An Introduction to Financial Mathematics*

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1 Introduction

This colloquium is an introduction to interest rate models, or more generally fixed income models. However, to achieve the goal, the very first step is inevitably understanding basic knowledge of financial mathematics. By basic knowledge, we mean as follows:

(i) probability space

(ii) conditional expectation

(iii) Brownian motion

(iv) Itô's formula

(v) arbitrage opportunity

(vi) risk-neutral price

However, understanding the above concepts with rigor takes a good amount of dedication. Therefore, this lecture will often come with rough explanations. Interested readers may want to refer to He et al. (2018); Shreve (2004).

Before the details, here is one question for you. If you can answer the following question, this rudiment part may not be for you:

> "For a fair value of derivative contracts, must you choose the risk-neutral price?"

The answer is "no". Sometime you must, but sometimes you don't have to. However, if you are a practitioner, there is no harm in choosing risk-neutral prices anyway. Well, does it sound so confusing? This lecture may help you answer the question clearly.

2 Probability Spaces

Most literature of mathematical finance begin with the following one magic sentence:

> " Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space."

Then it is followed by something like this: "Now, let us equipped with a filtration $\mathbb{F} := (\mathcal{F}_t)_{t\geq 0}$". Let's break the terms down together.

Definition 2.1. *A collection of Ω, denoted by $\mathcal{F}$, is called a σ-algebra of Ω if the following conditions are satisfied.*

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(i) $\emptyset \in \mathcal{F}$

(ii) *if* $A \in \mathcal{F}$, *then* $A^c \in \mathcal{F}$

(iii) *if* $A_i \in \mathcal{F}$, $i \in \mathbb{N}$, *then* $\cup_{i=1}^{\infty} A_i \in \mathcal{F}$.

σ-algebras is only for mathematical technicality. There is no financial meaning! (oh please don't try...). So to speak, we have to remove some bad sets from the power set. The main reason is that we take *the axiom of choice* (see Folland, 2013, p.20).

Definition 2.2. $\mathbb{F} := \{\mathcal{F}_t\}_{t\geq 0}$ *is called a filtration of* Ω *if, for any* $0 \leq s \leq t$, *we have* $\mathcal{F}_s \subseteq \mathcal{F}_t$, *and each* $\mathcal{F}_t$ *is a filtration of* Ω.

Definition 2.3. *A* σ*-algebra generated by* A *is the smallest* σ*-algebra containing* A.

Problem 2.4. Prove that

$$\cap_{i=1}^{\infty} \mathcal{A}_i, \quad A \subseteq \mathcal{A}_i, \tag{2.1}$$

is the σ-algebra generated by A.

Example 2.5. Consider two tosses of an unfair coin such that $\mathbb{P}(\{H\}) = 1/3$ and $\mathbb{P}(\{T\}) = 2/3$. Then we have $\Omega = \{HH, HT, TH, TT\}$. The elements in Ω are called events. Now, denote the information at i-th toss by $\mathcal{F}_i$, $i = 0, 1, 2$. At first, we just do not know what will happen, so we have to say "well...by two tosses, it will HH or HT or TH or TT", which is simply a meaningless statement. Mathematically, this can be interpreted as $\{\Omega\} \approx \mathcal{F}$. But, for $\mathcal{F}_0$ to be a σ-algebra, we should set $\mathcal{F}_0 = \{\emptyset, \Omega\}$. Well done!

Let's turn to the next toss. After the first toss, we will know what the first toss is. Mathematically, this means that $\{\{HH, HT\}, \{TH, TT\}\} \subseteq \mathcal{F}_1$. Again for $\mathcal{F}_1$ to be σ-algebra,

$$\mathcal{F}_1 = \{\emptyset, \{HH, HT\}, \{TH, TT\}, \Omega\} \tag{2.2}$$

Likewise, at the second toss,

$$\mathcal{F}_2 = \{\emptyset, \{HH\}, \{HT\}, \{TH,\}, \{TT\}, \{HH, HT\}, \{TH, TT\}, \Omega\}. \tag{2.3}$$

Nope. This is not a σ-algebra. Modify it.

Notice that $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \mathcal{F}_2$. This is describing the flow of information.

Definition 2.6. *Let* $(\Omega, \mathcal{F})$ *be a measurable space. A random variable* $\xi : \Omega \to \mathbb{R}$ *is* $\mathcal{F}$*-measurable if* $\xi^{-1}(B) \in \mathcal{F}$, *for any* $B \in \mathcal{B}$, *where* $\mathcal{B}$ *is Borel* σ*-algebra.*

If you know the value of a random variable ξ by the information $\mathcal{F}$, we say ξ is $\mathcal{F}$-measurable.

Example 2.7. Let $(\Omega, \mathcal{F})$ be a measurable space and

$$F(\omega) = \sum_{i=1}^{\infty} a_i \mathbb{1}_{A_i}, \quad A_i \in \mathcal{F}. \tag{2.4}$$

Then F is $\mathcal{F}$-measurable.

Definition 2.8. *Let* $\mathbb{F} := \{\mathcal{F}_t\}_{t\geq 0}$ *be a filtration of* Ω. *A stochastic process* X *is* $\mathbb{F}$*-adapted if* X_t *is* $\mathcal{F}_t$*-measurable for any* $t \geq 0$.

Definition 2.9 (conditional expectation)**.** *The conditional expectation of ξ on $\mathcal{F}$, denoted by $\mathbb{E}[\xi \mid \mathcal{F}]$, is an $\mathcal{F}$-measurable random variable such that*

$$\int_A \xi \, \mathrm{d}\mathbb{P} = \int_A \mathbb{E}[\xi \mid \mathcal{F}] \mathrm{d}\mathbb{P}, \;\; \forall A \in \mathcal{F}. \tag{2.5}$$

Before confused by the seemingly weird definition, let us ask some questions.

1. Given that you know the price of a stock today, what is the probability whether it rains tomorrow?
2. Given that you know the price of a stock today, what is the expectation of the stock price today?
3. Recall that a conditional expectation is a random variable, e.g., the expectation of a stock price given the information of day-1 is not known on day-0. Under the info of day-0, what is the expectation of the random variable (the expectation of conditional expectation)?

The above statements can be cast into the following theorem.

Theorem 2.10. *i. If ξ is independent with $\mathcal{F}$, $\mathbb{E}[\xi \mid \mathcal{F}] = \mathbb{E}[\xi]$.*

ii. If ξ is $\mathcal{F}$-measurable, then $\mathbb{E}[\xi \mid \mathcal{F}] = \xi$.

iii. Let $\mathcal{F} \subseteq \mathcal{G}$. Then $\mathbb{E}[\mathbb{E}[\xi \mid \mathcal{G}] \mid \mathcal{F}] = \mathbb{E}[\xi \mid \mathcal{F}]$.

Problem 2.11. Prove theorem 2.10.

Definition 2.12. *$(X_t)_{t\geq 0}$ is a sub-martingale (resp. super-martingale) if $X_t \in \mathbb{L}^1$ for any $t \geq 0$ and*

$$\mathbb{E}(X_s \mid \mathcal{F}_t) \geq X_t, \;\; t \leq s \quad (\textit{resp. } \mathbb{E}(X_s \mid \mathcal{F}_t) \leq X_t, \;\; t \leq s). \tag{2.6}$$

If $(X_t)_{t\geq 0}$ are both a sub-martingale and a super-martingale, then it is called a martingale.

3 Brownian Motions and their Stochastic Integration

Definition 3.1. *Let $(\Omega, \mathbb{F}, \mathcal{F}, \mathbb{P})$ be a filtered probability space.*
Let $(W_t)_{t\geq 0}$ be a stochastic process such that

- *W is $\mathbb{F}$-adapted*
- *$W_t - W_s \sim N(0, t-s)$*
- *W_t is continuous in t a.s*
- *$W_0 = 0$.*

Then $(W_t)_{t\geq 0}$ is called the standard Brownian motion.

Problem 3.2. Calculate $\mathbb{E}[W_t]$, $\mathbb{E}[W_t^2]$, and $\mathbb{E}[W_t^3]$.

Definition 3.3.

$$\int_0^t \xi_s \, \mathrm{d}W_s = \lim \sum_i \xi_{t_i} (W_{t_{i+1}} - W_t). \tag{3.1}$$

Theorem 3.4. *(1) $\int_0^t \xi_s \, \mathrm{d}W_s$ is a local-martingale.*

(2) If ξ is a deterministic process,

$$\int_0^t \xi_s \, \mathrm{d}W_s \sim N\left(0, \int_0^t \xi_s^2 \, \mathrm{d}s\right) \tag{3.2}$$

Theorem 3.5 (Itô's formual). *Let $F \in C^2(\mathbb{R})$ and X follows*

$$\mathrm{d}X_t = a(t, X_t)\,\mathrm{d}t + b(t, X_t)\,\mathrm{d}W_t. \tag{3.3}$$

Then we have

$$F(X_t) - F(X_0) = \int_0^t \left[F'(X_s)a(t, X_s) + \boldsymbol{F''(X_s)\frac{b^2(s, X_s)}{2}}\right]\mathrm{d}s + \int_0^t F'(X_s)b(s, X_s)\,\mathrm{d}W_s. \tag{3.4}$$

The formula is often put in the following formal representation:

$$\mathrm{d}F(X_t) = \left[F'(X_s)a(t, X_t) + F''(X_t)\frac{b^2(t, X_t)}{2}\right]\mathrm{d}t + F'(X_t)b(t, X_t)\,\mathrm{d}W_t \tag{3.5}$$

The above theorem is a stochastic version of fundamental theorem of calculus. If we do not have the middle term

$$\int_0^t F''(X_s)\frac{b^2(s, X_s)}{2}\,\mathrm{d}s, \tag{3.6}$$

(3.4) can be rewritten as

$$F(X_t) - F(X_0) = \int_0^t F'(X_s)\,\mathrm{d}X_s. \tag{3.7}$$

Notice the analogy between (3.7) and the classical version of fundamental theorem of calculus:

$$g(b) - g(a) = \int_a^b g'(x)\,\mathrm{d}x. \tag{3.8}$$

However, (3.6) came out since the quadratic variation of Brownian motions is not zero. Actually, a Brownian motions is not a realistic object at all. To explain the source of (3.6), I prefer using Bluementhal's 0-1 Law rather than bringing out the definition of *quadratic variation*.

Theorem 3.6 (Bluementhal's 0-1 Law). *Let $\mathbb{F}$ be the filtration generated by a Brownian motion $(W_t)_{t\geq 0}$. Then $\mathcal{F}_{0+}$ is trivial. More precisely, for any $A \in \mathcal{F}_{0+}$, we have either*

$$\mathbb{P}(A) = 1 \quad \text{or} \quad \mathbb{P}(A) = 0. \tag{3.9}$$

Corollary 3.7. *Let $\tau := \inf\{t \geq 0 \mid W_t > 0\}$ and $\nu := \inf\{t \geq 0 \mid W_t < 0\}$. Then*

$$\mathbb{P}(\tau = 0) = \mathbb{P}(\nu = 0) = 1. \tag{3.10}$$

By corollary 3.7, for any interval $(0, \epsilon)$ $\epsilon > 0$, there is zero of Brownian motions. In other words, there is no interval that Brownian motions are monotonic. This feature of Brownian motions produces its non-zero quadratic variation, i.e., Brownian motions move quite hard. This is a reason why we have the adjustment term (3.6).

Example 3.8. Find $\mathbb{E}[W_t^4]$.
Sol. Put $\mathrm{d}W_t = \mathrm{d}W_t$. You can put $a(t,x) = 0$ and $b(t,x) = 1$ in (3.3). Let $F(x) = x^4$ and we apply Itô's formual:

$$\mathrm{d}W_t^4 = 6W_t^2\,\mathrm{d}t + 4W_t^3\,\mathrm{d}W_t. \tag{3.11}$$

Then, it follows that

$$W_t^4 - W_0^4 = \int_0^t 6W_s^2\,\mathrm{d}s + \int_0^t 4W_s^3\,\mathrm{d}W_s. \tag{3.12}$$

Therefore, we have

$$\mathbb{E}[W_t^4] = \int_0^t 6\mathbb{E}[W_s^2]\,\mathrm{d}s \tag{3.13}$$

$$= \int_0^t 6s\,\mathrm{d}s \tag{3.14}$$

$$= 3t^2. \tag{3.15}$$

In what follows, we will use the very popular recipe:

$$\begin{aligned} \mathrm{d}W_t\,\mathrm{d}W_t &= \mathrm{d}t. \\ \mathrm{d}W_t\,\mathrm{d}t &= 0. \\ \mathrm{d}t\,\mathrm{d}t &= 0. \end{aligned}$$

References

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