南方科技大学
SOUTHERN UNIVERSITY OF SCIENCE AND TECHNOLOGY

MAE5009
Continuum Mechanics B
Session 10: Fluid Statics

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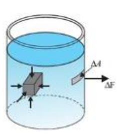
Introduction

- Fluid statics is a branch of fluid mechanics that deals with the behavior/response of fluid and the balance of forces which stabilize fluids when they are at rest or relative rest
- Since there is no relative movement within the fluid, no viscous shear stress exists
- Only pressure exists on the fluid surface
- Fluid statics mainly concerns with problems related to fluid pressure, the pressure on the solid surface, and their distributions



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Forces on fluids

- **Surface forces**: contact force; the forces acting on the surface of the fluids
  - Normal stress:

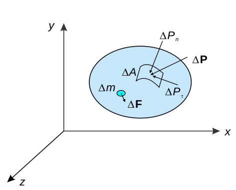
$$p = \lim_{\Delta A \to 0} \frac{\Delta P_n}{\Delta A} = \frac{dP_n}{dA}$$

  - Shear stress:

$$\tau = \lim_{\Delta A \to 0} \frac{\Delta P_\tau}{\Delta A} = \frac{dP_\tau}{dA}$$

- **Body forces**: non-contact force

$$\mathbf{f} = \lim_{\Delta m \to 0} \frac{\Delta \mathbf{F}}{\Delta m} = \frac{d\mathbf{F}}{dm} \quad \text{m/s}^2$$
$$= f_x\mathbf{i} + f_y\mathbf{j} + f_z\mathbf{k}$$



1 atm = 760 mmHg (at 273.15 K, g = 9.80665 m/s^2) = 101325 Pa

e.g., only gravity in *y* direction: f_x=0, f_y=-g; f_z=0

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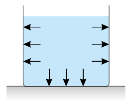
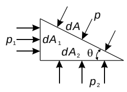
Pressure

- **Characteristics of pressure**:
  – The pressure of a fluid at rest always acts perpendicular to the contact surface and points to the inner normal direction
  – The pressures at any point in a fluid at rest are equal in every direction



$$p_1 dA_1 = p dA \sin\theta \qquad dA_1 = dA \sin\theta$$
$$p_2 dA_2 = p dA \cos\theta \qquad dA_2 = dA \cos\theta$$

$$p = p_1 = p_2$$



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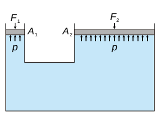
Pressure

- The fluid pressure applied to a fluid in a closed vessel is transmitted to all parts at the same pressure value as that applied (**Pascal's law**)



$$F_2 = F_1 \frac{A_2}{A_1}$$

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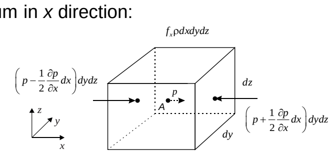
Differential equations of fluid statics

- Equilibrium in x direction:



$$\left(p - \frac{1}{2}\frac{\partial p}{\partial x}dx\right)dydz - \left(p + \frac{1}{2}\frac{\partial p}{\partial x}dx\right)dydz + f_x \rho dxdydz = 0$$

$$f_x - \frac{1}{\rho}\frac{\partial p}{\partial x} = 0 \rightarrow f_y - \frac{1}{\rho}\frac{\partial p}{\partial y} = 0 \quad f_z - \frac{1}{\rho}\frac{\partial p}{\partial z} = 0$$

$$f_i - \frac{1}{\rho} p_{,i} = 0 \qquad \mathbf{f} - \frac{1}{\rho}\nabla p = 0$$

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Differential equations of fluid statics

- Euler's equilibrium equation:

$$f_x - \frac{1}{\rho}\frac{\partial p}{\partial x} = 0 \qquad f_y - \frac{1}{\rho}\frac{\partial p}{\partial y} = 0 \qquad f_z - \frac{1}{\rho}\frac{\partial p}{\partial z} = 0$$

$$\Downarrow$$

$$f_x dx + f_y dy + f_z dz = \frac{1}{\rho}\left(\frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy + \frac{\partial p}{\partial z}dz\right) = \frac{1}{\rho}dp$$

$$\Downarrow$$

$$dp = \rho\left(f_x dx + f_y dy + f_z dz\right)$$

If only gravity exists in z direction:

$$\Downarrow$$

$$dp = -\rho gz$$

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Pressure of fluid at rest under gravity

$$pdA - \left(p + \frac{dp}{dz}dz\right)dA - \rho gAdz = 0$$

$$\Downarrow$$

$$dp = -\rho gdz$$

$$\Downarrow$$

$$p = -\rho g\int dz = -\rho gz + c$$

If the base point is set at z_0 below the upper surface, the upper surface has

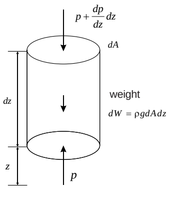
$$p = p_0 \qquad z = z_0$$

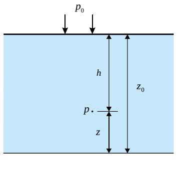
$$\Downarrow$$

$$c = p_0 + \rho gz_0$$

$$\Downarrow$$

$$p = p_0 + \rho g\left(z_0 - z\right) = p_0 + \rho gh$$





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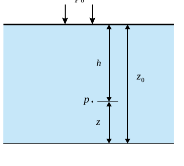
Pressure of fluid at rest under gravity

$$p = p_0 + \rho gh$$

- Under gravity, the pressure of fluid increases linearly with increasing depth
- Under gravity, the pressures of fluid at same depth are equal



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Measurement of pressure

- Absolute pressure uses vacuum as base pressure:

$$p = p_a + \rho g h$$

where p_a is the atmospheric pressure, absolute pressure only has positive values

- Gauge pressure (or relative pressure) uses atmospheric pressure as base pressure:

$$p_g = p - p_a$$

Gauge pressure could be positive and negative

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Measurement of pressure

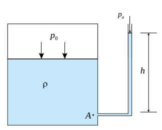
- Manometer
  - A device which measures the fluid pressure by the height of a liquid column

$$p = p_0 + \rho g h$$

- Single tube:

$$p = p_a + \rho g h$$

Disadvantage: the pipe should be long enough



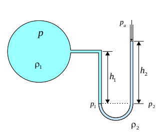
- U-shape tube:

$$p_1 = p + \rho_1 g h_1$$

$$p_2 = p_a + \rho_2 g h_2$$

$$p_1 = p_2$$

$$p = p_a + \rho_2 g h_2 - \rho_1 g h_1$$



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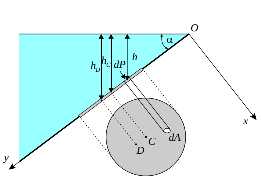
Hydrostatic forces on plane surfaces

- Pressure on the infinitesimal element:

$$p_e = \rho g h$$

$$dP = p_e dA = \rho g h dA = \rho g y \sin\alpha \cdot dA$$



- Overall forces:

$$P = \int_A dP = \rho g \sin\alpha \int_A y dA$$

$$\int_A y dA = y_c A$$ 1[st] moment of area

$$P = \rho g \sin\alpha \cdot y_c A = \rho g h_c A = p_c A$$

The total force P equals the product of the pressure at the centroid C and the underwater area A

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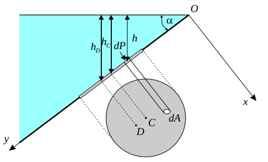
Hydrostatic forces on plane surfaces

- Force center:

$$P \cdot y_D = \int_A dP \cdot y$$



$$\int_A dP \cdot y = \int_A \rho g y \sin\alpha \cdot dA \cdot y$$
$$= \rho g \sin\alpha \int_A y^2 \cdot dA = \rho g \sin\alpha \cdot I_x$$

where $I_x = \int_A y^2 \cdot dA$ Area moment of inertia, or 2nd moment of area

since $I_x = I_{Cx} + y_C^2 A$

y_D is always greater than y_C

$$y_D = \frac{\int_A dP \cdot y}{P} = \frac{\rho g \sin\alpha \cdot I_x}{\rho g \sin\alpha \cdot y_C A} = \frac{I_x}{y_C A} = \frac{I_{Cx} + y_C^2 A}{y_C A} = y_C + \frac{I_{Cx}}{y_C A}$$

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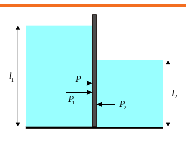
Example

- Overall forces:
  - Left: $P_1 = \rho g h_{C1} A_1 = \rho g \frac{l_1}{2} b l_1$
  - Right: $P_2 = \rho g h_{C2} A_2 = \rho g \frac{l_2}{2} b l_2$



$$P = P_1 - P_2$$

- Acting points:

$$I_{Cx} = \frac{1}{12} b l^3 \qquad y_D = y_C + \frac{I_{Cx}}{y_C A} = \frac{l}{2} + \frac{\frac{1}{12} b l^3}{\frac{l}{2} b l} = \frac{2}{3} l \rightarrow y_{D1} = \frac{2}{3} l_1 \qquad y_{D2} = \frac{2}{3} l_2$$

$$P \cdot L = P_1 \cdot \frac{1}{3} l_1 - P_2 \cdot \frac{1}{3} l_2 \rightarrow L = P_1 \cdot \frac{1}{3} l_1 - P_2 \cdot \frac{1}{3} l_2 = \frac{P_1 \cdot l_1 - P_2 \cdot l_2}{3(P_1 - P_2)}$$

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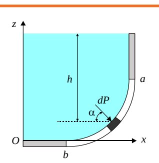
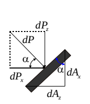
Hydrostatic forces on curved surfaces

- The horizontal force acting on the infinitesimal element:





$$dP = \rho g h \cdot dA$$

$$dP_x = dP \cos\alpha = \rho g h \cdot dA \cos\alpha$$

$$P_x = \int_{A_x} dP_x = \rho g \int_{A_x} h \cdot dA \cos\alpha = \rho g \int_{A_x} h \cdot dA_x = \rho g h_C A_x = p_C A_x$$

The horizontal component of the fluid force on the curved surface is equal to the overall force acting on the vertical projection of the curved surface

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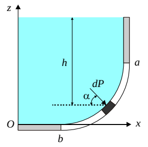
Hydrostatic forces on curved surfaces

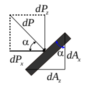
- The vertical force acting on the infinitesimal element:

$$dP = \rho g h \cdot dA$$

$$dP_z = dP \sin\alpha = \rho g h \cdot dA \sin\alpha$$

$$P_z = \int_{A_z} dP_z = \rho g \int_{A_z} h \cdot dA \sin\alpha = \rho g \int_{A_z} h \cdot dA_z = \rho g V_p$$





where $V_p = \int_{A_z} h \cdot dA_z$

Volume of the fluid body

The vertical component of the fluid force on the curved surface is equal to the weight of the fluid above the curved surface

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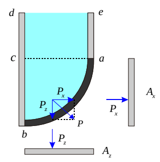
Hydrostatic forces on curved surfaces

- The overall force:

$$P = \sqrt{P_x^2 + P_z^2}$$

- The angle with vertical direction

$$\theta = \tan^{-1}\frac{P_x}{P_z}$$



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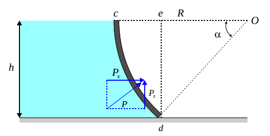
Example

- Horizontal force:

$$h = R\sin\alpha$$

$$P_x = \rho g h_C A_x = \rho g \frac{h}{2} bh$$



- Vertical force:

$$V_p = b\left(\pi R^2 \frac{\alpha}{2\pi} - \frac{1}{2} hR\cos\alpha\right) = \frac{b}{2}\left(\alpha R^2 - R^2 \sin\alpha\cos\alpha\right)$$

$$P_z = \rho g V_p$$

- Overall force:

$$P = \sqrt{P_x^2 + P_z^2}$$

$$\theta = \tan^{-1}\frac{P_x}{P_z}$$

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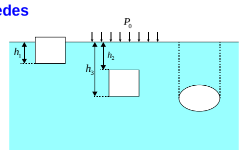
Buoyancy

Principle of Archimedes



$$P_1 = p_0 A \qquad P_1 = (p_0 + \rho g h_2) A$$
$$P_2 = (p_0 + \rho g h_1) A \qquad P_2 = (p_0 + \rho g h_3) A$$

$$P = P_2 - P_1 = \rho g h_1 A = \rho g V \qquad P = P_2 - P_1 = \rho g (h_3 - h_2) A = \rho g V$$

V is the volume of the body in the liquid

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