Homework 2

Due Oct 14 2021

1. Given a Cartesian basis, vectors and tensors can be completely represented by their components. State the components of the following quantities in terms of the components of $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{A}$, and $\boldsymbol{B}$.

   (a) $\boldsymbol{u} \otimes \boldsymbol{v}$;

   (b) $\boldsymbol{v} = \boldsymbol{A}\boldsymbol{u}$;

   (c) $\boldsymbol{A}^T$;

   (d) $\mathrm{tr}\boldsymbol{A}$;

   (e) $\boldsymbol{A}\boldsymbol{B}$;

   (f) $\boldsymbol{A}^T\boldsymbol{B}$;

   (g) $\boldsymbol{A} : \boldsymbol{B}$;

2. Given a matrix with components A_{ij}, verify that its determinant can be represented as $\varepsilon_{ijk} A_{1i} A_{2j} A_{3k}$.

3. Given an orthogonal tensor $\boldsymbol{Q}$, whose components in the Cartesian basis $\{\boldsymbol{e}_i\}$ is Q_{ij}. A new set of basis $\{\hat{\boldsymbol{e}}_i\}$ can be established by rotating $\{\boldsymbol{e}_i\}$ using $\boldsymbol{Q}$, that is

$$\hat{\boldsymbol{e}}_i = \boldsymbol{Q}\boldsymbol{e}_i.$$

   What are the components of $\boldsymbol{Q}$ in the coordinate frame $\{\hat{\boldsymbol{e}}_i\}$?

4. Consider the scalar field $\varphi(\boldsymbol{x}) = x_1^2 x_3 + x_2 x_3^2$ and the vector field $\boldsymbol{v}(\boldsymbol{x}) = x_3 \boldsymbol{e}_1 + x_2 \sin(x_1) \boldsymbol{e}_3$. Find the component of $\nabla\varphi(\boldsymbol{x})$, $\nabla\boldsymbol{v}(\boldsymbol{x})$, $\nabla \cdot \boldsymbol{v}(\boldsymbol{x})$, and $\mathrm{curl}\boldsymbol{v}(\boldsymbol{x})$.

5. Let $\{\boldsymbol{e}_i\}$ and $\{\hat{\boldsymbol{e}}_i\}$ be two coordinate frames and are related by a time-evolving orthogonal tensor $\boldsymbol{Q}(t)$,

$$\hat{\boldsymbol{e}}_i = \boldsymbol{Q}(t)\boldsymbol{e}_i.$$

   (a) Show that the time derivative $d\hat{\boldsymbol{e}}_i/dt$, as measured by an observer in the fixed frame $\{\boldsymbol{e}_i\}$, may be expressed as

$$\frac{d\hat{\boldsymbol{e}}_i}{dt} = \boldsymbol{\Omega}\hat{\boldsymbol{e}}_i = \boldsymbol{\omega} \times \hat{\boldsymbol{e}}_i,$$

   where $\boldsymbol{\Omega}(t)$ is a skew tensor defined by

$$\frac{d\boldsymbol{Q}}{dt} = \boldsymbol{\Omega}\boldsymbol{Q},$$

   and $\boldsymbol{\omega}$ is the axial vector of $\boldsymbol{\Omega}$.

   (b) For any vector $\boldsymbol{v} = v_i \boldsymbol{e}_i = \hat{v}_i \hat{\boldsymbol{e}}_i$, show that

$$\frac{d\hat{v}_i}{dt} = Q_{ji} \left[\frac{dv_j}{dt} - \Omega_{jk} v_k \right],$$

   where Q_{ij} and Ω_{jk} are the components of $\boldsymbol{Q}$ and $\boldsymbol{\Omega}$ in the fixed frame $\{\boldsymbol{e}_i\}$.

(c) Consider a tensor $\boldsymbol{S} = S_{ij}\boldsymbol{e}_i \otimes \boldsymbol{e}_j = \hat{S}_{ij}\hat{\boldsymbol{e}}_i \otimes \hat{\boldsymbol{e}}_j$. Show that

$$\frac{d\hat{S}_{ij}}{dt} = Q_{ki}Q_{lj}\left[\frac{dS_{kl}}{dt} - \Omega_{km}S_{ml} - \Omega_{lm}S_{km}\right],$$

where Q_{ij} and Ω_{jk} are the components of $\boldsymbol{Q}$ and $\boldsymbol{\Omega}$ in the fixed frame $\{\boldsymbol{e}_i\}$.