SOUTHERN UNIVERSITY OF SCIENCE AND TECHNOLOGY
地球与空间科学系 DEPARTMENT OF EARTH AND SPACE SCIENCES

MAE5009 Continuum Mechanics B Session 04: Formulation of Problems in Elasticity

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Now we have all the equations

Unknowns (15)

- Stress (6)

$$\begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ & \sigma_y & \tau_{yz} \\ sym. & & \sigma_z \end{bmatrix}(x,y,z)$$

- Strain (6)

$$\begin{bmatrix} \varepsilon_x & \varepsilon_{xy} & \varepsilon_{xz} \\ & \varepsilon_y & \varepsilon_{yz} \\ sym. & & \varepsilon_z \end{bmatrix}(x,y,z)$$

- Displacement (3)

$$[u,v,w](x,y,z)$$

Governing equations (field equations) (15)

- Equilibrium equations (3)

$$\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}+\frac{\partial\tau_{zx}}{\partial z}+f_x=0 \quad (x,y,z)$$

- Strain-displacement (6)

$$\varepsilon_x=\frac{\partial u}{\partial x} \quad \gamma_{xy}=\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x} \quad (x,y,z;u,v,w)$$

(Compatibility equations(3/6))

- Stress-strain relations

$$\varepsilon_x=\frac{1}{E}\left(\sigma_x-\nu\left(\sigma_y+\sigma_z\right)\right) \quad \gamma_{xy}=\frac{1}{G}\tau_{xy}$$

$$\sigma_x=2G\varepsilon_x+\lambda\varepsilon \quad \tau_{xy}=G\gamma_{xy} \quad (x,y,z)$$

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Solutions

- The 15 equations can be reduced to 6 equations in terms of stress, or 3 in terms of displacement

Goals:

- Determination of stress, strain and displacement functions based on field equations
- Satisfying the boundary conditions
- Solution satisfying all conditions for a given problem is unique

Governing equations (field equations) (15)

- Equilibrium equations (3)

$$\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}+\frac{\partial\tau_{zx}}{\partial z}+f_x=0 \quad (x,y,z)$$

- Strain-displacement (6)

$$\varepsilon_x=\frac{\partial u}{\partial x} \quad \gamma_{xy}=\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x} \quad (x,y,z;u,v,w)$$

(Compatibility equations(3/6))

- Stress-strain relations

$$\varepsilon_x=\frac{1}{E}\left(\sigma_x-\nu\left(\sigma_y+\sigma_z\right)\right) \quad \gamma_{xy}=\frac{1}{G}\tau_{xy}$$

$$\sigma_x=2G\varepsilon_x+\lambda\varepsilon \quad \tau_{xy}=G\gamma_{xy} \quad (x,y,z)$$

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*看一下

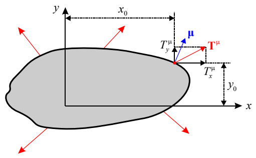
Boundary conditions

- **Stress boundary:**
  - Based on force equilibrium:

$$\begin{cases} T_x^{\mu} = \sigma_{x|0}\mu_x + \tau_{yx|0}\mu_y + \tau_{zx|0}\mu_z \\ T_y^{\mu} = \tau_{xy|0}\mu_x + \sigma_{y|0}\mu_y + \tau_{zy|0}\mu_z \\ T_z^{\mu} = \tau_{xz|0}\mu_x + \tau_{yz|0}\mu_y + \sigma_{z|0}\mu_z \end{cases}$$

- **Displacement boundary:**

$$\begin{cases} u(x_0, y_0, z_0) = u_b \\ v(x_0, y_0, z_0) = v_b \\ w(x_0, y_0, z_0) = w_b \end{cases}$$



Boundary point

(x_0, y_0, z_0)

Unit outward normal

$\boldsymbol{\mu}$

Surface force (stress)

$(T_x^{\mu}, T_y^{\mu}, T_z^{\mu})$

Direction cosines of $\boldsymbol{\mu}$ with respect to *x*, *y* and *z*

(μ_x, μ_y, μ_z)

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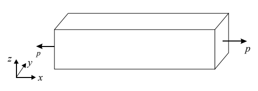
Boundary conditions

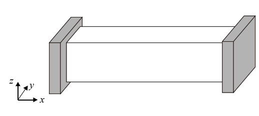
- **Stress boundary**
  - First boundary-value problem

$$T_x^{\mu} = \pm p, T_y^{\mu} = T_z^{\mu} = 0$$

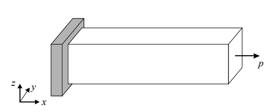


- **Displacement boundary**
  - Second boundary-value problem

$$u_b = v_b = w_b = 0$$



- **Stress & displacement boundary**
  - Mixed boundary-value problem



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z方向 $\varepsilon_z = \gamma_{yz} = \gamma_{xz} = 0$

Plane strain problems

- Displacements $\begin{cases} u = u(x,y) \\ v = v(x,y) \\ w = 0 \end{cases}$
- Strain-displacement

$$\varepsilon_x = \frac{\partial u}{\partial x} \quad \varepsilon_y = \frac{\partial v}{\partial y} \quad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

$$\varepsilon_z = \gamma_{yz} = \gamma_{xz} = 0$$

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- Stress-strain relations

$$\sigma_x = 2G\varepsilon_x + \lambda(\varepsilon_x + \varepsilon_y) \quad \tau_{xy} = G\gamma_{xy}$$

$$\sigma_y = 2G\varepsilon_y + \lambda(\varepsilon_x + \varepsilon_y) \quad \tau_{xz} = \tau_{yz} = 0$$

$$\sigma_z = \lambda(\varepsilon_x + \varepsilon_y) = \nu(\sigma_x + \sigma_y)$$

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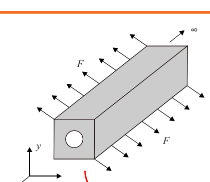
- Equilibrium equations

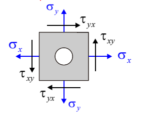
$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + f_x = 0$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + f_y = 0$$

$$f_z = 0$$

No body force in z since σ_z is only a function of *x* and *y*





For plane strain problems, ε_z vanishes, but σ_z does not

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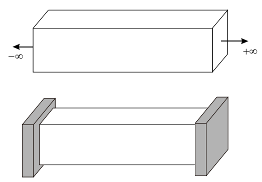
Plane strain problems

与能否存在之[illegible]边界问题

- **Boundary conditions**
  - Surface forces must be independent of *z*
  - The body in *z* direction may be either infinite or finite length with fixed ends



The only boundary conditions required for plane strain problems are those specified on the lateral surfaces

Stress boundary condition:

$$\begin{cases} T_x^{\mu} = T_x^{\mu}(x_0, y_0) \\ T_y^{\mu} = T_y^{\mu}(x_0, y_0) \\ T_z^{\mu} = 0 \end{cases} \Rightarrow \begin{cases} T_x^{\mu} = \sigma_{x|0}\mu_x + \tau_{yx|0}\mu_y \\ T_y^{\mu} = \tau_{xy|0}\mu_x + \sigma_{y|0}\mu_y \end{cases}$$

Displacement boundary condition:

$$\begin{cases} u(x_0, y_0) = u_b \\ v(x_0, y_0) = v_b \end{cases}$$

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Solution of plane strain problems – displacement formulation

- Displacements

$$\begin{cases} u = u(x,y) \\ v = v(x,y) \end{cases}$$

- Strain-displacement

$$\varepsilon_x = \frac{\partial u}{\partial x} \quad \varepsilon_y = \frac{\partial v}{\partial y} \quad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

- Stress-strain relations

$$\sigma_x = 2G\varepsilon_x + \lambda(\varepsilon_x + \varepsilon_y) \quad \tau_{xy} = G\gamma_{xy}$$

$$\sigma_y = 2G\varepsilon_y + \lambda(\varepsilon_x + \varepsilon_y)$$

- Equilibrium equations

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + f_x = 0$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + f_y = 0$$

$$\Rightarrow$$

$$G\nabla^2 u + (\lambda + G)\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + f_x = 0$$

$$G\nabla^2 v + (\lambda + G)\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + f_y = 0$$

where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$

Laplace operator

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Solution of plane strain problems – stress formulation

- Displacements

$$\begin{cases} u = u(x,y) \\ v = v(x,y) \end{cases}$$

- Strain-displacement

$$\varepsilon_x = \frac{\partial u}{\partial x} \quad \varepsilon_y = \frac{\partial v}{\partial y} \quad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \Rightarrow \frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}$$

- Stress-strain relations

$$\sigma_x = 2G\varepsilon_x + \lambda(\varepsilon_x + \varepsilon_y) \quad \tau_{xy} = G\gamma_{xy}$$

$$\sigma_y = 2G\varepsilon_y + \lambda(\varepsilon_x + \varepsilon_y)$$

$$\Rightarrow \quad \varepsilon_x = \frac{1}{E}\left((1-\nu^2)\sigma_x - \nu(1+\nu)\sigma_y\right)$$

$$\varepsilon_y = \frac{1}{E}\left((1-\nu^2)\sigma_y - \nu(1+\nu)\sigma_x\right)$$

$$\gamma_{xy} = \frac{1}{G}\tau_{xy}$$

- Equilibrium equations

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + f_x = 0$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + f_y = 0$$

$$\Downarrow$$

$$\nabla^2(\sigma_x + \sigma_y) = -\frac{1}{1-\nu}\left(\frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y}\right)$$

Compatibility equation in terms of stress

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Solution of plane strain problems

- Equilibrium equations $\frac{\partial \sigma_x}{\partial x}+\frac{\partial \tau_{yx}}{\partial y}+f_x=0 \ (x,y)$
- Hooke's law $\sigma_x=2G\varepsilon_x+\lambda\left(\varepsilon_x+\varepsilon_y\right) \ (x,y) \quad \tau_{xy}=G\gamma_{xy}$
- Strain-displacement $\varepsilon_x=\frac{\partial u}{\partial x} \quad \varepsilon_y=\frac{\partial v}{\partial y} \quad \gamma_{xy}=\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}$
- unknowns $\sigma_x,\sigma_y,\tau_{xy},\varepsilon_x,\varepsilon_y,\gamma_{xy},u,v$

Displacement formulation

1. Strain-displacement
2. Stress-strain relations
3. Equilibrium equations (displacement)

$$G\nabla^2 u+(\lambda+G)\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+f_x=0 \ (u,v)$$

u,v

Stress formulation

1. Strain-stress relations
2. Compatibility equation (stress)
3. Equilibrium equations

$$\nabla^2\left(\sigma_x+\sigma_y\right)=-\frac{1}{1-\nu}\left(\frac{\partial f_x}{\partial x}+\frac{\partial f_y}{\partial y}\right) \ \left(\sigma_x,\sigma_y,\tau_{xy}\right)$$

$\sigma_x,\sigma_y,\tau_{xy}$

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$$\begin{cases} \frac{\partial \sigma_x}{\partial x}+\frac{\partial \tau_{yx}}{\partial y}+\frac{\partial \tau_{zx}}{\partial z}+f_x=0 \\ \frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \sigma_y}{\partial y}+\frac{\partial \tau_{zy}}{\partial z}+f_y=0 \\ \frac{\partial \tau_{xz}}{\partial x}+\frac{\partial \tau_{yz}}{\partial y}+\frac{\partial \sigma_z}{\partial z}+f_z=0. \end{cases}$$

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$$\begin{cases} \varepsilon_x=\frac{\partial u}{\partial x} \\ \varepsilon_y=\frac{\partial v}{\partial y} \\ \varepsilon_z=\frac{\partial w}{\partial z} \end{cases} \quad \gamma_{xy}=\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x} \quad ② \quad \gamma_{yz}=\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y} \quad \gamma_{zx}=\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}$$

$$\begin{cases} \sigma_x=2G\varepsilon_x+\lambda\varepsilon \\ \sigma_y=2G\varepsilon_y+\lambda\varepsilon \\ \sigma_z=2G\varepsilon_z+\lambda\varepsilon \end{cases}$$

$$\begin{cases} \tau_{xy}=G\gamma_{xy} \\ \tau_{yz}=G\gamma_{yz} \\ \tau_{zx}=G\gamma_{zx} \end{cases}$$

Solution of 3D problems – displacement formulation

Governing equations

- Strain-displacement (6)

$$\varepsilon_x=\frac{\partial u}{\partial x} \quad \gamma_{xy}=\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x} \quad (x,y,z;u,v,w)$$

- Stress-strain relations (6)

$$\sigma_x=2G\varepsilon_x+\lambda\varepsilon \quad \tau_{xy}=G\gamma_{xy} \quad (x,y,z)$$

- Equilibrium equations (3)

$$\frac{\partial \sigma_x}{\partial x}+\frac{\partial \tau_{yx}}{\partial y}+\frac{\partial \tau_{zx}}{\partial z}+f_x=0 \quad (x,y,z)$$

- Substitute strain into Hooke's law

$$\sigma_x=2G\frac{\partial u}{\partial x}+\lambda\varepsilon \qquad \tau_{xy}=G\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)$$

$$\sigma_y=2G\frac{\partial v}{\partial y}+\lambda\varepsilon \qquad \tau_{yz}=G\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)$$

$$\sigma_z=2G\frac{\partial w}{\partial z}+\lambda\varepsilon \qquad \tau_{zx}=G\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)$$

- Substitute stress into equilibrium equations

$$\begin{cases} (\lambda+G)\frac{\partial \varepsilon}{\partial x}+G\nabla^2 u+f_x=0 \\ (\lambda+G)\frac{\partial \varepsilon}{\partial y}+G\nabla^2 v+f_y=0 \\ (\lambda+G)\frac{\partial \varepsilon}{\partial z}+G\nabla^2 w+f_z=0 \end{cases}$$

Navier's equations

where $\nabla^2 u=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$

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Solution of 3D problems – displacement formulation

Displacement boundary condition:

$$\begin{cases} u\left(x_0,y_0\right)=u_b \\ v\left(x_0,y_0\right)=v_b \\ w\left(x_0,y_0\right)=w_b \end{cases}$$

Stress boundary conditions: 可用displacement 转换

$$\begin{cases} T_x^{\mu}=\sigma_{x|0}\mu_x+\tau_{yx|0}\mu_y+\tau_{zx|0}\mu_z \\ T_y^{\mu}=\tau_{xy|0}\mu_x+\sigma_{y|0}\mu_y+\tau_{zy|0}\mu_z \\ T_z^{\mu}=\tau_{xz|0}\mu_x+\tau_{yz|0}\mu_y+\sigma_{z|0}\mu_z \end{cases}$$

$$\begin{cases} T_x^{\mu}=\lambda\varepsilon\mu_x+G\left(\frac{\partial u}{\partial x}\mu_x+\frac{\partial u}{\partial y}\mu_y+\frac{\partial u}{\partial z}\mu_z\right)+G\left(\frac{\partial u}{\partial x}\mu_x+\frac{\partial v}{\partial x}\mu_y+\frac{\partial w}{\partial x}\mu_z\right) \\ T_y^{\mu}=\lambda\varepsilon\mu_y+G\left(\frac{\partial v}{\partial x}\mu_x+\frac{\partial v}{\partial y}\mu_y+\frac{\partial v}{\partial z}\mu_z\right)+G\left(\frac{\partial u}{\partial y}\mu_x+\frac{\partial v}{\partial y}\mu_y+\frac{\partial w}{\partial y}\mu_z\right) \\ T_z^{\mu}=\lambda\varepsilon\mu_z+G\left(\frac{\partial w}{\partial x}\mu_x+\frac{\partial w}{\partial y}\mu_y+\frac{\partial w}{\partial z}\mu_z\right)+G\left(\frac{\partial u}{\partial z}\mu_x+\frac{\partial v}{\partial z}\mu_y+\frac{\partial w}{\partial z}\mu_z\right) \end{cases}$$

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Solution of 3D problems – stress formulation

Governing equations

- Stress-strain relations (6)

$$\varepsilon_x = \frac{1}{E}\left(\sigma_x - \nu\left(\sigma_y + \sigma_z\right)\right) \quad \gamma_{xy} = \frac{1}{G}\tau_{xy}$$

- Compatibility equations (3)

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}$$

$$2\frac{\partial^2 \varepsilon_x}{\partial y \partial z} = \frac{\partial}{\partial x}\left(-\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right)$$

- Equilibrium equations (3)

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + f_x = 0 \quad (x, y, z)$$

- Substitute strain into compatibility equations
- Together with the three equilibrium equations form the final six equations in terms of stress

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Solution of 3D problems – stress formulation

$$\boxed{\begin{aligned}
&\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}\\
&\frac{\partial^2 \varepsilon_y}{\partial z^2} + \frac{\partial^2 \varepsilon_z}{\partial y^2} = \frac{\partial^2 \gamma_{yz}}{\partial y \partial z}\\
&\frac{\partial^2 \varepsilon_z}{\partial x^2} + \frac{\partial^2 \varepsilon_x}{\partial z^2} = \frac{\partial^2 \gamma_{zx}}{\partial z \partial x}
\end{aligned}}
\qquad
\boxed{\begin{cases}
\varepsilon_x = \frac{1}{E}\left(\sigma_x - \nu\left(\sigma_y + \sigma_z\right)\right)\\
\varepsilon_y = \frac{1}{E}\left(\sigma_y - \nu\left(\sigma_z + \sigma_x\right)\right)\\
\varepsilon_z = \frac{1}{E}\left(\sigma_z - \nu\left(\sigma_x + \sigma_y\right)\right)
\end{cases}
\quad
\begin{cases}
\gamma_{xy} = \frac{1}{G}\tau_{xy}\\
\gamma_{yz} = \frac{1}{G}\tau_{yz}\\
\gamma_{zx} = \frac{1}{G}\tau_{zx}
\end{cases}}$$

Compatibility equations in terms of stress

$$\frac{\partial^2 \sigma_x}{\partial y^2} - \nu\frac{\partial^2\left(\sigma_y + \sigma_z\right)}{\partial y^2} + \frac{\partial^2 \sigma_y}{\partial x^2} - \nu\frac{\partial^2\left(\sigma_x + \sigma_z\right)}{\partial x^2} = 2(1+\nu)\frac{\partial^2 \tau_{xy}}{\partial x \partial y}$$

$$\frac{\partial^2 \sigma_y}{\partial z^2} - \nu\frac{\partial^2\left(\sigma_x + \sigma_z\right)}{\partial z^2} + \frac{\partial^2 \sigma_z}{\partial y^2} - \nu\frac{\partial^2\left(\sigma_x + \sigma_y\right)}{\partial y^2} = 2(1+\nu)\frac{\partial^2 \tau_{yz}}{\partial y \partial z}$$

$$\frac{\partial^2 \sigma_z}{\partial x^2} - \nu\frac{\partial^2\left(\sigma_x + \sigma_y\right)}{\partial x^2} + \frac{\partial^2 \sigma_x}{\partial z^2} - \nu\frac{\partial^2\left(\sigma_y + \sigma_z\right)}{\partial z^2} = 2(1+\nu)\frac{\partial^2 \tau_{xz}}{\partial x \partial z}$$

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Solution of 3D problems – stress formulation

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$$\frac{\partial^2 \sigma_x}{\partial x^2} + \frac{\partial^2 \tau_{yx}}{\partial x \partial y} + \frac{\partial^2 \tau_{zx}}{\partial x \partial z} + \frac{\partial f_x}{\partial x}$$

Eliminate shear stresses using equilibrium equations

$$\boxed{\begin{cases}
\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + f_x = 0\\
\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + f_y = 0\\
\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + f_z = 0
\end{cases}}$$

$$\begin{cases}
2\frac{\partial^2 \tau_{xy}}{\partial x \partial y} = -\left(+\frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} - \frac{\partial f_z}{\partial z} + \frac{\partial^2 \sigma_x}{\partial x^2} + \frac{\partial^2 \sigma_y}{\partial y^2} - \frac{\partial^2 \sigma_z}{\partial z^2}\right)\\
2\frac{\partial^2 \tau_{yz}}{\partial y \partial z} = -\left(-\frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} + \frac{\partial f_z}{\partial z} - \frac{\partial^2 \sigma_x}{\partial x^2} + \frac{\partial^2 \sigma_y}{\partial y^2} + \frac{\partial^2 \sigma_z}{\partial z^2}\right)\\
2\frac{\partial^2 \tau_{xz}}{\partial x \partial z} = -\left(+\frac{\partial f_x}{\partial x} - \frac{\partial f_y}{\partial y} + \frac{\partial f_z}{\partial z} + \frac{\partial^2 \sigma_x}{\partial x^2} - \frac{\partial^2 \sigma_y}{\partial y^2} + \frac{\partial^2 \sigma_z}{\partial z^2}\right)
\end{cases}$$

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Solution of 3D problems – stress formulation

$$\begin{cases}\frac{\partial^2\sigma_x}{\partial y^2}-\nu\frac{\partial^2(\sigma_y+\sigma_z)}{\partial y^2}+\frac{\partial^2\sigma_y}{\partial x^2}-\nu\frac{\partial^2(\sigma_x+\sigma_z)}{\partial x^2}=2(1+\nu)\frac{\partial^2\tau_{xy}}{\partial x\partial y}\\ \frac{\partial^2\sigma_y}{\partial z^2}-\nu\frac{\partial^2(\sigma_x+\sigma_z)}{\partial z^2}+\frac{\partial^2\sigma_z}{\partial y^2}-\nu\frac{\partial^2(\sigma_x+\sigma_y)}{\partial y^2}=2(1+\nu)\frac{\partial^2\tau_{yz}}{\partial y\partial z}\\ \frac{\partial^2\sigma_z}{\partial x^2}-\nu\frac{\partial^2(\sigma_x+\sigma_y)}{\partial x^2}+\frac{\partial^2\sigma_x}{\partial z^2}-\nu\frac{\partial^2(\sigma_y+\sigma_z)}{\partial z^2}=2(1+\nu)\frac{\partial^2\tau_{xz}}{\partial x\partial z}\end{cases}$$

$$\begin{cases}2\frac{\partial^2\tau_{xy}}{\partial x\partial y}=-\left(+\frac{\partial f_x}{\partial x}+\frac{\partial f_y}{\partial y}-\frac{\partial f_z}{\partial z}+\frac{\partial^2\sigma_x}{\partial x^2}+\frac{\partial^2\sigma_y}{\partial y^2}-\frac{\partial^2\sigma_z}{\partial z^2}\right)\\ 2\frac{\partial^2\tau_{yz}}{\partial y\partial z}=-\left(-\frac{\partial f_x}{\partial x}+\frac{\partial f_y}{\partial y}+\frac{\partial f_z}{\partial z}-\frac{\partial^2\sigma_x}{\partial x^2}+\frac{\partial^2\sigma_y}{\partial y^2}+\frac{\partial^2\sigma_z}{\partial z^2}\right)\\ 2\frac{\partial^2\tau_{xz}}{\partial x\partial z}=-\left(+\frac{\partial f_x}{\partial x}-\frac{\partial f_y}{\partial y}+\frac{\partial f_z}{\partial z}+\frac{\partial^2\sigma_x}{\partial x^2}-\frac{\partial^2\sigma_y}{\partial y^2}+\frac{\partial^2\sigma_z}{\partial z^2}\right)\end{cases}$$

$$\Downarrow$$

$$\frac{\partial^2\sigma_x}{\partial y^2}+\frac{\partial^2\sigma_y}{\partial z^2}+\frac{\partial^2\sigma_z}{\partial x^2}+\frac{\partial^2\sigma_x}{\partial z^2}+\frac{\partial^2\sigma_y}{\partial x^2}+\frac{\partial^2\sigma_z}{\partial y^2}-\nu\left(\frac{\partial^2(2\sigma_x+\sigma_y+\sigma_z)}{\partial x^2}+\frac{\partial^2(\sigma_x+2\sigma_y+\sigma_z)}{\partial y^2}+\frac{\partial^2(\sigma_x+\sigma_y+2\sigma_z)}{\partial z^2}\right)$$
$$=-(1+\nu)\left(\frac{\partial f_x}{\partial x}+\frac{\partial f_y}{\partial y}+\frac{\partial f_z}{\partial z}+\frac{\partial^2\sigma_x}{\partial x^2}+\frac{\partial^2\sigma_y}{\partial y^2}+\frac{\partial^2\sigma_z}{\partial z^2}\right)$$

$$\Downarrow$$

$$(1-\nu)\nabla^2(\sigma_x+\sigma_y+\sigma_z)=-(1+\nu)\left(\frac{\partial f_x}{\partial x}+\frac{\partial f_y}{\partial y}+\frac{\partial f_z}{\partial z}\right) \Rightarrow \nabla^2\Theta=-\left(\frac{1+\nu}{1-\nu}\right)\left(\frac{\partial f_x}{\partial x}+\frac{\partial f_y}{\partial y}+\frac{\partial f_z}{\partial z}\right)$$

$$\Theta=\sigma_x+\sigma_y+\sigma_z \qquad \nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$$

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Solution of 3D problems – stress formulation

$$\begin{cases}\frac{\partial^2\sigma_x}{\partial y^2}-\nu\frac{\partial^2(\sigma_y+\sigma_z)}{\partial y^2}+\frac{\partial^2\sigma_y}{\partial x^2}-\nu\frac{\partial^2(\sigma_x+\sigma_z)}{\partial x^2}=2(1+\nu)\frac{\partial^2\tau_{xy}}{\partial x\partial y}\\ \frac{\partial^2\sigma_y}{\partial z^2}-\nu\frac{\partial^2(\sigma_x+\sigma_z)}{\partial z^2}+\frac{\partial^2\sigma_z}{\partial y^2}-\nu\frac{\partial^2(\sigma_x+\sigma_y)}{\partial y^2}=2(1+\nu)\frac{\partial^2\tau_{yz}}{\partial y\partial z}\\ \frac{\partial^2\sigma_z}{\partial x^2}-\nu\frac{\partial^2(\sigma_x+\sigma_y)}{\partial x^2}+\frac{\partial^2\sigma_x}{\partial z^2}-\nu\frac{\partial^2(\sigma_y+\sigma_z)}{\partial z^2}=2(1+\nu)\frac{\partial^2\tau_{xz}}{\partial x\partial z}\end{cases}$$

$$\begin{cases}2\frac{\partial^2\tau_{xy}}{\partial x\partial y}=-\left(+\frac{\partial f_x}{\partial x}+\frac{\partial f_y}{\partial y}-\frac{\partial f_z}{\partial z}+\frac{\partial^2\sigma_x}{\partial x^2}+\frac{\partial^2\sigma_y}{\partial y^2}-\frac{\partial^2\sigma_z}{\partial z^2}\right)\\ 2\frac{\partial^2\tau_{yz}}{\partial y\partial z}=-\left(-\frac{\partial f_x}{\partial x}+\frac{\partial f_y}{\partial y}+\frac{\partial f_z}{\partial z}-\frac{\partial^2\sigma_x}{\partial x^2}+\frac{\partial^2\sigma_y}{\partial y^2}+\frac{\partial^2\sigma_z}{\partial z^2}\right)\\ 2\frac{\partial^2\tau_{xz}}{\partial x\partial z}=-\left(+\frac{\partial f_x}{\partial x}-\frac{\partial f_y}{\partial y}+\frac{\partial f_z}{\partial z}+\frac{\partial^2\sigma_x}{\partial x^2}-\frac{\partial^2\sigma_y}{\partial y^2}+\frac{\partial^2\sigma_z}{\partial z^2}\right)\end{cases}$$

$$\Downarrow$$

$$\begin{cases}\frac{\partial^2\sigma_x}{\partial y^2}-\nu\frac{\partial^2(\sigma_y+\sigma_z)}{\partial y^2}+\frac{\partial^2\sigma_y}{\partial x^2}-\nu\frac{\partial^2(\sigma_x+\sigma_z)}{\partial x^2}=-(1+\nu)\left(+\frac{\partial f_x}{\partial x}+\frac{\partial f_y}{\partial y}-\frac{\partial f_z}{\partial z}+\frac{\partial^2\sigma_x}{\partial x^2}+\frac{\partial^2\sigma_y}{\partial y^2}-\frac{\partial^2\sigma_z}{\partial z^2}\right)\\ \frac{\partial^2\sigma_y}{\partial z^2}-\nu\frac{\partial^2(\sigma_x+\sigma_z)}{\partial z^2}+\frac{\partial^2\sigma_z}{\partial y^2}-\nu\frac{\partial^2(\sigma_x+\sigma_y)}{\partial y^2}=-(1+\nu)\left(-\frac{\partial f_x}{\partial x}+\frac{\partial f_y}{\partial y}+\frac{\partial f_z}{\partial z}-\frac{\partial^2\sigma_x}{\partial x^2}+\frac{\partial^2\sigma_y}{\partial y^2}+\frac{\partial^2\sigma_z}{\partial z^2}\right)\\ \frac{\partial^2\sigma_z}{\partial x^2}-\nu\frac{\partial^2(\sigma_x+\sigma_y)}{\partial x^2}+\frac{\partial^2\sigma_x}{\partial z^2}-\nu\frac{\partial^2(\sigma_y+\sigma_z)}{\partial z^2}=-(1+\nu)\left(+\frac{\partial f_x}{\partial x}-\frac{\partial f_y}{\partial y}+\frac{\partial f_z}{\partial z}+\frac{\partial^2\sigma_x}{\partial x^2}-\frac{\partial^2\sigma_y}{\partial y^2}+\frac{\partial^2\sigma_z}{\partial z^2}\right)\end{cases}$$

$$\Downarrow$$

$\Theta=\sigma_x+\sigma_y+\sigma_z$

$$\begin{cases}\nabla^2\Theta-(1+\nu)\nabla^2\sigma_z-\frac{\partial^2\Theta}{\partial z^2}=-(1+\nu)\left(+\frac{\partial f_x}{\partial x}+\frac{\partial f_y}{\partial y}-\frac{\partial f_z}{\partial z}\right)\\ \nabla^2\Theta-(1+\nu)\nabla^2\sigma_x-\frac{\partial^2\Theta}{\partial x^2}=-(1+\nu)\left(-\frac{\partial f_x}{\partial x}+\frac{\partial f_y}{\partial y}+\frac{\partial f_z}{\partial z}\right)\\ \nabla^2\Theta-(1+\nu)\nabla^2\sigma_y-\frac{\partial^2\Theta}{\partial y^2}=-(1+\nu)\left(+\frac{\partial f_x}{\partial x}-\frac{\partial f_y}{\partial y}+\frac{\partial f_z}{\partial z}\right)\end{cases}$$

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Solution of 3D problems – stress formulation

$$\nabla^2\Theta=-\left(\frac{1+\nu}{1-\nu}\right)\left(\frac{\partial f_x}{\partial x}+\frac{\partial f_y}{\partial y}+\frac{\partial f_z}{\partial z}\right)$$

$$\begin{cases}\nabla^2\Theta-(1+\nu)\nabla^2\sigma_z-\frac{\partial^2\Theta}{\partial z^2}=-(1+\nu)\left(+\frac{\partial f_x}{\partial x}+\frac{\partial f_y}{\partial y}-\frac{\partial f_z}{\partial z}\right)\\ \nabla^2\Theta-(1+\nu)\nabla^2\sigma_x-\frac{\partial^2\Theta}{\partial x^2}=-(1+\nu)\left(-\frac{\partial f_x}{\partial x}+\frac{\partial f_y}{\partial y}+\frac{\partial f_z}{\partial z}\right)\\ \nabla^2\Theta-(1+\nu)\nabla^2\sigma_y-\frac{\partial^2\Theta}{\partial y^2}=-(1+\nu)\left(+\frac{\partial f_x}{\partial x}-\frac{\partial f_y}{\partial y}+\frac{\partial f_z}{\partial z}\right)\end{cases}$$

$$\Downarrow$$

$$\begin{cases}\nabla^2\sigma_z+\frac{1}{1+\nu}\frac{\partial^2\Theta}{\partial z^2}=-\left(\frac{\nu}{1-\nu}\right)\left(\frac{\partial f_x}{\partial x}+\frac{\partial f_y}{\partial y}+\frac{\partial f_z}{\partial z}\right)-2\frac{\partial f_z}{\partial z}\\ \nabla^2\sigma_x+\frac{1}{1+\nu}\frac{\partial^2\Theta}{\partial x^2}=-\left(\frac{\nu}{1-\nu}\right)\left(\frac{\partial f_x}{\partial x}+\frac{\partial f_y}{\partial y}+\frac{\partial f_z}{\partial z}\right)-2\frac{\partial f_x}{\partial x}\\ \nabla^2\sigma_y+\frac{1}{1+\nu}\frac{\partial^2\Theta}{\partial y^2}=-\left(\frac{\nu}{1-\nu}\right)\left(\frac{\partial f_x}{\partial x}+\frac{\partial f_y}{\partial y}+\frac{\partial f_z}{\partial z}\right)-2\frac{\partial f_y}{\partial y}\end{cases}$$

Compatibility equations in
terms of normal stress

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Solution of 3D problems – stress formulation

$$2\frac{\partial^2\varepsilon_x}{\partial y\partial z}=\frac{\partial}{\partial x}\left(-\frac{\partial\gamma_{yz}}{\partial x}+\frac{\partial\gamma_{zx}}{\partial y}+\frac{\partial\gamma_{xy}}{\partial z}\right)$$
$$2\frac{\partial^2\varepsilon_y}{\partial x\partial z}=\frac{\partial}{\partial y}\left(\frac{\partial\gamma_{yz}}{\partial x}-\frac{\partial\gamma_{zx}}{\partial y}+\frac{\partial\gamma_{xy}}{\partial z}\right)$$
$$2\frac{\partial^2\varepsilon_z}{\partial x\partial y}=\frac{\partial}{\partial z}\left(\frac{\partial\gamma_{yz}}{\partial x}+\frac{\partial\gamma_{zx}}{\partial y}-\frac{\partial\gamma_{xy}}{\partial z}\right)$$

$$\begin{cases}\varepsilon_x=\frac{1}{E}\left(\sigma_x-\nu\left(\sigma_y+\sigma_z\right)\right)\\ \varepsilon_y=\frac{1}{E}\left(\sigma_y-\nu\left(\sigma_z+\sigma_x\right)\right)\\ \varepsilon_z=\frac{1}{E}\left(\sigma_z-\nu\left(\sigma_x+\sigma_y\right)\right)\end{cases}\qquad\begin{cases}\gamma_{xy}=\frac{1}{G}\tau_{xy}\\ \gamma_{yz}=\frac{1}{G}\tau_{yz}\\ \gamma_{zx}=\frac{1}{G}\tau_{zx}\end{cases}$$

Compatibility equations
in terms of stress

$$\frac{\partial^2\sigma_x}{\partial y\partial z}=\left(-\frac{\partial^2\tau_{yz}}{\partial x^2}+\frac{\partial^2\tau_{zx}}{\partial x\partial y}+\frac{\partial^2\tau_{xy}}{\partial x\partial z}\right)+\nu\left(-\frac{\partial^2\tau_{yz}}{\partial x^2}+\frac{\partial^2\tau_{zx}}{\partial x\partial y}+\frac{\partial^2\tau_{xy}}{\partial x\partial z}+\frac{\partial^2\left(\sigma_y+\sigma_z\right)}{\partial y\partial z}\right)$$
$$\frac{\partial^2\sigma_y}{\partial x\partial z}=\left(+\frac{\partial^2\tau_{yz}}{\partial x\partial y}-\frac{\partial^2\tau_{zx}}{\partial y^2}+\frac{\partial^2\tau_{xy}}{\partial y\partial z}\right)+\nu\left(+\frac{\partial^2\tau_{yz}}{\partial x\partial y}-\frac{\partial^2\tau_{zx}}{\partial y^2}+\frac{\partial^2\tau_{xy}}{\partial y\partial z}+\frac{\partial^2\left(\sigma_z+\sigma_x\right)}{\partial x\partial z}\right)$$
$$\frac{\partial^2\sigma_z}{\partial x\partial y}=\left(+\frac{\partial^2\tau_{yz}}{\partial x\partial z}+\frac{\partial^2\tau_{zx}}{\partial y\partial z}-\frac{\partial^2\tau_{xy}}{\partial z^2}\right)+\nu\left(+\frac{\partial^2\tau_{yz}}{\partial x\partial z}+\frac{\partial^2\tau_{zx}}{\partial y\partial z}-\frac{\partial^2\tau_{xy}}{\partial z^2}+\frac{\partial^2\left(\sigma_x+\sigma_y\right)}{\partial x\partial y}\right)$$

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Solution of 3D problems – stress formulation

Eliminate normal
stresses using
equilibrium equations

$$\begin{cases}\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}+\frac{\partial\tau_{zx}}{\partial z}+f_x=0\\ \frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\sigma_y}{\partial y}+\frac{\partial\tau_{zy}}{\partial z}+f_y=0\\ \frac{\partial\tau_{xz}}{\partial x}+\frac{\partial\tau_{yz}}{\partial y}+\frac{\partial\sigma_z}{\partial z}+f_z=0\end{cases}$$

$$\frac{\partial^2\left(\sigma_y+\sigma_z\right)}{\partial y\partial z}=-\left(\frac{\partial^2\tau_{xy}}{\partial x\partial z}+\frac{\partial^2\tau_{zy}}{\partial z^2}+\frac{\partial^2\tau_{xz}}{\partial x\partial y}+\frac{\partial^2\tau_{yz}}{\partial y^2}+\frac{\partial f_y}{\partial z}+\frac{\partial f_z}{\partial y}\right)$$
$$\frac{\partial^2\left(\sigma_x+\sigma_z\right)}{\partial x\partial z}=-\left(\frac{\partial^2\tau_{yx}}{\partial y\partial z}+\frac{\partial^2\tau_{zx}}{\partial z^2}+\frac{\partial^2\tau_{xz}}{\partial x^2}+\frac{\partial^2\tau_{yz}}{\partial x\partial y}+\frac{\partial f_x}{\partial z}+\frac{\partial f_z}{\partial x}\right)$$
$$\frac{\partial^2\left(\sigma_x+\sigma_y\right)}{\partial x\partial y}=-\left(\frac{\partial^2\tau_{yx}}{\partial y^2}+\frac{\partial^2\tau_{zx}}{\partial z\partial y}+\frac{\partial^2\tau_{xy}}{\partial x^2}+\frac{\partial^2\tau_{zy}}{\partial x\partial z}+\frac{\partial f_x}{\partial y}+\frac{\partial f_y}{\partial x}\right)$$

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Solution of 3D problems – stress formulation

$$\frac{\partial^2\sigma_x}{\partial y\partial z}=\left(-\frac{\partial^2\tau_{yz}}{\partial x^2}+\frac{\partial^2\tau_{zx}}{\partial x\partial y}+\frac{\partial^2\tau_{xy}}{\partial x\partial z}\right)+\nu\left(-\frac{\partial^2\tau_{yz}}{\partial x^2}+\frac{\partial^2\tau_{zx}}{\partial x\partial y}+\frac{\partial^2\tau_{xy}}{\partial x\partial z}+\frac{\partial^2\left(\sigma_y+\sigma_z\right)}{\partial y\partial z}\right)$$
$$\frac{\partial^2\sigma_y}{\partial x\partial z}=\left(+\frac{\partial^2\tau_{yz}}{\partial x\partial y}-\frac{\partial^2\tau_{zx}}{\partial y^2}+\frac{\partial^2\tau_{xy}}{\partial y\partial z}\right)+\nu\left(+\frac{\partial^2\tau_{yz}}{\partial x\partial y}-\frac{\partial^2\tau_{zx}}{\partial y^2}+\frac{\partial^2\tau_{xy}}{\partial y\partial z}+\frac{\partial^2\left(\sigma_z+\sigma_x\right)}{\partial x\partial z}\right)$$
$$\frac{\partial^2\sigma_z}{\partial x\partial y}=\left(+\frac{\partial^2\tau_{yz}}{\partial x\partial z}+\frac{\partial^2\tau_{zx}}{\partial y\partial z}-\frac{\partial^2\tau_{xy}}{\partial z^2}\right)+\nu\left(+\frac{\partial^2\tau_{yz}}{\partial x\partial z}+\frac{\partial^2\tau_{zx}}{\partial y\partial z}-\frac{\partial^2\tau_{xy}}{\partial z^2}+\frac{\partial^2\left(\sigma_x+\sigma_y\right)}{\partial x\partial y}\right)$$

Compatibility equations

$$\frac{\partial^2\left(\sigma_y+\sigma_z\right)}{\partial y\partial z}=-\left(\frac{\partial^2\tau_{xy}}{\partial x\partial z}+\frac{\partial^2\tau_{zy}}{\partial z^2}+\frac{\partial^2\tau_{xz}}{\partial x\partial y}+\frac{\partial^2\tau_{yz}}{\partial y^2}+\frac{\partial f_y}{\partial z}+\frac{\partial f_z}{\partial y}\right)$$
$$\frac{\partial^2\left(\sigma_x+\sigma_z\right)}{\partial x\partial z}=-\left(\frac{\partial^2\tau_{yx}}{\partial y\partial z}+\frac{\partial^2\tau_{zx}}{\partial z^2}+\frac{\partial^2\tau_{xz}}{\partial x^2}+\frac{\partial^2\tau_{yz}}{\partial x\partial y}+\frac{\partial f_x}{\partial z}+\frac{\partial f_z}{\partial x}\right)$$
$$\frac{\partial^2\left(\sigma_x+\sigma_y\right)}{\partial x\partial y}=-\left(\frac{\partial^2\tau_{yx}}{\partial y^2}+\frac{\partial^2\tau_{zx}}{\partial z\partial y}+\frac{\partial^2\tau_{xy}}{\partial x^2}+\frac{\partial^2\tau_{zy}}{\partial x\partial z}+\frac{\partial f_x}{\partial y}+\frac{\partial f_y}{\partial x}\right)$$

Equilibrium equations

$$\frac{\partial^2\sigma_x}{\partial y\partial z}+\frac{\partial^2\left(\sigma_y+\sigma_z\right)}{\partial y\partial z}=\left(-\frac{\partial^2\tau_{yz}}{\partial x^2}+\frac{\partial^2\tau_{zx}}{\partial x\partial y}+\frac{\partial^2\tau_{xy}}{\partial x\partial z}\right)+\nu\left(-\frac{\partial^2\tau_{yz}}{\partial x^2}+\frac{\partial^2\tau_{zx}}{\partial x\partial y}+\frac{\partial^2\tau_{xy}}{\partial x\partial z}+\frac{\partial^2\left(\sigma_y+\sigma_z\right)}{\partial y\partial z}\right)+\frac{\partial^2\left(\sigma_y+\sigma_z\right)}{\partial y\partial z}$$
$$\frac{\partial^2\sigma_y}{\partial x\partial z}+\frac{\partial^2\left(\sigma_x+\sigma_z\right)}{\partial x\partial z}=\left(+\frac{\partial^2\tau_{yz}}{\partial x\partial y}-\frac{\partial^2\tau_{zx}}{\partial y^2}+\frac{\partial^2\tau_{xy}}{\partial y\partial z}\right)+\nu\left(+\frac{\partial^2\tau_{yz}}{\partial x\partial y}-\frac{\partial^2\tau_{zx}}{\partial y^2}+\frac{\partial^2\tau_{xy}}{\partial y\partial z}+\frac{\partial^2\left(\sigma_z+\sigma_x\right)}{\partial x\partial z}\right)+\frac{\partial^2\left(\sigma_x+\sigma_z\right)}{\partial x\partial z}$$
$$\frac{\partial^2\sigma_z}{\partial x\partial y}+\frac{\partial^2\left(\sigma_x+\sigma_y\right)}{\partial x\partial y}=\left(+\frac{\partial^2\tau_{yz}}{\partial x\partial z}+\frac{\partial^2\tau_{zx}}{\partial y\partial z}-\frac{\partial^2\tau_{xy}}{\partial z^2}\right)+\nu\left(+\frac{\partial^2\tau_{yz}}{\partial x\partial z}+\frac{\partial^2\tau_{zx}}{\partial y\partial z}-\frac{\partial^2\tau_{xy}}{\partial z^2}+\frac{\partial^2\left(\sigma_x+\sigma_y\right)}{\partial x\partial y}\right)+\frac{\partial^2\left(\sigma_x+\sigma_y\right)}{\partial x\partial y}$$

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Solution of 3D problems – stress formulation

$$\frac{\partial^2\sigma_x}{\partial y\partial z}=\left(-\frac{\partial^2\tau_{yz}}{\partial x^2}+\frac{\partial^2\tau_{zx}}{\partial x\partial y}+\frac{\partial^2\tau_{xy}}{\partial x\partial z}\right)+\nu\left(-\frac{\partial^2\tau_{yz}}{\partial x^2}+\frac{\partial^2\tau_{zx}}{\partial x\partial y}+\frac{\partial^2\tau_{xy}}{\partial x\partial z}+\frac{\partial^2(\sigma_y+\sigma_z)}{\partial y\partial z}\right)$$

$$\frac{\partial^2\sigma_y}{\partial x\partial z}=\left(+\frac{\partial^2\tau_{yz}}{\partial x\partial y}-\frac{\partial^2\tau_{zx}}{\partial y^2}+\frac{\partial^2\tau_{xy}}{\partial y\partial z}\right)+\nu\left(+\frac{\partial^2\tau_{yz}}{\partial x\partial y}-\frac{\partial^2\tau_{zx}}{\partial y^2}+\frac{\partial^2\tau_{xy}}{\partial y\partial z}+\frac{\partial^2(\sigma_z+\sigma_x)}{\partial x\partial z}\right)$$

$$\frac{\partial^2\sigma_z}{\partial x\partial y}=\left(+\frac{\partial^2\tau_{yz}}{\partial x\partial z}+\frac{\partial^2\tau_{zx}}{\partial y\partial z}-\frac{\partial^2\tau_{xy}}{\partial z^2}\right)+\nu\left(+\frac{\partial^2\tau_{yz}}{\partial x\partial z}+\frac{\partial^2\tau_{zx}}{\partial y\partial z}-\frac{\partial^2\tau_{xy}}{\partial z^2}+\frac{\partial^2(\sigma_x+\sigma_y)}{\partial x\partial y}\right)$$

Compatibility equations

$$\frac{\partial^2(\sigma_y+\sigma_z)}{\partial y\partial z}=-\left(\frac{\partial^2\tau_{xy}}{\partial x\partial z}+\frac{\partial^2\tau_{zy}}{\partial z^2}+\frac{\partial^2\tau_{xz}}{\partial x\partial y}+\frac{\partial^2\tau_{yz}}{\partial y^2}+\frac{\partial f_y}{\partial z}+\frac{\partial f_z}{\partial y}\right)$$

$$\frac{\partial^2(\sigma_x+\sigma_z)}{\partial x\partial z}=-\left(\frac{\partial^2\tau_{yx}}{\partial y\partial z}+\frac{\partial^2\tau_{zx}}{\partial z^2}+\frac{\partial^2\tau_{xz}}{\partial x^2}+\frac{\partial^2\tau_{yz}}{\partial x\partial y}+\frac{\partial f_x}{\partial z}+\frac{\partial f_z}{\partial x}\right)$$

$$\frac{\partial^2(\sigma_x+\sigma_y)}{\partial x\partial y}=-\left(\frac{\partial^2\tau_{yx}}{\partial y^2}+\frac{\partial^2\tau_{zx}}{\partial z\partial y}+\frac{\partial^2\tau_{xy}}{\partial x^2}+\frac{\partial^2\tau_{zy}}{\partial x\partial z}+\frac{\partial f_x}{\partial y}+\frac{\partial f_y}{\partial x}\right)$$

Equilibrium equations

$$\frac{\partial^2\Theta}{\partial y\partial z}=-\nu\left(\nabla^2\tau_{yz}+\left(\frac{\partial f_y}{\partial z}+\frac{\partial f_z}{\partial y}\right)\right)-\left(\nabla^2\tau_{yz}+\frac{\partial f_y}{\partial z}+\frac{\partial f_z}{\partial y}\right)$$

$$\frac{\partial^2\Theta}{\partial x\partial z}=-\nu\left(\nabla^2\tau_{zx}+\left(\frac{\partial f_x}{\partial z}+\frac{\partial f_z}{\partial x}\right)\right)-\left(\nabla^2\tau_{zx}+\frac{\partial f_x}{\partial z}+\frac{\partial f_z}{\partial x}\right)$$

$$\frac{\partial^2\Theta}{\partial x\partial y}=-\nu\left(\nabla^2\tau_{xy}+\left(\frac{\partial f_x}{\partial y}+\frac{\partial f_y}{\partial x}\right)\right)-\left(\nabla^2\tau_{xy}+\frac{\partial f_x}{\partial y}+\frac{\partial f_y}{\partial x}\right)$$

Compatibility equations in terms of shear stress

$$\nabla^2\tau_{yz}+\frac{1}{1+\nu}\frac{\partial^2\Theta}{\partial y\partial z}=-\left(\frac{\partial f_y}{\partial z}+\frac{\partial f_z}{\partial y}\right)$$

$$\nabla^2\tau_{zx}+\frac{1}{1+\nu}\frac{\partial^2\Theta}{\partial x\partial z}=-\left(\frac{\partial f_x}{\partial z}+\frac{\partial f_z}{\partial x}\right)$$

$$\nabla^2\tau_{xy}+\frac{1}{1+\nu}\frac{\partial^2\Theta}{\partial x\partial y}=-\left(\frac{\partial f_x}{\partial y}+\frac{\partial f_y}{\partial x}\right)$$

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Solution of 3D problems – stress formulation

Compatibility equations in terms of stress
Or Beltrami-Michell compatibility equations

$$\begin{cases}\nabla^2\sigma_z+\frac{1}{1+\nu}\frac{\partial^2\Theta}{\partial z^2}=-\left(\frac{\nu}{1-\nu}\right)\left(\frac{\partial f_x}{\partial x}+\frac{\partial f_y}{\partial y}+\frac{\partial f_z}{\partial z}\right)-2\frac{\partial f_z}{\partial z}\\ \nabla^2\sigma_x+\frac{1}{1+\nu}\frac{\partial^2\Theta}{\partial x^2}=-\left(\frac{\nu}{1-\nu}\right)\left(\frac{\partial f_x}{\partial x}+\frac{\partial f_y}{\partial y}+\frac{\partial f_z}{\partial z}\right)-2\frac{\partial f_x}{\partial x}\\ \nabla^2\sigma_y+\frac{1}{1+\nu}\frac{\partial^2\Theta}{\partial y^2}=-\left(\frac{\nu}{1-\nu}\right)\left(\frac{\partial f_x}{\partial x}+\frac{\partial f_y}{\partial y}+\frac{\partial f_z}{\partial z}\right)-2\frac{\partial f_y}{\partial y}\end{cases}$$

Compatibility equations in terms of normal stress

$$\begin{cases}\nabla^2\tau_{yz}+\frac{1}{1+\nu}\frac{\partial^2\Theta}{\partial y\partial z}=-\left(\frac{\partial f_y}{\partial z}+\frac{\partial f_z}{\partial y}\right)\\ \nabla^2\tau_{zx}+\frac{1}{1+\nu}\frac{\partial^2\Theta}{\partial x\partial z}=-\left(\frac{\partial f_x}{\partial z}+\frac{\partial f_z}{\partial x}\right)\\ \nabla^2\tau_{xy}+\frac{1}{1+\nu}\frac{\partial^2\Theta}{\partial x\partial y}=-\left(\frac{\partial f_x}{\partial y}+\frac{\partial f_y}{\partial x}\right)\end{cases}$$

Compatibility equations in terms of shear stress

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Solution of 3D problems

- Equilibrium equations (3) $\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}+\frac{\partial\tau_{zx}}{\partial z}+f_x=0\ (x,y,z)$
- Hooke's law (6) $\sigma_x=2G\varepsilon_x+\lambda(\varepsilon_x+\varepsilon_y+\varepsilon_z)\ (x,y,z)\quad \tau_{xy}=G\gamma_{xy}$
- Strain-displacement (6) $\varepsilon_x=\frac{\partial u}{\partial x}\quad \gamma_{xy}=\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}$
- Unknowns (15) $\sigma_x,\sigma_y,\sigma_z,\tau_{xy},\tau_{xz},\tau_{yz},\varepsilon_x,\varepsilon_y,\varepsilon_z,\gamma_{xy},\gamma_{xz},\gamma_{yz},u,v,w$

Displacement formulation

1. Strain-displacement
2. Stress-strain relations
3. Equilibrium equations (displacement)

$$G\nabla^2u+(\lambda+G)\frac{\partial\varepsilon}{\partial x}+f_x=0\ (u,v,w)$$

u,v,w

Stress formulation

1. Strain-stress relations
2. Compatibility equation (stress)
3. Equilibrium equations

$$\nabla^2\sigma_x+\frac{1}{1+\nu}\frac{\partial^2\Theta}{\partial x^2}=-\frac{\nu}{1-\nu}\left(\frac{\partial f_x}{\partial x}+\frac{\partial f_y}{\partial y}+\frac{\partial f_z}{\partial z}\right)-2\frac{\partial f_x}{\partial x}\ (\sigma_x,\sigma_y,\sigma_z)$$

$$\nabla^2\tau_{xy}+\frac{1}{1+\nu}\frac{\partial^2\Theta}{\partial x\partial y}=-\left(\frac{\partial f_x}{\partial y}+\frac{\partial f_y}{\partial x}\right)\ (\tau_{xy},\tau_{yz},\tau_{xz})$$

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Miscellaneous

- **Principal of superposition**
  - Two or more stresses fields may be superposed to yield the results for combined loads
  - Only when displacements and strains are small and the strain-displacement, stress-strain equations are linear
- **Uniqueness of elasticity solutions**
  - For a given surface force and body force distribution, there is only one solution for the stress components consistent with equilibrium and compatibility

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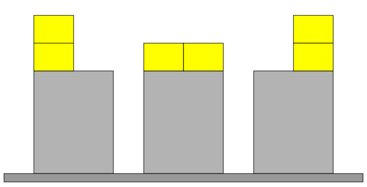
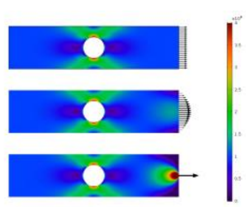
Saint-Venant's principle

- The stresses due to two statically equivalent loadings applied over a small area are significantly different only in the vicinity of the area on which the loadings are applied
- At distance which are large in comparison with the linear dimension of the area on which the loadings are applied, the effects due to these two loadings are the same





Stress distribution around a hole

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