Assignment 1: Stress

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1. Given $\boldsymbol{\sigma_x = -14MPa}$, $\boldsymbol{\sigma_y = 6MPa}$, and $\boldsymbol{\tau_{xy} = -17MPa}$, determine both by formulas and by the Mohr's circle,

(a) the principal stresses and their directions

Solution:

Let α_1, α_2 are the angles between this stress state and the principal stresses.

$\tan(2\alpha) = \frac{2\times(-17)}{-14-6} = \frac{17}{10}$, so $\alpha_1 = 29.77°, \alpha_2 = 119.77°$

The principal stresses:

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = (-4 + \sqrt{389})MPa$$

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = (-4 - \sqrt{389})MPa$$

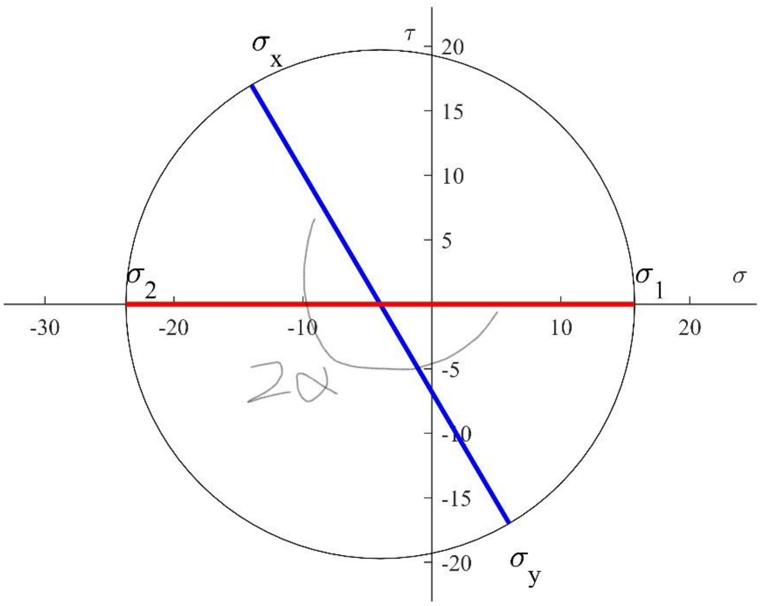
The corresponding directions are $\alpha_1 = 119.77°, \alpha_2 = 29.77°$.

$\frac{\sigma_x+\sigma_y}{2} = -4MPa$, the center of Mohr's circle is $(-4,0)$.

The radius of Mohr's circle is $\sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2 + \tau_{xy}^2} = \sqrt{389}$.

The stress state presented on the Mohr's circle are points, $(-14,17)$ and $(6,-17)$.

The principal stresses and their directions are shown in the figure.



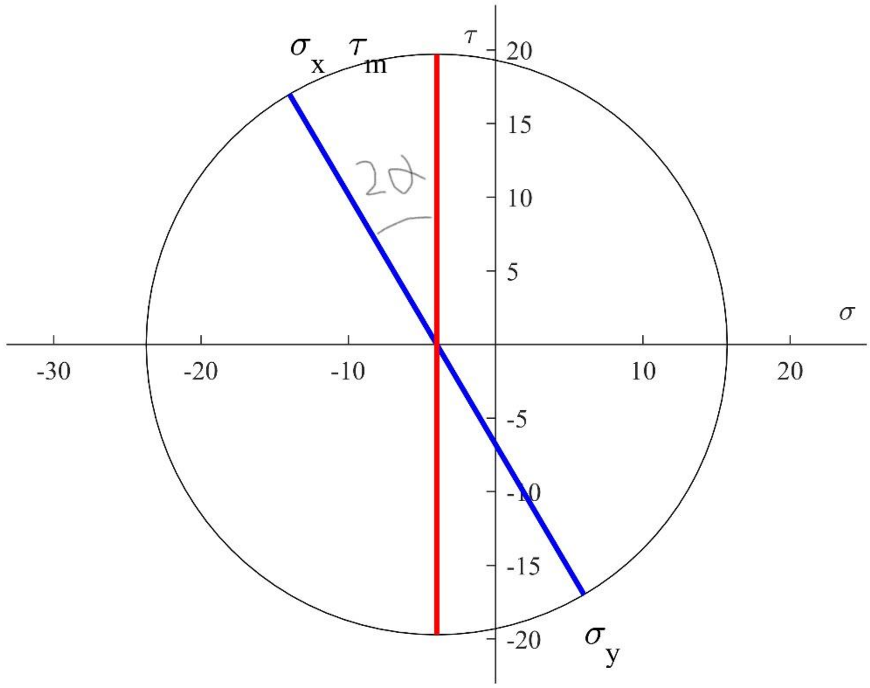
(b) the direction having the maximum shear stress and the corresponding shear and normal stress magnitudes

Solution:

$$\tan(2\alpha) = -\frac{\sigma_x - \sigma_y}{2\tau_{xy}} = -\frac{10}{17}, \alpha = -15.23°$$

$$\tau_{x'y'max} = \pm\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}{}^2} = \pm\sqrt{389}MPa$$

$$\sigma_{x'} = \sigma_{y'} = \frac{\sigma_x + \sigma_y}{2} = -4MPa$$



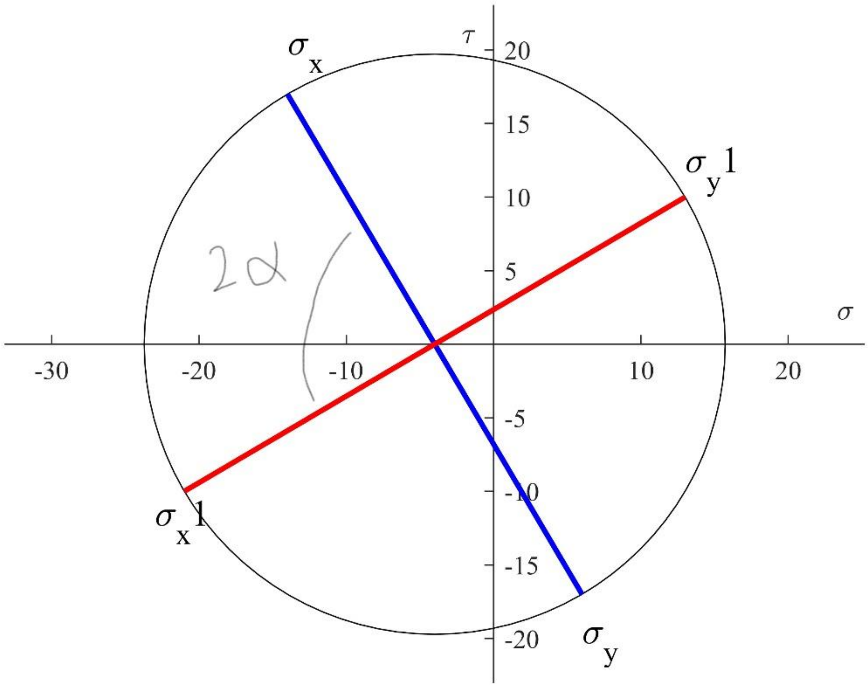
(c) the stress components on the x' and y' planes when $\alpha = 45°$

Solution:

$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos(2\alpha) + \tau_{xy}\sin(2\alpha) = -21MPa$$

$$\sigma_{y'} = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos(2\alpha) - \tau_{xy}\sin(2\alpha) = 13Mpa$$

$$\tau_{x'y'} = -\frac{\sigma_x - \sigma_y}{2}\sin(2\alpha) + \tau_{xy}\cos(2\alpha) = 10MPa$$



2, Given a three-dimensional stress state with

$$\boldsymbol{\sigma_x = 10MPa, \sigma_y = 20MPa, \sigma_z = -10MPa}$$

$$\boldsymbol{\tau_{xy} = 5MPa, \tau_{xz} = -10MPa, \tau_{yz} = -15MPa}$$

(a) find the magnitude and direction of the stress vector $\boldsymbol{p}$ on the $\boldsymbol{x'}$ plane where the $\boldsymbol{x'}$ direction is defined by $\boldsymbol{\cos(x',x) = \frac{1}{2}, \cos(x',y) = \frac{1}{\sqrt{2}}}$, and $\boldsymbol{\cos(x',z)}$ is positive.

Solution:

$$S = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix} = \begin{bmatrix} 10 & 5 & -10 \\ 5 & 20 & -15 \\ -10 & -15 & -10 \end{bmatrix}$$

$\cos^2(x',x) + \cos^2(x',y) + \cos^2(x',z) = \frac{1}{4} + \frac{1}{2} + \cos^2(x',z) = 1$ and $\cos(x',z)$ is positive. So $\cos(x',z) = \frac{1}{2}$, then:

$$P_x = \sigma_x \cos(x',x) + \tau_{yx} \cos(x',y) + \tau_{zx} \cos(x',z) = \frac{5\sqrt{2}}{2} MPa$$

$$P_y = \tau_{xy} \cos(x',x) + \sigma_y \cos(x',y) + \tau_{zy} \cos(x',z) = \left(-5 + 10\sqrt{2}\right) MPa$$

$$P_z = \tau_{xz} \cos(x',x) + \tau_{yz} \cos(x',y) + \sigma_z \cos(x',z) = (-10 - \frac{15\sqrt{2}}{2}) MPa$$

$$|P| = \sqrt{P_x^2 + P_y^2 + P_z^2} = 22.82MPa$$

$$\vec{P} = (\frac{5\sqrt{2}}{2}, -5 + 10\sqrt{2}, -10 - \frac{15\sqrt{2}}{2})$$

(b) find σ and τ on this plane

Solution:

$$\sigma = P_x \cos(x', x) + P_y \cos(x', y) + P_z \cos(x', z) = -2.07MPa$$
$$\tau^2 = P^2 - \sigma^2, \tau = 22.72MPa$$

(c) determine the angle between P and σ

Solution:

According to (a) and (b), let θ be the angle between P and σ, then $cos\theta = \frac{\sigma}{P} = -\frac{2.07}{22.82}$ and $\theta = 95.20°$.

(d) solve for $\tau_{x'y'}$ and $\tau_{x'z'}$, if $\cos(x, y') = \frac{1}{2}$ and $\cos(z, y')$ is negative.

Solution:

$$\cos^2(x, y') + \cos^2(y, y') + \cos^2(z, y') = 1$$
$$\cos(x', x)\cos(x, y') + \cos(x', y)\cos(y, y') + \cos(x', z)\cos(z, y') = 0$$
$$\cos(z, y') < 0$$

So, we can get $\cos(z, y') = -\frac{5}{6}, \cos(y', y') = \frac{\sqrt{2}}{6}$. Then according to the relationship between these direction cosines, then we can get:

$$\cos(x, z') = -\frac{\sqrt{2}}{2}, \cos(y, z') = \frac{2}{3}, \cos(z, z') = -\frac{\sqrt{2}}{6}$$

$$\tau_{x'y'} = P_x \cos(x, y') + P_y \cos(y, y') + P_z \cos(z, y') = 21.09MPa$$

$$\tau_{x'z'} = P_x \cos(x, z') + P_y \cos(y, z') + P_z \cos(z, z') = 8.45MPa$$

(e) evaluate all of the stress components acting on the x', y' and z' planes.

Solution:

The transformation matrix is:

$$R = \begin{bmatrix} \cos(x', x) & \cos(x', y) & \cos(x', z) \\ \cos(y', x) & \cos(y', y) & \cos(y', z) \\ \cos(z', x) & \cos(z', y) & \cos(z', z) \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{2}}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{2}}{6} & -\frac{5}{6} \\ -\frac{\sqrt{2}}{2} & \frac{2}{3} & \frac{\sqrt{2}}{6} \end{bmatrix}$$

All of the stress components acting on the x', y' and z' planes are shown as bellow:

$$R \cdot S \cdot R^T = \begin{bmatrix} -2.07 & 21.09 & 8.45 \\ 21.09 & 12.07 & 2.93 \\ 8.45 & 2.93 & 10 \end{bmatrix}$$

(f) determine the principal stresses and the direction cosines of the principal axes

Solution:

The principal stresses σ_p has the property that:

$$\begin{bmatrix} \sigma_x - \sigma_p & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y - \sigma_p & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z - \sigma_p \end{bmatrix} = 0$$

We let the determinate of this matrix be zero, then $\sigma_p^3 - I_1\sigma_p^2 + I_2\sigma_p - I_3 = 0$ where

$$I_1 = \sigma_x + \sigma_y + \sigma_z = 20$$

$$I_2 = \sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{zx}^2 = -450$$

$$I_3 = \sigma_x\sigma_y\sigma_z + 2\tau_{xy}\tau_{yz}\tau_{zx} - \sigma_z\tau_{xy}^2 - \sigma_x\tau_{yz}^2 - \sigma_y\tau_{zx}^2 = -4500$$

So $\sigma_1 = 30MPa, \sigma_2 = \left(5\sqrt{7} - 5\right)MPa, \sigma_3 = \left(-5\sqrt{7} - 5\right)MPa$ and they are the eigenvalue of S. What is more, the corresponding eigenvector are $\begin{bmatrix} 0.2650 \\ 0.3220 \\ 0.9089 \end{bmatrix}$, $\begin{bmatrix} 0.8736 \\ -0.4792 \\ -0.0849 \end{bmatrix}$ and $\begin{bmatrix} -0.4082 \\ -0.8165 \\ 0.4082 \end{bmatrix}$

On x_p plane, the direction cosines are: $\cos(x_p, x) = 0.2650, \cos(x_p, y) = 0.3220, \cos(x_p, z) = 0.9089$,

On y_p plane, the direction cosines are: $\cos(y_p, x) = 0.8736, \cos(y_p, y) = -0.4792, \cos(x_p, z) = -0.0849$,

On z_p plane, the direction cosines are: $\cos(x_p, x) = -0.4082, \cos(x_p, y) = -0.8165, \cos(x_p, z) = 0.4082$.

3, Given the following stress functions,

$$\boldsymbol{\sigma_x = 3x^2 + 3y^2 - z \qquad \tau_{xy} = z - 6xy - \frac{3}{4}}$$

$$\boldsymbol{\sigma_y = 3y^2 \qquad \tau_{xz} = x + y - \frac{3}{2}}$$

$$\boldsymbol{\sigma_z = 3x + y - z + \frac{5}{4} \qquad \tau_{yz} = 0}$$

(a) show that the above stress state is in equilibrium

Solution:

$$\tau_{yx} = \tau_{xy} = z - 6xy - \frac{3}{4}, \tau_{zx} = \tau_{xz} = x + y - \frac{3}{2}, \tau_{zy} = \tau_{yz} = 0$$

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = 6x - 6x + 0 = 0$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} = -6y + 6y + 0 = 0$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} = 1 + 0 - 1 = 0$$

So, this stress state is in equilibrium.

(b) for the stress state at point x = 1/2, y = 1, and z = 3/4, determine the principal stresses.

Solution:

At point $(\frac{1}{2}, 1, \frac{3}{4})$:

$$\sigma_x = 3, \sigma_y = 3, \sigma_z = 3, \tau_{xy} = -3, \tau_{xz} = 0, \tau_{yz} = 0$$

$$I_1 = \sigma_x + \sigma_y + \sigma_z = 9$$

$$I_2 = \sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{zx}^2 = 18$$

$$I_3 = \sigma_x\sigma_y\sigma_z + 2\tau_{xy}\tau_{yz}\tau_{zx} - \sigma_z\tau_{xy}^2 - \sigma_x\tau_{yz}^2 - \sigma_y\tau_{zx}^2 = 0$$

$$\sigma_p^3 - I_1\sigma_p^2 + I_2\sigma_p - I_3 = 0$$

We get $\sigma_1 = 6, \sigma_2 = 3, \sigma_3 = 0.$