1. In how many ways can we pick 2 books from a collection of 3 mathematics books, 5 computer science books, and 7 programming books if

   (a) both books are to be on the same subject?

   (b) the books are to be on different subjects?

   Solution:

   (a) There are $\binom{3}{2}$, $\binom{5}{2}$ and $\binom{7}{2}$ ways to pick 2 mathematics books, computer science books and programming books respectively. Therefore there are $\binom{3}{2} + \binom{5}{2} + \binom{7}{2}$ ways to pick two books on the same subject.

   (b) There are $\binom{3}{1}\binom{5}{1}$ ways to pick a mathematics book and a computer science book; $\binom{5}{1}\binom{7}{1}$ ways to pick a computer science book and a programming book; $\binom{7}{1}\binom{3}{1}$ ways to pick a programming book and a mathematics book; so in total there are $\binom{3}{1}\binom{5}{1} + \binom{5}{1}\binom{7}{1} + \binom{7}{1}\binom{3}{1}$ to pick two books on different subjects.

2. On any given week, Alice is 70% likely to attend Monday class, 80% likely to attend Thursday tutorial, and 60% likely to attend both. Given that she wasn't in class on Monday, what is the probability that she will show up to Thursday tutorial?

   Solution: Let M and T be the events that Alice attends Monday class and Thursday tutorial, respectively. By the definition of conditional probability and the axioms,

$$\mathrm{P}(T \mid M^c) = \frac{\mathrm{P}(T \cap M^c)}{\mathrm{P}(M^c)} = \frac{\mathrm{P}(T) - \mathrm{P}(T \cap M)}{1 - \mathrm{P}(M)} = \frac{0.8 - 0.6}{1 - 0.7} \approx 0.666.$$

3. Alice draws cards one by one from a shuffled 52-card deck. Find the PMF of the turn T at which she has drawn the fourth (and last) ace.

   Solution: The sample space consists of all $\binom{52}{4}$ arrangements of four (indistinguishable) aces and 48 other cards under equally likely outcomes. The event $T = t$ occurs when the first $t-1$ cards contain 3 aces and the t-th card is an ace. By the product rule there are $\binom{t-1}{3}$ such arrangements. By the equally likely outcomes formula

$$\mathrm{P}(T = t) = \frac{\binom{t-1}{3}}{\binom{52}{4}} = \frac{4 \cdot (t-1)(t-2)(t-3)}{52 \cdot 51 \cdot 50 \cdot 49} \tag{1}$$

   for $1 \leq t \leq 52$.

   Alternative solution: The event $T = t$ occurs if no ace is drawn in turns $t+1$ up to 52 and an ace is drawn in turn t. Let A_i be the event that an ace is drawn in turn i. By the multiplication rule

$$\begin{aligned}
\mathrm{P}(T = t) &= \mathrm{P}(A_t \cap A_{t+1}^c \cap \cdots \cap A_{52}^c) \\
&= \mathrm{P}(A_{52}^c) \cdot \mathrm{P}(A_{51}^c | A_{52}^c) \cdots \mathrm{P}(A_{t+1}^c | A_{t+2}^c \cap \cdots \cap A_{52}^c) \cdot \mathrm{P}(A_t | A_{t+1}^c \cap \cdots \cap A_{52}^c) \\
&= \frac{48}{52} \cdot \frac{47}{51} \cdots \frac{t-3}{t+1} \cdot \frac{4}{t}
\end{aligned}$$

   after cancellation this expression reduces to (1).

Yet another solution: The sample space consists of the $52 \cdot 51 \cdot 50 \cdot 49$ possible positions of the four aces in the deck under equally likely outcomes. The event consists of those outcomes in which three of the four positions are less than t and the fourth is equal to t. By the generalized product rule, there are four ways to choose the ace in position t, $t-1$ ways to choose the position of the first remaining ace, $t-2$ ways to choose the position of the second remaining ace, and $t = 3$ ways to choose the position of the third remaining ace. By the equally likely outcomes formula we obtain (1) again.

4. Eight boys and eight girls are randomly seated at a round table. What is the expected number of boys that are seated between two girls?

   Solution: Let X_i be the indicator random variable for the event "boy i is seated between two girls", and L_i, R_i be the events "the person to the left, right of boy i is a girl". By the multiplication rule:
$$\mathrm{P}(X_i = 1) = \mathrm{P}(L_i)\,\mathrm{P}(R_i|L_i) = \frac{8}{15} \cdot \frac{7}{14} = \frac{4}{15}.$$
   Therefore $\mathrm{E}[X_i]$ is also $4/15$ for all i. By linearity of expectation, the expected number of boys seated between two girls is
$$\mathrm{E}[X_1 + \cdots + X_8] = \mathrm{E}[X_1] + \cdots + \mathrm{E}[X_8] = 8 \cdot \frac{4}{15} = \frac{32}{15}.$$

5. You go to the casino with \$3 to play roulette. (Note: A roulette has 18 black slots, 18 red slots, and 1 green slot which is for the house only.) Calculate the expected value and standard deviation of the amount you lose under the following two gambling strategies:

   (a) You play for 3 rounds, where in every round you bet \$1 on red.

   (b) You bet all your money on red. If you win, you bet everything on red again. If you win again, you bet everything on red one last time.

   Solution:

   (a) **Solution 1:** Let L be the amount you lose after playing 3 rounds. The probability mass function of L is

x:	-3	-1	1	3
$f(x)$:	p^3	$3p^2(1-p)$	$3p(1-p)^2$	$(1-p)^3$

   where $p = 19/37$ is the probability that you lose a particular round. From here we can calculate
$$E[L] = (-3) \cdot p^3 + (-1) \cdot 3p^2(1-p) + 1 \cdot 3p(1-p)^2 + 3 \cdot (1-p)^3 = 0.081$$
   and
$$E[L^2] = 9 \cdot (p^3 + (1-p)^3) + 1 \cdot (3p^2(1-p) + 3p(1-p)^2) \approx 3.004$$
   so
$$Var[L] = E[L^2] - E[L]^2 \approx 2.997$$
   and the standard deviation is $\sigma = \sqrt{2.997} \approx 1.731$.

   Solution 2: Let L be the amount you lose and Y be the number of rounds you win. Then Y is Binomial(n, p) with $n = 3$, $p = 18/37$, and $L = 3 - 2Y$. So we have
$$E[Y] = np = 54/37 \qquad Var[Y] = np(1-p) = 1026/1369.$$
   Because $L = 3 - 2Y$, $E[L] = 3 - 2E[Y] = 3/37$ and
$$Var[L] = Var[3 - 2Y] = Var[2Y] = 2^2 \cdot Var[Y] = 4104/1369.$$
   so the standard deviation is $\sigma = \sqrt{4104/1369} \approx 1.7314$.

(b) Let Z be the amount you walk out of the casino with. The probability mass function of Z is

z:	0	24
$g(z)$:	$1-p^3$	p^3

So $E[Z] = 24p^3 = 139968/50653$, $E[Z^2] = 24^2 p^3 = 3359232/50653$, and $Var[X] = E[X^2] - (E[X])^2 \approx 58.6828$.

If L is the amount you lose, then $L = 3 - Z$ so the expectation is $\sigma = E[L] = 3 - E[X] \approx 0.237$, the variance is $Var[L] = Var[X]$ and the standard deviation is $\sigma = \sqrt{Var[L]} \approx 7.660$.

6. Let X and Y be random variables that take values from the set $\{-1, 0, 1\}$.

(a) Find a joint probability mass function for which X and Y are independent, and confirm that X^2 and Y^2 are also independent.

(b) (Hard) Find a joint pmf for which X and Y are **not** independent, but X^2 and Y^2 are independent.

Solution:

(a) We assign a joint probability mass function for X and Y as shown in the table below. The values are designed to observe the relations: $P_{XY}(x_k, y_j) = P_X(x_k) P_Y(y_j)$ for all k, j. Hence, the independence property of X and Y is enforced in the assignment.

$P_{XY}(x_k, y_j)$	$x_1 = -1$	$x_2 = 0$	$x_3 = 1$	$P_Y(y_j)$
$y_1 = -1$	$\frac{1}{12}$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{2}$
$y_2 = 0$	$\frac{1}{18}$	$\frac{1}{9}$	$\frac{1}{6}$	$\frac{1}{3}$
$y_3 = 1$	$\frac{1}{36}$	$\frac{1}{18}$	$\frac{1}{12}$	$\frac{1}{6}$
$P_X(x_k)$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$	

Given the above assignment for X and Y , the corresponding joint probability mass function for the pair X^2 and Y^2 is seen to be

$P_{X^2Y^2}(\widetilde{x}_k, \widetilde{y}_j)$	$\widetilde{x}_1 = 1$	$\widetilde{x}_2 = 0$	$P_{Y^2}(\widetilde{y}_j)$
$\widetilde{y}_1 = 1$	$\frac{1}{12} + \frac{1}{4} + \frac{1}{36} + \frac{1}{12} = \frac{4}{9}$	$\frac{1}{6} + \frac{1}{18} = \frac{2}{9}$	$\frac{2}{3}$
$\widetilde{y}_2 = 0$	$\frac{1}{18} + \frac{1}{6} = \frac{2}{9}$	$\frac{1}{9}$	$\frac{1}{3}$
$P_{X^2}(\widetilde{x}_k)$	$\frac{2}{3}$	$\frac{1}{3}$	

Note that $P_{X^2,Y^2}(\widetilde{x}_k, \widetilde{y}_j) = P_{X^2}(\widetilde{x}_k) P_{Y^2}(\widetilde{y}_j)$ for all k and j, so X^2 and Y^2 are also independent.

(b) Suppose we take the same joint pmf assignment for X^2 and Y^2 as in the second table, but modify the joint pmf for X and Y as shown in the following table.

$P_{XY}(x_k, y_j)$	$x_1 = -1$	$x_2 = 0$	$x_3 = 1$	$P_Y(y_j)$
$y_1 = -1$	$\frac{1}{4}$	$\frac{1}{6}$	$\frac{1}{12}$	$\frac{1}{2}$
$y_2 = 0$	$\frac{1}{18}$	$\frac{1}{9}$	$\frac{1}{6}$	$\frac{1}{3}$
$y_3 = 1$	$\frac{1}{12}$	$\frac{1}{18}$	$\frac{1}{36}$	$\frac{1}{6}$
$P_X(x_k)$	$\frac{7}{18}$	$\frac{1}{3}$	$\frac{5}{18}$	

This new joint pmf assignment for X and Y can be seen to give rise to the same joint pmf assignment for X^2 and Y^2 in the second table. However, in this new assignment, we observe that

$$\frac{1}{4} = P_{XY}(x_1, y_1) \neq P_X(x_1) P_Y(y_1) = \frac{7}{18} \cdot \frac{1}{2} = \frac{7}{36},$$

and the inequality of values can be observed also for $P_{XY}(x_1, y_3), P_{XY}(x_3, y_1)$ and $P_{XY}(x_3, y_3)$, etc. Hence, X and Y are **not** independent.

Remark

1. Since -1 and 1 are the two positive square roots of 1, we have

$$P_X(1) + P_X(-1) = P_{X^2}(1) \quad \text{and} \quad P_Y(1) + P_Y(-1) = P_{Y^2}(1)$$

therefore

$$\begin{aligned} P_{X^2}(1)P_{Y^2}(1) &= [P_X(1) + P_X(-1)]\,[P_Y(1) + P_Y(-1)] \\ &= P_X(1)P_Y(1) + P_X(-1)P_Y(1) + P_X(1)P_Y(-1) + P_X(-1)P_Y(-1). \end{aligned}$$

On the other hand, $P_{X^2Y^2}(1,1) = P_{XY}(1,1) + P_{XY}(-1,1) + P_{XY}(1,-1) + P_{XY}(-1,-1)$. Given that X^2 and Y^2 are independent, we have $P_{X^2Y^2}(1,1) = P_{X^2}(1)\ P_{Y^2}(1)$, that is,

$$\begin{aligned} &P_{XY}(1,1) + P_{XY}(-1,1) + P_{XY}(1,-1) + P_{XY}(-1,-1) \\ =&P_X(1)P_Y(1) + P_X(-1)P_Y(1) + P_X(1)P_Y(-1) + P_X(-1)P_Y(-1) \end{aligned}$$

However, there is no guarantee that $P_{XY}(1,1) = P_X(1)P_Y(1), P_{XY}(1,-1) = P_X(1)$ $P_Y(-1)$, etc., though their sums are equal.

2. Suppose X^3 and Y^3 are considered instead of X^2 and Y^2. Can we construct a pmf assignment where X^3 and Y^3 are independent but X and Y are not?
3. If the set of values assumed by X and Y is $\{0, 1, 2\}$ instead of $\{-1, 0, 1\}$, can we construct a pmf assignment for which X^2 and Y^2 are independent but X and Y are not?