Homework 3

Due Oct 28 2021

1. Given an arbitrary real tensor $\boldsymbol{A}$, we can always obtain its symmetric and anti-symmetric parts,

$$\boldsymbol{A} = \boldsymbol{A}^s + \boldsymbol{A}^w, \quad \boldsymbol{A}^s := \frac{1}{2}\left(\boldsymbol{A} + \boldsymbol{A}^T\right), \quad \boldsymbol{A}^w := \frac{1}{2}\left(\boldsymbol{A} - \boldsymbol{A}^T\right).$$

   (a) What are the three possible forms of $\boldsymbol{A}^s$?

   (b) The tensor $\boldsymbol{A}^w$ can be represented by its dual vector. How do one determine the dual vector?

2. Let us denote the acceleration defined on the initial configuration and current configuration as $\boldsymbol{A}(\boldsymbol{X}, t)$ and $\boldsymbol{a}(\boldsymbol{x}, t)$. Similarily, we denote the velocity on the two configurations as $\boldsymbol{V}(\boldsymbol{X}, t)$ and $\boldsymbol{v}(\boldsymbol{x}, t)$, respectively. We know that points between the two configurations are related by $\boldsymbol{x} = \boldsymbol{\varphi}(\boldsymbol{X}, t)$. Therefore, the quantites are related by

$$\boldsymbol{V}(\boldsymbol{X}, t) = \boldsymbol{v}(\varphi(\boldsymbol{X}, t), t), \quad \boldsymbol{A}(\boldsymbol{X}, t) = \boldsymbol{a}(\varphi(\boldsymbol{X}, t), t).$$

   And we also know that

$$\boldsymbol{A} = \frac{D\boldsymbol{V}}{Dt},$$

   in which D/Dt means the material time derivative. Show that

$$\boldsymbol{a}(\boldsymbol{x}, t) = \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{t}} + (\nabla_{\boldsymbol{x}} \boldsymbol{v})\, \boldsymbol{v}.$$

3. Suppose the deformation gradient at a point in a body has components

$$[\boldsymbol{F}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}.$$

   Determine the components of the right Cauchy-Green deformation tensor $\boldsymbol{C}$ and the Green-Lagrangian strain tensor $\boldsymbol{E}$.

4. Consider the motion given by

$$\boldsymbol{x} = \boldsymbol{X} + X_1 k \boldsymbol{e}_1.$$

   Let $d\boldsymbol{X}^1 = \left(dS_1/\sqrt{2}\right)(\boldsymbol{e}_1 + \boldsymbol{e}_2)$ and $d\boldsymbol{X}^2 = \left(dS_2/\sqrt{2}\right)(-\boldsymbol{e}_1 + \boldsymbol{e}_2)$ be two material elements in the initial configuration.

   (a) Determine the deformed elements $d\boldsymbol{x}^1$ and $d\boldsymbol{x}^2$.

   (b) Determine the relative length change of these two line elements.

   (c) Determine the angle change between them.

(d) Let k be 0.01 and 1, respectively. Compare the results of (b) and (c) predicted by the Green-Lagrange strain and infinitesimal strain tensor.

5. In the infinitesimal strain theory, we do not distinguish the difference between Lagrangian and Eulerian description, as the displacement is very small compared to the dimension of the body. Thus, the infinitesimal strain tensor can be written as $\boldsymbol{\varepsilon} := \frac{1}{2}\left(\nabla_{\boldsymbol{x}}\boldsymbol{u} + \nabla_{\boldsymbol{x}}\boldsymbol{u}^T\right)$. Apparently $\boldsymbol{\varepsilon}$ is real and symmetric, and the results of Problem 1 (a) can be straigntforwardly applied here. Here we consider a specific example. Let the strain be in a plane state, that is,

$$[\boldsymbol{\varepsilon}] = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & 0 \\ \varepsilon_{12} & \varepsilon_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

We may choose a different frame $\left\{\boldsymbol{e}_i^{'}\right\}$ by rotating $\{\boldsymbol{e}_i\}$, i.e., $\boldsymbol{e}_i^{'} = \boldsymbol{Q}\boldsymbol{e}_i$. In particular, we only rotate on the $e_1 - e_2$ plane, and the orthogonal tensor $\boldsymbol{Q}$ takes the following form

$$[\boldsymbol{Q}] = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

In the above, θ is the angle of the rotation.

(a) Show that in the new frame $\left\{\boldsymbol{e}_i^{'}\right\}$, the components of the tensor $\boldsymbol{\varepsilon}$ adopts the following form,

$$[\boldsymbol{\varepsilon}]_{\boldsymbol{e}_i^{'}} = \begin{bmatrix} \varepsilon_{11}^{'} & \varepsilon_{12}^{'} & 0 \\ \varepsilon_{12}^{'} & \varepsilon_{22}^{'} & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

and

$$\begin{aligned} \varepsilon_{11}^{'} &= \frac{1+\cos(2\theta)}{2}\varepsilon_{11} + \frac{1-\cos(2\theta)}{2}\varepsilon_{22} + \sin(2\theta)e_{12}, \\ \varepsilon_{22}^{'} &= \frac{1-\cos(2\theta)}{2}\varepsilon_{11} + \frac{1+\cos(2\theta)}{2}\varepsilon_{22} - \sin(2\theta)e_{12}, \\ \varepsilon_{12}^{'} &= -\frac{\sin(2\theta)}{2}\left(\varepsilon_{11} - \varepsilon_{22}\right) + \cos(2\theta)\varepsilon_{12}. \end{aligned}$$

(b) Determine the angle θ_1 that makes $\varepsilon_{12}^{'} = 0$.

(c) Determine the values of $\varepsilon_{11}^{'}$ and $\varepsilon_{22}^{'}$ when $\theta = \theta_1$.

(d) Determine the angle θ_2 that maximize the shear strain $\varepsilon_{12}^{'}$.

(e) What is the difference between θ_1 and θ_2?