MAE5009: Continuum Mechanics B

Assignment 02: Strain and Displacement

Due October 12, 2020

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1. Derive the six second-order and three fourth-order compatibility equations based on the six strain-displacement equations.

Solution:

$$\varepsilon_x = \frac{\partial u}{\partial x}, \varepsilon_y = \frac{\partial v}{\partial y}, \varepsilon_z = \frac{\partial w}{\partial z}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}, \gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$$

(1) We can find the second-order differential of $\varepsilon_x, \varepsilon_y$, which are:

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} = \frac{\partial^3 u}{\partial y^2 \partial x}, \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^3 v}{\partial x^2 \partial y}$$

Then we need find the second-order differential of γ_{xy} by x and y

$$\frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = \frac{\partial^3 u}{\partial y^2 \partial x} + \frac{\partial^3 v}{\partial x^2 \partial y}$$

So,

$$\frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = \frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2}$$

With the same way, we can get

$$\frac{\partial^2 \gamma_{yz}}{\partial y \partial z} = \frac{\partial^2 \varepsilon_y}{\partial z^2} + \frac{\partial^2 \varepsilon_z}{\partial y^2}$$

$$\frac{\partial^2 \gamma_{zx}}{\partial z \partial x} = \frac{\partial^2 \varepsilon_z}{\partial x^2} + \frac{\partial^2 \varepsilon_x}{\partial z^2}$$

For another three second-order compatibility equations, we need get the second-order differential of $\varepsilon_x, \varepsilon_y, \varepsilon_z$ firstly.

$$\frac{\partial^2 \varepsilon_x}{\partial y \partial z} = \frac{\partial^3 u}{\partial x \partial y \partial z}, \frac{\partial^2 \varepsilon_y}{\partial z \partial x} = \frac{\partial^3 v}{\partial x \partial y \partial z}, \frac{\partial^2 \varepsilon_z}{\partial x \partial y} = \frac{\partial^3 w}{\partial x \partial y \partial z}$$

Then we need fond the first-order differential of $\gamma_{xy}, \gamma_{yz}, \gamma_{zx}$

$$\frac{\partial \gamma_{xy}}{\partial z} = \frac{\partial^2 u}{\partial y \partial z} + \frac{\partial^2 v}{\partial z \partial x}, \frac{\partial \gamma_{yz}}{\partial x} = \frac{\partial^2 v}{\partial z \partial x} + \frac{\partial^2 w}{\partial x \partial y}, \frac{\partial \gamma_{zx}}{\partial y} = \frac{\partial^2 w}{\partial x \partial y} + \frac{\partial^2 u}{\partial y \partial z}$$

Then we get:

$$\frac{\partial}{\partial x}\left(-\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right) = 2\frac{\partial^2 \varepsilon_x}{\partial y \partial z}$$

$$\frac{\partial}{\partial y}(\frac{\partial\gamma_{yz}}{\partial x}-\frac{\partial\gamma_{zx}}{\partial y}+\frac{\partial\gamma_{xy}}{\partial z})=2\frac{\partial^2\varepsilon_y}{\partial z\partial x}$$

$$\frac{\partial}{\partial x}(\frac{\partial\gamma_{yz}}{\partial x}+\frac{\partial\gamma_{zx}}{\partial y}-\frac{\partial\gamma_{xy}}{\partial z})=2\frac{\partial^2\varepsilon_z}{\partial x\partial y}$$

(2) According to (1), we can know that:

$$\frac{\partial^3}{\partial x\partial y\partial z}(-\frac{\partial\gamma_{yz}}{\partial x}+\frac{\partial\gamma_{zx}}{\partial y}+\frac{\partial\gamma_{xy}}{\partial z})=2\frac{\partial^4\varepsilon_x}{\partial y^2\partial z^2}$$

$$\frac{\partial^3}{\partial x\partial y\partial z}(\frac{\partial\gamma_{yz}}{\partial x}-\frac{\partial\gamma_{zx}}{\partial y}+\frac{\partial\gamma_{xy}}{\partial z})=2\frac{\partial^4\varepsilon_y}{\partial z^2\partial x^2}$$

$$\frac{\partial^3}{\partial x\partial y\partial z}(\frac{\partial\gamma_{yz}}{\partial x}+\frac{\partial\gamma_{zx}}{\partial y}-\frac{\partial\gamma_{xy}}{\partial z})=2\frac{\partial^4\varepsilon_z}{\partial x^2\partial y^2}$$

$$\frac{\partial^4\gamma_{xy}}{\partial x\partial y\partial z^2}=\frac{\partial^4\varepsilon_x}{\partial y^2\partial z^2}+\frac{\partial^4\varepsilon_y}{\partial x^2\partial z^2}$$

$$\frac{\partial^4\gamma_{yz}}{\partial x^2\partial y\partial z}=\frac{\partial^4\varepsilon_y}{\partial z^2\partial x^2}+\frac{\partial^4\varepsilon_z}{\partial x^2\partial y^2}$$

$$\frac{\partial^4\gamma_{zx}}{\partial x\partial y^2\partial z}=\frac{\partial^4\varepsilon_z}{\partial x^2\partial y^2}+\frac{\partial^4\varepsilon_x}{\partial y^2\partial z^2}$$

2. The following displacement field is applied to a certain body

$$u=k(2x+y^2), v=k(x^2-3y^2), w=0$$

Where $k=10^{-4}$,

(a) show the distorted configuration of a two-dimensional element with sides dx and dy and its lower left corner (point A) initially at the point (2,1,0), i.e., determine and sketch the new length and angular position of each side. You may exaggerate the plot to facilitate visualization;

Solution:

Since point A is (2, 1, 0), which means $x=2, y=1, z=0$.

$$\frac{\partial u}{\partial x}=2k, \frac{\partial u}{\partial y}=2ky=2k, \frac{\partial v}{\partial x}=2kx=4k, \frac{\partial v}{\partial y}=-6ky=-6k$$

$$\frac{\partial v}{\partial x}dx=2kxdx=4kdx, \frac{\partial u}{\partial x}dx=2kdx,$$

$$\frac{\partial v}{\partial y}dy=-6kydy=-6kdy, \frac{\partial u}{\partial y}dy=2kydy=2kdy$$

For $A'B'$, $D'C'$

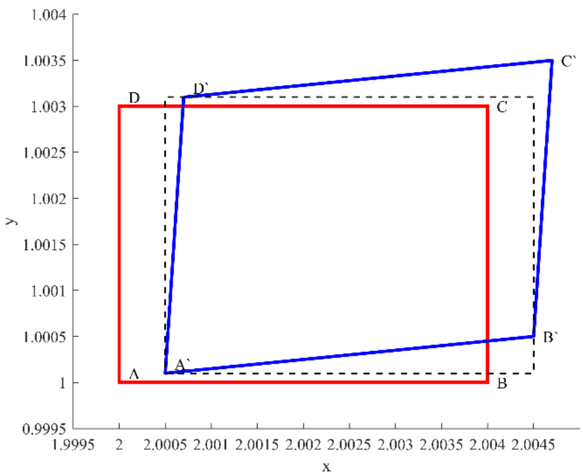
$$A'B'=D'C'=\sqrt{(dx+2kdx)^2+(2kxdx)^2}=dx\sqrt{20k^2+4k+1}$$

For $A'D'$, $B'C'$

$$A'D'=B'C'=\sqrt{(dy-6kydy)^2+(2kydy)^2}=dy\sqrt{40k^2-12k+1}$$

The angular positions of $A'D'$ and $A'B'$ are:

$$\theta = tan\theta = \frac{\partial v}{\partial x} = 2kx = 4 \times 10^{-4}, -\lambda = -tan\lambda = \frac{\partial u}{\partial y} = 2ky = 2 \times 10^{-4}$$



(b) determine the coordinates of point A after the displacement field is applied;

Solution:

Since point A $(2, 1, 0)$ and the side lengths are dx and dy, we can know

$$u = 5k, v = k, w = 0$$

The coordinates of point A' is $(2 + 5k, 1 + k, 0)$

(c) find ω_z at this point;

Solution:

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \frac{1}{2}(2kx - 2ky) = kx - ky = k = 10^{-4}$$

(d) find the maximum, minimum normal strain and maximum shear strain at this point.

Solution:

$$\varepsilon_x = \frac{\partial u}{\partial x} = 2k, \varepsilon_y = \frac{\partial v}{\partial y} = -6k, \varepsilon_{xy} = \frac{1}{2}\gamma_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = ky + kx = 3k$$

The maximum normal strain is $\frac{\varepsilon_x+\varepsilon_y}{2} + \sqrt{\left(\frac{\varepsilon_x-\varepsilon_y}{2}\right)^2 + \varepsilon_{xy}^2} = 3k$

The minimum normal strain is $\frac{\varepsilon_x+\varepsilon_y}{2} - \sqrt{\left(\frac{\varepsilon_x-\varepsilon_y}{2}\right)^2 + \varepsilon_{xy}^2} = -7k$

The maximum shear strain is $\sqrt{\left(\frac{\varepsilon_x-\varepsilon_y}{2}\right)^2 + \varepsilon_{xy}^2} = 5k$

The minimum shear strain is $-\sqrt{\left(\frac{\varepsilon_x-\varepsilon_y}{2}\right)^2 + \varepsilon_{xy}^2} = -5k$

3. Given the following system of strains,

$$\varepsilon_x = 5 + x^2 + y^2 + x^4 + y^4$$

$$\varepsilon_y = 6 + 3x^2 + 3y^2 + x^4 + y^4$$

$$\gamma_{xy} = 10 + 4xy(x^2 + y^2 + 2)$$

$$\varepsilon_z = \gamma_{xz} = \gamma_{yz} = 0$$

determine if the system of strains is possible.

Solution:

Since $\varepsilon_x = \frac{\partial u}{\partial x} = 5 + x^2 + y^2 + x^4 + y^4$, we can get $\frac{\partial^2 \varepsilon_x}{\partial y^2} = 2 + 12y^2$

Since $\varepsilon_y = \frac{\partial v}{\partial y} = 6 + 3x^2 + 3y^2 + x^4 + y^4$, we can get $\frac{\partial^2 \varepsilon_y}{\partial x^2} = 6 + 12x^2$

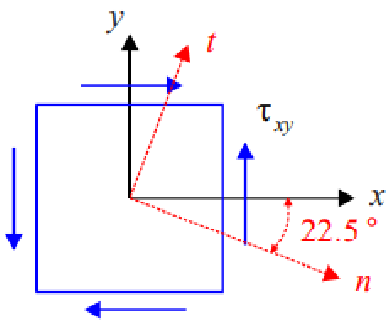
Since $\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 10 + 4xy(x^2 + y^2 + 2)$, we can get:

$$\frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = 12x^2 + 12y^2 + 8$$

Then $\frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = \frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = 12x^2 + 12y^2 + 8$

Therefore, this system of strain is possible.

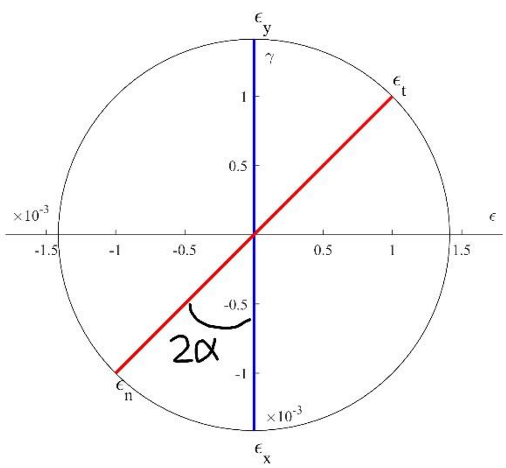
4. For the strain $\varepsilon_x = \varepsilon_y = 0$, $\gamma_{xy} = 0.002828$ (in x-y coordinate system) at a specific point in an isotropic material, using the Mohr's circle of strain to determine the strain components ε_n, ε_t, and γ_{tn} in n-t coordinate system shown in the following figure.



Solution:

The angle $\alpha = 22.5°$, the Mohr's circle is shown below.

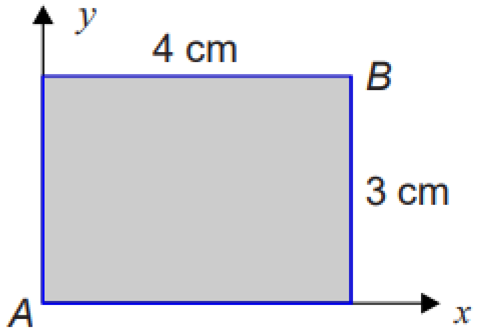
$$\varepsilon_n = -0.0009998, \varepsilon_t = 0.0009998, \gamma_{tn} = 0.0009998$$



5. A thin rectangular plate with dimensions 3 cm × 4 cm is acted upon by a stress distribution which results in the uniform strains

$$\varepsilon_x = 0.0025, \varepsilon_y = 0.0050, \varepsilon_z = 0, \gamma_{xy} = 0.001875, \gamma_{xz} = \gamma_{yz} = 0$$

as shown in the following figure. Determine the change in length of diagonal AB.



Solution:

Method 1:

The length of AB is 5 cm. Because of the uniform strains, $\omega_z = 0$. Then:

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = \varepsilon_x dx + \frac{1}{2}\gamma_{xy}dy - \omega_z dy = 0.0128$$

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy = \varepsilon_y dy + \frac{1}{2}\gamma_{xy}dx + \omega_z dx = 0.0187$$

The distance between A and B after deformation is:

$$\sqrt{(dx + du)^2 + (dy + dv)^2} = 5.0215$$

The change of AB is 0.0215 cm.

Method 2:

Let x' axis coincide with AB and the angle between AB and x axis be α, then we can get:

$$sin\alpha = \frac{3}{5}, cos\alpha = \frac{4}{5}, \sin(2\alpha) = \frac{24}{25}, \cos(2\alpha) = \frac{7}{25}$$

Therefore, the strain in x' direction is:

$$\varepsilon_{x'} = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2}\cos(2\alpha) + \frac{1}{2}\gamma_{xy}\sin(2\alpha) = 0.0043$$

Since $AB = \sqrt{3^2 + 4^2} = 5$ cm, the change in length of diagonal AB is $\Delta AB = AB \times \varepsilon_{x'} = 0.0215$ cm.