A BRIEF INTRODUCTION TO FINANCIAL MATHEMATICS*

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1. Introduction. The colloquium is an introduction to interest rate models, or more generally fixed income models, e.g., IR, credit, commodity, etc. However, to achieve the goal, the very first step is inevitably understanding basic knowledge of financial mathematics. By basic knowledge, we mean as follows:

(i) (filtered) probability space
(ii) conditional expectation
(iii) Brownian motion
(iv) Itô's formula
(v) self-financing portfolio
(vi) arbitrage opportunity
(vii) risk-neutral price

But, understanding the above concepts with rigor takes a good amount of dedication, which does not meet the most practitioners' taste. Therefore, this lecture will often come with (illegitimately) rough explanations. Before going into details, let us ask one question. If you can answer the following question, this rudiment part is not for you:

"Must the fair value of derivatives be risk-neutral?"

Actually, the answer is "yes or no". It depends on the considered market. If you consider Black-Scholes model (more generally a complete market), then yes, no other value but risk-neutral price is fair to both contractors (you and your counterparty). If you consider an incomplete market, e.g., jump models, there are infinitely many fair values.

Now, say you want to use a stochastic volatility model. Is it a complete market? Risk-neutral price? Actually, the risk-neutral price may or may not be a unique choice. Then, why are all practitioners doing risk-neutral prices? What does it mean in incomplete market models? If these questions confuse you, this lecture could be for you.

2. Probability Spaces. Most literature of mathematical finance begins with the following one magic sentence:

" Let $(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathcal{F}_\infty, \mathbb{P})$ be a (filtered) probability space."

Definition 2.1. *A collection of Ω, denoted by $\mathcal{F}$, is called a σ-algebra of Ω if the following conditions are satisfied.*

(i) $\emptyset \in \mathcal{F}$
(ii) *if* $A \in \mathcal{F}$, *then* $A^c \in \mathcal{F}$
(iii) *if* $A_i \in \mathcal{F}$, $i \in \mathbb{N}$, *then* $\cup_{i=1}^{\infty} A_i \in \mathcal{F}$.

σ-algebras is for technicality of mathematics. There is no financial meaning (please don't try...). More precisely, we have to remove some bad sets from the power set because of *the axiom of choice* [1, p.20].

Definition 2.2. $\{\mathcal{F}_t\}_{t\geq 0}$ *is called a filtration of* Ω *if, for any* $0 \leq s \leq t$, *we have* $\mathcal{F}_s \subseteq \mathcal{F}_t$, *and each* $\mathcal{F}_t$ *is a filtration of* Ω.

Example 2.3. Consider two tosses of an unfair coin such that $\mathbb{P}(\{H\}) = 1/3$ and $\mathbb{P}(\{T\}) = 2/3$. Then we have $\Omega = \{HH,\ HT,\ TH,\ TT\}$. The elements in Ω are called events.

Now, denote the information at i-th toss by $\mathcal{F}_i$, $i = 0, 1, 2$. At first, we just do not know what will happen, so we have to say "well...by two tosses, it will HH or HT or TH or TT", which is

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simply a meaningless statement. Mathematically, this can be interpreted as $\{\Omega\} \approx \mathcal{F}$. But, for $\mathcal{F}_0$ to be σ-algebra, we should set $\mathcal{F}_0 = \{\emptyset, \Omega\}$. Well done!

Let's turn to the next toss. After the first toss, we will know what the first toss is. Mathematically, this means that $\{\{HH, HT\}, \{TH, TT\}\} \subseteq \mathcal{F}_1$. Again for $\mathcal{F}_1$ to be σ-algebra,

$$\mathcal{F}_1 = \{\emptyset, \{HH, HT\}, \{TH, TT\}, \Omega\} \tag{2.1}$$

Likewise, at the second toss,

$$\mathcal{F}_2 = \{\emptyset, \{HH\}, \{HT\}, \{TH, \}, \{TT\}, \{HH, HT\}, \{TH, TT\}, \Omega\}. \tag{2.2}$$

Notice that $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \mathcal{F}_2$.

Definition 2.4. *Let $(\Omega, \mathcal{F})$ be a measurable space. A random variable $\xi\colon \Omega \to \mathbb{R}$ is $\mathcal{F}$-measurable if $\xi^{-1}(B) \in \mathcal{F}$, for any $B \in \mathcal{B}$, where $\mathcal{B}$ is Borel σ-algebra.*

Definition 2.5. *Let $\mathbb{F} \coloneqq \{\mathcal{F}_t\}_{t\geq 0}$ be a filtration of Ω. A stochastic process X is $\mathbb{F}$-adapted if X_t is $\mathcal{F}_t$-measurable for any $t \geq 0$.*

Definition 2.6 (conditional expectation). $\mathbb{E}[\xi \mid \mathcal{F}]$.

References

[1] G. B. Folland, *Real analysis: modern techniques and their applications*, John Wiley & Sons, 2013.