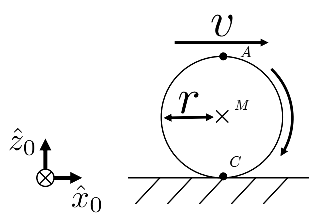
1. **(7 points)** A cylinder rolls without slipping in the $\hat{x}_0$ direction. The cylinder has a radius of r and a constant forward speed of v. What is the spatial acceleration of this cylinder expressed in $\{o\}$, ${}^o\mathcal{A}$ and expressed in $\{C\}$,${}^C\mathcal{A}$, where frame $\{C\}$ has the same orientation as frame $\{o\}$ and its origin is at the contact point C.



Solution: In frame $\{o\}$

We know that

$$ {}^o\mathcal{V}_{body} = \begin{bmatrix}0 & v/r & 0 & 0 & 0 & vC_x(t)/r\end{bmatrix}^T \tag{1}$$

so

$$\begin{aligned} {}^o\mathcal{A}_{body} &= {}^o\mathring{\mathcal{V}}_{body} \\ &= \begin{bmatrix}0 & \dot{v}/r & 0 & 0 & 0 & (\dot{v}C_x(t) + v^2)/r\end{bmatrix} \\ &= \begin{bmatrix}0 & 0 & 0 & 0 & 0 & v^2/r\end{bmatrix} \end{aligned} \tag{2}$$

In frame $\{C\}$

We know that

$$ {}^C\mathcal{V}_{body} = \begin{bmatrix}0 & v/r & 0 & 0 & 0 & 0\end{bmatrix}^T \tag{3}$$

so

$$\begin{aligned} {}^C\mathcal{A}_{body} &= {}^C\mathring{\mathcal{V}}_{body} + {}^C\mathcal{V}_C \times {}^C\mathcal{V}_{body} \\ &= \begin{bmatrix}0\\ \dot{v}/r\\ 0\\ 0\\ 0\\ 0\end{bmatrix} + \begin{bmatrix}\mathbf{0} & \mathbf{0}\\ [{}^Cv_C] & \mathbf{0}\end{bmatrix}\begin{bmatrix}0\\ v/r\\ 0\\ 0\\ 0\\ 0\end{bmatrix} \end{aligned} \tag{4}$$

That is

$$\begin{aligned} {}^C\mathcal{A}_{body} &= \begin{bmatrix}0 & \dot{v}/r & 0 & 0 & 0 & v^2/r\end{bmatrix}^T \\ &= \begin{bmatrix}0 & 0 & 0 & 0 & 0 & v^2/r\end{bmatrix} \end{aligned} \tag{5}$$

Or you can use adjoint transformation

$$ {}^C\mathcal{A}_{body} = {}^CX_o\,{}^o\mathcal{A}_{body} \tag{6}$$

□

2. **(7 points)** Let ${}^O T_A = (R, p)$ be the pose of frame A. Suppose A is moving with velocity ${}^O\mathcal{V}_A = (\omega, v)$. Show that

$$\frac{d}{dt}[{}^O X_A^*] = \begin{bmatrix} [\omega] & [v] \\ 0 & [\omega] \end{bmatrix} {}^O X_A^*$$

Solution:

$$ {}^O X_A^* = \begin{bmatrix} R & [p]R \\ 0 & R \end{bmatrix} \tag{7}$$

$$\frac{d}{dt}[{}^O X_A^*] = \begin{bmatrix} \dot{R} & \frac{d}{dt}[p]R \\ 0 & \dot{R} \end{bmatrix} \tag{8}$$

We know that $\dot{R} = [\omega]R$ and

$$\begin{aligned} \frac{d}{dt}[p]R &= \dot{[p]}R + p\dot{R} \\ &= [\dot{p}R] + [p][\omega]R \\ &= [v + \omega \times p]R + [p][\omega]R \\ &= [v]R + [\omega][p]R - [p][\omega]R + [p][\omega]R \\ &= [v]R + [\omega][p]R \end{aligned} \tag{9}$$

So

$$\begin{aligned} \frac{d}{dt}[{}^O X_A^*] &= \begin{bmatrix} [\omega]R & [v]R + [\omega][p]R \\ 0 & [\omega]R \end{bmatrix} \\ &= \begin{bmatrix} [\omega] & [v] \\ 0 & [\omega] \end{bmatrix} \begin{bmatrix} R & [p]R \\ 0 & R \end{bmatrix} \\ &= \begin{bmatrix} [\omega] & [v] \\ 0 & [\omega] \end{bmatrix} {}^O X_A^* \end{aligned} \tag{10}$$

□

3. **(7 points)** A rigid body is a collection of point masses m_i as location p_i. Given a reference point o, the angular momentum of point mass i is $\overrightarrow{op_i} \times m_i v_i$. Given the definition of the angular momentum of the rigid body $\phi_o = \sum_i \overrightarrow{op_i} \times m_i v_i$, show that for any reference point o and q, we have

$$\phi_q = \phi_o + \overrightarrow{qo} \times L \tag{11}$$

where L is the linear momentum of the rigid body.

Solution:

$$\phi_q = \sum_i \overrightarrow{qp_i} \times m_i v_i \tag{12}$$

Since $\overrightarrow{qp_i} = \overrightarrow{qo} + \overrightarrow{op_i}$, we have

$$\begin{aligned}\phi_q &= \sum_i \overrightarrow{op_i} \times m_i v_i + \sum_i \overrightarrow{qo} \times m_i v_i \\ &= \phi_o + \overrightarrow{qo} \times \sum_i m_i v_i\end{aligned} \tag{13}$$

Given $L = \sum_i m_i v_i$, we have

$$\phi_q = \phi_o + \overrightarrow{qo} \times L \tag{14}$$

$\square$

4. **(7 points)** Given our derivation in class, we have $M(\theta) = \sum_i J_i^T \mathcal{I}_i J_i$ and $c(\theta, \dot{\theta}) = \sum_i J_i^T (\mathcal{I}_i \dot{J}_i + \mathcal{I}_i V_i \times J_i + V_i \times^* \mathcal{I}_i J_i)$. Prove that $\dot{M} - 2c$ is skew symmetric.

Solution: We have

$$\dot{M} = \dot{J}_i \mathcal{I}_i J_i + J_i^T \mathcal{I}_i \dot{J}_i \tag{15}$$

which leads

$$\dot{M} - 2c = \dot{J}_i \mathcal{I}_i J_i - J_i^T \mathcal{I}_i \dot{J}_i - 2J_i^T \left(\mathcal{I}_i[\mathcal{V}_i \times] + [\mathcal{V}_i \times^*]\mathcal{I}_i\right) J_i \tag{16}$$

Using $\mathcal{I}_i^T = \mathcal{I}_i$ and $[\mathcal{V}_i \times^*] = -[\mathcal{V}_i \times]^T$

$$\left(\dot{J}_i \mathcal{I}_i J_i - J_i^T \mathcal{I}_i \dot{J}_i\right)^T = -\left(\dot{J}_i \mathcal{I}_i J_i - J_i^T \mathcal{I}_i \dot{J}_i\right) \tag{17}$$

and

$$\begin{aligned}\left(\mathcal{I}_i[\mathcal{V}_i \times] + [\mathcal{V}_i \times^*]\mathcal{I}_i\right)^T &= [\mathcal{V}_i \times]^T \mathcal{I}_i - \mathcal{I}_i[\mathcal{V} \times] \\ &= -\left(\mathcal{I}_i[\mathcal{V} \times] + [\mathcal{V}_i \times^*]\mathcal{I}_i\right)\end{aligned} \tag{18}$$

So $\dot{M} - 2c$ is skew-symmetric. $\square$