Continuum Mechanics (B)

Session 03: Stress Strain Relations

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- Generalized Hooke’s law
- Bulk Modulus of Elasticity
- Strain and Stress decomposition
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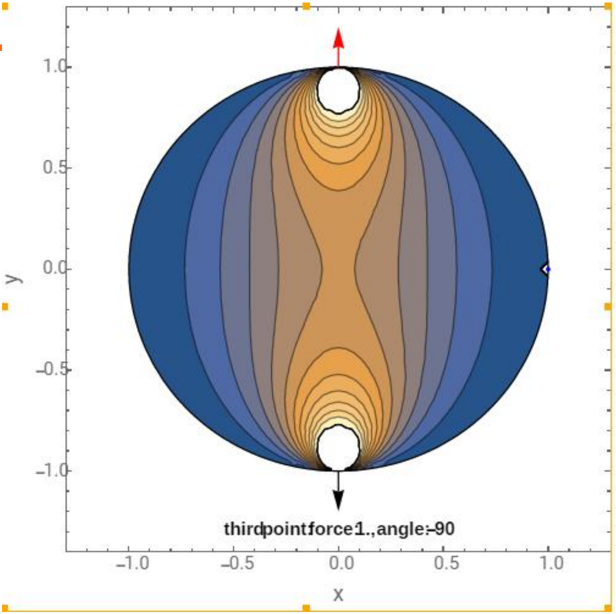
Generalized Hooke's law (广义胡克定律)

When a body is subjected to external loads (加载)

- induced internal forces
  - stress: σ_x, σ_y, σ_z, τ_{xy}, τ_{xz}, τ_{yz}
- induced internal deformation
  - strain: ε_x, ε_y, ε_z, γ_{xy}, γ_{xz}, γ_{yz}
- What is the relationship between the internal forces and deformations?
- Constitutive equations (本构方程):
  - The equations relating internal forces (stress, stress-rate), and internal deformation (strain, strain-rate).
  - It is a material property (物质属性) of the medium
    - It depends on temperature, pressure, grain size,...
- Generalized Hooke's law (广义胡克定律): the constitutive equations for linear elastic solids



strain (σ_{max}) distribution in a circular plate with concentrated loadings (https://demonstrations.wolfram.com/StressDistributionInACircularPlateWithConcentratedRadialLoad/)

Generalized Hooke's law

Generalized Hooke's law for 3D elastic solids

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} \\ c_{21} & c_{22} & c_{23} & c_{24} & c_{25} & c_{26} \\ c_{31} & c_{32} & c_{33} & c_{34} & c_{35} & c_{36} \\ c_{41} & c_{42} & c_{43} & c_{44} & c_{45} & c_{46} \\ c_{51} & c_{52} & c_{53} & c_{54} & c_{55} & c_{56} \\ c_{61} & c_{62} & c_{63} & c_{64} & c_{65} & c_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{bmatrix}$$

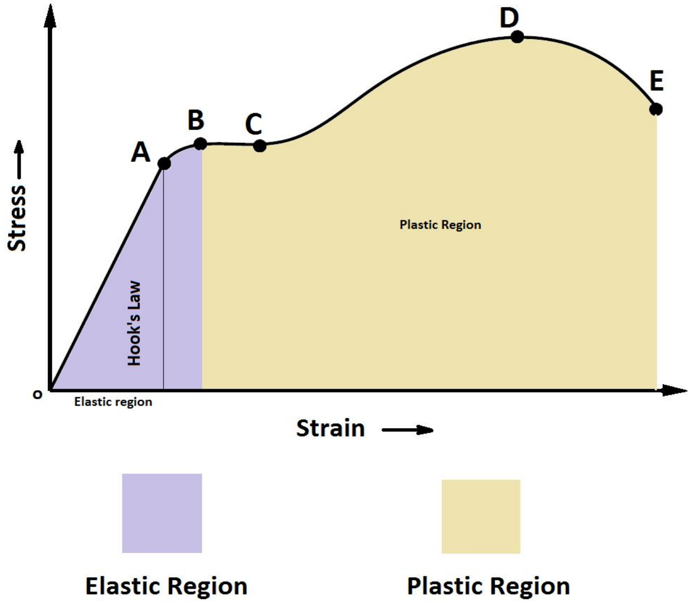
- Matrix C may vary spatially for arbitrary 3D solid
- Matrix C is symmetric and has 21 independent coefficients for fully anisotropic 3D solid

Homogeneous isotropic linear elastic solid assumption (均匀各向同性线弹性固体假设)

- Isotropic material
  - the elastic properties are the same in any direction at a point.
  - the 21 elastic constants can be reduced to 2 independent constants
- Homogeneous material
  - material properties independent of position
    - c_{ij} are thus constants (elastic constants, 弹性常数)

Generalized Hooke’s law

- Linear elastic solid assumption （线弹性假设)
  - Elastic solid model:
    - regains its original dimensions after the forces removed.
      - The constitutive law involes only stress and strain
  - Linear elastic solid model:
    - stress-strain relations are linear.
  - Elastic range (弹性范围/极限σ_B):
    - The range of stress and strain for which the behavior is elastic
    - ≈ proportional limit (比例极限σ_A), i.e., $\sigma_B \approx \sigma_A$



Generalized Hooke's law——shear stress and shear strain

Deduce the Generalized Hooke's law for isotropic elastic solid

- use experimental evidence and strain superposition principle
  - experimental observations:
    - Normal stress does not produce shear strain
    - Shear stress does not cause normal strain
    - A shear stress component only cause one shear strain component (e.g., $\tau_{xy} \rightarrow \gamma_{xy}$)

shear strain due to shear stress

$$\gamma_{xy} = \frac{1}{G}\tau_{xy}$$

$$\gamma_{yz} = \frac{1}{G}\tau_{yz}$$

$$\gamma_{zx} = \frac{1}{G}\tau_{zx}$$

G: shear modulus, or modulus of rigidity (Unit: Pa or GPa)

μ is also often used

Generalized Hooke's law——normal stress and normal strain

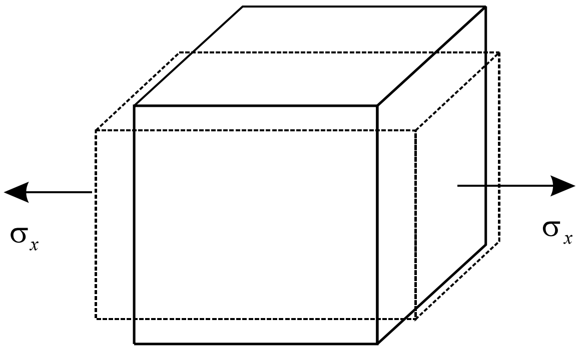
- Consider an element under uniaxial normal stress (单轴正应力) σ_x
  - The normal strain is proportional to the normal stress

$$\sigma_x = E\varepsilon_x$$

  - *E* is Young's modulus (unit: Pa or GPa)
- There are usually contractions in the y and z directions
  - ε_y and ε_z equal and are proportional to ε_x

$$\varepsilon_y = \varepsilon_z = -\nu\varepsilon_x = -\nu(\sigma_x / E)$$

  - ν is a constant called the Poisson's ratio
    - the Unit of ν
  - The Poisson's ratio must be -1.0 ≤ ν ≤ 0.5, and is usually larger than zero (0.0 ≤ ν ≤ 0.5)
    - Some materials, e.g. some polymer foams exhibit negative Poisson's ratio



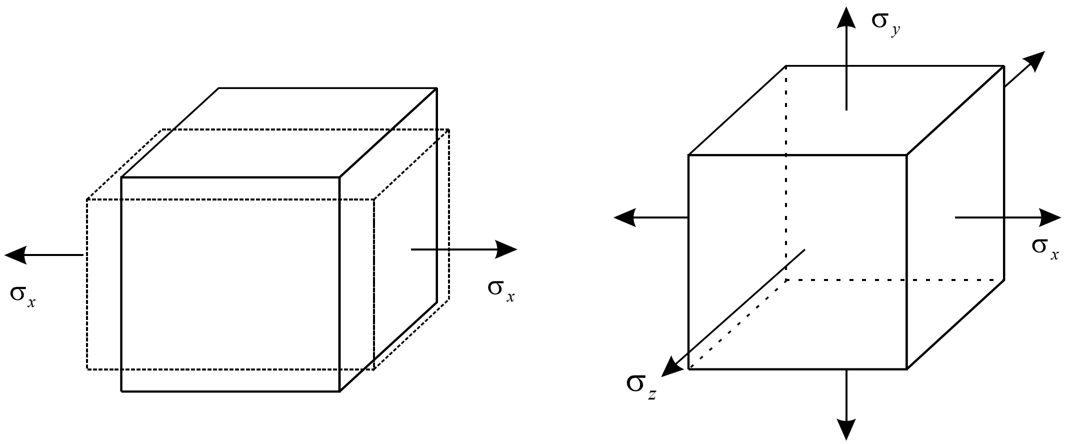
Material	Poisson's ratio
Rubber	0.4999
Gold	0.42–0.44
Rock	0.15–0.40
Cork	0.0

Generalized Hooke's law——normal stress and normal strain in 3D

For an element subjected to triaxial stress, calculate the strain ε_x

ε_x incorporates the contribution of all normal stresses:

- The contribution of σ_x: σ_x/E
- The contribution of σ_y: $-\nu\sigma_y/E$
- The contribution of σ_z: $-\nu\sigma_z/E$



Element under Triaxial stress

Based on the superposition principle of strain, the total normal strain along the x direction is

$$\varepsilon_x = \frac{1}{E}\left(\sigma_x - \nu\left(\sigma_y + \sigma_z\right)\right)$$

Similarly, normal strain along y and z directions are

$$\varepsilon_y = \frac{1}{E}\left(\sigma_y - \nu\left(\sigma_x + \sigma_z\right)\right)$$

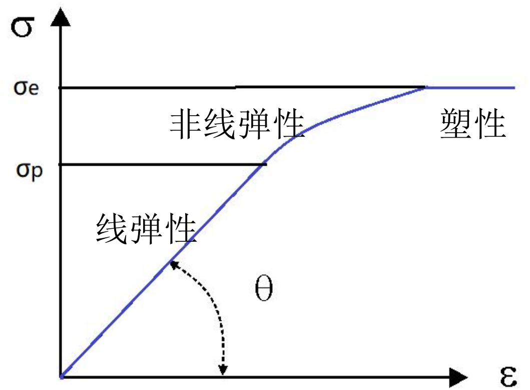
$$\varepsilon_z = \frac{1}{E}\left(\sigma_z - \nu\left(\sigma_x + \sigma_y\right)\right)$$

Classroom exercise

- Check that the dilatation (unit volume change) is

  $$\varepsilon_x + \varepsilon_y + \varepsilon_z$$

- Check that a perfectly incompressible isotropic material would have a Poisson's ratio of 0.5
- Prove that the volume either keeps constant or increases under uniaxial extension



stress-strain curve (应力应变曲线)

The Young’s modulus and elastic proportional limit (比例极限σ_p) of mild steel (生铁) is 200 GPa and 200 MPa, respectively. Calculate the maximum elastic strain in mild steel

0.001

The elastic strain we are dealing with is small.

Review

Constitutive equations (本构方程):

$\rightarrow$

Generalized Hooke's law (广义胡克定律):

- The constitutive equations for linearly elastic solids (线弹性体)
  - it involves only stress and strain
  - linear relation between stress and strain

Hooke's law for 3D anisotropic elastic solids

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} \\ c_{21} & c_{22} & c_{23} & c_{24} & c_{25} & c_{26} \\ c_{31} & c_{32} & c_{33} & c_{34} & c_{35} & c_{36} \\ c_{41} & c_{42} & c_{43} & c_{44} & c_{45} & c_{46} \\ c_{51} & c_{52} & c_{53} & c_{54} & c_{55} & c_{56} \\ c_{61} & c_{62} & c_{63} & c_{64} & c_{65} & c_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{bmatrix}$$

homogeneous isotropic linearly elastic solid (均匀各向同性线弹性固体假设)

- **only two** indepdent parameters (called elastic constants).

Experimental observations:

- Normal stress does not produce shear strain
- Shear stress does not cause normal strain
- A shear stress component only cause one shear strain component (e.g., $\tau_{xy} \rightarrow \gamma_{xy}$)

$$\begin{cases} \gamma_{xy} = \dfrac{1}{G}\tau_{xy} \\ \gamma_{yz} = \dfrac{1}{G}\tau_{yz} \\ \gamma_{zx} = \dfrac{1}{G}\tau_{zx} \end{cases}$$

$$\sigma_x = E\varepsilon_x$$

$$\varepsilon_y = \varepsilon_z = -\nu\varepsilon_x = -\nu(\sigma_x / E)$$

$$\varepsilon_x = \frac{1}{E}\left(\sigma_x - \nu\left(\sigma_y + \sigma_z\right)\right)$$