NAME：董骏博

SID: 12432995

4-7:

(a) $\frac{\partial}{\partial y}\nabla^2 u = \frac{\partial}{\partial x}\nabla^2 v$

$\because G\nabla^2 u + (\lambda+G)\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right) + f_x = 0$

$G\nabla^2 v + (\lambda+G)\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right) + f_y = 0$

$\therefore G\frac{\partial}{\partial y}\nabla^2 u + (\lambda+G)\frac{\partial^2}{\partial x\partial y}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right) = 0$

$\Downarrow$

$\frac{\partial}{\partial y}\nabla^2 u = -\frac{\lambda+G}{G}\frac{\partial^2}{\partial x\partial y}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)$

Similarly $\frac{\partial}{\partial x}\nabla^2 v = -\frac{\lambda+G}{G}\frac{\partial^2}{\partial x\partial y}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)$

$\therefore \frac{\partial}{\partial y}\nabla^2 u = \frac{\partial}{\partial x}\nabla^2 v$

(b) $\frac{\partial}{\partial x}\nabla^2 u = -\frac{\partial}{\partial y}\nabla^2 v$

$\because G\frac{\partial}{\partial x}\nabla^2 u + (\lambda+G)\frac{\partial^2}{\partial^2 x}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right) = 0$

$\frac{\partial}{\partial x}\nabla^2 u = -\frac{\lambda+G}{G}\frac{\partial^2}{\partial^2 x}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)$

Similarly $\frac{\partial}{\partial y}\nabla^2 v = -\frac{\lambda+G}{G}\frac{\partial^2}{\partial^2 y}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)$

$$\frac{\partial}{\partial x}\nabla^2 u + \frac{\partial}{\partial y}\nabla^2 v = -\frac{1+G}{G}\left(\underbrace{\left(\frac{\partial^2}{\partial^2 x}+\frac{\partial^2}{\partial^2 y}\right)}_{\nabla^2}\underbrace{\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)}_{\frac{\partial}{\partial x}u+\frac{\partial}{\partial y}v}\right)$$

$\therefore \frac{\partial}{\partial x}\nabla^2 u + \frac{\partial}{\partial y}\nabla^2 v = -\frac{1+G}{G}\left(\frac{\partial}{\partial x}\nabla^2 u + \frac{\partial}{\partial y}\nabla^2 v\right)$

$\because \frac{\lambda+G}{G} \neq 0 \quad \therefore \frac{\partial}{\partial x}\nabla^2 u + \frac{\partial}{\partial y}\nabla^2 v = 0$

$\therefore$ We can get: $\frac{\partial}{\partial x}\nabla^2 u = -\frac{\partial}{\partial y}\nabla^2 v$

(c) $\nabla^2 \varepsilon = \nabla^2 w_z = 0$

$\because \varepsilon = \varepsilon_x + \varepsilon_y \quad , \varepsilon_x = \frac{\partial u}{\partial x} \quad , \varepsilon_y = \frac{\partial v}{\partial y}$

$\therefore \nabla^2 \varepsilon = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$

$\because G \nabla^2 u \frac{\partial}{\partial x} + (\lambda + G) \frac{\partial^2}{\partial x^2}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$

$G \nabla^2 v \frac{\partial}{\partial y} + (\lambda + G) \frac{\partial^2}{\partial y^2}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$

$\therefore \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = -\frac{G}{\lambda + G}\left(\frac{\partial}{\partial x} \nabla^2 u + \frac{\partial}{\partial y} \nabla^2 v\right)$

Due to: $\frac{\partial}{\partial x} \nabla^2 u + \frac{\partial}{\partial y} \nabla^2 v = 0$

$\therefore \nabla^2 \varepsilon = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$

~~(d) $\nabla^4 u = \nabla^4 v = 0$~~

$\because w_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$

$\nabla^2 w_z = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$

Due to: $\frac{\partial}{\partial y} \nabla^2 u = \frac{\partial}{\partial x} \nabla^2 v \Rightarrow \nabla^2 \frac{\partial u}{\partial y} = \nabla^2 \frac{\partial v}{\partial x}$

$\therefore \nabla^2 w_z = \nabla^2 \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = 0$

(d) $\nabla^4 u = \nabla^4 v = 0$

$\because \frac{\partial}{\partial x} \nabla^2 u + \frac{\partial}{\partial y} \nabla^2 v = 0$

$\therefore \begin{cases} \frac{\partial^2}{\partial x^2} \nabla^2 u + \frac{\partial^2}{\partial x \partial y} \nabla^2 v = 0 \\ \frac{\partial^2}{\partial x \partial y} \nabla^2 u + \frac{\partial^2}{\partial y^2} \nabla^2 v = 0 \end{cases}$

$\because \frac{\partial}{\partial y} \nabla^2 u = \frac{\partial^2}{\partial x} \nabla^2 v$

$\therefore \begin{cases} \frac{\partial^2}{\partial x^2} \nabla^2 u + \frac{\partial^2}{\partial y^2} \nabla^2 u = 0 \\ \frac{\partial^2}{\partial x^2} \nabla^2 v + \frac{\partial^2}{\partial y^2} \nabla^2 v = 0 \end{cases}$

$\Downarrow$

$\therefore \nabla^4 u = \nabla^4 v = 0$

Governing equations:

$$\cancel{\because \frac{\partial \sigma_x}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \partial \sigma_z}$$

$$\because \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + f_x = 0$$

$$\sigma_x = 2G\varepsilon_x + \lambda\varepsilon \quad , \quad \varepsilon_x = \frac{\partial u}{\partial x}$$

$$\tau_{yx} = \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) G$$

$$\tau_{zx} = \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) G$$

$$\therefore 2G\frac{\partial^2 u}{\partial x^2} + \lambda\frac{\partial \varepsilon}{\partial x} + G\frac{\partial^2 u}{\partial y^2} + G\frac{\partial^2 v}{\partial x \partial y} + G\frac{\partial^2 u}{\partial z^2} + G\frac{\partial^2 w}{\partial z \partial x} + f_x = 0$$

$$G\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x \partial y} + \frac{\partial^2 w}{\partial z \partial x}\right) + \lambda\frac{\partial \varepsilon}{\partial x} + G\nabla^2 u + f_x = 0$$

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

$$\because \varepsilon = \varepsilon_x + \varepsilon_y + \varepsilon_z$$

$$\frac{\partial^2 v}{\partial x \partial y} = \frac{\partial \varepsilon_y}{\partial x} \qquad \frac{\partial^2 w}{\partial z \partial x} = \frac{\partial \varepsilon_z}{\partial x} \qquad \frac{\partial^2 u}{\partial x^2} = \frac{\partial \varepsilon_x}{\partial x}$$

$$\therefore \cancel{G\,(\lambda+G)}\; G\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x \partial y} + \frac{\partial^2 w}{\partial z \partial x}\right) = G\,(\varepsilon_x + \varepsilon_y + \varepsilon_z)\frac{\partial}{\partial x} = G\frac{\partial \varepsilon}{\partial x}$$

$$\therefore (\lambda+G)\frac{\partial \varepsilon}{\partial x} + G\nabla^2 u + f_x = 0$$

Similarly: $\begin{cases} (\lambda+G)\dfrac{\partial \varepsilon}{\partial y} + G\nabla^2 v + f_y = 0 \\ (\lambda+G)\dfrac{\partial \varepsilon}{\partial z} + G\nabla^2 w + f_z = 0 \end{cases}$

4-2.

$$\because \begin{cases} T_x^u = \sigma_x|_0 u_x + \tau_{yx}|_0 u_y + \tau_{zx}|_0 u_z \\ T_y^u = \tau_{xy}|_0 u_x + \sigma_y|_0 u_y + \tau_{zy}|_0 u_z \\ T_z^u = \tau_{xz}|_0 u_x + \tau_{yz}|_0 u_y + \sigma|_0 u_z \end{cases}$$

$$\because \sigma_x = 2G\varepsilon_x + \lambda\varepsilon = 2G\frac{\partial u}{\partial x} + \lambda\varepsilon$$

$$\tau_{yx} = G\gamma_{yx} = G\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

$$\tau_{zx} = G\gamma_{zx} = G\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$$

$$\therefore \cancel{T_x^u = \lambda\varepsilon (2G\varepsilon_x + \lambda}$$

$$\therefore T_x^u = (2G\varepsilon_x + \lambda\varepsilon)u_x + G\gamma_{yx}u_y + G\gamma_{zx}u_z$$

$$= \left(2G\frac{\partial u}{\partial x} + \lambda\varepsilon\right)u_x + G\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)u_y + G\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)u_z$$

$$= \lambda\varepsilon u_x + G\left(\frac{\partial u}{\partial x}u_x + \frac{\partial u}{\partial y}u_y + \frac{\partial u}{\partial z}u_z\right) + G\left(\frac{\partial u}{\partial x}u_x + \frac{\partial v}{\partial x}u_y + \frac{\partial w}{\partial x}u_z\right)$$

Similarly: T_y^u . T_z^u

- Wave equation:

$$\because A\sigma_{xx}(x+dx) - A\sigma_{xx}(x) = \rho\, dx A\frac{\partial^2 u}{\partial t^2}$$

$$\therefore \cancel{A}\frac{\partial\sigma_{xx}}{\partial x}dx = \rho\, dx \cancel{A}\frac{\partial^2 u}{\partial t^2}$$

$$\frac{\partial\sigma_{xx}}{\partial x} = \rho\frac{\partial^2 u}{\partial t^2}$$

$$\therefore \frac{\partial^2 u}{\partial t^2} = \frac{1}{\rho}\frac{\partial\sigma_{xx}}{\partial x}$$

Due to: $\sigma_{xx} = E\varepsilon_{xx}$. $\varepsilon_{xx} = \frac{\partial u}{\partial x}$

$$\therefore \frac{\partial u}{\partial t^2} = \frac{E}{\rho}\frac{\partial^2 u}{\partial x^2} = a^2\frac{\partial^2 u}{\partial x^2}$$