# Continuum Mechanics B Assignment#04

### **Answers for Assignment 4:**

If you find mistakes in these answers, please contact us and thank you for your correction.

Due to the calculation accuracy, some answers will be slightly different from your answers after the decimal point.

## **Question 1**

For a strain plane problem, we can easily obtained the following equations:

$$egin{cases} G
abla^2 u + (\lambda + G)rac{\partial}{\partial x}igg(rac{\partial u}{\partial x} + rac{\partial v}{\partial y}igg) + f_x = 0 \ G
abla^2 v + (\lambda + G)rac{\partial}{\partial y}igg(rac{\partial u}{\partial x} + rac{\partial v}{\partial y}igg) + f_y = 0 \end{cases}$$

(a).

 $\therefore f_x$  and  $f_y$  are constant

$$\therefore \frac{\partial f_x}{\partial y} = \frac{\partial f_y}{\partial x} = 0$$

٠.

$$Grac{\partial}{\partial y}
abla^2 u + (\lambda + G)rac{\partial^2}{\partial x \partial y}igg(rac{\partial u}{\partial x} + rac{\partial v}{\partial y}igg) + rac{\partial f_x}{\partial y} = Grac{\partial}{\partial x}
abla^2 v + (\lambda + G)rac{\partial^2}{\partial x \partial y}igg(rac{\partial u}{\partial x} + rac{\partial v}{\partial y}igg) + rac{\partial f_y}{\partial x} + rac{\partial^2}{\partial y}
abla^2 v + (\lambda + G)rac{\partial^2}{\partial x \partial y}igg(rac{\partial u}{\partial x} + rac{\partial v}{\partial y}igg) + rac{\partial f_y}{\partial x} + rac{\partial^2}{\partial y}
abla^2 v + (\lambda + G)rac{\partial^2}{\partial x \partial y}igg(rac{\partial u}{\partial x} + rac{\partial v}{\partial y}igg) + rac{\partial f_y}{\partial x} + rac{\partial^2}{\partial y}
abla^2 v + (\lambda + G)rac{\partial^2}{\partial x \partial y}igg(rac{\partial u}{\partial x} + rac{\partial v}{\partial y}igg) + rac{\partial f_y}{\partial x} + rac{\partial^2}{\partial y}
abla^2 v + (\lambda + G)rac{\partial^2}{\partial x \partial y} igg(rac{\partial u}{\partial x} + rac{\partial v}{\partial y}igg) + rac{\partial f_y}{\partial x} + rac{\partial^2}{\partial y} \nabla^2 v + (\lambda + G)rac{\partial^2}{\partial x \partial y} igg(rac{\partial u}{\partial x} + rac{\partial v}{\partial y}igg) + rac{\partial f_y}{\partial x} + rac{\partial f_y}{\partial y} + rac{\partial f_y}{\partial y} - rac{\partial f_y}{\partial y} + rac{\partial f_y}{\partial y} + rac{\partial f_y}{\partial y} + rac{\partial f_y}{\partial y} + rac{\partial f_y}{\partial y} - rac{\partial f_y}{\partial y} + rac{\partial f_y}{\partial$$

(b).

 $\because f_x$  and  $f_y$  are constant

$$\therefore \frac{\partial f_x}{\partial x} = \frac{\partial f_y}{\partial y} = 0$$

$$\begin{array}{l} \ddots \\ G\frac{\partial}{\partial x}\nabla^2 u + G\frac{\partial}{\partial y}\nabla^2 v + (\lambda + G)\frac{\partial^2}{\partial x^2}\bigg(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\bigg) + (\lambda + G)\frac{\partial^2}{\partial y^2}\bigg(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\bigg) + \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} = 0 \end{array}$$

$$G\left(rac{\partial}{\partial x}
abla^2 u + rac{\partial}{\partial y}
abla^2 v
ight) + (\lambda + G)\left(rac{\partial}{\partial x}
abla^2 u + rac{\partial}{\partial y}
abla^2 v
ight) = 0$$

$$\frac{\partial}{\partial x} \nabla^2 u + \frac{\partial}{\partial y} \nabla^2 v = 0$$

(c). Using informations that have been proved in (a) and (b), we have:

$$\because rac{\partial}{\partial x} 
abla^2 u + rac{\partial}{\partial y} 
abla^2 v = 0$$

$$egin{aligned} \therefore \left\{ egin{aligned} rac{\partial^2}{\partial x^2} 
abla^2 u + rac{\partial^2}{\partial x \partial y} 
abla^2 v = 0 \ rac{\partial^2}{\partial x \partial y} 
abla^2 u + rac{\partial^2}{\partial y^2} 
abla^2 v = 0 \end{aligned} 
ight.$$

$$\because rac{\partial}{\partial y} 
abla^2 u = rac{\partial}{\partial x} 
abla^2 v$$

$$egin{aligned} \therefore egin{cases} rac{\partial^2}{\partial x^2} 
abla^2 u + rac{\partial^2}{\partial y^2} 
abla^2 u = 0 \ rac{\partial^2}{\partial x^2} 
abla^2 v + rac{\partial^2}{\partial y^2} 
abla^2 v = 0 \end{cases} \implies egin{cases} 
abla^4 u = 0 \ 
abla^4 v = 0 \end{cases} \end{aligned}$$

$$\therefore 
abla^4 u = 
abla^4 v$$

## **Question 2**

(a). The 15 governing equations can be reduced into 7 equations:

$$egin{aligned} arepsilon_x &= -rac{
u}{E} \sigma_z \ arepsilon_y &= -rac{
u}{E} \sigma_z \ arepsilon_z &= rac{\partial u}{\partial x} \ arepsilon_y &= rac{\partial v}{\partial y} \ arepsilon_z &= rac{\partial w}{\partial z} \ rac{\partial \sigma_z}{\partial z} + f_z &= 0 \end{aligned}$$

(b).

The outwards normal unit vector of surface z=l is  $\mu=[0,0,1].$  If  $\sigma_z=
ho gz$ , then:

$$egin{cases} T_x^u = \mu_x \sigma_x + \mu_y au_{yx} + \mu_z au_{zx} = 0 \ T_y^u = \mu_x au_{xy} + \mu_y \sigma_y + \mu_z au_{zy} = 0 \ T_z^u = \mu_x au_{xz} + \mu_y au_{yz} + \mu_z \sigma_z = 
ho g l = \sigma_0 \end{cases}$$

, which means the prescribed boundary conditions are satisfied by this solution.

(c). By the solution of  $\sigma_z$  in section (b) and the generalized Hooke's law:

$$egin{cases} arepsilon_x = -rac{
u}{E}\sigma_z = -rac{
u
ho gz}{E} \ arepsilon_y = -rac{
u}{E}\sigma_z = -rac{
u
ho gz}{E} \ arepsilon_z = rac{
u
ho gz}{E} \ arepsilon_z = rac{
u
ho gz}{E} \end{cases}$$

(d). 
$$\begin{cases} u = \int du = \int \frac{\partial u}{\partial x} dx + \int \frac{\partial u}{\partial y} dy + \int \frac{\partial u}{\partial z} dz = \int \frac{\partial u}{\partial x} dx = -\frac{\nu \rho g}{E} xz \\ v = \int dv = \int \frac{\partial v}{\partial x} dx + \int \frac{\partial v}{\partial y} dy + \int \frac{\partial v}{\partial z} dz = \int \frac{\partial v}{\partial y} dy = -\frac{\nu \rho g}{E} yz \end{cases}$$

(e).

$$egin{aligned} w &= \int dw \ &= \int rac{\partial w}{\partial x} dx + \int rac{\partial w}{\partial y} dy + \int rac{\partial w}{\partial z} dz \ &= \int -rac{\partial u}{\partial z} dx + \int -rac{\partial v}{\partial z} dy + \int rac{\partial w}{\partial z} dz \ &= \int rac{
u 
ho g}{E} x dx + \int rac{
u 
ho g}{E} y dy + \int rac{
ho gz}{E} dz \ &= rac{
u 
ho g}{2E} x^2 + rac{
u 
ho g}{2E} y^2 + rac{
ho g}{2E} z^2 + C \end{aligned}$$

 $\therefore$  at point (0,0,l), w=0

$$\therefore C = -\frac{\rho g}{2E}l^2$$

$$\therefore w = \frac{\nu \rho g}{2E} x^2 + \frac{\nu \rho g}{2E} y^2 + \frac{\rho g}{2E} z^2 - \frac{\rho g}{2E} l^2$$

# **Question 3**

At side AB:

$$x=0,\,y\in[0,10\,cm]$$

$$\begin{cases} u = 0 \\ v = 0 \\ w = 0 \end{cases}$$

#### At side BC:

$$y=0,\,x\in[0,10\,cm],\,\mu=(0,-1\,cm,0)$$

$$egin{cases} T_{x}^{\mu}=- au_{yx}=x\,(MPa)\ T_{y}^{\mu}=-\sigma_{y}=-\sqrt{3}\,x\,(MPa) \implies egin{cases} au_{yx}=-x\,(MPa)\ \sigma_{y}=\sqrt{3}\,x\,(MPa)\ au_{yz}=0 \end{cases}$$

#### At side AC:

$$y+x=10\,cm,\,x\in[0,5\,cm],\,\mu=(rac{\sqrt{2}}{2}\,cm,rac{\sqrt{2}}{2}\,cm,0)$$

$$egin{cases} T_x^\mu = rac{\sqrt{2}}{2}\sigma_x + rac{\sqrt{2}}{2} au_{yx} = 10\left(MPa
ight) \ T_y^\mu = rac{\sqrt{2}}{2} au_{xy} + rac{\sqrt{2}}{2}\sigma_y = 0 \end{cases}$$

$$y+x=10\,cm,\,x\in(5\,cm,\,10\,cm],\,\mu=(rac{\sqrt{2}}{2}\,cm,rac{\sqrt{2}}{2}\,cm,0)$$

$$egin{cases} T_x^\mu = rac{\sqrt{2}}{2}\sigma_x + rac{\sqrt{2}}{2} au_{yx} = 0 \ T_y^\mu = rac{\sqrt{2}}{2} au_{xy} + rac{\sqrt{2}}{2}\sigma_y = 0 \ T_z^\mu = 0 \end{cases}$$