Two-Player Zero-Sum Games

Coin game: L = Left hand has \$1, right hand nothing
R = Left hand nothing, right hand \$2.

You guess L or R, and get money in that hand.

How much are you willing to pay for this game?

☹ 0, because I may always lose

☺: 2, I can get lucky & win \$2.

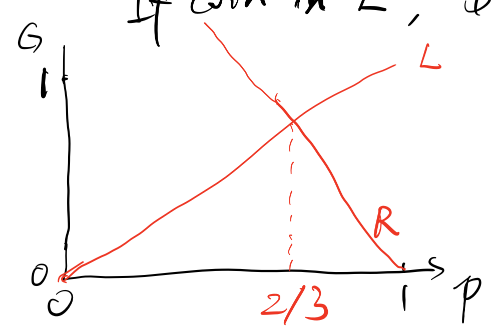
😐: 1, always choose L as I know you're stingy.

How would a rational person think?

Two pure strategies
- L (If dealer R, I lose)
- R (If dealer L, I lose)

Mixed strategy: I flip a coin, with heads p.
→ L with prob. p, R with prob. $1-p$.

⇒ Expected gain of game:
If coin in L, \$$p$; if coin in R, \$$2(1-p)$



At the point of $p = 2(1-p)$, no matter what the dealer does, your expected gain is guaranteed to be \$2/3.
In other cases, it depends on the dealer.

Now let's look at the game from the dealer's perspective.

L with prob. q, R with $1-q$. (mixed strategy)

My move	Expected loss
L	$\$q$
R	$\$2(1-q)$

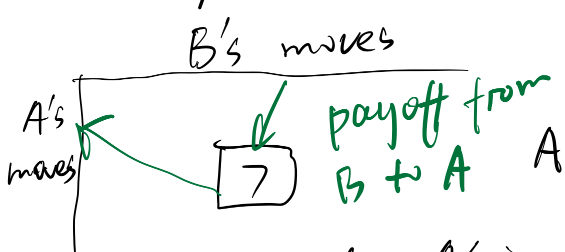
To minimize the exp. loss regardless of what I may do (dealer doesn't know what I'm going to do), the same line of reasoning works here.

$q = 2(1-q)$

$\boxed{\text{Dealer's expected loss} = \text{My expected gain}}$

Minimax Theorem (John von Neumann)

2 players. A, B.



B - Dealer

$$
\begin{array}{c|cc}
 & L & R \\
\hline
L & 1 & 0 \\
R & 0 & 2
\end{array}
$$

A

A strategy for A/B is a probability assignment to her actions.

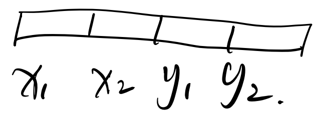
A's minimum expected gain is her expected gain for the worst possible move from B.

B's maximum expected loss is the symmetric notion.

Minimax Theorem. There exist strategies for which min exp gain = max exp loss.

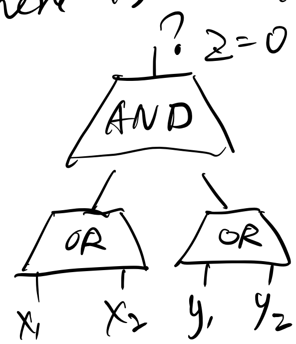
A or B cannot play better than this.

Applications



There's a 4-bit database. $x_1\ x_2\ y_1\ y_2$.

Need to pay \$1 to see one bit in this db. How to minimize the total cost?

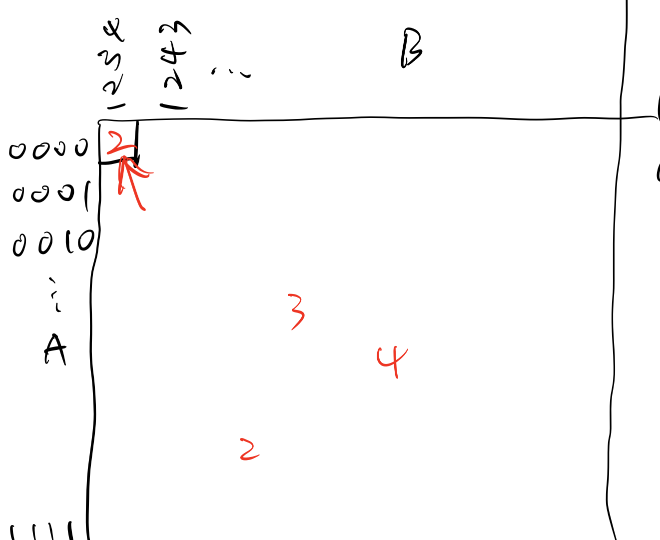


Naive: 4 bits, \$4.

A possible randomized strategy for B

① randomly choose an OR to inspect,

② randomly & independently choose order of its inputs.



Case 1: $z=0$.

$$E[\text{cost of } B] = \underbrace{\frac{1}{2}\cdot 2}_{\text{lucky}} + \frac{1}{2}\cdot\left(\frac{1}{2}\cdot 3 + \frac{1}{2}\cdot 4\right) = 2\frac{3}{4}$$

for the "1" OR, may only need to open 1 bit.

Case $z = 1$:

$$E[B' \text{ cost}] = \underset{\uparrow}{2} \cdot \left(\tfrac{1}{2} \cdot 1 + \tfrac{1}{2} \cdot 2\right) = 3$$

has to check

2 OR

So B is guaranteed to see expected cost of 3.
(worst-case). better than naive strategy.

From A's perspective: max her gain $\rightarrow$ max B's cost.

$\Rightarrow$ Set $x_1 x_2 y_1 y_2 \in \{0101, 0110, 1010, 1001\}$
1/4 1/4 1/4 1/4

For each of B's actions, has to check 2 OR gates.

$E[A\text{'s gain}] = 3$

$\Rightarrow$ So 3 is the value of the game.

Zero-sum games.