MEE5114 Spring 2023 **Homework # 2**
Assigned March 1, 2023 Due: March 8th, 2023

1. **(5 points)** Given a linear system $\dot{x} = Ax + Bu$. Its solution is given by

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

Now assume we have $u(t) \equiv u_k$ for $t \in [k\delta t, (k+1)\delta t)$. Please derive the Zero-order-hold discretization rule, namely, derive expressions for A_d and B_d such that

$$x_{k+1} = A_d x_k + B_d u_k$$

where $x_k \triangleq x(k \cdot \delta t)$ and $u_k = u(k \cdot \delta t)$

Solution: We know

$$x_k = e^{Ak\delta t}x(0) + \int_0^{k\delta t} e^{A(k\delta t-\tau)}Bu(\tau)d\tau \tag{1}$$

and

$$\begin{aligned} x_{k+1} &= e^{A(k+1)\delta t}x(0) + \int_0^{(k+1)\delta t} e^{A((k+1)\delta t-\tau)}Bu(\tau)d\tau \\ &= e^{A\delta t}\left(e^{Ak\delta t}x(0) + \int_0^{k\delta t} e^{A(k\delta t-\tau)}Bu(\tau)d\tau\right) + \int_{k\delta t}^{(k+1)\delta t} e^{A((k+1)\delta t-\tau}Bu(\tau)d\tau \end{aligned} \tag{2}$$

Substitute (1) into (2) and let $(k+1)\delta t - \tau = \alpha$ we can get

$$\begin{aligned} x_{k+1} &= e^{A\delta t}x_k + \int_0^{\delta t} e^{A\alpha}Bu((k+1)\delta t-\alpha)d\alpha \\ &= e^{A\delta t}x_k + \int_0^{\delta t} e^{A\tau}d\tau Bu_k \end{aligned} \tag{3}$$

So

$$A_d = e^{A\delta t} \qquad B_d = \int_0^{\delta t} e^{A\tau}d\tau B$$

and

$$\begin{aligned} \int_0^{\delta t} e^{A\tau}d\tau &= \int_0^{\delta t}\left(I + A\tau + \frac{A^2}{2!}\tau^2\right)d\tau \\ &= \delta t I + \frac{A}{2!}\delta t^2 + \frac{A^2}{3!}\delta t^3 + \dots \end{aligned} \tag{4}$$

□

2. **Spatial Velocity:** **(2×6 points)** A cylinder rolls without slipping in the $\hat{x}_0$ direction on the $\hat{x}_0 - \hat{y}_0$ plane. The cylinder has a radius of r and a constant forward speed of v. Let ${}^0C=[C_x(t),0,0]^T$ be the position of the contact point at time t. Let ${}^0A=[A_x(t),0,0]^T$ be the position of the instantaneous top of the cylinder at time t.

   (a) What is the linear velocity of the point C? (hint: just need to compute $\frac{d}{dt}C_x(t)$)?

(b) What is the linear velocity of the point A?

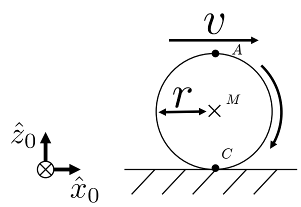
(c) What is velocity of the body-fixed point currently coincides with C?

(d) What is velocity of the body-fixed point currently coincides with A?

(e) What is the spatial velocity of the cylinder in {0}-frame?

(f) What is the spatial velocity of the cylinder in frame {C}? ({C} has the same orientation as {0}, while its origin is at the contact point C)

Note: The first 4 questions are all referring to the inertia frame {0}



Solution: (a): The constant forward speed is v. So

$$\frac{dC_x(t)}{dt} = v \tag{5}$$

The linear velocity of the point C is $[v, 0, 0]^T$.

(b): Similarly to (a)

$$\frac{dA_x(t)}{dt} = v \tag{6}$$

The linear velocity of the point A is $[v, 0, 0]^T$.

(c): We can get the angular velocity of the cylinder as

$$^0\omega = \begin{bmatrix} 0 \\ \frac{v}{r} \\ 0 \end{bmatrix} \tag{7}$$

So the velocity of the body-fixed point currently coincides with C is

$$^0v_C = \begin{bmatrix} v - |^0\omega|r \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{8}$$

Or $^0v_C = {^0v_M} + {^0\omega} \times {^0(\overrightarrow{MC})} = [0, 0, 0]^T$

(d):Similarly to (c), the velocity of the body-fixed point currently coincides with A is

$$ {}^0v_A = \begin{bmatrix} 2v \\ 0 \\ 0 \end{bmatrix} \tag{9} $$

Or ${}^0v_A = {}^0v_M + {}^0\omega \times {}^0(\overrightarrow{MA}) = [2v, 0, 0]^T$

(e): The velocity of the body-fixed point currently coincides with the origin of frame $\{0\}$ is

$$ 0v_0 = \begin{bmatrix} v - |{}^0\omega|r \\ 0 \\ |{}^0\omega|C_x(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \frac{vC_x(t)}{r} \end{bmatrix} \tag{10} $$

So the spatial velocity of the cylinder in $\{0\}$-frame is

$$ {}^0\mathcal{V} = \begin{bmatrix} 0 \\ \frac{v}{r} \\ 0 \\ 0 \\ 0 \\ \frac{vC_x(t)}{r} \end{bmatrix} \tag{11} $$

(f): Similarly to (e), the spatial velocity of the cylinder in frame $\{C\}$ is

$$ {}^C\mathcal{V} = \begin{bmatrix} 0 \\ \frac{v}{r} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{12} $$

□

3. **Spatial Velocity:** (2×8 **points)** Modern Robotics: Exercise 5.5

Solution: We can construct the twist (in spatial coordinates) for the revolute joint as

$$ \mathcal{V} = \begin{bmatrix} \omega \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} L \\ L \\ 0 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ L \\ -L \\ 0 \end{bmatrix} \tag{13} $$

(a): The position of P is

$$\begin{aligned}
{}^{s}P(t) &= e^{[\mathcal{V}]\theta s}P(0) \\
&= \begin{bmatrix} \cos\theta & -\sin\theta & 0 & L + L\sin\theta - L\cos\theta \\ \sin\theta & \cos\theta & 0 & L - L\sin\theta - L\cos\theta \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} L \\ L \\ 0 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} L + d\sin\theta \\ L - d\cos\theta \\ 0 \\ 1 \end{bmatrix}
\end{aligned} \tag{14}$$

(b): The velocity of point P in terms of the fixed frame is

$$\begin{aligned}
\dot{P} &= [V_s]P(t) \\
&= \dot{\theta} \begin{bmatrix} 0 & -1 & 0 & L \\ 1 & 0 & 0 & -L \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} L + d\sin\theta \\ L - d\sin\theta \\ 0 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} d\cos\theta \\ d\sin\theta \\ 0 \\ 0 \end{bmatrix}
\end{aligned} \tag{15}$$

or

$$\dot{P} = \frac{d^{s}P(t)}{dt} = \begin{bmatrix} d\cos\theta \\ d\sin\theta \\ 0 \\ 0 \end{bmatrix} \tag{16}$$

(c):

$$\begin{aligned}
T_{sb}(t) &= e^{[\mathcal{V}]\theta}T_{sb}(0) \\
&= \begin{bmatrix} \cos\theta & -\sin\theta & 0 & L + d\sin\theta \\ \sin\theta & \cos\theta & 0 & L - d\cos\theta \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned} \tag{17}$$

(d): The twist of T_{sb} in body coordinates is

$$\mathcal{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ d \\ 0 \\ 0 \end{bmatrix} \tag{18}$$

(e): The twist of T_{sb} in spatial coordinates is

$$\mathcal{V} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ L \\ -L \\ 0 \end{bmatrix} \tag{19}$$

(f):

$$\begin{aligned} \mathcal{V} &= \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & L - d\cos\theta & \cos\theta & -\sin\theta & 0 \\ 0 & 0 & -L - d\sin\theta & \sin\theta & \cos\theta & 0 \\ L(\sin\theta - \cos\theta) + d & L(\cos\theta + \sin\theta) & 0 & 0 & 0 & 1 \end{bmatrix} \mathcal{B} \\ &= \begin{bmatrix} R & 0 \\ [p]R & R \end{bmatrix} \mathcal{B} \\ &= [\mathrm{Ad}_T]\mathcal{B} \end{aligned} \tag{20}$$

(g): Let

$$\mathcal{B} = \begin{bmatrix} \omega^b \\ v^b \end{bmatrix} \tag{21}$$

We have

$${}^aR_b^T \dot{P} = v^b \tag{22}$$

And we also have

$$T_{sb}^{-1} \dot{P} = [\mathcal{B}]^b P(t) \tag{23}$$

(h): Let

$$\mathcal{V} = \begin{bmatrix} \omega \\ v \end{bmatrix} \tag{24}$$

We have

$$-[\omega]^s P(t) + \dot{P} = v \tag{25}$$

And we also have

$$\dot{P} = [\mathcal{V}]^s P(t) \tag{26}$$

$\square$

4. **Screw axis and its transformation:** **(3×3 points)**

   (a) Draw the screw axis for the twist $\mathcal{V} = (0, 2, 2, 4, 0, 0)$

   (b) Consider an arbitrary screw axis $\mathcal{S}$. Suppose the axis has gone through a rigid body transformation $T = (R, p)$ and the resulting new screw axis is $\mathcal{S}'$. Show that

   $$\mathcal{S}' = [\mathrm{Ad}_T]\,\mathcal{S}$$

   (we have given the proof in class, you need to go through it on your own again)

   (c) Consider a rigid body motion: rotation about z axis counterclockwise by 90^o and then translate along negative y-axis by 1m. All the axes are with respect to the fixed inertia frame.

      i. Find the numerical values of the corresponding transformation matrix T;

      ii. Move the screw axis in part (a) using T. Find the new screw axis $\mathcal{S}'$ after the motion.

Solution: (a): The axis is

$$l = \left\{ \frac{\omega \times v}{\|w\|^2} + \lambda\omega : \lambda \in \mathbb{R} \right\}$$

(b): We define the original frame is {A} and there is a screw axis $\mathcal{S}$ in {A}. After the transformation (a rigid motion) {A} becomes {B}, the corresponding screw axis is $\mathcal{S}'$. But the coordinate represented in their own frame is the same which is

$$^A\mathcal{S} = {}^B\mathcal{S}' \tag{27}$$

We already know there exist a twist corresponding to a screw motion. In some sense we can regard it as a "same" thing which means the adjoint transformation for change of coordinates can be applied on a screw axis. With slight abuse of notation, left multiply $[\mathrm{Ad}_{^AT_B}]$ in (27)

$$[\mathrm{Ad}_{^AT_B}]^A\mathcal{S} = [\mathrm{Ad}_{^AT_B}]^B\mathcal{S}' \tag{28}$$

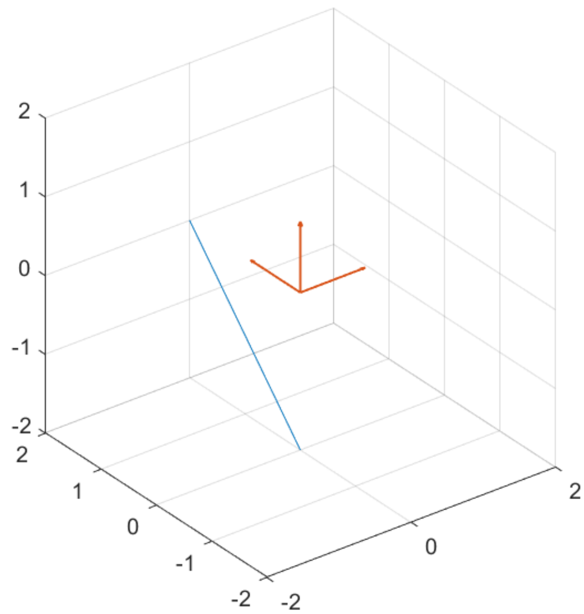


Figure 1: Axis for $\mathcal{V}$

So we have

$$[\mathrm{Ad}_{{}^AT_B}]^A\mathcal{S} = {}^A\mathcal{S}' \tag{29}$$

Omit superscript and subscript

$$^A\mathcal{S}' = [\mathrm{Ad}_T]\mathcal{S} \tag{30}$$

(c):

(i):

$$T = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{31}$$

(ii):

$$\mathcal{S}' = [\mathrm{Ad}_T]\mathcal{S} = \begin{bmatrix} R & 0 \\ [p]R & R \end{bmatrix} \mathcal{S} = \begin{bmatrix} -2 \\ 0 \\ 2 \\ -2 \\ 4 \\ -2 \end{bmatrix} \tag{32}$$

The screw coordinates of $\mathcal{S}'$ is

$$\begin{aligned} &\text{Pitch:} \quad h = \frac{\omega^T v}{\|\omega\|^2} = 0 \\ &\text{Axis:} \quad l = \{[-1,-1,-1]^T + \lambda[-2,0,2] : \lambda \in \mathbb{R}\} \\ &\text{Magnitude:} \quad M = 2\sqrt{2} \end{aligned} \tag{33}$$

See Fig. 2.

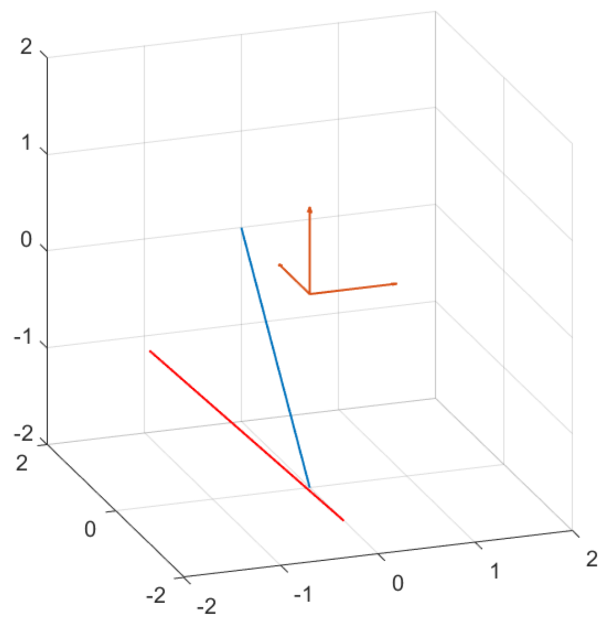


Figure 2: Screw axis after transformation

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