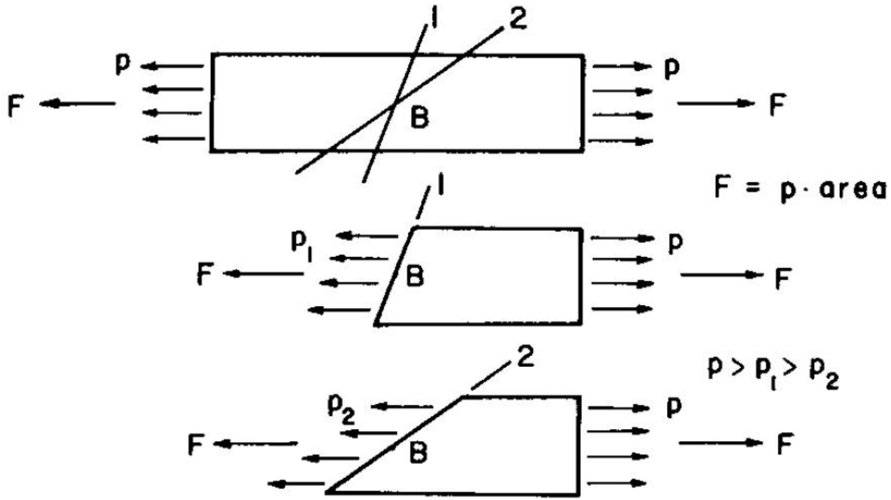
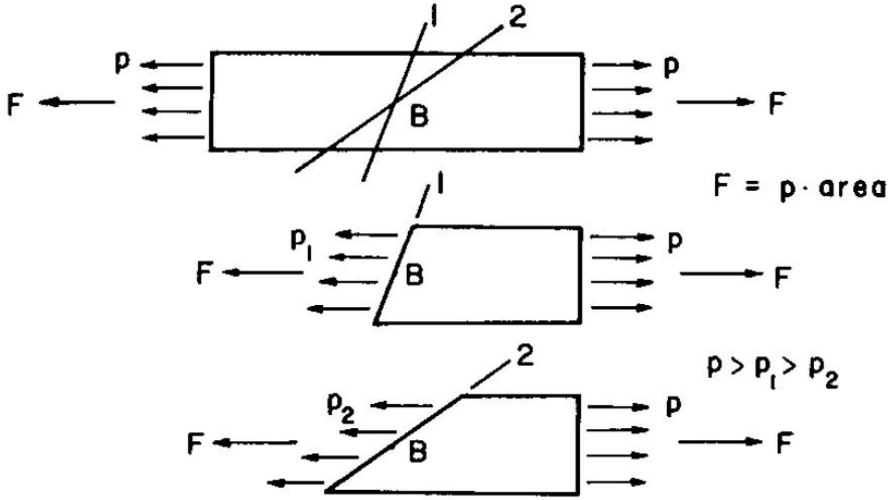
Quick Questions



- The microscopic origin (微观成因) of the **stress vector** at point B inside the material:
  - Intermolecular / interatomic forces between the left and right part
- What is the unit of stress vector?
- meaning of τ_{xy}, τ_{yz}, σ_x
- Can stress vectors of different planes through a point be added up?

Transformation of Stress Components for Uniform Stress Distribution

The stress vector at point B depends on the orientation of the plane it acts on.



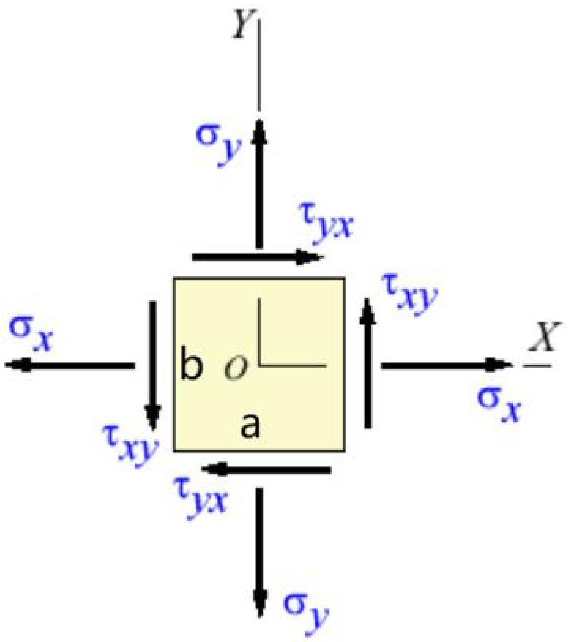
We now prove:

Given the stress vectors on the x and y planes, stress vectors on any plane can be determined.

Transformation of Stress Components for Uniform Stress Distribution

- **Uniform state of stress (均匀应力分布)**:
  - The state of stress in a body is the same at all points
  - Stress vectors still depend on the orientation of the plane on which they act

- Consider a 2D (XOY plane) case
  - Given stress vectors on X and Y planes: σ_x, σ_y, and τ_{xy}, τ_{yx},
  - we first demonstrate that $\tau_{xy} = \tau_{yx}$,
    - reciprocity of shear stress (剪应力互等定理)

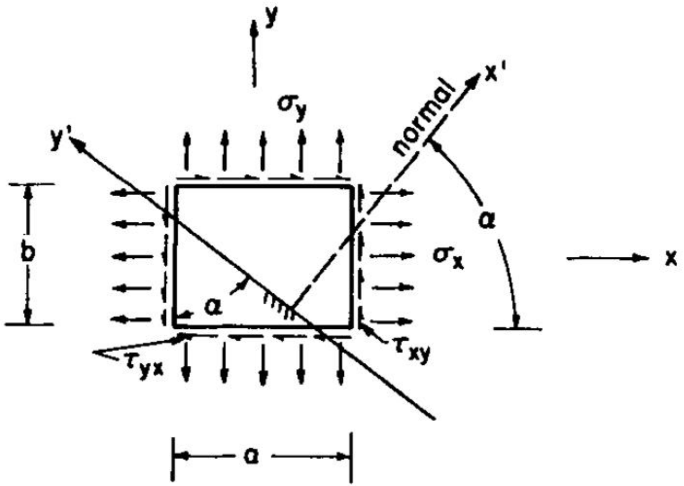


Take moments (力矩) about the lower left corner of the rectangle, we have

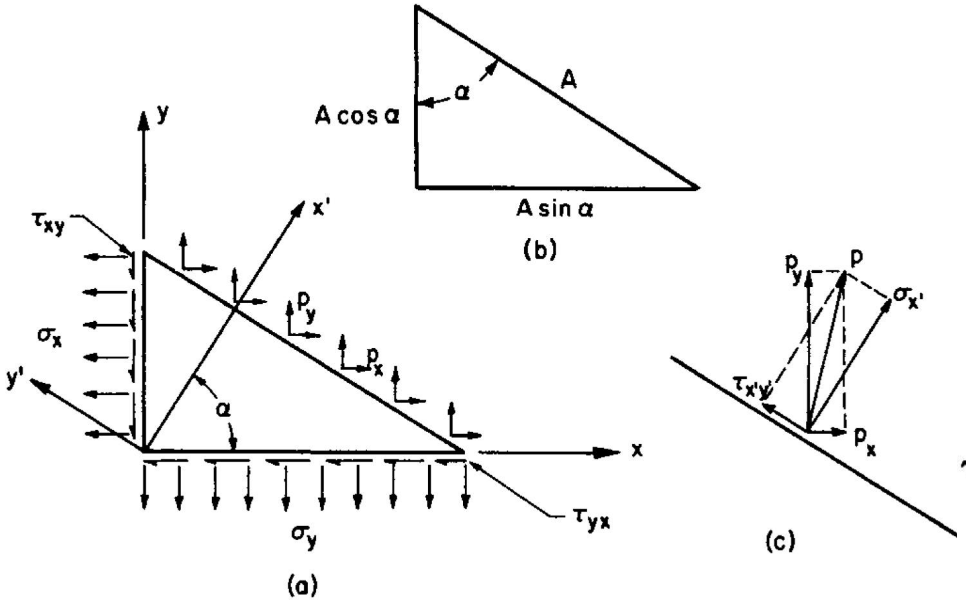
$$(\sigma_x b)(b/2) - (\sigma_x b)(b/2) + (\sigma_y a)(a/2) - (\sigma_y a)(a/2) - (\tau_{xy} b)a + (\tau_{yx} a)b = 0 \Rightarrow \tau_{xy} = \tau_{yx}$$

Transformation of Stress Components for Uniform Stress Distribution

- Given stress vectors on x and y planes: σ_x, σ_y, and τ_{xy}, τ_{yx} on the right,
- we now calculate $\sigma_{x'}$, $\sigma_{y'}$, and $\tau_{x'y'}$, $\tau_{y'x'}$ at an arbitrary new coordinate x'oy'
- We first calculate stress on an arbitrary plane x'
  - α, the angle from x to x',
    - positive for counterclockwise direction



Transformation of Stress Components for Uniform Stress Distribution



Consider the free body above, the stress vector on the x' plane $p=(p_x, p_y) = (\sigma_{x'}, \tau_{x'y'})$ can be determined by force balance:

$$\sum F_x = 0 \quad \Rightarrow \quad p_x = \sigma_x \cos\alpha + \tau_{xy} \sin\alpha$$

Similarly,

$$p_y = \sigma_y \sin\alpha + \tau_{xy} \cos\alpha$$

Normal stress on the x' plane is

$$\begin{aligned}\sigma_{x'} &= p_x \cos\alpha + p_y \sin\alpha \\ &= \sigma_x \cos^2\alpha + \sigma_y \sin^2\alpha + 2\tau_{xy} \sin\alpha \cos\alpha\end{aligned}$$

Shear stress on the x' plane is

$$\begin{aligned}\tau_{x'y'} &= p_y \cos\alpha - p_x \sin\alpha \\ &= (\sigma_y - \sigma_x) \sin\alpha \cos\alpha + \tau_{xy}(\cos^2\alpha - \sin^2\alpha)\end{aligned}$$

Normal stress on the y' plane is derived by substituting $\alpha+\pi/2$ for α

$$\sigma_{y'} = \sigma_x \sin^2\alpha + \sigma_y \cos^2\alpha - 2\tau_{xy} \sin\alpha \cos\alpha$$

Transformation of Stress Components for Uniform Stress Distribution

Transformation of stress equations (应力变换方程)

$$\begin{cases} \sigma_{x'} = \sigma_x \cos^2\alpha + \sigma_y \sin^2\alpha + 2\tau_{xy}\sin\alpha\cos\alpha \\ \sigma_{y'} = \sigma_x \sin^2\alpha + \sigma_y \cos^2\alpha - 2\tau_{xy}\sin\alpha\cos\alpha \\ \tau_{x'y'} = \left(\sigma_y - \sigma_x\right)\sin\alpha\cos\alpha + \tau_{xy}\left(\cos^2\alpha - \sin^2\alpha\right) \end{cases}$$

For two mutually orthogonal planes x and y:

- There are only three independent stress components
  - σ_x, σ_y, and $\tau_{xy} = \tau_{yx}$
- Stress on any plane can be determined (the state of stress is completely determined)

The stress state at a point is usually expressed as:

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{yx} & \sigma_y \end{bmatrix}$$

or for 3D:

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix}$$

Transformation of Stress Components for Uniform Stress Distribution

Transformation of stress equations
(应力变换方程)

$$\begin{cases}\sigma_{x'} = \sigma_x \cos^2\alpha + \sigma_y \sin^2\alpha + 2\tau_{xy}\sin\alpha\cos\alpha \\ \sigma_{y'} = \sigma_x \sin^2\alpha + \sigma_y \cos^2\alpha - 2\tau_{xy}\sin\alpha\cos\alpha \\ \tau_{x'y'} = \left(\sigma_y - \sigma_x\right)\sin\alpha\cos\alpha + \tau_{xy}\left(\cos^2\alpha - \sin^2\alpha\right)\end{cases}$$

$$\sin 2\alpha = 2\sin\alpha\cos\alpha$$
$$\sin^2\alpha = \tfrac{1}{2}(1-\cos 2\alpha)$$
$$\cos^2\alpha = \tfrac{1}{2}(1+\cos 2\alpha)$$

$\Rightarrow$

$$\begin{cases}\sigma_{x'} = \dfrac{\sigma_x+\sigma_y}{2} + \dfrac{\sigma_x-\sigma_y}{2}\cos 2\alpha + \tau_{xy}\sin 2\alpha \\ \sigma_{y'} = \dfrac{\sigma_x+\sigma_y}{2} - \dfrac{\sigma_x-\sigma_y}{2}\cos 2\alpha - \tau_{xy}\sin 2\alpha \\ \tau_{x'y'} = -\dfrac{\sigma_x-\sigma_y}{2}\sin 2\alpha + \tau_{xy}\cos 2\alpha\end{cases}$$

Questions:

- Body force must vanish in a uniform state of stress, right or wrong
- How do σ_x, σ_y, and τ_{xy}, τ_{yx} vary with z for 2D XOY plane cases?
- Prove that the sum of normal stresses on any two orthogonal planes is a constant

$$\sigma_{x'} + \sigma_{y'} = \sigma_x + \sigma_y$$

Principal Stresses and Maximum Shear Stress

Stress transformation equations

$$\begin{cases} \sigma_{x'} = \dfrac{\sigma_x + \sigma_y}{2} + \dfrac{\sigma_x - \sigma_y}{2}\cos 2\alpha + \tau_{xy}\sin 2\alpha \\ \sigma_{y'} = \dfrac{\sigma_x + \sigma_y}{2} - \dfrac{\sigma_x - \sigma_y}{2}\cos 2\alpha - \tau_{xy}\sin 2\alpha \\ \tau_{x'y'} = -\dfrac{\sigma_x - \sigma_y}{2}\sin 2\alpha + \tau_{xy}\cos 2\alpha \end{cases}$$

We now determine the orientation of the planes of extreme (maximum and minimum) normal stress

Differentiate $\sigma_{x'}$ with respect to α and the derivative is zero:

$$d\sigma_{x'}/d\alpha = -(\sigma_x - \sigma_y)\sin 2\alpha + 2\tau_{xy}\cos 2\alpha = 0$$

$$\Rightarrow \tan 2\alpha = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}\left(\alpha \in [0, \pi]\right)$$ not $\alpha \in [0, 2\pi]$

Principal Stresses and Maximum Shear Stress

Stress transformation equations

$$\begin{cases} \sigma_{x'} = \dfrac{\sigma_x + \sigma_y}{2} + \dfrac{\sigma_x - \sigma_y}{2}\cos 2\alpha + \tau_{xy}\sin 2\alpha \\ \sigma_{y'} = \dfrac{\sigma_x + \sigma_y}{2} - \dfrac{\sigma_x - \sigma_y}{2}\cos 2\alpha - \tau_{xy}\sin 2\alpha \\ \tau_{x'y'} = -\dfrac{\sigma_x - \sigma_y}{2}\sin 2\alpha + \tau_{xy}\cos 2\alpha \end{cases}$$

$$\tan 2\alpha = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}\left(\alpha \in [0, \pi]\right)$$ orientation of the plane of extreme normal stress

- Check shear stress on these planes
  - **Principal planes:** the planes on which shear stress vanish.
  - **principal stresses:** the normal stress on principal planes

- The principal planes are perpendicular to each other

Principal Stresses and Maximum Shear Stress

Maximum and minimum principal stresses.

$\sigma_1 = \sigma_{max}$, $\sigma_2 = \sigma_{min}$, (by convention, $\sigma_1 > \sigma_2$)

$$\tan 2\alpha = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} \left(\alpha \in [0, \pi]\right)$$

$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\alpha + \tau_{xy} \sin 2\alpha$$

$$\sin 2\alpha = \pm \frac{2\tau_{xy}}{\sqrt{4\tau_{xy}^2 + \left(\sigma_x - \sigma_y\right)^2}}$$

$$\cos 2\alpha = \pm \frac{\sigma_x - \sigma_y}{\sqrt{4\tau_{xy}^2 + \left(\sigma_x - \sigma_y\right)^2}}$$

Signs assigned for sin2α and cos2α are either both positive or both negative.

$$\Rightarrow \quad \sigma_{max} = \sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \qquad \sigma_{min} = \sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

Principal Stresses and Maximum Shear Stress

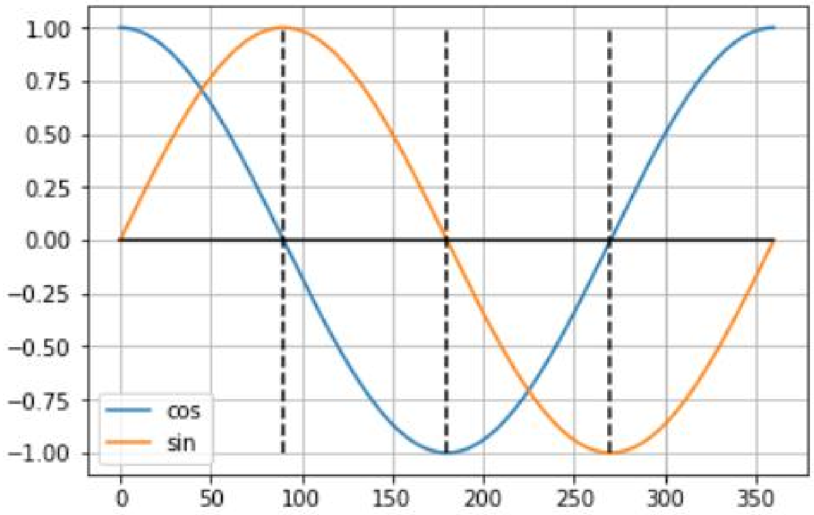
$$\tan 2\alpha = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}\left(\alpha \in [0,\pi]\right)$$

$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\alpha + \tau_{xy}\sin 2\alpha \qquad \alpha \in [0, \pi]$$

We now determine which α corresponds to the maximum principal stress

The extent of 2α for $\sigma_{max}=\sigma_1$ depends on the sign of τ_{xy} and $(\sigma_x-\sigma_y)$

sin2α and cos2α



$$\begin{array}{llllll}
0 < 2\alpha < \pi/2, & \text{if} & \tau_{xy} > 0 & \text{and} & (\sigma_x - \sigma_y) > 0 \\
\pi/2 < 2\alpha < \pi, & \text{if} & \tau_{xy} > 0 & \text{and} & (\sigma_x - \sigma_y) < 0 \\
\pi < 2\alpha < (3/2)\pi, & \text{if} & \tau_{xy} < 0 & \text{and} & (\sigma_x - \sigma_y) < 0 \\
(3/2)\pi < 2\alpha < 2\pi, & \text{if} & \tau_{xy} < 0 & \text{and} & (\sigma_x - \sigma_y) > 0
\end{array}$$

α: the angle between the x axis and the direction of σ_1

Principal Stresses and Maximum Shear Stress

We now determine the plane of max shear stress:

$$\tau_{x'y'} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\alpha + \tau_{xy}\cos 2\alpha$$

$$d\tau_{x'y'}/d\alpha = -\left(\sigma_x - \sigma_y\right)\cos 2\alpha - 2\tau_{xy}\sin 2\alpha = 0$$

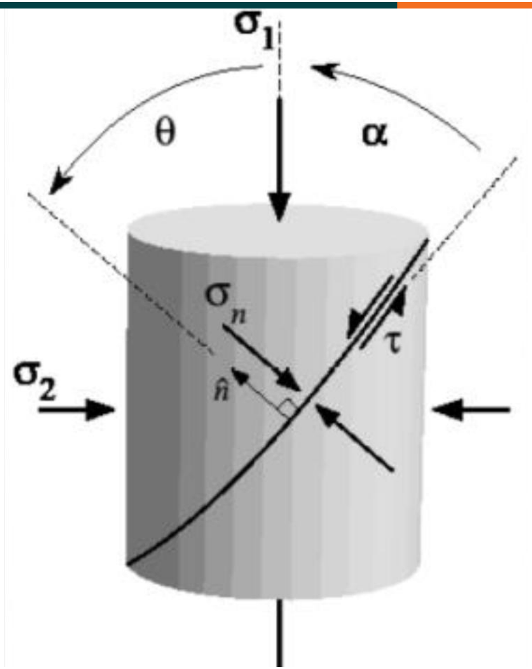
$$\Rightarrow \tan 2\alpha = -\frac{\sigma_x - \sigma_y}{2\tau_{xy}} \qquad \alpha \in [0, \pi]$$

maximum shear stress is

$$\tau_{x'y'\max} = \pm\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

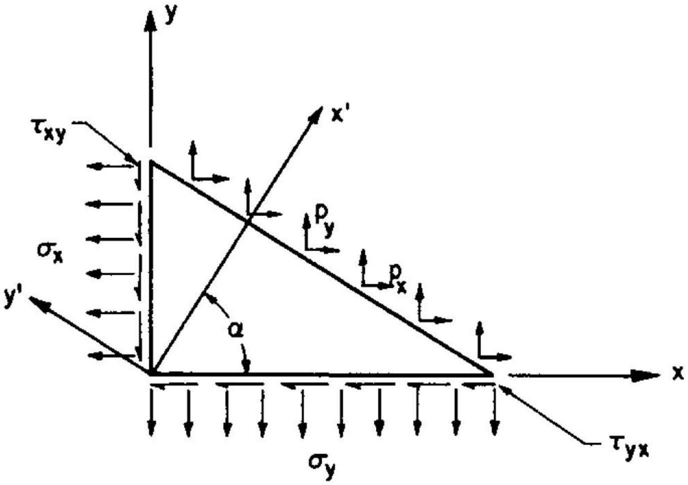
- The shear stresses reach the maximum on two perpendicular planes
  - The principal plane is 45º apart from the plane having max shear stresses
  - Normal stresses on the maximum shear stress planes are equal
  - Maximum shear stress are equal on these two planes
  - If $\alpha<\pi/2$ is taken for x' plane, the negative sign in the maximum shear stress formula is chosen if $\sigma_x>\sigma_y$ and the positive sign if $\sigma_y>\sigma_x$.

Classroom exercise



(1) Assume σ_1 = 5 MPa, σ_2 = 1 MPa, α= 30° , calculate the normal and shear stresses on the fracture plane

(2) Is the fracture slip left-lateral or right-lateral?



Given σ_x, σ_y, and τ_{xy}, τ_{yx},

Prove that p_y and $\sigma_{x'}$ can be expressed as:

$$p_y = \sigma_y \sin\alpha + \tau_{xy}\cos\alpha$$

$$\begin{aligned}\sigma_{x'} &= p_x \cos\alpha + p_y \sin\alpha \\ &= \sigma_x \cos^2\alpha + \sigma_y \sin^2\alpha + 2\tau_{xy}\sin\alpha\cos\alpha\end{aligned}$$

Review

- State of stress on a point is fully depicted in 2D by stress vectors on two planes:

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{yx} & \sigma_y \end{bmatrix}$$

  – moment equilibrium
    - $\tau_{yx} = \tau_{xy}$

  – force equilibrium
    - stress transformation equations (应力变换方程)

$$\begin{cases} \sigma_{x'} = \dfrac{\sigma_x + \sigma_y}{2} + \dfrac{\sigma_x - \sigma_y}{2}\cos 2\alpha + \tau_{xy}\sin 2\alpha \\ \sigma_{y'} = \dfrac{\sigma_x + \sigma_y}{2} - \dfrac{\sigma_x - \sigma_y}{2}\cos 2\alpha - \tau_{xy}\sin 2\alpha \\ \tau_{x'y'} = -\dfrac{\sigma_x - \sigma_y}{2}\sin 2\alpha + \tau_{xy}\cos 2\alpha \end{cases}$$

principal stress and principal plane

- **Principal planes**: shear stress vanishes on these planes
- Two perpendicular principal planes corresponding to $\sigma_1=\sigma_{max}$ and $\sigma_2=\sigma_{min}$, respectively.

Maximum shear stress

- Maximum shear stress planes are 45° apart from the two principal planes.
- Normal stresses on the maximum shear stress planes are equal