MAE5009 · Continuum Mechanics B

Homework 1

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1-1

(a) 主应力公式为:

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

将已知代入算得

$$\sigma_1 = \frac{-14000 + 6000}{2} + \sqrt{\left(\frac{-14000 - 6000}{2}\right)^2 + (-17320)^2} = 16012.8\,\text{psi}$$

$$\sigma_2 = \frac{-14000 + 6000}{2} - \sqrt{\left(\frac{-14000 - 6000}{2}\right)^2 + (-17320)^2} = -24012.8\,\text{psi}$$

主应力方向为:

$$\tan(2\alpha) = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

$$\tan(2\alpha) = \frac{2(-17320)}{-14000 - 6000} = \frac{-34,640}{-20000} = 1.732$$

$$2\alpha = \tan^{-1}(1.732) = 60^\circ$$

$$\alpha = \frac{60^\circ}{2} = 30^\circ$$

因此，主应力的方向为 $\alpha = 30^\circ$。

(b) 旋转后的应力分量公式为:

$$\sigma'_x = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos(2\alpha) + \tau_{xy}\sin(2\alpha)$$

$$\sigma'_y = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos(2\alpha) - \tau_{xy}\sin(2\alpha)$$

$$\tau_{x'y'} = -\frac{\sigma_x - \sigma_y}{2}\sin(2\alpha) + \tau_{xy}\cos(2\alpha)$$

当 $\alpha = 45^\circ$ 时应力分量为:

$$\sigma'_x = \frac{-14000 + 6000}{2} + \frac{-14000 - 6000}{2} \cdot 0 + (-17320) \cdot 1 = -4000 + 0 - 17320 = -21320\,\text{psi}$$

$$\sigma'_y = \frac{-14000 + 6000}{2} - \frac{-14000 - 6000}{2} \cdot 0 - (-17320) \cdot 1 = -4000 + 0 + 17320 = 13320\,\text{psi}$$

$$\tau_{x'y'} = -\frac{-14000 - 6000}{2} \cdot 1 + (-17320) \cdot 0 = 10000 + 0 = 10000\,\text{psi}$$

1-3

由题目已知可得：$\sigma_x = 10000\,\text{psi}, \sigma_y = 0\,\text{psi}, \tau_{xy} = 0\,\text{psi}$
使用平面应力变换公式来计算斜面上的应力分量：

$$\sigma'_x = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos(2\alpha) + \tau_{xy}\sin(2\alpha)$$

$$\sigma'_y = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos(2\alpha) - \tau_{xy}\sin(2\alpha)$$

$$\tau_{x'y'} = -\frac{\sigma_x - \sigma_y}{2}\sin(2\alpha) + \tau_{xy}\cos(2\alpha)$$

因为初始角度为 45°，所以 $\alpha = 15°$，代入上式可得：

$$\sigma'_x = \frac{10000+0}{2} + \frac{10000-0}{2}\cos(2*15) + 0*\sin(2*15) = 9330\text{psi}$$

$$\sigma'_y = \frac{10000+0}{2} - \frac{10000-0}{2}\cos(2*15) - 0*\sin(2*15) = 670\text{psi}$$

$$\tau_{x'y'} = -\frac{10000-0}{2}\sin(2*15) + 0\cos(2*15) = -2500\text{psi}$$

1-13

(a) 因为 σ'_x 与 σ'_y 之间的夹角为 90°。因此，在 Mochr's circle 上，它们之间的夹角为 180°，因此，$(\sigma'_x, \tau_{x'y'})$ 与 $(\sigma'_y, \tau_{y'x'})$ 的连线为通过 Mochr's circle 圆心的直径。因此，$\sigma'_x + \sigma'_y$ 为 $\sigma_1 + \sigma_2$ 为定值。

(b) 根据莫尔圆公式，旋转角度后的剪应力为：

$$\tau_{x'y'} = -\frac{\sigma_x - \sigma_y}{2}\sin(2\alpha) + \tau_{xy}\cos(2\alpha)$$

计算 $\sigma'_x\sigma'_y - \tau^2_{x'y'}$：

$$\sigma'_x\sigma'_y = \left(\frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos(2\alpha) + \tau_{xy}\sin(2\alpha)\right) \times \left(\frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos(2\alpha) - \tau_{xy}\sin(2\alpha)\right)$$

$$= \left(\frac{\sigma_x + \sigma_y}{2}\right)^2 - \left(\frac{\sigma_x - \sigma_y}{2}\right)^2 - \tau^2_{xy}$$

$$\tau^2_{x'y'} = \left(-\frac{\sigma_x - \sigma_y}{2}\sin(2\alpha) + \tau_{xy}\cos(2\alpha)\right)^2$$

因此可以得出：

$$\sigma'_x\sigma'_y - \tau^2_{x'y'} = \left(\frac{\sigma_x + \sigma_y}{2}\right)^2 - \left(\frac{\sigma_x - \sigma_y}{2}\right)^2 - \tau^2_{xy}$$

为不变量

1-15

(a) 由已知可得：

$$\sigma = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix} = \begin{bmatrix} 10 & 5 & -10 \\ 5 & 20 & -15 \\ -10 & -15 & -10 \end{bmatrix}$$

因为

$$a^2_{11} + a^2_{21} + a^2_{31} = 1$$

所以

$$a_{31}^2 = \frac{1}{4}, \quad a_{31} = \frac{1}{2} \quad \text{(题目条件给出 } a_{31} \text{ is positive).}$$

于是

$$\mathbf{a} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{bmatrix}$$

于是可以得到 $\mathbf{p}$:

$$\mathbf{p} = \begin{bmatrix} 3.54 \\ 8.64 \\ -20.61 \end{bmatrix} \text{psi}$$

于是可以求得 The magnitude of $\mathbf{p}$ is:

$$|\mathbf{p}| = \sqrt{p_1^2 + p_2^2 + p_3^2} = \sqrt{3.54^2 + 8.64^2 + (-20.61)^2} \approx 22.62\,\text{psi}$$

(b) 因为:

$$\sigma = \mathbf{p} \cdot \mathbf{a} = p_1 a_{11} + p_2 a_{21} + p_3 a_{31}$$
$$\sigma = -2.43\,\text{psi}$$

τ 的大小为:

$$\tau = \sqrt{|\mathbf{p}|^2 - \sigma^2} = \sqrt{22.62^2 - (-2.43)^2} = \sqrt{511.92 - 5.9} = \sqrt{506.02} \approx 22.49\,\text{psi}$$

(c) 因为:

$$\cos\theta = \frac{\sigma}{|\mathbf{p}|}$$
$$\cos\theta = \frac{-2.43}{22.62} = -0.107$$
$$\theta = \cos^{-1}(-0.107) = 96.13^\circ$$

由上式可得，P 和 σ 之间的角度为 96.13°.