Homework 4

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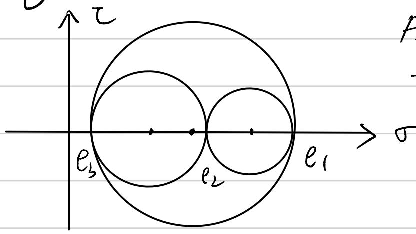
1\. (a) $\bar{\bar{e}} = \frac{1}{2}[\nabla\vec{u} + (\nabla\vec{u})^T]$

(b) principal strains are: when we choose the directions of eigenvectors as our basis, there is only elongation, no angle change.

principal planes are planes whose normal vectors are eigenvectors.

(c)



As it shows in the picture, the maximum shear strain is the radius of the biggest circle, which is $\frac{1}{2}(e_1 - e_3)$.

(d) Sometimes there may not exist displacement functions u_1, u_2, u_3 satisfying six equations defining the strain displacement relationships.

(e) $\frac{\partial^2 e_x}{\partial y^2} + \frac{\partial^2 e_y}{\partial x^2} = \frac{1}{2}\frac{\partial^2 \gamma_{xy}}{\partial x \partial y}$, $\frac{\partial^2 e_y}{\partial z^2} + \frac{\partial^2 e_z}{\partial y^2} = \frac{1}{2}\frac{\partial^2 \gamma_{yz}}{\partial y \partial z}$

$\frac{\partial^2 e_x}{\partial z^2} + \frac{\partial^2 e_z}{\partial x^2} = \frac{\partial^2 \gamma_{xz}}{2 \cdot \partial x \partial z}$, $\frac{\partial^2 e_x}{\partial y \partial z} - \frac{1}{2}\frac{\partial}{\partial x}\left(-\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{xz}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right) = 0$

$$\frac{\partial^2 e_y}{\partial x \partial z} - \frac{1}{2}\frac{\partial}{\partial y}\left(-\frac{\partial \gamma_{xz}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z} + \frac{\partial \gamma_{yz}}{\partial x}\right) = 0$$

$$\frac{\partial^2 e_z}{\partial x \partial y} - \frac{1}{2}\frac{\partial}{\partial z}\left(-\frac{\partial \gamma_{xy}}{\partial z} + \frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{xz}}{\partial y}\right) = 0.$$

(f) e_x, e_y, e_z: length change of an element in $\vec{e}_x, \vec{e}_y, \vec{e}_z$ directions.

$\gamma_{xy}, \gamma_{xz}, \gamma_{yz}$: angle change of two element along $\vec{e}_x$ and $\vec{e}_y$ ($\vec{e}_x$ and $\vec{e}_z$, $\vec{e}_y$ and $\vec{e}_z$) directions.

(g) They satisfy the 6 equations in (e), so they are compatible.

2. (a) $\frac{\partial T_{ij}}{\partial x_j} + \rho B_i = 0$, we get $B_i = -\frac{1}{\rho} \cdot \frac{\partial T_{ij}}{\partial x_j}$,

so $B_1 = (-\frac{1}{\rho})(y+5)$, $B_2 = 0$, $B_3 = 0$, that's $\vec{B} = [(-\frac{1}{\rho})(y+5), 0, 0]^T$.

(b) $T = \begin{bmatrix} 1 & 5 & 0 \\ 5 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\sigma_{11} = 1$, $\sigma_{22} = 4$,

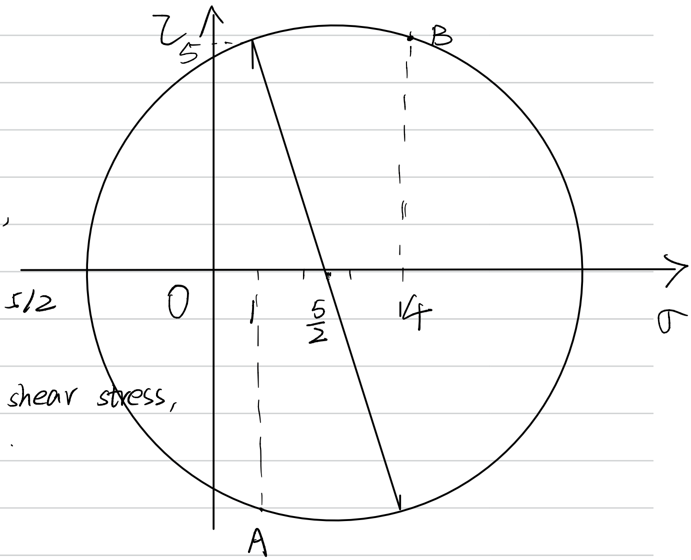
$\sigma_{12} = \sigma_{21} = 5$.

$c = (\sigma_{11} + \sigma_{22})/2 = 5/2$

$R = \sqrt{(\frac{3}{2})^2 + 5^2} = \frac{\sqrt{109}}{2}$

principal stress means no shear stress,

so, that is $\frac{5}{2} \pm \frac{\sqrt{109}}{2}$.



3. (a) $Q^T = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $Q = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$Q^T[\bar{\bar{\sigma}}]Q = \begin{bmatrix} \sigma_{11}\cos^2\theta + \sigma_{12}\sin 2\theta + \sigma_{22}\sin^2\theta & -\frac{\sigma_{11}}{2}\sin 2\theta + \sigma_{12}\cos 2\theta + \frac{1}{2}\sigma_{22}\sin 2\theta & 0 \\ -\frac{\sigma_{11}}{2}\sin 2\theta + \sigma_{12}\cos 2\theta + \frac{\sigma_{22}}{2}\sin 2\theta & \sigma_{11}\sin^2\theta - \sigma_{12}\sin 2\theta + \sigma_{22}\cos^2\theta & 0 \\ 0 & 0 & \sigma_{33} \end{bmatrix}$$

(b) $\sigma_{11}' + \sigma_{22}' = \sigma_{11}(\cos^2\theta + \sin^2\theta) + \sigma_{22}(\cos^2\theta + \sin^2\theta) = \sigma_{11} + \sigma_{22}$

Due to $|Q| = |Q^T| = 1$, so $\det([\bar{\bar{\sigma}}]) = \det([\bar{\bar{\sigma}}'])$,

so, $\sigma_{11} \cdot \sigma_{22}\sigma_{33} - \sigma_{12}^2\sigma_{33} = \sigma_{11}' \cdot \sigma_{22}' \cdot \sigma_{33}' - (\sigma_{12}')^2\sigma_{33}'$,

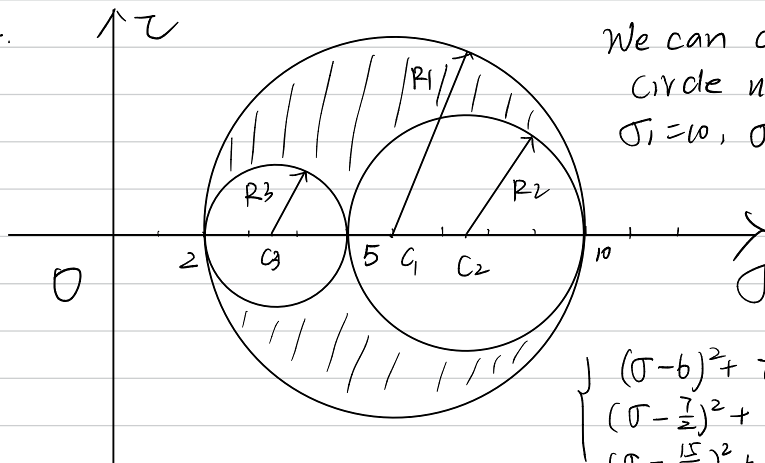
$\sigma_{33} = \sigma_{33}'$, so, $\sigma_{11} \cdot \sigma_{22} - \sigma_{12}^2 = \sigma_{11}' \cdot \sigma_{22}' - (\sigma_{12}')^2$

So, $\sigma_{11}'\sigma_{22}' + \sigma_{11}'\sigma_{33}' + \sigma_{22}'\sigma_{33}' - (\sigma_{12}')^2 = \sigma_{11}\sigma_{22} + \sigma_{11}\sigma_{33} + \sigma_{22}\sigma_{33} - (\sigma_{12})^2$

(c) It have been proved in (b), using $\det([\bar{\bar{\sigma}}]) = \det([\bar{\bar{\sigma}}'])$.

(d) Make some rotation on the basis's, there can be some invariant relationship in the Cauchy stress components.

4.



We can draw a Mohr's circle using $\sigma_1 = 10$, $\sigma_2 = 5$, $\sigma_3 = 2$.

$$\begin{cases} (\sigma - 6)^2 + \tau^2 \leq 16 & ① \\ (\sigma - \frac{7}{2})^2 + \tau^2 \geq \frac{9}{4} & ② \\ (\sigma - \frac{15}{2})^2 + \tau^2 \geq \frac{25}{4} & ③ \end{cases}$$

(a) $\sigma = 10$, $\tau = 1$ is out of the region, so not possible.

(b) $\sigma = 5$, $\tau = 4$ not satisfy equation ①, so not possible.

(c) $\sigma = 3$, $\tau = 1$ not satisfy equation ②, so not possible.