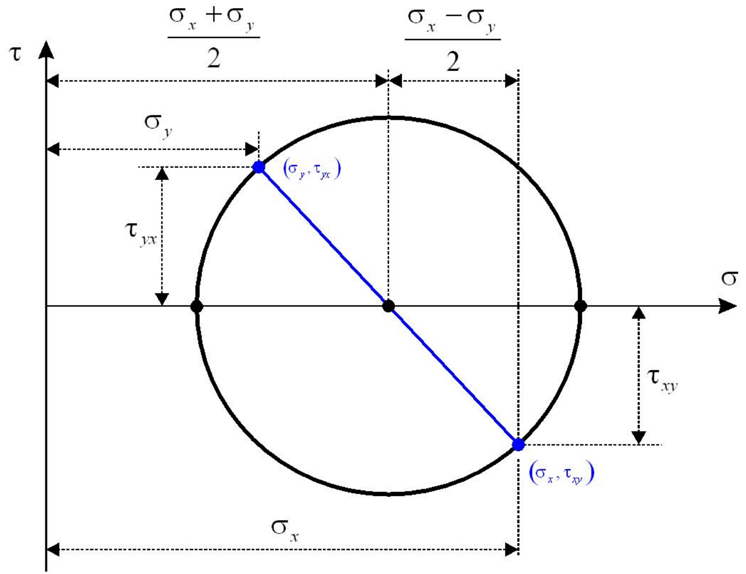
Review

Mohr's Circle of Stress

$$\left(\sigma-\frac{\sigma_x+\sigma_y}{2}\right)^2+\tau^2=\left(\frac{\sigma_x-\sigma_y}{2}\right)^2+\tau_{xy}^2$$

center: $\left(\frac{\sigma_x+\sigma_y}{2},0\right)$ radius: $\sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2+\tau_{xy}^2}$



The following relations under uniform stress distribution are also correct under a nonuniform stress distribution:

1. $\tau_{xy}=\tau_{yx}$
2. Transformation of stress equations (应力变换方程)
   1. Mohr's stress circle, formulations for principal stress and maximum shear stress

The differential equations of equilibrium (平衡微分方程):

2D

$$\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}+F_x=0$$

$$\frac{\partial\sigma_y}{\partial y}+\frac{\partial\tau_{xy}}{\partial x}+F_y=0$$

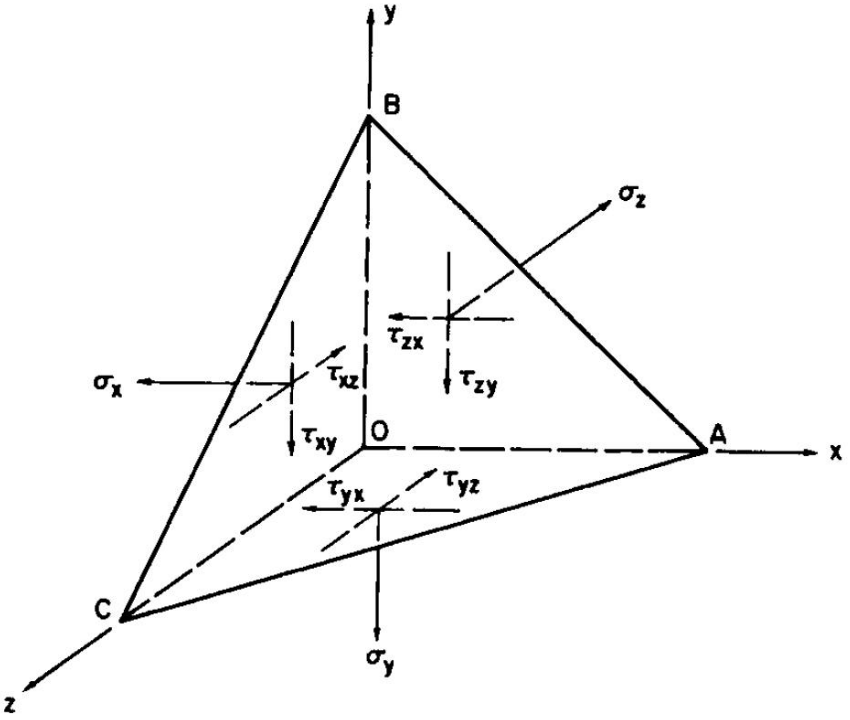
Three-Dimensional State of Stress at a Point (3D stress transformation)

- Given stress components on x, y, and z planes, we now get stress components on the arbitrary plane ABC.
  - The direction of plane ABC
  - = the direction of x'
  - $=(a_{11}, a_{21}, a_{31})$.

$a_{21}=\cos(y,x')=\cos$(the angle between y and x')

Similarly, the direction of y' is (a_{12}, a_{22}, a_{32})
the direction of z' is (a_{13}, a_{23}, a_{33})

Stress components on a tetrahedron



Three-Dimensional State of Stress at a Point

Notations for direction cosines

	x'	y'	z'
x	a_{11}	a_{12}	a_{13}
y	a_{21}	a_{22}	a_{23}
z	a_{31}	a_{32}	a_{33}

a_{21}=cos(y,x')=cos(the angle between y and x')

- The sum of the squares of the cosines in any column equals unity.
- The sum of the products of the corresponding cosines in any two columns is zero.
- Same rules apply for the rows in the table

$$a_{11}^2 + a_{21}^2 + a_{31}^2 = 1$$
$$a_{12}^2 + a_{22}^2 + a_{32}^2 = 1$$
$$a_{13}^2 + a_{23}^2 + a_{33}^2 = 1$$

$$a_{11} a_{12} + a_{21} a_{22} + a_{31} a_{32} = 0$$
$$a_{12} a_{13} + a_{22} a_{23} + a_{32} a_{33} = 0$$
$$a_{11} a_{13} + a_{21} a_{23} + a_{31} a_{33} = 0$$

Three-Dimensional State of Stress at a Point

We now calculate stress vector on plane x':

$p=(p_x, p_y, p_z)=(\sigma_{x'}, \tau_{x'y'}, \tau_{x'z'})$

With the force equilibrium conditions

$$\sum F_x = \sum F_y = \sum F_z = 0$$

We have

$$p_x = \sigma_x a_{11} + \tau_{yx} a_{21} + \tau_{zx} a_{31}$$

$$p_y = \tau_{xy} a_{11} + \sigma_y a_{21} + \tau_{zy} a_{31}$$

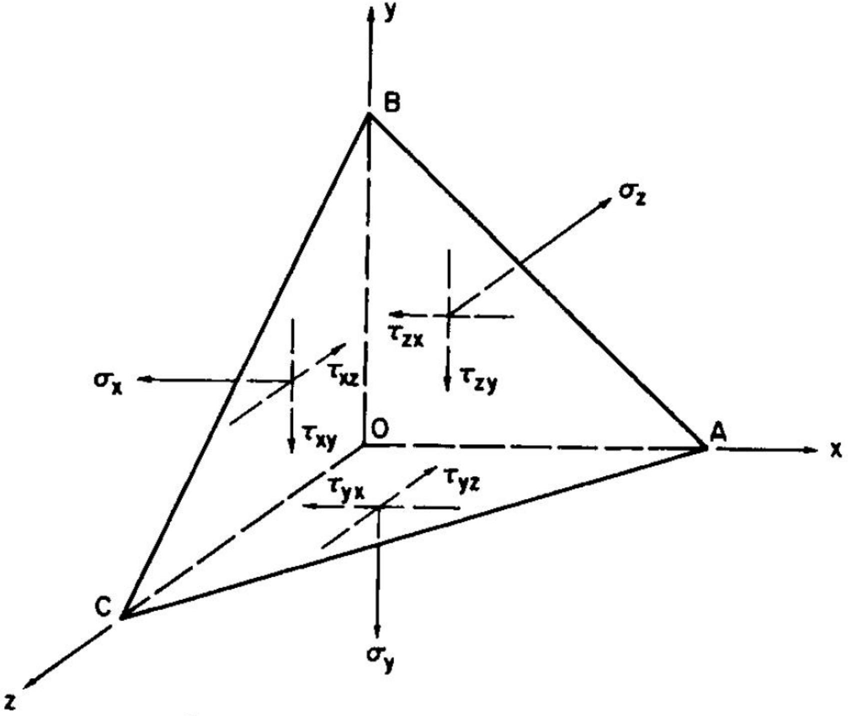
$$p_z = \tau_{xz} a_{11} + \tau_{yz} a_{21} + \sigma_z a_{31}$$

The normal stress $\sigma_{x'}$ is

$$\sigma_{x'} = (p_x, p_y, p_z) \cdot (a_{11}, a_{21}, a_{31})$$

$$\sigma_{x'} = \sigma_x a_{11}^2 + \sigma_y a_{21}^2 + \sigma_z a_{31}^2$$
$$+ 2\tau_{xy} a_{11} a_{21} + 2\tau_{yz} a_{21} a_{31} + 2\tau_{zx} a_{31} a_{11}$$

Stress components on a tetrahedron



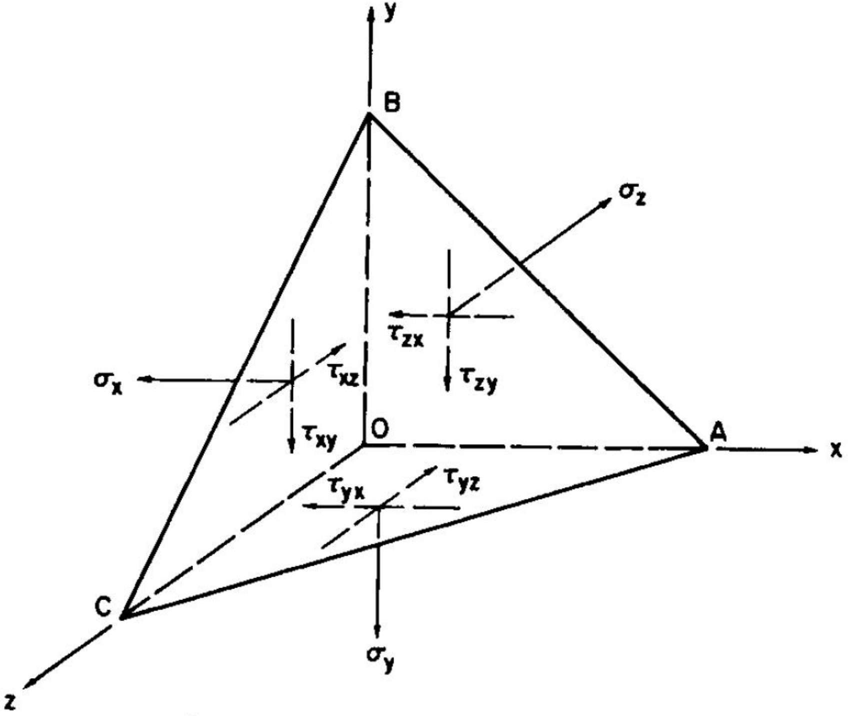
$$A_{AOC} = A_{ABC} \cos(y, x') = A a_{21}$$

A_{AOC}: area of AOC $A=A_{ABC}$: area of ABC

Similarly, $A_{AOB} = A a_{31}$ $A_{BOC} = A a_{11}$

Three-Dimensional State of Stress at a Point

Stress components on a tetrahedron



- Similarly, we can get the shear stresses $\tau_{x'y'}$, $\tau_{x'z'}$.

$$\tau_{x'y'} = (p_x, p_y, p_z)\cdot(a_{12}, a_{22}, a_{32})$$

$$\begin{aligned}\tau_{x'y'} = {} & \sigma_x a_{11} a_{12} + \sigma_y a_{21} a_{22} + \sigma_z a_{31} a_{32} \\ & + \tau_{xy}(a_{11} a_{22} + a_{21} a_{12}) \\ & + \tau_{yz}(a_{21} a_{32} + a_{31} a_{22}) \\ & + \tau_{zx}(a_{31} a_{12} + a_{11} a_{32})\end{aligned}$$

$$\tau_{x'z'} = (p_x, p_y, p_z)\cdot(a_{13}, a_{23}, a_{33})$$

$$\begin{aligned}\tau_{x'z'} = {} & \sigma_x a_{11} a_{13} + \sigma_y a_{21} a_{23} + \sigma_z a_{31} a_{33} \\ & + \tau_{xy}(a_{11} a_{23} + a_{21} a_{13}) \\ & + \tau_{yz}(a_{21} a_{33} + a_{31} a_{23}) \\ & + \tau_{zx}(a_{31} a_{13} + a_{11} a_{33})\end{aligned}$$

$$A_{AOC} = A_{ABC}\cos(y, x') = Aa_{21}$$

A_{AOC}: area of AOC　　$A=A_{ABC}$: area of ABC

Similarly, $A_{AOB} = Aa_{31}$　　$A_{BOC} = Aa_{11}$

Three-Dimensional State of Stress at a Point

$$p_x = \sigma_x a_{11} + \tau_{yx} a_{21} + \tau_{zx} a_{31}$$
$$p_y = \tau_{xy} a_{11} + \sigma_y a_{21} + \tau_{zy} a_{31}$$
$$p_z = \tau_{xz} a_{11} + \tau_{yz} a_{21} + \sigma_z a_{31}$$

Express the above transformation equations with the matrix multiplication:

$$\begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} = \begin{bmatrix} \sigma_x & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \sigma_y & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix} \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix}$$

$$\mathbf{p} = \boldsymbol{\sigma}^T \mathbf{n} = \boldsymbol{\sigma} \mathbf{n}$$

Let’s now check the principal plane and principale stresses.

$$\mathbf{p} = \boldsymbol{\sigma} \mathbf{n} = \sigma_p \mathbf{n}$$

There **always exist three mutually perpendicular principal planes** on which the shear stress vanishes in 3D.

The three principal stresses are named as:

$$\sigma_1 \geqslant \sigma_2 \geqslant \sigma_3$$

The three principal stresses are the roots of the equation:

$$\begin{aligned} &\sigma_p{}^3 - (\sigma_x + \sigma_y + \sigma_z)\sigma_p{}^2 \\ &+ (\sigma_x \sigma_y + \sigma_y \sigma_z + \sigma_z \sigma_x - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{zx}^2)\sigma_p \\ &- (\sigma_x \sigma_y \sigma_z + 2\tau_{xy}\tau_{yz}\tau_{zx} - \sigma_x \tau_{yz}^2 - \sigma_y \tau_{xz}^2 - \sigma_z \tau_{xy}^2) = 0 \end{aligned}$$

Three-Dimensional State of Stress at a Point

The three principal stresses are the roots of the equation:

$$\begin{aligned}\sigma_p{}^3 &- (\sigma_x + \sigma_y + \sigma_z)\sigma_p{}^2 \\ &+ (\sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{zx}^2)\sigma_p \\ &- (\sigma_x\sigma_y\sigma_z + 2\tau_{xy}\tau_{yz}\tau_{zx} - \sigma_x\tau_{yz}^2 - \sigma_y\tau_{xz}^2 - \sigma_z\tau_{xy}^2) = 0\end{aligned}$$

σ_p does not depend on the coordinate chosen,

$\Rightarrow$

the coefficients of the equation must be invariant (independent of the coordinate):

$$\sigma_p^3 - I_1\sigma_p^2 + I_2\sigma_p - I_3 = 0$$

The three stress invariants

$$I_1 = \sigma_x + \sigma_y + \sigma_z = \sigma_1 + \sigma_2 + \sigma_3$$

$$I_2 = \begin{vmatrix} \sigma_x & \tau_{xy} \\ \tau_{yx} & \sigma_y \end{vmatrix} + \begin{vmatrix} \sigma_y & \tau_{yz} \\ \tau_{zy} & \sigma_z \end{vmatrix} + \begin{vmatrix} \sigma_z & \tau_{zx} \\ \tau_{xz} & \sigma_x \end{vmatrix}$$

$$= \sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{zx}^2 = \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1$$

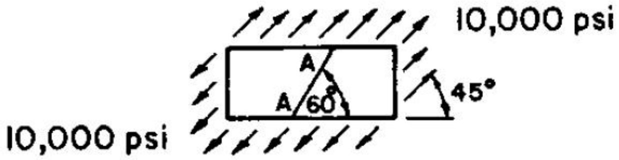
$$I_3 = \begin{vmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{vmatrix}$$

$$= \sigma_x\sigma_y\sigma_z + 2\tau_{xy}\tau_{yz}\tau_{zx} - \sigma_x\tau_{yz}^2 - \sigma_y\tau_{zx}^2 - \sigma_z\tau_{xy}^2 = \sigma_1\sigma_2\sigma_3$$

homework 1 (5 points)

1-1 Given $\sigma_x = -14{,}000$ psi, $\sigma_y = 6{,}000$ psi, and $\tau_{xy} = -17{,}320$ psi, determine by formulas, (a) the principal stresses and their directions and (b) the stress components on the x' and y' planes when $\alpha = 45°$.

1-3 A rectangular block is under a uniformly distributed load as shown in the figure. Find the stress components on the plane $A - A$.



1-13 By using Mohr's circle, show that the following quantities are invariant for a two-dimensional state of stress with $\sigma_z = \tau_{xz} = \tau_{yz} = 0$;

(a) $\sigma_{x'} + \sigma_{y'}$

(b) $\sigma_{x'}\sigma_{y'} - \tau_{x'y'}^2$.

homework 1 (5 points)

1-15 Given a three-dimensional state of stress with

(2 points)

$$\sigma_x = +10 \text{ psi} \qquad \tau_{xy} = +5 \text{ psi}$$
$$\sigma_y = +20 \text{ psi} \qquad \tau_{xz} = -10 \text{ psi}$$
$$\sigma_z = -10 \text{ psi} \qquad \tau_{yz} = -15 \text{ psi}$$

(a) Find the magnitude and direction of the stress vector p on the x' plane where the x' direction is defined by

$$a_{11} = +1/2 \qquad a_{21} = +1/\sqrt{2} \qquad a_{31} \text{ is positive.}$$

(b) Find σ and τ on this plane.

(c) Determine the angle between p and σ.