Review

The alternating tensor (交错张量) ε_{ikm}: an **isotropic third-order tensor**

$$\varepsilon_{ikm} = \begin{cases} +1 & i, k, m \text{ in cyclic order} \\ -1 & i, k, m \text{ in noncyclic order} \\ 0 & \text{any of the subscripts are equal} \end{cases}$$

- Tensor $w_{ik} = \varepsilon_{ikm} u_m$ is a 2nd-order antisymmetric tensor
- Vector $\varepsilon_{mik} u_i v_k$ represents vector (cross) product (矢量积、叉乘) $\mathbf{u} \times \mathbf{v}$

$\varepsilon_{mik} \nabla_i u_k$ represents the curl of u_k $\quad v_m = \nabla \times u = \varepsilon_{mik} \nabla_i u_k = \varepsilon_{mik} \dfrac{\partial u_k}{\partial x_i}$

$$v_1 = \frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3} \qquad v_2 = \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1} \qquad v_3 = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}$$

Review

Indicial expresssion of stress components: τ_{ik}, σ_{ik}

$$\begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix} = \tau_{ik} = \begin{bmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{bmatrix}$$

Stress τ_{ik} is a second-order symmetric tensor:

$$\boxed{\tau'_{jn} = a_{ij} a_{kn} \tau_{ik}}$$

The stress vector on the $\boldsymbol{\mu}$ plane τ_k^{μ} is

$$\boxed{\tau_k^{\mu} = \tau_{ik} \mu_i}$$

Equilibrium Equations

$$\begin{cases} \dfrac{\partial \sigma_x}{\partial x} + \dfrac{\partial \tau_{yx}}{\partial y} + \dfrac{\partial \tau_{zx}}{\partial z} + f_x = 0 \\ \dfrac{\partial \tau_{xy}}{\partial x} + \dfrac{\partial \sigma_y}{\partial y} + \dfrac{\partial \tau_{zy}}{\partial z} + f_y = 0 \\ \dfrac{\partial \tau_{xz}}{\partial x} + \dfrac{\partial \tau_{yz}}{\partial y} + \dfrac{\partial \sigma_z}{\partial z} + f_z = 0 \end{cases}$$

$$\downarrow$$

$$\frac{\partial \tau_{ik}}{\partial x_i} + f_k = 0 \qquad \tau_{ik,i} + f_k = 0$$

$$\begin{cases} T_x^{\mu} = \sigma_x \mu_x + \tau_{yx} \mu_y + \tau_{zx} \mu_z \\ T_y^{\mu} = \tau_{xy} \mu_x + \sigma_y \mu_y + \tau_{zy} \mu_z \\ T_z^{\mu} = \tau_{xz} \mu_x + \tau_{yz} \mu_y + \sigma_z \mu_z \end{cases}$$

$$\downarrow$$

$$T_j^{\mu} = \tau_{ij} \mu_i$$

We can also derive the equilibrium equations with the Gaussian theorem

The force on an arbitrary finite body is in equilibrium:

$$\int_V F_k \, dV + \int_S \tau_k^{\mu} \, dS = 0$$

body force inside body V

stress vector on boundary S

represent stress vector with stress tensor

$$\int_V F_k \, dV + \int_S \tau_{ik} \mu_i \, dS = 0$$

$$\int_S \tau_{ik} \mu_i \, dS = \int_V \frac{\partial \tau_{ik}}{\partial x_i} \, dV$$

Gaussian Theorem

$$\int_V \left(F_k + \frac{\partial \tau_{ik}}{\partial x_i} \right) dV = 0 \quad \rightarrow \quad \frac{\partial \tau_{ik}}{\partial x_i} + F_k = 0$$

Equilibrium Equations

Also, the moment about the origin equals to zero for any finite body:

$$M_i = \int_V \epsilon_{ijk} x_j F_k \, dV + \int_S \epsilon_{ijk} x_j \overset{\mu}{\tau}_k \, dS = 0$$

$$\int_S \epsilon_{ijk} x_j \overset{\mu}{\tau}_k \, dS \xlongequal{\text{represent traction with stress}} \int_S \epsilon_{ijk} x_j \mu_\ell \tau_{\ell k} \, dS \xlongequal{\text{Gauss theorem}} \int_V \frac{\partial(\epsilon_{ijk} x_j \tau_{\ell k})}{\partial x_\ell} dV = \int_V \epsilon_{ijk} \left(\delta_{j\ell} \tau_{\ell k} + x_j \frac{\partial \tau_{\ell k}}{\partial x_\ell} \right) dV$$

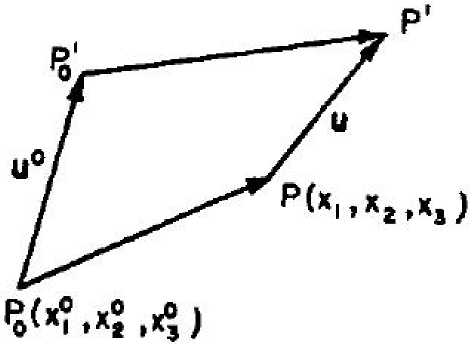
$$\begin{aligned} M_i &= \int_V \epsilon_{ijk} x_j F_k \, dV + \int_V \epsilon_{ijk} \left(\delta_{j\ell} \tau_{\ell k} + x_j \frac{\partial \tau_{\ell k}}{\partial x_\ell} \right) dV \\ &= \int_V \epsilon_{ijk} \Big(x_j F_k + \tau_{jk} + x_j \frac{\partial \tau_{\ell k}}{\partial x_\ell} \Big) dV \end{aligned}$$

$$\int_V \epsilon_{ijk} \tau_{jk} \, dV = 0 \qquad \epsilon_{ijk} \tau_{jk} = 0$$

Or

$$\begin{aligned} \tau_{12} - \tau_{21} &= 0 \\ \tau_{13} - \tau_{31} &= 0 \\ \tau_{23} - \tau_{32} &= 0 \end{aligned} \qquad \tau_{ik} = \tau_{ki}$$

Strain-Displacement Relations, General displacement (总位移)



Points P_0 and P moves to P'_0 and P' after deformation,

Given displacement at P_0 :u_i^0, calculate displacement at P: u_i

$$u_i = u_i^0 + \frac{\partial u_i}{\partial x_j}(x_j - x_j^0) + \cdots$$

$$= u_i^0 + \frac{\partial u_i}{\partial x_j} dx_j \qquad dx_j = x_j - x_j^0$$

Gradient of vector u_i can be decomposed into a symmetric and an antisymmetric tensor:

$$\frac{\partial u_i}{\partial x_j} = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}) + \frac{1}{2}(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}) = \varepsilon_{ij} + \omega_{ij}$$

Strain tensor: 2nd-order symmetric tensor

$$\varepsilon_{ij} = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}) = \frac{1}{2}(u_{i,j} + u_{j,i})$$

$$\varepsilon_{11} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_1}{\partial x_1}\right) = \varepsilon_x$$

$$\varepsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) = \frac{1}{2}\gamma_{xy}$$

$$\varepsilon_{13} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right) = \frac{1}{2}\gamma_{xz} \cdots$$

Rotation tensor(转动张量): 2nd-order antisymmetric tensor

$$\omega_{ij} = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}) = \frac{1}{2}(u_{i,j} - u_{j,i})$$

Strain-Displacement Relations, General displacement

$$u_i = u_i^0 + \varepsilon_{ij} dx_j + \omega_{ij} dx_j$$

$$du_i = u_i - u_i^0 = \varepsilon_{ij} dx_j + \omega_{ij} dx_j$$

General displacement (总位移) of point P is due to

① displacement components of the point P_0

② relative movement of P due to deformation (strain)

③ relative movement of P due to a rotation around P_0.

The body movement includes:

- Translation, rotation and deformation
- body deformation = Normal strain + Shear strain

Classroom exercise

1. The strain tensor ε_{ij} is expressed as

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) = \frac{1}{2}(u_{i,j} + u_{j,i})$$

Prove that E_{ij} whose non-diagonal elements are engineering shear strain is not a 2nd-order tensor

$$E_{ij} = \begin{cases} \varepsilon_{ij} & \mathrm{i} = \mathrm{j} \\ 2\varepsilon_{ij} & \mathrm{i} \neq \mathrm{j} \end{cases}$$

$$E_{ij} = \begin{bmatrix} \varepsilon_{11} & 2\varepsilon_{12} & 2\varepsilon_{13} \\ 2\varepsilon_{21} & \varepsilon_{22} & 2\varepsilon_{23} \\ 2\varepsilon_{31} & 2\varepsilon_{32} & \varepsilon_{33} \end{bmatrix} = \begin{bmatrix} \varepsilon_x & \gamma_{xy} & \gamma_{xz} \\ \gamma_{yx} & \varepsilon_y & \gamma_{yz} \\ \gamma_{zx} & \gamma_{zy} & \varepsilon_z \end{bmatrix}$$

Classroom Exercise

The total derivative (全微分) of displacement can be expressed as a tensor equation:

$$du_i = \varepsilon_{ij} dx_j + \omega_{ij} dx_j$$

Show that the above expression for the total derivative of displacement is equivalent to

$$du = \epsilon_x dx + \tfrac{1}{2}\gamma_{xy} dy + \tfrac{1}{2}\gamma_{xz} dz - \omega_z dy + \omega_y dz$$

$$dv = \epsilon_y dy + \tfrac{1}{2}\gamma_{xy} dx + \tfrac{1}{2}\gamma_{yz} dz - \omega_x dz + \omega_z dx$$

$$dw = \epsilon_z dz + \tfrac{1}{2}\gamma_{xz} dx + \tfrac{1}{2}\gamma_{yz} dy - \omega_y dx + \omega_x dy$$

$$\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) \quad \omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) \quad \omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

Components of rotation: angles of rotation about x, y, and z axes, respectively.
Positive counterclockwise if viewed from the positive rotation axis direction toward the origin.

Rigid body displacement (rotation and translation) can be expressed as below (the rotation is infinitesimal):

$$u^* = u_0 - \omega_{z0} y + \omega_{y0} z$$

$$v^* = v_0 - \omega_{x0} z + \omega_{z0} x$$

$$w^* = w_0 - \omega_{y0} x + \omega_{x0} y$$

$u_0, v_0, w_0, w_{x0}, w_{y0}, w_{z0}$ are constants

Check its strain

Classroom Exercise

$$\varepsilon'_{jn} = a_{ij}a_{kn}\varepsilon_{ik}$$

$$\varepsilon'_{11} = a_{i1}a_{k1}\varepsilon_{ik}$$

$$= \varepsilon_{11}a^2_{11} + \varepsilon_{22}a^2_{21} + \varepsilon_{33}a^2_{31} + 2\varepsilon_{12}a_{11}a_{21} + 2\varepsilon_{23}a_{21}a_{31} + 2\varepsilon_{31}a_{31}a_{11}$$

$$E'_{11} = \varepsilon'_{11}$$

$$= \varepsilon_{11}a^2_{11} + \varepsilon_{22}a^2_{21} + \varepsilon_{33}a^2_{31} + 2\varepsilon_{12}a_{11}a_{21} + 2\varepsilon_{23}a_{21}a_{31} + 2\varepsilon_{31}a_{31}a_{11}$$

$$\neq a_{i1}a_{k1}E_{ik}$$

Generalized Hooke's law

$$\begin{cases}\sigma_x = 2G\varepsilon_x + \lambda\left(\varepsilon_x+\varepsilon_y+\varepsilon_z\right)\\ \sigma_y = 2G\varepsilon_y + \lambda\left(\varepsilon_x+\varepsilon_y+\varepsilon_z\right)\\ \sigma_z = 2G\varepsilon_z + \lambda\left(\varepsilon_x+\varepsilon_y+\varepsilon_z\right)\end{cases} \quad \begin{cases}\tau_{xy} = G\gamma_{xy}\\ \tau_{yz} = G\gamma_{yz}\\ \tau_{zx} = G\gamma_{zx}\end{cases} \rightarrow \begin{cases}\sigma_x = 2G\varepsilon_x + \lambda\left(\varepsilon_x+\varepsilon_y+\varepsilon_z\right)\\ \sigma_y = 2G\varepsilon_y + \lambda\left(\varepsilon_x+\varepsilon_y+\varepsilon_z\right)\\ \sigma_z = 2G\varepsilon_z + \lambda\left(\varepsilon_x+\varepsilon_y+\varepsilon_z\right)\end{cases} \quad \begin{cases}\tau_{xy} = 2G\varepsilon_{xy}\\ \tau_{yz} = 2G\varepsilon_{yz}\\ \tau_{zx} = 2G\varepsilon_{zx}\end{cases}$$

$\rightarrow$ $\tau_{ij} = 2G\varepsilon_{ij} + \lambda\varepsilon\delta_{ij}$ where $\varepsilon = \varepsilon_{kk}$

$$\begin{cases}\varepsilon_x = \frac{1}{E}\left(\sigma_x - \nu\left(\sigma_y+\sigma_z\right)\right)\\ \varepsilon_y = \frac{1}{E}\left(\sigma_y - \nu\left(\sigma_z+\sigma_x\right)\right)\\ \varepsilon_z = \frac{1}{E}\left(\sigma_z - \nu\left(\sigma_x+\sigma_y\right)\right)\end{cases} \quad \begin{cases}\gamma_{xy} = \frac{1}{G}\tau_{xy}\\ \gamma_{yz} = \frac{1}{G}\tau_{yz}\\ \gamma_{zx} = \frac{1}{G}\tau_{zx}\end{cases} \xrightarrow{G=\frac{E}{2(1+\nu)}} \begin{cases}\varepsilon_x = \frac{1}{E}\left((1+\nu)\sigma_x - \nu\left(\sigma_x+\sigma_y+\sigma_z\right)\right)\\ \varepsilon_y = \frac{1}{E}\left((1+\nu)\sigma_y - \nu\left(\sigma_x+\sigma_y+\sigma_z\right)\right)\\ \varepsilon_z = \frac{1}{E}\left((1+\nu)\sigma_z - \nu\left(\sigma_x+\sigma_y+\sigma_z\right)\right)\end{cases} \quad \begin{cases}\varepsilon_{xy} = \frac{1+\nu}{E}\tau_{xy}\\ \varepsilon_{yz} = \frac{1+\nu}{E}\tau_{yz}\\ \varepsilon_{zx} = \frac{1+\nu}{E}\tau_{zx}\end{cases}$$

$\rightarrow$ $\varepsilon_{ij} = \frac{1+\nu}{E}\tau_{ij} - \frac{\nu}{E}\delta_{ij}\Theta$ where $\Theta = \tau_{kk}$

Generalized Hooke's law: elastic constant C_{ijkl} is a 4_th order tensor

Generalized Hooke's law

$$\tau_{ij} = C_{ijkl}\varepsilon_{kl}$$
$$= C_{ij11}\varepsilon_{11} + C_{ij12}\varepsilon_{12} + C_{ij13}\varepsilon_{13}$$
$$+ C_{ij21}\varepsilon_{21} + C_{ij22}\varepsilon_{22} + C_{ij23}\varepsilon_{23}$$
$$+ C_{ij31}\varepsilon_{31} + C_{ij32}\varepsilon_{32} + C_{ij33}\varepsilon_{33}$$

Sometimes we use e to represent strain

$$e_{kl} = \varepsilon_{kl}$$

The elastic constant C_{ijkl} is a fourth order tensor , i.e.

$$C'_{mpqr} = a_{im}a_{jp}a_{kq}a_{lr}C_{ijkl}$$

or $C_{ijkl} = a_{im}a_{jp}a_{kq}a_{lr}C'_{mpqr}$

$$\tau_{ij} = a_{im}a_{jp}\tau'_{mp}$$
$$= a_{im}a_{jp}C'_{mpqr}\,\varepsilon'_{qr}$$
$$= a_{im}a_{jp}C'_{mpqr}\,a_{kq}a_{lr}\varepsilon_{kl}$$

$$\tau_{ij} = C_{ijkl}\varepsilon_{kl}$$

$$C_{ijkl}\varepsilon_{kl} = a_{im}a_{jp}a_{kq}a_{lr}C'_{mpqr}\,\varepsilon_{kl}$$

$$C_{ijkl} = a_{im}a_{jp}a_{kq}a_{lr}C'_{mpqr}$$

The elastic constant C_{ijkl} is a fourth order tensor

Generalized Hooke's law: elastic constant C_{ijkl} has only 21 independent constants

Generalized Hooke's law

$$\tau_{ij} = C_{ijkl}\varepsilon_{kl}$$

Only 21 independent parameters needed for the elastic constant C_{ijkl}

1. Because of the symmetry in the strain components, we can set

$$C_{ijkl} = C_{ijlk}$$

which reduces the number of independent constants to 54.

2. Because of the symmetry in the stress components, we can set

$$C_{ijkl} = C_{jikl}$$

$$C_{ijkl} = C_{jikl} = C_{ijlk} = C_{jilk}$$

The number of independent constants reduces to 36.

$$\begin{bmatrix} \tau_{11} \\ \tau_{22} \\ \tau_{33} \\ \tau_{23} \\ \tau_{31} \\ \tau_{12} \end{bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1123} & C_{1131} & C_{1112} \\ C_{2211} & C_{2222} & C_{2233} & C_{2223} & C_{2231} & C_{2212} \\ C_{3311} & C_{3322} & C_{3333} & C_{3323} & C_{3331} & C_{3312} \\ C_{2311} & C_{2322} & C_{2333} & C_{2323} & C_{2331} & C_{2312} \\ C_{3111} & C_{3122} & C_{3133} & C_{3123} & C_{3131} & C_{3112} \\ C_{1211} & C_{1222} & C_{1233} & C_{1223} & C_{1231} & C_{1212} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{23} \\ \varepsilon_{31} \\ \varepsilon_{12} \end{bmatrix}$$

Generalized Hooke's law: elastic constant C_{ijkl} has only 21 independent constants

3. The strain energy density is

$$U_0 = \frac{1}{2}\left(\sigma_x\varepsilon_x + \sigma_y\varepsilon_y + \sigma_z\varepsilon_z + \tau_{xy}\gamma_{xy} + \tau_{yz}\gamma_{yz} + \tau_{zx}\gamma_{zx}\right)$$

$$U_0 = \frac{1}{2}\sigma_{ij}\varepsilon_{ij} = \frac{1}{2}C_{ijkl}\varepsilon_{kl}\varepsilon_{ij} = \frac{1}{2}\sigma_{kl}\varepsilon_{kl} = \frac{1}{2}C_{klij}\varepsilon_{kl}\varepsilon_{ij}$$

Generally, we can set $C_{ijkl} = C_{klij}$ which further reduces the number of independent constants to 21. e.g.,

$$C_{ijkl} = C_{jikl} = C_{ijlk} = C_{jilk} =$$
$$C_{klij} = C_{klji} = C_{lkij} = C_{lkji}$$

$$\begin{bmatrix} \tau_{11} \\ \tau_{22} \\ \tau_{33} \\ \tau_{23} \\ \tau_{31} \\ \tau_{12} \end{bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1123} & C_{1131} & C_{1112} \\ & C_{2222} & C_{2233} & C_{2223} & C_{2231} & C_{2212} \\ & & C_{3333} & C_{3323} & C_{3331} & C_{3312} \\ & & & C_{2323} & C_{2331} & C_{2312} \\ & Sym. & & & C_{3131} & C_{3112} \\ & & & & & C_{1212} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{23} \\ \varepsilon_{31} \\ \varepsilon_{12} \end{bmatrix}$$

For fully anisotropic body, the independent elastic constants are 21

Generalized Hooke's law: strain and stress principal axes coicide for isotropic material

For isotropic material, elastic property is the same for any coordinate systems

$$C'_{ijkl} = C_{ijkl}$$

Let x_1, x_2, x_3 coincide with the principal axes of strain, i.e., $\varepsilon_{12}=\varepsilon_{13}=\varepsilon_{23}=0$.

We now prove that $\tau_{12}=\tau_{13}=\tau_{23}=0$.

First, we prove that $\tau_{23}=0$

$$\tau_{23} = C_{23ij}\varepsilon_{ij} = C_{2311}\varepsilon_{11} + C_{2322}\varepsilon_{22} + C_{2333}\varepsilon_{33}$$

Now, let new coordinate x'_1, x'_2, x'_3 be obtained by rotating the x_1, x_2 axes through 180° about x_3. Then a_{ij} is:

	x'_1	x'_2	x'_3
x_1	-1	0	0
x_2	0	-1	0
x_3	0	0	1

$$a_{ij} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\tau'_{23} = a_{k2}a_{l3}\tau_{kl} = -\tau_{23}$$

$$\varepsilon'_{ij} = a_{ki}a_{lj}\varepsilon_{kl} = \begin{bmatrix} \varepsilon_{11} & 0 & 0 \\ 0 & \varepsilon_{22} & 0 \\ 0 & 0 & \varepsilon_{33} \end{bmatrix}$$

$$\tau'_{23} = C_{23ij}\varepsilon'_{ij} = C_{2311}\varepsilon_{11} + C_{2322}\varepsilon_{22} + C_{2333}\varepsilon_{33} = \tau_{23}$$

$$\tau_{23} = 0$$

Similarly, we can prove $\tau_{12}=\tau_{13}=0$.