THE PDE TO PRICE RANGE ACCRUAL NOTE*

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Abstract.

Key words.

AMS subject classifications.

1. Preliminary.

PROPOSITION 1.1. *Let $X = (X^1, X^2, \ldots, X^d)$ be a d-dimensional semi-martingale, $\overline{X}$ be the continuous part of X, and F be a twice continuously differentiable function. Then we have (formally)*

$$\begin{aligned} \mathrm{d}F(t, X_t) =& \partial_t F(t, X_t) + \sum_{i=1}^{d} \partial_{x^i} F(t, X_t)\, \mathrm{d}\overline{X}_t^i + \sum_{i,j=1}^{d} \frac{1}{2} \partial_{x^i} \partial_{x^j} F(t, X_t)\, \mathrm{d}[\overline{X}_t^i, \overline{X}_t^j] \\ &+ (F(t, X_t) - F(t, X_{t-})). \end{aligned} \tag{1.1}$$

2. Range Accrual Type Notes. Consider a certain interest rate denoted by $(F_t)_{t\geq 0}$. Let $(A_t)_{t\geq 0}$ denote the process to count the number of days that the interest rate is in a range $\mathcal{R}$. In other words, we can write

$$A_t = \sum_{k=1}^{N} \mathbb{1}_{t_k \leq t} \mathbb{1}_{F_{t_k} \in \mathcal{R}}, \tag{2.1}$$

where t_k represents each day. Then, we can formally have

$$\mathrm{d}A_t = \sum_{k=1}^{N} \mathbb{1}_{F_{t_k} \in \mathcal{R}} \boldsymbol{\delta}_{t_k}(\mathrm{d}t). \tag{2.2}$$

For simplicity, consider one factor short rate process $(r_t)_{t\geq 0}$ that follows

$$\mathrm{d}r_t = \mu(t, r_t)\, \mathrm{d}t + \sigma(t, r_t)\, \mathrm{d}W_t. \tag{2.3}$$

Let $\{V(t, r_t, A_t)\}_{0\leq t\leq T}$ be the value of a range accrual type note. Moreover, we denote

$$\begin{aligned} B_t \coloneqq& \exp\left(\int_0^t r_s\, \mathrm{d}s\right), \\ \tilde{V}(t, r_t, A_t) \coloneqq& B_t^{-1} V(t, r_t, A_t). \end{aligned}$$

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Then by applying Itô's formula to $\tilde{V}$, we have

$$\begin{aligned} \mathrm{d}\tilde{V}(t,r_t,A_t) =& B_t^{-1}\Big[-r_tV(t,r_t,A_t)+\partial_tV(t,r_t,A_t) \\ & +\partial_rV(t,r_t,A_t)\mu(t,r_t)+\frac{\sigma^2(t,r_t)}{2}\partial_r^2V(t,r_t,A_t)\Big]\,\mathrm{d}t \\ & +\sum_{k=1}^{N}(\tilde{V}(t,r_t,A_t)-\tilde{V}(t,r_t,A_{t-}))\mathbb{1}_{F_{t_k}\in\mathcal{R}}\boldsymbol{\delta}_{t_k}(\mathrm{d}t) \\ & +B_t^{-1}\sigma(t,r_t)\partial_rV(t,r_t,A_t)\,\mathrm{d}W_t. \end{aligned} \tag{2.4}$$

Note that $(A_t)_{t\geq 0}$ jumps only at $t=t_k$, $k=1,\dots,N$, and

$$\Delta_{t_k}A = A_{t_k}-A_{t_k-1}=\begin{cases}1 & \text{where } F_{t_k}\in\mathcal{R},\\ 0 & \text{otherwise.}\end{cases} \tag{2.5}$$

Therefore, where $F_{t_k}\in\mathcal{R}$, we have $A_{t_k}=A_{t_{k-1}}+1$. Moreover, we assume F_t can be represented by (t,r_t) and denote the range for (t,r_t) by $\mathscr{R}$. In other words,

$$F_t\in\mathcal{R}\Leftrightarrow(t,r_t)\in\mathscr{R}. \tag{2.6}$$

Then, by (2.4), so that $\tilde{V}$ is a martingale, it suffices to find a function $V(t,r,A)$ that satisfies

$$\begin{cases}\partial_tV(t,r,A)+\partial_rV(t,r,A)\mu(t,r)+\frac{\sigma^2(t,r)}{2}\partial_r^2V(t,r,A)=rV(t,r,A),\\ V(t,r,A+1)=V(t,r,A), \quad \text{where } (t,r)\in\mathscr{R}.\end{cases} \tag{2.7}$$