Review

The solution of a problem in elasticity (弹性力学的定解问题):

- determination of the stress, strain and displacement functions satisfying the governing equations and boundary conditions.

- 15 unknowns
- 15 independent equations (governing equations, 控制方程)
  - Hooke's law
  - strain-displacement relation
  - equilibrium equations
- appropriate boundary conditions
  - stress
  - displacement

Plane strain:

$$\varepsilon_z = \varepsilon_{xz} = \varepsilon_{yz} = 0$$

$$\varepsilon_x = \varepsilon_x(x, y) \quad \varepsilon_y = \varepsilon_y(x, y) \quad \gamma_{xy} = \gamma_{xy}(x, y)$$

- 8 unknowns
- 8 governing equations

Plane stress:

$$\sigma_z = \tau_{xz} = \tau_{yz} = 0$$

$$\sigma_x = \sigma_x(x, y) \quad \sigma_y = \sigma_y(x, y) \quad \tau_{xy} = \tau_{xy}(x, y)$$

- 8 unknowns
- 8 governing equations

Review

Formulations of elastic problems (弹性力学问题的解法):

- The displacement formulation (位移解法)
  - Take displacement as the basic unknown quantities
    - express the governing equations with displacement
  - solve displacement first, then strain and stress
- The stress formulation (应力解法)
  - Take stress as the bais unknown quantities
    - express the governing equations with stress
  - solve stress first, then strain and displacement

Why is there no strain boundary condition or strain formulation?

For plane strain problem, we have deduced:

Displacement formulation

$$G\nabla^2 u + (\lambda + G)\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + f_x = 0$$

$$G\nabla^2 v + (\lambda + G)\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + f_y = 0$$

$$u, v$$

Stress formulation

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + f_x = 0$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + f_y = 0$$

$$\nabla^2(\sigma_x + \sigma_y) = -\frac{1}{1-\nu}\left(\frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y}\right)$$

Displacement formulation and stress formulation: 3D Problems

- Equilibrium equations (3) $$\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}+\frac{\partial\tau_{zx}}{\partial z}+f_x=0\ \ (x,y,z)$$
- Hooke's law (6) $$\sigma_x=2G\varepsilon_x+\lambda(\varepsilon_x+\varepsilon_y+\varepsilon_z)\ \ (x,y,z)\qquad \tau_{xy}=G\gamma_{xy}$$
- Strain-displacement (6) $$\varepsilon_x=\frac{\partial u}{\partial x}\qquad \gamma_{xy}=\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}$$

15 equations,
15 Unknowns:
$\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{xz}$
$\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{xz}$
u, v, w

Displacement formulation

Navier equations

$$\begin{cases}(\lambda+G)\dfrac{\partial\varepsilon}{\partial x}+G\nabla^2u+f_x=0\\(\lambda+G)\dfrac{\partial\varepsilon}{\partial y}+G\nabla^2v+f_y=0\\(\lambda+G)\dfrac{\partial\varepsilon}{\partial z}+G\nabla^2w+f_z=0\end{cases}$$

u,v,w

3 equations, 3 unknowns

Stress formulation

$$\nabla^2\sigma_x+\frac{1}{(1+\nu)}\frac{\partial^2\Theta}{\partial x^2}=-\frac{\nu}{1-\nu}\left(\frac{\partial f_x}{\partial x}+\frac{\partial f_y}{\partial y}+\frac{\partial f_z}{\partial z}\right)-2\frac{\partial f_x}{\partial x}\qquad (x,y,z)$$

$$\nabla^2\tau_{yz}+\frac{1}{(1+\nu)}\frac{\partial^2\Theta}{\partial y\partial z}=-\left(\frac{\partial f_x}{\partial y}+\frac{\partial f_y}{\partial x}\right)\qquad (x,y,z)$$

$$\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}+\frac{\partial\tau_{zx}}{\partial z}+f_x=0\qquad (x,y,z)$$

$\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{xz}$

$$\varepsilon=\varepsilon_x+\varepsilon_y+\varepsilon_z$$

$$\Theta=\sigma_x+\sigma_y+\sigma_z$$

9 equations, 6 unknowns

Plane stress problems

Governing equations of Plane stress

8 equations for 8 unknowns
(σ_x, σ_y, τ_{xy}, ε_x, ε_y, γ_{xy}, u, and v)

The Hooke's law

$$\varepsilon_x = \frac{1}{E}\left(\sigma_x - \nu\sigma_y\right) \qquad \gamma_{xy} = \frac{1}{G}\tau_{xy}$$

$$\varepsilon_y = \frac{1}{E}\left(\sigma_y - \nu\sigma_x\right)$$

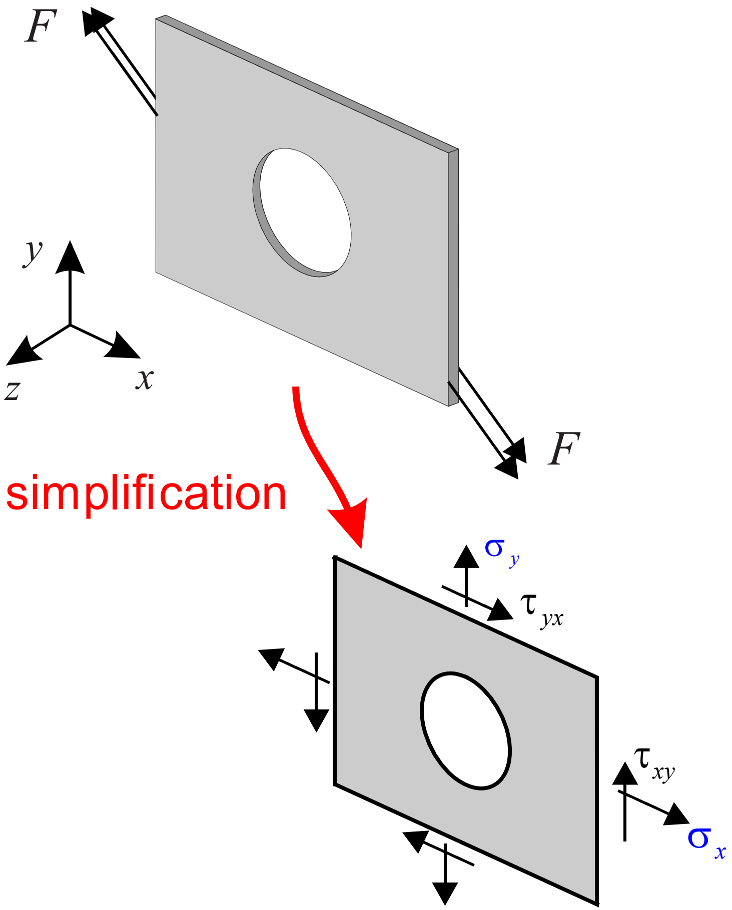
The equilibrium equations:

$$\frac{\partial\sigma_x}{\partial x} + \frac{\partial\tau_{yx}}{\partial y} + f_x = 0$$

$$\frac{\partial\tau_{xy}}{\partial x} + \frac{\partial\sigma_y}{\partial y} + f_y = 0$$

The strain-displacement relationship:

$$\varepsilon_x = \frac{\partial u}{\partial x} \qquad \varepsilon_y = \frac{\partial v}{\partial y} \qquad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$



Plane stress problems

displacement formulation

1. Represent stress with displacement

$$\sigma_x = \frac{E}{1-\nu^2}(\varepsilon_x + \nu\varepsilon_y) = \frac{E}{1-\nu^2}(\frac{\partial u}{\partial x} + \nu\frac{\partial v}{\partial y})$$

$$\sigma_y = \frac{E}{1-\nu^2}(\varepsilon_y + \nu\varepsilon_x) = \frac{E}{1-\nu^2}(\frac{\partial v}{\partial y} + \nu\frac{\partial u}{\partial x})$$

$$\tau_{xy} = G\gamma_{xy} = G(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x})$$

2. Get equilibrium equations in terms of displacement (Navier's Equations)

$$G\nabla^2 u + \frac{E}{2(1-\nu)}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + f_x = 0$$

$$G\nabla^2 v + \frac{E}{2(1-\nu)}\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + f_y = 0$$

$$G = \frac{E}{2(1+\nu)}$$

stress formulation

1. only one compatible equation left

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}$$

2. Compatibility equation in terms of stress

$$\nabla^2(\sigma_x + \sigma_y) = -(1+\nu)\left(\frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y}\right)$$

3. Combine with force equilibrium equation

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + f_x = 0$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + f_y = 0$$

$$\nabla^2(\sigma_x + \sigma_y) = -(1+\nu)\left(\frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y}\right)$$

Plane Stress Problems vs plane strain problems

stress formulation of plane stress

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + f_x = 0$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + f_y = 0$$

$$\nabla^2 (\sigma_x + \sigma_y) = -(1+\nu)\left(\frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y}\right)$$

$$G = \frac{E}{2(1+\nu)}$$

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}$$

stress formulation of plane strain

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + f_x = 0$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + f_y = 0$$

$$\nabla^2 (\sigma_x + \sigma_y) = -\frac{1}{1-\nu}\left(\frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y}\right)$$

Displacement formulation for plane stress

$$G\nabla^2 u + \frac{E}{2(1-\nu)}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + f_x = 0$$

$$G\nabla^2 v + \frac{E}{2(1-\nu)}\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + f_y = 0$$

Displacement formulation for plane strain

$$G\nabla^2 u + (\lambda + G)\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + f_x = 0$$

$$G\nabla^2 v + (\lambda + G)\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + f_y = 0$$

The plane stress equations can be changed into corresponding plane strain equations by replacing E and ν with E_1 and ν_1:

$$E_1 = \frac{E}{1-\nu^2}, \nu_1 = \frac{\nu}{1-\nu}$$

The plane strain equations can be changed into plane stress equations by replacing E and ν with E_2 and ν_2:

$$E_2 = \frac{E(1+2\nu)}{(1+\nu)^2}, \nu_2 = \frac{\nu}{1+\nu}$$

The solution method for the plane stress and the plane strain problems can be unified.

Plane Stress Problems——(Airy) stress function

stress formulation of plane stress

$$\frac{\partial \sigma_x}{\partial x}+\frac{\partial \tau_{yx}}{\partial y}+f_x=0$$

$$\frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \sigma_y}{\partial y}+f_y=0$$

$$\nabla^2\left(\sigma_x+\sigma_y\right)=-\left(1+\nu\right)\left(\frac{\partial f_x}{\partial x}+\frac{\partial f_y}{\partial y}\right)$$

Assume the body force is conservative (保守力), then we have a potential function *V*:

$$f_x=\frac{\partial V}{\partial x},\ f_y=\frac{\partial V}{\partial y}$$

Conservative force (保守力): the total work done in moving a particle between two points is independent of the path taken.

Let's introduce a stress function $\phi=\phi(x,y)$ (***Airy's stress function***):

$$\sigma_x+V=\frac{\partial^2\phi}{\partial y^2}$$

$$\sigma_y+V=\frac{\partial^2\phi}{\partial x^2}$$

$$\tau_{xy}=-\frac{\partial^2\phi}{\partial x\partial y}$$

The equilibrium equations are **always satisfied** by introduction of ϕ.

The compatibility equation in terms of stress becomes $\nabla^4\phi=\nabla^2\left(\nabla^2\phi\right)=\left(1-\nu\right)\nabla^2 V$

$$\nabla^4=\nabla^2\left(\nabla^2\right)=\frac{\partial^4}{\partial x^4}+2\frac{\partial^4}{\partial x^2\partial y^2}+\frac{\partial^4}{\partial y^4}$$

Biharmonic operator

The governing equation for plane stress is reduced to **only one** equation. We solve it to get stress function ϕ, then get stress.

Plane Stress Problems——(Airy) stress function

The governing equation for plane stress for conservative body force:

$$\nabla^4\phi = \nabla^2\left(\nabla^2\phi\right) = (1-\nu)\nabla^2 V$$

The corresponding stress function equation for plane strain can be derived by replacing ν with ν_1:

$$\nabla^4\phi = \frac{1-2\nu}{1-\nu}\nabla^2 V \qquad \nu_1 = \frac{\nu}{1-\nu}$$

If the body force is constant, or if $\nabla^2 V = 0$, both plane strain and plane stress problems are reduced to

V is a harmonic function

$$\nabla^4\phi = 0$$

Biharmonic equation

Plane Stress Problems——Approximate Character of Plane Stress Equations

Plane stress:

$$\sigma_x = \sigma_x(x, y) \qquad \tau_{xy} = \tau_{xy}(x, y)$$
$$\sigma_y = \sigma_y(x, y) \qquad \tau_{xz} = \tau_{yz} = \sigma_z = 0$$

We only constrain the solution with one compatibility equation

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}$$

- Other compatible equations are usually not satisfied.
- If we require all the compatibility equations are satisfied, then we will always have the results as below, which has little practical meaning

$$\varepsilon_z = -\nu(\varepsilon_x + \varepsilon_y) = Ax + By + C$$

A, B, C are arbitrary constants

Saint-Venant compatibility equations

$$\boxed{\begin{aligned}
2\frac{\partial^2 \varepsilon_x}{\partial y \partial z} &= \frac{\partial}{\partial x}\left(-\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right) \\
2\frac{\partial^2 \varepsilon_y}{\partial x \partial z} &= \frac{\partial}{\partial y}\left(\frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right) \\
2\frac{\partial^2 \varepsilon_z}{\partial x \partial y} &= \frac{\partial}{\partial z}\left(\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} - \frac{\partial \gamma_{xy}}{\partial z}\right)
\end{aligned}}
\qquad
\boxed{\begin{aligned}
\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} &= \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} \\
\frac{\partial^2 \varepsilon_y}{\partial z^2} + \frac{\partial^2 \varepsilon_z}{\partial y^2} &= \frac{\partial^2 \gamma_{yz}}{\partial y \partial z} \\
\frac{\partial^2 \varepsilon_z}{\partial x^2} + \frac{\partial^2 \varepsilon_x}{\partial z^2} &= \frac{\partial^2 \gamma_{zx}}{\partial z \partial x}
\end{aligned}}$$

Our solution is thus only an approximation for Plane stress and the solution is good only when:

- The body **must** be a thin plate
- The two *z* surfaces **must** be free from load
- The external forces (body forces and surface forces) have **no** *z* component and should be either independent of *z*, or distributed symmetrically with respect to the middle plane.

Uniqueness of Elasticity Solutions

For a given surface force and body force distribution, the stress and strain distribution inside an elastic body should be unique.

We now prove that this is true for the solved **elasticity solutions**

- unique elasticity solution <=>
  only one stress solution consistent with equilibrium and compatibility with given boundary conditions.

Governing equations and boundary conditions for general elastic problems

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + F_x = 0 \qquad (x, y, z)$$

$$\nabla^2 \sigma_x + \frac{1}{1+\nu} \frac{\partial^2 \Theta}{\partial x^2} = \cdots - 2 \frac{\partial F_x}{\partial x} \qquad (x, y, z)$$

$$\nabla^2 \tau_{yz} + \frac{1}{1+\nu} \frac{\partial^2 \Theta}{\partial y \partial z} = -\left(\frac{\partial F_z}{\partial y} + \frac{\partial F_y}{\partial z} \right) \qquad (x, y, z)$$

$$T_x^{\mu} = \sigma_{x0} \mu_x + \tau_{xy0} \mu_y + \tau_{xz0} \mu_z \qquad (x, y, z)$$

Uniqueness of Elasticity Solutions

Assume that there are two sets of stress components, each satisfy the same governing equations and boundary conditions:

$$\sigma'_x \ldots \tau'_{xz} \text{ and } \sigma''_x \ldots \tau''_{xz}$$

Now the first solution must satisfy the equilibrium equations

$$\frac{\partial \sigma'_x}{\partial x} + \frac{\partial \tau'_{xy}}{\partial y} + \frac{\partial \tau'_{xz}}{\partial z} + F_x = 0 \qquad (x, y, z)$$

the compatibility equations (4.25),

$$\nabla^2 \sigma'_x + \cdots = \cdots - 2\frac{\partial F_x}{\partial x} \qquad (x, y, z)$$

$$\nabla^2 \tau'_{yz} + \cdots = -\left(\frac{\partial F_z}{\partial y} + \frac{\partial F_y}{\partial z}\right) \qquad (x, y, z)$$

and the boundary conditions

$$T_x^{\mu} = \sigma'_{x0}\mu_x + \tau'_{xy0}\mu_y + \tau'_{xz0}\mu_z \qquad (x, y, z)$$

the second solution must satisfy the relations

$$\frac{\partial \sigma''_x}{\partial x} + \frac{\partial \tau''_{xy}}{\partial y} + \frac{\partial \tau''_{xz}}{\partial z} + F_x = 0 \qquad (x, y, z)$$

$$\nabla^2 \sigma''_x + \cdots = \cdots - 2\frac{\partial F_x}{\partial x} \qquad (x, y, z)$$

$$\nabla^2 \tau''_{yz} + \cdots = -\left(\frac{\partial F_z}{\partial y} + \frac{\partial F_y}{\partial z}\right) \qquad (x, y, z)$$

and

$$T_x^{\mu} = \sigma''_{x0}\mu_x + \tau''_{xy0}\mu_y + \tau''_{xz0}\mu_z \qquad (x, y, z)$$

Uniqueness of Elasticity Solutions

Set

$$\sigma_x = \sigma'_x - \sigma''_x, \quad \sigma_{x0} = \sigma'_{x0} - \sigma''_{x0}, \quad \text{etc.},$$

Substract the two equation systems, we have

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} = 0 \qquad (x, y, z)$$

$$\nabla^2 \sigma_x + \frac{1}{1+\nu} \frac{\partial^2 \Theta}{\partial x^2} = 0 \qquad (x, y, z)$$

$$\nabla^2 \tau_{yz} + \frac{1}{1+\nu} \frac{\partial^2 \Theta}{\partial y \partial z} = 0 \qquad (x, y, z)$$

$$\sigma_{x0} \mu_x + \tau_{xy0} \mu_y + \tau_{xz0} \mu_z = 0 \qquad (x, y, z)$$

The body forces and surface forces do not appear in the above equations

➡ $\sigma_x \ldots \tau_{zx}$: the state of stress in the body with no body forces and no surface forces.

$$\sigma_x = \sigma'_x - \sigma''_x = 0 \qquad (x, y, z)$$

➡ $$\tau_{xz} = \tau'_{xz} - \tau''_{xz} = 0 \qquad (x, y, z)$$

The two states of stress are identical so that the solution is unique.

Classroom exercise

Governing equations

- Strain-displacement (6)

$$\varepsilon_x = \frac{\partial u}{\partial x} \qquad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \quad (x, y, z; u, v, w)$$

- Stress-strain relations (6)

$$\sigma_x = 2G\varepsilon_x + \lambda\varepsilon \qquad \tau_{xy} = G\gamma_{xy} \quad (x, y, z)$$

- Equilibrium equations (3)

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + f_x = 0 \quad (x, y, z)$$

$$\varepsilon = \varepsilon_x + \varepsilon_y + \varepsilon_z = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

Show that the governing equations for the elastic problems using displacement is as below:

$$\begin{cases} (\lambda + G)\dfrac{\partial \varepsilon}{\partial x} + G\nabla^2 u + f_x = 0 \\ (\lambda + G)\dfrac{\partial \varepsilon}{\partial y} + G\nabla^2 v + f_y = 0 \\ (\lambda + G)\dfrac{\partial \varepsilon}{\partial z} + G\nabla^2 w + f_z = 0 \end{cases}$$

Equilibrium equations in terms of displacement, or Navier’s equations

$$\nabla^2 u = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

Laplace operator