ENGG 2760A / ESTR 2018: Probability for Engineers

4. Random Variables

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Random variable

A discrete random variable assigns a discrete value to every outcome in the sample space.

Example

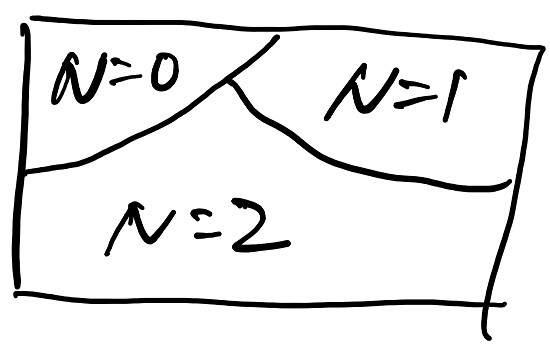
$$\begin{matrix} 1/4 & 1/4 & 1/4 & 1/4 \\ \{\,\mathrm{HH}, & \mathrm{HT}, & \mathrm{TH}, & \mathrm{TT}\,\} \end{matrix}$$

X, Y

N = number of Hs

Random variable

$N \in \{0, 1, 2\}$



$P(N=0) = P\{TT\} = 1/4$

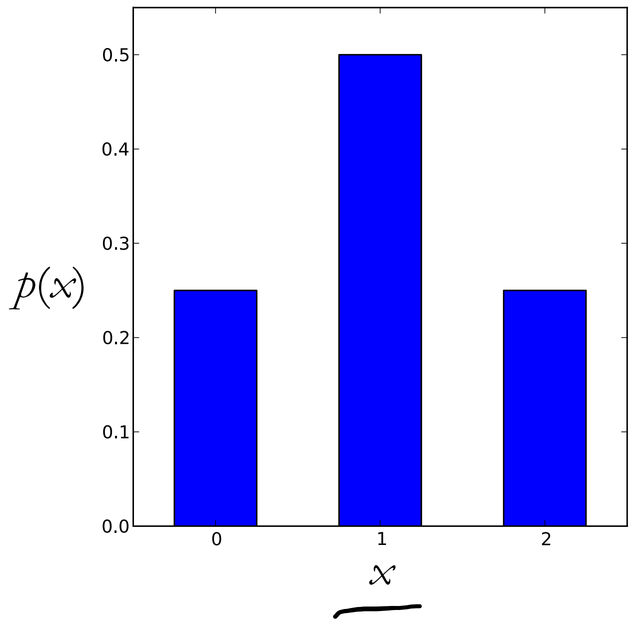
$P(N=1) = P\{HT, TH\} = 1/2$

$P(N=2) = P\{HH\} = 1/4$

Probability mass function PMF

We can describe the PMF by a table or by a chart.

x	0	1	2
$p(x)$	$1/4$	$1/2$	$1/4$



Two 3-sided dice are tossed. Calculate the PMF of the difference D of the rolls.

1st − 2nd

```
11 12 13
21 22 23
31 32 33
```

d	−2	−1	0	1	2
$P(D=d)$	$\frac{1}{9}$	$\frac{2}{9}$	$\frac{3}{9}$	$\frac{2}{9}$	$\frac{1}{9}$

PMF

What is the probability that $D \geq 1$? D is odd?

$P(D \geq 1) = P(1) + P(2) = \frac{2}{9} + \frac{1}{9} = \frac{3}{9}$

$P(D \text{ is odd}) = P(1) + P(-1) = \frac{4}{9}$

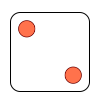
The binomial random variable

$\mathrm{Binomial}(n, p)$: **Perform n independent trials, each of which succeeds with probability p.**

$X =$ **number of successes** , $X \in \{0, 1, 2, \dots, n\}$

Examples

Toss n coins. "number of heads" is $\mathrm{Binomial}(n, \frac{1}{2})$**.**

Toss n dice. "Number of  **s" is** $\mathrm{Binomial}(n, 1/6)$**.**

A less obvious example

Flip n coins. Let C be the number of consecutive changes (HT or TH). flip

Examples:

ω	$C(\omega)$
HHHHHHH	0
THHHHHT	2
HTHHHHT	3

$n = 7$

Binomial(6, 1/2)

Then C is Binomial(n – 1, ½). Binomial$(n-1, \frac{1}{2})$

1/2

Draw a 10-card hand from a 52-card deck.

Let $N =$ number of aces among the drawn cards

Is N a $\text{Binomial}(10, 1/13)$ random variable? NO

$P(N = 6)$

"ith card is an Ace" NOT IND.

Probability mass function

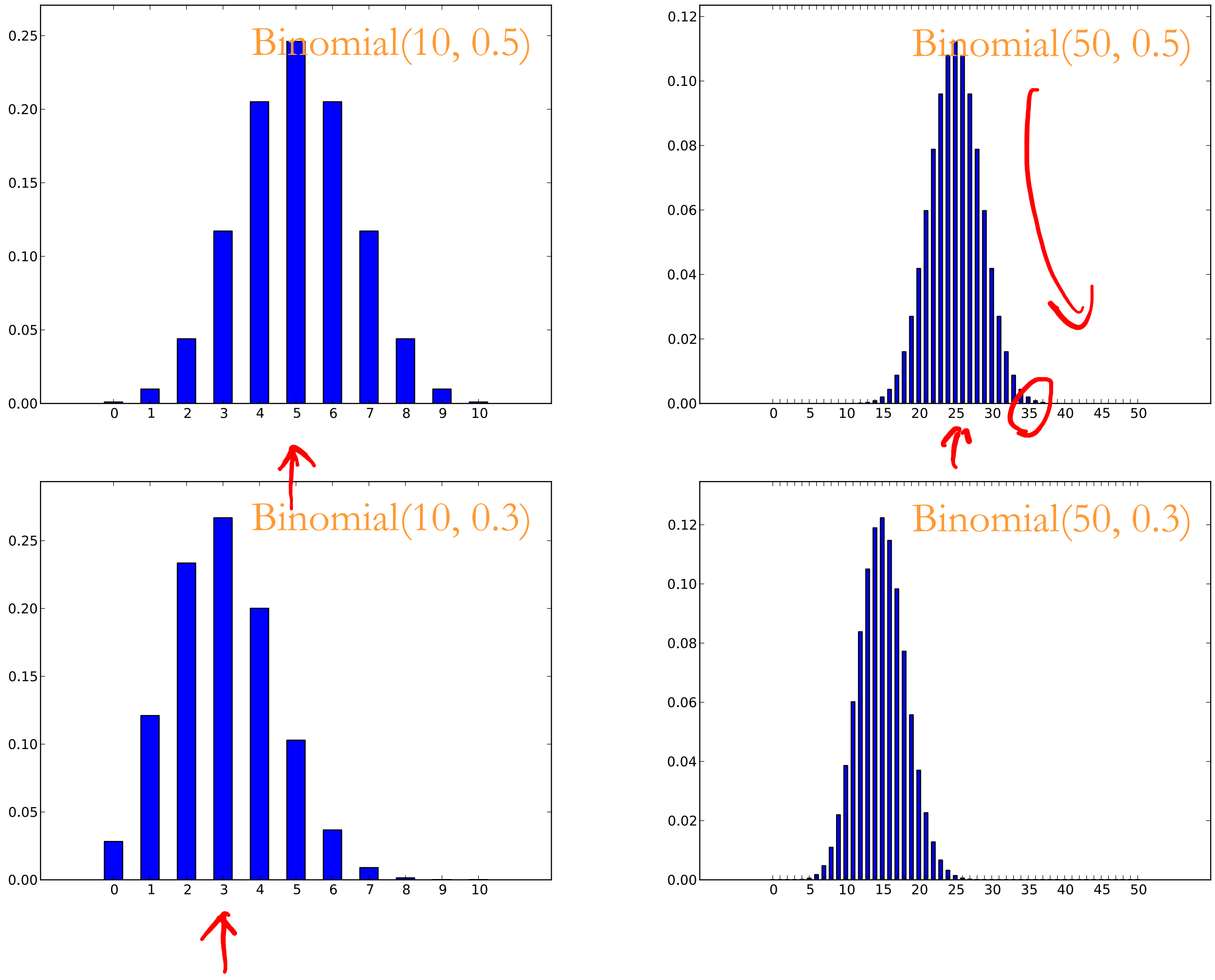
If X is $\mathrm{Binomial}(n, p)$, its PMF is

$$\boxed{p(k) = P(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}}$$

OUTCOMES $\{S, F\}^n = \{SS\cdots S, SFS\cdots S, \cdots, FF\cdots F\}$

$P(X=k) = P(k \text{ successes out of } n)$

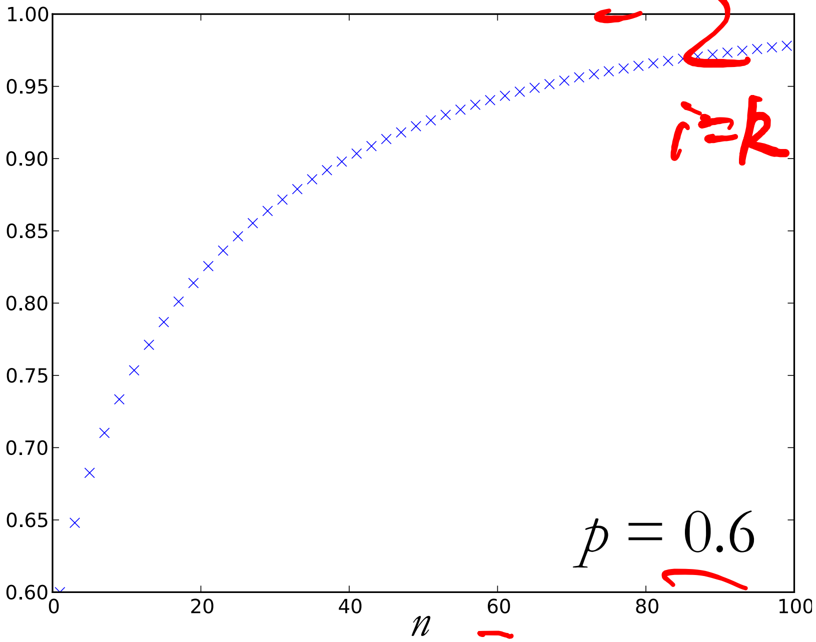
$= \binom{n}{k} \cdot p^k \cdot (1-p)^{n-k}$

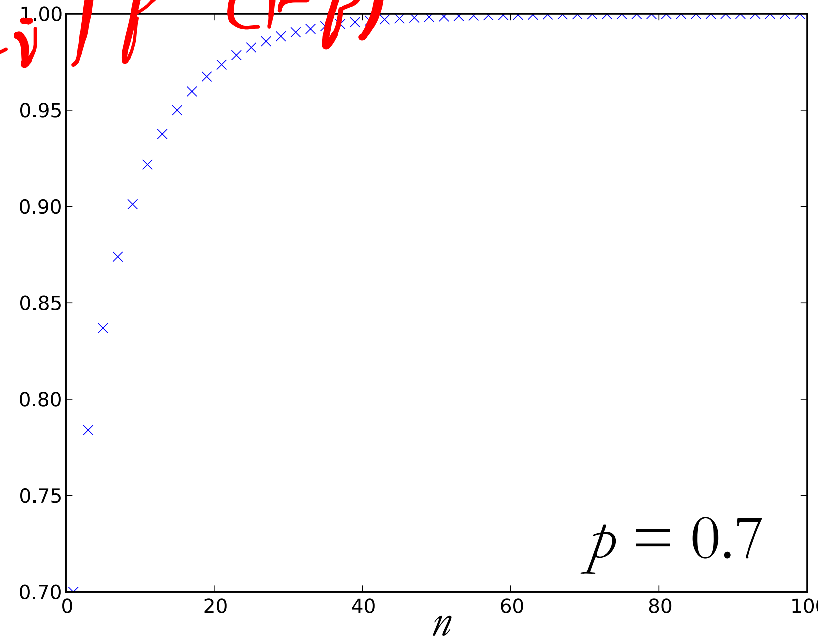

Binomial(10, 0.5)
Binomial(50, 0.5)
Binomial(10, 0.3)
Binomial(50, 0.3)
0.00
0.05
0.10
0.15
0.20
0.25
0.02
0.04
0.06
0.08
0.10
0.12
0 1 2 3 4 5 6 7 8 9 10
0 5 10 15 20 25 30 35 40 45 50

Binomial random variable

$$\mathbf{P}(\text{Binomial}(n,p) \geq k) = \sum_{i=k}^{n} P(\text{Binomial}(n,p) = i)$$

$$= \sum_{i=k}^{n} \binom{n}{i} p^{i} (1-p)^{n-i}$$





$$\mathbf{P}(\text{Binomial}(n,p) \geq (n-1)/2)$$

The Lakers and the Celtics meet for a 7-game playoff.

Lakers win 60% of the time. What is the IND. probability that all 7 games games are played?

$\Omega = \{C, L\}^7$. E.g.: CCLLCLC ; CCCLC

N = Length of series. $N \overset{?}{\sim} \text{Binomial}(\cdot \cdots)$

$P(N=7) = P(3\text{-}3 \text{ tie after } G6)$

$= P(\text{Binomial}(6, 0.6) = 3)$

$= \binom{6}{3} \cdot 0.6^3 \cdot 0.4^3 \approx 0.276$

What is the probability that Lakers win in game 6?

A = "Lakers win in game 6"

B = "L -- win game 6"

$P(A) = P(B) \cdot \underbrace{P(A|B)}_{} \rightarrow$ L win 3 out of the first 5

$= 0.6 \cdot P(\text{Binomial}(5, 0.6) = 3)$

$= 0.6 \cdot \binom{5}{3} \cdot 0.6^3 \cdot 0.4^2 \doteq 21\%$

https://stattrek.com/online-calculator/binomial.aspx

Geometric random variable

Let $X_1, X_2, \ldots$ be independent trials with success p.

A $\text{Geometric}(p)$ random variable N is the time of the first success among $X_1, X_2, \ldots$:

n

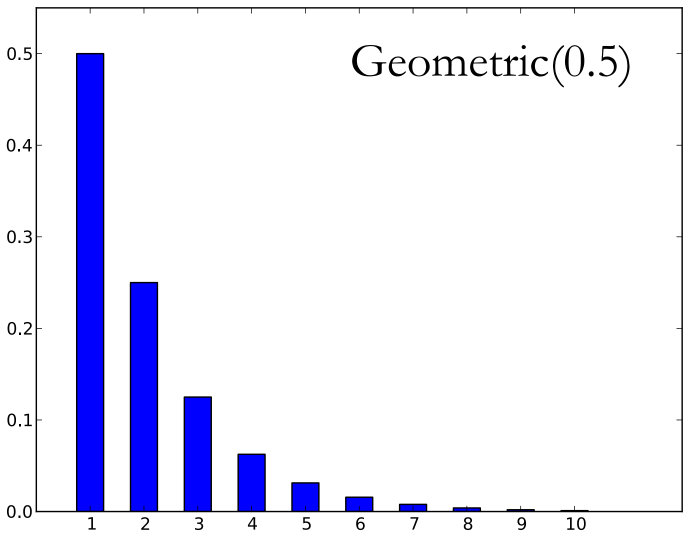
$$N = \text{first (smallest) } \cancel{n}\ i \text{ such that } X_{\cancel{n}\, i} = 1.$$

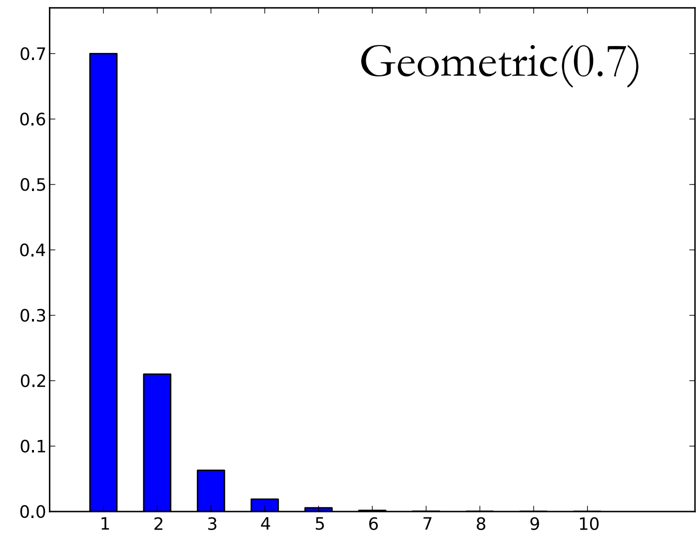
$P(N=1) = P(\{S\}) = p$

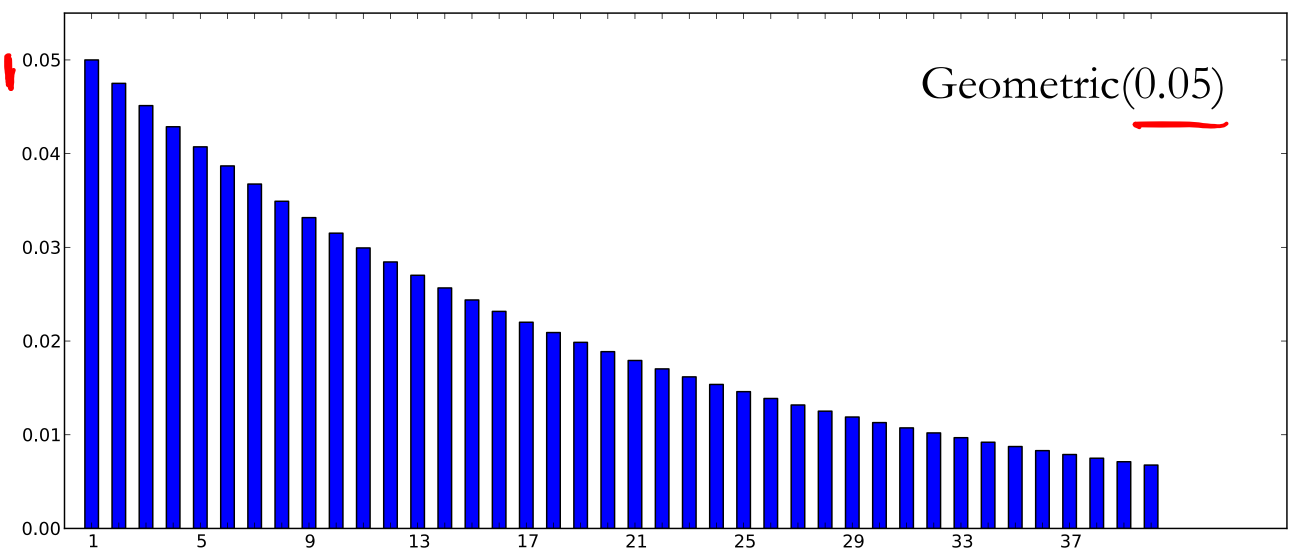
$P(N=2) = P(\{FS\}) = (1-p)p.$

i

$P(N=n) = P(\{FF\cdots FS\}) = (1-p)^{n-1}\cdot p \leftarrow \text{PMF}$


Geometric(0.5)
0.0
0.1
0.2
0.3
0.4
0.5
1
2
3
4
5
6
7
8
9
10


Geometric(0.7)
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
1
2
3
4
5
6
7
8
9
10


Geometric(0.05)
0.00
0.01
0.02
0.03
0.04
0.05
1
5
9
13
17
21
25
29
33
37

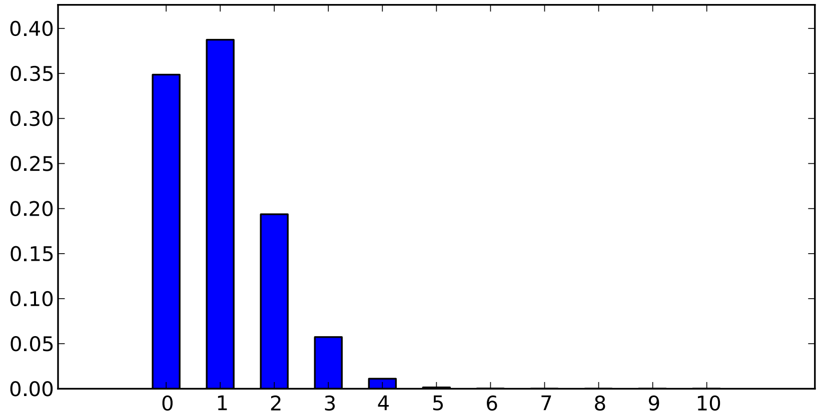
Apples

About 10% of the apples on your farm are rotten.

You sell 10 apples. How many are rotten?

Probability model

N**umber of rotten apples you sold is**
$\mathrm{Binomial}(n = 10, p = 1/10)$.

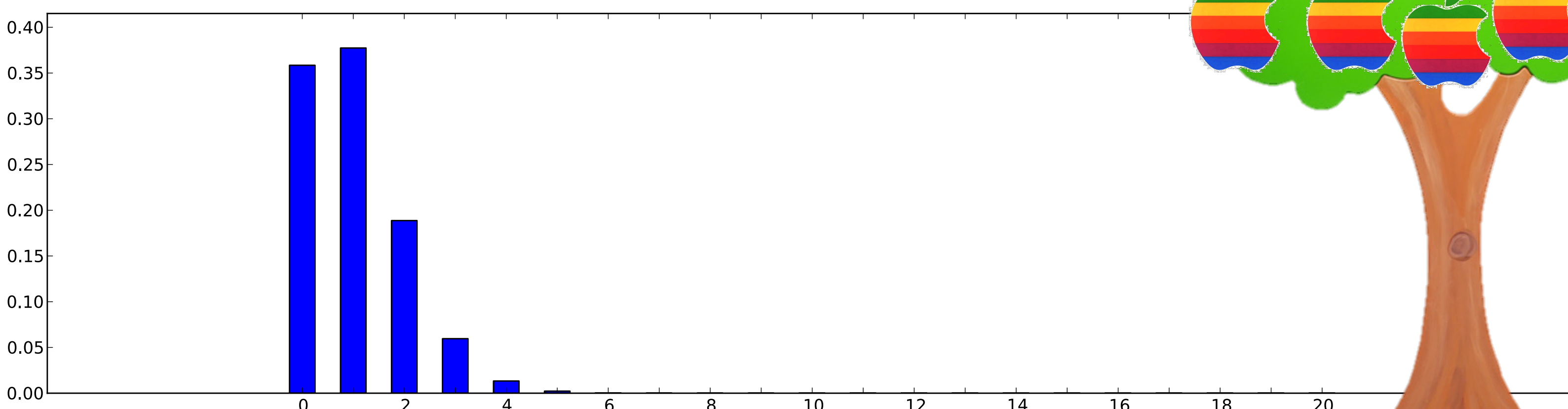


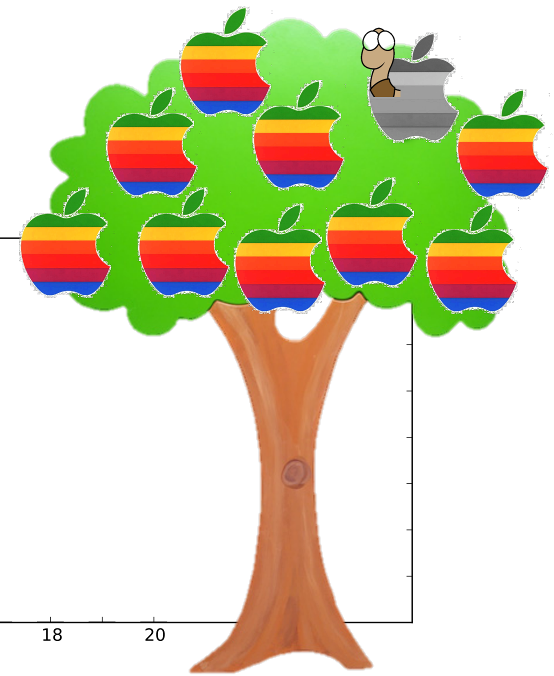
Apples

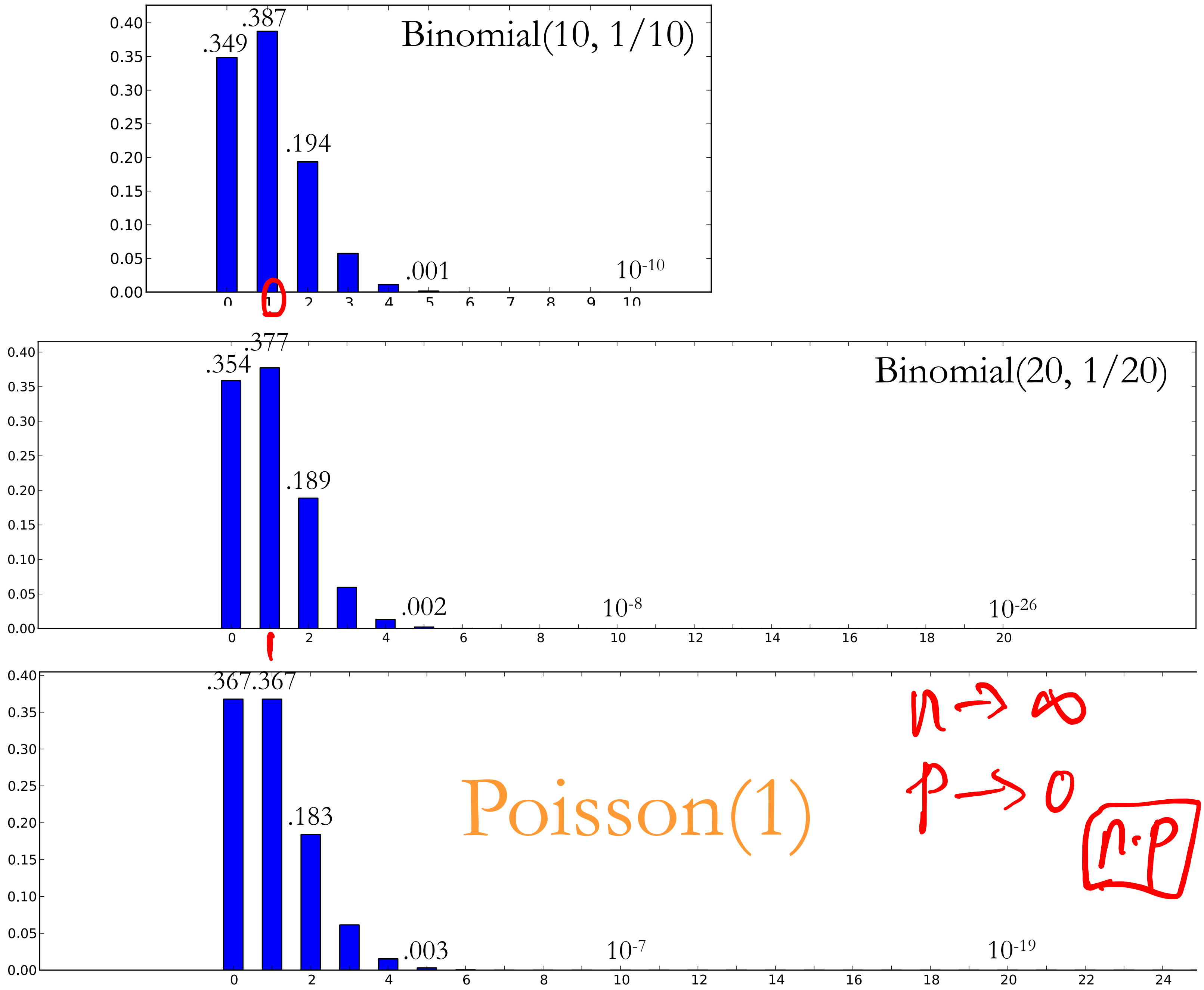
You improve productivity; now only 5% apples rot.

You can now sell 20 apples.

N **is now** $\mathrm{Binomial}(20, 1/20)$.






Binomial(10, 1/10)
.349
.387
.194
.001
10^{-10}
Binomial(20, 1/20)
.354
.377
.189
.002
10^{-8}
10^{-26}
Poisson(1)
.367
.367
.183
.003
10^{-7}
10^{-19}
$n \to \infty$
$p \to 0$
$\boxed{n \cdot p}$

The Poisson random variable

A $\text{Poisson}(\lambda)$ **random variable has PMF**

$\lambda = \text{RATE}$

$$\boxed{p(k) = e^{-\lambda} \lambda^k / k!}$$

$p(0) = e^{-\lambda}$

$k = 0, 1, 2, 3, \ldots$

$p(1) = \lambda \cdot e^{-\lambda}$

$p(2) = \frac{\lambda^2}{2} \cdot e^{-\lambda}$

Poisson random variables do not occur "naturally" in the sample spaces we have seen.

The Poisson random variable

Poisson(λ) approximates Binomial(n, p)
when $\lambda = np$ is fixed and n is large (p is small)

$$P(\text{Poisson}(\lambda) = k) = \lim_{n\to\infty} P(\text{Binomial}(n, \lambda/n) = k)$$

$$P(\text{Poisson}(\lambda) = 0) = \lim_{n\to\infty} P(\text{Binomial}(n, \lambda/n) = 0)$$

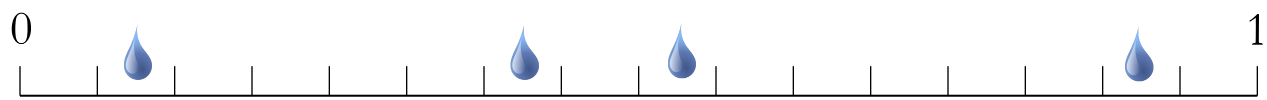
$$= \lim_{n\to\infty} \left(1 - \frac{\lambda}{n}\right)^n = e^{-\lambda}$$

$$P(\text{Poisson}(\lambda) = 1) = \lim_{n\to\infty} P(\text{Bino}(n, \lambda/n) = 1)$$

$$= \lim_{n\to\infty} \cancel{n} \cdot \frac{\lambda}{\cancel{n}} \cdot \left(1 - \frac{\lambda}{n}\right)^{n-1} = \lambda \cdot e^{-\lambda}$$

The Poisson random variable

Rain is falling on your head at a rate of 3 drops/sec.



k slots

$E_1, E_2, \dots, E_k$ = "Rain drop in slot i".

IND. events $\{E_i\}$.

$\Rightarrow$ # drops N_k is Binomial$(k, \frac{3}{k})$

$$P(N=n) = \lim_{k\to\infty} P(N_k = n) \text{ is Poisson}(3).$$

$$P(N=3) = e^{-3}\cdot 3^3/3!$$

$$P(N=4) = e^{-3}\cdot 3^4/4!$$

$$\vdots$$

Functions of random variables

PMF of X:

x	0	1	2
$p(x)$	$1/3$	$1/3$	$1/3$

PMF of $X - 1$? $\to Y$

y	-1	0	1
$p(y)$	$1/3$	$1/3$	$1/3$

PMF of $(X - 1)^2$? $\to Z$

z	0	1
$p(z)$	$1/3$	$2/3$

If X is a random variable with p.m.f. p_X,
then $Y = f(X)$ is a random variable with p.m.f.

$$p_Y(y) = \sum_{x:\, f(x) = y} p_X(x).$$

D **is the difference of two 3-sided dice rolls. Calculate the PMF of** $|D|$**.**

d	-2	-1	0	1	2
$P(D=d)$	$\frac{1}{9}$	$\frac{2}{9}$	$\frac{3}{9}$	$\frac{2}{9}$	$\frac{1}{9}$

x	0	1	2	
$P(\lvert D\rvert=x)$	$3/9$	$+\ 4/9$	$+\ 2/9$	$=1$

Expected value

The **expected value (expectation)** of a random variable X with p.m.f. p is

$$E[X] = \sum_x \underline{x\, p(x)}$$

Example

$N =$ **number of** H**s**

$$E[N] = 0 \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} = \frac{1}{2}$$

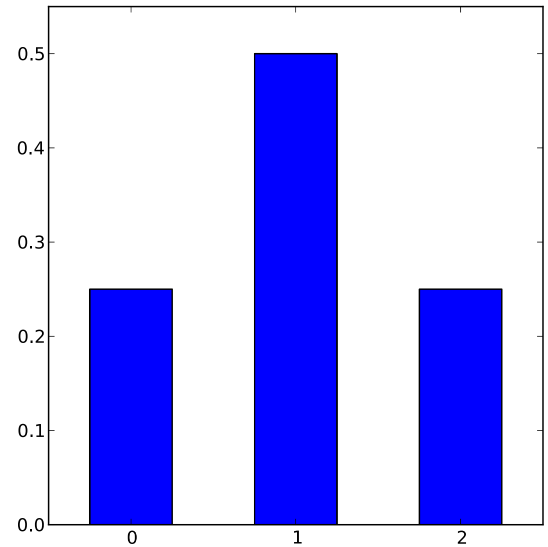
n	0	1
$p(n)$	$1/2$	$1/2$

Expected value

Example

$N =$ **number of** H**s**

n	0	1	2
$p(n)$	$1/4$	$1/2$	$1/4$



$$E[N] = 0 \cdot \frac{1}{4} + 1 \cdot \frac{1}{2} + 2 \cdot 1/4 = 1$$

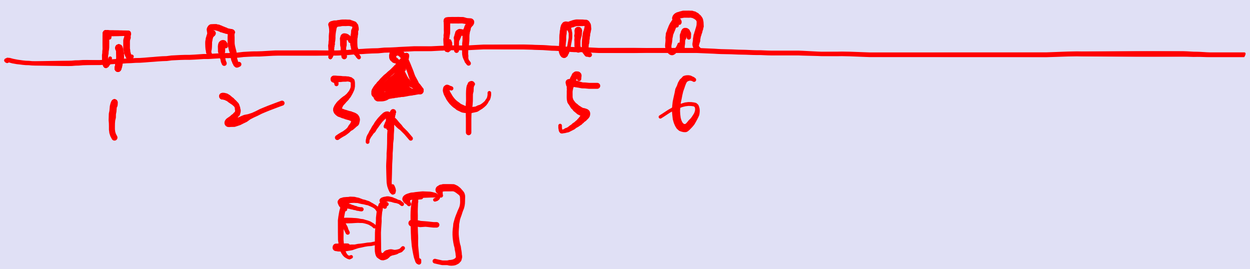
The expectation is the average value the random variable takes when experiment is done many times

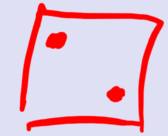
$F =$ **face value of fair 6-sided die**

Equal likelihood : $\{1, 2, 3, 4, 5, 6\}$

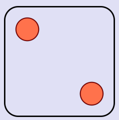
$$E[F] = \frac{1}{6} \cdot (1+2+ \cdots +6)$$

$$= 3.5$$



N : # 

$N \sim \text{Binomial}(3, 1/6)$

If  appears k times, you win $\$k$.

If it doesn't appear, you lose $\$1$.

G : Gain

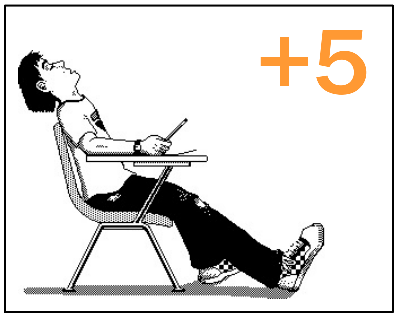
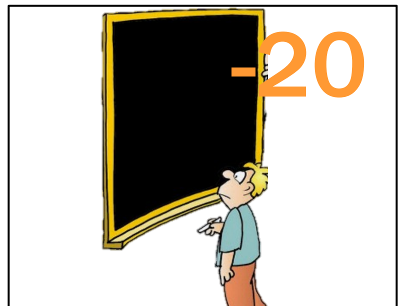
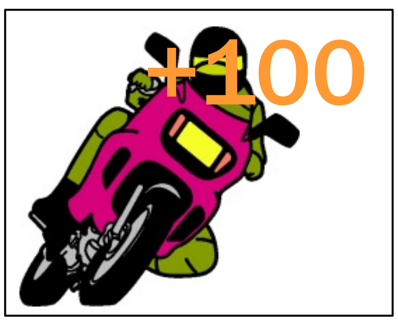
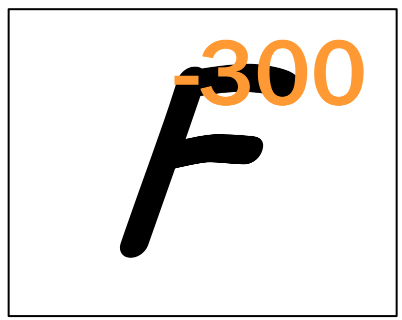
g	-1	1	2	3
$P(g)$	$(\frac{5}{6})^3$	$3\cdot\frac{1}{6}(\frac{5}{6})^2$	$3(\frac{1}{6})^2\cdot\frac{5}{6}$	$(\frac{1}{6})^3$

$$E[G] = -1\cdot(\tfrac{5}{6})^3 + \cdot 1 + \cdot 2 + \cdot 3$$

≈ -0.03 Not profitable

Utility

Should I go to tutorial?

	not called	called
*C*ome	 +5	 -20
*S*kip	 +100	 -300
	35/40	5/40

40 students

5 called

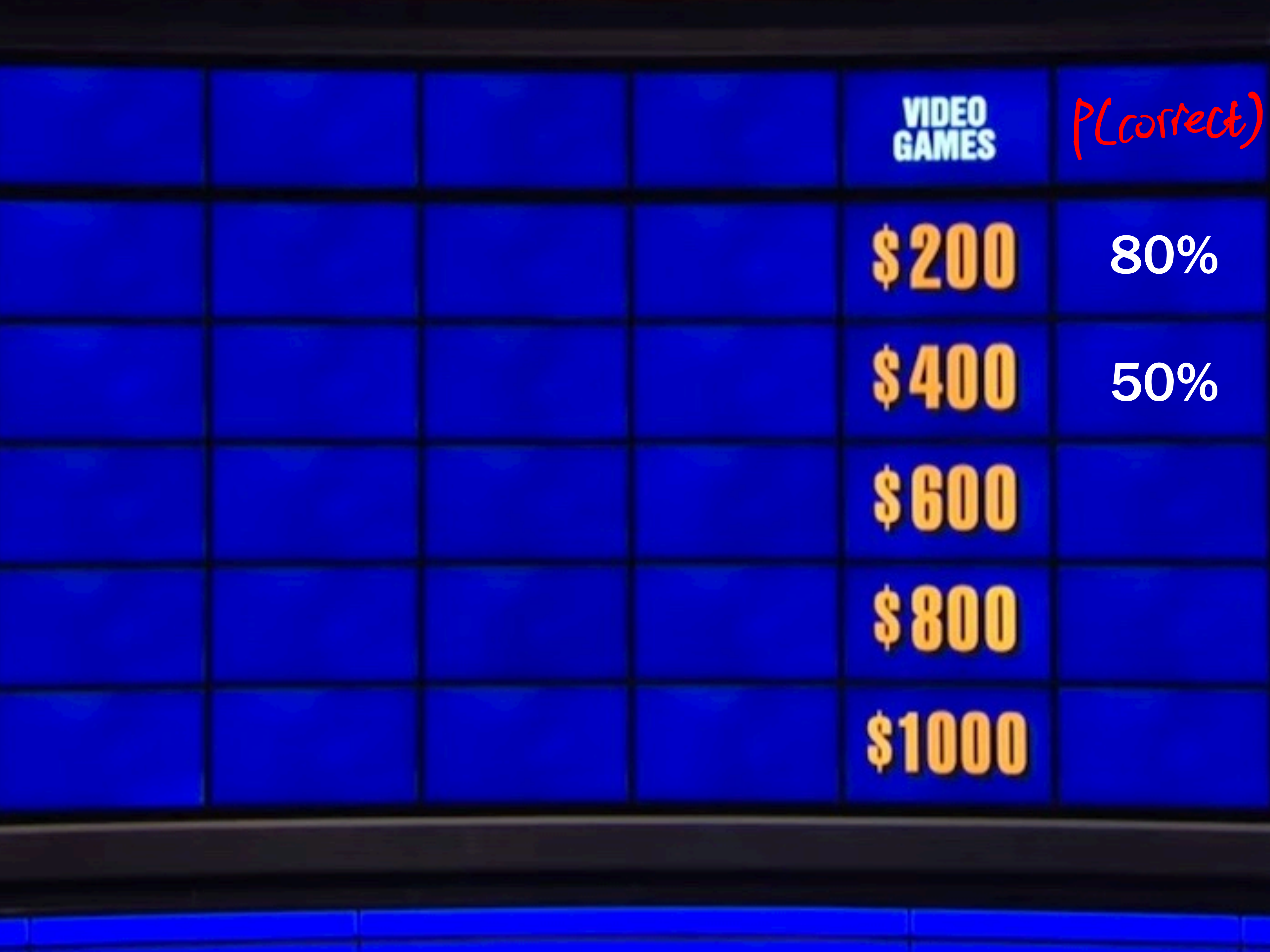
$$E[C] = 5 \cdot \frac{35}{40} - 20 \cdot \frac{5}{40}$$

$$= 1.9$$

$$E[S] = 100 \cdot \frac{35}{40} - 300 \cdot \frac{5}{40}$$

$$= 50$$

$$E[S] > E[C]$$


VIDEO GAMES
P(correct)
$200
80%
$400
50%
$600
$800
$1000