Continuum Mechanics (B)

Session 03: Stress Strain Relations

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Generalized Hooke's law

Generalized Hooke's law for 3D elastic solids

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} \\ c_{21} & c_{22} & c_{23} & c_{24} & c_{25} & c_{26} \\ c_{31} & c_{32} & c_{33} & c_{34} & c_{35} & c_{36} \\ c_{41} & c_{42} & c_{43} & c_{44} & c_{45} & c_{46} \\ c_{51} & c_{52} & c_{53} & c_{54} & c_{55} & c_{56} \\ c_{61} & c_{62} & c_{63} & c_{64} & c_{65} & c_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{bmatrix}$$

- Matrix C may vary spatially for arbitrary 3D solid
- Matrix C is symmetric and has 21 independent coefficients for fully anisotropic 3D solid

Homogeneous isotropic linear elastic solid assumption (均匀各向同性线弹性固体假设)

- Isotropic material
  - the elastic properties are the same in any direction at a point.
  - the 21 elastic constants can be reduced to 2 independent constants
- Homogeneous material
  - material properties independent of position
    - c_{ij} are thus constants (elastic constants, 弹性常数)

Generalized Hooke's law——shear stress and shear strain

Deduce the Generalized Hooke's law for isotropic elastic solid

- use experimental evidence and strain superposition principle
  - experimental observations:
    - Normal stress does not produce shear strain
    - Shear stress does not cause normal strain
    - A shear stress component only cause one shear strain component (e.g., $\tau_{xy} \rightarrow \gamma_{xy}$)

shear strain due to shear stress

$$\gamma_{xy} = \frac{1}{G}\tau_{xy}$$

$$\gamma_{yz} = \frac{1}{G}\tau_{yz}$$

$$\gamma_{zx} = \frac{1}{G}\tau_{zx}$$

G: shear modulus, or modulus of rigidity (Unit: Pa or GPa)

μ is also often used

Generalized Hooke's law——normal stress and normal strain

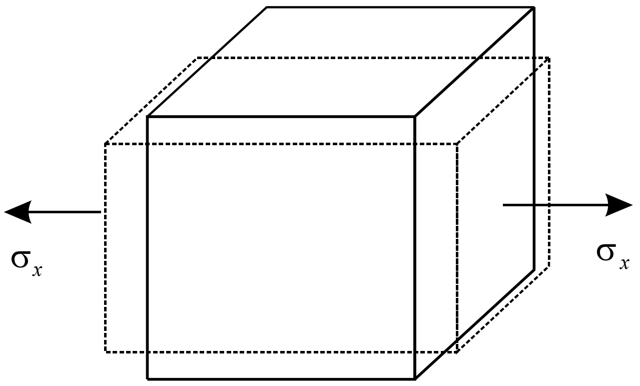
- Consider an element under uniaxial normal stress (单轴正应力) σ_x
  - The normal strain is proportional to the normal stress

$$\sigma_x = E\varepsilon_x$$

  - *E* is Young's modulus (unit: Pa or GPa)
- There are usually contractions in the y and z directions
  - ε_y and ε_z equal and are proportional to ε_x

$$\varepsilon_y = \varepsilon_z = -\nu\varepsilon_x = -\nu(\sigma_x / E)$$

  - ν is a constant called the Poisson's ratio
    - the Unit of ν
  - The Poisson's ratio must be $-1.0 \le \nu \le 0.5$, and is usually larger than zero ($0.0 \le \nu \le 0.5$)
    - Some materials, e.g. some polymer foams exhibit negative Poisson's ratio



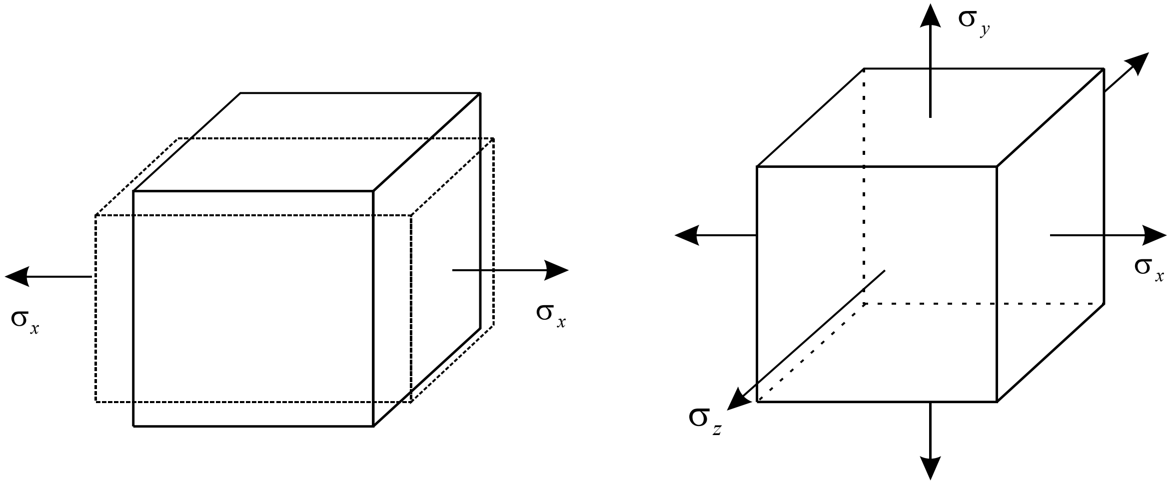
Material	Poisson's ratio
Rubber	0.4999
Gold	0.42–0.44
Rock	0.15–0.40
Cork	0.0

Generalized Hooke's law——normal stress and normal strain in 3D

For an element subjected to triaxial stress, calculate the strain ε_x

ε_x incorporates the contribution of all normal stresses:

- The contribution of σ_x: σ_x/E
- The contribution of σ_y: $-\nu\sigma_y/E$
- The contribution of σ_z: $-\nu\sigma_z/E$



Element under Triaxial stress

Based on the superposition principle of strain, the total normal strain along the x direction is

$$\varepsilon_x = \frac{1}{E}\left(\sigma_x - \nu\left(\sigma_y + \sigma_z\right)\right)$$

Similarly, normal strain along y and z directions are

$$\varepsilon_y = \frac{1}{E}\left(\sigma_y - \nu\left(\sigma_x + \sigma_z\right)\right)$$

$$\varepsilon_z = \frac{1}{E}\left(\sigma_z - \nu\left(\sigma_x + \sigma_y\right)\right)$$

Bulk Modulus of Elasticity——physical meanings of elastic modulus

stress expressed by strain

$$\begin{cases} \sigma_x = 2G\varepsilon_x + \lambda\left(\varepsilon_x + \varepsilon_y + \varepsilon_z\right) \\ \sigma_y = 2G\varepsilon_y + \lambda\left(\varepsilon_x + \varepsilon_y + \varepsilon_z\right) \\ \sigma_z = 2G\varepsilon_z + \lambda\left(\varepsilon_x + \varepsilon_y + \varepsilon_z\right) \end{cases} \qquad \begin{cases} \tau_{xy} = G\gamma_{xy} \\ \tau_{yz} = G\gamma_{yz} \\ \tau_{zx} = G\gamma_{zx} \end{cases}$$

strain expressed by stress

$$\begin{cases} \varepsilon_x = \frac{1}{E}\left(\sigma_x - \nu\left(\sigma_y + \sigma_z\right)\right) \\ \varepsilon_y = \frac{1}{E}\left(\sigma_y - \nu\left(\sigma_z + \sigma_x\right)\right) \\ \varepsilon_z = \frac{1}{E}\left(\sigma_z - \nu\left(\sigma_x + \sigma_y\right)\right) \end{cases} \qquad \begin{cases} \gamma_{xy} = \frac{1}{G}\tau_{xy} \\ \gamma_{yz} = \frac{1}{G}\tau_{yz} \\ \gamma_{zx} = \frac{1}{G}\tau_{zx} \end{cases}$$

$$\nu = \frac{\lambda}{2(\lambda + \mu)} \quad E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu} \quad \mu = \frac{E}{2(1+\nu)} \quad \lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \quad K = \frac{3\lambda + 2\mu}{3} = \frac{E}{3(1-2\nu)}$$

The physical meaning of E and ν: uniaxial stress $G=\mu$

The physical meaning of shear modulus μ (G): shear strain and shear stress

The physical meaning of bulk modulus K: average normal stress and volume change

The physical meaning of λ: uniaxial strain

Strain and Stress decomposition

Strain decomposition (应变分解): Decompose the strain state into a spherical component plus a deviatoric component.

$$\begin{bmatrix} \varepsilon_x & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_y & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_z \end{bmatrix} = \begin{bmatrix} \varepsilon_m & & \\ & \varepsilon_m & \\ & & \varepsilon_m \end{bmatrix} + \begin{bmatrix} \varepsilon_x - \varepsilon_m & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_y - \varepsilon_m & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_z - \varepsilon_m \end{bmatrix}$$

The strain state represented by a matrix

Spherical component

Deviatoric component

- The spherical component does not incorporate shear strain
- The deviatoric component does not include volume change.
- Does the deviatoric strain component incorporate normal strain?

$$\varepsilon_m = \frac{\varepsilon_x + \varepsilon_y + \varepsilon_y}{3} = \frac{\varepsilon_1 + \varepsilon_2 + \varepsilon_3}{3} = \frac{1}{3} I_1'$$

Strain and Stress decomposition

Stress decomposition (应力分解):
Similarly, decompose stress into a spherical component plus a deviatoric component.

$$\begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix} = \begin{bmatrix} \sigma_m & & \\ & \sigma_m & \\ & & \sigma_m \end{bmatrix} + \begin{bmatrix} \sigma_x - \sigma_m & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y - \sigma_m & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z - \sigma_m \end{bmatrix}$$

Stress state represented by a matrix

Spherical component

Deviatoric component

$$\sigma_m = \frac{\sigma_x + \sigma_y + \sigma_z}{3} = \frac{\sigma_1 + \sigma_2 + \sigma_3}{3} = \frac{1}{3} I_1$$

- The spherical component of stress does not induce shear strain
- The deviatoric component does not induce a volume change.

– Check that the dilatation (unit volume change) is

$\varepsilon_x + \varepsilon_y + \varepsilon_z$

$$\varepsilon = \frac{A'B - AB}{AB} = \frac{\partial u}{\partial x} \Rightarrow \Delta V \approx V \cdot (\varepsilon_x + \varepsilon_y + \varepsilon_z)$$

$$\frac{\Delta V}{V} = \varepsilon_x + \varepsilon_y + \varepsilon_z .$$

– Check that a perfectly incompressible isotropic material would have a Poisson's ratio of 0.5

$$\begin{cases} \varepsilon_x = \dfrac{\sigma_x}{E} \\ \varepsilon_y = \varepsilon_z = -\nu\varepsilon_x = -\nu(\sigma_x/E) \end{cases}$$

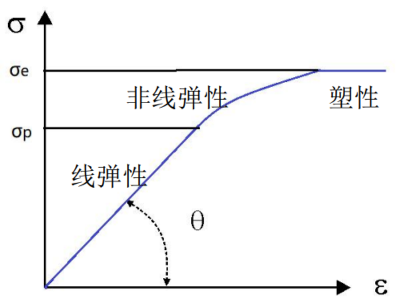
$$\frac{\Delta V}{V} = \varepsilon_x + \varepsilon_y + \varepsilon_z = \frac{\sigma_x}{E}(1-2\nu) = 0$$

$$\Rightarrow \nu = 0.5$$

– Prove that the volume either keeps constant or increases under uniaxial extension

$$\frac{\Delta V}{V} = \frac{\sigma_x}{E}(1-2\nu) \qquad 0.0 \le \nu \le 0.5$$

$$\frac{\Delta V}{V} = \begin{cases} 0 & , \nu = 0.5 \\ >0 & , \nu < 0.5 \\ <0 & , \nu > 0.5 \quad \times \text{ 超出范围.} \end{cases}$$



stress-strain curve (应力应变曲线)

$$\varepsilon = \frac{\sigma_x}{E} = \frac{200 \times 10^6}{200 \times 10^9} = 0.001$$

The Young’s modulus and elastic proportional limit (比例极限σ_p) of mild steel (生铁) is 200 GPa and 200 MPa, respectively. Calculate the maximum elastic strain in mild steel

0.001

The elastic strain we are dealing with is small.

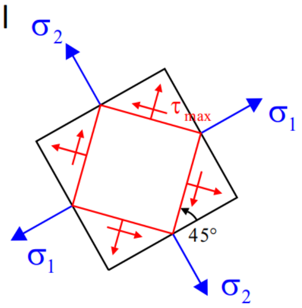
For isotropic elastic materials, the principal axes of stress and strain **coincide.**
Does the orientation of the maximum shear stress and maximum shear strain coincide?

maximum shear stress is

$$\tau_{x'y'\max} = \pm\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

Maximum shear stress

- Maximum shear stress planes are 45° apart from the two principal planes.
- Normal stresses on the maximum shear stress planes are equal



最大剪应力方向与主应力面夹角为45°.

由 Hooke's law

$$\left.\begin{aligned}\gamma_{xy} &= \frac{1}{G}\tau_{xy}\\ \gamma_{yz} &= \frac{1}{G}\tau_{yz}\\ \gamma_{zx} &= \frac{1}{G}\tau_{zx}\end{aligned}\right\} \Rightarrow$$

最大剪应力与最大剪应变都与主应力面夹角45°.

3-10 Determine the slope of the σ_x vs. ϵ_x curve in the elastic range if a material is tested under the following state of stress:

$$\sigma_x = 2\sigma_y = 3\sigma_z.$$

$$\begin{aligned}\varepsilon_x &= \frac{1}{E}\left[\sigma_x - \nu(\sigma_y + \sigma_z)\right]\\ \varepsilon_y &= \frac{1}{E}\left[\sigma_y - \nu(\sigma_x + \sigma_z)\right]\\ \varepsilon_z &= \frac{1}{E}\left[\sigma_z - \nu(\sigma_x + \sigma_y)\right]\end{aligned} \Leftarrow \left\{\begin{aligned}\sigma_y &= \frac{1}{2}\sigma_x\\ \sigma_z &= \frac{1}{3}\sigma_x\end{aligned}\right.$$

$$\varepsilon_x = \frac{1}{E}\left[\sigma_x - \nu\left(\frac{1}{2}\sigma_x + \frac{1}{3}\sigma_x\right)\right]$$

$$= \frac{1}{E}\left(1 - \frac{5}{6}\nu\right)\sigma_x$$

$$\frac{\sigma_x}{\varepsilon_x} = \frac{E}{1 - \frac{5}{6}\nu}$$

Classroom exercise

- Assume the stress components of hydrostatic stress state in x-y-z coordinate is:

$$\sigma_x = \sigma_y = \sigma_z = -p$$
$$\tau_{xy} = \tau_{yz} = \tau_{zx} = 0$$

$$\Longrightarrow$$

$$\sigma_{x'} = \sigma_x a_{11}^2 + \sigma_y a_{21}^2 + \sigma_z a_{31}^2 + 2\tau_{xy} a_{11} a_{21} + 2\tau_{yz} a_{21} a_{31} + 2\tau_{zx} a_{31} a_{11}$$

$$\tau_{x'y'} = \sigma_x a_{11} a_{12} + \sigma_y a_{21} a_{22} + \sigma_z a_{31} a_{32} + \tau_{xy}(a_{11} a_{22} + a_{21} a_{12}) + \tau_{yz}(a_{21} a_{32} + a_{31} a_{22}) + \tau_{zx}(a_{31} a_{12} + a_{11} a_{32})$$

$$\tau_{x'z'} = \sigma_x a_{11} a_{13} + \sigma_y a_{21} a_{23} + \sigma_z a_{31} a_{33} + \tau_{xy}(a_{11} a_{23} + a_{21} a_{13}) + \tau_{yz}(a_{21} a_{33} + a_{31} a_{23}) + \tau_{zx}(a_{31} a_{13} + a_{11} a_{33})$$

  - Check the normal stress and shear stress components of any coordinate system (x'-y'-z')

- Check the stress-strain relations for spherical and deviatoric stresses, respectively

$$\sigma_x' = -p(a_{11}^2 + a_{21}^2 + a_{31}^2) = -p$$

同理: $\sigma_x' = \sigma_y' = \sigma_z' = -p$.

$$\tau_{x'y'} = -p(a_{11}a_{12} + a_{21}a_{22} + a_{31}a_{32}) = 0$$

在静水应力条件下，更改坐标系不会改变应力状态。

$$\varepsilon = \frac{\Delta V}{V} = \varepsilon_x + \varepsilon_y + \varepsilon_z = \frac{1-2\nu}{E} \cdot \sigma_m = -\frac{(1-2\nu)}{E} p$$

球形应力仅影响材料的体积变化.

偏应力: deviatoric stresses:

$$\sigma_x' - p = \sigma_y' - p = \sigma_z' - p = 0$$

在静水应力状态下，无形变产生.

$$\begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix} = \begin{bmatrix} \sigma_m & & \\ & \sigma_m & \\ & & \sigma_m \end{bmatrix} + \begin{bmatrix} \sigma_x - \sigma_m & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y - \sigma_m & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z - \sigma_m \end{bmatrix}$$

Stress state represented by a matrix — Spherical component — Deviatoric component

$$\sigma_m = \frac{\sigma_x + \sigma_y + \sigma_z}{3} = \frac{\sigma_1 + \sigma_2 + \sigma_3}{3} = \frac{1}{3} I_1$$

Classroom Exercise

The three invariants of stress are the three coefficients of the eigenequation below

$$\sigma_p^3-(\sigma_x+\sigma_y+\sigma_z)\sigma_p^2+(\sigma_x\sigma_y+\sigma_y\sigma_z+\sigma_z\sigma_x-\tau_{xy}^2-\tau_{yz}^2-\tau_{zx}^2)\sigma_p-(\sigma_x\sigma_y\sigma_z+2\tau_{xy}\tau_{yz}\tau_{zx}-\sigma_x\tau_{yz}^2-\sigma_y\tau_{xz}^2-\sigma_z\tau_{xy}^2)=0$$

$$\sigma_p^3-I_1\sigma_p^2+I_2\sigma_p-I_3=0$$

$$I_1=\sigma_x+\sigma_y+\sigma_z=\sigma_1+\sigma_2+\sigma_3$$

$$I_2=\begin{vmatrix}\sigma_x & \tau_{xy}\\ \tau_{yx} & \sigma_y\end{vmatrix}+\begin{vmatrix}\sigma_y & \tau_{yz}\\ \tau_{zy} & \sigma_z\end{vmatrix}+\begin{vmatrix}\sigma_z & \tau_{zx}\\ \tau_{xz} & \sigma_x\end{vmatrix}$$

$$=\sigma_x\sigma_y+\sigma_y\sigma_z+\sigma_z\sigma_x-\tau_{xy}^2-\tau_{yz}^2-\tau_{zx}^2=\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_1$$

$$I_3=\begin{vmatrix}\sigma_x & \tau_{xy} & \tau_{xz}\\ \tau_{yx} & \sigma_y & \tau_{yz}\\ \tau_{zx} & \tau_{zy} & \sigma_z\end{vmatrix}$$

$$=\sigma_x\sigma_y\sigma_z+2\tau_{xy}\tau_{yz}\tau_{zx}-\sigma_x\tau_{yz}^2-\sigma_y\tau_{zx}^2-\sigma_z\tau_{xy}^2=\sigma_1\sigma_2\sigma_3$$

Assume the invariants J_1, J_2, and J_3 of the deviatoric stress are the coefficients of the corresponding eigenequation,

$$S_p^3-J_1S_p^2-J_2S_p-J_3=0$$

prove that J_1, J_2, and J_3 are as below

$$J_1=0$$

$$J_2=\frac{1}{2}(\sigma_x'^2+\sigma_y'^2+\sigma_z'^2+2\tau_{xy}'^2+2\tau_{yz}'^2+2\tau_{zx}'^2)$$

$$=\frac{1}{6}[(\sigma_x-\sigma_y)^2+(\sigma_y-\sigma_z)^2+(\sigma_z-\sigma_x)^2]$$

$$+\tau_{xy}^2+\tau_{yz}^2+\tau_{zx}^2$$

解: 应力分解:

$$\begin{bmatrix}\sigma_x & \tau_{xy} & \tau_{xz}\\ \tau_{yx} & \sigma_y & \tau_{yz}\\ \tau_{zx} & \tau_{zy} & \sigma_z\end{bmatrix}=\begin{bmatrix}\sigma_m & & \\ & \sigma_m & \\ & & \sigma_m\end{bmatrix}+\begin{bmatrix}\sigma_x-\sigma_m & \tau_{xy} & \tau_{xz}\\ \tau_{yx} & \sigma_y-\sigma_m & \tau_{yz}\\ \tau_{zx} & \tau_{zy} & \sigma_z-\sigma_m\end{bmatrix}$$

Stress state represented by a matrix

Spherical component

Deviatoric component

$$\sigma_m=\frac{\sigma_x+\sigma_y+\sigma_z}{3}=\frac{\sigma_1+\sigma_2+\sigma_3}{3}=\frac{1}{3}I_1$$

$$S=\sigma-\sigma_m I$$

$S_x=\sigma_x-\sigma_m$, $S_y=\sigma_y-\sigma_m$, $S_z=\sigma_z-\sigma_m$.

$J_1=S_x+S_y+S_z=\sigma_x+\sigma_y+\sigma_z-3\sigma_m=0.$

代入.

$$J_2=\begin{vmatrix}S_y & \tau_{yz}\\ \tau_{yz} & S_z\end{vmatrix}+\begin{vmatrix}S_x & \tau_{zx}\\ \tau_{zx} & S_z\end{vmatrix}+\begin{vmatrix}S_x & \tau_{xy}\\ \tau_{xy} & S_y\end{vmatrix}$$

$$=S_yS_z-\tau_{yz}^2+S_xS_z-\tau_{zx}^2+S_xS_y-\tau_{xy}^2-\frac{1}{2}(J_1)^2$$

$$=\frac{1}{2}(S_x^2+S_y^2+S_z^2)-\tau_{yz}^2-\tau_{zx}^2-\tau_{xy}^2.$$

$$S_yS_z=(\sigma_y-\sigma_m)(\sigma_z-\sigma_m)=\left(\sigma_y-\frac{\sigma_x+\sigma_y+\sigma_z}{3}\right)\left(\sigma_z-\frac{\sigma_x+\sigma_y+\sigma_z}{3}\right)$$

$$=\sigma_y\sigma_z-\frac{\sigma_y(\sigma_x+\sigma_y+\sigma_z)}{3}-\frac{\sigma_z(\sigma_x+\sigma_y+\sigma_z)}{3}+\frac{(\sigma_x+\sigma_y+\sigma_z)^2}{9}$$

同理: $$S_xS_z=\sigma_x\sigma_z-\frac{\sigma_x(\sigma_x+\sigma_y+\sigma_z)}{3}-\frac{\sigma_z(\sigma_x+\sigma_y+\sigma_z)}{3}+\frac{(\sigma_x+\sigma_y+\sigma_z)^2}{9}$$

$$S_xS_y=\sigma_x\sigma_y-\frac{\sigma_x(\sigma_x+\sigma_y+\sigma_z)}{3}-\frac{\sigma_y(\sigma_x+\sigma_y+\sigma_z)}{3}+\frac{(\sigma_x+\sigma_y+\sigma_z)^2}{9}$$

代入 J_2:

$$=\sigma_y\sigma_z+\sigma_x\sigma_z+\sigma_x\sigma_y-\frac{2\sigma_y(\sigma_x+\sigma_y+\sigma_z)}{3}-\frac{2\sigma_z(\sigma_x+\sigma_y+\sigma_z)}{3}-\frac{2\sigma_x(\sigma_x+\sigma_y+\sigma_z)}{3}$$

$$=\sigma_y\sigma_z+\sigma_x\sigma_z+\sigma_x\sigma_y-\frac{2(\sigma_x+\sigma_y+\sigma_z)^2}{3}+\frac{(\sigma_x+\sigma_y+\sigma_z)^2}{3}$$

$$=\sigma_y\sigma_z+\sigma_x\sigma_z+\sigma_x\sigma_y-\frac{1}{3}\left[\sigma_x^2+\sigma_x\sigma_y+\sigma_x\sigma_z+\sigma_y\sigma_x+\sigma_y^2+\sigma_y\sigma_z+\sigma_z\sigma_x+\sigma_z\sigma_y+\sigma_z^2\right]$$

$$=\frac{1}{3}\sigma_x\sigma_y+\frac{1}{3}\sigma_y\sigma_z+\frac{1}{3}\sigma_x\sigma_z-\frac{1}{3}\sigma_x^2-\frac{1}{3}\sigma_y^2-\frac{1}{3}\sigma_z^2$$

$$=-\frac{1}{6}\left[(\sigma_x-\sigma_y)^2+(\sigma_y-\sigma_z)^2+(\sigma_x-\sigma_z)^2\right]$$

$$-\tau_{xy}^2-\tau_{yz}^2-\tau_{xz}^2$$

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Homework 3 (5 points)

3-2 If a medium is initially unstrained and is then subjected to a constant positive temperature change, the normal strains are expressed by

$$\epsilon_x = \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)] + \alpha T$$

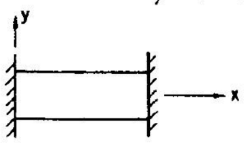
$$\cdots$$

$$\cdots$$

where α is the coefficient of linear expansion and T is the temperature rise. The temperature change does not affect the shear strain components.

A bar restrained in the x direction only, and free to expand in the y and z directions as shown, is subjected to a uniform temperature rise T. Show that the only nonvanishing stress (the bar is in a state of uniform stress) and strain components are

$$\sigma_x = -E\alpha T$$
$$\epsilon_y = \epsilon_z = \alpha T(1+\nu).$$



解：

已知：

$$\varepsilon_x = \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)] + \alpha T \qquad \varepsilon_x = 0 \quad (\text{restrained})$$

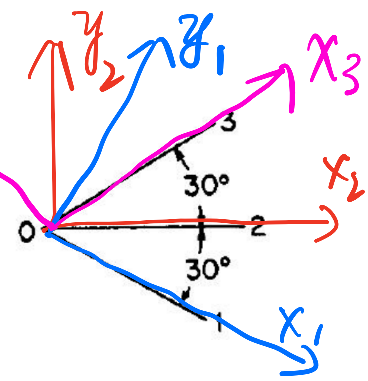
$$\varepsilon_y = \frac{1}{E}[\sigma_y - \nu(\sigma_x + \sigma_z)] + \alpha T \qquad \sigma_y = \sigma_z = 0. \quad (\text{free})$$

$$\varepsilon_z = \frac{1}{E}[\sigma_z - \nu(\sigma_x + \sigma_y)] + \alpha T.$$

$$\varepsilon_x = \frac{1}{E}[\sigma_x - 0] + \alpha T = 0 \Rightarrow \sigma_x = -E\alpha T$$

$$\varepsilon_y = \varepsilon_z = \frac{1}{E}[-\nu\sigma_x] + \alpha T = \nu\alpha T + \alpha T = (1+\nu)\alpha T.$$

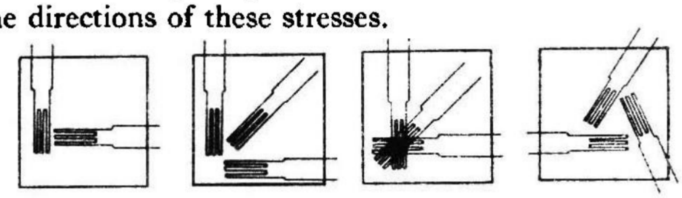
Homework 3 (5 points)



3-1 One of the applications of generalized Hooke's law is found in the use of rosette gages in the field of experimental stress analysis. The state of strain at a point is determined experimentally by determining the rosette gage readings, which give the normal strain in three directions in a plane. Since rosette gages are applied to a free surface, the stress components σ_z, τ_{zx}, and τ_{zy} are zero, where the z axis is normal to the free surface. By using generalized Hooke's law, then, one can define the state of stress.

By using the rosette gage shown in the figure in one experiment, the following strains are recorded on the surface of a steel bar: $\epsilon_{0-1} = +10^{-4}$, $\epsilon_{0-2} = +4 \times 10^{-4}$, $\epsilon_{0-3} = +6 \times 10^{-4}$. Given the material properties of this steel, $E = 30 \times 10^6$ psi, $\nu = 0.25$, determine the principal stresses and maximum shear stress at point O. Give the directions of these stresses.

Rosette strain gage (花形应变计，应变花): A single strain gauge (应变仪) can only measure strain in one direction. Rosette strain gauge (gage) is an arrangement of two or more strain gauges that are positioned closely to measure strains along different directions.



a 二轴90° 应变花　c 三轴45° 应变花　d 三轴45° 应变花　e 三轴60° 应变花

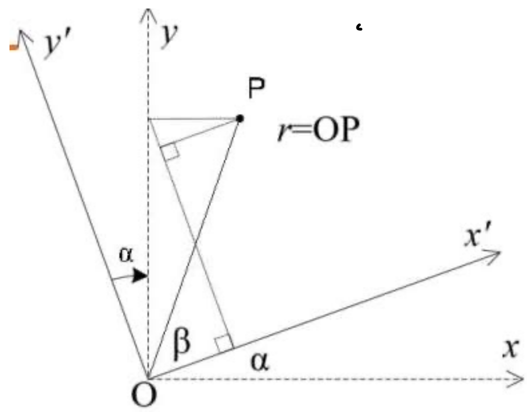
解：

2D strain transformation

$$\epsilon_{x'} = \frac{\epsilon_x + \epsilon_y}{2} + \frac{\epsilon_x - \epsilon_y}{2}\cos 2\alpha + \frac{\gamma_{xy}}{2}\sin 2\alpha$$

$$\epsilon_{y'} = \frac{\epsilon_x + \epsilon_y}{2} - \frac{\epsilon_x - \epsilon_y}{2}\cos 2\alpha - \frac{\gamma_{xy}}{2}\sin 2\alpha$$

$$\gamma_{x'y'} = (\epsilon_y - \epsilon_x)\sin 2\alpha + \gamma_{xy}\cos 2\alpha$$



对于 X_2OY_2 坐标系：

$\varepsilon_x = \varepsilon_{0-2} = 4\times 10^{-4}$, $\varepsilon_y, \gamma_{xy}$ 未知. ①

对于 X_3OY_3 坐标系：$\alpha = 30°$

$$\varepsilon_{0-3} = \varepsilon_{x'} = \frac{\varepsilon_x+\varepsilon_y}{2} + \frac{\varepsilon_x-\varepsilon_y}{2}\cdot\frac{1}{2} + \frac{\gamma_{xy}}{2}\cdot\frac{\sqrt{3}}{2} = 6\times 10^{-4} \quad ②$$

对于 X_1OY_1 坐标系：$\alpha = -30°$

$$\varepsilon_{0-1} = \varepsilon_{x'} = \frac{\varepsilon_x+\varepsilon_y}{2} + \frac{\varepsilon_x-\varepsilon_y}{2}\cdot\frac{1}{2} - \frac{\gamma_{xy}}{2}\cdot\frac{\sqrt{3}}{2} = 10^{-4} \quad ③$$

联立 ①, ②, ③ 可得 $\varepsilon_y, \gamma_{xy}$.

② − ③: $\gamma_{xy}\cdot\frac{\sqrt{3}}{2} = 5\times 10^{-4} \Rightarrow \gamma_{xy} = \frac{10^{-3}}{\sqrt{3}}$

② + ③: $\varepsilon_x + \varepsilon_y + \frac{\varepsilon_x - \varepsilon_y}{2} = 7\times 10^{-4} \Rightarrow \varepsilon_y = 2\times 10^{-4}$

⊕ Hooke's Law.

strain expressed by stress

$$\begin{cases} \varepsilon_x = \dfrac{1}{E}\left(\sigma_x - \nu\left(\sigma_y + \sigma_z\right)\right) \\ \varepsilon_y = \dfrac{1}{E}\left(\sigma_y - \nu\left(\sigma_z + \sigma_x\right)\right) \\ \varepsilon_z = \dfrac{1}{E}\left(\sigma_z - \nu\left(\sigma_x + \sigma_y\right)\right) \end{cases} \qquad \begin{cases} \gamma_{xy} = \dfrac{1}{G}\tau_{xy} \\ \gamma_{yz} = \dfrac{1}{G}\tau_{yz} \\ \gamma_{zx} = \dfrac{1}{G}\tau_{zx} \end{cases}$$

stress expressed by strain

$$\begin{cases} \sigma_x = 2G\varepsilon_x + \lambda\left(\varepsilon_x + \varepsilon_y + \varepsilon_z\right) \\ \sigma_y = 2G\varepsilon_y + \lambda\left(\varepsilon_x + \varepsilon_y + \varepsilon_z\right) \\ \sigma_z = 2G\varepsilon_z + \lambda\left(\varepsilon_x + \varepsilon_y + \varepsilon_z\right) \end{cases} \qquad \begin{cases} \tau_{xy} = G\gamma_{xy} \\ \tau_{yz} = G\gamma_{yz} \\ \tau_{zx} = G\gamma_{zx} \end{cases}$$

$$E = 30\times10^6 \quad \nu = 0.25 \quad G = \frac{E}{2(1+\nu)} = 1.2\times10^7 \quad \sigma_z = 0$$

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} = 12\times10^6 \quad \varepsilon_z = 0$$

$$\left.\begin{matrix} E\varepsilon_x = \sigma_x - \nu\sigma_y \\ E\varepsilon_y = \sigma_y - \nu\sigma_x \end{matrix}\right\} \Rightarrow \begin{cases} \sigma_x = 1.44\times10^4 \text{ psi} \\ \sigma_y = 0.96\times10^4 \text{ psi} \end{cases}$$

$$\tau_{xy} = G\gamma_{xy} = 1.2\times10^7 \cdot \frac{10^{-3}}{\sqrt{3}} = 6.928\times10^3 \text{ psi}$$

Maximum and minimum principal stresses.

$\sigma_1 = \sigma_{max}$, $\sigma_2 = \sigma_{min}$, (by convention, $\sigma_1 > \sigma_2$)

$$\tan 2\alpha = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}\left(\alpha \in [0,\pi]\right)$$

$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\alpha + \tau_{xy}\sin 2\alpha$$

$$\sin 2\alpha = \pm\frac{2\tau_{xy}}{\sqrt{4\tau_{xy}^2 + \left(\sigma_x - \sigma_y\right)^2}}$$

$$\cos 2\alpha = \pm\frac{\sigma_x - \sigma_y}{\sqrt{4\tau_{xy}^2 + \left(\sigma_x - \sigma_y\right)^2}}$$

Signs assigned for sin2α and cos2α are either both positive or both negative.

➡ $$\sigma_{max} = \sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \qquad \sigma_{min} = \sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\tan 2\theta = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} = \frac{2\times 6.328\times 10^3}{1.44\times10^4 - 0.36\times10^4} = 2.845$$

$$\Rightarrow 2\theta = \arctan(2.845) \Rightarrow \theta_1 = 35.45^\circ \quad \theta_2 = 125.45^\circ$$

代入 $\sigma_{x'} = \dfrac{\sigma_x+\sigma_y}{2} + \dfrac{\sigma_x-\sigma_y}{2}\cos 2\theta + \tau_{xy}\cdot\sin 2\theta$

当 $\theta_1 = 35.45^\circ$ 时:

$$\sigma_{x'_1} = \frac{1.44\times10^4 + 0.36\times10^4}{2} + \frac{1.44\times10^4 - 0.36\times10^4}{2}\cdot 0.3272 + 6.328\times10^3\cdot 0.9449$$

$$= 1.2\times10^4 + 0.07853\times10^4 + 0.6546\times10^4$$

$$= 1.933\times10^4 \text{ psi}$$

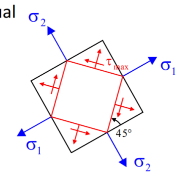
当 $\theta_2 = 125.45^\circ$ 时. $\sigma_{x'_2} = 0.4668\times10^4$ psi

因此: $\theta_1 = 35.45^\circ$ 为最大主应力轴, 对应 $\sigma_{max} = 1.933$ psi

$\theta_2 = 125.45^\circ$ 为最小主应力轴, 对应 $\sigma_{min} = 0.4668$ psi

Maximum shear stress

- Maximum shear stress planes are 45° apart from the two principal planes.
- Normal stresses on the maximum shear stress planes are equal



maximum shear stress is

$$\tau_{x'y'\max} = \pm\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\tau_{xy\max} = \pm\sqrt{\left(\frac{1.44\times10^4 - 0.36\times10^4}{2}\right)^2 + (6.328\times10^3)^2}$$

$$= \pm 7.332\times10^3 \text{ psi}.$$

对于 $\tau_{xy\max}$, $\theta_1 = 80.45^\circ$ 或 $\theta_2 = 170.45^\circ$

3-3 Determine the stress and strain components if the bar in the preceding (3.2 in CE) problem is restrained in the x and y directions but is free to expand in the z direction.

解: $\varepsilon_x = \frac{1}{E}(\sigma_x - \nu(\sigma_y + \sigma_z)) + \alpha T$

$\varepsilon_y = \frac{1}{E}(\sigma_y - \nu(\sigma_x + \sigma_z)) + \alpha T$

$\varepsilon_z = \frac{1}{E}(\sigma_z - \nu(\sigma_x + \sigma_y)) + \alpha T$

$\varepsilon_x = \varepsilon_y = 0$

$\sigma_z = 0$

$$\begin{cases} \frac{1}{E}(\sigma_x - \nu\sigma_y) + \alpha T = 0 \\ \frac{1}{E}(\sigma_y - \nu\sigma_x) + \alpha T = 0 \end{cases} \qquad \begin{aligned} \sigma_x - \nu\sigma_y &= -\alpha E T \\ \sigma_y - \nu\sigma_x &= -\alpha E T \end{aligned}$$

$\sigma_x + \sigma_y - \nu(\sigma_x + \sigma_y) = -2\alpha E T \quad (1-\nu)(\sigma_x + \sigma_y) = -2\alpha E T$ ①

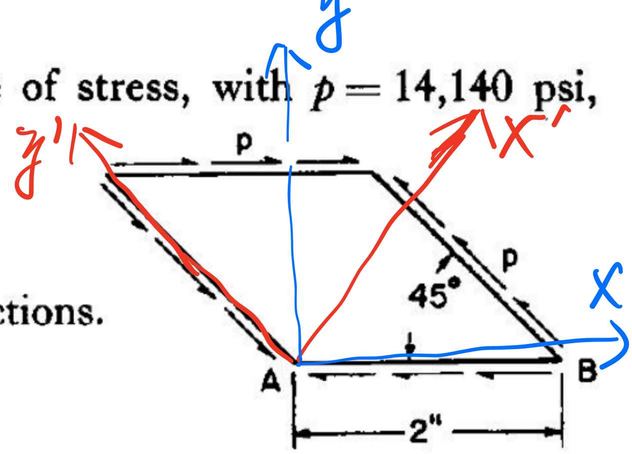
$\sigma_x - \sigma_y - \nu\sigma_y + \nu\sigma_x = 0 \quad (1+\nu)\sigma_x = \sigma_y(1+\nu) \Rightarrow \sigma_x = \sigma_y$ ②

$$\Rightarrow \sigma_x = \sigma_y = \frac{-\alpha E T}{1-\nu}$$

$$\varepsilon_z = \frac{1}{E}\left(-\nu \cdot \frac{-2\alpha E T}{1-\nu}\right) + \alpha T = \frac{2\alpha\nu T}{1-\nu} + \frac{(1-\nu)\alpha T}{1-\nu}$$

$$= \frac{1+\nu}{1-\nu}\alpha T$$

3-9 A thin plate is under a uniform state of stress, with $p = 14{,}140$ psi, $E = 30 \times 10^6$ psi, and $\nu = 0.3$, as shown.



(a) Find the change in length of AB.
(b) Find the principal strains and their directions.

解:

$$\begin{cases} \sigma_{x'} = \dfrac{\sigma_x+\sigma_y}{2} + \dfrac{\sigma_x-\sigma_y}{2}\cos 2\alpha + \tau_{xy}\sin 2\alpha \\ \sigma_{y'} = \dfrac{\sigma_x+\sigma_y}{2} - \dfrac{\sigma_x-\sigma_y}{2}\cos 2\alpha - \tau_{xy}\sin 2\alpha \\ \tau_{x'y'} = -\dfrac{\sigma_x-\sigma_y}{2}\sin 2\alpha + \tau_{xy}\cos 2\alpha \end{cases}$$

(a)

已知: $\sigma_y = 0,\ \sigma_{x'} = 0,\ \tau_{xy} = P = 14140\,\text{psi}$

$$\sigma_{x'} = \frac{\sigma_x}{2} + \frac{\sigma_x}{2}\cos 90^\circ + 14140 \cdot \sin 90^\circ = 0$$

$$\sigma_x = -28280\,\text{psi}$$

$$\varepsilon_x = \sigma_x / E = \frac{1}{30\times10^6}\cdot(-28280) = -0.0009427$$

$$\Delta L_{AB} = L_{AB}\cdot\varepsilon_x = -0.0018853''$$

(b)

Show that the principal strain direction for 2D are given by

$$\tan 2\alpha = \frac{\gamma_{xy}}{\epsilon_x - \epsilon_y}$$

and the magnitudes of the principal strains are

$$\left.\begin{matrix}\epsilon_1\\ \epsilon_2\end{matrix}\right\} = \frac{\epsilon_x+\epsilon_y}{2} \pm \tfrac{1}{2}\sqrt{(\epsilon_x-\epsilon_y)^2+\gamma_{xy}^2}$$

$$\varepsilon_y = -\nu\varepsilon_x = 0.3\cdot 0.0009427 = 0.00028281$$

$$\gamma_{xy} = \frac{1}{G}\tau_{xy} = \frac{2(1+\nu)}{E}\tau_{xy} = \frac{2(1+0.3)}{30\times10^6}\cdot 14140 = 0.00122547$$

$$\left.\begin{matrix}\varepsilon_1\\ \varepsilon_2\end{matrix}\right\} = \frac{-0.0009427+0.00028281}{2} \pm \frac{1}{2}\sqrt{(-0.0009427-0.00028281)^2 + 0.00122547^2}$$

$$\varepsilon_1 = 0.0005366 \qquad \varepsilon_2 = -0.001196$$

$$\tan 2\alpha = \frac{\gamma_{xy}}{\varepsilon_x - \varepsilon_y} = \frac{0.00122547}{-0.0009427 - 0.0002828|} = -1$$

$$\alpha_1 = 67.5 \quad \alpha_2 = 157.5.$$

3-11 Given $G = \dfrac{E}{2(1+\nu)}$ $\lambda = \dfrac{E\nu}{(1+\nu)(1-2\nu)}$ $K = \dfrac{E}{3(1-2\nu)}$

Prove the following relations among various elastic constants:

$$\nu = \frac{3K-E}{6K} \qquad \lambda = \frac{3K-2G}{3} \qquad E = \frac{\lambda(1+\nu)(1-2\nu)}{\nu}$$

① 由 $K = \dfrac{E}{3(1-2\nu)}$ $\quad 3K - 6K\nu = E \Rightarrow \nu = \dfrac{3K-E}{6K}$

②

$$\left.\begin{array}{l} G = \dfrac{E}{2(1+\nu)} \\ K = \dfrac{E}{3(1-2\nu)} \end{array}\right\} \Rightarrow \frac{3K-2G}{3} = \frac{\dfrac{E}{(1-2\nu)} - \dfrac{E}{1+\nu}}{3}$$

$$= \frac{\dfrac{E(1+\nu)}{(1-2\nu)(1+\nu)} - \dfrac{E(1-2\nu)}{(1-2\nu)(1+\nu)}}{3}$$

$$= \frac{3E\nu}{3(1-2\nu)(1+\nu)} = \frac{E\nu}{(1-2\nu)(1+\nu)}$$

得证.

③ $\lambda = \dfrac{E\nu}{(1-2\nu)(1+\nu)} \Rightarrow E = \dfrac{\lambda(1-2\nu)(1+\nu)}{\nu}$