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Homework 5.

1. (a) $A_{vol}^T = \frac{1}{3}(\text{tr}\bar{\bar{A}})\bar{\bar{I}}$, $\quad B_{dev} = \bar{\bar{B}} - \frac{1}{3}\text{tr}(\bar{\bar{B}})\bar{\bar{I}}$

$$A_{vol}^T B_{dev} = \frac{1}{3}(\text{tr}\bar{\bar{A}})\bar{\bar{B}} - \frac{1}{9}\text{tr}(\bar{\bar{A}})\text{tr}(\bar{\bar{B}})\bar{\bar{I}}$$

$$A_{vol} : B_{dev} = \text{tr}(A_{vol}^T B_{dev}) = \text{tr}\left[\frac{1}{3}(\text{tr}\bar{\bar{A}})\bar{\bar{B}} - \frac{1}{9}\text{tr}(\bar{\bar{A}})\text{tr}(\bar{\bar{B}})\bar{\bar{I}}\right]$$

$$= \frac{1}{3}\text{tr}(\bar{\bar{A}})\text{tr}(\bar{\bar{B}}) - \frac{1}{3}\text{tr}(\bar{\bar{A}})\text{tr}(\bar{\bar{B}}) = 0$$

(b) $A_{dev}^T = \bar{\bar{A}}^T - \frac{1}{3}\text{tr}(\bar{\bar{A}})\bar{\bar{I}}$, $\quad B_{dev} = \bar{\bar{B}} - \frac{1}{3}\text{tr}(\bar{\bar{B}})\bar{\bar{I}}$

$$A_{dev}^T B_{dev} = \bar{\bar{A}}^T\bar{\bar{B}} - \frac{1}{3}(\text{tr}\bar{\bar{A}})\bar{\bar{B}} - \frac{1}{3}\text{tr}(\bar{\bar{B}})\bar{\bar{A}} + \frac{1}{9}\text{tr}(\bar{\bar{A}})\text{tr}(\bar{\bar{B}})\bar{\bar{I}}$$

$$\text{tr}(A_{dev}^T B_{dev}) = \text{tr}(\bar{\bar{A}}^T\bar{\bar{B}}) - \frac{1}{3}\text{tr}(\bar{\bar{A}})\text{tr}(\bar{\bar{B}}) - \frac{1}{3}\text{tr}(\bar{\bar{A}})\text{tr}(\bar{\bar{B}}) + \frac{1}{3}\text{tr}(\bar{\bar{A}})\text{tr}(\bar{\bar{B}})$$

So, right $= \text{tr}(\bar{\bar{A}}^T\bar{\bar{B}}) = \bar{\bar{A}} : \bar{\bar{B}}$

(c) $W = \frac{1}{2}\bar{\bar{e}} : (\mathbb{C}\bar{\bar{e}}) = \frac{1}{2}C_{ijkl}e_{ij}e_{kl}$

$$= \frac{1}{2}[\lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})]e_{ij}e_{kl}$$

$$= \frac{1}{2}\lambda e_{ii}e_{jj} + \frac{1}{2}\mu(e_{ij})^2 + \frac{1}{2}\mu(e_{ij})^2$$

$$= \frac{1}{2}\lambda[\text{tr}(\bar{\bar{e}})]^2 + \mu(e_{ij})^2$$

$$= \frac{1}{2}\lambda[\text{tr}(\bar{\bar{e}})]^2 + \mu\,\bar{\bar{e}} : \bar{\bar{e}}$$

$$= \left(\frac{\lambda}{2} + \frac{\mu}{3}\right)(\text{tr}\,\bar{\bar{e}})^2 + \mu(\bar{\bar{e}}_{dev})^2$$

(d) $\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}$, $\quad \mu = \frac{E}{2(1+\nu)}$

$$\left\{\begin{array}{l} \frac{\lambda}{2} + \frac{\mu}{3} = \frac{\nu E}{2(1+\nu)(1-2\nu)} + \frac{E}{6(1+\nu)} = \frac{(1+\nu-6\nu^2)E}{6(1+\nu)(1-2\nu)} > 0 \\ \mu = \frac{E}{2(1+\nu)} > 0 \end{array}\right.$$

E represents the ratio of uniaxial stress to the uniaxial strain.

ν represents the ratio of transverse strain to the axial strain.

So, $E > 0$, $\quad 1+\nu > 0$, that's $\nu > -1$

① when $1-2\nu > 0$, that's $\nu < \frac{1}{2}$, $\quad 1+\nu-6\nu^2 > 0$, that's $-\frac{1}{3} < \nu < \frac{1}{2}$

② when $1-2\nu < 0$, that's $\nu > \frac{1}{2}$, $\quad 1+\nu-6\nu^2 < 0$, that's $\nu > \frac{1}{2}$或$\nu < -\frac{1}{3}$

So, $\nu > -1$.

(e) As $\bar{\bar{\sigma}}_{dev} = \begin{bmatrix} \sigma_{11} - \frac{tr\bar{\bar{\sigma}}}{3} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} - \frac{tr\bar{\bar{\sigma}}}{3} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} - \frac{tr\bar{\bar{\sigma}}}{3} \end{bmatrix}$

$C_1 = \frac{1}{2}(\sigma_{11}+\sigma_{22} - \frac{2}{3} tr\bar{\bar{\sigma}})$, $C_2 = \frac{1}{2}(\sigma_{22}+\sigma_{33} - \frac{1}{3} tr\bar{\bar{\sigma}})$, $C_3 = \frac{1}{2}(\sigma_{11}+\sigma_{33} - \frac{2}{3} tr\bar{\bar{\sigma}})$

$R_1 = \frac{1}{2}(\sigma_{11}-\sigma_{22})$, $R_2 = \frac{1}{2}(\sigma_{22}-\sigma_{33})$, $R_3 = \frac{1}{2}(\sigma_{11}-\sigma_{33})$

前一个摩尔圆是后一个摩尔圆向右平移 $\frac{1}{3} tr\bar{\bar{\sigma}}$ 的结果.

(2) (a) $[\bar{\bar{\sigma}}] = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix} = \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} = \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix} = \{\sigma_I\}$

$[\bar{\bar{e}}] = \begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{12} & e_{22} & e_{23} \\ e_{13} & e_{23} & e_{33} \end{bmatrix} = \begin{Bmatrix} e_{11} \\ e_{22} \\ e_{33} \\ e_{23} \\ e_{13} \\ e_{12} \end{Bmatrix} = \begin{Bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \end{Bmatrix} = \{e_I\}$

(b) $C_{ijkl} = C_{IJ}$. As $C_{ijkl} = C_{klij}$, so $C_{IJ} = C_{JI}$.

(c) $\bar{\bar{\sigma}} : \bar{\bar{e}} = e_{ij} e_{ij} = \sigma_I e_I$.

3. ① when $\nu = 0$, $\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} = 0$, $G = \frac{E}{2(1+\nu)} = \frac{E}{2}$, $K = \frac{E}{3(1-2\nu)} = \frac{E}{3}$

② when $\nu = 0.3$, $\lambda = 0.58E$, $G = 0.38E$, $K = 0.83E$

③ when $\nu = 0.5$, $\lambda = \infty$, $G = \frac{E}{3}$, $K = \infty$

$\nu = 0.5$ 对应的情况是：无论用多大的力都无法让该物体产生形变.

(4) (a) $\bar{\bar{e}} = \frac{1+\nu}{E}\bar{\bar{\sigma}} = \frac{\nu}{E}\,tr(\bar{\bar{\sigma}})\,\bar{\bar{I}}$

① simple tension. $\bar{\bar{e}} = \begin{bmatrix} 0.0022 & 0 & 0 \\ 0 & -0.0007 & 0 \\ 0 & 0 & -0.0007 \end{bmatrix}$

② pure sheer. $\bar{\bar{e}} = \begin{bmatrix} 0 & 0.0015 & 0 \\ 0.0015 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

③ hydrostatic compression. $\bar{\bar{e}} = \begin{bmatrix} 0.0023 & 0 & 0 \\ 0 & 0.0023 & 0 \\ 0 & 0 & 0.0023 \end{bmatrix}$

(b) ① simple tension : $\bar{\bar{e}} = \begin{bmatrix} 0.0014 & 0 & 0 \\ 0 & -0.0004 & 0 \\ 0 & 0 & -0.0004 \end{bmatrix}$

② pure sheer. $\bar{\bar{e}} = \begin{bmatrix} 0 & 0.0009 & 0 \\ 0.0009 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

③ hydrostatic compression. $\bar{\bar{e}} = \begin{bmatrix} -0.001 & 0 & 0 \\ 0 & -0.001 & 0 \\ 0 & 0 & -0.001 \end{bmatrix}$

(c) ① simple tension, $\bar{\bar{e}} = \begin{bmatrix} 7.895 & 0 & 0 \\ 0 & -3.940 & 0 \\ 0 & 0 & -3.940 \end{bmatrix}$

② pure sheer, $\bar{\bar{e}} = \begin{bmatrix} 0 & 5.5226 & 0 \\ 5.5226 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

③ hydrostatic compression, $\bar{\bar{e}} = \begin{bmatrix} -0.5263 & 0 & 0 \\ 0 & -0.5263 & 0 \\ 0 & 0 & 0.5263 \end{bmatrix}$

(d) $[\bar{\bar{\sigma}}] = \begin{bmatrix} 20 & 0 & 0 \\ 0 & 30 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\bar{\bar{e}} = \begin{bmatrix} 5.46\times10^{-5} & 0 & 0 \\ 0 & 1.17\times10^{-5} & 0 \\ 0 & 0 & -7\times10^{-5} \end{bmatrix}$

change in x-axis: $5.46\times10^{-5}\times300 = 0.0164\,mm$

change in y-axis: $1.17\times10^{-5}\times200 = 0.0023\,mm$

change in z-axis: $-7\times10^{-5}\times4 = -0.0003\,mm$

(5) $\nabla\cdot\bar{\bar{\sigma}} + \vec{b} = \vec{0}$, as $\vec{b}=0$, so $\nabla\cdot\bar{\bar{\sigma}} = \vec{0}$

(a) $\nabla\cdot\bar{\bar{\sigma}} = (\lambda+\mu)\nabla(\nabla\cdot\vec{u}) + \mu\nabla^2\vec{u} = 0$

$\because \lambda+\mu>0$, $\mu>0$, so $\nabla(\nabla\cdot\vec{u})=0$, $\nabla^2\vec{u}=0$

$\because \nabla\cdot\vec{u} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = e_1+e_2+e_3 = tr\bar{\bar{e}}$

$\therefore \nabla tr\bar{\bar{e}} = 0$. $\therefore \nabla^2 tr\bar{\bar{e}} = 0$.

(b) $\because \nabla^2\vec{u}=0$, $\therefore \nabla^2(\nabla^2\vec{u})=0$, so $\nabla^2(\nabla^2 u_i)=0$.

6. $\nabla^2\sigma_x + \frac{1}{1+\nu}\frac{\partial^2\Theta}{\partial x^2} = -2\frac{\partial f_x}{\partial x} - \frac{\nu}{1-\nu}\left(\frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} + \frac{\partial f_z}{\partial z}\right)$

$\nabla^2\sigma_y + \frac{1}{1+\nu}\frac{\partial^2\Theta}{\partial y^2} = -2\frac{\partial f_y}{\partial y} - \frac{\nu}{1-\nu}\left(\frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} + \frac{\partial f_z}{\partial z}\right)$

$\nabla^2\sigma_z + \frac{1}{1+\nu}\frac{\partial^2\Theta}{\partial z^2} = -2\frac{\partial f_z}{\partial z} - \frac{\nu}{1-\nu}\left(\frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} + \frac{\partial f_z}{\partial z}\right)$

$\nabla^2\tau_{yz} + \frac{1}{1+\nu}\frac{\partial^2\Theta}{\partial y\partial z} = -\left(\frac{\partial f_y}{\partial z} + \frac{\partial f_z}{\partial y}\right)$

$\nabla^2\tau_{zx} + \frac{1}{1+\nu}\frac{\partial^2\Theta}{\partial z\partial x} = -\left(\frac{\partial f_x}{\partial x} + \frac{\partial f_z}{\partial z}\right)$

$\nabla^2\tau_{xy} + \frac{1}{1+\nu}\frac{\partial^2\Theta}{\partial x\partial y} = -\left(\frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y}\right)$

$\Theta = I_1 = \sigma_x + \sigma_y + \sigma_z$.

$f_x = -\frac{\partial\sigma_x}{\partial x} - \frac{\partial\tau_{xy}}{\partial y} - \frac{\partial\tau_{xz}}{\partial z}$

$f_y = -\frac{\partial\sigma_y}{\partial y} - \frac{\partial\tau_{yx}}{\partial x} - \frac{\partial\tau_{yz}}{\partial z}$

$f_z = -\frac{\partial\sigma_z}{\partial z} - \frac{\partial\tau_{xz}}{\partial x} - \frac{\partial\tau_{yz}}{\partial y}$