Question 1

1. Verify the tensor notations of the 2nd and 3rd stress invariants

$$I_1 = \tau_{11} + \tau_{22} + \tau_{33} = \tau_{ii}$$

$$I_2 = \tau_{11}\tau_{22} + \tau_{22}\tau_{33} + \tau_{33}\tau_{11} - \tau_{12}^2 - \tau_{23}^2 - \tau_{31}^2 = \frac{1}{2}(\tau_{ii}\tau_{kk} - \tau_{ik}\tau_{ki})$$

$$I_3 = \tau_{11}\tau_{22}\tau_{33} + 2\tau_{12}\tau_{23}\tau_{31} - \tau_{11}\tau_{23}^2 - \tau_{22}\tau_{31}^2 - \tau_{33}\tau_{12}^2 = \varepsilon_{ijk}\tau_{1i}\tau_{2j}\tau_{3k} = \boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2 \times \boldsymbol{\tau}_3 = \det(\boldsymbol{\tau}\text{ij})$$

The definition of the principal invariants is from the well-known **Cayley-Hamilton eqution**, as:

$$\tau^3 - I_1\tau^2 + I_2\tau - I_3 = 0,$$

where

$$I_1 = \mathrm{tr}\boldsymbol{\tau} = \boldsymbol{\tau} : \boldsymbol{I} = \tau_{ij}\delta_{ij} = \tau_{ii},$$

$$\begin{aligned} I_2 &= \frac{1}{2}\left((\mathrm{tr}\boldsymbol{\tau})^2 - \mathrm{tr}(\boldsymbol{\tau}^2)\right) = \frac{1}{2}\left((\boldsymbol{\tau} : \boldsymbol{I})^2 - \boldsymbol{\tau}^2 : \boldsymbol{I}\right) \\ &= \frac{1}{2}\left((\boldsymbol{\tau} : \boldsymbol{I})^2 - \boldsymbol{\tau} : \boldsymbol{\tau}^{\mathrm{T}}\right) = \frac{1}{2}(\tau_{ii}\tau_{kk} - \tau_{ik}\tau_{ki}) \\ &= \tau_{11}\tau_{22} + \tau_{22}\tau_{33} + \tau_{33}\tau_{11} - \tau_{12}^2 - \tau_{23}^2 - \tau_{31}^2, \end{aligned}$$

and

$$\begin{aligned} I_3 &= \det\boldsymbol{\tau} = (\boldsymbol{u} \times \boldsymbol{v}) \cdot \boldsymbol{w} \quad \text{with} \quad \boldsymbol{\tau} = [\boldsymbol{u} \quad \boldsymbol{v} \quad \boldsymbol{w}] \\ &= \varepsilon_{ijk} tau_{1i}\tau_{2j}\tau_{3k} \\ &= \tau_{11}\tau_{22}\tau_{33} + 2\tau_{12}\tau_{23}\tau_{31} - \tau_{11}\tau_{23}^2 - \tau_{22}\tau_{31}^2 - \tau_{33}\tau_{12}^2. \end{aligned}$$

Question 2

2. Use the tensor notation of Hooke's law above to (1) establish the relation below and (2) establish the new alternative Hooke's law below

$$e_{ii} = \frac{1}{K}\frac{\tau_{ii}}{3} = \frac{1}{K}\frac{\Theta}{3}$$

K is the bulk modulus of elasticity

$$\tau_{ij} = 2G\varepsilon_{ij} + \lambda\varepsilon\delta_{ij} \qquad \text{where} \quad \varepsilon = \varepsilon_{kk}$$

The tensor notation of the Hooke's law is as follows:

$$\varepsilon_{ij} = \frac{1+\nu}{E}\tau_{ij} - \frac{\nu}{E}\delta_{ij}\Theta \quad \text{with} \quad \Theta = \tau_{kk} = \tau_{11} + \tau_{22} + \tau_{33}$$

$\delta_{ij} = 3$ when taking $i = j$, then the Hooke's law turns into

$$\varepsilon_{ii} = \frac{1+\nu}{E}\tau_{ii} - \frac{3\nu}{E}\Theta = \frac{1-2\nu}{E}\Theta = \frac{1}{K}\frac{\tau_{ii}}{3} = \frac{1}{K}\frac{\Theta}{3} \quad \text{with} \quad K = \frac{E}{3(1-2\nu)}.$$

Use the equality of $\Theta = \tau_{ii}$ and $\tau_{ii} = 3K\varepsilon_{ii}$ to substitute into the Hooke's law, which is restructured as

$$\begin{aligned}
\varepsilon_{ij} = \frac{1+\nu}{E}\tau_{ij} - \frac{3K\nu}{E}\delta_{ij}\varepsilon_{ii} &= \frac{1+\nu}{E}\tau_{ij} - \frac{\nu}{1-2\nu}\delta_{ij}\varepsilon \quad \text{with} \quad \varepsilon = \varepsilon_{ii} \\
\varepsilon_{ij} + \frac{\nu}{1-2\nu}\delta_{ij}\varepsilon &= \frac{1+\nu}{E}\tau_{ij} \\
\tau_{ij} &= \frac{E}{1+\nu}\varepsilon_{ij} + \frac{E\nu}{(1-2\nu)(1+\nu)}\delta_{ij}\varepsilon \\
\tau_{ij} = 2G\varepsilon_{ij} + \lambda\varepsilon\delta_{ij} \quad \text{with} \quad G &= \frac{E}{2(1+\nu)} \quad \text{and} \quad \lambda = \frac{E\nu}{(1-2\nu)(1+\nu)}.
\end{aligned}$$

Question 3

3. Given the governing equations of elasticity in the red box, derive the Navier's equation (equilibrium in the form of displacement) below:

Governing equations

$$\boxed{\begin{gathered}
\tau_{ik,i} + f_k = 0 \\
\tau_{\text{ij}} = \delta_{ij}\lambda e + 2Ge_{ij} \\
\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) = \frac{1}{2}(u_{i,j} + u_{j,i})
\end{gathered}}$$

Navier's equation

$$\rightarrow \quad (\lambda + G)u_{i,ik} + G\nabla^2 u_k + f_k = 0$$

Rewrite the constitutive eqution, as

$$\tau_{ik} = \delta_{ik}\lambda\varepsilon_{jj} + 2G\varepsilon_{ik},$$

and implement the divergence operator to both sides of the above eqution, as

$$\tau_{ik,i} = \delta_{ik,i}\lambda\varepsilon_{jj} + \delta_{ik}\lambda\varepsilon_{jj,i} + 2G\varepsilon_{ik,i} = \delta_{ik}\lambda\varepsilon_{jj,i} + 2G\varepsilon_{ik,i},$$

so that the equilibrium equation becomes:

$$\delta_{ik}\lambda\varepsilon_{jj,i} + 2G\varepsilon_{ik,i} + f_k = 0. \tag{1}$$

The displacement equation is

$$\varepsilon_{ik} = \frac{1}{2}(u_{i,k} + u_{k,i}),$$

and substitute it into equation (1), and we get

$$\begin{aligned}
G(u_{i,ki} + u_{k,ik}) + \delta_{ik}\lambda u_{j,ji} + f_k &= 0 \\
Gu_{i,ki} + Gu_{j,ij} + \lambda u_{j,ji} + f_k &= 0 \\
(G+\lambda)u_{j,jk} + G\nabla \cdot \nabla \boldsymbol{u} + f_k &= 0 \\
(G+\lambda)u_{i,ik} + G\nabla^2 \boldsymbol{u} + f_k &= 0.
\end{aligned}$$

Question 4

The rotation tensor w_{ik} is an antisymmetric tensor $w_{ik} = (\partial u_k/\partial x_i - \partial u_i/\partial x_k)/2$

Prove that the curl of displacement u_k: $v_m = \varepsilon_{mik}\nabla_i u_k$ is related to the rotation tensor w_{ik}

$$v_m = \varepsilon_{mik} w_{ik} = \varepsilon_{mik}(\frac{\partial u_k}{\partial x_i} - \frac{\partial u_i}{\partial x_k})/2$$

$$w_{ik} = \varepsilon_{mik}(\frac{1}{2}v_m) = \varepsilon_{ikm}(\frac{1}{2}v_m) = \frac{1}{2}\varepsilon_{ikm}\varepsilon_{mqr}\frac{\partial u_r}{\partial x_q}$$

Note: the alternating tensor and Kronecker delta have the following relation:

$$\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

Prove that (1) the infinitesimal rotation angle vector $\boldsymbol{w}_m$ equals half the curl of displacement u_k $\quad \mathrm{w}_m = \frac{1}{2}\xi_{mik}\nabla_i \mathrm{u}_k$

(2) rotation displacement can be calculated with cross product between $\boldsymbol{w}_m$ and $\boldsymbol{dx}$

$$d\mathbf{u} = d\mathbf{u}_D + d\mathbf{u}_R \qquad d\mathbf{u}_R = \omega_{ij}dx_j = \mathbf{w}_m \times d\mathbf{x}$$

Deformation Rotation

infinitesimal rotation angle vector

The rotation tensor is the anti-symmetric part of the deformation gradient:

$$\boldsymbol{F} = \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{x}}, \quad \boldsymbol{F} = \boldsymbol{D} + \boldsymbol{W}, \quad F_{ij} = \frac{\partial u_i}{\partial x_j} = u_{i,j}$$

$$\boldsymbol{W} = \frac{1}{2}(\boldsymbol{F} - \boldsymbol{F}^{\mathrm{T}}), \quad W_{ij} = \frac{1}{2}(F_{ij} - F_{ji}), \quad \boldsymbol{W} = -\boldsymbol{W}^{\mathrm{T}}$$

$$\boldsymbol{D} = \frac{1}{2}(\boldsymbol{F} + \boldsymbol{F}^{\mathrm{T}}), \quad D_{ij} = \frac{1}{2}(F_{ij} + F_{ji}), \quad \boldsymbol{D} = \boldsymbol{D}^{\mathrm{T}}.$$

The curl of the displacement is defined as

$$\boldsymbol{v} = \nabla \times \boldsymbol{u}, \quad v_m = \varepsilon_{mik}u_{k,i}.$$

The deformation gradient $u_{k,i}$ can be decomposed into the (anti-)symmetric parts, as

$$v_m = \varepsilon_{mik}\left[\frac{1}{2}(u_{i,k} + u_{k,i}) + \frac{1}{2}(u_{i,k} - u_{k,i})\right] = \varepsilon_{mik}\left(D_{ik} + W_{ik}\right).$$

Due to the symmetry of $D_k i$, we have the fact that

$$\varepsilon_{mki}D_{ki} = \varepsilon_{mik}D_{ik} = \varepsilon_{mik}D_{ki} = -\varepsilon_{mki}D_{ki} \Rightarrow \varepsilon_{mik}D_{ki} = 0.$$

So that we can prove that

$$v_m = \varepsilon_{mik}W_{ik} = \varepsilon_{mik}\cdot\frac{1}{2}(u_{i,k} - u_{k,i}).$$

Multiplicate $\frac{1}{2}\varepsilon_{mqr}$ to both sides of the above, as

$$\varepsilon_{mqr}v_m = \varepsilon_{qrm}\varepsilon_{mik}W_{ik} = (\delta_{qi}\delta_{rk} - \delta_{qk}\delta_{ir})W_{ik} = W_{qr} - W_{rq} = 2W_{qr} \Rightarrow W_{qr} = \frac{1}{2}\varepsilon_{mqr}v_m$$

We can substitute $v_m = \varepsilon_{mik} u_{i,k}$ into it, as

$$W_{qr} = \frac{1}{2} \varepsilon_{mqr} \varepsilon_{mik} u_{i,k} \Rightarrow W_{ik} = \frac{1}{2} \varepsilon_{mik} \varepsilon_{mqr} u_{q,r}.$$

The next prove begins:

$$d\boldsymbol{u} = d\boldsymbol{u}_{\mathrm{D}} + d\boldsymbol{u}_{\mathrm{R}} = \boldsymbol{D} \cdot d\boldsymbol{x} + \boldsymbol{W} \cdot d\boldsymbol{x}$$

$$w_m = \frac{1}{2} \varepsilon_{mik} u_{i,k} = \frac{1}{2} v_m \Rightarrow \boldsymbol{w} = \frac{1}{2} \boldsymbol{v}$$

$$d\boldsymbol{u}_{\mathrm{R}} = \boldsymbol{W} \cdot d\boldsymbol{x}, \quad du_{\mathrm{R},i} = W_{i,k} dx_k = \frac{1}{2} \varepsilon_{mik} v_m dx_k \Rightarrow d\boldsymbol{u}_{\mathrm{R}} = \boldsymbol{w} \times d\boldsymbol{x}.$$