## **Question 1**

# 1. Verify the tensor notations of the 2nd and 3rd stress invariants

$$\begin{split} I_1 &= \tau_{11} + \tau_{22} + \tau_{33} = \tau_{ii} \\ I_2 &= \tau_{11} \tau_{22} + \tau_{22} \tau_{33} + \tau_{33} \tau_{11} - \tau_{12}^2 - \tau_{23}^2 - \tau_{31}^2 = \frac{1}{2} (\tau_{ii} \tau_{kk} - \tau_{ik} \tau_{ki}) \\ I_3 &= \tau_{11} \tau_{22} \tau_{33} + 2 \tau_{12} \tau_{23} \tau_{31} - \tau_{11} \tau_{23}^2 - \tau_{22} \tau_{31}^2 - \tau_{33} \tau_{12}^2 = \boldsymbol{\varepsilon}_{iik} \boldsymbol{\tau}_{1i} \boldsymbol{\tau}_{2i} \boldsymbol{\tau}_{3k} = \boldsymbol{\tau}_1 \cdot \boldsymbol{\tau}_2 \times \boldsymbol{\tau}_3 = \det(\boldsymbol{\tau} i j) \end{split}$$

The definition of the principal invariants is from the well-known Cayley-Hamilton eqution, as:

$$\tau^3 - I_1 \tau^2 + I_2 \tau - I_3 = 0,$$

where

$$egin{aligned} I_1 &= \mathrm{tr} oldsymbol{ au} = oldsymbol{ au}: oldsymbol{I} = au_{ij} \delta_{ij} = au_{ii}, \ I_2 &= rac{1}{2} ig( (\mathrm{tr} oldsymbol{ au})^2 - \mathrm{tr} (oldsymbol{ au}^2) ig) = rac{1}{2} ig( (oldsymbol{ au}: oldsymbol{I})^2 - oldsymbol{ au}: oldsymbol{ au}^{\mathrm{T}} ig) = rac{1}{2} ig( (oldsymbol{ au}: oldsymbol{I})^2 - oldsymbol{ au}: oldsymbol{ au}^{\mathrm{T}} ig) = rac{1}{2} ig( au_{ii} au_{kk} - au_{ik} au_{ki} ig) \\ &= au_{11} au_{22} + au_{22} au_{33} + au_{33} au_{11} - au_{12}^2 - au_{23}^2 - au_{31}^2, \end{aligned}$$

and

$$egin{aligned} I_3 &= \det m{ au} = (m{u} imes m{v}) \cdot m{w} & ext{with} \quad m{ au} = [m{u} \quad m{v} \quad m{w}] \ &= arepsilon_{ijk} ta u_{1i} au_{2j} au_{3k} \ &= au_{11} au_{22} au_{33} + 2 au_{12} au_{23} au_{31} - au_{11} au_{23}^2 - au_{22} au_{31}^2 - au_{33} au_{12}^2. \end{aligned}$$

#### **Question 2**

2. Use the tensor notation of Hooke's law above to (1) establish the relation below and (2) establish the new alternative Hooke's law below

$$e_{ii} = \frac{1}{K} \frac{\tau_{ii}}{3} = \frac{1}{K} \frac{\Theta}{3}$$
  $\qquad \qquad \tau_{ij} = 2G\varepsilon_{ij} + \lambda \varepsilon \delta_{ij} \qquad \text{where } \varepsilon = \varepsilon_{kk}$ 

K is the bulk modulus of elasticity

The tensor notation of the Hooke's law is as follows:

$$arepsilon_{ij} = rac{1+
u}{E} au_{ij} - rac{
u}{E}\delta_{ij}\Theta \quad ext{with} \quad \Theta = au_{kk} = au_{11} + au_{22} + au_{33}$$

 $\delta_{ij}=3$  when taking i=j, then the Hooke's law turns into

$$arepsilon_{ii} = rac{1+
u}{E} au_{ii} - rac{3
u}{E}\Theta = rac{1-2
u}{E}\Theta = rac{1}{K}rac{ au_{ii}}{3} = rac{1}{K}rac{\Theta}{3} \quad ext{with} \quad K = rac{E}{3(1-2
u)}.$$

Use the equality of  $\Theta = \tau_{ii}$  and  $\tau_{ii} = 3K\varepsilon_{ii}$  to substitute into the Hooke's law, which is restructured as

$$\varepsilon_{ij} = \frac{1+\nu}{E} \tau_{ij} - \frac{3K\nu}{E} \delta_{ij} \varepsilon_{ii} = \frac{1+\nu}{E} \tau_{ij} - \frac{\nu}{1-2\nu} \delta_{ij} \varepsilon \quad \text{with} \quad \varepsilon = \varepsilon_{ii}$$

$$\varepsilon_{ij} + \frac{\nu}{1-2\nu} \delta_{ij} \varepsilon = \frac{1+\nu}{E} \tau_{ij}$$

$$\tau_{ij} = \frac{E}{1+\nu} \varepsilon_{ij} + \frac{E\nu}{(1-2\nu)(1+\nu)} \delta_{ij} \varepsilon$$

$$\tau_{ij} = 2G\varepsilon_{ij} + \lambda \varepsilon \delta_{ij} \quad \text{with} \quad G = \frac{E}{2(1+\nu)} \quad \text{and} \quad \lambda = \frac{E\nu}{(1-2\nu)(1+\nu)}.$$

#### **Question 3**

3. Given the governing equations of elasticity in the red box, derive the Navier's equation (equilibrium in the form of displacement) below:

# Governing equations

$$\tau_{ik,i} + f_k = 0$$

$$\tau_{ij} = \delta_{ij}\lambda e + 2Ge_{ij}$$

$$\varepsilon_{ij} = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}) = \frac{1}{2}(u_{i,j} + u_{j,i})$$
Navier's equation
$$(\lambda + G)u_{i,ik} + G\nabla^2 u_k + f_k = 0$$

Rewrite the constitutive eqution, as

$$au_{ik} = \delta_{ik} \lambda arepsilon_{jj} + 2G arepsilon_{ik},$$

and implement the divergence operator to both sides of the above eqution, as

$$au_{ik,i} = \delta_{ik,i} \lambda \varepsilon_{jj} + \delta_{ik} \lambda \varepsilon_{jj,i} + 2G \varepsilon_{ik,i} = \delta_{ik} \lambda \varepsilon_{jj,i} + 2G \varepsilon_{ik,i},$$

so that the equilibrium equation becomes:

$$\delta_{ik}\lambda\varepsilon_{jj,i} + 2G\varepsilon_{ik,i} + f_k = 0. \tag{1}$$

The displacement equation is

$$arepsilon_{ik} = rac{1}{2}(u_{i,k} + u_{k,i}),$$

and substitute it into equation (1), and we get

$$G(u_{i,ki}+u_{k,ik})+\delta_{ik}\lambda u_{j,ji}+f_k=0 \ Gu_{i,ki}+Gu_{j,ij}+\lambda u_{j,ji}+f_k=0 \ (G+\lambda)u_{j,jk}+G
abla\cdot
abla V oldsymbol{u}+f_k=0 \ (G+\lambda)u_{i,ik}+G
abla^2oldsymbol{u}+f_k=0.$$

### **Question 4**

The rotation tensor  $w_{ik}$  is an antisymmetric tensor  $w_{ik} = (\partial u_k / \partial x_i - \partial u_i / \partial x_k)/2$ 

Prove that the curl of displacement  $u_k$ :  $v_m = \varepsilon_{mik} \nabla_i u_k$  is related to the rotation tensor  $w_{ik}$ 

$$v_{m} = \varepsilon_{mik} w_{ik} = \varepsilon_{mik} \left( \frac{\partial u_{k}}{\partial x_{i}} - \frac{\partial u_{i}}{\partial x_{k}} \right) / 2$$

$$w_{ik} = \varepsilon_{mik}(\frac{1}{2}v_m) = \varepsilon_{ikm}(\frac{1}{2}v_m) = \frac{1}{2}\varepsilon_{ikm}\varepsilon_{mqr}\frac{\partial u_r}{\partial x_q}$$

Note: the alternating tensor and Kronecker delta have the following relation:

$$\epsilon_{ijk}\epsilon_{klm}=\delta_{il}\delta_{jm}-\delta_{im}\delta_{jl}$$

Prove that (1) the infinitesimal rotation angle vector  $\mathbf{w}_m$  equals half the curl of displacement  $u_k$   $\mathbf{w}_m = \frac{1}{2} \xi_{mik} \nabla_i \mathbf{u}_k$ 2) rotation displacement can be calculated with cross product between  $\mathbf{w}_m$  and  $\mathbf{d}\mathbf{x}$ 

$$d\mathbf{u} = d\mathbf{u}_D + d\mathbf{u}_R \qquad d\mathbf{u}_R = \omega_{ij} dx_j = \mathbf{w}_m \times d\mathbf{x}$$

Deformation Rotation infinitesimal rotation angle vector

The rotation tensor is the anti-symmetric part of the deformation gradient:

$$oldsymbol{F} = rac{\partial oldsymbol{u}}{\partial oldsymbol{x}}, \quad oldsymbol{F} = oldsymbol{D} + oldsymbol{W}, \quad F_{ij} = rac{\partial u_i}{\partial x_j} = u_{i,j} \ oldsymbol{W} = rac{1}{2}ig(oldsymbol{F} - oldsymbol{F}^{\mathrm{T}}ig), \quad W_{ij} = rac{1}{2}(F_{ij} - F_{ji}), \quad oldsymbol{W} = -oldsymbol{W}^{\mathrm{T}} \ oldsymbol{D} = rac{1}{2}ig(oldsymbol{F} + oldsymbol{F}^{\mathrm{T}}ig), \quad oldsymbol{D} = oldsymbol{D}^{\mathrm{T}}.$$

The curl of the displacement is defined as

$$oldsymbol{v} = 
abla imes oldsymbol{u}, \quad v_m = arepsilon_{mik} u_{k,i}.$$

The deformation gradient  $u_{k,i}$  can be decomposed into the (anti-)symmetric parts, as

$$v_m = arepsilon_{mik} \left[rac{1}{2}(u_{i,k} + u_{k,i}) + rac{1}{2}(u_{i,k} - u_{k,i})
ight] = arepsilon_{mik} \left(D_{ik} + W_{ik}
ight).$$

Due to the symmetry of  $D_k i$ , we have the fact that

$$arepsilon_{mki}D_{ki}=arepsilon_{mik}D_{ik}=arepsilon_{mik}D_{ki}=-arepsilon_{mki}D_{ki}\Rightarrowarepsilon_{mik}D_{ki}=0.$$

So that we can prove that

$$v_m = arepsilon_{mik} W_{ik} = arepsilon_{mik} \cdot rac{1}{2} (u_{i,k} - u_{k,i}).$$

Multiplicate  $\frac{1}{2}\varepsilon_{mqr}$  to both sides of the above, as

$$arepsilon_{mqr}v_m = arepsilon_{qrm}arepsilon_{mik}W_{ik} = (\delta_{qi}\delta_{rk} - \delta_{qk}\delta_{ir})W_{ik} = W_{qr} - W_{rq} = 2W_{qr} \Rightarrow W_{qr} = rac{1}{2}arepsilon_{mqr}v_m$$

We can substitute  $v_m = arepsilon_{mik} u_{i,k}$  into it, as

$$W_{qr} = rac{1}{2}arepsilon_{mqr}arepsilon_{mik}u_{i,k} \Rightarrow W_{ik} = rac{1}{2}arepsilon_{mik}arepsilon_{mqr}u_{q,r}.$$

The next prove begins:

$$egin{align*} doldsymbol{u} = doldsymbol{u}_{
m D} + doldsymbol{u}_{
m R} = oldsymbol{D} \cdot doldsymbol{x} + oldsymbol{W} \cdot doldsymbol{x} \ w_m = rac{1}{2}arepsilon_{mik}u_{i,k} = rac{1}{2}v_m \Rightarrow oldsymbol{w} = rac{1}{2}oldsymbol{v} \ doldsymbol{u}_{
m R} = oldsymbol{W} \cdot doldsymbol{x}, \quad du_{
m R,i} = W_{i,k}dx_k = rac{1}{2}arepsilon_{mik}v_m dx_k \Rightarrow doldsymbol{u}_{
m R} = oldsymbol{w} imes doldsymbol{x}. \end{gathered}$$