ENGG 2760A / ESTR 2018: Probability for Engineers

3. Conditional Probability and Independence

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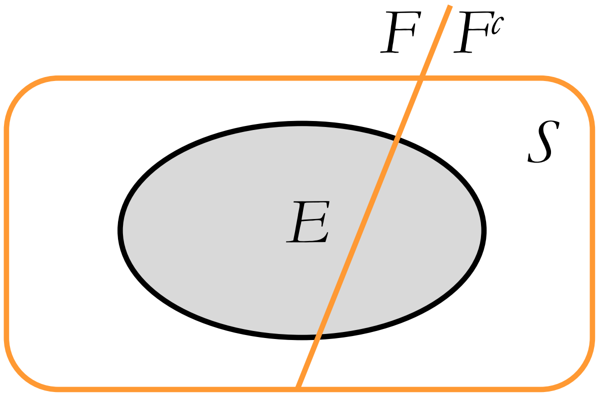
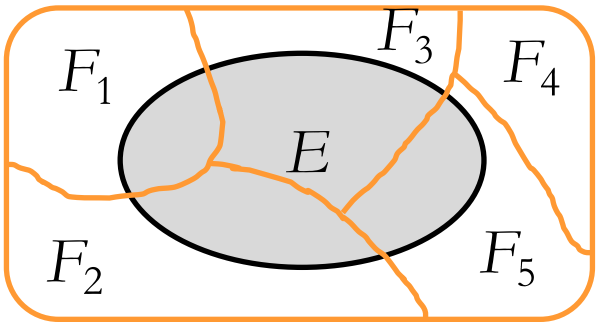
Total probability theorem

$$\mathbf{P}(E) = \mathbf{P}(E \cap F) + \mathbf{P}(E \cap F^c)$$
$$= \mathbf{P}(E \mid F)\mathbf{P}(F) + \mathbf{P}(E \mid F^c)\mathbf{P}(F^c)$$





More generally, if $F_1, \ldots, F_n$ partition Ω then

$$\mathbf{P}(E) = \mathbf{P}(E \mid F_1)\mathbf{P}(F_1) + \ldots + \mathbf{P}(E \mid F_n)\mathbf{P}(F_n)$$

An urn has 10 white balls and 20 black balls. You draw two at random. What is the probability that their colors are different?

Geography quiz...

What is the capital of Romania?

A: Brasov

B: Budapest

C: Bucharest

D: Bratislava

Did you know or were you lucky?

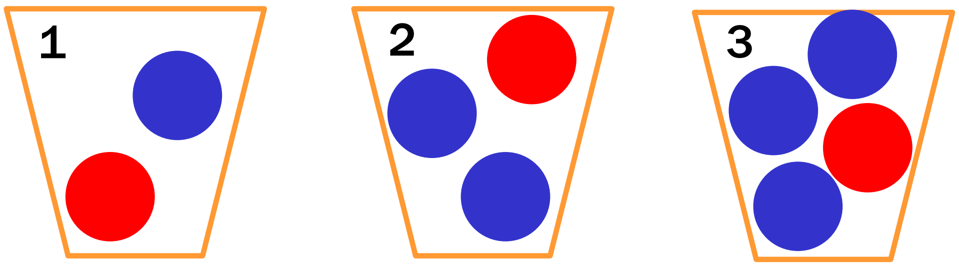
Geography quiz

Probability model

There are two types of students:

Type K: Knows the answer

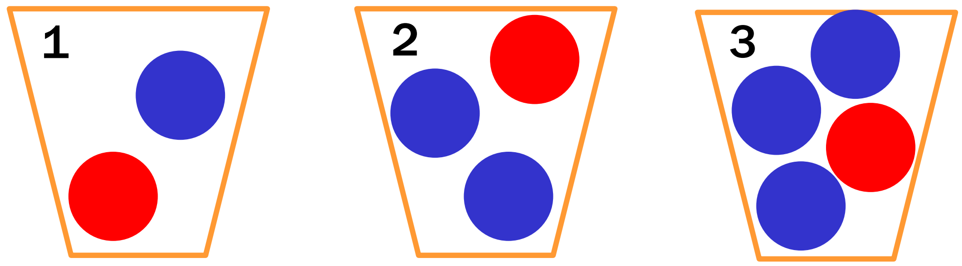
Type K^c: Picks a random answer



I choose a cup at random and then a random ball from that cup. The ball is red. You need to guess where the ball came from.

Which cup would you guess?

Cause and effect



cause: C_1 C_2 C_3

effect: R

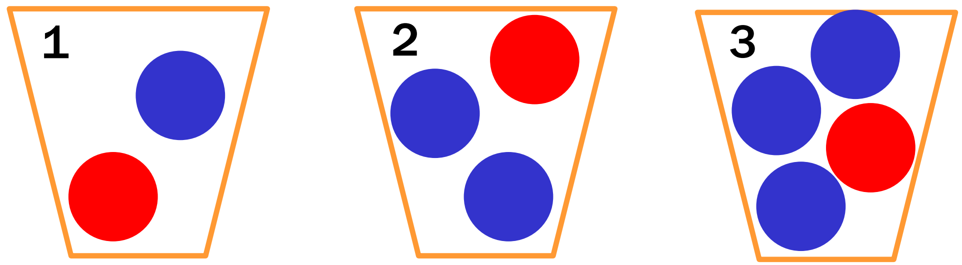
Bayes' rule

$$\mathbf{P}(C|E) = \frac{\mathbf{P}(E|C)\,\mathbf{P}(C)}{\mathbf{P}(E)} = \frac{\mathbf{P}(E|C)\,\mathbf{P}(C)}{\mathbf{P}(E|C)\,\mathbf{P}(C) + \mathbf{P}(E|C^c)\,\mathbf{P}(C^c)}$$

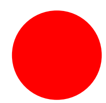
More generally, if $C_1, \ldots, C_n$ partition S then

$$\mathbf{P}(C_i|E) = \frac{\mathbf{P}(E|C_i)\,\mathbf{P}(C_i)}{\mathbf{P}(E|C_1)\,\mathbf{P}(C_1) + \ldots + \mathbf{P}(E|C_n)\,\mathbf{P}(C_n)}$$

Cause and effect



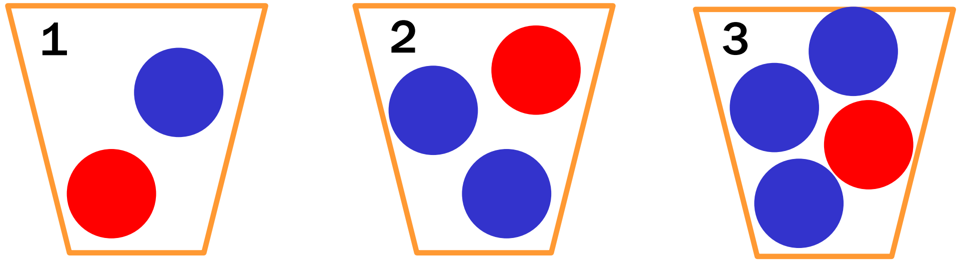
cause: C_1 C_2 C_3



effect: R

$$\mathbf{P}(C_i \,|\, R) = \frac{\mathbf{P}(R \,|\, C_i)\ \mathbf{P}(C_i)}{\mathbf{P}(R \,|\, C_1)\ \mathbf{P}(C_1) + \mathbf{P}(R \,|\, C_2)\ \mathbf{P}(C_2) + \mathbf{P}(R \,|\, C_3)\ \mathbf{P}(C_3)}$$

Cause and effect



$$\Omega =$$

$$\mathbf{P}(C_i) =$$

$$\mathbf{P}(R \mid C_i) =$$

$$\mathbf{P}(R) =$$

$$\mathbf{P}(C_i \mid R) =$$

Two classes take place in Lady Shaw Building.

ENGG2430 has 100 students, 20% are girls.

NURS2400 has 10 students, 80% are girls.

A girl walks out. What are the chances that she is from the engineering class?

Summary of conditional probability

Conditional probabilities are used:

① When there are causes and effects

to estimate the probability of a cause when we observe an effect

② To calculate ordinary probabilities

Conditioning on the right event can simplify the description of the sample space

Independence of two events

Let E_1 **be "first coin comes up** $\mathtt{H}$**"**

E_2 **be "second coin comes up** $\mathtt{H}$**"**

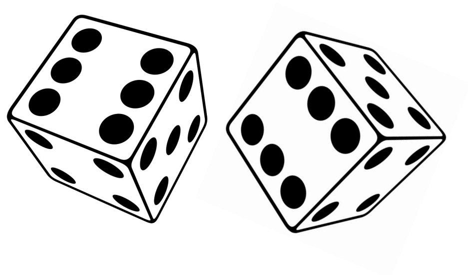
Then $\mathbf{P}(E_2 \mid E_1) = \mathbf{P}(E_2)$

$$\mathbf{P}(E_2 \cap E_1) = \mathbf{P}(E_2)\mathbf{P}(E_1)$$

Events A **and** B **are independent if**

$$\mathbf{P}(A \cap B) = \mathbf{P}(A)\, \mathbf{P}(B)$$

Examples of (in)dependence



Let E_1 be "first die is a 4"

S_6 be "sum of dice is a 6"

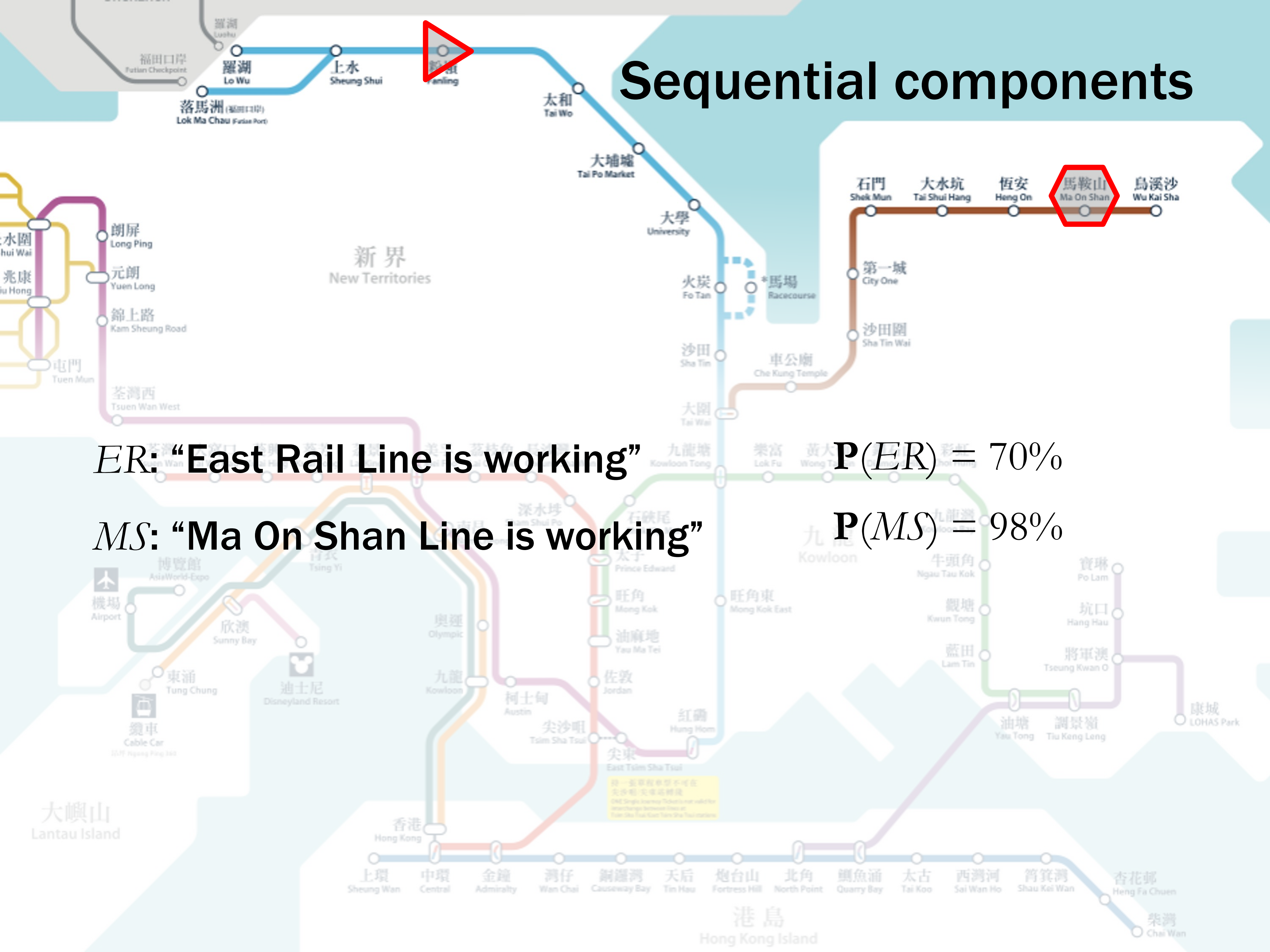
S_7 be "sum of dice is a 7"

E_1 **and** S_6**?**

E_1 **and** S_7**?**

S_6 **and** S_7**?**

Sequential components



ER: “East Rail Line is working” $\mathbf{P}(ER) = 70\%$

MS: “Ma On Shan Line is working” $\mathbf{P}(MS) = 98\%$

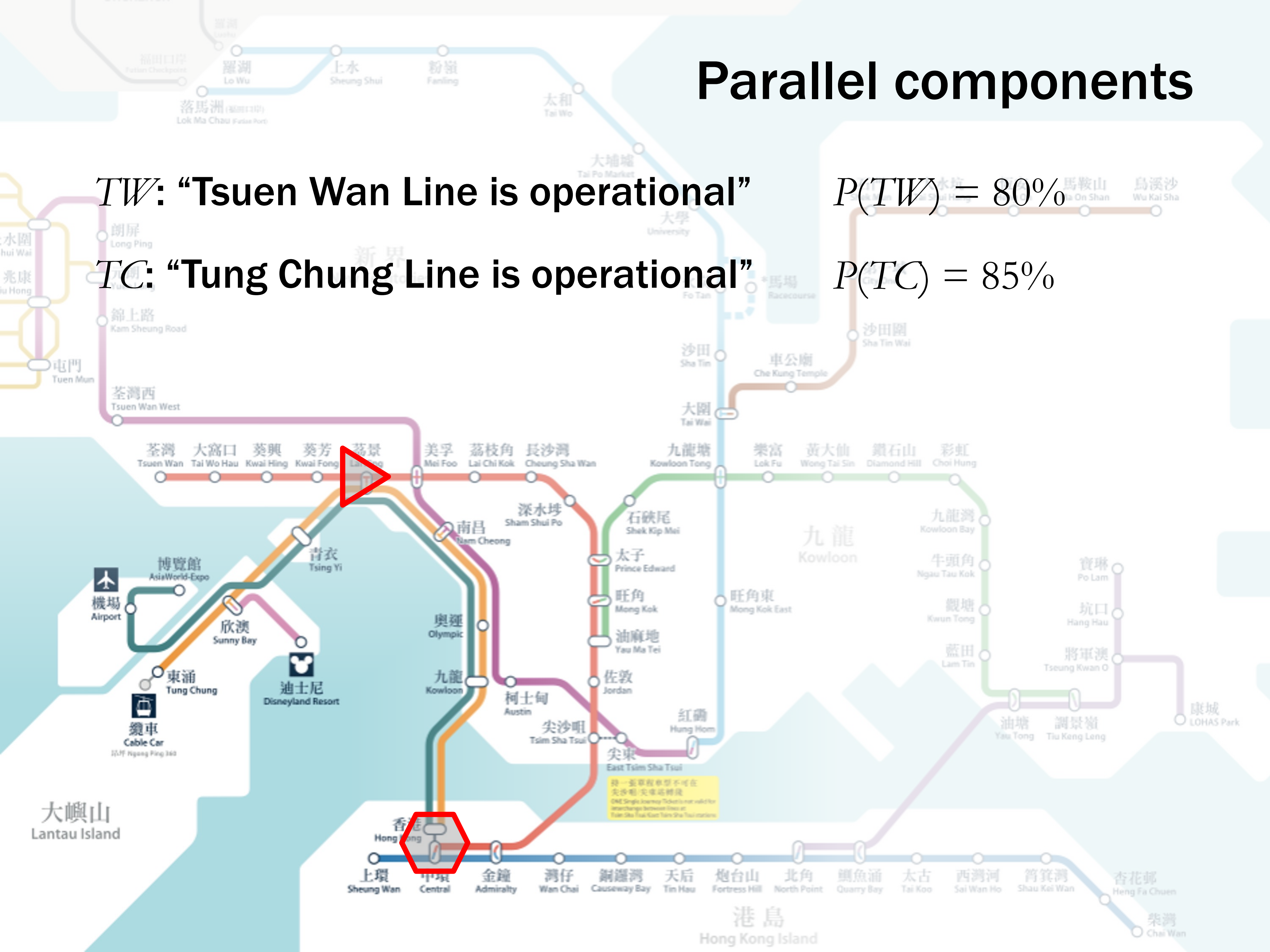
Algebra of independent events

If A and B are independent, then A and B^c are also independent.

Parallel components

TW: "Tsuen Wan Line is operational" $P(TW) = 80\%$

TC: "Tung Chung Line is operational" $P(TC) = 85\%$



Independence of three events

Events A, B, and C are **independent** if

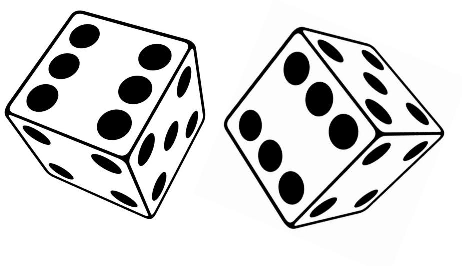
$$\mathbf{P}(A \cap B) = \mathbf{P}(A)\,\mathbf{P}(B)$$

$$\mathbf{P}(B \cap C) = \mathbf{P}(B)\,\mathbf{P}(C)$$

$$\mathbf{P}(A \cap C) = \mathbf{P}(A)\,\mathbf{P}(C)$$

and $\mathbf{P}(A \cap B \cap C) = \mathbf{P}(A)\,\mathbf{P}(B)\,\mathbf{P}(C)$.

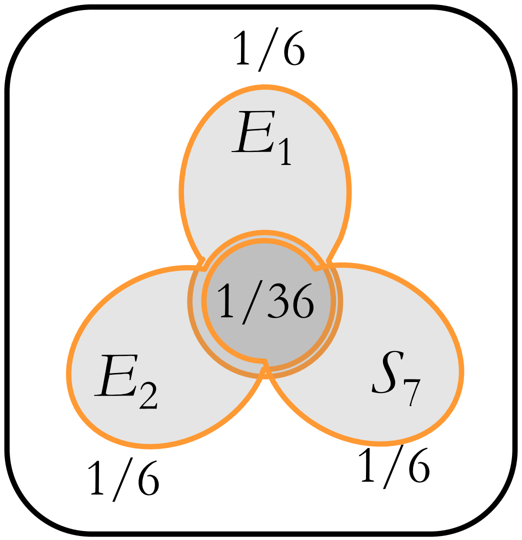
(In)dependence of three events



Let E_1 be "first die is a 4"

E_2 be "second die is a 3"

S_7 be "sum of dice is a 7"



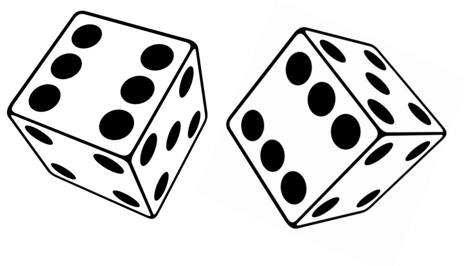
E_1, E_2?

E_1, S_7?

E_2, S_7?

E_1, E_2, S_7?

(In)dependence of three events



Let A be "first roll is 1, 2, or 3 "

B be "first roll is 3, 4, or 5"

C be "sum of rolls is 9"

A, B?

A, C?

B, C?

A, B, C?

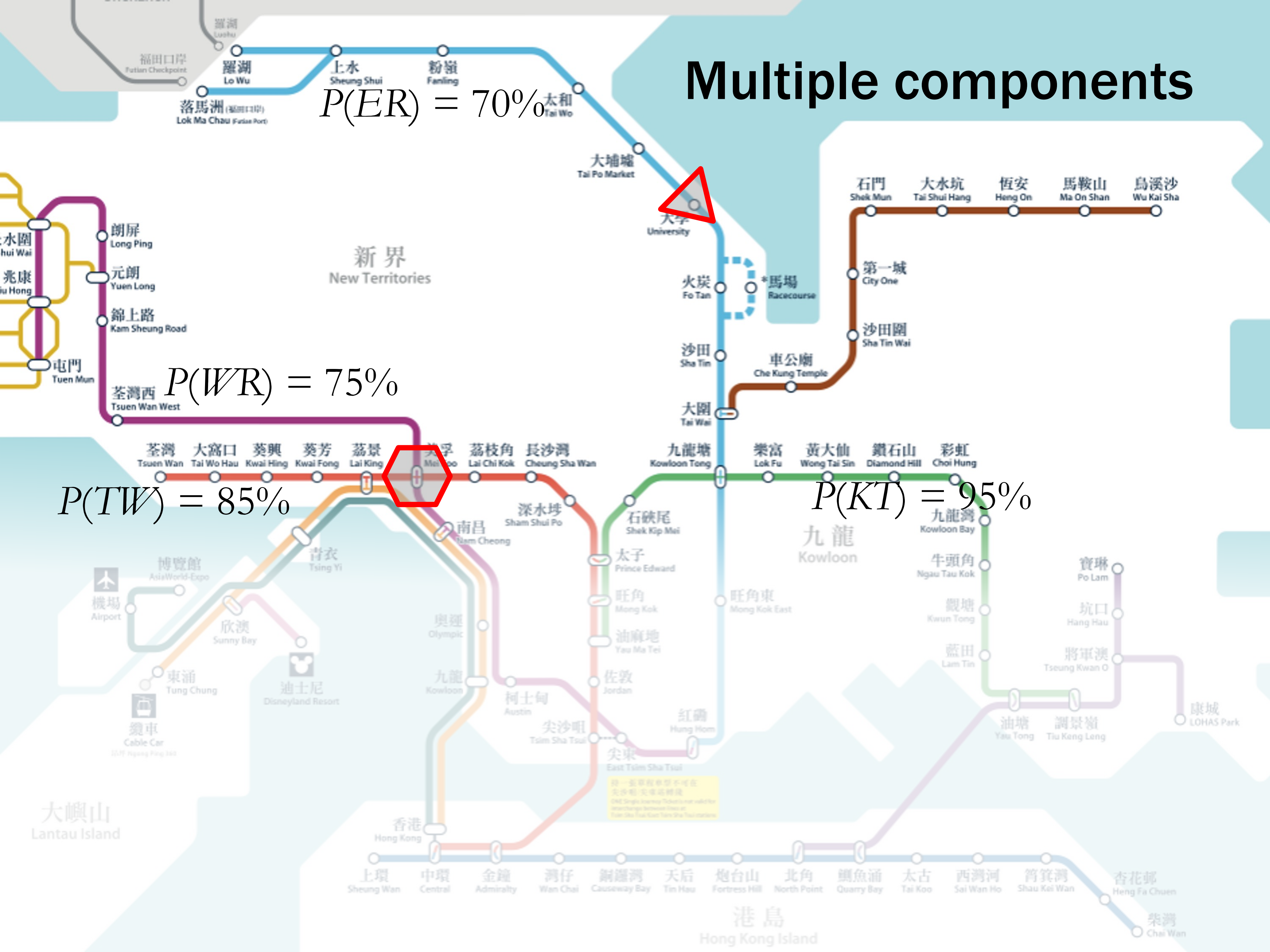
Independence of many events

Events $A_1, A_2, \ldots$ are independent if for every subset of the events, the probability of the intersection is the product of their probabilities.

Algebra of independent events

Independence is preserved if we replace some event(s) by their complements, intersections, unions

Multiple components



$P(ER) = 70\%$

$P(WR) = 75\%$

$P(TW) = 85\%$

$P(KT) = 95\%$

Multiple components

$$\boxed{\begin{aligned} P(ER) &= 70\% \\ P(WR) &= 75\% \\ P(KT) &= 95\% \\ P(TW) &= 85\% \end{aligned}}$$

Conditional independence

A and B are independent conditioned on F if

$$\mathbf{P}(A \cap B \mid F) = \mathbf{P}(A \mid F)\,\mathbf{P}(B \mid F)$$

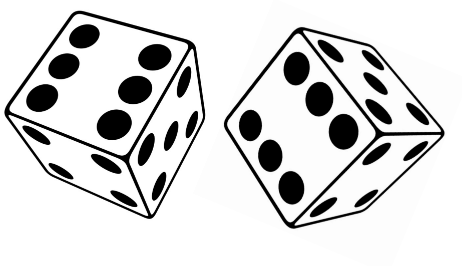
Alternative definition:

$$\mathbf{P}(A \mid B \cap F) = \mathbf{P}(A \mid F)$$

today	tomorrow
☀️	80% ☀️, 20% 🌧️
🌧️	40% ☀️, 60% 🌧️

It is ☀️ on Monday. Will it 🌧️ on Wednesday?

Conditioning does not preserve independence



Let E_1 be "first die is a 4"

E_2 be "second die is a 3"

S_7 be "sum of dice is a 7"

The New York Times

‘Crazy Rich Asians’ Has Soared, but It May Not Fly in China

Conditioning may destroy dependence

Probability model