南方科技大学
SOUTHERN UNIVERSITY OF SCIENCE AND TECHNOLOGY

MAE5009
Continuum Mechanics B
Session 06: Energy Methods

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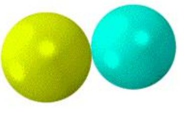
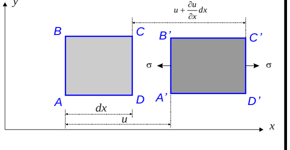
Strain energy

- Strain energy: the energy absorbed in the body due to external work

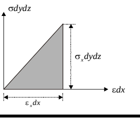


$$dU = \int_{\sigma=0}^{\sigma=\sigma_x} -\sigma dydz \cdot du + \int_{\sigma=0}^{\sigma=\sigma_x} \sigma dydz \cdot d\left(u + \frac{\partial u}{\partial x}dx\right)$$
$$= \int_{\sigma=0}^{\sigma=\sigma_x} \sigma d\left(\frac{\partial u}{\partial x}\right) dxdydz$$
$$= \int_{\sigma=0}^{\sigma=\sigma_x} \sigma d\varepsilon dxdydz = \int_{\sigma=0}^{\sigma=\sigma_x} \sigma \frac{d\sigma}{E} dxdydz$$
$$= \frac{\sigma_x^{\ 2}}{2E} dxdydz$$



Normal strain energy density:

$$U_0 = \frac{\sigma_x^{\ 2}}{2E} = \frac{1}{2}\sigma_x\varepsilon_x = \frac{1}{2}E\varepsilon_x^{\ 2}$$



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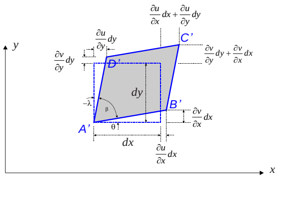
Strain energy

Shear strain energy:

$$dU = \frac{1}{2}\left(\tau_{xy}dydz\right)\left(\frac{\partial v}{\partial x}dx\right) + \frac{1}{2}\left(\tau_{yx}dxdz\right)\left(\frac{\partial u}{\partial y}dy\right)$$
$$= \frac{1}{2}\tau_{xy}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)dxdydz$$
$$= \frac{1}{2}\tau_{xy}\gamma_{xy}dxdydz$$



Shear strain energy density:

$$U_0 = \frac{1}{2}\tau_{xy}\gamma_{xy} = \frac{1}{2G}\tau_{xy}^{\ 2}$$
$$= \frac{1}{2}\tau_{xy}\varepsilon_{xy} + \frac{1}{2}\tau_{yx}\varepsilon_{yx}$$

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Strain energy

- Strain energy due to σ_x and σ_y:

$$U_0 = \frac{1}{2}\sigma_x\varepsilon_x + \frac{1}{2}\sigma_y\varepsilon_y$$

- Under a general stress condition:

$$U_0 = \frac{1}{2}\left(\sigma_x\varepsilon_x + \sigma_y\varepsilon_y + \sigma_z\varepsilon_z + \tau_{xy}\gamma_{xy} + \tau_{yz}\gamma_{yz} + \tau_{zx}\gamma_{zx}\right)$$

In terms of stress components:

$$U_0 = \frac{1}{2}\left(\frac{1}{E}\left(\sigma_x^2 + \sigma_y^2 + \sigma_z^2\right) - \frac{2\nu}{E}\left(\sigma_x\sigma_y + \sigma_x\sigma_z + \sigma_y\sigma_z\right) + \frac{1}{G}\left(\tau_{xy}^2 + \tau_{xz}^2 + \tau_{yz}^2\right)\right)$$

In terms of strain components:

$$U_0 = \frac{1}{2}\left(\lambda\varepsilon^2 + 2G\left(\varepsilon_x^2 + \varepsilon_y^2 + \varepsilon_z^2\right) + G\left(\gamma_{xy}^2 + \gamma_{yz}^2 + \gamma_{zx}^2\right)\right)$$

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Strain energy

- The derivative of U_0 with respect to any stress component is equal to the corresponding strain component, and the reverse is true:

$$\frac{\partial U_0\left(\sigma_x, \sigma_y, \ldots, \tau_{xz}\right)}{\partial \sigma_x} = \varepsilon_x \qquad \frac{\partial U_0\left(\varepsilon_x, \varepsilon_y, \ldots, \gamma_{xz}\right)}{\partial \varepsilon_x} = \sigma_x$$

- The strain energy density is equal to the half of the dot product between the stress and strain vector:

$$U_0 = \frac{1}{2}\left(\sigma_x\varepsilon_x + \sigma_y\varepsilon_y + \sigma_z\varepsilon_z + \tau_{xy}\gamma_{xy} + \tau_{yz}\gamma_{yz} + \tau_{zx}\gamma_{zx}\right)$$
$$= \frac{1}{2}\mathrm{vec}\left(\begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix}\right) \cdot \mathrm{vec}\left(\begin{bmatrix} \varepsilon_x & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_y & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_z \end{bmatrix}\right)$$

where vec() is a vectorization function which stacks all the columns of a matrix together and forms a vector

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