1. A factory has 200 old widgets, and 500 new widgets in stock. We know that 15% of the old widgets are defective, and 5% of the new ones are defective as well. Alice randomly chooses a widget in the factory. Given that the widget turn out to be defective, what is the probability that it is an old widget?

   Solution: Let D be the event that the chosen widgets are defectives, O be the event that the widget is old and N be the event that the widget is new. Using Bayes' rule,

$$P(O \mid D) = \frac{P(O) \cdot P(D \mid O)}{P(O) \cdot P(D \mid O) + P(N) \cdot P(D \mid N)} = \frac{6}{11}$$

2. Alice usually takes a bus to her company. In summer, it is rainy with probability $\frac{1}{3}$. Given that it is rainy, there will be heavy traffic with probability $\frac{1}{2}$, and given that it is not rainy, there will be heavy traffic with probability $\frac{1}{5}$. If it's rainy and there is heavy traffic, Alice arrive late for work with probability $\frac{1}{2}$. On the other hand, the probability of being late is reduced to $\frac{1}{10}$ if it is not rainy and there is no heavy traffic. In other situations (rainy and no traffic, not rainy and traffic) the probability of being late is $\frac{1}{5}$. Suppose you were Alice, you pick a random day in summer:

   Solution: Let R be the event that it's rainy, T be the event that there is heavy traffic, and L be the event that I am late for work. By the multiplication rule, we can compute the probabilities of each outcome in the sample space.

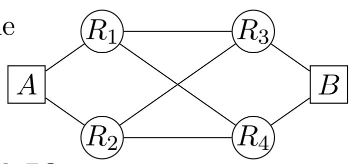
   (a) What is the probability that it's not raining and there is heavy traffic and Alice is not late?
   The required probability is $P(R^c \cap T \cap L^c) = P(R^c)P(T \mid R^c)P(L^c \mid R^c \cap T) = 2/3 \times 1/5 \times 4/5 = 8/75$.

   (b) What is the probability that Alice is late?
   The event that Alice is late includes several sub-events $R \cap T \cap L$, $R^c \cap T \cap L$, $R \cap T^c \cap L$ and $R^c \cap T^c \cap L$, and all these sub-events are disjoint. Therefore, the probability is

$$\begin{aligned} P(L) &= P(R \cap T \cap L) + P(R^c \cap T \cap L) + P(R \cap T^c \cap L) + P(R^c \cap T^c \cap L) \\ &= 1/3 \times 1/2 \times 1/2 + 2/3 \times 1/5 \times 1/5 + 1/3 \times 1/2 \times 1/5 + 2/3 \times 4/5 \times 1/10 \\ &= 1/12 + 2/75 + 1/30 + 8/150 \\ &= 59/300 \end{aligned}$$

   (c) Given that Alice arrived late at work, what is the probability that it rained that day?
   By definition of conditional probability, we have $P(R \mid L) = P(R \cap L)/P(L) = (P(R \cap T \cap L) + P(R \cap T^c \cap L))/P(L) = 35/59$.

3. Computers A and B are linked through routers R_1 to R_4 as in the picture. Each router fails independently with probability 10%.



   (a) What is the probability there is a connection between A and B?

Solution: Let R_i be the event that router i is operational. The event "there is a connection between A and B" is $(R_1 \cup R_2) \cap (R_3 \cup R_4)$. By independence

$$\begin{aligned}
\mathrm{P}((R_1 \cup R_2) \cap (R_3 \cup R_4)) &= \mathrm{P}(R_1 \cup R_2)\,\mathrm{P}(R_3 \cup R_4) \\
&= (1 - \mathrm{P}(R_1^c \cap R_2^c))(1 - \mathrm{P}(R_1^c \cap R_2^c)) \\
&= (1 - \mathrm{P}(R_1^c)\,\mathrm{P}(R_2^c))(1 - \mathrm{P}(R_3^c)\,\mathrm{P}(R_4^c)) \\
&= (1 - 0.1^2)^2 \\
&= 0.9801.
\end{aligned}$$

(b) Are the events "there is a connection between A and B" and "exactly two routers fail" independent? Justify your answer.

Solution: Let C be the event that there is a connection between A and B, F be the event that exactly two routers fail, and R_i be the event that router i is operational. So we need to determine whether $P(F \cap C) = P(C) \cdot P(F)$.

As shown in (a), we have $P(C) = (1 - 0.1^2)^2$. As for the event F. we need to choose two of the four routers to fail, the probability is:

$$P(F) = \binom{4}{2} \times \frac{1}{10^2} \times \frac{9^2}{10^2}$$

As for $C \cap F$, we have four cases: R_1R_3, R_1R_4, R_2R_3, R_2R_4. Therefore, the probability is that

$$P(C \cap F) = 4 \times \frac{1}{10^2} \times \frac{9^2}{10^2}$$

After calculation, we know that $P(C \cap F) \neq P(C) \cdot P(F)$, so these two events are not independent.

Alternative Solution No. The probability that there is a connection between A And B given that exactly two routers fail is $2/3$: given that exactly two routers fail, the failed routers are equally likely to be any of the 6 pairs $R_1R_3, R_1R_4, R_2R_3, R_2R_4, R_1R_2, R_3R_4$, and there is a connection between A and B in the first 4 out of these 6 possibilities. This probability is not equal to the unconditional probability from part (a) and so the two events are not independent.

4. In a certain business school, the ratio of the number of full-time students to part-time students is 15:10. At the end of their studies, all the school's 1700 students took a professional examination and 1100 passed. It is known that percentage of the full-time students passing the examination was twice that of the part-time students. A student chosen at random is found to have failed the examination. What is the probability that he was a part-time student?

Solution: We know that the number of full-time students is $1700 \times 15/25 = 1020$ and the number of part-time students is $1700 \times 10/25 = 680$. Let the pass percentage of full-time students be p. Then we have $1020p + 680(0.5p) = 1100$, and $p = \frac{55}{68}$. The pass percentage of part-time students is $0.5p = \frac{55}{136}$, and the fail percentage is $\frac{81}{136}$.

Let F and R be the event that the student is full-time and part-time respectively. Also, let A be the event that the student have failed the examination. Using Bayes' rule, we have

$$P(R \mid A) = \frac{P(A \mid R)P(R)}{P(A)} = \frac{\frac{81}{136} \cdot \frac{680}{1700}}{\frac{600}{1700}} = \frac{27}{40}.$$