Continuum Mechanics (B)

Session 06: Apply Cartesian Tensor to Elasticity

Lecturer: Ting Yang 杨亭
Department: ESS 地空系
E-mail: yangt3@sustech.edu.cn

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Stress Tensor

Index notation of stress as τ_{ik}, or σ_{ik}

$$\begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix} = \tau_{ik} = \begin{bmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{bmatrix}$$

- The first subscript i indicates the plane on which the stress acts
- The second subscript k gives the direction of the component

Now we prove that stress is a second-order tensor:

$$\boxed{\tau'_{jn} = a_{ij} a_{kn} \tau_{ik}}$$

Stress Tensor

Stress equilibrium of a tetrahedron in each direction

$$p_k = \tau_{ik} a_{i1} \qquad \mathbf{p} = \boldsymbol{\tau}^T \mathbf{n} = \boldsymbol{\tau}\mathbf{n}$$

a_{i1} is the normal unit vector of the plane x'.

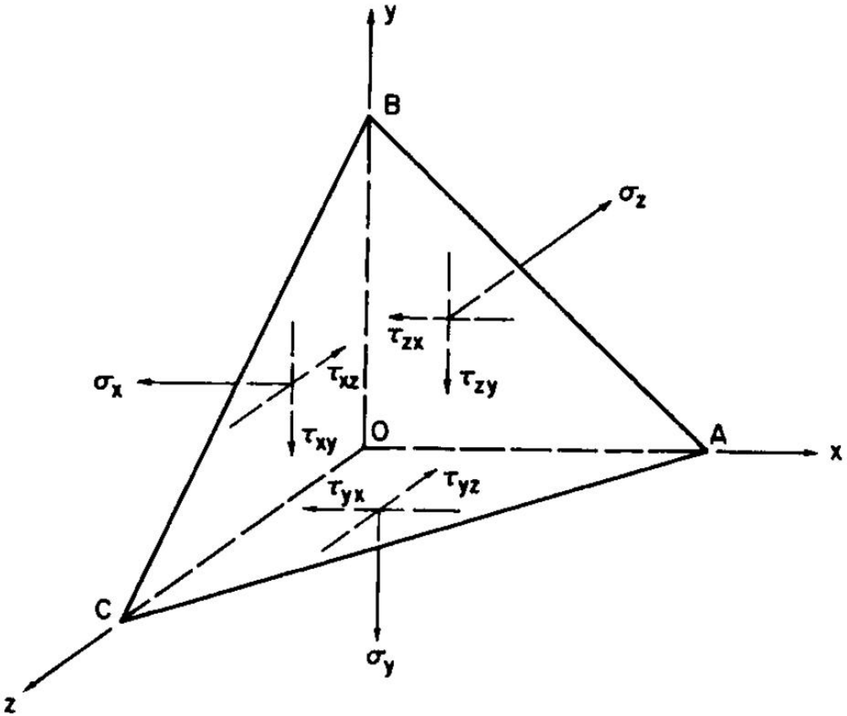
$$\tau'_{1n} = p_k a_{kn} = \tau_{ik} a_{i1} a_{kn}$$

$$\tau'_{jn} = \tau_{ik} a_{ij} a_{kn} = a_{ij} a_{kn} \tau_{ik}$$

So stress is a second-order tensor.

Since $\tau_{ij} = \tau_{ji}$, stress is a **symmetric second-order tensor**

Stress components on a tetrahedron



stress vector on plane ABC: (p_x, p_y, p_z)

Stress Tensor

The stress vector on the plane with $\boldsymbol{\mu}$ as the normal unit vector τ_k^{μ} can be expressed as

$$\tau_k^{\mu} = p_k = \tau_{ik}\mu_i \qquad \mathbf{p} = \boldsymbol{\tau}^T\mathbf{n} = \boldsymbol{\tau}\mathbf{n}$$

The stress component on the μ plane in the ν direction is equal to the stress component on the ν plane in the μ direction

i.e., $\tau_{ij} = \tau_{ji}$ is a special case for this rule

$$\begin{aligned}\boldsymbol{\tau}^{\mu}\cdot\boldsymbol{\nu} &= \tau_i^{\mu}\nu_i = \tau_{ji}\mu_j\nu_i \\ &= \tau_{ij}\nu_i\mu_j \\ &= \tau_j^{\nu}\mu_j = \boldsymbol{\tau}^{\nu}\cdot\boldsymbol{\mu}\end{aligned}$$

Stress Tensor: principal axes of the stress tensor

Principal axes of the stress tensor:

if **μ** is the principal axes of the stress tensor, then the stress vector acting on the surface defined by **μ** is parallel to **μ**:

$$\tau_j^{\mu} = \tau\mu_j \quad \rightarrow \quad (\tau_{ij} - \tau\delta_{ij})\mu_i = 0$$

$$\rightarrow \begin{vmatrix} \tau_{11}-\tau & \tau_{21} & \tau_{31} \\ \tau_{12} & \tau_{22}-\tau & \tau_{32} \\ \tau_{13} & \tau_{23} & \tau_{33}-\tau \end{vmatrix} = 0 \quad \text{or} \quad \tau^3 - I_1\tau^2 + I_2\tau - I_3 = 0$$

the coefficients I_1, I_2, I_3 must be invariant quantities (**The three stress invariants**): because the principal stress must be the same when referred to any coordinate system.

$$I_1 = \tau_{11} + \tau_{22} + \tau_{33} = \tau_{ii}$$

$$I_2 = \tau_{11}\tau_{22} + \tau_{22}\tau_{33} + \tau_{33}\tau_{11} - \tau_{12}^2 - \tau_{23}^2 - \tau_{31}^2 = \frac{1}{2}\left(\tau_{ii}\tau_{kk} - \tau_{ik}\tau_{ki}\right)$$

$$I_3 = \tau_{11}\tau_{22}\tau_{33} + 2\tau_{12}\tau_{23}\tau_{31} - \tau_{11}\tau_{23}^2 - \tau_{22}\tau_{31}^2 - \tau_{33}\tau_{12}^2 = \frac{1}{6}\left(2\tau_{ij}\tau_{jk}\tau_{ki} - 3\tau_{ij}\tau_{ji}\tau_{kk} + \tau_{ii}\tau_{jj}\tau_{kk}\right)$$

$$= \frac{1}{6}\left(\varepsilon_{ijk}\varepsilon_{pqr}\tau_{ip}\tau_{jq}\tau_{kr}\right)$$

Stress Tensor: principal axes of the stress tensor

the stress transformation equation

$$\tau'_{jn} = a_{ij} a_{kn} \tau_{ik}$$

can be expressed in terms of the principal stresses:

$$\tau'_{ij} = \tau_I \mu_i^I \mu_j^I + \tau_{II} \mu_i^{II} \mu_j^{II} + + \tau_{III} \mu_i^{III} \mu_j^{III}$$

τ_I, τ_{II}, τ_{III} represents the principal stresses
μ_i^{II} represents the directional cosine between the new plane i and the principal axis direction of τ_{II}
μ_j^{II} represents the directional cosine between the stress component direction j and the principal axis direction of τ_{II}

Represent the matrix rotation with eigenvalues and eigenvectors

Stress Tensor: stress ellipsoid

Components of the stress vector on any x' plane is

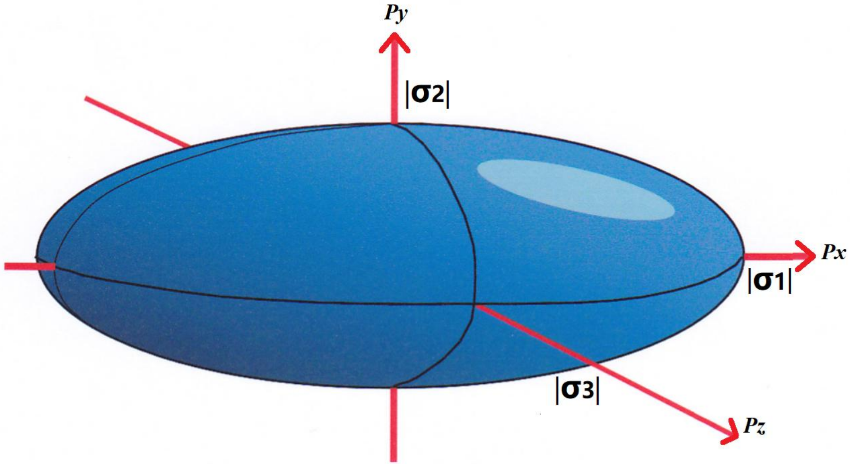
$$p_x = \tau_{i1} a_{i1} \quad = \sigma_1 a_{11}$$
$$p_y = \tau_{i2} a_{i1} \quad = \sigma_2 a_{21}$$
$$p_z = \tau_{i3} a_{i1} \quad = \sigma_3 a_{31}$$

when xyz coincide with principal axes

$$a_{i1}^2 = 1 \quad \rightarrow \quad \frac{p_x^2}{\sigma_1^2} + \frac{p_y^2}{\sigma_2^2} + \frac{p_z^2}{\sigma_3^2} = 1$$

Given any $\boldsymbol{x}'=(a_{11}, a_{21}, a_{31})$, we have (p_x, p_y, p_z) on the ellipsoid

The stress vectors on all inclinations compose the surface of an ellipsoid (the ellipsoid of Lamé, or the **stress ellipsoid**).



The Stress Ellipsoid of Lame (xyz coordinate in the principal stress directions)

For magnitude of stress vectors of all planes at a given point, σ_1 is the maximum one and σ_3 is the minimum one.

Stress Tensor: prove that stress is a second-order tensor

$$\tau'_{jn} = a_{ij} a_{kn} \tau_{ik}$$

Verify right stress expressions by setting $j = 1$ in the stress transformation equation.

3D stress transformation

$$\sigma_{x'} = \sigma_x a_{11}^2 + \sigma_y a_{21}^2 + \sigma_z a_{31}^2 + 2\tau_{xy} a_{11} a_{21} + 2\tau_{yz} a_{21} a_{31} + 2\tau_{zx} a_{31} a_{11}$$

$$\begin{aligned} \tau_{x'y'} = {} & \sigma_x a_{11} a_{12} + \sigma_y a_{21} a_{22} + \sigma_z a_{31} a_{32} \\ & + \tau_{xy}(a_{11} a_{22} + a_{21} a_{12}) \\ & + \tau_{yz}(a_{21} a_{32} + a_{31} a_{22}) \\ & + \tau_{zx}(a_{31} a_{12} + a_{11} a_{32}) \end{aligned}$$

$$\begin{aligned} \tau_{x'z'} = {} & \sigma_x a_{11} a_{13} + \sigma_y a_{21} a_{23} + \sigma_z a_{31} a_{33} \\ & + \tau_{xy}(a_{11} a_{23} + a_{21} a_{13}) \\ & + \tau_{yz}(a_{21} a_{33} + a_{31} a_{23}) \\ & + \tau_{zx}(a_{31} a_{13} + a_{11} a_{33}) \end{aligned}$$