Continuum Mechanics (B)

Session 02: Strain analysis

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Contents

- Strain-Displacement Relations
- State of Strain at a Point (strain transformation)
- Compatibility Equations
- Principle of Superposition

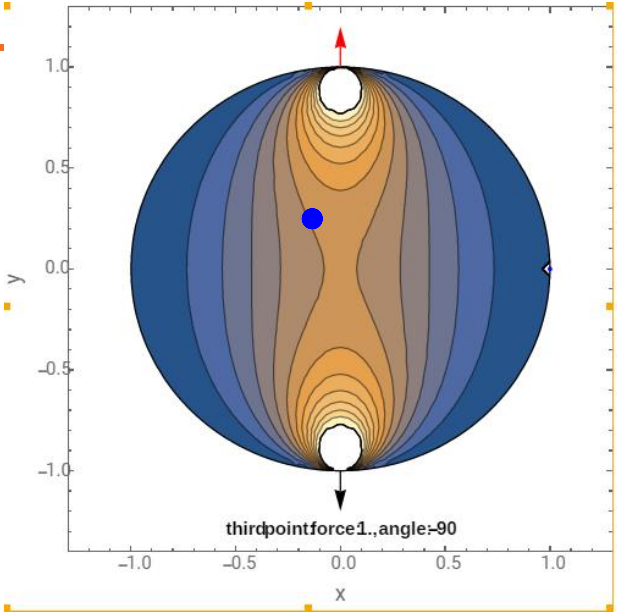
Strain-Displacement Relations（几何方程）

When a body is subjected to external loads (加载)

- induced internal forces
- **induced internal deformation**
  - changes in relative positions of continuum particles during deformation

Assumptions:

- **Continuum (continuous materials):**
  material is present at each point in the medium
- **Continuous displacement**
  Originally continuous material cannot contain gaps after it is displaced
- **Single-valued displacement function**
  A single material particle cannot occupy two positions in space after deformation
- **Infinitesimal deformation**
  Solid



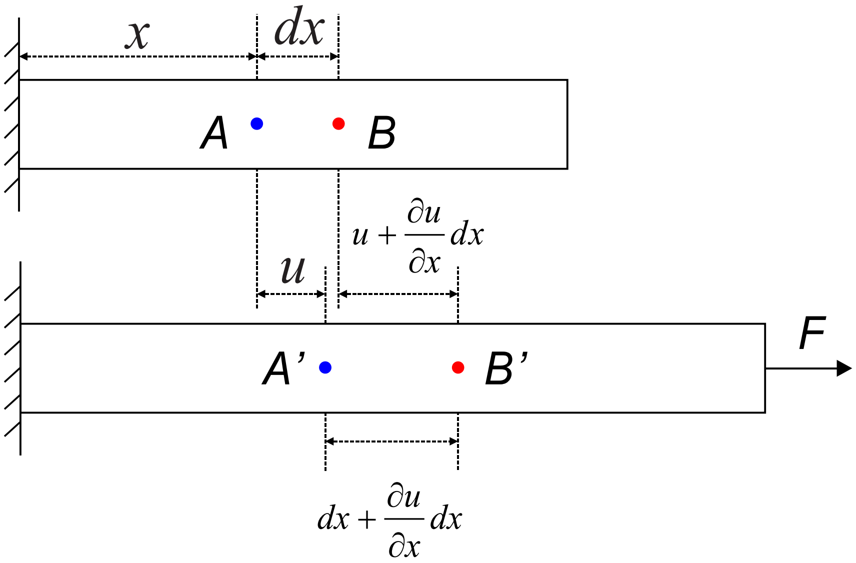
strain (ε_{max}) distribution in a circular plate with concentrated loadings (https://demonstrations.wolfram.com/StressDistributionInACircularPlateWithConcentratedRadialLoad/)

Strain-Displacement Relations（几何方程）

Deformation inside a 1D bar (杆)

- **uniaxial stress (单轴应力)**
  - a 1D state of stress in which normal stress acts along one direction only
- **Examine deformation at Point A**
  - deformation is associated with relative displacements
  - find Point B that is very close to A
  - Points A, B moves to A', B' after deformation
  - deformation at point A is defined as (normal) **strain ε: the length change in** unit distance

$$\varepsilon = \frac{A'B' - AB}{AB} = \frac{\left(dx + u + \frac{\partial u}{\partial x}dx - u\right) - dx}{dx} = \frac{\partial u}{\partial x}$$



Deformation of a bar subjected to uniaxial stress

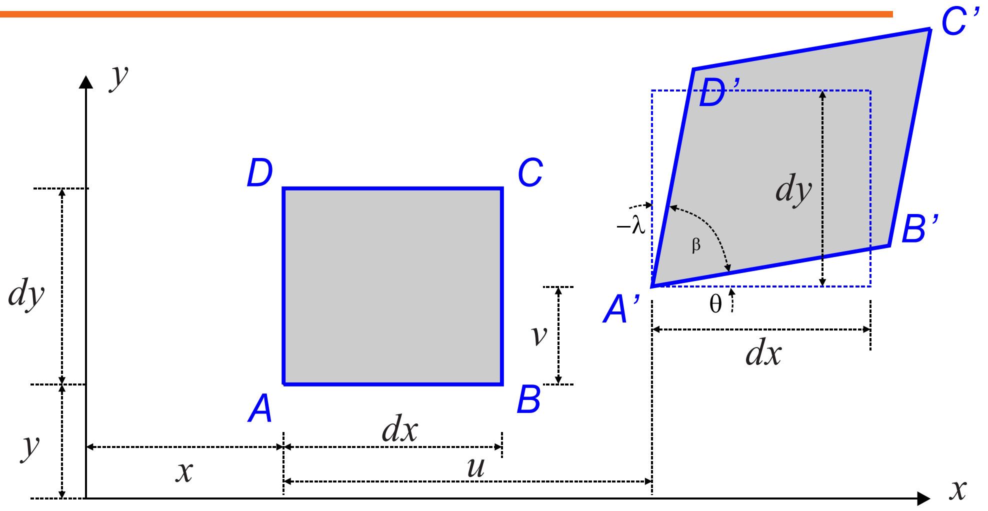
- Normal strain is positive for extension and negative for compression

Strain-Displacement Relations（几何方程）

Deformation of point A in 2D (plane strain)

$$u = u(x, y) \qquad v = v(x, y) \qquad w = 0$$

- Examine deformation at point A
- Deformation at point A can be fully depicted by
  - the sides's length change
  - the angle change between two sides
- **Normal strain ε** in a given direction
  - Length change in the unit distance of a line originally oriented in the given direction
- **Shear strain γ** (measured in radian)
  - the change in the original right angle between two axes



Translation and deformation of a 2D infinitesimal element

$$\varepsilon_x = \frac{A'B' - AB}{AB} = \frac{A'B' - dx}{dx}$$

$$\varepsilon_y = \frac{A'D' - AD}{AD} = \frac{A'D' - dy}{dy}$$

$$\gamma_{xy} = \frac{\pi}{2} - \beta = \theta - \lambda$$

counterclockwise angles of rotation are defined as positive

Strain-Displacement Relations（几何方程）

Normal strain

$$\varepsilon_x = \frac{A'B' - AB}{AB} = \frac{A'B' - dx}{dx}$$

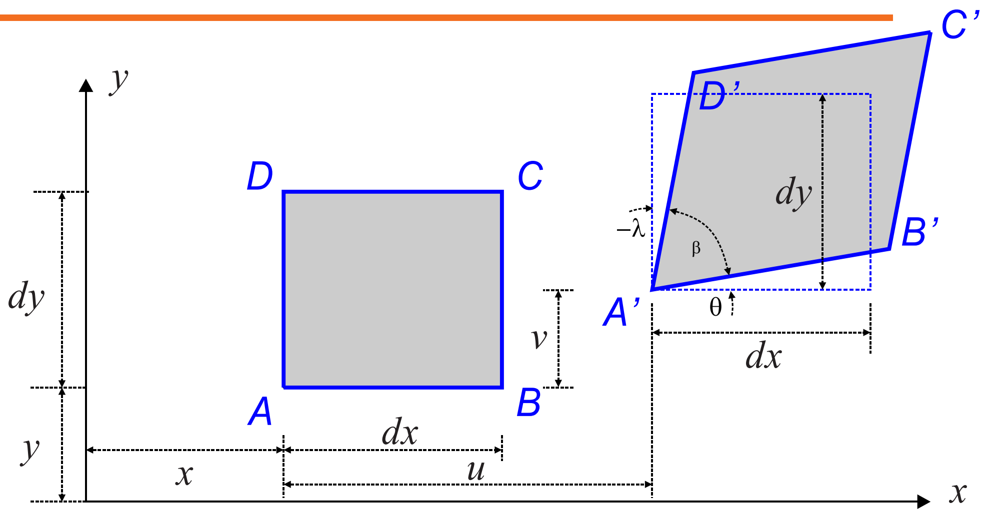
$$\varepsilon_y = \frac{A'D' - AD}{AD} = \frac{A'D' - dy}{dy}$$

$$A'B' = \sqrt{(dx + \frac{\partial u}{\partial x}dx)^2 + (\frac{\partial v}{\partial x}dx)^2} \approx (dx + \frac{\partial u}{\partial x}dx)$$

For small strains (displacement gradient << 1), the second term in the square root is negligible

$$\varepsilon_x = \frac{A'B' - AB}{AB} = \frac{\partial u}{\partial x}$$

Similarly $\varepsilon_y = \frac{\partial v}{\partial y}$



Translation and deformation of a 2D infinitesimal element

Strain-Displacement Relations（几何方程）

Shear strain

$$\gamma_{xy} = \frac{\pi}{2} - \beta = \theta - \lambda$$

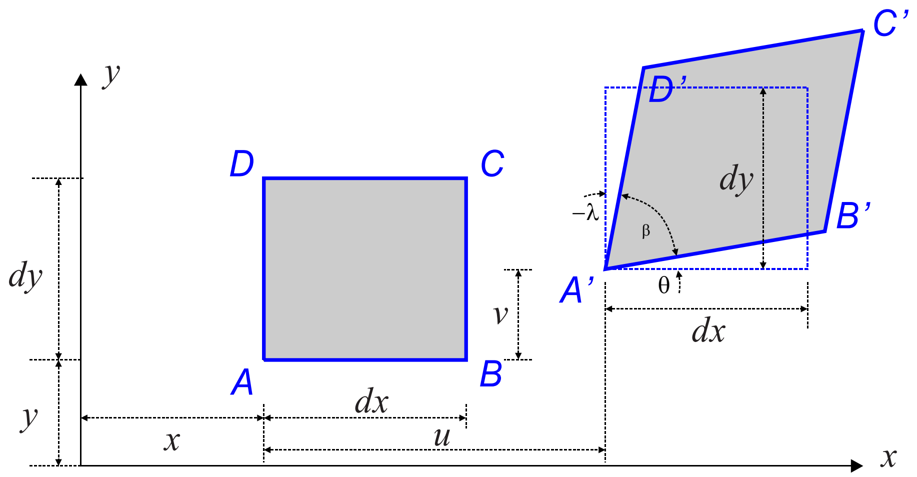
Assume small strain (displacement gradient << 1)

$$\theta = \tan\theta = \frac{\frac{\partial v}{\partial x}dx}{dx + \frac{\partial u}{\partial x}dx} = \frac{\frac{\partial v}{\partial x}}{1 + \frac{\partial u}{\partial x}} = \frac{\partial v}{\partial x}$$

$$-\lambda = -\tan\lambda = \frac{\frac{\partial u}{\partial y}dy}{dy + \frac{\partial v}{\partial y}dy} = \frac{\frac{\partial u}{\partial y}}{1 + \frac{\partial v}{\partial y}} = \frac{\partial u}{\partial y}$$

Thus the shear strain

$$\gamma_{xy} = \theta - \lambda = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$



Translation and deformation of a 2D element

- Shear strain is positive if
  - the right angle between two positive / negative directions of the two axes decreases, or
  - if the angle between a positive axis and a negative axis increases
- negative for other cases

Strain-Displacement Relations（几何方程）

3D strain-displacement relations (应变位移关系，几何方程)

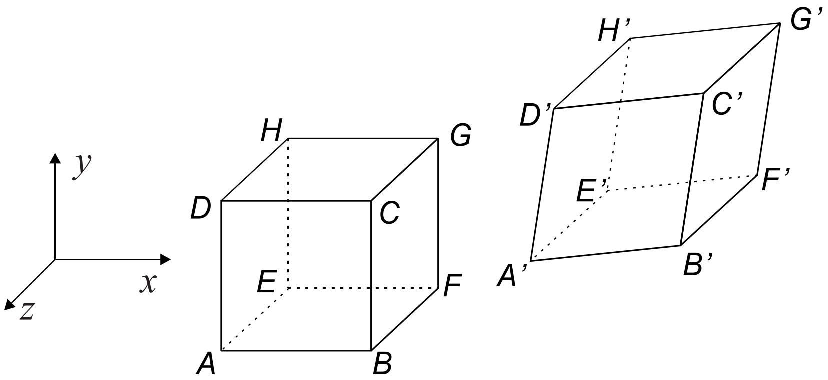
$$\varepsilon_x = \frac{\partial u}{\partial x} \qquad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

$$\varepsilon_y = \frac{\partial v}{\partial y} \qquad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}$$

$$\varepsilon_z = \frac{\partial w}{\partial z} \qquad \gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$$

Based on the definition:

$$\gamma_{xy} = \gamma_{yx} \qquad \gamma_{yz} = \gamma_{zy} \qquad \gamma_{zx} = \gamma_{xz}$$



strain in a 3D rectangular prism

$$\varepsilon_x = \frac{A'B' - AB}{AB} = \frac{D'C' - DC}{DC} = \frac{E'F' - EF}{EF} = \frac{H'G' - HG}{HG}$$

$$\gamma_{xy} = \frac{\pi}{2} - \angle B'A'D' = \frac{\pi}{2} - \angle F'E'H' = \frac{\pi}{2} - \angle B'C'D' = \frac{\pi}{2} - \angle F'G'H'$$

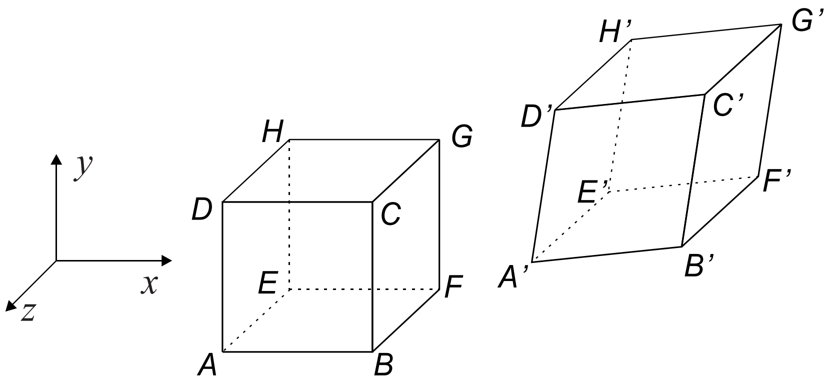
Engineering shear strain γ_{xy} vs tensor (mathematical) shear strain ε_{xy}:

$$\varepsilon_{xy} = 0.5\gamma_{xy}$$

Classroom exercises

1. Can we fully determine deformation inside the body with displacement?
2. The advantage of strain ε in describing the deformation at a point
3. Does the shear strain $\gamma_{xy} = \gamma_{yx}$?
4. What are the units of normal strain and shear strain, respectively?
5. Verify the following formulations for normal strain ε_y and shear strain γ_{xy} in 3D geometry

$$\varepsilon_y = \frac{\partial v}{\partial y} \qquad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$



strain in a 3D rectangular prism

Strain-Displacement Relations（几何方程）

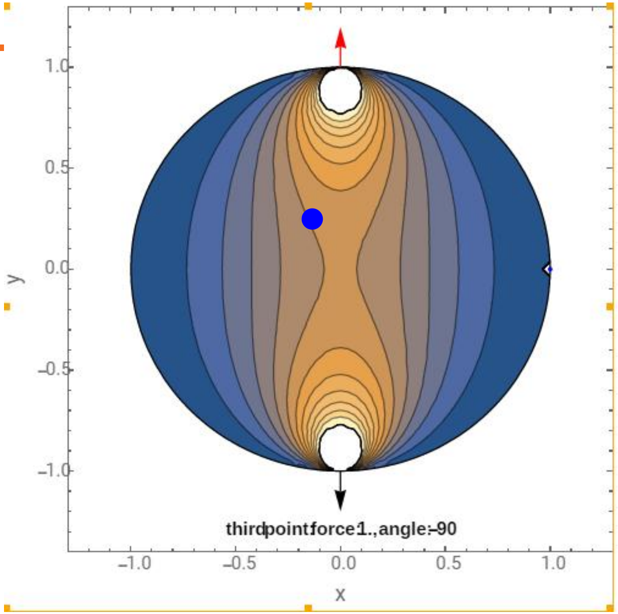
The stress state at a point represented by 9 components

$$\begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix}$$

The strain state at a point represented by 9 components

$$\begin{bmatrix} \varepsilon_x & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_y & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_z \end{bmatrix}$$

- The strain and stress components depend on the coordinate system.
- The stress and strain states do not depend on the coordinate.

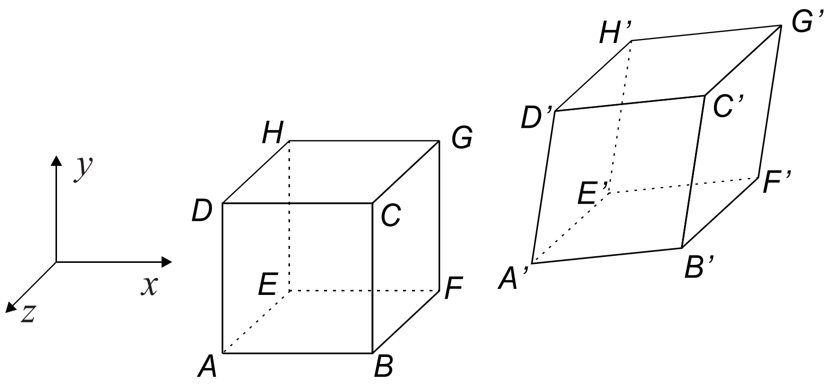


strain (ε_{max}) distribution in a circular plate with concentrated loadings (https://demonstrations.wolfram.com/StressDistributionInACircularPlateWithConcentratedRadialLoad/)

Cautious

Note that stress is defined over the **deformed body** while strain is defined over the **undeformed body**.

- This inconsistency is neglected under infinitesimal deformation.



strain in a 3D rectangular prism