

1. A six people committee is chosen at random from 5 girls and 15 boys. Conditioned on there being at least one girl on the committee, what is the expected number and the variance of boys?

**Solution:** Let  $G$  be the (unconditional) number of girls on the committee. By the conditional expectation formula, we have

$$E[G] = E[G|G=0]P(G=0) + E[G|G>0](1-P(G=0)).$$

By linearity of expectation,  $E[G] = E[G_1] + E[G_2] + E[G_3] + E[G_4] + E[G_5] + E[G_6]$ , where  $G_i, i = \{1, 2, 3, 4, 5, 6\}$  are random events of whether the  $i^{th}$  committee member is a girl.  $E[G] = 6 \cdot (5/20) = 3/2$ . On the other hand,  $G = 0$  occurs only when all six members are boys, so by the product formula:

$$P(G=0) = \frac{15}{20} \cdot \frac{14}{19} \cdot \frac{13}{18} \cdot \frac{12}{17} \cdot \frac{11}{16} \cdot \frac{10}{15} \approx 0.129.$$

Therefore :

$$E[G|G>0] = \frac{E[G]}{1-P(G=0)} \approx \frac{3/2}{1-0.129} \approx 1.722.$$

If  $B$  is the number of boys then  $B + G = 6$  so by linearity of expectation again

$$E[B|G>0] = 6 - E[G|G>0] \approx 4.278.$$

$$\begin{array}{c|ccccc} x & 1 & 2 & 3 & 4 & 5 \\ \hline P(B | G > 0) & \frac{3}{6751} & \frac{105}{6751} & \frac{910}{6751} & \frac{2730}{6751} & \frac{3003}{6751} \end{array}$$

$$\text{Var}(B | G > 0) = E(B^2 | G > 0) - E(B | G > 0)^2 \approx 0.569.$$

2. Let  $X$  and  $Y$  be independent random variables with PMFs  $P(X=1) = P(X=2) = P(X=3) = P(X=4) = 1/4$  and  $P(Y=6) = P(Y=7) = 1/2$ . Let  $M = X + Y$  and  $N = Y - X$ .

- (a) What is the PMF of  $M$  given  $N$ ?

**Solution:** We first calculate the joint PMF  $P_{MN}(m, n)$ : of  $M$  and  $N$ :

$m \backslash n$	2	3	4	5	6
7	0	0	0	$\frac{1}{8}$	0
8	0	0	$\frac{1}{8}$	0	$\frac{1}{8}$
9	0	$\frac{1}{8}$	0	$\frac{1}{8}$	0
10	$\frac{1}{8}$	0	$\frac{1}{8}$	0	0
11	0	$\frac{1}{8}$	0	0	0

Then we calculate the marginal PMF  $P_N(n)$  of  $N$ :

$n$ :	2	3	4	5	6
$P_N(n)$ :	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{8}$

The conditional PMF  $P_{M|N}(M=m|N=n)$  is then  $\frac{P_{MN}(m,n)}{P_N(n)}$ :

$m \backslash n$	2	3	4	5	6
7	0	0	0	$\frac{1}{2}$	0
8	0	0	$\frac{1}{2}$	0	1
9	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0
10	1	0	$\frac{1}{2}$	0	0
11	0	$\frac{1}{2}$	0	0	0

- (b) Are  $M$  and  $N$  independent? Justify your answer.

**Solution:** No. For example,  $P_{MN}(7, 2) = 0$ ,  $P_M(7) = 1/8$ , and  $P_N(2) = 1/8$ , so  $P_{MN}(7, 2) \neq P_M(7)P_N(2)$ .

- (c) What is the expectation of  $M$  given  $N < 2$ ?

**Solution:** Since the probability space of  $N < 2$  is  $\emptyset$ ,  $P(\emptyset) = 0$ .  $E(\emptyset) = 0$ .

- (d) What is the expectation of  $M$  given  $N < 4$ ?

**Solution:** We first calculate the joint PMF  $P_{MN|N<4}(m, n)$  of  $M$  and  $N$  given  $N < 4$ . This is obtained from the joint PMF of  $M$  and  $N$  by discarding the columns  $n = 4$ ,  $n = 5$  and  $n = 6$ . And rescaling so that the probabilities add up to one:

$m \backslash n$	2	3
7	0	0
8	0	0
9	0	$\frac{1}{3}$
10	$\frac{1}{3}$	0
11	0	$\frac{1}{3}$

Then, we obtain the conditional PMF  $P_{M|N<4}(m)$  by adding up the rows of this table:

$m$ :	9	10	11
$P_{M N<4}(m)$ :	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

The conditional expectation is:

$$E[M|N < 4] = 9 \cdot \frac{1}{3} + 10 \cdot \frac{1}{3} + 11 \cdot \frac{1}{3} = 10.$$

3. You roll a fair 6-sided die until you get an ordered sequence of  $\{3, 5, 5, 1\}$  in 4 recent rounds.

- (a) What is the probability of you stopping at round  $n$ ? Justify your answer.

**Solution:** We have  $P(n = 4) = \frac{1}{6^4}$ . For  $n = 5$ , since  $n = 4$  will never be achieved if the last four rolls are  $\{3, 5, 5, 1\}$ , then  $P(n = 5) = \frac{1}{6^4}$ . In this way, we can get  $P(n = 6) = \frac{1}{6^4}$ ,  $P(n = 7) = \frac{1}{6^4}$ . Similarly, we can get the conclusion of start from  $n = 4$ , the following four successive events have the same probability.

Let  $n \in \mathbf{N}^+$ ,  $n \bmod 4 = 0$  be the target event, we must ensure  $\{3, 5, 5, 1\}$  didn't emerge in the all subsequences  $n - 4$ ,  $n - 4 - 4$ , ..., which means  $P(n) = (1 - P(n - 4) - P(n - 4 - 4) - \dots) P_{seq}$ ,  $P_{seq} = \frac{1}{6^4}$ . And we have  $P(n) = P(n + 1) = P(n + 2) = P(n + 3)$ . E.g.,  $P(n = 8) = P(n = 9) = P(n = 10) = P(n = 11) = (1 - \frac{1}{6^4}) \frac{1}{6^4}$ .

Then we have:

$$P(n) = \begin{cases} 0, & 1 \leq n \leq 3 \\ (1 - \frac{1}{6^4})^{\lfloor \frac{n}{4} \rfloor - 1} \frac{1}{6^4}, & n \geq 4 \end{cases}$$

(Reminder: Since the sequence  $\{3, 5, 5, 1\}$  has no identical subsequences, the event of  $n$  is not independent of the event of  $n + 1$ .)

- (b) What is the expected value of the number of total rounds?

**Solution:** Let  $E$  be the expected number of rolls until  $\{3, 5, 5, 1\}$  and let  $E_{355}$  be the expected number of rolls until  $\{3, 5, 5, 1\}$  when we start with a rolled  $\{3, 5, 5\}$ , let  $E_{35}$  be the expected number of rolls until  $\{3, 5, 5, 1\}$  when we start with a rolled  $\{3, 5, 5, 1\}$ , let  $E_3$  be the expected number of rolls until  $\{3, 5, 5, 1\}$  when we start with a rolled 3.

1) Start from  $E_3$ : If the next dice value is 3, the current subsequence is still  $\{3\}$ , the event continues. If it is 5, the new subsequence is still  $\{3, 5\}$ , the event transforms into

the same with  $E_{35}$ . Otherwise, there are no valid subsequences, the event transforms into the same with  $E$ .

2) Start from  $E_{35}$ : If the next dice value is 3, the new subsequence is  $\{3\}$ , the event transforms into the same with  $E_3$ . If it is 5, the new subsequence is still  $\{3, 5, 5\}$ , the event transforms into the same with  $E_{355}$ . Otherwise, the event transforms into the same with  $E$ .

3) Start from  $E_{355}$ : If the next dice value is 3, the new subsequence is  $\{3\}$ , the event transforms into the same with  $E_3$ . If it is 1, the new subsequence is  $\{3, 5, 5, 1\}$ , terminated. Otherwise, the event transforms into the same with  $E$ .

4) Start from  $E$ : If the next dice value is 3, the new subsequence is  $\{3\}$ , the event transforms into the same with  $E_3$ . Otherwise, the event is still  $E$ .

Then we have:

$$\begin{aligned} E_3 &= \frac{1}{6}(E_3 + 1) + \frac{4}{6}(E + 1) + \frac{1}{6}(E_{35} + 1) \\ E_{35} &= \frac{1}{6}(E_3 + 1) + \frac{4}{6}(E + 1) + \frac{1}{6}(E_{355} + 1) \\ E_{355} &= \frac{1}{6}(E_3 + 1) + \frac{4}{6}(E + 1) + \frac{1}{6} \\ E &= \frac{1}{6}(E_3 + 1) + \frac{5}{6}(E + 1) \end{aligned}$$

Solve the equation, we have  $E = 1296, E_{355} = 1080, E_{35} = 1260, E_3 = 1290$ .

(c) What is the expected value if the number of total rounds is less than or equal to 10?

**Solution:**

$$\begin{aligned} P(n \leq 10) &= \frac{1}{6^4} \cdot 4 + (1 - \frac{1}{6^4}) \frac{1}{6^4} \cdot 3 \\ &= \frac{9069}{1679616} \approx 0.00539945. \end{aligned}$$

$$\begin{aligned} E(n \mid n \leq 10) &= [\frac{1}{6^4} \cdot (4 + 5 + 6 + 7) + (1 - \frac{1}{6^4}) \frac{1}{6^4} \cdot (8 + 9 + 10)] / P(n \leq 10) \\ &= \frac{63477}{9069} \approx 7. \end{aligned}$$

4. You play 10 rounds of roulette with 1 green, 18 reds, 18 blacks. If you choose green, you may win 3 times of what you bet or lose 3 times of what you bet. For red and black, you may win 1 time of what you bet or lose 1 time of what you bet. You invest \$100 and bet 10% of your balance on each color randomly (with the same probability) in every round. What is your average balance after 10 rounds?

**Solution:** Let  $X_i$  be the average balance after round  $i$ .  $X_0 = 100$ .  $W_i$  be the event of win in round  $i$ . By the conditional expectation formula, we have:

$$\begin{aligned} E(X_1) &= E(X_1 \mid A) P(A) + E(X_1 \mid A^c) P(A^c) \\ &= \frac{1}{3}(1.1 \times X_0 \times \frac{18}{37}) + \frac{1}{3}(1.1 \times X_0 \times \frac{18}{37}) + \frac{1}{3}(1.3 \times X_0 \times \frac{1}{37}) \\ &\quad + \frac{1}{3}(0.9 \times X_0 \times \frac{19}{37}) + \frac{1}{3}(0.9 \times X_0 \times \frac{19}{37}) + \frac{1}{3}(0.7 \times X_0 \times \frac{36}{37}) \\ &= \frac{1003}{1110} X_0 \approx 0.9036036 X_0 \end{aligned}$$

Then,  $E(X_{10}) = \frac{1003}{1110}^{10} X_0 = (\frac{1003}{1110})^{10} 100 \approx 36.28937949$ .

5. Let  $T$  be the number of times a 20-sided die is rolled until a 6 appears.

(a) What is your average value after 20 rounds?

**Solution:** Since dice value in the 20th round is known, if  $T \leq 20$  happened, rolling has been terminated. Otherwise,  $T > 20$ , continue to roll the dice. By the conditional expectation formula, we have:

$$E(T) = E(T \mid T \leq 20) P(T \leq 20) + E(T \mid T > 20) P(T > 20)$$

$$P(T \leq 20) = \sum_{i=0}^{19} \left(\frac{19}{20}\right)^i \left(\frac{1}{20}\right) = 1 - \left(\frac{19}{20}\right)^{20} \approx 0.64151408. \quad P(T > 20) = 1 - P(T \leq 20) = \left(\frac{19}{20}\right)^{20} \approx 0.35848592. \quad E(T) = 20. \quad \text{Then:}$$

$$E(T \mid T \leq 20) = \sum_{i=1}^{20} \left(\frac{19}{20}\right)^{i-1} \left(\frac{1}{20}\right) i / P(T \leq 20) = 20 - 2\left(\frac{19}{20}\right)^{20} \approx 19.28302816.$$

$$E(T \mid T > 20) = \frac{E(T) - E(T \mid T \leq 20) P(T \leq 20)}{P(T > 20)} = \frac{20 - (20 - 2\left(\frac{19}{20}\right)^{20})(1 - \left(\frac{19}{20}\right)^{20})}{\left(\frac{19}{20}\right)^{20}} = 22 - 2\left(\frac{19}{20}\right)^{20} \approx 21.28302816.$$

(b) What is the expected value of  $T$  conditioned on all rolls producing even numbers?

**Solution:** Let  $A$  be the event of all rolls producing even numbers.  $E(T \mid A) = \sum_{i=1}^{+\infty} i \cdot P(T = i \mid A)$ . Based on Bayes rule, we have:

$$\begin{aligned} P(A) &= \sum_{i=1}^{+\infty} P(A \cap (T = i)) \\ &= \sum_{i=1}^{+\infty} \left(\frac{9}{20}\right)^{i-1} \cdot \frac{1}{20} \\ &= \frac{1}{20} \cdot \frac{1}{1 - \frac{9}{20}} \\ &= \frac{1}{11} \end{aligned}$$

Then,

$$\begin{aligned} E(T \mid A) &= \sum_{i=1}^{+\infty} P(A \cap (T = i)) / P(A) \\ &= \sum_{i=1}^{+\infty} i \cdot 11 \cdot \left(\frac{9}{20}\right)^{i-1} \cdot \frac{1}{20} \\ &= \frac{20}{11} \approx 1.81818182. \end{aligned}$$