ENGG 2760A / ESTR 2018: Probability for Engineers

6. Conditional PMFs and Independent Random Variables

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Conditional PMF

Let X be a random variable and A be an event.

The conditional PMF of X given A is

$$P(X = x \mid A) = \frac{P(X = x \text{ and } A)}{P(A)}$$

What is the PMF of a 6-sided die roll given that the outcome is even?

Unconditional PMF.

x	~~1~~	2	~~3~~	4	~~5~~	6
$p(x)$	1/6	1/6	..			1/6

A: outcome is even. $P(A) = \frac{1}{2}$

x	2	4	6
$P_{X\|A}(x)$	1/3 +	1/3 +	1/3 = 1

You flip 3 coins. What is the PMF of number of heads given that there is at least one?

A : "At least one H". $P(A) = 1 - 1/8$

$= 7/8$

$N \sim \text{Binomial}(3, 1/2)$.

n	$\cancel{0}$	1	2	3
$p(n)$	$\cancel{1/8}$	3/8	3/8	1/8

n	1	2	3
$P_{N\mid A}(n)$	3/7	3/7	1/7

Conditioning on a random variable

The conditional PMF of X given Y is

$$\mathbf{P}(X = x \mid Y = y) = \frac{\mathbf{P}(X = x \text{ and } Y = y)}{\mathbf{P}(Y = y)}$$

cond. pmf

$$p_{X|Y}(x \mid y) = \frac{p_{XY}(x, y)}{p_Y(y)}$$

Joint pmf

marginal pmf

For fixed y, $p_{X|Y}$ is a PMF as a function of x.

$p_{X|Y}(x|y) \neq p_{Y|X}(y|x)$

Roll two 3-sided dice. What is the PMF of the sum S given the first roll? F

Joint PMF $P_{F,S}(f,s)$

Marginal PMF $P_F(f)$

	2	3	4	5	6	
1	1/9	1/9	1/9	0	0	1/3
2	0	1/9	1/9	1/9	0	1/3
3	0	0	1/9	1/9	1/9	1/3

for each f, each row is a PMF.

	2	3	4	5	6
1	1/3	1/3	1/3	0	0
2	0	1/3	1/3	1/3	0
3	0	0	1/3	1/3	1/3

conditional PMF $P_{S|F}(s|f)$

Roll two 3-sided dice. What is the PMF of the first roll given the sum?

F

S

$P_{F|S}(f|s)$

Joint PMF $P_{F,S}(f,s)$

	2	3	4	5	6
1	1/9	1/9	1/9	0	0
2	0	1/9	1/9	1/9	0
3	0	0	1/9	1/9	1/9

1/9	2/9	3/9	3/9	1/9

marginal PMF $P_S(s)$

$P_{F|S}(f|s)$

	1	2	3
2	1	0	0
3	1/2	1/2	0
4	1/3	1/3	1/3
5	0	1/2	1/2
6	0	0	1

Conditional Expectation

The conditional expectation of X given event A is

$$\mathbf{E}[X \mid A] = \textstyle\sum_x x \, \mathbf{P}(X = x \mid A)$$

The conditional expectation of X given $Y = y$ is

$$\mathbf{E}[X \mid Y = y] = \textstyle\sum_x x \, \mathbf{P}(\underbrace{X = x \mid Y = y}_{\text{Cond PMF}})$$

You flip 3 coins. What is the expected number of heads given that there is at least one?

$N \sim \text{Binomial}(3, 1/2)$

A: At least one Head

Cond. PMF

n	1	2	3
$P_{N\|A}(n\|A)$	3/7	3/7	1/7

$$E[N|A] = 1 \cdot \frac{3}{7} + 2 \cdot \frac{3}{7} + 3 \cdot \frac{1}{7} = \frac{12}{7}.$$

Total Expectation Theorem

$$\boxed{\mathbf{E}[X] = \mathbf{E}[X \mid A]\, \underline{\mathbf{P}(A)} + \mathbf{E}[X \mid A^c]\, \underline{\mathbf{P}(A^c)}}$$

Proof

$$E[X] = \sum_x x \cdot P(X=x)$$

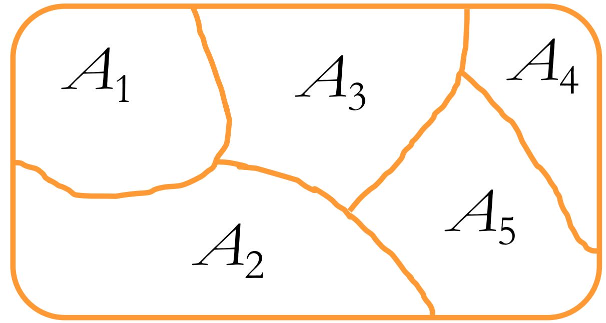
$$= \sum_x x \cdot \left(P(X=x \mid A) \cdot P(A) + P(X=x \mid A^c) \cdot P(A^c) \right)$$

$$= \underbrace{\left(\sum_x x \cdot P(X=x \mid A) \right)} \cdot P(A) + \underbrace{\left(\sum x \, P(X=x \mid A^c) \right)} \cdot P(A^c)$$

$$= E[X \mid A] \cdot P(A) + E[X \mid A^c] P(A^c)$$

Total Expectation Theorem (general form)

If $A_1, \ldots, A_n$ partition Ω
then



$$\mathbf{E}[X] = \mathbf{E}[X \mid A_1]\mathbf{P}(A_1) + \ldots + \mathbf{E}[X \mid A_n]\mathbf{P}(A_n)$$

In particular

$A = p(Y = y)$

$$\mathbf{E}[X] = \sum_y \mathbf{E}[X \mid Y = y]\, \mathbf{P}(Y = y)$$

type	S 	E 	W 
average time on facebook	30 min	60 min	10 min
% of visitors	60%	30%	10%

T $\quad E[T]$

average $\underbrace{\text{visitor time}}$ =

$$E[T] = E[T|S]\cdot P(S) + E[T|E]\cdot P(E) + E[T|W]\cdot P(W)$$

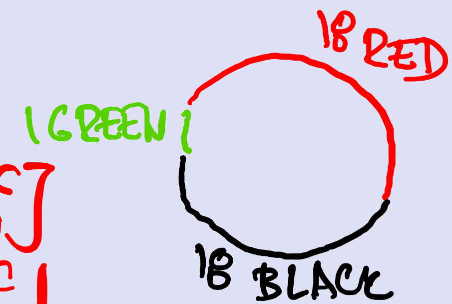
$$= 30\cdot 0.6 + 60\cdot 0.3 + 10\cdot 0.1$$

$$= 37$$

You play 10 rounds of roulette. You invest \$100 and bet 10% of your balance on red in every round.

What is your average balance after 10 rounds?

$W_i =$ "win in R_i". $P(W_i) = \frac{18}{37}$



$X_0 = 100.$

$E[X_1] = E[X_1 \mid W_1] \cdot P(W_1) + E[X_1 \mid W_1^c] \cdot P(W_1^c)$

$= 1.1\, W_0 \cdot \frac{18}{37} + 0.9\, W_0 \cdot \frac{19}{37} \approx 0.997 \cdot X_0$

$E[X_2] = 0.997 \cdot X_1 \approx 0.997^2 \cdot X_0$

$\vdots$

$E[X_{10}] = 0.997^{10} \cdot X_0 \approx 97.$

$E[X_{1000}] = 0.997^{1000} X_0 \approx 4.9$

$\lim_{n \to \infty} E[X_n] = 0$

Gambler's Ruin.

You flip 3 coins. What is the expected number of heads given that there is at least one?

$N \sim \text{Binomial}(3, 1/2)$. $A = $ "$N > 0$".

$E[N|A]$

$E[N] = E[N|A] \cdot P(A) + E[N|A^c] \cdot P(A^c)$

$3/2 = E[N|A] \cdot \frac{7}{8} + 0 \cdot 1/8$

$\Rightarrow E[N|A] = \frac{12}{7}$

Mean of the Geometric

$X = \text{Geometric}(p)$ **random variable**

W: 1st trial is success

$\mathbf{E}[X] = 1\cdot p + 2(1-p)\cdot p + 3\cdot(1-p)^2\cdot p + \cdots$

$X = 1 + Y \rightarrow$ # trials after 1st.

$Y \mid W$ is zero

$Y \mid W^c$ is a Geometric(p)

$E[X] = E[X \mid W]P(W) + E[X \mid W^c]P(W^c)$

$= 1\cdot p + E[1+Y \mid W^c]\cdot(1-p)$

$= p + (1 + E[Y \mid W^c])\cdot(1-p)$

$= p + (1 + E[X])(1-p)$

$= 1 + (1-p)E[X]$, $\Rightarrow E[X] = \frac{1}{p}$.

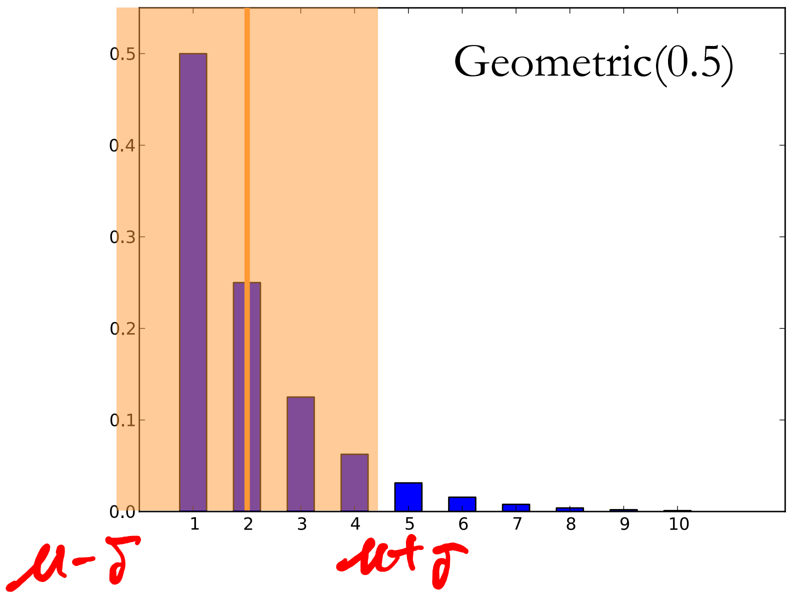
Variance of the Geometric

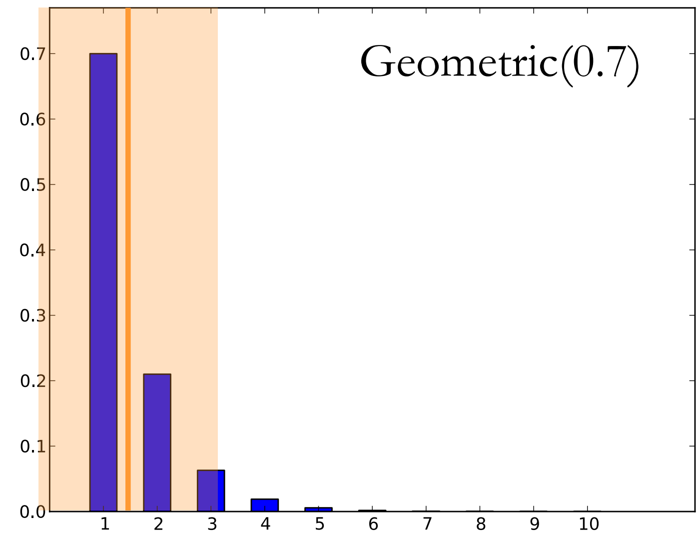
$X = \text{Geometric}(p)$ **random variable**

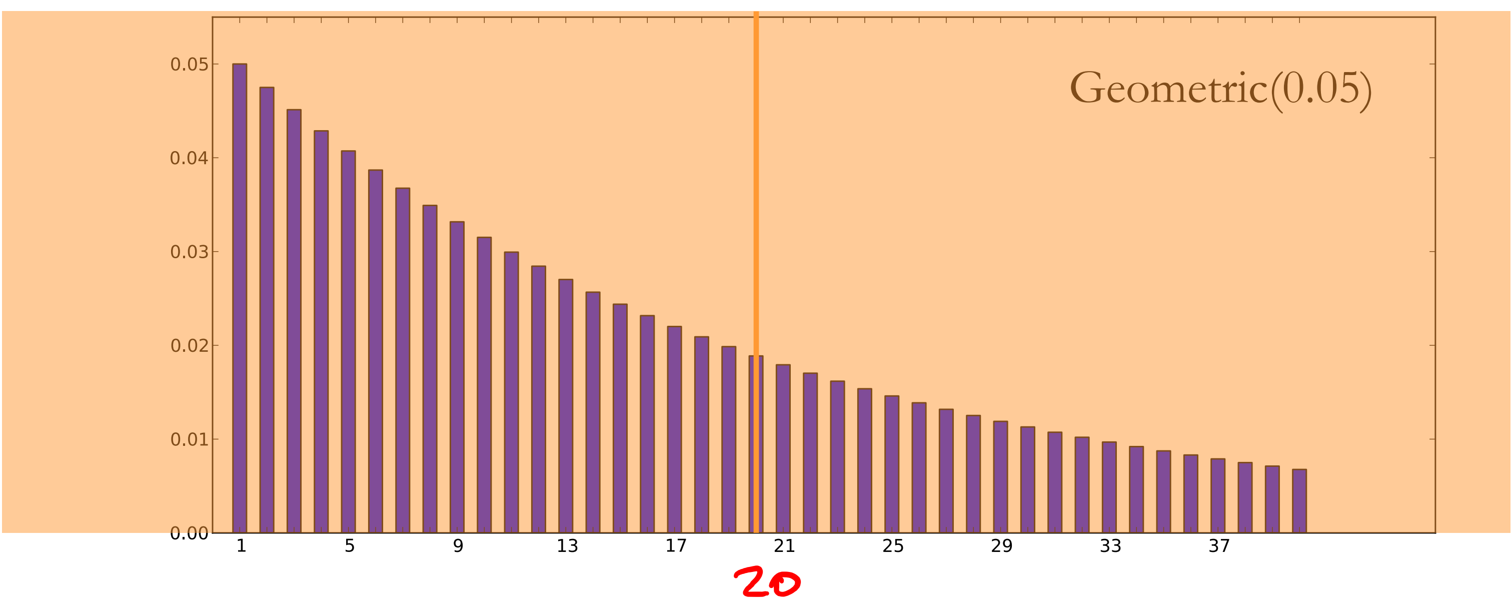
$\mathbf{Var}[X] = \dfrac{1-p}{p^2}$ $\qquad \sigma = \dfrac{\sqrt{1-p}}{p} < \dfrac{1}{p}.$

Proof (optional):

$$
\begin{aligned}
\text{Var}[X] &= E[(X-1/p)^2] \\
&= E[(X-1/p)^2 \mid W]\cdot p + E[(X-1/p)^2 \mid W^c]\cdot(1-p) \\
&= (1-1/p)^2\cdot p + E[(1+Y-1/p)^2 \mid W^c]\cdot(1-p) \\
&= (1-1/p)^2\cdot p + E[1+2(Y-1/p)+(Y-1/p)^2 \mid W^c](1-p) \\
&= (1-1/p)^2\cdot p + (1+\underbrace{\cancel{2E[Y-1/p \mid W]}}_{0}+\underbrace{E[(Y-1/p)^2 \mid W^c]}_{\text{Var}[X]})(1-p) \\
&= (1-1/p)^2\cdot p + (1+\text{Var}[X])\cdot(1-p)
\end{aligned}
$$


Geometric(0.5)
μ-σ
μ+σ


Geometric(0.7)


Geometric(0.05)
20

Independent random variables

Let X and Y be discrete random variables.

X and Y are independent if

$$\mathbf{P}(X = x,\, Y = y) = \mathbf{P}(X = x)\, \mathbf{P}(Y = y)$$

for all possible values of x and y.

In PMF notation, $\mathrm{p}_{XY}(x, y) = p_X(x)\, p_Y(y)$ for all x, y.

Joint = PROD. Marginal

Independent random variables

X **and** Y **are independent if**

$$\mathbf{P}(X = x \mid Y = y) = \mathbf{P}(X = x)$$

for all x **and** y **such that** $\mathbf{P}(Y = y) > 0$**.**

In PMF notation, $p_{X|Y}(x \mid y) = p_X(x)$ **if** $p_Y(y) > 0$**.**

Alice tosses three coins and so does Bob. Alice gets \$1 per head and Bob gets \$1 per tail.

Are their earnings independent?

A : Alice's earnings

B : Bob's "

$$P(A=3 \mid B=0) = P(A=3) = \frac{1}{8}$$

$$P(A=a \mid B=b) = P(A=a), \quad \text{IND.}$$

Now they toss the same coin three times. Are their earnings independent?

A = # Heads

B = # Tails

$A + B = 3$

$\Rightarrow P(A=3 \mid B=3) = 0$, but $P(A=3) = \frac{1}{8}$.

$\Rightarrow$ NOT IND.

Expectation and independence

X and Y are independent if and only if

$$\mathbf{E}[f(X)g(Y)] = \mathbf{E}[f(X)]\,\mathbf{E}[g(Y)]$$

for all real valued functions f and g.

e.g. $E[(X^2+3)e^Y] = E[X^2+7] \cdot E[e^Y]$

Expectation and independence

In particular, if X and Y are independent then

$$E[XY] = E[X]\,E[Y]$$

Not true in general!

Variance of a sum

X, Y are IND.

Recall $\mathbf{Var}[X] = \mathbf{E}[(X - \mathbf{E}[X])^2] = \underline{\mathbf{E}[X^2] - \mathbf{E}[X]^2}$

$$\mathbf{Var}[X + Y] = \underbrace{E[(X+Y)^2]}_{①} - \underbrace{E[X+Y]^2}_{②}$$

$$① = E[X^2+Y^2+2XY]$$
$$= E[X^2] + E[Y^2] + \cancel{2E[XY]}$$

$$② = (E[X]+E[Y])^2$$
$$= E[X]^2 + E[Y]^2 + \cancel{2E[X]\cdot E[Y]}$$

$$① - ② = E[X^2] - E[X]^2 + E[Y^2] - E[Y]^2$$

$$\mathrm{Var}[X+Y] = \mathrm{Var}[X] + \mathrm{Var}[Y]$$

Variance of a sum

$$\mathbf{Var}[X_1 + \ldots + X_n] = \mathbf{Var}[X_1] + \ldots + \mathbf{Var}[X_n]$$

if every pair X_i, X_j is independent.

Not true in general!

Keep tossing a fair coin until you get ten heads. On average, how many times T will you toss?

$\underbrace{TTH_1}_{T_1}\ \underbrace{TTTH_2}_{T_2}\ \underbrace{H_3}_{T_3}\ T \cdots \underbrace{TH_{10}}_{T_{10}}$

$T = T_1 + T_2 + \cdots + T_{10}.$

$E[T] = E[T_1] + E[T_2] + \cdots + E[T_{10}]$

$= 10 \cdot E[\mathrm{Geo}(1/2)]$

$= 20.$

What is the standard deviation of T?

$\rightarrow T_i, T_j$ are IND.

$\mathrm{Var}[T] = \mathrm{Var}[T_1] + \mathrm{Var}[T_2] + \cdots + \mathrm{Var}[T_{10}]$

$= 10 \cdot \frac{1 - 1/2}{(1/2)^2} = 20.$

$\sigma = \sqrt{20}.$

Variance of the Binomial (n, p)

$X = X_1 + \cdots + X_n$

Bernoulli(p)

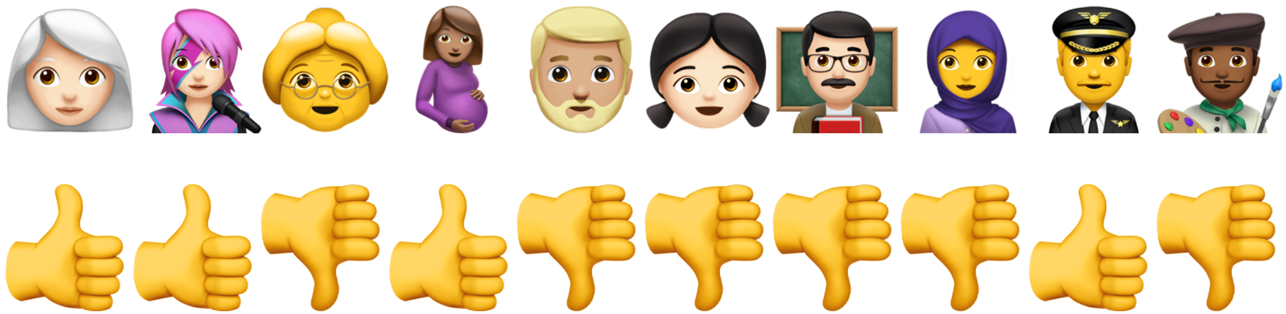
X_i, X_j, are IND.

$\mathrm{Var}[X] = \mathrm{Var}[X_1] + \cdots + \mathrm{Var}[X_n]$

$$\mathrm{Var}[X_i] = E[X_i^2] - E[X_i]^2$$
$$= 1 \cdot p + 0 - p^2$$
$$= p(1-p)$$

$$\boxed{\mathrm{Var}[X] = n \cdot p \cdot (1-p)}$$

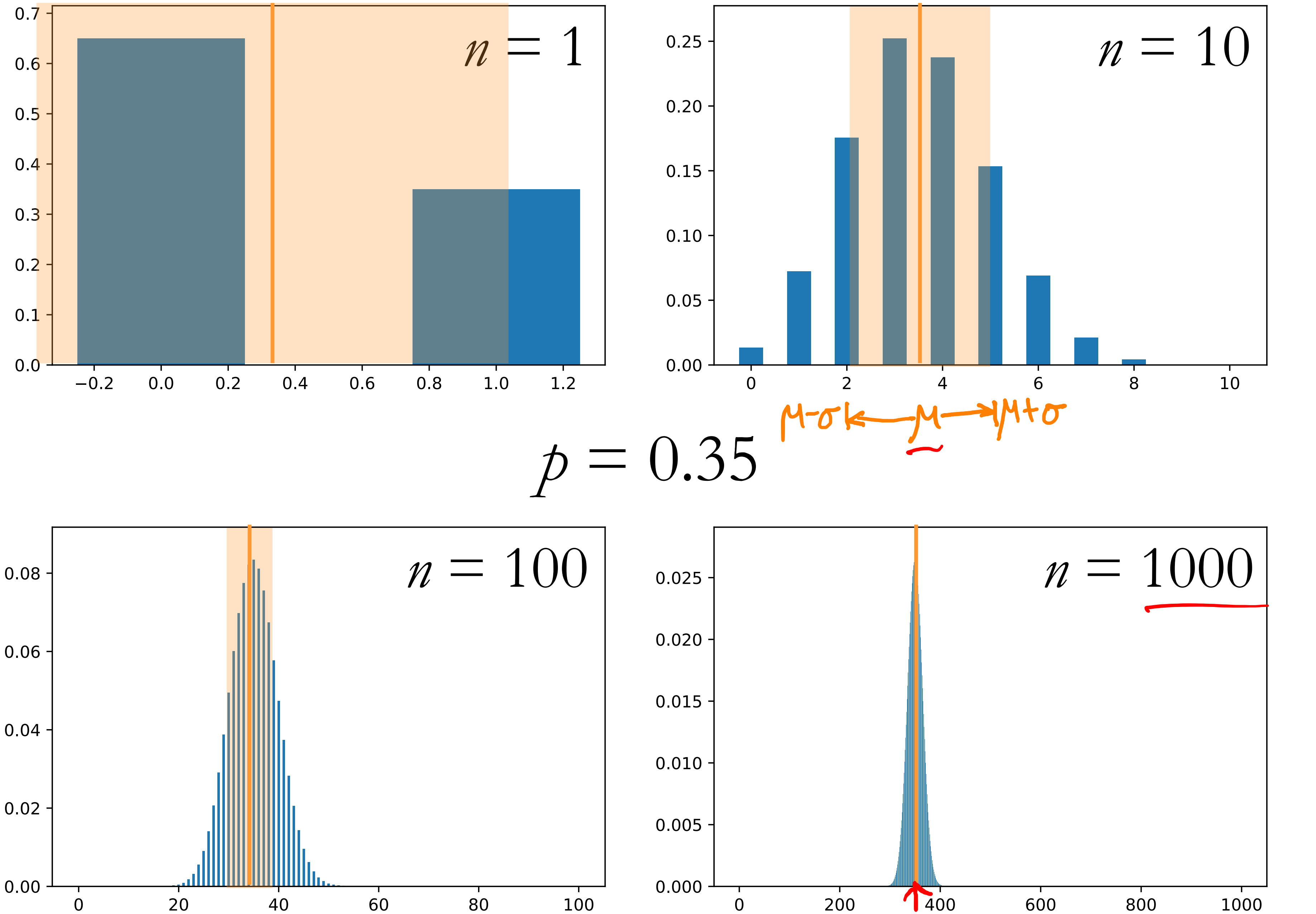
Sample mean



Randomly with repetition

$\underset{\sim}{X} = \text{Binomial}(\underset{\sim}{n}, \underline{p})$ $\quad$ p is unknown

n	p	$E[X]=np$		$\sigma=\sqrt{np(1-p)}$
100	0.5	50	$\pm$	5
100	0.25	25	$\pm$	4.33
1000	0.5	500	$\pm$	15.81
1000	0.25	250	$\pm$	13.69



$n = 1$
$n = 10$
$\mu-\sigma$
μ
$\mu+\sigma$
$p = 0.35$
$n = 100$
$n = 1000$

Variance of the Poisson

Poisson(λ) **approximates** Binomial(n, λ/n) **for large** n

$$\boxed{p(k) = e^{-\lambda}\, \lambda^k / k!} \qquad k = 0, 1, 2, 3, \ldots$$

$\mathrm{Var}[\mathrm{Binomial}(n, p)] = np(1-p)$

$\mathrm{Var}[\mathrm{Poisson}(\lambda)]$ $\qquad \cancel{n} \cdot \dfrac{\lambda}{\cancel{n}} \cdot (1 - \dfrac{\lambda}{n})$

$n \to \infty \longrightarrow \lambda.$

$\mathrm{Var}[\mathrm{Poisson}(\lambda)] = \lambda, \qquad \sigma = \sqrt{\lambda}.$

Independence of multiple random variables

X, Y, Z independent if

$$\mathbf{P}(X = x, Y = y, Z = z) = \mathbf{P}(X = x)\ \mathbf{P}(Y = y)\ \mathbf{P}(Z = z)$$

for all possible values of x, y, z.

X, Y, Z independent if and only if

$$\mathbf{E}[f(X)g(Y)h(Z)] = \mathbf{E}[f(X)]\ \mathbf{E}[g(Y)]\ \mathbf{E}[h(Z)]$$

for all f, g, h.

Usual warnings apply.