Martingales and Stopping Times

Gambling strategies

Get \$1 for each head ; lose \$1 for tail

Strategy One : Play for 5 rounds , stop.

$T=\begin{cases} 5, & 1/2 \\ 10, & 1/2 \end{cases}$ Two : If after 5 rounds, you are ahead, stop ; otherwise, play 5 more rounds

Three. Stop after HHHHH for the 1st time.

Four : Stop when either \$5 ahead or \$10 behind.

Five : Stop as soon as \$5 ahead.

Which strategy beats the house ?

Martingale $X_0 \quad X_1 \quad X_2 \quad \cdots$

X_t = winnings at time t , $\in \mathbb{R}$

$$E[X_t \mid X_{t-1}=x_{t-1}, X_{t-2}=x_{t-2}, \cdots, X_0=x_0]=x_{t-1}$$

$x_0=0$. $E[X_1]=0$. $X_2=\begin{cases} X_1+1, & 1/2 \\ X_1-1, & 1/2 \end{cases} \Rightarrow E[X_2]=E[X_1]$

$\Rightarrow E[X_t]=0$, $\forall t$.

Stopping time T takes value $0, 1, 2, \cdots$. is a R.V.

$E[X_T]=?$ can't depend on future

Assumption $P(T=t \mid X_0=x_0, X_1=x_1, \cdots, \cancel{X_{t+1}=x_{t+1}, \cdots})$

$=P(T=t \mid X_0=x_0, \cdots, X_t=x_t)$

Difference between ① ②, and ③ ④ ⑤

① ②: T is finite, deterministic

③ ④ ⑤: T can be infinite.

In S③ − ⑤: $P(T=\infty)=0$. Similar to E01.

And you will eventually achieve your goal.

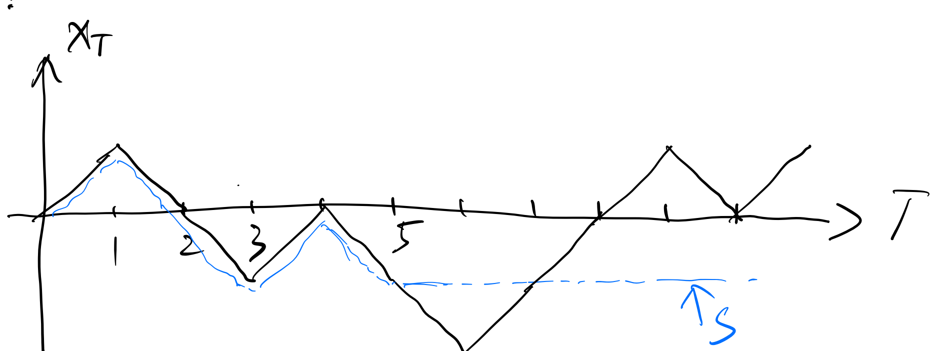
So why not play S5, which gives \$5 always?

△ Theorem: If T is a stopping time and $X_0, X_1, \cdots$, is a martingale, then if $E[T] < \infty$, then $E[X_T] = E[X_0]$.

(In other words, $E[T]$ for S5 is ∞. On average, you need to play ∞ round. That's the price of S5.)

Proof:

Idea 1:



Let $\bar{X}_0, \bar{X}_1, \bar{X}_2, \cdots$, be: $\bar{X}_t = X_{\min\{T,t\}}$.

$\Rightarrow$ Say $T=5$, then $\bar{X}_0 = X_0, \cdots, \bar{X}_5 = X_5$, $\bar{X}_6 = X_5, \cdots$.

Let's look at this R.V. instead. Clearly this is still a Martingale.

If for example $T \leq 10$ with prob. 1,
$E[X_T] = E[\bar{X}_T] = E[\bar{X}_{10}] = E[\bar{X}_0] = E[X_0] = 0.$

So S2 is pointless. $E_{S_2} = 0$.

What if. T unbounded? (k bounded)

Idea 2: Force to stop at say k rounds.

Let $\bar{T}$ be the same as T up until k rounds.

$$\bar{T} = \min\{T, k\}. \text{ — bounded.}$$

$\Rightarrow E[X_{\bar{T}}] = 0$ by above.

If one can prove $E[X_{\bar{T}}] = E[X_T]$, then it's done.

But this is not true in general.

(We can show that as $k \to \infty$, the difference between the two $\to 0$.)

So $E[X_T - X_{\bar{T}}] = \sum_{t=0}^{\infty} E[X_T - X_{\bar{T}} \mid T=t] \cdot P(T=t)$ ...... $\cancel{\star}$

Since $\bar{T} = \min\{T, k\}$, when $t \leq k$, the exp. is 0.

$\Rightarrow \cancel{\star} = \sum_{t=k+1}^{\infty} E[X_T - X_{\bar{T}} \mid T=t] \cdot P(T=t)$

Now $|X_T - X_{\bar{T}}| \leq t$ (diff can't be bigger than total # rounds played).

$\Rightarrow \cancel{\star} \leq \sum_{t=k+1}^{\infty} t \cdot P(T=t) \longrightarrow$ $E[T]$ with first k terms missing.

If $E[T]$ is finite, as $k \to \infty$,

$$\sum_{t=k+1}^{\infty} t \cdot P(T=t) \longrightarrow 0.$$

$$\Rightarrow \lim_{k \to 0} \left| E\left[X_T - X_{\min\{T,k\}}\right] \right| = 0$$

$$= \lim_{k \to 0} \left| \underbrace{E[X_T]}_{\to 0} - E\left[\cancelto{0}{X_{\min\{T,k\}}}\right] \right|$$