南方科技大学
SOUTHERN UNIVERSITY OF SCIENCE AND TECHNOLOGY

MAE5009
Continuum Mechanics B
Session 02: Strain-displacement

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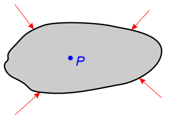
Why strain?

- Stress distribution in a body under an external force system also depends on material properties
- Equations of equilibrium are not sufficient to obtain the full stress components in a body
  – Six independent unknowns, only three equations so far
  – Strain-displacement and generalized Hooke's law are needed



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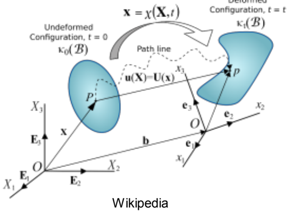
Assumptions

- Infinitesimal deformation
- Continuous material:
  – Material is present at each point in the medium
  – Continuous displacements
  – Original material can not contain gaps after displacement
- Displacement functions must be single-valued



Wikipedia

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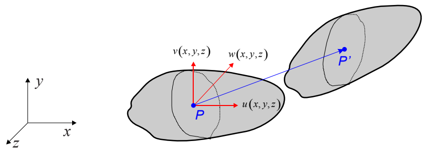
Strain & displacement

- A body is strained or deformed when the relative positions of points in the body are changed
- Displacement of a point is defined as the **vector** distance from the initial to the its final location



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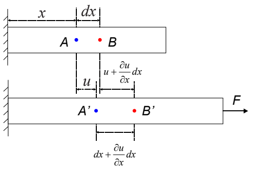
Strain & displacement

- Normal strain: unit change in length

$$\varepsilon = \frac{\partial u}{\partial x}dx \Big/ dx = \frac{\partial u}{\partial x}$$



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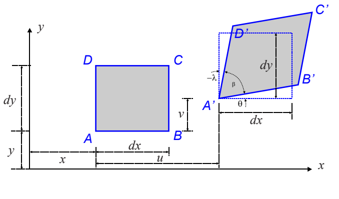
Strain & displacement

- The body movement includes:
  - Translation, rotation and deformation
- The body deformation includes:
  - The sides change length
  - The sides rotate with respect to each other



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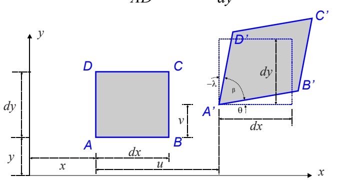
Normal strain

- The normal strain ε in a given direction is defined as the unit change in length of a line which was originally oriented in the given direction
  - Positive if the line increases in length
  - Negative if the line is shortened

$$\varepsilon_x = \frac{A'B' - AB}{AB} = \frac{A'B' - dx}{dx} \Rightarrow (\varepsilon+1)dx = A'B'$$

$$\varepsilon_y = \frac{A'D' - AD}{AD} = \frac{A'D' - dy}{dy}$$



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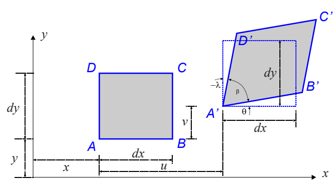
Shear strain

- The shear strain γ is associated with two orthogonal directions, and is defined as the change of angle between the two axes (measured in radian)
  - Positive if the angle between the two positive axes decreases

$$\gamma_{xy} = \frac{\pi}{2} - \beta = \theta - \lambda$$



Engineering shear strain

γ_{xy}

Mathematical shear strain

$\frac{1}{2}\gamma_{xy}$

Positive shear strain: the right angle between the positive extensions of the coordinate axes decreases

Sign of rotation angle:
- *counter-clockwise positive*

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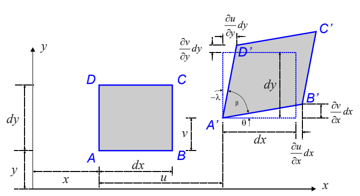
Strain

$$(A'B')^2 = \left(dx(1+\varepsilon_x)\right)^2 = \left(dx + \frac{\partial u}{\partial x}dx\right)^2 + \left(\frac{\partial v}{\partial x}dx\right)^2$$

$$\varepsilon_x^2 + 2\varepsilon_x + 1 = 1 + 2\frac{\partial u}{\partial x} + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2$$

$$\varepsilon_x = \frac{\partial u}{\partial x} \quad \xrightarrow{\text{similarly}} \quad \varepsilon_y = \frac{\partial v}{\partial y}$$



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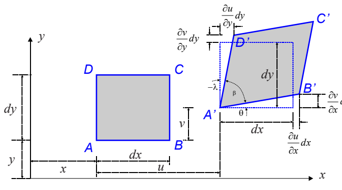
Strain

$$\theta = \tan\theta = \frac{\frac{\partial v}{\partial x}dx}{dx + \frac{\partial u}{\partial x}dx} = \frac{\frac{\partial v}{\partial x}}{1+\frac{\partial u}{\partial x}} = \frac{\partial v}{\partial x}$$

$$-\lambda = -\tan\lambda = \frac{\frac{\partial u}{\partial y}dy}{dy+\frac{\partial v}{\partial y}dy} = \frac{\frac{\partial u}{\partial y}}{1+\frac{\partial v}{\partial y}} = \frac{\partial u}{\partial y}$$

$$\gamma_{xy} = \theta - \lambda = \frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}$$



The two partial derivatives are positive if AB and AD rotate inward, i.e. u and v increase with increasing y and x, respectively

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Strain-displacement relations

$$\varepsilon_x = \frac{\partial u}{\partial x} \qquad \gamma_{xy} = \frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}$$

$$\varepsilon_y = \frac{\partial v}{\partial y} \qquad \gamma_{yz} = \frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}$$

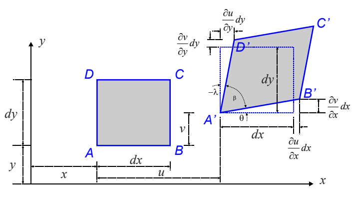
$$\varepsilon_z = \frac{\partial w}{\partial z} \qquad \gamma_{zx} = \frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}$$

(Cauchy)

$$\gamma_{xy} = \gamma_{yx}$$

$$\gamma_{yz} = \gamma_{zy}$$

$$\gamma_{zx} = \gamma_{xz}$$



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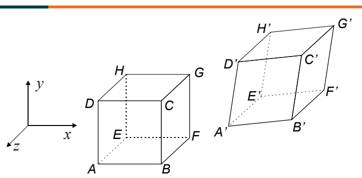
Strain in 3D element



$$\varepsilon_x = \frac{A'B'-AB}{AB} = \frac{D'C'-DC}{DC} = \frac{E'F'-EF}{EF} = \frac{H'G'-HG}{HG}$$

$$\gamma_{xy} = \frac{\pi}{2} - \angle B'A'D' = \frac{\pi}{2} - \angle F'E'H' = \frac{\pi}{2} - \angle B'C'D' = \frac{\pi}{2} - \angle F'G'H'$$

Inconsistency:

- *Stress components are distributed over a deformed body, and the coordinates x, y and z refer to deformed body*
- *In strain analysis, the coordinates x, y and z refer to undeformed body*
- *Error is minor under the assumption of infinitesimal deformation*

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Compatibility equations

- Six equations for the strain components are functions of only three displacement components
- If six strain components are known, we have six equations for only three unknowns
- There must be additional equations relate the six strain components

$$\varepsilon_x = \frac{\partial u}{\partial x} \qquad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

$$\varepsilon_y = \frac{\partial v}{\partial y} \qquad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}$$

$$\varepsilon_z = \frac{\partial w}{\partial z} \qquad \gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$$

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Compatibility equations

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} = \frac{\partial^3 u}{\partial x \partial y^2} \qquad \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^3 u}{\partial x^2 \partial y}$$

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^3 u}{\partial x \partial y^2} + \frac{\partial^3 u}{\partial x^2 \partial y}$$

$$\frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = \frac{\partial^3 u}{\partial x \partial y^2} + \frac{\partial^3 v}{\partial x^2 \partial y}$$

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}$$

$$\varepsilon_x = \frac{\partial u}{\partial x} \qquad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

$$\varepsilon_y = \frac{\partial v}{\partial y} \qquad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}$$

$$\varepsilon_z = \frac{\partial w}{\partial z} \qquad \gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$$

$$\frac{\partial^2 \varepsilon_y}{\partial z^2} + \frac{\partial^2 \varepsilon_z}{\partial y^2} = \frac{\partial^2 \gamma_{yz}}{\partial y \partial z}$$

$$\frac{\partial^2 \varepsilon_z}{\partial x^2} + \frac{\partial^2 \varepsilon_x}{\partial z^2} = \frac{\partial^2 \gamma_{zx}}{\partial z \partial x}$$

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Compatibility equations

$$\frac{\partial^2 \varepsilon_x}{\partial y \partial z} = \frac{\partial^3 u}{\partial x \partial y \partial z} \qquad \frac{\partial^2 \varepsilon_y}{\partial x \partial z} = \frac{\partial^3 v}{\partial x \partial y \partial z} \qquad \frac{\partial^2 \varepsilon_z}{\partial x \partial y} = \frac{\partial^3 w}{\partial x \partial y \partial z}$$

$$\frac{\partial \gamma_{xy}}{\partial z} = \frac{\partial^2 u}{\partial y \partial z} + \frac{\partial^2 v}{\partial x \partial z} \qquad \frac{\partial \gamma_{yz}}{\partial x} = \frac{\partial^2 v}{\partial x \partial z} + \frac{\partial^2 w}{\partial x \partial y}$$

$$\frac{\partial \gamma_{zx}}{\partial y} = \frac{\partial^2 w}{\partial x \partial y} + \frac{\partial^2 u}{\partial y \partial z}$$

$$\varepsilon_x = \frac{\partial u}{\partial x} \qquad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

$$\varepsilon_y = \frac{\partial v}{\partial y} \qquad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}$$

$$\varepsilon_z = \frac{\partial w}{\partial z} \qquad \gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$$

$$2\frac{\partial^2 \varepsilon_x}{\partial y \partial z} = \frac{\partial}{\partial x}\left(-\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right)$$

$$2\frac{\partial^2 \varepsilon_y}{\partial x \partial z} = \frac{\partial}{\partial y}\left(\frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right)$$

$$2\frac{\partial^2 \varepsilon_z}{\partial x \partial y} = \frac{\partial}{\partial z}\left(\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} - \frac{\partial \gamma_{xy}}{\partial z}\right)$$

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}$$

$$\frac{\partial^2 \varepsilon_y}{\partial z^2} + \frac{\partial^2 \varepsilon_z}{\partial y^2} = \frac{\partial^2 \gamma_{yz}}{\partial y \partial z}$$

$$\frac{\partial^2 \varepsilon_z}{\partial x^2} + \frac{\partial^2 \varepsilon_x}{\partial z^2} = \frac{\partial^2 \gamma_{zx}}{\partial z \partial x}$$

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Saint-Venant compatibility equations

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}$$

$$\frac{\partial^2 \varepsilon_y}{\partial z^2} + \frac{\partial^2 \varepsilon_z}{\partial y^2} = \frac{\partial^2 \gamma_{yz}}{\partial y \partial z}$$

$$\frac{\partial^2 \varepsilon_z}{\partial x^2} + \frac{\partial^2 \varepsilon_x}{\partial z^2} = \frac{\partial^2 \gamma_{zx}}{\partial z \partial x}$$

$$2\frac{\partial^2 \varepsilon_x}{\partial y \partial z} = \frac{\partial}{\partial x}\left(-\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right)$$

$$2\frac{\partial^2 \varepsilon_y}{\partial x \partial z} = \frac{\partial}{\partial y}\left(\frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right)$$

$$2\frac{\partial^2 \varepsilon_z}{\partial x \partial y} = \frac{\partial}{\partial z}\left(\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} - \frac{\partial \gamma_{xy}}{\partial z}\right)$$

Compatibility equations in terms of strain

The strain components must satisfy these expressions in order that the solutions for the displacement components exist

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Saint-Venant compatibility equations

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}$$

$$\frac{\partial^2 \varepsilon_y}{\partial z^2} + \frac{\partial^2 \varepsilon_z}{\partial y^2} = \frac{\partial^2 \gamma_{yz}}{\partial y \partial z}$$

$$\frac{\partial^2 \varepsilon_z}{\partial x^2} + \frac{\partial^2 \varepsilon_x}{\partial z^2} = \frac{\partial^2 \gamma_{zx}}{\partial z \partial x}$$

$$2\frac{\partial^2 \varepsilon_x}{\partial y \partial z} = \frac{\partial}{\partial x}\left(-\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right)$$

$$2\frac{\partial^2 \varepsilon_y}{\partial x \partial z} = \frac{\partial}{\partial y}\left(\frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right)$$

$$2\frac{\partial^2 \varepsilon_z}{\partial x \partial y} = \frac{\partial}{\partial z}\left(\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} - \frac{\partial \gamma_{xy}}{\partial z}\right)$$

$$\frac{\partial^4 \varepsilon_x}{\partial y^2 \partial z^2} + \frac{\partial^4 \varepsilon_y}{\partial x^2 \partial z^2} = \frac{\partial^4 \gamma_{xy}}{\partial x \partial y \partial z^2}$$

$$\frac{\partial^4 \varepsilon_y}{\partial x^2 \partial z^2} + \frac{\partial^4 \varepsilon_z}{\partial x^2 \partial y^2} = \frac{\partial^4 \gamma_{yz}}{\partial x^2 \partial y \partial z}$$

$$\frac{\partial^4 \varepsilon_z}{\partial x^2 \partial y^2} + \frac{\partial^4 \varepsilon_x}{\partial y^2 \partial z^2} = \frac{\partial^4 \gamma_{zx}}{\partial z \partial x \partial y^2}$$

$$2\frac{\partial^4 \varepsilon_x}{\partial y^2 \partial z^2} = \frac{\partial^3}{\partial x \partial y \partial z}\left(-\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right)$$

$$2\frac{\partial^4 \varepsilon_y}{\partial x^2 \partial z^2} = \frac{\partial^3}{\partial x \partial y \partial z}\left(\frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right)$$

$$2\frac{\partial^4 \varepsilon_z}{\partial x^2 \partial y^2} = \frac{\partial^3}{\partial x \partial y \partial z}\left(\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} - \frac{\partial \gamma_{xy}}{\partial z}\right)$$

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Saint-Venant compatibility equations

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}$$

$$\frac{\partial^2 \varepsilon_y}{\partial z^2} + \frac{\partial^2 \varepsilon_z}{\partial y^2} = \frac{\partial^2 \gamma_{yz}}{\partial y \partial z}$$

$$\frac{\partial^2 \varepsilon_z}{\partial x^2} + \frac{\partial^2 \varepsilon_x}{\partial z^2} = \frac{\partial^2 \gamma_{zx}}{\partial z \partial x}$$

$$2\frac{\partial^2 \varepsilon_x}{\partial y \partial z} = \frac{\partial}{\partial x}\left(-\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right)$$

$$2\frac{\partial^2 \varepsilon_y}{\partial x \partial z} = \frac{\partial}{\partial y}\left(\frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right)$$

$$2\frac{\partial^2 \varepsilon_z}{\partial x \partial y} = \frac{\partial}{\partial z}\left(\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} - \frac{\partial \gamma_{xy}}{\partial z}\right)$$

- *Only three compatibility equations are independent*
- *If the displacement components are single-valued, continuous functions, the strain components will automatically satisfy the compatibility equations*

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2D strain transformation

- Same as stress, strain is also a point property
- Given the strain components ε_x, ε_y, γ_{xy} at a point in specific directions, it's possible to determine the strain at the point in any direction

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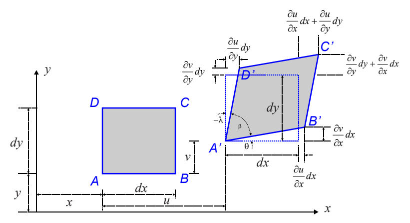
2D strain transformation



Displacement of *C* in *x* and *y* direction:

$$u + du = u + \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy \qquad v + dv = v + \frac{\partial v}{\partial y}dy + \frac{\partial v}{\partial x}dx$$

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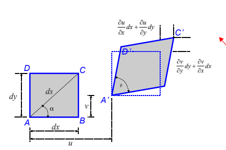
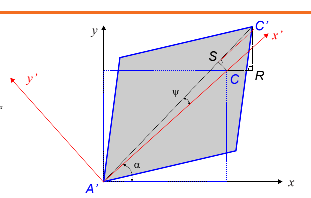
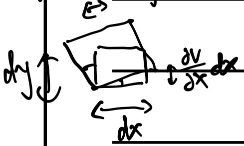
2D strain transformation

$\tan\theta = -\frac{\partial u}{\partial y}$







$\tan\theta = \frac{\partial v}{\partial x}$

$$CR = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

$$C'R = \frac{\partial v}{\partial y}dy + \frac{\partial v}{\partial x}dx$$

$$C'S = \frac{CR\cos\alpha + C'R\sin\alpha}{\cos\psi} = CR\cos\alpha + C'R\sin\alpha$$

$$\varepsilon_{x'} = \frac{C'S}{ds} = \frac{CR\cos\alpha + C'R\sin\alpha}{ds} = \left(\frac{\partial u}{\partial x}\frac{dx}{ds} + \frac{\partial u}{\partial y}\frac{dy}{ds}\right)\cos\alpha + \left(\frac{\partial v}{\partial y}\frac{dy}{ds} + \frac{\partial v}{\partial x}\frac{dx}{ds}\right)\sin\alpha$$

$$= \varepsilon_x\cos^2\alpha + \varepsilon_y\sin^2\alpha + \gamma_{xy}\cos\alpha\sin\alpha = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\alpha + \frac{\gamma_{xy}}{2}\sin 2\alpha$$

$$= \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\alpha + \varepsilon_{xy}\sin 2\alpha$$

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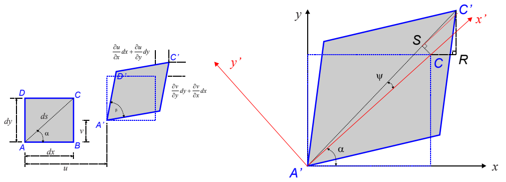
2D strain transformation



$$\varepsilon_{x'} = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\alpha + \frac{\gamma_{xy}}{2}\sin 2\alpha$$

$$= \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\alpha + \varepsilon_{xy}\sin 2\alpha$$

Substituting $(\alpha+\pi/2)$ for α

$$\varepsilon_{y'} = \frac{\varepsilon_x + \varepsilon_y}{2} - \frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\alpha - \frac{\gamma_{xy}}{2}\sin 2\alpha$$

$$= \frac{\varepsilon_x + \varepsilon_y}{2} - \frac{\varepsilon_x - \varepsilon_y}{2}\cos 2\alpha - \varepsilon_{xy}\sin 2\alpha$$

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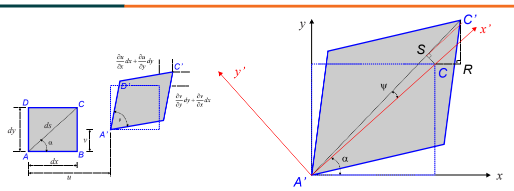
2D strain transformation



Angular displacement for ψ:

$$\tan\psi = \psi = \frac{SC}{ds} = \frac{C'R\cos\alpha - CR\sin\alpha - C'S\tan\psi}{ds}$$

$$= -\left(\varepsilon_x - \varepsilon_y\right)\sin\alpha\cos\alpha + \frac{\partial v}{\partial x}\cos^2\alpha - \frac{\partial u}{\partial y}\sin^2\alpha$$

Angular displacement for $\psi|_{\alpha+\pi/2}$ (substituting $(\alpha+\pi/2)$ for α):

$$\psi\big|_{\alpha+\pi/2} = \left(\varepsilon_x - \varepsilon_y\right)\sin\alpha\cos\alpha + \frac{\partial v}{\partial x}\sin^2\alpha - \frac{\partial u}{\partial y}\cos^2\alpha$$

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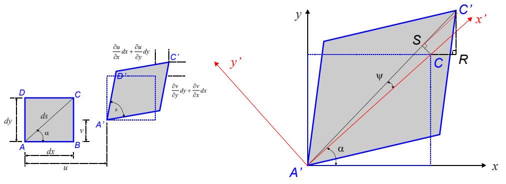
2D strain transformation



Shear strain $\gamma_{x'y'}$:

$$\gamma_{x'y'} = \psi - \psi\big|_{\alpha+\pi/2} = -2\left(\varepsilon_x - \varepsilon_y\right)\sin\alpha\cos\alpha + \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\left(\cos^2\alpha - \sin^2\alpha\right)$$

$$= -\left(\varepsilon_x - \varepsilon_y\right)\sin 2\alpha + \gamma_{xy}\cos 2\alpha$$

$$\frac{1}{2}\gamma_{x'y'} = -\frac{\varepsilon_x - \varepsilon_y}{2}\sin 2\alpha + \frac{1}{2}\gamma_{xy}\cos 2\alpha$$

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2D strain transformation

Strain transformation:

$$\begin{cases} \varepsilon_{x'} = \dfrac{\varepsilon_x+\varepsilon_y}{2} + \dfrac{\varepsilon_x-\varepsilon_y}{2}\cos 2\alpha + \varepsilon_{xy}\sin 2\alpha \\ \varepsilon_{y'} = \dfrac{\varepsilon_x+\varepsilon_y}{2} - \dfrac{\varepsilon_x-\varepsilon_y}{2}\cos 2\alpha - \varepsilon_{xy}\sin 2\alpha \\ \varepsilon_{x'y'} = -\dfrac{\varepsilon_x-\varepsilon_y}{2}\sin 2\alpha + \varepsilon_{xy}\cos 2\alpha \end{cases}$$

Recall stress transformation:

$$\begin{cases} \sigma_{x'} = \dfrac{\sigma_x+\sigma_y}{2} + \dfrac{\sigma_x-\sigma_y}{2}\cos 2\alpha + \tau_{xy}\sin 2\alpha \\ \sigma_{y'} = \dfrac{\sigma_x+\sigma_y}{2} - \dfrac{\sigma_x-\sigma_y}{2}\cos 2\alpha - \tau_{xy}\sin 2\alpha \\ \tau_{x'y'} = -\dfrac{\sigma_x-\sigma_y}{2}\sin 2\alpha + \tau_{xy}\cos 2\alpha \end{cases}$$

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2D strain transformation

Directions of principal strain (directions for which γ vanishes):

$$\tan 2\alpha = \frac{2\varepsilon_{xy}}{\varepsilon_x-\varepsilon_y}$$

Magnitudes of the principal strains:

$$\left.\begin{matrix}\varepsilon_1\\ \varepsilon_2\end{matrix}\right\} = \frac{\varepsilon_x+\varepsilon_y}{2} \pm \sqrt{\left(\frac{\varepsilon_x-\varepsilon_y}{2}\right)^2 + \varepsilon_{xy}^2}$$

- *2D strain transformation can also be represented by the Mohr's circle of strain*
- *Strain transformation follows the same rule as stress*
- *For isotropic material, principal directions of stress and strain are the same*

Strain tensor:

$$\begin{bmatrix} \varepsilon_x & \varepsilon_{xy} & \varepsilon_{xz} \\ & \varepsilon_y & \varepsilon_{yz} \\ sym. & & \varepsilon_z \end{bmatrix}$$

Strain transformation:

$$\begin{bmatrix} \varepsilon'_x & \varepsilon'_{xy} & \varepsilon'_{xz} \\ & \varepsilon'_y & \varepsilon'_{yz} \\ sym. & & \varepsilon'_z \end{bmatrix} = \mathbf{R}\cdot\begin{bmatrix} \varepsilon_x & \varepsilon_{xy} & \varepsilon_{xz} \\ & \varepsilon_y & \varepsilon_{yz} \\ sym. & & \varepsilon_z \end{bmatrix}\cdot\mathbf{R}^T$$

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Strain tensor and strain invariants

Strain tensor:
$$\begin{bmatrix} \varepsilon_x & \varepsilon_{xy} & \varepsilon_{xz} \\ & \varepsilon_y & \varepsilon_{yz} \\ sym. & & \varepsilon_z \end{bmatrix}$$

Strain invariants:

$$I'_1 = \varepsilon_x+\varepsilon_y+\varepsilon_z = \varepsilon_1+\varepsilon_2+\varepsilon_3$$

$$I'_2 = \begin{vmatrix} \varepsilon_x & \varepsilon_{xy} \\ \varepsilon_{yx} & \varepsilon_y \end{vmatrix} + \begin{vmatrix} \varepsilon_y & \varepsilon_{yz} \\ \varepsilon_{zy} & \varepsilon_z \end{vmatrix} + \begin{vmatrix} \varepsilon_z & \varepsilon_{zx} \\ \varepsilon_{xz} & \varepsilon_x \end{vmatrix}$$
$$= \varepsilon_x\varepsilon_y + \varepsilon_y\varepsilon_z + \varepsilon_z\varepsilon_x - \varepsilon_{xy}^2 - \varepsilon_{yz}^2 - \varepsilon_{zx}^2 = \varepsilon_1\varepsilon_2 + \varepsilon_2\varepsilon_3 + \varepsilon_3\varepsilon_1$$

$$I'_3 = \begin{vmatrix} \varepsilon_x & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_y & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_z \end{vmatrix}$$
$$= \varepsilon_x\varepsilon_y\varepsilon_z + 2\varepsilon_{xy}\varepsilon_{yz}\varepsilon_{zx} - \varepsilon_x\varepsilon_{yz}^2 - \varepsilon_y\varepsilon_{zx}^2 - \varepsilon_z\varepsilon_{xy}^2 = \varepsilon_1\varepsilon_2\varepsilon_3$$

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Displacement functions

- Given displacement functions *u*, *v* and *w*, we can get the complete strain components
- If strain components are known, we cannot obtain the complete displacement functions
- By integrating strain-displacement relations to obtain displacements, we need to introduce constants, which are equivalent to rigid body translations and rotations

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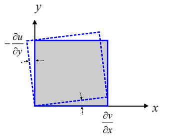
Displacement functions

Rigid body angular displacement:

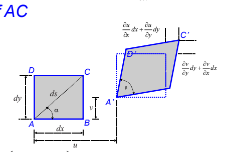
$$\omega_{z0} = \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$



Rigid body displacement + deformation:

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$ *Angular displacement of AC*

Displacement of point *C*:



$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$$

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = \frac{\partial u}{\partial x}dx + \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)dy + \frac{1}{2}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right)dy$$

$$= \varepsilon_x dx + \frac{1}{2}\gamma_{xy}dy - \omega_z dy$$

$$dv = \varepsilon_y dy + \frac{1}{2}\gamma_{xy}dx + \omega_z dx$$

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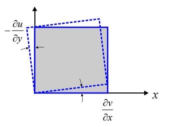
Displacement functions

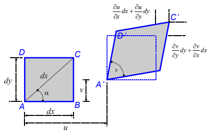
$$du = \varepsilon_x dx + \frac{1}{2}\gamma_{xy}dy - \omega_z dy$$

$$dv = \varepsilon_y dy + \frac{1}{2}\gamma_{xy}dx + \omega_z dx$$



Arbitrary parts in displacement function:

$$u^* = u_0 - \omega_{z0}y$$
$$v^* = v_0 + \omega_{z0}x$$



- *The arbitrary parts produce no strain*
- *They are formulas for the displacement of a rigid body by a transition (u_0,v_0) and a small rotation angle ω_{z0}*
- *If (u_0,v_0) and ω_{z0} are known, the displacement functions are unique*

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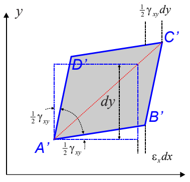
Pure deformation



$$du = \varepsilon_x dx + \frac{1}{2}\gamma_{xy} dy - \omega_z dy \qquad du = \varepsilon_x dx + \frac{1}{2}\gamma_{xy} dy$$

$$dv = \varepsilon_y dy + \frac{1}{2}\gamma_{xy} dx + \omega_z dx \qquad dv = \varepsilon_y dy + \frac{1}{2}\gamma_{xy} dx$$

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = 0$$

Angular displacement of AC is zero

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Principle of superposition

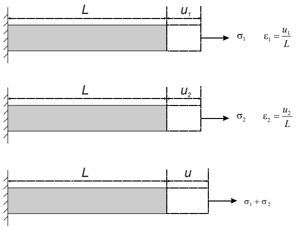
$\sigma_1 + \sigma_2$:

$$\begin{aligned} u &= u_1 + \varepsilon_2 (L + u_1) \\ &= u_1 + \varepsilon_2 L + \varepsilon_2 \varepsilon_1 L \\ &= u_1 + u_2 + \varepsilon_2 \varepsilon_1 L \end{aligned}$$

$\sigma_2 + \sigma_1$:

$$\begin{aligned} u &= u_2 + \varepsilon_1 (L + u_2) \\ &= u_2 + \varepsilon_1 L + \varepsilon_1 \varepsilon_2 L \\ &= u_2 + u_1 + \varepsilon_1 \varepsilon_2 L \end{aligned}$$

$$u = u_1 + u_2$$



- *$\varepsilon_1\varepsilon_2$ is too small compared to either ε_1 or ε_2*
- *Two strain fields can be combined by direct superposition, and the order of the two has no effect to final configuration*

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