一、Solution:

The six strain-displacement equations are

$$\varepsilon_x = \frac{\partial u}{\partial x} \qquad \varepsilon_y = \frac{\partial v}{\partial y} \qquad \varepsilon_z = \frac{\partial w}{\partial z}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \quad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \quad \gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$$

And we can derive the six second-order

$$\frac{\partial \varepsilon_x}{\partial y} = \frac{\partial^2 u}{\partial x \partial y} \qquad \frac{\partial^2 \varepsilon_x}{\partial y^2} = \frac{\partial^3 u}{\partial x \partial^2 y}$$

$$\Rightarrow$$

$$\frac{\partial \varepsilon_y}{\partial x} = \frac{\partial^2 v}{\partial x \partial y} \qquad \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^3 v}{\partial^2 x \partial y}$$

$$\frac{\partial \gamma_{xy}}{\partial x} = \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 v}{\partial x^2} \qquad \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = \frac{\partial^3 u}{\partial x \partial y^2} + \frac{\partial^3 v}{\partial x^2 \partial y}$$

$$\Rightarrow \frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} \quad \cdots\cdots\cdots (1)$$

Similarly,

$$\frac{\partial^2 \varepsilon_x}{\partial z^2} + \frac{\partial^2 \varepsilon_z}{\partial x^2} = \frac{\partial^2 \gamma_{zx}}{\partial x \partial z} \quad \cdots\cdots\cdots (2)$$

$$\frac{\partial^2 \varepsilon_y}{\partial z^2} + \frac{\partial^2 \varepsilon_z}{\partial y^2} = \frac{\partial^2 \gamma_{yz}}{\partial y \partial z} \quad \cdots\cdots\cdots (3)$$

For another three second-order compatibility equations, we need to get the second-order differential of $\varepsilon_x, \varepsilon_y, \varepsilon_z$ firstly.

$$\frac{\partial^2 \varepsilon_x}{\partial y \partial z} = \frac{\partial^3 u}{\partial x \partial y \partial z} \qquad \frac{\partial^2 \varepsilon_y}{\partial x \partial z} = \frac{\partial^3 v}{\partial x \partial y \partial z} \qquad \frac{\partial^2 \varepsilon_z}{\partial x \partial y} = \frac{\partial^3 w}{\partial x \partial y \partial z}$$

Then we need to get the first-order of differential of γ_{xy}, γ_{yz}, γ_{zx}.

$$\frac{\partial \gamma_{xy}}{\partial z} = \frac{\partial^2 u}{\partial y \partial z} + \frac{\partial^2 v}{\partial x \partial z} \quad \frac{\partial \gamma_{yz}}{\partial x} = \frac{\partial^2 v}{\partial z \partial x} + \frac{\partial^2 w}{\partial y \partial x} \quad \frac{\partial \gamma_{xz}}{\partial y} = \frac{\partial^2 u}{\partial y \partial z} + \frac{\partial^2 w}{\partial x \partial y}$$

Then we can get

$$2\frac{\partial^2 \varepsilon_x}{\partial y \partial z} = \frac{\partial}{\partial x}\left(-\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{xz}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right) \cdots\cdots ④$$

$$2\frac{\partial^2 \varepsilon_y}{\partial z \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{xz}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right) \cdots\cdots ⑤$$

$$2\frac{\partial^2 \varepsilon_z}{\partial x \partial y} = \frac{\partial}{\partial z}\left(\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{xz}}{\partial y} - \frac{\partial \gamma_{xy}}{\partial z}\right) \cdots\cdots ⑥$$

And the three fourth-order compatibility equations are

$$\frac{\partial^3}{\partial x \partial y \partial z}\left(-\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right) = 2\frac{\partial^4 \varepsilon_x}{\partial y^2 \partial z^2} \cdots ①$$

$$\frac{\partial^3}{\partial x \partial y \partial z}\left(\frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right) = 2\frac{\partial^4 \varepsilon_y}{\partial x^2 \partial z^2} \cdots ②$$

$$\frac{\partial^3}{\partial x \partial y \partial z}\left(\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} - \frac{\partial \gamma_{xy}}{\partial z}\right) = 2\frac{\partial^4 \varepsilon_z}{\partial x^2 \partial y^2} \cdots ③$$

$$\frac{\partial^4 \gamma_{xy}}{\partial x \partial y \partial z} = \frac{\partial^4 \varepsilon_x}{\partial y^2 \partial z^2} + \frac{\partial^4 \varepsilon_y}{\partial x^2 \partial z^2} \quad \cdots\cdots \text{④}$$

$$\frac{\partial^4 \gamma_{yz}}{\partial x \partial y \partial z} = \frac{\partial^4 \varepsilon_y}{\partial x^2 \partial z^2} + \frac{\partial^4 \varepsilon_z}{\partial x^2 \partial y^2} \quad \cdots\cdots \text{⑤}$$

$$\frac{\partial^4 \gamma_{zx}}{\partial x \partial y \partial z} = \frac{\partial^4 \varepsilon_z}{\partial x^2 \partial y^2} + \frac{\partial^4 \varepsilon_x}{\partial y^2 \partial z^2} \quad \cdots\cdots \text{⑥}$$

二、(a) Solution:

Since A is (2,1,0), which means $x=2, y=1, z=0$.

$$\frac{\partial u}{\partial x}=2k \quad \frac{\partial v}{\partial x}=2kx=4k \quad \frac{\partial u}{\partial y}=2ky=2k \quad \frac{\partial v}{\partial y}=-6ky=-6k$$

$$\frac{\partial u}{\partial x}dx=2kdx \quad \frac{\partial v}{\partial x}dx=4kdx$$

$$\frac{\partial u}{\partial y}dy=2kdy \quad \frac{\partial v}{\partial y}dy=-6kdy$$

For A'B', D'C'

$$A'B'=D'C'=\sqrt{(dx+\frac{\partial u}{\partial x}dx)^2+(\frac{\partial v}{\partial x}dx)^2}$$

$$=\sqrt{(dx+2kdx)^2+(4kdx)^2}$$

$$=dx\sqrt{20k^2+4k+1}$$

For A'D', B'C'

$$A'D'=B'C'=\sqrt{(dy+\frac{\partial v}{\partial y}dy)^2+(\frac{\partial u}{\partial y}dy)^2}$$

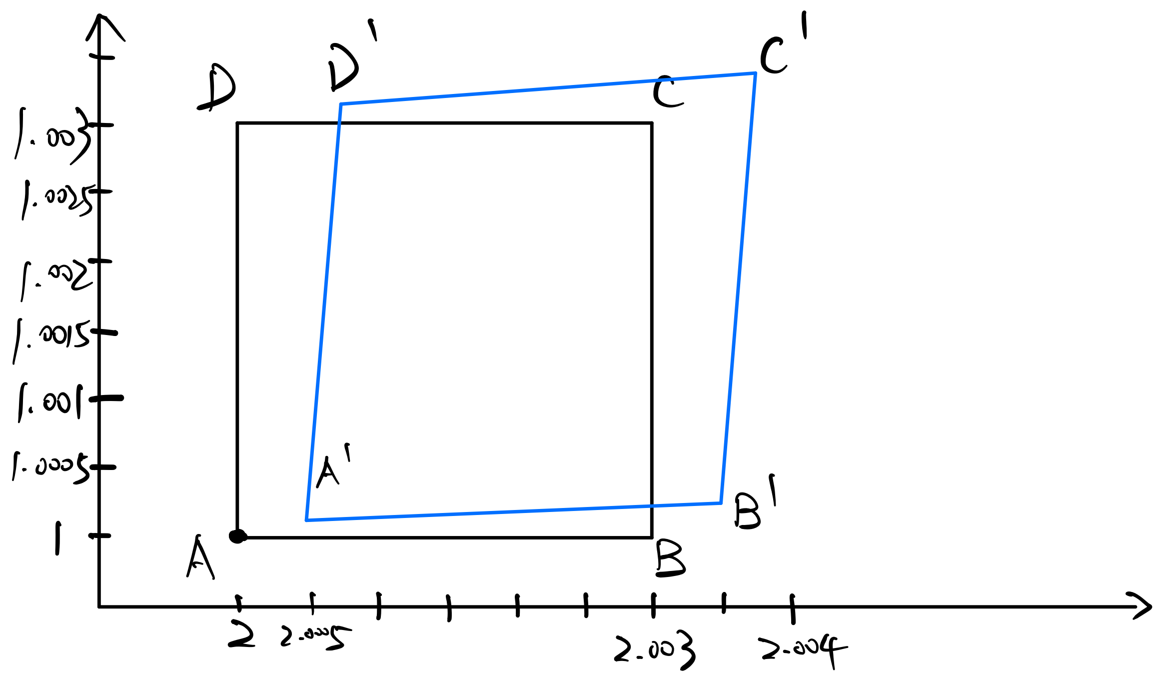
$$=\sqrt{(dy-6kdy)^2+(2kdy)^2}$$

$$=dy\sqrt{40k^2-12k+1}$$

The angular positions of A'D' and A'B' are

$$\theta = \tan\theta = \frac{\partial v}{\partial x} = 4k = 4\times10^{-4}$$

$$-\lambda = -\tan\lambda = \frac{\partial u}{\partial y} = 2k = 2\times10^{-4}$$



(PS: assume the $dx = 0.003$
$dy = 0.003$)

(b) Solution:

Since the point A(2,1,0) and the side length are dx and dy, we can know

$u = 5k$, $v = k$, $w = 0$

So the coordinates of point A is

$(2+5k,\ 1+k,\ 0)$

(c) Solution:

$$w_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = k = 1\times10^{-4}$$

(d) Solution:

$$\varepsilon_x = \frac{\partial u}{\partial x} = 2k \qquad \varepsilon_y = \frac{\partial v}{\partial y} = -6k$$

$$\varepsilon_{xy} = \frac{1}{2}\gamma_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = 3k$$

The maximum normal strain is

$$\frac{\varepsilon_x+\varepsilon_y}{2} + \sqrt{\left(\frac{\varepsilon_x-\varepsilon_y}{2}\right)^2 + \varepsilon_{xy}^2} = 3k \quad \cdots\cdots ①$$

The minimum normal strain is

$$\frac{\varepsilon_x+\varepsilon_y}{2} - \sqrt{\left(\frac{\varepsilon_x-\varepsilon_y}{2}\right)^2 + \varepsilon_{xy}^2} = -7k \quad \cdots\cdots ②$$

The maximum shear strain is

$$\sqrt{\left(\frac{\varepsilon_x-\varepsilon_y}{2}\right)^2 + \varepsilon_{xy}^2} = 5k \quad \cdots\cdots ③$$

The minimum shear strain is

$$-\sqrt{\left(\frac{\varepsilon_x-\varepsilon_y}{2}\right)^2 + \varepsilon_{xy}^2} = -5k \quad \cdots\cdots ④$$

3. Solution:

Since $\varepsilon_x = \frac{\partial u}{\partial x} = 5 + x^2 + y^2 + x^4 + y^4$

we can get $\frac{\partial^2 \varepsilon_x}{\partial y^2} = 2 + 12y^2$ ------ ①

Since $\varepsilon_y = \frac{\partial v}{\partial y} = 6 + 3x^2 + 3y^2 + x^4 + y^4$

we can get $\frac{\partial^2 \varepsilon_y}{\partial x^2} = 6 + 12x^2$ ------ ②

Since $\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 10 + 4xy(x^2 + y^2 + 2)$

we can get $\frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = 12x^2 + 12y^2 + 8$ -- ③

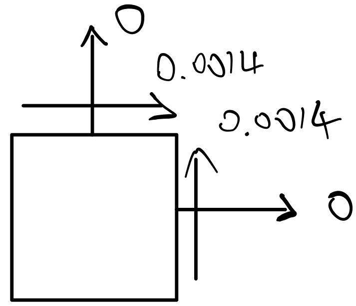
Then we get get

$$\frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = \frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2}$$

Therefore, the system of strain is possible.

4、Solution:

The angle $\alpha = 22.5°$, the Mohr's circle is shown below

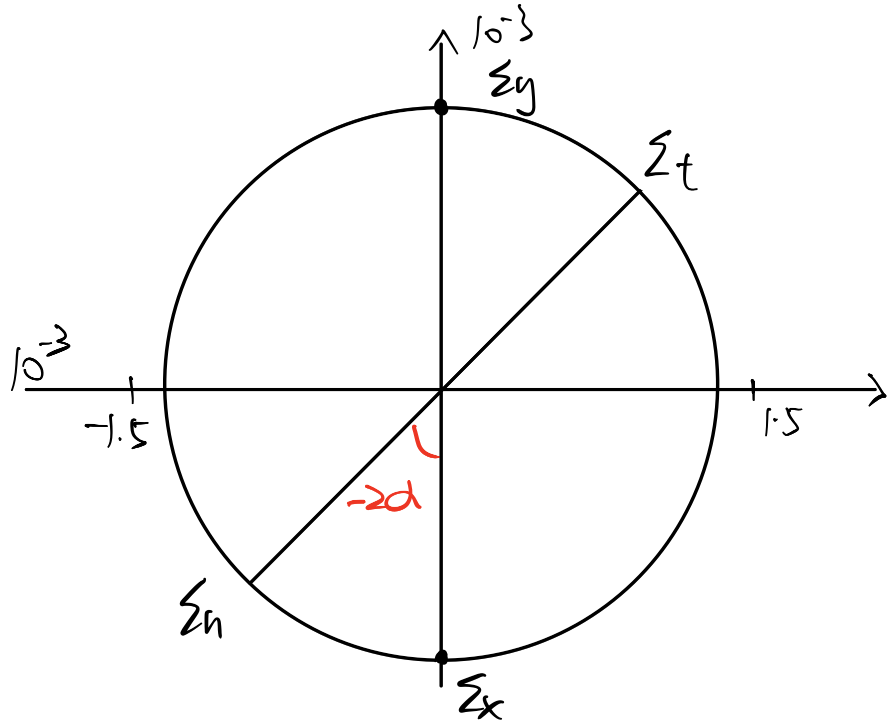


the center (0, 0)

radius = 0.002828

$\varepsilon_n = -0.0009998$, $\varepsilon_t = 0.0009998$

$\gamma_{tn} = 0.0009998$



5. Solution:

According to the Pythagorean theorem, we can get $AB = 5\ cm$

Because of the uniform strains, $w_z = 0$.

Then:

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$$

$$= \varepsilon_x dx + \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) dy + \frac{1}{2}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) dy$$

$$= \varepsilon_x dx + \frac{1}{2}\gamma_{xy} dy - w_z dy$$

$$= 0.0128$$

Similarly,

$$dv = \frac{\partial v}{\partial y} dy + \frac{\partial v}{\partial x} dx$$

$$= \varepsilon_y dy + \frac{1}{2}\gamma_{xy} dx + w_z dx = 0.0187$$

So we can get $A'B' = \sqrt{(dx+du)^2 + (dy+dv)^2} = 5.0215$

The change of AB is $0.0215\ cm$