MAE5009: Continuum Mechanics B

Assignment 04: Formulation of Problems in Elasticity

Due November 2, 2021

1. Verify the following equations for plane strain problems with constant f_x and f_y:

(a) $\frac{\partial}{\partial y}\nabla^2 u = \frac{\partial}{\partial x}\nabla^2 v$

Solution:

According to the displacement formulation of the plane strain problem, we can know:

$$G\nabla^2 u + (\lambda+G)\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right) + f_x = 0$$

$$G\nabla^2 v + (\lambda+G)\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right) + f_y = 0$$

Then:

$$G\frac{\partial}{\partial y}\nabla^2 u = -(\lambda+G)\frac{\partial^2}{\partial x\partial y}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)$$

$$G\frac{\partial}{\partial x}\nabla^2 u = -(\lambda+G)\frac{\partial^2}{\partial x\partial y}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)$$

Thus:

$$\frac{\partial}{\partial y}\nabla^2 u = -\frac{\lambda+G}{G}\frac{\partial^2}{\partial x\partial y}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)$$

$$\frac{\partial}{\partial x}\nabla^2 v = -\frac{\lambda+G}{G}\frac{\partial^2}{\partial x\partial y}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)$$

$$\Longrightarrow \therefore \frac{\partial}{\partial y}\nabla^2 u = \frac{\partial}{\partial x}\nabla^2 v$$

(b) $\frac{\partial}{\partial x}\nabla^2 u + \frac{\partial}{\partial y}\nabla^2 v = 0$

Solution:

According to (a)

$$\nabla^2 u = -\frac{\lambda+G}{G}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right) - \frac{f_x}{G}$$

$$\nabla^2 v = -\frac{\lambda+G}{G}\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right) - \frac{f_y}{G}$$

Then:

$$\frac{\partial}{\partial x}\nabla^2 u = -\frac{\lambda+G}{G}\frac{\partial^2}{\partial x^2}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)$$

$$\frac{\partial}{\partial y}\nabla^2 v = -\frac{\lambda+G}{G}\frac{\partial^2}{\partial y^2}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)$$

And:

$$\frac{\partial}{\partial x}\nabla^2 u + \frac{\partial}{\partial y}\nabla^2 v = -\frac{\lambda+G}{G}\left(\frac{\partial}{\partial x}\nabla^2 u + \frac{\partial}{\partial y}\nabla^2 v\right)$$

Since $-\frac{\lambda+G}{G} \neq 0$

So $\frac{\partial}{\partial x}\nabla^2 u + \frac{\partial}{\partial y}\nabla^2 v = 0$

(c) $\nabla^4 u = \nabla^4 v = 0$, where $\nabla^4 = \nabla^2\left(\nabla^2\right)$.

Solution:

$$\nabla^4 u = \nabla^2(\nabla^2 u) = \frac{\partial^2}{\partial x^2}\Delta^2 u + \frac{\partial^2}{\partial y^2}\nabla^2 u$$

$$\nabla^4 u = \nabla^2(\nabla^2 v) = \frac{\partial^2}{\partial x^2}\Delta^2 v + \frac{\partial^2}{\partial y^2}\nabla^2 v$$

According to (a)

$$\frac{\partial}{\partial y}\nabla^2 u = \frac{\partial}{\partial x}\nabla^2 v, \text{ then } \underline{\frac{\partial^2}{\partial y^2}\nabla^2 u} = \frac{\partial^2}{\partial x \partial y}\nabla^2 v,\ \underline{\frac{\partial^2}{\partial x^2}\nabla^2 v} = \frac{\partial^2}{\partial x \partial y}\nabla^2 u$$

According to (b)

$$\frac{\partial}{\partial x}\nabla^2 u = -\frac{\partial}{\partial y}\nabla^2 v, \text{ then } \underline{\frac{\partial^2}{\partial x^2}\nabla^2 u} = -\frac{\partial^2}{\partial x \partial y}\nabla^2 v,\ \underline{\frac{\partial^2}{\partial y^2}\nabla^2 v} = -\frac{\partial^2}{\partial x \partial y}\nabla^2 u$$

Thus:

$$\nabla^4 u = -\frac{\partial^2}{\partial x \partial y}\nabla^2 v + \frac{\partial^2}{\partial x \partial y}\nabla^2 v = 0$$

$$\nabla^4 v = \frac{\partial^2}{\partial x \partial y}\nabla^2 u - \frac{\partial^2}{\partial x \partial y}\nabla^2 u = 0$$

$$\therefore \nabla^4 u = \nabla^4 v = 0$$

2. A bar of constant mass density ρ hangs under its own weight and is supported by the uniform stress σ_0 as shown in the figure. Assume that the stresses σ_x, σ_y, τ_{xy}, τ_{xz} and τ_{yz} are all zero,

(a) based on the above assumption, reduce 15 governing equations to seven equations in terms of σ_z, ε_x, ε_y, ε_z, u, v and w

Solution:

From the stress-strain relations:

$$\sigma_x = 2G\varepsilon_x + \lambda(\varepsilon_x + \varepsilon_y + \varepsilon_z) = 0$$

$$\sigma_y = 2G\varepsilon_y + \lambda(\varepsilon_x + \varepsilon_y + \varepsilon_z) = 0$$

$$\sigma_z = 2G\varepsilon_z + \lambda(\varepsilon_x + \varepsilon_y + \varepsilon_z) = E\varepsilon_z$$

we can get:

$$\varepsilon_x = \varepsilon_y = -\varepsilon_z,\ \gamma_{xy} = \gamma_{xz} = \gamma_{yz} = 0$$

According to the equailibrium equations:

$$f_x = f_y = 0$$

$$\frac{\partial \sigma_z}{\partial z} = f_z$$

For the strain displacement:

$$\varepsilon_x = \frac{\partial u}{\partial v} \quad \varepsilon_y = \frac{\partial v}{\partial y} \quad \varepsilon_z = \frac{\partial w}{\partial z}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0 \quad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = 0 \quad \gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} = 0$$

(b) integrate the equilibrium equation to show that

$\sigma_z = \rho g z$

where g is the acceleration due to gravity. Also show that the prescribed boundary conditions are satisfied by this solution

Solution:

Let the cross-section area be A, the force equilibrium for any z should be satisfied.

$$\sigma_z \cdot A - mg = 0$$

Then $mg = \rho V \cdot g = \rho g \cdot Az \Rightarrow \sigma_z \cdot A - \rho g A z = 0$

We can prove $\sigma_z = \rho g z$

When $z = l$, $\sigma_z = \rho g l = \sigma_0$

When $z = 0$, $\sigma_z = 0$

(c) find ε_x, ε_y, ε_z from the generalized Hooke's law

Solution:

$$\sigma_x = 2G\varepsilon_x + \lambda(\varepsilon_x+\varepsilon_y+\varepsilon_z) = 0$$
$$\sigma_y = 2G\varepsilon_y + \lambda(\varepsilon_x+\varepsilon_y+\varepsilon_z) = 0$$

We can get:

$$\varepsilon_x = \varepsilon_y = -\nu\varepsilon_z \quad \sigma_z = 2G\varepsilon_z + \lambda(\varepsilon_x+\varepsilon_y+\varepsilon_z) = E\varepsilon_z$$

So

$$\varepsilon_z = \frac{\sigma_z}{E} = \frac{\rho g z}{E}, \quad \varepsilon_x = \varepsilon_y = -\nu\varepsilon_z = -\nu\frac{\rho g z}{E}$$

(d) if the displacement and rotation components are zero at the point $(0,0,l)$, determine the displacement component u and v

Solution:

According to (a)(b)(c):

$$\varepsilon_x = \varepsilon_y = \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = -\nu\cdot\frac{\rho g z}{E} \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \gamma_{xy} = 0$$

Then we can make assumptions that

$$u = -\nu\frac{\rho g z}{E}x + C_1(y,z) \quad v = -\nu\frac{\rho g z}{E}y + C_2(x,z)$$
$$w = \frac{\rho g}{2E}\cdot z^2 + C_3(x,y)$$

Since the displacement and rotation components are zero at the point $(0,0,l)$, we can get

$$C_1(y,z)\big|_{(0,l)} = 0, \quad C_2(x,z)\big|_{(0,l)} = 0$$
$$C_3(x,y)\big|_{(0,0)} = -\frac{\rho g}{2E}l^2$$

According to $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$,

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x\partial y}, \quad \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial x\partial y} \Rightarrow \frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 u}{\partial y^2}$$

Because $\frac{\partial^2 u}{\partial x^2} = 0$, then $\frac{\partial^2 u}{\partial y^2} = 0 = \frac{\partial^2 C_1(y,z)}{\partial y^2}$

The $\frac{\partial C_1(y,z)}{\partial y}$ is no function of y. With the

Since $\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0$, then $\frac{\partial C_1(y,z)}{\partial y} = -\frac{\partial C_2(x,z)}{\partial x} = f_1(z)$

$$\frac{\partial C_1(y,z)}{\partial y}\bigg|_{(0,l)} = 0 \quad f_1(z)\big|_{z=l} = 0$$

Then $C_1(y,z) = f_1(z)y$, $C_2(x,z) = -f_1(z)\cdot x$

$$u = -\frac{\nu\rho g}{E}xz + f_1(z)y, \quad v = -\frac{\nu\rho g}{E}yz - f_1(z)x$$

With the same way about u and w, where A_1 is a constant, So $f_1(z) = A_1 z - A_1 l$ for the boundary condition that $f_1(z)\big|_{z=l} = 0$

Since:

$$\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = \left(-\frac{\nu\rho g}{E}x + A_1 y\right) + \frac{\partial C_3(x,y)}{\partial x}, \quad C_3(x,y) = \frac{\nu\rho g}{2E}x^2 - A_1 xy + f_2(y)$$

$$\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = \left(-\frac{\nu\rho g}{E}y - A_1 x\right) + \frac{C_3(x,y)}{\partial y}, \quad C_3(x,y) = \frac{\nu\rho g}{2E}y^2 + A_1 xy + f_3(x)$$

The boundary condition $C_3(x,y)\big|_{(0,0)} = -\frac{\rho g}{2E}l^2$

Then $A_1 = 0$, $f_1(z) = 0$, $f_2(y) = \frac{\nu\rho g}{2E}y^2 - \frac{\rho g}{2E}l^2$, $f_3(x) = \frac{\nu\rho g}{2E}x^2 - \frac{\rho g}{2E}l^2$

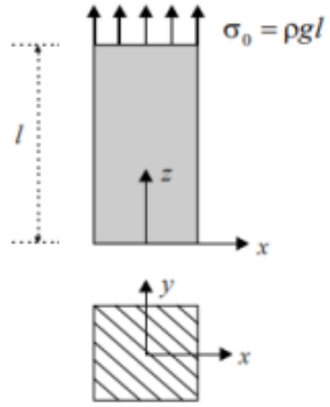
4/2

$$C_3(x,y) = \frac{\nu\rho g}{2E}x^2 + \frac{\nu\rho g}{2E}y^2 - \frac{\rho g}{2E}l^2$$

same way, $\frac{\partial C_2(x,z)}{\partial x}$ is not a function of x $u=-\frac{\nu\rho g}{E}xz,\ v=-\frac{\nu\rho g}{E}yz$

(e) prove that

$$w=\frac{\rho g}{2E}(z^2+\nu x^2+\nu y^2-l^2)$$



Solution:

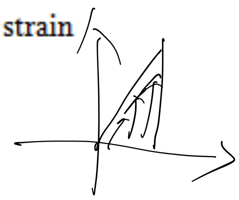
According to (d)

$$C_3(x,y)=\frac{\nu\rho g}{2E}x^2+\frac{\nu\rho g}{2E}y^2-\frac{\rho g}{2E}l^2$$

Then

$$w=\frac{\rho g}{2E}z^2+C_3(x,y)$$
$$=\frac{\rho g}{2E}z^2+\frac{\nu\rho g}{2E}x^2+\frac{\nu\rho g}{2E}y^2-\frac{\rho g}{2E}l^2$$
$$=\frac{\rho g}{2E}(z^2+\nu x^2+\nu y^2-l^2)$$

3. Express the boundary conditions for the following plate subjected to plate strain condition. The surface forces are functions of x and y only.

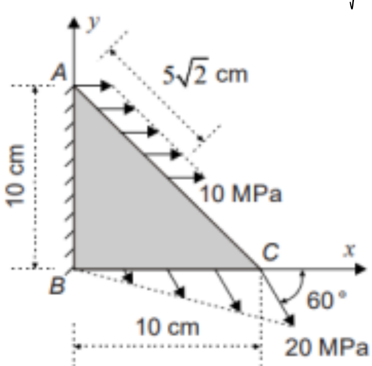


Solution:

Surface $f(x,y)$ for plane strain problem

$\varepsilon_z=\gamma_{yz}=\gamma_{xz}=0$

The boundary condition: (The surface is in balance)



AB : $\varepsilon_x=\varepsilon_y=0,\ x=0$

$$F_1=\int_0^{5\sqrt{2}}6u\,du=50\sqrt{2}\text{ MPa}$$

$$F_2=\int_0^{10}2x\,dx=100\text{ MPa}$$

$$\tau_x=\frac{F_{ABx}}{A}=\frac{F_1}{A}=\frac{50\sqrt{2}}{10}=5\sqrt{2}\text{ MPa}$$

$$\tau_y=\frac{F_{ABy}}{A}=\frac{F_2}{A}=10\text{ MPa}$$

$$\therefore\ AB:\begin{cases}\tau_x=5\sqrt{2}\text{ MPa}, & x=0,\ 0\le y\le 10\\ \tau_y=10\text{ MPa}, & x=0,\ 0\le y\le 0\end{cases}$$

$$BC:\begin{cases}\tau_x=2x\cos 60^\circ, & y=0,\ 0\le x\le 10\\ \tau_y=2x\sin 60^\circ, & y=0,\ 0\le x\le 0\end{cases}$$

$$AC_1:\begin{cases}\tau_x=10\text{ MPa}, & y=10-x\\ \tau_y=0\text{ MPa}, & y=10-x\end{cases}$$

$$AC_2:\begin{cases}\tau_x=0\text{ MPa}, & y=10-x\\ \tau_y=0\text{ MPa}, & y=10-x\end{cases}$$