MAE5009: Continuum Mechanics B
Assignment 05: Notation
Due November 30, 2020

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1. Please expand the following Cartesian tensor notations and give final values if possible:

(a) B_{ijj}

$$i = 1, j = 1,2,3, B_{1jj} = B_{111} + B_{122} + B_{133}$$
$$i = 2, j = 1,2,3, B_{2jj} = B_{211} + B_{222} + B_{233}$$
$$i = 3, j = 1,2,3, B_{3jj} = B_{311} + B_{322} + B_{333}$$
$$B_{ijj} = B_{111} + B_{122} + B_{133} + B_{211} + B_{222} + B_{233} + B_{311} + B_{322} + B_{333}$$

(b) $a_i T_{ij}$

$$j = 1, a_i T_{i1} = a_1 T_{11} + a_2 T_{21} + a_3 T_{31}$$
$$j = 2, a_i T_{i2} = a_1 T_{12} + a_2 T_{22} + a_3 T_{32}$$
$$j = 3, a_i T_{i3} = a_1 T_{13} + a_2 T_{23} + a_3 T_{33}$$
$$\begin{bmatrix} T_{11} & T_{21} & T_{31} \\ T_{12} & T_{22} & T_{32} \\ T_{13} & T_{23} & T_{33} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

(c) $a_i b_j S_{ij}$

$$\begin{bmatrix} a_1 b_1 S_{11} & a_1 b_2 S_{12} & a_1 b_3 S_{13} \\ a_2 b_1 S_{21} & a_2 b_2 S_{22} & a_2 b_3 S_{23} \\ a_3 b_1 S_{31} & a_3 b_2 S_{32} & a_3 b_3 S_{33} \end{bmatrix}$$

(d) δ_{ii}

$$i = 1, \delta_{11} = 1, i = 2, \delta_{22} = 1, i = 3, \delta_{33} = 1$$
$$\delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 3$$

(e) $\delta_{ij}\delta_{ij}$

$$i = j, \delta_{ij} = 1, i \neq j, \delta_{ij} = 0$$
$$i = j = 1, \delta_{11}\delta_{11} = 1; i = j = 2, \delta_{22}\delta_{22} = 1; i = j = 3, \delta_{33}\delta_{33} = 1$$
$$\delta_{ij}\delta_{ij} = \delta_{11}\delta_{11} + \delta_{22}\delta_{22} + \delta_{33}\delta_{33} = 3$$

(f) $\delta_{ij}\delta_{ik}\delta_{jk}$

$$i = j, \delta_{ij} = 1, i \neq j, \delta_{ij} = 0$$
$$i = j = k = 1, \delta_{11}\delta_{11}\delta_{11} = 1; i = j = k = 2, \delta_{22}\delta_{22}\delta_{22} = 1; i = j = k = 3, \delta_{33}\delta_{33}\delta_{33} = 1$$
$$\delta_{ij}\delta_{ik}\delta_{jk} = \delta_{11}\delta_{11}\delta_{11} + \delta_{22}\delta_{22}\delta_{22} + \delta_{33}\delta_{33}\delta_{33} = 3$$

(g) $\varepsilon_{ijk}\varepsilon_{kij}$

if any two of i, j, k are equal, $\varepsilon_{ijk} = 0$

$$i = 1, j = 2, k = 3, \varepsilon_{123}\varepsilon_{312} = 1 \times 1 = 1$$

$$i = 1, j = 3, k = 2, \varepsilon_{132}\varepsilon_{213} = -1 \times (-1) = 1$$
$$i = 2, j = 1, k = 3, \varepsilon_{213}\varepsilon_{321} = -1 \times (-1) = 1$$
$$i = 2, j = 3, k = 1, \varepsilon_{231}\varepsilon_{123} = 1 \times 1 = 1$$
$$i = 3, j = 1, k = 2, \varepsilon_{312}\varepsilon_{231} = 1 \times 1 = 1$$
$$i = 3, j = 2, k = 1, \varepsilon_{321}\varepsilon_{132} = -1 \times (-1) = 1$$
$$\varepsilon_{ijk}\varepsilon_{kij} = \varepsilon_{123}\varepsilon_{312} + \varepsilon_{132}\varepsilon_{213} + \varepsilon_{213}\varepsilon_{321} + \varepsilon_{231}\varepsilon_{123} + \varepsilon_{312}\varepsilon_{231} + \varepsilon_{321}\varepsilon_{132} = 6$$

2. Prove the following:

(a) $\delta_{ik}\varepsilon_{ikm} = 0$

When $i = k$, $\varepsilon_{ikm} = 0$. When $i \neq k$, $\delta_{ik} = 0$. Thus, $\delta_{ik}\varepsilon_{ikm} = 0$ for any i and j.

(b) $\varepsilon_{ijk}\varepsilon_{ijk} = 6$

if any two of i, j, k are equal, $\varepsilon_{ijk} = 0$

$$i = 1, j = 2, k = 3, \varepsilon_{123}\varepsilon_{312} = 1 \times 1 = 1$$
$$i = 1, j = 3, k = 2, \varepsilon_{132}\varepsilon_{213} = -1 \times (-1) = 1$$
$$i = 2, j = 1, k = 3, \varepsilon_{213}\varepsilon_{321} = -1 \times (-1) = 1$$
$$i = 2, j = 3, k = 1, \varepsilon_{231}\varepsilon_{123} = 1 \times 1 = 1$$
$$i = 3, j = 1, k = 2, \varepsilon_{312}\varepsilon_{231} = 1 \times 1 = 1$$
$$i = 3, j = 2, k = 1, \varepsilon_{321}\varepsilon_{132} = -1 \times (-1) = 1$$
$$\varepsilon_{ijk}\varepsilon_{kij} = \varepsilon_{123}\varepsilon_{312} + \varepsilon_{132}\varepsilon_{213} + \varepsilon_{213}\varepsilon_{321} + \varepsilon_{231}\varepsilon_{123} + \varepsilon_{312}\varepsilon_{231} + \varepsilon_{321}\varepsilon_{132} = 6$$

(c) $\varepsilon_{ijk}\varepsilon_{ijp} = 2\delta_{kp}$

If any two of i, j, k or i, j, p are equal, $\varepsilon_{ijk}\varepsilon_{ijp} = 0$. So, $k = p$ when $\varepsilon_{ijk}\varepsilon_{ijp} \neq 0$.
If $k = p$, the $\delta_{kp} = 1$. Otherwise, $\delta_{kp} = 0$. Thus, $\varepsilon_{ijk}\varepsilon_{ijp} = 2\delta_{kp}$ when $k \neq p$.
When $k = p$:

$k = p = 1$:

$$i = 2, j = 3, k = 1, \varepsilon_{231}\varepsilon_{231} = 1 \times 1 = 1$$
$$i = 3, j = 2, k = 1, \varepsilon_{321}\varepsilon_{321} = -1 \times (-1) = 1$$
$$\varepsilon_{ij1}\varepsilon_{ij1} = \varepsilon_{231}\varepsilon_{231} + \varepsilon_{321}\varepsilon_{321} = 2$$
$$\delta_{11} = 1$$
$$\varepsilon_{ij1}\varepsilon_{ij1} = 2\delta_{11} = 2$$

$k = p = 2$:

$$i = 1, j = 3, k = 2, \varepsilon_{132}\varepsilon_{132} = -1 \times (-1) = 1$$
$$i = 3, j = 1, k = 2, \varepsilon_{312}\varepsilon_{312} = 1 \times 1 = 1$$
$$\varepsilon_{ij2}\varepsilon_{ij2} = \varepsilon_{132}\varepsilon_{132} + \varepsilon_{312}\varepsilon_{312} = 2$$
$$\delta_{22} = 1$$
$$\varepsilon_{ij2}\varepsilon_{ij2} = 2\delta_{22} = 2$$

$k = p = 3$:

$$i = 1, j = 2, k = 3, \varepsilon_{123}\varepsilon_{123} = 1 \times 1 = 1$$
$$i = 2, j = 1, k = 3, \varepsilon_{213}\varepsilon_{213} = -1 \times (-1) = 1$$
$$\varepsilon_{ij3}\varepsilon_{ij3} = \varepsilon_{123}\varepsilon_{123} + \varepsilon_{213}\varepsilon_{213} = 2$$
$$\delta_{33} = 1$$
$$\varepsilon_{ij3}\varepsilon_{ij3} = 2\delta_{33} = 2$$

$$\varepsilon_{ijk}\varepsilon_{ijp} = \varepsilon_{231}\varepsilon_{231} + \varepsilon_{321}\varepsilon_{321} + \varepsilon_{132}\varepsilon_{132} + \varepsilon_{312}\varepsilon_{312} + \varepsilon_{123}\varepsilon_{123} + \varepsilon_{213}\varepsilon_{213}$$
$$= 2\left(\delta_{1q} + \delta_{2q} + \delta_{3q}\right) = 2\delta_{pq} = 6$$

(d) $\delta_{ij}\delta_{jk}\delta_{km} = \delta_{im}$

$i = 1$:
$$\delta_{11}\delta_{1k} + \delta_{12}\delta_{2k} + \delta_{13}\delta_{3k} = \delta_{11}\delta_{1k} = \delta_{1k}$$
$i = 2$:
$$\delta_{21}\delta_{1k} + \delta_{22}\delta_{2k} + \delta_{23}\delta_{3k} = \delta_{22}\delta_{2k} = \delta_{2k}$$
$i = 3$:
$$\delta_{31}\delta_{1k} + \delta_{32}\delta_{2k} + \delta_{33}\delta_{3k} = \delta_{33}\delta_{3k} = \delta_{3k}$$
Then
$$\delta_{ij}\delta_{jk} = \delta_{ik}$$
With the same way, $\delta_{ik}\delta_{km} = \delta_{im}$.

So, $\delta_{ij}\delta_{jk}\delta_{km} = \delta_{ik}\delta_{km} = \delta_{im}$