Continuum Mechanics (B)

Session 04: Formulation of Problems in Elasticity (弹性问题的解法)

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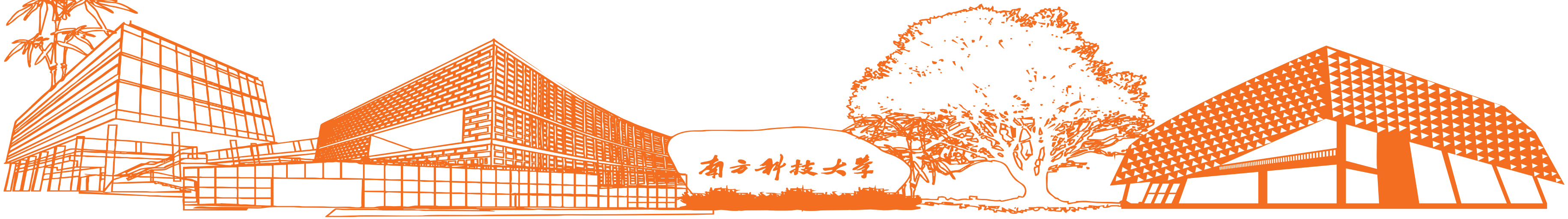

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Boundary Conditions

When a body is subjected to external loads (加载)

Stress

$$\begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ & \sigma_y & \tau_{yz} \\ sym. & & \sigma_z \end{bmatrix}$$

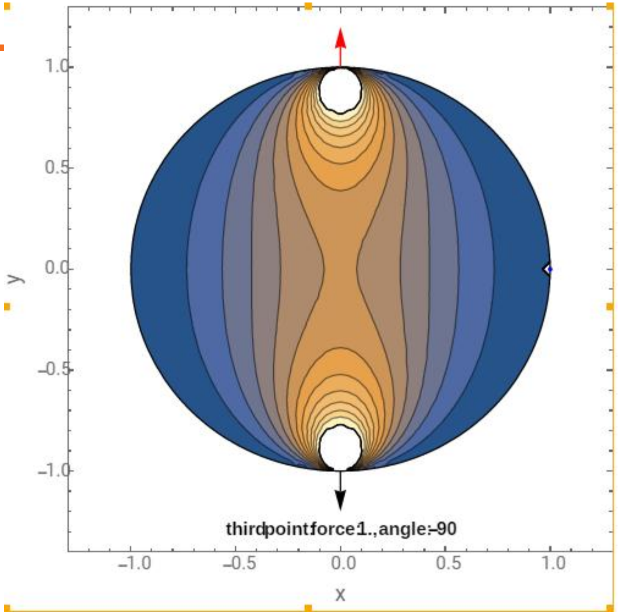
Strain

$$\begin{bmatrix} \varepsilon_x & \gamma_{xy} & \gamma_{xz} \\ & \varepsilon_y & \gamma_{yz} \\ sym. & & \varepsilon_z \end{bmatrix}$$

Displacement

$$\begin{bmatrix} u & v & w \end{bmatrix}$$

How do we determine the stress, strain, and displacement components inside the elastic body?



stress (σ_{max}) distribution in a circular plate with concentrated loadings (https://demonstrations.wolfram.com/StressDistributionInACircularPlateWithConcentratedRadialLoad/)

Boundary Conditions

Unknowns (15)

- Stress (6)

$$\begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ & \sigma_y & \tau_{yz} \\ sym. & & \sigma_z \end{bmatrix}$$

- Strain (6)

$$\begin{bmatrix} \varepsilon_x & \gamma_{xy} & \gamma_{xz} \\ & \varepsilon_y & \gamma_{yz} \\ sym. & & \varepsilon_z \end{bmatrix}$$

- Displacement (3)

$$\begin{bmatrix} u & v & w \end{bmatrix}$$

Governing equations (15)

- Equilibrium equations (3)

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + f_x = 0 \quad (x,y,z)$$

(x,y, z) indicates that there are two more equations obtainable by cyclic permutation (循环排列) of x, y, z.

- Strain-displacement (6)

$$\varepsilon_x = \frac{\partial u}{\partial x} \qquad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \quad (x,y,z;u,v,w)$$

- Stress-strain relations (6)

$$\sigma_x = 2G\varepsilon_x + \lambda\varepsilon \qquad \tau_{xy} = G\gamma_{xy} \quad (x,y,z)$$

We have already gotten 15 equations for the 15 unknowns

Boundary Conditions

- There are infinite stress, strain, and displacement distributions satisfying the governing equations.
  - each stress, strain, and displacement distribution represent a solution to some elasticity problem
- How to get the specific results for a specific problem?
  - boundary conditions
    - surface forces, displacement constraints

The solution of a problem in elasticity (弹性力学的定解问题) consists of the determination of the stress, strain and displacement functions satisfying the governing equations and boundary conditions.

Boundary Conditions

The boundary conditions can be either prescribed stress **vector**, displacement **vector**, or a combination of these on each surface of the body.

- **Stress boundary condition:**

  Surface stress vector/tractions

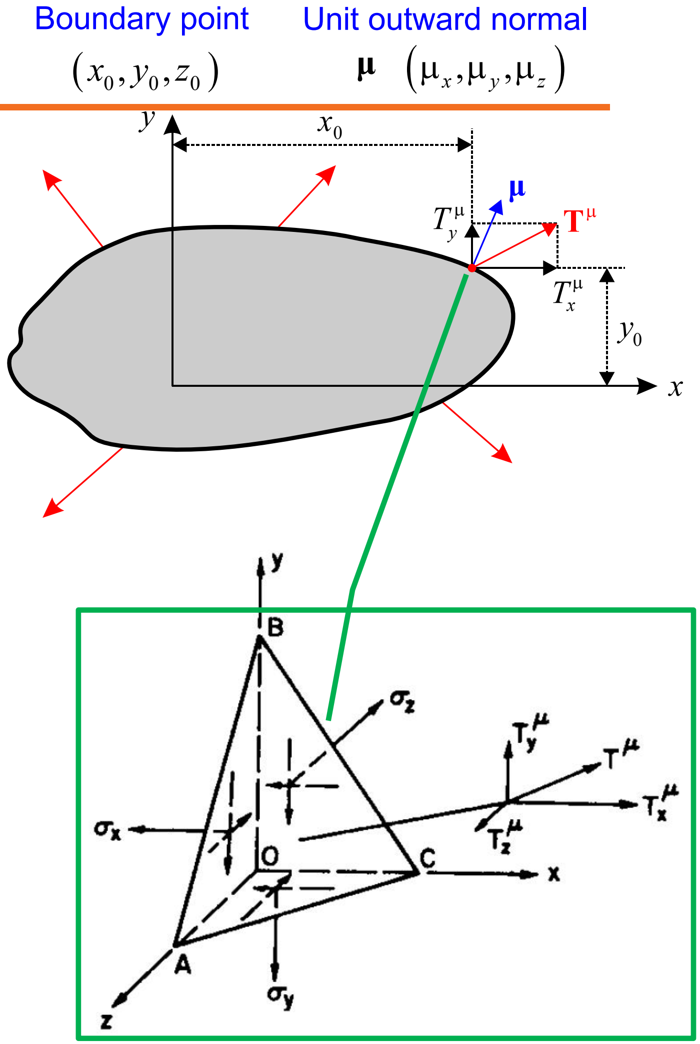
  $\boldsymbol{T}^{\boldsymbol{\mu}} \quad \left(T_x^{\mu}, T_y^{\mu}, T_z^{\mu}\right)$

  – Based on force equilibrium:

$$\begin{cases} T_x^{\mu} = \sigma_{x|0}\mu_x + \tau_{yx|0}\mu_y + \tau_{zx|0}\mu_z \\ T_y^{\mu} = \tau_{xy|0}\mu_x + \sigma_{y|0}\mu_y + \tau_{zy|0}\mu_z \\ T_z^{\mu} = \tau_{xz|0}\mu_x + \tau_{yz|0}\mu_y + \sigma_{z|0}\mu_z \end{cases}$$

- **Displacement boundary condition:**

$$\begin{cases} u\left(x_0, y_0, z_0\right) = u_b \\ v\left(x_0, y_0, z_0\right) = v_b \\ w\left(x_0, y_0, z_0\right) = w_b \end{cases}$$



Boundary Conditions

- **Stress boundary-value problem**
  - First boundary-value problem, stress is prescribed over the **entire** boundary surface.
- **Displacement boundary-value problem**
  - Second boundary-value problem, displacement is prescribed over the **entire** boundary surface.

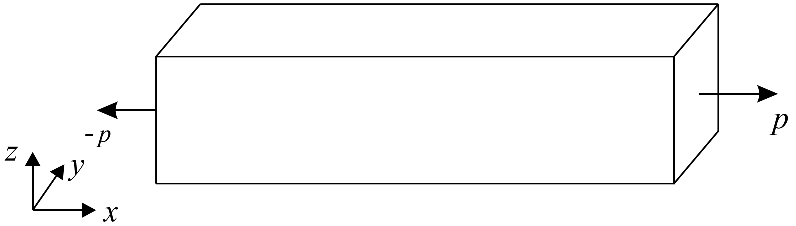
$$\text{e.g.} \quad u_b = v_b = w_b = 0$$

- **Mixed boundary-value problem**
  - Third boundary-value problem, the displacement components are prescribed over part of the boundary and the stress components over the **rest** of the boundary.

Boundary Conditions

- **Example**

What kind of boundary condition does this example belong to?



First boundary-value problem

Let's check $\boldsymbol{T^{\mu}}$ at each boundary

At left/right boundaries $(\mu_x, \mu_y, \mu_z) = (\pm 1, 0, 0)$

$$T_x^{\mu} = \pm p,\ T_y^{\mu} = T_z^{\mu} = 0 \Rightarrow \sigma_{xo} = p,\ \tau_{xyo} = \tau_{xzo} = 0$$

At front/rear boundaries

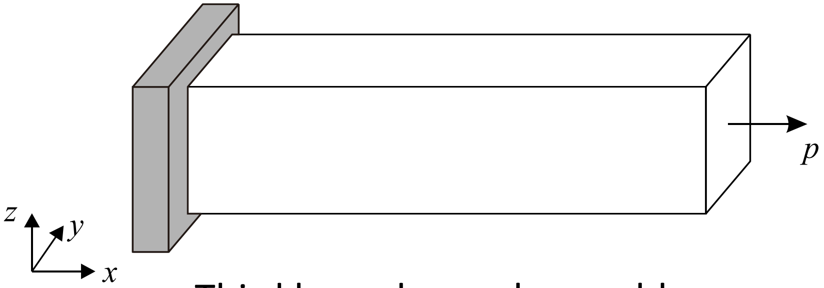
$$T_x^{\mu} = T_y^{\mu} = T_z^{\mu} = 0 \Rightarrow \sigma_{yo} = \tau_{xyo} = \tau_{yzo} = 0$$

At top/bottom boundaries

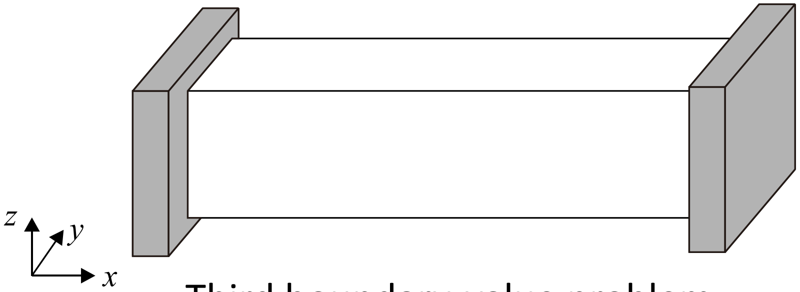
$$T_x^{\mu} = T_y^{\mu} = T_z^{\mu} = 0 \Rightarrow \sigma_{yo} = \tau_{xyo} = \tau_{yzo} = 0$$

Boundary Conditions

What kind of boundary conditions do these two examples belong to?



Third boundary-value problem



Third boundary-value problem

Given the differential equation, you have the first (Dirichlet, 狄利克雷), the second (Neumann, 诺依曼), or the third (mixed, 混合) boundary conditions for each **single part** of the domain boundary.

Note: 弹性力学里的应力、位移、混合边值问题（作用于整个区域）并不等同于微分方程求解中的边界条件（作用于单个边界点）