南方科技大学
SOUTHERN UNIVERSITY OF SCIENCE AND TECHNOLOGY

MAE5009
Continuum Mechanics B
Session 12: Fluid Dynamics

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Fall 2021

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Analysis approaches

- **Basic physical laws**:
  - Conservation of mass
  - Newton's second law (conservation of momentum, moment of momentum)
  - First law of thermodynamics (conservation of energy)
    - For incompressible fluid flow, only the first two will be used
    - For compressible fluid flow, all three may be needed

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System and control volume

- **System**:
  - A specified collection of fluid particles
  - Corresponding to Lagrangian approach
- **Control volume**:
  - A fixed volume in space
  - Could be a real or imaginary volume
  - Corresponding to Eulerian approach

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Integral and differential approaches

- **Differential approach**:
  - The basic equations are written in differential form
  - Reflects parameter change in neighborhood areas
  - Gives detailed physical parameter change in space
  - Differential equations are difficult to solve
- **Integral approach:**
  - The basic equations are written in integral form
  - Reflects parameter change in finite control volume
  - Does not need to know details of the fluid field

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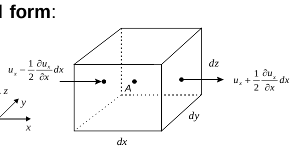
Continuity equation

- The mass of fluid flows into a specific control volume during a unit time span should be equal to the summation of fluid mass flowed out and the increased fluid mass in the control volume
- **Differential form**:



Left: $u_x - \frac{1}{2}\frac{\partial u_x}{\partial x}dx, \rho - \frac{1}{2}\frac{\partial \rho}{\partial x}dx$

Right: $u_x + \frac{1}{2}\frac{\partial u_x}{\partial x}dx, \rho + \frac{1}{2}\frac{\partial \rho}{\partial x}dx$

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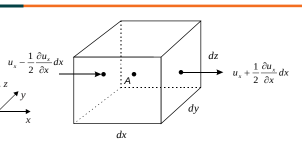
Continuity equation



The mass flows in the control volume per unit time from left:

$$\left(\rho - \frac{1}{2}\frac{\partial \rho}{\partial x}dx\right)\left(u_x - \frac{1}{2}\frac{\partial u_x}{\partial x}dx\right)dydz$$

The mass flows out the control volume per unit time from right:

$$\left(\rho + \frac{1}{2}\frac{\partial \rho}{\partial x}dx\right)\left(u_x + \frac{1}{2}\frac{\partial u_x}{\partial x}dx\right)dydz$$

The net mass flows per unit time in the control volume in *x* direction:

$$\left(\rho - \frac{1}{2}\frac{\partial \rho}{\partial x}dx\right)\left(u_x - \frac{1}{2}\frac{\partial u_x}{\partial x}dx\right)dydz - \left(\rho + \frac{1}{2}\frac{\partial \rho}{\partial x}dx\right)\left(u_x + \frac{1}{2}\frac{\partial u_x}{\partial x}dx\right)dydz = -\frac{\partial(\rho u_x)}{\partial x}dxdydz$$

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Continuity equation

The net mass flows in the control volume in *y* and *z* direction:

$$-\frac{\partial(\rho u_y)}{\partial y}dxdydz \qquad -\frac{\partial(\rho u_z)}{\partial z}dxdydz$$

The overall net mass flow into the control volume:

$$-\left(\frac{\partial(\rho u_x)}{\partial x}+\frac{\partial(\rho u_y)}{\partial y}+\frac{\partial(\rho u_z)}{\partial z}\right)dxdydz$$

The overall mass increase in the control volume:

$$\lim_{\Delta t\to 0}\frac{\rho(x,y,z,t+\Delta t)dxdydz-\rho(x,y,z,t)dxdydz}{\Delta t}=\frac{\partial\rho}{\partial t}dxdydz$$

$$\Downarrow$$

$$\frac{\partial\rho}{\partial t}+\frac{\partial(\rho u_x)}{\partial x}+\frac{\partial(\rho u_y)}{\partial y}+\frac{\partial(\rho u_z)}{\partial z}=0$$

$$\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\mathbf{u})=0 \qquad \frac{\partial\rho}{\partial t}+\mathrm{div}(\rho\mathbf{u})=0 \qquad \frac{\partial\rho}{\partial t}+(\rho u_i)_{,i}=0$$

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Continuity equation

- For steady flow:

$$\frac{\partial\rho}{\partial t}=0 \qquad \frac{\partial(\rho u_x)}{\partial x}+\frac{\partial(\rho u_y)}{\partial y}+\frac{\partial(\rho u_z)}{\partial z}=0$$

Masses of fluid flowing in and out are equal

- For incompressible flow:

$$\rho=\mathrm{const} \qquad \frac{\partial u_x}{\partial x}+\frac{\partial u_y}{\partial y}+\frac{\partial u_z}{\partial z}=0$$

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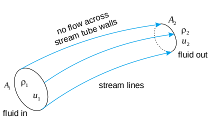
Continuity equation

- **Integral form – steady flow**:

For steady flow, the position and shape of stream tube will not change with time, then the fluid mass inside the control volume is constant:



$$\int_{A_1}\rho_1 u_1 dA_1=\int_{A_2}\rho_2 u_2 dA_2$$

$$\Downarrow$$

$$\rho_1\bar{u}_1 A_1=\rho_2\bar{u}_2 A_2 \qquad \bar{u}=\frac{Q}{A}$$

For impressible fluid:

$$\rho_1=\rho_2=\mathrm{const} \qquad \bar{u}_1 A_1=\bar{u}_2 A_2=Q$$

When flow rate is constant, the smaller the effective area, the greater the flow velocity

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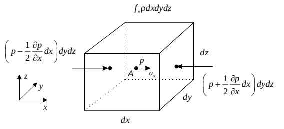
Differential equation of motion of ideal flow

- Ideal fluid has no viscosity



$$\left(p-\frac{1}{2}\frac{\partial p}{\partial x}dx\right)dydz-\left(p+\frac{1}{2}\frac{\partial p}{\partial x}dx\right)dydz+f_x\rho dxdydz=\rho dxdydz\frac{du_x}{dt}$$

$$f_x-\frac{1}{\rho}\frac{\partial p}{\partial x}=\frac{du_x}{dt}$$

$$f_y-\frac{1}{\rho}\frac{\partial p}{\partial y}=\frac{du_y}{dt}$$

$$f_z-\frac{1}{\rho}\frac{\partial p}{\partial z}=\frac{du_z}{dt}$$

$$\mathbf{f}-\frac{1}{\rho}\nabla p=\frac{d\mathbf{u}}{dt}$$

Euler's equation

This equation is appropriate for both compressible and incompressible fluid

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Differential equation of ideal flow

$$f_x-\frac{1}{\rho}\frac{\partial p}{\partial x}=\frac{du_x}{dt}=\frac{\partial u_x}{\partial t}+u_x\frac{\partial u_x}{\partial x}+u_y\frac{\partial u_x}{\partial y}+u_z\frac{\partial u_x}{\partial z}$$

$$f_y-\frac{1}{\rho}\frac{\partial p}{\partial y}=\frac{du_y}{dt}=\frac{\partial u_y}{\partial t}+u_x\frac{\partial u_y}{\partial x}+u_y\frac{\partial u_y}{\partial y}+u_z\frac{\partial u_y}{\partial z}$$

$$f_z-\frac{1}{\rho}\frac{\partial p}{\partial z}=\frac{du_z}{dt}=\frac{\partial u_z}{\partial t}+u_x\frac{\partial u_z}{\partial x}+u_y\frac{\partial u_z}{\partial y}+u_z\frac{\partial u_z}{\partial z}$$

$$\mathbf{f}-\frac{1}{\rho}\nabla p=\frac{d\mathbf{u}}{dt}=\frac{\partial \mathbf{u}}{\partial t}+(\mathbf{u}\cdot\nabla)\mathbf{u}$$

Four unknows for ideal & incompressible flow: u_x, u_y, u_z, p

*Plus one continuity equations, theoretically, we can solve the unknows for **ideal** & **incompressible** fluid flow problems*

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Differential equation of ideal flow

$$\begin{aligned} f_x-\frac{1}{\rho}\frac{\partial p}{\partial x}=\frac{du_x}{dt}&=\frac{\partial u_x}{\partial t}+u_x\frac{\partial u_x}{\partial x}+u_y\frac{\partial u_x}{\partial y}+u_z\frac{\partial u_x}{\partial z}\\ &=\frac{\partial u_x}{\partial t}+\left(u_x\frac{\partial u_x}{\partial x}+u_y\frac{\partial u_y}{\partial x}+u_z\frac{\partial u_z}{\partial x}\right)+u_y\frac{\partial u_x}{\partial y}-u_y\frac{\partial u_y}{\partial x}+u_z\frac{\partial u_x}{\partial z}-u_z\frac{\partial u_z}{\partial x}\\ &=\frac{\partial u_x}{\partial t}+\frac{\partial}{\partial x}\left(\frac{u_x^2+u_y^2+u_z^2}{2}\right)-u_y\left(\frac{\partial u_y}{\partial x}-\frac{\partial u_x}{\partial y}\right)+u_z\left(\frac{\partial u_x}{\partial z}-\frac{\partial u_z}{\partial x}\right)\\ &=\frac{\partial u_x}{\partial t}+\frac{\partial}{\partial x}\left(\frac{u^2}{2}\right)+2\left(u_z\omega_y-u_y\omega_z\right)\end{aligned}$$

where $u^2=u_x^2+u_y^2+u_z^2$

$$f_x-\frac{1}{\rho}\frac{\partial p}{\partial x}-\frac{\partial}{\partial x}\left(\frac{u^2}{2}\right)=\frac{\partial u_x}{\partial t}+2\left(u_z\omega_y-u_y\omega_z\right)$$

$$f_y-\frac{1}{\rho}\frac{\partial p}{\partial y}-\frac{\partial}{\partial y}\left(\frac{u^2}{2}\right)=\frac{\partial u_y}{\partial t}+2\left(u_x\omega_z-u_z\omega_x\right)$$

$$f_z-\frac{1}{\rho}\frac{\partial p}{\partial z}-\frac{\partial}{\partial z}\left(\frac{u^2}{2}\right)=\frac{\partial u_z}{\partial t}+2\left(u_y\omega_x-u_x\omega_y\right)$$

$$\mathbf{f}-\frac{1}{\rho}\nabla p-\nabla\left(\frac{u^2}{2}\right)=\frac{\partial \mathbf{u}}{\partial t}+2\left(\boldsymbol{\omega}\times\mathbf{u}\right)$$

Lamb's equation

For unspinning flow, $\boldsymbol{\omega}=0$, *the second component on the RHS is zero*

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Differential equation of ideal flow

- **Potential body force**:
  - If the body force vector **f** can be described by the gradient of a scalar function ($-\pi$), i.e.

$$\mathbf{f} = \nabla(-\pi)$$

$-\pi$ is the body force potential function

If only gravity exists, we have

$$\pi = gz$$
$$\mathbf{f} = 0\mathbf{i} + 0\mathbf{j} - g\mathbf{k}$$

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Differential equation of ideal flow

- **Barotropic fluid**:
  - The density of fluid is a function of pressure only, i.e.

$$\rho = \rho(p)$$

  - Introducing a pressure function:

$$P_F = \int \frac{dp}{\rho(p)} \quad \text{i.e.} \quad dP_F = \frac{dp}{\rho}$$

Since we have:

$$dP_F = \frac{\partial P_F}{\partial x}dx + \frac{\partial P_F}{\partial y}dy + \frac{\partial P_F}{\partial z}dz \qquad dp = \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy + \frac{\partial p}{\partial z}dz$$

Therefore, we have

$$\frac{\partial P_F}{\partial x} = \frac{1}{\rho}\frac{\partial p}{\partial x}, \frac{\partial P_F}{\partial y} = \frac{1}{\rho}\frac{\partial p}{\partial y}, \frac{\partial P_F}{\partial z} = \frac{1}{\rho}\frac{\partial p}{\partial z} \quad \text{i.e.} \quad \nabla P_F = \frac{\nabla p}{\rho}$$

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Differential equation of ideal flow

- **Barotropic fluid**:
  - For incompressible fluid:

$$\rho = \text{const}$$

therefore

$$P_F = \frac{p}{\rho}$$

- **Baroclinic fluid**:
  - Density is a function of not only pressure, e.g.,

$$\rho = \rho(p, T)$$

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Differential equation of ideal flow

- **Velocity potential**:
  - If a flow is non-spinning, i.e.,

$$\boldsymbol{\omega} = 0$$

  - There must be a scalar function called the velocity potential function

$$\varphi(x, y, z, t)$$

which gives the velocity field by

$$\mathbf{u} = \nabla\varphi$$

Non-spinning flow is called potential flow

The introduce of velocity potential function φ reduces the number of unknows

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整个过程要懂得怎么去阐述，可能会出一个阐述题

Bernoulli integral

- If the body force of a fluid flow is potential function based, and the fluid is a barotropic fluid, **for steady flow**, we have

$$\frac{\partial \mathbf{u}}{\partial t} = 0$$

- The Lamb's equation becomes:

$$\mathbf{f} - \frac{1}{\rho}\nabla p - \nabla\left(\frac{u^2}{2}\right) = 2(\boldsymbol{\omega}\times\mathbf{u})$$

$$\downarrow$$

$$\nabla\left(-\pi - P_F - \frac{u^2}{2}\right) = 2(\boldsymbol{\omega}\times\mathbf{u})$$

$$\nabla\left(\pi + P_F + \frac{u^2}{2}\right) = -2(\boldsymbol{\omega}\times\mathbf{u})$$

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Bernoulli integral

- Based on the integration along streamlines, multiplying an infinitesimal sector on the streamline on the two sides of the equation:

$$\nabla\left(\pi + P_F + \frac{u^2}{2}\right)\cdot d\mathbf{s} = -2(\boldsymbol{\omega}\times\mathbf{u})\cdot d\mathbf{s}$$

where

$$d\mathbf{s} = dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k}$$

$$\downarrow$$

Since $d\mathbf{s} \perp (\boldsymbol{\omega}\times\mathbf{u})$

$$\nabla\left(\pi + P_F + \frac{u^2}{2}\right)\cdot d\mathbf{s} = 0$$

$$= \frac{\partial\left(\pi + P_F + \frac{u^2}{2}\right)}{\partial x}dx + \frac{\partial\left(\pi + P_F + \frac{u^2}{2}\right)}{\partial y}dy + \frac{\partial\left(\pi + P_F + \frac{u^2}{2}\right)}{\partial z}dz$$

$$= d\left(\pi + P_F + \frac{u^2}{2}\right) = 0$$

$$\downarrow$$

$$\pi + P_F + \frac{u^2}{2} = C_l$$

- Ideal fluid (compressible & incompressible)
- Body force is potential function based
- Barotropic fluid
- Steady flow
- Integral along streamlines

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Lagrangian integral

- If the body force of a fluid flow is potential function based, and the fluid is a barotropic fluid and, **the flow is non-spinning**, we have

$$\boldsymbol{\omega} = 0$$

- The Lamb's equation becomes:

$$\mathbf{f} - \frac{1}{\rho}\nabla p - \nabla\left(\frac{u^2}{2}\right) = \frac{\partial \mathbf{u}}{\partial t}$$

$$\downarrow$$

$$\nabla\left(-\pi - P_F - \frac{u^2}{2}\right) = \frac{\partial \mathbf{u}}{\partial t}$$

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla\left(\pi + P_F + \frac{u^2}{2}\right) = 0$$

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Lagrangian integral

- For non-spinning flow, there exists a velocity potential function

$$\mathbf{u} = \nabla\varphi$$

- Then,

$$\frac{\partial \mathbf{u}}{\partial t} = \frac{\partial(\nabla\varphi)}{\partial t} = \nabla\frac{\partial\varphi}{\partial t}$$

$$\downarrow$$

$$\nabla\left(\frac{\partial\varphi}{\partial t} + \pi + P_F + \frac{u^2}{2}\right) = 0$$

Expression in the parenthesis is independent of x, y and z

$$\downarrow$$

$$\frac{\partial\varphi}{\partial t} + \pi + P_F + \frac{u^2}{2} = C(t)$$

For steady flow, we have:

$$\pi + P_F + \frac{u^2}{2} = C$$

- Ideal fluid (compressible & incompressible)
- Body force is potential function based
- Barotropic fluid
- Non-spinning flow

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Bernoulli's equation

- For **steady** and **incompressible** fluid, if the body force only contains gravity, we have:

$$\pi = gz,\ P_F = \frac{p}{\rho}$$

- Then the Bernoulli integral becomes

$$z + \frac{p}{\rho g} + \frac{u^2}{2g} = C_l$$ along a streamline

$$z_1 + \frac{p_1}{\rho g} + \frac{u_1^2}{2g} = z_2 + \frac{p_2}{\rho g} + \frac{u_2^2}{2g}$$ at two points on a streamline

- *For ideal and incompressible flow, the energy equation is equivalent to the motion equation*
- *Bernoulli's equation reflects the energy conservation and conversion in fluid mechanics*

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Bernoulli's equation

- Along a streamline:

$$z+\frac{p}{\rho g}+\frac{u^2}{2g}=C_l$$

position head
gravity potential energy

pressure head
pressure potential energy

velocity head
kinetic energy

The overall energy is constant along a streamline

Suitable for:
- *Ideal fluid*
- *Incompressible fluid*
- *Body force is gravity*
- *Steady flow*
- *Along streamline*

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Bernoulli's equation

- Based on Lagrangian integral, on the whole fluid field:

$$z+\frac{p}{\rho g}+\frac{u^2}{2g}=C$$

position head
gravity potential energy

pressure head
pressure potential energy

velocity head
kinetic energy

The overall energy is constant on the whole fluid field

Suitable for:
- *Ideal fluid*
- *Incompressible fluid*
- *Body force is gravity*
- *Steady flow*
- *Non-spinning flow*

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Application of Bernoulli's equation

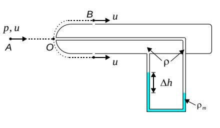
- **Pitot tube**:

A & O:

$$z_A+\frac{p}{\rho g}+\frac{u^2}{2g}=z_0+\frac{p_0}{\rho g}+\frac{u_0{}^2}{2g}$$

$$\Downarrow$$

$$p_0=p+\frac{1}{2}\rho u^2$$



O & B:

$$z_0+\frac{p_0}{\rho g}+\frac{u_0{}^2}{2g}=z_B+\frac{p_B}{\rho g}+\frac{u_B{}^2}{2g}$$

$$\Downarrow$$

$$p_0=p+\frac{1}{2}\rho u^2$$

$$\Rightarrow \quad p_0-p=\frac{1}{2}\rho u^2=(\rho_m-\rho)g\Delta h$$

$$\Downarrow$$

$$u=\sqrt{\frac{2(\rho_m-\rho)}{\rho}g\Delta h}$$

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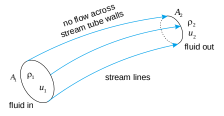
Total flow Bernoulli's equation

- The overall energy on the effective cross-section is constant:



$$\int_A \left(z + \frac{p}{\rho g} + \frac{u^2}{2g} \right) \rho dQ = \text{const}$$

$$\int_A \left(z + \frac{p}{\rho g} \right) \rho dQ + \int_A \frac{u^2}{2g} \rho dQ = \text{const}$$

For gradually varied flow:

$$z + \frac{p}{\rho g} = \text{const}$$

$$\int_A \frac{u^2}{2g} \rho dQ = \alpha \frac{\bar{u}^2}{2g} \rho Q \qquad \int_A \left(z + \frac{p}{\rho g} \right) \rho dQ = \left(z + \frac{p}{\rho g} \right) \rho \int_A dQ = \left(z + \frac{p}{\rho g} \right) \rho Q$$

$$\left(z + \frac{p}{\rho g} \right) \rho Q + \frac{\bar{u}^2}{2g} \rho Q = \text{const} \qquad z + \frac{p}{\rho g} + \frac{\bar{u}^2}{2g} = \text{const}$$

(along total flow)

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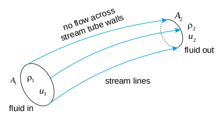
Total flow Bernoulli's equation

$$z_1 + \frac{p_1}{\rho g} + \frac{\bar{u}_1^2}{2g} = z_2 + \frac{p_2}{\rho g} + \frac{\bar{u}_2^2}{2g}$$

(along total flow)



Suitable for:
- *Ideal fluid*
- *Incompressible fluid*
- *Body force is gravity*
- *Steady flow*
- *Cross-sections are chosen in gradually varied flow areas*

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Momentum equation

- The previous **continuity** (mass) and **Bernoulli**'s (energy) equation cannot reflect the interactive forces between fluid and solid
- The increase of momentum in unit time is equal to the overall external forces:

$$\sum \mathbf{F} = \frac{m\mathbf{u}_2 - m\mathbf{u}_1}{\Delta t} = \frac{\Delta \mathbf{K}}{\Delta t}$$

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Momentum equation

- For steady and incompressible flow

$$\rho_1 = \rho_2 = \rho$$

- At time t:

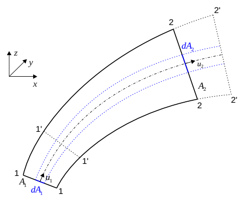
$$\mathbf{K}_{1-2} = \mathbf{K}_{1-1'} + \mathbf{K}_{1'-2}$$

- At time $t+\Delta t$:

$$\mathbf{K}_{1'-2'} = \mathbf{K}_{1'-2} + \mathbf{K}_{2-2'}$$

- The increase of momentum:

$$\Delta\mathbf{K} = \mathbf{K}_{1'-2'} - \mathbf{K}_{1-2} = \mathbf{K}_{2-2'} - \mathbf{K}_{1-1'}$$



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Momentum equation

- The momentum of fluid flows into 1-1 during Δt:

$$\rho \cdot u_1 dA_1 \Delta t \cdot \mathbf{u}_1$$

- The momentum of fluid flows out 2-2 during Δt:

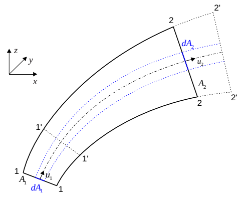
$$\rho \cdot u_2 dA_2 \Delta t \cdot \mathbf{u}_2$$

- Therefore:

$$\mathbf{K}_{1-1'} = \int_{A_1} \rho \cdot u_1 dA_1 \Delta t \cdot \mathbf{u}_1$$

$$\mathbf{K}_{2-2'} = \int_{A_2} \rho \cdot u_2 dA_2 \Delta t \cdot \mathbf{u}_2$$

$$\Delta\mathbf{K} = \mathbf{K}_{2-2'} - \mathbf{K}_{1-1'} = \int_{A_2} \rho \cdot u_2 dA_2 \Delta t \cdot \mathbf{u}_2 - \int_{A_1} \rho \cdot u_1 dA_1 \Delta t \cdot \mathbf{u}_1$$



The momentum increase of the fluid system has been converted to the momentum difference between the fluid mass flowing in and flowing out the control volume

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Momentum equation

- The overall external force:

$$\sum \mathbf{F} = \frac{\Delta\mathbf{K}}{\Delta t} = \int_{A_2} \mathbf{u}_2 \rho u_2 dA_2 - \int_{A_1} \mathbf{u}_1 \rho u_1 dA_1$$

Σ**F** represents the surface forces and body forces

- The integrations:

$$\int_A \mathbf{u} \rho u dA = \beta \bar{\mathbf{u}} \rho \bar{u} A = \rho Q \bar{\mathbf{u}} = \dot{m} \bar{\mathbf{u}}$$

- Therefore,

$$\sum \mathbf{F} = \dot{m}(\bar{\mathbf{u}}_2 - \bar{\mathbf{u}}_1) \rightarrow \begin{aligned} \sum F_x &= \rho Q(\bar{u}_{2x} - \bar{u}_{1x}) = \dot{m}(\bar{u}_{2x} - \bar{u}_{1x}) \\ \sum F_y &= \rho Q(\bar{u}_{2y} - \bar{u}_{1y}) = \dot{m}(\bar{u}_{2y} - \bar{u}_{1y}) \\ \sum F_z &= \rho Q(\bar{u}_{2z} - \bar{u}_{1z}) = \dot{m}(\bar{u}_{2z} - \bar{u}_{1z}) \end{aligned}$$

The overall external forces the control volume subjected to is equal to the momentum difference between the fluid mass flowing out and flowing in the control volume in unit time

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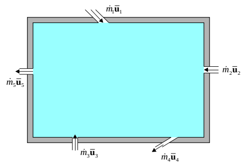
Momentum equation

- For steady and incompressible flow:

$$\sum \mathbf{F} = \left(\dot{m}_4\bar{\mathbf{u}}_4 + \dot{m}_5\bar{\mathbf{u}}_5\right) - \left(\dot{m}_1\bar{\mathbf{u}}_1 + \dot{m}_2\bar{\mathbf{u}}_2 + \dot{m}_3\bar{\mathbf{u}}_3\right)$$



Suitable for:
- *Steady flow*
- *Incompressible fluid*
- *Cross-sections are chosen in gradually varied flow areas*
- *No restriction on fluid type (ideal fluid or viscous fluid)*

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Example

- For steady and incompressible flow:

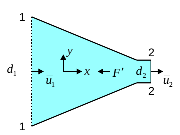
$$z_1 + \frac{p_1}{\rho g} + \frac{\bar{u}_1^{\,2}}{2g} = z_2 + \frac{p_a}{\rho g} + \frac{\bar{u}_2^{\,2}}{2g}$$

$$p_1 = p_a + \frac{\rho \bar{u}_2^{\,2}}{2} - \frac{\rho \bar{u}_1^{\,2}}{2}$$

$$p_{e1} = p_1 - p_a = \frac{\rho \bar{u}_2^{\,2}}{2} - \frac{\rho \bar{u}_1^{\,2}}{2}$$

$$p_{e1}A_1 - F' = \rho Q\left(\bar{u}_2 - \bar{u}_1\right)$$

$$F' = p_{e1}A_1 - \rho Q\left(\bar{u}_2 - \bar{u}_1\right)$$



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Example

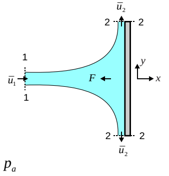
- For steady and incompressible flow:

$$z_1 + \frac{p_1}{\rho g} + \frac{\bar{u}_1^{\,2}}{2g} = z_2 + \frac{p_2}{\rho g} + \frac{\bar{u}_2^{\,2}}{2g}$$

$$p_1 + \frac{\rho \bar{u}_1^{\,2}}{2} = p_2 + \frac{\rho \bar{u}_2^{\,2}}{2}$$

$\bar{u}_1 = \bar{u}_2 = \bar{u}$ since $p_1 = p_2 = p_a$

$$-F = \rho Q\left(\bar{u}_{2x} - \bar{u}_{1x}\right) = \rho Q\left(0 - \bar{u}\right)$$



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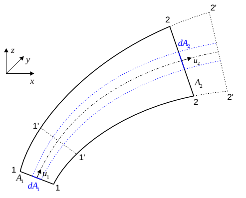
Angular momentum equation

- The increase of angular momentum in unit time is equal to the momentum of the overall force:

$$\mathbf{M} = \mathbf{r} \times \sum \mathbf{F} = m(\mathbf{r}_2 \times \mathbf{u}_2 - \mathbf{r}_1 \times \mathbf{u}_1)$$

- For steady and incompressible flow:

$$\mathbf{M} = \mathbf{r} \times \sum \mathbf{F} = \int_{A_2} (\mathbf{r}_2 \times \mathbf{u}_2) \rho u_2 dA_2 - \int_{A_1} (\mathbf{r}_1 \times \mathbf{u}_1) \rho u_1 dA_1 = \rho Q(\mathbf{r}_2 \times \bar{\mathbf{u}}_2 - \mathbf{r}_1 \times \bar{\mathbf{u}}_1)$$



The momentum of the overall external forces is equal to the angular momentum difference between the fluid mass flowing in and flowing out the control volume in unit time

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Example

$$z_1 + \frac{p_1}{\rho g} + \frac{\bar{u}_1^2}{2g} = z_2 + \frac{p_2}{\rho g} + \frac{\bar{u}_2^2}{2g}$$

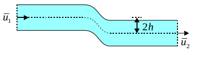
$$\downarrow$$

$$\bar{u}_1 = \bar{u}_2 = \bar{u}$$

$$\downarrow$$

$$\mathbf{M} = \rho Q(\mathbf{r}_2 \times \bar{\mathbf{u}}_2 - \mathbf{r}_1 \times \bar{\mathbf{u}}_1) = -\rho Q(\mathbf{r}_1 \times \bar{\mathbf{u}}_1)$$

$$M = \rho Q 2h\bar{u}$$



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Example

- 1. Bernoulli's equation

$$z_0 + \frac{p_0}{\rho g} + \frac{\bar{u}_0^2}{2g} = z_1 + \frac{p_1}{\rho g} + \frac{\bar{u}_1^2}{2g} = z_2 + \frac{p_2}{\rho g} + \frac{\bar{u}_2^2}{2g}$$

$$\bar{u}_0 = \bar{u}_1 = \bar{u}_2$$

- 2. Continuity equation:

$$\bar{u}_0 b_0 = \bar{u}_1 b_1 + \bar{u}_2 b_2 \rightarrow b_0 = b_1 + b_2$$

- 3. Momentum equation:

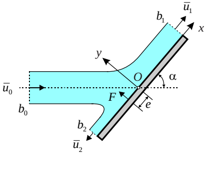
x: $0 = \rho b_1 \bar{u}_1 \bar{u}_1 - \rho b_2 \bar{u}_2 \bar{u}_2 - \rho b_0 \bar{u}_0 \bar{u}_0 \cos\alpha$ → $b_1 - b_2 - b_0 \cos\alpha = 0$

y: $F = 0 - \rho b_0 \bar{u}_0 \bar{u}_0 (-\sin\alpha) = \rho b_0 \bar{u}_0 \bar{u}_0 \sin\alpha$ → $F = \rho b_0 \bar{u}_0^2 \sin\alpha$

- 4. Angular momentum equation:

$$-Fe = -\rho b_1 \bar{u}_1 \bar{u}_1 \frac{b_1}{2} + \rho b_2 \bar{u}_2 \bar{u}_2 \frac{b_2}{2} - 0$$

$$e = \rho b_1 \bar{u}_1 \bar{u}_1 \frac{b_1}{2F} - \rho b_2 \bar{u}_2 \bar{u}_2 \frac{b_2}{2F} = \frac{\rho \bar{u}_0^2}{2F}(b_1^2 - b_2^2) = \frac{b_1^2 - b_2^2}{b_0 \sin\alpha} = \frac{b_0}{2} \cot\alpha$$



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Navier-Stokes equation

- Stress tensor in fluid flow:

$$\sigma_{ij} = \begin{bmatrix} \sigma'_x & \tau_{xy} & \tau_{xz} \\ & \sigma'_y & \tau_{yz} \\ sym. & & \sigma'_z \end{bmatrix} = \begin{bmatrix} \sigma_x - p & \tau_{xy} & \tau_{xz} \\ & \sigma_y - p & \tau_{yz} \\ sym. & & \sigma_z - p \end{bmatrix} = -p\delta_{ij} + \tau_{ij}$$

$$\tau_{ij} = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ & \sigma_y & \tau_{yz} \\ sym. & & \sigma_z \end{bmatrix}$$

τ_{ij} is the viscous stress tensor

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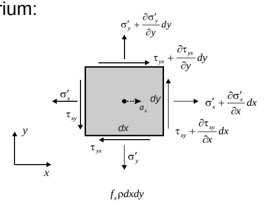
Navier-Stokes equation

- Force equilibrium:



$$\sum F_x = 0: \quad \left(\sigma'_x + \frac{\partial \sigma'_x}{\partial x} dx\right) dy - \sigma'_x dy + \left(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y} dy\right) dx - \tau_{yx} dx + f_x \rho dxdy = \rho dxdy \frac{du_x}{dt}$$

$$f_x + \frac{1}{\rho}\frac{\partial \sigma'_x}{\partial x} + \frac{1}{\rho}\frac{\partial \tau_{yx}}{\partial y} = \frac{du_x}{dt} \quad \rightarrow \quad f_x - \frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{1}{\rho}\left(\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y}\right) = \frac{du_x}{dt}$$

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Navier-Stokes equation

- 3D:

$$f_x - \frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{1}{\rho}\left(\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right) = \frac{du_x}{dt} = \frac{\partial u_x}{\partial t} + u_x\frac{\partial u_x}{\partial x} + u_y\frac{\partial u_x}{\partial y} + u_z\frac{\partial u_x}{\partial z}$$

$$f_y - \frac{1}{\rho}\frac{\partial p}{\partial y} + \frac{1}{\rho}\left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}\right) = \frac{du_y}{dt} = \frac{\partial u_y}{\partial t} + u_x\frac{\partial u_y}{\partial x} + u_y\frac{\partial u_y}{\partial y} + u_z\frac{\partial u_y}{\partial z}$$

$$f_z - \frac{1}{\rho}\frac{\partial p}{\partial z} + \frac{1}{\rho}\left(\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z}\right) = \frac{du_z}{dt} = \frac{\partial u_z}{\partial t} + u_x\frac{\partial u_z}{\partial x} + u_y\frac{\partial u_z}{\partial y} + u_z\frac{\partial u_z}{\partial z}$$

valid for any fluid in any general motion

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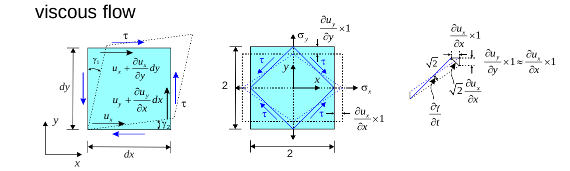
Navier-Stokes equation

- **Viscous stress tensor** components for **incompressible** viscous flow



$$\tau_{xy} = \tau = \mu \frac{\partial \gamma}{\partial t} = \mu\left(\frac{\partial \gamma_1}{\partial t} + \frac{\partial \gamma_2}{\partial t}\right)$$

$$= \mu\left(\left(\frac{\partial u_x}{\partial y} dy\right)\frac{dt}{dydt} + \left(\frac{\partial u_y}{\partial x} dx\right)\frac{dt}{dxdt}\right)$$

$$= \mu\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right)$$

$$\tau_{yz} = \mu\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right) \quad \tau_{xz} = \mu\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right)$$

$$\tau = \mu \frac{\partial \gamma}{\partial t} = \mu \frac{\sqrt{2}\frac{\partial u}{\partial x}}{\sqrt{2}} = \mu \frac{\partial u_x}{\partial x}$$

$$\sigma_x = 2\sqrt{2}\tau \cos 45° = 2\tau = 2\mu \frac{\partial u_x}{\partial x}$$

$$\sigma_y = 2\mu \frac{\partial u_y}{\partial y}$$

$$\sigma_z = 2\mu \frac{\partial u_z}{\partial z}$$

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Navier-Stokes equation

- When μ is constant:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = \mu\left(\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2} + \frac{\partial^2 u_x}{\partial z^2}\right) + \mu\frac{\partial}{\partial x}\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}\right) = \mu\left(\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2} + \frac{\partial^2 u_x}{\partial z^2}\right)$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} = \mu\left(\frac{\partial^2 u_y}{\partial x^2} + \frac{\partial^2 u_y}{\partial y^2} + \frac{\partial^2 u_y}{\partial z^2}\right) + \mu\frac{\partial}{\partial y}\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}\right) = \mu\left(\frac{\partial^2 u_y}{\partial x^2} + \frac{\partial^2 u_y}{\partial y^2} + \frac{\partial^2 u_y}{\partial z^2}\right)$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} = \mu\left(\frac{\partial^2 u_z}{\partial x^2} + \frac{\partial^2 u_z}{\partial y^2} + \frac{\partial^2 u_z}{\partial z^2}\right) + \mu\frac{\partial}{\partial z}\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}\right) = \mu\left(\frac{\partial^2 u_z}{\partial x^2} + \frac{\partial^2 u_z}{\partial y^2} + \frac{\partial^2 u_z}{\partial z^2}\right)$$

since $\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} = 0$ (continuity equation for incompressible fluid)

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Navier-Stokes equation

$$f_x - \frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2} + \frac{\partial^2 u_x}{\partial z^2}\right) = \frac{\partial u_x}{\partial t} + u_x\frac{\partial u_x}{\partial x} + u_y\frac{\partial u_x}{\partial y} + u_z\frac{\partial u_x}{\partial z}$$

$$f_y - \frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 u_y}{\partial x^2} + \frac{\partial^2 u_y}{\partial y^2} + \frac{\partial^2 u_y}{\partial z^2}\right) = \frac{\partial u_y}{\partial t} + u_x\frac{\partial u_y}{\partial x} + u_y\frac{\partial u_y}{\partial y} + u_z\frac{\partial u_y}{\partial z}$$

$$f_z - \frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left(\frac{\partial^2 u_z}{\partial x^2} + \frac{\partial^2 u_z}{\partial y^2} + \frac{\partial^2 u_z}{\partial z^2}\right) = \frac{\partial u_z}{\partial t} + u_x\frac{\partial u_z}{\partial x} + u_y\frac{\partial u_z}{\partial y} + u_z\frac{\partial u_z}{\partial z}$$

Kinematic viscosity $\nu = \frac{\mu}{\rho}$ $\quad \mathbf{f} - \frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{u} = \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}$

Suitable for:

- *Incompressible fluid*
- *Homogeneous fluid, μ is constant*

Four unknows for incompressible and viscous flow: u_x, u_y, u_z, p

Plus one incompressible continuity equations, theoretically, we could find all the unknowns

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Comparison

- Navier-Stokes equation (**incompressible** viscous fluid):

$$\mathbf{f}-\frac{1}{\rho}\nabla p+\nu\nabla^2\mathbf{u}=\frac{\partial\mathbf{u}}{\partial t}+(\mathbf{u}\cdot\nabla)\mathbf{u}$$

- Euler's differential equation of fluid motion (ideal and compressible/incompressible fluid):

$$\mathbf{f}-\frac{1}{\rho}\nabla p=\frac{\partial\mathbf{u}}{\partial t}+(\mathbf{u}\cdot\nabla)\mathbf{u}$$

- Euler's equilibrium equation of fluid statics:

$$\mathbf{f}-\frac{1}{\rho}\nabla p=0$$

- Equilibrium equation of static solid mechanics:

$$\mathbf{f}+\nabla\cdot\mathbf{\Sigma}=0$$

$$\Downarrow$$

$$\mathbf{f}+G\nabla^2\mathbf{u}+(\lambda+G)\nabla\nabla\cdot\mathbf{u}=0$$ Navier's equation

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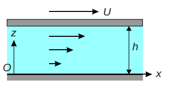
Application examples

- Parallel laminar flow:
  - All fluid particle are flowing in the same direction



$$u_x\neq 0,\, u_y=u_z=0$$

- 1. continuity equation (incompressible):

$$\frac{\partial u_x}{\partial x}+\frac{\partial u_y}{\partial y}+\frac{\partial u_z}{\partial z}=0 \rightarrow \frac{\partial u_x}{\partial x}=0$$

u_x does not change in x direction, i.e. $u_x=u_x(y,z,t)$

- 2. Navier-Stokes equation:

$$f_x-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2 u_x}{\partial y^2}+\frac{\partial^2 u_x}{\partial z^2}\right)=\frac{\partial u_x}{\partial t}$$

$$f_y-\frac{1}{\rho}\frac{\partial p}{\partial y}=0$$

$$f_z-\frac{1}{\rho}\frac{\partial p}{\partial z}=0$$

$$f_x=f_y=0,\; f_z=-g \rightarrow$$

$$-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2 u_x}{\partial y^2}+\frac{\partial^2 u_x}{\partial z^2}\right)=\frac{\partial u_x}{\partial t}$$

$$\frac{\partial p}{\partial y}=0$$

$$\frac{\partial p}{\partial z}=\rho g$$

Pressure does not change with y; pressure follows rules of fluid statics

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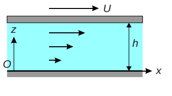
Couette flow

- Steady flow: $\frac{\partial u_x}{\partial t}=0$



- u_x does change in y:

$$u_x=u_x(z)$$

- Navier-Stokes equation changes to:

$$\frac{\partial p}{\partial x}=\mu\frac{\partial^2 u_x}{\partial z^2}=\mu\frac{d^2 u_x}{dz^2}$$

Boundary condition: $z=0,\, u_x=0$; $z=h,\, u_x=U$

- If $\partial p/\partial x$ = const, after integration:

$$u_x=\frac{1}{2\mu}\frac{\partial p}{\partial x}z^2+C_1 z+C_2$$

$z=0,\, u_x=0$; $z=h,\, u_x=U$ $\rightarrow$ *Using boundary condition:*

$$u_x=\frac{U}{h}z+\frac{z}{2\mu}\frac{\partial p}{\partial x}(z-h)$$

$\partial p/\partial x = 0$: $u_x=\frac{U}{h}z$ *velocity is linearly increasing with z*

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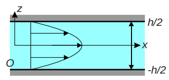
Poiseuille flow

- Steady flow:

$$\frac{\partial p}{\partial x} = \mu \frac{d^2 u_x}{dz^2}$$



- Boundary condition:

$$z = \frac{h}{2}, u_x = 0;\ z = -\frac{h}{2}, u_x = 0 \quad \rightarrow \quad u_x = \frac{1}{2\mu}\frac{\partial p}{\partial x}\left(z^2 - \frac{h^2}{4}\right)$$

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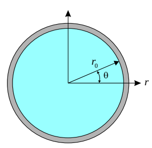
Laminar flow in tube

- Similar to Poiseuille flow:

$$u_r = \frac{1}{4\mu}\frac{\partial p}{\partial x}\left(r^2 - r_0^{\ 2}\right)$$



Boundary condition: $r = r_0, u_r = 0$

$$r = 0 \qquad u_r = u_{max} = -\frac{1}{4\mu}\frac{\partial p}{\partial x} r_0^{\ 2}$$

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