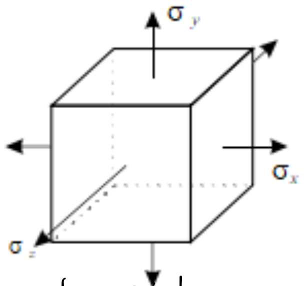
MAE5009: Continuum Mechanics B

Assignment 03: Stress Strain Relations

Due October 22, 2021

1. Derive the relations between the normal stresses and normal strains by adding the normal stresses on the cube in the following consecutive order: σ_z, σ_y and σ_x.



Solution:

Let the initial length of each sides are l_{x0}, l_{y0}, l_{z0}

Apply σ_z:

$\varepsilon_z = \frac{\sigma_z}{E}$, $l_{z1} = (1+\varepsilon_z) l_{z0} = (1+\frac{\sigma_z}{E}) l_{z0}$

$\varepsilon_x = \varepsilon_y = -\nu \varepsilon_z = -\nu \frac{\sigma_z}{E}$ $l_{x1} = (1+\varepsilon_x) l_{x0} = (1-\nu\frac{\sigma_z}{E}) l_{x0}$ $l_{y1} = (1-\nu\frac{\sigma_y}{E}) l_{y0}$

Apply σ_y:

$\varepsilon_y = \frac{\sigma_y}{E}$, $l_{y2} = (1+\varepsilon_y) l_{y1} = (1+\frac{\sigma_y}{E})(1-\nu\frac{\sigma_y}{E}) l_{y0}$

$\varepsilon_x = \varepsilon_z = -\nu\varepsilon_y = -\nu\frac{\sigma_y}{E}$, $l_{x2} = (1+\varepsilon_x) l_{x1} = (1-\nu\frac{\sigma_y}{E})(1-\nu\frac{\sigma_z}{E}) l_{x0}$

$l_{z2} = (1+\varepsilon_z) l_{z1} = (1-\nu\frac{\sigma_y}{E})(1+\nu\frac{\sigma_z}{E}) l_{z0}$

Apply σ_x:

$\varepsilon_x = \frac{\sigma_x}{E}$, $l_{x1} = (1+\varepsilon_x) l_{x0} = (1+\frac{\sigma_x}{E}) l_{x0}$

$\varepsilon_y = \varepsilon_z = -\nu\varepsilon_x = -\nu\frac{\sigma_x}{E}$, $l_{y3} = (1+\varepsilon_y) l_{y2} = (1-\nu\frac{\sigma_x}{E})(1+\frac{\sigma_y}{E})(1-\nu\frac{\sigma_y}{E}) l_{y0}$

$l_{z3} = (1+\varepsilon_z) l_{z2} = (1-\nu\frac{\sigma_x}{E})(1-\nu\frac{\sigma_y}{E})(1+\nu\frac{\sigma_z}{E}) l_{z0}$

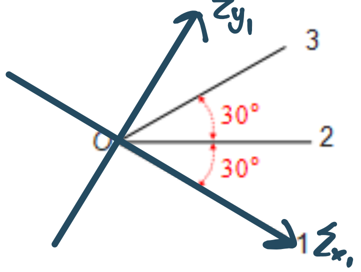
Neglect the negligible items:

$$\varepsilon_x = \frac{L_{x3}-L_{x0}}{L_{x0}} = \frac{1}{E}[\sigma_x - \nu(\sigma_y+\sigma_z)]$$

$$\varepsilon_y = \frac{L_{y3}-L_{y0}}{L_{y0}} = \frac{1}{E}[\sigma_y - \nu(\sigma_x+\sigma_z)]$$

$$\varepsilon_z = \frac{L_{z3}-L_{z0}}{L_{z0}} = \frac{1}{E}[\sigma_z - \nu(\sigma_x+\sigma_y)]$$

2. For a given x-y plane, the normal strains at point O in the O-1, O-2 and O-3 directions are respectively $\varepsilon_{O-1} = 10^{-4}$, $\varepsilon_{O-2} = 4\times10^{-4}$ and $\varepsilon_{O-3} = 6\times10^{-4}$. Given the material properties $E = 30$ GPa, $\nu = 0.25$, determine the principal stresses and maximum shear stress at point O and their directions (only consider the stresses and strains in the x-y plane, i.e., a pure 2D problem)



Solution:

Let $\varepsilon_{O-1}=\varepsilon_{x_1}, \varepsilon_{O-2}=\varepsilon_{x_2}, \varepsilon_{O-3}=\varepsilon_{x_3}$

According to the known conditions, we can get

$$\begin{cases} \sigma_x = 2G\varepsilon_{x_1} + \lambda(\varepsilon_{x_1}+\varepsilon_{y_1}) \cdots ① \\ \sigma_y = 2G\varepsilon_{y_1} + \lambda(\varepsilon_{x_1}+\varepsilon_{y_1}) \cdots ② \\ \tau_{xy} = G\gamma_{xy} = 2G\varepsilon_{xy} \cdots ③ \end{cases}$$

but ε_{y_1} and ε_{xy} are unknow.

We know $\left[\varepsilon_{x'} = \frac{\varepsilon_x+\varepsilon_y}{2} + \frac{\varepsilon_x-\varepsilon_y}{2}\cos 2\alpha + \varepsilon_{xy}\sin 2\alpha\right]$

then, we can get

$$\varepsilon_{O-2}=\varepsilon_{x_2} = \frac{\varepsilon_{x_1}+\varepsilon_{y_1}}{2} + \frac{\varepsilon_{x_1}-\varepsilon_{y_1}}{2}\cos(60^\circ) + \varepsilon_{xy}\sin 60^\circ \cdots ④$$

$$\varepsilon_{O-3}=\varepsilon_{x_3} = \frac{\varepsilon_{x_1}+\varepsilon_{y_1}}{2} + \frac{\varepsilon_{x_1}-\varepsilon_{y_1}}{2}\cos(120^\circ) + \varepsilon_{xy}\sin 120^\circ \cdots ⑤$$

So, $\varepsilon_{y_1} = 5\times10^{-4}$, $\varepsilon_{xy} = \frac{4\sqrt{3}}{3}\times10^{-4}$

$\sigma_x = 96\times10^{-4}$ GPa, $\sigma_y = 192\times10^{-4}$ GPa, $\tau_{xy} = 32\sqrt{3}\times10^{-4}$ GPa

The principle stress:

$$\sigma_{max} = \frac{\sigma_x+\sigma_y}{2} + \sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2+\tau_{xy}^2} = (144+16\sqrt{21})\times10^{-4}\text{GPa} = 21.73\text{MPa}$$

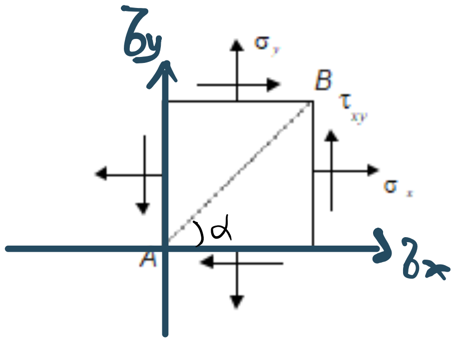
$$\sigma_{min} = \frac{\sigma_x+\sigma_y}{2} - \sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2+\tau_{xy}^2} = 7.07\text{MPa}$$

The direction of principle stress: $\tan 2\alpha = \frac{2\tau_{xy}}{\sigma_x-\sigma_y} \Rightarrow \alpha = 65.45^\circ$ or -24.55°

The maximum shear stress: $\tau_{xy\,max} = \sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2+\tau_{xy}^2} = 7.33\text{MPa}$

The direction of maximum shear stress: $\tan 2\alpha = -\frac{\sigma_x-\sigma_y}{2\tau_{xy}} = \frac{\sqrt{3}}{2} \Rightarrow \alpha = 69.55^\circ$ or -20.45°

3. A homogeneous and isotropic square plate is loaded as shown, where $\sigma_x = \sigma_y = \tau_{xy} =$ 15 MPa. If E = 10 GPa, ν = 0.3, determine the change in length of the diagonal AB.



Solution:

According to the known conditions

the change in length of AB is $\Delta AB = \varepsilon_{x'} \cdot AB$

In this square, we can know $\alpha = 45^\circ$

$$\begin{cases} \varepsilon_{x'} = \frac{\varepsilon_x+\varepsilon_y}{2} + \frac{\varepsilon_x-\varepsilon_y}{2}\cos 2\alpha + \varepsilon_{xy}\sin 2\alpha \\ \varepsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y) = 1.05\times10^{-3} \\ \varepsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x) = 1.05\times10^{-3} \\ \varepsilon_{xy} = \frac{1}{2}\gamma_{xy} \\ \gamma_{xy} = \frac{1}{G}\tau_{xy} \end{cases}$$

$$\gamma_{xy} = \frac{1}{G}\tau_{xy} = \frac{2.6\times15}{1\times10^4}$$

then we can get $\gamma_{xy} = 3.9\times10^{-3}$ $\varepsilon_{xy} = 1.95\times10^{-3}$ $\varepsilon_{x'} = 3\times10^{-3}$

So $\Delta AB = \varepsilon_{x'} \cdot AB$

$= 3\times10^{-3} AB$

4. Prove the following relations among various elastic constants:

$$\nu = \frac{3K - E}{6K}$$

$$\lambda = \frac{3K - 2G}{3}$$

$$E = \frac{9K(K - \lambda)}{3K - \lambda}$$

$$G = \frac{3KE}{9K - E}$$

$$K = \frac{EG}{3(3G - E)}$$

Solution:

We already know that

$$G = \frac{E}{2(1+\nu)}, \quad \lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}, \quad K = \frac{E}{3(1-2\nu)}$$

since $K = \frac{E}{3(1-2\nu)} \Rightarrow 1 - 2\nu = \frac{E}{3K} \Rightarrow \nu = \frac{3K-E}{6K}$

Since $G = \frac{E}{2(1+\nu)}$, $K = \frac{E}{3(1-2\nu)} \Rightarrow$

$$\frac{3K-2G}{3} = \frac{\frac{E}{1-2\nu} - \frac{E}{1+\nu}}{3} = \frac{\nu E}{(1-2\nu)(1+\nu)} = \lambda$$

since $K = \frac{E}{3(1-2\nu)} \Rightarrow 3K = \frac{E}{1-2\nu}$, $\lambda = \frac{3K\nu}{1+\nu}$, due to $\nu = \frac{3K-E}{6K}$

then we can get $E = \frac{9K(K-\lambda)}{3K-\lambda}$

Since $G = \frac{E}{2(1+\nu)}$ and $\nu = \frac{3K-E}{6K}$, then $G = \frac{3KE}{9K-E}$

$9KG - GE = 3KE$, $K(9G - 3E) = EG \Rightarrow$ then we can get $K = \frac{EG}{9G-3E} = \frac{EG}{3(3G-E)}$