Review

3D stress transformation:

Given $\boldsymbol{\sigma}$ at xyz coordinate, calculate stress vector $\boldsymbol{p}$ at any plane $\boldsymbol{n}$

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_x & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \sigma_y & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix}$$

$$\mathbf{p} = \boldsymbol{\sigma}^T \mathbf{n} = \boldsymbol{\sigma} \mathbf{n}$$

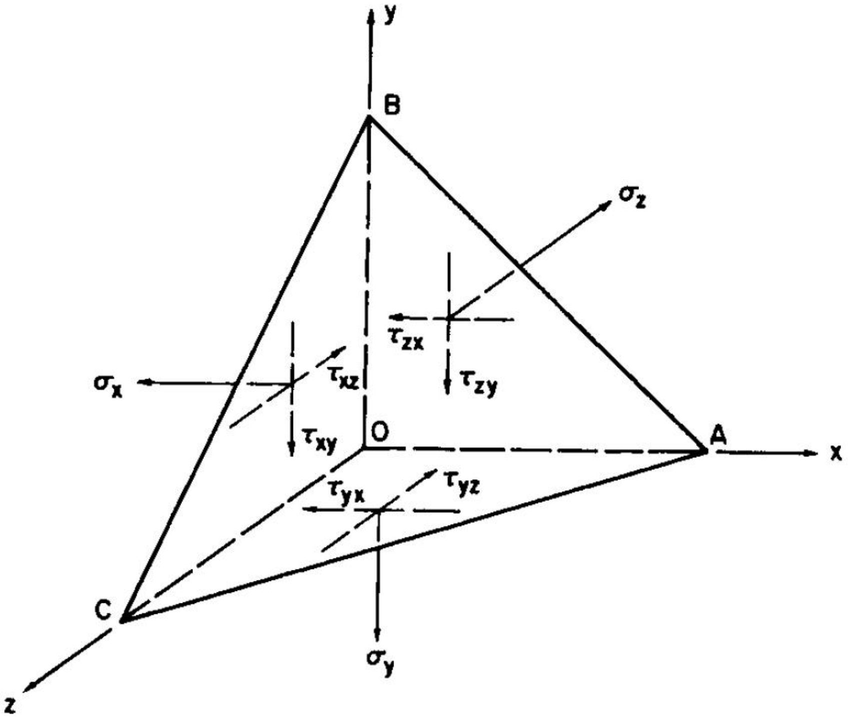
$$\sigma_{x'} = (p_x, p_y, p_z) \cdot (a_{11}, a_{21}, a_{31})$$

$$\tau_{x'y'} = (p_x, p_y, p_z) \cdot (a_{12}, a_{22}, a_{32})$$

$$\tau_{x'z'} = (p_x, p_y, p_z) \cdot (a_{13}, a_{23}, a_{33})$$

Principal stresses and stress invariants

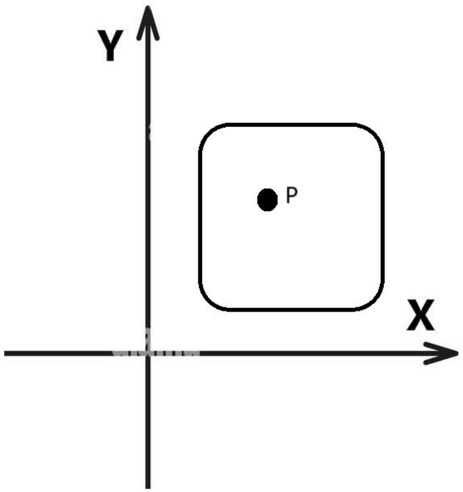
Stress components on a tetrahedron



Review

Strain: a quantity that measures the deformation that a material experiences in response to an external force

- **Normal strain** ε in a given direction
  - length change in unit distance of a line originally oriented in the given direction
- **Shear strain γ** between two axes:
  - the change in the original right angle between two axes



Strain-displacement relations (应变位移关系，几何方程):

$$\varepsilon_x = \frac{\partial u}{\partial x} \qquad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

$$\varepsilon_y = \frac{\partial v}{\partial y} \qquad \gamma_{xy} = \gamma_{yx}$$

Strain-Displacement Relations（几何方程）

3D strain-displacement relations (应变位移关系，几何方程)

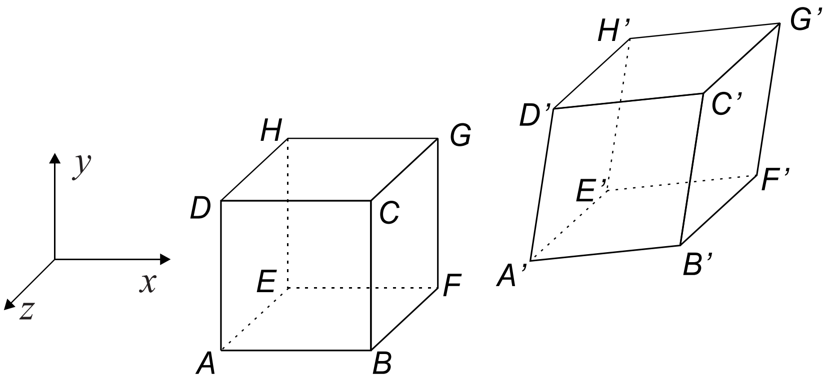
$$\varepsilon_x = \frac{\partial u}{\partial x} \qquad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

$$\varepsilon_y = \frac{\partial v}{\partial y} \qquad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}$$

$$\varepsilon_z = \frac{\partial w}{\partial z} \qquad \gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$$

Based on the definition:

$$\gamma_{xy} = \gamma_{yx} \qquad \gamma_{yz} = \gamma_{zy} \qquad \gamma_{zx} = \gamma_{xz}$$



strain in a 3D rectangular prism

$$\varepsilon_x = \frac{A'B' - AB}{AB} = \frac{D'C' - DC}{DC} = \frac{E'F' - EF}{EF} = \frac{H'G' - HG}{HG}$$

$$\gamma_{xy} = \frac{\pi}{2} - \angle B'A'D' = \frac{\pi}{2} - \angle F'E'H' = \frac{\pi}{2} - \angle B'C'D' = \frac{\pi}{2} - \angle F'G'H'$$

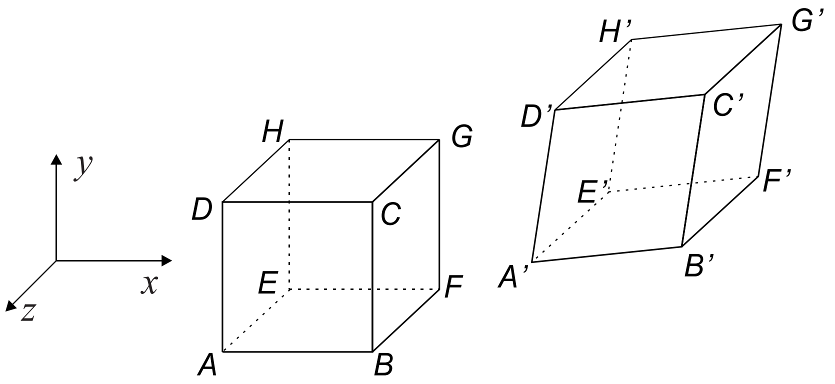
Engineering shear strain γ_{xy} vs tensor (mathematical) shear strain ε_{xy}:

$$\varepsilon_{xy} = 0.5\gamma_{xy}$$

Classroom exercises

1. Can we fully determine deformation inside the body with displacement at every point?
2. The advantage of strain ε in describing the deformation at a point than displacement
3. Does the shear strain $\gamma_{xy} = \gamma_{yx}$?
4. What are the units of normal strain and shear strain, respectively?
5. Verify the following formulations for normal strain ε_y and shear strain γ_{xy} in 3D geometry

$$\varepsilon_y = \frac{\partial v}{\partial y} \qquad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

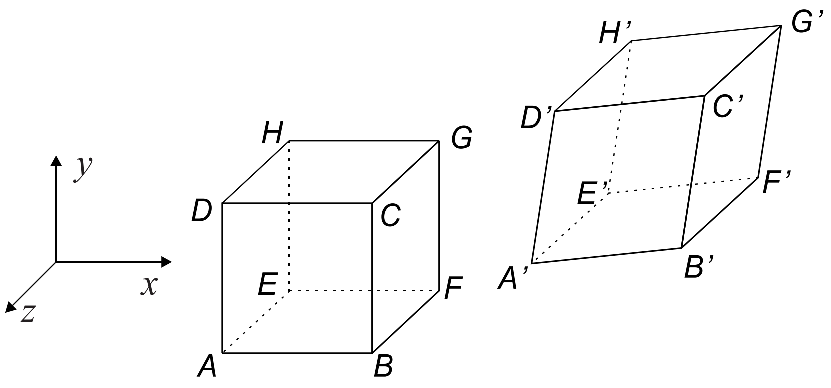


strain in a 3D rectangular prism

Cautious

Note that stress is defined over the **deformed body** while strain is defined over the **undeformed body**.

- This inconsistency is neglected under infinitesimal deformation.



strain in a 3D rectangular prism

Strain-Displacement Relations（几何方程）

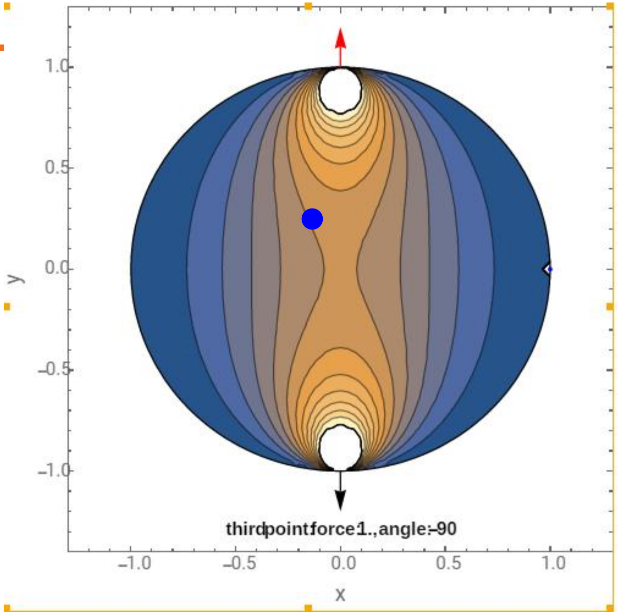
The stress state at a point represented by 9 components

$$\begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix}$$

The strain state at a point represented by 9 components

$$\begin{bmatrix} \varepsilon_x & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_y & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_z \end{bmatrix}$$

- The strain and stress components depend on the coordinate system.
- The stress and strain states do not depend on the coordinate.
- Note that stress components are defined on the plane, while strain components are defined on the axes.



strain (ε_{max}) distribution in a circular plate with concentrated loadings

(https://demonstrations.wolfram.com/StressDistributionInACircularPlateWithConcentratedRadialLoad/)

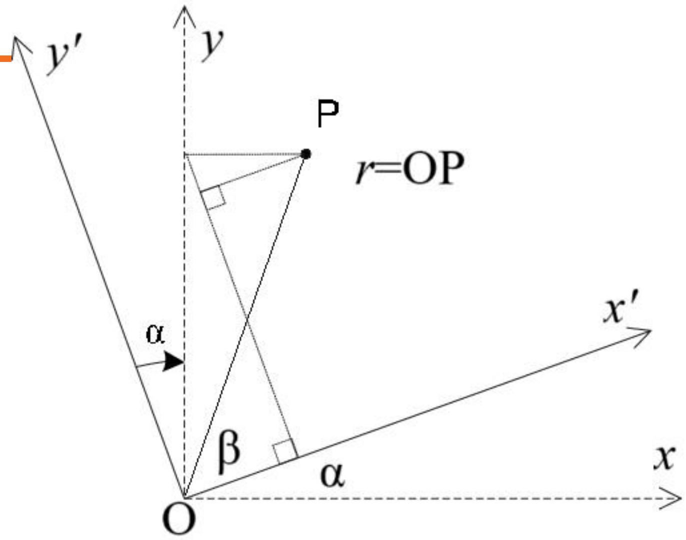
State of Strain at a Point (strain transformation)

- The stress vector depends on the direction of the plane it acts on
- The state of stress at a point is uniquely determined if the stress components on two (three) planes are given for 2D(3D) cases.

$$\mathbf{p} = \boldsymbol{\sigma}^T \mathbf{n} = \boldsymbol{\sigma} \mathbf{n}$$

- Similar to the stress, the state of strain at a point is uniquely determined if the strain components on two (three) planes are given for 2D (3D) cases.
  – Given the strain components ε_x, ε_y, γ_{xy} at a point —>the strain in any direction

Given ε_x, ε_y, γ_{xy} at point P —> $\varepsilon_{x'}$, $\varepsilon_{y'}$, $\gamma_{x'y'}$ in the new coordiante



Based on definition,

$$\varepsilon_{x'} = \frac{\partial u'}{\partial x'}$$

$$\varepsilon_{y'} = \frac{\partial v'}{\partial y'}$$

$$\gamma_{x'y'} = \frac{\partial u'}{\partial y'} + \frac{\partial v'}{\partial x'}$$

$$x = r\cos(\alpha+\beta) = r\cos\beta\cos\alpha - r\sin\beta\sin\alpha = x'\cos\alpha - y'\sin\alpha$$
$$y = r\sin(\alpha+\beta) = r\sin\beta\cos\alpha + r\cos\beta\sin\alpha = x'\sin\alpha + y'\cos\alpha$$

$$x = x'\cos\alpha - y'\sin\alpha$$
$$y = x'\sin\alpha + y'\cos\alpha$$
$$u = u'\cos\alpha - v'\sin\alpha$$
$$v = u'\sin\alpha + v'\cos\alpha$$

$$\varepsilon_{x'} = \frac{\partial u'}{\partial x'} = \frac{\partial u'}{\partial x}\frac{\partial x}{\partial x'} + \frac{\partial u'}{\partial y}\frac{\partial y}{\partial x'}$$
$$= \left(\frac{\partial u}{\partial x}\cos\alpha - \frac{\partial v}{\partial x}\sin\alpha\right)\cos\alpha - \left(\frac{\partial u}{\partial y}\cos\alpha - \frac{\partial v}{\partial y}\sin\alpha\right)\sin\alpha$$
$$= \varepsilon_x\cos^2\alpha + \varepsilon_y\sin^2\alpha - \gamma_{xy}\sin\alpha\cos\alpha$$

$$u' = u\cos\alpha - v\sin\alpha$$
$$v' = u\sin\alpha + v\cos\alpha$$

State of Strain at a Point

2D strain transformation

$$\epsilon_{x'} = \frac{\epsilon_x + \epsilon_y}{2} + \frac{\epsilon_x - \epsilon_y}{2}\cos 2\alpha + \frac{\gamma_{xy}}{2}\sin 2\alpha$$

$$\epsilon_{y'} = \frac{\epsilon_x + \epsilon_y}{2} - \frac{\epsilon_x - \epsilon_y}{2}\cos 2\alpha - \frac{\gamma_{xy}}{2}\sin 2\alpha$$

$$\gamma_{x'y'} = (\epsilon_y - \epsilon_x)\sin 2\alpha + \gamma_{xy}\cos 2\alpha$$

2D stress transformation

$$\begin{cases} \sigma_{x'} = \dfrac{\sigma_x + \sigma_y}{2} + \dfrac{\sigma_x - \sigma_y}{2}\cos 2\alpha + \tau_{xy}\sin 2\alpha \\ \sigma_{y'} = \dfrac{\sigma_x + \sigma_y}{2} - \dfrac{\sigma_x - \sigma_y}{2}\cos 2\alpha - \tau_{xy}\sin 2\alpha \\ \tau_{x'y'} = -\dfrac{\sigma_x - \sigma_y}{2}\sin 2\alpha + \tau_{xy}\cos 2\alpha \end{cases}$$

By simply replacing σ by ε and τ by γ/2, the stress transformation equations are converted to the strain relations.

Check the strain transformation formulas using tensor shear strain

The 3D strain transformation equations can be deduced from the corresponding stress relations by replacing σ by ε and τ by γ/2

Classroom exercise

Show that the principal strain direction for 2D are given by

$$\tan 2\alpha = \frac{\gamma_{xy}}{\epsilon_x - \epsilon_y}$$

and the magnitudes of the principal strains are

$$\left.\begin{matrix}\epsilon_1 \\ \epsilon_2\end{matrix}\right\} = \frac{\epsilon_x + \epsilon_y}{2} \pm \frac{1}{2}\sqrt{(\epsilon_x - \epsilon_y)^2 + \gamma_{xy}^2}$$