Notes on Trade, Macro, Spatial, and IO

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Contents

Chapter A

Statistics

A.1 Properties of Fréchet Random Variables

Definition Fréchet . A random variable, Z, has a Fréchet distribution if there exists a scale parameter $T > 0$ and shape parameter $\theta > 0$ such that $\mathbb{P}[Z \leq z] = e^{-Tz^{-\theta}}$.

Let $\Gamma(x) \equiv \int_0^\infty t^{x-1}e^{-t}\ \mathrm{d}t$ denote the Gamma function.

Expectation of Fréchet . If Z is Fréchet with scale T and shape $\theta > 1$, then $\mathbb{E}[Z] = \Gamma(1 - 1/\theta)T^{1/\theta}$.
Proof. $\mathbb{E}[Z] = \int_0^\infty z\theta T z^{-\theta-1}e^{-Tz^{-\theta}}\mathrm{d}z = \int_0^\infty t^{-1/\theta}e^{-t}\ \mathrm{d}tT^{1/\theta} = \Gamma(1-1/\theta)T^{1/\theta}$.

Copula . A function $C : [0,1]^N \to [0,1]$ is a copula if there exists a random vector $(U_1, \ldots, U_N)$ on $[0,1]^N$ such that $C(u_1, \ldots, u_N) = \mathbb{P}[U_1 \leq u_1, \ldots, U_N \leq u_N]$ for each $(u_1, \ldots, u_N) \in [0,1]^N$. Given a random vector $(Z_1, \ldots, Z_N)$, its copula is

$$C(u_1, \ldots, u_N) \equiv \mathbb{P}[F_1(Z_1) \leq u_1, \ldots, F_N(Z_N) \leq u_N]$$

where $F_i(z) \equiv \mathbb{P}[Z_i \leq z]$ for each $i = 1, \ldots, N$.

Chapter B

International Trade

B.1 Gains from Trade

B.1.1 "New trade models, same old gains?" (Arkolakis, Costinot, and Rodríguez-Clare, 2012)

B.1.2 "Trade with Correlation" (Lind and Ramondo, 2023)

B.1.2.1 Motivation

An insight from Ricardo (1817): Two countries gain more from trade when they have dissimilar production productivities. However, the models in the recent quatitative trade literature, building on EK (2002), relies on independence assumptions, which, although leading to convenient functional forms for estimation, restrict **expenditure substitution patterns** and **impact inference on the gains from trade**.

B.1.2.2 Takeaways and Contribution

Takeaways

1. This paper proposes a cross-nested CES (CNCES) structure for productivity draws, which treating the nests as latent factors in the multisector trade data. In the context of Ricardian theory, these latent factors have a natural interpretation that production and transportation technology may be shared across countries and sectors.

2. This paper presents an alternative to the BLP procedure in (Adão, Costinot, and Donaldson, 2017), but shares the same goal of departing from IIA.

Contribution This paper generalizes the ACR's gains from trade sufficient statistics to a GEV class. Assuming productivities are drawn from CNCES, the gains from trade (GFT) relative to autarky are

$$\frac{W_d/P_d}{W_d^A/P_d^A} = \pi_{dd}^{-1/\theta} \left[\sum_{k=1}^{K} \frac{\left(\pi_{kdd}^W\right)^{1-\rho_k} \pi_{kd}^B}{\pi_{dd}} \right]^{-\frac{1}{\theta}} \tag{B.1.1}$$

Notably, the GFT not only depends on own share π_{dd}, and shape parameter of Fréchet θ, but also nest-own trade share π_{kd} and correlation coefficients ρ_s.

B.1.2.3 Theory

Preferences Consumers have identical CES preferences over continuum of goods, $v \in [0,1]$ with elasticity of substitution $\eta > 1$. Thus, the expenditure share on v is

$$X_d(v) = \left(\frac{P_d(v)}{P_d}\right)^{1-\eta} X_d, \tag{B.1.2}$$

where the subscript d denotes destination countries. $P_d(v) = \left(\int_0^1 P_d(v)^{1-\eta} dv\right)^{\frac{1}{1-\eta}}$ is the price level, and X_d is total expenditure in country d.

Production Each good v is produced with a labor-only technology that features constant returns to scale:

$$Y_{od}(v) = Z_{od}(v)L_{od}(v), \tag{B.1.3}$$

where $Z_{od}(v)$ is a generalization of the standard iceberg trade costs assumption that assumes $Z_{od} = \frac{Z_o(v)}{\tau_{od}}$.

Max-Stable Multivariate Fréchet Productivity Given a destination country d, this paper assumes that the joint distribution of productivity across origin countries is given by

$$\Pr\left[Z_{1d}(v) \leq z_1, \ldots, Z_{Nd}(v) \leq z_N\right] = \exp\left[-G^d\left(T_{1d}z_1^{-\theta}, \ldots, T_{Nd}z_N^{-\theta}\right)\right] \tag{B.1.4}$$

where $T_{od} > 0$ is the scale parameter and $\theta > 0$ the shape parameter characterizing the marginal Fréchet distributions, $\Pr\left[Z_{od}(v) \leq z\right] = e^{-T_{od}z^{-\theta}}$. The scale parameters capture the absolute advantage of countries, while the shape parameter regulates the heterogeneity of iid productivity draws across the continuum of goods.[1]

The function G^d is a correlation function, also called tail dependence function in probability and statistics. This function allows for a flexible dependence structure across origin countries o serving destination d.[2] In EK (2002), productivities are independent across origin countries, so $G^d(x_1, x_2, \ldots, x_N) = \sum_{o=1}^{N} x_o$, and

$$\begin{aligned} \Pr\left[Z_{1d}(v) \leq z_1, \ldots, Z_{Nd}(v) \leq z_N\right] &= \prod_{o=1,\ldots,N} \Pr\left[Z_{od}(v) \leq z_o\right] \\ &= \exp\left(-\sum_{o=1}^{N} T_{od}z_o^{-\theta}\right) \end{aligned} \tag{B.1.5}$$

Remark: With independent productivities, θ alone governs the gains from trade in the EK model.

Proposition 1 (CNCES Approximation)**.** *Any correlation function can be approximated uniformly on compact sets using a cross-nested CES (CNCES) correlation function.*

$$G^d\left(x_1, \ldots, x_N\right) = \sum_{k=1}^{K}\left[\sum_{o=1}^{N}\left(\omega_{kod}x_o\right)^{\frac{1}{1-\rho_k}}\right]^{1-\rho_k}, \tag{B.1.6}$$

where $k = 1, 2, \ldots, K$ is the nest index. $\rho_k \in [0, 1)\forall k$ is correlation coefficient that governs the correlation in productivity across origins within nest k. For $\rho_k = 0$, productivity is independent and the k^{th} nest is additive. In contrast, as $\rho_k \to 1$, productivity becomes perfectly correlated within nest s, and the k^{th} nest converges to a max function. $\omega_{kod} > 0$, and $\sum_k \omega_{kod} = 1$. The

[1] The smaller θ is, the greater the heterogeneity.

[2] See Gudendorf, Gordon, and Johan Segers. 2010. "Extreme-Value Copulas."

weight ω_{kod} indicates the relative importance of each nest s for a given trading pair od. If ω_{kod} is high, nest s is particularly productive in country o for delivery to d.

PROOF:

Remarks on CNCES in equation (B.1.6)

- Productivity draws can be correlated within nest k: $\rho_k \in [0, 1)$, and the correlation is homogeneous within nest.
- Correlation can be different across nests: $\rho_k \neq \rho_{k'}$.
- Productivity draws are **independent across nests**.

B.1.2.4 Quantitative Application

Productivity Lind and Ramondo (2023) generalize the production technology in equation (B.1.3) to the following specification where $Z_{sod}(v)$ — the productivity for a good v in country o sector s sold to d — is distributted according to

$$\Pr\left[Z_{sod}(v) \leq z_{so}, \forall s, o\right] = \exp\left[-\sum_{k=1}^{K}\left(\sum_{s=1}^{S}\sum_{o=1}^{N}\left(\omega_{ksod}T_{sod}z_{so}^{-\theta}\right)^{\frac{1}{1-\rho_k}}\right)^{1-\rho_k}\right], \tag{B.1.7}$$

where $\omega_{ksod} > 0, \quad \sum_k \omega_{ksod} = 1$. Within nest k (A.K.A. "factor"), correlation is symmetric across origin countries and sectors. Productivity draws are independent across factors.

Expenditure share Assuming destination country d sources goods by the lowest price, the sectoral share takes the following forms:

$$\begin{aligned}\pi_{sod} &= \sum_{k=1}^{K}\pi^{*}_{ksod} \\ &= \sum_{k=1}^{K}\underbrace{\left[\frac{\left[T^{*}_{ksod}\left(t_{sod}W_o\right)^{-\theta}\right]^{\frac{1}{1-\rho_k}}}{\sum_{s'=1}^{S}\sum_{o'=1}^{N}\left[T^{*}_{ks'o'd}\left(t_{s'o'd}W_{o'}\right)^{-\theta}\right]^{\frac{1}{1-\rho_k}}}\right]}_{\pi^{W}_{ksod}} \times \underbrace{\left[\frac{\left\{\sum_{s'=1}^{S}\sum_{o'=1}^{N}\left[T^{*}_{ks'o'd}\left(t_{s'o'd}W_{o'}\right)^{-\theta}\right]^{\frac{1}{1-\rho_k}}\right\}^{1-\rho_k}}{\sum_{k'=1}^{K}\left\{\sum_{s'=1}^{S}\sum_{o'=1}^{N}\left[T^{*}_{k's'o'd}\left(t_{s'o'd}W_{o'}\right)^{-\theta}\right]^{\frac{1}{1-\rho_k}}\right\}^{1-\rho_{k'}}}\right]}_{\pi^{B}_{kd}}\end{aligned} \tag{B.1.8}$$

where $T^{*}_{ksod} \equiv \omega^{*}_{ksod}T_{sod}$. W_o is nominal wage in country o. π^{W}_{ksod} is destination d's within-factor expenditure share across sectors and origins.

Additional assumption However, for the purpose of estimation, equation (B.1.8) is overparameterized because the productivity T^{*}_{ksod} has four dimensions. To ensure that the model is not underidentified, it is necessary to add some structure to the productivity distribution. Specifically, Lind and Ramondo (2023) impose a separability assumption as follows:

$$T^*_{ksod} = \left(B_{sk} A_{kod}\right)^{\theta}, \tag{B.1.9}$$

where B_{sk} captures the usefulness of factor k for sector s, while A_{kod} measures the productivity of origin o in factor k when delivering to destination d, capturing barriers to apply technologies in a country as well as geographical barriers to trade (e.g., distance).

B.2 Multi-country, Multi-sectoral Ricardian Model

B.2.1 "Estimates of the Trade and Welfare Effects of NAFTA" (Caliendo and Parro, 2015)

B.2.1.1 Model

The Caliendo-Parro model is a multi-industry extension of the EK (2002) Ricardian model of trade (Antràs and Chor, 2022).

Environment N countries. J sectors. **One factor of production, labor, which is mobile across sectors but not across countries.**[3] CRS production function. Output produced in all sectors can be used as intermediate input or a final good.

Preferences The representative consumer in each country has preferences over the output of the S sectors given by:

$$u(C_n) = \prod_{j=1}^{J} \left(C_n^j\right)^{\alpha_j}, \quad \text{where } \sum_j \alpha_j = 1 \tag{B.2.1}$$

where C_n^j denotes non-tradable final consumption goods.

Intermediate goods [4] A continuum of intermediate goods $\omega^j \in [0, 1]$ is produced in each sector j. Two types of inputs, labor and "materials from all sectors" are used for the production of each ω^j. The production technology of a good ω^j is

$$y_n^j(w^j) = z_n^j(w^j) \left[l_n^j(w^j)\right]^{\gamma_n^j} \prod_{k=1}^{J} \left[M_n^{k,j}(w^j)\right]^{\gamma_n^{k,j}} \tag{B.2.2}$$

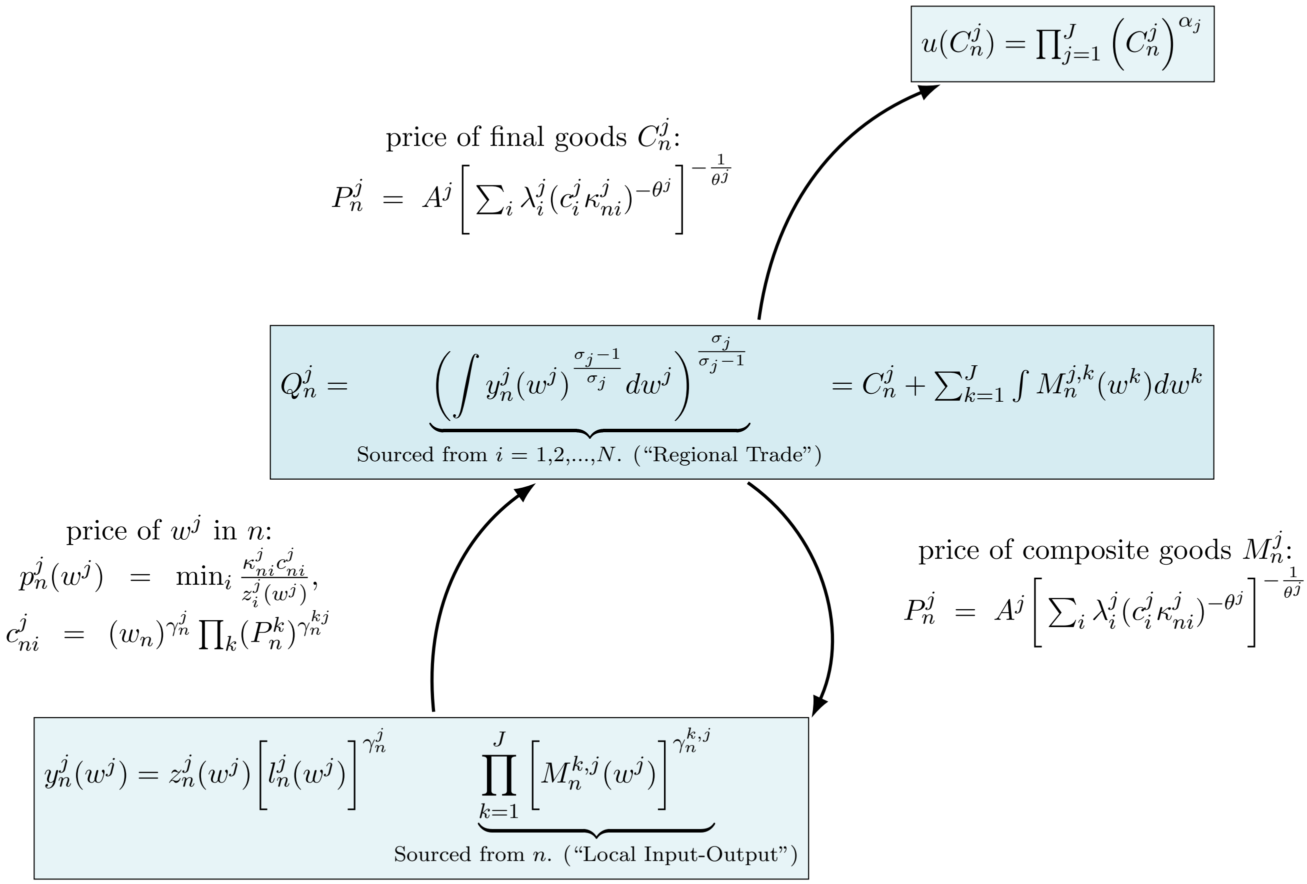
where

1. "$z_n^j(w^j) \sim$ Frechet" governs the efficiency of producing intermediate good ω^j in county n
2. $l_n^j(w^j)$ is labor
3. $M_n^{k,j}(w^j)$ is the materials from sector k used for the production of intermediate good ω^j (capturing the roundabout production)
4. $\gamma_n^{k,j} \geq 0$ is the share of the materials used from k in j (capturing the strength of the I-O linkages)

[3] See Caliendo et al. (2018) for an extension to a spatial setting with mobile labor across locations.

[4] Antràs and Chor (2022) term the production function in Caliendo and Parro (2015) as a "roundabout model" and suggest that "it has quickly become a benchmark model in the field [of modeling GVCs with macro approaches]. The macro approaches emphasize the role of trade in intermediate inputs and of global inter-sectoral linkages in shaping response of the world economy to various types of shocks."

Figure B.2.1: Roundabout Production



Composite goods: CES across goods Producers of composite goods in sector j and country n supply final consumptions C_n^j and materials $M_n^{k,j}$ using a CES technology across goods:

$$Q_n^j = \left(\int q_n^j(w^j)^{\frac{\sigma_j-1}{\sigma_j}} dw^j\right)^{\frac{\sigma_j}{\sigma_j-1}} \tag{B.2.3}$$

where $\sigma^j > 1$ is the elasticity of substitution across intermediate goods within sector j, and $q_n^j(w^j)$ is the demand of intermediate goods w^j from the lowest cost supplier.

Trade costs There are two types of trade costs: iceberg trade costs and an ad-valorem flat-rate tariffs:

$$\kappa_{ni}^j = (1+\tau_{ni}^j)d_{ni}^j = \tilde{\tau}_{ni}^j d_{ni}^j \tag{B.2.4}$$

Prices and Expenditure Since production of intermediate goods is at CRS and markets are perfectly competitive, firms price at unit cost, $c_n^j / z_n^j(w^j)$, where c_n^j denotes the costs of an input bundle. In particular,

$$c_n^j = \Gamma_n^j \times (w_n^j)^{\gamma_n^j} \prod_{k=1}^{J} (P_n^k)^{\gamma_n^{k,j}} \tag{B.2.5}$$

Given trade cost structure, the price of ω^j in n is

$$p_n^j(w^j) = \min_i \frac{\kappa_{ni}^j c_n^j}{z_i^j(w^j)} \tag{B.2.6}$$

To deliver a gravity equation for intermediate goods producers, this papaer follows EK (2002) to introduce a probablistic representation of technologies. In particular, idisynractic productivities for intermediate goods producers are iid drawn from a Fréchet distribution with a location parameter $\lambda_n^j \geq 0$ and a shape paramter θ^j. Final goods producers source from the least cost supplier, therefore, the price of final goods P_n^j is:

$$P_n^j = A^j \left[\sum_i \lambda_i^j (c_i^j \kappa_{ni}^j)^{-\theta^j} \right]^{-\frac{1}{\theta^j}} \tag{B.2.7}$$

Using the properties of the Fréchet distribution, we can derive expenditure shares as a function of technologies, prices, and trade costs

$$\pi_{ni}^j = \frac{\lambda_i^j \left[c_i^j \kappa_{ni}^j \right]^{-\theta^j}}{\sum_{h=1}^N \lambda_h^j \left[c_h^j \kappa_{nh}^j \right]^{-\theta^j}} \tag{B.2.8}$$

B.2.1.2 Calibration

Dispersion of productivities (or trade elasticities) The trade elasticities θ^j are the key parameters for quantitative trade policy evaluation. In both the EK (2002) model and Caliendo and Parro (2015), the trade elasticities are related to the dispersion of productivities: If productivty is less dispersed (larger value of θ^j), then a change in tariffs will not substantially change the trade flows. The reason is that goods are less substitutable.[5]

Per equation (B.2.8), we can derive a multiplicative trade flows of sector j for countries i, h, n:

$$\frac{X_{ni}^j X_{ih}^j X_{hn}^j}{X_{nh}^j X_{hi}^j X_{in}^j} = \left(\frac{\kappa_{ni}^j}{\kappa_{in}^j} \frac{\kappa_{ih}^j}{\kappa_{hi}^j} \frac{\kappa_{hn}^j}{\kappa_{nh}^j} \right)^{-\theta^j} \tag{B.2.9}$$

Note that the equation above only involves trade flows and trade costs.

[5]Note that, in the Armington model, the trade elasticity is given by the elasticity of substitution.

B.3 Multinational Production

In this section, I summarize the multinational production literature that explicitly model the role of platform by multinational firms.These flows are important because

1. A large share of foreign affiliates' production is exported to other countries, beyond exports back to their parent countries (Tintelnot, 2017).

2. They also lead to interesting implications for commercial policy and for the welfare gains from trade and multinational production.

The main challenge for modeling export platforms is the ugly corner solutions for production, consumption, and trade. However, it is not surprising that, the same solution that EK (2002) provided for extending the Ricardian trade model to multiple countries is also helpful for the extension of the multinational production models to multiple countries: **A probabilistic structure based on the Frechet distribution.**

$$\Pr(Z_i(j) \leq z) = F_i(z) = \exp\Big(-T_i z^{-\theta}\Big) \tag{B.3.1}$$

where

- $T_i > 0$ governs the **location** of $F_i(z)$. Higher T_i implies higher productivity draw is more likely for any good j. "Absolute advantage."

- $\theta > 1$ determines the dispersion of productivity, where a higher θ means there is less dispersion (common across countries). "Comparative advantage."

B.3.1 "Trade, Multinational Production, and the Gains from Openness" (Ramondo and Rodríguez-Clare, 2013)

Motivation The existing literature on the gains from inter-country interactions studies **trade in goods**, and **MP/FDI** in isolation. The omission of combining these two interactions in analysis is important because trade agreements often combine tariff reductions and removal of barriers to MP.

Some Notations

B.3.1.1 Model

MP Cost Unit cost of the MP input bundle for MP by i in l:

$$c_{li} = \Big[(1-a)(c_l h_{li})^{1-\xi} + (a)(c_i d_{li})^{1-\xi}\Big]^{\frac{1}{1-\xi}} \tag{B.3.2}$$

Table B.3.1: Notations in (Ramondo and Rodríguez-Clare, 2013)

Notation	Meaning
$i = \{1, 2, 3, \cdots, I\}$	country of headquarter (parent)
$l = \{1, 2, 3, \cdots, I\}$	country of production
$n = \{1, 2, 3, \cdots, I\}$	country of destination
$d_{nl} \geq 1$	trade costs
$h^g_{li} \geq 1$	MP costs
$v \in [0, 1]$	tradable intermediate goods
$u \in [0, 1]$	non-tradable final goods

NOTES: $h^g_{li} \geq 1$ implies that home production is more efficient that those of foreign affiliates.

Productivity Distributions For a home country, she faces a vector $\mathbf{z}^s_i = (z_{1i}, z_{2i}, \cdots z_{Ii}), s = g, f$ that is drawn independently across goods and countries from a Multivariate Frechet distribution:

$$F_i(\mathbf{z}^s_i; T_i) = \exp\left[-T_i\left(\sum_{l=1}^{I}(z^s_{li})^{\frac{-\theta}{1-\rho}}\right)^{1-\rho}\right] \tag{B.3.3}$$

where ρ governs the correlation of productivity across production locations.

Equilibrium Analysis Final goods are non-tradable, so i must produce them in destination country n to obtain positive market share. Therefore, the price of final good u in n is

$$p^f_n(u) = \min_i \frac{c^f_{ni}}{z^f_{ni}}$$

As $z^f_{ni} \sim F(z)$, the share of expenditure by country n on final goods produced in country n with country i technologies is

$$\pi^f_{ni} = \frac{T_i(c^f_{ni})^{-\theta}}{\sum_j T_j(c^f_{nj})^{-\theta}} \tag{B.3.4}$$

Compared to non-tradable final goods, intermediate goods are tradable and can be imported from production countries that might differ from the home countries and destination countries ($i \neq l \neq n$).

The price of intermediate good v in n is

$$p^g_n(v) = \min_{i,l} \frac{c^g_{ni} d_{nl}}{z^g_{li}}, \quad d_{nl} \geq 1$$

where z^g_{li} is home-production technology. As $z^g_{li} \sim F(z)$, the share of expenditures by country n on intermediate goods produced in country l with country i technology is:

$$\pi_{nli}^{g} = \frac{T_i \left(\tilde{c}_{ni}^{g}\right)^{-\theta}}{\sum_j T_j \left(\tilde{c}_{nj}^{g}\right)^{-\theta}} \frac{\left(c_{li}^{g} d_{nl}\right)^{-\theta/(1-\rho)}}{\sum_k \left(c_{ki}^{g} d_{nk}\right)^{-\theta/(1-\rho)}} \tag{B.3.5}$$

where $\tilde{c}_{ni}^{g} = \left(\sum_k \left(c_{ki}^{g} d_{nk}\right)^{-\theta/(1-\rho)}\right)^{-(1-\rho)/\theta}$ This expression has a natural interpretation: The first term on the right-hand side is the share of expenditures that country n allocates to intermediate goods produced with country i's technologies independently of the location where they are produced, while the second term on the right-hand side is the share of these goods that are produced in country l.

B.3.1.2 Calibration

Chapter C

Spatial Economics

C.1 Spatial Equilibrium

C.1.1 "On the Equilibrium Properties of Spatial Models" (Allen, Arkolakis, and Li, 2024)

C.1.1.1 Setting

Exponential growth in computational power and new micro-data allow the empirical implementation of models where large number of agents (e.g., locations) interact with each other in multiple ways (e.g., spatial linkages).

Consider a system of N locations across which there are H types of interactions whose equilibrium can be reduced to a set of $N \times H$ equations of the following forms:

$$x_{ih} = \sum_{j=1}^{N} f_{ijh}(x_{j1}, x_{j2}, \cdots, x_{jH}) \tag{C.1.1}$$

where $\{x_{ih}\} \in \mathbb{R}_{++}^{N\times H}$ reflect the (strictly positive) equilibrium outcome of each interaction in each locations, and $f_{ijh} : \mathbb{R}_{++}^{H} \to \mathbb{R}_{++}$ are known functions that govern the interactions across locations. In other words, f_{ijh} governs the impact of location j on i's h via H linkages.

C.1.1.2 Overview

This paper provides a new theorem that offers

1. an iterative algorithm for calculating an equilibrium and
2. sufficient and "globally necessary" conditions under which the equilibrium is unique.

In short, this paper approaches the analysis of $N^2 \times H$ functions $\{f_{ijh}\}$ in equation (C.1.1) to a $H \times H$ matrix $(\mathbf{A})_{hh'} \equiv \sup_{ij} \left(\mid \frac{\partial \ln f_{ijh}}{\partial \ln x_{jh'}} \mid \right)$ that characterize the strength of the economic interactions.

C.1.1.3 Perov Fixed Point Theorem

Perov FP Theorem is a multi-dimensional extension of the standard contraction mapping theorem (CMT).[1] Let $\{(X_h, d_h)\}_{h=1,2,\ldots,H}$ be H metric spaces, where X_h is a set and d_h is its corresponding metric. Define $\mathbf{X} \equiv X_1 \times X_2 \times \ldots \times X_H$, and $d: X \times X \rightarrow \mathbb{R}_+^H$ such that for $x = (x_{1,\ldots,} x_H)$, $x' = \left(x'_{1,\ldots,}, x'_H\right) \in \mathbf{X}$, $d(x, x') = \begin{pmatrix} d_1(x_1, x'_1) \\ \ldots \\ d_H(x_H, x'_H) \end{pmatrix}$. Given operator $T: X \rightarrow X$, suppose for any $x, x' \in X$

$$d\left(T(x), T\left(x'\right)\right) \leq \mathbf{A} d\left(x, x'\right)$$

where $\mathbf{A}$ is a non-negative matrix and the inequality is entry-wise. Denote $\rho(\mathbf{A})$ as the spectral radius of $\mathbf{A}$. If $\rho(\mathbf{A}) < 1$ and for all $h = 1, 2, \ldots, H, (X_h, d_h)$ is complete, there exists a unique fixed point of T, and for any $x \in X$, the sequence of $x, T(x), T(T(x)), \ldots$ converges to the fixed point of T.

C.1.1.4 The Theorem

Table C.1.1: Notations in Allen, Arkolakis, and Li (2024)

Notation	Meaning
f_{ijh}	type h-spatial linkage
x_{ih}	location i's equilibrium outcome h
$\epsilon_{ijh,jh'}(x_j)$	the impact of location j's outcome h' on i's outcome h
$\mathbf{A}$	bounds of the elasticities $\epsilon_{ijh,jh'}(x_j)$
$\rho(\mathbf{A})$	spectral radius of matrix $\mathbf{A}$

NOTES:

Theorem 1 Suppose there exists an H-by-H matrix $\mathbf{A}$ such that for all $i, j \in \mathcal{N}, h, h' \in \mathcal{H}$, and $x_j \in \mathbb{R}_{++}^H$, $\left|\epsilon_{ijh,jh'}(x_j)\right| \leq (\mathbf{A})_{hh'}$. Then:

(i) If $\rho(\mathbf{A}) < 1$, then there exists a unique solution to equation (1) which can be computed by iteratively applying equation (1) with a rate of convergence $\rho(\mathbf{A})$;

(ii) If $\rho(\mathbf{A}) = 1$ and: a. For all $i \in \mathcal{N}$ and $h, h' \in \mathcal{H}$ when $(\mathbf{A})_{hh'} \neq 0$ there exists some j such that for all $x_j \in \mathbb{R}_{++}^H$, $\left|\epsilon_{ijh,jh'}(x_j)\right| < (\mathbf{A})_{hh'}$, then equation (1) has at most one solution; b. For all $x_j, \epsilon_{ijh,jh'}(x_j) = \alpha_{hh'} \in \mathbb{R}$ where $|\alpha_{hh'}| = (\mathbf{A})_{hh'}$ for all $i, j \in \mathcal{N}$ and $h, h' \in \mathcal{H}$-i.e. $f_{ijh}(x_j) = K_{ijh} \prod_{h' \in \mathcal{H}} x_{jh'}^{\alpha_{hh'}}$ for some $K_{ijh} > 0$-then there is at most one columnwise up-to-scale solution, i.e. for every two solutions x and x' and $h \in \mathcal{H}$, it must be $x'_{.h} = c_h x_{.h}$ for some scalar $c_h > 0$;

[1] Recall that the standard CMT follows (Stokey and Lucas Jr, 1989):

(iii) If $\rho(\mathbf{A}) > 1$ and $N \geq 2H+1$, then there exists some $\{K_{ijh} > 0\}_{i,j\in\mathcal{N},h\in\mathcal{H}}$ such that for $f_{ijh}(x_j) = K_{ijh}\prod_{h'\in\mathcal{H}} x_{jh'}^{\alpha_{hh'}}$ where $\alpha_{hh'} \in \mathbb{R}$ and $|\alpha_{hh'}| = (\mathbf{A})_{hh'}$, equation (1) has multiple solutions that are column-wise up-to-scale different, i.e. it has two solutions x and x' such that for some $h \in \mathcal{H}, x'_{.h} \neq c_h x_{.h}$ with every $c_h > 0$.

Proof. Part (i): Notice that equation (C.1.1) can be written as $y_{ih} \equiv \ln x_{ih} = \ln \sum_{j\in\mathcal{N}} f_{ijh}(\exp \ln x_j)$ and furthermore denote its right side as function $g_{ih}(y)$ for matrix y, we thus have:

$$\frac{\partial g_{ih}}{\partial y_{jh'}} = \frac{\epsilon_{ijh,jh'}(\exp y_j)\, f_{ijh}(\exp y_j)}{\sum_{k\in\mathcal{N}} f_{ikh}(\exp y_j)} \tag{C.1.2}$$

Given any y and y', according to the mean value theorem, for each i and h, there exists $\hat{y} = (1-t_{ih})\, y + t_{ih} y'$ where $t_{ih} \in [0,1]$ such that:

$$g_{ih}(y) - g_{ih}(y') = \sum_{j\in\mathcal{N},h'\in\mathcal{H}} \frac{\partial g_{ih}(\hat{y})}{\partial y_{jh'}} \left(y_{jh'} - y'_{jh'}\right) \tag{C.1.3}$$

Equations (C.1.2) and (C.1.3) together with condition $\left|\epsilon_{ijh,jh'}(x_j)\right| \leq (\mathbf{A})_{hh'}$, imply

$$\left|g_{ih}(y) - g_{ih}(y')\right| \leq \sum_{h'\in\mathcal{H}} (\mathbf{A})_{hh'} \max_{j\in\mathcal{N}} \left|y_{jh'} - y'_{jh'}\right| \tag{C.1.4}$$

For any $h \in H$, define metric $d_h(y_h, y'_h) = \max_{j\in\mathcal{N}} \left|y_{jh} - y'_{jh}\right|$ on space $Y_h \equiv \mathbb{R}^N$. Furthermore, define $Y = Y_1 \times \ldots \times Y_H$ and $d(y,y') = [d_1(y_1,y'_1), \ldots, d_H(y_H,y'_H)]'$ for $y, y' \in Y$. Notice that inequality (C.1.4) then becomes $d(g(y), g(y')) \leqq \mathbf{A} d(y,y')$. Thus we can apply the Perov Fixed Point Theorem to obtain the desired results (existence, uniqueness and computation).

C.1.1.5 Application to Spatial Models with Input-output Linkages

C.2 Static Models

C.2.1 "The Impact of Regional and Sectoral Productivity Changes on the US Economy" (Caliendo et al., 2018)

C.2.1.1 Model

Notation

- J: Number of sectors
- N: Number of locations
- Superscript for sector, and subscript for location: γ_n^{jk} for I-O linkage between j and k in n. $(1-\beta_n)$ labor share in n. Q_n^j final goods in sector j produced in n.

C.2.1.2 Counterfactual Equilibrium in Relative Changes

Cost of input bundle (JN equations):

$$\hat{x}_n^j = (\hat{\omega}_n)^{\gamma_n^j} \prod_{k=1}^{J} \left(\hat{P}_n^k\right)^{\gamma_n^{jk}} \tag{C.2.1}$$

Price index (JN equations):

$$\hat{P}_n^j = \left(\sum_{i=1}^{N} \pi_{ni}^j \left[\hat{\kappa}_{ni}^j \hat{x}_i^j\right]^{-\theta^j} \hat{T}_i^{j\theta^j \gamma_i^j} \right)^{-1/\theta^j} \tag{C.2.2}$$

Trade shares (JN^2 equations)

$$\pi_{ni}^{j\prime} = \pi_{ni}^j \left(\frac{\hat{x}_i^j}{\hat{P}_n^j} \hat{\kappa}_{ni}^j \right)^{-\theta^j} \hat{T}_i^{j\theta^j \gamma_i^j} \tag{C.2.3}$$

Labor mobility condition (N equations):

$$\hat{L}_n = \frac{\left(\frac{\hat{\omega}_n}{\varphi_n \hat{P}_n \hat{U} + (1-\varphi_n)\hat{b}_n} \right)^{1/\beta_n}}{\sum_i L_i \left(\frac{\hat{\omega}_i}{\varphi_i \hat{P}_i \hat{U} + (1-\varphi_i)\hat{b}_i} \right)^{1/\beta_i}} L \tag{C.2.4}$$

Regional market clearing in final goods (JN equations):

$$X_n^{j\prime} = \sum_{k=1}^{J} \gamma_n^{k,j} \left(\sum_{i=1}^{N} \pi_{in}^{k\prime} X_i^{k\prime} \right) + \alpha^j \left(\hat{\omega}_n \left(\hat{L}_n\right)^{1-\beta_n} (I_n L_n + \Upsilon_n + S_n) - S_n^\prime - \Upsilon_n^\prime \right) \tag{C.2.5}$$

Labor market clearing (N equations)

$$\hat{\omega}_n \left(\hat{L}_n\right)^{1-\beta_n} (L_n I_n + \Upsilon_n + S_n) = \sum_j \gamma_n^j \sum_i \pi_{in}'^j X_i'^j \tag{C.2.6}$$

C.2.1.3 Calibration

Quantitative analysis on the effects of sectoral-regional shocks to the aggreagte economy requires

- **Data**: $\{I_n, L_n, \Upsilon_n, \pi_{ni}^j\}_{n=1,i=1,j=1}^{N,N,J}$.[2]
- **Values for parameters**: $\{\theta^j, \alpha^j, \beta_n, \gamma_n^j, \gamma_n^{jk}\}_{n=1,j=1,k=1}^{N,J,J}$.

The following table summarizes the details of calibration.

Table C.2.1: Calibration in Caliendo et al. (2018)

Baseline Shares/Parameters	Details
L_n (employment)	employment
$r_n H_n$ (the payment to structures and land)	$\beta_n VA_n$
I_n (Income per capita)	$\frac{VA_n}{L_n} - \frac{\Upsilon_n}{L_n}$
$\iota_n \in [0,1]$	$\text{argmin}_{\iota_n} (\Upsilon_n^{Data} - \Upsilon_n^{Model}(\iota))'(\Upsilon_n^{Data} - \Upsilon_n^{Model}(\iota))$
θ_j (dispersion of productivities)	Caliendo and Parro (2015)

NOTES: $\Upsilon_n^{Model}(\iota)$ is a function of $\{\iota_i\}_{i=1}^N$: $\Upsilon_n^{Model}(\iota) = \iota_n \times \underbrace{r_n H_n}_{\beta \times VA_n} - \frac{\sum_i \iota_i r_i H_i}{L} L_n$

[2]Recall that I_n is income per capita for residents in n: $I_n = w_n + \underbrace{\frac{\sum_{i=1}^N \iota_i r_i H_i}{L}}_{\equiv \chi} + (1-\iota_n)\frac{r_n H_n}{L_n}$. Υ_n is trade surplus, such that $I_n L_n + \chi L_n = w_n L_n + (1-\iota_n) r_n H_n + \underbrace{r_n \iota_n H_n}_{\Upsilon_n + \chi L_n}$

C.3 Dynamic Models

C.4 Spatial Growth

C.5 Optimal Spatial Policy

C.6 Transportation Networks

Chapter D

Econometrics

D.1 Shfit-Share Designs

D.1.1 "A Practical Guide to Shift-Share Instruments" (Borusyak, Hull, and Jaravel, 2024)

A recent econometric literature shows two distinct paths for identification with shift-share instruments, leveraging either many exogenous shifts (Borusyak, Hull, and Jaravel, 2022; Adão, Arkolakis, and Esposito, 2019) or exogenous shares (Goldsmith-Pinkham, Sorkin, and Swift, 2020).

This paper presents the core logic of both paths and practical takeaways.

D.1.1.1 ADH (2003)'s SSIV

The influential China shock paper by ADH constructs an instrumental variable with a shift-share structure:

$$\text{SSIV}_i = \sum_k \text{emp share}_{i,k} \times \text{avg. of growth in Chinese import among non-US countries}_k \tag{D.1.1}$$

where k denotes industry and j denotes commuting zone.

D.1.1.2 Definition of SSIV

A shift-share structure follows

$$z_i = \sum_{k=1}^{K} \underbrace{s_{ik}}_{\text{Share}} \underbrace{g_k}_{\text{Shift}} \tag{D.1.2}$$

Remarks

- Shifts vary at a different level (e.g. industries) than the unit of analysis (e.g. local labor markets).

- Shares vary across units but are usually predetermined (e.g., employment shares are measured in a pre-period).
- To argue convincingly that SSIV are exogenous, one must explain what properties of the shifts and shares make z_i uncorrelated with ϵ_i (rather than simply stating $\mathrm{Cov}[z_i, \epsilon_i] = 0$).
- $\sum_{k=1}^{K} s_{ik}$ is generally one. For incomplete share see Section D.1.1.3.

One strategy to ensure that the shift-share instrument z_i is exogenous is to have exogenous shift g_k. The key threat to identification in the exogenous shifts approach is the violation of the following condition: g_k **should be uncorrelated with an average of** ϵ_i **taken across units with weights** s_{ik}**.**

D.1.1.3 Incomplete Shift Share

In shift-share designs where the exposure shares $s_{i,k}$ do not add up to one, a special control must by included: the sum of shares, $S_i = \sum_k s_{i,k}$. This control remedies the bias arising from the correlation between S_i and the error.

D.1.1.4 A Checklist for the Shift-Based Approach

1. Thinking about what endogeneity bias is being addressed.
2. Bridge the gap between the observed and ideal shifts. Control for $\sum_k s_{ik} q_k$: shift-share aggregates of the industry-level confounders
3. Include the "incomplete share" control.
4. Lag shares to the beginning of the natural experiment.
5. Report descriptive statistics for shifts, such as mean and std. of z_i and g_k.
6. Implement balance tests for shifts in addition to the instrument.
7. Use correct standard errors.

Chapter E

Computational Economics

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