Homework 7

Due Dec 23 2021

1. Consider the polar coordinate system in 2D.

   (a) Express the bi-harmonic equation $\Delta^2\phi = 0$ using polar coordinates.

   (b) Consider problems that are axisymmetric, verify that the solution $\phi = Ar^2\ln(r) + Br^2 + C\ln(r) + D$ satisfies the bi-harmonic equation.

   (c) Let the above ϕ be the Airy stress function, what is the resulting Cauchy stress?

   (d) Consider the pressurized vessel shown in Figure 1 (left). Determine the stress in the vessel wall.

   (e) Comsider a stress free hole in an infinite medium under equal biaxial loading at infinity.Use the solution develop for a pressurized cylinder to obtain the stress distribution in the medium.

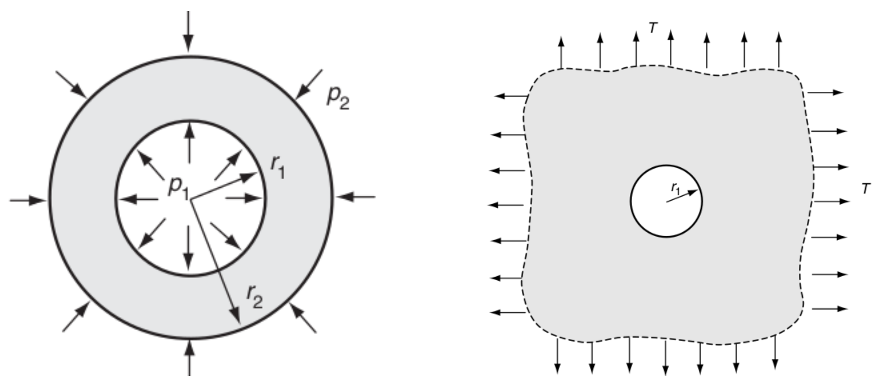

Figure 1: Thick-walled cylinder (left) and a stress-free hole in an infinite medium under bi-axial loading

2. Generally speaking, continuum mechanics also involves the study of electromagnetics. This is just a set of simple exercises that helps you review the divergence Theorem and the Stokes' Theorem discussed before.

   (a) The *Gauss's law* states the conservation of the charge of one body. The integral form of the Gauss's law reads as follows.

   $$\int_{\partial\Omega} \epsilon_0 \boldsymbol{E}\cdot\boldsymbol{n}dA = \int_{\Omega}\rho dV,$$

   in which Ω is a region, $\boldsymbol{E}$ is an electric field, $\epsilon_0 = 8.85\times 10^{-12}$ (Nm^2/C^2)$^{-1}$ is the permittivity constant, C is the SI measure of a unit charge called a Coulomb, $\boldsymbol{n}$ is the unit outward normal of the boundary $\partial\Omega$, ρ is the charge density. Derive the local form (i.e., the differential equation) for this integral equation.

   (b) The *Faraday's law* states that

   $$\int_{\partial\Gamma}\boldsymbol{E}\cdot d\boldsymbol{s} = -\frac{d}{dt}\int_{\Gamma}\boldsymbol{B}\cdot\boldsymbol{n}dA,$$

where Γ is a surface with a smooth boundary curve $\partial\Gamma$, $\boldsymbol{n}$ is a unit outward normal to Γ, $\boldsymbol{B}$ is the magnetic field. Show the local, differential form of the Faraday's law.

(c) The *Ampere-Maxwell Law* states that

$$\int_{\partial\Gamma} \boldsymbol{B} \cdot d\boldsymbol{s} = \mu_0 \int_{\Gamma} \boldsymbol{j} \cdot \boldsymbol{n} dA + \mu_0 \epsilon_0 \int_{\Gamma} \frac{\partial \boldsymbol{E}}{\partial t} \cdot \boldsymbol{n} dA,$$

where $\boldsymbol{j}$ is the current density and $\mu_0 = 4\pi \times 10^{-7}$ Tm/A is the permeability constant. Here the units T represents Tesla and A represents Ampere. Derive the local form for this law.

(d) The absence of magnetic monopoles states that

$$\int_{\partial\Omega} \boldsymbol{B} \cdot \boldsymbol{n} dA = 0,$$

where $\partial\Omega$ is a surface bounding a region Ω. Derive the local form for this law.

(e) The above four differential equations constitute the Maxwell equations for electromagnetics. Use the identity $\nabla \times \nabla \times \boldsymbol{E} = \nabla(\nabla \cdot \boldsymbol{E}) - \Delta \boldsymbol{E}$ to show that the electric field satisfies the following equation,

$$\frac{\partial^2 \boldsymbol{E}}{\partial t^2} - \frac{1}{\mu_0 \epsilon_0} \Delta \boldsymbol{E} = -\frac{1}{\epsilon_0} \nabla \rho.$$

The value $\sqrt{\frac{1}{\mu_0 \epsilon_0}}$ determines the propagation speed of electromagnetic disturbances. Determine its value.

3. Let $\boldsymbol{\alpha}(\boldsymbol{x}, t)$ be a vector field defined on $\Omega_{\boldsymbol{x}} = \boldsymbol{\varphi}_t(\Omega_{\boldsymbol{X}})$. Show the following transport theorem for $\boldsymbol{\alpha}$.

$$\frac{D}{Dt} \int_{\Omega_{\boldsymbol{x}}} \boldsymbol{\alpha}(\boldsymbol{x}, t) dv = \int_{\Omega_{\boldsymbol{x}}} \frac{\partial}{\partial t} \boldsymbol{\alpha}(\boldsymbol{x}, t) + \nabla_{\boldsymbol{x}} \cdot (\boldsymbol{\alpha}(\boldsymbol{x}, t) \otimes \boldsymbol{v}(\boldsymbol{x}, t)) \, dv.$$

4. For an ideal gas, determine its bulk thermal expansion coefficient α and isothermal compressibility coefficient β.

5. For a flow with velocity $\boldsymbol{v}(\boldsymbol{x}, t) = cx\boldsymbol{e}_x - cy\boldsymbol{e}_y$ with c being a constant, determine its vorticity and rate-of-strain.

6. For a vector field $\boldsymbol{a}(\boldsymbol{x}, t)$, it is called a *conservative vector field* if its line integrals over any closed curve is zero. What can you say about $\nabla \times \boldsymbol{a}$?