

# An Introduction to Financial Mathematics<sup>\*</sup>

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## 1 Introduction

This colloquium is an introduction to interest rate models, or more generally fixed income models. However, to achieve the goal, the very first step is inevitably understanding basic knowledge of financial mathematics. By basic knowledge, we mean as follows:

- (i) probability space
- (ii) conditional expectation
- (iii) Brownian motion
- (iv) Itô's formula
- (v) arbitrage opportunity
- (vi) risk-neutral price

However, understanding the above concepts with rigor takes a good amount of dedication. Therefore, this lecture will often come with rough explanations. Interested readers may want to refer to [He et al. \(2018\)](#); [Shreve \(2004\)](#).

Before the details, here is one question for you. If you can answer the following question, this rudiment part may not be for you:

“For a fair value of derivative contracts, must you choose the risk-neutral price?”

The answer is “no”. Sometime you must, but sometimes you don't have to. However, if you are a practitioner, there is no harm in choosing risk-neutral prices anyway. Well, does it sound so confusing? This lecture may help you answer the question clearly.

## 2 Probability Spaces

Most literature of mathematical finance begins with the following one magic sentence:

“Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space.”

Then it is followed by something like this: “Now, let us equipped with a filtration  $\mathbb{F} := (\mathcal{F}_t)_{t \geq 0}$ ”. Let's break the terms down together.

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**Definition 2.1.** A collection of  $\Omega$ , denoted by  $\mathcal{F}$ , is called a  $\sigma$ -algebra of  $\Omega$  if the following conditions are satisfied.

- (i)  $\emptyset \in \mathcal{F}$
- (ii) if  $A \in \mathcal{F}$ , then  $A^c \in \mathcal{F}$
- (iii) if  $A_i \in \mathcal{F}$ ,  $i \in \mathbb{N}$ , then  $\cup_{i=1}^{\infty} A_i \in \mathcal{F}$ .

$\sigma$ -algebras is only for mathematical technicality. There is no financial meaning! (oh please don't try...). So to speak, we have to remove some bad sets from the power set. The main reason is that we take *the axiom of choice* (see [Folland, 2013](#), p.20).

**Definition 2.2.**  $\mathbb{F} := \{\mathcal{F}_t\}_{t \geq 0}$  is called a filtration of  $\Omega$  if, for any  $0 \leq s \leq t$ , we have  $\mathcal{F}_s \subseteq \mathcal{F}_t$ , and each  $\mathcal{F}_t$  is a filtration of  $\Omega$ .

**Definition 2.3.** A  $\sigma$ -algebra generated by  $A$  is the smallest  $\sigma$ -algebra containing  $A$ .

**Problem 2.4.** Prove that

$$\cap_{i=1}^{\infty} \mathcal{A}_i, \quad A \subseteq \mathcal{A}_i, \quad (2.1)$$

is the  $\sigma$ -algebra generated by  $A$ .

**Example 2.5.** Consider two tosses of an unfair coin such that  $\mathbb{P}(\{H\}) = 1/3$  and  $\mathbb{P}(\{T\}) = 2/3$ . Then we have  $\Omega = \{HH, HT, TH, TT\}$ . The elements in  $\Omega$  are called events. Now, denote the information at  $i$ -th toss by  $\mathcal{F}_i$ ,  $i = 0, 1, 2$ . At first, we just do not know what will happen, so we have to say “well...by two tosses, it will  $HH$  or  $HT$  or  $TH$  or  $TT$ ”, which is simply a meaningless statement. Mathematically, this can be interpreted as  $\{\Omega\} \approx \mathcal{F}$ . But, for  $\mathcal{F}_0$  to be a  $\sigma$ -algebra, we should set  $\mathcal{F}_0 = \{\emptyset, \Omega\}$ . Well done!

Let's turn to the next toss. After the first toss, we will know what the first toss is. Mathematically, this means that  $\{\{HH, HT\}, \{TH, TT\}\} \subseteq \mathcal{F}_1$ . Again for  $\mathcal{F}_1$  to be  $\sigma$ -algebra,

$$\mathcal{F}_1 = \{\emptyset, \{HH, HT\}, \{TH, TT\}, \Omega\} \quad (2.2)$$

Likewise, at the second toss,

$$\mathcal{F}_2 = \{\emptyset, \{HH\}, \{HT\}, \{TH\}, \{TT\}, \{HH, HT\}, \{TH, TT\}, \Omega\}. \quad (2.3)$$

Nope. This is not a  $\sigma$ -algebra. Modify it.

Notice that  $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \mathcal{F}_2$ . This is describing the flow of information.

**Definition 2.6.** Let  $(\Omega, \mathcal{F})$  be a measurable space. A random variable  $\xi: \Omega \rightarrow \mathbb{R}$  is  $\mathcal{F}$ -measurable if  $\xi^{-1}(B) \in \mathcal{F}$ , for any  $B \in \mathcal{B}$ , where  $\mathcal{B}$  is Borel  $\sigma$ -algebra.

If you know the value of a random variable  $\xi$  by the information  $\mathcal{F}$ , we say  $\xi$  is  $\mathcal{F}$ -measurable.

**Example 2.7.** Let  $(\Omega, \mathcal{F})$  be a measurable space and

$$F(\omega) = \sum_{i=1}^{\infty} a_i \mathbb{1}_{A_i}, \quad A_i \in \mathcal{F}. \quad (2.4)$$

Then  $F$  is  $\mathcal{F}$ -measurable.

**Definition 2.8.** Let  $\mathbb{F} := \{\mathcal{F}_t\}_{t \geq 0}$  be a filtration of  $\Omega$ . A stochastic process  $X$  is  $\mathbb{F}$ -adapted if  $X_t$  is  $\mathcal{F}_t$ -measurable for any  $t \geq 0$ .

**Definition 2.9** (conditional expectation). *The conditional expectation of  $\xi$  on  $\mathcal{F}$ , denoted by  $\mathbb{E}[\xi | \mathcal{F}]$ , is an  $\mathcal{F}$ -measurable random variable such that*

$$\int_A \xi \, d\mathbb{P} = \int_A \mathbb{E}[\xi | \mathcal{F}] \, d\mathbb{P}, \quad \forall A \in \mathcal{F}. \quad (2.5)$$

Before confused by the seemingly weird definition, let us ask some questions.

1. Given that you know the price of a stock today, what is the probability whether it rains tomorrow?
2. Given that you know the price of a stock today, what is the expectation of the stock price today?

This statement can be cast into the following theorem.

**Theorem 2.10.** *i. If  $\xi$  is independent with  $\mathcal{F}$ ,  $\mathbb{E}[\xi | \mathcal{F}] = \mathbb{E}[\xi]$   
ii. If  $\xi$  is  $\mathcal{F}$ -measurable, then  $\mathbb{E}[\xi | \mathcal{F}] = \xi$ .*

**Problem 2.11.** Prove theorem 2.10.

**Definition 2.12.**  $(X_t)_{t \geq 0}$  is sub-martingale (resp. super-martingale) if

### 3 Brownian Motions and its Stochastic Integration

**Definition 3.1.** Let  $(\Omega, \mathbb{F}, \mathcal{F}, \mathbb{P})$  be a filtered probability space.  
Assume  $(W_t)_{t \geq 0}$  is a stochastic process such that

- $W$  is  $\mathbb{F}$ -adapted
- $W_t - W_s \sim N(0, t - s)$
- $W_t$  is continuous in  $t$  a.s
- $W_0 = 0$ .

Then  $(W_t)_{t \geq 0}$  is called the standard Brownian motion.

**Problem 3.2.** Calculate  $\mathbb{E}[W_t]$ ,  $\mathbb{E}[W_t^2]$ , and  $\mathbb{E}[W_t^3]$ .

### References

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- Shreve, S. E. (2004). *Stochastic calculus for finance ii: Continuous-time models* (Vol. 11). Springer Science & Business Media.