ENGG 2760A / ESTR 2018: Probability for Engineers

5. Expectation, Variance, Joint PMFs

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Expectation of a function

PMF of X:

x	0	1	2
$p(x)$	$1/3$	$1/3$	$1/3$

$$\mathbf{E}[X] =$$

$$\mathbf{E}[X-1] =$$

$$\mathbf{E}[(X-1)^2] =$$

Expectation of a function, again

p.m.f. of X**:**

x	0	1	2
$p(x)$	$1/3$	$1/3$	$1/3$

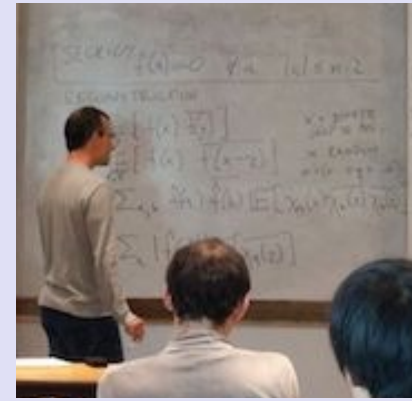
$\mathbf{E}[X] =$

$\mathbf{E}[X-1]=$

$\mathbf{E}[(X-1)^2]=$

$$\boxed{\mathbf{E}[f(X)] = \textstyle\sum_x f(x)\, p(x)}$$

1km



60%

40%

5km/h

30km/h

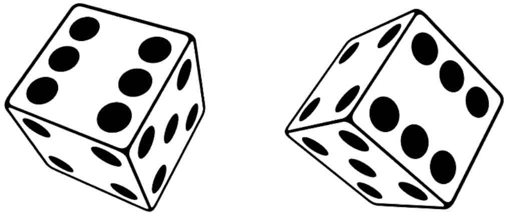
AVERAGE TIME?

Joint probability mass function

The joint PMF of random variables X, Y is the bivariate function

$$p(x, y) = \mathbf{P}(X = x, Y = y)$$

Example



There is a bag with 4 cards:

1	2	3	4

You draw two without replacement. What is the joint PMF of the face values?

What is the PMF of the sum? $Z = X + Y$

What is the expected value?

PMF and expectation of a function

$Z = f(X, Y)$ **has PMF**

$$p_Z(z) = \sum_{x,\, y:\, f(x,\, y) \,=\, z} p_{XY}(x, y)$$

and expected value

$$\mathbf{E}[Z] = \sum_{x,\, y} f(x, y)\, p_{XY}(x, y)$$

What if the cards are drawn with replacement?

Marginal probabilities

$Y \backslash X$	1	2	3	4	$P(Y = y) = \sum_x P(X = x, Y = y)$
1	0	1/12	1/12	1/12	1/4
2	1/12	0	1/12	1/12	1/4
3	1/12	1/12	0	1/12	1/4
4	1/12	1/12	1/12	0	1/4
	1/4	1/4	1/4	1/4	1

$$P(X = x) = \sum_y P(X = x, Y = y)$$

Linearity of expectation

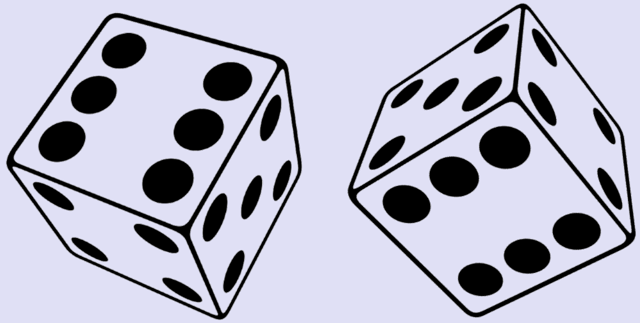
For every two random variables X and Y

$$\mathbf{E}[X + Y] = \mathbf{E}[X] + \mathbf{E}[Y]$$

1 2 3 4

$$\mathbf{E}[X + Y]$$

without replacement	with replacement

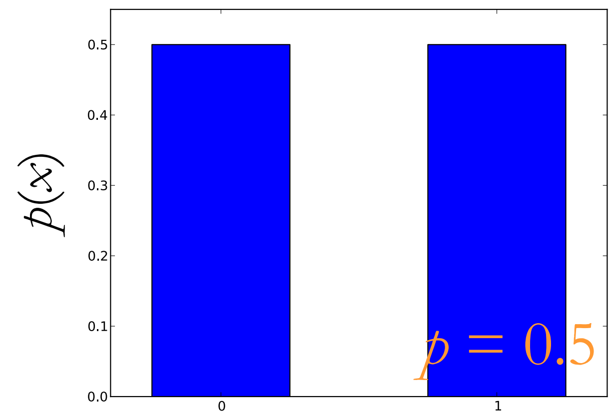


$$\mathbf{E}[X + Y] = ?$$

The indicator (Bernoulli) random variable

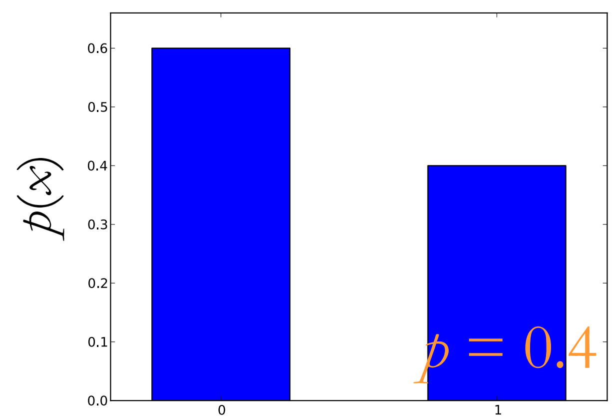
Perform a trial that succeeds with probability p and fails with probability $1-p$.

x	0	1
$p(x)$	$1-p$	p



If X is $\text{Bernoulli}(p)$ then

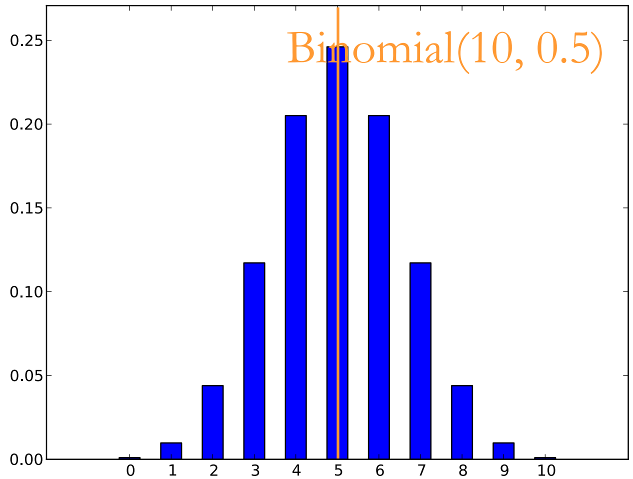
$$E[X] = p$$

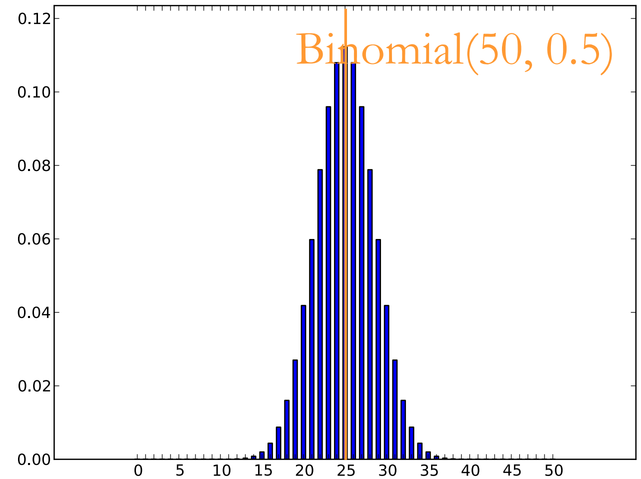


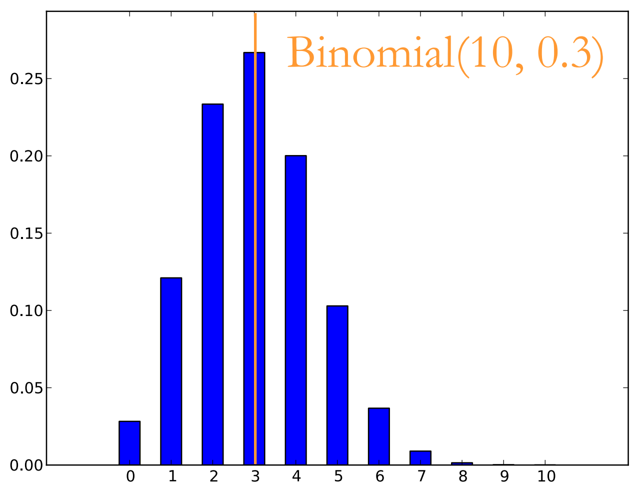
Mean of the Binomial

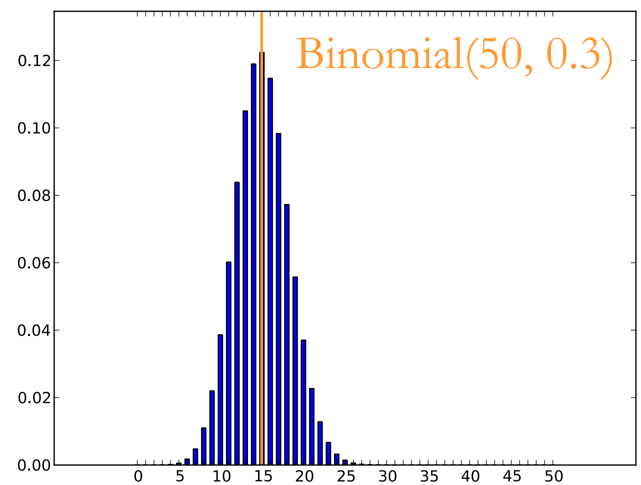
$\text{Binomial}(n, p)$: **Perform n independent trials, each of which succeeds with probability p.**

$X =$ **number of successes**


Binomial(10, 0.5)
0.00 0.05 0.10 0.15 0.20 0.25
0 1 2 3 4 5 6 7 8 9 10


Binomial(50, 0.5)
0.00 0.02 0.04 0.06 0.08 0.10 0.12
0 5 10 15 20 25 30 35 40 45 50


Binomial(10, 0.3)
0.00 0.05 0.10 0.15 0.20 0.25
0 1 2 3 4 5 6 7 8 9 10


Binomial(50, 0.3)
0.00 0.02 0.04 0.06 0.08 0.10 0.12
0 5 10 15 20 25 30 35 40 45 50

n people throw their hats in a box and each picks one out at random. On average, how many get back their own hat?

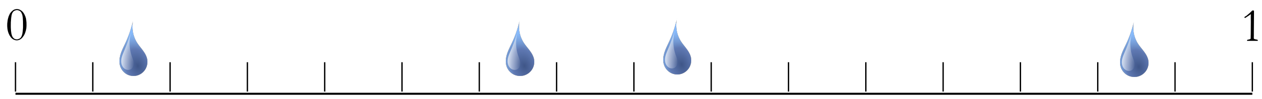
Mean of the Poisson

$\mathrm{Poisson}(\lambda)$ **approximates** $\mathrm{Binomial}(n, \lambda/n)$ **for large** n

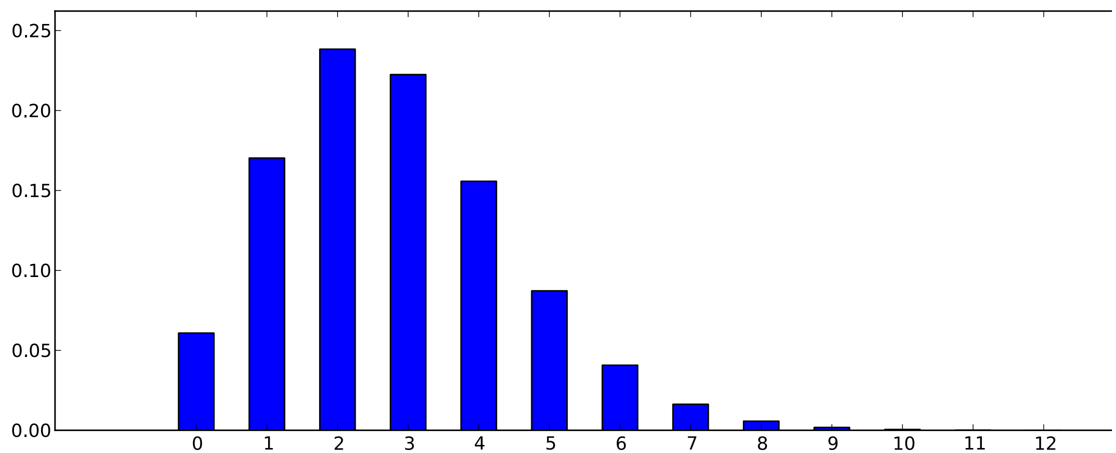
$$\boxed{p(k) = e^{-\lambda} \lambda^k / k!} \qquad k = 0, 1, 2, 3, \ldots$$

Raindrops

Rain is falling on your head at an average speed of 2.8 drops/second.



Number of drops N **is** $\text{Binomial}(n, 2.8/n)$



Rain falls on you at an average rate of 3 drops/sec.

When 100 drops hit you, your hair gets wet.

You walk for 30 sec from MTR to bus stop.

What is the probability your hair got wet?

Investments

You have three investment choices:

A: put \$25 in one stock

B: put \$½ in each of 50 unrelated stocks

C: keep your money in the bank

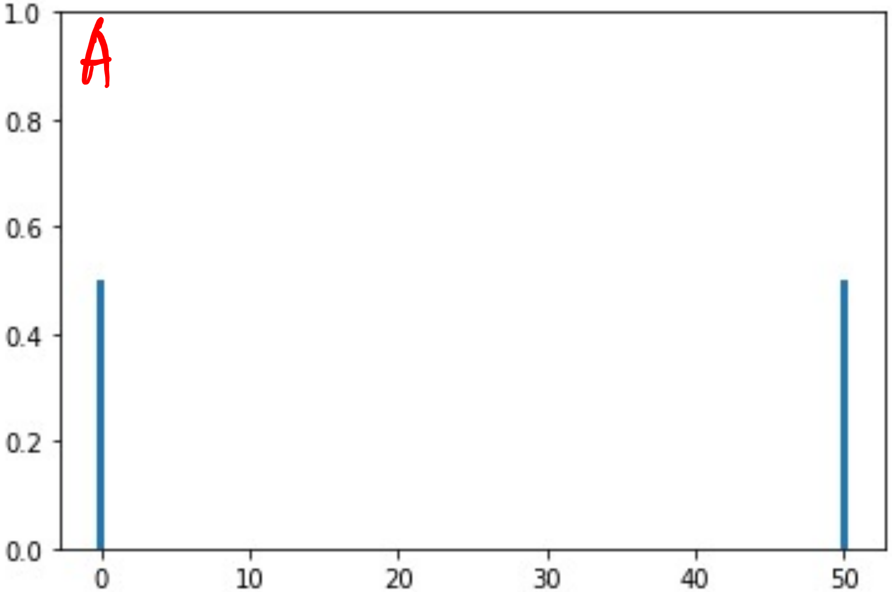
Which do you prefer?

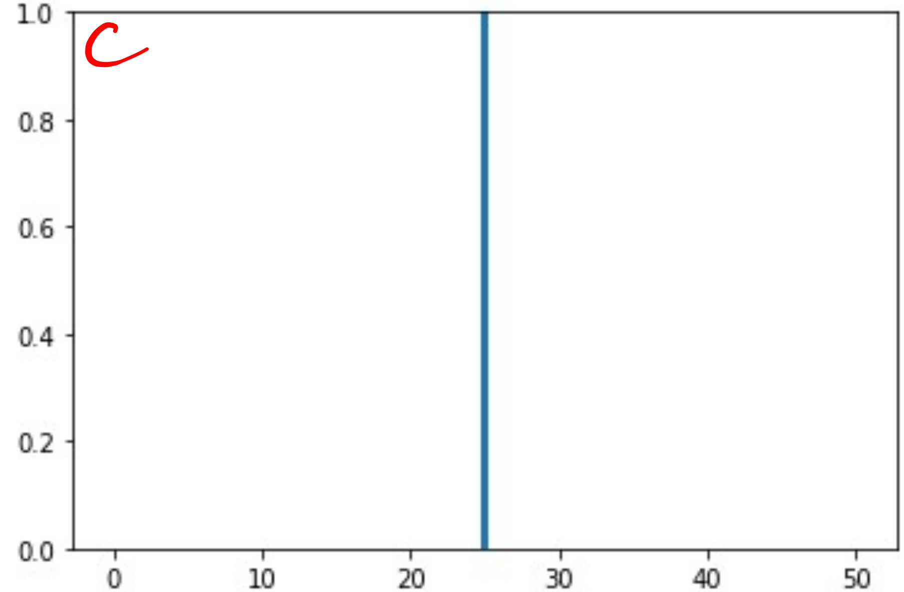
Investments

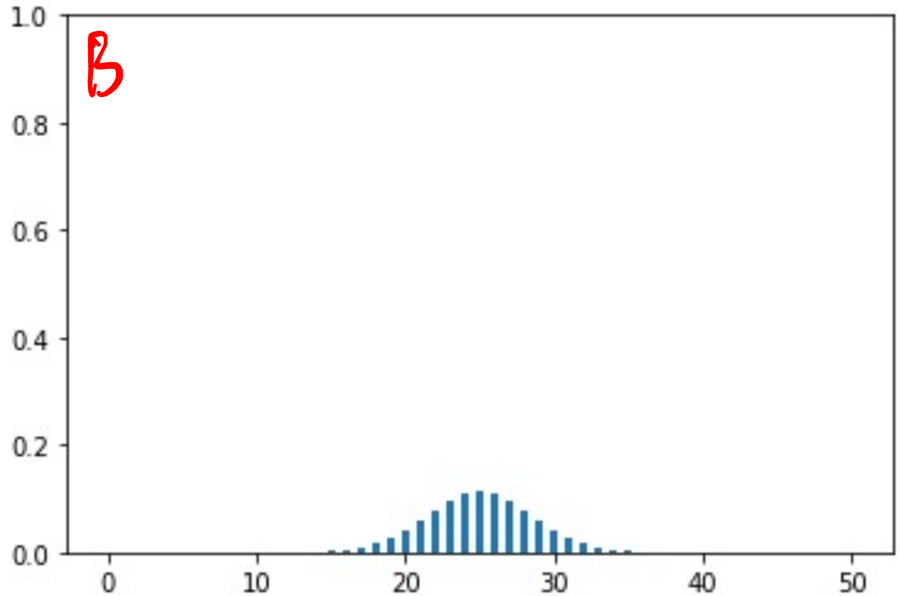
Probability model

Each stock $\begin{cases} \text{doubles in value with probability } 1/2 \\ \text{loses all value with probability } 1/2 \end{cases}$

Different stocks perform independently


A
1.0
0.8
0.6
0.4
0.2
0.0
0
10
20
30
40
50


C
1.0
0.8
0.6
0.4
0.2
0.0
0
10
20
30
40
50


B
1.0
0.8
0.6
0.4
0.2
0.0
0
10
20
30
40
50

Variance and standard deviation

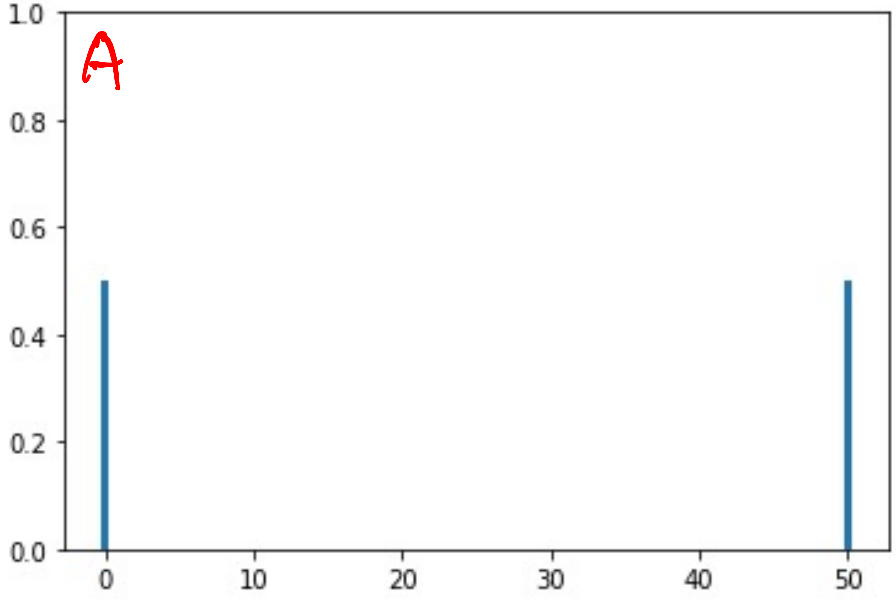
Let $\mu = \mathbf{E}[X]$ be the expected value of X.

The variance of X is the quantity

$$\mathbf{Var}[X] = E[(X - \mu)^2]$$

The standard deviation of X is $\sigma = \sqrt{\mathbf{Var}[X]}$

It measures how close X and μ are typically.


A
1.0
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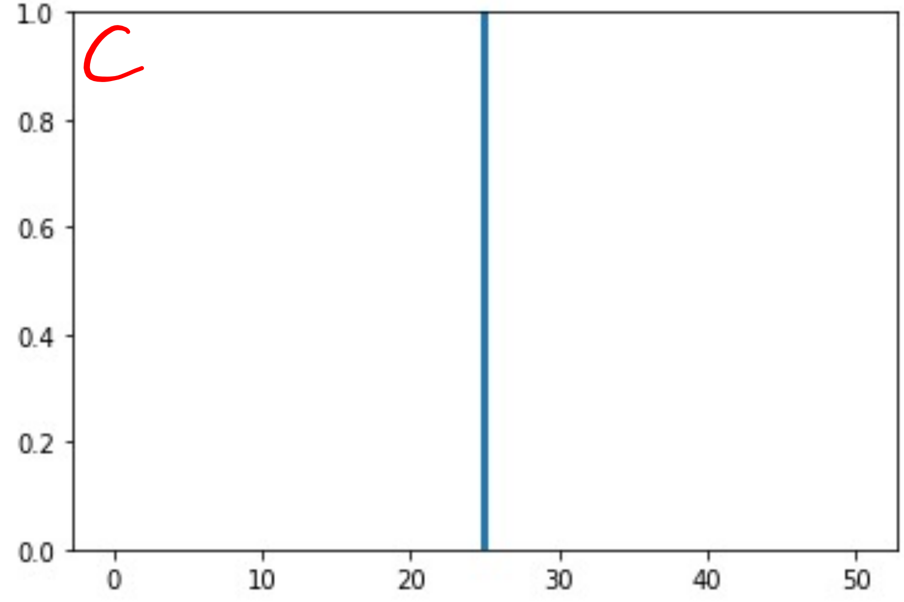
C
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10
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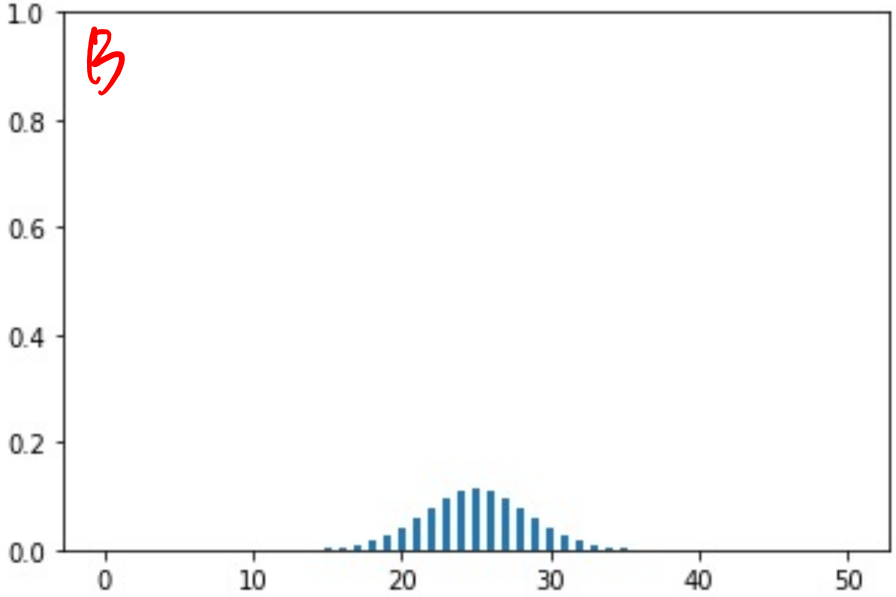
50

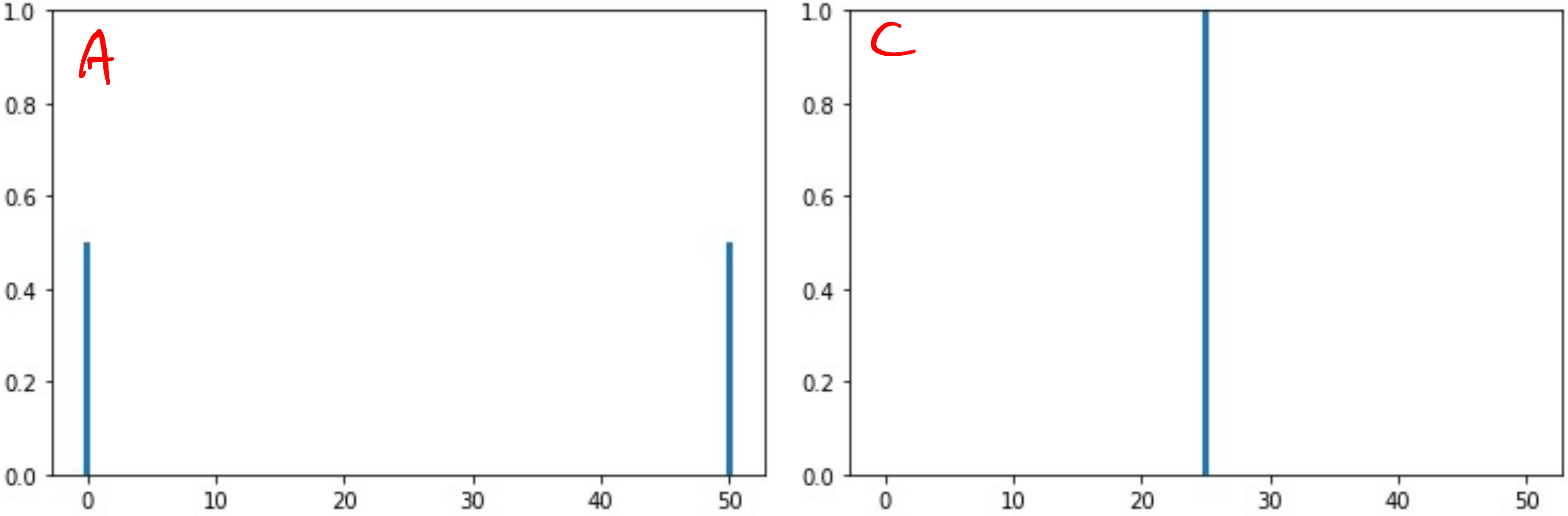


$$\boxed{\mathbf{Var}[\text{Binomial}(n, p)] = np(1 - p)}$$

***Most* of the probability mass is within *a few* σs from μ**

More on this in later lectures...

Another formula for variance


A
C
1.0
0.8
0.6
0.4
0.2
0.0
0
10
20
30
40
50

$$\mathbf{E}[X] = ?$$

$$\mathbf{Var}[X] = ?$$

x	1	2	3	4	5	6
$p(x)$	1/6	1/6	1/6	1/6	1/6	1/6

$E[X] = 3.5$