Review

Equilibrium equations:

$$\tau_{ik,i} + f_k = 0$$

$$\tau_{ij} = 2G\varepsilon_{ij} + \lambda\varepsilon\delta_{ij} \quad \text{where } \varepsilon = \varepsilon_{kk}$$

$$\varepsilon_{ij} = \frac{1+\nu}{E}\tau_{ij} - \frac{\nu}{E}\delta_{ij}\Theta \quad \text{where } \Theta = \tau_{kk}$$

$$\frac{\partial u_i}{\partial x_j} = \varepsilon_{ij} + \omega_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)$$

Strain tensor: Rotation tensor (转动张量):

$$\tau_{ij} = C_{ijkl}\varepsilon_{kl}$$

$$du_i = u_i - u_i^0 = \varepsilon_{ij}dx_j + \omega_{ij}dx_j$$

i.e. $d\mathbf{u} = d\mathbf{u}_D + d\mathbf{u}_R$

Deformation Rotation

Generalized Hooke's law: elastic constant C_{ijkl} has only 21 independent constants

Generalized Hooke's law

$$\tau_{ij} = C_{ijkl}\varepsilon_{kl}$$

Only 21 independent parameters needed for the elastic constant C_{ijkl}

1. Because of the symmetry in the strain components, we can set

$$C_{ijkl} = C_{ijlk}$$

which reduces the number of independent constants to 54.

2. Because of the symmetry in the stress components, we can set

$$C_{ijkl} = C_{jikl}$$

$$C_{ijkl} = C_{jikl} = C_{ijlk} = C_{jilk}$$

The number of independent constants reduces to 36.

$$\begin{bmatrix} \tau_{11} \\ \tau_{22} \\ \tau_{33} \\ \tau_{23} \\ \tau_{31} \\ \tau_{12} \end{bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1123} & C_{1131} & C_{1112} \\ C_{2211} & C_{2222} & C_{2233} & C_{2223} & C_{2231} & C_{2212} \\ C_{3311} & C_{3322} & C_{3333} & C_{3323} & C_{3331} & C_{3312} \\ C_{2311} & C_{2322} & C_{2333} & C_{2323} & C_{2331} & C_{2312} \\ C_{3111} & C_{3122} & C_{3133} & C_{3123} & C_{3131} & C_{3112} \\ C_{1211} & C_{1222} & C_{1233} & C_{1223} & C_{1231} & C_{1212} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{23} \\ \varepsilon_{31} \\ \varepsilon_{12} \end{bmatrix}$$

Generalized Hooke's law: elastic constant C_{ijkl} has only 21 independent constants

3. The strain energy density is

$$U_0 = \frac{1}{2}\left(\sigma_x\varepsilon_x + \sigma_y\varepsilon_y + \sigma_z\varepsilon_z + \tau_{xy}\gamma_{xy} + \tau_{yz}\gamma_{yz} + \tau_{zx}\gamma_{zx}\right)$$

$$U_0 = \frac{1}{2}\tau_{ij}\varepsilon_{ij} = \frac{1}{2}C_{ijkl}\varepsilon_{kl}\varepsilon_{ij} = \frac{1}{2}\tau_{kl}\varepsilon_{kl} = \frac{1}{2}C_{klij}\varepsilon_{kl}\varepsilon_{ij}$$

Generally, we can set $C_{ijkl} = C_{klij}$ which further reduces the number of independent constants to 21. e.g.,

$$C_{ijkl} = C_{jikl} = C_{ijlk} = C_{jilk} =$$
$$C_{klij} = C_{klji} = C_{lkij} = C_{lkji}$$

$$\begin{bmatrix} \tau_{11} \\ \tau_{22} \\ \tau_{33} \\ \tau_{23} \\ \tau_{31} \\ \tau_{12} \end{bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1123} & C_{1131} & C_{1112} \\ & C_{2222} & C_{2233} & C_{2223} & C_{2231} & C_{2212} \\ & & C_{3333} & C_{3323} & C_{3331} & C_{3312} \\ & & & C_{2323} & C_{2331} & C_{2312} \\ & Sym. & & & C_{3131} & C_{3112} \\ & & & & & C_{1212} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{23} \\ \varepsilon_{31} \\ \varepsilon_{12} \end{bmatrix}$$

For fully anisotropic body, the independent elastic constants are 21

Generalized Hooke's law: strain and stress principal axes coicide for isotropic material

For isotropic material, elastic property is the same for any coordinate systems

$$C'_{ijkl} = C_{ijkl}$$

Let x_1, x_2, x_3 coincide with the principal axes of strain, i.e., $\varepsilon_{12}=\varepsilon_{13}=\varepsilon_{23}=0$.

We now prove that $\tau_{12}=\tau_{13}=\tau_{23}=0$.

First, we prove that $\tau_{23}=0$

$$\tau_{23} = C_{23ij}\varepsilon_{ij} = C_{2311}\varepsilon_{11} + C_{2322}\varepsilon_{22} + C_{2333}\varepsilon_{33}$$

Now, let new coordinate x'_1, x'_2, x'_3 be obtained by rotating the x_1, x_2 axes through 180° about x_3. Then a_{ij} is:

	x'_1	x'_2	x'_3
x_1	-1	0	0
x_2	0	-1	0
x_3	0	0	1

$$a_{ij} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\tau'_{23} = a_{k2}a_{l3}\tau_{kl} = -\tau_{23}$$

$$\varepsilon'_{ij} = a_{ki}a_{lj}\varepsilon_{kl} = \begin{bmatrix} \varepsilon_{11} & 0 & 0 \\ 0 & \varepsilon_{22} & 0 \\ 0 & 0 & \varepsilon_{33} \end{bmatrix}$$

$$\tau'_{23} = C_{23ij}\varepsilon'_{ij} = C_{2311}\varepsilon_{11} + C_{2322}\varepsilon_{22} + C_{2333}\varepsilon_{33} = \tau_{23}$$

$$\tau_{23} = 0$$

Similarly, we can prove $\tau_{12}=\tau_{13}=0$.

Generalized Hooke's law: only two independent elastic constants for isotropic material

Let x_1, x_2, x_3 coincide with the principal axes of strain, the stress-strain relations are

$$\tau_{ij} = C_{ijkl} e_{kl}$$
$$= C_{ij11} e_{11} + C_{ij22} e_{22} + C_{ij33} e_{33}$$

$$\tau_{11} = C_{1111} e_{11} + C_{1122} e_{22} + C_{1133} e_{33}$$
$$\tau_{22} = C_{2211} e_{11} + C_{2222} e_{22} + C_{2233} e_{33}$$
$$\tau_{33} = C_{3311} e_{11} + C_{3322} e_{22} + C_{3333} e_{33}$$

For isotropic material, we now use coordinate transformations to show that the elastic constants above can be reduced to 2 constants.

The isotropic 4th-order tensor C_{ijkl} transforms to C_{mpqr} as below

$$C_{mpqr} = a_{mi} a_{pj} a_{qk} a_{rl} C'_{ijkl} = a_{mi} a_{pj} a_{qk} a_{rl} C_{ijkl}$$

For a coordinate system x'_i obtained by rotating the old coordinate system through 90° about x_1. The direction cosines are

$$a_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

and we have

$$C_{11qr} = a_{1i} a_{1j} a_{qk} a_{rl} C_{ijkl} = a_{qk} a_{rl} C_{11kl}$$
$$C_{22qr} = a_{2i} a_{2j} a_{qk} a_{rl} C_{ijkl} = a_{qk} a_{rl} C_{33kl}$$
$$C_{33qr} = a_{3i} a_{3j} a_{qk} a_{rl} C_{ijkl} = a_{qk} a_{rl} C_{22kl}$$

which yield

$$C_{1133} = C_{1122}, \; C_{2211} = C_{3311},$$
$$C_{2222} = C_{3333}, \; C_{2233} = C_{3322}$$

Generalized Hooke's law: only two independent elastic constants for isotropic material

Furthermore, for a coordinate system x'_i obtained by rotating the old coordinate system 90° clockwisely about x_2. The direction cosines are

$$a_{ij} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

We have

$$C_{11qr} = a_{1i}a_{1j}a_{qk}a_{rl}C_{ijkl} = a_{qk}a_{rl}C_{33kl}$$

$$C_{22qr} = a_{2i}a_{2j}a_{qk}a_{rl}C_{ijkl} = a_{qk}a_{rl}C_{22kl}$$

$$C_{33qr} = a_{3i}a_{3j}a_{qk}a_{rl}C_{ijkl} = a_{qk}a_{rl}C_{11kl}$$

which yields to

$$C_{1111} = C_{3333}, C_{1122} = C_{3322}$$

$$C_{1133} = C_{3311}, C_{2211} = C_{2233}$$

Combine 7 independent relations, we have only 2 independent constants:

$$C_{1111} = C_{2222} = C_{3333} = C$$

$$C_{1122} = C_{1133} = C_{2211} = C_{2233} = C_{3311} = C_{3322} = \lambda$$

The principal strain and stress relations

$$\tau_{11} = C_{1111}e_{11} + C_{1122}e_{22} + C_{1133}e_{33}$$

$$\tau_{22} = C_{2211}e_{11} + C_{2222}e_{22} + C_{2233}e_{33}$$

$$\tau_{33} = C_{3311}e_{11} + C_{3322}e_{22} + C_{3333}e_{33}$$

become

$$\tau_{11} = Ce_{11} + \lambda(e_{22} + e_{33}) = (C - \lambda)e_{11} + \lambda e = \lambda e + 2Ge_{11}$$

$$\tau_{22} = Ce_{22} + \lambda(e_{11} + e_{33}) = (C - \lambda)e_{22} + \lambda e = \lambda e + 2Ge_{22}$$

$$\tau_{33} = Ce_{33} + \lambda(e_{11} + e_{22}) = (C - \lambda)e_{33} + \lambda e = \lambda e + 2Ge_{33}$$

where

$$e = e_{ii} = e_{11} + e_{22} + e_{33}$$

$$2G = C - \lambda$$

Generalized Hooke's law: only two independent elastic constants for isotropic material

Only 2 elastic constants needed for the Hooke's law on the principal axes

$$\tau_{11} = \lambda e + 2Ge_{11}$$
$$\tau_{22} = \lambda e + 2Ge_{22}$$
$$\tau_{33} = \lambda e + 2Ge_{33}$$

Now we get the Hooke's law for an arbitrary coordinate with the transformation equation

$$\tau'_{ij} = a_{ki}a_{lj}\tau_{kl}$$

we have
$$\begin{aligned}\tau'_{\mathrm{ij}} &= a_{1i}a_{1j}\tau_{11} + a_{2i}a_{2j}\tau_{22} + a_{3i}a_{3j}\tau_{33} \\ &= a_{1i}a_{1j}(\lambda e + 2Ge_{11}) + a_{2i}a_{2j}(\lambda e + 2Ge_{22}) + a_{3i}a_{3j}(\lambda e + 2Ge_{33}) \\ &= (a_{1i}a_{1j} + a_{2i}a_{2j} + a_{3i}a_{3j})\lambda e + 2G(a_{1i}a_{1j}e_{11} + a_{2i}a_{2j}e_{22} + a_{3i}a_{3j}e_{33}) \\ &= a_{mi}a_{mj}\lambda e + 2Ge'_{ij} \\ &= \delta_{ij}\lambda e + 2Ge'_{ij}\end{aligned}$$

The expression is valid for any coordinate system, so we may drop the primes and get

$$\tau_{\mathrm{ij}} = \delta_{ij}\lambda e + 2Ge_{ij}$$

Equations of Compatibility

strain-displacement relation in tensor notation

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$$

$$\varepsilon_{il,jk} - \varepsilon_{jl,ik} + \varepsilon_{jk,il} - \varepsilon_{ik,jl} = 0$$

it represents a set of 81 equations

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}$$

$$\frac{\partial^2 \varepsilon_y}{\partial z^2} + \frac{\partial^2 \varepsilon_z}{\partial y^2} = \frac{\partial^2 \gamma_{yz}}{\partial y \partial z}$$

$$\frac{\partial^2 \varepsilon_z}{\partial x^2} + \frac{\partial^2 \varepsilon_x}{\partial z^2} = \frac{\partial^2 \gamma_{zx}}{\partial z \partial x}$$

$$2\frac{\partial^2 \varepsilon_x}{\partial y \partial z} = \frac{\partial}{\partial x}\left(-\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right)$$

$$2\frac{\partial^2 \varepsilon_y}{\partial x \partial z} = \frac{\partial}{\partial y}\left(\frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right)$$

$$2\frac{\partial^2 \varepsilon_z}{\partial x \partial y} = \frac{\partial}{\partial z}\left(\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} - \frac{\partial \gamma_{xy}}{\partial z}\right)$$

- Above tensor equation reduces to only the 6 compatible equations (Many equations vanish identically).
  - e.g., the last compatible equation can be derived by letting i=1, j=k=3, l=2 or i=2, j=k=3, l=1
- By contracting above tensor equation with respect to j and k, we get equation:

$$\varepsilon_{il,jj} - \varepsilon_{jl,ij} + \varepsilon_{jj,il} - \varepsilon_{ij,jl} = 0$$

- It represents a set of 9 equations and is also equivalent to the 6 compatible equations.
  - The first three compatible equations can be get by setting $i=l$
  - The last three compatible equations can be get by setting $i \neq l$

Equations of Compatibility

Compatibility equations in terms of stress
Or Beltrami-Michell compatibility equations

$$\begin{cases} \nabla^2\sigma_z + \dfrac{1}{1+\nu}\dfrac{\partial^2\Theta}{\partial z^2} = -\left(\dfrac{\nu}{1-\nu}\right)\left(\dfrac{\partial f_x}{\partial x} + \dfrac{\partial f_y}{\partial y} + \dfrac{\partial f_z}{\partial z}\right) - 2\dfrac{\partial f_z}{\partial z} \\ \nabla^2\sigma_x + \dfrac{1}{1+\nu}\dfrac{\partial^2\Theta}{\partial x^2} = -\left(\dfrac{\nu}{1-\nu}\right)\left(\dfrac{\partial f_x}{\partial x} + \dfrac{\partial f_y}{\partial y} + \dfrac{\partial f_z}{\partial z}\right) - 2\dfrac{\partial f_x}{\partial x} \\ \nabla^2\sigma_y + \dfrac{1}{1+\nu}\dfrac{\partial^2\Theta}{\partial y^2} = -\left(\dfrac{\nu}{1-\nu}\right)\left(\dfrac{\partial f_x}{\partial x} + \dfrac{\partial f_y}{\partial y} + \dfrac{\partial f_z}{\partial z}\right) - 2\dfrac{\partial f_y}{\partial y} \end{cases}$$

$$\begin{cases} \nabla^2\tau_{yz} + \dfrac{1}{1+\nu}\dfrac{\partial^2\Theta}{\partial y\partial z} = -\left(\dfrac{\partial f_y}{\partial z} + \dfrac{\partial f_z}{\partial y}\right) \\ \nabla^2\tau_{zx} + \dfrac{1}{1+\nu}\dfrac{\partial^2\Theta}{\partial x\partial z} = -\left(\dfrac{\partial f_x}{\partial z} + \dfrac{\partial f_z}{\partial x}\right) \\ \nabla^2\tau_{xy} + \dfrac{1}{1+\nu}\dfrac{\partial^2\Theta}{\partial x\partial y} = -\left(\dfrac{\partial f_x}{\partial y} + \dfrac{\partial f_y}{\partial x}\right) \end{cases}$$

Tensor notation

$$\nabla^2\tau_{ik} + \frac{1}{1+\nu}\Theta_{,ik} = -\frac{\nu}{1-\nu}\delta_{ik} f_{j,j} - \left(f_{i,k} + f_{k,i}\right)$$

where $\Theta = \tau_{jj}$

Strain Energy in tensor notations

Strain energy density:

$$U_0 = \frac{1}{2}\left(\sigma_x\varepsilon_x + \sigma_y\varepsilon_y + \sigma_z\varepsilon_z + \tau_{xy}\gamma_{xy} + \tau_{yz}\gamma_{yz} + \tau_{zx}\gamma_{zx}\right) \rightarrow U_0 = \frac{1}{2}\tau_{ij}\varepsilon_{ij}$$

In terms of stress components:

$$U_0 = \frac{1}{2}\left(\frac{1}{E}\left(\sigma_x{}^2 + \sigma_y{}^2 + \sigma_z{}^2\right) - \frac{2\nu}{E}\left(\sigma_x\sigma_y + \sigma_x\sigma_z + \sigma_y\sigma_z\right) + \frac{1}{G}\left(\tau_{xy}{}^2 + \tau_{xz}{}^2 + \tau_{yz}{}^2\right)\right) \rightarrow U_0 = -\frac{\nu}{2E}\tau^2{}_{kk} + \frac{1}{4G}\tau_{ij}\tau_{ij}$$

$$G = \frac{E}{2(1+\nu)} \qquad \nu = \frac{\lambda}{2(\lambda + G)}$$

In terms of strain components:

$$U_0 = \frac{1}{2}\left(\lambda\varepsilon^2 + 2G\left(\varepsilon_x{}^2 + \varepsilon_y{}^2 + \varepsilon_z{}^2\right) + G\left(\gamma_{xy}{}^2 + \gamma_{yz}{}^2 + \gamma_{zx}{}^2\right)\right) \rightarrow U_0 = \frac{\lambda}{2}\varepsilon^2{}_{kk} + G\varepsilon_{ij}\varepsilon_{ij}$$

Strain Energy in tensor notations: due to volume change and distortion

The stress and strain tensors can be decomposed into spherical and deviatoric stress and strain tensors.

$$\sigma_{ij} = \sigma_m \delta_{ij} + \sigma'_{ij} \qquad \varepsilon_{ij} = \varepsilon_m \delta_{ij} + \varepsilon'_{ij}$$

$$\sigma'_{ij} = \sigma_{ij} - \sigma_m \delta_{ij} \qquad \varepsilon'_{ij} = \varepsilon_{ij} - \varepsilon_m \delta_{ij}$$

The strain energy density $U_0 = \frac{1}{2}\sigma_{ij}\varepsilon_{ij}$ can also be decomposed into two parts: one due to volume change and the other due to distortion.

$$U_0 = \frac{1}{2}\sigma_{ij}\varepsilon_{ij} = \frac{1}{2}\lambda\varepsilon^2 + \mu\varepsilon_{ij}\varepsilon_{ij} = \frac{1}{2}(\lambda + \frac{2}{3}\mu)\varepsilon^2 + \mu\varepsilon'_{ij}\,\varepsilon'_{ij} = \frac{1}{2}K\varepsilon^2 + \mu\varepsilon'_{ij}\,\varepsilon'_{ij}$$

Strain energy density due to volume change, K is bulk modulus.

Strain energy density due to distortion.

von Mises yielding criteria: $\sqrt{\frac{3}{2}\sigma'_{ij}\,\sigma'_{ij}} \leq \sigma_y$

Equilibrium Equations in terms of Displacement (Navier's Equations)

Navier's equations

$$\begin{cases} (\lambda + G)\dfrac{\partial \varepsilon}{\partial x} + G\nabla^2 u + f_x = 0 \\ (\lambda + G)\dfrac{\partial \varepsilon}{\partial y} + G\nabla^2 v + f_y = 0 \\ (\lambda + G)\dfrac{\partial \varepsilon}{\partial z} + G\nabla^2 w + f_z = 0 \end{cases}$$

$$(\lambda + G)\varepsilon_{,k} + G\nabla^2 u_k + f_k = 0$$

where $\varepsilon = \varepsilon_{kk} = u_{k,k}$

$$\nabla^2 u_k = u_{k,ii}$$

Governing Equations of Elasticity: 3D problems

Equilibrium equations (3) $\tau_{ij,j} + f_i = 0$

Hooke's law (6) $\varepsilon_{ij} = \frac{1+\nu}{E}\tau_{ij} - \frac{\nu}{E}\delta_{ij}\Theta \qquad \tau_{ij} = 2G\varepsilon_{ij} + \lambda\varepsilon\delta_{ij}$

Strain-displacement (6) $\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$

Unknowns (15) $u_i, \tau_{ij}, \varepsilon_{ij}$

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}$$

$$G = \frac{E}{2(1+\nu)}$$

Displacement formulation

$$\left(\lambda + G\right)u_{j,ji} + G\nabla^2 u_i + f_i = 0$$

$$u_i$$

Stress formulation

$$\tau_{ij,j} + f_i = 0 \qquad \tau_{ij}$$

$$\nabla^2\tau_{ik} + \frac{1}{1+\nu}\tau_{jj,ik} = -\frac{\nu}{1-\nu}\delta_{ik}f_{j,j} - \left(f_{i,k} + f_{k,i}\right)$$

Governing Equations of Elasticity: plane stress

Equilibrium equations $\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}+f_x=0\ (x,y)$

Hooke's law $\varepsilon_x=\frac{1}{E}(\sigma_x-\nu\sigma_y)\quad (\mathrm{x},\mathrm{y})\qquad \gamma_{xy}=\frac{1}{G}\tau_{xy}$

Strain-displacement $\varepsilon_x=\frac{\partial u}{\partial x}\quad \varepsilon_y=\frac{\partial v}{\partial y}\quad \gamma_{xy}=\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}$

8 governing equationgs for 8 unknowns

$\sigma_x,\sigma_y,\tau_{xy},\varepsilon_x,\varepsilon_y,\gamma_{xy},u,v$

Equilibrium equations (2) $\tau_{ij,j}+f_i=0$

Hooke's law (3) $\varepsilon_{ij}=\frac{1+\nu}{E}\tau_{ij}-\frac{\nu}{E}\delta_{ij}\Theta$

Strain-displacement (3) $\varepsilon_{ij}=\frac{1}{2}\left(u_{i,j}+u_{j,i}\right)$

Unknowns (8) $u_i,\tau_{ij},\varepsilon_{ij}$

Governing Equations of Elasticity: plane stress

Equilibrium equations (2) $\tau_{ij,j} + f_i = 0$

Hooke's law (3) $\varepsilon_{ij} = \frac{1+\nu}{E}\tau_{ij} - \frac{\nu}{E}\delta_{ij}\Theta \quad \tau_{ij} = 2G\varepsilon_{ij} + \lambda\varepsilon\delta_{ij}$

Strain-displacement (3) $\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$

Unknowns (8) $u_i, \tau_{ij}, \varepsilon_{ij}$

Displacement formulation

$$\frac{E}{2(1-\nu)}u_{j,ji} + G\nabla^2 u_i + f_i = 0$$

$$u_i$$

Stress formulation

$$\tau_{ij,j} + f_i = 0$$

$$\nabla^2\tau_{ii} = -(\nu+1)f_{i,i} \qquad \tau_{ij}$$

Airy Stress Function

$$\nabla^4\phi = (1-\nu)\nabla^2 V$$

$$V_{,i} = f_i \qquad \phi_{,11} = \tau_{22} + V$$

$$\phi_{,12} = -\tau_{12} \qquad \phi_{,22} = \tau_{11} + V$$

The plane stress equations can be transformed into those for plane strain by replacing E and ν by E_1 and ν_1, respectively.

$$E_1 = \frac{E}{1-\nu^2} \qquad \nu_1 = \frac{\nu}{1-\nu}$$

Governing Equations of Elasticity: plane strain

Equilibrium equations (2) $\tau_{ij,j} + f_i = 0$

Hooke's law (3) $\varepsilon_{ij} = \frac{1+\nu}{E}\tau_{ij} - \frac{\nu}{E}\delta_{ij}\Theta \quad \tau_{ij} = 2G\varepsilon_{ij} + \lambda\varepsilon\delta_{ij}$

Strain-displacement (3) $\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$

Unknowns (8) $u_i, \tau_{ij}, \varepsilon_{ij}$

Displacement formulation

$$(\lambda + G)u_{j,ji} + G\nabla^2 u_i + f_i = 0$$

$$u_i$$

Stress formulation

$$\tau_{ij,j} + f_i = 0$$

$$\nabla^2\tau_{ii} = \frac{1}{\nu - 1} f_{i,i} \qquad \tau_{ij}$$

Airy Stress Function

$$\nabla^4\phi = \frac{1-2\nu}{1-\nu}\nabla^2 V$$

$$V_{,i} = f_i \qquad \phi_{,11} = \tau_{22} + V$$

$$\phi_{,12} = -\tau_{12} \qquad \phi_{,22} = \tau_{11} + V$$

The plane strain equations can be transformed into those for plane stress by replacing E and ν by E_2 and ν_2, respectively. $E_2 = \frac{E(1+2\nu)}{(1+\nu)^2} \qquad \nu_2 = \frac{\nu}{1+\nu}$

Betti reciprocal theorem (功的互换定理，互易定理)***

Assume two sets of external forces acting on the same elastic body, each produces the corresponding stress, strain and displacement under the action of the forces:
Group 1: surface force T'_i, body force f'_i; displacement u'_i, strain ε'_i, stress σ'_i
Group 2: surface force T''_i, body force f''_i; displacement u''_i, strain ε''_i, stress σ''_i

There is the following reciprocity theorem:
The work done by the first group through the displacements produced by the second group is equal to the work done by the second group through the displacements produced by the first group.

$$W_{12} = \iiint_V f'_i u''_i \, dV + \oint\!\!\oint_S T'_i u''_i \, dS = \iiint_V f''_i u'_i \, dV + \oint\!\!\oint_S T''_i u'_i \, dS = W_{21}$$

$$W_{12} = \iiint_V f'_i u''_i \, dV + \oint\!\!\oint_S \sigma'_{ij} n_j u''_i \, dS = \iiint_V [f'_i u''_i + \frac{\partial(\sigma'_{ij} u''_i)}{\partial x_j}] dV$$

$$= \iiint_V \sigma'_{ij} \frac{\partial u''_i}{\partial x_j} dV = \iiint_V \sigma'_{ij} \varepsilon''_{ij} \, dV$$

Similarly, $W_{21} = \iiint_V \sigma''_{ij} \varepsilon'_{ij} \, dV$

Thus, $W_{12} = W_{21}$

Betti reciprocal theorem (功的互换定理，互易定理)***

例 1 一等截面弹性杆，在横向受到一对互相平衡的力 P_1 的作用，如图 4-5 所示。求杆的端部伸长 δ_1。设 h，P_1 及杆的横截面面积 F 及杆的杨氏弹性模量 E，泊松比 ν 均已知。

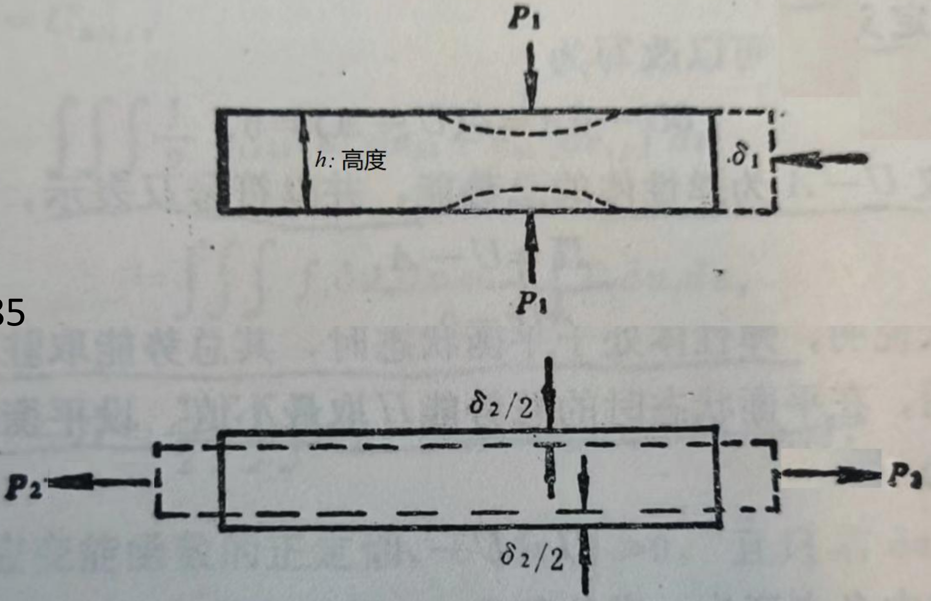


图 4-5 应用贝蒂定理的例子

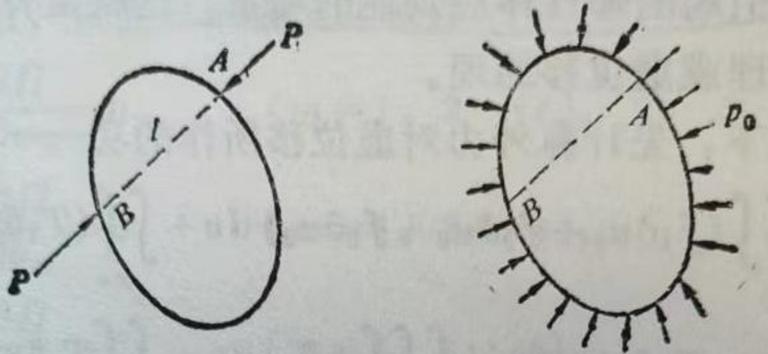
上图——第一组力作用；下图——第二组力作用

尹祥础, 1985

$$\delta_1 = \frac{\nu h}{EF} P_1$$

Betti reciprocal theorem (功的互换定理，互易定理)

例 2 如图 4-6 所示，一弹性体受到一对平衡力的作用，已知两受力点 A、B 间的距离为 l，弹性体的体积弹性模量为 K。求弹性体的体积缩小量。



尹祥础, 1985

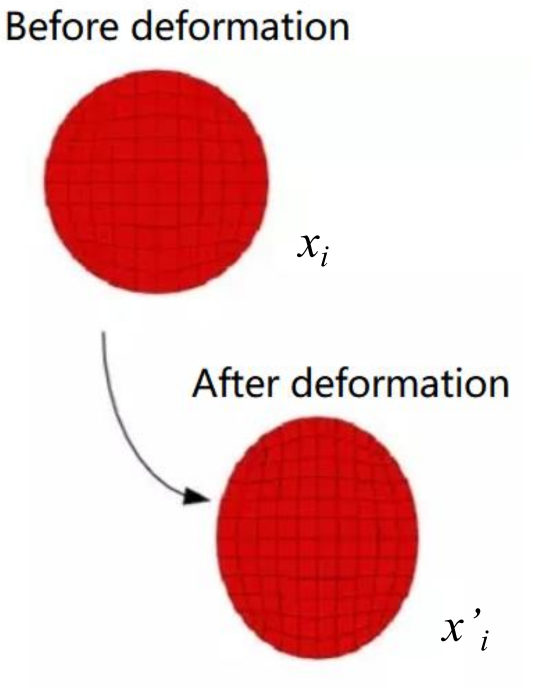
图 4-6

假定第二组外力为均匀静水压强p_0，则有 $$P\delta_2 = p_0\delta_1 = p_0\Delta V$$

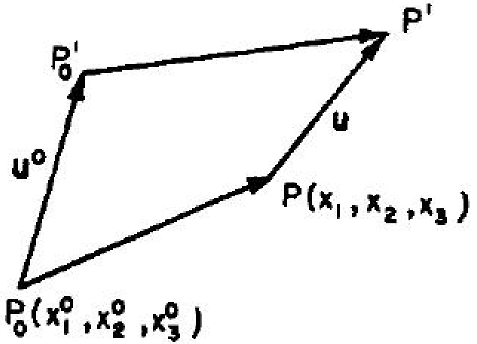
$$\delta_2 = \varepsilon_{AB} l = \frac{\varepsilon}{3} l = \frac{p_0}{3K} l \Rightarrow \Delta V = \frac{Pl}{3K}$$

Finite strain (有限应变、大应变)***

Finite strain: the strain is not significantly smaller than 1 (usually choose ε>0.05)



For a small element P_0P, it moves to P'_0P' after deformation



$$u_i = u_i(x_i) = u_i(x'_i) = x'_i - x_i$$

$$dl^2 = |P_0P|^2 = dx_i dx_i$$

$$dl'^2 = |P'_0 P'|^2 = dx'_i dx'_i$$

$$dl'^2 = (dx_i + du_i)(dx_i + du_i)$$

Finite strain (有限应变、大应变)***

We now check dl'^2-dl^2:

$$dl^2 = dx_i dx_i$$

$$dl'^2 = (dx_i + du_i)(dx_i + du_i) = (dx_i + \frac{\partial u_i}{\partial x_m} dx_m)(dx_i + \frac{\partial u_i}{\partial x_n} dx_n) \rightarrow dl'^2 - dl^2 = [\frac{\partial u_i}{\partial x_m}\frac{\partial u_i}{\partial x_n} + \frac{\partial u_m}{\partial x_n} + \frac{\partial u_n}{\partial x_m}] dx_m dx_n$$

Define Green-Lagrangian strain tensor E_{mn}: $$E_{mn} = \frac{dl'^2 - dl^2}{2dx_m dx_n} = \frac{1}{2}(\frac{\partial u_i}{\partial x_m}\frac{\partial u_i}{\partial x_n} + \frac{\partial u_m}{\partial x_n} + \frac{\partial u_n}{\partial x_m})$$

For small strain: $$E_{mn} = \frac{1}{2}(\frac{\partial u_i}{\partial x_m}\frac{\partial u_i}{\partial x_n} + \frac{\partial u_m}{\partial x_n} + \frac{\partial u_n}{\partial x_m}) \approx \frac{1}{2}(\frac{\partial u_m}{\partial x_n} + \frac{\partial u_n}{\partial x_m}) = \varepsilon_{mn}$$ Cauchy strain

Similarly, define Almansi strain E'_{mn}: $$E'_{mn} = \frac{dl'^2 - dl^2}{2dx'_m dx'_n} = \frac{1}{2}(-\frac{\partial u_i}{\partial x'_m}\frac{\partial u_i}{\partial x'_n} + \frac{\partial u_m}{\partial x'_n} + \frac{\partial u_n}{\partial x'_m})$$

For small strain: $E'_{mn}=E_{mn}=\varepsilon_{mn}$

Classroom exercise

The stress and strain tensors can be decomposed into spherical and deviatoric stress and strain tensors.

$$\sigma_{ij} = \sigma_m \delta_{ij} + \sigma'_{ij} \qquad \varepsilon_{ij} = \varepsilon_m \delta_{ij} + \varepsilon'_{ij}$$

$$\sigma'_{ij} = \sigma_{ij} - \sigma_m \delta_{ij} \qquad \varepsilon'_{ij} = \varepsilon_{ij} - \varepsilon_m \delta_{ij}$$

Prove that the constitutive relation for spherical stress and strain tensors are:

$$\sigma_m \delta_{ij} = 3K \varepsilon_m \delta_{ij}$$

While the constitutive relation for deviatoric stress and strain tensors are,

$$\sigma'_{ij} = 2\mu \varepsilon'_{ij}$$

Homework (4 points)

1. Verify the tensor notations of the 2nd and 3rd stress invariants

$$I_1 = \tau_{11} + \tau_{22} + \tau_{33} = \tau_{ii}$$

$$I_2 = \tau_{11}\tau_{22} + \tau_{22}\tau_{33} + \tau_{33}\tau_{11} - \tau_{12}^2 - \tau_{23}^2 - \tau_{31}^2 = \frac{1}{2}(\tau_{ii}\tau_{kk} - \tau_{ik}\tau_{ki})$$

$$I_3 = \tau_{11}\tau_{22}\tau_{33} + 2\tau_{12}\tau_{23}\tau_{31} - \tau_{11}\tau_{23}^2 - \tau_{22}\tau_{31}^2 - \tau_{33}\tau_{12}^2 = \varepsilon_{ijk}\tau_{1i}\tau_{2j}\tau_{3k} = \boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2 \times \boldsymbol{\tau}_3 = \det(\boldsymbol{\tau}\mathrm{ij})$$

$\varepsilon_{ij} = \frac{1+\nu}{E}\tau_{ij} - \frac{\nu}{E}\delta_{ij}\Theta$ where $\Theta = \tau_{kk}$

2. Use the tensor notation of Hooke's law above to (1) establish the relation below and (2) establish the new alternative Hooke's law below

$$e_{ii} = \frac{1}{K}\frac{\tau_{ii}}{3} = \frac{1}{K}\frac{\Theta}{3}$$

K is the bulk modulus of elasticity

$\tau_{ij} = 2G\varepsilon_{ij} + \lambda\varepsilon\delta_{ij}$ where $\varepsilon = \varepsilon_{kk}$

Homework (4 points)

3. Given the governing equations of elasticity in the red box, derive the Navier's equation (equilibrium in the form of displacement) below:

Governing equations

$$\boxed{\begin{gathered} \tau_{ik,i} + f_k = 0 \\ \tau_{\mathrm{ij}} = \delta_{ij}\lambda e + 2Ge_{ij} \\ \varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) = \frac{1}{2}(u_{i,j} + u_{j,i}) \end{gathered}}$$

Navier's equation

$$(\lambda + G)u_{i,ik} + G\nabla^2 u_k + f_k = 0$$

Homework (4 points)

The rotation tensor w_{ik} is an antisymmetric tensor $w_{ik} = (\partial u_k/\partial x_i - \partial u_i/\partial x_k)/2$

Prove that the curl of displacement u_k : $v_m = \varepsilon_{mik} \nabla_i u_k$ is related to the rotation tensor w_{ik}

$$v_m = \varepsilon_{mik} w_{ik} = \varepsilon_{mik} (\frac{\partial u_k}{\partial x_i} - \frac{\partial u_i}{\partial x_k})/2$$

$$w_{ik} = \varepsilon_{mik} (\frac{1}{2} v_m) = \varepsilon_{ikm} (\frac{1}{2} v_m) = \frac{1}{2} \varepsilon_{ikm} \varepsilon_{mqr} \frac{\partial u_r}{\partial x_q}$$

Note: the alternating tensor and Kronecker delta have the following relation:

$$\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

Prove that (1) the infinitesimal rotation angle vector $\boldsymbol{w}_m$ equals half the curl of displacement u_k $\quad \mathrm{w}_m = \frac{1}{2} \xi_{mik} \nabla_i \mathrm{u}_k$

(2) rotation displacement can be calculated with cross product between $\boldsymbol{w}_m$ and $\boldsymbol{dx}$

$$d\mathbf{u} = d\mathbf{u}_D + d\mathbf{u}_R \qquad d\mathbf{u}_R = \omega_{ij} dx_j = \mathbf{w}_m \times d\mathbf{x}$$

Deformation Rotation

infinitesimal rotation angle vector

Reading material

The indicial notation (tensor notation) is a shorthand for equations and physical quantities written in terms of Cartesian coordinates, it represents a convenient method of condensing algebraic, differential and integral equations. The use of indicial notation does not require the definition of any new algebraic operations (代数运算). With a few simple conventions, we can readily write most of our equations in tensor notation. This is in sharp contrast to vector notation, which requires a redefinition of the operations of addition, subtraction, and multiplication, and becomes increasingly unmanageable for higher-order tensors due to the large number of operations needing to be defined.

In cases, one may have trouble in digesting the physical meaning of an equations written in indicial notation, or he may have doubts in the mathematical operations. In these cases, one should not hesitate to write the equation in expanded (unabridged component) form.

There are, however, some drawbacks in using Cartesian tensor notation. Equations derived with Cartesian tensor notation cannot be changed readily into curvilinear coordinates. One need to derive all the equations involving spacial derivatives in each particular curvilinear coordinate system separately, which is obviously very tedious. Therefore, in treating equations in curvilinear coordinates, only general tensor and symbolic (vector and dyadic) approaches are practical. The great advantage of the symbolic method is that relations expressed in this way are independent of any coordinate system.