1. There are 5 cards in a box, one of which has the number '1' on it, one has the number '2' on it, and the other three cards have the number '3' on it. Assuming that the probability of each card being drawn is equal, now randomly draw a card from the box. Please calculate: 1) What is the expectation of the number on the card? 2) What is the variance of the number on the card?

   Solution: Let X be the number on the drawn card. Then we can have PMF of X as:

x	1	2	3
$P(x)$	1/5	1/5	3/5

   Then, $E(X) = 1 \times \frac{1}{5} + 2 \times \frac{1}{5} + 3 \times \frac{3}{5} = \frac{12}{5}$. $Var(X) = E\left[(X - E(X))^2\right] = (-\frac{7}{5})^2 \times \frac{1}{5} + (-\frac{2}{5})^2 \times \frac{1}{5} + (\frac{3}{5}^2) \times \frac{3}{5} = \frac{16}{25}$.

2. Roll a fair six-sided dice 100 times. What is the expected number of times T that four consecutive number '3' occur? (For the case where the "consecutive number '3'" is more than four times, we consider it as multiple "four consecutive number '3'" situations. For example, in sequence "1123333345", we think it contains two cases because the "four consecutive number '3'" case can either start at the 4th or 5th place of the sequence.)

   Solution: Let T_i be:

$$T_i = \begin{cases} 1, & \text{if the result of roll } i, i+1, i+2 \text{ and } i+3 \text{ are all number } '3'. \\ 0, & \text{else.} \end{cases}$$

   And we have $P(T_i = 1) = (\frac{1}{6})^4$, so $E(T_i) = (\frac{1}{6})^4 \times 1 + 0 = (\frac{1}{6})^4$.

   Let $T = T_1 + T_2 + T_3 + \cdots + T_{97}$, so the final result is:

$$E(T) = E(T_1 + T_2 + \cdots + T_{97}) = E(T_1) + E(T_2) + \cdots + E(T_{97}) = 97 * (\frac{1}{6})^4 = \frac{97}{1296}$$

3. A box contains 10 red and 8 blue balls. Two balls in the box are drawn randomly without replacement. If the two balls are in different colors, you win \$1; if they are in the same color, you lose \$1. Suppose you have \$0 at the beginning. Calculate the following problems after you draw two balls from the box (The money in your hand can be negative):

   (a) Calculate the probability that you have more than \$0.

   (b) What is the expectation of the money you win?

   (c) What is the variance of the money you win?

   Solution: Let random variable X be:

$$X = \begin{cases} 1, & \text{if the two balls are in different color.} \\ -1, & \text{else.} \end{cases}$$

   (a) Actually, we should calculate $P(X = 1)$ because only if the two balls are in different color, we can earn money, so we have:

$$P(X = 1) = \frac{10}{18} \times \frac{8}{17} + \frac{8}{18} \times \frac{10}{17} = \frac{80}{153}$$

(b) We can have PMF of X as:

x	-1	1
$P(x)$	$73/153$	$80/153$

So $E(X) = -1 \times \frac{73}{153} + 1 \times \frac{80}{153} = \frac{7}{153}$.

(c) $Var(X) = E\left[(X - E(X))^2\right] = \frac{73}{153} \times (-\frac{160}{153})^2 + \frac{80}{153} \times (\frac{146}{153})^2 = \frac{23360}{23409} \approx 0.998$

4. Roll a fair six-sided dice twice. Let X and Y be the minimum and maximum of the two rolls, respectively. Please calculate the following questions:

(a) $P((Y - X) = 3)$.

(b) The joint PMF of X and Y and their marginal PMFs.

(c) The expected value of $X + Y$.

Solution:

(a) For $(Y - X) = 3$, there are three conditions: $x = 1, y = 4$; $x = 2, y = 5$ and $x = 3, y = 6$. We have:

$$P(x = 1, y = 4) = P(x = 2, y = 5) = P(x = 3, y = 6) = 2 \times \frac{1}{6} \times \frac{1}{6} = \frac{1}{18}$$

So, $P((Y - X) = 3) = 3 \times \frac{1}{18} = \frac{1}{6}$.

(b) For the cases that $x = y$, it means the tow rolls come out the same result. So for a certain x and y, its probability is $\frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$.

For the cases that $x < y$, there are two conditions: the first roll comes out x, the second roll comes out y; or the first roll comes out y, the second roll comes out x. so for a certain x and y, its probability is $2 \times \frac{1}{6} \times \frac{1}{6} = \frac{1}{18}$.

For the cases that $x > y$, it is impossible because the minimum of two rolls can not be larger than the maximum, so its probability is 0.

We can have the PMF of X and Y as:

$y \backslash x$	1	2	3	4	5	6
1	1/36	0	0	0	0	0
2	1/18	1/36	0	0	0	0
3	1/18	1/18	1/36	0	0	0
4	1/18	1/18	1/18	1/36	0	0
5	1/18	1/18	1/18	1/18	1/36	0
6	1/18	1/18	1/18	1/18	1/18	1/36

For $P(X = x)$, we should add up each column in PMF table above; for $P(Y = y)$, we should add up each row in the PMF table above. So we can have the marginal PMFs:

x	1	2	3	4	5	6
$P(x)$	11/36	1/4	7/36	5/36	1/12	1/36

y	1	2	3	4	5	6
$P(y)$	1/36	1/12	5/36	7/36	1/4	11/36

(c) For $E(X + Y) = E(X) + E(Y)$, and:

$$E(X) = 1 \times \frac{11}{36} + 2 \times \frac{1}{4} + 3 \times \frac{7}{36} + 4 \times \frac{5}{36} + 5 \times \frac{1}{12} + 6 \times \frac{1}{36} = \frac{91}{36}$$

$$E(Y) = 1 \times \frac{1}{36} + 2 \times \frac{1}{12} + 3 \times \frac{5}{36} + 4 \times \frac{7}{36} + 5 \times \frac{1}{4} + 6 \times \frac{11}{36} = \frac{161}{36}$$

So we have:

$$E(X+Y) = \frac{91}{36} + \frac{161}{36} = 7$$

5. Throw 50 balls to 30 bins. Each ball is equally likely to land in any of the bins and the throw of each ball is independent.

   (a) What is the probability that there exists at least one empty box?

   (b) What is the expected number of bins that receive exactly one ball?

   (c) What is the expected number of balls that are not alone in their bin?

   Solution:

   (a) Let A_i be the event that the ith box is empty, so we should calculate $P(\bigcup_{i=1}^{30} A_i)$. According to inclusion-exclusion principle, we have:

$$P(\bigcup_{i=1}^{30} A_i) = \sum_{k=1}^{30} \left((-1)^{k-1} \sum_{I \subseteq \{1,\cdots,30\}, |I|=k} P(A_I) \right) \text{, where } A_I = \bigcap_{i \in I} A_i$$

   For k = 1, $P(A_I) = (1-\frac{1}{30})^{50}$; for k = 2, $P(A_I) = (1-\frac{2}{30})^{50}$; $\cdots$; for k = 29, $P(A_I) = (1-\frac{29}{30})^{50}$; for k = 30, $P(A_i) = (1-\frac{30}{30})^{50}$.

   So, the result is:

$$\begin{aligned} P(\bigcup_{i=1}^{30} A_i) &= \binom{30}{1} \times (1-\frac{1}{30})^{50} - \binom{30}{2} \times (1-\frac{2}{30})^{50} + \cdots - \binom{30}{30} \times (1-\frac{30}{30})^{50} \\ &= \sum_{k=1}^{30} (-1)^{k-1} \times \binom{30}{k} \times (1-\frac{k}{30})^{50} \end{aligned}$$

   Proof of the inclusion-exclusion principle. We should prove:

$$P(\bigcup_{i=1}^{n} A_i) = \sum_{k=1}^{n} \left((-1)^{k-1} \sum_{I \subseteq \{1,\cdots,n\}, |I|=k} P(A_I) \right) \text{, where } A_I = \bigcap_{i \in I} A_i$$

   Let the notation $|\cdot|$ be the number of elements in the set. For example, $|S|$ means the number of elements in Set S. So actually we should prove:

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{k=1}^{n} \left((-1)^{k-1} \sum_{I \subseteq \{1,\cdots,n\}, |I|=k} \left|\bigcap_{i \in I} A_i\right| \right) \quad (*)$$

   Assume a random element α is in $\bigcup_{i=1}^{n} A_i$, and it is in the intersection of m sets ($1 \leq m \leq n$, and depends on the where the certain α is) in A_i ($\alpha \in \bigcap_{i \in I_m} A_i$, where $I_m \subseteq \{1, \cdots, n\}, |I_m| = m$).
   Assume T_α is the times that α has been calculated in $(*)$, so T_α will be:

$$T_\alpha = \sum_{k_m=1}^{m} \left((-1)^{k_m - 1} |J_{k_m}| \right), \quad \text{where } J_{k_m} \subseteq I_m, \ |J_{k_m}| = k_m$$

So for $k_m = 1$, $|J_{k_m}| = \binom{m}{1}$; for $k_m = 2$, $|J_{k_m}| = \binom{m}{2}$; $\cdots$; for $k_m = m$, $|J_{k_m}| = \binom{m}{m}$. So T_α is:

$$T_\alpha = \binom{m}{1} - \binom{m}{2} + \binom{m}{3} - \cdots + (-1)^{m-1}\binom{m}{m}$$

Note that:

$$0 = (1-1)^m = \sum_{i=0}^{m} \binom{m}{i} \times (-1)^m = \binom{m}{0} - T_\alpha$$

So we have:

$$T_\alpha = \binom{m}{0} = 1$$

That means for random $\alpha \in \bigcup_{i=1}^{n} A_i$, in $(*)$, we always count it for only one time. So it proves that $(*)$ is right.

(b) Let X_i be:

$$X_i = \begin{cases} 1, & \text{the } i\text{th bin receive exactly one ball.} \\ 0, & \text{else.} \end{cases}$$

So, for the total 30 bins, $X = X_1 + X_2 + \cdots + X_{30}$. We should calculate $E(X)$ to get the expected number of bins that receive exactly one ball.
For $E(X_i)$, we have:

$$E(X_i) = P(X_i = 1) \times 1 + 0 = \binom{50}{1} \times \frac{1}{30} \times (\frac{29}{30})^{49}$$

So, the final result is:

$$E(X) = E(X_1+X_2+\cdots+X_{30}) = E(X_1)+E(X_2)+\cdots+E(X_{30}) = 30 \times E(X_i) = 50 \times (\frac{29}{30})^{49}$$

(c) i. **Method 1:** Let random variable Y be the number of balls that are not alone in a bin. And $X + Y$ will be the number of all the balls we have (i.e., 50) because X represent the total ball number in the bins that there is one ball, and Y represents the total ball number in the bins there are more than one balls (for those bins with 0 balls, they do not contribute to the total ball number). So $X + Y = 50$. The final solution is: $E(Y) = E(50 - X) = 50 - E(X) = 50 - 50 \times (\frac{29}{30})^{49}$

ii. **Method 2:** Let Y_i be the number of balls in ith bin where the bin has more than two balls. So:

$$Y_i = \begin{cases} x, & \text{there are } x \text{ balls in the } i\text{th bin and } x \geq 2. \\ 0, & \text{else.} \end{cases}$$

For $x = 2, 3, \cdots, 50$, we have $P(Y_i = x) = \binom{50}{x} \times (\frac{1}{30})^x \times (\frac{29}{30})^{50-x}$. So:

$$E(Y_i) = \sum_{x=2}^{50} x \times P(Y_i = x) = \sum_{x=2}^{50} x \times \binom{50}{x} \times (\frac{1}{30})^x \times (\frac{29}{30})^{50-x}$$

If we want to calculate the total number of balls that are not alone in their bin, we have: $Y = Y_1 + Y_2 + \cdots + Y_{30}$, so:

$$E(Y) = E(Y_1 + Y_2 + \cdots + Y_{30}) = \sum_{i=1}^{30} E(Y_i) = 30 \times E(Y_i)$$

$$= 30 \times \sum_{x=2}^{50} x \times \binom{50}{x} \times (\frac{1}{30})^x \times (\frac{29}{30})^{50-x} = 50 - 50 \times (\frac{29}{30})^{49}$$

6. Suppose you will have a snack with the probability of 50% on any given day between Monday and Saturday. And you will have a snack on Sunday *if and only if* you didn't have one in any of the previous six days. Please calculate: 1) What is the expected number of times that you will have a snack in one week? 2) What is the variance of the number of times that you will have a snack in one week?

Solution: Let Y be the number of times you will have snack in one week. Let X be the number of times that you will have snack between Monday and Saturday. So we have:

$$Y = \begin{cases} x, & \text{if } x > 0. \\ 1, & \text{if } x = 0. \end{cases}$$

And $P(X = x) = \binom{6}{x} \times (\frac{1}{2})^x \times (\frac{1}{2})^{6-x} = \binom{6}{x} \times (\frac{1}{2})^6$, so the PMF of Y is:

y	1	2	3	4	5	6
$P(y)$	$\frac{\binom{6}{0}+\binom{6}{1}}{2^6}$	$\frac{\binom{6}{2}}{2^6}$	$\frac{\binom{6}{3}}{2^6}$	$\frac{\binom{6}{4}}{2^6}$	$\frac{\binom{6}{5}}{2^6}$	$\frac{\binom{6}{6}}{2^6}$

We can calculate $E(Y)$ as:

$$E(Y) = 1 \times \frac{7}{2^6} + 2 \times \frac{15}{2^6} + 3 \times \frac{20}{2^6} + 4 \times \frac{15}{2^6} + 5 \times \frac{6}{2^6} + 6 \times \frac{1}{2^6} = \frac{193}{64}$$

For $Var(Y) = E\left[(X - E(X))^2\right]$, so the result is:

$$Var(Y) = (-\frac{129}{64})^2 \times \frac{7}{64} + (-\frac{65}{64})^2 \times \frac{15}{64} + (-\frac{1}{64})^2 \times \frac{20}{64} + (\frac{63}{64})^2 \times \frac{15}{64} + (\frac{127}{64})^2 \times \frac{6}{64} + (\frac{191}{64})^2 \times \frac{1}{64} = \frac{5823}{4096} \approx 1.42$$