南方科技大学 SOUTHERN UNIVERSITY OF SCIENCE AND TECHNOLOGY

地球与空间科学系 DEPARTMENT OF EARTH AND SPACE SCIENCES

MAE5009
Continuum Mechanics B
Session 08: Symbolic Notation

Ke Gao
PhD, Associate Professor
Department of Earth and Space Sciences
Fall 2021

1

Symbolic (vector and dyadic) notation

- For vector (1st order tensor) quantities, the symbolic notation is most suitable
- For 2nd order tensor quantities, both symbolic and tensor notations could be used
- For higher-order tensors, tensor notation is more applicable

- Tensor notation is coordinate system based, equations derived with tensor notation cannot be changed readily into curvilinear coordinates
- For the purpose of treating linear elasticity in curvilinear coordinates, the symbolic notation is more convenient than the tensor notation
- Both tensor and symbolic notations are being used in the literature today

2

Basic vector analysis

- Vector:

$\mathbf{A} = (A_x, A_y, A_z)$ magnitude $|\mathbf{A}|$

Unit vector $|\mathbf{i}| = |\mathbf{j}| = |\mathbf{k}| = 1$

$\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}$

- Scalar (dot) product:

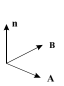
$$\mathbf{A}\cdot\mathbf{B} = |\mathbf{A}||\mathbf{B}|\cos(\mathbf{A},\mathbf{B})$$

$$\mathbf{A}\cdot\mathbf{B} = (A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k})\cdot(B_x\mathbf{i} + B_y\mathbf{j} + B_z\mathbf{k}) = A_xB_x + A_yB_y + A_zB_z$$

$$\mathbf{A}\cdot\mathbf{B} = \mathbf{B}\cdot\mathbf{A}$$

- Vector (cross) product:

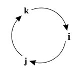
$$\mathbf{A}\times\mathbf{B} = |\mathbf{A}||\mathbf{B}|\sin(\mathbf{A},\mathbf{B})\mathbf{n}$$



3

Basic vector analysis

- Vector (cross) product:



$$\mathbf{i}\times\mathbf{j}=\mathbf{k}\qquad \mathbf{j}\times\mathbf{k}=\mathbf{i}\qquad \mathbf{k}\times\mathbf{i}=\mathbf{j}$$

$$\mathbf{j}\times\mathbf{i}=-\mathbf{k}\quad \mathbf{k}\times\mathbf{j}=-\mathbf{i}\quad \mathbf{i}\times\mathbf{k}=-\mathbf{j}$$

?

$$\mathbf{i}\times\mathbf{i}=\mathbf{j}\times\mathbf{j}=\mathbf{k}\times\mathbf{k}=0$$

$$\mathbf{A}\times\mathbf{B}=\left(A_x\mathbf{i}+A_y\mathbf{j}+A_z\mathbf{k}\right)\times\left(B_x\mathbf{i}+B_y\mathbf{j}+B_z\mathbf{k}\right)$$

$$=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ A_x & A_y & A_z\\ B_x & B_y & B_z\end{vmatrix}=\left(A_yB_z-B_yA_z\right)\mathbf{i}+\left(A_zB_x-B_zA_x\right)\mathbf{j}+\left(A_xB_y-B_xA_y\right)\mathbf{j}$$

$$\mathbf{A}\times\mathbf{B}=-\mathbf{B}\times\mathbf{A}$$

4

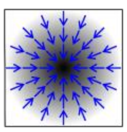
Basic vector analysis

升一级

- Vector operator ∇ (del or nabla), or gradient operator:

✱ $$\nabla=\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}$$

$$\operatorname{grad}A=\nabla A=\mathbf{i}\frac{\partial A}{\partial x}+\mathbf{j}\frac{\partial A}{\partial y}+\mathbf{k}\frac{\partial A}{\partial z}\quad\left(A=A(x,y,z)\right)$$



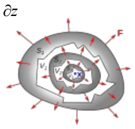
The gradient of a scalar field is a vector field, which gives the maximum rate of change of *A* at a point

降一级

- Divergence of a vector field is a scalar field:

$$\operatorname{div}\mathbf{A}=\nabla\cdot\mathbf{A}=\left(\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}\right)\cdot\left(A_x\mathbf{i}+A_y\mathbf{j}+A_z\mathbf{k}\right)=\frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z}$$

The divergence represents the volume density of the outward flux of a vector field from an infinitesimal volume around a given point.



5

$$\left(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z}\right)\times\left(A_x\vec{i}+A_y\vec{j}+A_z\vec{k}\right)$$

Basic vector analysis

- The curl of a vector field:

$$\operatorname{curl}\mathbf{A}=\nabla\times\mathbf{A}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ A_x & A_y & A_z\end{vmatrix}=\left(\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z}\right)\mathbf{i}+\left(\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x}\right)\mathbf{j}+\left(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right)\mathbf{k}$$

The curl is a vector operator that describes the infinitesimal rotation of a vector field in three-dimensional Euclidean space

- Another useful operator:

$$\operatorname{div}\nabla=\nabla\cdot\nabla=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}=\nabla^2$$

The Laplace operator

6

Dyadic – 2nd order tensor

- Stress dyadic

$$\begin{aligned}\wp &= \mathbf{BC}=\mathbf{B}\otimes\mathbf{C}^T\\ &= B_xC_x\mathbf{ii}+B_xC_y\mathbf{ij}+B_xC_z\mathbf{ik}\\ &\quad +B_yC_x\mathbf{ji}+B_yC_y\mathbf{jj}+B_yC_z\mathbf{jk}\\ &\quad +B_zC_x\mathbf{ki}+B_zC_y\mathbf{kj}+B_zC_z\mathbf{kk}\end{aligned}$$

where

$$\mathbf{B}=B_x\mathbf{i}+B_y\mathbf{j}+B_z\mathbf{k}$$
$$\mathbf{C}=C_x\mathbf{i}+C_y\mathbf{j}+C_z\mathbf{k}$$

Tensor notation: $S_{ij}=B_iC_j$

Dyadic can also be written as:

$$\begin{aligned}\wp &= S_{xx}\mathbf{ii}+S_{xy}\mathbf{ij}+S_{xz}\mathbf{ik}\\ &\quad +S_{yx}\mathbf{ji}+S_{yy}\mathbf{jj}+S_{yz}\mathbf{jk}\\ &\quad +S_{zx}\mathbf{ki}+S_{zy}\mathbf{kj}+S_{zz}\mathbf{kk}\end{aligned}$$

Its conjugate dyadic:

$$\begin{aligned}\wp_c &= S_{xx}\mathbf{ii}+S_{xy}\mathbf{ji}+S_{xz}\mathbf{ki}\\ &\quad +S_{yx}\mathbf{ij}+S_{yy}\mathbf{jj}+S_{yz}\mathbf{kj}\\ &\quad +S_{zx}\mathbf{ik}+S_{zy}\mathbf{jk}+S_{zz}\mathbf{kk}\\ &= \mathbf{CB}\end{aligned}$$

If a dyadic is equal to its conjugate, the dyadic is symmetric:

$$\wp=\wp_c$$

7

Dyadic – 2nd order tensor

- If a dyadic is antisymmetric:

$$\wp=-\wp_c$$

$$S_{xx}=S_{yy}=S_{zz}=0$$
$$S_{xy}=-S_{yx},S_{xz}=-S_{zx},S_{yz}-S_{zy}$$

- A dyadic may also be expressed in terms of three vectors:

$$\wp=\mathbf{i}\mathbf{S}_x+\mathbf{j}\mathbf{S}_y+\mathbf{k}\mathbf{S}_z$$

where

$$\mathbf{S}_x=S_{xx}\mathbf{i}+S_{xy}\mathbf{j}+S_{xz}\mathbf{k}$$
$$\mathbf{S}_y=S_{yx}\mathbf{i}+S_{yy}\mathbf{j}+S_{yz}\mathbf{k}$$
$$\mathbf{S}_z=S_{zx}\mathbf{i}+S_{zy}\mathbf{j}+S_{zz}\mathbf{k}$$

Its conjugate dyadic:

$$\wp_c=\mathbf{S}_x\mathbf{i}+\mathbf{S}_y\mathbf{j}+\mathbf{S}_z\mathbf{k}$$

8

Vector products between vector and dyadic

$$\begin{aligned}\mathbf{A}\cdot\wp &= \mathbf{A}\cdot\mathbf{BC}=\left(A_x\mathbf{i}+A_y\mathbf{j}+A_z\mathbf{k}\right)\cdot\left(\mathbf{i}\mathbf{S}_x+\mathbf{j}\mathbf{S}_y+\mathbf{k}\mathbf{S}_z\right)\\ &= A_x\mathbf{S}_x+A_y\mathbf{S}_y+A_z\mathbf{S}_z\\ &= (\mathbf{A}\cdot\mathbf{B})\mathbf{C}\end{aligned}$$

$$\mathbf{A}\cdot\wp_c=\mathbf{A}\cdot\mathbf{CB}=(\mathbf{A}\cdot\mathbf{S}_x)\mathbf{i}+\left(\mathbf{A}\cdot\mathbf{S}_y\right)\mathbf{j}+(\mathbf{A}\cdot\mathbf{S}_z)\mathbf{k}=(\mathbf{A}\cdot\mathbf{C})\mathbf{B}=\wp\cdot\mathbf{A}$$

$$\mathbf{A}\times\wp=\mathbf{A}\times\mathbf{BC}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ A_x & A_y & A_z\\ \mathbf{S}_x & \mathbf{S}_y & \mathbf{S}_z\end{vmatrix}$$

$$\begin{aligned}\operatorname{div}\wp &= \nabla\cdot\wp=\left(\frac{\partial}{\partial x}\mathbf{i}+\frac{\partial}{\partial y}\mathbf{j}+\frac{\partial}{\partial z}\mathbf{k}\right)\cdot\left(\mathbf{i}\mathbf{S}_x+\mathbf{j}\mathbf{S}_y+\mathbf{k}\mathbf{S}_z\right)\\ &= \frac{\partial}{\partial x}\mathbf{S}_x+\frac{\partial}{\partial y}\mathbf{S}_y+\frac{\partial}{\partial z}\mathbf{S}_z\end{aligned}$$

9

Vector products between vector and dyadic

$$
\begin{aligned}
\operatorname{div}\wp_c = \nabla\cdot\wp_c = \nabla\cdot\mathbf{CB} &= \left(\frac{\partial}{\partial x}\mathbf{i}+\frac{\partial}{\partial y}\mathbf{j}+\frac{\partial}{\partial z}\mathbf{k}\right)\cdot\left(\mathbf{S}_x\mathbf{i}+\mathbf{S}_y\mathbf{j}+\mathbf{S}_z\mathbf{k}\right)\\
&= (\nabla\cdot\mathbf{S}_x)\mathbf{i}+(\nabla\cdot\mathbf{S}_y)\mathbf{j}+(\nabla\cdot\mathbf{S}_z)\mathbf{k}\\
&= (\nabla\cdot\mathbf{C})\mathbf{B}
\end{aligned}
$$

$$
\operatorname{curl}\wp = \nabla\times\wp = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ \mathbf{S}_x & \mathbf{S}_y & \mathbf{S}_z\end{vmatrix} = \mathbf{i}\times\frac{\partial\wp}{\partial x}+\mathbf{j}\times\frac{\partial\wp}{\partial y}+\mathbf{k}\times\frac{\partial\wp}{\partial z}
$$

Vector products (dot and cross) between a vector and a dyadic are always made between the vector and the adjacent vector in the dyadic

10

Comparison between tensor and symbolic notations

Quantities & operations	Tensor notation	Symbolic notation
scalar	ϕ	ϕ
1st order tensor, or vector	u_i	$\mathbf{u}$
2nd order tensor, or dyadic	S_{ij}	$\wp$
Conjugate of a dyadic	S_{ji} for S_{ij}	$\wp_c$ for $\wp$
Scalar product	$u_i v_i$	$\mathbf{u}\cdot\mathbf{v}$
Vector product	$\varepsilon_{ijk}u_i v_j$	$\mathbf{u}\times\mathbf{v}$
Gradient	$\phi_{,i}$	$\nabla\phi$
Divergence	$u_{i,i}$	$\nabla\cdot\mathbf{u}$
Curl	$\varepsilon_{ijk}u_{k,j}$	$\nabla\times\mathbf{u}$
Gradient of a vector	$u_{i,j}$	$\nabla\mathbf{u}$
Divergence of a dyadic	$S_{ij,i}$	$\nabla\cdot\wp$
Curl of a dyadic	$\omega_{il}=\varepsilon_{ijk}S_{kl,j}$	$\nabla\times\wp$
* Laplace	$\phi_{,ii}$	$\nabla^2\phi$
Biharmonic	$\phi_{,iijj}$	$\nabla^4\phi$ or $\nabla^2\nabla^2\phi$

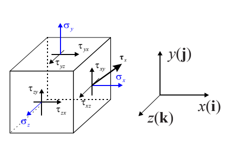
$\nabla^2(\nabla^2\phi)$

11

Vector representation of stress on a plane

- Stress components acting on the corresponding planes:

$$
\begin{aligned}
\boldsymbol{\tau}_x &= \sigma_x\mathbf{i}+\tau_{xy}\mathbf{j}+\tau_{xz}\mathbf{k}\\
\boldsymbol{\tau}_y &= \tau_{yx}\mathbf{i}+\sigma_y\mathbf{j}+\tau_{yz}\mathbf{k}\\
\boldsymbol{\tau}_z &= \tau_{zx}\mathbf{i}+\tau_{zy}\mathbf{j}+\sigma_z\mathbf{k}
\end{aligned}
$$



- The relations between the stress vector and its components are:

$$
\begin{aligned}
\sigma_x &= |\boldsymbol{\tau}_x|\cos(\boldsymbol{\tau}_x,\mathbf{i}) = \boldsymbol{\tau}_x\cdot\mathbf{i}\\
\tau_{xy} &= |\boldsymbol{\tau}_x|\cos(\boldsymbol{\tau}_x,\mathbf{j}) = \boldsymbol{\tau}_x\cdot\mathbf{j}\\
\tau_{xz} &= |\boldsymbol{\tau}_x|\cos(\boldsymbol{\tau}_x,\mathbf{k}) = \boldsymbol{\tau}_x\cdot\mathbf{k}
\end{aligned}
$$

12

Equations of transformation of stress

- The unit vector $\boldsymbol{\mu}_1$ normal to the x' plane is:

$$\boldsymbol{\mu}_1 = \left[\cos(x',x) \quad \cos(x',y) \quad \cos(x',z)\right] = \left[a_{11} \quad a_{12} \quad a_{13}\right]$$

- Force equilibrium:

$$A\mathbf{p} = A_x\boldsymbol{\tau}_x + A_y\boldsymbol{\tau}_y + A_z\boldsymbol{\tau}_z$$

$$\mathbf{p} = \frac{A_x}{A}\boldsymbol{\tau}_x + \frac{A_y}{A}\boldsymbol{\tau}_y + \frac{A_z}{A}\boldsymbol{\tau}_z$$

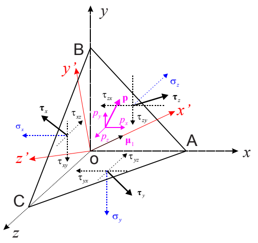
$$\cos(x',x) = \frac{A_x}{A} = a_{11}$$

$$\cos(x',y) = \frac{A_y}{A} = a_{12}$$

$$\cos(x',z) = \frac{A_z}{A} = a_{13}$$

$$\Downarrow$$

$$\mathbf{p} = a_{11}\boldsymbol{\tau}_x + a_{12}\boldsymbol{\tau}_y + a_{13}\boldsymbol{\tau}_z$$



Transformational matrix **R**

	x	y	z
x'	a_{11}	a_{12}	a_{13}
y'	a_{21}	a_{22}	a_{23}
z'	a_{31}	a_{32}	a_{33}

13

Equations of transformation of stress

- **p** can be decomposed into:

$$\mathbf{p} = a_{11}\boldsymbol{\tau}_x + a_{12}\boldsymbol{\tau}_y + a_{13}\boldsymbol{\tau}_z$$

$$\Downarrow$$

$$p_x = \sigma_x a_{11} + \tau_{yx}a_{12} + \tau_{zx}a_{13} = \boldsymbol{\tau}_x \cdot \boldsymbol{\mu}_1$$
$$p_y = \tau_{xy} a_{11} + \sigma_y a_{12} + \tau_{zy}a_{13} = \boldsymbol{\tau}_y \cdot \boldsymbol{\mu}_1$$
$$p_z = \tau_{xz} a_{11} + \tau_{yz} a_{12} + \sigma_z a_{13} = \boldsymbol{\tau}_z \cdot \boldsymbol{\mu}_1$$

$$\Downarrow$$

$$\mathbf{p} = p_x\mathbf{i} + p_y\mathbf{j} + p_z\mathbf{k} = \mathbf{i}(\boldsymbol{\tau}_x\cdot\boldsymbol{\mu}_1) + \mathbf{j}(\boldsymbol{\tau}_y\cdot\boldsymbol{\mu}_1) + \mathbf{k}(\boldsymbol{\tau}_z\cdot\boldsymbol{\mu}_1) = (\mathbf{i}\boldsymbol{\tau}_x + \mathbf{j}\boldsymbol{\tau}_y + \mathbf{k}\boldsymbol{\tau}_z)\cdot\boldsymbol{\mu}_1$$

$$\Downarrow$$

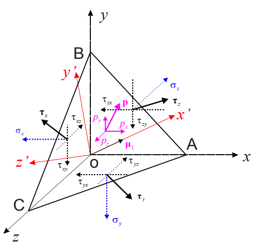
$$\mathbf{p} = \boldsymbol{\Sigma}\cdot\boldsymbol{\mu}_1$$

where $\boldsymbol{\Sigma} = \mathbf{i}\boldsymbol{\tau}_x + \mathbf{j}\boldsymbol{\tau}_y + \mathbf{k}\boldsymbol{\tau}_z$ $\Rightarrow$

$$\boldsymbol{\Sigma} = \sigma_x\mathbf{ii} + \tau_{xy}\mathbf{ij} + \tau_{xz}\mathbf{ik} + \tau_{yx}\mathbf{ji} + \sigma_y\mathbf{jj} + \tau_{yz}\mathbf{jk} + \tau_{zx}\mathbf{ki} + \tau_{zy}\mathbf{kj} + \sigma_z\mathbf{kk}$$

$$\boldsymbol{\Sigma} = \boldsymbol{\Sigma}_c$$

Stress dyadic, symbolic representation of stress tensor



14

Equations of transformation of stress

- The normal stress on the x' plane is:

$$\sigma_{x'} = \boldsymbol{\mu}_1\cdot\mathbf{p} = \boldsymbol{\mu}_1\cdot\boldsymbol{\Sigma}\cdot\boldsymbol{\mu}_1 = a_{11}^2\sigma_x + a_{12}^2\sigma_y + a_{13}^2\sigma_z + 2a_{11}a_{12}\tau_{xy} + 2a_{11}a_{13}\tau_{xz} + 2a_{11}a_{13}\tau_{yz}$$

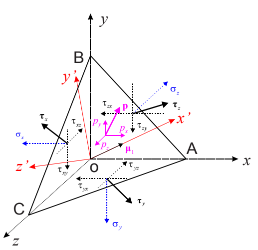
- Similarly:

$$\tau_{x'y'} = \boldsymbol{\mu}_2\cdot\mathbf{p} = \boldsymbol{\mu}_2\cdot\boldsymbol{\Sigma}\cdot\boldsymbol{\mu}_1$$
$$\tau_{x'z'} = \boldsymbol{\mu}_3\cdot\mathbf{p} = \boldsymbol{\mu}_3\cdot\boldsymbol{\Sigma}\cdot\boldsymbol{\mu}_1$$

where $\boldsymbol{\mu}_2$ and $\boldsymbol{\mu}_3$ are unit vectors in the y' and z' directions, respectively

- If x, y and z directions coincides with principal directions:

$$\boldsymbol{\Sigma} = \mathbf{i}\boldsymbol{\tau}_1 + \mathbf{j}\boldsymbol{\tau}_2 + \mathbf{k}\boldsymbol{\tau}_3 = \sigma_1\mathbf{ii} + \sigma_2\mathbf{jj} + \sigma_3\mathbf{kk}$$



Transformational matrix **R**

	x	y	z
x'	a_{11}	a_{12}	a_{13}
y'	a_{21}	a_{22}	a_{23}
z'	a_{31}	a_{32}	a_{33}

15

Equilibrium equations

$$\begin{cases}\frac{\partial \sigma_x}{\partial x}+\frac{\partial \tau_{yx}}{\partial y}+\frac{\partial \tau_{zx}}{\partial z}+f_x=0\\ \frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \sigma_y}{\partial y}+\frac{\partial \tau_{zy}}{\partial z}+f_y=0\\ \frac{\partial \tau_{xz}}{\partial x}+\frac{\partial \tau_{yz}}{\partial y}+\frac{\partial \sigma_z}{\partial z}+f_z=0\end{cases} \rightarrow \begin{cases}\frac{\partial \sigma_x}{\partial x}+\frac{\partial \tau_{xy}}{\partial y}+\frac{\partial \tau_{xz}}{\partial z}+f_x=0\\ \frac{\partial \tau_{yx}}{\partial x}+\frac{\partial \sigma_y}{\partial y}+\frac{\partial \tau_{yz}}{\partial z}+f_y=0\\ \frac{\partial \tau_{zx}}{\partial x}+\frac{\partial \tau_{zy}}{\partial y}+\frac{\partial \sigma_z}{\partial z}+f_z=0\end{cases}$$

$$\begin{aligned}\boldsymbol{\tau}_x&=\sigma_x\mathbf{i}+\tau_{xy}\mathbf{j}+\tau_{xz}\mathbf{k}\\ \boldsymbol{\tau}_y&=\tau_{yx}\mathbf{i}+\sigma_y\mathbf{j}+\tau_{yz}\mathbf{k}\\ \boldsymbol{\tau}_z&=\tau_{zx}\mathbf{i}+\tau_{zy}\mathbf{j}+\sigma_z\mathbf{k}\end{aligned} \rightarrow \begin{aligned}\operatorname{div}\boldsymbol{\tau}_x&=\nabla\cdot\boldsymbol{\tau}_x=\frac{\partial \sigma_x}{\partial x}+\frac{\partial \tau_{yx}}{\partial y}+\frac{\partial \tau_{zx}}{\partial z}\\ \operatorname{div}\boldsymbol{\tau}_y&=\nabla\cdot\boldsymbol{\tau}_y=\frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \sigma_y}{\partial y}+\frac{\partial \tau_{zy}}{\partial z}\\ \operatorname{div}\boldsymbol{\tau}_z&=\nabla\cdot\boldsymbol{\tau}_z=\frac{\partial \tau_{xz}}{\partial x}+\frac{\partial \tau_{yz}}{\partial y}+\frac{\partial \sigma_z}{\partial z}\end{aligned}$$

$$\downarrow$$

$$\begin{aligned}\operatorname{div}\boldsymbol{\tau}_x+f_x&=0\\ \operatorname{div}\boldsymbol{\tau}_y+f_y&=0\\ \operatorname{div}\boldsymbol{\tau}_z+f_z&=0\end{aligned} \rightarrow \begin{aligned}&\mathbf{i}(\operatorname{div}\boldsymbol{\tau}_x+f_x)+\mathbf{j}(\operatorname{div}\boldsymbol{\tau}_y+f_y)+\mathbf{k}(\operatorname{div}\boldsymbol{\tau}_z+f_z)=0\\ &\operatorname{div}(\mathbf{i}\boldsymbol{\tau}_x+\mathbf{j}\boldsymbol{\tau}_y+\mathbf{k}\boldsymbol{\tau}_z)+\mathbf{f}=0\\ &\operatorname{div}\boldsymbol{\Sigma}+\mathbf{f}=0 \quad \text{where} \quad \mathbf{f}=f_x\mathbf{i}+f_y\mathbf{j}+f_z\mathbf{k}\end{aligned}$$

16

Displacement and strain

- Displacement vector

$$\mathbf{u}=u\mathbf{i}+v\mathbf{j}+w\mathbf{k}$$

- The divergence of u gives the volumetric strain (dilation):

$$\operatorname{div}\mathbf{u}=\nabla\cdot\mathbf{u}=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=\varepsilon_x+\varepsilon_y+\varepsilon_z=\varepsilon$$

- Strain vector:

$$\begin{aligned}\mathbf{e}_x&=\varepsilon_x\mathbf{i}+\tfrac{1}{2}\gamma_{xy}\mathbf{j}+\tfrac{1}{2}\gamma_{xz}\mathbf{k}\\ &=\frac{\partial u}{\partial x}\mathbf{i}+\frac{1}{2}\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\mathbf{j}+\frac{1}{2}\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)\mathbf{k}\\ &=\frac{1}{2}\left(\nabla u+\frac{\partial \mathbf{u}}{\partial x}\right)\end{aligned}$$

$$\mathbf{e}_y=\frac{1}{2}\left(\nabla v+\frac{\partial \mathbf{u}}{\partial y}\right)$$

$$\mathbf{e}_z=\frac{1}{2}\left(\nabla w+\frac{\partial \mathbf{u}}{\partial z}\right)$$

17

Displacement and strain

- Strain dyadic:

$$\mathbf{E}=\mathbf{i}\mathbf{e}_x+\mathbf{j}\mathbf{e}_y+\mathbf{k}\mathbf{e}_z$$

$$\mathbf{E}=\frac{1}{2}(\nabla\mathbf{u}+\mathbf{u}\nabla)$$

where
$$\nabla\mathbf{u}=\mathbf{i}\frac{\partial}{\partial x}\mathbf{u}+\mathbf{j}\frac{\partial}{\partial y}\mathbf{u}+\mathbf{k}\frac{\partial}{\partial z}\mathbf{u}$$

$$\mathbf{u}\nabla=\frac{\partial}{\partial x}\mathbf{u}\mathbf{i}+\frac{\partial}{\partial y}\mathbf{u}\mathbf{j}+\frac{\partial}{\partial z}\mathbf{u}\mathbf{k}$$

18

Generalized Hooke's law and Navier's equation

- Stress-displacement relations:

$$\sigma_x = 2G\frac{\partial u}{\partial x} + \lambda \,\mathrm{div}\,\mathbf{u} \qquad \tau_{xy} = G\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) \qquad \tau_{xz} = G\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)$$

$$\downarrow$$

$$\boldsymbol{\tau}_x = \sigma_x\mathbf{i} + \tau_{xy}\mathbf{j} + \tau_{xz}\mathbf{k} = G\left(\nabla u + \frac{\partial \mathbf{u}}{\partial x}\right) + (\lambda \,\mathrm{div}\,\mathbf{u})\mathbf{i} = 2G\mathbf{e}_x + (\lambda \,\mathrm{div}\,\mathbf{u})\mathbf{i}$$

$$\boldsymbol{\tau}_y = \tau_{yx}\mathbf{i} + \sigma_y\mathbf{j} + \tau_{yz}\mathbf{k} = G\left(\nabla v + \frac{\partial \mathbf{u}}{\partial y}\right) + (\lambda \,\mathrm{div}\,\mathbf{u})\mathbf{j} = 2G\mathbf{e}_y + (\lambda \,\mathrm{div}\,\mathbf{u})\mathbf{j}$$

$$\boldsymbol{\tau}_z = \tau_{zx}\mathbf{i} + \tau_{zy}\mathbf{j} + \sigma_z\mathbf{k} = G\left(\nabla w + \frac{\partial \mathbf{u}}{\partial z}\right) + (\lambda \,\mathrm{div}\,\mathbf{u})\mathbf{k} = 2G\mathbf{e}_z + (\lambda \,\mathrm{div}\,\mathbf{u})\mathbf{k}$$

$$\downarrow$$

$$\boldsymbol{\Sigma} = 2G\mathbf{E} + \mathbf{I}\lambda\varepsilon$$

where

$$\mathbf{I} = \mathbf{ii} + \mathbf{jj} + \mathbf{kk}$$ is the unit dyadic

19

Generalized Hooke's law and Navier's equation

- Stress-displacement relations:

$$\boldsymbol{\Sigma} = 2G\mathbf{E} + \mathbf{I}\lambda\varepsilon = G(\nabla\mathbf{u} + \mathbf{u}\nabla) + \mathbf{I}\lambda\nabla\cdot\mathbf{u}$$

$$\Big\downarrow \; \mathrm{div}\,\boldsymbol{\Sigma} + \mathbf{f} = 0$$

$$G\nabla^2\mathbf{u} + G\nabla\cdot(\mathbf{u}\nabla) + \nabla\cdot(\mathbf{I}\lambda\nabla\cdot\mathbf{u}) + \mathbf{f} = 0$$

$$\downarrow$$

$$\boxed{G\nabla^2\mathbf{u} + (\lambda + G)\nabla\nabla\cdot\mathbf{u} + \mathbf{f} = 0}$$

since $\nabla\cdot(\mathbf{u}\nabla) = \nabla\cdot\mathbf{u}\nabla = (\nabla\cdot\mathbf{u})\nabla = \nabla(\nabla\cdot\mathbf{u}) = \nabla\nabla\cdot\mathbf{u}$

$$\nabla\cdot(\mathbf{I}\nabla\cdot\mathbf{u}) = \nabla(\nabla\cdot\mathbf{u})$$

20

Compatibility equations

- Curl and conjugate curl of dyadic:

$$\nabla\times\wp = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \mathbf{F}_x & \mathbf{F}_y & \mathbf{F}_z \end{vmatrix} = \mathbf{i}\times\frac{\partial\wp}{\partial x} + \mathbf{j}\times\frac{\partial\wp}{\partial y} + \mathbf{k}\times\frac{\partial\wp}{\partial z}$$

where $\wp = \mathbf{iF}_x + \mathbf{jF}_y + \mathbf{kF}_z$

$$\wp\times\nabla = \frac{\partial\wp}{\partial x}\times\mathbf{i} + \frac{\partial\wp}{\partial y}\times\mathbf{j} + \frac{\partial\wp}{\partial z}\times\mathbf{k}$$

$$\downarrow$$

$$\nabla\times\nabla\wp + \wp\nabla\times\nabla = 0$$

21

Compatibility equations

- Curl and conjugate curl of dyadic:

$$\mathbf{E}=\frac{1}{2}(\nabla\mathbf{u}+\mathbf{u}\nabla) \qquad \mathbf{\Omega}=\frac{1}{2}(\nabla\mathbf{u}-\mathbf{u}\nabla)$$ Rotation dyadic

$$\downarrow$$

$$\mathbf{E}+\mathbf{\Omega}=\nabla\mathbf{u} \qquad \nabla\times\mathbf{E}+\nabla\times\mathbf{\Omega}=0$$

$$\downarrow$$

$$\nabla\times\mathbf{E}\times\nabla=0$$ Compatibility equations

$$\downarrow$$

$$\nabla^2\mathbf{\Sigma}+\frac{1}{1+\nu}\nabla\nabla\Theta=-\frac{\nu}{1-\nu}\mathbf{I}\nabla\cdot\mathbf{F}+\mathbf{F}\nabla+\nabla\mathbf{F}=0$$

Compatibility equations in terms of stress

22

南方科技大学 SOUTHERN UNIVERSITY OF SCIENCE AND TECHNOLOGY

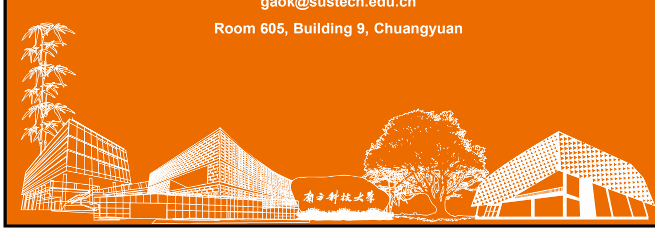
地球与空间科学系 DEPARTMENT OF EARTH AND SPACE SCIENCES

Ke Gao

0755-88018649

gaok@sustech.edu.cn

Room 605, Building 9, Chuangyuan



23