MAE5009: Continuum Mechanics B

Assignment 04: Formulation of Problems in Elasticity

Due November 9, 2020

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1. Verify the following equations for plane strain problems with constant f_x and f_y:

(a) $\frac{\partial}{\partial y}\nabla^2 u = \frac{\partial}{\partial x}\nabla^2 v$

Solution:

According to the displacement formation of the solution of plane strain problems, we can know:

$$G\nabla^2 u + (\lambda + G)\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + f_x = 0$$

$$G\nabla^2 v + (\lambda + G)\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + f_y = 0$$

Then:

$$G\frac{\partial}{\partial y}\nabla^2 u = -(\lambda + G)\frac{\partial^2}{\partial x \partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$

$$G\frac{\partial}{\partial x}\nabla^2 v = -(\lambda + G)\frac{\partial^2}{\partial x \partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$

Thus:

$$\frac{\partial}{\partial y}\nabla^2 u = \frac{\partial}{\partial x}\nabla^2 v = -\frac{(\lambda + G)}{G}\frac{\partial^2}{\partial x \partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$

(b) $\frac{\partial}{\partial x}\nabla^2 u + \frac{\partial}{\partial y}\nabla^2 v = 0$

Solution:

According to (a),

$$\nabla^2 u = -\frac{(\lambda + G)}{G}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) - \frac{f_x}{G}$$

$$\nabla^2 v = -\frac{(\lambda + G)}{G}\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) - \frac{f_y}{G}$$

Then:

$$\frac{\partial}{\partial x}\nabla^2 u = -\frac{(\lambda + G)}{G}\frac{\partial^2}{\partial x^2}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$

$$\frac{\partial}{\partial y}\nabla^2 v = -\frac{(\lambda + G)}{G}\frac{\partial^2}{\partial y^2}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$

And:

$$\frac{\partial}{\partial x}\nabla^2 u + \frac{\partial}{\partial y}\nabla^2 v = -\frac{(\lambda + G)}{G}\left(\frac{\partial}{\partial x}\nabla^2 u + \frac{\partial}{\partial y}\nabla^2 v\right)$$

$$\frac{(\lambda+2G)}{G}\left(\frac{\partial}{\partial x}\nabla^2 u+\frac{\partial}{\partial y}\nabla^2 v\right)=0$$

Since $\frac{(\lambda+2G)}{G}$ is not zero, we can prove

$$\frac{\partial}{\partial x}\nabla^2 u+\frac{\partial}{\partial y}\nabla^2 v=0$$

(c) $\nabla^4 u=\nabla^4 v=0$, where $\nabla^4=\nabla^2(\nabla^2)$.

Solution:

$$\nabla^4 u=\nabla^2(\nabla^2 u)=\frac{\partial^2}{\partial x^2}\nabla^2 u+\frac{\partial^2}{\partial y^2}\nabla^2 u$$
$$\nabla^4 v=\nabla^2(\nabla^2 v)=\frac{\partial^2}{\partial x^2}\nabla^2 v+\frac{\partial^2}{\partial y^2}\nabla^2 v$$

According to (a), $\frac{\partial}{\partial y}\nabla^2 u=\frac{\partial}{\partial x}\nabla^2 v$, then $\frac{\partial^2}{\partial y^2}\nabla^2 u=\frac{\partial^2}{\partial x\partial y}\nabla^2 v$, $\frac{\partial^2}{\partial x^2}\nabla^2 v=\frac{\partial^2}{\partial x\partial y}\nabla^2 u$.

According to (b), $\frac{\partial}{\partial x}\nabla^2 u=-\frac{\partial}{\partial y}\nabla^2 v$, then $\frac{\partial^2}{\partial x^2}\nabla^2 u=-\frac{\partial^2}{\partial x\partial y}\nabla^2 v$, $\frac{\partial^2}{\partial y^2}\nabla^2 v=-\frac{\partial^2}{\partial x\partial y}\nabla^2 u$.

Thus,

$$\nabla^4 u=\nabla^2(\nabla^2 u)=\frac{\partial^2}{\partial x^2}\nabla^2 u+\frac{\partial^2}{\partial y^2}\nabla^2 u=-\frac{\partial^2}{\partial x\partial y}\nabla^2 v+\frac{\partial^2}{\partial x\partial y}\nabla^2 v=0$$
$$\nabla^4 v=\nabla^2(\nabla^2 v)=\frac{\partial^2}{\partial x^2}\nabla^2 v+\frac{\partial^2}{\partial y^2}\nabla^2 v=\frac{\partial^2}{\partial x\partial y}\nabla^2 u-\frac{\partial^2}{\partial x\partial y}\nabla^2 u=0$$
$$\nabla^4 u=\nabla^4 v=0$$

2. A bar of constant mass density ρ hangs under its own weight and is supported by the uniform stress σ_0 as shown in the figure. Assume that the stresses σ_x, σ_y, τ_{xy}, τ_{xz} and τ_{yz} are all zero,

(a) based on the above assumption, reduce 15 governing equations to seven equations in terms of σ_z, ε_x, ε_y, ε_z, u, v and w.

Solution:

For the stress-strain relations:

$$\sigma_x=2G\varepsilon_x+\lambda(\varepsilon_x+\varepsilon_y+\varepsilon_z)=0$$

$$\sigma_y=2G\varepsilon_y+\lambda(\varepsilon_x+\varepsilon_y+\varepsilon_z)=0$$

$$\sigma_z=2G\varepsilon_z+\lambda(\varepsilon_x+\varepsilon_y+\varepsilon_z)=E\varepsilon_z$$

We can get $\varepsilon_x=\varepsilon_y=-\nu\varepsilon_z$, $\gamma_{xy}=\gamma_{xz}=\gamma_{yz}=0$.

According to the equilibrium equations:

$$f_x=f_y=0$$
$$\frac{\partial\sigma_z}{\partial z}+f_z=0$$

For the strain-displacement relations:

$$\varepsilon_x = \varepsilon_y = -\nu\varepsilon_z = \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = -\nu\frac{\partial w}{\partial z}$$

$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0$, $\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = 0$ and $\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = 0$.

(b) integrate the equilibrium equation to show that $\sigma_z = \rho g z$ where g is the acceleration due to gravity. Also show that the prescribed boundary conditions are satisfied by this solution

Solution:

Let the cross-section area be A, the force equilibrium for any z should be satisfied.

$$\sigma_z A - mg = 0$$

Then $mg = \rho g V = \rho g A z$ and $\sigma_z A - \rho g A z = 0$, we can prove:

$$\sigma_z = \rho g z$$

When $z = l$, $\sigma_z = \rho g l = \sigma_0$. When $z = 0$, $\sigma_z = 0$.

(c) find ε_x, ε_y, ε_z from the generalized Hooke's law

Solution:

$$\sigma_x = 2G\varepsilon_x + \lambda(\varepsilon_x + \varepsilon_y + \varepsilon_z) = 0$$

$$\sigma_y = 2G\varepsilon_y + \lambda(\varepsilon_x + \varepsilon_y + \varepsilon_z) = 0$$

We can get $\varepsilon_x = \varepsilon_y = -\nu\varepsilon_z$, then

$$\sigma_z = 2G\varepsilon_z + \lambda(\varepsilon_x + \varepsilon_y + \varepsilon_z) = E\varepsilon_z$$

So $\varepsilon_z = \frac{\rho g z}{E}, \varepsilon_x = \varepsilon_y = -\nu\varepsilon_z = -\nu\frac{\rho g z}{E}$.

(d) if the displacement and rotation components are zero at the point $(0,0,l)$, determine the displacement component u and v

Solution:

According to (a) (b) and (c), $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = -\frac{\nu\rho g z}{E}, \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0$, then we can make assumptions that

$$u = -\frac{\nu\rho g z}{E}x + C_1(y,z), v = -\frac{\nu\rho g z}{E}y + C_2(x,z), w = \frac{\rho g}{2E}z^2 + C_3(x,y)$$

Since the displacement and rotation components are zero at point $(0,0,l)$, we can get

$$C_1(y,z)|_{(0,l)} = 0, C_2(x,z)|_{(0,l)} = 0, C_3(x,y)|_{(0,0)} = -\frac{\rho g}{2E}l^2$$

$$\frac{\partial u}{\partial y} = \frac{\partial C_1(y,z)}{\partial y}|_{(0,l)} = 0, \frac{\partial v}{\partial x} = \frac{\partial C_2(x,z)}{\partial x}|_{(0,l)} = 0$$

$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{\partial C_1(y,z)}{\partial y} + \frac{\partial C_2(x,z)}{\partial x} = 0$$

According to $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0$, we know that

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y}, \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial x \partial y}$$

Because $\frac{\partial^2 u}{\partial x^2} = 0$, then $\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 C_1(y,z)}{\partial y^2} = -\frac{\partial^2 u}{\partial x^2} = 0$. The $\frac{\partial C_1(y,z)}{\partial y}$ is not a function of y. With the same way, $\frac{\partial C_2(x,z)}{\partial x}$ is not a function of x.

Since $\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0$, then $\frac{\partial C_1(y,z)}{\partial y} = -\frac{\partial C_2(x,z)}{\partial x} = f_1(z)$

$$\frac{\partial C_1(y,z)}{\partial y}|_{(0,l)} = 0$$

$$f_1(z)|_{z=l} = 0$$

Then $C_1(y,z) = f_1(z)y, C_2(x,z) = -f_1(z)x$

$$u = -\frac{\nu\rho g}{E}xz + f_1(z)y, v = -\frac{\nu\rho g}{E}yz - f_1(z)x$$

With the same way about u and w, $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial z^2} = \frac{\partial^2 f_1(z)}{\partial z^2}y = 0$

Then $\frac{\partial^2 f_1(z)}{\partial z^2} = 0$ and $\frac{\partial f_1(z)}{\partial z} = A_1$, where A_1 is a constant.

So $f_1(z) = A_1 z - A_1 l$ for the boundary condition that $f_1(z)|_{z=l} = 0$

Since:

$$\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = \left(-\frac{\nu\rho g}{E}x + A_1 y\right) + \frac{\partial C_3(x,y)}{\partial x} = 0, C_3(x,y) = \frac{\nu\rho g}{2E}x^2 - A_1 xy + f_2(y)$$

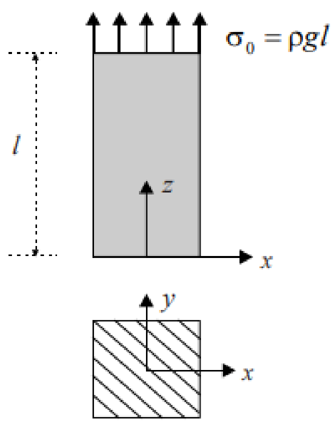
$$\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = \left(-\frac{\nu\rho g}{E}y - A_1 x\right) + \frac{\partial C_3(x,y)}{\partial y} = 0, C_3(x,y) = \frac{\nu\rho g}{2E}y^2 + A_1 xy + f_3(x)$$

The boundary condition $C_3(x,y)|_{(0,0)} = -\frac{\rho g}{2E}l^2$

Then $A_1 = 0$, $f_1(z) = 0, f_2(y) = \frac{\nu\rho g}{2E}y^2 - \frac{\rho g}{2E}l^2, f_3(x) = \frac{\nu\rho g}{2E}x^2 - \frac{\rho g}{2E}l^2$

$$C_3(x,y) = \frac{\nu\rho g}{2E}x^2 + \frac{\nu\rho g}{2E}y^2 - \frac{\rho g}{2E}l^2$$

$$u = -\frac{\nu\rho g}{E}xz, v = -\frac{\nu\rho g}{E}yz$$



(e) prove that $w = \frac{\rho g}{2E}(z^2 + \nu x^2 + \nu y^2 - l^2)$

Solution:

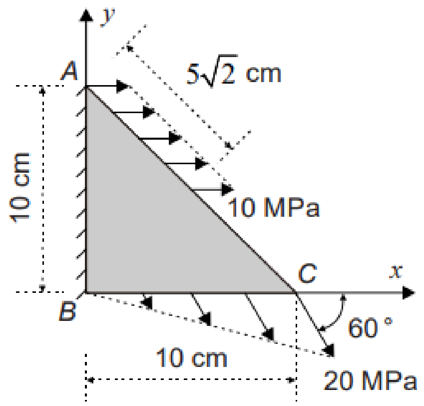
According to (d),

$$C_3(x,y) = \frac{\nu\rho g}{2E}x^2 + \frac{\nu\rho g}{2E}y^2 - \frac{\rho g}{2E}l^2$$

Then

$$w = \frac{\rho g}{2E}z^2 + C_3(x,y) = \frac{\rho g}{2E}z^2 + \frac{\nu\rho g}{2E}x^2 + \frac{\nu\rho g}{2E}y^2 - \frac{\rho g}{2E}l^2 = \frac{\rho g}{2E}(z^2 + \nu x^2 + \nu y^2 - l^2)$$

3. Express the boundary conditions for the following plate subjected to plate strain condition. The surface forces are functions of x and y only.



Solution:

Surface $f(x, y)$, for plane strain problem $\varepsilon_z = \gamma_{xz} = \gamma_{yz} = 0$

The boundary conditions:

$$\varepsilon_x = \varepsilon_y = 0, \quad x = 0$$

AB :

$$\begin{cases} \sigma_x = 2x\cos(60°), y = 0, 0 \le x \le 10 \\ \sigma_y = -2x\sin(60°), y = 0, 0 \le x \le 10 \end{cases}$$

$$\begin{cases} \sigma_x = 10\ MPa, 0 \le x \le 5, y = 10 - x \\ \sigma_y = 0\ MPa, 0 \le x \le 5, y = 10 - x \end{cases}$$

$$\begin{cases} \sigma_x = 0\ MPa, 5 < x < 10, y = 10 - x \\ \sigma_y = 0\ MPa, 5 \le x < 10, y = 10 - x \end{cases}$$