# Interest Rate Modeling

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# Outline

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   - Derivation of PDE using martingale method

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    - Bond Price ⇒ forward rate fitting parameters to market curve, etc
    - Bond Option ⇒ Swaption price (Jamshidian decomposition)
    - Calibration by diagonal swaption price quote (in OTC database, refer to "otcora:FICC_SWAPTION_ATM")

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**A Crash Course on Semi-martingale Theory**

# Why Semi-martingale?

Denote each day by $\{\tau_1, \ldots, \tau_N\}$, and let $(A_t)_{t\geq 0}$ be the process counting the number of day that an interest rate $F$ is in a range $\mathcal{R}$:

$$A_t := \sum_{i=1}^{N} \mathbb{1}_{t\geq\tau_i} \mathbb{1}_{F_{\tau_i}\in\mathcal{R}}. \tag{1}$$

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Note that $\tau_i$ is a deterministic time, so it is not a Poisson process.

**Remark**

*If there exsits $\lambda \in \mathbb{R}$ such that $\{\mathbb{1}_{\tau\geq t} - \lambda t\}$ is a martingale, then $\lambda$ is called the intensity of $\tau$. For $\tau$ to have an intensity, $\tau$ must be a totally inaccessible stopping time. However, deterministic time is a predictable stopping time which is not totally inaccessible.*

It means that we do not have yet Itô's formula with the process. Even before Itô's formula, we first have to define stochastic integral with a good integrator including processes like $(A_t)_{t\geq 0}$.

# Semi-martingale (Warning! I said “Semi” not “Sub”)

We want to define a good class of process for integrator $X$: $\int_0^t \xi_s \, \mathrm{d}X_s$. Here are something to consider.

1. $X$ should include all processes we have known, e.g., Brownian motion, Poisson process, default indicator, etc.
2. More or less, $X$ comes with surprise, like Poisson process, thus càdlàg.

# Semi-martingale (Warning! I said “Semi” not “Sub”)

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2. More or less, $X$ comes with surprise, like Poisson process, thus càdlàg.

**Definition**

$X$ is semi-martingale if $X = M + A$ for some local-martingale $M$ and adapted càdlàg process $A$ with bounded variation.

# Examples of Semi-martingale

**Example**

Brownian motion is a martingale, so is a semi-martingale.

**Example**

A Poisson process $\sum \mathbb{1}_{\tau_i \leq t}$ is a semi-martingale.

**Example**

The counting process for range accrual type note:

$$A_t := \sum_{i=1}^{N} \mathbb{1}_{t \geq \tau_i} \mathbb{1}_{F_{\tau_i} \in \mathcal{R}}. \tag{2}$$

is a semi-martingale.

# Itô's formula

## Proposition

*Let $X = (X^1, X^2, \dots, X^d)$ be a d-dimensional semi-martingale, $\overline{X}$ be the continuous part of $X$, and $F$ be a twice continuously differentiable function. Then for any stopping time $\tau$ and $a \geq 0$, we have*

$$\begin{aligned} F(\tau, X_\tau) - F(a, X_a) = &\int_a^\tau \partial_t F(t, X_t)\,\mathrm{d}t + \sum_{i=1}^d \int_a^\tau \partial_{x^i} F(t, X_t)\,\mathrm{d}\overline{X}_t^i \\ &+ \sum_{i,j=1}^d \int_a^\tau \frac{1}{2} \partial_{x^i}\partial_{x^j} F(t, X_t)\,\mathrm{d}[\overline{X}_t^i, \overline{X}_t^j] \\ &+ \sum_{s \leq \tau} [F(s, X_s) - F(s, X_{s-})]. \end{aligned} \tag{3}$$

The last jump term can be written in a dirac-delta expression:

$$\sum_{s \leq \tau} [F(s, X_s) - F(s, X_{s-})] = \int_0^t \Delta_s F(s, X_s)\,\mathrm{d}X_s^J \tag{4}$$

## Example

Let $X_t = W_t + \mathbb{1}_{\tau \le t}$. Then $X$ is a semi-martingale and the continuous part is $\overline{X}_t = W_t$. Hence for a twice continuously differentiable function $F : \mathbb{R} \to \mathbb{R}$,

$$F(X_t) - F(X_0) = \int_0^t F'(X_s)\,\mathrm{d}W_s\,\mathrm{d}s + \int_0^t \frac{1}{2}F''(X_s)\,\mathrm{d}s \\ + \mathbb{1}_{\tau \le t}[F(X_{\tau-} + 1) - F(X_{\tau-})] \tag{5}$$

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+ \mathbb{1}_{\tau \le t}[F(X_{\tau-} + 1) - F(X_{\tau-})]
\end{aligned}
\tag{5}
$$

By using dirac-delta expression, the last term can be put as

$$
\begin{aligned}
\mathbb{1}_{\tau \le t}[F(X_{\tau-} + 1) - F(X_{\tau-})] &= \int_0^t [F(X_{s-} + 1) - F(X_{s-})]\boldsymbol{\delta}_\tau(\mathrm{d}s) \\
&=: \int_0^t \Delta_s F(X)\,\mathrm{d}X_s^J.
\end{aligned}
$$

Note that $X^J$ means the pure jump part of $X$, i.e.,

$$
X^J := X - \overline{X} = \mathbb{1}_{\tau \le t}. \tag{6}
$$

**PDE Derivation using Martingale Argument**

# PDE derivation using martingale argument

## Proposition

*Let $S$ be a tradable asset and $V$ is its self-financing portfolio. If $N^{-1}S$ is a $\mathbb{P}$-martingale for a Numéraire $N$, then $N^{-1}V$ is also a $\mathbb{P}$-martingale.*

# PDE derivation using martingale argument

## Proposition

*Let $S$ be a tradable asset and $V$ is its self-financing portfolio. If $N^{-1}S$ is a $\mathbb{P}$-martingale for a Numéraire $N$, then $N^{-1}V$ is also a $\mathbb{P}$-martingale.*

Let us verify the proposition for a special case where $N$ is the money market account $B$. Under a risk-neutral probability,

$$\mathrm{d}B_t = rB_t\,\mathrm{d}t, \qquad \mathrm{d}S_t = rS_t\,\mathrm{d}t + \sigma S_t\,\mathrm{d}W_t. \tag{7}$$

Define a self financing portfolio by the two asset:

$$\begin{aligned} V_t :=& \eta_t B_t + \xi_t S_t \\ \mathrm{d}V_t =& \eta_t\,\mathrm{d}B_t + \xi_t\,\mathrm{d}S_t \\ =& r(\eta_t B_t + \xi_t S_t)\,\mathrm{d}t + \sigma V_t\,\mathrm{d}W_t \\ =& rV_t\,\mathrm{d}t + \sigma V_t\,\mathrm{d}W_t \end{aligned} \tag{8}$$

Let us guess $V_t = V(t, S_t)$. Recall from the previous slide, $\{B_t^{-1}V(t, S_t)\}$ is a martingale. Appying Itô's formula yields

$$
\begin{aligned}
\mathrm{d}(B_t^{-1}V(t,S_t)) =& -rB_t^{-1}V(t,S_t)\,\mathrm{d}t + B_t^{-1}\,\mathrm{d}V(t,S_t) \\
=& [-rB_t^{-1}V(t,S_t) + B_t^{-1}\partial_t V(t,S_t)]\mathrm{d}t \\
&+ [B_t^{-1}\partial_s V(t,S_t) r S_t + B_t^{-1}\frac{\sigma^2 S_t^2}{2}\partial_s^2 V(t,S_t)]\mathrm{d}t \\
&+ B_t^{-1}\sigma S_t \partial_s V(t,S_t)\,\mathrm{d}W_t
\end{aligned} \tag{9}
$$

Let us guess $V_t = V(t, S_t)$. Recall from the previous slide, $\{B_t^{-1}V(t, S_t)\}$ is a martingale. Appying Itô's formula yields

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Therefore, so that $B^{-1}V(t, S_\cdot)$ is a martingale, we should have

$$\partial_t V(t,x) + rx\partial_x V(t,x) + \frac{\sigma^2 x^2}{2}\partial_x^2 V(t,x) - rV(t,x) = 0 \tag{10}$$

Forgive me. Painful time has finally gone. Now, let us get the PDE for range accrual type note. Let

$$\mathrm{d}x_t = \mu(t, x_t)\,\mathrm{d}t + \sigma(t, x_t)\,\mathrm{d}W_t, \tag{11}$$

$$A_t := \sum_{i=1}^{N} \mathbb{1}_{t \geq \tau_i} \mathbb{1}_{(\tau_i, x_{\tau_i}) \in \mathscr{R}}, \tag{12}$$

and let $V(t, x_t, A_t)$ be the value of range accrual note. From (12), we can formaly write

$$\mathrm{d}A_t = \sum_{i=1}^{N} \mathbb{1}_{(t,\ x_t) \in \mathscr{R}} \boldsymbol{\delta}_{\tau_i}(\mathrm{d}t).$$

We will apply Itô's formula (of semi-martingale) to $\{B_t^{-1} V(t, x_t, A_t)\}$ to obtain the PDE.

Note that $\{(x_t, A_t)\}$ is a 2-dimensional semi-martingale whose continuous part is $\{(x_t, 0)\}$. Then, applying Itô's formula to $V(t, x_t, A_t)$ yields

$$
\begin{aligned}
\mathrm{d}V(t, x_t, A_t) =& \big[\partial_t V(t, x_t, A_t) + \partial_x V(t, x_t, A_t)\mu(t, x_t)\big]\,\mathrm{d}t \\
& + \frac{\sigma^2(t, x_t)}{2}\partial_x^2 V(t, x_t, A_t)\mathrm{d}t + \partial_x V(t, x_t, A_t)\sigma(t, x_t)\,\mathrm{d}W_t \\
& + [V(t, x_t, A_t) - V(t, x_t, A_{t-})]\mathrm{d}A_t \\
=& [\dots]\mathrm{d}t + [\dots]\,\mathrm{d}W_t \\
& + \sum_{i=1}^{N}[V(t, x_t, A_t) - V(t, x_t, A_{t-})]\mathbb{1}_{(t,\ x_t)\in\mathscr{R}}\boldsymbol{\delta}_{\tau_i}(\mathrm{d}t) \qquad (13)
\end{aligned}
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\mathrm{d}V(t, x_t, A_t) =& \big[\partial_t V(t, x_t, A_t) + \partial_x V(t, x_t, A_t)\mu(t, x_t)\big]\,\mathrm{d}t \\
&+ \frac{\sigma^2(t, x_t)}{2}\partial_x^2 V(t, x_t, A_t)\mathrm{d}t + \partial_x V(t, x_t, A_t)\sigma(t, x_t)\,\mathrm{d}W_t \\
&+ [V(t, x_t, A_t) - V(t, x_t, A_{t-})]\mathrm{d}A_t \\
=& [\dots]\mathrm{d}t + [\dots]\,\mathrm{d}W_t \\
&+ \sum_{i=1}^{N}[V(t, x_t, A_t) - V(t, x_t, A_{t-})]\mathbb{1}_{(t,\ x_t)\in\mathscr{R}}\boldsymbol{\delta}_{\tau_i}(\mathrm{d}t)
\end{aligned} \tag{13}$$

Recall that the jump size of $A$ is always 1. Hence, the last term in (13) can be written as

$$\sum_{i=1}^{N}[V(t, x_t, A_{t-} + 1) - V(t, x_t, A_{t-})]\mathbb{1}_{(t,\ x_t)\in\mathscr{R}}\boldsymbol{\delta}_{\tau_i}(\mathrm{d}t) \tag{14}$$

By applying Itô's formula to $B_t^{-1}V(t, x_t, A_t)$, we have

$$
\begin{aligned}
\mathrm{d}[B_t^{-1}V] =& B_t^{-1}\big[-r_t V(t,x_t,A_t) + \partial_t V(t,x_t,A_t)\big]\mathrm{d}t \\
&+ B_t^{-1}\left[\frac{\sigma^2(t,x_t)}{2}\partial_x^2 V(t,x_t,A_t)\right]\mathrm{d}t \\
&+ B_t^{-1}\partial_x V(t,x_t,A_t)\mu(t,x_t)\mathrm{d}t \\
&+ \sum_{i=1}^{N}[V(t,x_t,A_{t-}+1) - V(t,x_t,A_{t-})]\mathbb{1}_{(t,\ x_t)\in\mathscr{R}}\boldsymbol{\delta}_{\tau_i}(\mathrm{d}t) \\
&+ B_t^{-1}\partial_x V(t,x_t,A_t)\sigma(t,x_t)\,\mathrm{d}W_t.
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& + B_t^{-1}\left[\frac{\sigma^2(t,x_t)}{2}\partial_x^2 V(t,x_t,A_t)\right]\mathrm{d}t \\
& + B_t^{-1}\partial_x V(t,x_t,A_t)\mu(t,x_t)\mathrm{d}t \\
& + \sum_{i=1}^{N}[V(t,x_t,A_{t-}+1) - V(t,x_t,A_{t-})]\mathbb{1}_{(t,\ x_t)\in\mathscr{R}}\boldsymbol{\delta}_{\tau_i}(\mathrm{d}t) \\
& + B_t^{-1}\partial_x V(t,x_t,A_t)\sigma(t,x_t)\,\mathrm{d}W_t.
\end{aligned}
$$

Therefore, it suffices to find a function $V(t, x, A)$ such that

$$
\begin{cases}
\partial_t V(t,x,A) + \mu(t,x)\partial_x V(t,x,A) + \frac{\sigma^2(t,x)}{2}\partial_x^2 V(t,x,A) = rV(t,x,A), \\
V(t,x,A+1) = V(t,x,A), \quad \text{where } (t,x)\in\mathscr{R}.
\end{cases}
$$

# **Hull-White One Factor Model**

# Hull-White One Factor Model

$$\mathrm{d}r_t = (\theta_t - a r_t)\,\mathrm{d}t + \sigma_t\,\mathrm{d}W_t, \quad \sigma_t = \sigma_i,\ t_i \leq t < t_{i+1}$$

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- We fix $0.03 < a < 0.1$.

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- We fix $0.03 < a < 0.1$. Question time! You choose a bigger $a > 0$, then does volatility become bigger or smaller?
- $\{\sigma_t\}_{t\ge 0}$ can be calibrated by (diagonal) swaption quotes.
- Then $\{\theta_t\}_{t\ge 0}$ is given by market curve, $a$, and $\{\sigma_t\}_{t\ge 0}$.

To this end, we first have to find the formula of bond prices:

$$P(t,T) := \mathbb{E}\left[\exp\left(-\int_t^T r_s\,\mathrm{d}s\right)\middle|\mathcal{F}_t\right] \tag{15}$$

# Hull-White One Factor Model:

## Bond Formula and Calibrating the Curve

$$
\begin{aligned}
\mathrm{d}(e^{at}r_t) =& a e^{at} r_t \,\mathrm{d}t + e^{at}\,\mathrm{d}r_t \\
=& a e^{at} r_t \,\mathrm{d}t + e^{at}(\theta_t - a r_t)\,\mathrm{d}t + e^{at}\sigma_t \,\mathrm{d}W_t
\end{aligned}
\tag{16}
$$

Thus, we have for $t \leq s$

$$
\begin{aligned}
e^{as} r_s =& e^{-at} r_t + \int_t^s e^{au}\theta_u \,\mathrm{d}u + \int_t^s e^{au}\sigma_u \,\mathrm{d}W_u \\
\Rightarrow \quad r_s =& e^{-a(s-t)} r_t + \int_t^s e^{-a(s-u)}\theta_u \,\mathrm{d}u + \int_t^s e^{-a(s-u)}\sigma_u \,\mathrm{d}W_u
\end{aligned}
\tag{17}
$$

We will continue to get $\int_t^T r_s \,\mathrm{d}s$.

Recall

$$r_s = e^{-a(s-t)} r_t + \int_t^s e^{-a(s-u)} \theta_u \,\mathrm{d}u + \int_t^s e^{-a(s-u)} \sigma_u \,\mathrm{d}W_u \tag{18}$$

Therefore,

$$\begin{aligned}\int_t^T r_s \,\mathrm{d}s =& r_t e^{at} \int_t^T e^{-as} \,\mathrm{d}s + \int_t^T \int_t^s e^{-a(s-u)} \theta_u \,\mathrm{d}u \,\mathrm{d}s \\ &+ \int_t^T \int_t^s e^{-a(s-u)} \sigma_u \,\mathrm{d}W_u \,\mathrm{d}s\end{aligned} \tag{19}$$

Recall

$$r_s = e^{-a(s-t)} r_t + \int_t^s e^{-a(s-u)} \theta_u \, \mathrm{d}u + \int_t^s e^{-a(s-u)} \sigma_u \, \mathrm{d}W_u \tag{18}$$

Therefore,

$$\begin{aligned} \int_t^T r_s \, \mathrm{d}s = & r_t e^{at} \int_t^T e^{-as} \, \mathrm{d}s + \int_t^T \int_t^s e^{-a(s-u)} \theta_u \, \mathrm{d}u \, \mathrm{d}s \\ & + \int_t^T \int_t^s e^{-a(s-u)} \sigma_u \, \mathrm{d}W_u \, \mathrm{d}s \end{aligned} \tag{19}$$

We will reformulate the last two terms by Fubini's theorem. Before doing so, let us recall one lemma:

## Lemma

*Let* $(\Delta_t)_{t \geq 0}$ *is a (measurable) deterministic process, then*

$$\int_0^t \Delta_s \, \mathrm{d}W_s \sim N\left(0, \int_0^t \Delta_s^2 \, \mathrm{d}s\right) \tag{20}$$

$$\begin{aligned}\int_t^T \int_t^s e^{-a(s-u)}\sigma_u \, \mathrm{d}W_u \, \mathrm{d}s &= \int_t^T \int_u^T e^{-a(s-u)}\sigma_u \, \mathrm{d}s \, \mathrm{d}W_u \\ &= \int_t^T e^{au}\sigma_u \int_u^T e^{-as} \, \mathrm{d}s \, \mathrm{d}W_u \\ &= \int_t^T \frac{1}{a}(1 - e^{-a(T-u)})\sigma_u \, \mathrm{d}W_u. \end{aligned} \tag{21}$$

Hence, good news is that $\int_t^T r_s \, \mathrm{d}s$ follows a normal distribution. Moreover

$$\begin{aligned}\mathbb{E}\left[\int_t^T r_s \, \mathrm{d}s \middle| \mathcal{F}_t\right] &= \frac{r_t}{a}(1 - e^{-a(T-t)}) + \int_t^T \frac{\theta_u}{a}(1 - e^{-a(T-u)}) \, \mathrm{d}u \\ \mathbf{Var}\left[\int_t^T r_s \, \mathrm{d}s \middle| \mathcal{F}_t\right] &= \int_t^T \frac{\sigma_u^2}{a^2}(1 - e^{-a(T-u)})^2 \, \mathrm{d}u. \end{aligned} \tag{22}$$

$$
\begin{aligned}
P(t,T) &= \mathbb{E}\left[\exp\left(-\int_t^T r_s\,\mathrm{d}s\right)\middle|\mathcal{F}_t\right] \\
&= \exp\left(-\mathbb{E}\left[\int_t^T r_s\,\mathrm{d}s\middle|\mathcal{F}_t\right] + \frac{1}{2}\mathbf{Var}\left[\int_t^T r_s\,\mathrm{d}s\middle|\mathcal{F}_t\right]\right) \\
&=: \exp\Big(-r_t A^a(t,T) + C^{a,\sigma,\theta}(t,T)\Big). \qquad (23)
\end{aligned}
$$

$$
\begin{aligned}
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&=: \exp\left(-r_t A^a(t,T) + C^{a,\sigma,\theta}(t,T)\right).
\end{aligned} \tag{23}
$$

**Remark**

*Let $F(t,T^*,T)$, $T^* < T$, denote the simple forward rate seen from $t \geq 0$. Then $F(t,T^*,T)$ can be represented by the bond formula (23):*

$$
F(t,T^*,T) = \frac{1}{(T-T^*)}\left(\frac{P(t,T^*)}{P(t,T)} - 1\right). \tag{24}
$$

In the next slide, we continue to obtain the form of $\{\theta_t\}_{t\geq 0}$ by using bond formula (23).

First, we fit $\{\theta_t\}_{t\geq 0}$ so that $P^M(0,T) = P(0,T)$, where

$$P^M(0,T) = \exp\left( -\int_0^T f^M(0,t)\,\mathrm{d}t \right). \tag{25}$$

In other words,

$$f^M(0,T) = -\frac{\partial}{\partial T} \ln P^M(0,T).$$

So, we obtained $\{\theta_t\}_{t\geq 0}$. Then finding $\{\sigma_t\}_{t\geq 0}$ will be our last job. Let us finish the calculation for $\{\theta_t\}_{t\geq 0}$ in the next slide.

Let us summarize what we have gotten so far:

$$
\begin{aligned}
P(t,T) =& \exp\Big( - r_t A(t,T) + C(t,T) \Big) \\
A(t,T) =& \frac{1}{a}(1 - e^{-a(T-t)}) \\
C(t,T) =& -\int_t^T \frac{\theta_u}{a}(1 - e^{-a(T-u)})\,\mathrm{d}u + \frac{1}{2}\int_t^T \frac{\sigma_u^2}{a^2}(1 - e^{-a(T-u)})^2\,\mathrm{d}u.
\end{aligned}
$$

Let us summarize what we have gotten so far:

$$
\begin{aligned}
P(t,T) =& \exp\Big( - r_t A(t,T) + C(t,T)\Big)\\
A(t,T) =& \frac{1}{a}(1-e^{-a(T-t)})\\
C(t,T) =& -\int_t^T \frac{\theta_u}{a}(1-e^{-a(T-u)})\,\mathrm{d}u + \frac{1}{2}\int_t^T \frac{\sigma_u^2}{a^2}(1-e^{-a(T-u)})^2\,\mathrm{d}u.
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
\partial_t A(0,t) =& e^{-at}\\
\partial_t C(0,t) =& -\int_0^t \partial_t\left(\frac{\theta_u}{a}(1-e^{-a(t-u)})\right)\mathrm{d}u\\
&+\frac{1}{2}\int_0^t \partial_t\left(\frac{\sigma_u^2}{a^2}(1-e^{-a(t-u)})^2\right)\mathrm{d}u.
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
f(0,t) =& -\partial_t \ln P(0,t) \\
=& r_0 \partial_t A(0,t) - \partial_t C(0,t) \\
=& r_0 e^{-at} + \int_0^t \boldsymbol{\theta_u} e^{-a(t-u)}\,\mathrm{d}u - \int_0^t \frac{\sigma_u^2}{a}\left(e^{-a(t-u)} - e^{-2a(t-u)}\right)\mathrm{d}u.
\end{aligned}
$$

By differentiating again both sides,

$$
\begin{aligned}
\partial_t f(0,t) =& -a r_0 e^{-at} + \theta_t - a\int_0^t \theta_u e^{-a(t-u)}\,\mathrm{d}u \\
&+ \int_0^t \sigma_u^2\left(e^{-a(t-u)} - 2e^{-2a(t-u)}\right)\mathrm{d}u \\
=& \theta_t - a f(0,t) - \int_0^t \sigma_u^2 e^{-2a(t-u)}\,\mathrm{d}u
\end{aligned}
$$

Finally!

$$\theta_t = \partial_t f^M(0,t) + a f^M(0,t) + \int_0^t \sigma_u^2 e^{-2a(t-u)} \, \mathrm{d}u \tag{26}$$

Finally!

$$\theta_t = \partial_t f^M(0,t) + af^M(0,t) + \int_0^t \sigma_u^2 e^{-2a(t-u)}\,\mathrm{d}u \tag{26}$$

Oh no.... $f^M(0,t)$ may not be differentiable.

Finally!

$$\theta_t = \partial_t f^M(0,t) + af^M(0,t) + \int_0^t \sigma_u^2 e^{-2a(t-u)}\,\mathrm{d}u \tag{26}$$

Oh no.... $f^M(0,t)$ may not be differentiable. No problem!

Finally!

$$\theta_t = \partial_t f^M(0,t) + af^M(0,t) + \int_0^t \sigma_u^2 e^{-2a(t-u)}\,\mathrm{d}u \tag{26}$$

Oh no.... $f^M(0,t)$ may not be differentiable. No problem! We will put $r = \varphi + x$, for some (deterministic & absolutely continuous) $\varphi$ and $x$, where $x_0 = 0$. Then, recalling (18)

$$\begin{aligned} r_t =& e^{-at} r_0 + \int_0^t e^{-a(t-u)}\theta_u\,\mathrm{d}u + \int_0^t e^{-a(t-u)}\sigma_u\,\mathrm{d}W_u \\ =& \varphi_t + x_t. \end{aligned} \tag{27}$$

Note that $r_0 = \varphi_0 = f^M(0,0)$ and $\mathrm{d}x_t = -ax_t\,\mathrm{d}t + \sigma_t\,\mathrm{d}W_t$. This is called Gaussian model. The term $\partial_t f^M(0,t)$ will hopefully disappear by integration in (27). Indeed, it will.

$$
\begin{aligned}
&e^{-at}\int_0^t e^{au}(\partial_u f^M(0,u) + a f^M(0,u))\,\mathrm{d}u \\
&\qquad = e^{-at}\int_0^t (e^{au}\partial_u f^M(0,u) + \partial_u e^{au} f^M(0,u))\,\mathrm{d}u \\
&\qquad = e^{-at}(e^{at} f^M(0,t) - f^M(0,0)) = f^M(0,t) - e^{-at} r_0. \qquad (28)
\end{aligned}
$$

$$
\begin{aligned}
e^{-at}\int_0^t e^{au}(\partial_u f^M(0,u) + af^M(0,u))\,\mathrm{d}u \\
&= e^{-at}\int_0^t (e^{au}\partial_u f^M(0,u) + \partial_u e^{au} f^M(0,u))\,\mathrm{d}u \\
&= e^{-at}(e^{at}f^M(0,t) - f^M(0,0)) = f^M(0,t) - e^{-at}r_0. \qquad (28)
\end{aligned}
$$

$$
\begin{aligned}
\int_0^t \left(\int_0^u \sigma_s^2 e^{-2a(u-s)}\,\mathrm{d}s\right) e^{-a(t-u)}\,\mathrm{d}u \\
&= e^{-at}\int_0^t e^{2as}\sigma_s^2 \left(\int_s^t e^{-au}\,\mathrm{d}u\right)\mathrm{d}s \\
&= e^{-at}\int_0^t \sigma_s^2 \left(\frac{e^{as} - e^{2as-at}}{a}\right)\mathrm{d}s \\
&= e^{-at}\left(\sigma_s^2 \frac{2e^{as} - e^{2as-at}}{2a^2}\bigg|_0^t - \int_0^t \frac{\mathrm{d}\sigma_s^2}{\mathrm{d}s}\frac{2e^{as} - e^{2as-at}}{2a^2}\,\mathrm{d}s\right) \qquad \text{(Cont...)} \\
&=: I + II \qquad (29)
\end{aligned}
$$

Let $t_k$ denote the time such that $t_k \le t$ but $t_{k+1} > t$. Then

$$I = \frac{\sigma_k^2}{2a^2} - \frac{\sigma_0^2(2e^{-at} - e^{-2at})}{2a^2}. \tag{30}$$

Recall that

$$(\sigma_t^2)' = \begin{cases} \sigma_0^2 & t = 0, \\ \sigma_i^2 - \sigma_{i-1}^2 & t = t_i, \ i \neq 0. \end{cases}$$

Therefore, we have

$$II = \sum_{i=1}^{k} \frac{2e^{-a(t-t_i)} - e^{-2a(t-t_i)}}{2a^2}(\sigma_i^2 - \sigma_{i-1}^2) \tag{31}$$

From (30) and (31), we have

$$I + II = \frac{\sigma_0^2(1 - e^{-at})^2}{2a^2} + \sum_{1 \le i, t_i \le t} \frac{\sigma_i^2 - \sigma_{i-1}^2}{2a^2}\left(1 - e^{-a(t-t_i)}\right)^2 \tag{32}$$

Finally, we obtain

$$\varphi_t = f^M(0,t) + \frac{\sigma_0^2(1-e^{-at})^2}{2a^2} + \sum_{1\le i, t_i \le t} \frac{\sigma_i^2 - \sigma_{i-1}^2}{2a^2}\left(1 - e^{-a(t-t_i)}\right)^2. \tag{33}$$

## Form 1

$r = \varphi + x$, where $\varphi$ is defined as (33) and $\{x_t\}$ follows

$$x_t = e^{-at}\int_0^t e^{au}\sigma_u \,\mathrm{d}W_u, \quad \sigma_t = \sigma_i \;\text{ for }\; t_i \le t < t_{i+1}. \tag{34}$$

## Form 2

In other forms, one is able to write $r = f^M(0,t) + y$, where

$$
\begin{aligned}
\mathrm{d}y_t =& (\Theta_t - a y_t)\,\mathrm{d}t + \sigma_t W_t, \quad y_0 = 0, \\
\Theta_t =& \int_0^t \sigma_s^2 e^{-2a(t-s)}\,\mathrm{d}s
\end{aligned}
$$

With this from, the bond formula becomes

$$
\begin{aligned}
P(t,T) =& P^M(t,T) \exp\left( - y_t g(t,T) - \frac{\Theta_t}{2} g^2(t,T) \right) \\
g(t,T) =& \int_t^T e^{-a(t-s)}\,\mathrm{d}s.
\end{aligned}
$$

# Hull-White One Factor Model:

## Bond Option Formula and Calibrating the Volatility

# Bond Option Formula

Let us consider a bond call option with the maturity $T^* > 0$ on a bond with the maturity $T > T^*$, i.e.,

$$B_t \mathbb{E}\left[B_{T^*}^{-1}(P(T^*, T) - K)^+ \middle| \mathcal{F}_t\right]. \tag{35}$$

The typical approach is transforming (35) by using $T^*$-forward measure, but the calculation is still heavy. Instead, we will use a little trick using Black-Scholes PDE. To this end, we first define

$$S_t := \frac{P(t, T)}{P(t, T^*)}. \tag{36}$$

Then the bond option can be seen as a product on $S_t$:

$$v_c(t, S_t) := B_t \mathbb{E}\left[B_{T^*}^{-1}(S_{T^*} - K)^+ \middle| \mathcal{F}_t\right]. \tag{37}$$

Instead of directly dealing with the bond option, we consider a slightly different product

$$\omega(t, S_t) := \frac{v_c(t, S_t)}{P(t, T^*)} \tag{38}$$

and hedge $\omega$ using $S$. Then the hedge portfolio with the hedge ratio $\Delta$ can be written as

$$\begin{aligned} \Pi &:= \omega - \Delta S \\ \mathrm{d}\Pi &= \mathrm{d}\omega - \Delta\,\mathrm{d}S. \end{aligned}$$

For the SDE governing $(S_t)_{t\geq 0}$, recall that

$$P(t,T) = \exp\big(-r_t A(t,T) + C(t,T)\big). \tag{39}$$

Thus, $S_t$ is given by

$$S_t = \exp\big\{-r_t\big(A(t,T) - A(t,T^*)\big) + C(t,T) - C(t,T^*)\big\}. \tag{40}$$

It follows that $(S_t)_{t\geq 0}$ satisfies

$$\mathrm{d}S_t = S_t\Big[\alpha(t,S_t)\,\mathrm{d}t - \sigma_t\big(A(t,T) - A(t,T^*)\big)\,\mathrm{d}W_t\Big], \tag{41}$$

for some $\alpha\colon \mathbb{R}^2 \to \mathbb{R}$. For the time being, let's not care for what $\alpha$ is. In what follows, we denote

$$\begin{aligned}\Sigma_t &:= \sigma_t\big(A(t,T) - A(t,T^*)\big)\\ \Gamma(t^*,t) &:= \int_{t*}^{t} \Sigma_s^2\,\mathrm{d}s.\end{aligned}$$

$$
\begin{aligned}
\mathrm{d}\Pi_t =& \mathrm{d}\omega(t, S_t) - \Delta_t \, \mathrm{d}S_t \\
=& \left[\partial_t \omega(t, S_t) + \frac{\Sigma_t^2}{2} \partial_s^2 \omega(t, S_t)\right] \mathrm{d}t \\
&+ \left[\partial_s \omega(t, S_t) \alpha(t, S_t) - \Delta_t \alpha(t, S_t)\right] \mathrm{d}t \\
&- \Sigma_t \left[\partial_s \omega(t, S_t) - \Delta_t\right] \mathrm{d}W_t.
\end{aligned}
$$

Observing that the proper hedge ratio is $\Delta_t = \partial_s \omega(t, S_t)$, we obtain

$$
\begin{cases}
\partial_t \omega(t, s) + \frac{\Sigma_t^2}{2} \partial_s^2 \omega(t, s) = 0 \\
\omega(T^*, s) = (s - K)^+.
\end{cases}
$$

The solution of the heat equation (or Black-Scholes with $r = 0$),

$$
\begin{aligned}
\omega(t, s) =& s\phi(d_+(t, s)) - K\phi(d_-(t, s)) \\
d_+(t, s) =& \frac{\ln(s/K) + \frac{\Gamma(t, T^*)}{2}}{\sqrt{\Gamma(t, T^*)}} \\
d_-(t, s) =& d_+(t, s) - \Gamma(t, T^*).
\end{aligned}
$$

Recall that the value of call option is $v(t, S_t) = P(t, T^*)\omega(t, S_t)$,

$$v_c(t, S_t) = P(t, T)\phi(d_+(t, S_t)) - KP(t, T^*)\phi(d_-(t, S_t)). \tag{42}$$

For the precise form of (42), note that

$$d_+(t, S_t) = \frac{\ln(P(t, T)) - \ln(P(t, T^*)K) + \frac{\Gamma(t, T^*)}{2}}{\sqrt{\Gamma(t, T^*)}}$$

$$d_-(t, S_t) = \frac{\ln(P(t, T)) - \ln(P(t, T^*)K) - \frac{\Gamma(t, T^*)}{2}}{\sqrt{\Gamma(t, T^*)}}$$

Then we can easily derive the bond put option formula:

$$v_p(t, S_t) = KP(t, T^*)\phi(-d_-(t, S_t)) - P(t, T)\phi(-d_+(t, S_t)) \tag{43}$$

In what follows, we will specify the maturities and exercise price for denoting the call option value, e.g.,

$$v_c(T^*, T, K, t, S_t^{T^*, T}) := v_c(t, S_t). \tag{44}$$

# The Final Touch: Swaption Formula

## Jamshidian's Trick

Unfortunately, bond option markets are dead. For now, the most liquid interest rate option is the swaption. Since swaption can be seen as a coupon bearing bond, we first consider a (call) pay-off of a coupon bearing bond:

$$\left[\sum_{i=1}^{N} c_i P(T^*, T_i) - K\right]^+. \tag{45}$$

Recall that in a short rate model, a bond price at $t \geq 0$ can be represented by $t$, $r_t$ and its maturity. We denote the function by $\Pi$, namely

$$\Pi(t, T, r) := \exp\big(-rA(t, T) + C(t, T)\big). \tag{46}$$

Then we denote by $r^*$ the short rate that (45) becomes zero:

$$\sum_{i=1}^{N} c_i \Pi(T^*, T_i, r^*) = K. \tag{47}$$

By observing $\partial_r \Pi(T^*, T, r) < 0$,

$$\left[\sum_{i=1}^{N} c_i(\Pi(T^*, T_i, r_{T^*}) - \Pi(T^*, T_i, r^*))\right]^+$$
$$= \sum_{i=1}^{N} c_i \left[(\Pi(T^*, T_i, r_{T^*}) - \Pi(T^*, T_i, r^*))\right]^+. \tag{48}$$

Hence, the call option on a coupon bearing bond is the sum of zero coupon bond options:

$$\sum_{i=1}^{N} c_i v_c(T^*, T_i, \Pi(T^*, T_i, r^*), t, S_t^{T^*, T_i}). \tag{49}$$

# (Payer) Swaption

A swaption is an option on IRS. Recall that the value of the payer swap starting at $T_\alpha$ with an exercise $K > 0$ is

$$\sum_{i=\alpha+1}^{\beta} (F(T_\alpha, T_{i-1}, T_i) - K)\tau_i P(T_\alpha, T_i), \tag{50}$$

where $\tau_i = T_i - T_{i-1}$.

## Remark

*Let $S_{\alpha,\beta}(t)$ be the par swap rate that the coupon payments are made $\{T_{\alpha+1}, \ldots T_\beta\}$. Then, in a classical single curve model,*

$$S_{\alpha,\beta}(t) = \frac{P(t,T_\alpha) - P(t,T_\beta)}{\sum_{i=\alpha+1}^{\beta} (T_i - T_{i-1}) P(t, T_i)}. \tag{51}$$

Then the payer (resp. receiver) swaption is the option on the payer (resp. receiver) swap that its pay-off is

$$
\begin{aligned}
&\left[\sum_{i=\alpha+1}^{\beta}(F(T_\alpha,T_{i-1},T_i)-K)\tau_i P(T_\alpha,T_i)\right]^+ \\
&\qquad\qquad =\left[1-P(T_\alpha,T_\beta)-\sum_{i=\alpha+1}^{\beta}K\tau_i P(T_\alpha,T_i)\right]^+ \\
&\qquad\qquad =\left[1-\sum_{i=\alpha+1}^{\beta}c_i P(T_\alpha,T_i)\right]^+,
\end{aligned}
\tag{52}
$$

where $c_i = K\tau_i$, $i<\beta$, and $c_\beta = 1+K\tau_\beta$. Defining $r^*$ such that $\sum_{i=\alpha+1}^{\beta} c_i \Pi(T_\alpha,T_i,r^*) = 1$, the value of the payer swaption is

$$
\sum_{i=\alpha+1}^{\beta} c_i v_p(T_\alpha,T_i,\Pi(T_\alpha,T_i,r^*),t,S_t^{T_\alpha,T_i}). \tag{53}
$$

# Two Factor Model: G2++

# The Model

Let $(W_t)$ be a $d$-dimensional Browian motion with the correlation Matrix $R_t = [\rho_t^{i,j}]_{i,j}$. We consider a short rate $r_t \in \mathbb{R}$ and a $d$-dimensional gaussian process $\mathbf{x}_t$ such that

$$\begin{aligned} r_t =& f^M(0,t) + \mathbb{1}^\top \mathbf{x}_t && (54) \\ \mathrm{d}\mathbf{x}_t =& (\Theta_t \mathbb{1} - \boldsymbol{\lambda}_t \circ \mathbf{x}_t) + \boldsymbol{\sigma}_t^\top \,\mathrm{d}W_t, \quad \mathbf{x}_0 = 0 && (55) \\ \Theta_t :=& \left[ \int_0^t \rho_s^{i,j} \sigma_s^i \sigma_s^j m^i(s,t) m^j(s,t)\,\mathrm{d}s \right]_{i,j} && (56) \\ \mathbf{m}(s,t) :=& [m^i(s,t)]_i = \left[ e^{-\int_s^t \lambda_u^i \,\mathrm{d}u} \right]_i && (57) \end{aligned}$$

In this setup, we have the following bond formula

$$\begin{aligned} P(t,T) =& P^M(t,T) \exp\left( -\mathbf{g}(t,T)^\top \mathbf{x}_t - \frac{1}{2} \mathbf{g}(t,T)^\top \Theta_t \mathbf{g}(t,T) \right) && (58) \\ \mathbf{g}(t,T) :=& \int_t^T \mathbf{m}(t,s)\,\mathrm{d}s. && (59) \end{aligned}$$

# Two Factor Model: Swaption Approximation

## Hyperplane Method

Thank You!

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