ENGG 2760A / ESTR 2018: Probability for Engineers

5. Expectation, Variance, Joint PMFs

Prof. Hong Xu

Credit to Prof. Andrej Bogdanov

Expectation of a function

PMF of X:

x	0	1	2
$p(x)$	$1/3$	$1/3$	$1/3$

$\mathbf{E}[X] = 1$

$\mathbf{E}[\underbrace{X-1}_{\text{func}}] = -1\cdot\frac{1}{3} + 0\cdot + 1\cdot\frac{1}{3} = 0$

$y = x-1$	-1	0	1
$p(y)$	$1/3$	$1/3$	$1/3$

$\mathbf{E}[\underline{(X-1)^2}] = 0\cdot\frac{1}{3}\cdot + 1\cdot\frac{2}{3} = \frac{2}{3}$

$z = (x-1)^2$	0	1
$p(z)$	$1/3$	$2/3$

Expectation of a function, again

p.m.f. of X:

x	0	1	2
$p(x)$	$1/3$	$1/3$	$1/3$

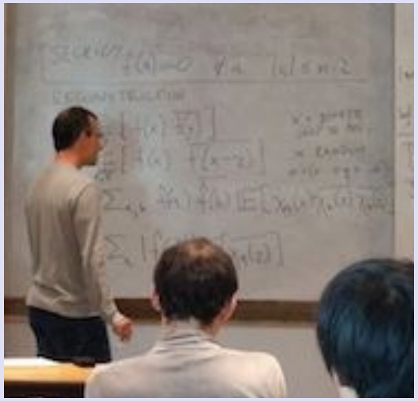
$\mathbf{E}[X] =$ 1

$\mathbf{E}[X-1] = (-1)\cdot\frac{1}{3} + (1-1)\cdot\frac{1}{3} + (2-1)\cdot\frac{1}{3} = 0$

$\mathbf{E}[(X-1)^2] = (0-1)^2\cdot\frac{1}{3} + ((1-1))^2\cdot\frac{1}{3} + (2-1)^2\frac{1}{3} = \frac{2}{3}$

$$\boxed{\mathbf{E}[f(X)] = \sum_x \underline{f(x)}\ \underline{p(x)}}$$

1km



60%

40%

5km/h

30km/h

AVERAGE TIME?

$$T = \frac{\text{Distance}}{\text{Speed}} = \frac{1}{V}.$$

$$E[V] = 0.6 \cdot 5 + 0.4 \cdot 30 = 15,$$

$$\frac{1}{E[V]} = \frac{1}{15} \approx 0.067 \text{ hr.}$$

$$E[T] = E\left[\frac{1}{V}\right] = 0.6 \cdot \frac{1}{5} + 0.4 \cdot \frac{1}{30}$$

$$\approx 0.133 \text{ hr.}$$

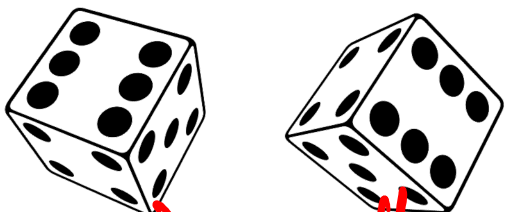
$$E\left[\frac{1}{V}\right] \neq \frac{1}{E[V]}$$

Joint probability mass function

The joint PMF of random variables X, Y is the bivariate function

$$p(x, y) = \mathbf{P}(X = x, Y = y)$$

Example



X, Y = 1st, 2nd roll

$p(x, y) = 1/36.$

$Y \backslash X$	1	2	3	4	5	6
1	1/36	- -	1/36	-	-	
2						
3						
4						
5						
6						

There is a bag with 4 cards:

1 2 3 4

You draw two without replacement. What is the joint PMF of the face values?

X, Y. 4 x 3

Y \ X	1	2	3	4
1	0	1/12	1/12	1/12
2	1/12	0	-	- -
3	⋮		0	
4				0

What is the PMF of the sum?

$Z = X + Y$

$X, Y.$ 4×3

Y \ X	1	2	3	4
1	0	1/12	1/12	1/12
2	1/12	0	-	- -
3	1		0	
4				0

Z	3	4	5	6	7
p(z)	1/6	1/6	1/3	1/6	1/6

What is the expected value?

$$E[Z] = 3 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + \cdots = \underline{5}$$

PMF and expectation of a function

$Z = f(X, Y)$ **has PMF**

$$\mathcal{P}_Z(z) = \sum_{x,y:f(x,y)=z} \mathcal{P}_{XY}(x,y)$$

and expected value

$$\mathbf{E}[Z] = \sum_{x,y} f(x,y)\, p_{XY}(x,y)$$

What if the cards are drawn with replacement?

Y \ X	1	2	3	4
1	1/16	1/16	---	
2	1/16	1/16		
3				
4				

$$E[X+Y]$$

$$= \frac{1}{16}\Big((1+1)+(1+2)+\cdots + (4+4)\Big)$$

$$= \frac{1}{16}\cdot(1+2+3+4)\cdot 8$$

$$= 5$$

Marginal probabilities

Y \ X	1	2	3	4	
1	0	1/12	1/12	1/12	1/4
2	1/12	0	1/12	1/12	1/4
3	1/12	1/12	0	1/12	1/4
4	1/12	1/12	1/12	0	1/4
	1/4	1/4	1/4	1/4	1

$P(Y = y) = \sum_x P(X = x, Y = y)$

~~p(y|x)~~

p(y)

p(X=1) =

$P(X = x) = \sum_y P(X = x, Y = y)$

Linearity of expectation

$$E[X+Y] = \sum_{x,y} (x+y)\cdot p_{XY}(x,y)$$

$$= \sum_{x,y} x\cdot p_{XY}(x,y) + \sum_{x,y} y\cdot p_{XY}(x,y)$$

$$= \sum_{x} x \sum_{y} p_{XY}(x,y) + \sum_{y} y\cdot \sum_{x} p_{XY}(x,y)$$

$p(x)$ $p(y)$

For every two random variables X and Y

$$\mathbf{E}[X+Y] = \mathbf{E}[X] + \mathbf{E}[Y]$$

$$\sum_x x\cdot p(x) + \sum_y y\cdot p(y)$$

$$= E[X] + E[Y]$$

1 2 3 4

$$\mathbf{E}[X + Y]$$

without replacement

P_X $1/4$

P_Y $1/4$

$E[X] + E[Y]$

$= 2.5 + 2.5 = 5$

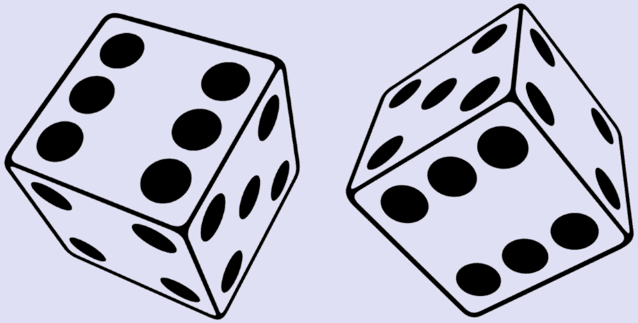
with replacement

P_X $1/4$

P_Y $1/4$

$E[X] = 2.5$ $E[Y] = 2.5$

$E[X] + E[Y] = 5$



$$\mathbf{E}[X+Y] = ?$$

$$= E[X] + E[Y]$$

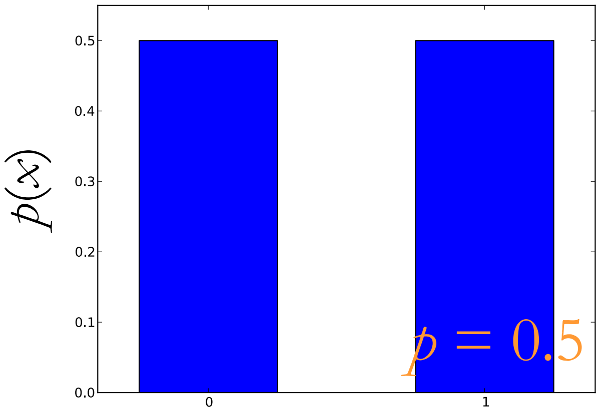
$$= \frac{1+2+3+\cdots+6}{6} \cdot 2$$

$$= 7$$

The indicator (Bernoulli) random variable

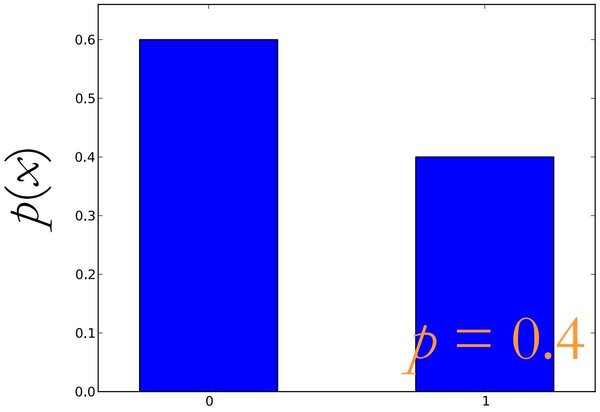
Perform a trial that succeeds with probability p and fails with probability $1 - p$.

x	0	1
$p(x)$	$1-p$	p



If X is $\mathrm{Bernoulli}(p)$ then

$$E[X] = p$$



Mean of the Binomial

$\mathrm{Binomial}(n, p)$: **Perform** n **independent trials, each of which succeeds with probability** p.

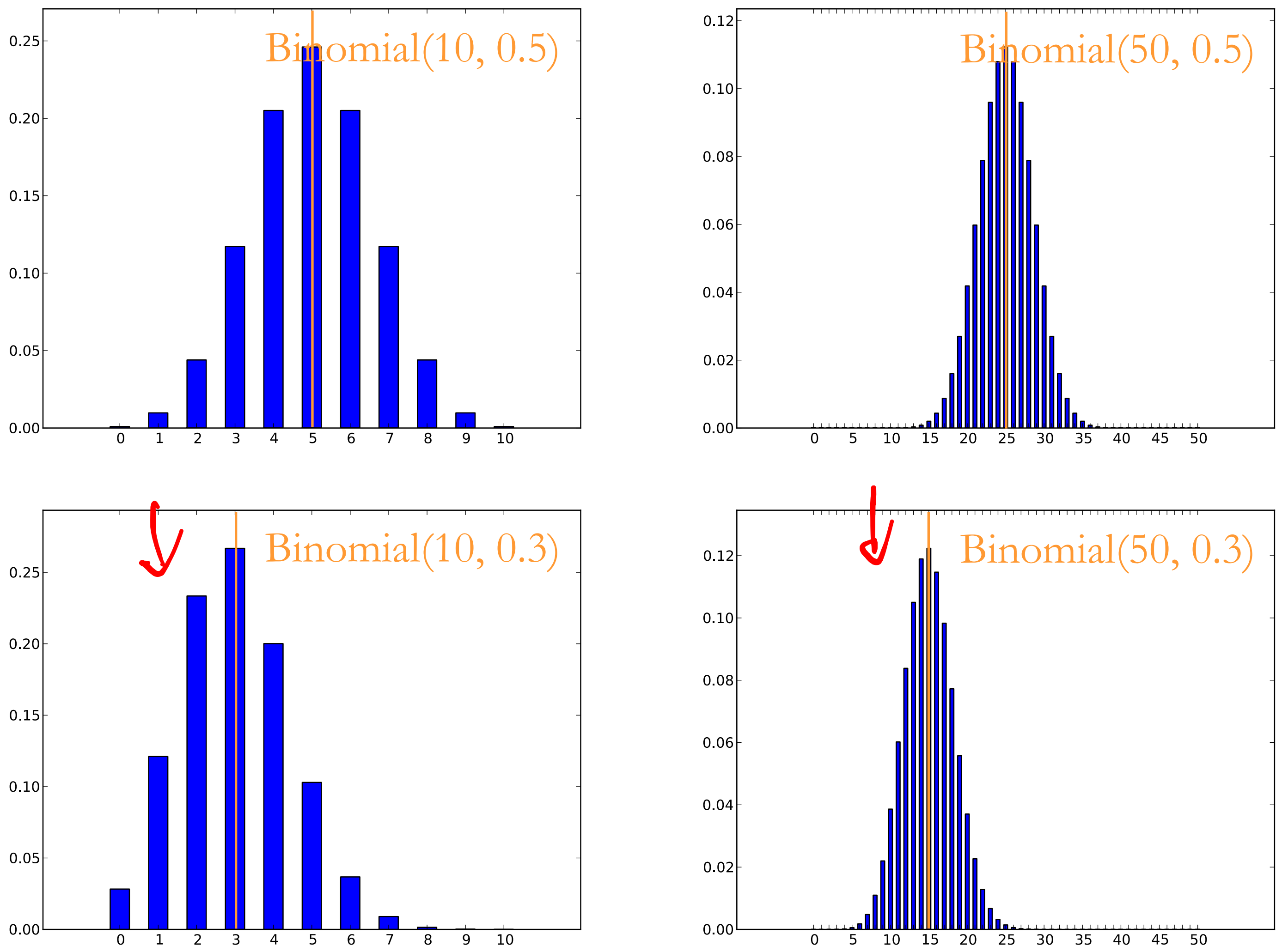
$X =$ **number of successes**

$X = X_1 + X_2 + \cdots + X_n, \quad X_i \sim \mathrm{Bernoulli}(p)$

$\Rightarrow E[X] = E[X_1] + E[X_2] + \cdots + E[X_n]$

$= n \cdot p$

$$\boxed{E[\mathrm{Binomial}(n,p)] = n \cdot p}$$



Binomial(10, 0.5)
Binomial(50, 0.5)
Binomial(10, 0.3)
Binomial(50, 0.3)

n people throw their hats in a box and each picks one out at random. On average, how many get back their own hat?

X = # People who get back own hat.

$$X = X_1 + \cdots + X_n, \quad X_i = \begin{cases} 1, & \text{if } P_i \text{ gets own hat} \\ 0, & \text{otherwise.} \end{cases}$$

$$E[X] = E[X_1] + E[X_2] + \cdots + E[X_n]$$
$$= P(X_1=1) + P(X_2=1) + \cdots$$
$$= \frac{1}{n} + \frac{1}{n} + \cdots = 1$$

Events "$X_i = 1$" NOT IND.

Mean of the Poisson

Poisson(λ) **approximates** Binomial(n, λ/n) **for large** n

$$\boxed{p(k) = e^{-\lambda}\, \lambda^k / k!} \qquad k = 0, 1, 2, 3, \ldots$$

$$\cancel{7}\ p(k) = \lim_{n\to\infty} P(\text{Binomial}(n, \lambda/n) = k)$$

$$E[\text{Poisson}(\lambda)] = \lambda \longleftarrow E[\text{Binomial}(n, \lambda/n)]$$

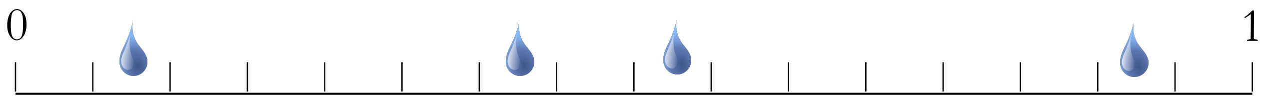
$$= n \cdot \lambda/n = \lambda$$

$$E[\text{Poisson}(\lambda)] = \sum_{k=0}^{\infty} k \cdot e^{-\lambda} \frac{\lambda^k}{k!} = e^{-\lambda} \cdot \lambda \cdot \underline{\sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!}}$$

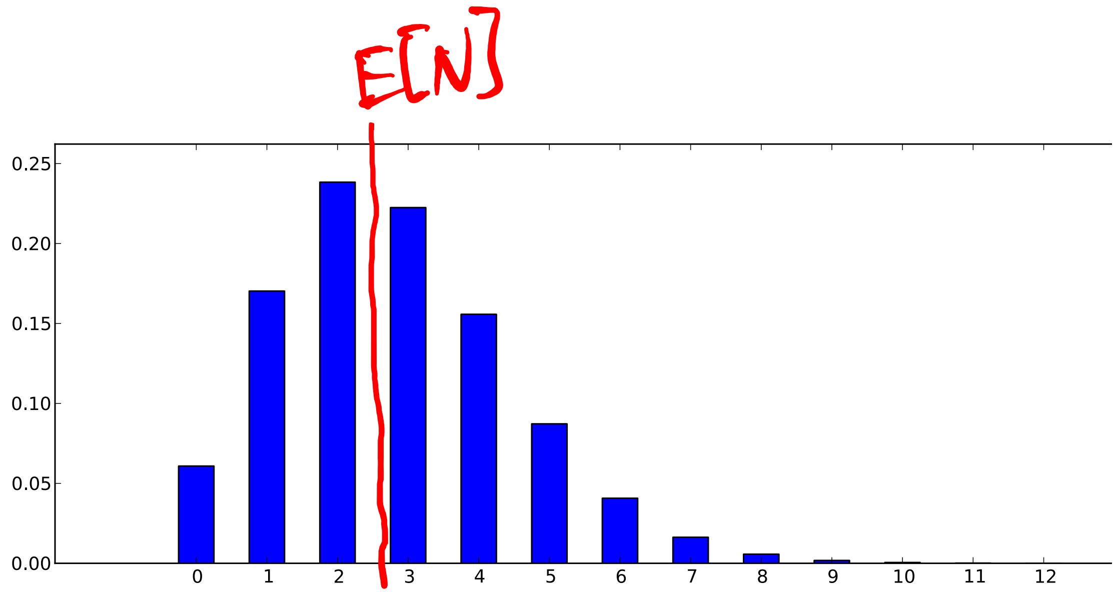
$$= e^{-\lambda} \cdot \lambda \cdot e^{\lambda} = \lambda$$

Raindrops

Rain is falling on your head at an average speed of 2.8 drops/second.



Number of drops N **is** $\mathrm{Binomial}(n, 2.8/n)$

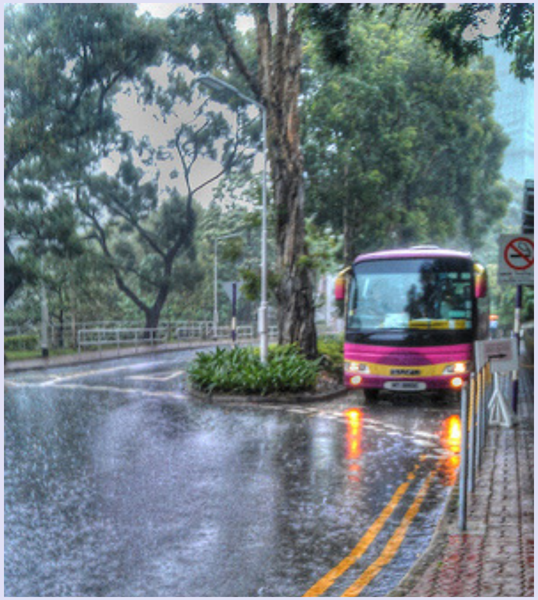


Rain falls on you at an average rate of 3 drops/sec.

When 100 drops hit you, your hair gets wet.

You walk for 30 sec from MTR to bus stop.

What is the probability your hair got wet?



X = # drops I'm hit by during 30s.

$X \sim \text{Poisson}(\lambda)$, $\lambda = 3 \Rightarrow 90$

$\lambda = E[X] = E[X_1 + \cdots + X_{30}]$. X_i: # drops in sec i.

$= E[X_1] + E[X_2] + \cdots$

$= 3 + \cdots = 90$.

$$P(X \geq 100) = 1 - P(X < 100)$$

$$= 1 - \sum_{0}^{99} e^{-k} \cdot \frac{\lambda^k}{k!}$$

$$\approx 0.1582$$

Investments

You have three investment choices:

A: put $25 in one stock

B: put $½ in each of 50 unrelated stocks

C: keep your money in the bank

Which do you prefer?

Investments

Probability model

Each stock
- doubles in value with probability ½
- loses all value with probability ½

Different stocks perform independently

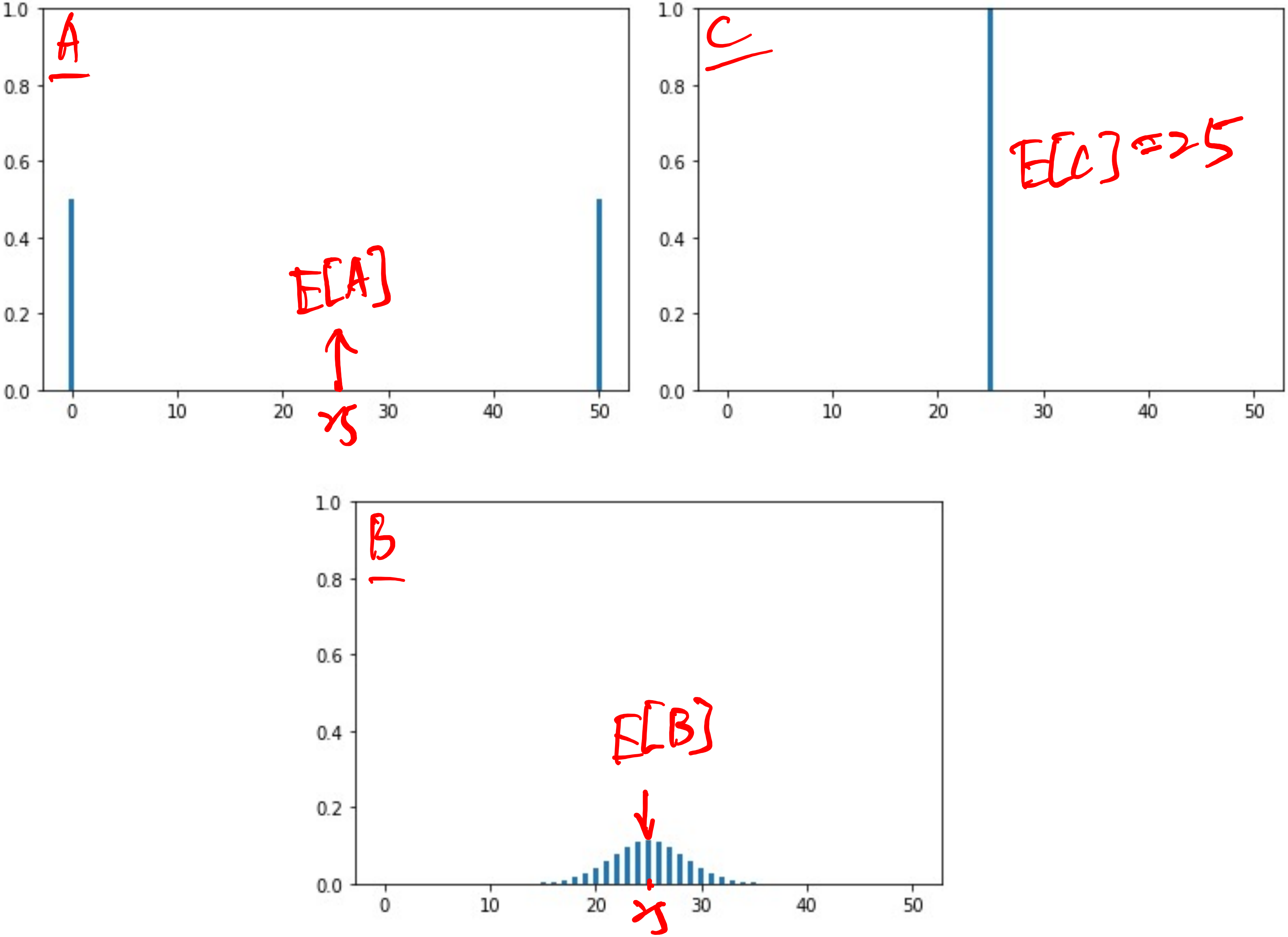
\$25

A. PMF:

0	50
50%	50%

B. $B = B_1 + B_2 + \cdots + B_{50}$, Binomial(50, 1/2)

C = 25. with prob. 1


A
E[A]
25
C
E[C] = 25
B
E[B]
25
0.0
0.2
0.4
0.6
0.8
1.0
0
10
20
30
40
50

Variance and standard deviation

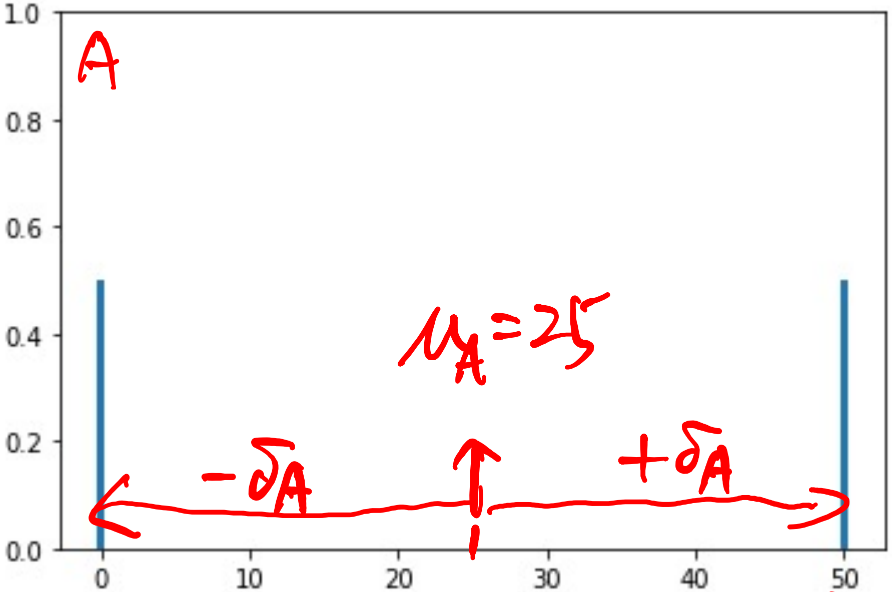
Let $\mu = \mathbf{E}[X]$ be the expected value of X.

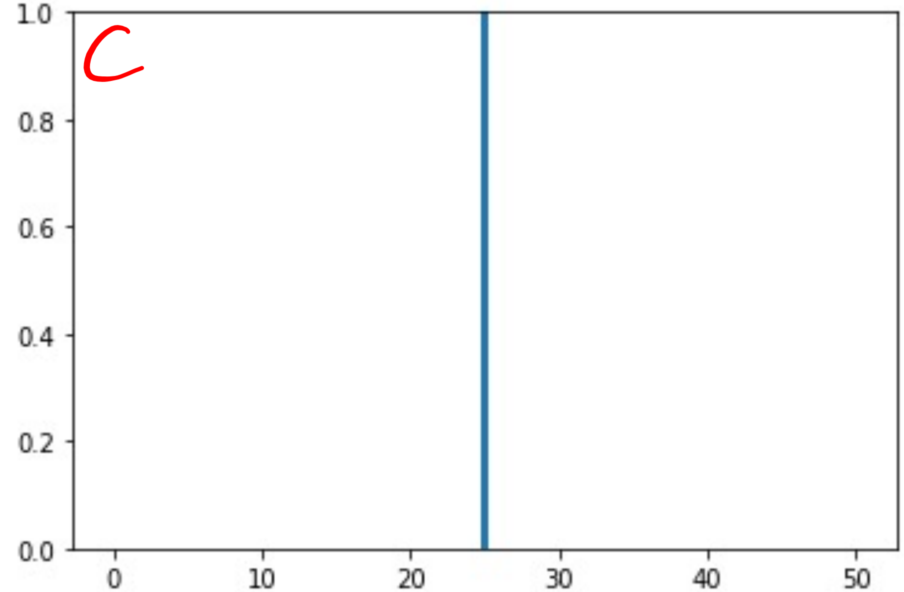
The variance of X is the quantity

$$\mathbf{Var}[X] = E[(X - \mu)^2]$$

The standard deviation of X is $\sigma = \sqrt{\mathbf{Var}[X]}$

It measures how close X and μ are typically.





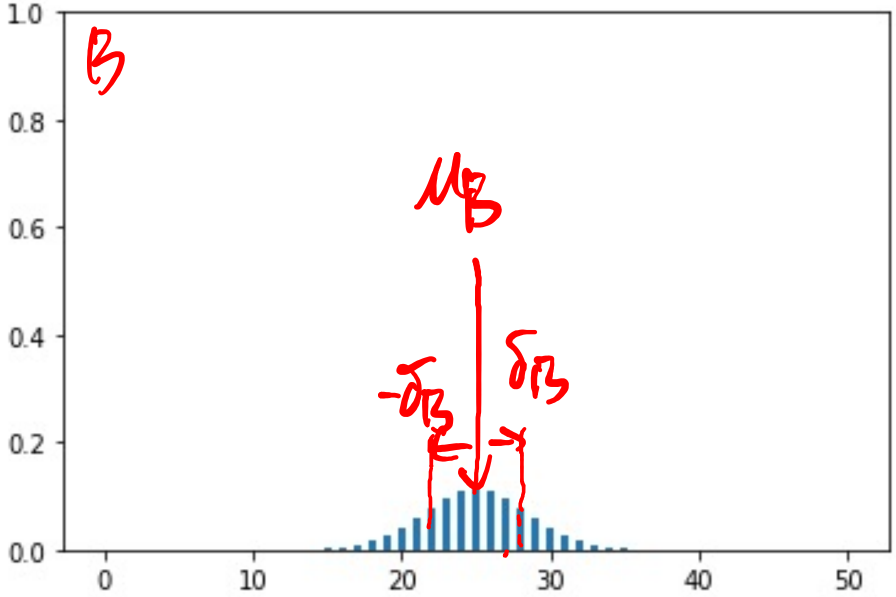
$$Var(A) = E[(A - \mu_A)^2]$$

$$= \frac{1}{2}(-25)^2 + \frac{1}{2}(25)^2$$

$$= 25^2$$

$$\sigma_A = 25.$$

$$Var(B) = 0$$



$$\mathbf{Var}[\text{Binomial}(n, p)] = np(1-p)$$

$Var(B) = 50 \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{50}{4}$

$\sigma_B \approx 3.54$

***Most* of the probability mass is within *a few* σs from μ**

More on this in later lectures…

Another formula for variance

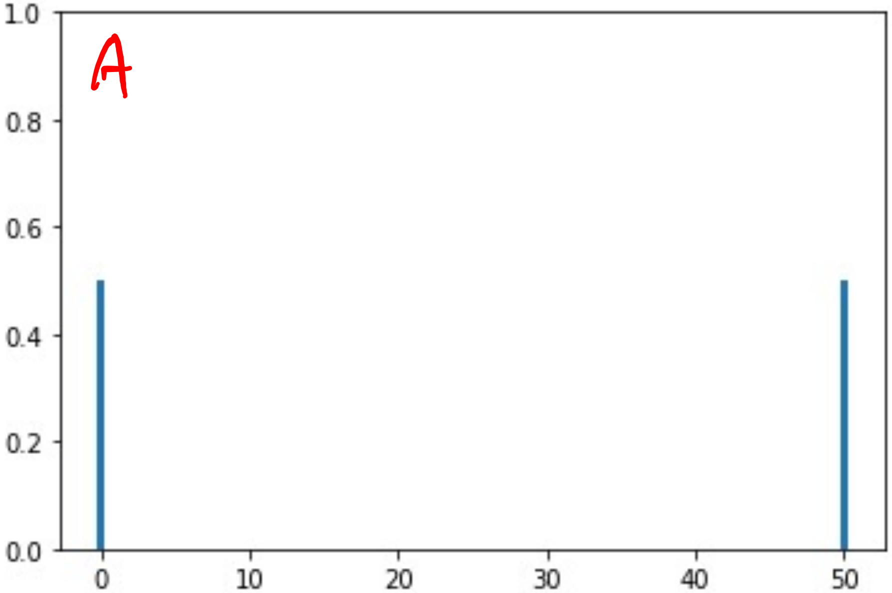
$$Var[X] = E[(X-\mu)^2]$$

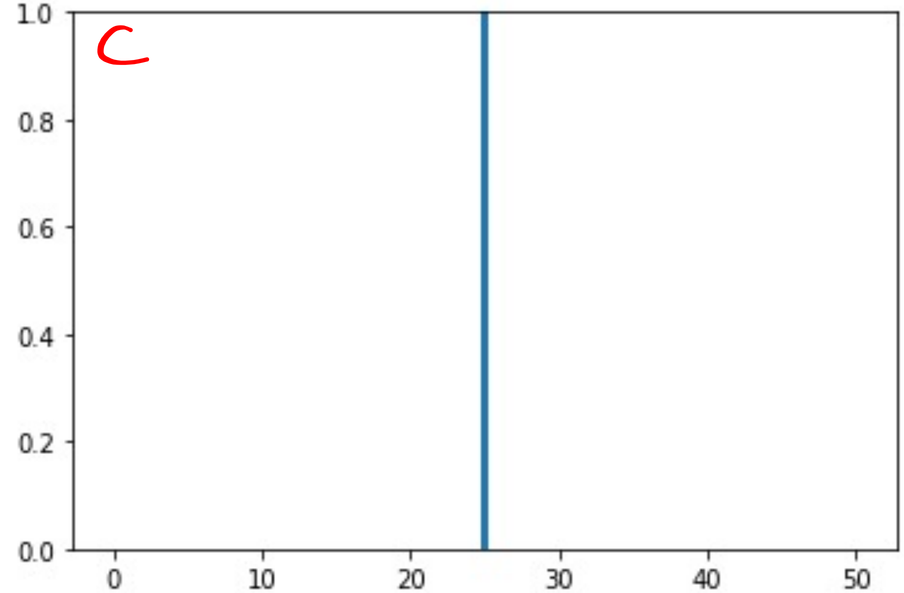
$$= E[(X^2 - 2\mu X + \mu^2)]$$

$$= E[X^2] - 2\mu \cdot E[X] + \mu^2$$

$$= E[X^2] - 2\mu^2 + \mu^2$$

$$\underline{\underline{= E[X^2] - E[X]^2}} \geq 0$$





$E[A^2] = \frac{1}{2} \cdot 50^2$

$E[A]^2 = 25^2$

$Var[A] = \frac{1}{2} 50^2 - 25^2 = 25^2$

$$\mathbf{E}[X] = ?$$

$$\mathbf{Var}[X] = ?$$

x	1	2	3	4	5	6
$p(x)$	$1/6$	$1/6$	$1/6$	$1/6$	$1/6$	$1/6$

pmf

$$\sum_{i}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$E[X] = 3.5$$

$$E[X^2] = \frac{1}{6}\cdot 1^2 + \frac{1}{6}\cdot 2^2 + \cdots + \frac{1}{6}\cdot 6^2$$

$$= \frac{1}{6}(1^2 + 2^2 + 3^2 + \cdots + 6^2)$$

$$\approx 15.167$$

$$\mathrm{Var}[X] = 15.167 - 3.5^2 \approx 2.917, \quad \sigma \doteq 1.707$$