# A Market Vega Matrix under Local Volatility *

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**Abstract**

We introduce an algorithm to find a vega matrix on implied volatility surfaces. Instead of directly bumping implied volatility surfaces, we first derive local volatility vega and this can be converted to the market vega matrix. The re-calibarion issue regarding the arbitrage-free condition does not appear in this algorithm. Thus, this may produce better vega matrix by improving locality. The conversion ratio between the market/model vega is the sensitivity of implied volatility with respect to local volatility. Once this ratio matrix is calculated, then this can be cached, i.e., this matrix does not need to be re-calculated for each ELS-vega. The computational performance can be further improved by the *most likely path approach*.

## 1 The Main Algorithm

For the vega conversion, one may want to refer to Cahpter 26 in Andersen & Piterbarg (2010). For implied volatility conversion, Gatheral & Wang (2012); Guyon & Henry-Labordere (2010); Reghai (2012) are good references. The main idea is simple. Let $V$ denote the value of a derivative portfolio and $\sigma^{imp}$, $\sigma^{loc}$ denote implied/local volatility respectively. We then formally have:

$$\frac{\partial V}{\partial \sigma^{loc}} = \frac{\partial V}{\partial \sigma^{imp}} \frac{\partial \sigma^{imp}}{\partial \sigma^{loc}}. \tag{1.1}$$

Hence, the implied vega matrix $\nu^{imp}$ can be obtained through the other two terms:

$$\frac{\partial \sigma^{imp}}{\partial \sigma^{loc}}, \quad \frac{\partial V}{\partial \sigma^{loc}}. \tag{1.2}$$

The both terms do not require any arbitrage-free condition, so the re-calibration issue will not arise. we will focus on how to find (1.2) in what follows.

Note that (1.1) is only a formal representation, since a rank mismatch issue in (1.1) still remains. The local volatility grid is much finer than the bumping grid of implied volatility surface. The vega on the grid is called *block vega matrix*. Similarly, a change of one grid in $\sigma^{loc}$ impacts many implied volatility blocks. Furthermore, one block of $\sigma^{imp}$ contains many call options with $(K, T)$, thus it is unclear in this stage how to define:

$$\frac{\partial \sigma^{imp}}{\partial \sigma^{loc}}. \tag{1.3}$$

For the rank mismatch issue, we will define sensitivities mainly by least-square minimization fashion. To be more precise, we first introduce some notations. We consider implied/local volatility

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surfaces $\{\sigma_{i,j}^{imp}\} \in \mathbb{R}^{I\times J}$, $\{\sigma_{n,m}^{loc}\} \in \mathbb{R}^{N\times M}$. Mostly in practice, $I \times J << N \times M$. However, for the purpose of reducing the computational cost in bumping, we can consider a reduced set of block of local volatility, e.g., consider a bumping set of local volatility $\{\Delta_{n',m'}^{loc}\}$, where $n' \le N' < N$, $m' \le M' < M$ such that for some $(k,l) \in \mathbb{N}^2$:

$$\Delta_{n',m'}^{loc} = \begin{cases} 1 \text{ bp}, & k(n'-1) \le n \le kn' \;\;\&\;\; l(m'-1) \le m \le lm' \\ 0, & \text{otherwise.} \end{cases}$$

Moreover, let $\{\Delta_{i,j}^{imp}\}$ denote the change of the implied volatility blocks. Now, by least-square minimization fashion, we define our implied volatility vega matrix $\{\hat{\nu}_{i,j}\}$ based on (1.1) as follows:

$$\hat{\nu} := \operatorname*{arg\,min}_{\{\nu_{i,j}^{imp}\}} \sum_{n',m'} \left\{ \nu_{n',m'}^{loc} - \sum_{i,j} \nu_{i,j}^{imp} \left( \frac{\partial \sigma_{i,j}^{imp}}{\partial \sigma_{n',m'}^{loc}} \right) \right\} \tag{1.4}$$

In (1.4), $\nu_{n',m'}^{loc}$ represents each ELS-vega by perturbing $\Delta_{n',m'}^{loc}$. The local volatility vega can be computed by an FDM or MC algorithm, or one may try to reduce the computation using a pathwise differentiation technique.

The subtle part is $\{\Delta_{i,j}^{imp}\}$ sensitivity, because there are many grids of call options in one block $(i,j)$. Formally, we let $C_{n,m}(\sigma^{imp})$ denote the call option value on each grid and $C_{n,m}^{\Delta_{n',m'}^{loc}}$ denote the changed call option value after bumping of $\Delta_{n',m'}^{loc}$. Now, we define:

$$\hat{\Delta}^{n',m'} := \operatorname*{arg\,min}_{\{\Delta_{i,j}\}} \left\{ \sum_{i,j} \sum_{n,m} \left\{ C_{n,m}(\sigma^{imp} + \Delta_{i,j}) - C_{n,m}^{\Delta_{n',m'}^{loc}} \right\} \right\} \tag{1.5}$$

Therefore, $\{\hat{\Delta}_{i,j}^{n',m'}\}$ is the matrix of market volatility change by $\Delta_{n',m'}^{loc}$. We then take:

$$\frac{\partial \sigma_{i,j}^{imp}}{\partial \sigma_{n',m'}^{loc}} = \frac{\hat{\Delta}_{i,j}^{n',m'}}{\Delta_{n'.m'}^{loc}}. \tag{1.6}$$

The computation of this brute force approach may be costly. In the next section, we discuss one example to improve the computational cost for finding $\hat{\Delta}$. The key ingredient is the reverse engineering from the local to implied volatility.

## 2 From Local to Implied Volatilities

To grasp the idea, consider the case of no skew, i.e.,

$$\frac{\partial w}{\partial k} = \frac{\partial^2 w}{\partial k^2} = 0, \tag{2.1}$$

where $w$ is the forward BS variance. In this case, we have

$$\sigma_{BS}^2(K,T) = \frac{1}{T} \int_0^T \sigma_{loc}^2(0,t)\, \mathrm{d}t. \tag{2.2}$$

While (2.2) is intuitive, it is obviously not accurate. There have been many attempts to improve the accuracy of the reverse algorithm Gatheral & Wang (2012); Guyon & Henry-Labordere (2010); Reghai (2012). One classic choice is the *most likely path* method.

To understand the idea, we let $F_t^T$ denote the $T$-forward price of $S_t$. Then

$$\begin{aligned}\sigma_{imp}^2(K,T) &\approx \frac{1}{T}\int_0^T \mathbb{E}[\sigma_{loc}^2(F_t^T,t) \mid F_T^T = K]\,\mathrm{d}t \\ &\approx \frac{1}{T}\int_0^T \sigma_{loc}^2(\mathbb{E}[F_t^T \mid F_T^T = K],\ t)\,\mathrm{d}t. \end{aligned} \tag{2.3}$$

Thus, for given $T > 0, K > 0$, we define the most likely path $X_t^{(K,T)}$:

$$X_t^{(K,T)} := \mathbb{E}[F_t^T \mid F_T^T = K]. \tag{2.4}$$

The path is given by a fixed point representation (see Reghai (2012)):

$$\begin{aligned} X_t^{(K,T)} &= F_0^T \exp\left\{\frac{\lambda(t)}{\lambda(T)}\ln\frac{K}{F_0^T} + \frac{\lambda(t)}{2}\left(1-\frac{\lambda(t)}{\lambda(T)}\right)\right\} \\ \lambda(t) &= \int_0^t \sigma(s)^2\,\mathrm{d}s, \quad \sigma(t) = \sigma_{loc}(t, X_t^{(K,T)}). \end{aligned} \tag{2.5}$$

This can be obtained by $X_t^{(K,T)} = \lim_{n\to\infty} X_t^n$ where

$$\begin{aligned} X_t^n &= F_0^T \exp\left\{\frac{\lambda^n(t)}{\lambda^n(T)}\ln\frac{K}{F_0^T} + \frac{\lambda^n(t)}{2}\left(1-\frac{\lambda^n(t)}{\lambda^n(T)}\right)\right\} \\ \lambda^n(t) &= \int_0^t \sigma_n^2(s)\,\mathrm{d}s, \quad \sigma_{n+1}(t) = \sigma_{loc}(t, X_t^n), \quad \sigma_0(t) = \sigma_{loc}(F_0^T, t). \end{aligned} \tag{2.6}$$

The above calculation can be gotten by only a few iteration. The approximated implied BS volatility is then given by:

$$\sigma_{imp}^2(K,T) \approx \frac{1}{T}\int_0^T \sigma_{loc}^2(X_t^{(K,T)},\ t)\,\mathrm{d}t. \tag{2.7}$$

Do note that such approximation like (2.7) is a trade-off between computation and accuracy. Some simpler approaches are BBF formula and its naïve extension Berestycki et al. (2002); Gatheral & Wang (2012):

$$\frac{1}{\sigma_{imp}(k,T)} \approx \frac{1}{T}\int_0^T \frac{1}{\sigma_{loc}(k\frac{t}{T},0)}\,\mathrm{d}t \tag{2.8}$$

$$\frac{1}{\sigma_{imp}(k,T)} \approx \int_0^1 \frac{1}{\sigma_{loc}(\alpha k, \alpha T)}\,\mathrm{d}\alpha \tag{2.9}$$

In (2.8)-(2.9), $k$ represents the log-strike, i.e., $k = \ln\frac{K}{S_0}$. It is recommended to start from (2.8)-(2.9) together with (1.4), then try to upgrade the implied volatility algorithm to acquire accuracy.

## References


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