1. The surface temperature of a lake changes from one location to another as $T(x_1, x_2)$. If you attach a thermometer to a boat and take a path through the lake given by $x_i = b_i(t)$, find an expression for the rate of change of the thermometer temperature in terms of the lake temperature.

The position of the boat can be specified as

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1(t) \\ b_2(t) \end{bmatrix},$$

So that the temperature at (x_1, x_2) is

$$T(x_1, x_2) = T(b_1(t), b_2(t)) = T(t).$$

The rate of the temperature change is

$$\frac{dT}{dt} = \frac{\partial T}{\partial x_1}\dot{b}_1 + \frac{\partial T}{\partial x_2}\dot{b}_2.$$

2. In a table of vector differential operators, look up the expressions for $\nabla\times\boldsymbol{v}$ in a cylindrical coordinate system. (a) Compute the vorticity for the flow in a round tube where the velocity profile is

$$v_z = v_0\left[1 - \left(\frac{r}{R}\right)^2\right]$$

(b) Given Γ = 10, compute the vorticity for an ideal vortex where the velocity is

$$v_\theta = \frac{\Gamma}{2\pi r}$$

(a)

The velocity under the cylinderical coordinate is

$$\boldsymbol{v} = v_r\boldsymbol{e}_r + v_\theta\boldsymbol{e}_\theta + v_z\boldsymbol{e}_z = v_0\left[1 - \left(\frac{r}{R}\right)^2\right] \quad \text{with} \quad v_r = v_\theta = 0,$$

and the curl of the velocity is

$$\nabla\times\boldsymbol{v} = \frac{1}{r^2\sin\theta}\det\begin{bmatrix} \boldsymbol{e}_r & r\boldsymbol{e}_\theta & \boldsymbol{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial z} \\ v_r & v_\theta & v_z \end{bmatrix}.$$

The compoenets of $\nabla\times\boldsymbol{v}$ is

$$(\nabla \times \boldsymbol{v})_r = \frac{1}{r}\left(\frac{\partial v_z}{\partial \theta} - \frac{\partial v_\theta}{\partial z}\right) = 0,$$
$$(\nabla \times \boldsymbol{v})_\theta = \frac{1}{r}\left(\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}\right) = \frac{2v_0}{R^2},$$
$$(\nabla \times \boldsymbol{v})_z = \frac{1}{r}\left(\frac{\partial r v_\theta}{\partial r} - \frac{\partial v_r}{\partial \theta}\right) = 0.$$

So that the vorticity is

$$\nabla \times \boldsymbol{v} = \frac{2v_0}{R}\boldsymbol{e}_\theta.$$

(b)

We can derive the vorticity through the same process with the velocity of $\boldsymbol{v} = v_\theta \boldsymbol{e}_\theta$, as

$$(\nabla \times \boldsymbol{v})_z = \frac{1}{r}\frac{\partial r v_\theta}{\partial r} = \frac{1}{r}\frac{\partial \Gamma / 2\pi}{\partial r} = 0.$$

so that the vorticity of an ideal vortex is 0 or the Dirac function $\delta(r)$.

3. Consider a two-dimensional flow with velocity components $v_1 = cx_1$, $v_2 = -cx_2$. Give each component of the strain rate tensor.

The definition of the strain rate tensor components is

$$S_{ij} = \frac{1}{2}(v_{j,i} + v_{i,j}) = \frac{1}{2}\left(\frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j}\right)$$

By take i, j to be $1, 2$ respectively, we can determine the strain rate tensor as

$$\boldsymbol{S} = \begin{bmatrix} c & 0 \\ 0 & -c \end{bmatrix}$$

4. Discuss the difference between streamline, pathline, and streakline

1. Streamlines

Definition:
Streamlines are lines that are instantaneously tangent to the velocity vector of the flow at every point. They represent the direction a fluid element will move at a particular instant in time.

Characteristics:

- **Instantaneous:** They represent the flow pattern at a specific moment.
- **No Crossing:** Streamlines never intersect each other.
- **Tangency:** At every point on a streamline, the velocity vector of the flow is tangent to the streamline.

Formula:
In a steady flow, where the velocity field $\mathbf{v} = (u, v, w)$ does not change with time, the streamline can be described by the differential equation:

$$\frac{dy}{dx} = \frac{v}{u}, \quad \frac{dz}{dx} = \frac{w}{u}$$

2. Pathlines

Definition:
Pathlines are the trajectories that individual fluid particles follow over a period of time. They show the actual path taken by a specific fluid element.

Characteristics:

- **Time-Dependent:** Pathlines are traced over time as fluid particles move.
- **Same as Streamlines in Steady Flow:** In steady flows, streamlines and pathlines coincide.

Formula:
The pathline of a fluid particle can be determined by integrating the velocity field over time:

$$\mathbf{r}(t) = \mathbf{r}_0 + \int_{t_0}^{t} \mathbf{v}(\mathbf{r}(t'), t')\, dt'$$

where:

- $\mathbf{r}(t)$ is the position vector at time t,
- $\mathbf{r}_0$ is the initial position at time t_0.

3. Streaklines

Definition:
Streaklines are the loci of all fluid particles that have previously passed through a specific point in the flow field. They are often visualized by introducing a tracer (like dye) at a fixed point.

Characteristics:

- **History-Dependent:** Streaklines reflect the history of the flow.
- **Visible in Experiments:** Commonly observed in experiments using dye or smoke.

Formula:
Mathematically, a streakline can be represented as the set of positions occupied by particles that have passed through a point (\mathbf{r}_s) at different times:

$$\mathbf{r}(t) = \mathbf{r}_0 + \int_{t_0}^{t} \mathbf{v}(\mathbf{r}(t'), t')\, dt'$$

where:

- $\mathbf{r}_s$ is the fixed point where particles enter the flow,
- t_s is the time when particles passed through $\mathbf{r}_s$.

Comparison of Streamlines, Pathlines, and Streaklines

Feature	Streamlines	Pathlines	Streaklines
Definition	Instantaneous flow direction lines	Trajectories of individual fluid particles	Loci of fluid particles passing through a point
Dependence	Instant in time	Over time	History of the flow
Visualization	Instantaneous snapshot	Particle tracking	Tracer introduction (e.g., dye)
In Steady Flow	Coincide with Pathlines and Streaklines	Coincide with Streamlines and Streaklines	Coincide with Streamlines and Pathlines
Equations	$\frac{dy}{dx} = \frac{v}{u}$ etc.	$\mathbf{r}(t) = \mathbf{r}_0 + \int \mathbf{v}\, dt$	$\mathbf{r}(t) = \mathbf{r}_s + \int \mathbf{v}\, dt$

Steady vs. Unsteady Flow

- **Steady Flow:** Velocity field $\mathbf{v}(\mathbf{r})$ does not change with time.
  - **Streamlines = Pathlines = Streaklines**
- **Unsteady Flow:** Velocity field $\mathbf{v}(\mathbf{r}, t)$ changes with time.
  - **Streamlines, Pathlines, and Streaklines are Distinct**

Summary

- **Streamlines** provide an instantaneous view of the flow direction.
- **Pathlines** trace the actual path of individual fluid particles over time.
- **Streaklines** show the collection of fluid particles that have passed through a specific point, reflecting the history of the flow.

In **steady flows**, all three lines coincide, offering a unified representation. However, in **unsteady flows**, they diverge, each providing unique insights into the flow dynamics.