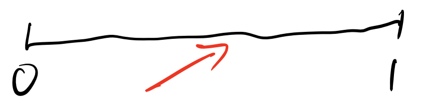
E02: COUNTABLE, UNCOUNTABLE, AND MEASURABLE SETS



$P(\text{IRRATIONAL}) = ?$

Infinitely many numbers in $[0,1]$,

$P(\text{any particular number}) = 0$

$\Rightarrow P(\text{IRRATIONAL}) = P(\pi) + P(\sqrt{2}) + \cdots = 0$??

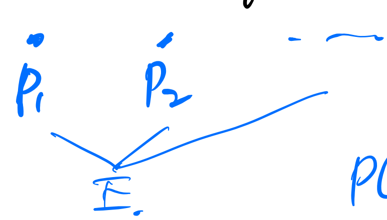
$\boxed{\text{fishy}}$ by this reasoning,

$P(\text{anything}) = 0$ Absurd!

How to get out of this ??

Finite Sample Space

outcomes



$$P(E) = \sum_{\omega \in E} P(\omega) \leftarrow \text{Axioms } ✓$$

Infinite Sample Space:

Can we set up probabilities $\Omega = [0,1]$ so that the axioms work?

i.e., events $E \subseteq [0,1]$, we have $P(E)$ as a number

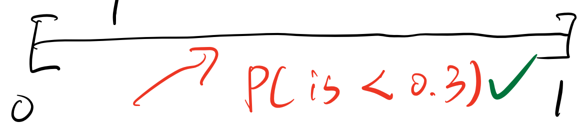
(something mathematicians has studied long)

Attempt 1: Allow events to be arbitrary subsets of [0, 1]. ☹ axioms inconsistent

$$\underbrace{P(A\cup B)}_{1} = \underline{P(A)} + \underbrace{P(B)}_{0} - \underline{P(A\cap B)}$$

too much freedom, need to restrict.

Attempt 2: (the other extreme)



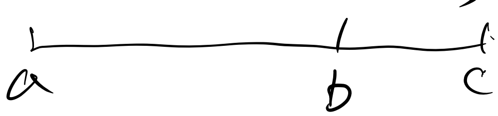
$P(\text{is} < 0.3)$ ✓

$P(0.9 < ? < 10/11)$ ✓

At minimum, we should allow subintervals of [0, 1].

$$P([a,b]) = b-a, \quad \forall a,b \in [0,1]. \checkmark$$

$$\Rightarrow P([a,c]) = P([a,b]) + P([b,c]) \checkmark$$



but this is still limited, only subintervals

What about this: 

$$P([0.1, 0.2] \cup [0.7, 0.8]) = P([0.1, 0.2]) + P([0.7, 0.8])$$

- Start with 'atomic' events $[a, b]$
- Take complements, $\cup$, $\cap$ to get new events

if these are finite, then axioms hold

BOREL: Countable $\cup, \cap$ of events, axioms ✓

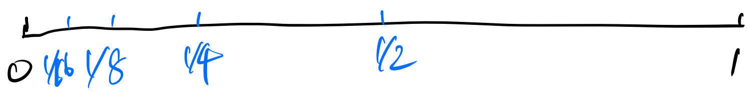
$E = [\frac{1}{2}, 1] \cup [\frac{1}{8}, \frac{1}{4}] \cup [\frac{1}{32}, \frac{1}{16}] \cup \cdots$ can list them $\to$ countable

$P(E) = P(\ ) + P(\ ) + \cdots = \frac{1}{2} + \frac{1}{8} + \frac{1}{32} + \cdots$

$\Rightarrow$ Now, I can repeat and take countable $\cup/\cap/C$ of countable $\cup/\cap/C$ of events. axioms ✓

Now we can attack our question.

E = " is a negative power of 2"



$P(E) = P([1/2, 1/2]) + P([1/4, 1/4]) + \cdots$; $\cup$ of intervals

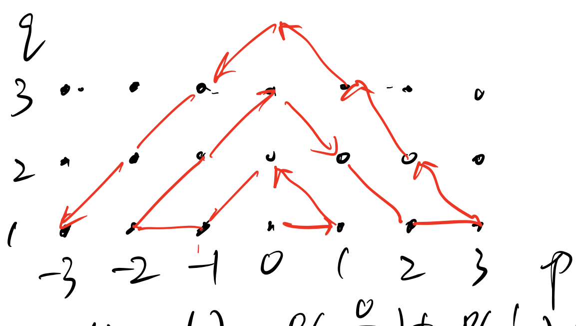
$= 0 + 0 + \cdots = 0$ ✓ legitimate use of axioms

So what is P(Rational)?

Theorem: The set of rational numbers is countable. (can list them)

rational num: p/q. q: positive integer. p: any integer.

how to list rational numbers?



$\frac{0}{1}, \frac{1}{1}, \frac{0}{2}, \frac{-1}{1}, \cdots$

redundant

list of rational num.

$P(\text{rational}) = P(\frac{0}{1}) + P(\frac{1}{1}) + \cdots = 0.$ ✓

By axioms,

$P(\text{Irrational}) = 1 - P(\text{rational}) = 1$ → irrational # uncountable

One result of Borel's theorem is that any events of non-zero probability must be uncountable.

Does there exist an event of prob. 0 that is uncountable? YES

B = " Base-3 expansion of a num. contains only 0s and 2s "

E.g. $0.0020220 \in B$ $\quad 0.00120 \notin B$.

$0.0012_{\text{Base }3} = \frac{0}{3} + \frac{0}{3^2} + \frac{1}{3^3} + \cdots$

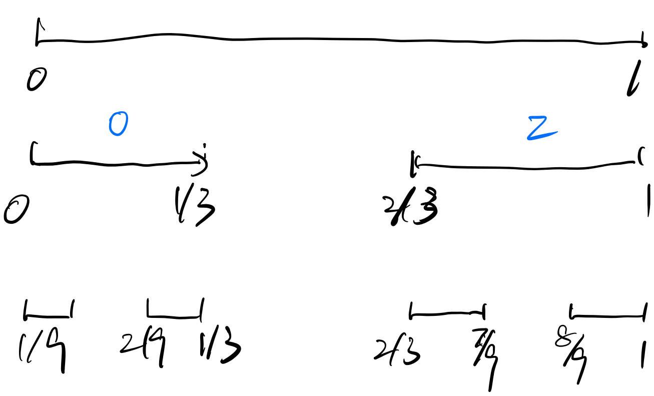
$P(B) = ?$

First, is B legitimate? can it be formed as intervals via $\cap$, $\cup$, etc.

Let B_n = " The first n tri-gits are 0 or 2 "
digits

so $0.0012 \in B_1, B_2, \notin B_3$

B_0 $\quad$ B_1 $\quad$ B_2



$P(B_0) = 1$ ...

$P(B_1) = \frac{2}{3}$

$P(B_2) = \frac{4}{9}$

$P(B_n) = \left(\frac{2}{3}\right)^n$

$B = B_0 \cap B_1 \cap B_2 \cap \cdots$

$\Rightarrow P(B) \leq P(B_n) = (\frac{2}{3})^n \longrightarrow 0$ going to 0.

But,

<u>Theorem</u>: B is uncountable.

<u>Proof (by Cantor)</u>: Suppose B is countable.

$\Rightarrow$ There is a way to list all of B.

e.g.:
$0.\underline{2}202\cdots$
$0.0\underline{0}22\cdots$
$0.20\underline{0}0\cdots$
$0.222\underline{2}\cdots$
$\vdots$

flip-it: $0.0220\cdots$

it cannot be in the list!

$\Rightarrow$ The list cannot be complete.

$\Rightarrow$ B is uncountable.