南方科技大学
SOUTHERN UNIVERSITY OF SCIENCE AND TECHNOLOGY

MAE5009
Continuum Mechanics B
Session 07: Tensor Notation

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Notation

- Tensor notation, indicial notation:
  - If one of the equations is written, the other expressions may be derived by cyclic permutation
  - We could use the tensor notation to condense the governing equations
- Symbolic notation:
  - Vector-dyadic notations, vector notation

Equilibrium equations

$$\begin{cases} \dfrac{\partial \sigma_x}{\partial x} + \dfrac{\partial \tau_{yx}}{\partial y} + \dfrac{\partial \tau_{zx}}{\partial z} + f_x = 0 \\ \dfrac{\partial \tau_{xy}}{\partial x} + \dfrac{\partial \sigma_y}{\partial y} + \dfrac{\partial \tau_{zy}}{\partial z} + f_y = 0 \\ \dfrac{\partial \tau_{xz}}{\partial x} + \dfrac{\partial \tau_{yz}}{\partial y} + \dfrac{\partial \sigma_z}{\partial z} + f_z = 0 \end{cases}$$

Compatibility equations

$$\begin{cases} \dfrac{\partial^2 \varepsilon_x}{\partial y^2} + \dfrac{\partial^2 \varepsilon_y}{\partial x^2} = \dfrac{\partial^2 \gamma_{xy}}{\partial x \partial y} \\ \dfrac{\partial^2 \varepsilon_y}{\partial z^2} + \dfrac{\partial^2 \varepsilon_z}{\partial y^2} = \dfrac{\partial^2 \gamma_{yz}}{\partial y \partial z} \\ \dfrac{\partial^2 \varepsilon_z}{\partial x^2} + \dfrac{\partial^2 \varepsilon_x}{\partial z^2} = \dfrac{\partial^2 \gamma_{zx}}{\partial z \partial x} \end{cases}$$

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Indicial notation and vector transformations

- Using x_1, x_2 and x_3 to represent the three Cartesian coordinates:

$$\begin{cases} x \\ y \\ z \end{cases} \rightarrow \begin{cases} x_1 \\ x_2 \\ x_3 \end{cases}$$

- For a vector transformed to another coordinate system x'_1, x'_2 and x'_3 :

$$\begin{bmatrix} x'_1 \\ x'_2 \\ x'_3 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \rightarrow \begin{bmatrix} x'_1 \\ x'_2 \\ x'_3 \end{bmatrix} = \mathbf{R} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\downarrow$$

$$\begin{cases} x'_1 = x_1 a_{11} + x_2 a_{12} + x_3 a_{13} \\ x'_2 = x_1 a_{21} + x_2 a_{22} + x_3 a_{23} \\ x'_3 = x_1 a_{31} + x_2 a_{32} + x_3 a_{33} \end{cases}$$

Direction cosines
(Transformational matrix **R**)

	x_1	x_2	x_3
x'_1	a_{11}	a_{12}	a_{13}
x'_2	a_{21}	a_{22}	a_{23}
x'_3	a_{31}	a_{32}	a_{33}

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Indicial notation and vector transformations

- The three equations can be condensed to:

$$\begin{cases} x_1' = x_1 a_{11} + x_2 a_{12} + x_3 a_{13} \\ x_2' = x_1 a_{21} + x_2 a_{22} + x_3 a_{23} \\ x_3' = x_1 a_{31} + x_2 a_{32} + x_3 a_{33} \end{cases} \rightarrow \begin{cases} x_1' = \sum_{i=1}^{3} x_i a_{1i} \\ x_2' = \sum_{i=1}^{3} x_i a_{2i} \\ x_3' = \sum_{i=1}^{3} x_i a_{3i} \end{cases} \rightarrow x_j' = \sum_{i=1}^{3} x_i a_{ji},\ j = 1,2,3$$

题目一：要会转换

Tensor notation

$$x_j' = x_i a_{ji} \qquad i, j = 1,2,3$$

Cartesian tensor notation:

- *If a repeated alphabetic subscript appears in one monomial, automatic summation over the range of this subscript is required*

① $b_{i1}b_{1k} + b_{i2}b_{2k} + b_{i3}b_{3k}$ 给定 j, i.k $3\times3=9$

$3^2=9$

☆ 必考

4 $b_{ij}b_{jk}$

dummy index: $i=1,2,3$, 加起来

$a_{ii} = a_{11}+a_{22}+a_{33}$

free index: $V_i = V_1, V_2, V_3$ 列出来

V_{ij} : 3×3

V_{ijklmn} : 3^6

Cartesian tensor notation

- A repeated subscript is called **dummy** index
- Non-repeated subscript is called **free** index
- Both dummy and free index can be arbitrary
- A subscript may appear no more than twice in each monomial
- If a subscript appears only once in a monomial, it must appear just once in each other monomial

$$x_j' = x_i a_{ji} \qquad i, j = 1,2,3$$

i is dummy index 重复的 (哑标)

j is free index 不重复的

$$x_k' = x_i a_{ki} \qquad i, k = 1,2,3$$

$$x_j' = x_l a_{jl} \qquad l, j = 1,2,3$$

$$b_{ij}b_{jk} \quad i,j,k = 1,2,3 \rightarrow \begin{array}{lll} i=1, k=1: b_{11}b_{11} + b_{12}b_{21} + b_{13}b_{31} & i=2, k=1 & i=3, k=1 \\ k=2: b_{11}b_{12} + b_{12}b_{22} + b_{13}b_{32} & k=2 & k=2 \\ k=3: b_{11}b_{13} + b_{12}b_{23} + b_{13}b_{33} & k=3 & k=3 \end{array}$$

$$\boxed{b_{i1}b_{1k} + b_{i2}b_{2k} + b_{i3}b_{3k}}$$

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Cartesian tensor notation

- For a vector transformed back from x'_1, x'_2 and x'_3 to coordinate system x_1, x_2 and x_3:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} \begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} \rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{R}^T \begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix}$$

$$\downarrow$$

$$\begin{cases} x_1 = x_1' a_{11} + x_2' a_{21} + x_3' a_{31} \\ x_2 = x_1' a_{12} + x_2' a_{22} + x_3' a_{32} \\ x_3 = x_1' a_{13} + x_2' a_{23} + x_3' a_{33} \end{cases}$$

$$\downarrow$$

$$x_j = x_i' a_{ij} \qquad i, j = 1,2,3$$

Direction cosines (Transformational matrix $\mathbf{R}^T$)

	x'_1	x'_2	x'_3
x_1	a_{11}	a_{21}	a_{31}
x_2	a_{12}	a_{22}	a_{32}
x_3	a_{13}	a_{23}	a_{33}

Direction cosines (Transformational matrix $\mathbf{R}$)

	x_1	x_2	x_3
x'_1	a_{11}	a_{12}	a_{13}
x'_2	a_{21}	a_{22}	a_{23}
x'_3	a_{31}	a_{32}	a_{33}

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Cartesian tensor notation

- For a vector transformed back from x'_1, x'_2 and x'_3 to coordinate system x_1, x_2 and x_3:

$$x_j = x'_i a_{ij} \qquad i,j = 1,2,3$$

$$\Big\downarrow \; i \rightarrow k$$

$$x_j = x'_i a_{ij} = x_i a_{ki} a_{kj} \qquad i,j,k = 1,2,3$$

$$\Big\downarrow$$

$$a_{ki} a_{kj} = \mathbf{I} \qquad i,j,k = 1,2,3$$

$$\Big\downarrow$$

$$a_{ki} a_{kj} = 1 \quad \text{if } i = j$$
$$a_{ki} a_{kj} = 0 \quad \text{if } i \neq j$$

$$\Big\downarrow$$

$$a_{11}^2 + a_{21}^2 + a_{31}^3 = 1 \qquad a_{11}a_{12} + a_{21}a_{22} + a_{21}a_{32} = 0$$
$$a_{12}^2 + a_{22}^2 + a_{32}^3 = 1 \qquad a_{11}a_{13} + a_{21}a_{23} + a_{31}a_{33} = 0$$
$$a_{13}^2 + a_{23}^2 + a_{33}^3 = 1 \qquad a_{12}a_{13} + a_{22}a_{23} + a_{32}a_{33} = 0$$

Direction cosines
(Transformational matrix **R**)

	x_1	x_2	x_3
x'_1	a_{11}	a_{12}	a_{13}
x'_2	a_{21}	a_{22}	a_{23}
x'_3	a_{31}	a_{32}	a_{33}

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \mathbf{R}^T \mathbf{R} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\Big\downarrow$$

$$\mathbf{R}^T \mathbf{R} = \mathbf{I}$$

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Definition of vector

- A set of three quantities F_i referred to a coordinate system x_i and transformed to another coordinate system x'_i by the following equation is defined as a vector:

$$F'_j = F_i a_{ji} \qquad i,j = 1,2,3$$

Direction cosines
(Transformational matrix **R**)

	x_1	x_2	x_3
x'_1	a_{11}	a_{12}	a_{13}
x'_2	a_{21}	a_{22}	a_{23}
x'_3	a_{31}	a_{32}	a_{33}

Not any set of three scalar quantities attached to the x_1, x_2, x_3 system can be called a vector

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Higher order tensors

- Consider any two vectors u_i and v_k, in another coordinate system their coordinates are:

$$u'_j = u_i a_{ji}$$
$$v'_l = v_k a_{lk}$$

$$\Big\downarrow$$

$$u'_j v'_l = a_{ji} a_{lk} u_i v_k \qquad A'_{jl} = a_{ji} a_{lk} A_{ik}$$

Any group of nine scalar quantities A_{ik} referred to a coordinate system x_i and which transforms to a group of nine quantities referred to another coordinate system x'_i by the above rule is called a tensor of second order

Third order tensor: $w'_{ijk} = a_{ip} a_{jq} a_{kr} w_{pqr}$

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Gradient of scalar and vector field

- The gradient of a scalar field is a vector field:

$$U = U(x_1, x_2, x_3)$$

- and its gradient in two coordinate systems x_i and x'_i:

$$\frac{\partial U}{\partial x_i}, \frac{\partial U}{\partial x'_j} \rightarrow \frac{\partial U}{\partial x'_j} = \frac{\partial U}{\partial x_i}\frac{\partial x_i}{\partial x'_j} = a_{ij}\frac{\partial U}{\partial x_i}$$

- The gradient of a vector is a second order tensor:

$$u_i, u'_j \rightarrow \frac{\partial u_i}{\partial x_k}, \frac{\partial u'_j}{\partial x'_l}$$

$$\downarrow$$

$$\frac{\partial u'_j}{\partial x'_l} = \frac{\partial u'_j}{\partial x_k}\frac{\partial x_k}{\partial x'_l} = a_{kl}\frac{\partial u'_j}{\partial x_k} = a_{kl}\frac{\partial (u_i a_{ji})}{\partial x_k} = a_{ji}a_{kl}\frac{\partial u_i}{\partial x_k}$$

Since: $\frac{\partial a_{ji}}{\partial x_k} = \frac{\partial}{\partial x_k}\left(\frac{\partial x_j}{\partial x'_i}\right) = 0$ a_{ji} is constant

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The Kronecker delta

- The gradient of the position vector x_i is a second order tensor:

$$\frac{\partial x_i}{\partial x_k} \rightarrow \frac{\partial x_1}{\partial x_1} = 1 \quad \frac{\partial x_1}{\partial x_2} = 0 \quad \frac{\partial x_1}{\partial x_3} = 0$$

$$\downarrow$$

$$\frac{\partial x_i}{\partial x_k} = 1 \quad \text{if } i = k$$
$$\frac{\partial x_i}{\partial x_k} = 0 \quad \text{if } i \neq k$$

$$\rightarrow \quad \delta_{ik} = 1 \quad \text{if } i = k, \qquad \delta_{ik} = 0 \quad \text{if } i \neq k$$

Kronecker delta

$$\downarrow$$

$$\delta_{ik} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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The Kronecker delta

- The Kronecker delta has identical components in any coordinate systems. Applying the transformation:

$$\delta'_{jl} = a_{ij}a_{kl}\delta_{ik} = a_{ij}a_{il}$$

$$\downarrow$$

$$a_{ij}a_{il} = 1 \quad \text{if } j = l, \qquad a_{ij}a_{il} = 0 \quad \text{if } j \neq l$$

$$\rightarrow \quad \delta'_{jl} = 1 \quad \text{if } j = l, \qquad \delta'_{jl} = 0 \quad \text{if } j \neq l$$

$$\downarrow$$

$$\delta'_{jl} = \delta_{jl}$$

Tensors with identical components in any coordinate system are called isotropic tensors

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The Kronecker delta

- The Kronecker delta has identical components in any coordinate systems. Applying the transformation:

$$\delta_{ik}u_k = \delta_{i1}u_1 + \delta_{i2}u_2 + \delta_{i3}u_3$$

① $\delta_{11}U_1 + \delta_{12}U_2 + \delta_{13}U_3$

$= U_1 + 0 + 0$

$$i=1:\ \delta_{ik}u_k = u_1$$
$$i=2:\ \delta_{ik}u_k = u_2 \quad \rightarrow \quad \delta_{ik}u_k = u_i = U_1$$
$$i=3:\ \delta_{ik}u_k = u_3$$

Kronecker delta is sometimes called the ***substitution tensor***

$$\frac{\partial X_i}{\partial X_k} = \delta_{ik}\underline{\frac{\partial u_j}{\partial x_k} = \frac{\partial u_j}{\partial x_i}} = u_{j,i} \qquad \delta_{il}\frac{\partial^2 u_j}{\partial x_l \partial x_k} = \delta_{il}u_{j,lk} = \frac{\partial^2 u_j}{\partial x_i \partial x_k} = u_{j,ik}$$

$$\delta_{ik}w_{ik} = w_{ii} = w_{kk}$$

where:

$$u_{,i} = \frac{\partial u}{\partial x_i} \quad \text{vector} \qquad u_{,ik} = \frac{\partial^2 u}{\partial x_i \partial x_k}$$

$$\frac{\partial U_i}{\partial X_k} = U_{j,k}.$$

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The alternating tensor

- Another isotropic tensor, ε_{ikm}

$$\varepsilon_{ikm} = \begin{cases} 0 & \text{If any two of the subscripts } i, k, m \text{ are equal} \\ +1 & \text{If the subscripts } i, k, m \text{ are unequal but in cyclic order of 1 2 3} \\ -1 & \text{If the subscripts } i, k, m \text{ are unequal and in noncyclic order of 1 2 3} \end{cases}$$

- The ε_{ikm} is a third order tensor:

$$\begin{aligned} \varepsilon'_{jln} &= a_{ij}a_{kl}a_{mn}\varepsilon_{ikm} \\ &= a_{1j}a_{2l}a_{3n} + a_{2j}a_{3l}a_{1n} + a_{3j}a_{1l}a_{2n} \\ &\quad - a_{1j}a_{3l}a_{2n} - a_{2j}a_{1l}a_{3n} - a_{3j}a_{2l}a_{1n} \\ &= \begin{vmatrix} a_{1j} & a_{1l} & a_{1n} \\ a_{2j} & a_{2l} & a_{2n} \\ a_{3j} & a_{3l} & a_{3n} \end{vmatrix} \end{aligned}$$

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State of stress at a point

- Stress can be described as a tensor:

$$\tau_{ik} = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix} = \begin{bmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{bmatrix}$$

$$\tau'_{jn} = a_{ij}a_{kn}\tau_{ik}$$

Stress transformation:

$$\begin{bmatrix} \sigma'_x & \tau'_{xy} & \tau'_{xz} \\ & \sigma'_y & \tau'_{yz} \\ sym. & & \sigma'_z \end{bmatrix} = \mathbf{R} \cdot \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ & \sigma_y & \tau_{yz} \\ sym. & & \sigma_z \end{bmatrix} \cdot \mathbf{R}^T$$

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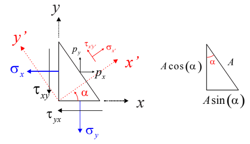
Utilization of tensor notation

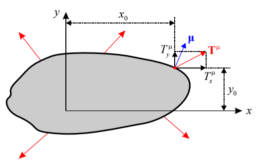
- Stress transformation on an inclined surface:

$$\begin{cases} T_x^{\mu} = \sigma_{x|0}\mu_x + \tau_{yx|0}\mu_y + \tau_{zx|0}\mu_z \\ T_y^{\mu} = \tau_{xy|0}\mu_x + \sigma_{y|0}\mu_y + \tau_{zy|0}\mu_z \\ T_z^{\mu} = \tau_{xz|0}\mu_x + \tau_{yz|0}\mu_y + \sigma_{z|0}\mu_z \end{cases}$$

$$\Downarrow$$

$$T_i^{\mu} = \tau_{ji}\mu_j$$





$$\frac{\partial \tau_{11}}{\partial x_1} + \frac{\partial \tau_{21}}{\partial x_2} + \frac{\partial \tau_{31}}{\partial x_3} + f_x = 0$$

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$$\frac{\partial \tau_{i1}}{\partial x_i} + f_x = 0$$

$$\Downarrow$$

$$\tau_{ij,i} + f_x = 0$$

Principal axes of stress tensor

$$\begin{vmatrix} \tau_{11}-\tau & \tau_{21} & \tau_{31} \\ \tau_{12} & \tau_{22}-\tau & \tau_{32} \\ \tau_{13} & \tau_{23} & \tau_{33}-\tau \end{vmatrix} = 0 \qquad \tau^3 - I_1\tau^2 + I_2\tau - I_3 = 0$$

$$\Downarrow$$

$$I_1 = \tau_{11} + \tau_{22} + \tau_{33} = \tau_{ii}$$

$$I_2 = \tau_{11}\tau_{22} + \tau_{22}\tau_{33} + \tau_{33}\tau_{11} - \tau_{12}^2 - \tau_{23}^2 - \tau_{31}^2 = \frac{1}{2}\left(\tau_{ii}\tau_{kk} - \tau_{ik}\tau_{ki}\right)$$

$$I_3 = \tau_{11}\tau_{22}\tau_{33} + 2\tau_{12}\tau_{23}\tau_{31} - \tau_{11}\tau_{23}^2 - \tau_{22}\tau_{31}^2 - \tau_{33}\tau_{12}^2 = \frac{1}{6}\left(2\tau_{ij}\tau_{jk}\tau_{ki} - 3\tau_{ij}\tau_{ji}\tau_{kk} + \tau_{ii}\tau_{jj}\tau_{kk}\right)$$

$$= \frac{1}{6}\left(\varepsilon_{ijk}\varepsilon_{pqr}\tau_{ip}\tau_{jq}\tau_{kr}\right)$$

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Equations of equilibrium

??

$$\begin{cases} \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + f_x = 0 \\ \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + f_y = 0 \\ \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + f_z = 0 \end{cases}$$

$$\Downarrow$$

$$\frac{\partial \tau_{ik}}{\partial x_i} + f_k = 0 \qquad \tau_{ik,i} + f_k = 0$$

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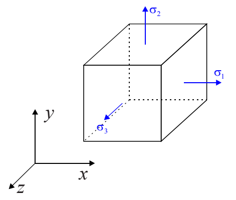
The stress ellipsoid

$$p_x = \sigma_1 a_{11}$$
$$p_y = \sigma_2 a_{12}$$
$$p_z = \sigma_3 a_{13}$$

$$\downarrow$$

$$\left(\frac{p_x}{\sigma_1}\right)^2 + \left(\frac{p_y}{\sigma_2}\right)^2 + \left(\frac{p_z}{\sigma_3}\right)^2 = 1$$



p_x, p_y and p_z are the components of the stress vector on the x'_1 plane, and the intercepts on the coordinate axes are $\pm\sigma_1$, $\pm\sigma_2$ and $\pm\sigma_2$

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Displacement and strain

- Displacement: $u_i = u_i^0 + \frac{\partial u_i}{\partial x_j} dx_j$ Taylor series

$$= u_i^0 + \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) dx_j + \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) dx_j$$
$$= u_i^0 + \varepsilon_{ij} dx_j + \omega_{ij} dx_j$$

✓ Strain component:

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) \rightarrow$$

$$\varepsilon_{11} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_1}{\partial x_1}\right) = \varepsilon_x$$
$$\varepsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) = \frac{1}{2}\gamma_{xy}$$
$$\varepsilon_{13} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right) = \frac{1}{2}\gamma_{xz} \cdots$$

了解 Rotation component:

$$\omega_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) = \frac{1}{2}\left(u_{i,j} - u_{j,i}\right) \rightarrow$$

$$\omega_{12} = -\omega_{21} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1}\right)$$
$$\omega_{13} = -\omega_{31} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right)$$
$$\omega_{23} = -\omega_{32} = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} - \frac{\partial u_3}{\partial x_2}\right) \cdots$$

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Generalized Hooke's law

$$\begin{bmatrix} \tau_{11} \\ \tau_{22} \\ \tau_{33} \\ \tau_{12} \\ \tau_{23} \\ \tau_{31} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} \\ c_{21} & c_{22} & c_{23} & c_{24} & c_{25} & c_{26} \\ c_{31} & c_{32} & c_{33} & c_{34} & c_{35} & c_{36} \\ c_{41} & c_{42} & c_{43} & c_{44} & c_{45} & c_{46} \\ c_{51} & c_{52} & c_{53} & c_{54} & c_{55} & c_{56} \\ c_{61} & c_{62} & c_{63} & c_{64} & c_{65} & c_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{bmatrix}$$

$$C_{ijkl} = a_{im} a_{jp} a_{kq} a_{lr} C'_{mpqr}$$

C_{ijkl} is a fourth order tensor

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$\varepsilon_x = \frac{1}{E}\left[\sigma_x - \nu(\sigma_y + \sigma_z)\right]$ $\gamma_{xy} = \frac{1}{G}\tau_{xy} = \frac{1}{2G}\tau_{xy} = \varepsilon_{xy} = \frac{1+\nu}{E}\tau_{xy}$

$\varepsilon_y = \frac{1}{E}\left[\sigma_y - \nu(\sigma_x + \sigma_z)\right]$ $\gamma_{yz} = \frac{1}{G}\tau_{yz} = \frac{1}{2G}\tau_{yz} = \varepsilon_{yz} = \frac{1+\nu}{E}\tau_{xy}$

$\varepsilon_{xx} = \frac{1}{E}\left[\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})\right]$

$= \frac{1}{E}\left[(1+\nu)\sigma_{xx} - \nu(\sigma_{xx} + \sigma_{yy} + \sigma_{zz})\right]$

$\varepsilon_{ij} = \frac{1+\nu}{E}\tau_{ij} - \delta_{ij}\frac{\nu}{E}\Theta$ τ_{kk}

Generalized Hooke's law

$$\begin{cases} \varepsilon_x = \frac{1}{E}\left(\sigma_x - \nu\left(\sigma_y + \sigma_z\right)\right) \\ \varepsilon_y = \frac{1}{E}\left(\sigma_y - \nu\left(\sigma_z + \sigma_x\right)\right) \\ \varepsilon_z = \frac{1}{E}\left(\sigma_z - \nu\left(\sigma_x + \sigma_y\right)\right) \end{cases} \qquad \begin{cases} \gamma_{xy} = \frac{1}{G}\tau_{xy} \\ \gamma_{yz} = \frac{1}{G}\tau_{yz} \\ \gamma_{zx} = \frac{1}{G}\tau_{zx} \end{cases}$$

$$\downarrow$$

$$\begin{cases} \varepsilon_x = \frac{1}{E}\left((1+\nu)\sigma_x - \nu\left(\sigma_x + \sigma_y + \sigma_z\right)\right) \\ \varepsilon_y = \frac{1}{E}\left((1+\nu)\sigma_y - \nu\left(\sigma_x + \sigma_y + \sigma_z\right)\right) \\ \varepsilon_z = \frac{1}{E}\left((1+\nu)\sigma_z - \nu\left(\sigma_x + \sigma_y + \sigma_z\right)\right) \end{cases} \qquad \begin{cases} \varepsilon_{xy} = \frac{1+\nu}{E}\tau_{xy} \\ \varepsilon_{yz} = \frac{1+\nu}{E}\tau_{yz} \\ \varepsilon_{zx} = \frac{1+\nu}{E}\tau_{zx} \end{cases}$$

$$\downarrow$$

$$\varepsilon_{ij} = \frac{1+\nu}{E}\tau_{ij} - \frac{\nu}{E}\delta_{ij}\Theta \qquad \varepsilon_{ij} = \frac{1}{2G}\left(\tau_{ij} - \lambda\delta_{ij}\frac{\tau_{kk}}{3\lambda + 2G}\right)$$

where $\Theta = \tau_{kk}$

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Generalized Hooke's law

$$\begin{cases} \sigma_x = 2G\varepsilon_x + \lambda\left(\varepsilon_x + \varepsilon_y + \varepsilon_z\right) \\ \sigma_y = 2G\varepsilon_y + \lambda\left(\varepsilon_x + \varepsilon_y + \varepsilon_z\right) \\ \sigma_z = 2G\varepsilon_z + \lambda\left(\varepsilon_x + \varepsilon_y + \varepsilon_z\right) \end{cases} \qquad \begin{cases} \tau_{xy} = G\gamma_{xy} \\ \tau_{yz} = G\gamma_{yz} \\ \tau_{zx} = G\gamma_{zx} \end{cases}$$

$$\downarrow$$

$\tau_x = 2G\varepsilon_x + \lambda\varepsilon$

$\tau_{xy} = G\gamma_{xy} = 2G\varepsilon_{xy}$

$$\begin{cases} \tau_{xy} = 2G\varepsilon_{xy} \\ \tau_{yz} = 2G\varepsilon_{yz} \\ \tau_{zx} = 2G\varepsilon_{zx} \end{cases}$$

$$\downarrow$$

$$\tau_{ij} = 2G\varepsilon_{ij} + \lambda\varepsilon\delta_{ij}$$

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Compatibility equations

$$\boxed{\begin{aligned} \frac{\partial^2\varepsilon_x}{\partial y^2} + \frac{\partial^2\varepsilon_y}{\partial x^2} &= \frac{\partial^2\gamma_{xy}}{\partial x\partial y} \\ \frac{\partial^2\varepsilon_y}{\partial z^2} + \frac{\partial^2\varepsilon_z}{\partial y^2} &= \frac{\partial^2\gamma_{yz}}{\partial y\partial z} \\ \frac{\partial^2\varepsilon_z}{\partial x^2} + \frac{\partial^2\varepsilon_x}{\partial z^2} &= \frac{\partial^2\gamma_{zx}}{\partial z\partial x} \end{aligned}} \qquad \boxed{\begin{aligned} 2\frac{\partial^2\varepsilon_x}{\partial y\partial z} &= \frac{\partial}{\partial x}\left(-\frac{\partial\gamma_{yz}}{\partial x} + \frac{\partial\gamma_{zx}}{\partial y} + \frac{\partial\gamma_{xy}}{\partial z}\right) \\ 2\frac{\partial^2\varepsilon_y}{\partial x\partial z} &= \frac{\partial}{\partial y}\left(\frac{\partial\gamma_{yz}}{\partial x} - \frac{\partial\gamma_{zx}}{\partial y} + \frac{\partial\gamma_{xy}}{\partial z}\right) \\ 2\frac{\partial^2\varepsilon_z}{\partial x\partial y} &= \frac{\partial}{\partial z}\left(\frac{\partial\gamma_{yz}}{\partial x} + \frac{\partial\gamma_{zx}}{\partial y} - \frac{\partial\gamma_{xy}}{\partial z}\right) \end{aligned}}$$

$$\varepsilon_{il,jk} - \varepsilon_{jl,ik} + \varepsilon_{jk,il} - \varepsilon_{ik,jl} = 0$$

$$\varepsilon_{il,ij} - \varepsilon_{jl,ij} + \varepsilon_{jj,il} - \varepsilon_{ij,jl} = 0$$

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Compatibility in terms of stress

$$\begin{cases} \nabla^2\sigma_z + \dfrac{1}{1+\nu}\dfrac{\partial^2\Theta}{\partial z^2} = -\left(\dfrac{\nu}{1-\nu}\right)\left(\dfrac{\partial f_x}{\partial x}+\dfrac{\partial f_y}{\partial y}+\dfrac{\partial f_z}{\partial z}\right) - 2\dfrac{\partial f_z}{\partial z} \\ \nabla^2\sigma_x + \dfrac{1}{1+\nu}\dfrac{\partial^2\Theta}{\partial x^2} = -\left(\dfrac{\nu}{1-\nu}\right)\left(\dfrac{\partial f_x}{\partial x}+\dfrac{\partial f_y}{\partial y}+\dfrac{\partial f_z}{\partial z}\right) - 2\dfrac{\partial f_x}{\partial x} \\ \nabla^2\sigma_y + \dfrac{1}{1+\nu}\dfrac{\partial^2\Theta}{\partial y^2} = -\left(\dfrac{\nu}{1-\nu}\right)\left(\dfrac{\partial f_x}{\partial x}+\dfrac{\partial f_y}{\partial y}+\dfrac{\partial f_z}{\partial z}\right) - 2\dfrac{\partial f_y}{\partial y} \end{cases}$$

$$\begin{cases} \nabla^2\tau_{yz} + \dfrac{1}{1+\nu}\dfrac{\partial^2\Theta}{\partial y\partial z} = -\left(\dfrac{\partial f_y}{\partial z}+\dfrac{\partial f_z}{\partial y}\right) \\ \nabla^2\tau_{zx} + \dfrac{1}{1+\nu}\dfrac{\partial^2\Theta}{\partial x\partial z} = -\left(\dfrac{\partial f_x}{\partial z}+\dfrac{\partial f_z}{\partial x}\right) \\ \nabla^2\tau_{xy} + \dfrac{1}{1+\nu}\dfrac{\partial^2\Theta}{\partial x\partial y} = -\left(\dfrac{\partial f_x}{\partial y}+\dfrac{\partial f_y}{\partial x}\right) \end{cases}$$

Compatibility equations in terms of stress
Or Beltrami-Michell compatibility equations

$$\nabla^2\tau_{ik} + \frac{1}{1+\nu}\Theta_{,ik} = -\frac{\nu}{1-\nu}\delta_{ik}f_{j,j} - \left(f_{i,k}+f_{k,i}\right)$$

where

$$\Theta = \tau_{kk}$$

$$\nabla^2\tau_{ik} = \tau_{ik,jj}$$

Boundary:

$$T_i^{\mu} = \tau_{ji}\mu_j$$

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Equilibrium equations in terms of displacement

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$$\begin{cases} (\lambda+G)\dfrac{\partial\varepsilon}{\partial x} + G\nabla^2 u + f_x = 0 \\ (\lambda+G)\dfrac{\partial\varepsilon}{\partial y} + G\nabla^2 v + f_y = 0 \\ (\lambda+G)\dfrac{\partial\varepsilon}{\partial z} + G\nabla^2 w + f_z = 0 \end{cases}$$

Navier's equations

$$(\lambda+G)\varepsilon_{,i} + G\nabla^2 u_i + f_i = 0$$

where

$$\varepsilon = \varepsilon_{kk} = u_{k,k}$$

$$\nabla^2\tau_{ik} = \tau_{ik,jj}$$

Boundary:

$$u_i^{\,0} = u_i^{\,b}$$

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Strain energy density

$$U_0 = \frac{1}{2}\left(\sigma_x\varepsilon_x + \sigma_y\varepsilon_y + \sigma_z\varepsilon_z + \tau_{xy}\gamma_{xy} + \tau_{yz}\gamma_{yz} + \tau_{zx}\gamma_{zx}\right)$$

$$= \frac{1}{2}\mathrm{vec}\left(\begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix}\right)\cdot\mathrm{vec}\left(\begin{bmatrix} \varepsilon_x & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_y & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_z \end{bmatrix}\right)$$

In terms of stress components:

$$U_0 = \frac{1}{2}\left(\frac{1}{E}\left(\sigma_x^{\,2}+\sigma_y^{\,2}+\sigma_z^{\,2}\right) - \frac{2\nu}{E}\left(\sigma_x\sigma_y + \sigma_x\sigma_z + \sigma_y\sigma_z\right) + \frac{1}{G}\left(\tau_{xy}^{\,2}+\tau_{xz}^{\,2}+\tau_{yz}^{\,2}\right)\right)$$

In terms of strain components:

$$U_0 = \frac{1}{2}\left(\lambda\varepsilon^2 + 2G\left(\varepsilon_x^{\,2}+\varepsilon_y^{\,2}+\varepsilon_z^{\,2}\right) + G\left(\gamma_{xy}^{\,2}+\gamma_{yz}^{\,2}+\gamma_{zx}^{\,2}\right)\right)$$

$$\Downarrow$$

$$U_0 = \frac{1}{2}\tau_{ij}\varepsilon_{ij} \qquad U_0 = -\frac{\nu}{2E}\tau^2_{kk} + \frac{1}{4G}\tau_{ij}\tau_{ij} \qquad U_0 = \frac{\lambda}{2}\varepsilon^2_{kk} + G\varepsilon_{ij}\varepsilon_{ij}$$

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Solution of 3D problems

- Equilibrium equations (3) $\tau_{ij,j} + f_i = 0$
- Hooke's law (6) $\varepsilon_{ij} = \frac{1+\nu}{E}\tau_{ij} - \frac{\nu}{E}\delta_{ij}\Theta$ $\quad \tau_{ij} = 2G\varepsilon_{ij} + \lambda\varepsilon\delta_{ij}$
- Strain-displacement (6) $\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$
- Unknowns (15) $u_i, \tau_{ij}, \varepsilon_{ij}$

Displacement formulation

$$\tau_{ij} = G\left(u_{i,j} + u_{j,i}\right) + \lambda\delta_{ij}u_{k,k}$$
$$\tau_{ij,j} + f_i = 0$$
$$u_i, \tau_{ij}$$

$$(\lambda + G)u_{j,ji} + G\nabla^2 u_i + f_i = 0$$
$$u_i$$

Stress formulation

$$\tau_{ij,j} + f_i = 0$$
$$\varepsilon_{il,jj} - \varepsilon_{jl,ij} + \varepsilon_{jj,il} - \varepsilon_{ij,jl} = 0$$
$$\varepsilon_{ij} = \frac{1+\nu}{E}\tau_{ij} - \frac{\nu}{E}\delta_{ij}\tau_{kk} \qquad \tau_{ij}, \varepsilon_{ij}$$

$$\tau_{ij,j} + f_i = 0 \qquad \tau_{ij}$$
$$\nabla^2\tau_{ik} + \frac{1}{1+\nu}\tau_{jj,ik} = -\frac{\nu}{1-\nu}\delta_{ik}f_{j,j} - \left(f_{i,k} + f_{k,i}\right)$$

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Solution of plane strain problems

- Equilibrium equations (2) $\tau_{ij,j} + f_i = 0$
- Hooke's law (3) $\varepsilon_{ij} = \frac{1+\nu}{E}\tau_{ij} - \frac{\nu}{E}\delta_{ij}\Theta$ $\quad \tau_{ij} = 2G\varepsilon_{ij} + \lambda\varepsilon\delta_{ij}$
- Strain-displacement (3) $\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$
- Unknowns (8) $u_i, \tau_{ij}, \varepsilon_{ij}$

Displacement formulation

$$\tau_{ij} = G\left(u_{i,j} + u_{j,i}\right) + \lambda\delta_{ij}u_{k,k}$$
$$\tau_{ij,j} + f_i = 0$$
$$u_i, \tau_{ij}$$

$$(\lambda + G)u_{j,ji} + G\nabla^2 u_i + f_i = 0$$
$$u_i$$

Stress formulation

$$\tau_{ij,j} + f_i = 0 \qquad \varepsilon_{ij} = \frac{1}{2G}\left(\tau_{ij} - \nu\delta_{ij}\tau_{kk}\right)$$
$$\varepsilon_{11,22} - \varepsilon_{22,11} = 2\varepsilon_{12,12}$$
$$\tau_{ij}, \varepsilon_{ij}$$

$$\tau_{ij,j} + f_i = 0$$
$$\nabla^2\tau_{ii} = \frac{1}{\nu - 1}f_{i,i} \qquad \tau_{ij}$$

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Solution of plane stress problems

- Equilibrium equations (2) $\tau_{ij,j} + f_i = 0$
- Hooke's law (3) $\varepsilon_{ij} = \frac{1+\nu}{E}\tau_{ij} - \frac{\nu}{E}\delta_{ij}\Theta$ $\quad \tau_{ij} = 2G\varepsilon_{ij} + \frac{\nu E}{1-\nu^2}\varepsilon\delta_{ij}$
- Strain-displacement (3) $\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$
- Unknowns (8) $u_i, \tau_{ij}, \varepsilon_{ij}$

Displacement formulation

$$\tau_{ij} = G\left(u_{i,j} + u_{j,i}\right) + \frac{\nu E}{1-\nu^2}\delta_{ij}u_{k,k}$$
$$\tau_{ij,j} + f_i = 0$$
$$u_i, \tau_{ij}$$

$$\frac{E}{2(1-\nu)}u_{j,ji} + G\nabla^2 u_i + f_i = 0$$
$$u_i$$

Stress formulation

$$\tau_{ij,j} + f_i = 0 \qquad \varepsilon_{ij} = \frac{1+\nu}{E}\tau_{ij} - \frac{\nu}{E}\delta_{ij}\tau_{kk}$$
$$\varepsilon_{11,22} - \varepsilon_{22,11} = 2\varepsilon_{12,12}$$
$$\tau_{ij}, \varepsilon_{ij}$$

$$\tau_{ij,j} + f_i = 0$$
$$\nabla^2\tau_{ii} = -(\nu + 1)f_{i,i} \qquad \tau_{ij}$$

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