Generalized Hooke’s law——only two independent elastic modulus

generalized Hooke’s law for isotropic linear elastic solid:

$$\begin{cases} \varepsilon_x = \dfrac{1}{E}\left(\sigma_x - \nu\left(\sigma_y + \sigma_z\right)\right) \\ \varepsilon_y = \dfrac{1}{E}\left(\sigma_y - \nu\left(\sigma_z + \sigma_x\right)\right) \\ \varepsilon_z = \dfrac{1}{E}\left(\sigma_z - \nu\left(\sigma_x + \sigma_y\right)\right) \end{cases} \qquad \begin{cases} \gamma_{xy} = \dfrac{1}{G}\tau_{xy} \\ \gamma_{yz} = \dfrac{1}{G}\tau_{yz} \\ \gamma_{zx} = \dfrac{1}{G}\tau_{zx} \end{cases}$$

- Only two indepdent elastic constants for isotropic linear elastic solid
- Three elastic modulus (E, ν, G) appear above
  - We can express any one with the other two modulus

We now demonstrate this in the state of plane stress

First, for isotropic elastic materials, the principal axes of stress and strain **coincide**

$$\gamma_{xy} = \gamma_{yz} = \gamma_{zx} = 0 \Rightarrow \tau_{xy} = \tau_{yz} = \tau_{zx} = 0$$

Generalized Hooke's law——only two independent elastic modulus

Plane stress: $\sigma_z = \tau_{xz} = \tau_{yz} = 0$

The generalized Hooke's law in the state of plane stress is

$$\varepsilon_x = \frac{1}{E}\left(\sigma_x - \nu\sigma_y\right)$$

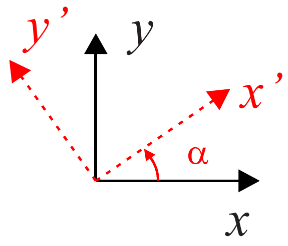
$$\varepsilon_y = \frac{1}{E}\left(\sigma_y - \nu\sigma_x\right)$$

$$\gamma_{xy} = \frac{1}{G}\tau_{xy}$$

In new coordinate:

$$\begin{cases} \varepsilon_{x'} = \frac{1}{E}\left(\sigma_{x'} - \nu\sigma_{y'}\right) \\ \varepsilon_{y'} = \frac{1}{E}\left(\sigma_{y'} - \nu\sigma_{x'}\right) \end{cases}$$

$$\gamma_{x'y'} = \frac{1}{G}\tau_{x'y'}$$



Let x and y be the principal axes, we have

$$\tau_{x'y'} = (\sigma_y - \sigma_x)\sin\alpha\cos\alpha$$

$$\gamma_{x'y'} = 2(\varepsilon_y - \varepsilon_x)\sin\alpha\cos\alpha$$

$$\frac{\tau_{x'y'}}{\gamma_{x'y'}} = G = \frac{\sigma_y - \sigma_x}{2(\varepsilon_y - \varepsilon_x)}$$

$$\Rightarrow \quad G = \frac{E}{2(1+\nu)}$$

Generalized Hooke's law——only two independent elastic modulus

strain expressed by stress

$$\begin{cases} \varepsilon_x = \frac{1}{E}\left(\sigma_x - \nu\left(\sigma_y + \sigma_z\right)\right) \\ \varepsilon_y = \frac{1}{E}\left(\sigma_y - \nu\left(\sigma_z + \sigma_x\right)\right) \\ \varepsilon_z = \frac{1}{E}\left(\sigma_z - \nu\left(\sigma_x + \sigma_y\right)\right) \end{cases} \qquad \begin{cases} \gamma_{xy} = \frac{1}{G}\tau_{xy} \\ \gamma_{yz} = \frac{1}{G}\tau_{yz} \\ \gamma_{zx} = \frac{1}{G}\tau_{zx} \end{cases}$$

$$\Longleftrightarrow$$

stress expressed by strain

$$\begin{cases} \sigma_x = 2G\varepsilon_x + \lambda\left(\varepsilon_x + \varepsilon_y + \varepsilon_z\right) \\ \sigma_y = 2G\varepsilon_y + \lambda\left(\varepsilon_x + \varepsilon_y + \varepsilon_z\right) \\ \sigma_z = 2G\varepsilon_z + \lambda\left(\varepsilon_x + \varepsilon_y + \varepsilon_z\right) \end{cases} \qquad \begin{cases} \tau_{xy} = G\gamma_{xy} \\ \tau_{yz} = G\gamma_{yz} \\ \tau_{zx} = G\gamma_{zx} \end{cases}$$

where $G = \frac{E}{2(1+\nu)}$ $\quad \lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}$ Lame's constants

Bulk modulus of elasticity

For general stress state, the mean of the normal stress is

$$\sigma_m = \frac{\sigma_x + \sigma_y + \sigma_z}{3}$$

σ_m is called hydrostatic (spherical) component of stress.

Pressure is usually defined as p = -σ_m

Dilatation (volumetric strain) ε: volume change per unit volume

$$\varepsilon = \frac{\Delta V}{V} = \varepsilon_x + \varepsilon_y + \varepsilon_z$$

The volume strain is

$$\varepsilon = \frac{\Delta V}{V} = \frac{3(1-2\nu)}{E}\sigma_m = -\frac{3(1-2\nu)}{E}p$$

$$K = \frac{E}{3(1-2\nu)}$$

K: bulk modulus (体积模量)

$$-p = K\varepsilon = K\frac{\Delta V}{V}$$

We also have $\frac{\Delta V}{V} = -\beta p$

$K = \frac{1}{\beta}$ β: compressibility (压缩系数)

Bulk Modulus of Elasticity——physical meanings of elastic modulus

stress expressed by strain

$$\begin{cases}\sigma_x = 2G\varepsilon_x + \lambda(\varepsilon_x + \varepsilon_y + \varepsilon_z) \\ \sigma_y = 2G\varepsilon_y + \lambda(\varepsilon_x + \varepsilon_y + \varepsilon_z) \\ \sigma_z = 2G\varepsilon_z + \lambda(\varepsilon_x + \varepsilon_y + \varepsilon_z)\end{cases} \qquad \begin{cases}\tau_{xy} = G\gamma_{xy} \\ \tau_{yz} = G\gamma_{yz} \\ \tau_{zx} = G\gamma_{zx}\end{cases}$$

strain expressed by stress

$$\begin{cases}\varepsilon_x = \frac{1}{E}\left(\sigma_x - \nu(\sigma_y + \sigma_z)\right) \\ \varepsilon_y = \frac{1}{E}\left(\sigma_y - \nu(\sigma_z + \sigma_x)\right) \\ \varepsilon_z = \frac{1}{E}\left(\sigma_z - \nu(\sigma_x + \sigma_y)\right)\end{cases} \qquad \begin{cases}\gamma_{xy} = \frac{1}{G}\tau_{xy} \\ \gamma_{yz} = \frac{1}{G}\tau_{yz} \\ \gamma_{zx} = \frac{1}{G}\tau_{zx}\end{cases}$$

$$\nu = \frac{\lambda}{2(\lambda + \mu)} \quad E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu} \quad \mu = \frac{E}{2(1+\nu)} \quad \lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \quad K = \frac{3\lambda + 2\mu}{3} = \frac{E}{3(1-2\nu)}$$

The physical meaning of E and ν: uniaxial stress $G=\mu$

The physical meaning of shear modulus μ (G): shear strain and shear stress

The physical meaning of bulk modulus K: average normal stress and volume change

The physical meaning of λ: uniaxial strain

Bulk Modulus of Elasticity——physical meanings of elastic modulus

$$\nu = \frac{\lambda}{2(\lambda + \mu)} \quad E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu} \quad \mu = \frac{E}{2(1+\nu)} \quad \lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \quad K = \frac{3\lambda + 2\mu}{3} = \frac{E}{3(1-2\nu)}$$

$$G=\mu$$

E, K, and μ are alwalys positive, constraining ν to be between -1.0 and 0.5

Note that λ has the same sign of ν, that is, λ can be negative

Strain and Stress decomposition

Strain decomposition (应变分解): Decompose the strain state into a spherical component plus a deviatoric component.

- The spherical component does not incorporate shear strain
- The deviatoric component does not include volume change.
- Does the deviatoric strain component incorporate normal strain?

$$\begin{bmatrix} \varepsilon_x & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_y & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_z \end{bmatrix} = \begin{bmatrix} \varepsilon_m & & \\ & \varepsilon_m & \\ & & \varepsilon_m \end{bmatrix} + \begin{bmatrix} \varepsilon_x - \varepsilon_m & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_y - \varepsilon_m & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_z - \varepsilon_m \end{bmatrix}$$

The strain state represented by a matrix

Spherical component

Deviatoric component

$$\varepsilon_m = \frac{\varepsilon_x + \varepsilon_y + \varepsilon_y}{3} = \frac{\varepsilon_1 + \varepsilon_2 + \varepsilon_3}{3} = \frac{1}{3} I_1'$$

Strain and Stress decomposition

Stress decomposition (应力分解):
Similarly, decompose stress into a spherical component plus a deviatoric component.

$$\begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix} = \begin{bmatrix} \sigma_m & & \\ & \sigma_m & \\ & & \sigma_m \end{bmatrix} + \begin{bmatrix} \sigma_x - \sigma_m & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y - \sigma_m & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z - \sigma_m \end{bmatrix}$$

Stress state represented by a matrix

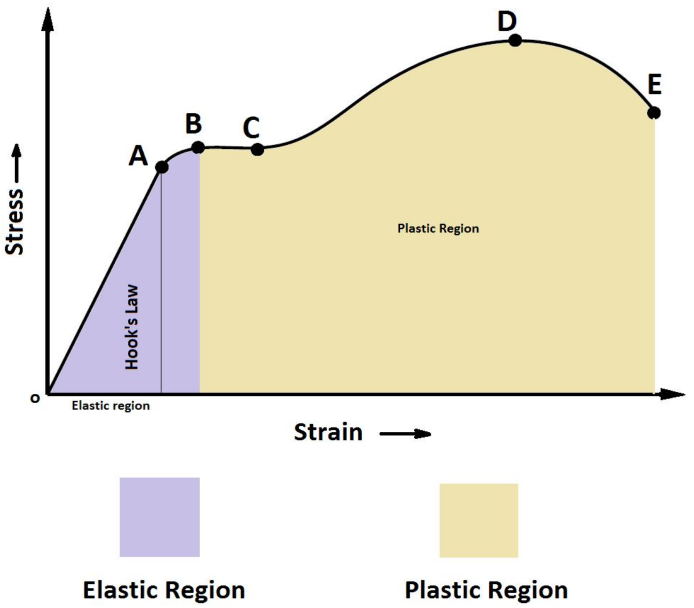
Spherical component

Deviatoric component

$$\sigma_m = \frac{\sigma_x + \sigma_y + \sigma_z}{3} = \frac{\sigma_1 + \sigma_2 + \sigma_3}{3} = \frac{1}{3} I_1$$

- The spherical component of stress does not induce shear strain
- The deviatoric component does not induce a volume change.

Stress and Strain Invariants and Plastic Yielding



Solids yield (屈服) under large strain or stress, but how do we know if the strain or stress is large or small?

The stress state at a point represented by a matrix

$$\begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix}$$

The strain state at a point represented by a matrix

$$\begin{bmatrix} \varepsilon_x & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_y & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_z \end{bmatrix}$$

Equivalent stress (等效应力) is usually used.

- ***Definition****: A scalar value that describes the magnitude of the stress a material suffers.*

$$\sigma^* < \sigma_y = \sigma_B$$

Equivalent stress should be independent of the coordinate chosen.

Stress and Strain Invariants and Plastic Yielding

Equivalent stress (等效应力) is constructed with the combination of the three principal stresses or stress invariants

The three invariants of stress

$$I_1 = \sigma_x + \sigma_y + \sigma_z = \sigma_1 + \sigma_2 + \sigma_3$$

$$I_2 = \begin{vmatrix} \sigma_x & \tau_{xy} \\ \tau_{yx} & \sigma_y \end{vmatrix} + \begin{vmatrix} \sigma_y & \tau_{yz} \\ \tau_{zy} & \sigma_z \end{vmatrix} + \begin{vmatrix} \sigma_z & \tau_{zx} \\ \tau_{xz} & \sigma_x \end{vmatrix}$$

$$= \sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{zx}^2 = \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1$$

$$I_3 = \begin{vmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{vmatrix}$$

$$= \sigma_x\sigma_y\sigma_z + 2\tau_{xy}\tau_{yz}\tau_{zx} - \sigma_x\tau_{yz}^2 - \sigma_y\tau_{zx}^2 - \sigma_z\tau_{xy}^2 = \sigma_1\sigma_2\sigma_3$$

Tresca yield criterion (特雷斯卡屈服准则)

- Yielding will occur when the greatest maximum shear stress reaches a critical value

$$\sigma^* = \tau_{\max} = \frac{\sigma_1 - \sigma_3}{2} < \sigma_y = \sigma_B$$

In a 2D case that incorporate the σ_1 and σ_3 principal stress axes

$$\tau_{\max} = \frac{\sqrt{I_1^2 - 4I_2}}{2}$$

Stress and Strain Invariants and Plastic Yielding

The three invariants of stress

$$I_1 = \sigma_x + \sigma_y + \sigma_z = \sigma_1 + \sigma_2 + \sigma_3$$

$$I_2 = \begin{vmatrix} \sigma_x & \tau_{xy} \\ \tau_{yx} & \sigma_y \end{vmatrix} + \begin{vmatrix} \sigma_y & \tau_{yz} \\ \tau_{zy} & \sigma_z \end{vmatrix} + \begin{vmatrix} \sigma_z & \tau_{zx} \\ \tau_{xz} & \sigma_x \end{vmatrix}$$

$$= \sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{zx}^2 = \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1$$

$$I_3 = \begin{vmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{vmatrix}$$

$$= \sigma_x\sigma_y\sigma_z + 2\tau_{xy}\tau_{yz}\tau_{zx} - \sigma_x\tau_{yz}^2 - \sigma_y\tau_{zx}^2 - \sigma_z\tau_{xy}^2 = \sigma_1\sigma_2\sigma_3$$

$$J_2 = \frac{1}{2}(\sigma_x'^2 + \sigma_y'^2 + \sigma_z'^2 + 2\tau_{xy}'^2 + 2\tau_{yz}'^2 + 2\tau_{zx}'^2)$$

Von Mises Yield Criterion (米塞斯屈服准则)

- yielding will begin when the second invariant of deviatoric stress J_2 reaches a critical value

$$\sigma^* = \sqrt{3J_2} \leq \sigma_y = \sigma_B$$

Define Von Mises equivalent stress

$$\sigma_v = \sqrt{3J_2} = \sqrt{\frac{3}{2}\sigma_{ij}'\sigma_{ij}'}$$

The yield criterion is:

$$\sigma_v \leq \sigma_y = \sigma_B$$

Stress and Strain Invariants and Plastic Yielding

The three invariants of stress

$$I_1 = \sigma_x + \sigma_y + \sigma_z = \sigma_1 + \sigma_2 + \sigma_3$$

$$I_2 = \begin{vmatrix} \sigma_x & \tau_{xy} \\ \tau_{yx} & \sigma_y \end{vmatrix} + \begin{vmatrix} \sigma_y & \tau_{yz} \\ \tau_{zy} & \sigma_z \end{vmatrix} + \begin{vmatrix} \sigma_z & \tau_{zx} \\ \tau_{xz} & \sigma_x \end{vmatrix}$$

$$= \sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{zx}^2 = \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1$$

$$I_3 = \begin{vmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{vmatrix}$$

$$= \sigma_x\sigma_y\sigma_z + 2\tau_{xy}\tau_{yz}\tau_{zx} - \sigma_x\tau_{yz}^2 - \sigma_y\tau_{zx}^2 - \sigma_z\tau_{xy}^2 = \sigma_1\sigma_2\sigma_3$$

Drucker-Prager yield criterion (德鲁克-普拉格屈服准则)

- yielding will begin when the second invariant of deviatoric stress J_2 reaches a critical value that depends on I_1

$$\sigma^* = \sqrt{3J_2} \le \sigma_y = \frac{I_1}{3} f + C$$

f: friction coefficient C: cohesion

With the von Mises equivalent stress

$$\sigma_v = \sqrt{3J_2} = \sqrt{\frac{3}{2}\sigma'_{ij}\sigma'_{ij}}$$

The yield criterion is:

$$\sigma_v \le \sigma_y = -\frac{I_1}{3} f + C$$

Classroom Exercise

For isotropic elastic materials, the principal axes of stress and strain **coincide.**
Does the orientation of the maximum shear stress and maximum shear strain coincide?

3-10 Determine the slope of the σ_x vs. ϵ_x curve in the elastic range if a material is tested under the following state of stress:

$$\sigma_x = 2\sigma_y = 3\sigma_z.$$

Classroom exercise

- Assume the stress components of hydrostatic stress state in x-y-z coordinate is:

$$\sigma_x = \sigma_y = \sigma_z = -p$$

$$\tau_{xy} = \tau_{yz} = \tau_{zx} = 0$$

  – Check the normal stress and shear stress components of any coordinate system (x'-y'-z')

- Check the stress-strain relations for spherical and deviatoric stresses, respectively

$$\sigma_{x'} = \sigma_x a_{11}^2 + \sigma_y a_{21}^2 + \sigma_z a_{31}^2 + 2\tau_{xy} a_{11} a_{21} + 2\tau_{yz} a_{21} a_{31} + 2\tau_{zx} a_{31} a_{11}$$

$$\begin{aligned}\tau_{x'y'} = \sigma_x a_{11} a_{12} + \sigma_y a_{21} a_{22} + \sigma_z a_{31} a_{32} \\ + \tau_{xy}(a_{11} a_{22} + a_{21} a_{12}) \\ + \tau_{yz}(a_{21} a_{32} + a_{31} a_{22}) \\ + \tau_{zx}(a_{31} a_{12} + a_{11} a_{32})\end{aligned}$$

$$\begin{aligned}\tau_{x'z'} = \sigma_x a_{11} a_{13} + \sigma_y a_{21} a_{23} + \sigma_z a_{31} a_{33} \\ + \tau_{xy}(a_{11} a_{23} + a_{21} a_{13}) \\ + \tau_{yz}(a_{21} a_{33} + a_{31} a_{23}) \\ + \tau_{zx}(a_{31} a_{13} + a_{11} a_{33})\end{aligned}$$

Classroom Exercise

The three invariants of stress are the three coefficients of the eigenequation below

$$\sigma_p^3-(\sigma_x+\sigma_y+\sigma_z)\sigma_p^2+(\sigma_x\sigma_y+\sigma_y\sigma_z+\sigma_z\sigma_x-\tau_{xy}^2-\tau_{yz}^2-\tau_{zx}^2)\sigma_p-(\sigma_x\sigma_y\sigma_z+2\tau_{xy}\tau_{yz}\tau_{zx}-\sigma_x\tau_{yz}^2-\sigma_y\tau_{xz}^2-\sigma_z\tau_{xy}^2)=0$$

$$\sigma_p^3-I_1\sigma_p^2+I_2\sigma_p-I_3=0$$

$$I_1=\sigma_x+\sigma_y+\sigma_z=\sigma_1+\sigma_2+\sigma_3$$

$$I_2=\begin{vmatrix}\sigma_x & \tau_{xy}\\ \tau_{yx} & \sigma_y\end{vmatrix}+\begin{vmatrix}\sigma_y & \tau_{yz}\\ \tau_{zy} & \sigma_z\end{vmatrix}+\begin{vmatrix}\sigma_z & \tau_{zx}\\ \tau_{xz} & \sigma_x\end{vmatrix}$$

$$=\sigma_x\sigma_y+\sigma_y\sigma_z+\sigma_z\sigma_x-\tau_{xy}^2-\tau_{yz}^2-\tau_{zx}^2=\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_1$$

$$I_3=\begin{vmatrix}\sigma_x & \tau_{xy} & \tau_{xz}\\ \tau_{yx} & \sigma_y & \tau_{yz}\\ \tau_{zx} & \tau_{zy} & \sigma_z\end{vmatrix}$$

$$=\sigma_x\sigma_y\sigma_z+2\tau_{xy}\tau_{yz}\tau_{zx}-\sigma_x\tau_{yz}^2-\sigma_y\tau_{zx}^2-\sigma_z\tau_{xy}^2=\sigma_1\sigma_2\sigma_3$$

Assume the invariants J_1, J_2, and J_3 of the deviatoric stress are the coefficients of the corresponding eigenequation,

$$S_p^3-J_1S_p^2-J_2S_p-J_3=0$$

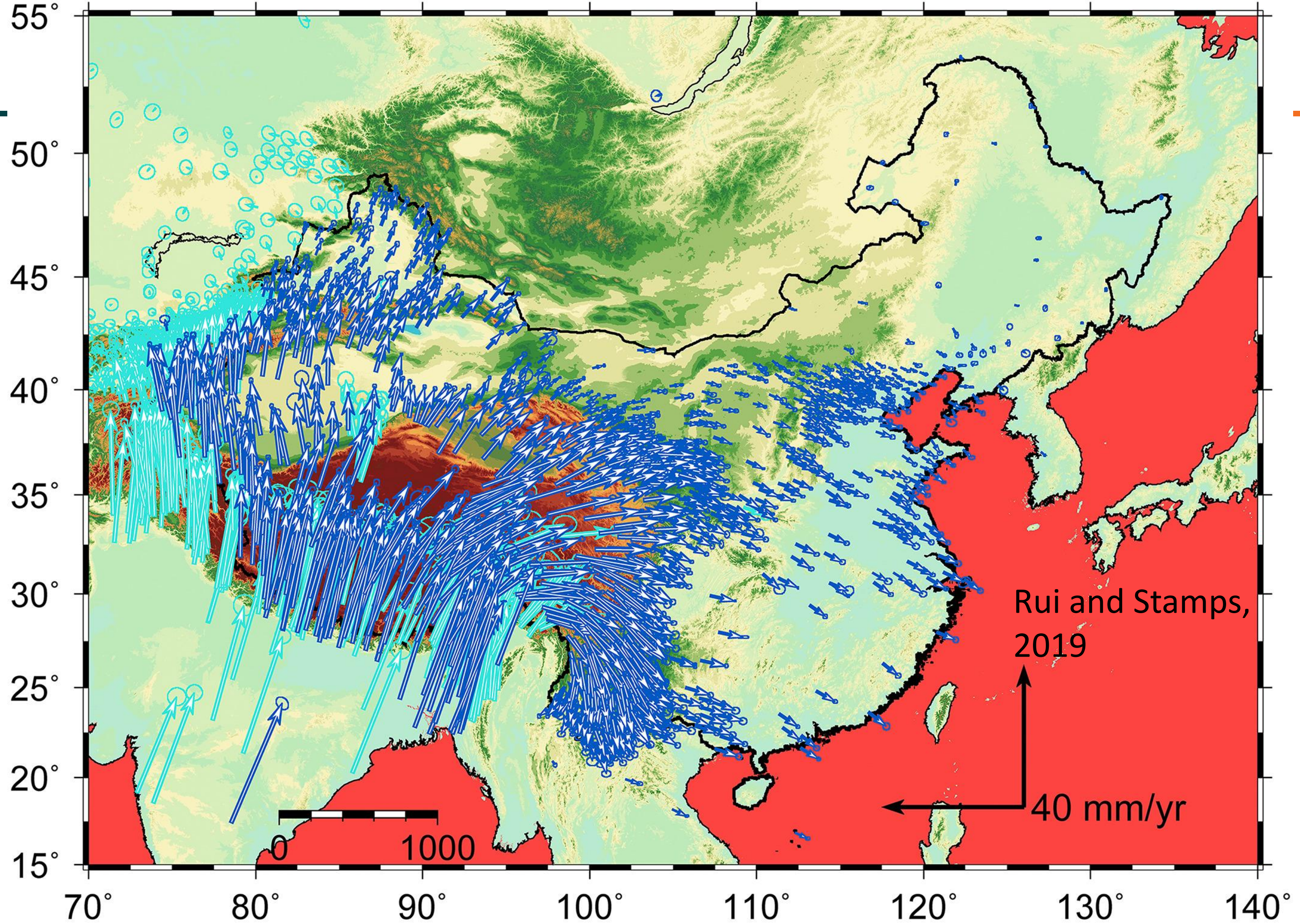
prove that J_1, J_2, and J_3 are as below

$$J_1=0$$

$$J_2=\frac{1}{2}(\sigma_x'^2+\sigma_y'^2+\sigma_z'^2+2\tau_{xy}'^2+2\tau_{yz}'^2+2\tau_{zx}'^2)$$

$$=\frac{1}{6}[(\sigma_x-\sigma_y)^2+(\sigma_y-\sigma_z)^2+(\sigma_z-\sigma_x)^2]$$

$$+\tau_{xy}^2+\tau_{yz}^2+\tau_{zx}^2$$



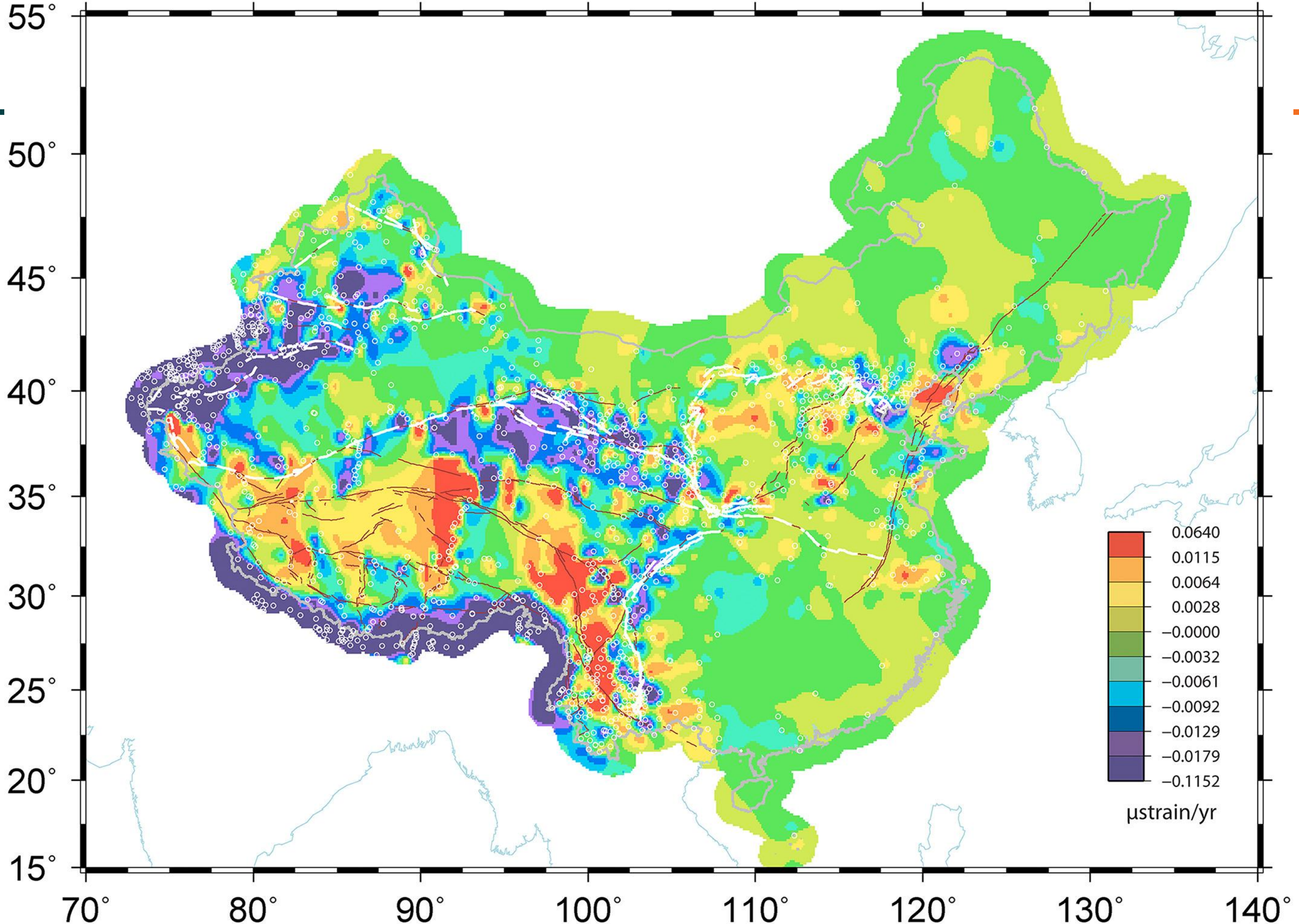
55°
50°
45°
40°
35°
30°
25°
20°
15°
70°
80°
90°
100°
110°
120°
130°
140°
Rui and Stamps, 2019
40 mm/yr
0
1000

Rate of Dilatation

$$\dot{\varepsilon}_{xx} + \dot{\varepsilon}_{yy}$$

or

$$\dot{\varepsilon}_{\theta\theta} + \dot{\varepsilon}_{\phi\phi}$$

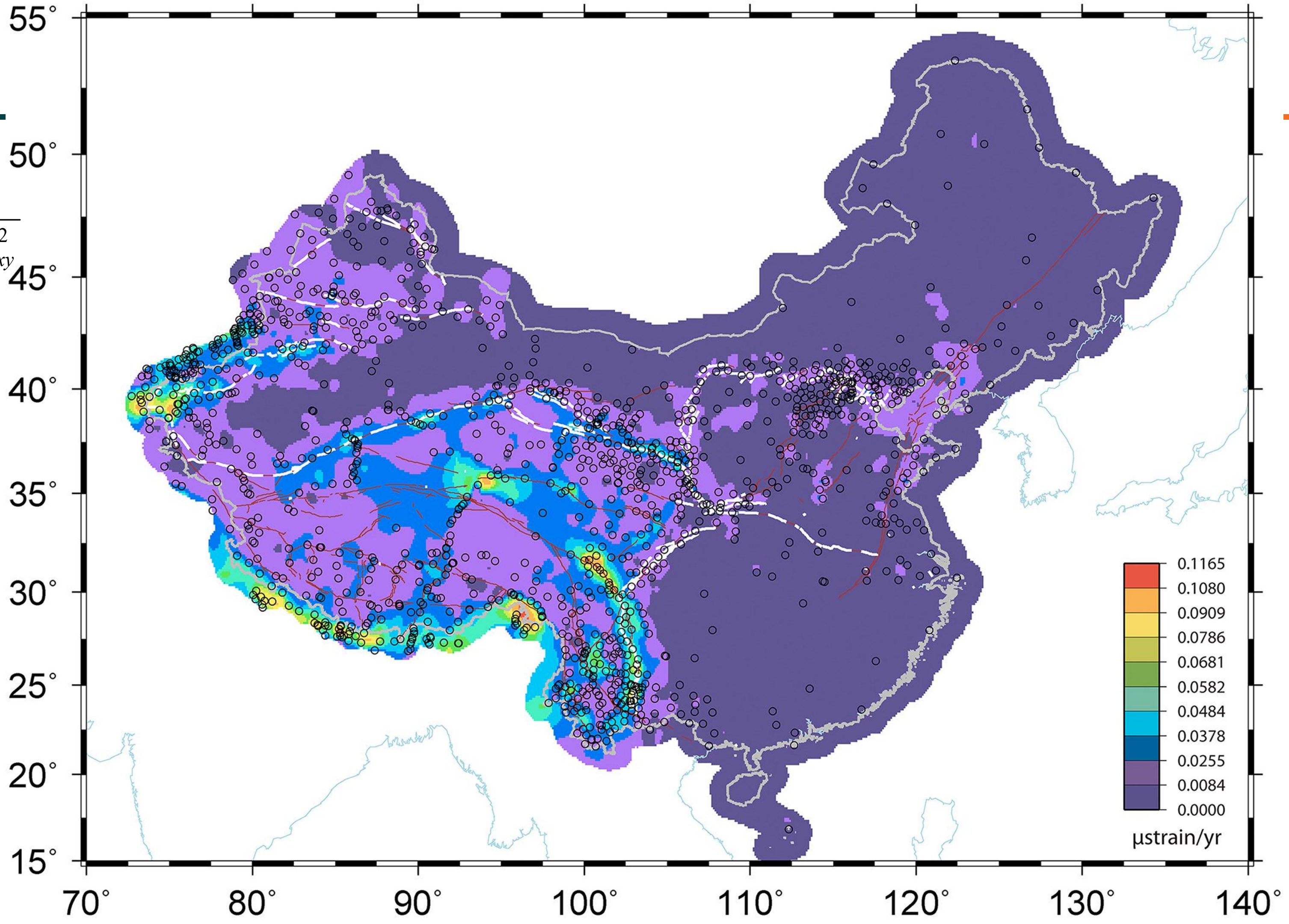


Second invariant of strain rate:

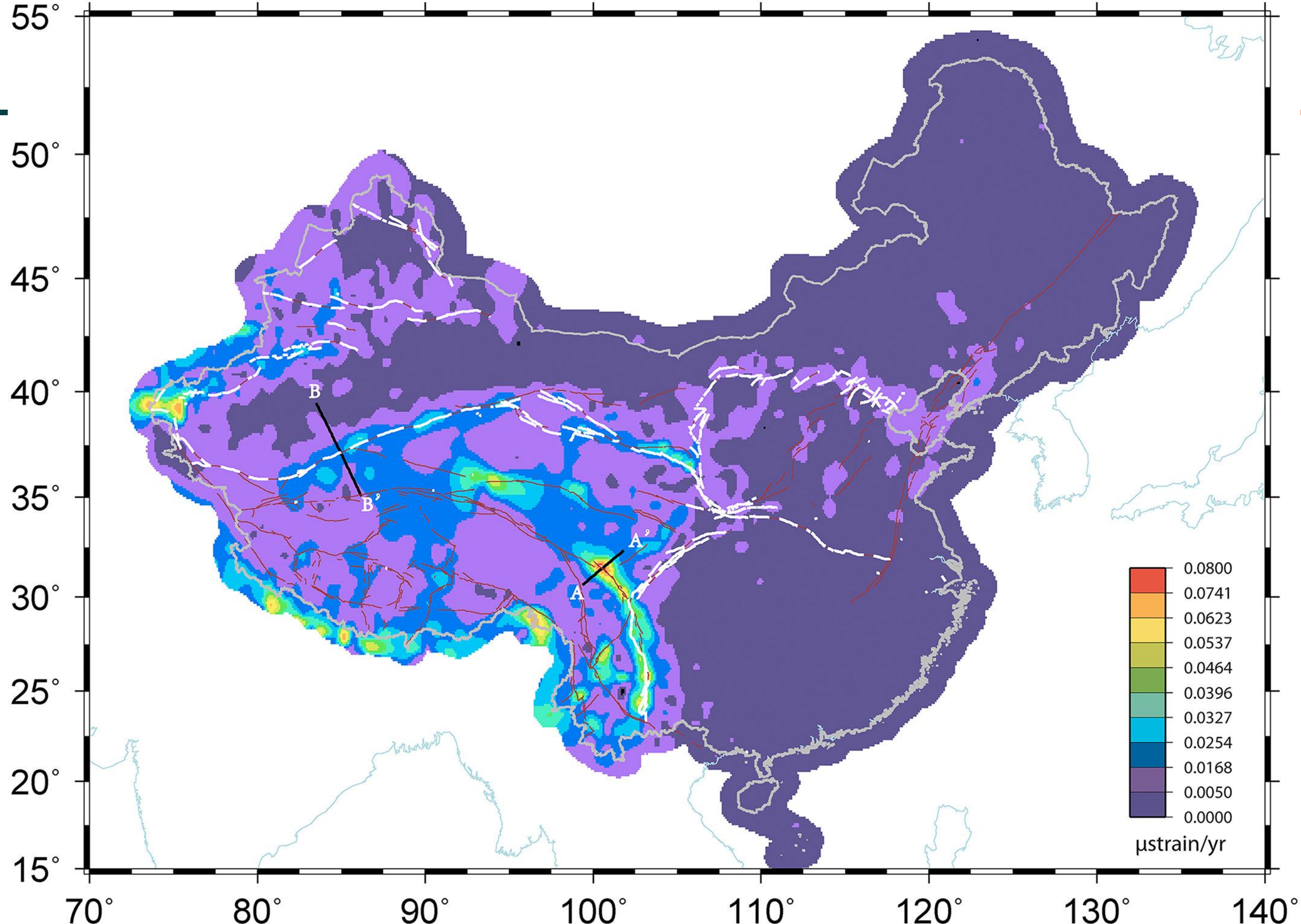
$$\sqrt{\dot{\varepsilon}_{xx}^2 + \dot{\varepsilon}_{yy}^2 + 2\dot{\varepsilon}_{xy}^2}$$

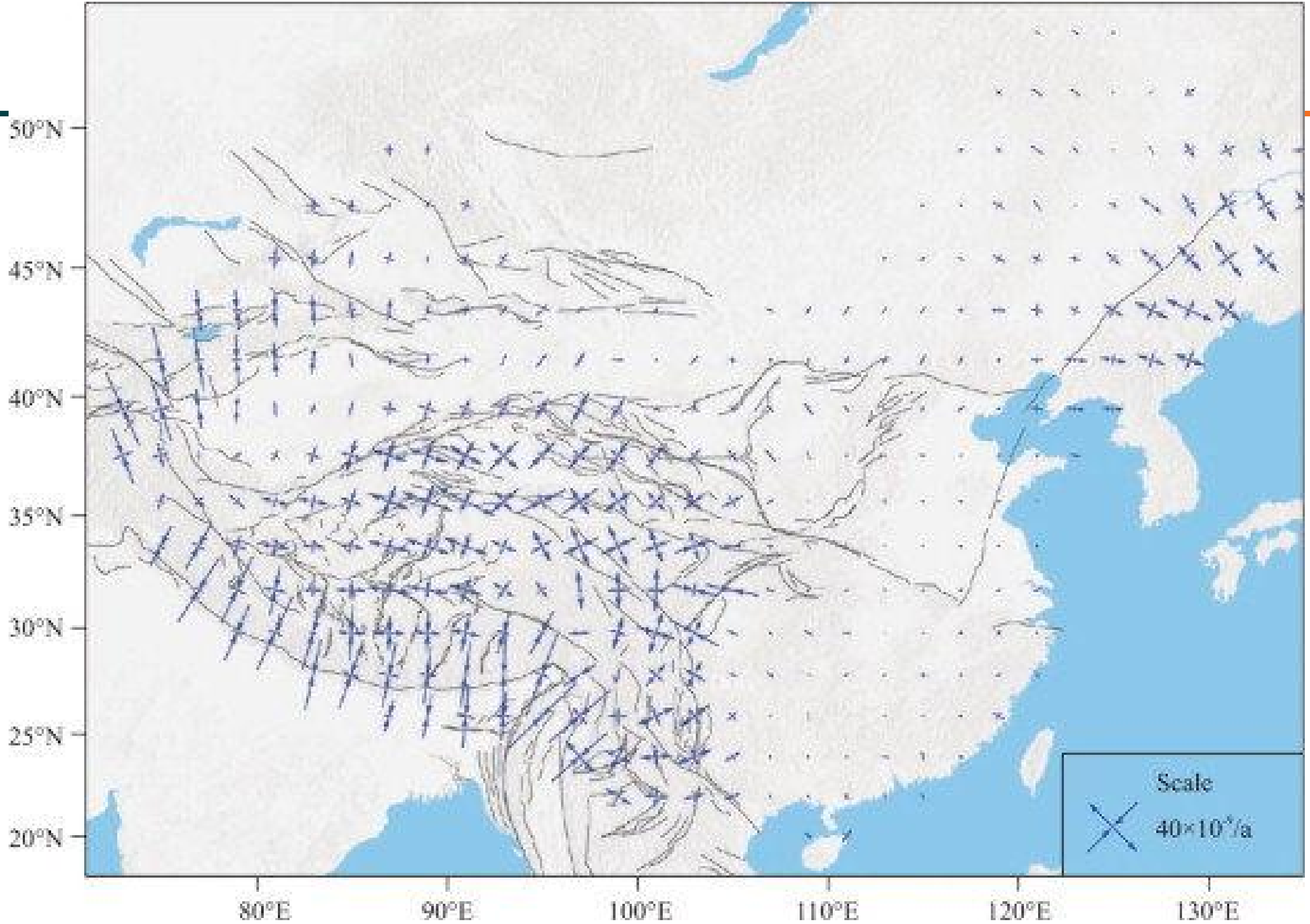
or

$$\sqrt{\dot{\varepsilon}_{\varphi\varphi}^2 + \dot{\varepsilon}_{\theta\theta}^2 + 2\dot{\varepsilon}_{\varphi\theta}^2}$$



Maximum shear strain rate




50°N
45°N
40°N
35°N
30°N
25°N
20°N
80°E
90°E
100°E
110°E
120°E
130°E
Scale
40×10⁻⁹/a

Homework 3 (5 points)

3-2 If a medium is initially unstrained and is then subjected to a constant positive temperature change, the normal strains are expressed by

$$\epsilon_x = \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)] + \alpha T$$

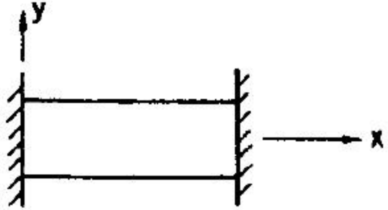
$$\cdots$$

$$\cdots$$

where α is the coefficient of linear expansion and T is the temperature rise. The temperature change does not affect the shear strain components.

A bar restrained in the x direction only, and free to expand in the y and z directions as shown, is subjected to a uniform temperature rise T. Show that the only nonvanishing stress (the bar is in a state of uniform stress) and strain components are

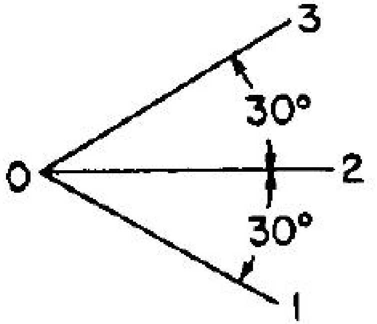
$$\sigma_x = -E\alpha T$$
$$\epsilon_y = \epsilon_z = \alpha T(1 + \nu).$$



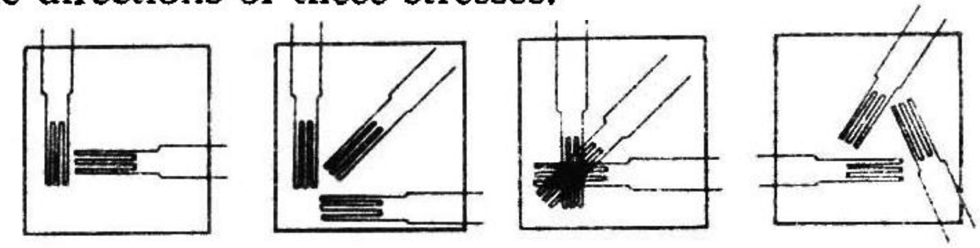
Homework 3 (5 points)

3-1 One of the applications of generalized Hooke's law is found in the use of rosette gages in the field of experimental stress analysis. The state of strain at a point is determined experimentally by determining the rosette gage readings, which give the normal strain in three directions in a plane. Since rosette gages are applied to a free surface, the stress components σ_z, τ_{zx}, and τ_{zy} are zero, where the z axis is normal to the free surface. By using generalized Hooke's law, then, one can define the state of stress.

By using the rosette gage shown in the figure in one experiment, the following strains are recorded on the surface of a steel bar: $\epsilon_{0-1} = +10^{-4}$, $\epsilon_{0-2} = +4 \times 10^{-4}$, $\epsilon_{0-3} = +6 \times 10^{-4}$. Given the material properties of this steel, $E = 30 \times 10^6$ psi, $\nu = 0.25$, determine the principal stresses and maximum shear stress at point O. Give the directions of these stresses.



Rosette strain gage (花形应变计， 应变花): A single strain gauge (应变仪) can only measure strain in one direction. Rosette strain gauge (gage) is an arrangement of two or more strain gauges that are positioned closely to measure strains along different directions.



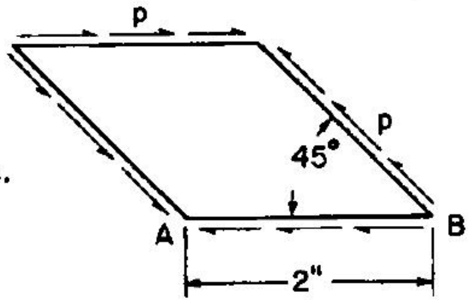
a 二轴90° 应变花 c 三轴45° 应变花 d 三轴45° 应变花 e 三轴60° 应变花

Homework 3 (5 points)

3-3 Determine the stress and strain components if the bar in the preceding (3.2 in CE) problem is restrained in the x and y directions but is free to expand in the z direction.

3-9 A thin plate is under a uniform state of stress, with $p = 14{,}140$ psi, $E = 30 \times 10^6$ psi, and $\nu = 0.3$, as shown.

(a) Find the change in length of AB.
(b) Find the principal strains and their directions.



3-11 Given $G = \dfrac{E}{2(1+\nu)}$ $\quad \lambda = \dfrac{E\nu}{(1+\nu)(1-2\nu)}$ $\quad K = \dfrac{E}{3(1-2\nu)}$

Prove the following relations among various elastic constants:

$$\nu = \frac{3K - E}{6K} \qquad \lambda = \frac{3K - 2G}{3} \qquad E = \frac{\lambda(1+\nu)(1-2\nu)}{\nu}$$

Reading material

Under the assumption of infinitesimal deformation, the differential equations of equilibrium and strain-displacement relationships are valid for any solid. The relationships between internal force and internal deformation, however, depend upon the material properties of the particular medium under discussion. The relationships between internal force (stress, stress rate) and internal deformation (strain, strain-rate) are called the constitutive equations.

The constitutive equations of elastic solids are called generalized Hooke's law and involve only stress and strain and is independent of stress-rate or strain-rate. In linear elastic model, the solid regains its original dimensions after the forces acting on it are removed and the stress-strain relationship is linear.

Materials are seldom isotropic or homogeneous in the crystal scale, because the crystalline or molecular structure of material is not continuous and may not be randomly oriented. However, the assumptions of isotropy and homogeneity usually lead to reasonable results. This is because the stresses and strains are averaged over dimensions that are much larger than the dimensions of crystals and molecules.

The equilibrium equations, the strain-displacement relations, and the generalized Hooke's law constitute 15 equations. Along with the prescribed boundary conditions, the strain, stress, and displacement distribution (15 unknown quantities in total) inside the elastic solid can be uniquely determined.

Reading material

Elastic Modulus is the measurement of a material's elasticity. Elastic modulus quantifies a material's resistance to non-permanent, or elastic, deformation. When under stress, materials will first exhibit elastic properties: the stress causes them to deform, but the material will return to its previous state after the stress is removed. After passing through the elastic region and through their yield point, materials enter a plastic region, where they exhibit permanent deformation even after the tensile stress is removed.

Modulus is defined as being the slope of the straight-line portion of a stress (σ) strain (ε) curve. Stress σ is force divided by the specimen's cross-sectional area and strain ε is the change in length of the material divided by the material's original gauge length.

Since both stress and strain are normalized measurements, modulus is a consistent material property that can be compared between specimens of different sizes. A large steel specimen will have the same modulus as a small steel specimen, although the large specimen will require a higher maximum force to deform the material. For most typical metals the magnitude of this modulus ranges between 45 GPa, for magnesium (镁), and 407 GPa, for tungsten (钨). Values of the elastic modulus for ceramic materials (陶瓷材料) are about the same as for metals; for polymers (高分子材料) they are lower. These differences are a direct consequence of the different types of atomic bonding in the three materials. Furthermore, with increasing temperature, the modulus of elasticity diminishes.