NAME: 董骏博

SID: 12432995

1. $\lambda E-A_1=\begin{bmatrix}\lambda-3 & -2 & 5\\ -2 & \lambda-6 & 10\\ -1 & -2 & \lambda+3\end{bmatrix}$ $\quad$ $\lambda E-A_2=\begin{bmatrix}\lambda-6 & -20 & 34\\ -6 & \lambda-32 & 51\\ -4 & -20 & \lambda+32\end{bmatrix}$

$\xrightarrow{r_3\leftrightarrow r_1}\begin{bmatrix}-1 & -2 & \lambda+3\\ -2 & \lambda-6 & 10\\ \lambda-3 & -2 & 5\end{bmatrix}$ $\quad$ $\xrightarrow{r_3\leftrightarrow r_1}\begin{bmatrix}-4 & -20 & \lambda+32\\ -6 & \lambda-32 & 51\\ \lambda-6 & -20 & 34\end{bmatrix}$

$\xrightarrow{r_3+(\lambda-3)r_1,\ r_2-2r_1}\begin{bmatrix}-1 & -2 & \lambda+3\\ 0 & \lambda-2 & -2\lambda+4\\ 0 & -2\lambda+4 & \lambda^2-4\end{bmatrix}$ $\quad$ $\xrightarrow[r_3+\frac{1}{4}(\lambda-6)]{r_2-\frac{3}{2}r_1}\begin{bmatrix}-4 & -20 & \lambda+32\\ 0 & \lambda-2 & -\frac{3}{2}\lambda+3\\ 0 & -5(\lambda-2) & \frac{1}{4}(\lambda-6)(\lambda+32)+34\end{bmatrix}$

$\xrightarrow[c_3+2c_2]{c_2-2c_1,\ c_3+(\lambda+3)c_1}\begin{bmatrix}-1 & 0 & 0\\ 0 & \lambda-2 & 0\\ 0 & -2\lambda+4 & (\lambda-2)^2\end{bmatrix}$ $\quad$ $\xrightarrow[c_1\times-\frac{1}{4}]{c_2-5c_1,\ c_3+\frac{1}{4}(\lambda+32)}\begin{bmatrix}1 & 0 & 0\\ 0 & \lambda-2 & -\frac{3}{2}\lambda+3\\ 0 & -5(\lambda-2) & \frac{1}{4}(\lambda-6)(\lambda+32)+34\end{bmatrix}$

$\xrightarrow{r_1\times-1,\ c_3+2c_2}\begin{bmatrix}1 & 0 & 0\\ 0 & \lambda-2 & 0\\ 0 & 0 & (\lambda-2)^2\end{bmatrix}$ $\quad$ $\xrightarrow{r_3+5r_2,\ c_3+\frac{3}{2}c_2}\begin{bmatrix}1 & 0 & 0\\ 0 & \lambda-2 & 0\\ 0 & 0 & (\lambda-2)^2\end{bmatrix}$

$A_1, A_2$的不变因子为 $d_1(\lambda):1$

Smith标准型一样, $d_2(\lambda):\lambda-2$

不变因子与行列式因子也一致，为 $d_3(\lambda):(\lambda-2)^2$

行列式因子:

$D_1(\lambda)=1$

$D_2(\lambda)=\lambda-2$

$D_3(\lambda)=(\lambda-2)^3$

2. 由题目已知得

$$d_4(\lambda) = \lambda^2(\lambda-1)(\lambda+1)^3$$

$$d_3(\lambda) = \lambda(\lambda-1)(\lambda+1)$$

$$d_2(\lambda) = \lambda$$

$$d_1(\lambda) = 1$$

$$\therefore A(\lambda) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 & 0 \\ 0 & 0 & \lambda(\lambda-1)(\lambda+1) & 0 & 0 \\ 0 & 0 & 0 & \lambda^2(\lambda-1)(\lambda+1)^3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

3. 将a矩阵化为Jordan标准型得

$$J = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

又$\because AP = PJ$

$$A(x_1, x_2, x_3) = (x_1, x_2, x_3)J$$

$$\therefore (E-A)x_1 = 0$$

$$(E-A)x_2 = 0$$

$$(E-A)x_3 = -x_2$$

$$x_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix},\ x_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},\ x_3 = \cancel{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\therefore P = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

代入$AP = PJ$验证.

$$\begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \\ -1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \\ -1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\therefore P = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

b. Smith标准型为

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \lambda-2 & 0 \\ 0 & 0 & 0 & (\lambda-2)^3 \end{pmatrix}.$$

初等因子组为 $\cancel{\lambda-2,\ (\lambda-2)^3}$ $(\lambda-2)^{\cancel{4}3}$, $(\lambda-2)$

验证 $P^{-1}AP = J$ 成立

$\therefore$ Jordan标准型为:

$$\begin{pmatrix} 2 & \cancel{0}0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

故P为

$A(x_1, x_2, x_3, x_4) = (x_1, x_2, x_3, x_4)J$

$$\therefore P = (x_1, x_2, x_3, x_4) = \begin{bmatrix} 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

4. Smith标准型为:

a.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \lambda-1 & 0 \\ 0 & 0 & (\lambda-1)(\lambda-2) \end{bmatrix}$$

初等因子为 $(\lambda-1)$, $(\lambda-2)$, $(\lambda-1)$

$\therefore$ Jordan标准型为:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

b. Smith标准型为

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & (\lambda-1)^4 \end{bmatrix}$$ ，初等因子组为 $(\lambda-1)^4$.

∴ Jordan 标准型为：

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

5. 初等因子组可为：

① $(4-\lambda)^3, (\lambda+1)^2$

② $(4-\lambda), (4-\lambda)^2, (\lambda+1)^2$

③ $(4-\lambda)^3, (-\lambda-1), (-\lambda-1)$

④ $(4-\lambda), (4-\lambda)^2, (-\lambda-1), (-\lambda-1)$

⑤ $(4-\lambda), (4-\lambda), (4-\lambda), (-\lambda-1), (-\lambda-1)$

⑥ $(4-\lambda), (4-\lambda), (4-\lambda), (-\lambda-1)^2$

∴ Jordan 标准型为：

① $\begin{bmatrix} 4 & 1 & & & \\ & 4 & 1 & & \\ & & 4 & & \\ & & & -1 & 1 \\ & & & & -1 \end{bmatrix}$ ② $\begin{bmatrix} 4 & & & & \\ & 4 & 1 & & \\ & & 4 & & \\ & & & -1 & 1 \\ & & & & -1 \end{bmatrix}$ ③ $\begin{bmatrix} -1 & & & & \\ & -1 & & & \\ & & 4 & 1 & \\ & & & 4 & 1 \\ & & & & 4 \end{bmatrix}$

④ $\begin{bmatrix} 4 & & & & \\ & 4 & 1 & & \\ & & 4 & & \\ & & & -1 & \\ & & & & -1 \end{bmatrix}$ ⑤ $\begin{bmatrix} 4 & & & & \\ & 4 & & & \\ & & 4 & & \\ & & & -1 & \\ & & & & -1 \end{bmatrix}$ ⑥ $\begin{bmatrix} 4 & & & & \\ & 4 & & & \\ & & 4 & & \\ & & & -1 & 1 \\ & & & & -1 \end{bmatrix}$

6. $\dfrac{dx}{dt}=\underbrace{\begin{bmatrix}-1&1&0\\-4&3&0\\-8&8&-1\end{bmatrix}}_{A}\underbrace{\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}}_{x}$

$\therefore A=\begin{bmatrix}-1&1&0\\-4&3&0\\-8&8&-1\end{bmatrix}$ ，A的Smith标准型为: $\begin{bmatrix}1&0&0\\0&1&0\\0&0&(\lambda+1)(\lambda-1)^2\end{bmatrix}$

~~A的Jordan~~ A的初等因子组为 $(\lambda+1),(\lambda-1)^2$

∴A的Jordan标准型为:

$$\begin{bmatrix}-1&0&0\\0&1&1\\0&0&1\end{bmatrix}$$

$$P=\begin{bmatrix}0&1&-1\\0&2&-1\\1&4&-2\end{bmatrix}$$

令 $x=PY$

$\dfrac{dY}{dt}=JY \Rightarrow \dfrac{dy_1}{dt}=y_1,\ \dfrac{dy_2}{dt}=y_2+y_3,\ \dfrac{dy_3}{dt}=y_3$

$\therefore\ y_1=k_1e^{-t},\ y_2=(k_3t+k_2)e^{t},\ y_3=k_3e^{t}$

代入 $x=PY$ 得

$$x_1=y_2-y_3=(k_3t+k_2-k_3)e^{t}$$

$$x_2=2y_2-y_3=(2k_3t+2k_2-k_3)e^{t}$$

$$x_3=y_1+4y_2-2y_3=k_1e^{t}+(4k_3t+4k_2-2k_3)e^{t}$$

将 $t=0,\ x=\begin{bmatrix}1\\2\\5\end{bmatrix}$ 代入得

$k_1=1\quad k_2=1\quad k_3=0$

$\therefore x_1=e^{t}$

$x_2=2e^{t}$

$x_3=e^{-t}+4e^{t}$