Review

Displacement formulation (位移解法)
Navier's equation for 3D problems

$$\boxed{\begin{array}{c}\begin{cases}(\lambda+G)\dfrac{\partial\varepsilon}{\partial x}+G\nabla^2 u+f_x=0\\(\lambda+G)\dfrac{\partial\varepsilon}{\partial y}+G\nabla^2 v+f_y=0\\(\lambda+G)\dfrac{\partial\varepsilon}{\partial z}+G\nabla^2 w+f_z=0\end{cases}\\ u,v,w\end{array}}$$

3 equations, 3 unknowns

- The governing equations of plane stress and plane strain can be unified.
- Uniqueness of Elasticity Solutions

With ***Airy's stress function*** $\phi = \phi(x,y)$:

$$\sigma_x + V = \frac{\partial^2\phi}{\partial y^2}$$

$$\sigma_y + V = \frac{\partial^2\phi}{\partial x^2}$$

$$\tau_{xy} = -\frac{\partial^2\phi}{\partial x\partial y}$$

The governing equation for plane stress:

$$\nabla^4\phi = \nabla^2\left(\nabla^2\phi\right) = (1-\nu)\nabla^2 V$$

The governing equation for plane strain:

$$\nabla^4\phi = \frac{1-2\nu}{1-\nu}\nabla^2 V$$

Principle of Superposition of Elastic Problems

- Linear differential equations
  - An equation is linear if it contains only terms of up to the first degree in the dependent variables (因变量, stress, strain, displacement in our case) and their derivatives. (方程只包含因变量及其导数的零次或一次方)

1D wave equation

$$\frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial x^2} = f(x, t)$$

1D Burgers equation (Euler equation without source or pressure gradient, 冲击波方程)

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = 0$$

2D Navier-Stokes equation

$$\rho\left(\frac{\partial u_x}{\partial t} + u_x \frac{\partial u_x}{\partial x} + u_y \frac{\partial u_x}{\partial y}\right) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2}\right) + \rho g_x$$

$$\rho\left(\frac{\partial u_y}{\partial t} + u_x \frac{\partial u_y}{\partial x} + u_y \frac{\partial u_y}{\partial y}\right) = -\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 u_y}{\partial x^2} + \frac{\partial^2 u_y}{\partial y^2}\right) + \rho g_y.$$

Principle of Superposition of Elastic Problems

- Linear differential equations meets the principle of superposition (叠加原理)

Linear differential operator (线性微分算子) L, e.g.

$$L = \frac{\partial^2}{\partial t^2} - a^2 \frac{\partial^2}{\partial x^2} \quad \text{then} \quad Lu = \frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial x^2}$$

If function u_1, u_2, u_3... are the solutions of the equations $Lu = f_i$, i.e.,

$$Lu_1 = f_1 \quad Lu_2 = f_2 \quad Lu_3 = f_3 \ldots$$

Then $\sum_i u_i$ is the solution of the equation $Lu = \sum_i f_i$

note the inhomogeneous terms (非齐次项)

Ex: Show that the Burgers equation (nonlinear equation) below does not satisfy the principle of superposition

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$$

Principle of Superposition of Elastic Problems

Check if these governing equations for the elastic problems are linear?

Stress formula

$$\begin{cases} \nabla^2\sigma_z + \dfrac{1}{1+\nu}\dfrac{\partial^2\Theta}{\partial z^2} = -\left(\dfrac{\nu}{1-\nu}\right)\left(\dfrac{\partial f_x}{\partial x}+\dfrac{\partial f_y}{\partial y}+\dfrac{\partial f_z}{\partial z}\right) - 2\dfrac{\partial f_z}{\partial z} \\ \nabla^2\sigma_x + \dfrac{1}{1+\nu}\dfrac{\partial^2\Theta}{\partial x^2} = -\left(\dfrac{\nu}{1-\nu}\right)\left(\dfrac{\partial f_x}{\partial x}+\dfrac{\partial f_y}{\partial y}+\dfrac{\partial f_z}{\partial z}\right) - 2\dfrac{\partial f_x}{\partial x} \\ \nabla^2\sigma_y + \dfrac{1}{1+\nu}\dfrac{\partial^2\Theta}{\partial y^2} = -\left(\dfrac{\nu}{1-\nu}\right)\left(\dfrac{\partial f_x}{\partial x}+\dfrac{\partial f_y}{\partial y}+\dfrac{\partial f_z}{\partial z}\right) - 2\dfrac{\partial f_y}{\partial y} \end{cases}$$

$$\begin{cases} \nabla^2\tau_{yz} + \dfrac{1}{1+\nu}\dfrac{\partial^2\Theta}{\partial y\partial z} = -\left(\dfrac{\partial f_y}{\partial z}+\dfrac{\partial f_z}{\partial y}\right) \\ \nabla^2\tau_{zx} + \dfrac{1}{1+\nu}\dfrac{\partial^2\Theta}{\partial x\partial z} = -\left(\dfrac{\partial f_x}{\partial z}+\dfrac{\partial f_z}{\partial x}\right) \\ \nabla^2\tau_{xy} + \dfrac{1}{1+\nu}\dfrac{\partial^2\Theta}{\partial x\partial y} = -\left(\dfrac{\partial f_x}{\partial y}+\dfrac{\partial f_y}{\partial x}\right) \end{cases}$$

$$\begin{cases} \dfrac{\partial\sigma_x}{\partial x}+\dfrac{\partial\tau_{yx}}{\partial y}+\dfrac{\partial\tau_{zx}}{\partial z}+f_x=0 \\ \dfrac{\partial\tau_{xy}}{\partial x}+\dfrac{\partial\sigma_y}{\partial y}+\dfrac{\partial\tau_{zy}}{\partial z}+f_y=0 \\ \dfrac{\partial\tau_{xz}}{\partial x}+\dfrac{\partial\tau_{yz}}{\partial y}+\dfrac{\partial\sigma_z}{\partial z}+f_z=0 \end{cases}$$

Displacement formula

$$\begin{cases} (\lambda+G)\dfrac{\partial\varepsilon}{\partial x}+G\nabla^2 u+f_x=0 \\ (\lambda+G)\dfrac{\partial\varepsilon}{\partial y}+G\nabla^2 v+f_y=0 \\ (\lambda+G)\dfrac{\partial\varepsilon}{\partial z}+G\nabla^2 w+f_z=0 \end{cases}$$

- Elastic deformation satisfies the superposition principle
  - i.e., it is a linear system
  - The dependent variables (因变量, stress, strain, displacement) determined due to each set of external loads (body force, surface force, and prescribed surface displacement) acting separately, may be superposed to give the total values due to the combined external loads.

Principle of Superposition of Elastic Problems

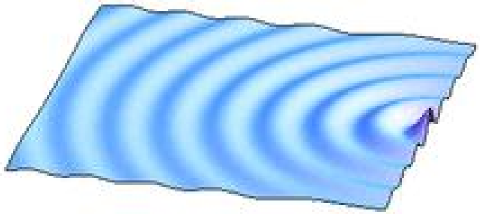
Let σ_x, ... τ_{xz} denote the stress components satisfying the governing equations and prescribed boundary conditions for a certain elastic body under body forces Fx, Fy, Fz, and surface forces, T_x^{μ},....

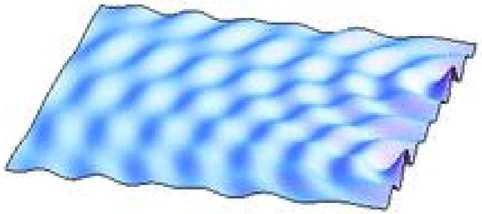
Furthermore, let the stress components in the same body under Fx', Fy', Fz', and surface forces, T_x^{μ}',.... denoted by σ_x', ... τ_{xz}'.

The two stress fields may be superposed to yield the results for combined loads.

Principle of Superposition of Elastic Problems

- Examples that the superposition principle applies
  - the superposition principle of gravitational field (引力场)
    - If several point particles (质点) exist at the same time, their gravitational fields are superimposed on each other, forming a combined gravitational field
  - the superposition principle of electric fields (电场)
    - If several electric charges (电荷) exist at the same time, their electric fields are superimposed on each other, forming a combined electric field.
  - the superposition principle of waves
    - when multiple waves meet at a point, the resultant displacement at that point is the algebraic sum of the displacements produced by each wave individually.
- Examples that the superposition principle cannot apply
  - Fluid motion (流体运动), finite strain (有限应变)

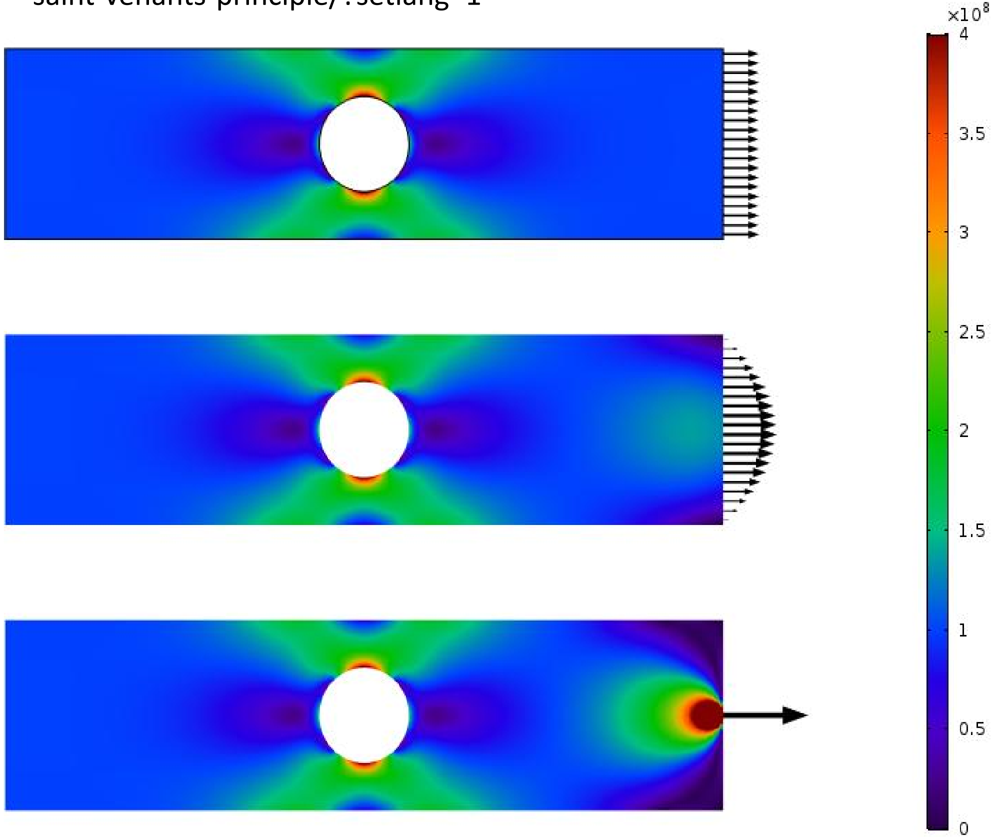




Saint-Venant’s principle

- The stresses due to two statically equivalent loadings (静态等效载荷) applied over a small area are significantly different only in the vicinity of the loading area
- The difference decreases with $O(L^2)$ to $O(L^3)$
- We can replace complicated stress distributions or weak boundary conditions with simple ones,
  - if the boundary is geometrically short.
  - The stress boundary condition can be simplified.

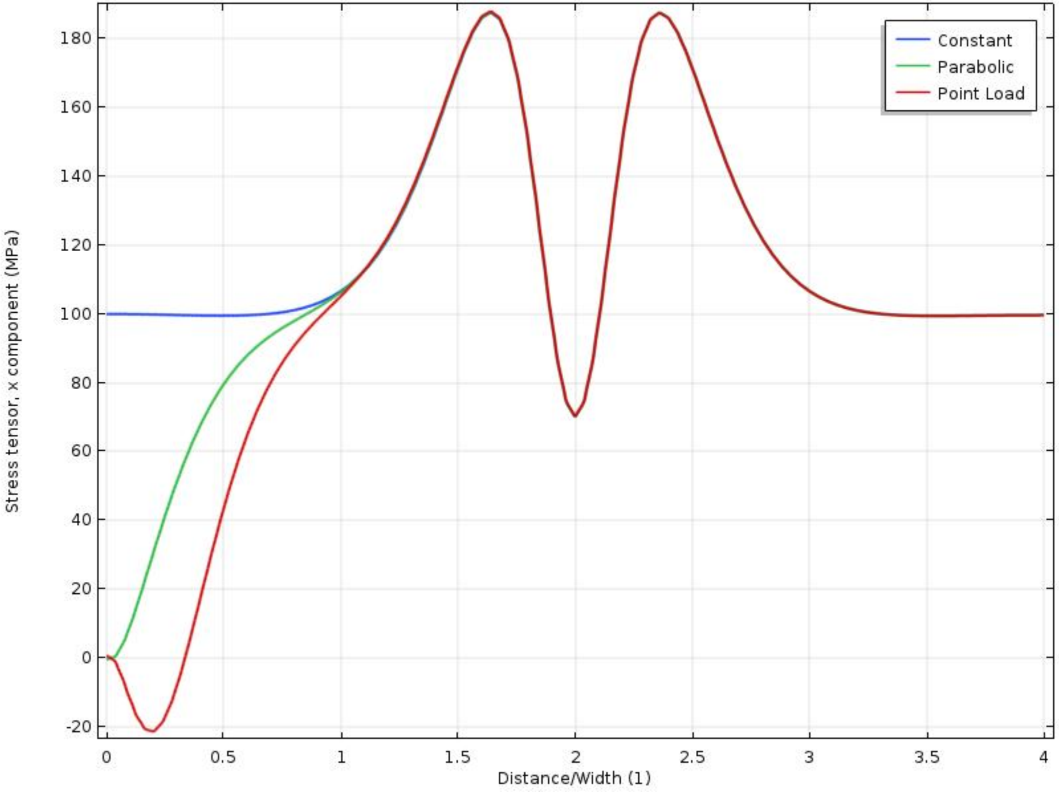
https://www.comsol.it/blogs/applying-and-interpreting-saint-venants-principle/?setlang=1



Von Mises stress contours for three load cases.

Saint-Venant’s principle

Stress along the upper edge as a function of the distance from the loaded boundary.



The distance is normalized by the width of the plate.

Classroom exercise

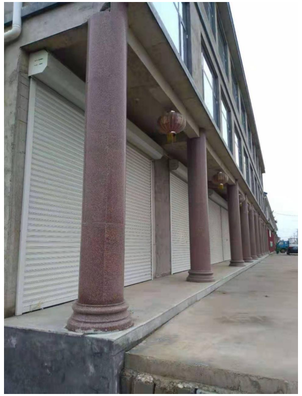
For a linear differential equation under the initial and boundary conditions:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - a^2 \dfrac{\partial^2 u}{\partial x^2} = f(x,t), \quad x \in (0,L),\ t \in (0,\infty) \\ u(x,t=0) = \varphi(x) \\ u(x=0) = 0 \\ u(x=L) = b(t) \end{cases}$$

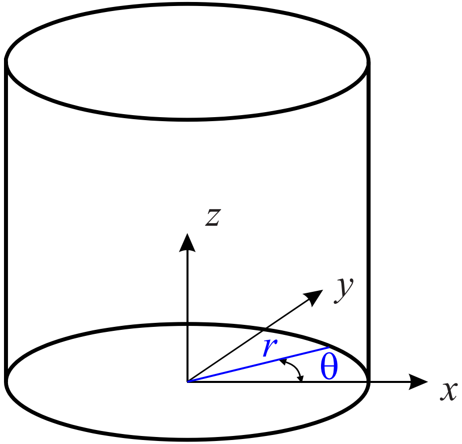
If function u_1, u_2, u_3 are the solutions of the following three problems, prove that $u=u_1 + u_2 + u_3$ is the solution to the left problem

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - a^2 \dfrac{\partial^2 u}{\partial x^2} = f(x,t) \\ u(x,t=0) = 0 \\ u(x=0) = 0 \\ u(x=L) = 0 \end{cases} \qquad \begin{cases} \dfrac{\partial^2 u}{\partial t^2} - a^2 \dfrac{\partial^2 u}{\partial x^2} = 0 \\ u(x,t=0) = \varphi(x) \\ u(x=0) = 0 \\ u(x=L) = 0 \end{cases} \qquad \begin{cases} \dfrac{\partial^2 u}{\partial t^2} - a^2 \dfrac{\partial^2 u}{\partial x^2} = 0 \\ u(x,t=0) = 0 \\ u(x=0) = 0 \\ u(x=L) = b(t) \end{cases}$$

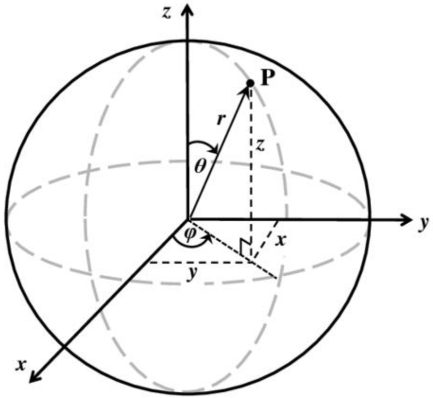
Polar Coordinates in 2D Problems



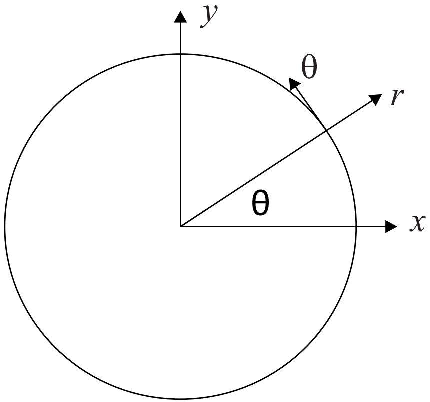
- Orthogonal curvilinear coordinates (正交曲线坐标系):
  - 3D
    - 3D Cartesian coordinates
    - Cylindrical coordinates, r, θ, z
    - Spherical coordinates, r, θ, ϕ
  - 2D
    - 2D Cartesian coordinates
    - **Polar coordinates, r, θ**



Cylindrical coordinate system



Spherical coordinate system



Polar coordinate system

Polar Coordinates in 2D Problems

- We must employ different governing equations for different coordinate systems.
- 4 ways to derive governing equations for any curvilinear coordinate
  - The use of transformations from Cartesian equations
    - first find the functional relations between the particular coordinates and the Cartesian coordinates
    - then tansform all our equations in Cartesian coordinates accordingly
    - Usually **very tedious** and lengthy 非常冗长枯燥，但符号运算可能可以帮助
  - Derive every equations in the particular coordinate system from basic physical laws
    - derive the strain, stress, and displacement relations for an infinitesimal element in the particular coordinate
  - The vector (and dyadic) approach
    - we write all equations in vector form
    - has the advantage of being independent of coordinates selected.
    - need to find the expressions for different vector operators in that system
  - The general tensor approach
    - all equations are derived in tensor (general tensor, not Cartesian tensor) form, which is applicable to any (orthogonal or nonorthogonal) coordinate system.

Polar Coordinates in 2D Problems

- For relations which do not contain any space derivatives, the equations in orthogonal curvilinear coordinats can be obtained directly from the corresponding equations in Cartesian coordinates by changing x, y into, say, r, θ.

Plane strain:

Hooke's law:

$$\begin{cases} \sigma_r = 2G\varepsilon_r + \lambda\left(\varepsilon_r + \varepsilon_\theta\right) \\ \sigma_\theta = 2G\varepsilon_\theta + \lambda\left(\varepsilon_r + \varepsilon_\theta\right) \\ \tau_{r\theta} = G\gamma_{r\theta} \end{cases}$$

Stress transformation:

Cartesian to polar

$$\begin{cases} \sigma_r = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\tau_{xy}\sin\theta\cos\theta \\ \sigma_\theta = \sigma_x \sin^2\theta + \sigma_y \cos^2\theta - 2\tau_{xy}\sin\theta\cos\theta \\ \tau_{r\theta} = \left(\sigma_y - \sigma_x\right)\sin\theta\cos\theta + \tau_{xy}\left(\cos^2\theta - \sin^2\theta\right) \end{cases}$$

Polar to Cartesian

$$\begin{cases} \sigma_x = \sigma_r \cos^2\theta + \sigma_\theta \sin^2\theta - 2\tau_{r\theta}\sin\theta\cos\theta \\ \sigma_y = \sigma_r \sin^2\theta + \sigma_\theta \cos^2\theta + 2\tau_{r\theta}\sin\theta\cos\theta \\ \tau_{xy} = \left(\sigma_r - \sigma_\theta\right)\sin\theta\cos\theta + \tau_{r\theta}\left(\cos^2\theta - \sin^2\theta\right) \end{cases}$$

Polar Coordinates in 2D Problems--transformation from Cartesian equations

Governing equations for plane elastic problems in Cartesian coordinates

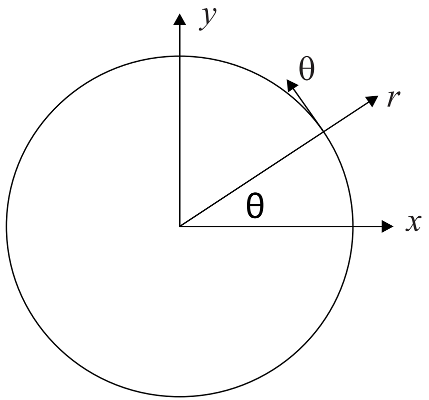
Equilibrium equations $$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + f_x = 0 \ (x, y)$$

Hooke's law $$\varepsilon_x = \frac{1}{E}(\sigma_x - \nu \sigma_y) \quad (\mathrm{x}, \mathrm{y}) \qquad \gamma_{xy} = \frac{1}{G}\tau_{xy}$$

Strain-displacement $$\varepsilon_x = \frac{\partial u}{\partial x} \quad \varepsilon_y = \frac{\partial v}{\partial y} \quad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

?

Governing equations in polar coordinates



Polar coordinates

Polar Coordinates in 2D Problems--transformation from Cartesian equations

We first use coordinate transformation to get Strain-displacement equations in polar coordinate systems

$$\begin{cases} \varepsilon_r = \varepsilon_x \cos^2\theta + \varepsilon_y \sin^2\theta + \gamma_{xy} \sin\theta\cos\theta \\ \varepsilon_\theta = \varepsilon_x \sin^2\theta + \varepsilon_y \cos^2\theta - \gamma_{xy} \sin\theta\cos\theta \\ \gamma_{r\theta} = 2\left(\varepsilon_y - \varepsilon_x\right)\sin\theta\cos\theta + \gamma_{xy}\left(\cos^2\theta - \sin^2\theta\right) \end{cases}$$

where

$$\varepsilon_x = \frac{\partial u}{\partial x} \qquad \varepsilon_y = \frac{\partial v}{\partial y} \qquad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

$$\boxed{\begin{cases} \varepsilon_r = \dfrac{\partial u_r}{\partial r} \\ \varepsilon_\theta = \dfrac{u_r}{r} + \dfrac{1}{r}\dfrac{\partial u_\theta}{\partial \theta} \\ \gamma_{r\theta} = \dfrac{1}{r}\dfrac{\partial u_r}{\partial \theta} + \dfrac{\partial u_\theta}{\partial r} - \dfrac{u_\theta}{r} \end{cases}}$$

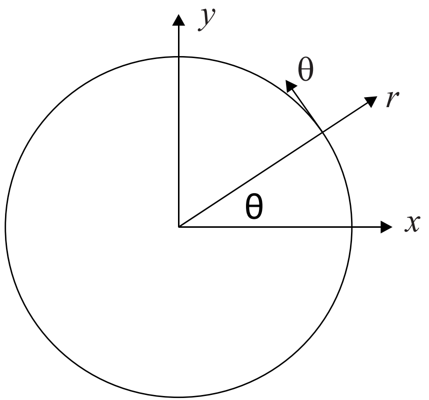
Strain-displacement equations in polar coordinate system

Polar Coordinates in 2D Problems--transformation from Cartesian equations

relations between polar and cartesian coordinates

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \qquad \begin{cases} \theta = \theta(x,y) = \tan^{-1}\dfrac{y}{x} \\ r^2 = x^2 + y^2 \end{cases}$$

$$\begin{cases} \dfrac{\partial r}{\partial x} = \dfrac{x}{r} = \cos\theta \\[2ex] \dfrac{\partial r}{\partial y} = \dfrac{y}{r} = \sin\theta \end{cases} \qquad \begin{cases} \dfrac{\partial \theta}{\partial x} = -\dfrac{y}{r^2} = -\dfrac{\sin\theta}{r} \\[2ex] \dfrac{\partial \theta}{\partial y} = \dfrac{x}{r^2} = \dfrac{\cos\theta}{r} \end{cases}$$



Polar coordinates

Any derivatives with respect to x and y can be transformed into derivatives with respect to r and θ

$$\begin{cases} \dfrac{\partial}{\partial x} = \dfrac{\partial r}{\partial x}\dfrac{\partial}{\partial r} + \dfrac{\partial \theta}{\partial x}\dfrac{\partial}{\partial \theta} = \cos\theta\dfrac{\partial}{\partial r} - \dfrac{\sin\theta}{r}\dfrac{\partial}{\partial \theta} \\[2ex] \dfrac{\partial}{\partial y} = \dfrac{\partial r}{\partial y}\dfrac{\partial}{\partial r} + \dfrac{\partial \theta}{\partial y}\dfrac{\partial}{\partial \theta} = \sin\theta\dfrac{\partial}{\partial r} + \dfrac{\cos\theta}{r}\dfrac{\partial}{\partial \theta} \end{cases}$$

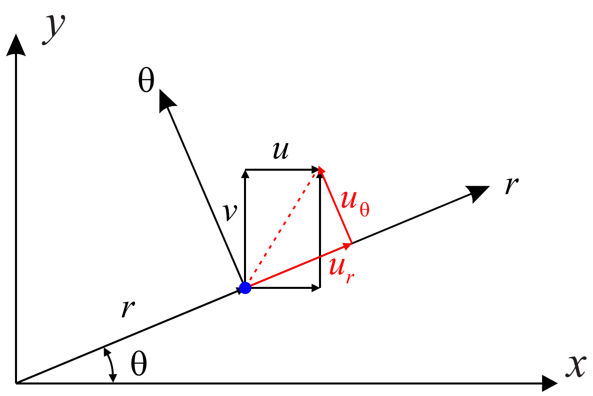
Polar Coordinates in 2D Problems--transformation from Cartesian equations

relations between displacement components

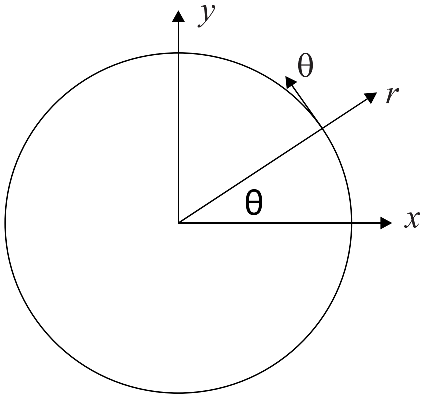
$$u_r = \mathbf{u}\cdot\hat{\mathbf{r}} = (u\hat{\mathbf{x}} + v\hat{\mathbf{y}})\cdot\hat{\mathbf{r}} = u\cos\theta + v\sin\theta$$

$$u_\theta = u\cos(\theta + \frac{\pi}{2}) + v\sin(\theta + \frac{\pi}{2}) = -u\sin\theta + v\cos\theta$$

$$\begin{cases} u = u_r\cos\theta - u_\theta\sin\theta \\ v = u_r\sin\theta + u_\theta\cos\theta \end{cases}$$



Displacement components



Polar coordinates

Polar Coordinates in 2D Problems--transformation from Cartesian equations

Strain-displacement equations in polar coordinate system

$$\begin{cases} \varepsilon_r = \varepsilon_x \cos^2\theta + \varepsilon_y \sin^2\theta + \gamma_{xy} \sin\theta\cos\theta \\ \varepsilon_\theta = \varepsilon_x \sin^2\theta + \varepsilon_y \cos^2\theta - \gamma_{xy} \sin\theta\cos\theta \\ \gamma_{r\theta} = 2(\varepsilon_y - \varepsilon_x)\sin\theta\cos\theta + \gamma_{xy}(\cos^2\theta - \sin^2\theta) \end{cases}$$

where

$$\varepsilon_x = \frac{\partial u}{\partial x} \qquad \varepsilon_y = \frac{\partial v}{\partial y} \qquad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

and $\begin{cases} u = u_r\cos\theta - u_\theta \sin\theta \\ v = u_r \sin\theta + u_\theta\cos\theta \end{cases}$

$$\varepsilon_x = \left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)u \qquad \varepsilon_y = \left(\sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\right)v$$

$$\gamma_{xy} = \left(\sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\right)u + \left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)v$$

$$\varepsilon_x = \left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)(u_r\cos\theta - u_\theta\sin\theta)$$
$$= \cos^2\theta\frac{\partial u_r}{\partial r} + \sin^2\theta\left(\frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial\theta}\right) - \cos\theta\sin\theta\left(\frac{\partial u_\theta}{\partial r} + \frac{1}{r}\frac{\partial u_r}{\partial\theta} - \frac{u_\theta}{r}\right)$$

$$\varepsilon_y = \left(\sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\right)(u_r\sin\theta + u_\theta\cos\theta)$$
$$= \sin^2\theta\frac{\partial u_r}{\partial r} + \cos^2\theta\left(\frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial\theta}\right) + \cos\theta\sin\theta\left(\frac{\partial u_\theta}{\partial r} + \frac{1}{r}\frac{\partial u_r}{\partial\theta} - \frac{u_\theta}{r}\right)$$

$$\gamma_{xy} = \left(\sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\right)(u_r\cos\theta - u_\theta\sin\theta) + \left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)(u_r\sin\theta + u_\theta\cos\theta)$$
$$= \sin^2\theta\left(\frac{\partial u_r}{\partial r} - \frac{1}{r}\frac{\partial u_\theta}{\partial\theta} - \frac{u_r}{r}\right) + \cos^2\theta\left(\frac{\partial u_\theta}{\partial\theta} + \frac{1}{r}\frac{\partial u_r}{\partial\theta} - \frac{u_\theta}{r}\right)$$

$$\boxed{\begin{cases} \varepsilon_r = \dfrac{\partial u_r}{\partial r} \\ \varepsilon_\theta = \dfrac{u_r}{r} + \dfrac{1}{r}\dfrac{\partial u_\theta}{\partial\theta} \\ \gamma_{r\theta} = \dfrac{1}{r}\dfrac{\partial u_r}{\partial\theta} + \dfrac{\partial u_\theta}{\partial r} - \dfrac{u_\theta}{r} \end{cases}}$$

Strain-displacement equations in polar coordinate system

$$\varepsilon = \varepsilon_r + \varepsilon_\theta = \frac{1}{r}\left[\frac{\partial(ru_r)}{\partial r} + \frac{\partial u_\theta}{\partial\theta}\right]$$

Polar Coordinates in 2D Problems--based on elemental analysis

We now derive the equilibrium equations **based on basic physical laws**

- Force in *r* direction:

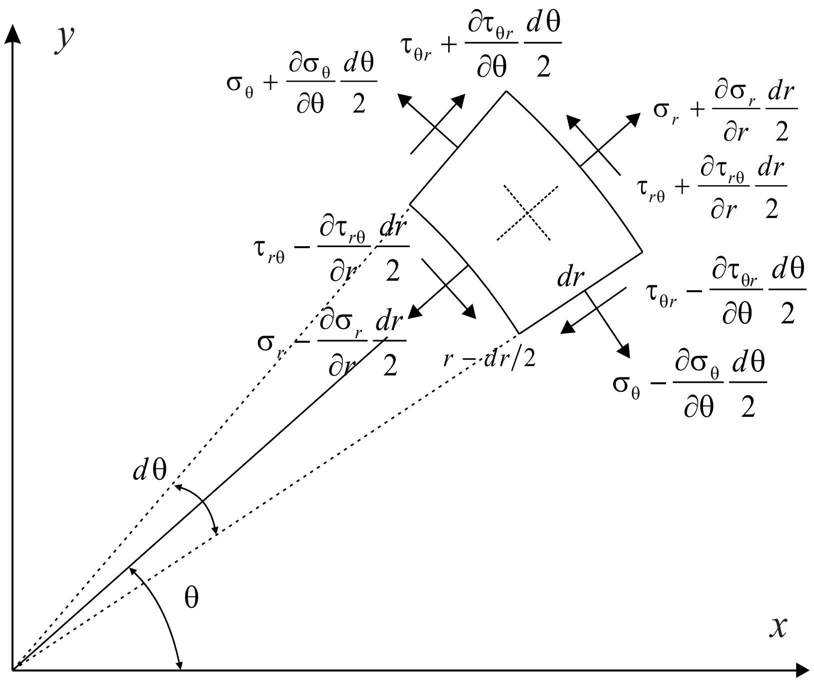
$$\left(\sigma_r+\frac{\partial\sigma_r}{\partial r}\frac{dr}{2}\right)\left(r+\frac{dr}{2}\right)d\theta-\left(\sigma_r-\frac{\partial\sigma_r}{\partial r}\frac{dr}{2}\right)\left(r-\frac{dr}{2}\right)d\theta$$
$$+\left(\tau_{\theta r}+\frac{\partial\tau_{\theta r}}{\partial\theta}\frac{d\theta}{2}\right)dr\cos\frac{d\theta}{2}-\left(\tau_{\theta r}-\frac{\partial\tau_{\theta r}}{\partial\theta}\frac{d\theta}{2}\right)dr\cos\frac{d\theta}{2}$$
$$-\left(\sigma_\theta+\frac{\partial\sigma_\theta}{\partial\theta}\frac{d\theta}{2}\right)dr\sin\frac{d\theta}{2}-\left(\sigma_\theta-\frac{\partial\sigma_\theta}{\partial\theta}\frac{d\theta}{2}\right)dr\sin\frac{d\theta}{2}+f_r r\,d\theta\,dr=0$$

Equilibrium equation in r direction

$$\frac{\partial\sigma_r}{\partial r}+\frac{1}{r}\frac{\partial\tau_{\theta r}}{\partial\theta}+\frac{\sigma_r-\sigma_\theta}{r}+f_r=0$$

- Force in θ direction:

$$\left(\tau_{r\theta}+\frac{\partial\tau_{r\theta}}{\partial r}\frac{dr}{2}\right)\left(r+\frac{dr}{2}\right)d\theta-\left(\tau_{r\theta}-\frac{\partial\tau_{r\theta}}{\partial r}\frac{dr}{2}\right)\left(r-\frac{dr}{2}\right)d\theta$$
$$+\left(\sigma_\theta+\frac{\partial\sigma_\theta}{\partial\theta}\frac{d\theta}{2}\right)dr\cos\frac{d\theta}{2}-\left(\sigma_\theta-\frac{\partial\sigma_\theta}{\partial\theta}\frac{d\theta}{2}\right)dr\cos\frac{d\theta}{2}$$
$$+\left(\tau_{\theta r}+\frac{\partial\tau_{\theta r}}{\partial\theta}\frac{d\theta}{2}\right)dr\sin\frac{d\theta}{2}+\left(\tau_{\theta r}-\frac{\partial\tau_{\theta r}}{\partial\theta}\frac{d\theta}{2}\right)dr\sin\frac{d\theta}{2}+f_\theta r\,dr\,d\theta=0$$



Stress on an element in polar coordinates

Equilibrium equation in θ direction

$$\frac{1}{r}\frac{\partial\sigma_\theta}{\partial\theta}+\frac{\partial\tau_{r\theta}}{\partial r}+\frac{2\tau_{r\theta}}{r}+f_\theta=0$$

Polar Coordinates in 2D Problems--displacement and stress formulations

strain-displacement equations

$$\begin{cases} \varepsilon_r = \dfrac{\partial u_r}{\partial r} \\ \varepsilon_\theta = \dfrac{u_r}{r} + \dfrac{1}{r}\dfrac{\partial u_\theta}{\partial \theta} \\ \gamma_{r\theta} = \dfrac{1}{r}\dfrac{\partial u_r}{\partial \theta} + \dfrac{\partial u_\theta}{\partial r} - \dfrac{u_\theta}{r} \end{cases}$$

Equilibrium equation

$$\frac{\partial \sigma_r}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta r}}{\partial \theta} + \frac{\sigma_r - \sigma_\theta}{r} + f_r = 0$$ r direction

$$\frac{1}{r}\frac{\partial \sigma_\theta}{\partial \theta} + \frac{\partial \tau_{r\theta}}{\partial r} + \frac{2\tau_{r\theta}}{r} + f_\theta = 0$$ θ direction

Plane strain Hooke's law:

$$\begin{cases} \sigma_r = 2G\varepsilon_r + \lambda(\varepsilon_r + \varepsilon_\theta) \\ \sigma_\theta = 2G\varepsilon_\theta + \lambda(\varepsilon_r + \varepsilon_\theta) \\ \tau_{r\theta} = G\gamma_{r\theta} \end{cases}$$

Displacement formulation (plane strain):

$$(\lambda + 2G)\frac{\partial \varepsilon}{\partial r} - \frac{2G}{r}\frac{\partial \omega}{\partial \theta} + f_r = 0$$

$$(\lambda + 2G)\frac{1}{r}\frac{\partial \varepsilon}{\partial \theta} + 2G\frac{\partial \omega}{\partial r} + f_\theta = 0$$

Stress formulation (plane strain):

Compatibility equation:

$$\frac{\partial^2 \varepsilon_\theta}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 \varepsilon_r}{\partial \theta^2} + \frac{2}{r}\frac{\partial \varepsilon_\theta}{\partial r} - \frac{1}{r}\frac{\partial \varepsilon_r}{\partial r} = \frac{1}{r}\frac{\partial^2 \gamma_{r\theta}}{\partial r \partial \theta} + \frac{1}{r^2}\frac{\partial \gamma_{r\theta}}{\partial \theta}$$

Introduce the stress function ϕ

$$\sigma_r = \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} \qquad \sigma_\theta = \frac{\partial^2 \phi}{\partial r^2} \qquad \tau_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \phi}{\partial \theta}\right)$$

We get compatibility equation in terms of stress function:

$$\nabla^4 \phi = \nabla^2 \nabla^2 \phi = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\phi = 0$$

Derive the stress function formulation directly

Classroom exercise

- Show that the lateral strain ε_θ accompanying an uplift Δy of large areas on a circle is given by

$$\varepsilon_\theta=\Delta y/R$$

where R is the radius of the circle.

- Show that the lateral surface strain ε_s accompanying an uplift Δy of large areas on a sphere is given by

$$\varepsilon_s=\Delta S/S=2\Delta y/R$$

where R is the radius of the sphere.

Classroom exercise

4-6 An irregularly shaped body is subjected to a constant pressure p at all points on its surface. Determine the stress, strain, and displacement components in the body. Assume that the displacement and rotation components at the origin are zero. Hint: Make a reasonable assumption regarding the stress components, and show that the assumed stresses satisfy the governing equations and the boundary conditions.

Stress formulation is more appropriate

$$\nabla^2\sigma_x + \frac{1}{(1+\nu)}\frac{\partial^2\Theta}{\partial x^2} = 0 \qquad (x, y, z)$$

$$\nabla^2\tau_{yz} + \frac{1}{(1+\nu)}\frac{\partial^2\Theta}{\partial y\partial z} = 0 \qquad (x, y, z)$$

$$\frac{\partial\sigma_x}{\partial x} + \frac{\partial\tau_{yx}}{\partial y} + \frac{\partial\tau_{zx}}{\partial z} + f_x = 0 \qquad (x, y, z)$$

boundary condition

$$\begin{cases} T_x^{\mu} = \sigma_{x|0}\mu_x + \tau_{yx|0}\mu_y + \tau_{zx|0}\mu_z \\ T_y^{\mu} = \tau_{xy|0}\mu_x + \sigma_{y|0}\mu_y + \tau_{zy|0}\mu_z \\ T_z^{\mu} = \tau_{xz|0}\mu_x + \tau_{yz|0}\mu_y + \sigma_{z|0}\mu_z \end{cases}$$