Practice questions

1. Alice rolls three 3-sided dice. Calculate the PMFs and the expected values of

   (a) The maximum of the three rolls.

   Solution: Call this random variable MAX. The sample space has $3^3 = 27$ equally likely outcomes consisting of the three rolls. Out of these, the event $Max = 1$ happens for a single outcome 111, while $MAX = 2$ happens for all but one of the eight outcomes $\{1, 2\} \times \{1, 2\} \times \{1, 2\}$. Therefore $\mathrm{P}(MAX = 1) = 1/27$ and $\mathrm{P}(MAX = 2) = 7/27$. Since probabilities must add up to one, $\mathrm{P}(MAX = 3) = 19/27$. The PMF is

x	1	2	3
$\mathrm{P}(MAX = x)$	1/27	7/27	19/27

   Therefore, we have the expected value as:

   $$E[MAX] = (1/27) * 1 + (7/27) * 2 + (19/27) * 3 = 8/3$$

   (b) The minimum of the three rolls.

   Solution: You can calculate this as in part (a) or reason it out like this: If we replaced roll x by $4 - x$, the minimum MIN would become $4 - MAX$. Since the replacement preserves the probabilities of all outcomes, MIN and $4 - MAX$ must have the same PMF, which is

x	1	2	3
$\mathrm{P}(MIN = x)$	19/27	7/27	1/27.

   Therefore, we have the expected value as:

   $$E[MIN] = (19/27) * 1 + (7/27) * 2 + (1/27) * 3 = 4/3$$

   (c) The sum of the three rolls.

   Solution: Let SUM be the random variable. This variable can take values $3, 4, 5, 6, 7, 8,$ or 9. The event $SUM = 3$ consists of the single outcome 111, $SUM = 4$ consists of the three outcomes 211, 121, 112, and $SUM = 5$ consists of six outcomes: Three with one 3 roll and two 1 rolls, and three with two 2 rolls and one 1 roll. So $\mathrm{P}(SUM = 3) = 1/27, \mathrm{P}(SUM = 4) = 3/27$, and $\mathrm{P}(SUM = 5) = 6/27$. By the same argument is in part (b), if we replace roll x by $4 - x$, SUM becomes $12 - SUM$ and so we can deduce the PMF values at 7, 8, and 9. It remains to determine $\mathrm{P}(SUM = 6)$ which must then equal $1 - 2(1/27 - 3/27 - 6/27) = 7/27$. The PMF is

x	3	4	5	6	7	8	9
$\mathrm{P}(SUM = x)$	1/27	3/27	6/27	7/27	6/27	3/27	1/27.

   Therefore, we have the expected value as:

   $$\begin{aligned} E[SUM] &= (1/27) * 3 + (3/27) * 4 + (6/27) * 5 + (7/27) * 6 \\ &\quad + (6/27) * 7 + (3/27) * 8 + (1/27) * 9 \\ &= 6 \end{aligned}$$

2. You flip two bias coins. The probabilities of obtaining head for the two coins are $2/3$ and $3/4$ respectively. If they both come out with the same result, you stop. If not, you try again until they do. Let F be the total number of coin flips you performed. For example if the outcome is `TH HT HH` then $F = 6$. If the outcome is `TH TT` then $F = 4$. What is the PMF (probability mass function) of F?

Solution: F can never be odd as you always perform an even number of flips. To perform a total of $2k$ flips (k rounds), the first $k-1$ rounds must have all resulted in failure and the last one in success. The probability of each round succeeding is $\frac{2}{3} \times \frac{3}{4} + \frac{1}{3} \times \frac{1}{4} = \frac{7}{12}$ and the successes are independent of one another, so

$$\mathrm{P}(F = 2k) = (\frac{5}{12})^{k-1}(\frac{7}{12}),$$

where k ranges over $1, 2, \dots,$ or if you prefer

$$\mathrm{P}(F = f) = (\frac{5}{12})^{f/2-1}(\frac{7}{12}),$$

where f ranges over the positive even numbers.

3. Suppose the number of school bus arriving at the Sir Run Run Shaw Hall in any time interval is a Poisson random variable, with a rate of 1 bus in 5 minutes.

(a) What is the probability that no bus arrives in an interval of 30 minutes?

Solution: The rate of bus arrivals is 6 in 30 minutes, so the number of buses that arrive in a 30-minute interval is a Poisson(6) random variable X. We are interested in the probability of the event $X = 0$, which equals $e^{-6} \approx 0.002479$.

(b) What is the probability that there are at least 5 buses in an interval of 10 minutes?

Solution: The rate of arrivals is 2 in 10 minutes, so we want to know what is the probability that a Poisson(2) random variable Y takes value 5 or more. So we need to calculate

$$\mathrm{P}(Y \geq 5) = \mathrm{P}(Y = 6) + \mathrm{P}(Y = 7) + \cdots,$$

which is an infinite sum. By the axioms of probability, we can instead calculate

$$\begin{aligned}
\mathrm{P}(Y \geq 5) &= 1 - \mathrm{P}(Y < 5) \\
&= 1 - \mathrm{P}(Y = 0) - \mathrm{P}(Y = 1) - \mathrm{P}(Y = 2) - \mathrm{P}(Y = 3) - \mathrm{P}(Y = 4) \\
&= 1 - \frac{e^{-2} \cdot 2^0}{0!} - \frac{e^{-2} \cdot 2^1}{1!} - \frac{e^{-2} \cdot 2^2}{2!} - \frac{e^{-2} \cdot 2^3}{3!} - \frac{e^{-2} \cdot 2^4}{4!}, \\
&= 1 - 7e^{-2}
\end{aligned}$$

which is about 0.0527.

4. You go to a party with 500 guests.

(a) What is the probability that exactly one other guest has the same birthday as you? (For simplicity, exclude birthdays on February 29.) (The result should be rounded to 4 decimal places.)

(b) Now model the number of other guests that share your birthday as a Poisson(λ) random variable N. What is the rate λ? What is the the probability that N equals 1? (The result should be rounded to 4 decimal places.)

Solution:

(a) We can model the number of guests having *your* birthday as a Binomial($n = 499, p = 1/365$) random variable X. The probability that $X = 1$ is $\binom{499}{1} \cdot p \cdot (1-p)^{499-1} \approx 0.3487$.

(b) We can model this process as a Poisson(λ) random variable N with $\lambda = np = 499/365$. Then the probability of $N = 1$ is $\lambda \cdot e^{-\lambda} \approx 0.3484$.