Homework 1 Solution

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1-1 Given $\sigma_x = -14{,}000$ psi, $\sigma_y = 6{,}000$ psi, and $\tau_{xy} = -17{,}320$ psi, determine by formulas, (a) the principal stresses and their directions and (b) the stress components on the x' and y' planes when $\alpha = 45°$.

Solution

The unit of stress is psi, which is ignored for simplification.

The Cauchy stress can be constructed as

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{yx} & \sigma_y \end{bmatrix} = \begin{bmatrix} -14000 & -17320 \\ -17320 & 6000 \end{bmatrix}. \tag{1}$$

The transformation of stress can be written as

$$\boldsymbol{\sigma}' = \boldsymbol{Q}\boldsymbol{\sigma}\boldsymbol{Q}^{\mathrm{T}} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{yx} & \sigma_y \end{bmatrix} \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} = \begin{bmatrix} \sigma_{x'} & \tau_{x'y'} \\ \tau_{y'x'} & \sigma_{y'} \end{bmatrix}, \tag{2}$$

through the rotation tensor $\boldsymbol{Q}$. Then we can write the component as

$$\begin{cases} \sigma_{x'} = \sigma_x \cos^2\alpha + \sigma_y \sin^2\alpha + 2\tau_{xy}\sin\alpha\cos\alpha, & (3) \\ \sigma_{y'} = \sigma_x \sin^2\alpha + \sigma_y \cos^2\alpha - 2\tau_{xy}\sin\alpha\cos\alpha, & (4) \\ \tau_{x'y'} = (\sigma_y - \sigma_x)\sin\alpha\cos\alpha + \tau_{xy}(\cos^2\alpha - \sin^2\alpha). & (5) \end{cases}$$

(a)

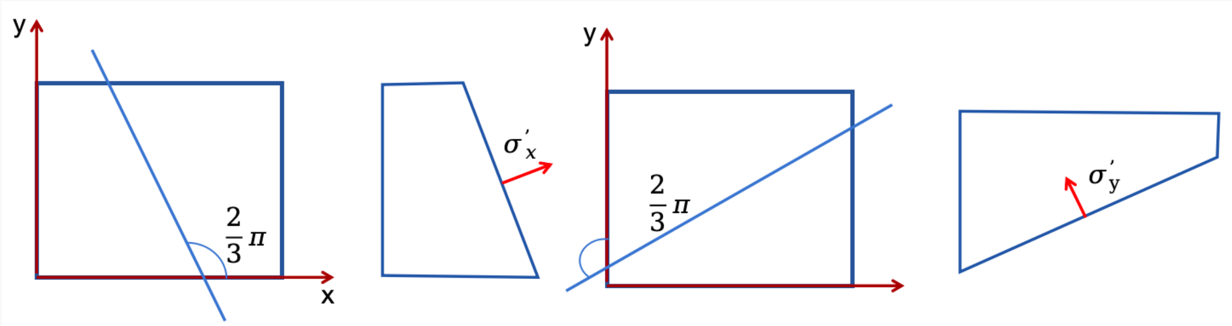
Then

$$\frac{d\sigma_{x'}}{d\alpha} = 0 \Rightarrow \tan 2\alpha = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} = 1.732 \quad \text{with} \quad \alpha \in [0, \pi] \tag{6}$$

We can solve that $\alpha = \pi/6$ or $2\pi/3$, where the shear stresses vanish at both these two planes. The principal stress can be solved by taking either principal plane, and the result is obtained by taking $\alpha = 2\pi/3$, as follows

$$\begin{cases} \sigma_{x'} = \sigma_x \cos^2\alpha + \sigma_y \sin^2\alpha + 2\tau_{xy}\sin\alpha\cos\alpha = 1.6 \times 10^4, \\ \sigma_{y'} = \sigma_x \sin^2\alpha + \sigma_y \cos^2\alpha - 2\tau_{xy}\sin\alpha\cos\alpha = -2.4 \times 10^4. \end{cases}$$

The unit of principal stresses are psi, and the maximum stress is $\sigma_{y'}$, which acts as a compression. The detailed stress directions are as follows,



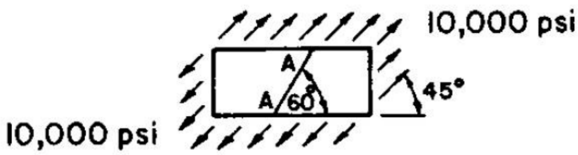
(b)

Lets set $\alpha = \pi/4$ and substitute in formula (3), (4) and (5), as follows:

$$\begin{cases} \sigma_{x'} = \sigma_x \cos^2\alpha + \sigma_y \sin^2\alpha + 2\tau_{xy}\sin\alpha\cos\alpha = -21320, \\ \sigma_{y'} = \sigma_x \sin^2\alpha + \sigma_y \cos^2\alpha - 2\tau_{xy}\sin\alpha\cos\alpha = 13320, \\ \tau_{x'y'} = (\sigma_y - \sigma_x)\sin\alpha\cos\alpha + \tau_{xy}(\cos^2\alpha - \sin^2\alpha) = 10000. \end{cases}$$

With the unit of psi.

1-3 A rectangular block is under a uniformly distributed load as shown in the figure. Find the stress components on the plane A —A.

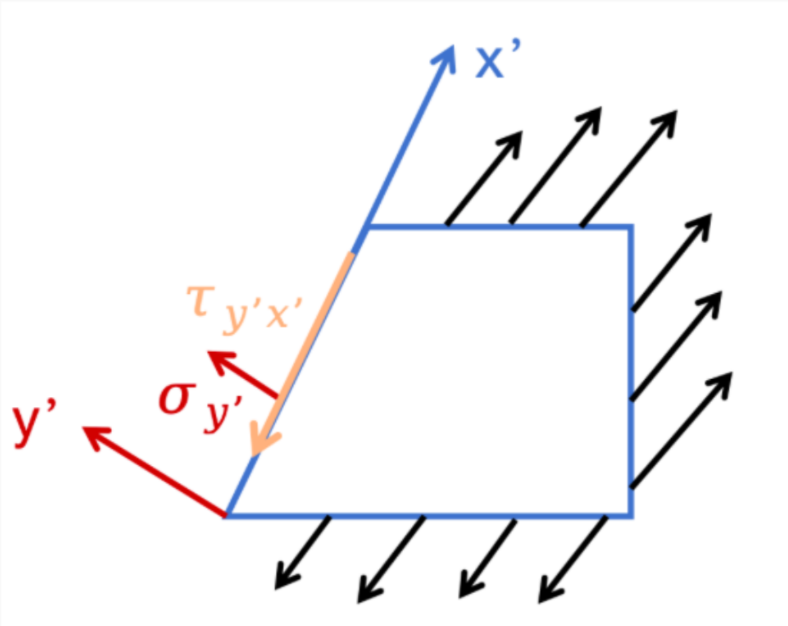


Solution

We may construct the coordinate with horizontal line as x axis and vertical line as y axis, and then define a $B - B$ and $C - C$ plane firstly. Through the construction of $B - B$ and $C - C$ plane, we can demonstrate the stress state at any point of this block at the prescribed coordinate as

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{yx} & \sigma_y \end{bmatrix} = \begin{bmatrix} 5000\sqrt{2} & 5000\sqrt{2} \\ 5000\sqrt{2} & 5000\sqrt{2} \end{bmatrix}. \tag{7}$$

Then use the transformation of stress (formula (3), (4), and (5)) with $\alpha = \pi/3$ to obtain the stress compoents of $A - A$ plane, as follows:



$$\begin{cases} \sigma_{y'} = \sigma_x \cos^2\alpha + \sigma_y \sin^2\alpha - 2\tau_{xy} \sin\alpha\cos\alpha = 947.34, \\ \tau_{x'y'} = (\sigma_y - \sigma_x)\sin\alpha\cos\alpha + \tau_{xy}(\cos^2\alpha - \sin^2\alpha) = -3535.53, \end{cases}$$

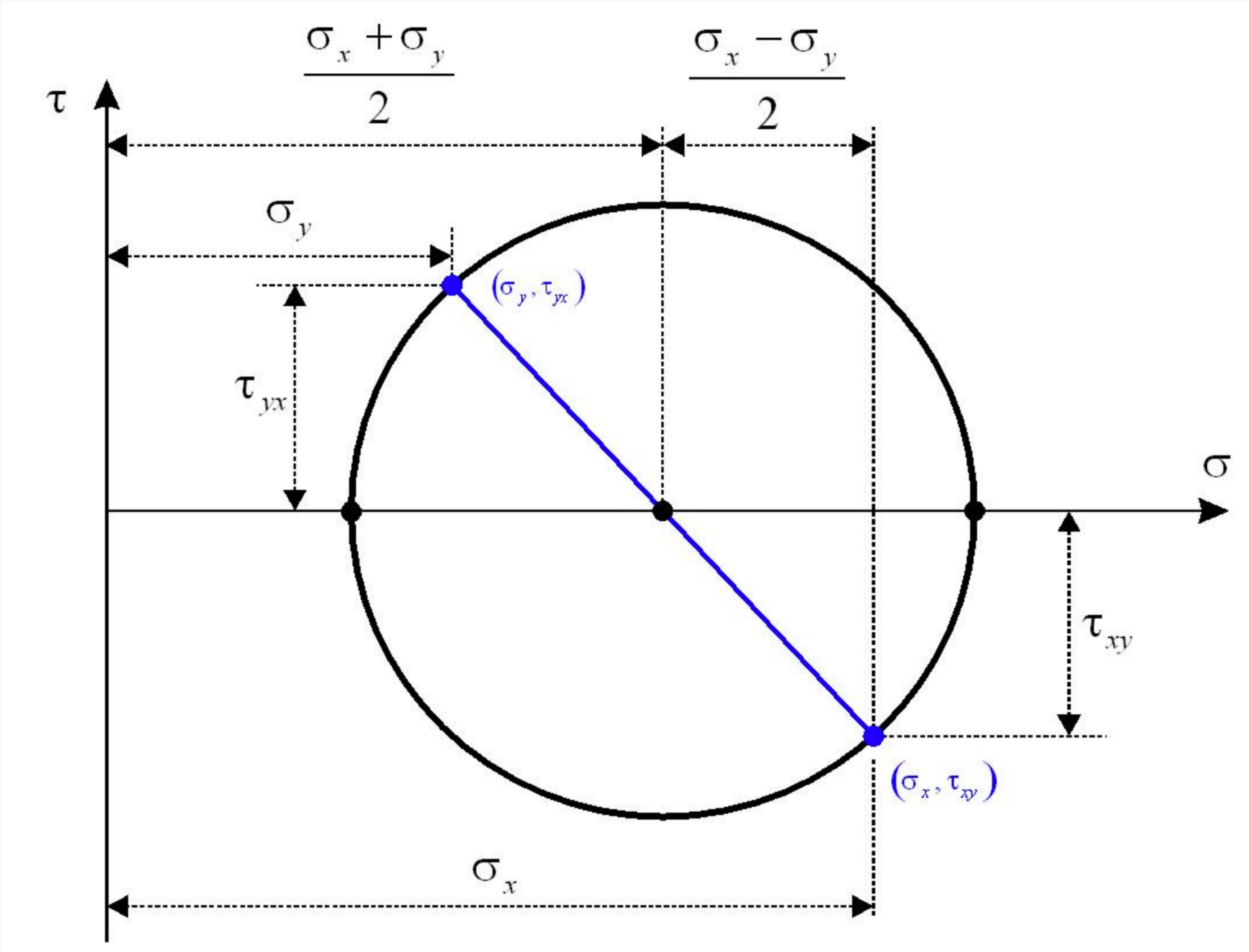
with the unit of psi.

1-13 By using Mohr's circle, show that the following quantities are invariant for a two-dimensional state of stress with $\sigma_z = \tau_{xz} = \tau_{yz} = 0$;

(a) $\sigma_{x'} + \sigma_{y'}$

(b) $\sigma_{x'}\sigma_{y'} - \tau_{x'y'}^2$.

Solution



(a)

The Mohr's circle expresses the stress transformation equations with principal stresses. Assume we have a two-dimensional state of stress as follows:

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{yx} & \sigma_y \end{bmatrix} \rightarrow \boldsymbol{\sigma}' = \begin{bmatrix} \sigma'_x & \tau'_{xy} \\ \tau'_{yx} & \sigma'_y \end{bmatrix} \tag{8}$$

through any abstract rotation. We have the relation with the form of

$$\left(\sigma'_{(\cdot)} - \frac{\sigma_x + \sigma_y}{2}\right)^2 + {\tau'_{(\cdot)}}^2 = \left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2. \tag{9}$$

The invariant can be determined by the mid-point rule of the circle center as

$$\sigma'_x + \sigma'_y = \sigma_x + \sigma_y. \tag{10}$$

(b)

The values of principal stresses are obtained as

$$\sigma_{1,2} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \quad \text{with} \quad \sigma_1 > \sigma_2. \tag{11}$$

Through any rotation of 2β, we can obtain the stress components as

$$\begin{aligned} \sigma_x &= \frac{\sigma_1 + \sigma_2}{2} + R\cos 2\beta, \\ \sigma_y &= \frac{\sigma_1 + \sigma_2}{2} - R\cos 2\beta, \\ \tau_{xy} &= R\sin 2\beta, \end{aligned}$$

with the radius of $R = \frac{\sigma_1 - \sigma_2}{2}$, then we can obtain

$$\sigma_x\sigma_y - \tau_{xy}^2 = \left(\frac{\sigma_1 + \sigma_2}{2}\right)^2 - R^2 = \sigma_1\sigma_2 \tag{12}$$

So, $\sigma_x\sigma_y = \tau_{xy}^2$ is the invariant.

1-15 Given a three-dimensional state of stress with

(2 points)

$$\begin{aligned} \sigma_x &= +10 \text{ psi} & \tau_{xy} &= +5 \text{ psi} \\ \sigma_y &= +20 \text{ psi} & \tau_{xz} &= -10 \text{ psi} \\ \sigma_z &= -10 \text{ psi} & \tau_{yz} &= -15 \text{ psi} \end{aligned}$$

(a) Find the magnitude and direction of the stress vector p on the x' plane where the x' direction is defined by

$$a_{11} = +1/2 \qquad a_{21} = +1/\sqrt{2} \qquad a_{31} \text{ is positive.}$$

(b) Find σ and τ on this plane.
(c) Determine the angle between p and σ.

Solution

(a)

The Cauchy stress is defined as

$$\boldsymbol{\sigma} = \begin{bmatrix} 10 & 5 & -10 \\ 5 & 20 & -15 \\ -10 & -15 & -10 \end{bmatrix} \text{psi}. \tag{13}$$

The direction vector of x' plane can be determined by the fact that the magnitude of direction vector is 1, as follows

$$a_{11}^2 + a_{21}^2 + a_{31}^2 = 1 \quad \text{with} \quad a_{11} = \frac{1}{2} \quad \text{and} \quad a_{21} = \frac{1}{\sqrt{2}}. \tag{14}$$

We can solve $a_{31} = \frac{1}{2} > 0$. The the direction vector $\boldsymbol{n}$ can be defined as

$$\boldsymbol{n} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{bmatrix}. \tag{15}$$

We can find the stress vector $\boldsymbol{p}$ on x' plane as

$$\boldsymbol{p} = \boldsymbol{\sigma n} = \begin{bmatrix} 10 & 5 & -10 \\ 5 & 20 & -15 \\ -10 & -15 & -10 \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{bmatrix} \text{psi} = \begin{bmatrix} \frac{5\sqrt{2}}{2} \\ 10\sqrt{2} - 5 \\ -\frac{15\sqrt{2}}{2} - 10 \end{bmatrix} \text{psi} \approx \begin{bmatrix} 3.54 \\ 9.14 \\ -20.61 \end{bmatrix} \text{psi}. \tag{16}$$

Then we can write $\boldsymbol{p}$ in the form of magnitude and direction:

$$\boldsymbol{p} = 22.81 \begin{bmatrix} 0.1551 \\ 0.4007 \\ -0.9034 \end{bmatrix} \tag{17}$$

(b)

We can find the normal and shear stress by projecting the $\boldsymbol{p}$ to the direction of x' plane, as follows

$$\sigma = \boldsymbol{p} \cdot \boldsymbol{n} = 22.81 \begin{bmatrix} 0.1551 \\ 0.4007 \\ -0.9034 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{2} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{bmatrix} = -2.07. \tag{18}$$

Then the shear stress can be obtained by

$$\tau^2 + \sigma^2 = |\boldsymbol{p}|^2 = 22.81^2 \Rightarrow \tau = \sqrt{22.81^2 - (-2.07)^2} = 22.72, \tag{19}$$

with unit of psi.

(c)

The direction of σ is same as the plane known as $\boldsymbol{n}$, and we define the direction of $\boldsymbol{p}$ as $\boldsymbol{N} = [0.1551, 0.4007, -0.9034]$. The angle θ between $\boldsymbol{n}$ and $\boldsymbol{N}$ can be determined as

$$\theta = \cos^{-1} \frac{\boldsymbol{n} \cdot \boldsymbol{N}}{|\boldsymbol{n}|\,|\boldsymbol{N}|} = 95.21^o \tag{20}$$