Continuum Mechanics (B)

Session 05: Cartesian Tensor Notation

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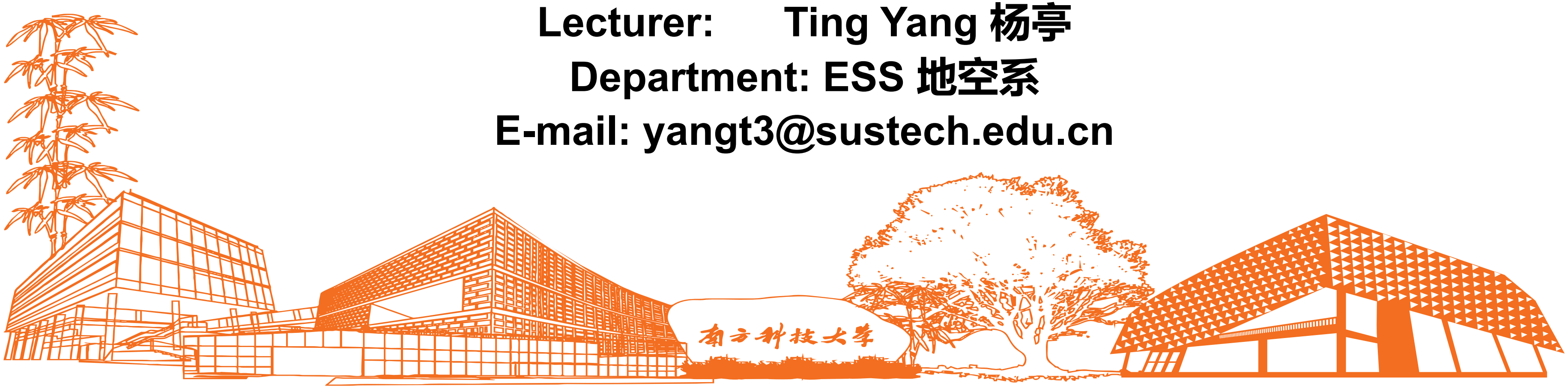

Contents

Indicial Notation (下标表示，张量表示)

Using 1, 2 and 3 to represent the three components in Cartesian coordinates:

$$\begin{cases} x \\ y \\ z \end{cases} \rightarrow \begin{cases} x_1 \\ x_2 \\ x_3 \end{cases} \qquad \begin{cases} u_x \\ u_y \\ u_z \end{cases} \rightarrow \begin{cases} u_1 \\ u_2 \\ u_3 \end{cases}$$

$$\begin{cases} \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{cases} \rightarrow \begin{cases} \tau_{12} \\ \tau_{13} \\ \tau_{23} \end{cases} \qquad \begin{cases} \sigma_x \\ \sigma_y \\ \sigma_z \end{cases} \rightarrow \begin{cases} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \end{cases}$$

Indicial Notation (下标表示，张量表示)

$$\begin{cases} x'_1 = a_{11}x_1 + a_{21}x_2 + a_{31}x_3 \\ x'_2 = a_{12}x_1 + a_{22}x_2 + a_{32}x_3 \\ x'_3 = a_{13}x_1 + a_{23}x_2 + a_{33}x_3 \end{cases}$$

$$x'_1 = \sum_{i=1}^{3} a_{i1} x_i \quad x'_2 = \sum_{i=1}^{3} a_{i2} x_i \quad x'_3 = \sum_{i=1}^{3} a_{i3} x_i$$

$$x'_j = \sum_{i=1}^{3} a_{ij} x_i \quad j = 1, 2, 3$$

$$x'_j = a_{ij} x_i \quad i, j = 1, 2, 3$$

summation convention

$$x'_j = a_{ij} x_i$$

$$i, j = 1, 2, 3$$

can be neglected

Summation convention: if a subscript appears twice in one monomial (单项式), automatic summation over the range of this subscript is required.

Exercise:

write down the expanded form of the following equations

$$a_{ii} = 0 \qquad y_i = a_{ij} x_j$$

Indicial Notation

Tensor notation

$$x'_j = a_{ij} x_i$$

free index *j* dummy index *i*

dummy index（哑标，哑指标）：
repeated subscripts
free index（自由标，自由指标）：
nonrepeated subscripts

Example:

1. Find the dummy and free indices in the equation below

$$\sigma_{ji,j} + f_i = 0$$

2. write down the expanded form of the following equations

$$b_{ij} b_{jk} = 0$$

$\boxed{b_{i1}b_{1k} + b_{i2}b_{2k} + b_{i3}b_{3k}}$ has 9 components

Rules for summation convention:
In a tensor equation,

- a subscript may appear no more than twice in each monomial;
- if a subscript appears only once in a monomial, it must appear and just appear **once** in every monomial.

E.g.

$$a_i = b_{ijj} c_j$$

$$b_j = A_{ij} x_i + f_j + g_k$$

Indicial Notation

The letter used both for the free index and dummy index can be replaced by other letters

$$x'_j = a_{ij} x_i \quad \Longleftrightarrow \quad x'_k = a_{ik} x_i$$

replace free index
j with letter *k*

$$x'_j = a_{ij} x_i = a_{\ell j} x_\ell$$

replace dummy index *i* with *l*

$$b_j = A_{ij} x_i + f_j \quad \Longleftrightarrow \quad b_k = A_{jk} x_j + f_k$$

Example:

Prove that the trace of AB equals to the trace of BA where A, B are two matrices

Vector Transformation (向量变换)

Assume a position vector **P** is represented by (x_1, x_2, x_3) in *xyz* coordinate and (x'_1, x'_2, x'_3) in a new coordinate system *x'y'z'*, we have

$$\begin{cases} x'_1 = a_{11}x_1 + a_{21}x_2 + a_{31}x_3 \\ x'_2 = a_{12}x_1 + a_{22}x_2 + a_{32}x_3 \\ x'_3 = a_{13}x_1 + a_{23}x_2 + a_{33}x_3 \end{cases}$$

Coefficients are direction cosines

$$a_{ij} = \cos(x_i, x'_j)$$

Direction cosines

	x'_1	x'_2	x'_3
x_1	a_{11}	a_{12}	a_{13}
x_2	a_{21}	a_{22}	a_{23}
x_3	a_{31}	a_{32}	a_{33}

a_{ij}, the first index i and the second index j come from xyz and x'y'z' system, respectively

In general, direction cosine $a_{12} \neq a_{21}$.

In indical notation

$$x'_j = a_{ij}x_i$$

Conversely, vector components (x_1, x_2, x_3) can be represented by (x'_1, x'_2, x'_3):

$$\begin{cases} x_1 = a_{11}x'_1 + a_{12}x'_2 + a_{13}x'_3 \\ x_2 = a_{21}x'_1 + a_{22}x'_2 + a_{23}x'_3 \\ x_3 = a_{31}x'_1 + a_{32}x'_2 + a_{33}x'_3 \end{cases}$$

$$x_m = a_{mk}x'_k$$

Vector Transformation (向量变换)

$x'_j = a_{ij} x_i$ $\qquad$ $x_m = a_{mk} x'_k$

change free index m to i, we have

$$x_i = a_{ik} x'_k$$

substitute this equation into coordinate transformation equation

$$x'_j = a_{ij} x_i$$

we have

$$x'_j = a_{ij} a_{ik} x'_k$$

we have

$$a_{ij} a_{ik} = 1 \qquad \text{if} \quad j = k$$

$$a_{ij} a_{ik} = 0 \qquad \text{if} \quad j \neq k$$

$\Longleftrightarrow$

Direction cosines

	x'_1	x'_2	x'_3
x_1	a_{11}	a_{12}	a_{13}
x_2	a_{21}	a_{22}	a_{23}
x_3	a_{31}	a_{32}	a_{33}

a_{ij} represents the unit vector of x'_j and a_{ik} represents the unit vector of x'_k

$$a_{11}^2 + a_{21}^2 + a_{31}^2 = 1 \qquad a_{11}a_{12} + a_{21}a_{22} + a_{21}a_{32} = 0$$

$$a_{12}^2 + a_{22}^2 + a_{32}^2 = 1 \qquad a_{11}a_{13} + a_{21}a_{23} + a_{31}a_{33} = 0$$

$$a_{13}^2 + a_{23}^2 + a_{33}^2 = 1 \qquad a_{12}a_{13} + a_{22}a_{23} + a_{32}a_{33} = 0$$

Vector Transformation

Similar to the position vector, for any vector quantity **F**, its components transform to a primed coordinate system according to the rule

$$F'_j = a_{ij} F_i$$

Definition of a vector based on coordinate transformation:

A set of three quantities F_i referred to a coordinate system xyz and transformed to another coordinate system x'y'z' by the following equation is defined as a vector

$$F'_j = a_{ij} F_i$$

Assume the components of a quantity in the xyz and x'y'z' coordinates are:

$$A'_j = A_i = [1, 1, 1]$$

Show that it is not a vector:

$$A'_j \neq a_{ij} A_i$$

Review

Indical (Tensor) notation (下标表示，张量表示)

$$x_i,\ u_i,\ p_i,\ \tau_{ij}$$

Summation convention (爱因斯坦求和约定)

$$x'_j = a_{ij} x_i$$

free index j dummy index i

Definition of a vector based on coordinate transformation:

$$F'_j = a_{ij} F_i \qquad \text{also} \quad F_i = a_{ij} F'_j$$

Direction cosines

$$a_{ij} = \cos(x_i, x'_j)$$

Classroom exercise

show that

$$a_{ji}a_{ki} = 1 \qquad \text{if} \quad j = k$$

$$a_{ji}a_{ki} = 0 \qquad \text{if} \quad j \neq k$$

8-1 Demonstrate that the dot product of vectors u_i and v_m may be written in all of the following forms

$$\begin{aligned} \mathbf{u}\cdot\mathbf{v} &= u_i v_i \\ &= a_{ij}a_{mj}u_i v_m \\ &= u'_j v'_j \\ &= a_{ij}a_{ik}u'_j v'_k \end{aligned}$$

Further understanding of position vector transformation (if needed)

$$\vec{P} = x_1\hat{x}_1 + x_2\hat{x}_2 + x_3\hat{x}_3 = x'_1\,\hat{x}'_1 + x'_2\,\hat{x}'_2 + x'_3\,\hat{x}'_3$$

$$\begin{aligned} x'_j &= \vec{P}\cdot\hat{x}'_j = (x_1\hat{x}_1 + x_2\hat{x}_2 + x_3\hat{x}_3)\cdot\hat{x}'_j \\ &= x_1\hat{x}_1\cdot\hat{x}'_j + x_2\hat{x}_2\cdot\hat{x}'_j + x_3\hat{x}_3\cdot\hat{x}'_j \\ &= x_1 a_{1j} + x_2 a_{2j} + x_3 a_{3j} \end{aligned}$$

$$x'_j = a_{ij}x_i$$

$$\begin{aligned} x_m &= \vec{P}\cdot\hat{x}_m = (x'_1\,\hat{x}'_1 + x'_2\,\hat{x}'_2 + x'_3\,\hat{x}'_3)\cdot\hat{x}_m \\ &= x'_1\,\hat{x}'_1\cdot\hat{x}_m + x'_2\,\hat{x}'_2\cdot\hat{x}_m + x'_3\,\hat{x}'_3\cdot\hat{x}_m \\ &= x'_1\,a_{m1} + x'_2\,a_{m2} + x'_3\,a_{m3} \end{aligned}$$

$x_m = a_{mk}x'_k$ or

$x_i = a_{ij}x'_j$