1. ρ is density, and v_i is flow velocity. Prove the following two equations (hint: the continuity equation is used):

(a) $$\frac{\partial(\rho Q_{lm..})}{\partial t}+\frac{\partial(\rho v_j Q_{lm..})}{\partial x_j}=\rho\frac{DQ_{lm..}}{Dt}$$

(b) $$v_i\frac{\partial(\rho v_i)}{\partial t}+v_i\frac{\partial(\rho v_j v_i)}{\partial x_j}=\frac{\partial(\frac{1}{2}\rho v^2)}{\partial t}+\frac{\partial(\frac{1}{2}\rho v_j v^2)}{\partial x_j}$$

The proof of equation (a) begins by the defition of the material derivative, as

$$\frac{D(\cdot)}{Dt}=\frac{\partial(\cdot)}{\partial t}+\boldsymbol{v}\cdot\nabla(\cdot).$$

We shall take into the above equation, as

$$\rho\frac{D\boldsymbol{Q}}{Dt}=\rho\frac{\partial\boldsymbol{Q}}{\partial t}+\rho\boldsymbol{v}\cdot\nabla\boldsymbol{Q},$$

and

$$\left[\frac{\partial\rho\boldsymbol{Q}}{\partial t}\right]_{lm}=\frac{\partial(\rho Q_{lm})}{\partial t}\quad\text{and}\quad[\nabla\cdot(\rho\boldsymbol{Q}\otimes\boldsymbol{v})]_{lm}=\frac{\partial(\rho Q_{lm}v_j)}{\partial x_j}.$$

$$\frac{\partial(\rho\boldsymbol{Q})}{\partial t}=\frac{\partial\rho}{\partial t}\boldsymbol{Q}+\rho\frac{\partial\boldsymbol{Q}}{\partial t}\quad\text{and}\quad\nabla\cdot(\boldsymbol{Q}\otimes\rho\boldsymbol{v})=(\nabla\cdot(\rho\boldsymbol{v}))\boldsymbol{Q}+\nabla\boldsymbol{Q}\cdot\rho\boldsymbol{v}.$$

$$\frac{\partial(\rho\boldsymbol{Q})}{\partial t}+\nabla\cdot(\boldsymbol{Q}\otimes\rho\boldsymbol{v})=\rho\frac{\partial\boldsymbol{Q}}{\partial t}+\left(\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\boldsymbol{v})\right)\boldsymbol{Q}+\nabla\boldsymbol{Q}\cdot\rho\boldsymbol{v}=\rho\frac{D\boldsymbol{Q}}{Dt},$$

with the continuity equation used. So that

$$\rho\frac{DQ_{lm}}{Dt}=\frac{\partial(\rho Q_{lm})}{\partial t}+\frac{\partial\rho v_j Q_{lm}}{\partial x_j}.$$

The proof of equation (b) begins by

$$\frac{\partial\left(\frac{1}{2}\rho v^2\right)}{\partial t}=\frac{1}{2}\rho\frac{\partial v^2}{\partial t}+|\boldsymbol{v}|^2\frac{\partial(\frac{1}{2}\rho)}{\partial t}=v_i\rho\frac{\partial v_i}{\partial t}+|\boldsymbol{v}|^2\frac{\partial(\frac{1}{2}\rho)}{\partial t}$$

$$v_i\frac{\partial(\rho v_i)}{\partial t}=|\boldsymbol{v}|^2\frac{\partial\rho}{\partial t}+\rho v_i\frac{\partial v_i}{\partial t}$$

$$\frac{\partial\left(\frac{1}{2}\rho v_j v^2\right)}{\partial x_j}=\nabla\cdot\left(\frac{1}{2}\rho\boldsymbol{v}|\boldsymbol{v}|^2\right)=|\boldsymbol{v}|^2\nabla\cdot(\frac{1}{2}\rho\boldsymbol{v})+\boldsymbol{v}\cdot\nabla(\frac{1}{2}\rho|\boldsymbol{v}|^2)=|\boldsymbol{v}|^2\nabla\cdot(\frac{1}{2}\rho\boldsymbol{v})+\boldsymbol{v}\cdot\nabla(\rho\boldsymbol{v})\cdot\boldsymbol{v}.$$

$$v_i\frac{\partial(\rho v_j v_i)}{\partial x_j}=\boldsymbol{v}\cdot\nabla\cdot(\rho\boldsymbol{v}\otimes\boldsymbol{v})=\boldsymbol{v}\cdot(\nabla(\rho\boldsymbol{v})\cdot\boldsymbol{v}+\boldsymbol{v}\nabla\cdot(\rho\boldsymbol{v}))=\boldsymbol{v}\cdot\nabla(\rho\boldsymbol{v})\cdot\boldsymbol{v}+|\boldsymbol{v}|^2\nabla\cdot(\rho\boldsymbol{v}).$$

Conbine them, we get

$$\frac{\partial\left(\frac{1}{2}\rho v^2\right)}{\partial t}+\frac{\partial\left(\frac{1}{2}\rho v_j v^2\right)}{\partial x_j}=v_i\rho\frac{\partial v_i}{\partial t}+|\boldsymbol{v}|^2\frac{\partial(\frac{1}{2}\rho)}{\partial t}+|\boldsymbol{v}|^2\nabla\cdot(\frac{1}{2}\rho\boldsymbol{v})+\boldsymbol{v}\cdot\nabla(\rho\boldsymbol{v})\cdot\boldsymbol{v}.$$

$$v_i\frac{\partial(\rho v_i)}{\partial t}+v_i\frac{\partial(\rho v_j v_i)}{\partial x_j}=\rho v_i\frac{\partial v_i}{\partial t}+|\boldsymbol{v}|^2\frac{\partial\rho}{\partial t}+|\boldsymbol{v}|^2\nabla\cdot(\rho\boldsymbol{v})+\boldsymbol{v}\cdot\nabla(\rho\boldsymbol{v})\cdot\boldsymbol{v}.$$

The purple parts are the continuity equation, which are vanished. So that

$$\frac{\partial\left(\frac{1}{2}\rho v^2\right)}{\partial t}+\frac{\partial\left(\frac{1}{2}\rho v_j v^2\right)}{\partial x_j}=v_i\frac{\partial(\rho v_i)}{\partial t}+v_i\frac{\partial(\rho v_j v_i)}{\partial x_j}.$$

2. The volume flow Q of a centrifugal pump is a function of the input power P, impeller diameter D, rotational rate Ω, and the density ρ and viscosity μ of the fluid:

$$Q = f(P, D, \Omega, \rho, \mu)$$

Rewrite this as a dimensionless relationship as below with Ω, ρ, and D as repeating variables.

$$\frac{Q}{\Omega D^3} = f(\frac{P}{\rho\Omega^3 D^5}, \frac{\mu}{\rho\Omega D^2})$$

Use the dimentional analysis, as

$$[Q] = \mathrm{m}^3/\mathrm{s}, \quad [P] = \mathrm{W} = \mathrm{J/s} = \mathrm{kg}\cdot\mathrm{m}^2/\mathrm{s}^3, \quad [D] = \mathrm{m}, \quad [\Omega] = \mathrm{s}^{-1}, \quad [\rho] = \mathrm{kg/m}^3, \quad [\mu] = \mathrm{kg}/(\mathrm{m}\cdot\mathrm{s}^3)$$

Choose three unrelated quantities (the dimensions are unrelated) as basis, as

$$[D] = \mathrm{m}, \quad [\Omega] = \mathrm{s}^{-1}, \quad [\rho] = \mathrm{kg/m}^3.$$

There exist three other quantities, we can construct the dimension equation respectively, as

$$[Q] = [D]^\alpha[\Omega]^\beta[\rho]^\gamma \Rightarrow \mathrm{m}^3/\mathrm{s} = \mathrm{m}^\alpha\mathrm{s}^{-\beta}(\mathrm{kg/m}^3)^\gamma \Rightarrow \alpha = 3, \beta = 1, \gamma = 0.$$

So that $Q/(D^3\Omega)$ is one dimensionaless quantity.

$$[P] = [D]^\alpha[\Omega]^\beta[\rho]^\gamma \Rightarrow \mathrm{kg}\cdot\mathrm{m}^2/\mathrm{s}^3 = \mathrm{m}^\alpha\mathrm{s}^{-\beta}(\mathrm{kg/m}^3)^\gamma \Rightarrow \alpha = 5, \beta = 3, \gamma = 1.$$

So that $P/(\rho D^5\Omega^3)$ is one dimensionaless quantity.

$$[\mu] = [D]^\alpha[\Omega]^\beta[\rho]^\gamma \Rightarrow \mathrm{kg}/(\mathrm{m}\cdot\mathrm{s}^3) = \mathrm{m}^\alpha\mathrm{s}^{-\beta}(\mathrm{kg/m}^3)^\gamma \Rightarrow \alpha = 2, \beta = 1, \gamma = 1.$$

So that $\mu/(\rho D^2\Omega)$ is one dimensionaless quantity. We can construct the dimensionless relation, as

$$\frac{Q}{\Omega D^3} = f\left(\frac{P}{\rho\Omega^3 D^5}, \frac{\mu}{\rho\Omega D^2}\right).$$

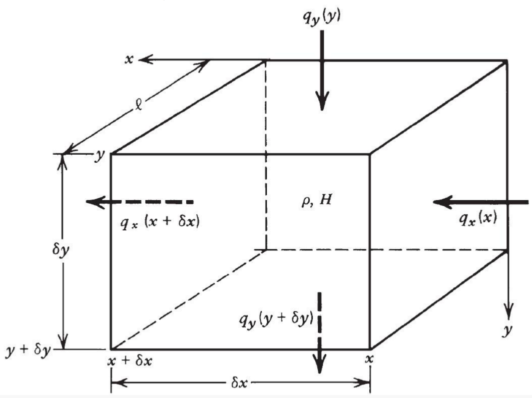
3. One important mathmatical physical equations is the thermal conduction equation (a type of parabolic equation) that describes temperature distribution due to thermal conduction inside a solid body. The thermal conduction equation can be derived by neglecting the advection from the energy conservation equation:

$$\rho c_p \frac{DT}{Dt} = -\frac{\partial q_i}{\partial x_i} + \tau_{ij}\dot{\varepsilon}_{ij} + \alpha T\frac{Dp}{Dt} + \rho H \quad \Rightarrow \quad \rho c_p \frac{\partial T}{\partial t} = -\frac{\partial q_i}{\partial x_i} + \rho H$$

Please derive the thermal conduction equation

$$\rho c_p \frac{\partial T}{\partial t} = k\nabla^2 T + \rho H$$

by conducting thermal balance analysis for a element on the right:



The heat change in this element is

$$\rho c_{\mathrm{p}} \frac{\partial T}{\partial t}\delta x\delta y\delta z = \rho c_{\mathrm{p}}\frac{\partial T}{\partial t}\delta V.$$

The energy change of the element due to the outflow is

$$\left(\frac{q_x(x) - q_x(x+\delta x)}{\delta x} + \frac{q_y(y) - q_y(y+\delta y)}{\delta y} + \frac{q_z(z) - q_z(z+\delta z)}{\delta z}\right)\delta x\delta y\delta z + \rho H\delta x\delta y\delta z = -(q_{,x} + q_{,y} + q_{,z} + \rho$$

So that, the energy conservation reads

$$\rho c_{\mathrm{p}} \frac{\partial T}{\partial t} \delta V = (-\nabla \cdot \boldsymbol{q} + \rho H) \delta V.$$

The Fourier's law reads

$$\boldsymbol{q} = -k \nabla T.$$

The final conservation becomes

$$\rho c_{\mathrm{p}} \frac{\partial T}{\partial t} = k \nabla^2 T + \rho H.$$