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1. (a) $A^H = \begin{bmatrix} 4-3i & 4i & 6-2i \\ -4i & 4+3i & -2+6i \\ -6+2i & -2+6i & 0 \end{bmatrix}$

$A^H A = \begin{bmatrix} 81 & 0 & -6+2i \\ 0 & 81 & 2-6i \\ -6-2i & 2+6i & 80 \end{bmatrix}$ $AA^H = \begin{bmatrix} 81 & 0 & 6+2i \\ 0 & 81 & 2+6i \\ 6-2i & 2-6i & 80 \end{bmatrix}$

$A^H A \neq AA^H$. ∴ (a) 不是正规矩阵

(b) $A^H = \begin{bmatrix} -1 & i & 0 \\ -i & 0 & -i \\ 0 & i & -1 \end{bmatrix}$

$A^H A = \begin{bmatrix} 2 & -i & 1 \\ i & 2 & i \\ 1 & -i & 2 \end{bmatrix}$ $AA^H = \begin{bmatrix} 2 & -i & 1 \\ i & 2 & i \\ 1 & -i & 2 \end{bmatrix}$

$A^H A = AA^H$ (也可以 $A = A^H$. ∴ $A^H A = AA^H$)

∴ (b) 为正规矩阵.

$|\lambda I - A| = 0$ 时.

$$\begin{vmatrix} \lambda+1 & -i & 0 \\ i & \lambda & i \\ 0 & -i & \lambda+1 \end{vmatrix} = \lambda(\lambda+1)^2 - 2(\lambda+1) = 0$$

∴ $\lambda_1 = 1$, $\lambda_2 = -2$, $\lambda_3 = -1$ $\Lambda = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

$(\lambda_1 I - A)x = 0$

$x_1 = \begin{bmatrix} 1 \\ -2i \\ 1 \end{bmatrix}$ 同理 $x_2 = \begin{bmatrix} 1 \\ i \\ 1 \end{bmatrix}$ $x_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$$U = \begin{bmatrix} 1 & 1 & -1 \\ -2i & i & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\beta_1 = \alpha_1$$

$$\beta_2 = \alpha_2 - \beta_1 \square \qquad \square = \frac{\langle \beta_1, \alpha_2 \rangle}{\langle \beta_1, \beta_1 \rangle}$$

$$\beta_3 = \alpha_3 - (\beta_1 \square + \beta_2 \square)$$

$$\therefore \beta_1 = \begin{bmatrix} 1 \\ -2i \\ 1 \end{bmatrix}, \quad \beta_2 = \begin{bmatrix} \cancel{10}\ \frac{1}{3} \\ \frac{7}{3}i \\ \frac{1}{3} \end{bmatrix} \quad \beta_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$\tilde{\beta}_1 = \frac{\beta_1}{\|\beta_1\|} = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -2i \\ 1 \end{bmatrix}$$

$$\tilde{\beta}_2 = \frac{\beta_2}{\|\beta_2\|} = \frac{1}{\sqrt{\frac{17}{3}}} \begin{bmatrix} \frac{1}{3} \\ \frac{7}{3}i \\ \frac{1}{3} \end{bmatrix}$$

$$\tilde{\beta}_3 = \frac{\beta_3}{\|\beta_3\|} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$\therefore U = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{51}} & \frac{-1}{\sqrt{2}} \\ \frac{-2}{\sqrt{6}}i & \frac{7i}{\sqrt{51}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{51}} & \frac{1}{\sqrt{2}} \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

验证

$$U^H A U = D$$

2. (a) Gram-Schmidt 正交求Q.

$$q_1 = \alpha_1$$

$$q_2 = \alpha_2 - q_1 \square \qquad \square = \frac{\langle q_1, \alpha_2 \rangle}{\langle q_1, q_1 \rangle}$$

$$q_3 = \alpha_3 - (q_1 \square + q_2 \square)$$

$$\therefore \tilde{q}_1 = \frac{q_1}{\|q_1\|} = \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \qquad \tilde{q}_2 = \frac{q_2}{\|q_2\|} = \begin{bmatrix} \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \end{bmatrix} \qquad \tilde{q}_3 = \frac{q_3}{\|q_3\|} = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$$

$$\therefore Q = [\tilde{q}_1, \tilde{q}_2, \tilde{q}_3] = \begin{bmatrix} 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

$$R = Q^T A = \begin{bmatrix} \sqrt{2} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{\sqrt{3}}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ 0 & 0 & \frac{2}{\sqrt{3}} \end{bmatrix}$$

(b) 同理

$$Q = \begin{bmatrix} \frac{1}{3} & -\frac{2}{\sqrt{129}} & \frac{2}{\sqrt{14}} \\ \frac{2}{3} & \frac{5}{\sqrt{129}} & -\frac{1}{\sqrt{14}} \\ \frac{2}{3} & -\frac{10}{\sqrt{129}} & \frac{3}{\sqrt{14}} \end{bmatrix}$$

$$R = Q^T A = \begin{bmatrix} 9 & 30 & 18 \\ 0 & \sqrt{129} & -\frac{153}{\sqrt{129}} \\ 0 & 0 & \sqrt{14} \end{bmatrix}$$

3、(a)

$$A^TA=\begin{bmatrix}5&2&2\\2&4&0\\2&0&1\end{bmatrix},\qquad AA^T=\begin{bmatrix}5&2\\2&5\end{bmatrix}$$

$$|\lambda I-A^TA|=\begin{vmatrix}\lambda-5&-2&-2\\-2&\lambda-4&0\\-2&0&\lambda-1\end{vmatrix}=0\qquad |\lambda I-AA^T|=\begin{vmatrix}\lambda-5&-2\\-2&\lambda-5\end{vmatrix}=0$$

$\therefore \lambda_1=0$, $\lambda_2=3$, $\lambda_3=7$ $\xrightarrow[\text{求得}]{\text{代入}(\lambda I-A^TA)x=0}$ $v_1=[-\frac{1}{2},\frac{1}{4},1]^T$, $v_2=[1,-2,1]^T$, $v_3=[3,2,1]^T$

$\therefore \lambda_1=3$, $\lambda_2=7$ $\longrightarrow$ $v_1=\begin{bmatrix}1\\-1\end{bmatrix}$, $v_2=\begin{bmatrix}1\\1\end{bmatrix}$

$$\therefore V=\begin{bmatrix}-\frac{1}{2}&1&3\\\frac{1}{4}&-2&2\\1&1&1\end{bmatrix}\qquad U=\begin{bmatrix}1&1\\-1&1\end{bmatrix}$$

对其Gram-Schmidt正交求得

$$V=\begin{bmatrix}\frac{-2\sqrt{21}}{21}&\frac{\sqrt{6}}{6}&\frac{3\sqrt{14}}{14}\\\frac{\sqrt{21}}{21}&-\frac{\sqrt{6}}{3}&\frac{\sqrt{14}}{7}\\\frac{4\sqrt{21}}{21}&\frac{\sqrt{6}}{6}&\frac{\sqrt{14}}{14}\end{bmatrix}\qquad U=\begin{bmatrix}\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\\-\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\end{bmatrix}$$

$\therefore \Sigma=\begin{bmatrix}\sqrt{7}&0&0\\0&\sqrt{3}&0\end{bmatrix}$ 奇异值为$\sqrt{7}$、$\sqrt{3}$.

(b) 可以从题目直接得出

$$\cancel{U=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\quad V=[}$$

$$U=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\qquad V^T=\begin{bmatrix}1&0\\0&1\end{bmatrix}\qquad \Sigma=\begin{bmatrix}2&0\\0&3\\0&0\end{bmatrix}$$

$$V=\begin{bmatrix}0&1\\1&0\end{bmatrix}$$

$\therefore$奇异值为2、3.

4. 证明：① A的列空间$\mathbb{R}^m$的基$[c_1, c_2, \cdots c_r]$可以经过 Gram-schmidt正交化变为

$U=[u_1, u_2, \cdots u_m]$

同理，A的行空间$\mathbb{R}^n$的基$[b_1, b_2, \cdots b_r]$经过 Gram-schmidt 正交化变为

$V=[v_1, v_2, \cdots, v_n]$

② $\because A = U\Sigma V^T = [U_1\ U_2]\begin{bmatrix}\Sigma_1 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}V_1^T\\ V_2^T\end{bmatrix} = U_1\Sigma_1 V_1^T$

其中$\Sigma_1 \in \mathbb{R}^{r\times r}$

③ 因为可以得出 $C=U_1$, $B=V_1$, $M=\Sigma_1$

则 $A=CMB^T$ 对任意满足条件的矩阵A都成立.

5. (a) 由题目得

$$A=\begin{bmatrix}1&2&3\\1&0&1\\2&0&2\\1&4&6\end{bmatrix} \quad b=\begin{bmatrix}1\\0\\1\\3\end{bmatrix} \quad \bar{A}=[A \vdots b]$$

$\mathrm{rank}(A)=3$ $\quad \mathrm{rank}(A)\neq \mathrm{rank}(\bar{A})$

$\mathrm{rank}(\bar{A})=4$ $\quad \therefore$ 方程无解

(b) $$X=(A^TA)^{-1}\cdot(A^TB) = \begin{bmatrix}7&6&14\\6&20&30\\14&30&38\end{bmatrix}^{-1}\begin{bmatrix}6\\14\\23\end{bmatrix} = \begin{bmatrix}-1\\-\frac{11}{10}\\\frac{7}{5}\end{bmatrix}$$

A^T~~A~~

(c) $\|X\|_2 = \sqrt{x_1^2+x_2^2+x_3^2} = \sqrt{\frac{417}{196}}$

$$R = b-AX = \begin{bmatrix}1\\0\\1\\3\end{bmatrix} - \begin{bmatrix}1&2&3\\1&0&1\\2&0&2\\1&4&6\end{bmatrix}\begin{bmatrix}-1\\-\frac{11}{10}\\\frac{7}{5}\end{bmatrix} = \begin{bmatrix}0\\-\frac{2}{5}\\\frac{1}{5}\\0\end{bmatrix} \quad \|R\|_2=\frac{1}{\sqrt{5}}$$

4×1 $\quad 4\times3$ $\quad 3\times1$

6. 由题目已知

$$A=\begin{bmatrix}1&2&0\\0&0&2\\2&4&1\end{bmatrix}\quad b=\begin{bmatrix}1\\1\\3\end{bmatrix}$$

$$A^TA=\begin{pmatrix}5&10&2\\10&20&4\\2&4&5\end{pmatrix}$$ 其中 c_1, c_2 线性相关，故 A^TA 不可逆。

可用伪逆求解。

$x=A^+b$，其中 $A^+=C^H(CC^H)^{-1}(B^HB)^{-1}B^H$.（$A=BC$ 为A的满秩分解）

$$=\frac{1}{50}\begin{bmatrix}2&0&4\\4&0&8\\0&25&0\end{bmatrix}\begin{bmatrix}1\\1\\3\end{bmatrix}=\frac{1}{50}\begin{bmatrix}14\\28\\25\end{bmatrix}$$