MAE5009: Continuum Mechanics B

Assignment 03: Stress Strain Relations

Due October 21, 2020

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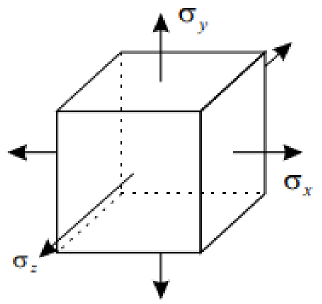
1. Derive the relations between the normal stresses and normal strains by adding the normal stresses on the cube in the following consecutive order: σ_z, σ_y and σ_x.



Solution:

Let the initial length of each sides are l_{x0}, l_{y0} and l_{z0}.

Apply σ_z:

$\varepsilon_z = \frac{\sigma_z}{E}$, $l_{z1} = l_{z0}(1 + \varepsilon_z) = l_{z0}\left(1 + \frac{\sigma_z}{E}\right)$

$\varepsilon_x = \varepsilon_y = -\nu\varepsilon_z = -\frac{\nu\sigma_z}{E}$, $l_{x1} = l_{x0}(1 + \varepsilon_x) = l_{x0}(1 - \frac{\nu\sigma_z}{E})$, $l_{y1} = l_{y0}(1 + \varepsilon_y) = l_{y0}(1 - \frac{\nu\sigma_z}{E})$

Apply σ_y:

$\varepsilon_y = \frac{\sigma_y}{E}$, $l_{y2} = l_{y1}(1 + \varepsilon_y) = l_{y1}\left(1 + \frac{\sigma_y}{E}\right) = l_{y0}(1 - \frac{\nu\sigma_z}{E})\left(1 + \frac{\sigma_y}{E}\right)$

$\varepsilon_x = \varepsilon_z = -\nu\varepsilon_y = -\frac{\nu\sigma_y}{E}$, $l_{x2} = l_{x1}(1 + \varepsilon_x) = l_{x1}\left(1 - \frac{\nu\sigma_y}{E}\right) = l_{x0}(1 - \frac{\nu\sigma_z}{E})\left(1 - \frac{\nu\sigma_y}{E}\right)$, $l_{z2} = l_{z1}(1 + \varepsilon_z) = l_{z1}\left(1 - \frac{\nu\sigma_y}{E}\right) = l_{z0}\left(1 + \frac{\sigma_z}{E}\right)\left(1 - \frac{\nu\sigma_y}{E}\right)$

Apply σ_x:

$\varepsilon_x = \frac{\sigma_x}{E}$, $l_{x3} = l_{x2}(1 + \varepsilon_x) = l_{x2}\left(1 + \frac{\sigma_x}{E}\right) = l_{x0}(1 - \frac{\nu\sigma_z}{E})\left(1 - \frac{\nu\sigma_y}{E}\right)\left(1 + \frac{\sigma_x}{E}\right)$

$\varepsilon_y = \varepsilon_z = -\nu\varepsilon_x = -\frac{\nu\sigma_x}{E}$, $l_{y3} = l_{y2}(1 + \varepsilon_y) = l_{y2}\left(1 - \frac{\nu\sigma_x}{E}\right) = l_{y0}(1 - \frac{\nu\sigma_z}{E})\left(1 + \frac{\sigma_y}{E}\right)\left(1 - \frac{\nu\sigma_x}{E}\right)$, $l_{z3} = l_{z2}(1 + \varepsilon_z) = l_{z2}\left(1 - \frac{\nu\sigma_x}{E}\right) = l_{z0}\left(1 + \frac{\sigma_z}{E}\right)\left(1 - \frac{\nu\sigma_y}{E}\right)\left(1 - \frac{\nu\sigma_x}{E}\right)$

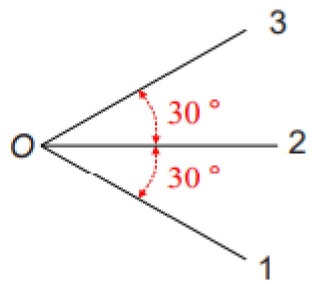
Neglect the negligible items:

$$\varepsilon_x = \frac{L_{x3} - L_{x0}}{L_{x0}} = \frac{1}{E}\left(\sigma_x - \nu(\sigma_y + \sigma_z)\right)$$

$$\varepsilon_y = \frac{L_{y3} - L_{y0}}{L_{y0}} = \frac{1}{E}\left(\sigma_y - \nu(\sigma_x + \sigma_z)\right)$$

$$\varepsilon_z = \frac{L_{z3} - L_{z0}}{L_{z0}} = \frac{1}{E}\left(\sigma_z - \nu(\sigma_x + \sigma_y)\right)$$

2. For a given x-y plane, the normal strains at point O in the $O-1$, $O-2$ and $O-3$ directions are respectively $\varepsilon_{o-1} = 10^{-4}$, $\varepsilon_{o-2} = 4 \times 10^{-4}$ and $\varepsilon_{o-3} = 6 \times 10^{-4}$. Given the material properties $E = 30\,GPa$, $\nu = 0.25$, determine the principal stresses and maximum shear stress at point O and their directions (only consider the stresses and strains in the x-y plane, i.e., a pure 2D problem)



Solution:

Let $\varepsilon_{o-1} = \varepsilon_{x1}$, $\varepsilon_{o-2} = \varepsilon_{x2}$, $\varepsilon_{o-3} = \varepsilon_{x3}$.

$$\varepsilon_{x'} = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2} cos2\alpha + \varepsilon_{xy} sin2\alpha$$

Then

$$\varepsilon_{o-2} = \varepsilon_{x2} = \frac{\varepsilon_{x1} + \varepsilon_{y1}}{2} + \frac{\varepsilon_{x1} - \varepsilon_{y1}}{2}\cos(60°) + \varepsilon_{xy1}\sin(60°)$$

$$\varepsilon_{o-3} = \varepsilon_{x3} = \frac{\varepsilon_{x1} + \varepsilon_{y1}}{2} + \frac{\varepsilon_{x1} - \varepsilon_{y1}}{2}\cos(120°) + \varepsilon_{xy1}\sin(120°)$$

So, we can get: $\varepsilon_{y1} = 5 \times 10^{-4}$, $\varepsilon_{xy1} = \frac{4\sqrt{3}}{3} \times 10^{-4}$.

$$G = \frac{E}{2(1+\nu)} = 12GPa, \lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} = 12GPa$$

$$\sigma_x = 2G\varepsilon_{x1} + \lambda(\varepsilon_{x1} + \varepsilon_{y1}) = 96 \times 10^{-4} GPa$$

$$\sigma_y = 2G\varepsilon_{y1} + \lambda(\varepsilon_{x1} + \varepsilon_{y1}) = 192 \times 10^{-4} GPa$$

$$\tau_{xy} = G\gamma_{xy} = 2G\varepsilon_{xy1} = 32\sqrt{3} \times 10^{-4} GPa$$

The principle stress:

$$\sigma_{max} = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \left(144 + 16\sqrt{21}\right) \times 10^{-4} GPa = 21.73MPa$$

$$\sigma_{min} = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \left(144 - 16\sqrt{21}\right) \times 10^{-4} GPa = 7.07MPa$$

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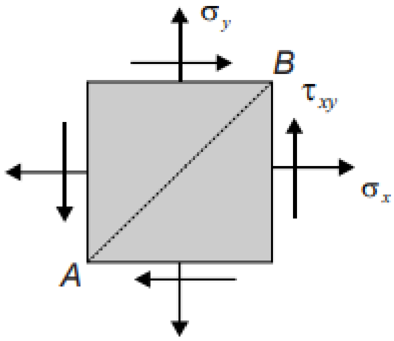
The direction of principle stress: $tan2\alpha = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} = -\frac{2\sqrt{3}}{3}$, $\alpha = -24.55°$

The maximum shear stress: $\tau_{xymax} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = 7.33MPa$

The direction of maximum shear stress: $tan2\alpha = -\frac{\sigma_x - \sigma_y}{2\tau_{xy}} = \frac{\sqrt{3}}{2}$, $\alpha = -20.45°$

3. A homogeneous and isotropic square plate is loaded as shown, where $\sigma_x = \sigma_y = \tau_{xy} = 15\ MPa$. If $E = 10\ GPa$, $\nu = 0.3$, determine the change in length of the diagonal AB.



Solution:

$$\varepsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y) = 1.05 \times 10^{-3}, \varepsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x) = 1.05 \times 10^{-3}$$

$$G = \frac{E}{2(1+\nu)}, \gamma_{xy} = \frac{1}{G}\tau_{xy} = \frac{2(1+\nu)}{E}\tau_{xy} = 3.9 \times 10^{-3}$$

Along AB direction:

$$\varepsilon_{x'} = \frac{\varepsilon_x + \varepsilon_y}{2} + \frac{\varepsilon_x - \varepsilon_y}{2}cos2\alpha + \varepsilon_{xy}sin2\alpha = \frac{\varepsilon_x + \varepsilon_y}{2} + \varepsilon_{xy} = \frac{\varepsilon_x + \varepsilon_y + \gamma_{xy}}{2}$$

$$\varepsilon_{x\prime} = 3 \times 10^{-3}$$

The change in length of AB is $\Delta AB = \varepsilon_{x\prime}AB = 3 \times 10^{-3}AB$.

4. Prove the following relations among various elastic constants:

$$\nu = \frac{3K - E}{6K}$$

$$\lambda = \frac{3K - 2G}{3}$$

$$E = \frac{9K(K - \lambda)}{3K - \lambda}$$

$$G = \frac{3KE}{9K - E}$$

$$K = \frac{EG}{3(3G - E)}$$

Solution:

We already know that: $G = \frac{E}{2(1+\nu)}$, $\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}$ and $K = \frac{E}{3(1-2\nu)}$.

Since $K = \frac{E}{3(1-2\nu)}$, then $1-2\nu = \frac{E}{3K}$, $-2\nu = \frac{-3K+E}{3K}$, we can get $\nu = \frac{3K-E}{6K}$.

Since $G = \frac{E}{2(1+\nu)}$ and $K = \frac{E}{3(1-2\nu)}$, then $2G = \frac{E}{(1+\nu)}$ and $3K = \frac{E}{(1-2\nu)}$, $3K - 2G = \frac{3\nu E}{(1+\nu)(1-2\nu)} = 3\lambda$, we can get $\lambda = \frac{3K-2G}{3}$.

Since $K = \frac{E}{3(1-2\nu)}$, then $3K = \frac{E}{(1-2\nu)}$, $\lambda = \frac{3K\nu}{1+\nu}$. Due to $\nu = \frac{3K-E}{6K}$, then $\lambda = 3K\frac{3K-E}{9K-E}$, we can get $E = \frac{9K(K-\lambda)}{3K-\lambda}$.

Since $G = \frac{E}{2(1+\nu)}$ and $\nu = \frac{3K-E}{6K}$, then $G = \frac{3KE}{9K-E}$.

$9KG - GE = 3KE$, $K(9G-3E) = GE$, then we can get $K = \frac{GE}{9G-3E} = \frac{EG}{3(3G-E)}$.