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1. Verify 2nd invariants.

$$\tau_{ii}\tau_{kk} = (\tau_{11}+\tau_{22}+\tau_{33})(\tau_{11}+\tau_{22}+\tau_{33})$$

$$= \tau_{11}^2+\tau_{22}^2+\tau_{33}^2+2(\tau_{11}\tau_{22}+\tau_{22}\tau_{33}+\tau_{33}\tau_{11})$$

$$\tau_{ik}\tau_{ki} = (\tau_{11}^2+\tau_{22}^2+\tau_{33}^2)+2(\tau_{12}^2+\tau_{23}^2+\tau_{31}^2)$$

$$\therefore \frac{1}{2}(\tau_{ii}\tau_{kk}-\tau_{ik}\tau_{ki}) = \frac{1}{2}[(\tau_{11}^2+\tau_{22}^2+\tau_{33}^2+2(\tau_{11}\tau_{22}+\tau_{22}\tau_{33}+\tau_{33}\tau_{11}))-((\tau_{11}^2+\tau_{22}^2+\tau_{33}^2)+2(\tau_{12}^2+\tau_{23}^2+\tau_{31}^2))]$$

$$= \tau_{11}\tau_{22}+\tau_{22}\tau_{33}+\tau_{33}\tau_{11}-\tau_{12}^2-\tau_{23}^2-\tau_{31}^2$$

3rd stress invariants:

$$\varepsilon_{ijk}\tau_{i1}\tau_{j2}\tau_{k3} = \tau_1\cdot(\tau_2\times\tau_3)$$

$$= \tau_{11}(\tau_{22}\tau_{33}-\tau_{23}\tau_{32})-\tau_{12}(\tau_{21}\tau_{33}-\tau_{23}\tau_{31})+\tau_{13}(\tau_{21}\tau_{32}-\tau_{22}\tau_{31})$$

$$= \tau_{11}\tau_{22}\tau_{33}+2\tau_{12}\tau_{23}\tau_{31}-\tau_{11}\tau_{23}^2-\tau_{22}\tau_{31}^2-\tau_{33}\tau_{12}^2$$

2. (1) Given: $\varepsilon_{ij} = \frac{1+\nu}{E}\tau_{ij} - \frac{\nu}{E}\delta_{ij}\Theta$

when $i=j$: $\varepsilon_{ii} = \frac{1+\nu}{E}\tau_{ii} - 3\frac{\nu}{E}\Theta$ $(\delta_{ii}=3)$

$\because K = \frac{E}{3(1-2\nu)} \Rightarrow E = 3K(1-2\nu)$

$\therefore \varepsilon_{ii} = \frac{1+\nu}{3K(1-2\nu)}\tau_{ii} - \frac{3\nu}{3K(1-2\nu)}\Theta = \frac{1}{K}\frac{\tau_{ii}}{3} = \frac{1}{K}\frac{\Theta}{3}$

(2) Given: $\tau_{ij} = 2G\varepsilon_{ij} + \lambda\varepsilon\delta_{ij}$

$\because G = \frac{E}{2(1+\nu)}$, $\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}$

$\therefore \tau_{ij} = \frac{E}{1+\nu}\varepsilon_{ij} + \frac{\nu E}{(1+\nu)(1-2\nu)}\varepsilon_{kk}\delta_{ij}$

3. substitute $\tau_{ij} = \delta_{ij}\lambda e + 2Ge_{ij}$ into $\tau_{ik,i} + f_k = 0$

$$(\delta_{ik}\lambda e + 2Ge_{ik})_{,j} + f_k = 0$$

$$(\delta_{ik}\lambda e + 2Ge_{ik})_{,j} = \delta_{ik}\lambda e_{,j} + G\left(\frac{\partial^2 u_i}{\partial x_j \partial x_k} + \frac{\partial^2 u_k}{\partial x_j \partial x_i}\right)$$

where $e = \varepsilon_{kk}$

$$\varepsilon_{kk} = \frac{1}{2}\left(\frac{\partial u_k}{\partial x_k} + \frac{\partial u_k}{\partial x_k}\right) = \frac{\partial u_k}{\partial x_k}$$

so, we can get : $e_{,j} = \dfrac{\partial^2 u_k}{\partial x_j \partial x_k}$

∴ substitute $e_{,j}$ into above equation

$$\underline{\delta_{ik}\lambda \frac{\partial^2 u_k}{\partial x_j \partial x_k} + G\left(\frac{\partial^2 u_i}{\partial x_j \partial x_k} + \frac{\partial^2 u_k}{\partial x_j \partial x_i}\right) + f_k = 0} \quad (\delta_{ik} \Rightarrow i=k)$$

$$\Downarrow$$

$$\cancel{(\lambda+G)\frac{\partial^2 u_i}{\partial x_j \partial x_k}} \quad (\lambda+G)\frac{\partial^2 u_i}{\partial x_j \partial x_i} + G\frac{\partial^2 u_k}{\partial x_j \partial x_k} + f_k = 0$$

$$\because \frac{\partial^2 u_k}{\partial x_j \partial x_k} = \nabla^2 u_k$$

$$\therefore (\lambda+G)u_{i,jk} + G\nabla^2 u_k + f_k = 0$$

4. (1) $v_m = \varepsilon_{mik}\frac{1}{2}\left(\frac{\partial u_k}{\partial x_i} - \frac{\partial u_i}{\partial x_k}\right)$

$$= \frac{1}{2}\varepsilon_{mik}\frac{\partial u_k}{\partial x_i} - \frac{1}{2}\varepsilon_{mik}\frac{\partial u_i}{\partial x_k}$$

$$= \frac{1}{2}\varepsilon_{mik}\frac{\partial u_k}{\partial x_i} + \frac{1}{2}\varepsilon_{mki}\frac{\partial u_k}{\partial x_i}$$

$\because \varepsilon_{ijk}\varepsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$

∴ Simplify above equation: $v_m = \frac{1}{2}\varepsilon_{mik}\frac{\partial u_k}{\partial x_i} = \frac{1}{2}\varepsilon_{mik}\nabla_j u_k.$

(2) $du_R = w_{ij} dx_j$

$w_{ij} = \varepsilon_{ikm} w_{km}$

$\because$ ~~w~~ w_{ik} is an antisymmetric,

so $du_R = \varepsilon_{ikm} w_{km} dx_j$

$= \frac{1}{2} \varepsilon_{ikm} \left(\frac{\partial u_m}{\partial x_k} - \frac{\partial u_k}{\partial x_m} \right) dx_j$

$= w_m \times dx.$