Review

Fluid: definition

The **Lagrangian viewpoint**: dependent on time and the fluid particle

$$T = T_L(x_i^0, t)$$

The **Eulerian viewpoint**: dependent on time and point in space.

$$T = T_E(x_i, t)$$

Substantial derivative $D(\ )/Dt$: the property changing rate of a specific particle.

The substantial derivative can be formulated with Eulerian viewpoints.

$$\frac{D(\ )}{Dt} \equiv \underbrace{\frac{\partial(\ )}{\partial t}}_{\text{local rate of change}} + \underbrace{v_i\ \partial_i(\ )}_{\text{convective change}}$$

$$\frac{\partial v_i}{\partial x_j} = \underbrace{\frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)}_{\text{Strain rate tensor (Symmetric 2nd tensor)}} + \underbrace{\frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right)}_{\text{Rotation rate tensor (Antisymmetric 2nd tensor)}} = \dot{\varepsilon}_{ij} + \dot{\omega}_{ij}$$

The Helmholtz velocity decomposition theorem

$$v_i^{P'} = v_i^P + \underbrace{S_{ij}dx_j}_{\text{Deform}} + \underbrace{\dot{\omega}_{ij}dx_j}_{\text{Rotation}}$$

$\dot{\omega}_k$: Vorticity (涡量、涡度)

$$\dot{\omega}_k = \nabla \times \mathbf{v} = \varepsilon_{kmn}\nabla_m v_n$$

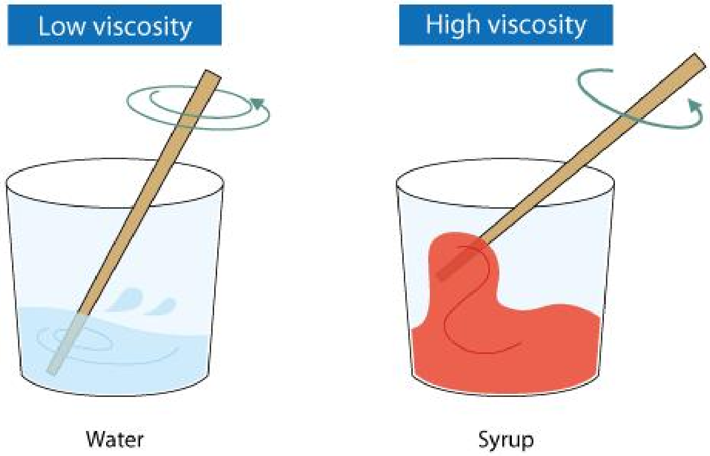
Vorticity (涡量) is linked with the angular andvelocity of the local spin (rigid rotation)

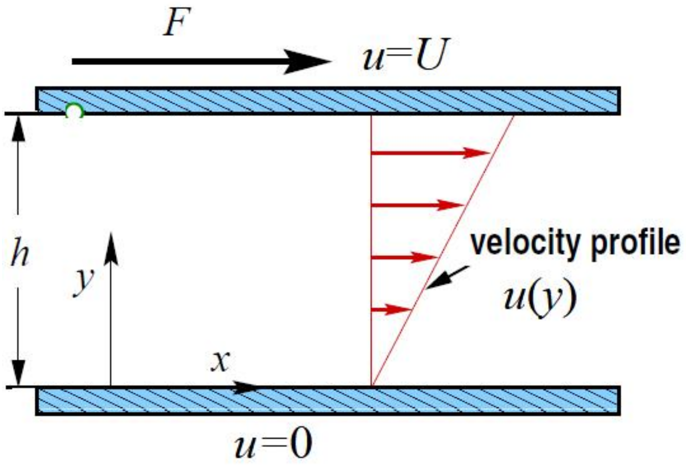
Ω the rotation rate tensor

$$\dot{\omega}_k = 2\Omega_k \qquad \dot{\omega}_{ij} = -\frac{1}{2}\varepsilon_{ijk}\dot{\omega}_k$$

Fluid Viscosity

- **Viscosity *μ*, *η***:
  - A quantitative measure of a fluid's resistance to flow
    - due to 'internal friction' between fluid elements.
- Viscosity experiments between two parallel plates spaced h apart (平板实验):





A: area of the plate
F: Applied force
h: spacing between the plates.
U: upper plate speed

Newton's law of viscosity $F = \mu \frac{AU}{h}$

μ: **(dynamic) viscosity**, Pa·s

Fluid Viscosity

Newton's law of viscosity

$$F = \mu \frac{AU}{h}$$

Define shear stress $\tau = F/A$, we have

$$\tau = \frac{F}{A} = \mu \frac{U}{h}$$

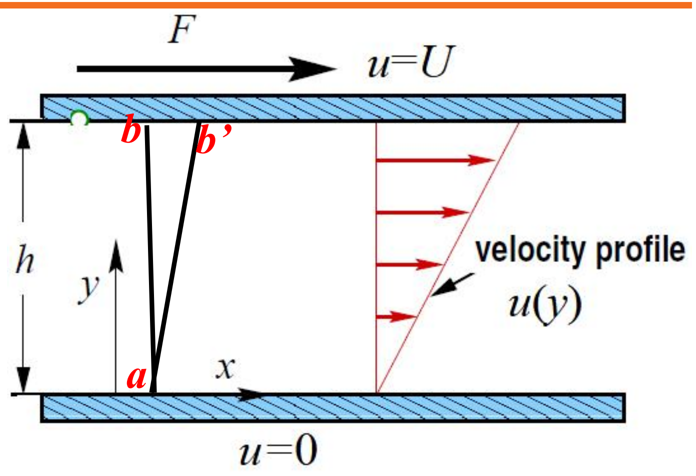
$U/h = du/dy$ (u varies linearly between two plates) is the angular velocity of the line *ab*

we have $\tau = \mu \dfrac{du}{dy}$

du/dy is (engineering) shear strain rate.

Generally, we have

$$\tau_{ij} = 2\mu\dot{\varepsilon}_{ij} = \mu(v_{i,j} + v_{j,i})$$



Newtonian fluids:

- μ is independent of stress
- stress is linearly proportional to strain rate

Non-Newtonian fluids:

- μ is dependent of stress (usually decreases with stress)
- stress is proportional to the power of strain rate

Fluid Viscosity

$$F = \mu \frac{AU}{h}$$

Deduce the unit of viscosity μ in SI system

Transport Properties of Some Common Fluids at 15°C and Atmospheric Pressure

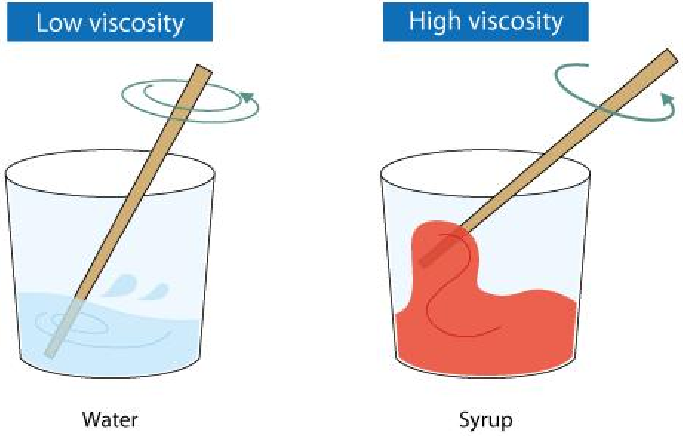
	Viscosity μ (Pa s)	Kinematic Viscosity ν ($m^2\ s^{-1}$)	Thermal Diffusivity κ ($m^2\ s^{-1}$)
Air	1.78×10^{-5}	1.45×10^{-5}	2.02×10^{-5}
Water	1.14×10^{-3}	1.14×10^{-6}	1.40×10^{-7}
Mercury	1.58×10^{-3}	1.16×10^{-7}	4.2×10^{-6}
Ethyl alcohol	1.34×10^{-3}	1.70×10^{-6}	9.9×10^{-8}
Carbon tetrachloride	1.04×10^{-3}	6.5×10^{-7}	8.4×10^{-8}
Olive oil	0.099	1.08×10^{-4}	9.2×10^{-8}
Glycerine (甘油)	2.33	1.85×10^{-3}	9.8×10^{-8}
Asthenosphere	1E20	3E16	
Lower mantle	3E22	1E19	

Kinematic viscosity ν:

$$\nu = \frac{\mu}{\rho}$$ where ρ is density

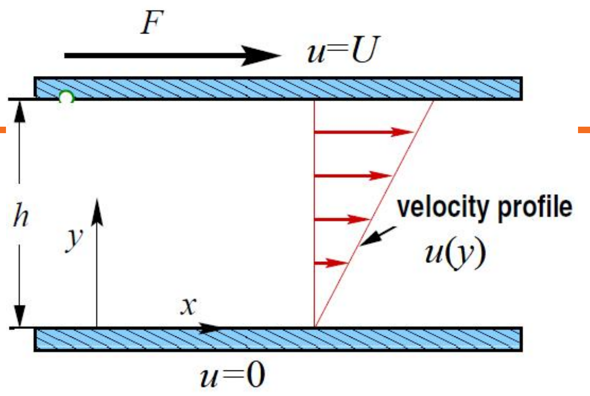
Deduce the unit of kinematic viscosity ν

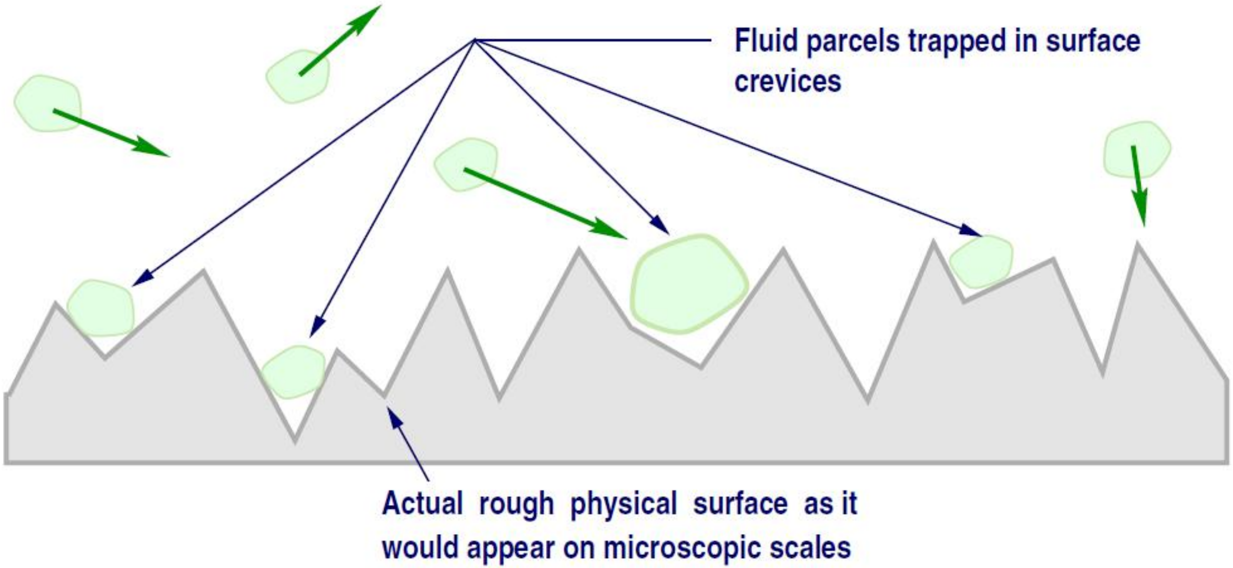
Kinematic viscosity describes the diffusion of momentum.



Fluid Viscosity

no-slip condition for viscous fluids: fluids always take the velocity of the surfaces to which they are adjacent.



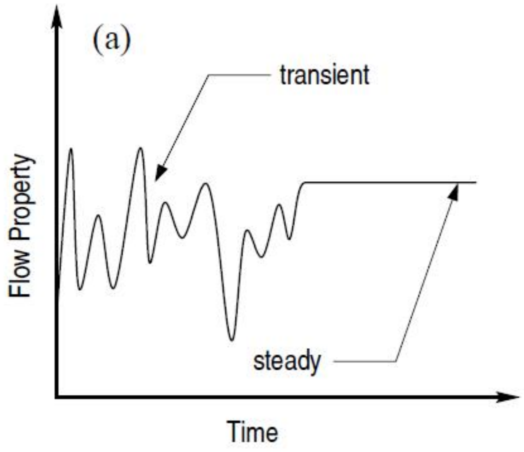


Physical situation giving rise to the no-slip condition

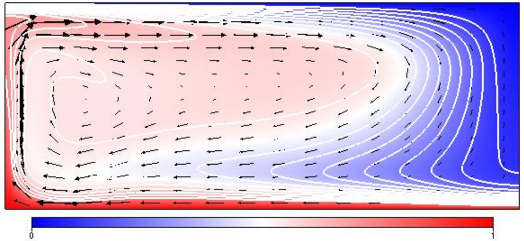
Classification of Flow Phenomena

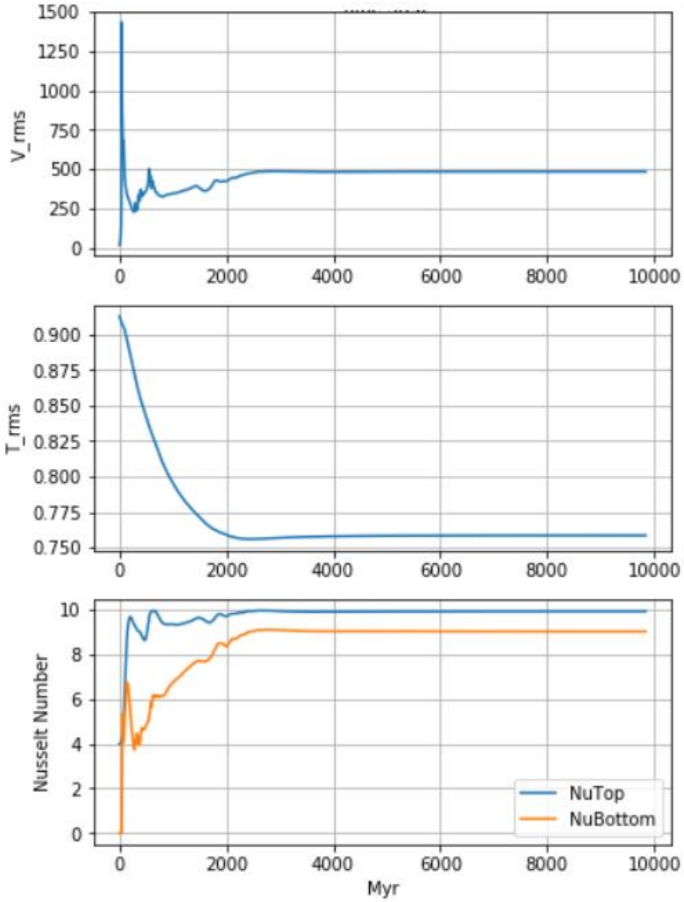
Steady and unsteady flows (定常流 vs 非定常流):

If all properties of a flow are independent of time, then the flow is steady; otherwise, it is unsteady.



Transient followed by steady state



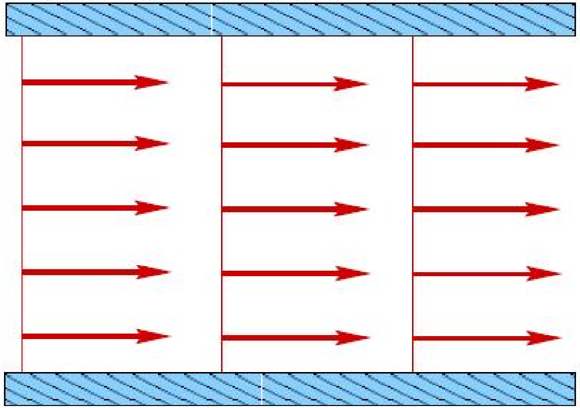


Classification of Flow Phenomena

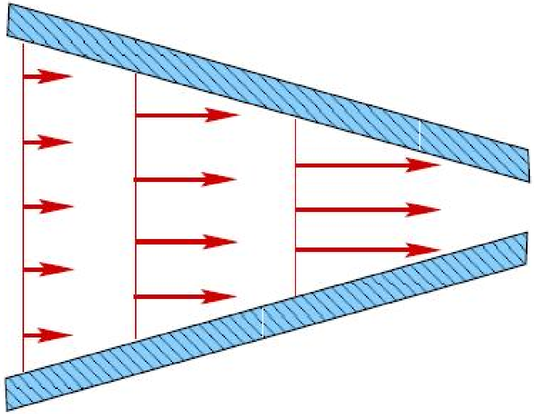
Uniform and non-uniform flows (均匀流 vs 非均匀流):

A uniform flow: velocity vectors are identical (in both direction and magnitude) at every point of the flow for any given instant of time. Flows for which this is not true are said to be nonuniform.

$$v_{i,j} = \partial v_i / \partial x_j = 0$$



(a) uniform flow



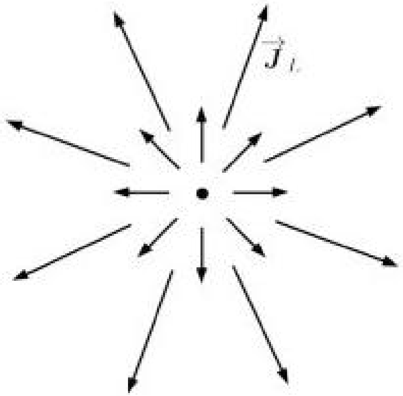
(b) non-uniform flow

Classification of Flow Phenomena

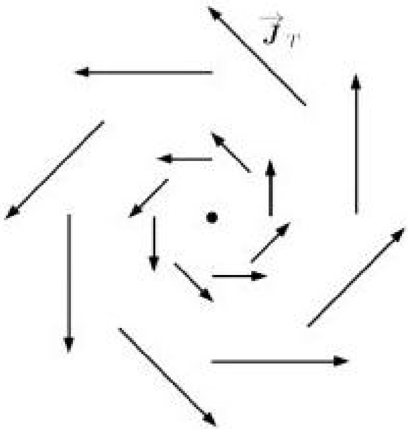
Rotational and irrotational flows (有旋流 vs 无旋流)，vortex (涡流):

vorticity (涡量、涡度) $\boldsymbol{\omega} = \mathrm{curl}\ \mathbf{v} = \nabla \times \mathbf{v}$ or $\omega_i = \varepsilon_{ijk} \nabla_j v_k$

A flow field with velocity vector **v** is said to be rotational if **ω** ≠ 0; otherwise, it is irrotational.



Irrotational or longitudinal current



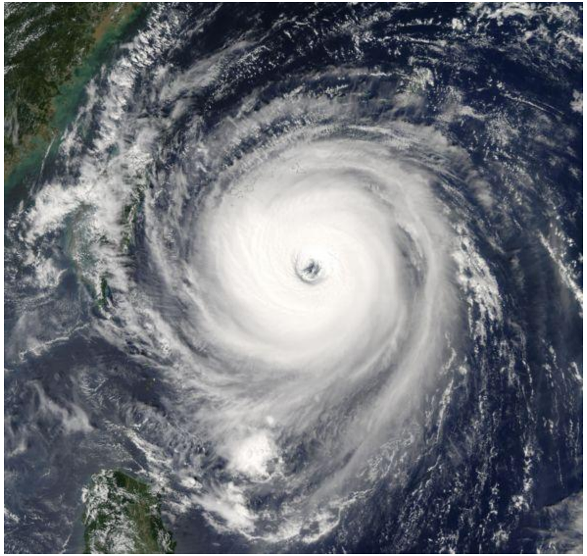
Solenoidal or tangential current

For 2D flows, only one vorticity component (涡度分量) exists

Vorticity (涡度) is a local property of the flow field, whereas the word vortex (涡流，涡旋，旋涡) describes any type of swirling flow pattern.

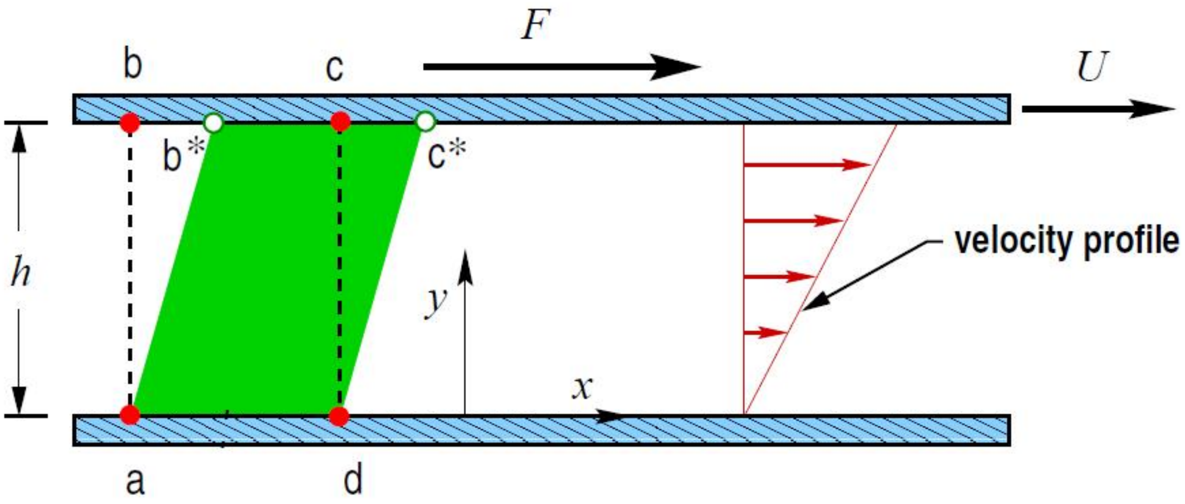
Classification of Flow Phenomena

Rotational and irrotational flows(有旋流 vs 无旋流)， vortex (涡流):



Classification of Flow Phenomena

Rotational and irrotational flows: 1D shear flows



1D flow between two horizontal, parallel plates with upper one moving at velocity U

$$u_x = v_x(y,t)$$

$$\boldsymbol{\omega} = \begin{pmatrix} 0 & 0 & -\dfrac{\partial v_x}{\partial y} \end{pmatrix}$$

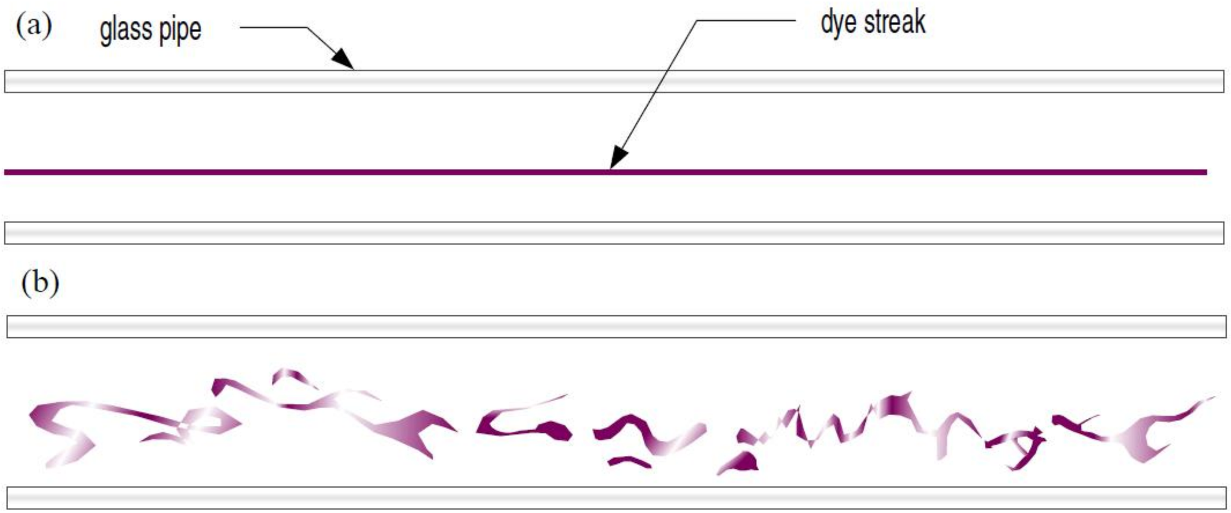
1D shear flow is rotational flow although it does not have vortices (旋涡)

Classification of Flow Phenomena

Laminar and turbulent flows (层流 vs 紊流、湍流):

Laminar flow: fluid particles follow smooth paths in layers, with each layer moving smoothly past the adjacent layers with no mixing.

Turbulent flow: has many time-dependent eddies with chaotic changes in pressure and flow velocity



Reynolds' experiment using water in a pipe to study transition to turbulence; (a) low-speed, (b) higher-speed.

With increasing flow speed, flow transits from laminar flow (层流) to turbulent flow (紊流).

Classification of Flow Phenomena

- Most laminar flow can be predicted with reasonable accuracy now.
- But except for the very simplest situations, it is not possible to predict details of turbulent fluid motion.
    - We can’t predict the speed at which an orderly, non-turbulent (“laminar”) flow will make the transition to a turbulent flow.

The hot air rising from a candle flame illustrate the often sudden transformation from a calm, orderly flow to a turbulent flow

Classification of Flow Phenomena

Viscous and inviscid flows (粘滞流vs无粘性流):

in some cases, the viscous forces (surface force, but not pressure) are small compared with other forces and can be neglected.

Incompressible and compressible flows (可压缩vs不可压缩流):

Gases are often quite compressible, but flows of gases can often be treated as incompressible flows

Ideal fluid (perfect fluid, 理想流体, 无粘不可压缩流体):

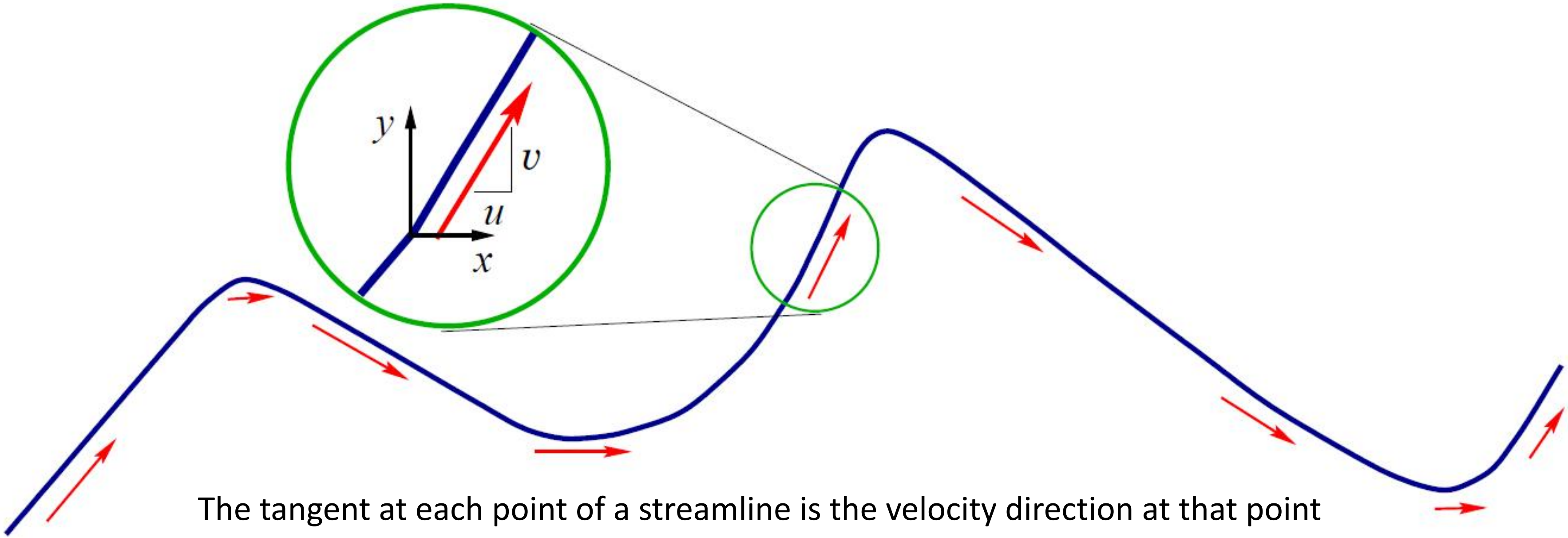
An ideal fluid is a fluid that has no viscosity (internal friction) and is incompressible.

Streamline (流线), Pathline (迹线), Streakline (脉线)

Streamline (流线):

Streamlines are defined as lines that at any instant are tangent to the velocity vectors. If dx_i is a differential along a streamline, then

$$\frac{dx_1}{v_1} = \frac{dx_2}{v_2} = \frac{dx_3}{v_3} \quad \text{or} \quad \begin{aligned} \varepsilon_{ijk} v_j \, dx_k &= 0 \quad \text{tensor notation} \\ \mathbf{v} \times d\mathbf{x} &= 0 \quad \text{symbolic notation} \end{aligned}$$



The tangent at each point of a streamline is the velocity direction at that point

Streamline, Pathline, Streakline

Physical attributes of streamlines

- Streamlines display a snapshot of the flow field at a single instant in time
    - Streamlines change with time for time-dependent flows
- Streamlines cannot cross each other
    - Each streamline starting from a different selected point in the flow field.
- Streamlines close to solid walls are parallel to the walls
    - The only nonzero component(s) of the velocity vector very close to the surface is (are) the tangential one(s).

Streamline, Pathline, Streakline

Pathlines (迹线):

A Pathline is the trajectory of an individual fluid element (particle).

$$x(t) = x(t = t_1) + \int_{t_1}^{t} u(x(t'), y(t'), z(t'), t')dt'$$

$$y(t) = y(t = t_1) + \int_{t_1}^{t} v(x(t'), y(t'), z(t'), t')dt'$$

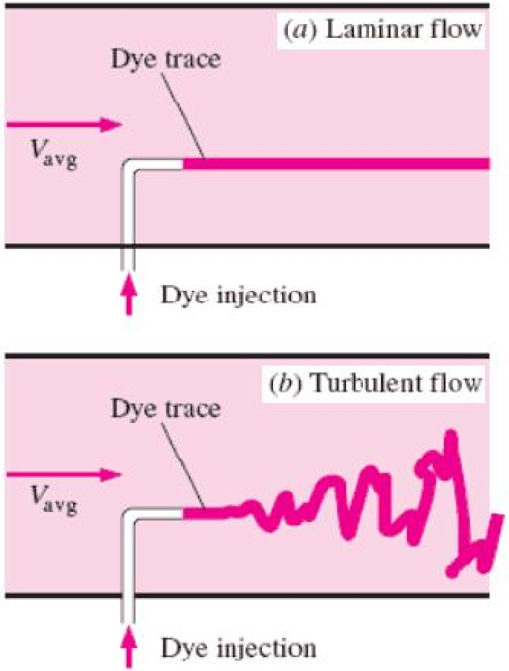
$$z(t) = z(t = t_1) + \int_{t_1}^{t} w(x(t'), y(t'), z(t'), t')dt'$$

or $$dx_i = v_i dt$$
$$x_i(t = t_1) = x_{1i}$$

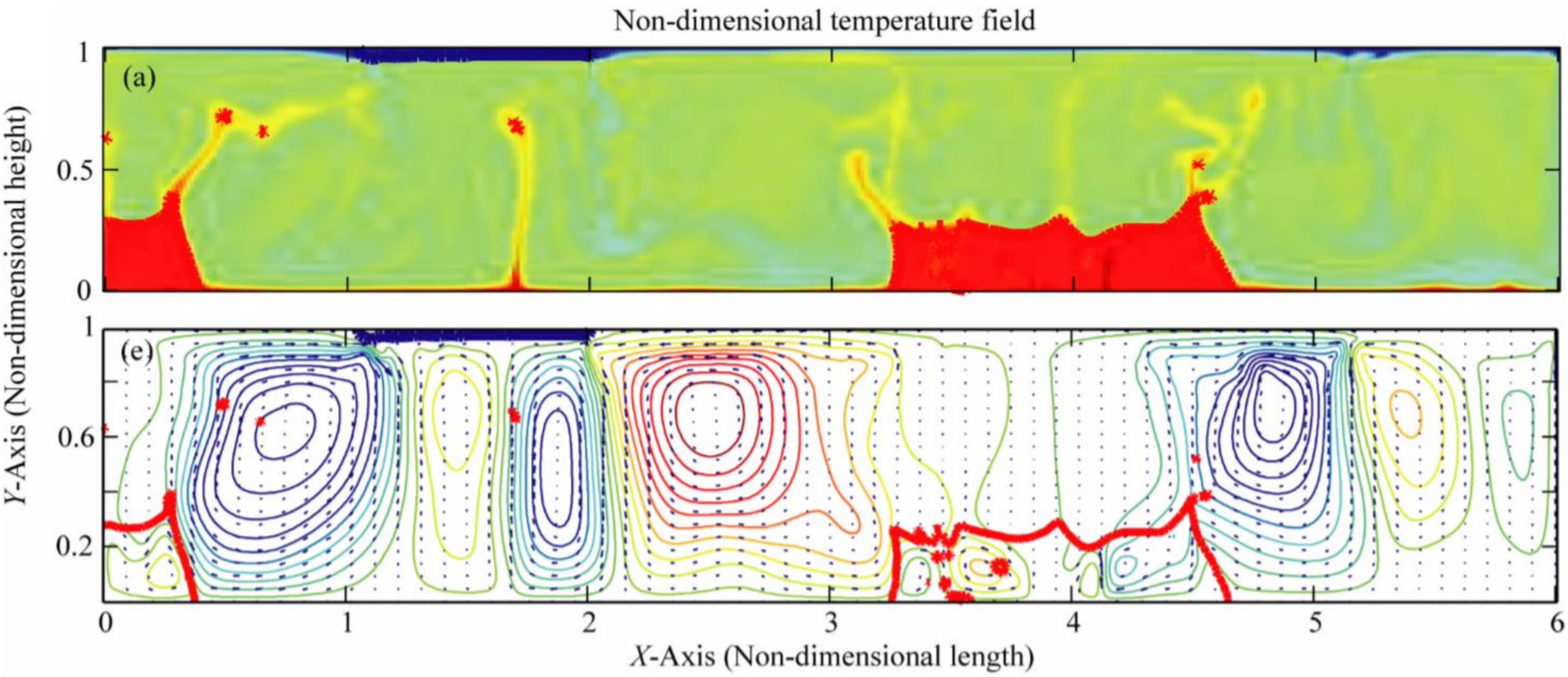
At each instant time, the end of pathline is parallel to the streamline at that moment

Streaklines (脉线，染色线):

A streakline is the locus of all fluid elements that have previously passed through a given point.



Streamline, Pathline, Streakline



Steamlines: counter-clockwise rotation for red color while clockwise rotation for blue color

The study of Fluids: theoretical, experimental, computational

The theoretical method (理论方法、分析方法):

Turbulent flow: 'the last unsolved problem of classical mathematical physics' termed by Werner Heisenberg (海森堡)

For turbulence to be considered a solved problem in physics, we would need to be able to demonstrate that we can start with the basic equation describing fluid motion and then solve it to predict, in detail, how a fluid will move under any particular set of conditions. That we cannot do this in general is the central reason that many physicists consider turbulence to be an unsolved problem.

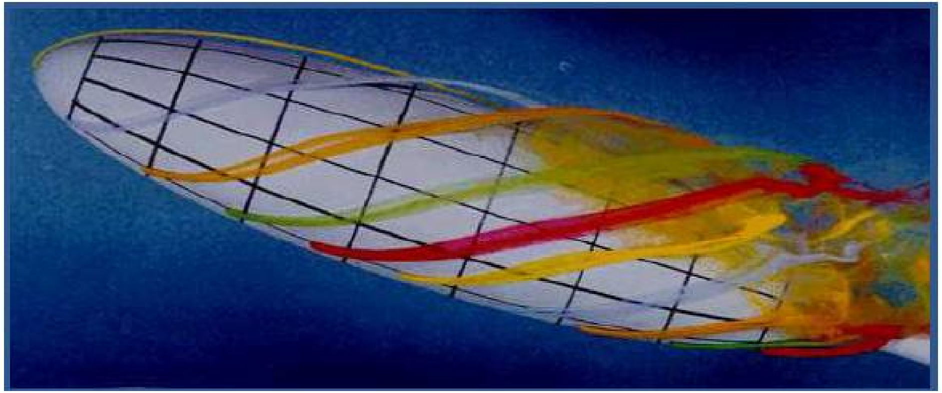
N-S equations: Millennium Prize Problems (千禧年大奖难题), Prove or give a counter-example of the following statement:

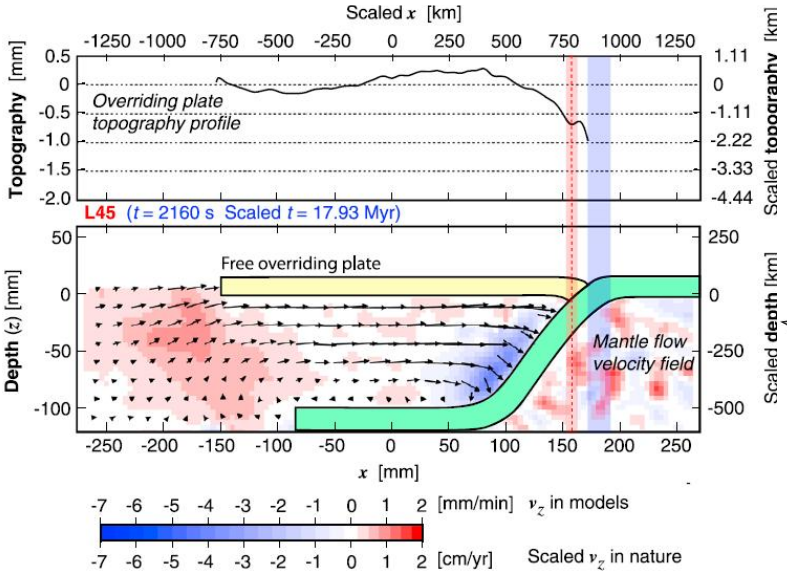
In three space dimensions and time, given an initial velocity field, there exists a vector velocity and a scalar pressure field, which are both smooth and globally defined, that solve the Navier–Stokes equations.

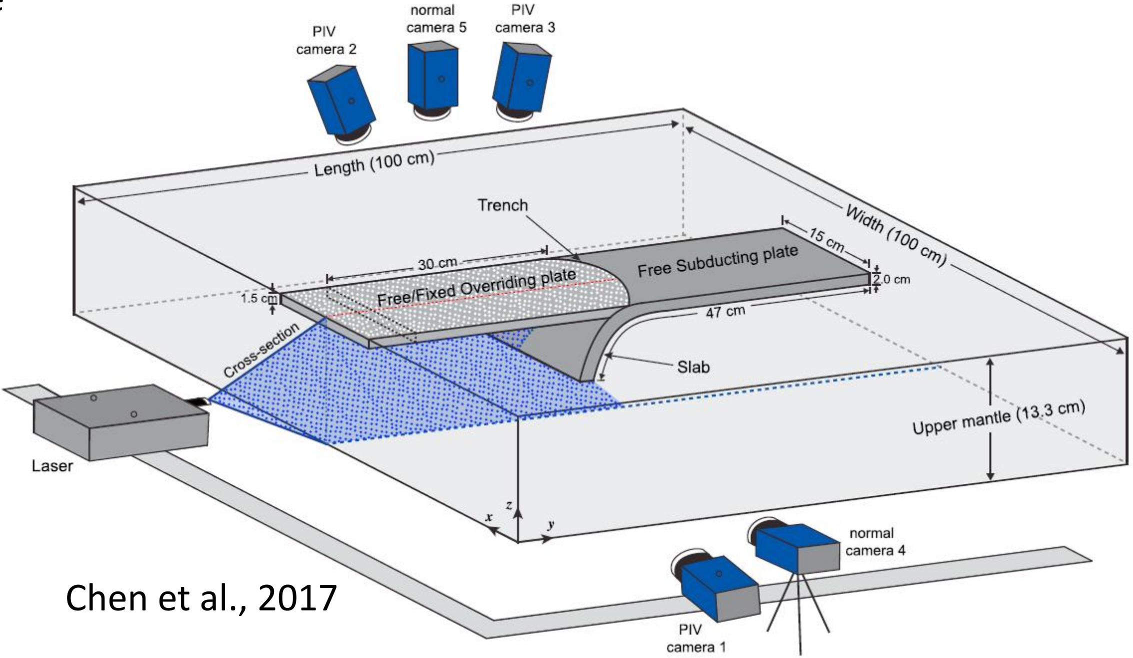
The study of Fluids

The experimental method:

Colored Streaks and pressure has long been measured to indicate basic flow patterns and gain some quantitative analysis.
Flow velocity, surface topography can also be measured now.





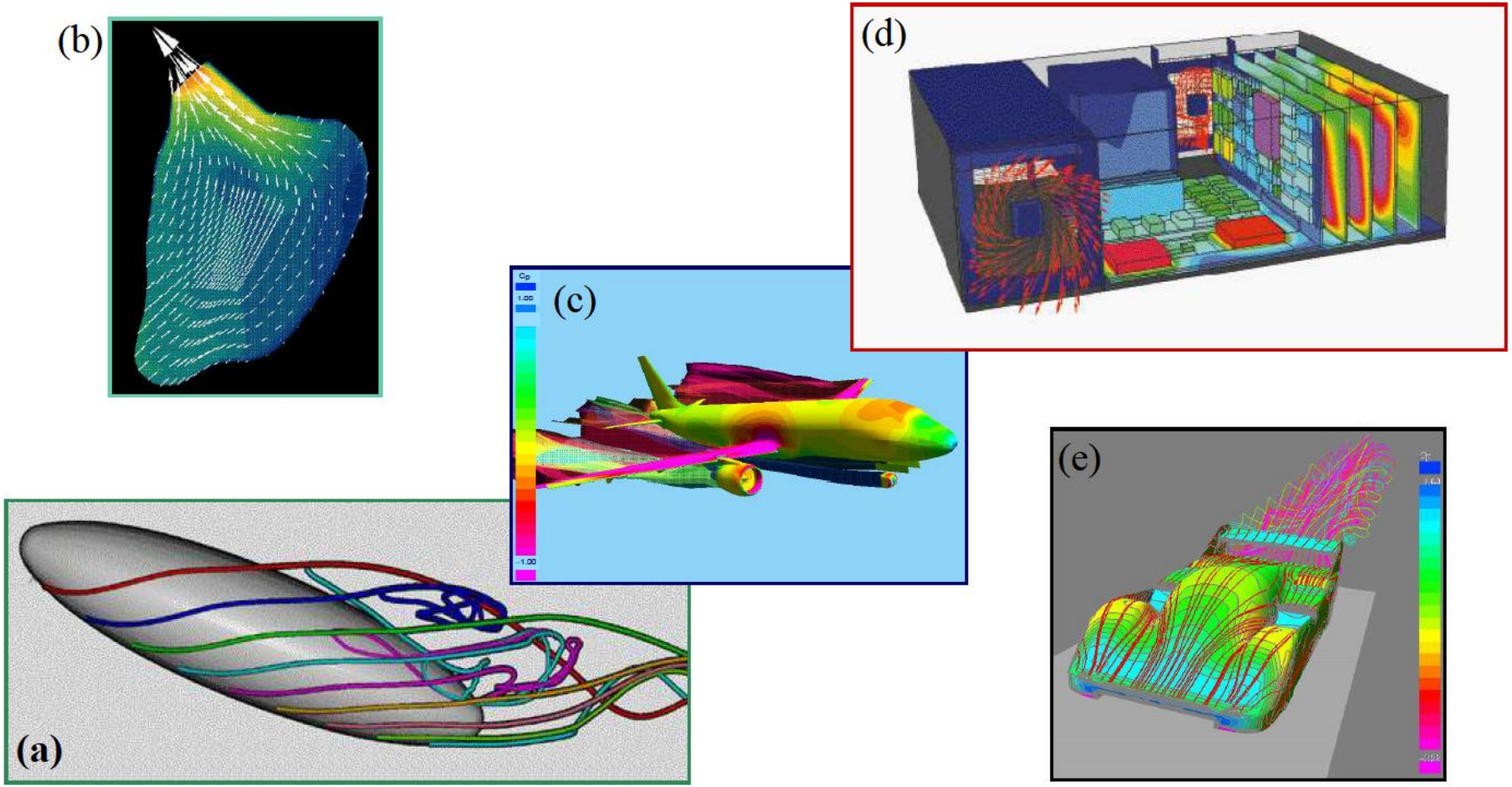


Chen et al., 2017

The study of Fluids

The computational method:

(a) Compares with experimental results; (b) predicted flow field in human heartc; (c) pressure field at a plane’s surface; (d) temperature field in a PC, velocity around the fan is shown; (e) pressure distribution at a race car’s surface and streamlines.



Classroom Exercise

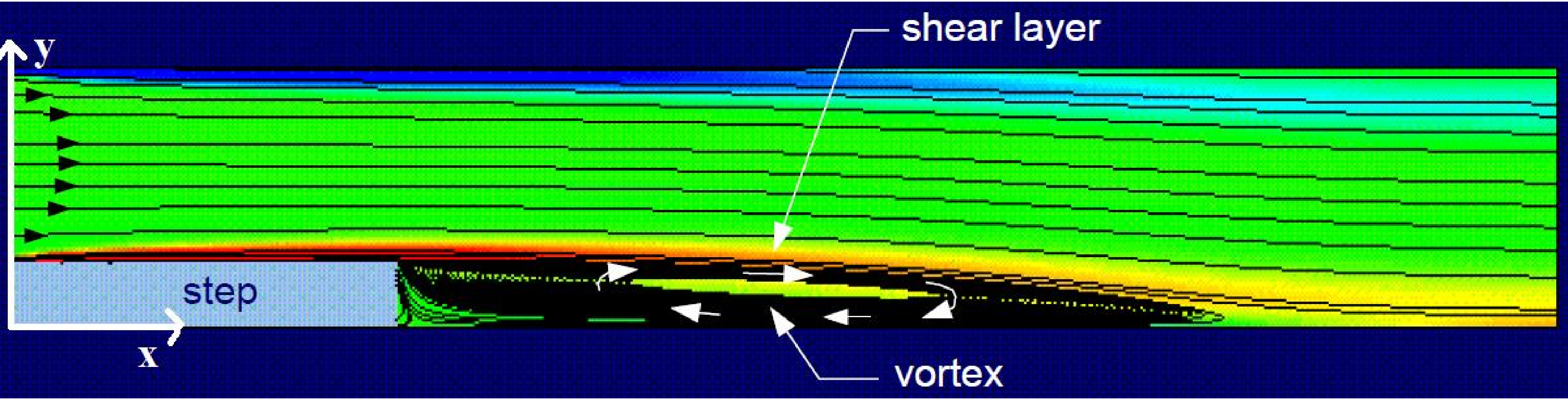
True or false:
Streamline, pathline, and streakline are all equivalent for steady flows.

The viscous stress tensor τ_{ij} within the viscous fluid is usually assumed to be linearly proportional to the flow velocity gradient $v_{k,l}$, i.e.,

$$\tau_{ij} = c_{ijkl} v_{k,l}$$

Assume $c_{ijkl} = \eta(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$
Prove that $\tau_{ij} = 2\eta\varepsilon_{i,j} = \eta(v_{i,j} + v_{j,i})$

Classroom Exercise

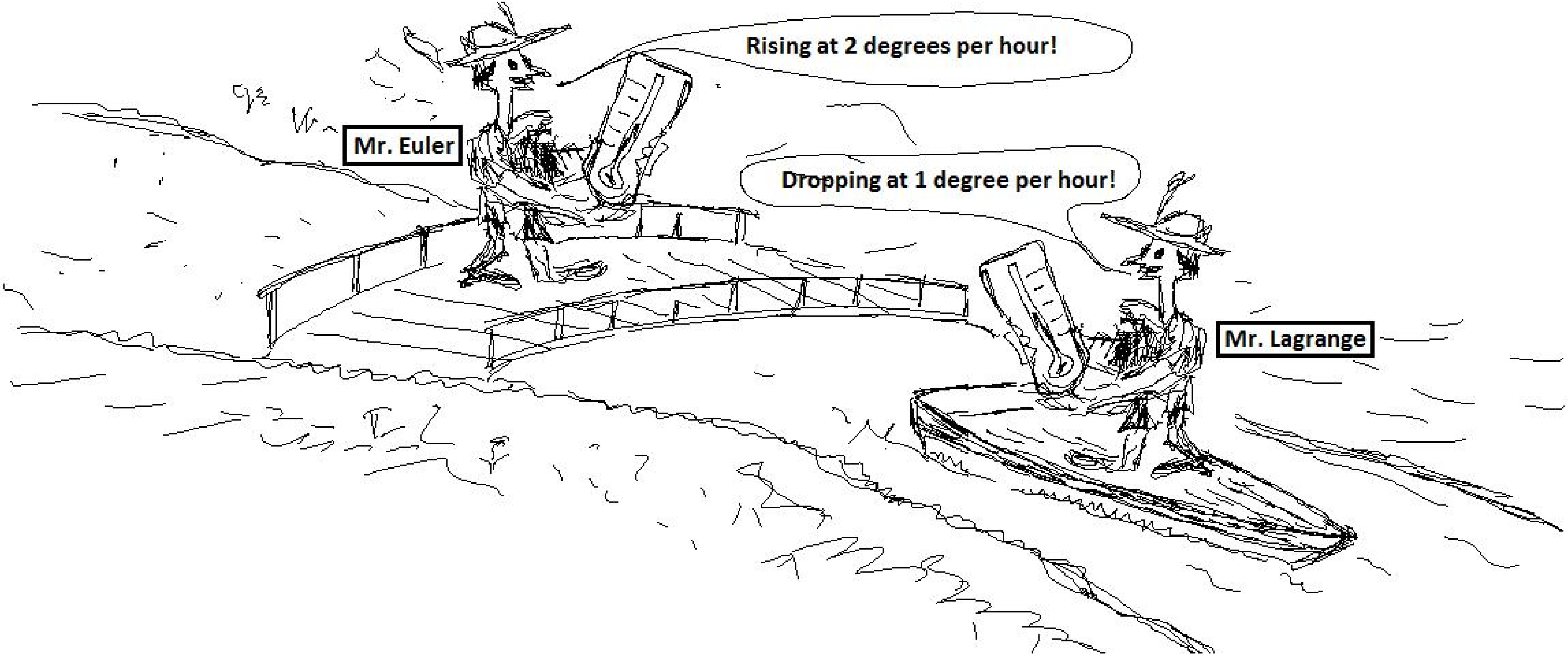


2D flow over a step generates a very prominent vortex (旋涡).

Color represents vorticity. Red: negative; Blue: positive; green: vorticity close to zero. Black lines: flow path

Question: explain why the vorticity is positive at the upper boundary while negative along the upper surface of the step for the fluid field above.

$$\boldsymbol{\omega} = \begin{pmatrix} 0 & 0 & \dfrac{\partial v_y}{\partial x} - \dfrac{\partial v_x}{\partial y} \end{pmatrix}$$


Rising at 2 degrees per hour!
Mr. Euler
Dropping at 1 degree per hour!
Mr. Lagrange

Reading Material

Imagine two people, Mr. Euler and Mr. Lagrange, making temperature observations in order to calculate the rate at which the temperature is varying. Mr. Euler is standing on a bridge, while Mr. Lagrange is in a boat carried by a river. When the boat passes the bridge, they discover that their observed rates differ. Why?

This happens because the temperature differs in different locations. To calculate the rate, they divide the difference between the two observed temperatures by the time interval between the measurements. Even if one of the measurements was made when the boat was passing the bridge, the other measurement was when the boat was away from the bridge. Then, even if they took this other measurements at the same instant, the results can differ.

In the example with the boat it was assumed that the boat was carried by the fluid, that is, it was moving with the velocity of the fluid itself. The Eulerian viewpoint consists in considering quantities as dependent on time and point in space. The Lagrangian viewpoint consists in considering quantities as dependent on time and the fluid particle. The two viewpoints differ essentially in the choice of independent variables. The dependent variables are the same for both descriptions. Lagrangian viewpoint is particularly useful when the laws of fluid motion are derived from the Newton laws. Once derived, however, these laws are more convenient to use from the Eulerian viewpoint.

Reading Material

Fluid flow is characterized by a velocity vector field in three-dimensional space. Streamlines, streaklines, and pathlines are field lines resulting from this vector field description of the flow. They differ only when the flow changes with time: that is, when the flow is not steady.

Streamlines are a family of curves that are instantaneously tangent to the velocity vector of the flow. These show the direction a fluid element will travel instantly in at any point in time. Streaklines are the locus of points of all the fluid particles that have passed continuously through a particular spatial point in the past. Dye steadily injected into the fluid at a fixed point extends along a streakline. Pathlines are the trajectories that individual fluid particles follow. These can be thought of as "recording" the path of a fluid element in the flow over a certain period.

By definition, different streamlines at the same instant in a flow do not intersect, because a fluid particle cannot have two different velocities at the same point. Similarly, streaklines cannot intersect themselves or other streaklines, because two particles cannot be present at the same location at the same instant of time. However, pathlines are allowed to intersect themselves or other pathlines (except the starting and end points of the different pathlines, which need to be distinct).

Homework (4 points)

1. The surface temperature of a lake changes from one location to another as $T(x_1, x_2)$. If you attach a thermometer to a boat and take a path through the lake given by $x_i = b_i\,(t)$, find an expression for the rate of change of the thermometer temperature in terms of the lake temperature.

2. In a table of vector differential operators, look up the expressions for $\nabla\times\boldsymbol{v}$ in a cylindrical coordinate system. (a) Compute the vorticity for the flow in a round tube where the velocity profile is

$$v_z = v_0\left[1 - \left(\frac{r}{R}\right)^2\right]$$

(b) Given $\Gamma = 10$, compute the vorticity for an ideal vortex where the velocity is

$$v_\theta = \frac{\Gamma}{2\pi r}$$

3. Consider a two-dimensional flow with velocity components $v_1 = cx_1$, $v_2 = -cx_2$. Give each component of the strain rate tensor.

4. Discuss the difference between streamline, pathline, and streakline