A Note on Development

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1 Payoff Decomposition

ELS Payoff, denoted by X is like:

$$
\begin{aligned}
X = &\sum_{i=1}^{N}\Big(\prod_{j<i}\mathbb{1}_{S_{T_j}<\mathcal{R}_j}\Big)\mathbb{1}_{S_{T_i}\geq\mathcal{R}_i}C_i + \sum_{i=1}^{L}\Big(\prod_{j<i}\mathbb{1}_{S_{T_j}<\mathcal{R}_j}\Big)\mathbb{1}_{S^*_{(T_{i-1},T_i]}\geq\mathcal{L}_i}C_i^L \\
&+\Big(\prod_{j<N}\mathbb{1}_{S_{T_j}<\mathcal{R}_j}\Big)\Big(\mathbb{1}_{S^*_T<\mathcal{B}}\phi(S_T)+\mathbb{1}_{S^*_T\geq\mathcal{B}}C^B\Big) \\
= &\sum_{i=1}^{N}\Big(\prod_{j<i}\mathbb{1}_{S_{T_j}<\mathcal{R}_j}\Big)\mathbb{1}_{S_{T_i}\geq\mathcal{R}_i}C_i + \sum_{i=1}^{L}\Big(\prod_{j<i}\mathbb{1}_{S_{T_j}<\mathcal{R}_j}\Big)\mathbb{1}_{S^*_{(T_{i-1},T_i]}\geq\mathcal{L}_i}C_i^L \\
&+\Big(\prod_{j<N}\mathbb{1}_{S_{T_j}<\mathcal{R}_j}\Big)\Big(\phi(S_T)+\mathbb{1}_{S^*_T\geq\mathcal{B}}(C^B-\phi(S_T))\Big) \\
= &\sum_{i=1}^{N-1}\Big(\prod_{j<i}\mathbb{1}_{S_{T_j}<\mathcal{R}_j}\Big)\mathbb{1}_{S_{T_i}\geq\mathcal{R}_i}C_i + \Big(\prod_{j<N-1}\mathbb{1}_{S_{T_j}<\mathcal{R}_j}\Big)\Big(\mathbb{1}_{S_{T_N}\geq\mathcal{R}_N}C_N+\mathbb{1}_{S_{T_N}<\mathcal{R}_N}\phi(S_T)\Big) \\
&+\sum_{i=1}^{L}\Big(\prod_{j<i}\mathbb{1}_{S_{T_j}<\mathcal{R}_j}\Big)\mathbb{1}_{S^*_{(T_{i-1},T_i]}\geq\mathcal{L}_i}C_i^L \\
&+\Big(\prod_{j<N}\mathbb{1}_{S_{T_j}<\mathcal{R}_j}\Big)\mathbb{1}_{S^*_T\geq\mathcal{B}}(C^B-\phi(S_T))
\end{aligned}
\tag{1.1}
$$

Hence there should be $L+2$ grids: one for ranges, L for the lizard barriers, one for the maturity barrier.

2 OIS Schedule

Let us denote the calculation period pair of fixing, value start, value end, payment dates by:

$$
I_i = \{\phi_i, \sigma_i, \epsilon_i\}. \tag{2.1}
$$

To explain the notation, we usually have $\epsilon_i - \sigma_i = \mathrm{Day}(1)$ in overnight index cases. These periods comprise the swap intervals $\{\{S_i, E_i, \Pi_i\}\}_i$ such that:

$$
(S_i, E_i] := (\sigma_{M_i}, \epsilon_{N_i}] = \cup(\sigma_k, \epsilon_k], \tag{2.2}
$$

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and Π_i represents the regarding payment date. Then, for a given evaluation date $t \geq 0$, find the earliest payment date and its valu end date:

$$t \mapsto \Pi_{i(t)} \mapsto E_{i(t)}. \tag{2.3}$$

The next step is to find the latest end date of the past data earlier than $E_{i(t)}$:

$$E_{i(t)} \geq \epsilon^*_{i(t)} := \{\epsilon_i \mid (\phi_i, \sigma_i, \epsilon_i) \text{ of the past data}\}. \tag{2.4}$$

In addition, find the earlist value start date (of the fixing data) and fixing date:

$$S_{i(t)} \mapsto (\phi^*_{i(t)}, \sigma^*_{i(t)}) \tag{2.5}$$

Then, the calculation starts from:

$$\sigma^*_{i(t)} \wedge S_{i(t)}. \tag{2.6}$$

For the forcasting, we obtain:

$$[1 + C(0, 0, E_{i(t)})(E_{i(t)} - \epsilon^*_{i(t)})] \tag{2.7}$$

For any case that the next value start date $S_{i(t)+1}$ needs a past data, we calculate:

$$C(t, S_i \vee 0, E_i)(E_i - S_i) \tag{2.8}$$

3 Local Volatility

3.1 No Dividends

Note that (S)SVI models represents the total variance with respect to:

$$y = \ln\Big(\frac{K}{F}\Big). \tag{3.1}$$

In other words:

$$w(y) = \sigma^2(T, Fe^k)T. \tag{3.2}$$

Recall the Dupire formula:

$$v_L = \frac{\partial_T w}{1 - \frac{y}{w}\partial_y w + \frac{1}{4}\Big(-\frac{1}{4} - \frac{1}{w} + \frac{y^2}{w^2}\Big)(\partial_y w)^2 + \frac{1}{2}\partial_y^2 w}. \tag{3.3}$$

Hence, recall that $\sigma_{BS}(T, Fe^y) = \sqrt{v_L(y)}$, in other words:

$$\sigma_{BS}(T, K) = \sqrt{v_L\Big[\ln\Big(\frac{K}{F}\Big)\Big]}. \tag{3.4}$$

For any $T < T_1$ and $T > T_N$, the possible extrapolation may be:

$$\sigma_{BS}\Big(T_1,\ K\frac{F_1}{F}\Big) \text{ and } \sigma_{BS}\Big(T_N,\ K\frac{F_N}{F}\Big) \tag{3.5}$$

3.2 Discrete Dividends

This recipe is borrowed from the section 2.3.1.2 in Bergomi (2015).

$$\begin{aligned}
\delta S(t) &:= \sum_{t_i<t} q_i \frac{t-t_i}{t} B_{t_i}^{-1} \\
\delta K(t) &:= \sum_{t_i<t} q_i \frac{t_i}{t} B_{t_i}^{-1} B_t \\
y &:= \ln\left(\frac{K+\Delta K(t)}{(S_0-\delta S(t))B_t}\right) \\
w(t,y) &:= \sigma_L^2(t,K)t.
\end{aligned}$$

Then, (3.3) implies directly, but:

$$\sigma_{BS}(t,K) = \sqrt{v_L\left[\ln\left(\frac{K+\Delta K(t)}{(S_0-\delta S(t))B_t}\right)\right]} \tag{3.6}$$

Moreover, the extrapolation accross time is:

$$\sigma_{BS}\left(t_1,\ e^{-y+y_1}\right) \text{ and } \sigma_{BS}\left(t_N,\ e^{-y+y_N}\right) \tag{3.7}$$

3.3 Referencing the Intial Price

Recall that the SDE of local volatility model follows:

$$\mathrm{d}S_t = r_t S_t\,\mathrm{d}t + \sigma_{BS}(t,S_t)S_t\,\mathrm{d}W_t - \mathrm{d}\mathfrak{D}_t \tag{3.8}$$

Let $X_t = S_t/S^*$ where S^* represents the reference price on the effective date. Then, we have:

$$\begin{aligned}
\mathrm{d}X_t =& r_t X_t\,\mathrm{d}t + \sigma_{BS}(t,X_tS^*)X_t\,\mathrm{d}W_t - \frac{1}{S^*}\mathrm{d}\mathfrak{D}_t \\
=& r_t X_t\,\mathrm{d}t + \sigma_{BS}(t,X_tS^*)X_t\,\mathrm{d}W_t - \sum \frac{q_i}{S^*}\delta_{t_i}(\mathrm{d}t)
\end{aligned} \tag{3.9}$$

4 Log BS with Discrete Dividends

Let $(S_t)_{t\geq 0}$ follow:

$$\begin{aligned}
\mathrm{d}S_t =& r_t S_t\,\mathrm{d}t + v_L\left(t, \ln\left(\frac{S_t}{F_0^t}\right)\right)^{\frac{1}{2}} S_t\,\mathrm{d}W_t - \mathrm{d}\mathfrak{D}_t \\
\mathfrak{D}_t =& \sum_{i\leq N} q_i \mathbb{1}_{T_i\leq t}.
\end{aligned}$$

We formally have:

$$\mathrm{d}\mathfrak{D}_t = \sum_{i\leq N} q_i \boldsymbol{\delta}_{T_i}(\mathrm{d}t). \tag{4.1}$$

Let $X = \ln(S/S^*)$. Then:

$$\mathrm{d}X_t = \left(r_t - \frac{\sigma_t^2}{2}\right)\mathrm{d}t + v_L\left(t, X_t + \ln\left(\frac{S^*}{F_0^t}\right)\right)^{\frac{1}{2}}\mathrm{d}W_t + \boldsymbol{\delta}_{T_i}(\mathrm{d}t)\left(\ln(e^{X_{t-}} - \frac{q_i}{S^*}) - \ln(e^{X_{t-}})\right) \tag{4.2}$$

Therefore:

$$
\begin{aligned}
X_{t+\Delta t} - X_t \approx & \left(r_t - \frac{\sigma_t^2}{2}\right)\Delta t + \sigma_t \Delta W_t + \ln\left(\frac{e^{X_t} - \sum \frac{q_i}{S^*} \mathbb{1}_{t<T_i\leq t+\Delta t}}{e^{X_t}}\right) \\
\sigma_t = & v_L\left(t, X_t + \ln\left(\frac{S^*}{F_0^t}\right)\right)^{\frac{1}{2}}
\end{aligned}
$$

References

Bergomi, L. (2015). *Stochastic volatility modeling*. CRC press.