Review

In fluid, stress tensor incorporates **pressure** and **viscous stress tensor**.

$$\sigma_{ij} = -p\delta_{ij} + \tau_{ij}$$

For isotropic Newtonian fluids, we have

$$\tau_{ij} = \lambda\delta_{ij}\dot{\varepsilon}_{kk} + 2\mu\dot{\varepsilon}_{ij}$$ Constitutive equation for isotropic fluids

μ, λ are termed dynamic (first) and second viscosity, respectively.

Stokes assumption: viscous stress is a deviatoric stress tensor, we have

$$\kappa = (\lambda + \frac{2\mu}{3}) = 0 \quad \lambda = -\frac{2\mu}{3}$$

The constitutive equation for Newtonian fluids is

$$\tau_{ij} = -\frac{2\mu}{3}\delta_{ij}\dot{\varepsilon}_{kk} + 2\mu\dot{\varepsilon}_{ij} \qquad \sigma_{ij} = -p\delta_{ij} - \frac{2\mu}{3}\delta_{ij}\dot{\varepsilon}_{kk} + 2\mu\dot{\varepsilon}_{ij}$$

Review

Reynolds transport theorem (雷诺输运定理)

Suppose $R(t)$ is a fluid region with surface S traveling at the flow velocity v_k, we have

$$\frac{D}{Dt}\int_{R(t)} T_{ij\ldots}(x_i, t)\, dV = \int_R \frac{\partial T_{ij\ldots}}{\partial t}\, dV + \int_S n_k v_k T_{ij\ldots}\, dS$$

The dilation rate (unit volume change rate, 单位体积变化率) is velocity's divergence

$$\lim_{V_{\mathrm{MR}}\to 0} \frac{1}{V_{\mathrm{MR}}} \frac{DV_{\mathrm{MR}}}{Dt} = \partial_i v_i = \nabla \cdot \mathbf{v}$$

Mass conservation:

$$\frac{d}{dt}\int_{\mathrm{MR}} \rho\, dV = 0$$

$\Rightarrow$

mass conservation in the differential form

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v_i)}{\partial x_i} = 0$$

mass change rate of the fixed region

net mass outflow rate of fixed region

mass conservation for incompressible fluid

$$\frac{\partial v_i}{\partial x_i} = 0$$

Review

Conservation of momentum:

$$\frac{d}{dt}\int_{\mathrm{MR}} \rho v_i \, dV = \int_{\mathrm{MR}} \rho F_i \, dV + \int_{\mathrm{MR}} R_i \, dS$$

The momentum equation can be reduced to:

$$\underbrace{\rho \frac{Dv_i}{Dt}}_{\text{momentum change rate}} = \underbrace{\rho F_i}_{\text{total body force}} - \underbrace{\frac{\partial p}{\partial x_i}}_{\text{net pressure force}} + \underbrace{\frac{\partial \tau_{ji}}{\partial x_j}}_{\text{net viscous force}}$$

The momentum change of a particle equates the net force acting on the particle.

Conservation of Energy

Three ways of heat transfer: radiation, conduction, convection

Fourier's law of thermal conduction (傅里叶定律): the conduction heat flow is proportional to the negative gradient of the temperature.

$$q_i = B_{ij}\, \partial_j T$$

q_i: heat flow W/m^2
T: temperature K, ℃
B_{ij}: thermal conductivity, a 2nd-order tensor, $W/m/K$

The thermal conductivity is often assumed to be isotropic, we have

$$q_i = -k\, \partial_i T$$ note the minus sign

Thermal diffusivity (热扩散系数, κ): m²/s $\kappa = \dfrac{k}{\rho c_p}$ c_p: isobaric heat capacity

Conservation of Energy

Conservation of Energy (based on the first law of thermodynamics): The change of energy within a material region equals to the heat transmitted to and work done to the region.

$$\Delta U = Q + W$$

ΔU: energy change of the material region
Q: heat transmitted to the system
W: work done to the system

The total energy of a material element includes **internal energy** (内能), **potential energy (势能)** and **kinetic energy (动能)**.
Kinetic energy: the energy possessed by a body due to its motion.0.5mv^2
Internal energy: the total energy associated with the random, disordered motion of molecules.

- It is the total of the kinetic energy due to the motion of molecules and the potential energy associated with the vibrational motion and electric energy of atoms within molecules.

Total energy of a material element dV is

$$\rho(e + E + \frac{1}{2}v^2)dV$$

e: internal energy in unit mass
E: potential energy in unit mass
v: bulk velocity

Conservation of Energy

Rate of Work: $\frac{dW}{dt}$

The rate of work under force f_i is:

$$\frac{dW}{dt} = f_i \frac{dl_i}{dt} = f_i v_i$$

In continuum mechanics, forces include both body force and surface forces.
The rate of work on an element is:

work rate due to gravity (body) force $\rho F_i v_i dV$

work rate due to surface force $R_i v_i dS = n_j \sigma_{ji} v_i dS$

Heat transfer rate: $\frac{dQ}{dt} = -n_i q_i dS$ (due to conduction, radiation)

q_i is heat flux vector (W/m²) which gives the magnitude and direction of the flux.
q_i is positive for heat flux in the positive axis direction.

Conservation of Energy

Conservation of Energy: $\dfrac{dU}{dt} = \dfrac{dQ}{dt} + \dfrac{dW}{dt}$ ➡

$$\frac{d}{dt}\int_{MR} \rho(e+\frac{1}{2}v^2)dV = -\int_{MR} n_i q_i dS + \int_{MR} \rho F_i v_i dV + \int_{MR} n_i \sigma_{ij} v_j dS$$

Using Reynolds transport theorems on the left-hand and convert all surface integrals to body integrals with Gauss's theorem, we get

$$\int_{MR} \frac{\partial}{\partial t}[\rho(e+\frac{1}{2}v^2)]dV + \int_{MR} \frac{\partial}{\partial x_i}[\rho v_i(e+\frac{1}{2}v^2)]dV = -\int_{MR} \frac{\partial q_i}{\partial x_i} dV + \int_{MR} \rho F_i v_i dV + \int_{MR} \frac{\partial(\sigma_{ij} v_j)}{\partial x_i} dV$$

$$\frac{\partial}{\partial t}[\rho(e+\frac{1}{2}v^2)] + \frac{\partial}{\partial x_i}[\rho v_i(e+\frac{1}{2}v^2)] = -\frac{\partial q_i}{\partial x_i} + \rho F_i v_i + \frac{\partial(\sigma_{ij} v_j)}{\partial x_i}$$

symbolic form

$$\underbrace{\frac{\partial}{\partial t}\left[\rho\left(e+\frac{1}{2}v^2\right)\right]}_{\text{energy increase rate per unit volume}} + \underbrace{\nabla\cdot\left[\rho\mathbf{v}\left(e+\frac{1}{2}v^2\right)\right]}_{\text{convection of energy out of a point by flow}} = \underbrace{-\nabla\cdot\mathbf{q}}_{\text{net heat flow in}} + \underbrace{\nabla\cdot(\boldsymbol{\sigma}\cdot\mathbf{v})}_{\text{work of surface forces}} + \underbrace{\rho\mathbf{v}\cdot\mathbf{F}}_{\text{work of body forces}}$$

Conservation of Energy

multiply v_i to both sides of the momentum conservation equation:

$$v_i \frac{\partial(\rho v_i)}{\partial t} + v_i \frac{\partial(\rho v_j v_i)}{\partial x_j} = \rho v_i F_i + v_i \frac{\partial \sigma_{ji}}{\partial x_j}$$

$$\Rightarrow \quad \rho \frac{D}{Dt}\left(\frac{1}{2} v^2\right) = \rho v_i F_i + v_i \frac{\partial \sigma_{ji}}{\partial x_j}$$

Kinetic Energy equation (动能、机械能方程):

Substract kinetic energy equation from energy conservation equation

$$\rho \frac{D}{Dt}\left(e + \frac{1}{2} v^2\right) = -\frac{\partial q_i}{\partial x_i} + \rho g_i v_i + \frac{\partial(\sigma_{ij} v_j)}{\partial x_i} \quad \Rightarrow$$

$$\rho \frac{De}{Dt} = -\frac{\partial q_i}{\partial x_i} + \sigma_{ij} \frac{\partial v_j}{\partial x_i}$$

Thermal energy equation (热能方程):

$$\tau_{ij} \frac{\partial v_j}{\partial x_i} = \tau_{ij}(\dot{\varepsilon}_{ij} + \dot{\omega}_{ij}) = \tau_{ij} \dot{\varepsilon}_{ij} \quad \Rightarrow \quad \rho \frac{De}{Dt} = -\frac{\partial q_i}{\partial x_i} - p \frac{\partial v_i}{\partial x_i} + \tau_{ij} \dot{\varepsilon}_{ij}$$

internal energy | heat flow in | pressure work | viscous work

Conservation of Energy

Thermodynamic identities give us the following equation

$$\frac{De}{Dt} = c_p \frac{DT}{Dt} - \rho^{-1}\alpha T \frac{Dp}{Dt} + \rho^{-2} p \frac{D\rho}{Dt}$$

α: bulk thermal expansivity (热胀系数),
c_p: isobaric heat capacity (等压热容)

$$\frac{D\rho}{Dt} = -\frac{\rho}{V}\frac{DV}{Dt} = -\rho \frac{\partial v_i}{\partial x_i} \Rightarrow \rho \frac{De}{Dt} = \rho c_p \frac{DT}{Dt} - \alpha T \frac{Dp}{Dt} - p \frac{\partial v_i}{\partial x_i}$$

Substitute above of $\rho De/Dt$ to the thermal equation,

$$\rho \frac{De}{Dt} = -\frac{\partial q_i}{\partial x_i} - p \frac{\partial v_i}{\partial x_i} + \tau_{ij}\dot{\varepsilon}_{ij}$$

we have Thermal energy equation in terms of temperature

$$\rho c_p \frac{DT}{Dt} = -\frac{\partial q_i}{\partial x_i} + \tau_{ij}\dot{\varepsilon}_{ij} + \alpha T \frac{Dp}{Dt}$$

When we say energy conservation equation, we usually refer to this one.

thermal energy change rate

outflow energy

viscous dissipation

adiabatic heating

Conservation of Energy

Represent internal energy variation with variations of temperature, pressure, and density

$$de = TdS - pdV = T\left((\frac{\partial S}{\partial T})_p dT + (\frac{\partial S}{\partial p})_T dp \right) - pdV$$

$$= c_p dT - \rho^{-1}\alpha T dp - pdV$$

$$= c_p dT - \rho^{-1}\alpha T dp + \rho^{-2} p d\rho$$

Refer to 'Introduction to the Physics of the Earth's Interior (2nd Ed.)' by Jean-Paul Poirier.

Navier-Stokes Equations

Governing equations of fluid dynamics:

mass conservation

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v_i)}{\partial x_i} = 0$$

momentum conservation

$$\rho \frac{Dv_i}{Dt} = \rho g_i - \frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ji}}{\partial x_j}$$

energy conservation

$$\rho c_p \frac{DT}{Dt} = -\frac{\partial q_i}{\partial x_i} + \tau_{ij}\dot{\varepsilon}_{ij} + \alpha T \frac{Dp}{Dt}$$

Navier–Stokes equations: In the governing equations of fluid dynamics, express the strain rate and viscous stress in terms of velocity, and the heat flux in terms of temperature.

The Navier-Stokes equations are (general compressible flow):

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v_i)}{\partial x_i} = 0$$

$$\rho \frac{Dv_i}{Dt} = \rho g_i - \frac{\partial p}{\partial x_i} + [\mu(v_{j,i} + v_{i,j} - \frac{2}{3} v_{k,k}\delta_{ji})]_{,j}$$

$$\rho c_p \frac{DT}{Dt} = \frac{\partial(kT_{,i})}{\partial x_i} - \frac{2}{3}\mu v_{i,i}^2 + 2\mu\dot{\varepsilon}_{ij}^2 + \alpha T \frac{Dp}{Dt}$$

5 variables

- 3 velocity components,
- 1 pressure,
- 1 temperature;

5 equations

- 1 mass conservation
- 3 momentum conservation
- 1 energy conservation

Navier-Stokes Equations

N-S equations for incompressible flow:

mass conservation

$$\frac{\partial v_i}{\partial x_i} = 0$$

momentum conservation

$$\rho \frac{Dv_i}{Dt} = \rho g_i - \frac{\partial p}{\partial x_i} + [\mu(v_{j,i} + v_{i,j})]_{,j} = \rho g_i - \frac{\partial p}{\partial x_i} + \mu v_{i,jj}$$

energy conservation

$$\rho c_p \frac{DT}{Dt} = \frac{\partial(kT_{,i})}{\partial x_i} + 2\mu\dot{\varepsilon}_{ij}^2 + \alpha T \frac{Dp}{Dt}$$

N-S equations for incompressible constant temperature flow:

mass conservation

$$\frac{\partial v_i}{\partial x_i} = 0$$

momentum conservation

$$\rho \frac{Dv_i}{Dt} = \rho g_i - \frac{\partial p}{\partial x_i} + \mu v_{i,jj}$$

In hydraulics (水力学) and some other subjects, temperature is usually not considered.

Compare between elastic solid and viscous fluid

Elastic deformation

15 unknowns:

- 6 stress, 6 strain, 3 displacement

15 equations

- 6 Strain-displacement relations

$$\varepsilon_{ij} = 0.5(u_{i,j} + u_{j,i})$$

- 3 Equilibrium equations

$$\sigma_{ji,j} + f_i = 0$$

- 6 Constitutive equations (Hooke's law)

$$\sigma_{ij} = \lambda\delta_{ij}\varepsilon_{kk} + 2\mu\varepsilon_{ij}$$

λ, μ are elastic modulus (Lame parameters)

Viscous deformation

16 unknowns:

- 6 viscous stress, 6 strain rate, 3 velocity, 1 pressure

16 equations

- 6 Strain rate-velocity relations

$$\dot{\varepsilon}_{ij} = 0.5(v_{i,j} + v_{j,i})$$

- 3 Conservation of momentum

$$\sigma_{ji,j} + f_i = \rho a_i$$

- 6 Constitutive equations (Newtonian viscosity law)

$$\sigma_{ij} = -p\delta_{ij} - \frac{2\mu}{3}\delta_{ij}\dot{\varepsilon}_{kk} + 2\mu\dot{\varepsilon}_{ij}$$

- 1 Conservation of mass

$$\frac{\partial\rho}{\partial t} + \frac{\partial(\rho v_i)}{\partial x_i} = 0$$

Compare between elastic solid and viscous fluid

Elastic deformation

Governing equations: Navier's equations

$$(\lambda + G)\varepsilon_{,k} + G\nabla^2 u_k + f_k = 0$$

Viscous deformation

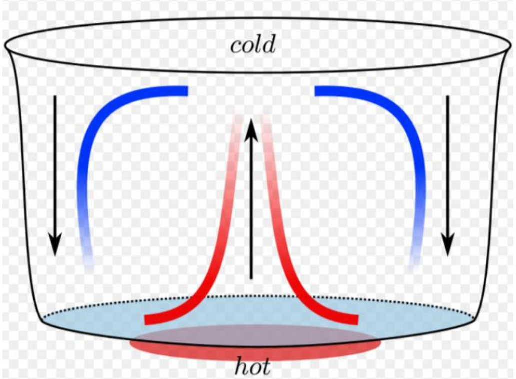
Governing equations: Navier-Stokes equations

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v_i)}{\partial x_i} = 0$$

$$\rho \frac{Dv_i}{Dt} = \rho g_i - \frac{\partial p}{\partial x_i} + [\mu(v_{j,i} + v_{i,j} - \frac{2}{3} v_{k,k}\delta_{ji})]_{,j}$$

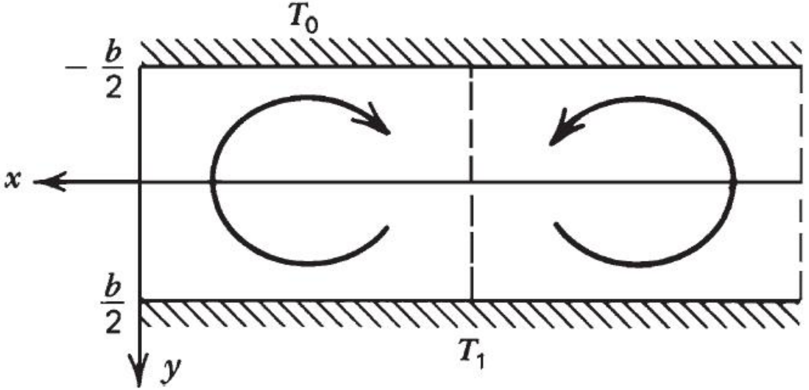
$$\rho c_p \frac{DT}{Dt} = \frac{\partial(kT_{,i})}{\partial x_i} - \frac{2}{3}\mu v_{i,i}^2 + 2\mu\dot{\varepsilon}_{ij}^2 + \alpha T \frac{Dp}{Dt}$$

Rayleigh Number and Onset of Thermal Convection



If the fluid (honey) is heated from below, under which condition will the convection start?

Rayleigh Number and Onset of Thermal Convection



Assume the fluid layer is heated from below. The top and bottom boundary temperature is fixed at T_1 and T_0.

For convection to occur, the convective thermal transport rate must be larger than the diffusive transport rate:

Diffusive thermal transport time: b^2 / κ

Convective thermal transport time: b / u

For viscous laminar flow:

$$\Delta\rho b^2 \delta y g = b \delta y \eta u / b$$

driving force drag force

$$u = \Delta\rho g b^2 / \eta$$

Convective thermal transport time:

$$b / u = \eta / \Delta\rho g b$$

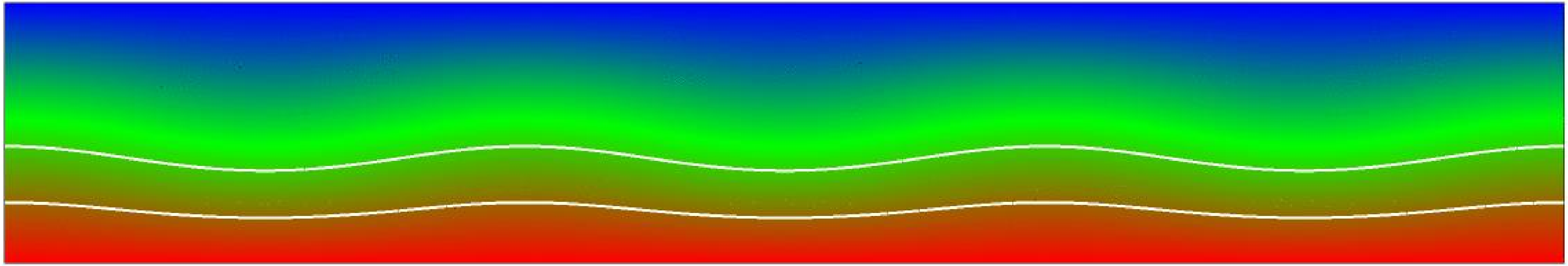
Define Rayleigh number as ratio between the diffusive and convective thermal transport time:

$$Ra = \frac{b^2 / \kappa}{\eta / \Delta\rho g b} = \frac{\Delta\rho g b^3}{\eta\kappa} = \frac{\rho\alpha_v (T_1 - T_0) g b^3}{\eta\kappa}$$

For thermal convection to occur, Ra must exceeds a critical value Ra_c:
$Ra > Ra_c$ = (700 - several thousands)

Rayleigh Number and Onset of Thermal Convection

Initial periodic thermal perturbation gradually fades away
because Ra=4.5E2 < critical Rayleigh Number

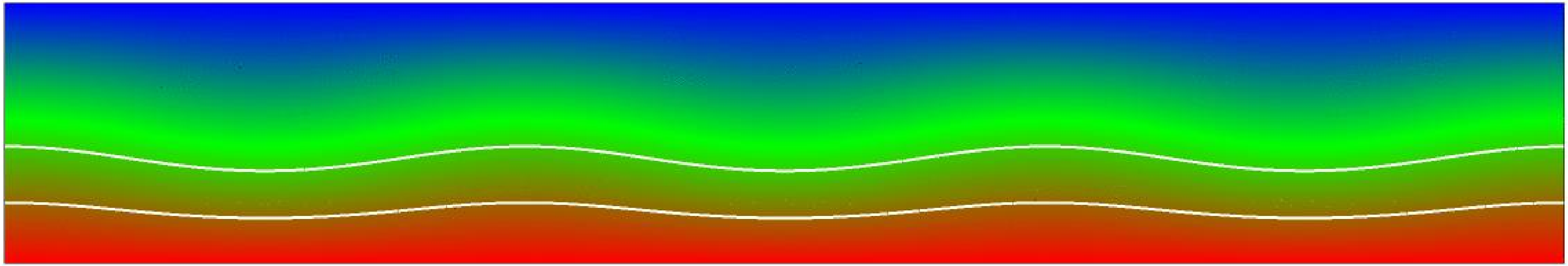


$$Ra = \frac{\rho \alpha_v (T_1 - T_0) g b^3}{\eta \kappa} = 450 < Ra_c$$

dimensionless temperature

Rayleigh Number and Onset of Thermal Convection

Initial periodic thermal perturbation triggers thermal convection because Ra=4.5E3 > critical Rayleigh Number



$$Ra = \frac{\rho \alpha_v (T_1 - T_0) g b^3}{\eta \kappa} = 4500 > Ra_c$$

dimensionless temperature

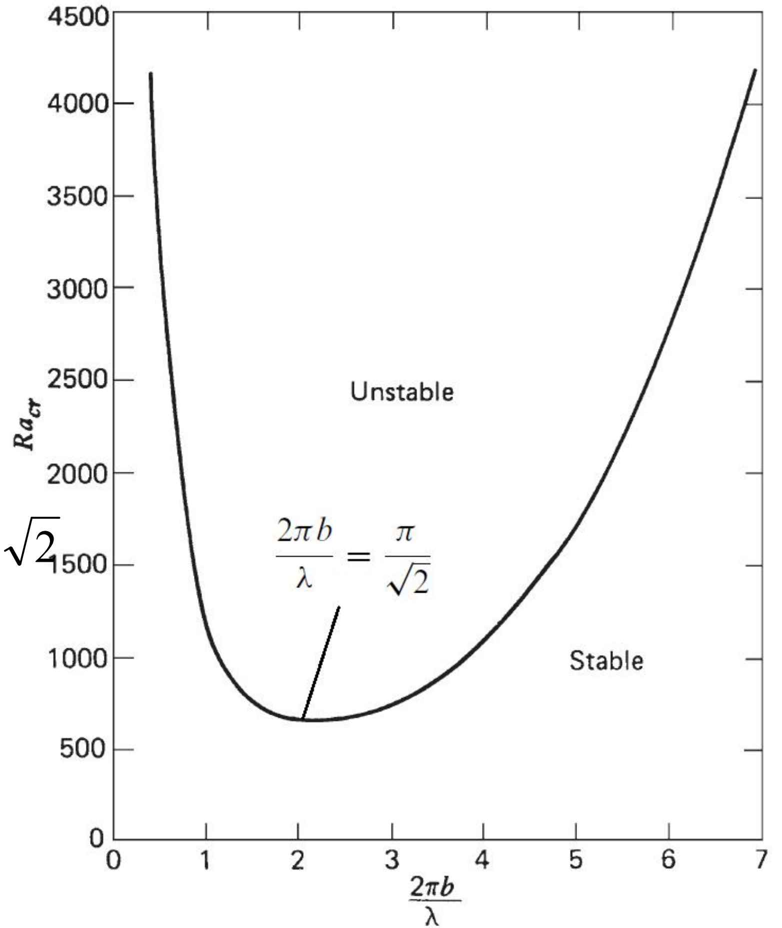
Rayleigh Number and Onset of Thermal Convection

The critical Rayleigh number depends on the wavelength of the initial temperature perturbation

The minimum critical Rayleigh number is at

$$\frac{2\pi b}{\lambda} = \frac{\pi}{\sqrt{2}} \quad \text{or} \quad \lambda = 2\sqrt{2}\, b$$

Corresponding to a convective cell aspect ratio of $\sqrt{2}$



Classroom exercise

Consider a Newtonian fluid with dynamic viscosity μ flowing between two parallel plates separated by a distance h. The upper plate moves at a velocity V, while the lower plate remains stationary. Determine the expression for the velocity distribution, strain rate, and stress of the fluid.