} \ ,$$

$$\nabla^4 f = \frac{\partial^4 f}{\partial R^4} + \frac{2}{R}\frac{\partial^3 f}{\partial R^3} - \frac{1}{R^2}\frac{\partial^2 f}{\partial R^2} + \frac{1}{R^3}\frac{\partial f}{\partial R}$$
$$+ \frac{2}{R^2}\frac{\partial^4 f}{\partial R^2 \partial \Theta^2} - \frac{2}{R^3}\frac{\partial^3 f}{\partial R \partial \Theta^2} + \frac{4}{R^4}\frac{\partial^2 f}{\partial \Theta^2} + \frac{1}{R^4}\frac{\partial^4 f}{\partial \Theta^4} \ .$$

Section 4.33

E4.32-1 Using (4.33.10), prove the validity of the restrictions (4.33.12).

Exercises for Chapter 5

Section 5.1

E5.1-1 Rederive the expression for $G^{1/2}$ in (5.1.15).

Section 5.2

E5.2-1 Using the definition (5.2.5), sketch the projection $\bar{\mathbf{d}}_1$ of the director $\mathbf{d}_1$.

E5.2-2 Prove the validity of (5.2.14).

Section 5.3

E5.3-1 Prove the validity of (5.3.14).

E5.3-2 Rederive the expression (5.3.16).

E5.3-3 Rederive the expression (5.3.25).

E5.3-4 Rederive the expression (5.3.32).

E5.3-5 Starting with (5.3.38), rederive the equation (5.3.39).

E5.3-6 Rederive the expression (5.6.9).

Section 5.6

E5.6-1 Rederive the expression (5.6.9).

E5.6-2 Rederive the expression (5.6.10).

Section 5.7

E5.7-1 Prove the validity of (5.7.3).

E5.7-2 Prove the validity of (5.7.8).

Section 5.8

E5.8-1 Prove the validity of (5.8.10).

Section 5.9

E5.9-1 Rederive the constraint (5.9.5).

E5.9-2 Rederive the constraint (5.9.9).

E5.9-3 Rederive the constraint responses (5.9.23).

E5.9-4 Starting with (5.9.31), rederive the expressions (5.9.32).

Section 5.11

E5.11-1 Prove the validity of (5.11.5).

E5.11-2 Prove the validity of (5.11.17).

E5.11-3 Prove the validity of (5.11.28).

Section 5.12

E5.12-1 Rederive the expanded form of the strain energy (5.12.12).

Section 5.13

E5.13-1 Starting with $\mathbf{FF}^{-1}=\mathbf{I}$, prove the validity of the expression for $\tilde{\mathbf{F}}^{-1}$ in (5.13.5).

E5.13-2 Prove the validity of the expression for ρ in (5.13.6).

Section 5.14

E5.14-1 Show that the expressions (5.14.17) and (5.14.25) satisfy the equilibrium equations (5.14.12).

E5.14-2 Use the displacements (5.14.21) to derive the strains (5.14.23).

E5.14-3 Rederive the expressions (5.14.32).

Section 5.15

E5.15-1 Sketch the deformation described by the displacements (5.15.1).

E5.15-2 Prove the validity of (5.15.3).

E5.15-3 Prove the validity of (5.15.8).

Section 5.16

E5.16-1 Sketch the deformation described by the displacements (5.16.1).

E5.16-2 Prove the validity of (5.16.3).

E5.16-3 Prove the validity of (5.16.9).

Section 5.17

E5.17-1 Use the displacements (5.17.8) to derive the results (5.17.9).

Section 5.18

E5.18-1 Use the displacements (5.18.7) to derive the results (5.18.8) and (5.18.10).

Section 5.19

E5.19-1 Use the displacements (5.19.7) to derive the results (5.19.9).

Section 5.21

E5.21-1 Use the displacements (5.21.8) to derive the results (5.21.11).

E5.21-2 Rederive the equations of motion (5.21.12).

E5.21-3 Show that (5.21.13) predict the correct displacement field for homogeneous deformations.

Section 5.22

E5.22-1 Rederive the equation of motion (5.22.4).

Section 5.23

E5.23-1 Rederive the expressions (5.23.4).

E5.23-2 Use the displacements (5.23.5) to derive the strains (5.23.6).

E5.23-3 Use the strains (5.23.6) to derive the constitutive responses (5.23.7).

Section 5.24

E5.24-1 Calculate the value of p associated with the strain energy function (5.24.8).

Section 5.25

E5.25-1 Prove the validity of equations (5.25.7) and (5.25.9).

E5.25-2 Prove the validity of equations (5.25.12) and (5.25.13).

E5.25-3 Prove the validity of equations (5.25.14) and (5.25.15).

Section 5.27

E5.27-1 Rederive the constitutive responses (5.27.2).

Section 5.28

E5.28-1 Rederive the equations of motion (5.28.6).

E5.28-2 Rederive the expressions (5.28.7) for the constraint responses.

E5.28-3 Use the strain energy function (5.28.18) to derive the expressions (5.28.20).

E5.28-4 Rederive the expressions (5.28.28) for the linearized theory.

E5.28-5 Rederive the equations of motion (5.28.33).

E5.28-6 Neglect assigned fields and determine the dispersion relationship for the differential equation associated with u_1 in (5.28.33).

Section 5.29

E5.29-1 Use the st.ain energy function (5.29.9) to rederive the expressions (5.29.11).

E5.29-2 Use the kinematics (5.29.17) to rederive the constitutive responses (5.29.21).

E5.29-3 Rederive the equations of motion (5.29.22).

Section 5.31

E5.31-1 Rederive the equations (5.31.3).

E5.31-2 Let the total strain energy of the string $\mathcal{U}$ be defined by

$$\mathcal{U} = \int \rho \, \Sigma \, ds \ ,$$

where Σ is the strain energy per unit mass of the string. Next, define the rate of material dissipation $\mathcal{D}$ per unit present arclength by the formula

$$\int_P \mathcal{D} \, ds = \mathcal{W} - \dot{\mathcal{K}} - \dot{\mathcal{U}} \geq 0 \ ,$$

where $\mathcal{W}$ and $\mathcal{K}$ are given by (5.31.7). Using these expressions, show that $\mathcal{D}$ becomes

$$\mathcal{D} = \mathbf{T} \bullet \mathbf{D} - \rho \, \dot{\Sigma} \ .$$

E5.31-3 Using the form (5.31.11) for the strain energy function and the results (5.31.3), show that $\mathcal{D}$ in E5.31-2 can be written in the form

$$\mathcal{D} = [\ \mathbf{T} - \rho \, \lambda \frac{\partial \Sigma}{\partial \lambda} \{ \frac{\mathbf{d}_3 \otimes \mathbf{d}_3}{d_{33}} \} \ ] \bullet \mathbf{D} \ .$$

For an elastic string, the rate of dissipation $\mathcal{D}$ vanishes for all motions and it can be shown using the usual argument that

$$\mathbf{T} = \rho \, \lambda \frac{\partial \Sigma}{\partial \lambda} \{ \frac{\mathbf{d}_3 \otimes \mathbf{d}_3}{d_{33}} \} \ ,$$

where use has been made of the form (5.31.9) for $\mathbf{T}$.

E5.31-4 Using the form (5.31.11) for the strain energy function and the results (5.31.3), show that $\mathcal{D}$ in E5.31-2 can be written in the alternative form

$$\mathcal{D} = [\ \mathbf{T} - \rho \, \lambda \frac{\partial \Sigma}{\partial \lambda} \mathbf{I} \ ] \bullet \mathbf{D} \ .$$

For an elastic string, the rate of dissipation $\mathcal{D}$ vanishes for all motions and it is tempting to conclude that the coefficient of $\mathbf{D}$ in this expression vanishes. To understand why this conclusion is incorrect, it is necessary to note that $\mathbf{D}$ is not a general symmetric three-dimensional tensor. To this end, let $\mathbf{d}_\alpha$ be vectors that are orthogonal to $\mathbf{d}_3$ such that $\mathbf{d}_i$ forms a right-handed triad. Show, that the components of $\mathbf{D}$ relative to these base vectors have the properties that

$$D_{\alpha\beta} = \mathbf{D} \bullet (\mathbf{d}_\alpha \otimes \mathbf{d}_\beta) = 0 \ .$$

It therefore follows that only three of the components of the coefficient of $\mathbf{D}$ appear explicitly in the expression for the rate of dissipation $\mathcal{D}$. In other words, no information can be obtained about the components of the coefficient of $\mathbf{D}$ that do not appear explicitly in this expression for $\mathcal{D}$.

Section 5.32

E5.32-1 Prove the validity of the results (5.32.5).

E5.32-2 Prove the validity of the results (5.32.8).

E5.32-3 Prove the validity of the results (5.32.10).

Section 5.33

E5.33-1 Use the kinematics (5.33.2) to rederive the results (5.33.3).

E5.33-2 Use the results (5.33.3) to rederive the constitutive responses (5.33.4).

E5.33-3 Use the results (5.33.4) to rederive the equations of motion (5.33.8).

E5.33-4 Use the results (5.33.4) to rederive the equations of motion (5.33.9).

E5.33-5 Use the results (5.33.4) to rederive the equations of motion (5.33.11).

E5.33-6 Use the results (5.33.16) to rederive the buckling equation (5.33.18).

Section 5.34

E5.34-1 Prove the validity of (5.34.13).

E5.34-2 Use the strain energy function (5.34.9) to prove the validity of (5.34.16).

E5.34-3 Use the results (5.34.16) to prove the validity of (5.34.18).

Section 5.35

E5.35-1 Use (5.35.10) to prove the validity of the restrictions (5.35.12).

Exercises for Chapter 6

Section 6.3

E6.3-1 Prove the validity of (6.3.11) and (6.3.18).

E6.3-2 Prove the validity of (6.3.20) and (6.3.21).

Section 6.4

E6.4-1 Rederive the reduced form (6.4.8) of the balance of angular momentum.

Section 6.6

E6.6-1 Prove the validity of (6.6.4).

Section 6.7

E6.7-1 Rederive the equation (6.7.3).

E6.7-2 Prove the validity of (6.7.7).

Section 6.9

E6.9-1 Prove the validity of (6.9.20).

E6.9-2 Prove the validity of (6.9.23) and (6.9.24).

E6.9-3 Use (6.9.26) to prove the validity of (6.9.25).

Section 6.13

E6.13-1 Starting with $\mathbf{F}\mathbf{F}^{-1}=\mathbf{I}$, prove the validity of the expression for $\tilde{\mathbf{F}}^{-1}$ in (6.13.5).

E6.13-2 Prove the validity of the expression for ρ in (6.13.6).

Section 6.14

E6.14-1 Use the displacements (6.14.20) to derive the expressions (6.14.21).

E6.14-2 Rederive the expressions (6.14.24).

Exercises for Chapter 7

Section 7.2

E7.2-1 Rederive the expressions (7.2.4).

E7.2-2 Rederive the expressions (7.2.8).

E7.2-3 Prove the validity of the expressions (7.2.13).

E7.2-4 Prove the validity of the equations of motion (7.2.20).

Section 7.3

E7.3-1 Rederive the equation (7.3.8).

E7.3-2 Rederive the equations (7.3.20).

Section 7.4

E7.4-1 Show that the expressions (7.4.39) satisfy the equations of motion (7.4.15).

E7.4-2 Rederive the expression (7.4.43) for the rate of work.

E7.4-3 Show that ${}_{I}\boldsymbol{\beta}_{\alpha}$ vanish for homogeneous deformation.

Section 7.5

E7.5-1 Rederive the expression (7.5.39) for the rate of work.

E7.5-2 Show that the expressions (7.5.41) satisfy the equations of motion (7.5.16).

Section 7.6

E7.6-1 Rederive the expression (7.6.39) for the rate of work.

E7.6-2 Show that the expressions (7.6.42) satisfy the equations of motion (7.6.16).

Exercises for Appendix A

EA-1 Prove the validity of the equations (A.4.8).

EA-2 Prove the validity of the equations (A.4.10).

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REFERENCES

The numbers in square brackets at the end of each reference indicate the sections where the reference is referred to in the test.

Alehossein, H. and Muhlhaus, H.B. (1994). "Computer modelling of comminuting granular media", Proceedings of the 1st Congress on Computing in Civil Engineering Part 1, Publ. by ASCE, New, 660-667. [1.1]

Antman, S.S. (1972). The theory of rods, In S. Flugge's Handbuch der Physik, Vol. VIa/2, (ed. C. Truesdell), Springer-Verlag, Berlin, 641-703. [1.1, 1.2, 5.28, 5.34]

Antman, S.S. (1995). *Nonlinear problems of elasticity*, Applied Mathematical Sciences 107, Springer-Verlag, Berlin. [1.1, 1.2, 4.30, 5.28, 5.33]

Antman, S.S. and Jordan, K.B. (1975). "Qualitative aspects of the spatial deformation of non-linearly elastic rods", *Proc. Roy. Soc. Edinburgh* **73A**, 85-105. [5.34]

Atkin, R.J. and Fox, N. (1980). *An introduction to the theory of elasticity.* Longman, New York. [3.11]

Axelrad, E.L. (1987). *Theory of flexible shells.* Vol. 28 in North-Holland Series in Applied Mathematics and Mechanics. Elsevier Science, Amsterdam. [1.2]

Bathe, K.J. (1982). *Finite element procedures in engineering analysis.* Prentice-Hall, New Jersey. [7.1]

Becker, E.B., Carey, G.F. and Oden, J.T. (1981). *Finite elements: an introduction, volume I.* Prentice-Hall, New Jersey. [7.1]

Biezeno, C.B. and Grammel, R. (1954). *Engineering dynamics - Volume III Steam turbines.* Blackie & Son Limited, London. [7.3]

Cairns, D.S., Ilcewicz, L.B., Walker, T. and Minguet, P.J. (1995). "Fracture scaling parameters of inhomogeneous microstructure in composite structures", *Composites Sci. and Tech.* **53**, 223-231. [1.1]

Carey, G.F. and Oden, J.T. (1983). *Finite elements: a second course, Volume II.* Prentice-Hall, New Jersey. [7.1]

Carey, G.F. and Oden, J.T. (1984). *Finite elements: computational aspects, Volume III.* Prentice-Hall, New Jersey. [7.1]

Calladine, C.R. (1983). *Theory of shell structures.* Cambridge University Press, Cambridge. [1.2]

Cheng, Z.Q. and He, L.H. (1995). "Micropolar elastic fields due to a spherical inclusion", *Int. J. Engn. Sci.* **33**, 389-397. [1.1]

Cohen, H. (1981). "Pseudo-rigid bodies", *Utilitas Math.* **20**, 221-247. [1.1]

Cohen, H. and Mac Sithigh, G.P. (1993). "Plane motions of elastic pseudo-rigid bodies: an example", *J. Elasticity* **32**, 51-59. [6.1]

Cohen, H. and Mac Sithigh, G.P. (1994). "Impulsive motions of elastic pseudo-rigid bodies II: further results and examples", *J. Elasticity* **34**, 149-166. [6.1]

Cohen, H. and Mac Sithigh, G.P. (1996). "Collisions of pseudo-rigid bodies: A Brach-type treatment", *Int. J. Engng. Sci.* **34**, 249-256. [6.1]

Cohen, H. and Muncaster, R.G. (1984). "The dynamics of pseudo-rigid bodies: general structure and exact solutions", *J. Elasticity* **14**, 127-154. [1.1]

Cohen, H. and Muncaster, R.G. (1988). *Theory of pseudo-rigid bodies.* Springer, Berlin. [1.1]

Cohen, H. and Sun, Q.-X. (1991). "Snapping of an elastic pseudo-rigid body", *Int. J. Solids Structures* **28**, 807-818. [6.1]

Cosserat, E. and F. (1909). "Theorie des corps deformables". Paris. vi+226pp. = Appendix, pp. 953-1173 of Chwolson's Traite de Physique, 2nd ed., Paris. Also in translated form, "Theory of deformable bodies", NASA TT F-11, 561 (Washington, D.C.). Clearinghouse for U.S. Federal Scientific and Technical Information, Springfield, Virginia. [Preface, 1.1]

Craine, R.E. and Newman, M.G. (1996). "Modelling creep failure in welded plates under uniaxial loading", *J. Strain Anal. Engn. Design* **31**, 117-124. [1.1]

Darlow, M.S. (1989). *Balancing of high-speed machinery.* Springer-Verlag, New York. [7.3]

De Borst, R. (1991). "Simulation of strain localization: a reappraisal of the Cosserat continuum", *Engineering Computations* **8**, 317-332. [1.1]

Desai, C.S. (1979). *Elementary finite element method.* Prentice-Hall, New Jersey. [7.1]

Duhem, P. (1893). "Le potentiel thermodynamique et la pression hydrostatic", *Ann. Ecole Norm.* **10**, 187-230. [1.1]

Ericksen, J. L. (1961). "Conservation laws for liquid crystals", *Trans. Soc. Rheol.* **5**, 23-34. [1.1]

Ericksen, J. L. (1970). "Simpler static problems in nonlinear theories of rods", *Int. J. Solids Structures* **6**, 371-377. [5.34]

Ericksen, J.L. and Truesdell C. (1958). "Exact theory of stress and strain in rods and shells", *Arch. Ration. Mech. Anal.* **1**, 295-323. [1.1]

Eringen, A.C. (1967). *Mechanics of continua*, John Wiley & Sons, New York. [2.1]

Eringen, A.C. and Suhubi, E. S. (1964). "Nonlinear theory of simple micro-elastic solids - I", *Int. J. Engng. Sci.* **2**, 189-203. [1.1]

Flory, P.J. (1961). Thermodynamic relations for high elastic materials. *Trans. Faraday Soc.* **57**, 829-838. [3.11]

Flugge, W. (1973). *Stresses in shells.* 2nd Ed., Springer-Verlag, Berlin. [1.2]

Gladwell, G.M.L (1998). Private communication. [3.15]

Gol'denveizer, A.L. (1961). *Theory of elastic thin shells* (translation from the 1953 Russian ed.). Pergamon Press, Oxford. [1.2]

Graff, K.F. (1975). *Wave motion in elastic solids*, Ohio State University Press. [5.27, 5.28, 5.33, 7.3]

Green, A.E. (1959). "The equilibrium of rods", *Arch. Ration. Mech. Anal.* **3**, 417-421. [1.1]

Green, A.E., and Adkins, J.E. (1960). *Large elastic deformations and nonlinear continuum mechanics*, Oxford University Press. [3.10]

Green, A.E., Knops, R.J. and Laws, N. (1968). "Large deformations, superposed small deformations and stability of elastic rods", *Int. J. Solids Structures* **4**, 555-577. [5.13]

Green, A.E. and Naghdi, P.M. (1971). "On superposed small deformations on a large deformation of an elastic Cosserat surface", *J. Elasticity* **1**, 1-17. [4.13]

Green, A.E. and Naghdi, P.M. (1979a). "On thermal effects in the theory of shells", *Proc. Royal Soc. London* **A365**, 161-190. [1.2]

Green, A.E. and Naghdi, P.M. (1979b). "On thermal effects in the theory of rods", *Int. J. Solids Structures* **15**, 829-853. [1.2]

Green, A.E. and Naghdi, P.M. (1982). "A theory of laminated composite plates", *IMA J. Appl. Math.* **29**, 1-23. [4.14, 4.15, 5.14]

Green, A.E. and Naghdi, P.M. (1983). "On electromagnetic effects in the theory of shells and plates", *Phil. Trans. Royal Soc. London* **A309**, 559-610. [1.1]

Green, A.E. and Naghdi, P.M. (1985). "Electromagnetic effects in the theory of rods", *Phil. Trans. Royal Soc. London* **A314**, 311-352. [1.1]

Green, A.E. and Naghdi, P.M. (1991). "A thermomechanical theory of a Cosserat point with application to composite materials", *Q. J. Mech. and Appl. Math.* **44**, 335-355. [1.1, 1.2, 6.2]

Green, A.E. and Naghdi, P.M. (1996). "On thermomechanical formulation of theories of continuous media", *Nonlinear effects in fluids and solids*, (edited by M.M. Carroll and M. Hayes), Plenum Press, New York, 161-187. [1.1, 1.2]

Green, A.E., Naghdi, P.M., and Trapp, J.A. (1970). "Thermodynamics of a continuum with internal constraints", *Int. J. Engng. Sci.* **8**, 891-908. [3.8]

Green, A.E., Naghdi, P.M., and Wenner, M.L. (1974a). "On the theory of rods - Part I: Derivations from the three-dimensional equations", *Proc. Royal Soc. London* **A337**, 451-483. [1.1]

Green, A.E., Naghdi, P.M., and Wenner, M.L. (1974b). "On the theory of rods - Part II: Developments by direct approach", *Proc. Royal Soc. London* **A337**, 485-507. [1.1, 5.12, 5.14, 5.15]

Hill, R. (1971). *The mathematical theory of plasticity*, Oxford University Press, Oxford. [A]

Huebner, K.H. (1975). *The finite element method for engineers*, John Wiley, New York. [7.1]

Hughes, T.J.R. (1987). *The finite element method: linear static and dynamic finite element analysis*. Prentice-Hall, New Jersey. [7.1]

Hutchinson, J.R. and Zillmer, S.D. (1983). "Vibration of a free rectangular parallelepiped", *ASME J. Appl. Mech.* **50**, 123-130. [3.17]

Iordache, M.M. and Willam, K. (1995). "Elastoplastic bifurcation in Cosserat continua", Proceedings of the 10th Conference on Engineering Mechanics, in Proceedings of Engineering Mechanics Part 1. Publ. by ASCE, New York, 525-528. [1.1]

Kolsky, H. (1963). *Stress waves in solids*, Dover, New York. [7.3, 7.4]

Kraus, H. (1967). *Thin elastic shells. An introduction to the theoretical foundations and the analysis of their static and dynamic behavior*. John Wiley, New York. [1.2]

Lakes, R. (1993). "Strongly Cosserat elastic lattice and foam materials for enhanced toughness", *Cellular Polymers* **12**, 17-30. [1.1]

Lakes, R.S., Nakamura, S., Behiri, J.C. and Bonfield, W. (1990). "Fracture mechanics of bone with short cracks", *J. of Biomechanics* **23**, 967-975. [1.1]

Lekhnitskii, S.G. (1963). *Theory of elasticity of an anisotropic elastic body*, Holden-Day, San Francisco. [3.14, 3.15]

Libai, A. and Simmonds, J.G. (1988). *The nonlinear theory of elastic shells - one spatial dimension*, Academic Press, Boston. [1.2]

Libai, A. and Simmonds, J.G. (1998). *The nonlinear theory of elastic shells*, Cambridge University Press, Cambridge. [1.2]

Love, A.E.H. (1944). *A treatise on the mathematical theory of elasticity*, fourth edition, Dover, New York. [4.21, 4.23, 5.34]

Lu, C.L. and Perkins, N.C. (1994). "Nonlinear spatial equilibria and stability of cables under uni-axial torque and thrust", *ASME J. Appl. Mech.* **61**, 879-886. [5.34]

Malvern, L.E. (1969). *Introduction to the mechanics of a continuous medium*, Prentice-Hall, New Jersey. [2.1, A.7]

Marshall, J.S. and Naghdi, P.M. (1989a). "A thermodynamical theory of turbulence, Part I: Basic developments", *Phil. Trans. Royal Soc. London* **A327**, 415-448. [1.1]

Marshall, J.S. and Naghdi, P.M. (1989b). "A thermodynamical theory of turbulence, Part II: Determination of constitutive coefficients and illustrative examples", *Phil. Trans. Royal Soc. London* **A327**, 449-475. [1.1]

Marshall, J.S., Naghdi, P.M. and Srinivasa, A.R. (1991). "A macroscopic theory of microcrack growth in brittle materials", *Phil. Trans. Royal Soc. London* **A335**, 455-485. [1.1]

Meseguer, J. and Perales, J.M. (1992). "Non steady phenomena in the vibration of viscous, cylindrical, long liquid bridges", *Microgravity Sci. Tech.* **5**, 69-72. [1.1]

Mindlin, R.D. (1951). "Influence of rotary inertia and shear on flexural vibration of isotropic, elastic plates", *ASME J. Appl. Mech.* **18**, 31-38. [4.12]

Mollmann, H. (1981). *Introduction to the theory of thin shells.* John Wiley, New York. [1.2]

Muncaster, R.G. (1984a). "Invariant manifolds in mechanics I: the general construction of coarse theories from fine theories", *Arch. Ration. Mech. Anal.* **84**, 353-373. [1.1]

Muncaster, R.G. (1984b). "Invariant manifolds in mechanics II: zero-dimensional elastic bodies with directors", *Arch. Ration. Mech. Anal.* **84**, 375-392. [1.1]

Naghdi, P.M. (1972). The theory of plates and shells, In S. Flugge's Handbuch der Physik, Vol. VIa/2, (ed. C. Truesdell), Springer-Verlag, Berlin, 425-640. [1.1, 1.2, 2.1, 4.9, 4.13, 4.32]

Naghdi, P.M. (1979). "Fluid jets and fluid sheets: a direct formulation", Proceedings of the 12th Symposium on Naval Hydrodynamics, National Academy of Sciences, Washington, D.C., 500-515. [1.1]

Naghdi, P.M. (1982). "Finite deformation of elastic rods and shells", Proceedings of the IUTAM Symposium on Finite Elasticity (Bethlehem, PA), Edited by D. E. Carlson and R. T. Shield, Martinus Nijhoff Pub., The Hague. 47-103. [1.1, 4.2, 5.2, 6.2]

Naghdi, P.M., and Rubin, M.B. (1984). "Constrained theories of rods", *J. Elasticity* **14**, 343-361. [5.2, 5.9]

Naghdi, P.M., and Rubin, M.B. (1989). "On the significance of normal cross-sectional extension in beam theory with application to contact problems", *Int. J. Solids Structures* **25**, 249-265. [5.14]

Naghdi, P.M., and Rubin, M.B. (1995). "Restrictions on nonlinear constitutive equations for elastic shells", *J. Elasticity* **39**, 133-163. [1.2, 4.2, 4.11, 4.13, 4.22, 6.11]

Naghdi, P.M. and Srinivasa, A.R. (1993a). "A dynamical theory of structured solids, Part I: Basic developments", *Phil. Trans. Royal Soc. London* **A345**, 425-458. [1.1]

Naghdi, P.M. and Srinivasa, A.R. (1993b). "A dynamical theory of structured solids, Part II: Special constitutive equations and special cases of the theory", *Phil. Trans. Royal Soc. London* **A345**, 459-476. [1.1]

Niordson, F.I. (1985). *Shell theory.* Vol. 29 in North-Holland Series in Applied Mathematics and Mechanics. Elsevier Science, Amsterdam. [1.2]

Noble, B. (1969). *Applied linear algebra.* Prentice Hall, New Jersey. [A.1]

Novozhilov, V.V. (1959). *The theory of thin shells* (transl. from the 1951 Russian ed.). P. Noordhoof Ltd, Groningen. [1.2]

Ogden, R.W. (1997). *Non-linear elastic deformations*, Dover, New York. [3.11]

Reddy, J.N. (1985). *An introduction to the finite element method.* McGraw-Hill, New York. [7.1]

Reissner, E. (1945). "The effect of transverse shear deformation on the bending of elastic plates", *ASME J. Appl. Mech.* **12**, 69-77. [4.12]

Rubin, M.B. (1985a). "On the theory of a Cosserat point and its application to the numerical solution of continuum problems", *ASME J. Appl. Mech.* **52**, 368-372. [1.1, 6.1, 7.1]

Rubin, M.B. (1985b). "On the numerical solution of one-dimensional continuum problems using the theory of a Cosserat point", *ASME J. Appl. Mech.* **52**, 373-378. [1.1, 6.1, 7.1]

Rubin, M.B. (1986). "Free vibration of a rectangular parallelepiped using the theory of a Cosserat point", *ASME J. Appl. Mech.* **53**, 45-50. [1.1, 3.16, 4.19, 6.1, 6.8, 6.15]

Rubin, M.B. (1987a). "On the numerical solution of nonlinear string problems using the theory of a Cosserat point", *Int. J. Solids Structures* **23**, 335-349. [1.1, 1.2, 4.19, 6.1, 7.1, 7.3]

Rubin, M.B. (1987b). "On the numerical solution of spherically symmetric problems using the theory of a Cosserat surface", *Int. J. Solids Structures* **23**, 335-349. [1.1, 4.14, 6.1, 7.1, 7.2]

Rubin, M.B. (1987c). "An elastic-viscoplastic model exhibiting continuity of solid and fluid states", *Int. J. Engng. Sci.* **25**, 1175-1191. [3.11]

Rubin, M.B. (1987d). "On the singular nature of linearization of certain constrained theories of shells", *J. Appl. Mech.* **54**, 472-474. [4.9, 5.28]

Rubin, M.B. (1995). "Numerical solution of two- and three-dimensional thermomechanical problems using the theory of a Cosserat point", *J. of Math. and Physics (ZAMP)* **46**, 308-334. [1.1, 1.2, 6.1]

Rubin, M.B. (1996). "Restrictions on nonlinear constitutive equations for elastic rods", *J. Elasticity* **44**, 9-36. [1.2, 5.2, 4.14, 5.11, 5.12, 5.14, 5.28, 6.11]

Rubin, M.B. (1997). "An intrinsic formulation for nonlinear elastic rods", *Int. J. Solids Structures* **34**, 4191-4212. [5.34]

Rubin, M.B. (2000). "Numerical solution procedures for nonlinear elastic rods using the theory of a Cosserat point", To appear in the *Int. J. Solids Structures*. [1.2,7.4]

Rubin, M.B. and Gottlieb, O. (1996). "Numerical solutions of forced vibration and whirling of a nonlinear string using the theory of a Cosserat point", *J. of Sound and Vibration* **197**, 85-101. [7.1, 7.3]

Slawianowski, J.J. (1974). "Analytical mechanics of finite homogeneous strains", *Arch. Mech.* **26**, 569-587. [1.1]

Slawianowski, J.J. (1975). "Newtonian dynamics of homogeneous strains", *Arch. Mech.* **27**, 93-102. [1.1]

Slawianowski, J.J. (1982). "The mechanics of the homogeneously-deformable body. Dynamical models with high symmetries", *Z. Angew. Math. Mech.* **62**, 229-240. [1.1]

Simo, J.C. and Fox, D.D. (1989). "On a stress resultant geometrically exact shell model. Part I: Formulation and optimal parameterization", *Comp. Meth. Appl. Mech. Engn.* **72**, 267-304. [1.1]

Simo, J.C., Fox, D.D. and Rifai, M.S. (1989). "On a stress resultant geometrically exact shell model. Part II: The linear theory; computational aspects", *Comp. Meth. Appl. Mech. Engn.* **73**, 53-92. [1.1]

Simo, J.C., Rifai, M.S. and Fox, D.D. (1990). "On a stress resultant geometrically exact shell model. Part IV: Variable thickness shells with through-the-thickness stretching", *Comp. Meth. Appl. Mech. Engn.* **81**, 91-126. [1.1,4.9]

Sokolnikoff, I.S. (1964). *Tensor analysis - Theory and applications to geometry and mechanics of continua*, John Wiley & Sons, New York. [2.1,A.1]

Sokolnikoff, I.S. (1956). *Mathematical theory of elasticity,* McGraw-Hill, New York. [4.9, 4.17, 4.22, 4.28, 7.2]

Strang, G. and Fix, G.J. (1973). *An analysis of the finite element method*, Prentice-Hall, New Jersey. [7.1]

Suhubi, E.S. (1968). "On the foundation of the theory of rods", *Int. J. Engng. Sci.* **6**, 169-191. [1.1]

Suhubi, E.S. and Eringen, A.C. (1964). "Nonlinear theory of micro-elastic solids - II", *Int. J. Engng. Sci.* **2**, 389-404. [1.1, 4.9]

Sulem, J. and Vardoulakis, I. (1990). "Bifurcation analysis of the triaxial test on rock specimens. A theoretical model for shape and size effect", *Acta Mechanica* **83**, 195-212. [1.1]

The MathWorks Inc., (1996). MATLAB 5.2.0, Natick, Massachusetts. [7.3, 7.4]

Timoshenko, S. (1921). "On the correction for shear of the differential equations for transverse vibrations of prismatic bars", *Phil. Mag.* **41**, 744-746. [5.28]

Timoshenko, S. and Gere, J.M. (1961). *Theory of elastic stability*, 2nd ed., McGraw-Hill, New York. [5.33]

Timoshenko, S. and Goodier, J.N. (1951). *Theory of elasticity*, 2nd ed., McGraw-Hill, New York. [3.15, 4.20]

Timoshenko, S. and Woinowsky-Kreiger, W. (1959). *Theory of plates and shells*, 2nd ed., McGraw-Hill, New York. [1.2]

Toupin, R.A. (1964). "Theories of elasticity with couple-stress", *Arch. Ration. Mech. Anal.* **17**, 85-112. [1.1]

Truesdell, C. and Toupin, R. (1960). The classical field theories. In Handbuch der Physik. Vol. III/1 (edit. by S. Flugge), pp. 226-793. Springer-Verlag, Berlin. [1.2]

Truesdell, C. and Noll, W. (1965). The non-linear field theories of mechanics. In Handbuch der Physik. Vol. III/3 (edit. by S. Flugge). Springer-Verlag, Berlin. [1.2]

Vardoulakis, I. (1989). "Shear-banding and liquefaction in granular materials on the basis of a Cosserat continuum theory", *Ingenieur-Archiv* **59**, 106-113. [1.1]

Villaggio, P. (1997). *Mathematical models for elastic structures*, Cambridge University Press, Cambridge. [1.1]

Vyalov, S.S. (1986). *Rheological fundamentals of soil mechanics*, Elsevier, Amsterdam. [A]

Watkins, B.O. (1969). *Introduction to control systems*, Macmillan, New York. [6.3]

Whitman, A.B. and DeSilva, C.N. (1974). "An exact solution in a nonlinear theory of rods", *J. Elasticity* **4**, 265-280. [5.34]

Zienkiewicz, O.C. and Taylor, R.L. (1989). *The finite element method, volume 1, basic formulation and linear problems*. McGraw-Hill, New York. [7.1]

Zienkiewicz, O.C. and Taylor, R.L. (1991). *The finite element method, volume 2, solid and fluid mechanics, dynamics and non-linearity*. McGraw-Hill, New York. [7.1]

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