MAE5009

Continuum Mechanics B

Session 05: Two-Dimensional Problems

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Fall 2021

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$$\begin{cases} \sigma_z = \tau_{xz} = \tau_{yz} = 0 \\ \varepsilon_z \neq 0 \end{cases}$$

Plane stress problems

- Displacements (2) $\begin{cases} u = u(x,y) \\ v = v(x,y) \\ w = w(x,y) \end{cases}$

- Strain-displacement (3)

$$\varepsilon_x = \frac{\partial u}{\partial x} \quad \varepsilon_y = \frac{\partial v}{\partial y} \quad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

$$\gamma_{yz} = \gamma_{xz} = 0 \quad \varepsilon_z = -\frac{\nu}{E}(\sigma_x + \sigma_y)$$

- Stress-strain relations (3)

$$\sigma_x = 2G\varepsilon_x + \lambda(\varepsilon_x + \varepsilon_y + \varepsilon_z) \quad \tau_{xy} = G\gamma_{xy}$$

$$\sigma_y = 2G\varepsilon_y + \lambda(\varepsilon_x + \varepsilon_y + \varepsilon_y) \quad \sigma_z = \tau_{xz} = \tau_{yz} = 0$$

- Equilibrium equations (2) $\sigma_z = 2G\varepsilon_z + \lambda(\varepsilon_x + \varepsilon_y + \varepsilon_z) = 0$ $\Rightarrow$

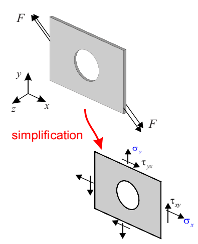
$$\underbrace{\frac{\partial \tau_{xz}}{\partial x}}_{=0} + \underbrace{\frac{\partial \tau_{yz}}{\partial y}}_{=0} + \underbrace{\frac{\partial \sigma_z}{\partial y}}_{=0} + f_z = 0$$

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + f_x = 0$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + f_y = 0$$

$f_z = 0$

No body force in z since all stresses in z are zero



For plane stress problems, σ_z vanishes, but ε_z does not

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$$\left.\begin{aligned} \varepsilon_x &= \frac{1}{E}(\sigma_x - \nu\sigma_y) \\ \varepsilon_y &= \frac{1}{E}(\sigma_y - \nu\sigma_x) \\ \varepsilon_z &= -\frac{\nu}{E}(\sigma_x + \sigma_y) \end{aligned}\right\}$$ 这两用这3个解出来

Solution of plane stress problems – stress formulation

- Displacements

$$\begin{cases} u = u(x,y) \\ v = v(x,y) \end{cases}$$

- Strain-displacement

$$\varepsilon_x = \frac{\partial u}{\partial x} \quad \varepsilon_y = \frac{\partial v}{\partial y} \quad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \rightarrow \frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}$$

- Stress-strain relations

$$\varepsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y) \quad \gamma_{xy} = \frac{1}{G}\tau_{xy}$$

$$\varepsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x)$$

- Equilibrium equations

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + f_x = 0$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + f_y = 0$$

$$\nabla^2(\sigma_x + \sigma_y) = -(1+\nu)\left(\frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y}\right)$$

Compatibility equation in terms of stress

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Solution of plane stress problems – stress formulation

$$\nabla^2(\sigma_x+\sigma_y) = -(1+\nu)\left(\frac{\partial f_x}{\partial x}+\frac{\partial f_y}{\partial y}\right)$$

Compatibility equation in terms of stress (plane stress)

$$\boxed{\nabla^2(\sigma_x+\sigma_y) = -\frac{1}{1-\nu}\left(\frac{\partial f_x}{\partial x}+\frac{\partial f_y}{\partial y}\right)}$$

Compatibility equation in terms of stress (plane strain)

For problems with no or constant body force intensities, the corresponding plane strain and plane stress problems are identical:

$$\nabla^2(\sigma_x+\sigma_y)=0$$

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Solution of plane stress problems – stress formulation

- If the body force intensity is conservative, there is a potential function V such that:

$$f_x = \frac{\partial V}{\partial x},\ f_y = \frac{\partial V}{\partial y}$$

- Let's introduce a stress function $\phi = \phi(x,y)$ (*Airy's stress function*):

$$\sigma_x + V = \frac{\partial^2\phi}{\partial y^2}$$

$$\sigma_y + V = \frac{\partial^2\phi}{\partial x^2}$$

$$\tau_{xy} = -\frac{\partial^2\phi}{\partial x\partial y}$$

The introduction of ϕ implies that the equilibrium equations are identically satisfied

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给出φ表达式，判断是否是有效的 stress formulation？代入 equilibrium

Solution of plane stress problems – stress formulation

- The compatibility equation in terms of stress becomes

$$\nabla^4\phi = \nabla^2(\nabla^2\phi) = (1-\nu)\nabla^2 V$$

where

$$\nabla^4 = \nabla^2(\nabla^2) = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2\partial y^2} + \frac{\partial^4}{\partial y^4}$$ *Biharmonic operator*

This is the governing field equation for plane stress problems in which the body forces are conservative

- If the body forces are constant, or if V is a harmonic function (i.e. $\nabla^2 V = 0$):

$$\nabla^4\phi = 0$$ *Biharmonic equation*

Then this equation is true for both plane strain and plane stress problems

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$(2G+\lambda)\varepsilon_z = -\lambda(\varepsilon_x+\varepsilon_y)$

$\varepsilon_z = -\frac{\lambda(\varepsilon_x+\varepsilon_y)}{2G+\lambda}$

$\sigma_z = 2G\varepsilon_z + \lambda(\varepsilon_x+\varepsilon_y+\varepsilon_z) = 0$

Solution of plane stress problems – displacement formulation

- Displacements

$$\begin{cases} u = u(x,y) \\ v = v(x,y) \end{cases}$$

- Strain-displacement

$$\varepsilon_x = \frac{\partial u}{\partial x} \quad \varepsilon_y = \frac{\partial v}{\partial y} \quad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

- Stress-strain relations

$$\sigma_x = 2G\varepsilon_x + \lambda(\varepsilon_x+\varepsilon_y+\varepsilon_z) \quad \tau_{xy} = G\gamma_{xy}$$
$$\sigma_y = 2G\varepsilon_y + \lambda(\varepsilon_x+\varepsilon_y+\varepsilon_z)$$

- Equilibrium equations

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + f_x = 0$$
$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + f_y = 0$$

$$\varepsilon_z = -\frac{\nu}{1-\nu}(\varepsilon_x+\varepsilon_y)$$

$$\begin{cases} \sigma_x = \dfrac{E}{1-\nu^2}(\varepsilon_x + \nu\varepsilon_y) \\ \sigma_y = \dfrac{E}{1-\nu^2}(\varepsilon_y + \nu\varepsilon_x) \end{cases}$$

$$G\nabla^2 u + \frac{E}{2(1-\nu)}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + f_x = 0$$
$$G\nabla^2 v + \frac{E}{2(1-\nu)}\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + f_y = 0$$

Equilibrium equations in terms of displacement

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Solution of plane stress problems – displacement formulation

$$G\nabla^2 u + \frac{E}{2(1-\nu)}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + f_x = 0$$
$$G\nabla^2 v + \frac{E}{2(1-\nu)}\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + f_y = 0$$

Equilibrium equation in terms of displacement (plane stress)

$$G\nabla^2 u + (\lambda+G)\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + f_x = 0$$
$$G\nabla^2 v + (\lambda+G)\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + f_y = 0$$

Equilibrium equation in terms of displacement (plane strain)

→ *Replace E and ν with E_1 and ν_1:* $E_1 = \frac{E}{1-\nu^2}, \nu_1 = \frac{\nu}{1-\nu}$

Replace E and ν with E_2 and ν_2: $E_2 = \frac{E(1+2\nu)}{(1+\nu)^2}, \nu_2 = \frac{\nu}{1+\nu}$ ←

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Solution of plane stress problems

- Equilibrium equations $\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + f_x = 0 \; (x,y)$
- Hooke's law $\sigma_x = 2G\varepsilon_x + \lambda(\varepsilon_x+\varepsilon_y+\varepsilon_z) \; (x,y) \quad \tau_{xy} = G\gamma_{xy}$
- Strain-displacement $\varepsilon_x = \frac{\partial u}{\partial x} \quad \varepsilon_y = \frac{\partial v}{\partial y} \quad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$
- unknowns $\sigma_x, \sigma_y, \tau_{xy}, \varepsilon_x, \varepsilon_y, \gamma_{xy}, u, v$

Displacement formulation

1. Strain-displacement
2. Stress-strain relations
3. Equilibrium equations (displacement)

$$G\nabla^2 u + \frac{E}{2(1-\nu)}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + f_x = 0 \; (u,v)$$

u, v

Stress formulation

1. Strain-stress relations
2. Compatibility equation (stress)
3. Equilibrium equations

$$\nabla^2(\sigma_x+\sigma_y) = -(1+\nu)\left(\frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y}\right) \; (\sigma_x, \sigma_y, \tau_{xy})$$

$\sigma_x, \sigma_y, \tau_{xy}$ if $f_x = \frac{\partial V}{\partial x}, f_y = \frac{\partial V}{\partial y}$ $\quad \nabla^4\phi = (1-\nu)\nabla^2 V$

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Approximate character of plane stress equations

- Assumptions:

$$\sigma_x = \sigma_x(x, y)$$
$$\sigma_y = \sigma_y(x, y)$$
$$\tau_{xy} = \tau_{xy}(x, y)$$
$$\tau_{xz} = \tau_{yz} = \sigma_z = 0$$

- If without body forces, the six equations for 3D in terms of stress become:

$$\begin{cases} \nabla^2\sigma_x + \dfrac{1}{1+\nu}\dfrac{\partial^2\Theta}{\partial x^2} = 0 \\ \nabla^2\sigma_y + \dfrac{1}{1+\nu}\dfrac{\partial^2\Theta}{\partial y^2} = 0 \\ \nabla^2\sigma_z + \dfrac{1}{1+\nu}\dfrac{\partial^2\Theta}{\partial z^2} = 0 \end{cases} \qquad \begin{cases} \nabla^2\tau_{yz} + \dfrac{1}{1+\nu}\dfrac{\partial^2\Theta}{\partial y\partial z} = 0 \\ \nabla^2\tau_{zx} + \dfrac{1}{1+\nu}\dfrac{\partial^2\Theta}{\partial x\partial z} = 0 \\ \nabla^2\tau_{xy} + \dfrac{1}{1+\nu}\dfrac{\partial^2\Theta}{\partial x\partial y} = 0 \end{cases}$$

- Equilibrium equations become:

$$\begin{cases} \dfrac{\partial\sigma_x}{\partial x} + \dfrac{\partial\tau_{yx}}{\partial y} + \dfrac{\partial\tau_{zx}}{\partial z} = 0 \\ \dfrac{\partial\tau_{xy}}{\partial x} + \dfrac{\partial\sigma_y}{\partial y} + \dfrac{\partial\tau_{zy}}{\partial z} = 0 \\ \dfrac{\partial\tau_{xz}}{\partial x} + \dfrac{\partial\tau_{yz}}{\partial y} + \dfrac{\partial\sigma_z}{\partial z} = 0 \end{cases}$$

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Approximate character of plane stress equations

- Introducing a stress function ϕ satisfying the first two equilibrium equations:

$$\sigma_x = \frac{\partial^2\phi}{\partial y^2}$$
$$\sigma_y = \frac{\partial^2\phi}{\partial x^2}$$
$$\tau_{xy} = -\frac{\partial^2\phi}{\partial x\partial y}$$

- They satisfy the biharmonic equation:

$$\nabla^4\phi = 0$$

- However, they cannot satisfy the first, second and sixth equation individually in the six equations for 3D in terms of stress

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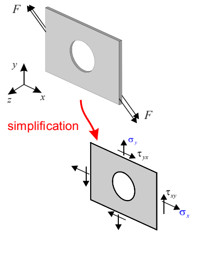
Restrictions of plane stress

- The body must be a thin plate
- The two *z* surfaces must be free from load
- The external forces have no *z* component
- Through the thickness of the plate, the external forces may be either uniformly distributed, or distributed symmetrically with respect to the middle plane



For thin plate, the error involved by the assumptions is negligible

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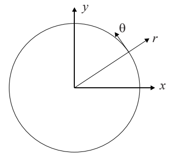
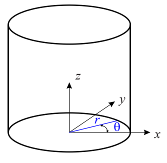
Polar coordinates in 2D problems

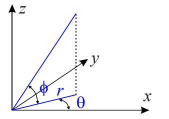
- Curvilinear orthogonal coordinates:
  - Polar coordinates, r, θ
  - Cylindrical coordinates, r, θ, z
  - Spherical coordinates, r, θ, ϕ



Polar coordinate system



Cylindrical coordinate system



Spherical coordinate system

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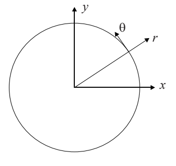
Polar coordinates in 2D problems

- **Polar coordinates**

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \qquad \begin{cases} \theta = \theta(x,y) = \tan^{-1}\dfrac{y}{x} \\ r^2 = x^2 + y^2 \end{cases}$$



$$\begin{cases} \dfrac{\partial r}{\partial x} = \dfrac{x}{r} = \cos\theta \\ \dfrac{\partial r}{\partial y} = \dfrac{y}{r} = \sin\theta \end{cases} \qquad \begin{cases} \dfrac{\partial \theta}{\partial x} = -\dfrac{y}{r^2} = -\dfrac{\sin\theta}{r} \\ \dfrac{\partial \theta}{\partial y} = \dfrac{x}{r^2} = \dfrac{\cos\theta}{r} \end{cases}$$

$$\boxed{\frac{d\left(\tan^{-1}x\right)}{dx} = \frac{1}{1+x^2}}$$

$$\begin{cases} \dfrac{\partial}{\partial x} = \dfrac{\partial r}{\partial x}\dfrac{\partial}{\partial r} + \dfrac{\partial \theta}{\partial x}\dfrac{\partial}{\partial \theta} = \cos\theta\dfrac{\partial}{\partial r} - \dfrac{\sin\theta}{r}\dfrac{\partial}{\partial \theta} \\ \dfrac{\partial}{\partial y} = \dfrac{\partial r}{\partial y}\dfrac{\partial}{\partial r} + \dfrac{\partial \theta}{\partial y}\dfrac{\partial}{\partial \theta} = \sin\theta\dfrac{\partial}{\partial r} + \dfrac{\cos\theta}{r}\dfrac{\partial}{\partial \theta} \end{cases}$$

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2D Polar solution based on coordinates transformation

- Relations governing properties at a point and which do not contain any space derivatives are not affected by the curvilinear nature of the coordinates:

  Stress transformation is not affected:

$$\begin{cases} \sigma_x = \sigma_r\cos^2\theta + \sigma_\theta\sin^2\theta - 2\tau_{r\theta}\sin\theta\cos\theta \\ \sigma_y = \sigma_r\sin^2\theta + \sigma_\theta\cos^2\theta + 2\tau_{r\theta}\sin\theta\cos\theta \\ \tau_{xy} = \left(\sigma_r - \sigma_\theta\right)\sin\theta\cos\theta + \tau_{r\theta}\left(\cos^2\theta - \sin^2\theta\right) \end{cases}$$

  Equilibrium equations:

$$\begin{cases} \dfrac{\partial \sigma_x}{\partial x} + \dfrac{\partial \tau_{yx}}{\partial y} = 0 \\ \dfrac{\partial \tau_{xy}}{\partial x} + \dfrac{\partial \sigma_y}{\partial y} = 0 \end{cases} \quad \xrightarrow{\text{Using chain rule}} \quad \begin{cases} \dfrac{\partial \sigma_r}{\partial r} + \dfrac{1}{r}\dfrac{\partial \tau_{r\theta}}{\partial \theta} + \dfrac{\sigma_r - \sigma_\theta}{r} = 0 \\ \dfrac{1}{r}\dfrac{\partial \sigma_\theta}{\partial \theta} + \dfrac{\partial \tau_{r\theta}}{\partial r} + \dfrac{2\tau_{r\theta}}{r} = 0 \end{cases}$$

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2D Polar solution based on coordinates transformation

Strain-displacement equations:

$$\begin{cases} \varepsilon_r = \dfrac{\partial u_r}{\partial r} \\ \varepsilon_\theta = \dfrac{u_r}{r} + \dfrac{1}{r}\dfrac{\partial u_\theta}{\partial \theta} \\ \gamma_{r\theta} = \dfrac{1}{r}\dfrac{\partial u_r}{\partial \theta} + \dfrac{\partial u_\theta}{\partial r} - \dfrac{u_\theta}{r} \end{cases}$$

Compatibility equations:

$$\frac{\partial^2 \varepsilon_\theta}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 \varepsilon_r}{\partial \theta^2} + \frac{2}{r}\frac{\partial \varepsilon_\theta}{\partial r} - \frac{1}{r}\frac{\partial \varepsilon_r}{\partial r} = \frac{1}{r}\frac{\partial^2 \gamma_{r\theta}}{\partial r \partial \theta} + \frac{1}{r^2}\frac{\partial \gamma_{r\theta}}{\partial \theta}$$

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2D Polar solution based on coordinates transformation

- Plane stress:

$$\begin{cases} \varepsilon_r = \dfrac{1}{E}(\sigma_r - \nu\sigma_\theta) \\ \varepsilon_\theta = \dfrac{1}{E}(\sigma_\theta - \nu\sigma_r) \\ \gamma_{r\theta} = \dfrac{1}{G}\tau_{r\theta} \end{cases}$$

$$\begin{cases} \sigma_r = \dfrac{E}{1-\nu^2}(\varepsilon_r + \nu\varepsilon_\theta) \\ \sigma_\theta = \dfrac{E}{1-\nu^2}(\varepsilon_\theta + \nu\varepsilon_r) \\ \tau_{r\theta} = G\gamma_{r\theta} \end{cases}$$

- Plane strain:

$$\begin{cases} \varepsilon_r = \dfrac{1-\nu^2}{E}\left(\sigma_r - \dfrac{\nu}{1-\nu}\sigma_\theta\right) \\ \varepsilon_\theta = \dfrac{1-\nu^2}{E}\left(\sigma_\theta - \dfrac{\nu}{1-\nu}\sigma_r\right) \\ \gamma_{r\theta} = \dfrac{1}{G}\tau_{r\theta} \end{cases}$$

$$\begin{cases} \sigma_r = 2G\varepsilon_r + \lambda(\varepsilon_r + \varepsilon_\theta) \\ \sigma_\theta = 2G\varepsilon_\theta + \lambda(\varepsilon_r + \varepsilon_\theta) \\ \tau_{r\theta} = G\gamma_{r\theta} \end{cases}$$

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2D Polar solution based on coordinates transformation

- Equilibrium equations – plane strain:

$$(\lambda + 2G)\frac{\partial \varepsilon}{\partial r} - \frac{2G}{r}\frac{\partial \omega}{\partial \theta} + f_r = 0$$

$$(\lambda + 2G)\frac{1}{r}\frac{\partial \varepsilon}{\partial \theta} + 2G\frac{\partial \omega}{\partial r} + f_\theta = 0$$

where

$$\omega = \frac{1}{2r}\left(\frac{\partial (r u_\theta)}{\partial r} - \frac{\partial u_r}{\partial \theta}\right)$$

- Compatibility equation in terms of stress function (no body force):

$$\nabla^4\phi = \nabla^2\nabla^2\phi = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\phi = 0$$

The stress components are given by

$$\sigma_r = \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} \qquad \sigma_\theta = \frac{\partial^2 \phi}{\partial r^2} \qquad \tau_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \phi}{\partial \theta}\right)$$

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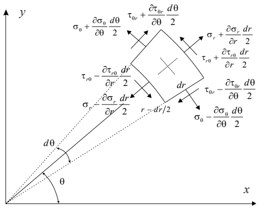
2D Polar solution based on physical laws – equilibrium eqs

- Force in *r* direction:

$$\left(\sigma_r+\frac{\partial\sigma_r}{\partial r}\frac{dr}{2}\right)\left(r+\frac{dr}{2}\right)d\theta-\left(\sigma_r-\frac{\partial\sigma_r}{\partial r}\frac{dr}{2}\right)\left(r-\frac{dr}{2}\right)d\theta$$
$$+\left(\tau_{\theta r}+\frac{\partial\tau_{\theta r}}{\partial\theta}\frac{d\theta}{2}\right)dr\cos\frac{d\theta}{2}-\left(\tau_{\theta r}-\frac{\partial\tau_{\theta r}}{\partial\theta}\frac{d\theta}{2}\right)dr\cos\frac{d\theta}{2}$$
$$-\left(\sigma_\theta+\frac{\partial\sigma_\theta}{\partial\theta}\frac{d\theta}{2}\right)dr\sin\frac{d\theta}{2}-\left(\sigma_\theta-\frac{\partial\sigma_\theta}{\partial\theta}\frac{d\theta}{2}\right)dr\sin\frac{d\theta}{2}+f_r r\,d\theta\,dr=0$$



Equilibrium equations

$$\frac{\partial\sigma_r}{\partial r}+\frac{1}{r}\frac{\partial\tau_{\theta r}}{\partial\theta}+\frac{\sigma_r-\sigma_\theta}{r}+f_r=0$$

- Force in θ direction:

$$\left(\tau_{r\theta}+\frac{\partial\tau_{r\theta}}{\partial r}\frac{dr}{2}\right)\left(r+\frac{dr}{2}\right)d\theta-\left(\tau_{r\theta}-\frac{\partial\tau_{r\theta}}{\partial r}\frac{dr}{2}\right)\left(r-\frac{dr}{2}\right)d\theta$$
$$+\left(\sigma_\theta+\frac{\partial\sigma_\theta}{\partial\theta}\frac{d\theta}{2}\right)dr\cos\frac{d\theta}{2}-\left(\sigma_\theta-\frac{\partial\sigma_\theta}{\partial\theta}\frac{d\theta}{2}\right)dr\cos\frac{d\theta}{2}$$
$$+\left(\tau_{\theta r}+\frac{\partial\tau_{\theta r}}{\partial\theta}\frac{d\theta}{2}\right)dr\sin\frac{d\theta}{2}+\left(\tau_{\theta r}-\frac{\partial\tau_{\theta r}}{\partial\theta}\frac{d\theta}{2}\right)dr\sin\frac{d\theta}{2}+f_\theta r\,dr\,d\theta=0$$

Equilibrium equations

$$\frac{1}{r}\frac{\partial\sigma_\theta}{\partial\theta}+\frac{\partial\tau_{r\theta}}{\partial r}+\frac{2\tau_{r\theta}}{r}+f_\theta=0$$

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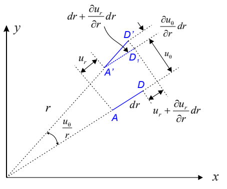
Strain-displacement equations in polar coordinate system

- Normal strain in *r* direction:

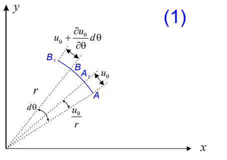


$$\varepsilon_r=\frac{A'D'-AD}{AD}=\frac{\sqrt{\left(dr+\frac{\partial u_r}{\partial r}dr\right)^2+\left(\frac{\partial u_\theta}{\partial r}dr\right)^2}-dr}{dr}=\frac{\partial u_r}{\partial r}$$

$$\gamma_{r\theta_1}=\frac{\frac{\partial u_\theta}{\partial r}dr}{dr}=\frac{\partial u_\theta}{\partial r}$$

- Normal strain in θ direction:

(1)



$$\varepsilon_{\theta_1}=\frac{A_1B_1-AB}{AB}=\frac{\left(u_\theta+\frac{\partial u_\theta}{\partial\theta}d\theta+rd\theta-u_\theta\right)-rd\theta}{rd\theta}=\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}$$

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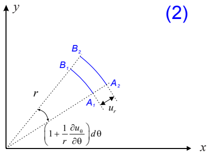
Strain-displacement equations in polar coordinate system

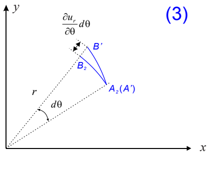
- Normal strain in θ direction:

(2)



$$\varepsilon_{\theta_2}=\frac{A_2B_2-A_1B_1}{A_1B_1}=\frac{(r+u_r)d\theta-rd\theta}{rd\theta}=\frac{u_r}{r}$$

(3)



$$\varepsilon_{\theta_3}=\frac{\sqrt{(rd\theta)^2+\left(\frac{\partial u_r}{\partial\theta}d\theta\right)^2}-rd\theta}{rd\theta}\cong 0$$

$$\gamma_{r\theta_3}=\frac{\frac{\partial u_r}{\partial\theta}d\theta}{rd\theta}=\frac{1}{r}\frac{\partial u_r}{\partial\theta}$$

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Strain-displacement equations in polar coordinate system

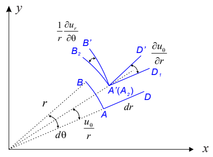
- Normal strain:

$$\varepsilon_r = \frac{\partial u_r}{\partial r}$$

$$\varepsilon_\theta = \varepsilon_{\theta_1} + \varepsilon_{\theta_2} + \varepsilon_{\theta_3} = \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}$$

- Shear strain:

$$\gamma_{r\theta} = \gamma_{r\theta_1} + \gamma_{r\theta_2} - \frac{u_\theta}{r} = \frac{\partial u_\theta}{\partial r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r}$$



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Axisymmetric plane problems

- All stress and displacement components are independent of θ

$$\gamma_{r\theta} = \gamma_{\theta r} = 0 \qquad \tau_{r\theta} = \tau_{\theta r} = 0$$

- Equilibrium equations:

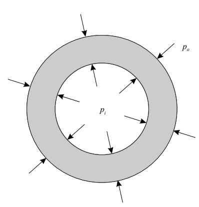
$$\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\theta}{r} = 0$$

- Strain components:

$$\varepsilon_r = \frac{du_r}{dr} \qquad \varepsilon_\theta = \frac{u_r}{r}$$

- Stress components:

$$\sigma_r = \frac{1}{r}\frac{d\phi}{dr} \qquad \sigma_\theta = \frac{d^2\phi}{dr^2}$$



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Axisymmetric plane problems

- Biharmonic equation:

$$\nabla^4\phi = \left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right)\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right)\phi = 0 \rightarrow \frac{d^4\phi}{dr^4} + \frac{2}{r}\frac{d^3\phi}{dr^3} - \frac{1}{r^2}\frac{d^2\phi}{dr^2} + \frac{1}{r^3}\frac{d\phi}{dr} = 0$$

If introduce: $r = e^t \rightarrow t = \log r$

$$\begin{cases} \dfrac{d\phi}{dr} = \dfrac{d\phi}{dt}\dfrac{dt}{dr} = \dfrac{1}{r}\dfrac{d\phi}{dt} \\ \dfrac{d^2\phi}{dr^2} = \dfrac{d}{dr}\left(\dfrac{d\phi}{dr}\right) = \dfrac{1}{r^2}\left(\dfrac{d^2\phi}{dt^2} - \dfrac{d\phi}{dt}\right) \\ \dfrac{d^3\phi}{dr^3} = \dfrac{d}{dr}\left(\dfrac{d^2\phi}{dr^2}\right) = \dfrac{1}{r^3}\left(\dfrac{d^3\phi}{dt^3} - 3\dfrac{d^2\phi}{dt^2} + 2\dfrac{d\phi}{dt}\right) \\ \dfrac{d^4\phi}{dr^4} = \dfrac{1}{r^4}\left(\dfrac{d^4\phi}{dt^4} - 6\dfrac{d^3\phi}{dt^3} + 11\dfrac{d^2\phi}{dt^2} - 6\dfrac{d\phi}{dt}\right) \end{cases} \rightarrow \frac{d^4\phi}{dt^4} - 4\frac{d^3\phi}{dt^3} + 4\frac{d^2\phi}{dt^2} = 0$$

General solution:

$$\phi = Ar^2\log r + Br^2 + C\log r + D$$

$$\sigma_r = \frac{1}{r}\frac{d\phi}{dr} = 2A\log r + \frac{C}{r^2} + A + 2B$$

$$\sigma_\theta = \frac{d^2\phi}{dr^2} = 2A\log r - \frac{C}{r^2} + 3A + 2B$$

$$\tau_{r\theta} = \tau_{\theta r} = 0$$

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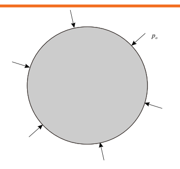
Axisymmetric plane problems

- If a region is simply connected:

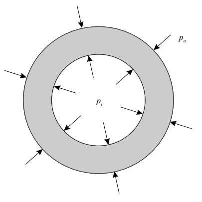
$$A = C = 0 \qquad \sigma_r = \sigma_\theta = 2B$$
$$\tau_{r\theta} = \tau_{\theta r} = 0$$

To make sure the stress remain finite at origin



- If a region is multiply connected, the compatibility equation is not sufficient to guarantee single-valued displacement, the displacement must be considered:

$$\varepsilon_r = \frac{du_r}{dr} \qquad \varepsilon_\theta = \frac{u_r}{r}$$



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Axisymmetric plane problems

- Based on Hooke's law, we have:

$$\begin{cases} \varepsilon_r = \dfrac{du_r}{dr} = K_1(\sigma_r - K_2\sigma_\theta) \\ \varepsilon_\theta = \dfrac{u_r}{r} = K_1(\sigma_\theta - K_2\sigma_r) \end{cases}$$

Plane stress: $K_1 = \dfrac{1}{E} \qquad K_2 = \nu$

Plane strain: $K_1 = \dfrac{1-\nu^2}{E} \qquad K_2 = \dfrac{\nu}{1-\nu}$

- For plane strain:

$$\begin{cases} \dfrac{du_r}{dr} = K_1\left(2A\log r + \dfrac{C}{r^2} + A + 2B - K_2\left(2A\log r - \dfrac{C}{r^2} + 3A + 2B\right)\right) \\ \dfrac{u_r}{r} = K_1\left(2A\log r - \dfrac{C}{r^2} + 3A + 2B - K_2\left(2A\log r + \dfrac{C}{r^2} + A + 2B\right)\right) \end{cases}$$

$$\Downarrow$$

$$4Ar - F = 0$$

Since the equation must be satisfied for all r in the region:

$$A = F = 0$$

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Axisymmetric plane problems

- B are C are to be determined from boundary conditions:

For displacement formulation:

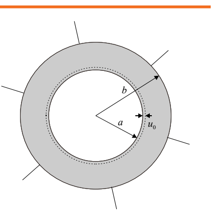
$$u_\theta = \omega = 0$$

$$\Downarrow$$

$$\frac{d^2u_r}{dr^2} + \frac{1}{r}\frac{du_r}{dr} - \frac{u_r}{r^2} = 0$$

$$\Downarrow$$

$$u_r(r) = c_1 r + c_2\frac{1}{r}$$



$$u_r\big|_{r=a} = u_0$$
$$u_r\big|_{r=b} = 0$$

$$\Downarrow$$

$$c_1 = \frac{a}{a^2-b^2}u_0 \qquad c_2 = \frac{-ab^2}{a^2-b^2}u_0$$

$$\Downarrow$$

$$u_r(r) = \frac{au_0}{a^2-b^2}\left(r - \frac{b^2}{r}\right)$$

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