Homework 3

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1. (a) 1° A^S has 3 different eigen values.

2° A^S has eigen values that satisfy $\lambda_1 = \lambda_2 \neq \lambda_3$

3° A^S has 3 eigen values that are equal.

(b) $T_{12} = \vec{e_1} \cdot \bar{\bar{A}}^w \vec{e_2} = \vec{e_1} \cdot \vec{t}^w \times \vec{e_2} = \vec{t}^w \cdot \vec{e_2} \times \vec{e_1} = -\vec{t}^w \vec{e_3} = -t^w_3$

$T_{23} = \vec{e_2} \cdot \bar{\bar{A}}^w \vec{e_3} = \vec{e_2} \cdot \vec{t}^w \times \vec{e_3} = \vec{t}^w \cdot \vec{e_3} \times \vec{e_2} = -\vec{t}^w \cdot \vec{e_1} = -t_1^w$

$T_{31} = \vec{e_3} \cdot \bar{\bar{A}}^w \vec{e_1} = \vec{e_3} \cdot \vec{t}^w \times \vec{e_1} = \vec{t}^w \cdot \vec{e_1} \times \vec{e_3} = -\vec{t}^w \cdot \vec{e_2} = -t_2^w$

So, $\vec{t}^w = [-T_{23}, -T_{31}, -T_{12}]^T = [T_{32}, T_{13}, T_{21}]^T$

2. $$a_1(\vec{x},t) = \frac{\partial v_1(x_1,x_2,x_3,t)}{\partial t} + \frac{\partial v_1(x_1,x_2,x_3,t)}{\partial x_1} \cdot \frac{\partial x_1}{\partial t} + \frac{\partial v_1(x_1,x_2,x_3,t)}{\partial x_2} \cdot \frac{\partial x_2}{\partial t} + \frac{\partial v_1(x_1,x_2,x_3,t)}{\partial x_3} \cdot \frac{\partial x_3}{\partial t}$$

$$= \frac{\partial v_1}{\partial t} + \frac{\partial v_1}{\partial x_1} \cdot v_1 + \frac{\partial v_1}{\partial x_2} \cdot v_2 + \frac{\partial v_1}{\partial x_3} \cdot v_3 = \frac{\partial v_1}{\partial t} + \frac{\partial v_1}{\partial x_i} v_i,$$

Similarly, $a_2(\vec{x},t) = \frac{\partial v_2}{\partial t} + \frac{\partial v_2}{\partial x_i} v_i$, $a_3(\vec{x},t) = \frac{\partial v_3}{\partial t} + \frac{\partial v_3}{\partial x_i} v_i$,

So, $\vec{a}(\vec{x},t) = \frac{\partial \vec{v}}{\partial t} + (\nabla_{\vec{x}} \vec{v}) \vec{v}$.

3. $$\bar{\bar{C}} = \bar{\bar{F}}^T \bar{\bar{F}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 4 \\ 0 & 4 & 5 \end{bmatrix}$$

$$\nabla \vec{u} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \quad \bar{\bar{E}} = \frac{1}{2}[\nabla \vec{u} + \nabla \vec{u}^T + (\nabla \vec{u})^T (\nabla \vec{u})] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 2 & 2 \end{bmatrix}$$

4. (a) $\vec{u} = \vec{x} - \vec{X} = X_1 k \vec{e_1}$, $\nabla \vec{u} = \begin{bmatrix} k & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\bar{\bar{F}} = \bar{\bar{I}} + \nabla \vec{u} = \begin{bmatrix} k+1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$d\vec{x} = \bar{\bar{F}} d\vec{X} = \begin{bmatrix} (k+1) dX_1 \\ dX_2 \\ dX_3 \end{bmatrix}$$, such is, $d\vec{x}^1 = [(k+1)\cdot\frac{dS_1}{\sqrt{2}}, \frac{dS_1}{\sqrt{2}}, 0]^T$

$d\vec{x}^2 = [-(k+1)\cdot\frac{dS_2}{\sqrt{2}}, \frac{dS_2}{\sqrt{2}}, 0]^T$.

(b) $$\frac{|d\vec{x}^1|}{|d\vec{X}^1|} = \frac{\sqrt{(\frac{(k+1)^2}{2} + \frac{1}{2})(ds)^2}}{ds} = \frac{\sqrt{2(k^2+2k+2)}}{2}$$

$$\frac{|d\vec{x}^2|}{|d\vec{X}^2|} = \frac{\sqrt{(\frac{[-(k+1)]^2}{2} + \frac{1}{2})(ds)^2}}{ds} = \frac{\sqrt{2(k^2+2k+2)}}{2}.$$

(c) $$\cos\theta = \frac{d\vec{x}^1 \cdot d\vec{x}^2}{|d\vec{x}^1|\,|d\vec{x}^2|} = \frac{\frac{-(k+1)^2+1}{2} dS_1 \cdot dS_2}{\sqrt{\frac{k^2+2k+2}{2}} \cdot \sqrt{\frac{k^2+2k+2}{2}} \cdot dS_1 dS_2} = \frac{-k^2-2k}{k^2+2k+2}$$

$$\cos\alpha = \frac{d\vec{X}^1 \cdot d\vec{X}^2}{|d\vec{X}^1|\,|d\vec{X}^2|} = \frac{-\frac{1}{2}+\frac{1}{2}}{1} = 0.$$

$\theta = \arccos\left(\frac{-k^2-2k}{k^2+2k+2}\right)$, $\alpha = \frac{\pi}{2}$, $\theta - \alpha = \arccos\left(\frac{-k^2-2k}{k^2+2k+2}\right) - \frac{\pi}{2}$.

(d) 1° when $k = 0.01$, then $\frac{|d\vec{x}^1|}{|d\vec{X}^1|} = \frac{|d\vec{x}^2|}{|d\vec{X}^2|} \approx \frac{\sqrt{4.04}}{2} \approx 1$

$\theta \approx \arccos\left(\frac{-0.0201}{2}\right) \approx \arccos(-0.01)$

2° when $k = 0.1$, then $\frac{|d\vec{x}^1|}{|d\vec{X}^1|} = \frac{|d\vec{x}^2|}{|d\vec{X}^2|} = \frac{\sqrt{4.42}}{2} \approx 1.05$

$\theta \approx \arccos\left(\frac{-0.21}{2.2}\right) \approx \arccos(-0.1)$

5. (a) $[\bar{\bar{\varepsilon}}]_{\vec{e}_i'} = [\bar{\bar{Q}}^T][\bar{\bar{\varepsilon}}]_{\vec{e}_i}[\bar{\bar{Q}}]$

$$= \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & 0 \\ \varepsilon_{12} & \varepsilon_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1+\cos 2\theta}{2}\varepsilon_{11} + \frac{1-\cos 2\theta}{2}\varepsilon_{22} + (\sin 2\theta)\varepsilon_{12} & -\frac{\sin 2\theta}{2}(\varepsilon_{11}-\varepsilon_{22}) + (\cos 2\theta)\varepsilon_{12} & 0 \\ -\frac{\sin 2\theta}{2}(\varepsilon_{11}-\varepsilon_{22}) + (\cos 2\theta)\varepsilon_{12} & \frac{1-\cos 2\theta}{2}\varepsilon_{11} + \frac{1+\cos 2\theta}{2}\varepsilon_{22} - \sin 2\theta\,\varepsilon_{12} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

So, $\varepsilon_{11}' = \frac{1+\cos 2\theta}{2}\varepsilon_{11} + \frac{1-\cos 2\theta}{2}\varepsilon_{22} + (\sin 2\theta)\varepsilon_{12}$,

$\varepsilon_{22}' = \frac{1-\cos 2\theta}{2}\varepsilon_{11} + \frac{1+\cos 2\theta}{2}\varepsilon_{22} - (\sin 2\theta)\varepsilon_{12}$,

$\varepsilon_{12}' = -\frac{\sin 2\theta}{2}(\varepsilon_{11}-\varepsilon_{22}) + (\cos 2\theta)\varepsilon_{12}$.

(b) $\varepsilon_{11}' = \frac{1+\cos 2\theta_1}{2}\varepsilon_{11} + \frac{1-\cos 2\theta_1}{2}\varepsilon_{22} + (\sin 2\theta_1)\varepsilon_{12}$.

$\varepsilon_{22}' = \frac{1-\cos 2\theta_1}{2}\varepsilon_{11} + \frac{1+\cos 2\theta_1}{2}\varepsilon_{22} - (\sin 2\theta_1)\varepsilon_{12}$.

(c) let $\frac{\partial \varepsilon_{12}'}{\partial \theta} = -\cos 2\theta(\varepsilon_{11}-\varepsilon_{22}) - 2\sin 2\theta\,\varepsilon_{12} = 0$.

$\theta = \frac{1}{2}\arctan\left(\frac{\varepsilon_{22}-\varepsilon_{11}}{2\varepsilon_{12}}\right)$

(d)