Review

strain-displacement relations
(应变位移关系，几何方程)

$$\boxed{\varepsilon_x = \frac{\partial u}{\partial x}} \qquad \boxed{\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}}$$

$$\boxed{\varepsilon_y = \frac{\partial v}{\partial y}} \qquad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}$$

$$\varepsilon_z = \frac{\partial w}{\partial z} \qquad \gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$$

2D transformation
of strain equations

$$\epsilon_{x'} = \frac{\epsilon_x + \epsilon_y}{2} + \frac{\epsilon_x - \epsilon_y}{2} \cos 2\alpha + \frac{\gamma_{xy}}{2} \sin 2\alpha$$

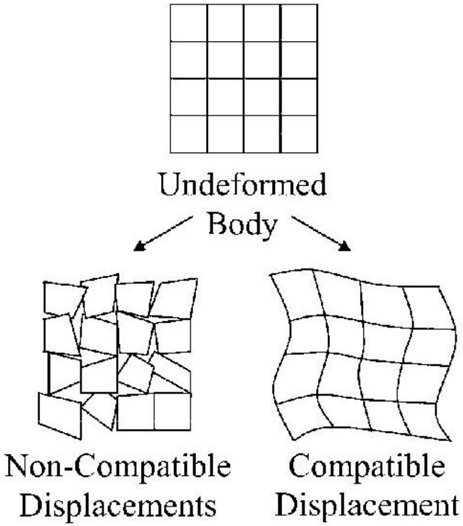
$$\epsilon_{y'} = \frac{\epsilon_x + \epsilon_y}{2} - \frac{\epsilon_x - \epsilon_y}{2} \cos 2\alpha - \frac{\gamma_{xy}}{2} \sin 2\alpha$$

$$\gamma_{x'y'} = (\epsilon_y - \epsilon_x) \sin 2\alpha + \gamma_{xy} \cos 2\alpha$$

Compatibility Equations of strain (应变协调方程、相容性方程)

- Strain-displacement relations:
  – 3 displacement components (u, v, w) ->
  – 6 strain components
- Sometimes we are given 6 strain components,
  – we want to make sure that the given 6 strain components are physically reasonable
    - i.e., strain between neighboring elements cannot be arbitrary
- Saint-Venant compatibility equations (圣维南应变协调方程) or Compatibility equations of strain:
  – a group of partial differential equations ensuring that the strain distribution within a deformable body is physically realizable and consistent with the principles of continuum mechanics.

$$\boxed{\begin{aligned}
\varepsilon_x &= \frac{\partial u}{\partial x} & \gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \\
\varepsilon_y &= \frac{\partial v}{\partial y} & \gamma_{yz} &= \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \\
\varepsilon_z &= \frac{\partial w}{\partial z} & \gamma_{zx} &= \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}
\end{aligned}}$$



Compatibility Equations (应变协调方程、相容性方程)

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} = \frac{\partial^3 u}{\partial x \partial y^2} \qquad \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^3 u}{\partial x^2 \partial y}$$

$$\Downarrow$$

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^3 u}{\partial x \partial y^2} + \frac{\partial^3 u}{\partial x^2 \partial y}$$

$$\frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = \frac{\partial^3 u}{\partial x \partial y^2} + \frac{\partial^3 v}{\partial x^2 \partial y}$$

$$\Downarrow$$

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}$$

$$\boxed{\begin{array}{ll} \varepsilon_x = \dfrac{\partial u}{\partial x} & \gamma_{xy} = \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \\ \varepsilon_y = \dfrac{\partial v}{\partial y} & \gamma_{yz} = \dfrac{\partial v}{\partial z} + \dfrac{\partial w}{\partial y} \\ \varepsilon_z = \dfrac{\partial w}{\partial z} & \gamma_{zx} = \dfrac{\partial w}{\partial x} + \dfrac{\partial u}{\partial z} \end{array}}$$

Similarly

$$\frac{\partial^2 \varepsilon_y}{\partial z^2} + \frac{\partial^2 \varepsilon_z}{\partial y^2} = \frac{\partial^2 \gamma_{yz}}{\partial y \partial z}$$

$$\frac{\partial^2 \varepsilon_z}{\partial x^2} + \frac{\partial^2 \varepsilon_x}{\partial z^2} = \frac{\partial^2 \gamma_{zx}}{\partial z \partial x}$$

Compatibility Equations (应变协调方程、相容性方程)

$$\frac{\partial^2 \varepsilon_x}{\partial y \partial z} = \frac{\partial^3 u}{\partial x \partial y \partial z}$$

$$\frac{\partial \gamma_{xy}}{\partial z} = \frac{\partial^2 u}{\partial y \partial z} + \frac{\partial^2 v}{\partial x \partial z} \qquad \frac{\partial \gamma_{yz}}{\partial x} = \frac{\partial^2 v}{\partial x \partial z} + \frac{\partial^2 w}{\partial x \partial y}$$

$$\frac{\partial \gamma_{zx}}{\partial y} = \frac{\partial^2 w}{\partial x \partial y} + \frac{\partial^2 u}{\partial y \partial z}$$

$$\varepsilon_x = \frac{\partial u}{\partial x} \qquad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

$$\varepsilon_y = \frac{\partial v}{\partial y} \qquad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}$$

$$\varepsilon_z = \frac{\partial w}{\partial z} \qquad \gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$$

$$\Downarrow$$

$$2\frac{\partial^2 \varepsilon_x}{\partial y \partial z} = \frac{\partial}{\partial x}\left(-\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right)$$

Similarly

$$2\frac{\partial^2 \varepsilon_y}{\partial x \partial z} = \frac{\partial}{\partial y}\left(\frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right)$$

$$2\frac{\partial^2 \varepsilon_z}{\partial x \partial y} = \frac{\partial}{\partial z}\left(\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} - \frac{\partial \gamma_{xy}}{\partial z}\right)$$

Compatibility Equations (应变协调方程、相容性方程)

Strain-Displacement Relations
（几何方程）

$$\varepsilon_x = \frac{\partial u}{\partial x} \qquad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

$$\varepsilon_y = \frac{\partial v}{\partial y} \qquad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}$$

$$\varepsilon_z = \frac{\partial w}{\partial z} \qquad \gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$$

Saint-Venant compatibility equations (圣维南应变协调方程) or
Compatibility equations in terms of strain

$$2\frac{\partial^2 \varepsilon_x}{\partial y \partial z} = \frac{\partial}{\partial x}\left(-\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right)$$

$$2\frac{\partial^2 \varepsilon_y}{\partial x \partial z} = \frac{\partial}{\partial y}\left(\frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right)$$

$$2\frac{\partial^2 \varepsilon_z}{\partial x \partial y} = \frac{\partial}{\partial z}\left(\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} - \frac{\partial \gamma_{xy}}{\partial z}\right)$$

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}$$

$$\frac{\partial^2 \varepsilon_y}{\partial z^2} + \frac{\partial^2 \varepsilon_z}{\partial y^2} = \frac{\partial^2 \gamma_{yz}}{\partial y \partial z}$$

$$\frac{\partial^2 \varepsilon_z}{\partial x^2} + \frac{\partial^2 \varepsilon_x}{\partial z^2} = \frac{\partial^2 \gamma_{zx}}{\partial z \partial x}$$

(1) The strain components must satisfy Saint-Venant compatibility equations in order that solutions for the displacement components exist.

(2) If the displacement components are single-valued, continuous functions, the strain components will automatically satisfy the compatibility equations

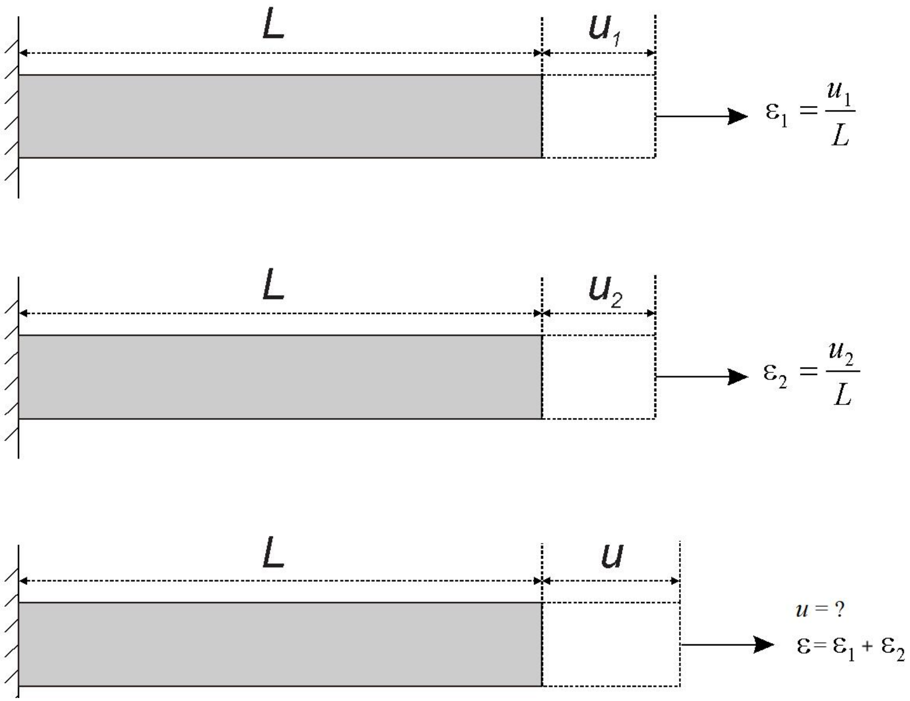
Classroom exercise

2-8 Given the following system of strains,

$$\epsilon_x = 5 + x^2 + y^2 + x^4 + y^4$$
$$\epsilon_y = 6 + 3x^2 + 3y^2 + x^4 + y^4$$
$$\gamma_{xy} = 10 + 4xy(x^2 + y^2 + 2)$$
$$\epsilon_z = \gamma_{xz} = \gamma_{yz} = 0$$

determine if the system of strains is possible.

Principle of Superposition (应变的叠加原理)



The principle of superposition of strain:

- The displacement fields due to several seperate strains can be added to give the displacement due to the combined strains.
  - The order of application of these strain has no effect on the final configuration (构形) of the body in question.
- under the infinitesimal deformation assumption

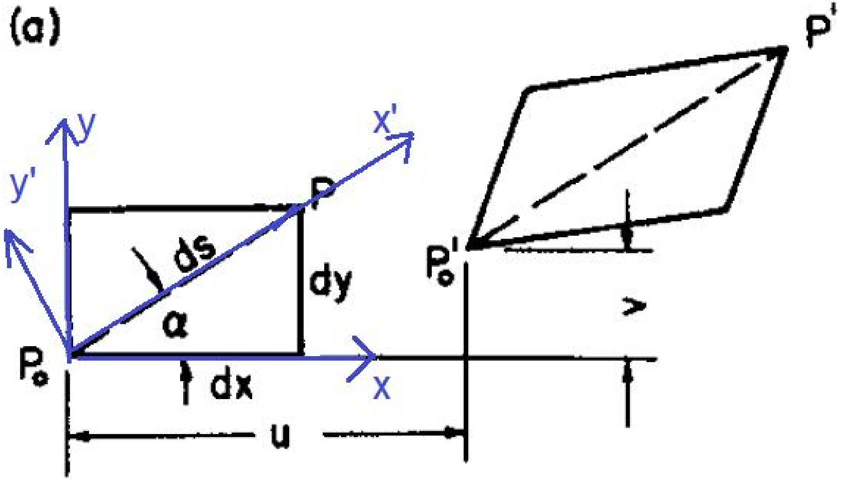
The principle of superposition applys to all the linear systems, e.g.,

system $y=ax$ vs system $y=ax^3$

Classroom exercise

Angular displacement ψ of line P_0P (x') is

$$\psi = -(\epsilon_x - \epsilon_y)\sin\alpha\cos\alpha + \frac{\partial v}{\partial x}\cos^2\alpha - \frac{\partial u}{\partial y}\sin^2\alpha$$



Prove that the average rotation at P_0 equals to the average rotation of x and y axis:

$$\int_0^{2\pi} \psi d\alpha \Big/ 2\pi = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \Big/ 2$$

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$ represents the average (rigid) rotation of the continuum at point A

Prove that the rotation angle of the two principal strain axes are ω_z.
Hint: the principal strain direction and ε_x, ε_y, γ_{xy} have the following relation:

$$\tan 2\alpha = \frac{\gamma_{xy}}{\epsilon_x - \epsilon_y}$$

Reading material

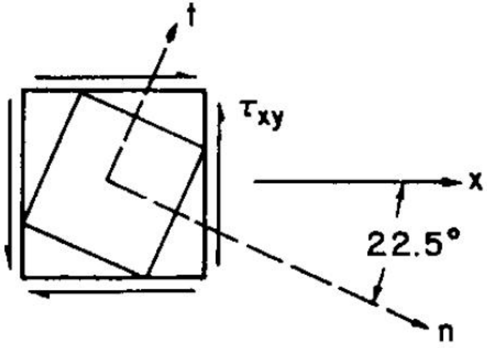
In continuum mechanics, the infinitesimal strain theory, sometimes called small deformation theory, small displacement theory, or small displacement-gradient theory, deals with infinitesimal deformations of a continuum body. For an infinitesimal deformation the displacements and the displacement gradients are small compared to unity, i.e., the non-linear or second-order terms of the finite strain tensor can be neglected.

For prescribed strain components the strain tensor equation (几何方程) represents a system of six differential equations for the determination of three displacements components, giving an over-determined system. Thus, a solution does not generally exist for an arbitrary choice of strain components. Therefore, some restrictions, named compatibility equations, are imposed upon the strain components. These constraints on strain were discovered by Saint-Venant, and are called the "Saint Venant compatibility equations". Although there are six compatibility equations, they are equivalent to three independent fourth-order equations. It is usually more convenient, however, to use the six second-order equations rather than the three fourth-order equations. With the addition of the compatibility equations the number of independent equations is reduced to three, matching the number of unknown displacement components.

The compatibility equations serve to assure a single-valued continuous displacement function. If the elastic medium is visualised as a set of infinitesimal cubes in the unstrained state, after the medium is strained, an arbitrary strain tensor may not yield a situation in which the distorted cubes still fit together without overlapping.

Homework (4 points)

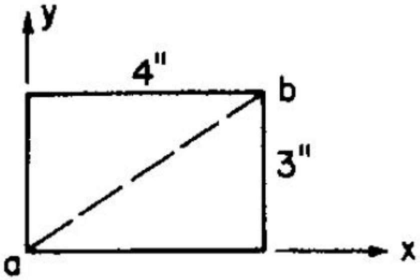
2-9 Determine ϵ_n, ϵ_t, and γ_{tn} if $\gamma_{xy}=0.002828$ and $\epsilon_x=\epsilon_y=0$, for the element shown.



2-10 A thin rectangular plate 3″ by 4″ is acted upon by a stress distribution which results in the uniform strains

$$\epsilon_x=0.0025, \quad \epsilon_y=0.0050, \quad \epsilon_z=0, \quad \gamma_{xy}=0.001875, \quad \gamma_{xz}=\gamma_{yz}=0$$

as shown in the figure. Determine the change in length of diagonal *ab*.



Homework (4 points)

2-6 Derive the equations which define the directions and magnitude of maximum shear strain at a point (two-dimensional). Check the relations by replacing σ by ϵ and τ by $\gamma/2$ in the corresponding stress equations.

2-7 The following displacement field is applied to a certain body

$$u = k(2x + y^2) \qquad v = k(x^2 - 3y^2) \qquad w = 0$$

where $k = 10^{-4}$.

(a) Show the distorted configuration of a two-dimensional element with sides dx and dy and its lower left corner (point A) initially at the point (2, 1, 0), i.e., determine the new length and angular position of each side.

(b) Determine the coordinates of point A after the displacement field is applied.

(c) Find ω_z at this point.

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$ represents the average (rigid) rotation of the continuum at point A

Review

Saint-Venant compatibility equations (圣维南应变协调方程)

$$\frac{\partial^2 \varepsilon_x}{\partial y^2}+\frac{\partial^2 \varepsilon_y}{\partial x^2}=\frac{\partial^2 \gamma_{xy}}{\partial x \partial y} \qquad 2\frac{\partial^2 \varepsilon_x}{\partial y \partial z}=\frac{\partial}{\partial x}\left(-\frac{\partial \gamma_{yz}}{\partial x}+\frac{\partial \gamma_{zx}}{\partial y}+\frac{\partial \gamma_{xy}}{\partial z}\right)$$

$$\frac{\partial^2 \varepsilon_y}{\partial z^2}+\frac{\partial^2 \varepsilon_z}{\partial y^2}=\frac{\partial^2 \gamma_{yz}}{\partial y \partial z} \qquad 2\frac{\partial^2 \varepsilon_y}{\partial x \partial z}=\frac{\partial}{\partial y}\left(\frac{\partial \gamma_{yz}}{\partial x}-\frac{\partial \gamma_{zx}}{\partial y}+\frac{\partial \gamma_{xy}}{\partial z}\right)$$

$$\frac{\partial^2 \varepsilon_z}{\partial x^2}+\frac{\partial^2 \varepsilon_x}{\partial z^2}=\frac{\partial^2 \gamma_{zx}}{\partial z \partial x} \qquad 2\frac{\partial^2 \varepsilon_z}{\partial x \partial y}=\frac{\partial}{\partial z}\left(\frac{\partial \gamma_{yz}}{\partial x}+\frac{\partial \gamma_{zx}}{\partial y}-\frac{\partial \gamma_{xy}}{\partial z}\right)$$

a group of partial differential equations ensuring that the strain distribution within a deformable body is physically realizable

The principle of superposition of strain:

- $\varepsilon_1 \to u_1$, $\varepsilon_2 \to u_2$ -->
- $\varepsilon_1+\varepsilon_2 \to u_1+u_2$