Homework 5

Due Nov 25 2021

1. Consider two tensors $\boldsymbol{A}$ and $\boldsymbol{B}$. They can be decomposed into volumetric parts and deviatoric parts as follows,

$$\boldsymbol{A} = \boldsymbol{A}_{\mathrm{vol}} + \boldsymbol{A}_{\mathrm{dev}}, \quad \boldsymbol{B} = \boldsymbol{B}_{\mathrm{vol}} + \boldsymbol{B}_{\mathrm{dev}}, \quad \boldsymbol{A}_{\mathrm{vol}} = \frac{1}{3}\left(\mathrm{tr}\boldsymbol{A}\right)\boldsymbol{I}, \quad \boldsymbol{B}_{\mathrm{vol}} = \frac{1}{3}\left(\mathrm{tr}\boldsymbol{B}\right)\boldsymbol{I}.$$

   (a) Show that $\boldsymbol{A}_{\mathrm{vol}} : \boldsymbol{B}_{\mathrm{dev}} = 0$.

   (b) Show that $\boldsymbol{A} : \boldsymbol{B} = \frac{1}{3}\left(\mathrm{tr}\boldsymbol{A}\right)\left(\mathrm{tr}\boldsymbol{B}\right) + \boldsymbol{A}_{\mathrm{dev}} : \boldsymbol{B}_{\mathrm{dev}}$.

   (c) Show that for isotropic linear elastic materials, the strain energy W can be represented as

$$W = \left(\frac{\lambda}{2} + \frac{\mu}{3}\right)\left(\mathrm{tr}\boldsymbol{e}\right)^2 + \mu|\boldsymbol{e}_{\mathrm{dev}}|^2.$$

   (d) The positive-definiteness of the strain energy requires that $\frac{\lambda}{2} + \frac{\mu}{3} > 0$ and $\mu > 0$ from the above representation. Use the two inequalities to determine the range of Young's modulus and Poisson's ratio.

   (e) What is the relationship between the Mohr's circle of the Cauchy stress $\boldsymbol{\sigma}$ and the Mohr's circle of its deviatoric part

$$\boldsymbol{\sigma}_{\mathrm{dev}} := \boldsymbol{\sigma} - \frac{1}{3}\left(\mathrm{tr}\boldsymbol{\sigma}\right)\boldsymbol{I}?$$

2. Consider the generalized Hooke's law,

$$\boldsymbol{\sigma} = \mathbb{C}\boldsymbol{e} = \lambda \mathrm{tr}\boldsymbol{e}\boldsymbol{I} + 2\mu\boldsymbol{e}.$$

   (a) Recall that the Voigt's notation allows one to exploit the minor symmetry properties and express $\boldsymbol{\sigma}$ and $\boldsymbol{e}$ as a 'vector' of six components. Express the Cauchy stress and the infinitesimal strain tensors in terms of the Voigt notation.

   (b) Express the elasticity tensor $\mathbb{C}$ in terms of the Voigt notation.

   (c) Show that $\boldsymbol{\sigma} : \boldsymbol{e} = \sigma_I e_I$, where σ_I and e_I are the components of the Voigt vectors, and I runs from 1 to 6.

3. Given the Young's modulus E, determine the elastic moduli λ, G, and K for the cases of Poisson's ratio $\nu = 0, 0.3, 0.5$, respectively. Comment on the case with $\nu = 0.5$.

4. The stress state represented from the following forms

$$[\boldsymbol{\sigma}] = \begin{bmatrix} \sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad [\boldsymbol{\sigma}] = \begin{bmatrix} 0 & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad [\boldsymbol{\sigma}] = \begin{bmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{bmatrix},$$

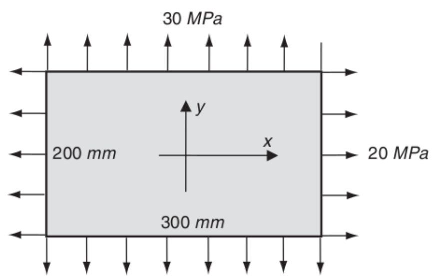
   are referred to as simple tension, pure shear, and hydrostatic compression (tension), respectively. The Young's moduli for aluminum, steel, and rubber are 68.9 GPa, 207 GPa, and 0.0019 GPa, respectively. The Poisson's ratios for aluminum, steel, and rubber are 0.34, 0.29, and 0.499, respectively.

(a) Calculate the strains in aluminum in the following cases: simple tension with $\sigma = 150$ MPa, pure shear with $\tau = 75$ MPa, and hydrostatic compression with $p = 500$ MPa.

(b) Calculate the strains in steel in the following cases: simple tension with $\sigma = 300$ MPa, pure shear with $\tau = 150$ MPa, and hydrostatic compression with $p = 500$ MPa.

(c) Calculate the strains in rubber in the following cases: simple tension with $\sigma = 15$ MPa, pure shear with $\tau = 7$ MPa, and hydrostatic compression with $p = 500$ MPa.

(d) A rectangular steel plate with thickness being 4 mm is subjected to a uniform biaxial stress field as shown in the following figure. Determine changes in the dimensions of the plate under this loading.



5. Assume that the body force vanishes and the body reaches equilibrium (i.e. $\boldsymbol{a} = \mathbf{0}$).

(a) Show that

$$\nabla^2 (\mathrm{tr}\boldsymbol{e}) = 0.$$

(b) Use the result in (a) to show that the displacement components u_i satisfy the following equation,

$$\nabla^2 \left(\nabla^2 u_i\right) = 0.$$

The above equation is called a biharmonic equation and its solution is called a biharmonic function. Hence, the displacement components are biharmonic.

6. Explicitly state the six Beltrami-Michell equations using the following notations,

$$[\boldsymbol{\sigma}] = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix}.$$