Review

Viscosity *μ*, *η*:

- A quantitative measure of a fluid's resistance to flow due to 'internal friction' between fluid elements.

Classification of Flow Phenomena

- Steady and unsteady flows (定常流 vs 非定常流)
- Uniform and non-uniform flows (均匀流 vs 非均匀流)
- Rotational and irrotational flows (有旋流 vs 无旋流)，Vortex (涡流)

  vorticity (涡量、涡度) $\boldsymbol{\omega} = \text{curl}\,\mathbf{v} = \nabla \times \mathbf{v}$ or $\omega_i = \varepsilon_{ijk} \nabla_j v_k$
- Laminar and turbulent flows (层流 vs 紊流、湍流)
- Viscous and inviscid flows (粘滞流vs无粘性流)
- Incompressible and compressible flows (可压缩vs不可压缩流)
- Ideal fluid (perfect fluid, 理想流体)

Streamline, Pathline, Streakline

The study of Fluids: theoretical, experimental, computational

Classroom exercises

Consider a two-dimensional incompressible flow with the velocity field given by the following expressions:

$$u(x, y) = -\omega y$$

$$v(x, y) = \omega x$$

where ω is a constant.

Determine direction of the streamline passing through the point (1, 1) at time t=0.

$$\frac{dy}{dx} = \frac{u}{v} = -1$$

Assume there is a one-dimensional flow that varies with time, with the velocity given by $u(t) = U_0 + A\sin(\omega t)$, where U_0, A, and ω are positive constants.

Find the pathline $x(t)$ of a fluid particle passing through the origin at time $t = 0$.

$$x(t) = \int_0^t u(t')dt' = U_0 t + A[1 - \cos(wt)]$$

Continuum Mechanics (B)
Session 08: Governing Equations of Fluid Mechanics

Lecturer: Ting Yang 杨亭
Department: ESS 地空系
E-mail: yangt3@sustech.edu.cn

Contents

- Pressure and Viscous Stress
- Constitutive Equation for Newtonian Fluids
- Leibnitz's theorem and Reynolds transport theorem
- Conservation of Mass (Continuity Equation)
- Conservation of Momentum
- Conservation of Energy
- Navier-Stokes Equations
- Rayleigh Number and Onset of Thermal Convection
- Scaling and Dimensional Analysis
- Control Volume (Integral) Analysis of Mass and Momentum Conservation***

Pressure and Viscous Stress Tensor

Forces acting on a fluid region include **body forces** and **surface forces**.

Stress vector at a point is:

$$R_i = \lim_{dS \to 0} \frac{dt_i}{dS}$$

Normal stress and **shear stress** components

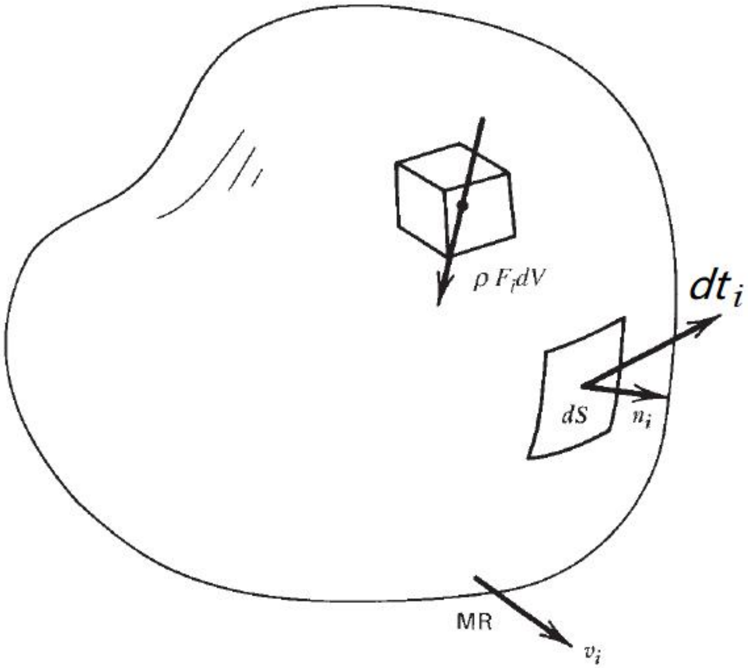
Stress vector at a point depends on the orientation of the plane it acts on.

- There are infinite stress vectors at a single point.

Stress tensor are used to depict the stress state at a point.

$$\sigma_{ij} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} = \begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix}$$

Given stress tensor, stress vector at any oriented plane n_i can be derived: $R_j = \sigma_{ij} n_i$



Calculate stress vector on the plane that is normal to vector (0, 2, 0)

Pressure and Viscous Stress Tensor

Stress tensor incorporates two contributions, **pressure** and **viscous stress tensor,** in fluid.

$$\sigma_{ij} = -p\delta_{ij} + \tau_{ij}$$

The stress tensor and viscous stress tensors are symmetric

$$\sigma_{12} = \sigma_{21} \quad \tau_{12} = \tau_{21}$$

The normal stress is the sum of pressure and normal viscous stress

$$\sigma_{11} = -p + \tau_{11}$$

$$\sigma_{22} = -p + \tau_{22}$$

$$\sigma_{33} = -p + \tau_{33}$$

Only viscous stress contributes to the shear stress.

- For fluid under rest, only pressure exists.
  - $\tau_{ij}(v_k) \equiv 0$, if $v_k = c$
- **Stokes assumption:** Viscous stress is a deviatoric stress tensor
  - The average normal viscous stress is zero

Constitutive Equation for Newtonian Fluids

Newtonian Fluids: the viscous stress is a linear function of the strain rate

$$\tau_{ij} = c_{ijkl}(T, P, C, d, f_{H_2O})\dot{\varepsilon}_{kl}$$ Constitutive equation for Newtonian fluids

Only 21 independent parameters for fully anisotropic fluid

Most fluids are isotropic, the viscosity parameters is independent of directions, we have

$$\tau_{ij} = \lambda\delta_{ij}\dot{\varepsilon}_{kk} + 2\mu\dot{\varepsilon}_{ij}$$ Constitutive equation for isotropic fluids

μ, λ are termed dynamic (first) and second viscosity, respectively.

Under the Stokes's assumption, the average normal viscous stress is zero. So we have

$$\tau_{ii} = 3\lambda\dot{\varepsilon}_{kk} + 2\mu\dot{\varepsilon}_{ii} = 3(\lambda + \frac{2\mu}{3})\dot{\varepsilon}_{ii} = 3\kappa\dot{\varepsilon}_{ii} = 0$$ κ is termed bulk viscosity

$$\Rightarrow \quad \kappa = (\lambda + \frac{2\mu}{3}) = 0 \quad \lambda = -\frac{2\mu}{3}$$

The constitutive equation for Newtonian fluids under the Stokes's assumption is

$$\tau_{ij} = -\frac{2\mu}{3}\delta_{ij}\dot{\varepsilon}_{kk} + 2\mu\dot{\varepsilon}_{ij} \qquad \sigma_{ij} = -p\delta_{ij} - \frac{2\mu}{3}\delta_{ij}\dot{\varepsilon}_{kk} + 2\mu\dot{\varepsilon}_{ij}$$

Discuss the difference in the constitutive equations between elastic solid and viscous fluid

Leibnitz's theorem and Reynolds transport theorem

Leibnitz's theorem (莱布尼茨定理)

Consider integration

$$I_{ij\ldots}(t) = \int_{R(t)} T_{ij\ldots}(x_i, t)\ dV$$

T_{ij} stands for a scalar, vector, or tensor function of interest.

Not only the integrand (被积函数) changes with time, but the region of integration *R(t)* may be moving by expanding, contracting, or translating the surface of R.

Let w_i be the velocity of the surface of R. The theorem of Leibnitz is

$$\frac{d}{dt} \int_{R(t)} T_{ij\ldots}(x_i, t)\ dV = \int_R \frac{\partial T_{ij\ldots}}{\partial t}\ dV + \int_S n_k w_k T_{ij\ldots}\ dS$$

i.e., we may move the derivative with respect to time inside the integral if we add a surface integral to compensate for the motion of the boundary.

- Surface integral: how fast T_{ij} is coming out of R because of the surface velocity w_i.
- If the boundary does not move, $w_i = 0$: it is permissible to interchange the order of differentiation and integration.

Leibnitz's theorem and Reynolds transport theorem

Leibnitz's theorem (莱布尼茨定理)

$$\frac{d}{dt}\int_{R(t)} T_{ij\dots}(x_i, t)\; dV = \int_R \frac{\partial T_{ij\dots}}{\partial t}\, dV + \int_S n_k w_k T_{ij\dots}\; dS$$

Reynolds transport theorem (雷诺输运定理)

Suppose $R(t)$ is a fluid element with surface S traveling at the flow velocity v_k, we have

$$\frac{D}{Dt}\int_{R(t)} T_{ij\dots}(x_i, t)\; dV = \int_R \frac{\partial T_{ij\dots}}{\partial t}\, dV + \int_S n_k v_k T_{ij\dots}\; dS$$

Differences from Leibnitz’s theorem:
(1) Substitute derivative is used on the left-hand side.
(2) surface velocity is changed to fluid velocity.

Leibnitz's theorem and Reynolds transport theorem

The volume of a material region is given by the integration

$$V_{\mathrm{MR}} = \int_{R(t)} 1 \, dV$$

With the Leibnitz's theorem, we have the volume change rate

$$\frac{DV_{\mathrm{MR}}}{Dt} = \frac{d}{dt}\int_{R(t)} 1 \, dV = \int \partial t \cdot 1 \, dV + \int n_i w_i \cdot 1 \, dS = \int_S n_i v_i \, dS$$

The surface velocity w_i of the region R equals to the local fluid velocity v_i.

The surface integral can be converted into a volume integral by Gauss's theorem:

$$\frac{DV_{\mathrm{MR}}}{Dt} = \int_R \partial_i v_i \, dV = \overline{(\partial_i v_i)} V_{\mathrm{MR}}$$

When the volume approaches zero about a specific point

$$\lim_{V_{\mathrm{MR}} \to 0} \frac{1}{V_{\mathrm{MR}}} \frac{DV_{\mathrm{MR}}}{Dt} = \partial_i v_i = \nabla \cdot \mathbf{v}$$

The divergence of the velocity represent the dilation rate

Conservation of Mass (Continuity Equation)

Conservation of mass: The time rate of change of the mass of a material region is zero.

In mathematical terms, we have:

$$\frac{dM_{\mathrm{MR}}}{dt} = \frac{d}{dt}\int_{\mathrm{MR}} \rho \, dV = 0$$

Using the Reynolds transport theorem, we have

$$\int_{MR} \frac{\partial \rho}{\partial t} dV + \int_S n_i v_i \rho dS = 0$$

With the Gauss's theorem, the surface integral is changed into a volume integral:

$$\int_{MR} [\frac{\partial \rho}{\partial t} + \frac{\partial (\rho v_i)}{\partial x_i}] dV = 0$$

Since the chosen of the integration region is arbitrary, we have

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho v_i)}{\partial x_i} = 0$$

mass conservation in the differential form

Symbolic notation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

Conservation of Mass (Continuity Equation)

Alternatively, we consider a small parcel.
Based on mass conservation, we have

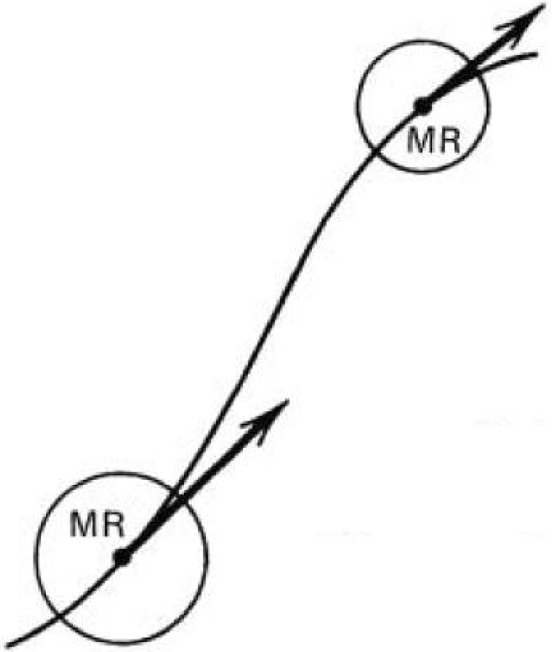
$$\frac{d(\rho V)}{dt} = 0$$

$$\frac{d\rho}{dt} + \frac{dV}{Vdt}\rho = 0$$

➡ $$\frac{d\rho}{dt} + \frac{\partial v_i}{\partial x_i}\rho = 0$$

Unit volume change rate equals to velocity divergence

➡ $$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho v_i)}{\partial x_i} = 0$$



An infinetesimal element in space

The density change of a particle is entirely due to changes in its volume.

Conservation of Mass (Continuity Equation)

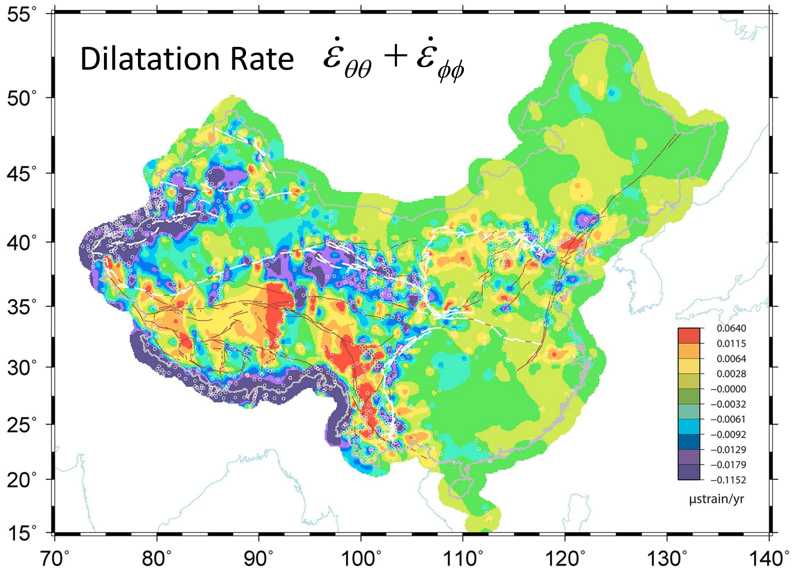
Incompressible flow: the density of a fluid element does not change during its motion.

$$\frac{D\rho}{Dt} = 0$$

mass conservation

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v_i)}{\partial x_i} = 0 \quad \Rightarrow \quad \frac{\partial v_i}{\partial x_i} = 0$$

For incompressible flow, the divergence of velocity is zero. i.e., the velocity field is 'divergence free' (无散场) or solenoidal (螺线场)



Discuss the crustal thickness variation of China

Conservation of Momentum

Conservation of momentum: The time rate of change of the linear momentum of a material region is equal to the sum of the forces on the region.

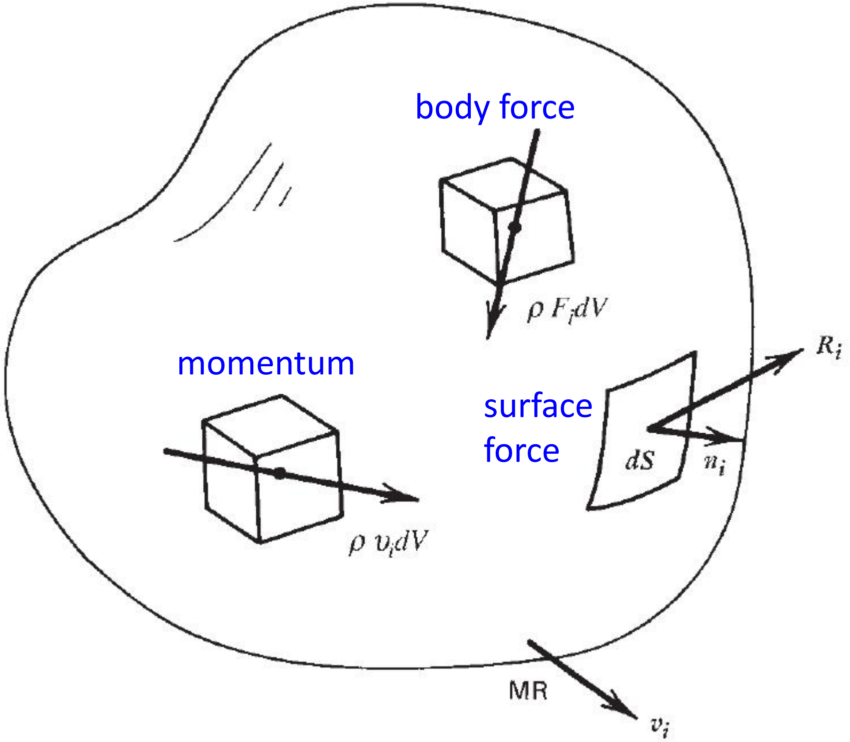
$$\text{Net force on material region} = \int_{MR} \rho F_i \, dV + \int_{MR} R_i \, dS$$

Momentum change rate of material region is

$$\frac{d}{dt}\int_{MR} \rho v_i \, dV$$

Conservation of momentum:

$$\frac{d}{dt}\int_{MR} \rho v_i \, dV = \int_{MR} \rho F_i \, dV + \int_{MR} R_i \, dS$$



Conservation of Momentum

Apply Leibnitz's and Gauss's theorems to the left-hand side of momentum conservation

$$\int_{MR}\left[\frac{\partial(\rho v_i)}{\partial t}+\frac{\partial(\rho v_j v_i)}{\partial x_j}\right]dV=\int_{MR}\rho F_i dV+\int_{MR}R_i dS$$

The surface stress vector can be expressed with stress tensor:

$$R_i=\sigma_{ji}n_j=(-p\delta_{ji}+\tau_{ji})n_j$$

Apply the Gauss's theorem to surface force integration, we have:

$$\int_{MR}\left[\frac{\partial(\rho v_i)}{\partial t}+\frac{\partial(\rho v_j v_i)}{\partial x_j}\right]dV=\int_{MR}\rho F_i dV+\int_{MR}\left(-\frac{\partial p}{\partial x_i}+\frac{\partial \tau_{ji}}{\partial x_j}\right)dV$$

Since the integration volume is arbitrary, we have

$$\frac{\partial(\rho v_i)}{\partial t}+\frac{\partial(\rho v_j v_i)}{\partial x_j}=\rho F_i-\frac{\partial p}{\partial x_i}+\frac{\partial \tau_{ji}}{\partial x_j}$$

differential form of the momentum conservation equation

momentum change rate

momentum flowed out

total body force

net pressure force

net viscous force

symbolic notation:

$$\frac{\partial(\rho\mathbf{v})}{\partial t}+\nabla\cdot(\rho\mathbf{v}\mathbf{v})=\rho\mathbf{F}-\nabla p+\nabla\cdot\tau$$

Conservation of Momentum

The left-hand side of the equation can be simplified:

$$\frac{\partial(\rho v_i)}{\partial t}+\frac{\partial(\rho v_j v_i)}{\partial x_j}=\rho\frac{\partial v_i}{\partial t}+v_i\frac{\partial \rho}{\partial t}+\rho v_j\frac{\partial v_i}{\partial x_j}+v_i\frac{\partial(\rho v_j)}{\partial x_j}$$

$$=\rho\frac{\partial v_i}{\partial t}+\rho v_j\frac{\partial v_i}{\partial x_j}=\rho\frac{Dv_i}{Dt}$$

The momentum equation becomes:

$$\rho\frac{Dv_i}{Dt}=\rho F_i-\frac{\partial p}{\partial x_i}+\frac{\partial \tau_{ji}}{\partial x_j}$$

The momentum change of a particle equates the net force acting on the particle.

momentum change rate

total body force

net pressure force

net viscous force