Review

- moment equilibrium
  - $\tau_{yx} = \tau_{xy}$
- force equilibrium
  - stress transformation equations (应力变换方程)

$$
\begin{cases}
\sigma_{x'} = \dfrac{\sigma_x + \sigma_y}{2} + \dfrac{\sigma_x - \sigma_y}{2}\cos 2\alpha + \tau_{xy}\sin 2\alpha \\
\sigma_{y'} = \dfrac{\sigma_x + \sigma_y}{2} - \dfrac{\sigma_x - \sigma_y}{2}\cos 2\alpha - \tau_{xy}\sin 2\alpha \\
\tau_{x'y'} = -\dfrac{\sigma_x - \sigma_y}{2}\sin 2\alpha + \tau_{xy}\cos 2\alpha
\end{cases}
$$

State of stress on a point is fully depicted in 2D by stress vectors on two planes:

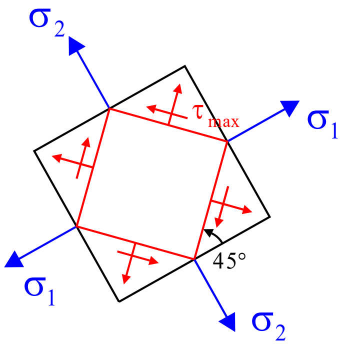
$$
\boldsymbol{\sigma} = \begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{yx} & \sigma_y \end{bmatrix}
$$

principal stress and principal plane

- **Principal planes**: shear stress vanishes on these planes
- Two perpendicular principal planes corresponding to $\sigma_1=\sigma_{max}$ and $\sigma_2=\sigma_{min}$, respectively.

Maximum shear stress

- Maximum shear stress planes are 45° apart from the two principal planes.
- Normal stresses on the maximum shear stress planes are equal



Mohr's Circle of Stress

$$\begin{cases} \sigma_{x'} - \dfrac{\sigma_x + \sigma_y}{2} = \dfrac{\sigma_x - \sigma_y}{2}\cos 2\alpha + \tau_{xy}\sin 2\alpha & (1) \\ \sigma_{y'} - \dfrac{\sigma_x + \sigma_y}{2} = -\dfrac{\sigma_x - \sigma_y}{2}\cos 2\alpha - \tau_{xy}\sin 2\alpha & (2) \\ \tau_{x'y'} = -\dfrac{\sigma_x - \sigma_y}{2}\sin 2\alpha + \tau_{xy}\cos 2\alpha & (3) \end{cases}$$

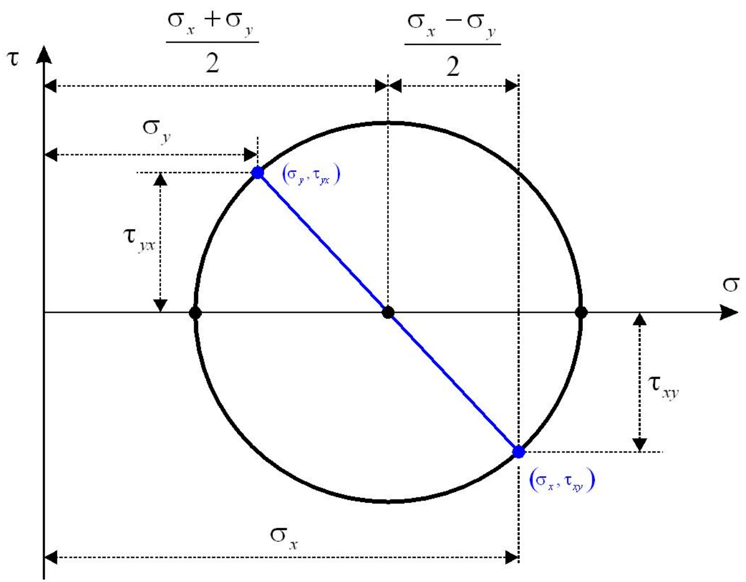
$(1)^2 + (3)^2 \Rightarrow$

$$\left(\sigma - \frac{\sigma_x + \sigma_y}{2}\right)^2 + \tau^2 = \left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2$$

$$(\sigma = \sigma_{x'} \qquad \tau = \tau_{x'y'})$$

center: $\left(\dfrac{\sigma_x + \sigma_y}{2}, 0\right)$ radius: $\sqrt{\left(\dfrac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$

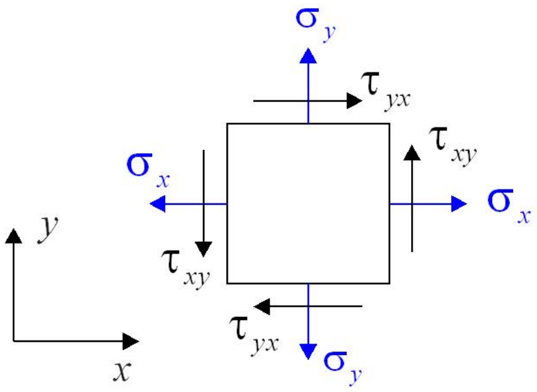
Mohr's circle of stress (应力莫尔圆)



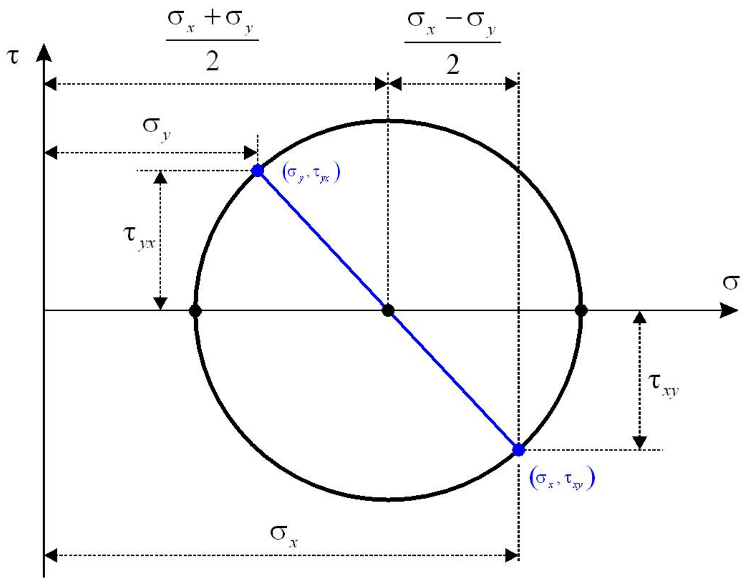
- Principal stresses
  - zero shear stress on principal plane
- maximum shear stress
- Circle Center: $\dfrac{\sigma_x + \sigma_y}{2} = \dfrac{\sigma_1 + \sigma_2}{2} = C$

Mohr's Circle of Stress

- **Sign convention in the Mohr-Circle space**
  - Normal stress convention is the same as the physical space
  - Shear stress causing clockwise moment is positive
    - τ_{yx} and τ_{xy} are both positive in the physical space, but τ_{xy} is negative in the Mohr circle
  - Whatever the rotation angle exists in physical space, it is doubled on Mohr's circle and with the same direction



Mohr's circle of stress (应力莫尔圆)

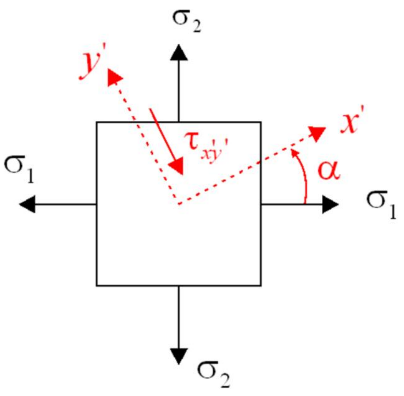


(σ_x, τ_{xy}) is beneath the x-axis and
(σ_y, τ_{yx}) is above the x-axis

Classroom Practice

1. Rewrite the stress transformation equations with principal stresses

$$\begin{cases} \sigma_{x'} - \dfrac{\sigma_x + \sigma_y}{2} = \dfrac{\sigma_x - \sigma_y}{2}\cos 2\alpha + \tau_{xy}\sin 2\alpha \\ \sigma_{y'} - \dfrac{\sigma_x + \sigma_y}{2} = -\dfrac{\sigma_x - \sigma_y}{2}\cos 2\alpha - \tau_{xy}\sin 2\alpha \\ \tau_{x'y'} = -\dfrac{\sigma_x - \sigma_y}{2}\sin 2\alpha + \tau_{xy}\cos 2\alpha \end{cases}$$

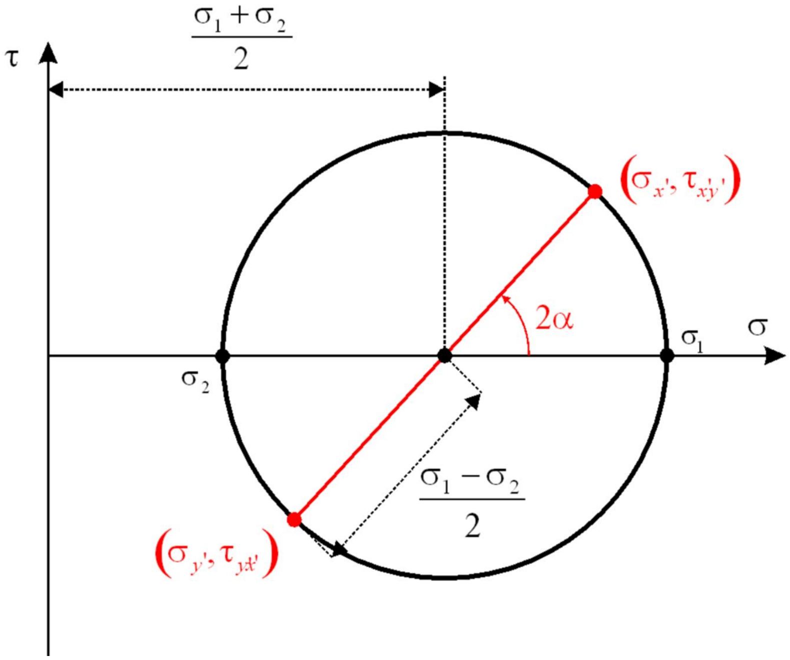


$$\sigma_{x'} - \frac{\sigma_1 + \sigma_2}{2} = \frac{\sigma_1 - \sigma_2}{2}\cos 2\alpha$$

$$\sigma_{y'} - \frac{\sigma_1 + \sigma_2}{2} = -\frac{\sigma_1 - \sigma_2}{2}\cos 2\alpha$$

$$\tau_{xy} = -\frac{\sigma_1 - \sigma_2}{2}\sin 2\alpha$$

2. Express the stress transformation equations with principal stresses with Mohr's circle



Classroom Practice

1-1 Given $\sigma_x = -14{,}000$ psi, $\sigma_y = 6{,}000$ psi, and $\tau_{xy} = -17{,}320$ psi, determine both by formulas and by the Mohr's circle, (a) the principal stresses and their directions

First, plot stress in the physical space (实体空间), then plot stress in the Mohr's circle space.

(a)

σ_1 = 1.6e4 psi

σ_2 = -2.4e4 psi

direction: rotate counterclockwisely for 120° to get σ1 and counterclockwisely for 30° to get σ2

Transformation of Stress Components for Nonuniform Stress distribution

Under uniform stress distribution, we have

(1)

$$\tau_{xy} = \tau_{yx}$$

(2)

$$\begin{cases} \sigma_{x'} = \dfrac{\sigma_x + \sigma_y}{2} + \dfrac{\sigma_x - \sigma_y}{2} \cos 2\alpha + \tau_{xy} \sin 2\alpha \\ \sigma_{y'} = \dfrac{\sigma_x + \sigma_y}{2} - \dfrac{\sigma_x - \sigma_y}{2} \cos 2\alpha - \tau_{xy} \sin 2\alpha \\ \tau_{x'y'} = -\dfrac{\sigma_x - \sigma_y}{2} \sin 2\alpha + \tau_{xy} \cos 2\alpha \end{cases}$$

These same relationships also apply to each point in a body under a nonuniform stress distribution, including the effects of body forces.

Transformation of Stress Components for Nonuniform Stress distribution

Given

σ_x, τ_{xy}, σ_y: stress components at point O

F_x, F_y: the body force components at O

We want to get

p_x, p_y: stress components on plane AB through point O

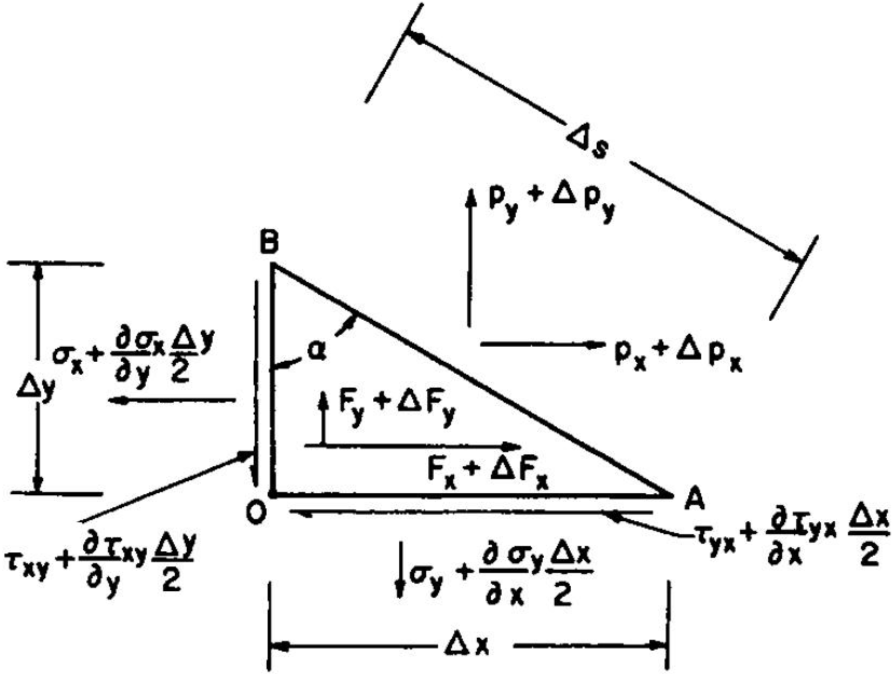
This is done with force equilibrium for the infinetesimal free body OAB:

Force equilibrium in the x direction.

$$(p_x + \Delta p_x)\Delta s = \left(\sigma_x + \frac{\partial \sigma_x}{\partial y}\frac{\Delta y}{2}\right)\Delta y + \left(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial x}\frac{\Delta x}{2}\right)\Delta x - (F_x + \Delta F_x)\frac{\Delta x \Delta y}{2}$$

Neglecting the small terms, we have

$$p_x = \sigma_x \cos\alpha + \tau_{yx}\sin\alpha$$



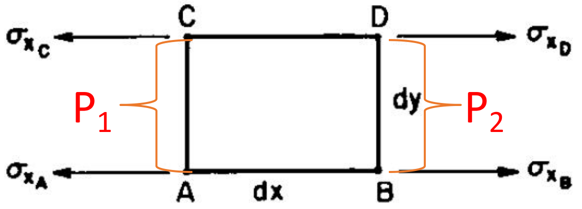
Free body of an infinitesimal element under a nonuniform state of stress

The formulation for p_x is the same as it is in uniform stress distribution

Differential Equations of Equilibrium (平衡微分方程)

Assume 2D cases:

- $\tau_{zx}=\tau_{zy}=\tau_{xz}=\tau_{yz}=0$
- σ_x, σ_y, τ_{xy}, F_x, and F_y do not depend on z.



set $\sigma_{xA}=\sigma_x$, we have horizontal normal stress at other vertexes:

$$\sigma_{x_B}=\sigma_x+\frac{\partial\sigma_x}{\partial x}dx \qquad \sigma_{x_C}=\sigma_x+\frac{\partial\sigma_x}{\partial y}dy$$

$$\sigma_{x_D}=\sigma_{x_B}+\frac{\partial\sigma_{x_B}}{\partial y}dy=\sigma_x+\frac{\partial\sigma_x}{\partial x}dx+\frac{\partial\sigma_x}{\partial y}dy$$

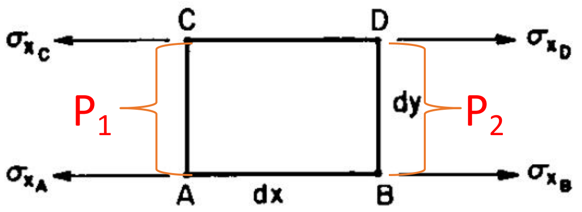
The net difference between the horizontal pull forces (unit: N) applied to the small element ABDC is P_2 - P_1 :

$$P_1=\int_A^C\sigma_x dy=\sigma_x dy+\frac{1}{2}\frac{\partial\sigma_x}{\partial y}dy^2$$

$$P_2=\int_C^D\sigma_x dy=\sigma_x dy+\frac{\partial\sigma_x}{\partial y}dxdy+\frac{1}{2}\frac{\partial\sigma_x}{\partial y}dy^2$$

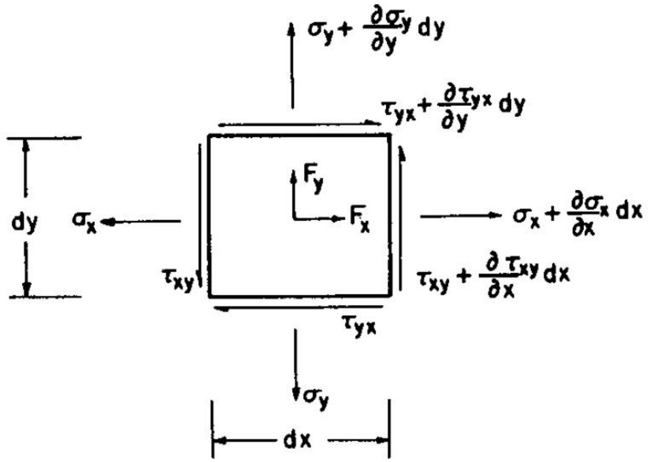
$$P_2-P_1=\frac{\partial\sigma_x}{\partial x}dx\,dy$$

Differential Equations of Equilibrium (平衡微分方程)



$$P_2 - P_1 = \frac{\partial \sigma_x}{\partial x}\, dx\, dy$$

Left result is reproduced assuming σx and σx + (∂σx/∂x)dx acting at the centers of the left and right faces, respectively.

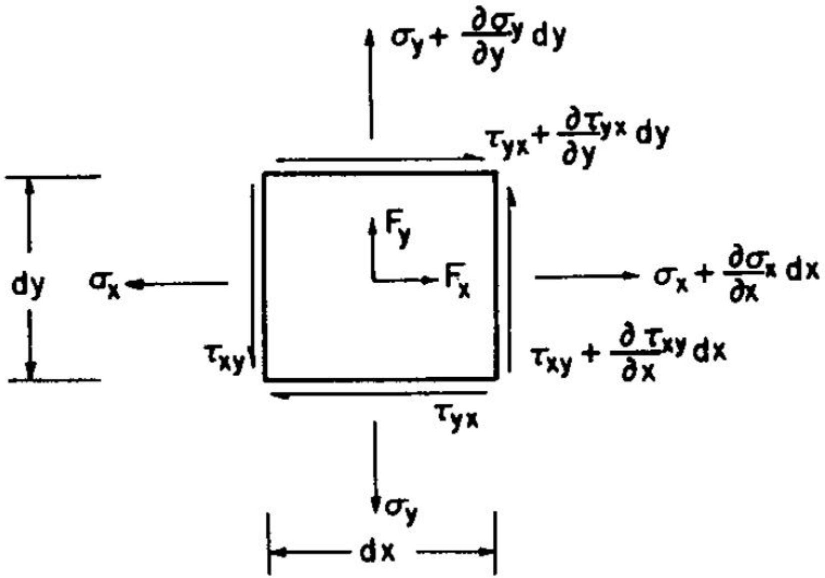


Uniform stress distribution on each surface is used in deriving the equilibrium equations.

Differential Equations of Equilibrium

We now check the moment (力矩) balance.

Take moments about the lower left corner, we have



$$\left(\frac{\partial \sigma_y}{\partial y} dy\, dx\right)\frac{dx}{2} - \left(\frac{\partial \sigma_x}{\partial x} dx\, dy\right)\frac{dy}{2} + \left(\tau_{xy} + \frac{\partial \tau_{xy}}{\partial x} dx\right) dy\, dx$$

$$-\left(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y} dy\right) dx\, dy + F_y\, dx\, dy\, \frac{dx}{2} - F_x\, dx\, dy\, \frac{dy}{2} = 0$$

Neglecting terms containing triple products of dx or dy, we have

$$\tau_{xy} = \tau_{yx}$$

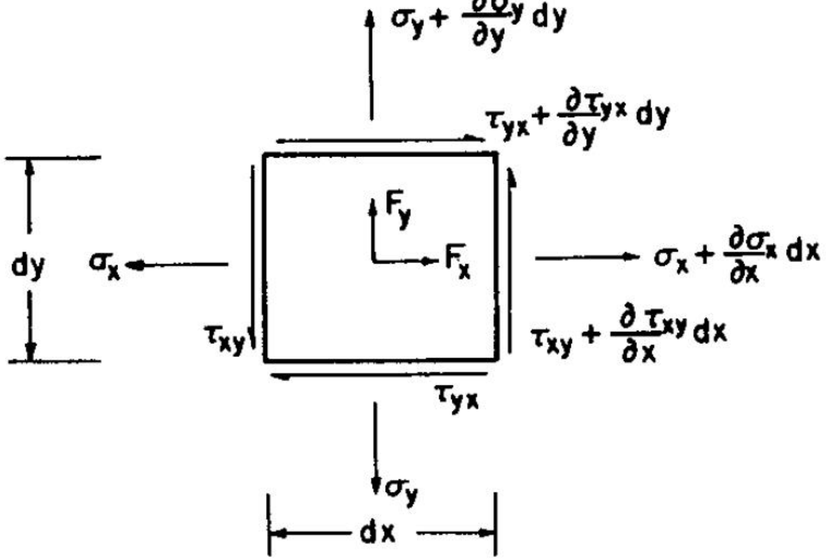
Differential Equations of Equilibrium

Consider force equilibrium in the x-direction:

$$F_x\,dx\,dy + \left[\sigma_x + \frac{\partial \sigma_x}{\partial x}dx\right]dy - \sigma_x\,dy$$

$$+\left[\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y}dy\right]dx - \tau_{yx}\,dx = 0 \quad \Rightarrow$$

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + F_x = 0$$



Similarly, in the y direction

$$\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + F_y = 0$$

Spatial variation of stress (stress's spatial gradient) always exists if the body force exists.

Differential Equations of Equilibrium

The differential equations of equilibrium (平衡微分方程):

2D

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + F_x = 0$$

$$\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + F_y = 0$$

3D

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + F_x = 0$$

$$\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{zy}}{\partial z} + F_y = 0$$

$$\frac{\partial \sigma_z}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + F_z = 0$$

Consider 2D momentum equation, we have

$$\tau_{xy} = \tau_{yx}$$

Consider the 3D case and write ΣM = 0 about x, y, and z axes, we have

$$\tau_{xy} = \tau_{yx}, \qquad \tau_{xz} = \tau_{zx}, \qquad \tau_{yz} = \tau_{zy}$$

Q: Three independent stress components are needed to fully depict the stress state in 2D plane. How many independent stress components are needed in 3D?

Classroom Practice

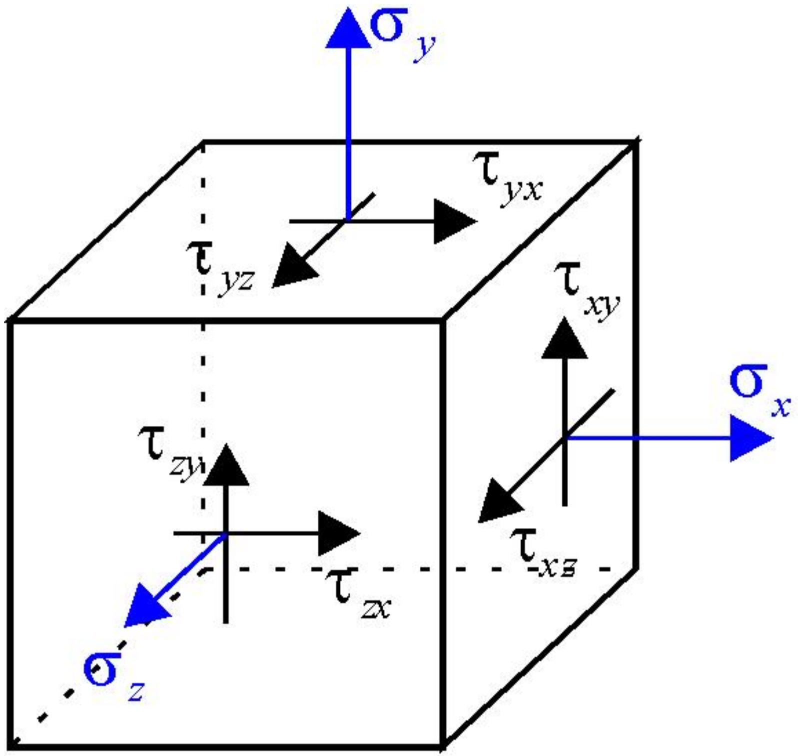
Verify the 3D equilibrium equations

3D

$$\frac{\partial \sigma_x}{\partial x}+\frac{\partial \tau_{yx}}{\partial y}+\frac{\partial \tau_{zx}}{\partial z}+F_x=0$$

$$\frac{\partial \sigma_y}{\partial y}+\frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \tau_{zy}}{\partial z}+F_y=0$$

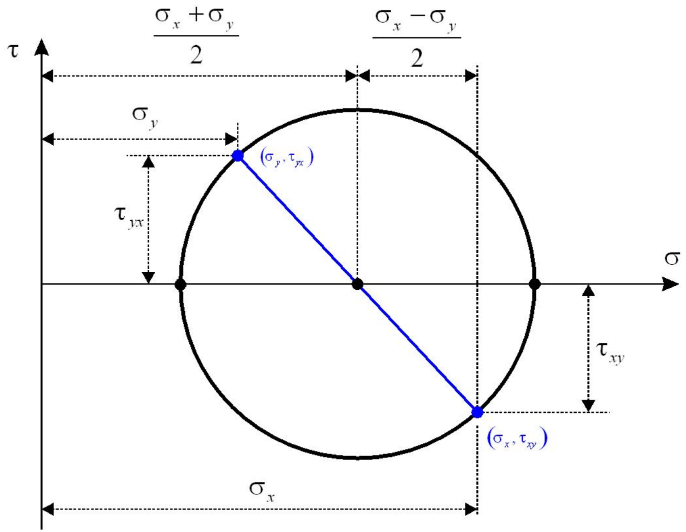
$$\frac{\partial \sigma_z}{\partial z}+\frac{\partial \tau_{xz}}{\partial x}+\frac{\partial \tau_{yz}}{\partial y}+F_z=0$$



Review

Mohr's Circle of Stress

center: $\left(\frac{\sigma_x+\sigma_y}{2}, 0\right)$ radius: $\sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2+\tau_{xy}^2}$



- Maximum and minimum principal stresses
- zero shear stress on principal plane
- maximum shear stress

The following relations under uniform stress distribution are also correct under a nonuniform stress distribution:

1. $\tau_{xy} = \tau_{yx}$
2. Transformation of stress equations (应力变换方程)
3. Mohr's stress circle, principal stress, and maximum shear stress deduced from (2)

The differential equations of equilibrium (平衡微分方程):

2D

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + F_x = 0$$

$$\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + F_y = 0$$