ENGG 2760A / ESTR 2018: Probability for Engineers

7. Continuous Random Variables

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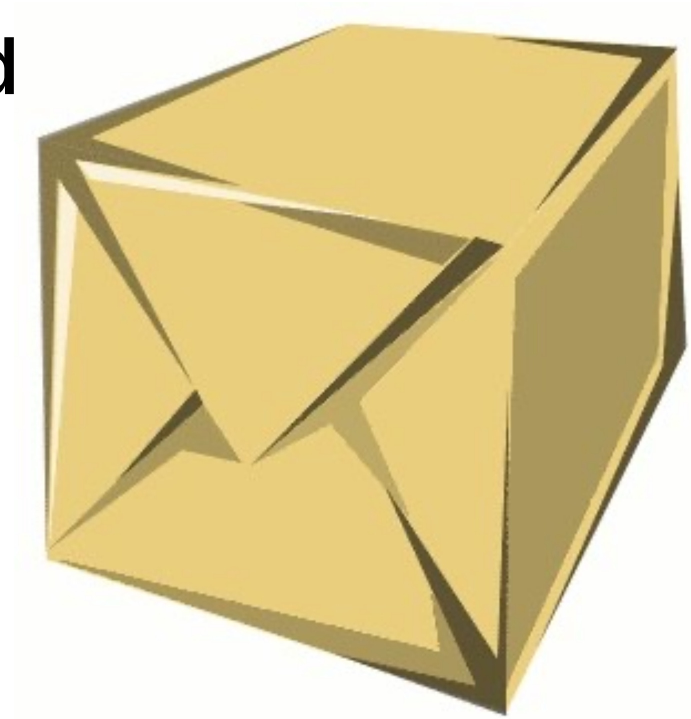
Credit to Prof. Andrej Bogdanov

Delivery time

A package is to be delivered between noon and 1pm.



You go out between 12:30 and 12:45.

What is the probability you missed the delivery?

Delivery time

1. Sample space:

2. Event of interest:

3. Probabilities?

Uncountable sample spaces

In Lecture 1 we said:

> "*The probability of an event is the sum of the probabilities of its elements*"

...but all elements have probability zero!

To specify and calculate probabilites, we have to work with the axioms of probability

Delivery time

From 12:08 - 12:12 and 12:54 - 12:57 the doorbell wasn’t working.

Event of interest:

Probability:

The uniform random variable

Sample space $\Omega = [0, 60)$

Events of interest: **intervals** $[x, y) \subseteq [0, 60)$

their intersections, unions, etc.

Probabilities: $P([x, y)) = (y - x)/60$

Random variable: $X(\omega) = \omega$

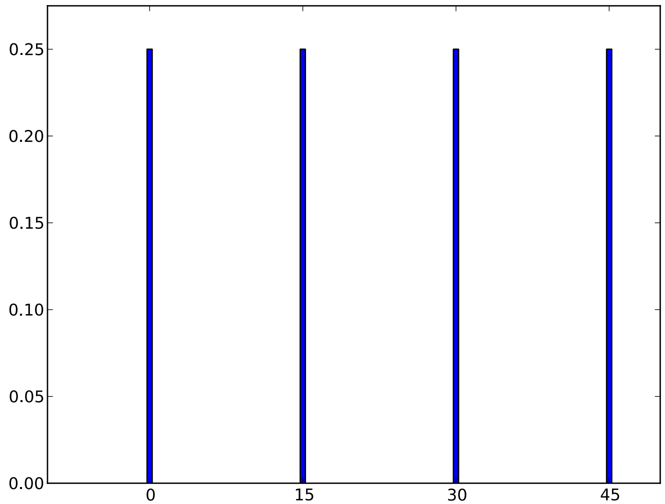
Cumulative distribution function

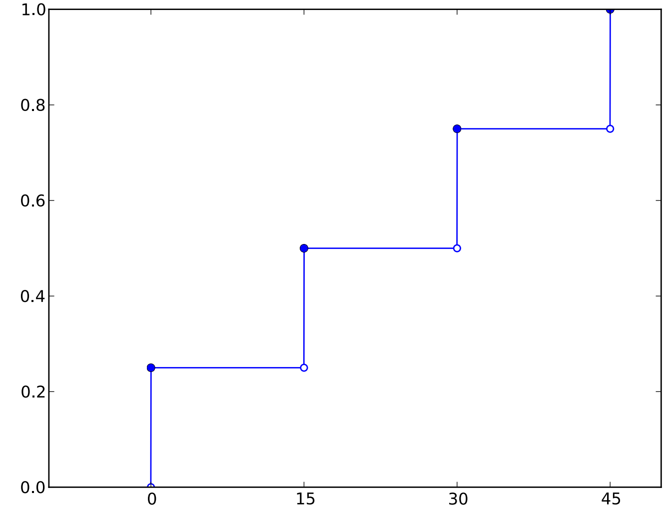
The probability mass function doesn't make much sense because $P(X = x) = 0$ for all x.

Instead, we can describe X by its cumulative distribution function (CDF) F:

$$F_X(x) = P(X \leq x)$$

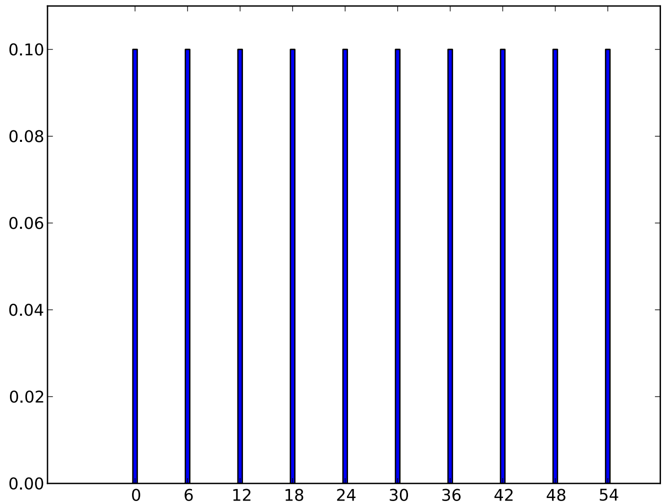
Cumulative distribution functions

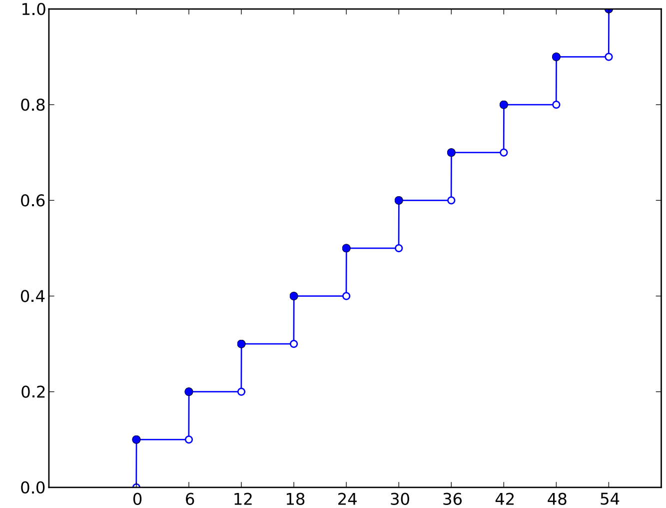




$$f_X(x) = P(X = x)$$

$$F_X(x) = P(X \leq x)$$

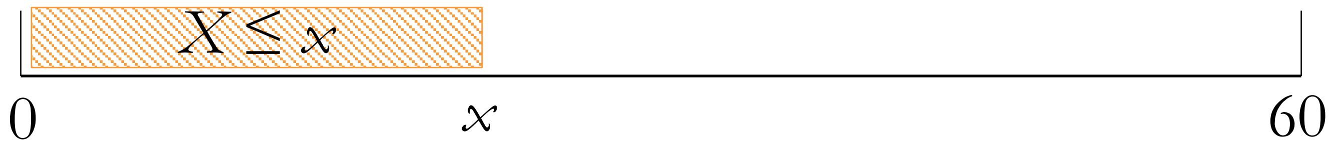


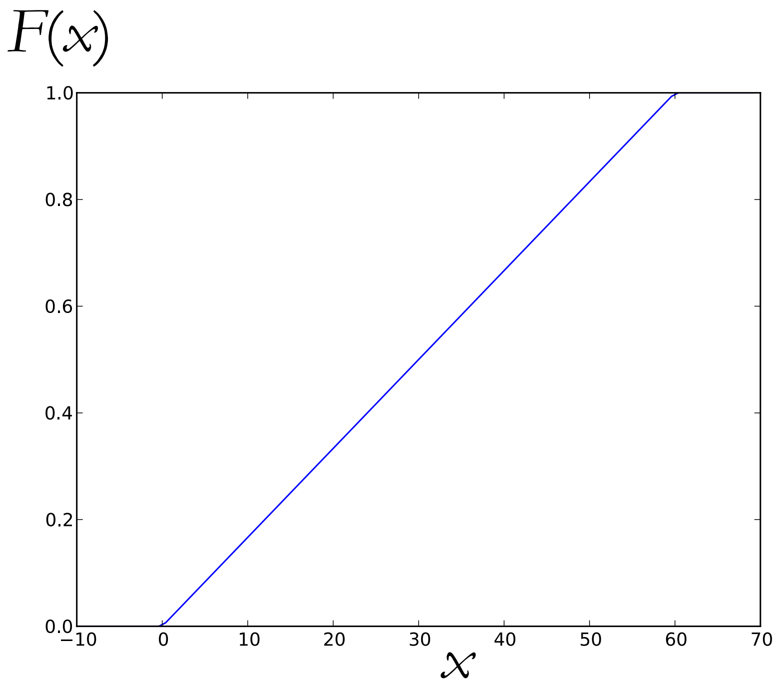


What is the $\mathrm{Geometric}(1/2)$ CDF?

Uniform random variable

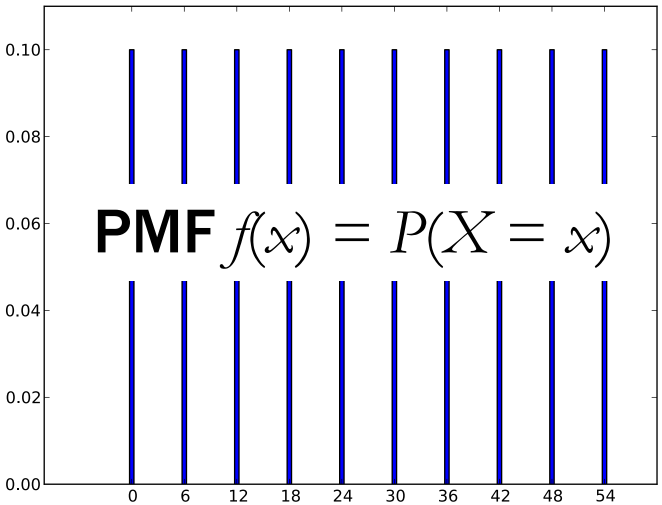
If X is uniform over $[0, 60)$ then

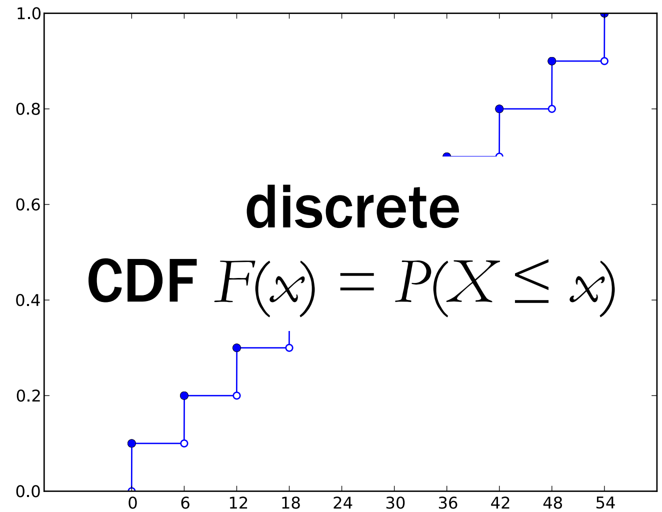




$$P(X \leq x) = \begin{cases} 0 & \textbf{for } x < 0 \\ x/60 & \textbf{for } x \in [0, 60) \\ 1 & \textbf{for } x > 60 \end{cases}$$

Cumulative distribution functions





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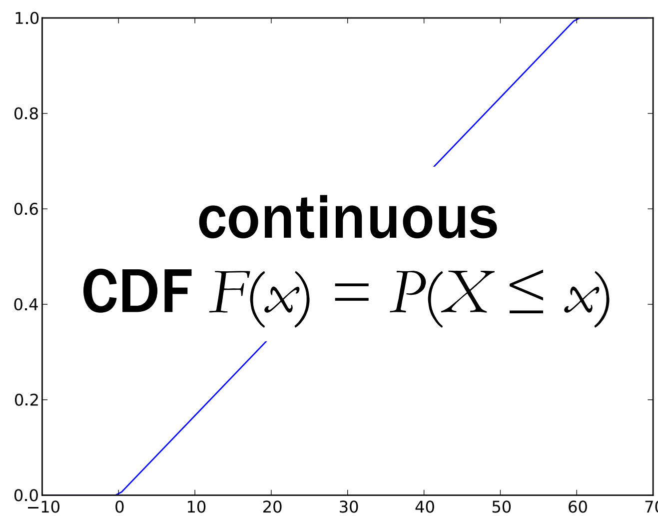


Discrete random variables:

PMF $f(x) = P(X = x)$

CDF $F(x) = P(X \leq x)$

$f(x) = F(x) - F(x - \delta)$

for small δ

$F(a) = \sum_{x \leq a} f(x)$

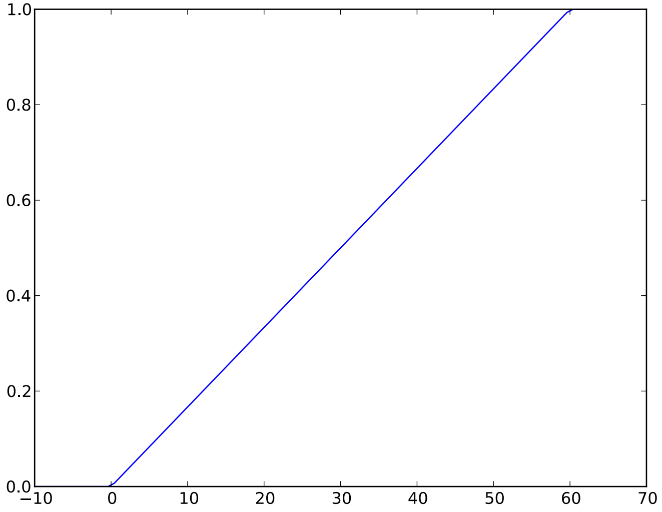
Continuous random variables:

The probability density function (PDF) of a random variable with CDF $F(x)$ **is**

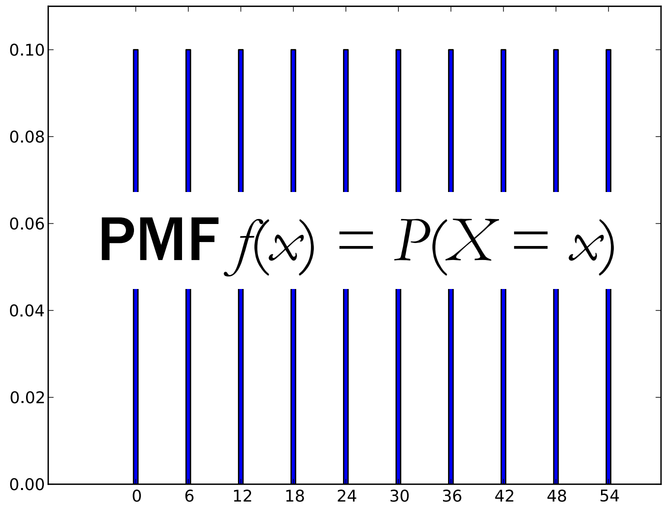
$$f(x) = \lim_{\delta \to 0} \frac{F(x) - F(x - \delta)}{\delta} = \frac{dF(x)}{dx}$$

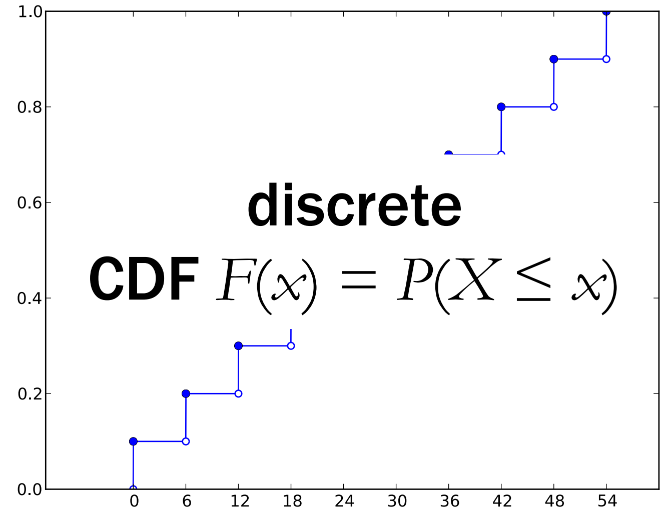
Uniform random variable

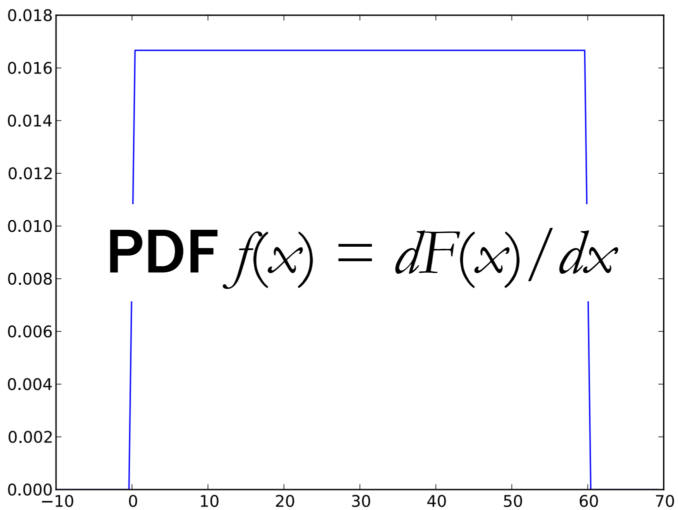
$$F(x) = \begin{cases} 0 & \textbf{if } x < 0 \\ x/60 & \textbf{if } x \in [0, 60) \\ 1 & \textbf{if } x \geq 60 \end{cases}$$

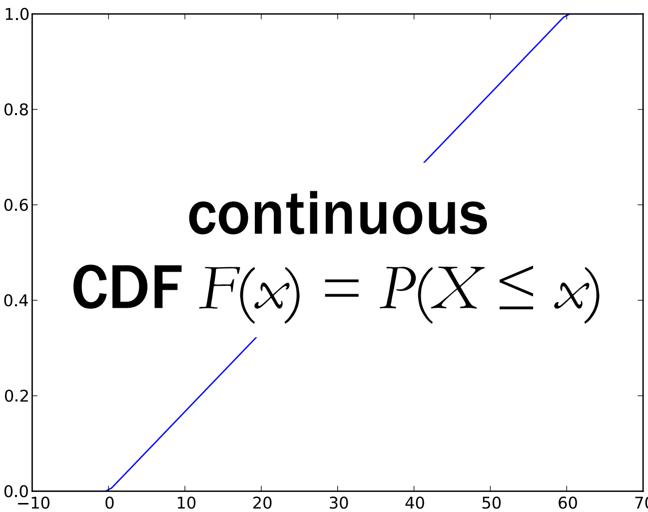


Probability density functions





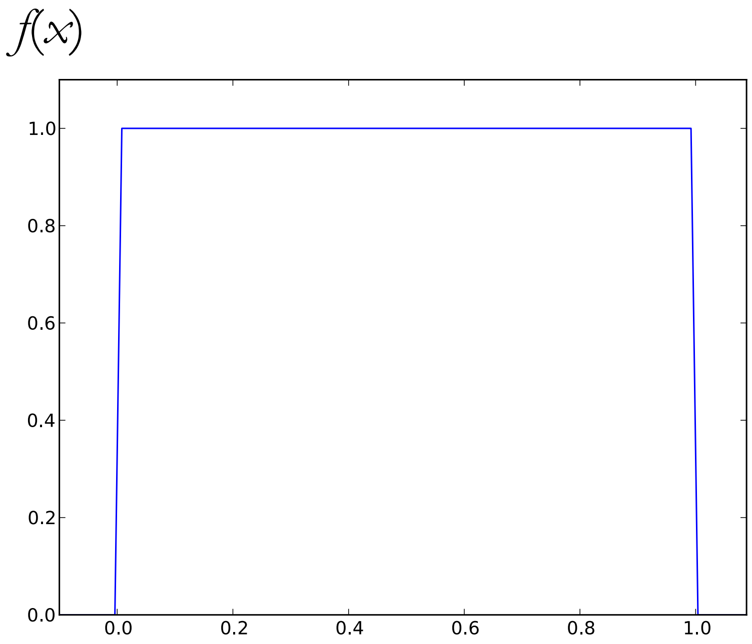




Uniform random variable

The $\mathrm{Uniform}(0, 1)$ **PDF is**

$$f(x) = \begin{cases} 1 & \textbf{if } x \in (0, 1) \\ 0 & \textbf{if } x < 0 \textbf{ or } x > 1 \end{cases}$$



The $\mathrm{Uniform}(a, b)$ **PDF is**

$$f(x) = \begin{cases} 1/(b - a) & \textbf{if } x \in (a, b) \\ 0 & \textbf{if } x < a \textbf{ or } x > b \end{cases}$$

Calculating the CDF

Discrete random variables:

PMF $f(x) = P(X = x)$

CDF $F(x) = P(X \leq x) = \sum_{t \leq x} f(t)$

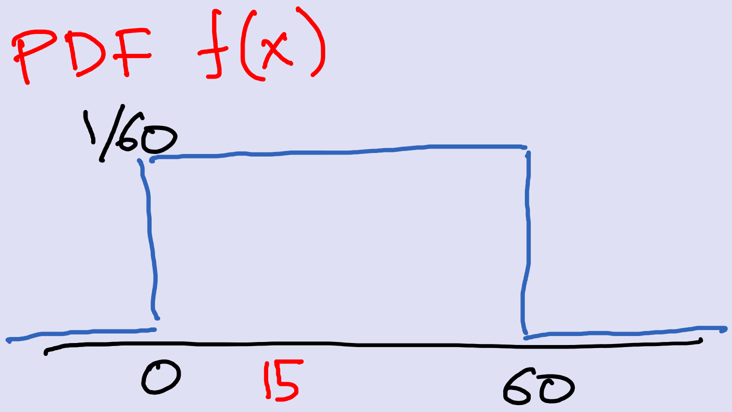
Continuous random variables:

PDF $f(x) = dF(x)/dx$

CDF $F(x) = P(X \leq x) = \int_{t \leq x} f(t)dt$

A package is to arrive between 12 and 1

What is the probability it arrived by 12.15?

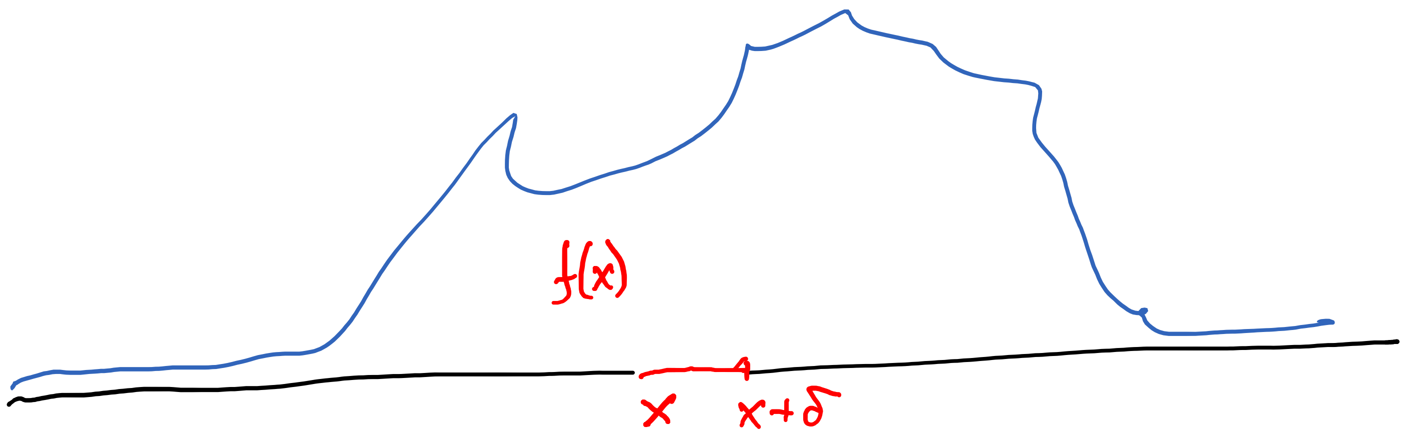


Interpretation of the PDF

The PDF value $f(x)\ \delta$ approximates the probability that X in an interval of length δ around x

$$P(x-\delta \leq X < x) = f(x)\ \delta + o(\delta)$$

$$P(x \leq X < x+\delta) = f(x)\ \delta + o(\delta)$$



Alice said she’ll show up between 7 and 8, probably around 7.30.

It is now 7.30. What is the probability Bob has to wait past 7.45?

Expectation and variance

	PMF $f(x)$	PDF $f(x)$
$\mathbf{P}(X \leq a)$ CDF	$\sum_{x \leq a} f(x)$	
$\mathbf{E}[X]$	$\sum_x x f(x)$	
$\mathbf{E}[X^2]$	$\sum_x x^2 f(x)$	
$\mathbf{Var}[X]$		

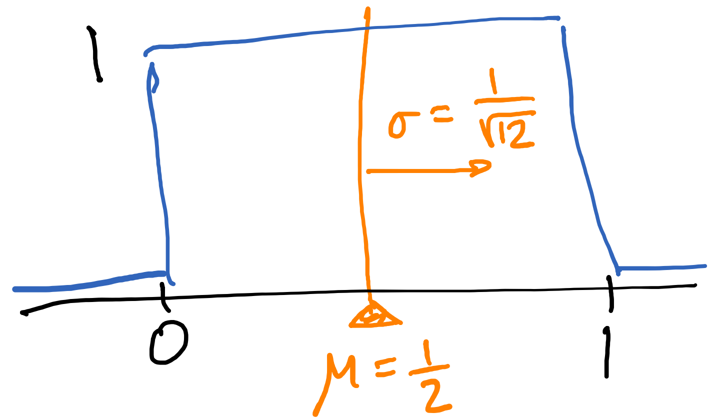
Mean and Variance of Uniform

Uniform(0,1)

$$\mu = E[X] = \int_0^1 x \cdot 1 \, dx = \frac{x^2}{2}\Big|_0^1 = \frac{1}{2}$$

$$E[X^2] = \int_0^1 x^2 \cdot 1 \, dx = \frac{x^3}{3}\Big|_0^1 = \frac{1}{3}$$

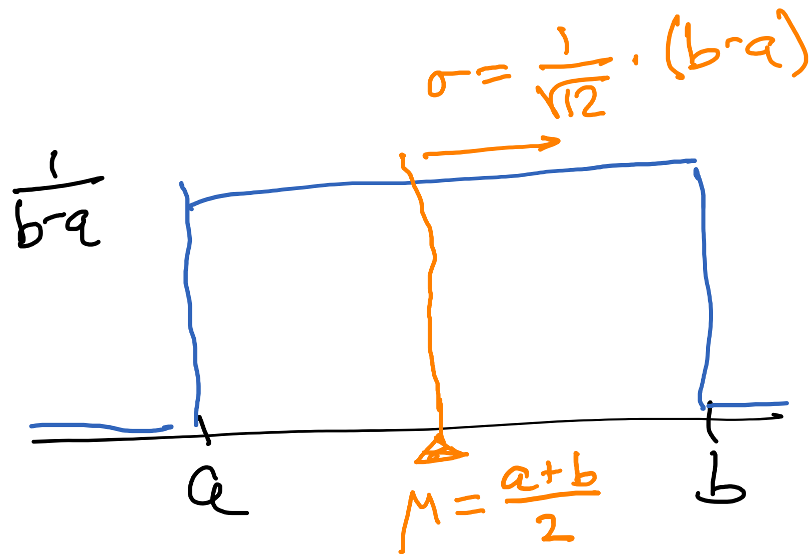
$$\sigma = \sqrt{Var[X]} = \sqrt{1/3 - (1/2)^2} = 1/\sqrt{12}.$$



Uniform(a, b)

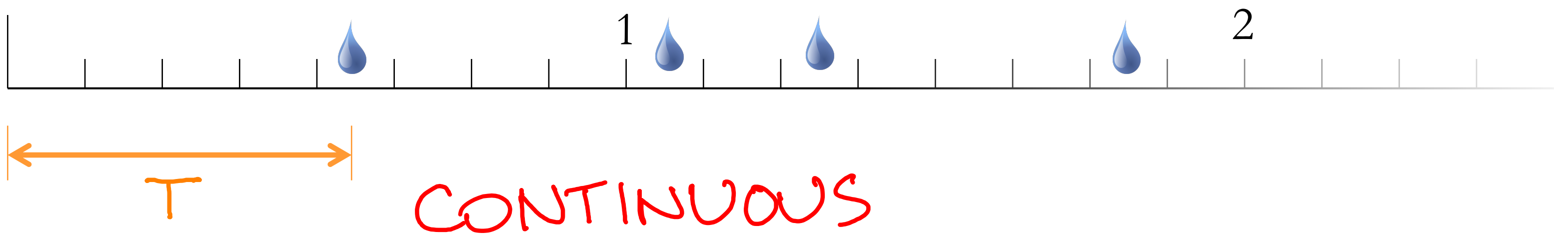
$$\mu = \frac{a+b}{2}$$

$$\sigma = \frac{b-a}{\sqrt{12}}.$$



Raindrops again

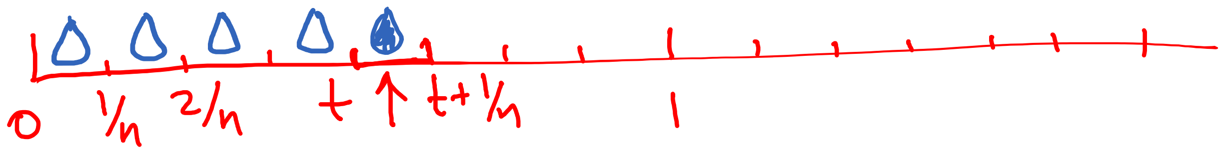
Rain is falling on your head at a rate of λ drops/sec.



CONTINUOUS

How long do we wait until the next drop?

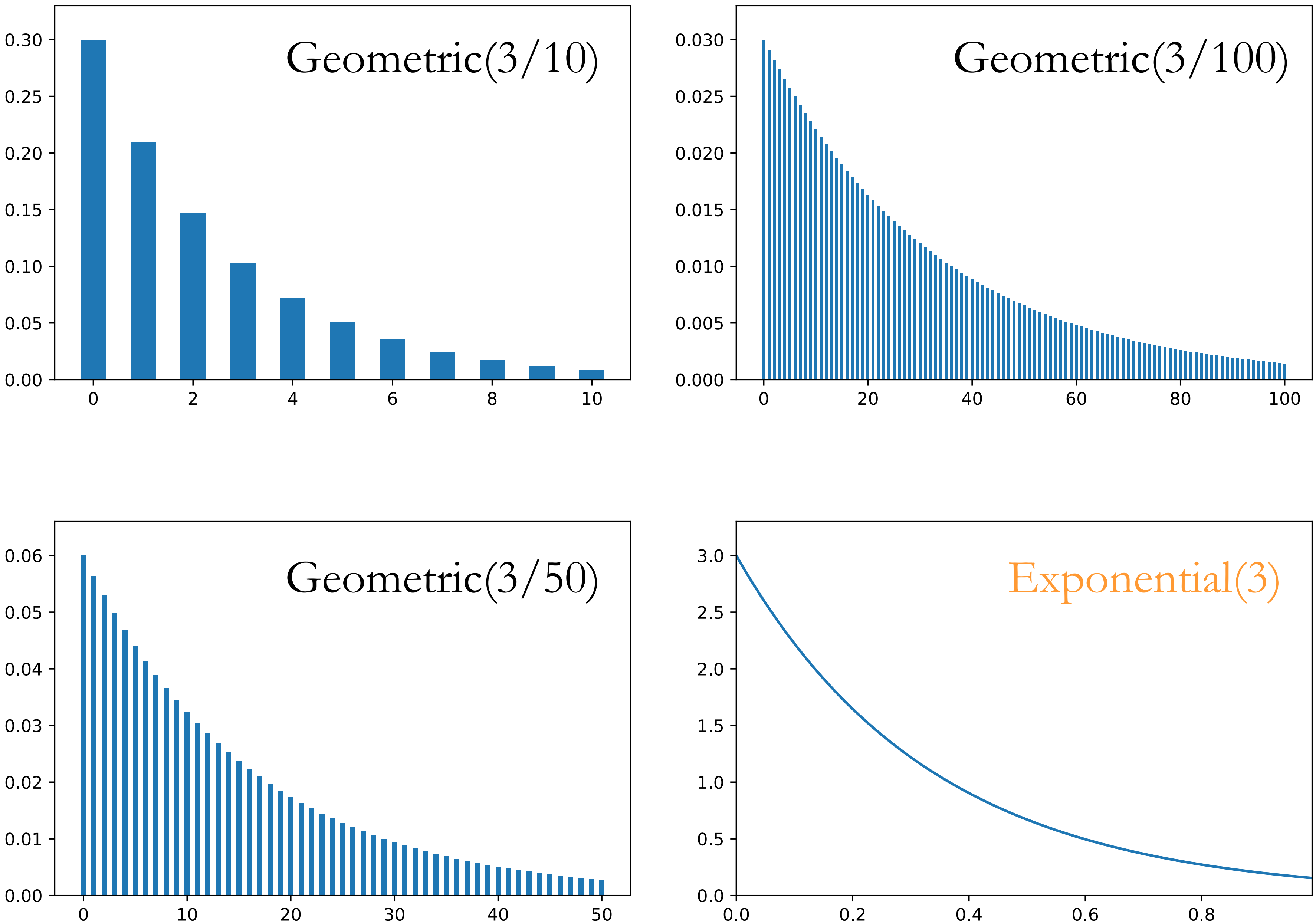
n CHUNKS/sec.



T = TIME OF 1ST RAINDROP.

$$P\left(t \le T < t + \frac{1}{n}\right) =$$

$$P(t \le T < t + \delta) =$$

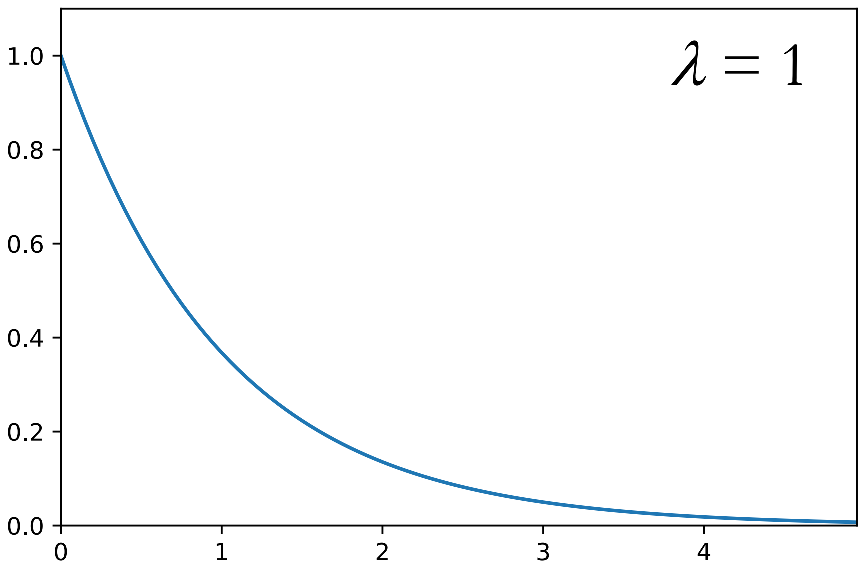


Geometric(3/10)
Geometric(3/100)
Geometric(3/50)
Exponential(3)

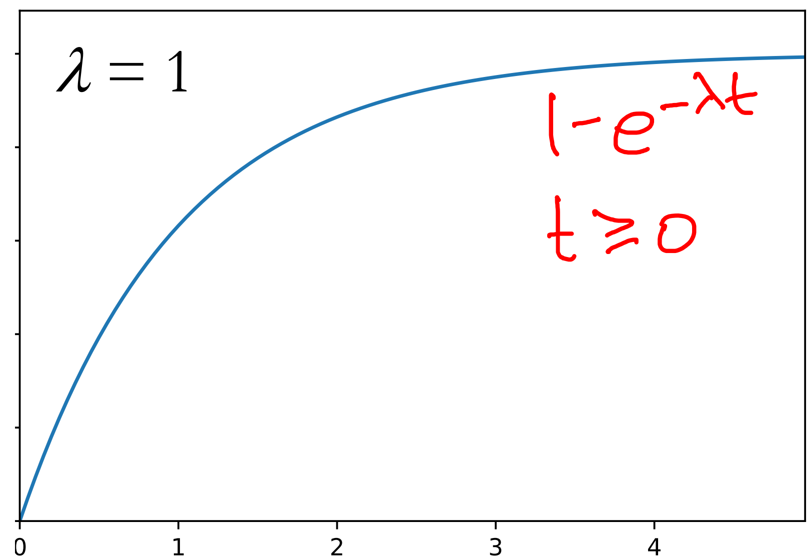
The exponential random variable

The $\text{Exponential}(\lambda)$ **PDF is**

$$f(t) = \begin{cases} \lambda\, e^{-\lambda t} & \textbf{if } x \geq 0 \\ 0 & \textbf{if } x < 0. \end{cases}$$



PDF $f(t)$



CDF $F(t) = P(T \leq t)$

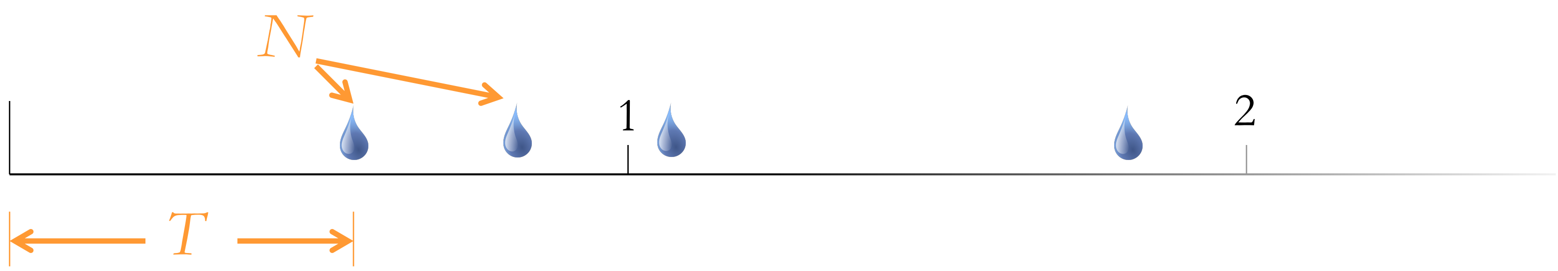
The exponential random variable

CDF of $\text{Exponential}(\lambda)$**:** $P(T \leq t) =$

$$\mathbf{E}[\text{Exponential}(\lambda)] =$$

$$\mathbf{Var}[\text{Exponential}(\lambda)] =$$

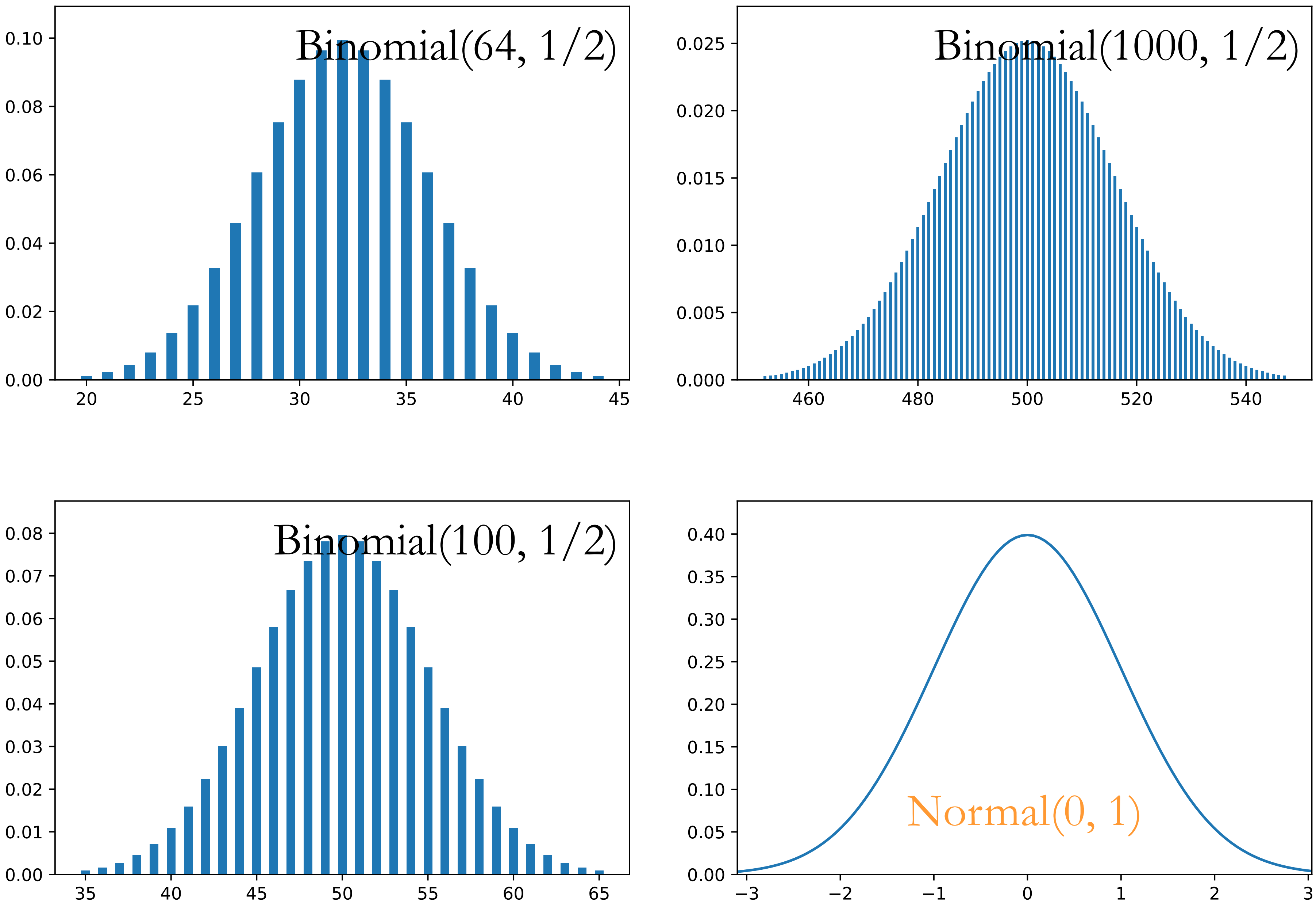
Poisson vs. exponential



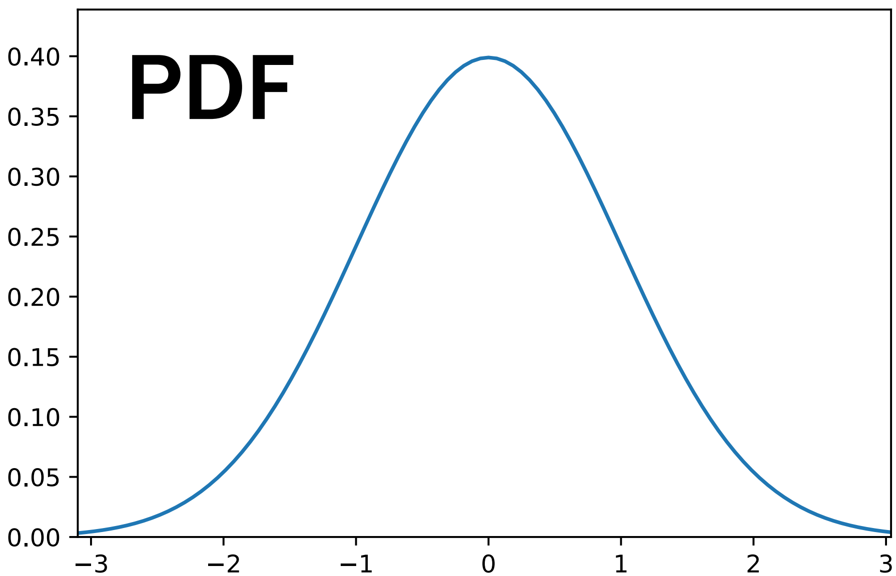
	Poisson(λ) N	Exponential(λ) T
description	**number of events within time unit** 0, 1, 2, ...	**time until first event happens** $[0, +\infty)$
expectation	λ	$1/\lambda$
std. deviation	λ	$1/\lambda$

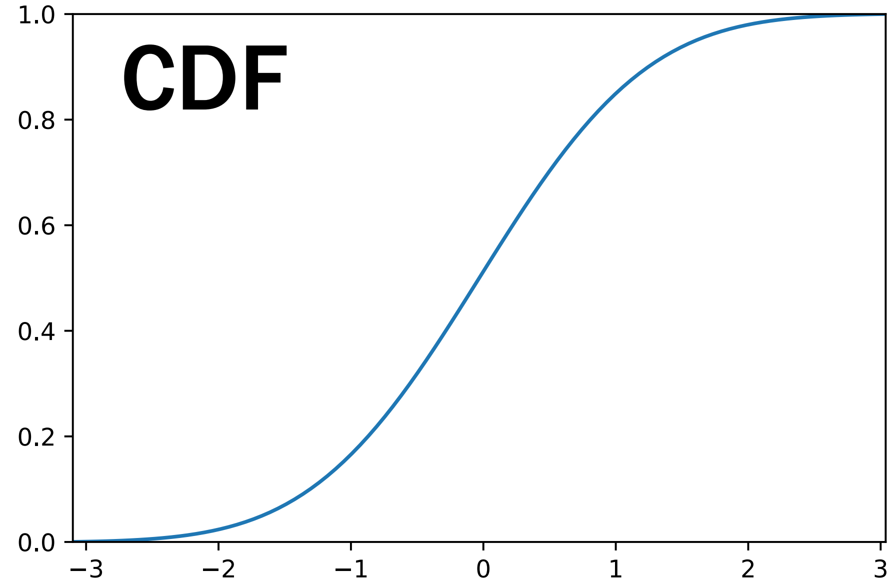
on average

A bus arrives once every 5 minutes. How likely are you to wait 5 to 10 minutes?


Binomial(64, 1/2)
Binomial(1000, 1/2)
Binomial(100, 1/2)
Normal(0, 1)

The $\mathrm{Normal}(0, 1)$ random variable





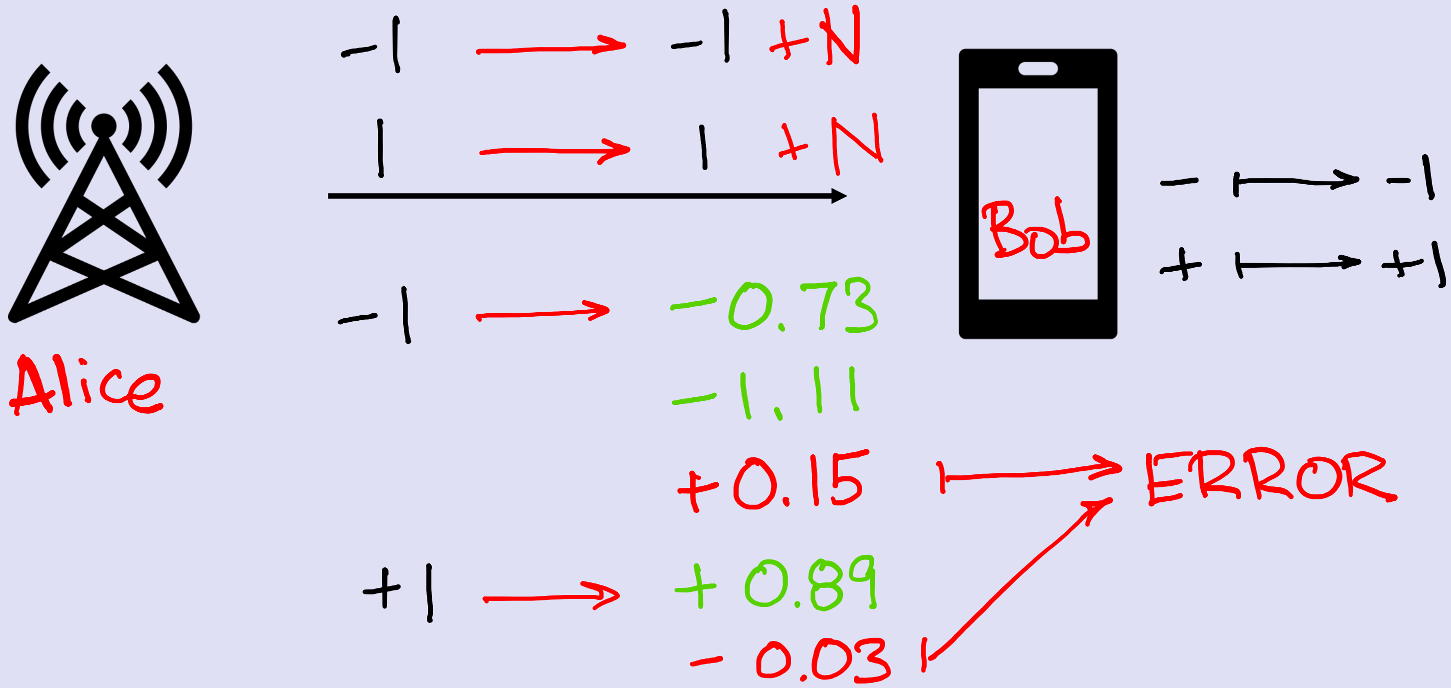
$$f(x) = (2\pi)^{-1/2}\, e^{-x^2/2}$$

$$F(x) = (2\pi)^{-1/2} \int_{t \le x} e^{-t^2/2}\, dt$$

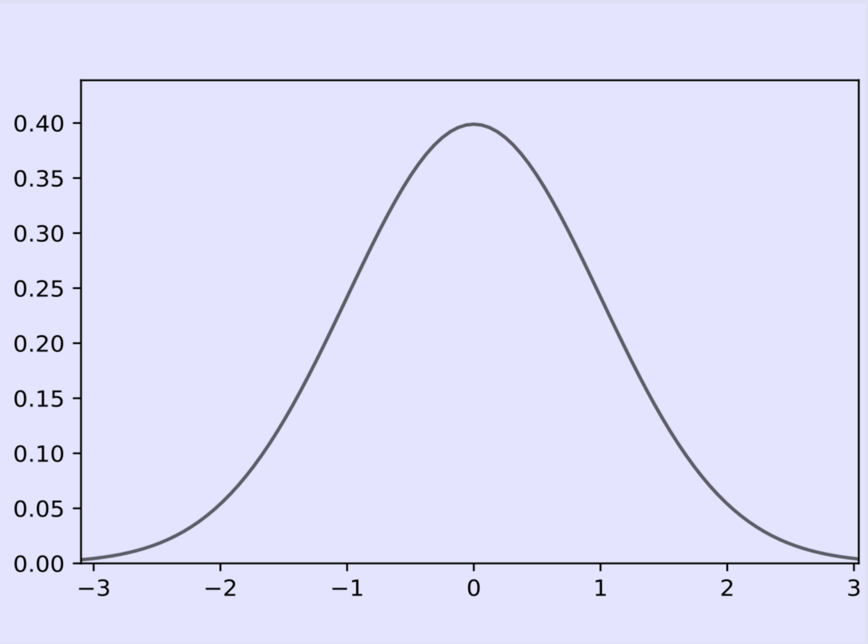
no closed form

$$\mathbf{E}[\mathrm{Normal}(0, 1)] =$$

$$\mathbf{Var}[\mathrm{Normal}(0, 1)] =$$



P(ERROR) =



https://stattrek.com/online-calculator/normal.aspx

The Normal(μ, σ) random variable

$$f(x) = (2\pi\sigma^2)^{-1/2}\, e^{-(x-\mu)^2/2\sigma^2}$$

$E[X] = \mu \qquad Var[X] = \sigma^2$

$$\boxed{\text{Normal}(\mu,\sigma) = \mu + \sigma \cdot \text{Normal}(0,1)}$$