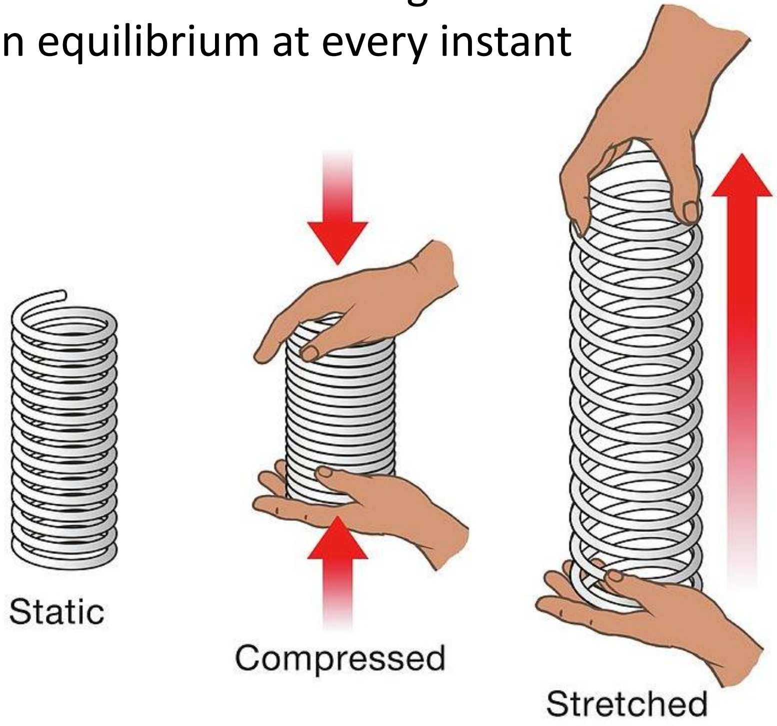
Strain Energy

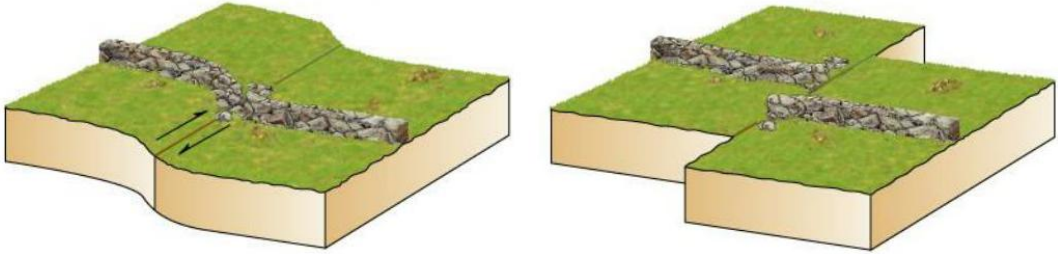
slow: the inertia terms can be neglected and the body is in equilibrium at every instant

- Consider an elastic body deformed by external forces
- Work done by these forces are **fully** transformed to energy stored in the body if the process is **slow**.
- **Strain energy (elastic potential energy, 应变能，弹性势能)**: the energy stored in the body due to external work
- Assume a spring has a stiffness factor (倔强系数) k, if it is streched **slowly** by displacement x under an external load. The work done by the external load is

$$W = \int_0^x F dx' = \int_0^x (kx') dx' = \frac{1}{2} kx^2$$

  - It equals to the stored elastic energy





Strain Energy: uniaxial stress

- Consider strain energy due to a **slow** uniform uniaxial stress condition as shown on the right.
  - $\sigma \rightarrow u, \varepsilon$
- During the process of deformation, the normal stress σ increases from zero to the final value of σ_x, the work done on the element (stored strain energy) is

$$dU = \int_{\sigma=0}^{\sigma=\sigma_x} -\sigma dydz \cdot du + \int_{\sigma=0}^{\sigma=\sigma_x} \sigma dydz \cdot d\left(u + \frac{\partial u}{\partial x}dx\right)$$

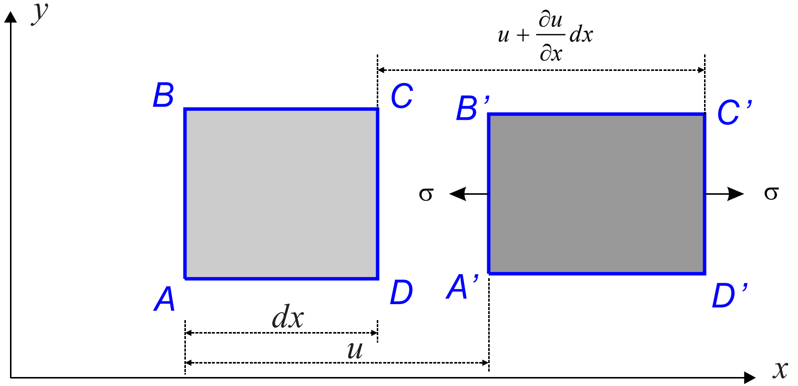
$$= \int_{\sigma=0}^{\sigma=\sigma_x} \sigma d\left(\frac{\partial u}{\partial x}\right) dxdydz$$

$$= \int_{\sigma=0}^{\sigma=\sigma_x} \sigma d\varepsilon dxdydz = \int_{\sigma=0}^{\sigma=\sigma_x} \sigma \frac{d\sigma}{E} dxdydz$$

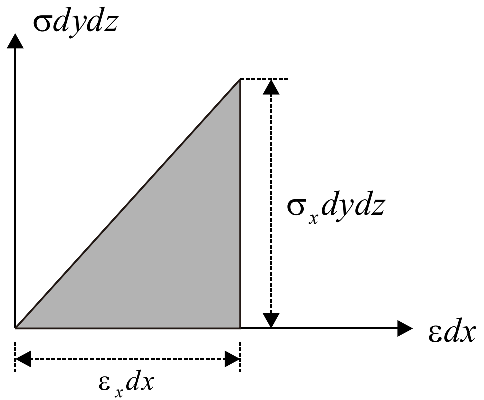
$$= \frac{\sigma_x^{\ 2}}{2E} dxdydz$$

strain energy density U_0: strain energy per unit volume

$$U_0 = \frac{\sigma_x^{\ 2}}{2E} = \frac{1}{2}\sigma_x \varepsilon_x = \frac{1}{2}E\varepsilon_x^{\ 2}$$



The work done by a force also equals to the area under its force-displacement curve



Strain Energy: shear stress

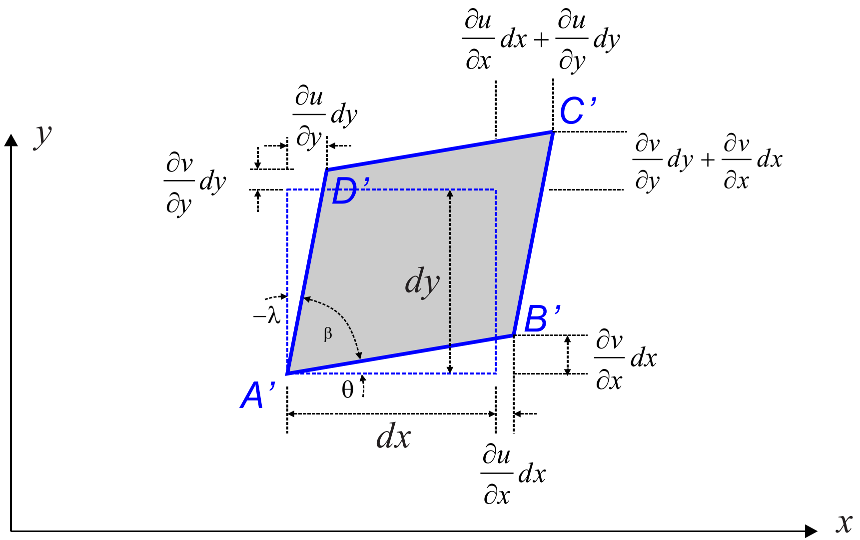
Shear strain energy:

Total work done by shear stress τ_{xy} on A'D' and B'C' planes are

$$dU_y = \frac{1}{2}(\tau_{xy}dydz)[\frac{1}{2}(\frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy)] - \frac{1}{2}\tau_{xy}[\frac{1}{2}(\frac{\partial v}{\partial y}dy)]$$
$$= \frac{1}{2}(\tau_{xy}dydz)\frac{\partial v}{\partial x}dx$$

Total work done by shear stress τ_{yx} on A'B' and D'C' planes are

$$dU_x = \frac{1}{2}(\tau_{yx}dxdz)\frac{\partial u}{\partial y}dy$$



Shear strain deformation due to shear stress

Shear strain energy: $$dU = \frac{1}{2}\tau_{xy}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)dxdydz$$
$$= \frac{1}{2}\tau_{xy}\gamma_{xy}dxdydz$$

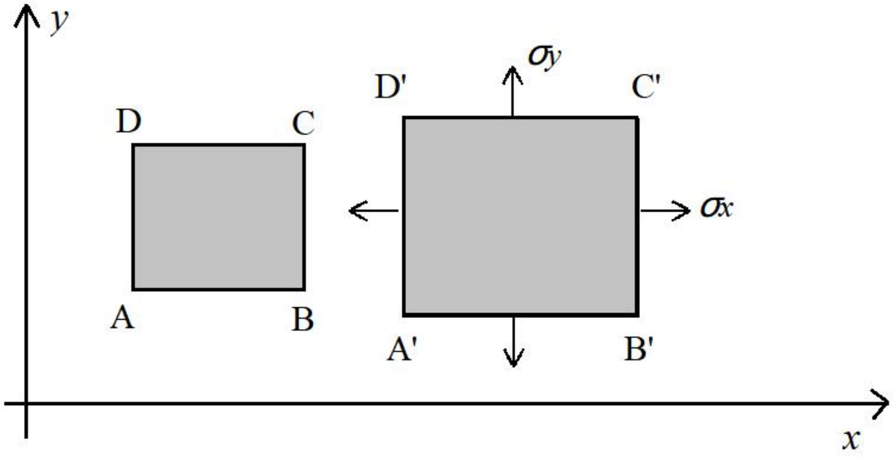
Shear strain energy density:

$$U_0 = \frac{1}{2}\tau_{xy}\gamma_{xy} = \frac{1}{2}\frac{\tau_{xy}^2}{G} = \frac{1}{2}G\gamma_{xy}^2$$

Strain Energy: general stress condition

Consider an element under the action of both σ_x and σ_y,
σ_x and σ_y are usually loaded at the same time, and the strains ε_x and ε_y are due to both σ_x and σ_y,

but stress in one direction does not do any work in the other direction perpendicular to it.



So the total strain energy can be calculated by direct summation of each strain energy:

$$U_0 = \frac{1}{2}\sigma_x\varepsilon_x + \frac{1}{2}\sigma_y\varepsilon_y$$

Under a general stress condition:

$$U_0 = \frac{1}{2}\left(\sigma_x\varepsilon_x + \sigma_y\varepsilon_y + \sigma_z\varepsilon_z + \tau_{xy}\gamma_{xy} + \tau_{yz}\gamma_{yz} + \tau_{zx}\gamma_{zx}\right)$$

Strain Energy: Variable Stress Distribution and Body Force

We now consider the strain energy in the case where the stress are not uniform and body force (may include inertia forces) exists.

The work done by normal stress σ_x on the the left plane AB is

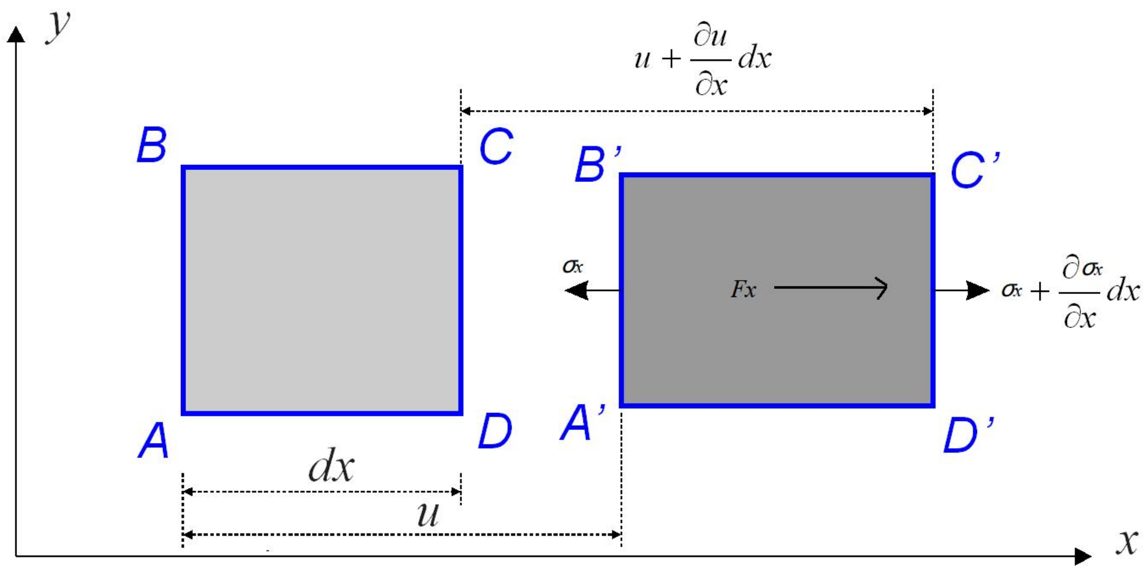
$$-\tfrac{1}{2}\sigma_x u\, dy\, dz$$

The work done by normal stress on the the right plane CD is

$$\frac{1}{2}\left(\sigma_x + \frac{\partial \sigma_x}{\partial x}dx\right)\left(u + \frac{\partial u}{\partial x}dx\right) dy\, dz$$

The net work done by the normal stress on the AB and CD planes is

$$\frac{1}{2}\left(\sigma_x + \frac{\partial \sigma_x}{\partial x}dx\right)\left(u + \frac{\partial u}{\partial x}dx\right)dy\, dz - \frac{1}{2}\sigma_x u\, dy\, dz = \frac{1}{2}\frac{\partial}{\partial x}(\sigma_x u)\, dx\, dy\, dz$$



Strain Energy: Variable Stress Distribution and Body Force

Similarly, the net work done by the shear stress components on xy plane is

$$\frac{1}{2}\frac{\partial(\tau_{xy}v)}{\partial x}dx\,dy\,dz+\frac{1}{2}\frac{\partial(\tau_{xz}w)}{\partial x}dx\,dy\,dz$$

The total work done by all stresses and body forces per unit volume (strain energy density) is

$$U_0=\frac{1}{2}\left[\frac{\partial}{\partial x}(\sigma_x u+\tau_{xy}v+\tau_{xz}w)+\frac{\partial}{\partial y}(\sigma_y v+\tau_{yx}u+\tau_{yz}w)+\frac{\partial}{\partial z}(\sigma_z w+\tau_{zx}u+\tau_{zy}v)+F_x u+F_y v+F_z w\right]$$

$$=\frac{1}{2}\left[\left(\sigma_x\frac{\partial u}{\partial x}+\sigma_y\frac{\partial v}{\partial y}+\sigma_z\frac{\partial w}{\partial z}\right)+\tau_{xy}\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)+\tau_{yz}\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)+\tau_{zx}\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)\right.$$

$$\left.+u\left(\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{xy}}{\partial y}+\frac{\partial\tau_{xz}}{\partial z}+F_x\right)+v\left(\frac{\partial\sigma_y}{\partial y}+\frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\tau_{yz}}{\partial z}+F_y\right)+w\left(\frac{\partial\sigma_z}{\partial z}+\frac{\partial\tau_{xz}}{\partial x}+\frac{\partial\tau_{yz}}{\partial y}+F_z\right)\right]$$

$$=\tfrac{1}{2}[\sigma_x\epsilon_x+\sigma_y\epsilon_y+\sigma_z\epsilon_z+\tau_{xy}\gamma_{xy}+\tau_{yz}\gamma_{yz}+\tau_{zx}\gamma_{zx}]$$

Identical to the strain energy density formula for uniform stress distribution without body force

Classroom exercise

4-8 A bar of constant mass density ρ hangs under its own weight and is supported by the uniform stress σ_0 as shown in the figure. Assume that the stresses σ_x, σ_y, τ_{xy}, τ_{xz}, and τ_{yz} vanish identically.

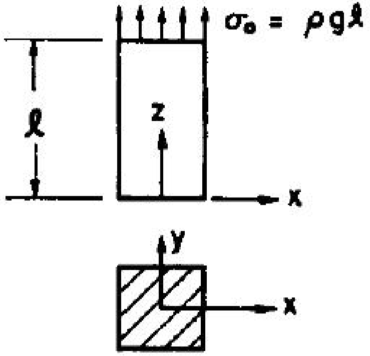
(a) Based on the above assumption, reduce the 15 governing equations to seven equations in terms of σ_z, ϵ_x, ϵ_y, ϵ_z, u, v, and w.

(b) Integrate the equilibrium equation to show that

$$\sigma_z = \rho g z$$

where g is the acceleration due to gravity. Also show that the precribed boundary conditions are satisfied by this solution.

(c) Find ϵ_x, ϵ_y, and ϵ_z from Hooke's law.



Equilibrium equations $\quad \dfrac{\partial \sigma_x}{\partial x} + \dfrac{\partial \tau_{yx}}{\partial y} + \dfrac{\partial \tau_{zx}}{\partial z} + f_x = 0 \quad (x, y, z)$

Hooke's Law $\quad \sigma_x = 2G\varepsilon_x + \lambda(\varepsilon_x + \varepsilon_y + \varepsilon_z) \quad (x, y, z) \qquad \tau_{xy} = G\gamma_{xy}$

Strain-displacement $\quad \varepsilon_x = \dfrac{\partial u}{\partial x} \qquad \gamma_{xy} = \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}$

Note that σ_z, ε_x, ε_y, ε_z vary with depth

- Think of the shape of a drop of water falling from a roof
- Compare this exercise with uniaxial stress.

Homework (5 points)

4-7 Verify the following relations for plane strain problems with F_x and F_y constant by considering Eqs. (4.14)

$$\frac{\partial}{\partial y}\nabla^2 u = \frac{\partial}{\partial x}\nabla^2 v \qquad \nabla^2\epsilon = \nabla^2\omega_z = 0$$

$$\frac{\partial}{\partial x}\nabla^2 u = -\frac{\partial}{\partial y}\nabla^2 v \qquad \nabla^4 u = \nabla^4 v = 0$$

where $\nabla^4 = \nabla^2\nabla^2 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$

$$G\nabla^2 u + (\lambda + G)\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + F_x = 0$$

$$G\nabla^2 v + (\lambda + G)\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + F_y = 0 \tag{4.14}$$

where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$

Homework (5 points)

4-2 Derive the stress boundary condition in terms of displacement equations

$$
\begin{aligned}
T_x^{\mu} &= \lambda\epsilon\mu_x + G\left(\frac{\partial u}{\partial x}\mu_x + \frac{\partial u}{\partial y}\mu_y + \frac{\partial u}{\partial z}\mu_z\right) + G\left(\frac{\partial u}{\partial x}\mu_x + \frac{\partial v}{\partial x}\mu_y + \frac{\partial w}{\partial x}\mu_z\right) \\
T_y^{\mu} &= \lambda\epsilon\mu_y + G\left(\frac{\partial v}{\partial y}\mu_y + \frac{\partial v}{\partial z}\mu_z + \frac{\partial v}{\partial x}\mu_x\right) + G\left(\frac{\partial v}{\partial y}\mu_y + \frac{\partial w}{\partial y}\mu_z + \frac{\partial u}{\partial y}\mu_x\right) \qquad (4.19) \\
T_z^{\mu} &= \lambda\epsilon\mu_z + G\left(\frac{\partial w}{\partial z}\mu_z + \frac{\partial w}{\partial x}\mu_x + \frac{\partial w}{\partial y}\mu_y\right) + G\left(\frac{\partial w}{\partial z}\mu_z + \frac{\partial u}{\partial z}\mu_x + \frac{\partial v}{\partial z}\mu_y\right)
\end{aligned}
$$

Homework (5 points)

Governing equations

- Strain-displacement (6)

$$\varepsilon_x = \frac{\partial u}{\partial x} \qquad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \qquad (x, y, z; u, v, w)$$

- Stress-strain relations (6)

$$\sigma_x = 2G\varepsilon_x + \lambda\varepsilon \qquad \tau_{xy} = G\gamma_{xy} \qquad (x, y, z)$$

- Equilibrium equations (3)

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + f_x = 0 \quad (x, y, z)$$

$$\varepsilon = \varepsilon_x + \varepsilon_y + \varepsilon_z = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

Show that the governing equations for the elastic problems using displacement is as below:

$$\begin{cases} (\lambda + G)\dfrac{\partial \varepsilon}{\partial x} + G\nabla^2 u + f_x = 0 \\ (\lambda + G)\dfrac{\partial \varepsilon}{\partial y} + G\nabla^2 v + f_y = 0 \\ (\lambda + G)\dfrac{\partial \varepsilon}{\partial z} + G\nabla^2 w + f_z = 0 \end{cases}$$

Equilibrium equations in terms of displacement, or Navier's equations

$$\nabla^2 u = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Laplace operator

Homework (5 points)

There are three main kinds of mathematical physics equations. One of them is the wave equation (波动方程). The 1D wave equation is:

2 points

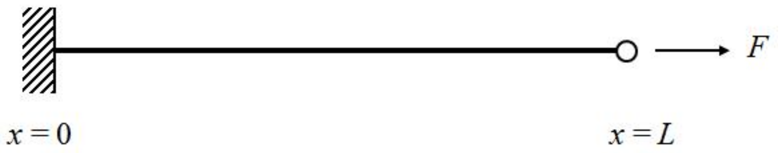
$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}$$

where u is particle displacement, a is wave propagation rate.

Consider 1D vibration along a uniform bar after initial excitation. The only non-zero stress component is σ_{xx}. Prove that $u(x, t)$ satisfies the 1D wave equation below if gravity is neglected:

$$\frac{\partial^2 u}{\partial t^2} = \frac{E}{\rho}\frac{\partial^2 u}{\partial x^2} = a^2 \frac{\partial^2 u}{\partial x^2}$$

where E is Young's modulus, ρ is string density, a is wave propagation rate.



All displacement components (u, v, w) exist

stress: $(\sigma_{xx}, \sigma_{yy}, \sigma_{zz}) = (\sigma_{xx}, 0, 0)$

Hint: (1) you cannot directly use the horizontal differential equation of equilibrium with the inertia term, which does not consider τyx=τzx=0:

$$(\lambda + G)\varepsilon_{,x} + G\nabla^2 u = \rho \frac{\partial^2 u}{\partial t^2}$$

(2) use elemental horizontal force balance:

$$A\sigma_{xx}(x+dx) - A\sigma_{xx}(x) = \rho dx A \frac{\partial^2 u}{\partial t^2}$$

Reading materials

The solution of a given problem in elasticity consists of the determination of the stress, strain, and displacement components as functions of the space coordinates of the elastic body. The necessary equations which the stress, strain, and displacement components must satisfy include the equilibrium equations, strain-displacement relations, and the generalized Hooke's law.

Any set of stress, strain, and displacement functions satisfying these 15 governing equations will represent the solution to some problem in elasticity. Suppose, however, that we are to solve a specific problem, for example, a plate subjected to prescribed surface forces. In this case, the stresses must not only satisfy the field equations, but the stresses evaluated at any point on the boundary surface must be in equilibrium with the prescribed surface force at the same point. Thus, the solution of a given problem in elasticity consists of the determination of stress, strain, and displacement components which satisfy the 15 governing field equations, and the prescribed stress or displacement boundary conditions.

Satisfaction of the boundary conditions demands that the stress components on the boundary must be in equilibrium with the applied surface forces in regions where the surface forces are prescribed, and the displacement components on the boundary are equal to the prescribed displacements in regions where displacement is prescribed. The solution satisfying all of these conditions for a given problem is unique, i.e., it represents the only solution to this problem.

Reading materials

The superposition principle states that, for all linear systems, the net response caused by two or more stimuli（激励）is the sum of the responses that would have been caused by each stimulus individually. So that if input A produces response X and input B produces response Y then input (A + B) produces response (X + Y).

Many physical systems can be modeled as linear systems. For example, a beam can be modeled as a linear system where the input stimulus is the load on the beam and the output response is the deflection of the beam.

The superposition principle applies to any linear system, including algebraic equations, linear differential equations, etc. The stimuli and responses could be numbers, functions, vectors, time-varying signals, etc. When vectors are involved, a superposition is interpreted as a vector sum.

By writing a general stimulus as the superposition of stimuli of a specific and simple form, often the response becomes easier to compute. For example, in Fourier analysis, the stimulus is written as the superposition of infinitely many sinusoids. Each of these sinusoids can be analyzed separately, and its individual response can be computed. The response is itself a sinusoid, with the same frequency as the stimulus, but generally a different amplitude and phase. In Green's function analysis, the stimulus is written as the superposition of infinitely many impulse functions, and the response is then a superposition of impulse responses.