1. Modern Robotics: Exercise 3.20

   Solution:

   According to the relationship of the frames, we have:

$$T_{ac} = T_{ab}T_{bc} \tag{1}$$

   Since the two wheels rolling the same distance, and the radius of the front wheel is twice that of the rear wheel, the rotation angle of the rear wheel is 2θ. Then we have:

$$R_{ab} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix} \tag{2}$$

   Suppose L is the wheel distance on y direction, we have $p_{ab} = [0, L, r]$, then we have:

$$T_{ab} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) & L\cos(2\theta) - r\sin(2\theta) \\ 0 & \sin(\theta) & \cos(\theta) & L\sin(2\theta) + r\cos(2\theta) \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3}$$

   Since there's no rotation between frame $\{b\}$ and $\{c\}$, we have:

$$T_{bc} = \begin{bmatrix} 1 & 0 & 0 & -D \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2r \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{4}$$

   Then the T_{ac} can be calculated by:

$$T_{ac} = T_{ab}T_{bc} = \begin{bmatrix} 1 & 0 & 0 & -D \\ 0 & \cos(\theta) & -\sin(\theta) & L\cos(2\theta) - 2r\sin(\theta) - r\sin(2\theta) \\ 0 & \sin(\theta) & \cos(\theta) & 2r\cos(\theta) + L\sin(2\theta) + r\cos(2\theta) \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{5}$$

   Note: the solution can be vary when the assumption of L is different (e.g. L can be the center distance of the two wheels). Answers with reasonable assumptions are all considered as correct. □

2. Modern Robotics: Exercise 3.23

   Solution:

   (a):

$$T_{01} = \begin{bmatrix} -\cos(\theta) & \sin(\theta) & 0 & L - R\sin(\theta) \\ -\sin(\theta) & -\cos(\theta) & 0 & L + R\cos(\theta) \\ 0 & 0 & 1 & H \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{6}$$

Since $L=1$, $R=2$, $\theta=\frac{v_1 t}{R}$, $H=2$:

$$T_{01}=\begin{bmatrix}-\cos(\frac{v_1t}{2}) & \sin(\frac{v_1t}{2}) & 0 & 1-2\sin(\frac{v_1t}{2})\\ -\sin(\frac{v_1t}{2}) & -\cos(\frac{v_1t}{2}) & 0 & 1+2\cos(\frac{v_1t}{2})\\ 0 & 0 & 1 & 2\\ 0 & 0 & 0 & 1\end{bmatrix} \tag{7}$$

$$T_{02}=\begin{bmatrix}\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & 1-\sqrt{2}+\frac{\sqrt{2}}{2}v_2t\\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & 1+\sqrt{2}-\frac{\sqrt{2}}{2}v_2t\\ 0 & 0 & 1 & 2\\ 0 & 0 & 0 & 1\end{bmatrix} \tag{8}$$

(b): According to $T_{12}=T_{01}^{-1}T_{02}$,

$$T_{01}^{-1}=\begin{bmatrix}R^T & -R^Tp\\ 0^T & 1\end{bmatrix}=\begin{bmatrix}-\cos(\frac{v_1t}{2}) & -\sin(\frac{v_1t}{2}) & 0 & \cos(\frac{v_1t}{2})+\sin(\frac{v_1t}{2})\\ \sin(\frac{v_1t}{2}) & -\cos(\frac{v_1t}{2}) & 0 & 2-\sin(\frac{v_1t}{2})+\cos(\frac{v_1t}{2})\\ 0 & 0 & 1 & -2\\ 0 & 0 & 0 & 1\end{bmatrix} \tag{9}$$

Then, we have:

$$T_{12}=\begin{bmatrix}\frac{\sqrt{2}}{2}[\sin(\frac{v_1t}{2})-\cos(\frac{v_1t}{2})] & -\frac{\sqrt{2}}{2}[\sin(\frac{v_1t}{2})+\cos(\frac{v_1t}{2})] & 0 & (\sqrt{2}-\frac{\sqrt{2}}{2}v_2t)[\cos(\frac{v_1t}{2})-\sin(\frac{v_1t}{2})]\\ \frac{\sqrt{2}}{2}[\sin(\frac{v_1t}{2})+\cos(\frac{v_1t}{2})] & \frac{\sqrt{2}}{2}[\sin(\frac{v_1t}{2})-\cos(\frac{v_1t}{2})] & 0 & 2-(\sqrt{2}-\frac{\sqrt{2}}{2}v_2t)[\cos(\frac{v_1t}{2})+\sin(\frac{v_1t}{2})]\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix} \tag{10}$$

The result can be further simplified if we transfer $\frac{\sqrt{2}}{2}$ to $\sin(\frac{\pi}{4})$ and $\cos(\frac{\pi}{4})$:

$$T_{12}=\begin{bmatrix}-\cos(\frac{v_1t}{2}+\frac{\pi}{4}) & -\sin(\frac{v_1t}{2}+\frac{\pi}{4}) & 0 & (v_2t-2)\cos(\frac{v_1t}{2}+\frac{\pi}{4})\\ \sin(\frac{v_1t}{2}+\frac{\pi}{4}) & \cos(\frac{v_1t}{2}+\frac{\pi}{4}) & 0 & 2+(v_2t-2)\sin(\frac{v_1t}{2}+\frac{\pi}{4})\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix} \tag{11}$$

□

3. Modern Robotics: Exercise 3.26

Solution: See Fig. 1

□

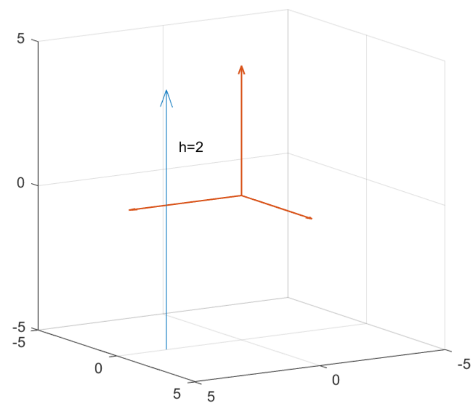

Figure 1: Screw axis for Exercise 3.26

4. Modern Robotics: Exercise 3.28

Solution:

$$\omega_b = R_{sb}^{-1}\omega_s \tag{12}$$

We have $\omega_s = [1, 2, 3]$ and:

$$R_{sb} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix} \tag{13}$$

Then:

$$\omega_b = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix} \tag{14}$$

□

5. Modern Robotics: Exercise 5.6 (Note: part-(a) is modified to compute the twist of frame b expressed in frame b, i.e., the ${}^b\mathcal{V}_b$)

Solution: (a): We have

$$\begin{aligned} {}^s\mathcal{S}_1 &= [0, 1, 0, 0, 0, 0]^T \\ {}^s\mathcal{S}_2 &= [0, 0, 1, 0, 0, 0]^T \end{aligned} \tag{15}$$

and

$$M = \begin{bmatrix} 1 & 0 & 0 & 10 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -20 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{16}$$

So

$$\begin{aligned} J_b &= [Ad^{-1}_{e^{\mathcal{S}_1\theta_1}e^{\mathcal{S}_2\theta_2}M}\mathcal{S}_1, Ad^{-1}_{e^{\mathcal{S}_2\theta_2}M}\mathcal{S}_2] \\ &= \begin{bmatrix} \sin\theta_2 & 0 \\ \cos\theta_2 & 0 \\ 0 & 1 \\ -20\cos\theta_2 & 0 \\ 20\sin\theta_2 & 10 \\ -10\cos\theta_2 & 0 \end{bmatrix} \end{aligned} \tag{17}$$

We have $\theta_1 = \theta_2 = t$ and $V = J\dot{\theta}$

$${}^b\mathcal{V} = [\sin t, \cos t, 1, -20\cos t, 20\sin t + 10, -10\cos t]^T \tag{18}$$

(b):

$$\begin{aligned} {}^s\mathcal{V} &= Ad_{T_{sb}}{}^b\mathcal{V} \\ &= [\sin t, 1, \cos t, 0, 0, 0]^T \end{aligned} \tag{19}$$

$$p(t) = e^{\mathcal{S}_1\theta_1}e^{\mathcal{S}_2\theta_2}p(0) \tag{20}$$

So

$$\dot{p} = \omega \times p + v = \begin{bmatrix} -10\sin 2t - 20\cos t \\ 10\cos t \\ 20\sin t - 10\cos 2t \end{bmatrix} \tag{21}$$

$\square$