Homework 4

Due Nov 11 2021

1. The infinitesimal strain $\boldsymbol{e}$ is a tensor that measures the shape change of a continuum body when the displacement is small relative to the dimension of the body.

   (a) Given the displacement $\boldsymbol{u}$, how to calculate the infinitesimal strain $\boldsymbol{e}$?

   (b) The strain tensor $\boldsymbol{e}$ is symmetric. Thus, it has three eigenvectors that form a Cartesian basis. These eigenvectors are called the principal axes of the strain $\boldsymbol{e}$. Following the definitions introduced in the stress analysis, state the definitions for the principal planes of the strain and the principal strains.

   (c) One can draw Mohr's circle for $\boldsymbol{e}$ given its three principal strains $e_3 \leq e_2 \leq e_1$. What is the value for the maximum shear strain?

   (d) What is the motivation for discussing the compatibility equations?

   (e) In the Cartesian system with basis $\{\boldsymbol{e}_x, \boldsymbol{e}_y, \boldsymbol{e}_z\}$, the strain $\boldsymbol{e}$ can be represented by

   $$\begin{bmatrix} e_{xx} & e_{xy} & e_{xz} \\ e_{yx} & e_{yy} & e_{yz} \\ e_{zx} & e_{zy} & e_{zz} \end{bmatrix}.$$

   In the engineering community, the strain is often represented by $e_x, e_y, e_z, \gamma_{xy}, \gamma_{xz}, \gamma_{yz}$, which are given by

   $$e_x = e_{xx}, \quad e_y = e_{yy}, \quad e_z = e_{zz}, \quad \gamma_{xy} = 2e_{xy}, \quad \gamma_{xz} = 2e_{xz}, \quad \gamma_{yz} = 2e_{yz}.$$

   State the compatibility equations in terms of the engineering notations.

   (f) The terms e_x, e_y, e_z are referred to as the *normal strians*; the terms $\gamma_{xy}, \gamma_{xz}, \gamma_{yz}$ are referred to as the *shear strains*. Give the geometrical interpretations of $e_x, e_y, e_z, \gamma_{xy}, \gamma_{xz}$, and γ_{yz}.

   (g) Determine if the following strain components are compatible,

   $$e_{11} = x_1 + x_2, \quad e_{22} = x_2 + x_3, \quad e_{33} = x_1 + x_3, \quad e_{12} = x_1, \quad e_{13} = x_2, \quad e_{23} = x_3.$$

2. The Cauchy stress in a continuum body is given by the following matrix form,

   $$\begin{bmatrix} xy & 5y & 0 \\ 5y & 4x & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

   (a) Determine the body force acting on the body.

   (b) Calculate the stress at the point $x = 1, y = 1$. Draw the Mohr's circle, and determine the principal stresses.

3. The Cauchy stress at a point has the following matrix form,

   $$\begin{bmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{21} & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{bmatrix}.$$

   On the $\boldsymbol{e}_1 - \boldsymbol{e}_2$ plane, we rotate the frame counter-clockwisely by an angle θ with $\boldsymbol{e}_3$ being the rotation axis. Let the new frame be $\{\boldsymbol{e}'_1, \boldsymbol{e}'_2, \boldsymbol{e}'_3 = \boldsymbol{e}_3\}$. Due to the symmetry of $\boldsymbol{\sigma}$, $\sigma_{12} = \sigma_{21}$.

(a) What is the matrix form of the Cauchy stress in the new frame?

(b) Let σ'_{11}, σ'_{12}, and σ'_{22} be the Cauchy stress components in the new frame. Show that

$$\sigma'_{11}\sigma'_{22} + \sigma'_{11}\sigma'_{33} + \sigma'_{22}\sigma'_{33} - \left(\sigma'_{12}\right)^2 = \sigma_{11}\sigma_{22} + \sigma_{11}\sigma_{33} + \sigma_{22}\sigma_{33} - \left(\sigma_{12}\right)^2$$

(c) Show that

$$\sigma'_{11}\sigma'_{22}\sigma'_{33} - \left(\sigma'_{12}\right)^2 \sigma'_{33} = \sigma_{11}\sigma_{22}\sigma_{33} - \left(\sigma_{12}\right)^2 \sigma_{33}$$

(d) Comment on the results of (b) and (c).

4. The principal stresses at a point are

$$\sigma_1 = 10, \qquad \sigma_2 = 5, \qquad \sigma_3 = 2.$$

The normal and shear stresses on a plane passing this point are σ and τ. Are the following values possible?

(a) $\sigma = 10$ and $\tau = 1$.

(b) $\sigma = 5$ and $\tau = 4$.

(c) $\sigma = 3$ and $\tau = 1$.