1. Bob tosses three fair coins. Given that at least one is a head, what is the probability that there are more heads than tails in the final outcome?

   Solution: Let A and B be the events that at least one is a head and there are more head than tails, respectively. We want to know the probability of $\mathrm{P}(B|A)$. Then,

   $$\mathrm{P}(A) = 1 - \mathrm{P}(A^c) = 1 - (\frac{1}{2})^3 = \frac{7}{8}$$

   and

   $$\mathrm{P}(A \cap B) = \mathrm{P}(B) = \binom{3}{2} \cdot (\frac{1}{2})^2 \cdot \frac{1}{2} + \binom{3}{3} \cdot (\frac{1}{2})^3 = \frac{1}{2}$$

   Therefore,

   $$\mathrm{P}(B \mid A) = \frac{\mathrm{P}(B \cap A)}{\mathrm{P}(A)} = \frac{4}{7}.$$

2. Alice tosses a six-sided dice, then she tosses R fair coins, where R is the outcome of the die. Let M be the event that all the coin tosses came out tails, and Y_i be the event that the outcome of the die is i. Calculate (a) $P(M \mid Y_3)$ (b) $P(M)$ (c) $P(Y_3 \mid M)$.

   Solution: Let Y_i be the event that the roll of the die is i, and A be the event that all the coin tosses are tails.

   (a) By the definition of conditional probability,

   $$P(M \mid Y_3) = \frac{P(M \cap Y_3)}{P(Y_3)} = \frac{\frac{1}{6} \cdot \frac{1}{2^3}}{\frac{1}{6}} = \frac{1}{2^3}$$

   (b) The event M consists of six smaller sub-events. Let M_i be the event that the outcome of the die is i and all i coin tosses come out tails. Then

   $$P(M) = P(M_1) + P(M_2) + P(M_3) + P(M_4) + P(M_5) + P(M_6)$$

   Since $P(M_i) = P(M \cap Y_i) = \frac{1}{6} \cdot \frac{1}{2^i}$, $P(M) = \frac{1}{6} \cdot \sum_{i=1}^{6} \frac{1}{2^i} = \frac{63}{384}$.

   (c) By the definition of conditional probability,

   $$P(Y_3 \mid M) = \frac{P(M \cap Y_3)}{P(M)} = \frac{8}{63}$$

3. There are 5 red balls and 2 blue balls. Each ball is randomly placed in one of two bins.

   (a) Find the probabilities that the first bin contains k balls for $k \in \{0, 1, 2, 3\}$.

   Solution: The sample space Ω consists of all sequences of length 7, where the value of each position can be either `1` or `2`, denoting which bin the ball goes to. Ω has size 2^7. Let E_k denote the event the first bin contains k balls where $k \in \{0, 1, 2, 3\}$. Then E_k consists of strings that contain k `1`s and $7 - k$ `2`s. Therefore $P(E_k) = \binom{7}{k}/2^7$. For $k \in \{0, 1, 2, 3\}$, we have

k	0	1	2	3
$P(E_k)$	1/128	7/128	21/128	35/128

(b) Suppose that the first bin contains 3 balls, what is the probability that they are all red balls?

Solution: Let A denote the event that all balls in the first bin are red. By the definition of conditional probability,

$$P(A \mid E_3) = \frac{P(A \cap E_3)}{P(E_3)}$$

Since $P(A) = \frac{\binom{5}{3}}{2^7}$ and $P(E_3) = \frac{35}{128}$, we have $P(A \mid E_3) = 2/7$.

4. A bag contains three fair coins and four bias coins and tossing a bias coin results in a head with probability 3/4. Alice randomly chooses a coin and toss them. Suppose she gets a head, what is the probability that Alice gets a fair coin?

Solution: Let A denote the event that Alice gets a fair coin, and H denote the event that Alice gets a head. By the definition of conditional probability,

$$P(A \mid H) = \frac{P(A \cap H)}{P(H)}$$

We know that $P(A \cap H)$ is the probability that Alice chooses a fair coin and tossing the coin comes out head, so $P(A \cap H) = \frac{3}{7} \times \frac{1}{2}$. Also, $P(H)$ is the probability that tossing the chosen coin comes out head, no matter whether Alice gets a fair coin or bias coin. Therefore, $P(H) = \frac{3}{7} \times \frac{1}{2} + \frac{4}{7} \times \frac{3}{4}$. Thus, we have $P(A \mid H) = \frac{1}{3}$.