Review

Fourier's law of thermal conduction (傅里叶定律): $q_i = -k\,\partial_i T$

Energy conservation equation

$$\rho \frac{D(e+\frac{1}{2}v^2)}{Dt} = -\frac{\partial q_i}{\partial x_i} + \rho F_i v_i + \frac{\partial(\sigma_{ij} v_j)}{\partial x_i}$$

both internal and kinetic energy

$$\rho c_p \frac{DT}{Dt} = -\frac{\partial q_i}{\partial x_i} + \tau_{ij}\dot{\varepsilon}_{ij} + \alpha T \frac{Dp}{Dt}$$

only internal energy, in temperature

Governing equations of fluid mechanics (Navier–Stokes equations):

mass conservation

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v_i)}{\partial x_i} = 0$$

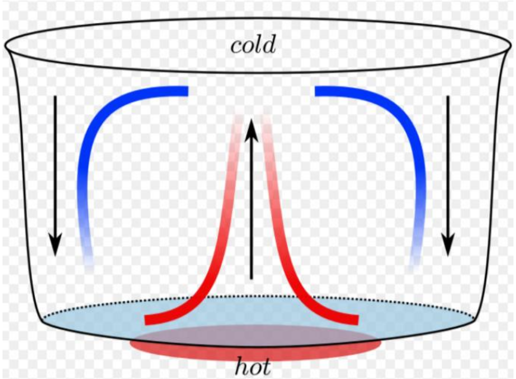
momentum conservation

$$\rho \frac{Dv_i}{Dt} = \rho g_i - \frac{\partial p}{\partial x_i} + [\mu(v_{j,i} + v_{i,j} - \frac{2}{3} v_{k,k}\delta_{ji})]_{,j}$$

energy conservation

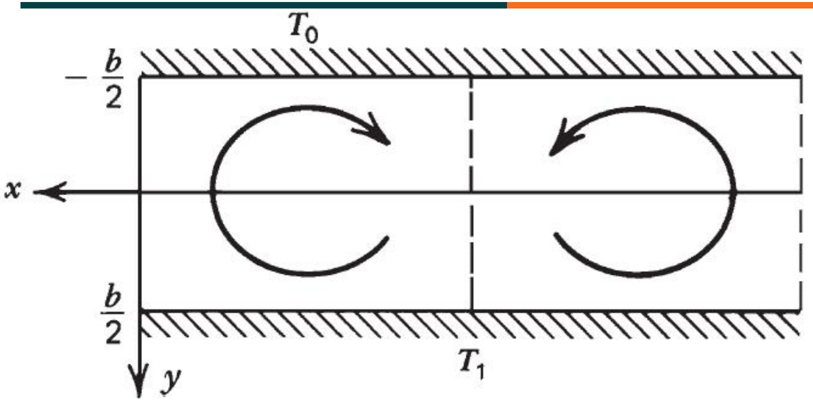
$$\rho c_p \frac{DT}{Dt} = \frac{\partial(kT_{,i})}{\partial x_i} - \frac{2}{3}\mu v_{i,i}^2 + 2\mu\dot{\varepsilon}_{ij}^2 + \alpha T \frac{Dp}{Dt}$$

Rayleigh Number and Onset of Thermal Convection



If the fluid (honey) is heated from below, under which condition will the convection start?

Rayleigh Number and Onset of Thermal Convection



Assume the fluid layer is heated from below. The top and bottom boundary temperature is fixed at T_1 and T_0.

For convection to occur, the convective thermal transport rate must be larger than the diffusive transport rate:

Diffusive thermal transport time: b^2 / κ

Convective thermal transport time: b / u

For viscous laminar flow:

$$\underbrace{\Delta\rho b^2 \delta y g}_{\text{driving force}} = \underbrace{b \delta y \eta u / b}_{\text{drag force}}$$

$$u = \Delta\rho g b^2 / \eta$$

Convective thermal transport time:

$$b / u = \eta / \Delta\rho g b$$

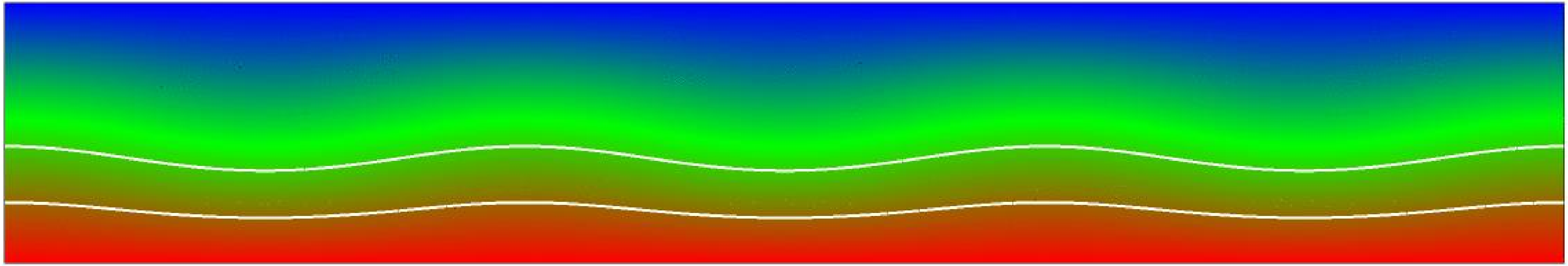
Define Rayleigh number as ratio between the diffusive and convective thermal transport time:

$$Ra = \frac{b^2/\kappa}{\eta/\Delta\rho g b} = \frac{\Delta\rho g b^3}{\eta\kappa} = \frac{\rho\alpha_v (T_1 - T_0) g b^3}{\eta\kappa}$$

For thermal convection to occur, Ra must exceeds a critical value Ra_c:
$Ra > Ra_c$ = (700 - several thousands)

Rayleigh Number and Onset of Thermal Convection

Initial periodic thermal perturbation gradually fades away
because Ra=4.5E2 < critical Rayleigh Number

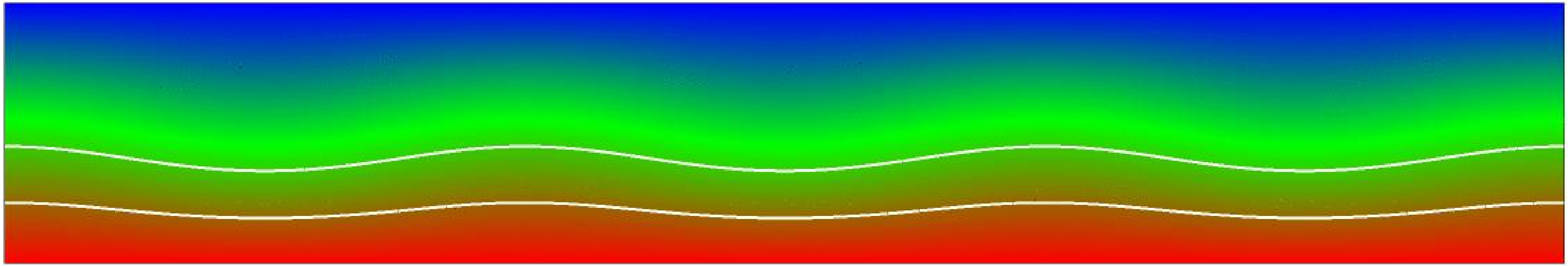


$$Ra = \frac{\rho \alpha_v (T_1 - T_0) g b^3}{\eta \kappa} = 450 < Ra_c$$

dimensionless temperature

Rayleigh Number and Onset of Thermal Convection

Initial periodic thermal perturbation triggers thermal convection because Ra=4.5E3 > critical Rayleigh Number



$$Ra = \frac{\rho \alpha_v (T_1 - T_0) g b^3}{\eta \kappa} = 4500 > Ra_c$$

dimensionless temperature

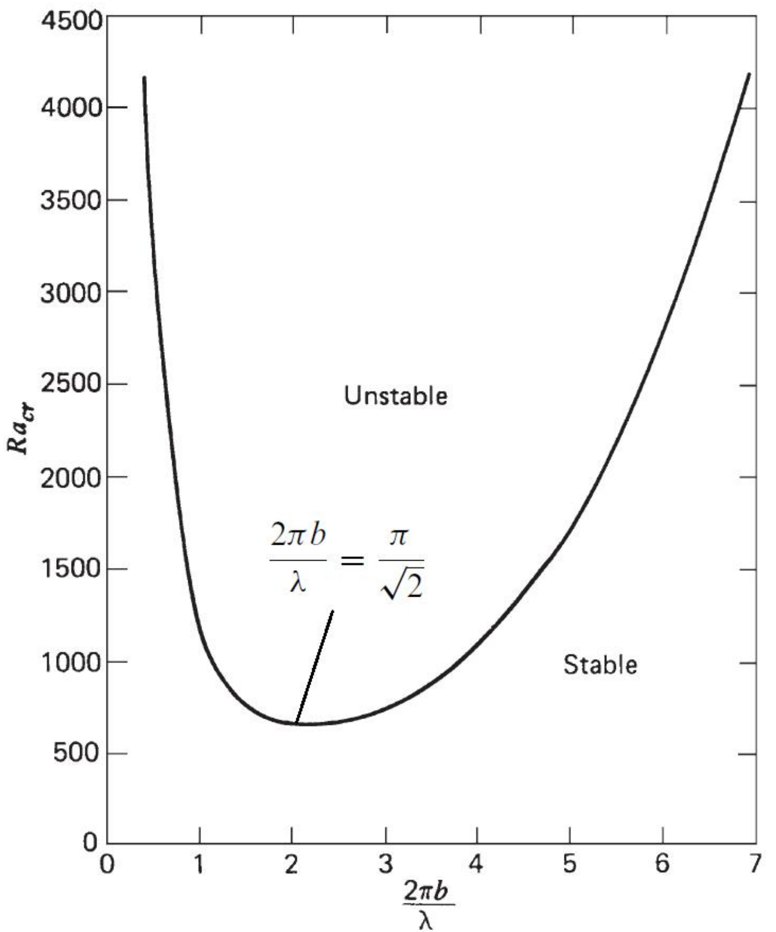
Rayleigh Number and Onset of Thermal Convection

The critical Rayleigh number depends on the wavelength of the initial temperature perturbation

The minimum critical Rayleigh number is at

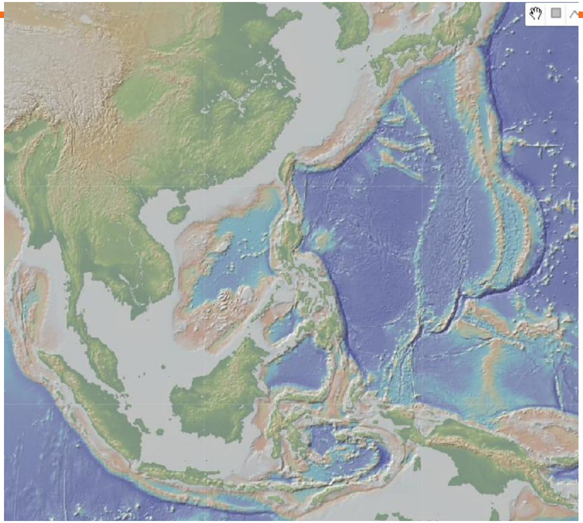
$$\frac{2\pi b}{\lambda} = \frac{\pi}{\sqrt{2}} \quad \text{or} \quad \lambda = 2\sqrt{2}\, b$$

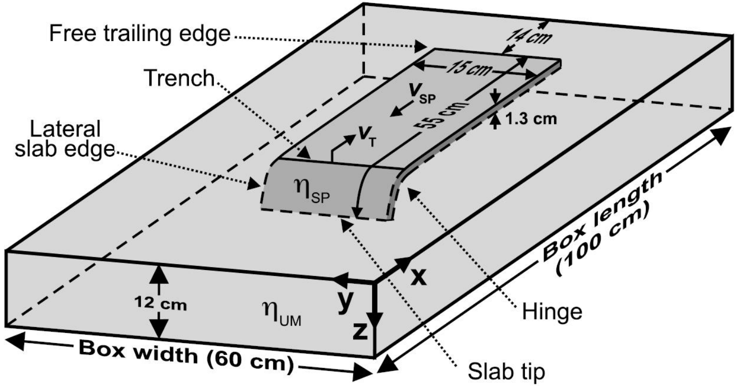
Corresponding to a convective cell aspect ratio of $\sqrt{2}$



Scaling and Dimensional Analysis (量纲分析): scaling

1. It is usually prohibitively expensive to build full-scale models for testing and subsequent modification of fluid dynamics laboratory experiments
    - e.g., flows around ships, trains and aircraft
2. Smaller models are used
    - under what conditions will the data obtained from a **model** be applicable to the full-size object?
3. **Geometric and dynamic similarities** must be maintained between scaled model and prototype





Variable η_{SP}/η_{UM} = 66-1375

schellart, 2010

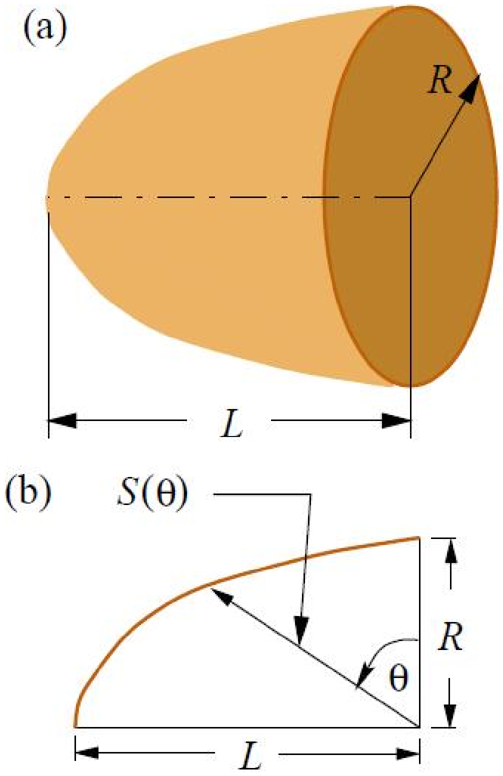
Scaling and Dimensional Analysis: scaling

Geometric similarity (几何相似): all linear length scales between the model and the prototype are fixed.

$$\frac{s(\theta)}{S(\theta)} = \alpha \qquad \text{for all} \quad \theta \in [0, \pi/2]$$

s: model
S: prototype
α: scale factor

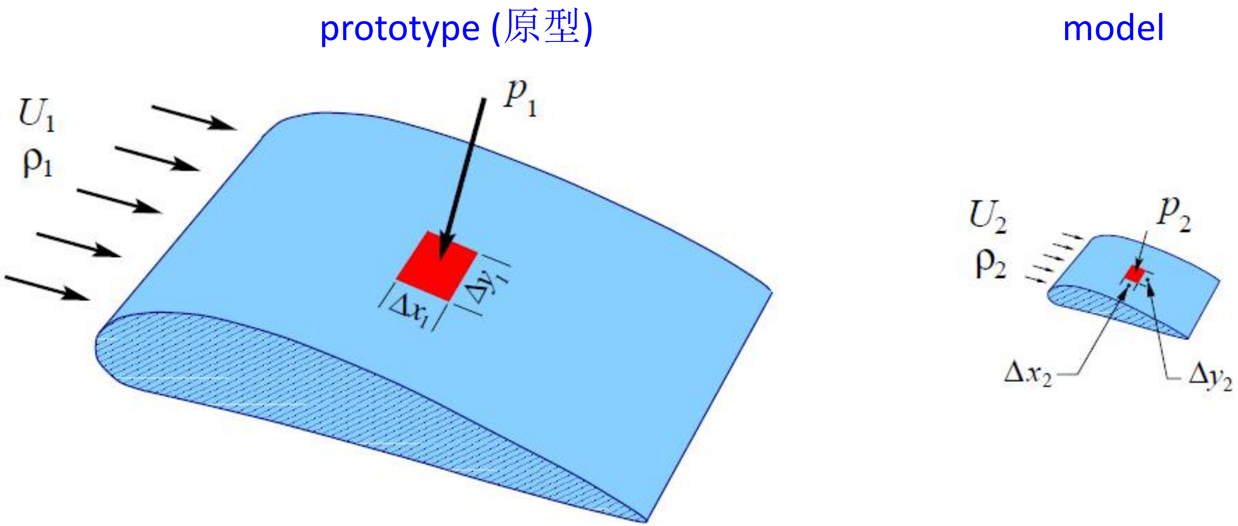
The surface areas ratio of the model to the actual object is α^2
The volume ratio of the model to the actual object is α^3



Missile nose cone (a) physical 3-D figure, (b) cross section indicating linear lengths.

Scaling and Dimensional Analysis: scaling

Dynamically similarity (动力相似): the forces acting at corresponding locations on two geometrically similar objects are everywhere in the same ratio.



- Requirement for dynamically similarity:
  - **dimensionless parameters** controlling flow around the two geometrically similar objects equal.

Scaling and Dimensional Analysis: dimensional analyis

Dimensional analyis (量纲分析):

- a process to identify the set of dimensionless parameters which completely characterizes behavior of the system.

Two ways to conduct dimensional analysis

- straightforward nondimensionalize the governing equations
    - when governing equations exist
- employ the Buckingham Π theorem (白金汉π定理)
    - whether the governing equations exist or not

Scaling and Dimensional Analysis: nondimensionalize the mass and momentum equations

Governing equations for 2D incompressible constant temperature flow:

mass conservation

$$\frac{\partial v_i}{\partial x_i} = 0$$

momentum conservation

$$\rho \frac{Dv_i}{Dt} = \rho g_i - \frac{\partial p}{\partial x_i} + \mu v_{i,jj}$$

- Independent variables of the system are x_i, t
- the dependent variables are v_i, p
- the parameters are g_i, density ρ, viscosity μ
- boundary and initial conditions will not introduce new independent or dependent variables, and thus not considered.

We now choose length scale H, velocity scale Us, and pressure scale p_s to scale the variables

$$x_i^* = \frac{x_i}{H} \quad t^* = \frac{U_s t}{H} \quad v_i^* = \frac{v_i}{U_s} \quad p^* = \frac{p}{p_s}$$

the "*" quantities are dimensionless

non-dimensionalize

$$\frac{\partial}{\partial x_i^*} = H \frac{\partial}{\partial x_i} \quad \frac{\partial}{\partial t^*} = \frac{H}{U_s} \frac{\partial}{\partial t}$$

Scaling and Dimensional Analysis: nondimensionalize the mass and momentum equations

Substitute the scaling relations to the governing equations, we have the dimensionless governing equations:

mass conservation

$$\frac{\partial v_i^*}{\partial x_i^*} = 0$$

momentum conservation

$$\frac{Dv_i^*}{Dt^*} = \frac{Hg_i}{U_s^2} - \frac{p_s}{\rho U_s^2}\frac{\partial p^*}{\partial x_i^*} + \frac{\mu}{\rho H U_s} v_{i,jj}^*$$

coefficients on the right-hand side are all dimensionless

Froude number (弗劳德数): square root of the ratio of the inertial force to gravitaty force

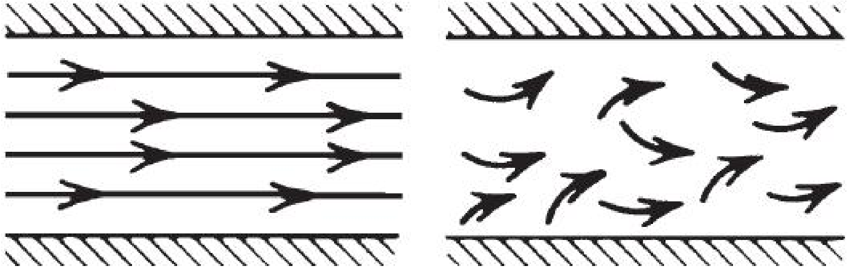
$$\frac{inertial}{gravity} \sim \frac{U}{\sqrt{Hg}} = Fr$$

H: characteristic scale of the fluid.

Reynold number (雷诺数): the ratio between the inertial forces (ρu^2) and the viscous forces ($\mu U/H$)

$$\frac{inertial}{viscous} \sim \frac{\rho U^2}{\mu U / H} = \frac{\rho U H}{\mu} = \mathrm{Re}$$

H: characteristic scale of the fluid.



laminar flow (层流), low *Re*

(*Re*<~2200)

turbulent flow (紊流), high *Re*

(*Re*>~3500)

Scaling and Dimensional Analysis: nondimensionalize the mass and momentum equations

Set the pressure scale as ρU_s^2 , the dimensionless momentum equation is reduced to

$$\frac{Dv_i^*}{Dt^*} = \frac{1}{Fr^2}\delta_{iy} - \frac{\partial p^*}{\partial x_i^*} + \frac{1}{\mathrm{Re}} v_{i,jj}^*$$

momentum conservation

$$\frac{\partial v_i^*}{\partial x_i^*} = 0$$

mass conservation

$$\rho \frac{Dv_i}{Dt} = \rho g_i - \frac{\partial p}{\partial x_i} + \mu v_{i,jj}$$

$$\frac{\partial v_i}{\partial x_i} = 0$$

The solution of the above dimensionless governing equations only depend on the parameters of Re and Fr, vs depending on g_i, ρ, μ, H etc.

Scaling and Dimensional Analysis: nondimensionalize the mass and momentum equations

The dimensionless governing equation is

$$\frac{Dv_i^*}{Dt^*} = \frac{1}{Fr^2}\delta_{iy} - \frac{\partial p^*}{\partial x_i^*} + \frac{1}{\mathrm{Re}} v_{i,jj}^*$$

$$\frac{\partial v_i^*}{\partial x_i^*} = 0$$

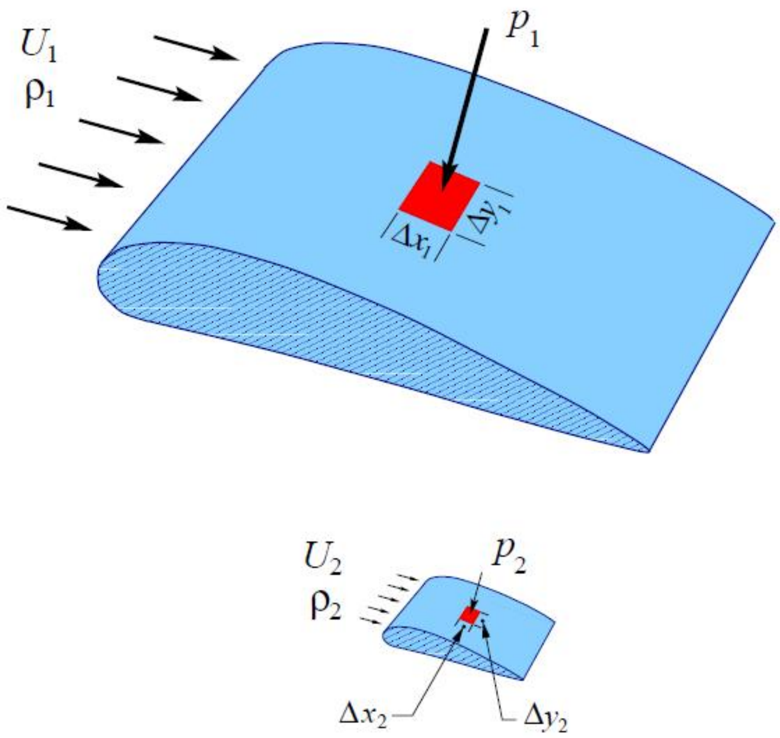
Assume the Froude number and Reynolds number is the same for the prototype and model,

- they have the same dimensionless velocity and pressure fields at corresponding locations.

$$p_1^* = p_2^*$$

- the unscaled forces will be in a constant ratio at all corresponding points of the two flow fields.

$$\frac{p_1}{p_2} = \frac{p_1^* \rho_1 U_1^2}{p_2^* \rho_2 U_2^2} = \frac{\rho_1 U_1^2}{\rho_2 U_2^2}$$



Prototype and model airfoils demonstrating dynamic similarity

Scaling and Dimensional Analysis: the Buckingham Π theorem

The Buckingham Π theorem (白金汉π定律、π定理): For any given physical problem that involves N_v dimensional variables (dependent variables + parameters), it can be reduced to a relation between only N_p dimensionless variables. The reduction $N_d = N_v - N_p$ is the number of independent dimensions (量纲) needed to describe the problem *(seldomly, Nd can be smaller than the number of independent dimensions).*

For example, suppose a convective system has 5 dependent variables and parameters: force F, length scale L, flow velocity V, gravity acceleration g, fluid density ρ and viscosity μ:

$$F = f(L, V, \rho, \mu, g)$$

We have $N_v = 5$, $N_d = 3$ (time *T*, mass *M*, length *L*).

Moreover, we can select N_d variables that do not form a pi (dimensionless quantity) among themselves to construct N_p independent pi groups. Each pi group will be a power product of these N_d variables and one of the remaining variables.

Assume the process involves five variables: $v_1 = f(v_2, v_3, v_4, v_5)$ and $N_d = 3$, we will get 2 pi groups.

Assume variables v_2, v_3, and v_4 do not form a pi, then we can construct two pi groups as:

$$\Pi_1 = (v_2)^a (v_3)^b (v_4)^c v_1 = M^0 L^0 T^0 \qquad \Pi_2 = (v_2)^a (v_3)^b (v_4)^c v_5 = M^0 L^0 T^0$$

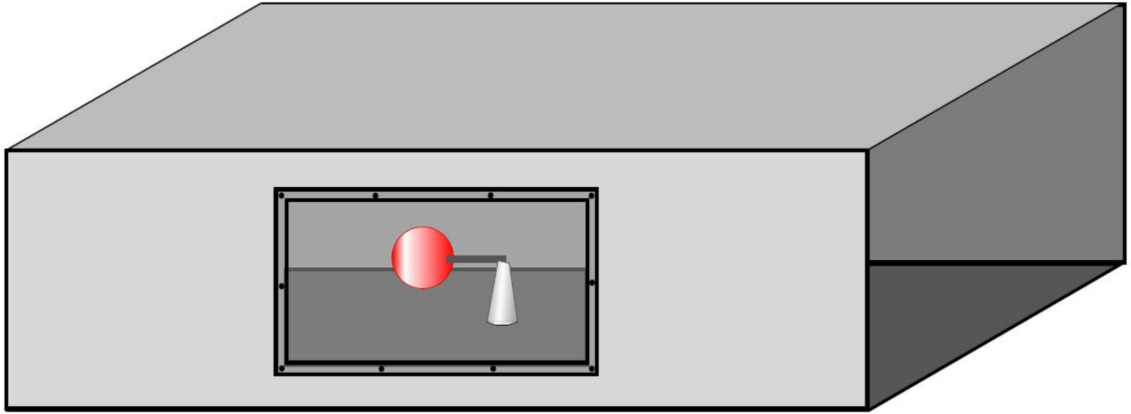
the powers a, b, c can be calculated for each pi group and Π_1 and Π_2 are independent.

Scaling and Dimensional Analysis: the Buckingham Π theorem

A sphere is immersed in a fluid flowing with speed U. Body forces are neglected.

We wish to analyze the force F acting on the sphere under various conditions.

We first wish to obtain the dimensionless correlation between the force exerted on the surface of a sphere and other parameters



Wind tunnel measurement of forces on sphere

Scaling and Dimensional Analysis: the Buckingham Π theorem

Step 1: determine all important variables associated with this problem.

$$F = f(D, U, \rho, \mu)$$

force F, sphere diameter D, flow velocity U, fluid viscosity μ, fluid density ρ. Thus $N_v = 5$

Step2: determine the number of independent dimensions (量纲) associated with the physical situation.

$$\begin{aligned} F &\sim M\frac{L}{T^2} \\ D &\sim L \\ U &\sim \frac{L}{T} \\ \rho &\sim \frac{M}{L^3} \\ \mu &\sim \frac{M}{LT}, \end{aligned}$$

we have 3 independent dimensions (3个独立量纲M: mass, L: length, T: time). $N_d = 3$

Thus $N_p = N_v - N_d = 2$ dimensionless parameters are needed to completely describe the forces acting on the sphere as a result of flow moving past it.

Scaling and Dimensional Analysis: the Buckingham Π theorem

Step3: select N_d scaling variables that do not form a pi among themselves

$$\mu^{\alpha 1}\rho^{\alpha 2}D^{\alpha 3} = M^0T^0L^0 \Rightarrow \alpha_1 = \alpha_2 = \alpha_3 = 0$$

Thus μ, ρ, D cannot form a pi group among themselves.

Step4: Sequencely construct each of the desired N_p pi groups. Find the exponents that make pi groups dimensionless.

$$\Pi_1 = \mu^{\alpha 1}\rho^{\alpha 2}D^{\alpha 3}F = M^0T^0L^0 \qquad M^{\alpha 1}L^{-\alpha 1}T^{-\alpha 1}M^{\alpha 2}L^{-3\alpha 2}L^{\alpha 3}MLT^{-2} = M^0T^0L^0$$

$$\Rightarrow \alpha_1 = -2 \quad \alpha_2 = 1 \quad \alpha_3 = 0 \qquad \Pi_1 = \mu^{-2}\rho F = \frac{\rho F}{\mu^2}$$

Similarly, we get

$$\Pi_2 = \mu^{\alpha 1}\rho^{\alpha 2}D^{\alpha 3}U = M^0T^0L^0 \qquad M^{\alpha 1}L^{-\alpha 1}T^{-\alpha 1}M^{\alpha 2}L^{-3\alpha 2}L^{\alpha 3}LT^{-1} = M^0T^0L^0 \Rightarrow$$

$$\alpha_1 = -1 \quad \alpha_2 = 1 \quad \alpha_3 = 1 \qquad \Pi_2 = \mu\rho DU = \frac{\rho DU}{\mu}$$

Scaling and Dimensional Analysis: the Buckingham Π theorem

Step5: write the final dimensionless function with all dimensionless pi groups

$$F = f\,(D, U, \rho, \mu) \qquad \Pi_1 = \mu^{-2}\rho F = \frac{\rho F}{\mu^2} \qquad \Pi_2 = \mu\rho D U = \frac{\rho D U}{\mu}$$

$$\Rightarrow \quad \Pi_1 = g\,(\Pi_2) \quad \text{or} \quad F = f\,(D, U, \rho, \mu) \Rightarrow \frac{\rho F}{\mu^2} = g\left(\frac{\rho D U}{\mu}\right)$$

Steps of the Buckingham π theorem:

1. determine all the N_v variables associated with the problem;
2. list the dimensions of each variable and determine the number of independent dimensions N_d and the number of dimensionless parameters N_p that must be found to characterize the problem;
3. select N_d scaling variables that do not form a pi product themselves. In general, these variables need to be as common as possible, e.g., density, length, instead of surface tension.
4. Squencely add one of the remaining N_p variables to the N_d scaling variables to construct the N_p pi groups as a power product of these variables. Algebraically find the exponents that make pi groups dimensionless. In this way, the N_p pi groups will be independent to each other.
5. write the final dimensionless function with all dimensionless pi groups. Check the terms to make sure all pi groups are dimensionless.

Classroom exercise

momentum conservation

$$\rho \frac{Dv_i}{Dt} = \rho g_i - \frac{\partial p}{\partial x_i} + \mu v_{i,jj}$$

Choose length scale H, velocity scale Us, and pressure scale ρU_s^2 to scale the variables

Prove that the non-dim momentum conservation is as below:

$$\frac{Dv_i^*}{Dt^*} = \frac{Hg_i}{U_s^2} - \frac{\partial p^*}{\partial x_i^*} + \frac{\mu}{\rho H U_s} v_{i,jj}^*$$

Homework (3 points)

1. ρ is density, and v_i is flow velocity. Prove the following two equations (hint: the continuity equation is used):

(a) $$\frac{\partial(\rho Q_{lm..})}{\partial t}+\frac{\partial(\rho v_j Q_{lm..})}{\partial x_j}=\rho\frac{DQ_{lm..}}{Dt}$$

(b) $$v_i\frac{\partial(\rho v_i)}{\partial t}+v_i\frac{\partial(\rho v_j v_i)}{\partial x_j}=\frac{\partial(\frac{1}{2}\rho v^2)}{\partial t}+\frac{\partial(\frac{1}{2}\rho v_j v^2)}{\partial x_j}$$

2. The volume flow Q of a centrifugal pump is a function of the input power P, impeller diameter D, rotational rate Ω, and the density ρ and viscosity μ of the fluid:

$$Q=f(P, D, \Omega, \rho, \mu)$$

Rewrite this as a dimensionless relationship as below with Ω, ρ, and D as repeating variables.

$$\frac{Q}{\Omega D^3}=f(\frac{P}{\rho\Omega^3 D^5},\frac{\mu}{\rho\Omega D^2})$$

Homework (3 points)

3. One important mathmatical physical equations is the thermal conduction equation (a type of parabolic equation) that describes temperature distribution due to thermal conduction inside a solid body. The thermal conduction equation can be derived by neglecting the advection from the energy conservation equation:

$$\rho c_p \frac{DT}{Dt} = -\frac{\partial q_i}{\partial x_i} + \tau_{ij}\dot{\varepsilon}_{ij} + \alpha T \frac{Dp}{Dt} + \rho H \quad \Rightarrow \quad \rho c_p \frac{\partial T}{\partial t} = -\frac{\partial q_i}{\partial x_i} + \rho H$$

Please derive the thermal conduction equation

$$\rho c_p \frac{\partial T}{\partial t} = k\nabla^2 T + \rho H$$

by conducting thermal balance analysis for a element on the right: