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Homework 2

1. (a) let $\bar{\bar{W}} = \vec{u} \otimes \vec{v}$, then $W_{ij} = u_i v_j$.

   (b) $v_i = A_{ij} u_j$

   (c) $[\bar{\bar{A}}^T] = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$

   (d) $tr\,\bar{\bar{A}} = A_{ii}$

   (e) $(AB)_{ij} = A_{im} B_{mj}$

   (f) $(A^T B)_{ij} = A^T_{im} B_{mj} = A_{mi} B_{mj}$

   (g) $\bar{\bar{A}} : \bar{\bar{B}} = tr(\bar{\bar{A}}^T \bar{\bar{B}}) = A_{mi} B_{mi}$

2. $$|A| = A_{11}A_{22}A_{33} + A_{12}A_{23}A_{31} + A_{13}A_{21}A_{32} - A_{13}A_{22}A_{31} - A_{12}A_{21}A_{32} - A_{11}A_{23}A_{32}$$
   $$= \varepsilon_{ijk} A_{1i} A_{2j} A_{3k}$$

3. Based on $\hat{\vec{e}}_i = \bar{\bar{Q}} \vec{e}_i$, we know that $\vec{e}_i = \bar{\bar{Q}}^T \hat{\vec{e}}_i = Q^T_{mi} \hat{\vec{e}}_m = Q_{im} \hat{\vec{e}}_m$,
   also we get $\vec{e}_j = Q_{jn} \hat{\vec{e}}_n$.
   So $Q_{ij} = \vec{e}_i \cdot \bar{\bar{Q}} \vec{e}_j = Q_{im} \hat{\vec{e}}_m \bar{\bar{Q}} Q_{jn} \hat{\vec{e}}_n$
   $= (Q_{im} Q_{jn}) \cdot \hat{\vec{e}}_m \cdot \bar{\bar{Q}} \hat{\vec{e}}_n$

4. $\nabla \varphi(\vec{x}) = 2x_1x_3 \vec{e}_1 + x_3^2 \vec{e}_2 + (x_1^2 + 2x_2x_3) \vec{e}_3$

   $$[\nabla \vec{V}(\vec{x})] = \begin{bmatrix} \frac{\partial V_1}{\partial x_1} & \frac{\partial V_1}{\partial x_2} & \frac{\partial V_1}{\partial x_3} \\ \frac{\partial V_2}{\partial x_1} & \frac{\partial V_2}{\partial x_2} & \frac{\partial V_2}{\partial x_3} \\ \frac{\partial V_3}{\partial x_1} & \frac{\partial V_3}{\partial x_2} & \frac{\partial V_3}{\partial x_3} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ x_2\cos(x_1) & \sin(x_1) & 0 \end{bmatrix}$$

   $\nabla \cdot \vec{V}(\vec{x})$

   $$curl\, \vec{V}(\vec{x}) = \left(\frac{\partial V_3}{\partial x_2} - \frac{\partial V_2}{\partial x_3}\right)\vec{e}_1 + \left(\frac{\partial V_1}{\partial x_3} - \frac{\partial V_3}{\partial x_1}\right)\vec{e}_2 + \left(\frac{\partial V_2}{\partial x_1} - \frac{\partial V_1}{\partial x_2}\right)\vec{e}_3$$
   $$= \sin(x_1) \cdot \vec{e}_1 + (1 - x_2\cos x_1)\vec{e}_2$$

5. (a) $\frac{d\hat{\vec{e}}_i}{dt} = \frac{d\bar{\bar{Q}}(t)}{dt}\vec{e}_i = \bar{\bar{\Omega}}(t)\bar{\bar{Q}}(t)\vec{e}_i = \bar{\bar{\Omega}}(t)\hat{\vec{e}}_i$

(b) $\vec{e}_i = \bar{\bar{Q}}(t)^{-1}\hat{\vec{e}}_i$,

as $\bar{\bar{Q}}(t)$ is an othogonal tensor, so $\bar{\bar{Q}}(t)^{-1} = \bar{\bar{Q}}(t)^T$

so, $\vec{e}_i = \bar{\bar{Q}}(t)^T\hat{\vec{e}}_i = Q_{im}\hat{\vec{e}}_m$

$\vec{v} = v_i\vec{e}_i = v_i Q_{im}\hat{\vec{e}}_m = \hat{v}_i\hat{\vec{e}}_i$ ①

① $\cdot\hat{\vec{e}}_i$ both in the left and right, we get $\hat{v}_i = v_i Q_{ii}$

so, $\frac{d\hat{v}_i}{dt} = \frac{d(v_i Q_{ii})}{dt} = \frac{dv_i}{dt}\cdot Q_{ii} + \frac{dQ_{ii}}{dt}\cdot v_i$

$= Q_{ji}\left[\frac{dv_j}{dt} - \Omega_{jk}v_k\right]$.

(c) $\vec{e}_i\otimes\vec{e}_j = [\vec{e}_i][\vec{e}_j]^T$, $\hat{\vec{e}}_i\otimes\hat{\vec{e}}_j = [\hat{\vec{e}}_i][\hat{\vec{e}}_j]^T = [\bar{\bar{Q}}][\vec{e}_i][\vec{e}_j]^T[\bar{\bar{Q}}]^T$

so, $S_{ij}[\vec{e}_i][\vec{e}_j]^T = \hat{S}_{ij}[\bar{\bar{Q}}][\vec{e}_i][\vec{e}_j]^T[\bar{\bar{Q}}]^T$

$\therefore \hat{S}_{ij} = [\bar{\bar{Q}}]^T S_{ij}[\bar{\bar{Q}}]$

$\therefore \frac{d\hat{S}_{ij}}{dt} = \frac{d([\bar{\bar{Q}}]^T S_{ij}[\bar{\bar{Q}}])}{dt}$

$= Q_{ki}Q_{lj}\left[\frac{dS_{kl}}{dt} - \Omega_{km}S_{ml} - \Omega_{lm}S_{km}\right]$

6. (a) $\operatorname{grad}\vec{u} = \begin{bmatrix} x_2x_3 & x_1x_3 & x_1x_2 \\ x_2 & x_1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, $\operatorname{div}(\vec{u}) = x_2x_3 + x_1$

$\operatorname{curl}\vec{u} = (x_1x_2 - 1)\vec{e}_2 + (x_2 - x_1x_3)\vec{e}_3$

(b) $\operatorname{grad}(\operatorname{div}\vec{u}) = \vec{e}_1 + x_3\vec{e}_2 + x_2\vec{e}_3$,

$\operatorname{curl}(\operatorname{curl}(\vec{u})) = \vec{e}_1 + x_3\vec{e}_2 + x_2\vec{e}_3$, $\therefore$ right $= \vec{0}$

$\nabla^2\vec{u} = \operatorname{div}(\nabla\vec{u}) = tr(\nabla(\nabla\vec{u})) = 0$;

so left = right.