Review

Definition of a vector based on coordinate transformation:

Direction cosines $a_{ij} = \cos(x_i, x'_j)$

$$F'_j = a_{ij} F_i \quad \text{also} \quad F_i = a_{ij} F'_j$$

Definition of a second-order tensor A_{ik}:

$$A'_{jl} = a_{ij} a_{kl} A_{ik}$$

- Scalar is a 0_th order tensor, vector is a 1st order tensor.
- Plus, minus, and product (dot product, cross product, outer product (外积、并矢积)) of two tensors are still tensors
  - vectors u_i and v_k, $u_i v_k$ is a 2nd-order tensor.
- Gradient of a nth-order tensor gets an (n+1)th-order tensor
- Kronecker delta (克罗内克符号) $\delta_{ik} = x_{i,k} = a_{ij} a_{kj} = a_{ji} a_{jk}$
  - Kronecker delta is a **isotropic 2nd-order tensor.**
  - Kronecker delta is also called the substitution tensor（替换张量）

$$\delta_{ik} u_k = u_i$$

The alternating tensor (交错张量, permutation tensor，置换张量/符号)

- The alternating tensor ε_{ikm}

$$\varepsilon_{ikm} = \begin{cases} +1 & \text{If the subscripts } i, k, m \text{ are in cyclic order of 1 2 3} \\ -1 & \text{If the subscripts } i, k, m \text{ are in noncyclic order of 1 2 3} \\ 0 & \text{any of the subscripts are equal} \end{cases}$$

- ε_{ikm} is an **isotropic third-order tensor**,
- The sign of the alternating tensor ε_{ikm} is changed by interchanging any two of the indices

  e.g., $\varepsilon_{ikm} = - \varepsilon_{kim}$

- Properties of the alternating tensor:
  – we can obtain an 2nd_order antisymmetric tensor by combining the alternating tensor with any vector

$$w_{ik} = \varepsilon_{ikm} u_m = \varepsilon_{ik1} u_1 + \varepsilon_{ik2} u_2 + \varepsilon_{ik3} u_3$$

$$= \begin{cases} \mathrm{u_m} & \text{i, k, m in cyclic order of 123} \\ \mathrm{-u_m} & \text{i, k, m in noncyclic orders} \\ 0 & \text{otherwise} \end{cases}$$

matrix form

$$w_{ik} = \begin{bmatrix} 0 & u_3 & -u_2 \\ -u_3 & 0 & u_1 \\ u_2 & -u_1 & 0 \end{bmatrix}$$

The alternating tensor (交错张量，置换符号)

- Properties of the alternating tensor:
  - we can always obtain an 2nd_order antisymmetric tensor by combining the alternating tensor with any vector
  - we can get a vector by combining the alternating tensor with any 2nd_order antisymmetric tensor

$$u_m = \varepsilon_{ikm} w_{ik} = \varepsilon_{12m} w_{12} + \varepsilon_{21m} w_{21} + \varepsilon_{32m} w_{32} + \varepsilon_{13m} w_{13} + \varepsilon_{23m} w_{23} + \varepsilon_{31m} w_{31}$$

$$u_1 = w_{23} - w_{32}$$
$$u_2 = w_{31} - w_{13}$$
$$u_3 = w_{12} - w_{21}$$

the components of u_m are twice those of w_{ik}

Show that $\varepsilon_{ikm} w_{ik}$ and $\varepsilon_{ikm} w_{km}$ represent the same vector

  - vector $\varepsilon_{mik} u_i v_k$ represents cross (vector) product (叉乘、矢量积) $\mathbf{u} \times \mathbf{v}$

$$w_m = \epsilon_{mik} u_i v_k$$

$$w_1 = u_2 v_3 - u_3 v_2$$
$$w_2 = u_3 v_1 - u_1 v_3$$
$$w_3 = u_1 v_2 - u_2 v_1$$

The alternating tensor (交错张量，置换符号)

- Properties of the alternating tensor:
  - we can always obtain an 2nd_order antisymmetric tensor by combining the alternating tensor with any vector
  - we can produce a vector by combining the alternating tensor with any 2nd_order antisymmetric tensor
  - vector $\varepsilon_{mik} u_i v_k$ represents vector (cross) product (矢量积、叉乘) $\mathbf{u} \times \mathbf{v}$
  - $\varepsilon_{mik} \nabla_i u_k$ gives the curl of u_k

$$v_m = \nabla \times u = \varepsilon_{mik} \nabla_i u_k = \varepsilon_{mik} \frac{\partial u_k}{\partial x_i}$$

$$v_1 = \frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}$$

$$v_2 = \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}$$

$$v_3 = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}$$

The alternating tensor (交错张量，置换符号)

- Properties of the alternating tensor:
  - we can always obtain an 2nd_order antisymmetric tensor by combining the alternating tensor with any vector
  - we can get a vector by combining the alternating tensor with any 2nd_order antisymmetric tensor
  - vector $\varepsilon_{mik} u_i v_k$ represents vector (cross) product (矢量积、叉乘) $\mathbf{u} \times \mathbf{v}$
  - $\varepsilon_{mik} \nabla_i u_k$ gives the curl of u_k
  - scalar $\varepsilon_{ikm} u_i v_k w_m$ represents the mixed triple product (混合三重积) $\boldsymbol{u} \cdot (\boldsymbol{v} \times \boldsymbol{w})$

$$\epsilon_{ikm} u_i v_k w_m = u_1 v_2 w_3 + u_2 v_3 w_1 + u_3 v_1 w_2 - u_2 v_1 w_3 - u_3 v_2 w_1 - u_1 v_3 w_2 = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

volume of a (parallelepiped)

The determinant of matrix $\boldsymbol{A}$, det ($\boldsymbol{A}$), can be expressed as $\varepsilon_{ikm} A_{1i} A_{2k} A_{3m} = \varepsilon_{ikm} A_{i1} A_{k2} A_{m3}$

Classroom exercise

if A_{ijkl} is the 81 components of a 4th-order tensor in the xyz coordinate, give its components A'_{pqrs} in the x'y'z' coordinate

8-10 Prove the following formulas:

$$\delta_{ik}\epsilon_{ikm} = 0$$
$$\epsilon_{ijk}\epsilon_{ijk} = 6$$
$$\epsilon_{ijp}\epsilon_{ijq} = 2\delta_{pq}$$

8-13 Are the nine direction cosines a_{ij} components of a second-order tensor?

yes $\quad a_{ij} = \dfrac{\partial x'_j}{\partial x_i}$

The coordinate transformation tensor a_{ij} is a rotation tensor（旋转张量）, a special kind of orthogonal tensor (正交张量), i.e.

$a_{ik}a_{jk} = \delta_{ij}$ $\quad \mathbf{a}\mathbf{a}^{\mathrm{T}} = \mathbf{I}$ (matrix form)

Show that det(a_{ij}) = 1

The Theorem of Gauss (高斯定理)

The theorem of Gauss (高斯定理、高斯散度定理): is a theorem which converts volume integrals to surface integrals, or vice versa.

$$\int_V \frac{\partial}{\partial x_i} A_{jk\ell\ldots} \, dV = \int_S \mu_i A_{jk\ell\ldots} \, dS$$

where S is the surface area of a region enclosing volume V, A_{jkl}... is a tensor of any order, and μ_i is the unit outward vector normal to S.

examples:

(a) $$\int_V \operatorname{grad} \phi \, dV = \int_S \boldsymbol{\mu} \phi \, dS$$

(b) $$\int_V \operatorname{div} \mathbf{U} \, dV = \int_S \boldsymbol{\mu} \cdot \mathbf{U} \, dS$$

Contravariant (逆变) and Covariant (协变) tensors***

A tensor in a general space may have contravariant (逆变， upper) or covariant (协变，lower) indices.

e.g., a_i^{jk} ← contravariant ← covariant

Contravariant second-rank tensors transform as

$$A'^{\,ij} = \frac{\partial x'_i}{\partial x_k}\,\frac{\partial x'_j}{\partial x_l}\,A^{kl}$$

The contravariant and covariant indices are equivalent for Cartesian tensors (tensors in 3D Euclidean space).

$$\frac{\partial x'_i}{\partial x_k}\frac{\partial x'_j}{\partial x_l} = \frac{\partial x_k}{\partial x'_i}\frac{\partial x_l}{\partial x'_j} = \frac{\partial x'_i}{\partial x_k}\frac{\partial x_l}{\partial x'_j} = a_{ki}a_{lj}$$

Covariant second-rank tensors transform as

$$C'_{ij} = \frac{\partial x_k}{\partial x'_i}\,\frac{\partial x_l}{\partial x'_j}\,C_{kl}$$

Thus we **do not discern between contravariant and covariant in this course.**

Mixed second-rank tensors transform as

$$B'_{\;j}{}^{i} = \frac{\partial x'_i}{\partial x_k}\,\frac{\partial x_l}{\partial x'_j}\,B^{k}{}_{l}$$

Homework (6 points)

8-11 Change the following integral theorems into Cartesian tensor notation. In these equations, dV is the elemental volume; dS the elemental surface area; dL the elemental length; $\boldsymbol{\mu}$ the unit vector along the outward normal direction of S or L.

(a) $$\int_V \operatorname{grad} \phi \, dV = \int_S \boldsymbol{\mu} \phi \, dS$$

(b) $$\int_V \operatorname{div} \mathbf{U} \, dV = \int_S \boldsymbol{\mu} \cdot \mathbf{U} \, dS$$

(c) $$\int_V \operatorname{curl} \mathbf{U} \, dV = \int_S \boldsymbol{\mu} \times \mathbf{U} \, dS$$

(d) $$\int_S \boldsymbol{\mu} \cdot \operatorname{curl} \mathbf{U} \, dS = \int_L \mathbf{U} \cdot d\mathbf{L}$$

(e) $$\int_V \nabla^2 \phi \, dV = \int_S \boldsymbol{\mu} \cdot \nabla \phi \, dS$$

Homework (6 points)

The integral form of the Maxwell's equations that specify the electric and magnetic fields and their time evolution for a given configuration are as below, deduce the differential form of these equations and express them with tensor notations.

Gauss's law for electricity (电场高斯定理) $\oiint_S \mathbf{E}\cdot d\mathbf{S} = \frac{Q}{\varepsilon_0}$ $\qquad \nabla\cdot\mathbf{E} = \frac{\rho}{\varepsilon_0}$

Gauss's law for magnetism (磁场高斯定理) $\oiint_S \mathbf{B}\cdot d\mathbf{S} = 0$ $\qquad \nabla\cdot\mathbf{B} = 0$

Faraday's law (法拉第电磁感应定理) $\oint_L \mathbf{E}\cdot d\mathbf{l} = -\iint_S \frac{\partial\mathbf{B}}{\partial t}\cdot d\mathbf{S}$ $\qquad \nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}$

Ampere's law (安培环路定理) $\oint_L \mathbf{B}\cdot d\mathbf{l} = \iint_S \mu_0(\mathbf{j}+\varepsilon_0\frac{\partial\mathbf{E}}{\partial t})\cdot d\mathbf{S}$ $\qquad \nabla\times\mathbf{B} = \mu_0(\mathbf{j}+\varepsilon_0\frac{\partial\mathbf{E}}{\partial t})$

Q is the amount of charge (电荷) enclosed by a closed surface and equals to

$Q = \iiint_V \rho dV$ $\quad \rho$ is bulk density of charge (charge per unit volume)

E: electrif field intensity (电场强度)

B: magnetic induction (磁感应强度)

j: current density (电流密度)

Homework (6 points)

8-2 Prove that $u_i v_j w_\ell$ is a third-order tensor, where u_i, v_j, and w_ℓ are arbitrary vectors.

8-12 Let u_i, u'_j, u''_k be the components of a vector in three coordinate systems, i.e., unprimed, primed, and double-primed coordinates, respectively. Show that the values of u''_k obtained by transforming from u_i are the same as those obtained by first transforming from u_i to u'_j and then from u'_j to u''_k. Hint: Use three different letters for the three sets of direction cosines, such as a_{ik}, b_{ij}, and c_{jk}.

Homework (6 points)

5:

The components of tensors A_{ij} and B_{ij} are as right.

(1) Calculate $A_{ij}B_{jk}$ and $A_{ij}B_{kj}$ and compare the results with the matrix multiplication $\boldsymbol{AB}$ and $\boldsymbol{AB}^T$.

(2) Calculate and compare $A_{ij}B_{ji}$ and $A_{ij}B_{ij}$

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}, B = \begin{pmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \\ 3 & 2 & 1 \end{pmatrix}$$

6:

The alternating tensor and Kronecker delta have the following relation:

$$\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

Prove the following identities with the relation above

$$\varepsilon_{ijk}\varepsilon_{ijk} = 6$$

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$$ (Lagrange formula)

Reading material

Tensors are a generalization of vectors. They are defined independently of any coordinate system or frame of reference. When the components of a tensor are specified in one coordinate system, they are determined in any other system (via a transformation law). Tensors can be found and used in many subjects. A known example is the stress tensor, which consists of the normal and shear stress components along three orthogonal planes.

Tensor notation provides a convenient and unified system for describing physical quantities. Scalars, vectors, second rank tensors (sometimes referred to loosely as tensors), and higher rank tensors can all be represented in tensor notation. In the most general representation, a tensor is denoted by a symbol followed by a collection of subscripts, e.g. a, X_j , σ_{ij} , U_l, β_{klm}, c_{ijkl} etc. The number of subscripts attached to a tensor defines the rank of the tensor. The number of subscripts ranges from zero (scalar) to four (4th-order tensor) in the above examples.

A tensor of rank zero (no subscripts) is nothing more than a familiar scalar. A tensor of rank one (one subscript) is simply a common vector. If we are working a problem in three-dimensional space, then the vector will contain three components. In the tensor notation, these three components are represented by stepping the subscripted index through the values 1,2, and 3. Thus an nth-order tensor has n subscripts and 3^n components.