1. A coin is tossed 10 times. What is the probability that

   (a) we get exactly 4 heads?

   Solution: The sample space consists of all 2^{10} outcomes of tossing a coin 10 times. The number of outcomes that there are exactly 4 heads is $\binom{4}{10}$, and its probability is $\binom{4}{10}/2^{10} \approx 0.2051$.

   (b) we get at least 3 heads?

   Solution: We are interested in the event A that we get at most 2 heads, which is the complement of the event that we get at least 3 heads. The event A is a union of three interest events A_0, A_1 and A_2 consisting of those outcomes in which there are exactly 0 head, 1 head and 2 heads. By a similar calculation as in part (a), we get

   $$|A| = |A_0| + |A_1| + |A_2| = 1 + \binom{1}{10} + \binom{2}{10} = 56$$

   Thus, the probability that we get at least 3 heads is $1 - P(A) = 1 - \frac{56}{2^{10}} \approx 0.945$.

2. A standard 52-card deck comprises 13 ranks in each of the four suits. Alice draws three cards without replacement. What is the probability that three cards have the same suit?

   Solution: The sample space consists of all $\binom{52}{3}$ outcomes. Let event S be the event that all three cards are spades, the total number of outcomes is $\binom{13}{3}$. The probability that all three cards are spades is $\binom{13}{3}/\binom{52}{3}$. Similarly, the probability that all three cards are clubs, diamonds or hearts are all $\binom{13}{3}/\binom{52}{3}$. All four events are mutually exclusive, and thus the probability that three cards have the same suit is $4 \times \binom{13}{3}/\binom{52}{3} \approx 0.05176$.

3. A six-sided die is rolled three times.

   (a) What is the probability that the face values are all different?

   Solution: The sample space consists of all 6^3 outcomes. The first die has 6 possible outcomes. For each of them, there are 5 possibilities for the second die that are different from the first, 4 possibilities for the third die different from the first two. The total number of possibilities is therefore $6\times5\times4 = 120$. The probability is $6\times5\times4/6^3 \approx 0.556$.

   (b) Which is more likely: the sum is even or the sum is odd?

   Solution: Let A and B be the events that the sum is even and odd, respectively. As the outcomes are equally likely, the probabilities of the two sums are $|A|/6^3$ and $|B|/6^3$ so we need to determine which of the sets A and B is bigger. Note that the sum is either even or odd, and $B = A^c$, i.e., B is a complementary event of A. The set A can be partitioned into A_1 and A_2, where A_1 is the event that there are three even numbers, and A_2 is the event that there are one even numbers. Therefore, $|A_1| = 3 \times 3 \times 3$, and $|A_2| = \binom{3}{2} \times 3 \times 3 \times 3$. The total number of outcomes is $|A| = |A_1| + |A_2| = 108$, and $|B| = 6^3 - |A^c| = 108$. Thus, event A and Event B are equally likely.

   Further Question: Are two events still equally likely if we toss the six-sided die four times? How about n times where $n \in \mathbb{Z}^+$?

4. Alice, Bob, and Charlie each toss a 6-sided die. What is the probability that Charlie's face value is strictly larger than both Alice's and Bob's?

Solution: The sample space consists of all $6^3 = 216$ possible outcomes (a, b, c) of Alice's, Bob's, and Charlie's dice. The event E of interest consists of those outcomes in which $c > a$ and $c > b$. We can write E as a disjoint union of $E_1, E_2, \ldots, E_6$ where E_c consists of those outcomes in which Charlie's toss is a c. Then E_c is a product set of size $(c-1)^2$ as Alice's and Bob's outcomes can have arbitrary values between 1 and $c - 1$. Therefore

$$|E| = |E_1| + |E_2| + \cdots + |E_6| = 0^2 + 1^2 + \cdots + 5^2 = 55,$$

so by the equally likely outcomes formula,

$$\mathrm{P}(E) = \frac{55}{216} \approx 0.255.$$