Review

Indical (Tensor) notation (下标表示，张量表示)

x_i, u_i, p_i, τ_{ij}

Summation convention (爱因斯坦求和约定)

$$\begin{cases} x'_1 = a_{11}x_1 + a_{21}x_2 + a_{31}x_3 \\ x'_2 = a_{12}x_1 + a_{22}x_2 + a_{32}x_3 \\ x'_3 = a_{13}x_1 + a_{23}x_2 + a_{33}x_3 \end{cases}$$

$$x'_j = a_{ij}x_i$$

free index *j* dummy index *i*

Express the equilibrium equations with indical notation:

τ_{yx} -> σ_{21}

σ_x -> σ_{11}

$$\begin{cases} \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + f_x = 0 \\ \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + f_y = 0 \\ \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + f_z = 0 \end{cases}$$

Vector Transformation (向量变换)

Assume a vector ***F*** is represented by (F_1, F_2, F_3) in xyz coordinate and (F'_1, F'_2, F'_3) in a new coordinate system $x'y'z'$,

we have

$$\begin{cases} F'_1 = a_{11}F_1 + a_{21}F_2 + a_{31}F_3 \\ F'_2 = a_{12}F_1 + a_{22}F_2 + a_{32}F_3 \\ F'_3 = a_{13}F_1 + a_{23}F_2 + a_{33}F_3 \end{cases}$$

Coefficients are direction cosines

$$a_{ij} = \cos(x_i, x'_j)$$

Direction cosines

	x'_1	x'_2	x'_3
x_1	a_{11}	a_{12}	a_{13}
x_2	a_{21}	a_{22}	a_{23}
x_3	a_{31}	a_{32}	a_{33}

a_{ij}, the first index i and the second index j come from xyz and x'y'z' system, respectively

In general, direction cosine $a_{12} \neq a_{21}$.

In indical notation

$$F'_j = a_{ij}F_i$$

Conversely, vector components (F_1, F_2, F_3) can be represented by (F'_1, F'_2, F'_3):

$$\begin{cases} F_1 = a_{11}F'_1 + a_{12}F'_2 + a_{13}F'_3 \\ F_2 = a_{21}F'_1 + a_{22}F'_2 + a_{23}F'_3 \\ F_3 = a_{31}F'_1 + a_{32}F'_2 + a_{33}F'_3 \end{cases}$$

$$F_i = a_{ij}F'_j$$

Vector Transformation

For any vector quantity **F**, its components transform to a primed coordinate system according to the rule

$$F'_j = a_{ij} F_i$$

Definition of a vector based on coordinate transformation:
A set of three quantities F_i referred to a coordinate system xyz and transformed to another coordinate system x'y'z' by the following equation is defined as a vector

$$F'_j = a_{ij} F_i$$

Assume the components of a quantity in the xyz and x'y'z' coordinates are:

$$A'_j = A_i = [1, 1, 1]$$

Show that it is not a vector:

$$A'_j \neq a_{ij} A_i$$

Classroom exercise

Prove the following equation (x' is the coordinate of a point in the x'y'z' coordinate)

$$x'_j = a_{ij} a_{ik} x'_k$$

show that

$$a_{ij} a_{ik} = 1 \qquad \text{if} \quad j = k$$

$$a_{ij} a_{ik} = 0 \qquad \text{if} \quad j \neq k$$

Expand the above equations:

$$a_{11}^2 + a_{21}^2 + a_{31}^2 = 1 \qquad a_{11}a_{12} + a_{21}a_{22} + a_{21}a_{32} = 0$$

$$a_{12}^2 + a_{22}^2 + a_{32}^2 = 1 \qquad a_{11}a_{13} + a_{21}a_{23} + a_{31}a_{33} = 0$$

$$a_{13}^2 + a_{23}^2 + a_{33}^2 = 1 \qquad a_{12}a_{13} + a_{22}a_{23} + a_{32}a_{33} = 0$$

Express the following equations using matrix multiplication

$$F'_j = a_{ij} F_i \qquad F_i = a_{ik} F'_k$$

8-1 Demonstrate that the dot product of vectors u_i and v_m may be written in all of the following forms

$$\begin{aligned} \mathbf{u} \cdot \mathbf{v} &= u_i v_i \\ &= a_{ij} a_{mj} u_i v_m \\ &= u'_j v'_j \\ &= a_{ij} a_{ik} u'_j v'_k \end{aligned}$$

Further understanding of position vector transformation (if needed)

$$\vec{F} = F_1\hat{x}_1 + F_2\hat{x}_2 + F_3\hat{x}_3 = F'_1\,\hat{x}'_1 + F'_2\,\hat{x}'_2 + F'_3\,\hat{x}'_3$$

$$\begin{aligned} F'_j &= \vec{F}\cdot\hat{x}'_j = (F_1\hat{x}_1 + F_2\hat{x}_2 + F_3\hat{x}_3)\cdot\hat{x}'_j \\ &= x_1\hat{x}_1\cdot\hat{x}'_j + x_2\hat{x}_2\cdot\hat{x}'_j + x_3\hat{x}_3\cdot\hat{x}'_j \\ &= x_1 a_{1j} + x_2 a_{2j} + x_3 a_{3j} \end{aligned}$$

$$F'_j = a_{ij}F_i$$

$$\begin{aligned} F_m &= \vec{F}\cdot\hat{x}_m = (F'_1\,\hat{x}'_1 + F'_2\,\hat{x}'_2 + F'_3\,\hat{x}'_3)\cdot\hat{x}_m \\ &= F'_1\,\hat{x}'_1\cdot\hat{x}_m + F'_2\,\hat{x}'_2\cdot\hat{x}_m + F'_3\,\hat{x}'_3\cdot\hat{x}_m \\ &= F'_1\,a_{m1} + F'_2\,a_{m2} + F'_3\,a_{m3} \end{aligned}$$

$$F_m = a_{mk}F'_k$$

Tensors

Definition of a vector: $F'_j = a_{ij} F_i$

Definition of a second-order tensor A_{ik}:

Any group of nine scalar quantities A_{ik} referred to a coordinate system xyz and transforms to a group of nine quantities referred to another coordinate system x’y’z’ by the rule below is called a tensor of second order

$$A'_{j\ell} = a_{ij} a_{k\ell} A_{ik}$$

Each component of A'_{jl} can be represented by a combination of A_{ik}

$$\begin{aligned} A'_{12} &= a_{i1} a_{k2} A_{ik} \\ &= a_{11} a_{k2} A_{1k} + a_{21} a_{k2} A_{2k} + a_{31} a_{k2} A_{3k} \\ &= (a_{11} a_{12} A_{11} + a_{11} a_{22} A_{12} + a_{11} a_{32} A_{13}) \\ &+ (a_{21} a_{12} A_{21} + a_{21} a_{22} A_{22} + a_{21} a_{32} A_{23}) \\ &+ (a_{31} a_{12} A_{31} + a_{31} a_{32} A_{32} + a_{31} a_{32} A_{33}) \end{aligned}$$

- The subscripts of A' appear as second subscripts of *a*'s while the first subscript of the a's appear as the subscripts of A

Tensors

Definition of tensors of order 3:

w_{ijk} is a 3rd-order tensor that contains 27 components if it meets the transformation rule:

$$w'_{pqr} = a_{ip} a_{jq} a_{kr} w_{ijk}$$

Definition of tensors of order n:

transformation of a nth-order tensor

$$w'_{pq\ldots} = a_{ip} a_{jq} \cdots w_{ij\ldots}$$

Vectors are first-order tensors and scalars are tensors of order zero.

How many components does a order-n tensor include? 3^n components

Tensors

The outer product (外积，并矢积) of two vectors u_i and v_k, $u_i v_k$ **is a tensor**

Consider xyz and x'y'z' coordinate systems:

$$u'_j = a_{ij} u_i$$

$$v'_\ell = a_{k\ell} v_k$$

$$u'_j v'_\ell = (a_{ij} u_i)(a_{k\ell} v_k)$$

$$= a_{ij} a_{k\ell} u_i v_k$$

If w_{ij} is a tensor, then

- its tranpose w_{ji} is also a tensor

If w_{ij} is a tensor, then

- Quantities ($w_{ij}+w_{ji}$) and ($w_{ij}-w_{ji}$) are also tensors
  - ($w_{ij} + w_{ji}$) is a symmetric tensor
  - ($w_{ij} - w_{ji}$) is an antisymmetric tensor
    - the elements on the main diagonal are zero.

Any second-order tensor can be expressed as the sum of a symmetric and antisymmetric tensor:

$$w_{ij} = \frac{1}{2}(w_{ij} + w_{ji}) + \frac{1}{2}(w_{ij} - w_{ji})$$

Gradient of a Tensor is a Tensor

- Generally, the gradient of an n-th order tensor is a tensor of order n+1

  e.g., $u \rightarrow u_{,i}$ $u_i \rightarrow u_{i,j}$, $u_{ij} \rightarrow u_{ij,k}$

- The gradient of a scalar $U = U(x_1, x_2, x_3)$ is a 1st-order tensor:
  - gradient U in two coordinate systems x_i and x'_i:

$$\frac{\partial U}{\partial x_i}, \frac{\partial U}{\partial x'_j} \qquad \frac{\partial U}{\partial x'_j} = \frac{\partial U}{\partial x_i}\frac{\partial x_i}{\partial x'_j} = a_{ij}\frac{\partial U}{\partial x_i} \qquad x_i = a_{ij}x'_j \rightarrow \frac{\partial x_i}{\partial x'_j} = a_{ij}$$

- The gradient of a vector u_i is a second order tensor:
  - vector **u** in two coordinate systems x_i and x'_i: u_i, u'_j
  - gradient of **u** in two coordinate systems: $\frac{\partial u_i}{\partial x_k}, \frac{\partial u'_j}{\partial x'_l}$

$$\frac{\partial u'_j}{\partial x'_\ell} = \frac{\partial u'_j}{\partial x_k}\frac{\partial x_k}{\partial x'_\ell} = a_{k\ell}\frac{\partial}{\partial x_k}(a_{ij}u_i) = a_{ij}a_{k\ell}\frac{\partial u_i}{\partial x_k}$$

The Kronecker Delta (克罗内克符号)

Check the components of tensor $\dfrac{\partial x_i}{\partial x_k}$

$$\delta_{ik} = \frac{\partial x_i}{\partial x_k} = \begin{cases} 1 & i = k \\ 0 & i \neq k \end{cases}$$

The Kronecker Delta

$$\delta_{11} = \frac{\partial x_1}{\partial x_1} = 1 \quad \delta_{12} = \frac{\partial x_1}{\partial x_2} = 0$$

Since $x_{i,k}$ is a second order tensor,
Kronecker delta is a **2nd-order tensor.**

Exericese:
Prove $\delta_{ii} = 3$

Prove that $\delta_{ik} = x_{i,k} = a_{ij}a_{kj}$

Isotropic tensors: tensors with identical components in any coordinate system

The Kronecker delta is an 2nd-order isotropic tensor.

Transform Kronecker delta from x_i to x'_i

$$\delta'_{j\ell} = a_{ij}a_{k\ell}\delta_{ik}$$

$$\rightarrow \quad \delta'_{j\ell} = a_{ij}a_{i\ell}$$

$$a_{ij}a_{i\ell} = \delta_{j\ell}$$

$$\rightarrow \quad \delta'_{j\ell} = \delta_{j\ell}$$

The Kronecker Delta

$$\delta_{ik} u_k = u_i$$

Above operation replaces the dummy index k by i. Thus Kronecker delta is sometimes called the **substitution tensor**（替换张量）

We also have the following identities:

$$\delta_{ik} \frac{\partial u_j}{\partial x_k} = \frac{\partial u_j}{\partial x_i} = u_{j,i}$$

$$\delta_{ik} w_{ik} = w_{ii} = w_{kk}$$

$$\delta_{il} \frac{\partial^2 u_j}{\partial x_l \partial x_k} = \delta_{il} u_{j,lk} = \frac{\partial^2 u_j}{\partial x_i \partial x_k} = u_{j,ik}$$

Tensor Contraction (张量缩并)

- The operation of equating two letter subscripts in a tensor and summing accordingly is known as contraction (缩并). E.g., C_{ij} —> C_{ii} = C_{jj}
  - Contraction gives another tensor of order **two less** than that of the original tensor.

- Example:
  - For a 4_th order tensor c_{ijkl}, lets i=j (contraction), we get a 2_nd order tensor $c_{iikl} = c_{11kl} + c_{22kl} + c_{33kl}$
  - Contraction of a second-order tensor w_{ik}, we get a scalar w_{ii} independent of the coordinate
  - contract the second-order tensor $u_i v_k$, we get the dot product between $\boldsymbol{u}$ and $\boldsymbol{v}$

$$u_i v_i = u_1 v_1 + u_2 v_2 + u_3 v_3$$

  - contract tensor $\partial u_k / \partial x_i$, we get the **divergence** of vector u_i

$$\frac{\partial u_i}{\partial x_i} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}$$

Classroom exercise

8-3 Prove that $w_{ij}u_k$ is a third-order tensor, where w_{ij} is any second-order tensor and u_k is any vector.

8-4 Prove that

$$(AB)_{,ii} = AB_{,ii} + 2A_{,i}B_{,i} + BA_{,ii}$$

where A and B are scalar functions.

Prove that

$$u'_m v'_m = u_i v_i \qquad u'_m v'_m = a_{im}u_i a_{jm}v_j = a_{im}a_{jm}u_i v_j = \delta_{ij}u_i v_j = u_i v_i$$

Prove that

$$\delta_{ij}\delta_{jk} = \delta_{ik} \qquad \delta_{ij}\delta_{jk}\delta_{kl} = \delta_{il}$$

Simplify the expressions:

$$T_{ijk}\delta_{jk}$$