Continuum Mechanics B Assignment#04

Answers for Assignment 4:

If you find mistakes in these answers, please contact us and thank you for your correction.

Due to the calculation accuracy, some answers will be slightly different from your answers after the decimal point.

Question 1

For a strain plane problem, we can easily obtained the following equations:

$$\begin{cases} G\nabla^2 u + (\lambda + G)\dfrac{\partial}{\partial x}\left(\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y}\right) + f_x = 0 \\ G\nabla^2 v + (\lambda + G)\dfrac{\partial}{\partial y}\left(\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y}\right) + f_y = 0 \end{cases}$$

(a).

$\because$ f_x and f_y are constant

$\therefore \dfrac{\partial f_x}{\partial y} = \dfrac{\partial f_y}{\partial x} = 0$

$\therefore$

$$G\frac{\partial}{\partial y}\nabla^2 u + (\lambda + G)\frac{\partial^2}{\partial x \partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \frac{\partial f_x}{\partial y} = G\frac{\partial}{\partial x}\nabla^2 v + (\lambda + G)\frac{\partial^2}{\partial x \partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \frac{\partial f_y}{\partial x}$$

$$\frac{\partial}{\partial y}\nabla^2 u = \frac{\partial}{\partial x}\nabla^2 v$$

(b).

$\because$ f_x and f_y are constant

$\therefore \dfrac{\partial f_x}{\partial x} = \dfrac{\partial f_y}{\partial y} = 0$

$\therefore$

$$G\frac{\partial}{\partial x}\nabla^2 u + G\frac{\partial}{\partial y}\nabla^2 v + (\lambda + G)\frac{\partial^2}{\partial x^2}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + (\lambda + G)\frac{\partial^2}{\partial y^2}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} = 0$$

$\therefore$

$$G\left(\frac{\partial}{\partial x}\nabla^2 u+\frac{\partial}{\partial y}\nabla^2 v\right)+(\lambda+G)\left(\frac{\partial}{\partial x}\nabla^2 u+\frac{\partial}{\partial y}\nabla^2 v\right)=0$$

$\therefore$

$$\frac{\partial}{\partial x}\nabla^2 u+\frac{\partial}{\partial y}\nabla^2 v=0$$

(c). Using informations that have been proved in (a) and (b), we have:

$$\because \frac{\partial}{\partial x}\nabla^2 u+\frac{\partial}{\partial y}\nabla^2 v=0$$

$$\therefore \begin{cases}\dfrac{\partial^2}{\partial x^2}\nabla^2 u+\dfrac{\partial^2}{\partial x\partial y}\nabla^2 v=0\\ \dfrac{\partial^2}{\partial x\partial y}\nabla^2 u+\dfrac{\partial^2}{\partial y^2}\nabla^2 v=0\end{cases}$$

$$\because \frac{\partial}{\partial y}\nabla^2 u=\frac{\partial}{\partial x}\nabla^2 v$$

$$\therefore \begin{cases}\dfrac{\partial^2}{\partial x^2}\nabla^2 u+\dfrac{\partial^2}{\partial y^2}\nabla^2 u=0\\ \dfrac{\partial^2}{\partial x^2}\nabla^2 v+\dfrac{\partial^2}{\partial y^2}\nabla^2 v=0\end{cases} \implies \begin{cases}\nabla^4 u=0\\ \nabla^4 v=0\end{cases}$$

$$\therefore \nabla^4 u=\nabla^4 v$$

Question 2

(a). The 15 governing equations can be reduced into 7 equations:

$$\begin{cases}\varepsilon_x=-\dfrac{\nu}{E}\sigma_z\\ \varepsilon_y=-\dfrac{\nu}{E}\sigma_z\\ \varepsilon_z=\dfrac{1}{E}\sigma_z\\ \varepsilon_x=\dfrac{\partial u}{\partial x}\\ \varepsilon_y=\dfrac{\partial v}{\partial y}\\ \varepsilon_z=\dfrac{\partial w}{\partial z}\\ \dfrac{\partial \sigma_z}{\partial z}+f_z=0\end{cases}$$

(b).

The outwards normal unit vector of surface $z=l$ is $\mu=[0,0,1]$. If $\sigma_z=\rho g z$, then:

$$\begin{cases} T_x^u = \mu_x\sigma_x + \mu_y\tau_{yx} + \mu_z\tau_{zx} = 0 \\ T_y^u = \mu_x\tau_{xy} + \mu_y\sigma_y + \mu_z\tau_{zy} = 0 \\ T_z^u = \mu_x\tau_{xz} + \mu_y\tau_{yz} + \mu_z\sigma_z = \rho g l = \sigma_0 \end{cases}$$

, which means the prescribed boundary conditions are satisfied by this solution.

(c). By the solution of σ_z in section (b) and the generalized Hooke's law:

$$\begin{cases} \varepsilon_x = -\dfrac{\nu}{E}\sigma_z = -\dfrac{\nu\rho g z}{E} \\ \varepsilon_y = -\dfrac{\nu}{E}\sigma_z = -\dfrac{\nu\rho g z}{E} \\ \varepsilon_z = \dfrac{1}{E}\sigma_z = \dfrac{\rho g z}{E} \end{cases}$$

(d).

$$\begin{cases} u = \int du = \int \dfrac{\partial u}{\partial x}dx + \int \dfrac{\partial u}{\partial y}dy + \int \dfrac{\partial u}{\partial z}dz = \int \dfrac{\partial u}{\partial x}dx = -\dfrac{\nu\rho g}{E}xz \\ v = \int dv = \int \dfrac{\partial v}{\partial x}dx + \int \dfrac{\partial v}{\partial y}dy + \int \dfrac{\partial v}{\partial z}dz = \int \dfrac{\partial v}{\partial y}dy = -\dfrac{\nu\rho g}{E}yz \end{cases}$$

(e).

$$\begin{aligned} w &= \int dw \\ &= \int \frac{\partial w}{\partial x}dx + \int \frac{\partial w}{\partial y}dy + \int \frac{\partial w}{\partial z}dz \\ &= \int -\frac{\partial u}{\partial z}dx + \int -\frac{\partial v}{\partial z}dy + \int \frac{\partial w}{\partial z}dz \\ &= \int \frac{\nu\rho g}{E}xdx + \int \frac{\nu\rho g}{E}ydy + \int \frac{\rho g z}{E}dz \\ &= \frac{\nu\rho g}{2E}x^2 + \frac{\nu\rho g}{2E}y^2 + \frac{\rho g}{2E}z^2 + C \end{aligned}$$

$\because$ at point $(0,0,l)$, $w=0$

$\therefore C = -\dfrac{\rho g}{2E}l^2$

$\therefore w = \dfrac{\nu\rho g}{2E}x^2 + \dfrac{\nu\rho g}{2E}y^2 + \dfrac{\rho g}{2E}z^2 - \dfrac{\rho g}{2E}l^2$

Question 3

At side AB:

$x=0,\ y \in [0, 10\,cm]$

$$\begin{cases} u = 0 \\ v = 0 \\ w = 0 \end{cases}$$

At side BC:

$y = 0,\ x \in [0, 10\,cm],\ \mu = (0, -1\,cm, 0)$

$$\begin{cases} T_x^\mu = -\tau_{yx} = x\,(MPa) \\ T_y^\mu = -\sigma_y = -\sqrt{3}\,x\,(MPa) \\ T_z^\mu = -\tau_{yz} = 0 \end{cases} \implies \begin{cases} \tau_{yx} = -x\,(MPa) \\ \sigma_y = \sqrt{3}\,x\,(MPa) \\ \tau_{yz} = 0 \end{cases}$$

At side AC:

$y + x = 10\,cm,\ x \in [0, 5\,cm],\ \mu = (\frac{\sqrt{2}}{2}\,cm, \frac{\sqrt{2}}{2}\,cm, 0)$

$$\begin{cases} T_x^\mu = \frac{\sqrt{2}}{2}\sigma_x + \frac{\sqrt{2}}{2}\tau_{yx} = 10\,(MPa) \\ T_y^\mu = \frac{\sqrt{2}}{2}\tau_{xy} + \frac{\sqrt{2}}{2}\sigma_y = 0 \end{cases}$$

$y + x = 10\,cm,\ x \in (5\,cm,\ 10\,cm],\ \mu = (\frac{\sqrt{2}}{2}\,cm, \frac{\sqrt{2}}{2}\,cm, 0)$

$$\begin{cases} T_x^\mu = \frac{\sqrt{2}}{2}\sigma_x + \frac{\sqrt{2}}{2}\tau_{yx} = 0 \\ T_y^\mu = \frac{\sqrt{2}}{2}\tau_{xy} + \frac{\sqrt{2}}{2}\sigma_y = 0 \\ T_z^\mu = 0 \end{cases}$$