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1. (a) $\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\left(\frac{\partial^2\phi}{\partial r^2}+\frac{1}{r}\frac{\partial \phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\phi}{\partial \theta^2}\right)=0.$

(b) $\Delta\phi=\frac{\partial^2\phi}{\partial r^2}+\frac{1}{r}\frac{\partial \phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\phi}{\partial \theta^2}=(4\ln r+4)A+4B.$

$\Delta^2\phi=\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\cdot\frac{\partial^2}{\partial \theta^2}\right)(\Delta\phi)=0$

which satisfies bi-harmonic equation.

(c) $\sigma_{rr}=\frac{1}{r}\frac{\partial \phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\phi}{\partial \theta^2}=(2\ln r+1)A+2B+\frac{C}{r^2}$

$\sigma_{\theta\theta}=\frac{\partial^2\phi}{\partial r^2}=(2\ln r+3)A+2B-\frac{C}{r^2}$

$\sigma_{r\theta}=-\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \phi}{\partial \theta}\right)=0.$

(d) $\varepsilon_r=\frac{du_r}{dr}=k_1(\sigma_r-k_2\sigma_\theta)$

$=k_1\left[2A\ln r+\frac{C}{r^2}+2B-k_2\left(2A\ln r-\frac{C}{r^2}+3A+2B\right)\right]$

$\varepsilon_\theta=\frac{u_r}{r}=k_1(\sigma_\theta-k_2\sigma_r)$

$=k_1\left[2A\ln r-\frac{C}{r^2}+3A+2B-k_2\left(2A\ln r+\frac{C}{r^2}+A+2B\right)\right]$

In plane stress: $k_1=\frac{1}{E}$, $k_2=\nu$

In plane strain: $k_1=\frac{1-\nu^2}{E}$, $k_2=\frac{\nu}{1-\nu}$.

$4Ar-F=0.$

The equation satisfied for $A=F=0$.

So, base on (c) we know $\sigma_{rr}=2B+\frac{C}{r^2}$, $\sigma_{\theta\theta}=2B-\frac{C}{r^2}$, $\sigma_{r\theta}=0$.

when $r=r_1$. $\sigma_{rr}=2B+\frac{C}{r_1^2}=-P_1$

when $r=r_2$, $\sigma_{rr}=2B+\frac{C}{r_2^2}=-P_2$.

So, $B=-\frac{P_1r_1^2-P_2r_2^2}{2(r_1^2-r_2^2)}$, $C=\frac{(P_1-P_2)r_1^2r_2^2}{r_1^2-r_2^2}$

$\therefore \sigma_{rr}=\left[\frac{(P_1-P_2)r_1^2r_2^2}{r_1^2-r_2^2}\right]/r^2+2\left(-\frac{P_1r_1^2-P_2r_2^2}{2(r_1^2-r_2^2)}\right)$

$\sigma_{\theta\theta}=-\left[\frac{(P_1-P_2)r_1^2r_2^2}{r_1^2-r_2^2}/r^2\right]/r^2+2\cdot\left(-\frac{P_1r_1^2-P_2r_2^2}{2(r_1^2-r_2^2)}\right)$

(e) $r_2\to\infty$, so $P_2=0$

$\therefore B=0$, $C=-P_1r_1^2$

$\sigma_{rr}=-P_1\left(\frac{r_1}{r}\right)^2$, $\sigma_{\theta\theta}=P_1\left(\frac{r_1}{r}\right)^2.$

2.(a) By Divergence Theorem, $\int_{\partial\Omega} \vec{V}\cdot\vec{n}\,dA = \int_{\Omega} \nabla\cdot\vec{V}\,dV$

$\because \rho = \nabla\cdot(\epsilon_0\vec{E})$, $\therefore \nabla\cdot\vec{E} = \frac{\rho}{\epsilon_0}$.

(b) By Curl Theorem, $\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$.

(c) By Curl Theorem, $\nabla\times\vec{B} = \mu_0\vec{j} + \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}$.

(d) By Divergence Theorem, $\int_{\partial\Omega}\vec{B}\cdot\vec{n}\,dA = \int_{\Omega}\nabla\cdot\vec{B}\,dv = 0$, $\therefore \nabla\cdot\vec{B} = 0$.

(e) $\nabla\cdot\vec{E} = \frac{\rho}{\epsilon_0}$ ① $\quad \nabla\times\vec{E} = \frac{-\partial B}{\partial t}$ ②

$\nabla\times\vec{B} = \mu_0\vec{j} + \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}$ ③ $\quad \nabla\cdot\vec{B} = 0$.

$\nabla\times\nabla\times\vec{E} = -\frac{\partial(\nabla\times\vec{B})}{\partial t} = -\mu_0\epsilon_0\frac{\partial^2\vec{E}}{\partial t^2}$

$\frac{\partial^2\vec{E}}{\partial t^2} - \frac{1}{\mu_0\epsilon_0}\Delta E = -\frac{1}{\mu_0\epsilon_0}(\nabla\times\nabla\times\vec{E} + \Delta\vec{E}) = -\frac{1}{\mu_0\epsilon_0}\nabla(\nabla\vec{E})$

$\because -\frac{1}{\epsilon_0}\nabla\rho = -\frac{1}{\mu_0\epsilon_0}\nabla(\rho\cdot\vec{E})$

$\therefore \frac{\partial^2\vec{E}}{\partial t^2} - \frac{1}{\mu_0\epsilon_0}\Delta\vec{E} = -\frac{1}{\epsilon_0}\nabla\rho$.

$\sqrt{\frac{1}{\mu_0\epsilon_0}} = \sqrt{\frac{1}{8.85\times10^{-12}\cdot 4\pi\times10^{-7}}} = 3\times10^{8}$.

3. $\because \frac{D\vec{\alpha}(\vec{x},t)}{Dt} = \frac{\partial\vec{\alpha}(\vec{x},t)}{\partial t} + \nabla_x\cdot\vec{\alpha}(\vec{x},t)\otimes\vec{v}(\vec{x},t)$

$\therefore \frac{D}{Dt}\int_{\Omega x}\vec{\alpha}(\vec{x},t)\,dv = \int_{\Omega x}\frac{\partial}{\partial t}\vec{\alpha}(\vec{x},t) + \nabla_x\cdot(\vec{\alpha}(\vec{x},t)\otimes\vec{v}(\vec{x},t))$

4. $\alpha = \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_p$, $\beta = -\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_T$

for ideal gas, $pV = nRT$, $\therefore \alpha = \frac{nR}{PV} = \frac{1}{T}$, $\beta = \left(-\frac{1}{V}\right)\cdot\left(-\frac{nRT}{P^2}\right) = \frac{1}{P}$.

5. ① vorticity: $w = \nabla\times\vec{v} = \begin{vmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix} = 0$.

② rate of strain: $l_{ij} = symm(v_{i,j}) = \frac{1}{2}(v_{i,j} + v_{j,i})$.

$\therefore l = \begin{bmatrix} C & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

6. for conservative vector field.

$\int_T(\nabla\times\vec{a})\cdot\vec{n}\,dA = \int_{\partial T}\vec{a}\,ds = 0$, $\therefore \nabla\times\vec{a} = 0$.