$\sigma_{x'}$ 法向应力 (正应力)

σ_x

南方科技大学
SOUTHERN UNIVERSITY OF SCIENCE AND TECHNOLOGY

MAE5009
Continuum Mechanics B
Session 01: Stress

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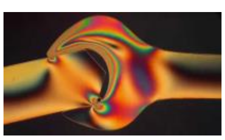
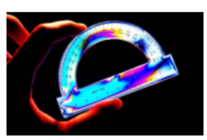
Contents

Why stress:





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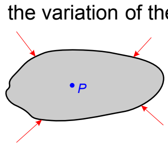
Two important aspects of stress:
- State of stress at a point: leads to the transformation of stress equations
- Equations governing the variation of the stress components in space



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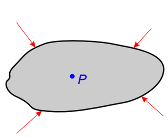
Forces

Behavior of a body is a function of internal force distribution, which depends upon the external forces

External forces:
- Body force: associated with body mass
  - Gravitational, magnetic and inertia forces
- Surface forces: induced by physical contact



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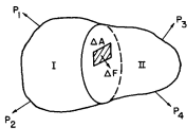
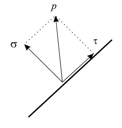
Stress



$$\text{stress} = \lim_{\Delta A \to 0} \frac{\Delta F}{\Delta A} = \frac{dF}{dA}$$

Stress referred to a given plane is a vector

Stress vector:
- **Magnitude**
- **Orientation**
- **Referring plane**



Normal & shear

Stress is a point property!

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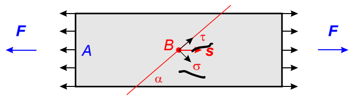
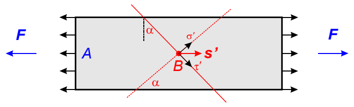
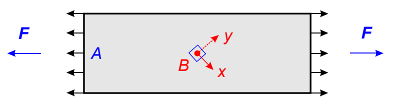
2D Stress



$A' = \dfrac{A}{\sin\alpha}$ $\quad s = \dfrac{F}{A'} = \dfrac{F}{A}\sin\alpha$ $\quad\rightarrow\quad$ $\sigma = \dfrac{F}{A}\sin^2\alpha$ $\quad \tau = \dfrac{F}{A}\sin\alpha\cos\alpha$



$A' = \dfrac{A}{\cos\alpha}$ $\quad s' = \dfrac{F}{A'} = \dfrac{F}{A}\cos\alpha$ $\quad\rightarrow\quad$ $\sigma' = \dfrac{F}{A}\cos^2\alpha$ $\quad \tau' = \dfrac{F}{A}\cos\alpha\sin\alpha$



For 2D analysis, essentially, we build a *x-y* coordinate system and consider an infinitesimal square of material

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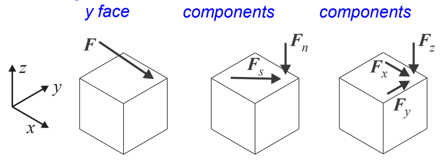
normal：垂直
shear：平行

3D Stress

Consider a cube of material, aligned with its faces parallel to the *x-y-z* axes:

Force applied at an arbitrary angle to the x-y face | *Resolved into normal and shear components* | *Shear components resolved into two Cartesian components*



$$\text{stress} = \lim_{\Delta A \to 0} \frac{\Delta F}{\Delta A} = \frac{dF}{dA}$$

Thus, every face has three stress components, each derived from a force component.

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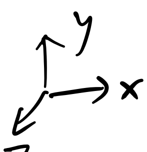
这是右手坐标系



对于 normal为x

x: normal σ_x

y: shear τ_{xy} → 以此类推 对于normal为y

z: shear τ_{xz}

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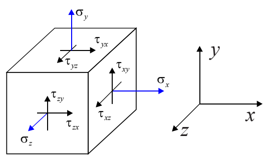
3D Stress

The stress components on an infinitesimal cube all have particular names



and are put in a matrix as 同一参考坐标系

同一方向

$$\begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix}$$

stress tension 应力张量 (定义比较的)

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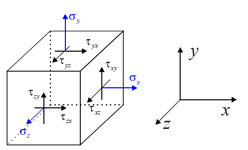
positive

Sign convention

- The direction of outward normal defined the sign of plane, e.g. positive *x* plane
- Two indices are needed to define a stress component:
  - The first refers to the plane
  - The second refers to the direction
  - Only one direction is needed for normal stress component
- On positive surfaces, all stress components which act in positive coordinate directions are positive



Engineering sign convention:
- *tension positive*
- *compression negative*

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Stress is a tensor

- Scalar has one property: magnitude,
  e.g. temperature, mass
- Vector has two properties: magnitude and direction
  e.g. force, velocity
- Tensor has three properties: magnitude, direction and a reference plane
  e.g. stress, strain

$$\begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix}$$

Mathematically, a tensor is a matrix that obeys certain transformation laws

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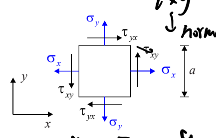
$$\sigma_x \cdot a \cdot \frac{a}{2} - \sigma_x \cdot a \cdot \frac{a}{2} + \sigma_y \cdot a \cdot \frac{a}{2} - \sigma_y \cdot a \cdot \frac{a}{2}$$

Stress tensor is symmetric

If we consider rotational equilibrium of the infinitesimal square, we calculate the moment with respect to lower left corner:



τ_{xy} ↓ normal

$$\sigma_x a(a/2) - \sigma_x a(a/2) + \sigma_y a(a/2) - \sigma_y a(a/2) + \tau_{yx} a \cdot a - \tau_{xy} a \cdot a = 0$$

$$\tau_{xy} a \cdot \frac{a}{2} - \tau_{yx} \cdot a \cdot \frac{a}{2} + \tau_{xy} a \cdot \frac{a}{2} - \tau_{yx} \cdot a \cdot \frac{a}{2} = 0$$

$$\tau_{xy} = \tau_{yx}$$

Likewise, for 3D stress:

$$\tau_{xy} = \tau_{yx}, \tau_{yz} = \tau_{zy}, \tau_{xz} = \tau_{zx}$$

which means that

SYMMETRIC ABOUT LEADING DIAGONAL

$$\begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix}$$

LEADING DIAGONAL

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Stress state at a point has 6 distinct components

$$\begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ & \sigma_y & \tau_{yz} \\ sym. & & \sigma_z \end{bmatrix}$$

3 NORMAL STRESS COMPONENTS

3 SHEAR STRESS COMPONENTS

Remember:

- A **SCALAR** has **ONE** property – magnitude
- A **VECTOR** has **TWO** properties – magnitude & direction – and needs **THREE** components in 3d
- A **TENSOR** has **THREE** properties – magnitude, direction & plane on which it acts – and needs **SIX** components in 3d

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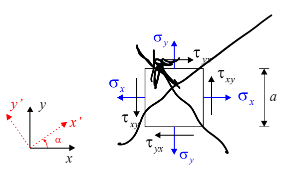
Stress transformation

- Stress state is a point property
- Tensor is an approach to demonstrate the stress state at a specific point
- Tensor is a coordinate-dependent quantity, and stress tensor components can be transformed



We do not actually transform stress; it is the coordinate system that we are transforming!

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$$P_x \cdot A - \sigma_x \cdot A\cos\alpha - \tau_{yx} \cdot A\sin\alpha = 0$$

$P_x = \sigma_x \cos\alpha + \tau_{yx}\sin\alpha$

$P_y \cdot A - \sigma_y \cdot A\sin\alpha - \tau_{xy}\cdot A\cos\alpha = 0$

$P_y = \sigma_y \sin\alpha + \tau_{xy}\cos\alpha$

$\sigma_{y'} = \sigma_x \sin\alpha + \sigma_y \cos\alpha$

Stress transformation equations

- The stress transformation equations may be derived based on force equilibrium analysis:

$\sum F_x = 0$

$p_x A - \sigma_x A\cos\alpha - \tau_{yx} A\sin\alpha = 0$

$p_x = \sigma_x\cos\alpha + \tau_{yx}\sin\alpha$

$\sum F_y = 0$

$p_y A - \sigma_y A\sin\alpha - \tau_{xy} A\cos\alpha = 0$

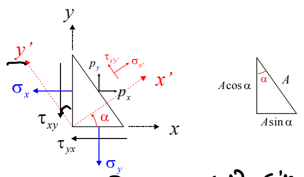
$p_y = \sigma_y\sin\alpha + \tau_{xy}\cos\alpha$

$\because \sigma_{x'} = p_x\cos\alpha + p_y\sin\alpha$

$\therefore \sigma_{x'} = \sigma_x\cos^2\alpha + \sigma_y\sin^2\alpha + 2\tau_{xy}\sin\alpha\cos\alpha$

$\because \tau_{x'y'} = p_y\cos\alpha - p_x\sin\alpha$

$\therefore \tau_{x'y'} = (\sigma_y - \sigma_x)\sin\alpha\cos\alpha + \tau_{xy}(\cos^2\alpha - \sin^2\alpha)$



$\sigma_{x'} = P_x\cos\alpha + P_y\sin\alpha$

$\sigma_{x'} = \sigma_x\cos^2\alpha + \sigma_y\sin^2\alpha + 2\tau_{yx}\sin\alpha\cos\alpha$

$\tau_{x'y'} = P_y\sin\alpha - P_x\sin\alpha$

$\tau_{x'y'} = \sigma_y\sin\alpha\cos\alpha + \tau_{xy}\cos^2\alpha - \sigma_x\sin\alpha\cos\alpha - \tau_{yx}\sin^2\alpha$

$= (\sigma_y - \sigma_x)\sin\alpha\cos\alpha + \tau_{xy}(\cos^2\alpha - \sin^2\alpha)$

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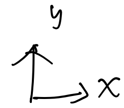
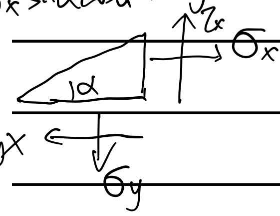
$P_x\sin\alpha + P_y\cos\alpha$

$= \sigma_x\sin\alpha\cos\alpha + \tau_{yx}\sin^2\alpha$





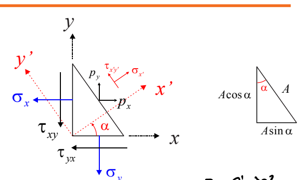
$P_{x'} = \sigma_x\cdot A\sin\alpha - \tau_{yx}\cdot\cos\alpha$

$P_{y'} = \sigma_y\cdot\cos\alpha - \tau_{xy}\cdot\sin\alpha$

Stress transformation equations

- By cutting the y' plane:

$\sigma_{y'} = \sigma_x\sin^2\alpha + \sigma_y\cos^2\alpha - 2\tau_{xy}\sin\alpha\cos\alpha$



$2\tau_{xy}\sin 2\alpha$

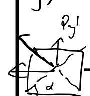
$$\begin{cases} \sigma_{x'} = \sigma_x\cos^2\alpha + \sigma_y\sin^2\alpha + 2\tau_{xy}\sin\alpha\cos\alpha \\ \sigma_{y'} = \sigma_x\sin^2\alpha + \sigma_y\cos^2\alpha - 2\tau_{xy}\sin\alpha\cos\alpha \\ \tau_{x'y'} = (\sigma_y - \sigma_x)\sin\alpha\cos\alpha + \tau_{xy}(\cos^2\alpha - \sin^2\alpha) \end{cases}$$

$\sigma_x(1-\sin^2\alpha) + \sigma_y\sin^2\alpha$

$\sigma_x - \sigma_x\sin^2\alpha + \sigma_y\sin^2\alpha$

$\sigma_x - \sin^2\alpha(\sigma_x - \sigma_y) + 2\tau_{xy}\sin 2\alpha$

$$\begin{cases} \sigma_{x'} = \dfrac{\sigma_x+\sigma_y}{2} + \dfrac{\sigma_x-\sigma_y}{2}\cos 2\alpha + \tau_{xy}\sin 2\alpha \\ \sigma_{y'} = \dfrac{\sigma_x+\sigma_y}{2} - \dfrac{\sigma_x-\sigma_y}{2}\cos 2\alpha - \tau_{xy}\sin 2\alpha \\ \tau_{x'y'} = -\dfrac{\sigma_x-\sigma_y}{2}\sin 2\alpha + \tau_{xy}\cos 2\alpha \end{cases}$$



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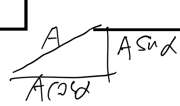
$\sigma_x(1-\sin^2\alpha) + \sigma_y\sin^2\alpha + \tau_{xy}\sin 2\alpha$

$= \sigma_x - \sigma_x\cdot\sin^2\alpha + \sigma_y\sin^2\alpha + \tau_{xy}\cdot\sin 2\alpha$

$= \sigma_x - \frac{1-\cos2\alpha}{2}\sigma_x + \sigma_y\cdot\frac{1-\cos2\alpha}{2} + \tau_{xy}\cdot\sin2\alpha$

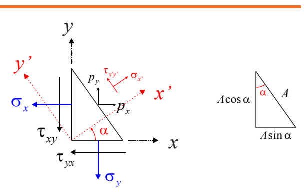
$= \frac12\sigma_x(1+\cos2\alpha) + \frac12\sigma_y(1-\cos2\alpha) = \frac12(\sigma_x + \sigma_x\cos2\alpha + \sigma_y - \sigma_y\cos2\alpha) + \tau_{xy}\cdot\sin2\alpha$

$= \frac{\sigma_x+\sigma_y}{2} + \frac{\sigma_x-\sigma_y}{2}\cdot\cos2\alpha + \tau_{xy}\cdot\sin2\alpha$



Stress transformation equations

- If σ_x, σ_y, τ_{xy} are known, for any given α, the $\sigma_{x'}$, $\sigma_{y'}$, $\tau_{x'y'}$ can be obtained
- In 2D, the state of stress is completely defined if the three stress components on two orthogonal planes are given
- The sum of normal stresses on two perpendicular planes is invariant, independent of α



$\sigma_{x'} + \sigma_{y'} = \sigma_x + \sigma_y$

The first stress invariant

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$(\sigma_y-\sigma_x)\cos 2\alpha = 2\tau_{xy}\sin 2\alpha$

$\frac{\sigma_y-\sigma_x}{2\tau_{xy}} = \tan 2\alpha$

Principal stresses

Stress transformation equations

$$\begin{cases} \sigma_{x'} = \frac{\sigma_x+\sigma_y}{2} + \frac{\sigma_x-\sigma_y}{2}\cos 2\alpha + \tau_{xy}\sin 2\alpha \\ \sigma_{y'} = \frac{\sigma_x+\sigma_y}{2} - \frac{\sigma_x-\sigma_y}{2}\cos 2\alpha - \tau_{xy}\sin 2\alpha \\ \tau_{x'y'} = -\frac{\sigma_x-\sigma_y}{2}\sin 2\alpha + \tau_{xy}\cos 2\alpha \end{cases}$$

Differentiate $\sigma_{x'}$ with respect to α to determine the plane of max normal stress:

$$d\sigma_{x'}/d\alpha = -(\sigma_x-\sigma_y)\sin 2\alpha + 2\tau_{xy}\cos 2\alpha = 0$$

or $$\tan 2\alpha = \frac{2\tau_{xy}}{\sigma_x-\sigma_y}$$

- This equation has two roots: the two values of α differ by 90°, i.e., the two planes of normal stresses (max, min) are perpendicular
- Shear stress on these planes are zero

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$\sigma_x = \frac{\sigma_x+\sigma_y}{2} + \frac{\sigma_x-\sigma_y}{2}\cdot\frac{\sigma_x-\sigma_y}{\sqrt{4\tau_{xy}^2+(\sigma_x-\sigma_y)^2}} + \tau_{xy}\cdot\frac{2\tau_{xy}}{\sqrt{4\tau_{xy}^2+(\sigma_x-\sigma_y)^2}}$

$= \frac{\sigma_x+\sigma_y}{2} + \frac{1}{\sqrt{4\tau_{xy}^2+(\sigma_x-\sigma_y)^2}}\left[\frac{(\sigma_x-\sigma_y)^2}{2}+4\tau_{xy}^2\right]$

$=$

$(\sigma_y-\sigma_x)\sin 2\alpha + 2\tau_{xy}\cos 2\alpha = 0$

$(\sigma_y-\sigma_x)\tan 2\alpha = 2\tau_{xy}$

Principal stresses

Principal planes:

$$\sin 2\alpha = \pm\frac{2\tau_{xy}}{\sqrt{4\tau_{xy}^2+(\sigma_x-\sigma_y)^2}}$$

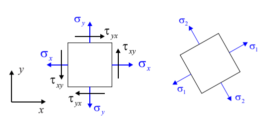
$$\cos 2\alpha = \pm\frac{\sigma_x-\sigma_y}{\sqrt{4\tau_{xy}^2+(\sigma_x-\sigma_y)^2}}$$

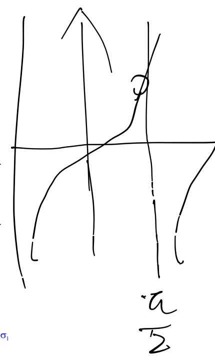
Principal stresses:

主应

$$\sigma_{max} = \sigma_1 = \frac{\sigma_x+\sigma_y}{2} + \sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2+\tau_{xy}^2}$$

$$\sigma_{min} = \sigma_2 = \frac{\sigma_x+\sigma_y}{2} - \sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2+\tau_{xy}^2}$$





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$\cancel{\tan 2\alpha = \frac{2\tau_{xy}}{\sigma_x-\sigma_y}}$

$$\begin{cases} \cos^2 2\alpha = 1-\sin^2 2\alpha \\ \frac{\sin 2\alpha}{\cos 2\alpha} = \tan 2\alpha \end{cases}$$

$\frac{\sin 2\alpha}{\sqrt{1-\sin^2 2\alpha}} = \frac{2\tau_{xy}}{\sigma_x-\sigma_y}$

$\frac{\sin^2 2\alpha}{1-\sin^2 2\alpha} = \frac{4\tau_{xy}^2}{(\sigma_x-\sigma_y)^2}$

$\sin^2 2\alpha\,(\sigma_x-\sigma_y)^2 = 4\tau_{xy}^2 - 4\tau_{xy}^2\sin^2 2\alpha$

$[(\sigma_x-\sigma_y)^2+4\tau_{xy}^2]\sin^2 2\alpha = 4\tau_{xy}^2$

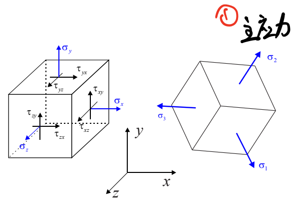
$\sin 2\alpha = \pm\frac{2\tau_{xy}}{\sqrt{(\sigma_x-\sigma_y)^2+4\tau_{xy}^2}}$

$\cos 2\alpha = \pm\frac{\sigma_x-\sigma_y}{\sqrt{4\tau_{xy}^2+(\sigma_x-\sigma_y)^2}}$

There are principal stresses and principal directions

- For any stress state there is an orientation in space at which the shear stresses vanish, and only normal stresses remain
- These normal stresses are called the ① principal stresses, and their orientations are the ② principal directions
- By convention, $\sigma_1 \geq \sigma_2 \geq \sigma_3$
- Note that a principal stress state comprises 6 distinct components: 3 stress magnitudes and 3 rotation angles

① 主应力



$$\begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ & \sigma_y & \tau_{yz} \\ \text{symmetric} & & \sigma_z \end{bmatrix} \qquad \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix}$$

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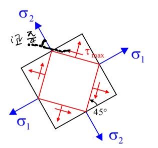
Planes of max shear stress

Differentiate $\tau_{x'y'}$ with respect to α to determine the plane of max normal stress:

求偏导 $d\tau_{x'y'}/d\alpha = -(\sigma_x - \sigma_y)\cos 2\alpha - 2\tau_{xy}\sin 2\alpha = 0$

or $\tan 2\alpha = -\dfrac{\sigma_x - \sigma_y}{2\tau_{xy}}$

- Two roots of α define a set of perpendicular planes that the shear stresses on them are equal
- Normal stresses on these planes are **NOT** necessary zero
- The principal plane is 45° apart from the plane having max shear stresses



$$\tau_{x'y'\max} = \pm\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

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① $\left(\sigma - \dfrac{\sigma_x+\sigma_y}{2}\right)^2 = \left(\dfrac{\sigma_x-\sigma_y}{2}\right)^2\cos^2 2\alpha + \tau_{xy}^2 \sin^2 2\alpha + (\sigma_x-\sigma_y)\tau_{xy}\sin 2\alpha\cos 2\alpha$

② $\tau^2 = \left(\dfrac{\sigma_x+\sigma_y}{2}\right)^2 \sin^2 2\alpha + \tau_{xy}^2 \sin^2 2\alpha - (\sigma_x-\sigma_y)\tau_{xy}\sin 2\alpha\cos 2\alpha$

① + ②

Mohr's circle of stress

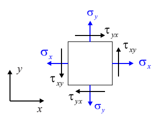
- 2D stress transformation can also be conveniently represented graphically in a circle:

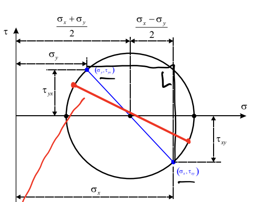
$$\begin{cases} \sigma_{x'} - \dfrac{\sigma_x+\sigma_y}{2} = \dfrac{\sigma_x-\sigma_y}{2}\cos 2\alpha + \tau_{xy}\sin 2\alpha \\ \sigma_{y'} - \dfrac{\sigma_x+\sigma_y}{2} = -\dfrac{\sigma_x-\sigma_y}{2}\cos 2\alpha - \tau_{xy}\sin 2\alpha \\ \tau_{x'y'} = -\dfrac{\sigma_x-\sigma_y}{2}\sin 2\alpha + \tau_{xy}\cos 2\alpha \end{cases}$$

$$\left(\sigma - \frac{\sigma_x+\sigma_y}{2}\right)^2 + \tau^2 = \left(\frac{\sigma_x-\sigma_y}{2}\right)^2 + \tau_{xy}^2$$

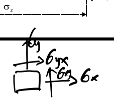
center: $\left(\dfrac{\sigma_x+\sigma_y}{2}, 0\right)$

radius: $\sqrt{\left(\dfrac{\sigma_x-\sigma_y}{2}\right)^2 + \tau_{xy}^2}$





对应莫尔坐标系。



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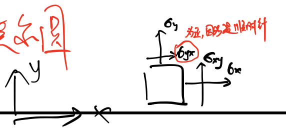
大 σ_{xy} σ_{yx} 要么头对头，要么尾对尾

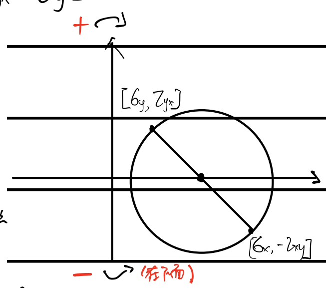
画莫尔圆

① :



$$\begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{yx} & \sigma_y \end{bmatrix}$$

②



容易找直径上两点

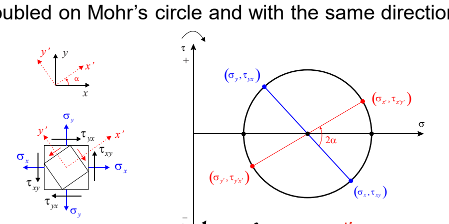
③ 先看第一个平面

$[\sigma_x, -\tau_{xy}]$

$[\sigma_y, \tau_{yx}]$

Sign convention

- **Physical space**
  - Normal stresses & shear stresses defined previously
- **Mohr-Circle space**
  - Normal stresses have the same sign convention as physical space
  - Shear stress causing clockwise moment is positive
  - Whatever the rotation angle exists in physical space, it is doubled on Mohr's circle and with the same direction



case:

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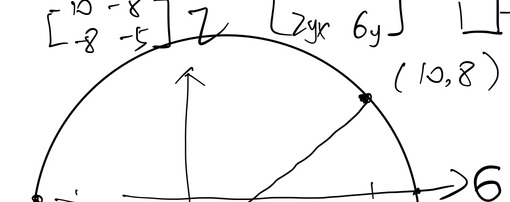
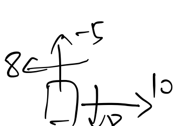
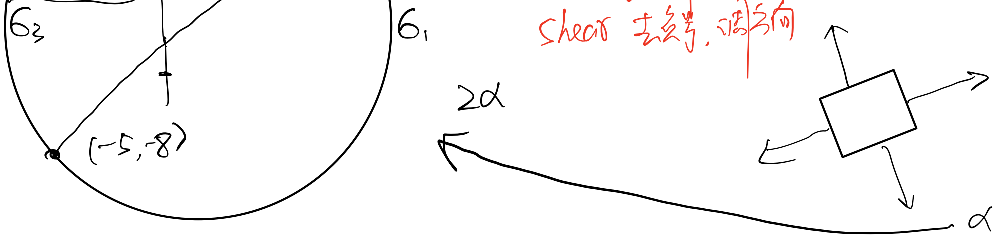
$$\begin{bmatrix} 10 & -8 \\ -8 & -5 \end{bmatrix}$$

$$\begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{yx} & \sigma_y \end{bmatrix}$$



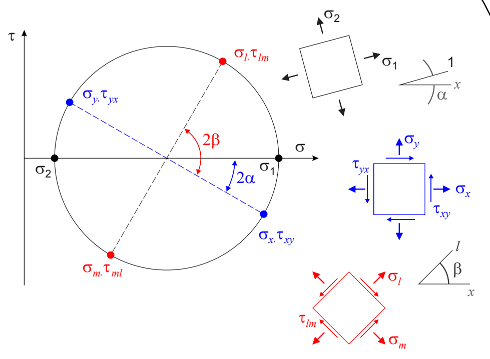


normal 正常的



?致 (变换方向?)

Mohr's circle of stress



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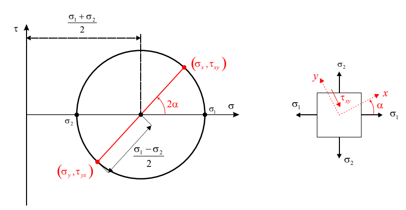
Mohr's circle of stress



$$\begin{cases} \sigma_{x'} - \dfrac{\sigma_x + \sigma_y}{2} = \dfrac{\sigma_x - \sigma_y}{2}\cos 2\alpha + \tau_{xy}\sin 2\alpha \\ \sigma_{y'} - \dfrac{\sigma_x + \sigma_y}{2} = -\dfrac{\sigma_x - \sigma_y}{2}\cos 2\alpha - \tau_{xy}\sin 2\alpha \\ \tau_{x'y'} = -\dfrac{\sigma_x - \sigma_y}{2}\sin 2\alpha + \tau_{xy}\cos 2\alpha \end{cases} \Rightarrow \begin{cases} \sigma_x = \dfrac{\sigma_1 + \sigma_2}{2} + \dfrac{\sigma_1 - \sigma_2}{2}\cos 2\alpha \\ \sigma_y = \dfrac{\sigma_1 + \sigma_2}{2} - \dfrac{\sigma_1 - \sigma_2}{2}\cos 2\alpha \\ \tau_{xy} = -\dfrac{\sigma_1 - \sigma_2}{2}\sin 2\alpha \end{cases}$$

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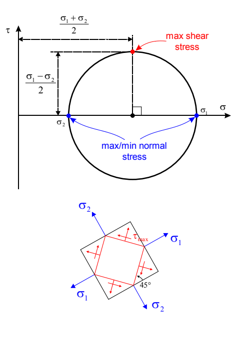
Mohr's circle of stress

- One planes which have the max/min normal stress (principal planes), the shear stress is minimum (zero)
- For planes having the max shear stress, the normal stress are **NOT** necessary zero
- The two types of planes are 45º apart



$$\begin{cases} \sigma_{max} = \sigma_1 \\ \sigma_{min} = \sigma_2 \end{cases} \Rightarrow \tau = 0$$

$$\tau_{max} = \frac{\sigma_1 - \sigma_2}{2} \Rightarrow \sigma = \frac{\sigma_1 + \sigma_2}{2}$$

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$A_{ABC} = A$

$A_{AOC} = A_{ABC} \cdot \cos(y, x') = A \cdot a_{12}$

$A_{AOB} = A_{ABC} \cdot \cos(x', z) = A \cdot a_{13}$

$A_{BOC} = A \cdot a_{11}$

3D stress transformation

$$\begin{cases} A_{AOC} = A_{ABC}\cos(y,x') = Aa_{12} \\ A_{AOB} = A_{ABC}\cos(z,x') = Aa_{13} \\ A_{BOC} = A_{ABC}\cos(x,x') = Aa_{11} \end{cases}$$

$$\sum F_x = \sum F_y = \sum F_z = 0$$

Force equilibrium

$P_x A = \tau_x a_{11} + \tau_{yx} a_{12} + \tau_{zx} a_{13}$

$[P_x\ P_y\ P_z] = [a_{11}\ a_{12}\ a_{13}] \begin{bmatrix} \sigma_x & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_y & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_z \end{bmatrix}$

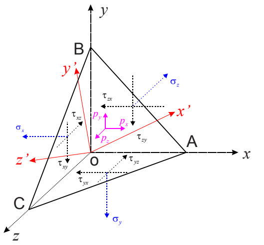
$$\begin{cases} p_x = \sigma_x a_{11} + \tau_{yx} a_{12} + \tau_{zx} a_{13} \\ p_y = \tau_{xy} a_{11} + \sigma_y a_{12} + \tau_{zy} a_{13} \\ p_z = \tau_{xz} a_{11} + \tau_{yz} a_{12} + \sigma_z a_{13} \end{cases}$$

Projection

$P_x \cos(x, x') + P_y \cos(y, x') + P_z \cos(z, x')$

$$\begin{cases} \sigma_{x'} = p_x a_{11} + p_y a_{12} + p_z a_{13} \\ \tau_{x'y'} = p_x a_{21} + p_y a_{22} + p_z a_{23} \\ \tau_{x'z'} = p_x a_{31} + p_y a_{32} + p_z a_{33} \end{cases}$$

$\tau_{x'y'} = P_x a_{21} + P_y a_{22} + P_z a_{23}$



Direction cosines

	x	y	z
x'	a_{11}	a_{12}	a_{13}
y'	a_{21}	a_{22}	a_{23}
z'	a_{31}	a_{32}	a_{33}

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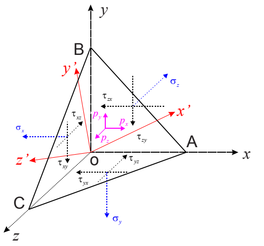
3D stress transformation

Transformational matrix **R**

	x	y	z
x'	a_{11}	a_{12}	a_{13}
y'	a_{21}	a_{22}	a_{23}
z'	a_{31}	a_{32}	a_{33}



$$\begin{bmatrix} \sigma'_x & \tau'_{xy} & \tau'_{xz} \\ & \sigma'_y & \tau'_{yz} \\ sym. & & \sigma'_z \end{bmatrix} = \mathbf{R} \cdot \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ & \sigma_y & \tau_{yz} \\ sym. & & \sigma_z \end{bmatrix} \cdot \mathbf{R}^T$$

Each component in transformation matrix is the direction cosine between the two corresponding axes

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Transformation matrix

Transformational matrix **R**

	x	y	z
x'	a_{11}	a_{12}	a_{13}
y'	a_{21}	a_{22}	a_{23}
z'	a_{31}	a_{32}	a_{33}

Transformational matrix **A**

	x'	y'	z'
x	a_{11}	a_{21}	a_{31}
y	a_{12}	a_{22}	a_{32}
z	a_{13}	a_{23}	a_{33}

$$\mathbf{R} = \mathbf{A}^T$$

$$\because \mathbf{S}' = \mathbf{R}\cdot\mathbf{S}\cdot\mathbf{R}^T \qquad \therefore \mathbf{S} = \mathbf{A}\cdot\mathbf{R}\cdot\mathbf{S}\cdot\mathbf{R}^T\cdot\mathbf{A}^T \qquad \mathbf{AR} = \mathbf{I}$$

$$\mathbf{S} = \mathbf{A}\cdot\mathbf{S}'\cdot\mathbf{A}^T \qquad = (\mathbf{AR})\cdot\mathbf{S}\cdot(\mathbf{AR})^T \qquad \mathbf{R}^T\mathbf{R} = \mathbf{I}$$

- The nine direction cosines are not independent
- **R** is orthogonal matrix, i.e. $\mathbf{R}^T = \mathbf{R}^{-1}$; det(**R**) = ± 1
- Each row/column contains the direction cosines of the new/old axis in terms of the old/new axes

Columns: $a_{11}^2 + a_{21}^2 + a_{31}^3 = 1, \ldots$

Rows: $a_{11}^2 + a_{12}^2 + a_{13}^3 = 1, \ldots$

Columns: $a_{11}a_{12} + a_{21}a_{22} + a_{31}a_{32} = 0, \ldots$

Rows: $a_{11}a_{21} + a_{12}a_{22} + a_{13}a_{23} = 0, \ldots$

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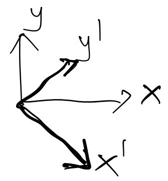
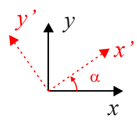


Recall: 2D stress transformation

Transformational matrix **R**

	x	y
x'	cosα	sinα
y'	-sinα	cosα



算一下

$$\begin{bmatrix} \sigma'_x & \tau'_{xy} \\ sym. & \sigma'_y \end{bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \cdot \begin{bmatrix} \sigma_x & \tau_{xy} \\ sym. & \sigma_y \end{bmatrix} \cdot \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}^T$$

$$\begin{cases} \sigma_{x'} = \sigma_x \cos^2\alpha + \sigma_y \sin^2\alpha + 2\tau_{xy} \sin\alpha\cos\alpha \\ \sigma_{y'} = \sigma_x \sin^2\alpha + \sigma_y \cos^2\alpha - 2\tau_{xy} \sin\alpha\cos\alpha \\ \tau_{x'y'} = \left(\sigma_y - \sigma_x\right)\sin\alpha\cos\alpha + \tau_{xy}\left(\cos^2\alpha - \sin^2\alpha\right) \end{cases}$$

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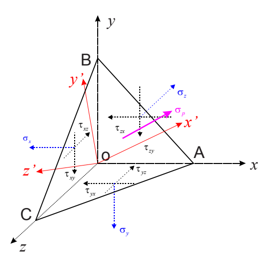
3D principal stresses

$$\begin{cases} p_x = \sigma_p a_{11} \\ p_y = \sigma_p a_{12} \\ p_z = \sigma_p a_{13} \end{cases}$$

$+$

$$\begin{cases} p_x = \sigma_x a_{11} + \tau_{yx} a_{12} + \tau_{zx} a_{13} \\ p_y = \tau_{xy} a_{11} + \sigma_y a_{12} + \tau_{zy} a_{13} \\ p_z = \tau_{xz} a_{11} + \tau_{yz} a_{12} + \sigma_z a_{13} \end{cases}$$



$$\begin{cases} \left(\sigma_x - \sigma_p\right) a_{11} + \tau_{yx} a_{12} + \tau_{zx} a_{13} = 0 \\ \tau_{xy} a_{11} + \left(\sigma_y - \sigma_p\right) a_{12} + \tau_{zy} a_{13} = 0 \\ \tau_{xz} a_{11} + \tau_{yz} a_{12} + \left(\sigma_z - \sigma_p\right) a_{13} = 0 \end{cases}$$

$$\because a_{11}^2 + a_{12}^2 + a_{13}^3 = 1$$

$$\therefore \begin{vmatrix} \sigma_x - \sigma_p & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \sigma_y - \sigma_p & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \sigma_z - \sigma_p \end{vmatrix} = 0$$

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3D principal stresses

算一下

$$\sigma_p^3 - \left(\sigma_x + \sigma_y + \sigma_z\right)\sigma_p^2 + \left(\sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{zx}^2\right)\sigma_p - \left(\sigma_x\sigma_y\sigma_z + 2\tau_{xy}\tau_{yz}\tau_{zx} - \sigma_x\tau_{yz}^2 - \sigma_y\tau_{zx}^2 - \sigma_z\tau_{xy}^2\right) = 0$$

$$\sigma_p^3 - I_1\sigma_p^2 + I_2\sigma_p - I_3 = 0$$

Stress invariants:

$$I_1 = \sigma_x + \sigma_y + \sigma_z = \sigma_1 + \sigma_2 + \sigma_3$$

$$I_2 = \begin{vmatrix} \sigma_x & \tau_{xy} \\ \tau_{yx} & \sigma_y \end{vmatrix} + \begin{vmatrix} \sigma_y & \tau_{yz} \\ \tau_{zy} & \sigma_z \end{vmatrix} + \begin{vmatrix} \sigma_z & \tau_{zx} \\ \tau_{xz} & \sigma_x \end{vmatrix} = \sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{zx}^2 = \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1$$

$$I_3 = \begin{vmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{vmatrix} = \sigma_x\sigma_y\sigma_z + 2\tau_{xy}\tau_{yz}\tau_{zx} - \sigma_x\tau_{yz}^2 - \sigma_y\tau_{zx}^2 - \sigma_z\tau_{xy}^2 = \sigma_1\sigma_2\sigma_3$$

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Eigenvalues and eigenvectors of stress tensor

- For symmetric matrix **A**, its eigenvalue and eigenvector satisfy:

$$\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$$

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \mathbf{0}$$

- The equation has non-zero **v** if and only if

$$|\mathbf{A} - \lambda\mathbf{I}| = 0$$

  - The principal stresses and principal directions are essentially eigenvalues and eigenvectors of stress tensor
  - Since stress tensor is real symmetric matrix, it has three real principal stresses
  - Three eigenvectors must be orthogonal

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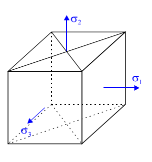
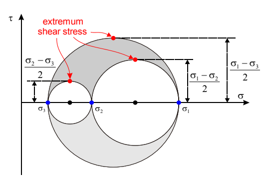
3D Mohr's circle

- The stress conditions on all planes must lie in the shaded region between the circles
- The max shear stress acts on the plane bisecting the planes of max and min principal stresses





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$\sigma_{x_A} = \sigma_x$

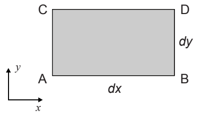
$\sigma_{x_B} = \sigma_x + \frac{\partial \sigma_x}{\partial x} dx$

$\sigma_{x_C} = \sigma_x + \frac{\partial \sigma_x}{\partial y} dy$

$\sigma_{x_D} = \sigma_{x_B} + \frac{\partial \sigma_{x_B}}{\partial y} dy = \sigma_x + \frac{\partial \sigma_x}{\partial x} dx + \frac{\partial x}{\partial y} dy$

Differential equation of equilibrium

- Stress varies from point to point in a stressed body, these variations are governed by the equilibrium conditions of statics



$$\sigma_{x_A} = \sigma_x$$

$$\sigma_{x_B} = \sigma_x + \frac{\partial \sigma_x}{\partial x} dx \qquad \sigma_{x_C} = \sigma_x + \frac{\partial \sigma_x}{\partial y} dy$$

$$\sigma_{x_D} = \sigma_{x_B} + \frac{\partial \sigma_{x_B}}{\partial y} dy = \sigma_x + \frac{\partial \sigma_x}{\partial x} dx + \frac{\partial}{\partial y}\left(\sigma_x + \frac{\partial \sigma_x}{\partial x} dx\right) dy$$

$$= \sigma_x + \frac{\partial \sigma_x}{\partial x} dx + \frac{\partial \sigma_x}{\partial y} dy$$

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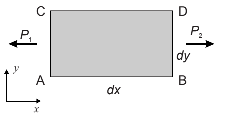
Differential equation of equilibrium



$$P_1 = \frac{\sigma_x + \sigma_x + \frac{\partial \sigma_x}{\partial y} dy}{2} dy = \sigma_x dy + \frac{1}{2}\frac{\partial \sigma_x}{\partial y} dy^2$$

$$P_2 = \frac{\sigma_x + \frac{\partial \sigma_x}{\partial x} dx + \sigma_x + \frac{\partial \sigma_x}{\partial x} dx + \frac{\partial \sigma_x}{\partial y} dy}{2} dy = \sigma_x dy + \frac{\partial \sigma_x}{\partial x} dxdy + \frac{1}{2}\frac{\partial \sigma_x}{\partial y} dy^2$$

$$\Downarrow$$

$$P_2 - P_1 = \frac{\partial \sigma_x}{\partial x} dxdy$$

Uniform stress distribution on each face can be used to derive equilibrium equations

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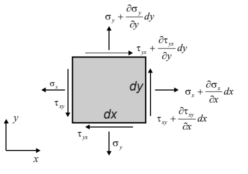
Differential equation of equilibrium



$$\sum Force_x = 0: \quad f_x dxdy + \left(\sigma_x + \frac{\partial \sigma_x}{\partial x} dx\right) dy - \sigma_x dy + \left(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y} dy\right) dx - \tau_{yx} dx = 0$$

$$\Downarrow$$

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + f_x = 0$$
$$\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + f_y = 0$$

$$\begin{cases} \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + f_x = 0 \\ \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + f_y = 0 \\ \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + f_z = 0 \end{cases}$$

f_x, f_y, f_z: body force intensities

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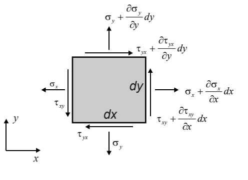
Differential equation of equilibrium



$$\sum M = 0: \quad \left(\sigma_x + \frac{\partial \sigma_x}{\partial x} dx\right) dy \frac{dy}{2} - \sigma_x dy \frac{dy}{2} + \sigma_y dx \frac{dx}{2} - \left(\sigma_y + \frac{\partial \sigma_y}{\partial y} dy\right) dx \frac{dx}{2}$$

wrt lower left corner

$$+ \left(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y} dy\right) dxdy - \left(\tau_{xy} + \frac{\partial \tau_{xy}}{\partial x} dx\right) dydx = 0$$

$$\Downarrow$$

$$\tau_{yx} = \tau_{xy}$$

3D: $\tau_{xy} = \tau_{yx}, \tau_{xz} = \tau_{zx}, \tau_{yz} = \tau_{zy}$

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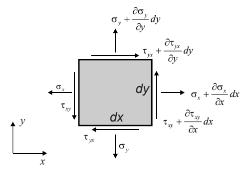
Differential equation of equilibrium

- Six equations are obtained based on force and moment equilibrium
- Only three force equilibrium equations remain to be satisfied by the stress components
- Additional equations are required for a complete solution of the stress distribution in a body:
  - Strain-displacement
  - Generalized Hooke's law



$$\begin{cases} \dfrac{\partial \sigma_x}{\partial x} + \dfrac{\partial \tau_{yx}}{\partial y} + \dfrac{\partial \tau_{zx}}{\partial z} + f_x = 0 \\ \dfrac{\partial \tau_{xy}}{\partial x} + \dfrac{\partial \sigma_y}{\partial y} + \dfrac{\partial \tau_{zy}}{\partial z} + f_y = 0 \quad \text{(Navier)} \\ \dfrac{\partial \tau_{xz}}{\partial x} + \dfrac{\partial \tau_{yz}}{\partial y} + \dfrac{\partial \sigma_z}{\partial z} + f_z = 0 \end{cases}$$

$$\tau_{xy} = \tau_{yx},\ \tau_{xz} = \tau_{zx},\ \tau_{yz} = \tau_{zy}$$

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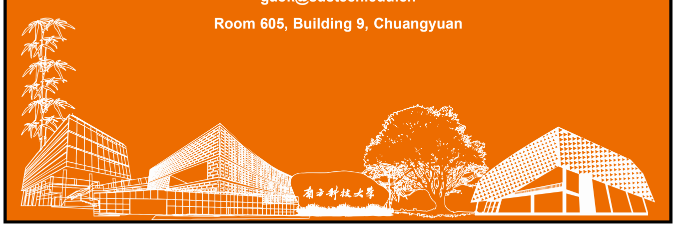

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