一、Given $\sigma_x = -14\,MPa$, $\sigma_y = 6\,MPa$, $\tau_{xy} = -17\,MPa$

(1) Solution: $\boxed{\text{Methos A:}}$

The principal stresses:

$$\begin{cases} \sigma_{max} = \dfrac{\sigma_x+\sigma_y}{2} + \sqrt{\left(\dfrac{\sigma_x-\sigma_y}{2}\right)^2+\tau_{xy}^2} = 15.72\ MPa \\ \sigma_{min} = \dfrac{\sigma_x+\sigma_y}{2} - \sqrt{\left(\dfrac{\sigma_x-\sigma_y}{2}\right)^2+\tau_{xy}^2} = -23.72\ MPa \end{cases}$$

The directions:

$$\tan 2\alpha = \frac{2\tau_{xy}}{\sigma_x-\sigma_y} \Rightarrow 2\alpha = 1.04\,rad = 59.59^\circ$$

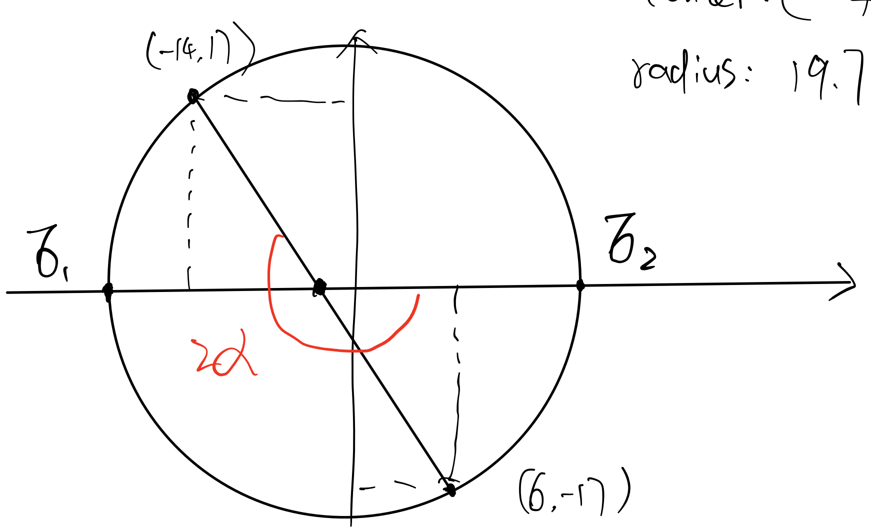
$$\Rightarrow \alpha_1 = 29.79^\circ$$

$$\alpha_2 = 119.79^\circ$$

$\boxed{\text{Methos B:}}$

center: $(-4, 0)$

radius: 19.72



$\therefore \sigma_1 = -23.72\ MPa$

$\sigma_2 = 15.72\ MPa$

The directions: As shown in the picture

(2) Solution:

$$\tan 2\alpha = -\frac{\sigma_x - \sigma_y}{2\tau_{xy}} \Rightarrow \alpha = -33.71^\circ$$

$$\tau_{x'y'} = \pm\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \pm\sqrt{389}\ \text{MPa}$$

$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\alpha + \tau_{xy}\sin 2\alpha$$

$$= -5.5\ \text{MPa}$$

(3) $$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\alpha + \tau_{xy}\sin 2\alpha = -21\ \text{MPa}$$

$$\sigma_{y'} = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos 2\alpha - \tau_{xy}\sin 2\alpha = 13\ \text{MPa}$$

$$\tau_{x'y'} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\alpha + \tau_{xy}\cos 2\alpha = 10\ \text{MPa}$$

二、Given a three-dimensional stress state with

$\sigma_x = 10\,MPa$, $\sigma_y = 20\,MPa$, $\sigma_z = -10\,MPa$

$\tau_{xy} = 5\,MPa$, $\tau_{xz} = -10\,MPa$, $\tau_{yz} = -15\,MPa$

(a) Solution:

$$\cos^2(x',x) + \cos^2(x',y) + \cos^2(x',z) = 1$$

$\because$ As we know $\cos(x',z)$ is positive

$\therefore \cos(x',z) = \frac{1}{2}$

$$P_x = \sigma_x \cdot \cos(x',x) + \tau_{yx} \cdot \cos(x',y) + \tau_{zx} \cdot \cos(x',z) = \frac{5\sqrt{2}}{2}\,MPa$$

$$P_y = \tau_{xy} \cdot \cos(x',x) + \sigma_y \cdot \cos(x',y) + \tau_{zy} \cdot \cos(x',z) = 10\sqrt{2} - 5\,MPa$$

$$P_z = \tau_{xz} \cos(x',x) + \tau_{yz} \cdot \cos(x',y) + \sigma_z \cdot \cos(x',z) = -\frac{15\sqrt{2}+20}{2}\,MPa$$

$$\therefore \vec{P} = \left(\frac{5\sqrt{2}}{2},\ 10\sqrt{2}-5,\ -\frac{15\sqrt{2}}{2} - 10\right)$$

(b) Solution:

$$\begin{cases} \sigma = P_x \cdot \cos(x',x) + P_y \cdot \cos(x',y) + P_z \cdot \cos(x',z) \\ \tau^2 = P^2 - \sigma^2 \end{cases}$$

And we can get

$$\begin{cases} \sigma = -2.07\,MPa \\ \tau = 22.72\,MPa \end{cases}$$

(c) Solution:

assume θ be the angle between p and σ

$$\cos\theta = \frac{\sigma}{p} \qquad |P| = \sqrt{P_x^2 + P_y^2 + P_z^2} = 22.82$$

$$\therefore \cos\theta = \frac{-2.07}{22.82} = 0.09$$

$$\Rightarrow \theta = 84.8^\circ$$

(d) Solution:

if $\cos(x, y') = \frac{1}{2}$ and $\cos(z, y')$ is negative. we can get

$$\begin{cases} \cos^2(x, y') + \cos^2(y, y') + \cos^2(z, y') = 1 & ① \\ \cos(x', x)\cdot\cos(x, y') + \cos(y, y')\cdot\cos(x', y) + \cos(z, y')\cdot\cos(x', z) = 0 & ② \\ \cos(z, y') < 0 & ③ \end{cases}$$

So we can get $\cos(z, y') = -\frac{5}{6}$ and $\cos(y, y') = \frac{\sqrt{2}}{6}$

And according to the relationship between these direction cosines, then we can get $\cos(x, z') = -\frac{\sqrt{2}}{2}$, $\cos(y, z') = \frac{2}{3}$, $\cos(z, z') = -\frac{\sqrt{2}}{6}$

$$\therefore \tau_{x'y'} = P_x \cos(x, y') + P_y \cos(y, y') + P_z \cos(z, y') = 21.09\ MPa$$

$$\tau_{x'z'} = P_x \cdot \cos(x, z') + P_y \cos(y, z') + P_z \cos(z, z') = 8.45\ MPa$$

(e) Solution:

As we know the transformational matrix

$$R = \begin{bmatrix} \cos(x,x') & \cos(x,y') & \cos(x,z') \\ \cos(x,y') & \cos(y,y') & \cos(z,y') \\ \cos(x,z') & \cos(y,z') & \cos(z,z') \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{2}}{6} & -\frac{5}{6} \\ -\frac{\sqrt{2}}{2} & \frac{2}{3} & -\frac{\sqrt{2}}{6} \end{bmatrix}$$

All of the stress components acting on the x', y', z' plane are shown as bellow:

$$R \cdot S \cdot R^T =$$

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{\sqrt{2}} & \frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{2}}{6} & -\frac{5}{6} \\ -\frac{\sqrt{2}}{2} & \frac{2}{3} & -\frac{\sqrt{2}}{6} \end{bmatrix} \cdot \begin{bmatrix} 10 & 5 & -10 \\ 5 & 20 & -15 \\ -10 & -15 & -10 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{\sqrt{2}}{2} \\ \frac{1}{\sqrt{2}} & \frac{\sqrt{2}}{6} & \frac{2}{3} \\ \frac{1}{2} & -\frac{5}{6} & -\frac{\sqrt{2}}{6} \end{bmatrix}$$

$$= \begin{bmatrix} -2.07 & 21.09 & 8.46 \\ 21.09 & 12.06 & 2.93 \\ 8.44 & 2.94 & 10 \end{bmatrix}$$

(f) Solution:

$$\begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix} = \begin{bmatrix} 10 & 5 & -10 \\ 5 & 20 & -15 \\ -10 & -15 & -10 \end{bmatrix}$$

We can get the

$$\begin{cases} \lambda_1 = 30 \\ \lambda_2 = 8.23 \\ \lambda_3 = -18.23 \end{cases} \quad \begin{cases} \vec{\alpha_1} = (0.4082, 0.8165, -0.4082) \\ \vec{\alpha_2} = (-0.8736, 0.4792, 0.0849) \\ \vec{\alpha_3} = (-0.2650, -0.3220, -0.9089) \end{cases}$$

So, $\begin{cases} \sigma_1 = 30\,MPa \\ \sigma_2 = 8.23\,MPa \\ \sigma_3 = -18.23\,MPa \end{cases}$

$$\begin{cases} \cos(x_p, x) = 0.4082 \\ \cos(x_p, y) = 0.8165 \\ \cos(x_p, z) = -0.4082 \\ \cos(y_p, x) = -0.8736 \\ \cos(y_p, y) = 0.4792 \\ \cos(y_p, z) = 0.0849 \\ \cos(z_p, x) = -0.2650 \\ \cos(z_p, y) = -0.3220 \\ \cos(z_p, z) = -0.9089 \end{cases}$$

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(1) Solution:

$$\begin{cases} \dfrac{\partial \sigma_x}{\partial x} + \dfrac{\partial \tau_{yx}}{\partial y} + \dfrac{\partial \tau_{zx}}{\partial z} + f_x = 0 \\ \dfrac{\partial \tau_{xy}}{\partial x} + \dfrac{\partial \sigma_y}{\partial y} + \dfrac{\partial \tau_{zy}}{\partial z} + f_y = 0 \\ \dfrac{\partial \tau_{xz}}{\partial x} + \dfrac{\partial \tau_{yz}}{\partial y} + \dfrac{\partial \sigma_z}{\partial z} + f_z = 0 \end{cases}$$

we can get

$$\begin{cases} 6x - 6x + 0 = 0 \\ 6y - 6y + 0 = 0 \\ 1 + 0 - 1 = 0 \end{cases}$$

So, we can know this stress state is in equilibrium.

(2) Solution:

At point $(\frac{1}{2}, 1, \frac{3}{4})$:

we can get
$$\begin{cases} \sigma_x = 3 \\ \sigma_y = 3 \\ \sigma_z = 3 \\ \tau_{xy} = -3 \\ \tau_{xz} = \tau_{yz} = 0 \end{cases}$$

$$I_1 = \sigma_x + \sigma_y + \sigma_z = \sigma_1 + \sigma_2 + \sigma_3 = 9$$

$$I_2 = \sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_x\sigma_z - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{xz}^2$$
$$= \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_1\sigma_3 = 18$$

$$I_3 = \sigma_x\sigma_y\sigma_z + \tau_{xy}\tau_{yz}\tau_{zx} + \tau_{xz}\tau_{yx}\tau_{zy}$$
$$- \sigma_y\tau_{xz}^2 - \sigma_x\tau_{zy}^2 - \sigma_z\tau_{xy}^2 = \sigma_1\sigma_2\sigma_3 = 0$$

So:
$$\begin{cases} \sigma_1 = 3 \\ \sigma_2 = 6 \\ \sigma_3 = 0 \end{cases}$$