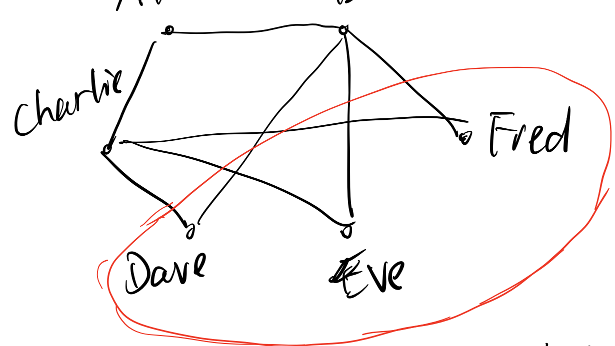
Ramsey Numbers.

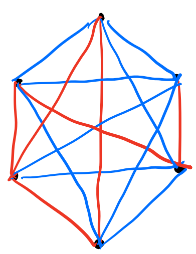
Consider the following claim:

In any group of 6 people, it must be that 3 of them know one another, or 3 of them are strangers to one another.

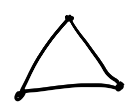


GRAPHS: use complete graphs, different colors to represent acquaintance or friend.

K_6



clique:

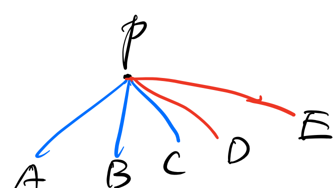


The claim is: for any 2-coloring K_6, there must be a blue clique on 3 vertices or a red clique on 3 vertices.

Proof:

Take any vertex, say P.

5 edges leaving P.



$\Rightarrow$ At least 3 of them are of the same color, say blue.

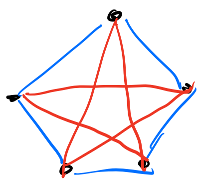
$\Rightarrow$ Then: ① if any one of AB, BC, CA is blue, you have a blue 3-clique.

② if no, then you have a red 3-clique.

$\Rightarrow$ Done.

So what if there're 5 people? Is this still true?

NO



$5 < R(3, 3) \le 6$ $\Rightarrow$ $R(3,3) = 6$.

blue red

$R(r, s)$: smallest graph size that guarantees blue r-clique or a red s-clique.

Turns out our reasoning before can be made more general to derive $R(r, s)$.

$R(1, s) = 1$ (trivial), $\Leftrightarrow$ $R(r, 1) = 1$.

$R(2, s) = s$. $R(r, 2) = r$

$R(3, 3) = 6 = R(3, 2) + R(2, 3)$

In general, $R(r, s) \le R(r, s-1) + R(r-1, s)$

Sketch of proof:

Consider a complete graph on $R(r, s-1) + R(r-1, s)$ vertices with 2-coloring. Pick a vertex v, partition the remaining vertices into 2 sets M and N s.t. for vertex w, $w \in M$ if (v, w) is blue, $w \in N$ if (v, w) is red.

$\Rightarrow$ The graph size is:

$$R(r-1, s) + R(r, s-1) = |M| + |N| + 1$$

$\Rightarrow$ Either $R(r-1, s) \le |M|$, or $R(r, s-1) \le |N|$

If $|M| \geq R(r-1, s)$,

① if M has a red K_s, then so does the original graph.

② if not, then M has a blue K_{r-1}, $M \cup \{v\}$ has a blue K_r by the definition of M.

If $|N| \geq R(r, s-1)$, can prove with the same Logic.

Now let $U(r,s) = R(r, s-1) + R(r-1, s)$. With boundary conditions $R(2,s) = s$ and $R(r,2) = r$, we can obtain $U(r,s)$. This is an upper bound for $R(r,s)$.

k	2	3	4	5	6	7	8	9
$U(k,k)$.	2	6	20	70	252	924	3432	12870.

This shows exponential growth. $\log U(k,k)$ is almost linear.

The base converges to between 3.75 to 4.

$\Rightarrow R(k,k) \leq \alpha^k$, $R(k,k) \leq 4^k$. In fact: $R(r,s) \leq 2^{r+s}$

Q: Is $R(k,k)$ really close to 4^k?

Is the bound tight / achievable? $R(k,k) \geq ?$

Now we can use probability! (finally ...).

Random graph

Prob. model. Fix n: # nodes/vertices.

Sample space: all 2-color complete graphs on n vertices.

Pick a graph uniformly at random.

So events "(u,v) is blue/red" are IND. and each has prob. $1/2$.

F = "Graph has no blue and red k-clique."

$P(F) = ?$

If n is large enough compared to k, e.g. $n = 4^k$,

$P(F) \to 0$.

If $P(F) > 0$, then $R(k,k) > n$.

$P(F)$ is difficult to calculate, but can be estimated.

F^c = "has blue or red k-clique (K_k)".

$= \bigcup_k (K_k^B \cup K_k^R) \to$ union of sets of k vertices with B/R K_k.

Note whether the vertices form B/R K_k is not independent (K_k^B, K_k^R not IND.).

Union rule: $P(A_1 \cup A_2 \cup A_3 \cdots) \le P(A_1) + P(A_2) \cdots$

$$\Rightarrow P(F^c) \le \sum_{S:\, k \text{ vertices}} \underbrace{P(K_k^B)}_{\frac{1}{2^{\binom{k}{2}}}} + P(K_k^R).$$

$\binom{k}{2}$ edges for K_k.

$$= \binom{n}{k} \cdot 2 \cdot 2^{-\binom{k}{2}}.$$ — increasing in n.

$\Rightarrow$ can find the largest n s.t. $P(F^c) < 1$, i.e. $P(F) > 0$, (by using the computer).

Since $\binom{n}{k} \le n^k$,

$$\binom{n}{k} \cdot 2 \cdot 2^{-\binom{k}{2}} \le 2 \cdot n^k \cdot 2^{-\binom{k}{2}} < 1$$

$$\Rightarrow n^k < \frac{1}{2} \cdot 2^{\binom{k}{2}} \Rightarrow k \log_2 n < \binom{k}{2} - 1.$$

$$\Rightarrow \log_2 n < \frac{\cancel{k}(k-1)}{2 \cdot \cancel{k}} - \frac{1}{k} \quad \text{i.e. } \frac{k}{2} - \frac{1}{2} - \frac{1}{k}$$

So n needs to be a bit smaller than $\sqrt{2}^k$.

In fact, we can have:

$$\boxed{\sqrt{2}^k \le R(k,k) \le 4^k}$$ big gap!!

$R(4,4)$ is known, but $R(5,5)$ is still unknown.