ENGG 2760A / ESTR 2018: Probability for Engineers

4. Random Variables

Prof. Hong Xu

Credit to Prof. Andrej Bogdanov

Random variable

A discrete random variable assigns a discrete value to every outcome in the sample space.

Example 

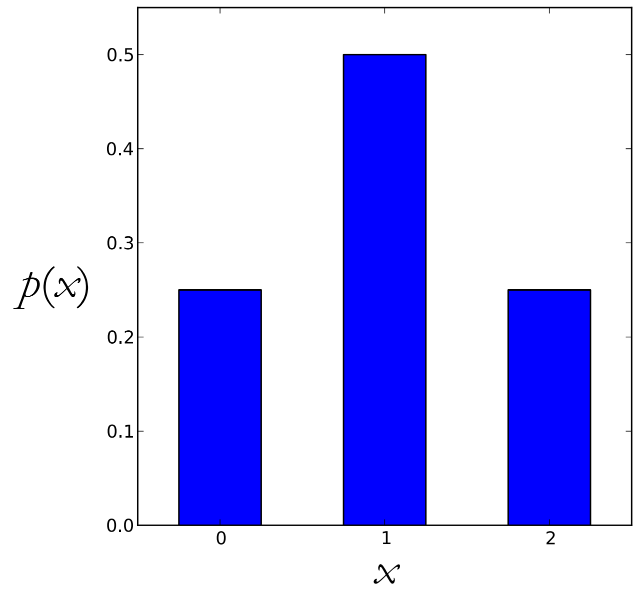
$\{\ \mathtt{HH},\ \mathtt{HT},\ \mathtt{TH},\ \mathtt{TT}\ \}$

$N =$ **number of** $\mathtt{H}$**s**

Probability mass function

We can describe the PMF by a table or by a chart.

x	0	1	2
$p(x)$	$1/4$	$1/2$	$1/4$



Two 3-sided dice are tossed. Calculate the PMF of the difference D of the rolls.

11	12	13
21	22	23
31	32	33

What is the probability that $D \geq 1$? D is odd?

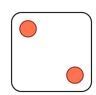
The binomial random variable

$\mathrm{Binomial}(n, p)$: **Perform n independent trials, each of which succeeds with probability p.**

$X =$ **number of successes**

Examples

Toss n coins. "number of heads" is $\mathrm{Binomial}(n, \frac{1}{2})$.

Toss n dice. "Number of  **s" is $\mathrm{Binomial}(n, 1/6)$.**

A less obvious example

Flip n coins. Let C be the number of consecutive changes ($\mathtt{HT}$ or $\mathtt{TH}$).

Examples:

ω	$C(\omega)$
HHHHHHH	0
THHHHHT	2
HTHHHHT	3

Then C is $\text{Binomial}(n - 1, ½)$.

Draw a 10-card hand from a 52-card deck.

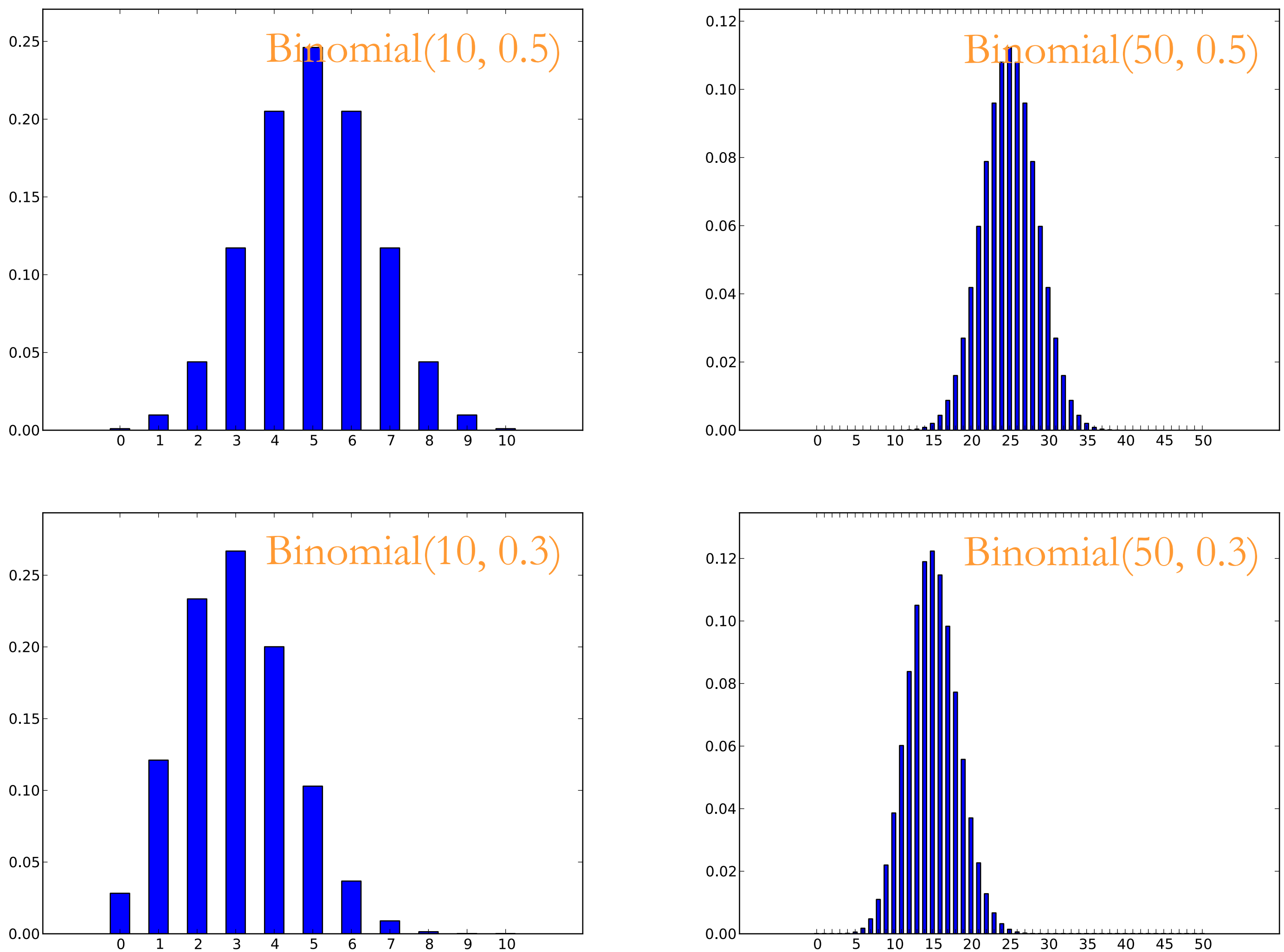
Let $N =$ number of aces among the drawn cards

Is N a $\mathrm{Binomial}(10, 1/13)$ random variable?

Probability mass function

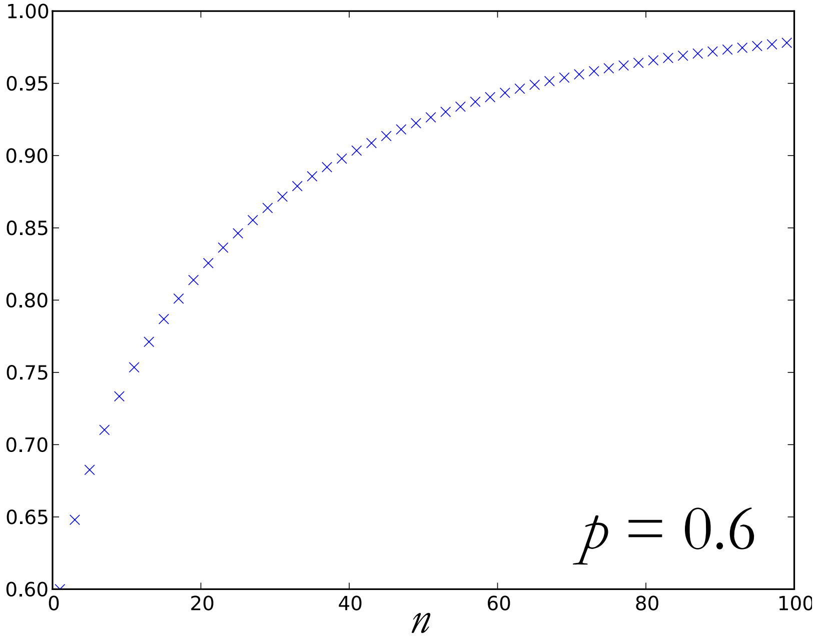
If X **is** $\mathrm{Binomial}(n, p)$**, its PMF is**

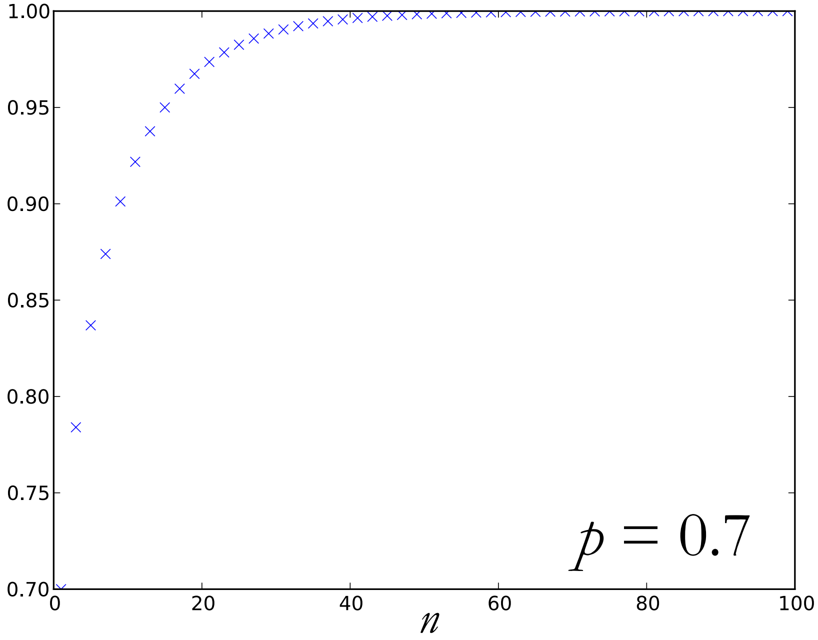
$$\boxed{p(k) = P(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}}$$


Binomial(10, 0.5)
Binomial(50, 0.5)
Binomial(10, 0.3)
Binomial(50, 0.3)
0.00
0.05
0.10
0.15
0.20
0.25
0.02
0.04
0.06
0.08
0.12
0 1 2 3 4 5 6 7 8 9 10
0 5 10 15 20 25 30 35 40 45 50

Binomial random variable

$$\mathbf{P}(\text{Binomial}(n, p) \geq k) =$$





$$\mathbf{P}(\text{Binomial}(n, p) \geq (n-1)/2)$$

The Lakers and the Celtics meet for a 7-game playoff.

Lakers win 60% of the time. What is the probability that all 7 games are played?

What is the probability that Lakers win in game 6?

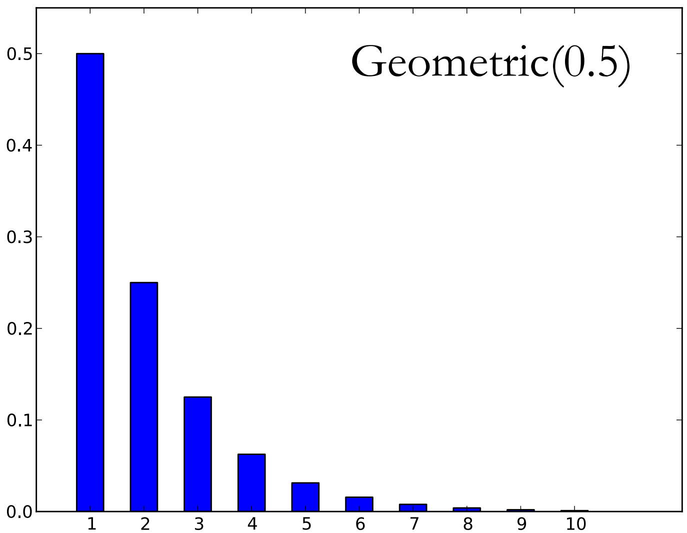
https://stattrek.com/online-calculator/binomial.aspx

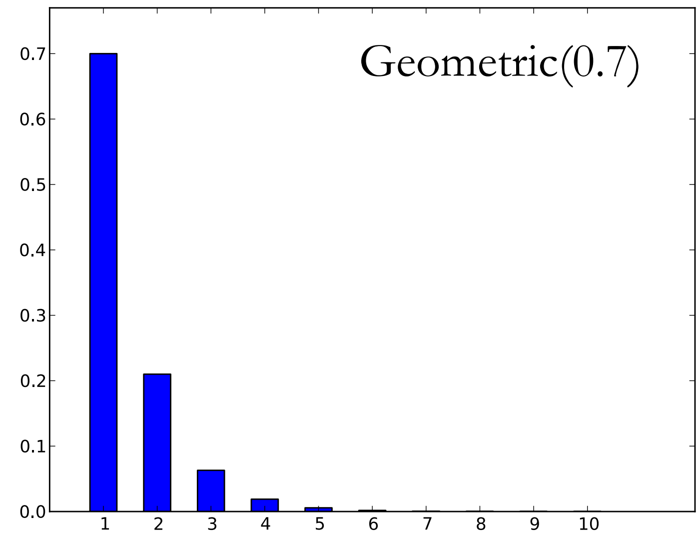
Geometric random variable

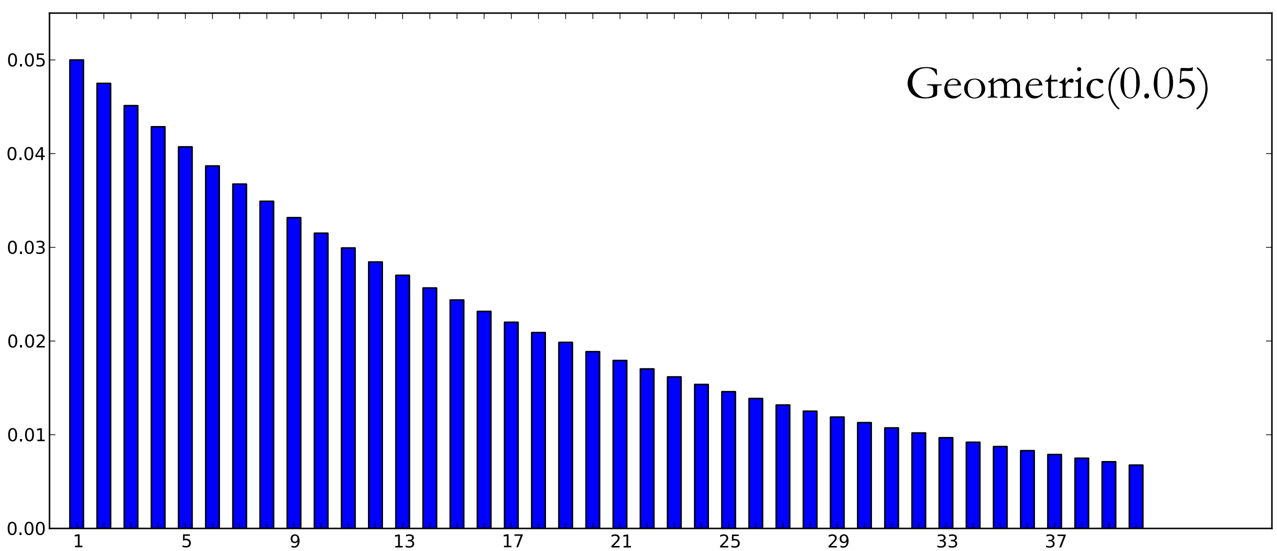
Let $X_1, X_2, \ldots$ be independent trials with success p.

A $\text{Geometric}(p)$ random variable N is the time of the first success among $X_1, X_2, \ldots$:

$N =$ first (smallest) n such that $X_n = 1$.


Geometric(0.5)
0.0 0.1 0.2 0.3 0.4 0.5
1 2 3 4 5 6 7 8 9 10


Geometric(0.7)
0.0 0.1 0.2 0.3 0.4 0.5 0.6 0.7
1 2 3 4 5 6 7 8 9 10


Geometric(0.05)
0.00 0.01 0.02 0.03 0.04 0.05
1 5 9 13 17 21 25 29 33 37

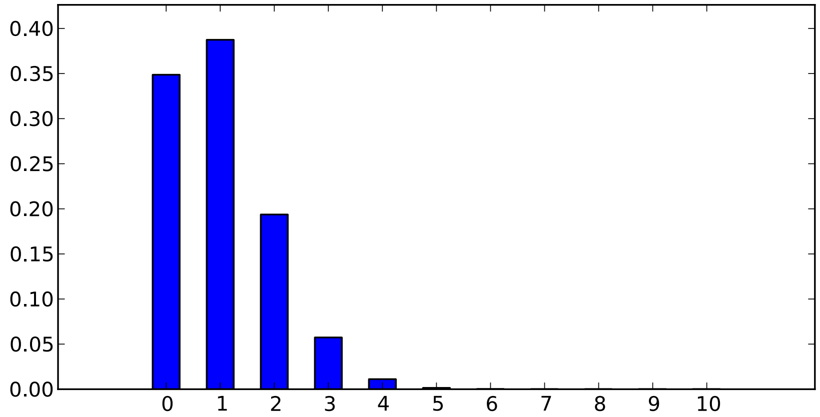
Apples

About 10% of the apples on your farm are rotten.

You sell 10 apples. How many are rotten?

Probability model

N**umber of rotten apples you sold is** $\text{Binomial}(n = 10, p = 1/10)$.

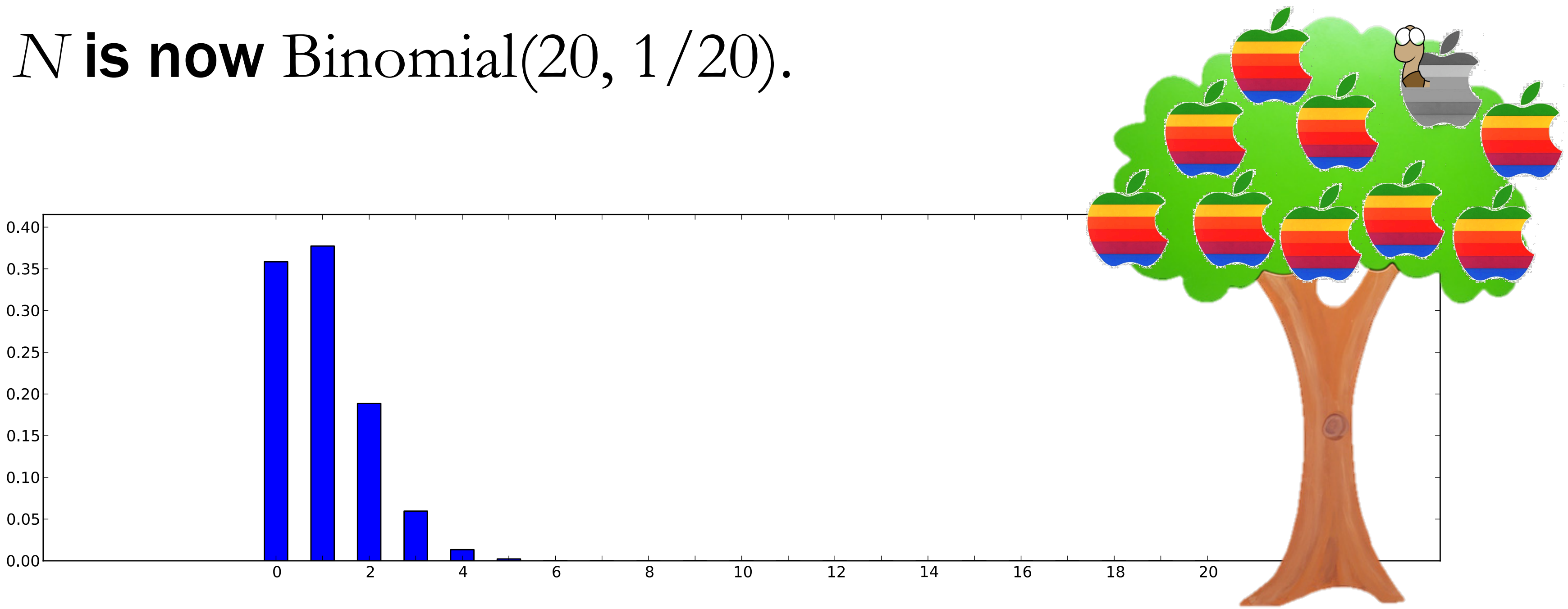


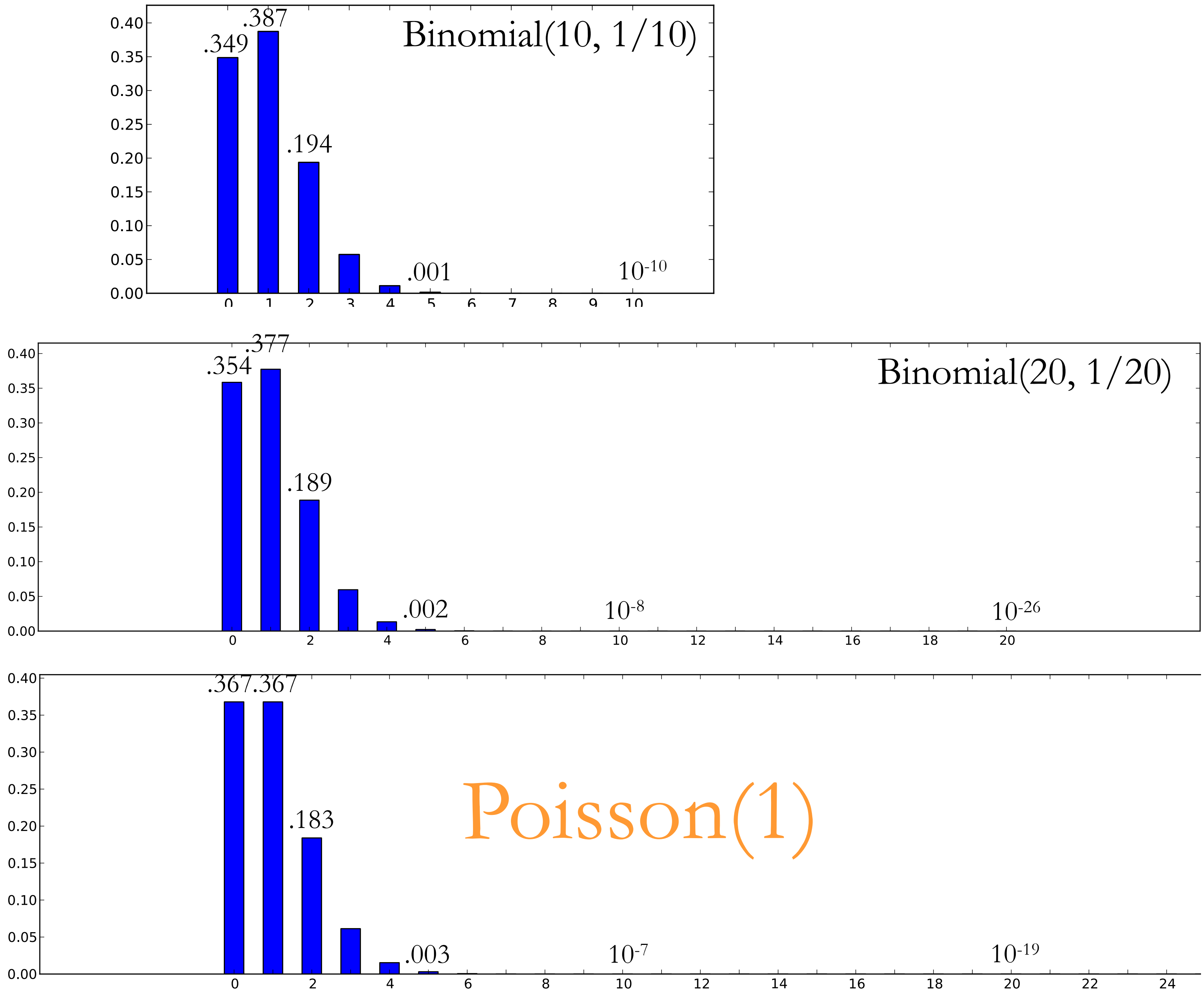
Apples

You improve productivity; now only 5% apples rot.

You can now sell 20 apples.

N **is now** $\mathrm{Binomial}(20, 1/20)$.




Binomial(10, 1/10)
.349
.387
.194
.001
10-10
Binomial(20, 1/20)
.354
.377
.189
.002
10-8
10-26
Poisson(1)
.367
.367
.183
.003
10-7
10-19

The Poisson random variable

A $\mathrm{Poisson}(\lambda)$ **random variable has PMF**

$$\boxed{p(k) = e^{-\lambda}\, \lambda^k / k!}$$

$k = 0, 1, 2, 3, \ldots$

Poisson random variables do not occur "naturally" in the sample spaces we have seen.

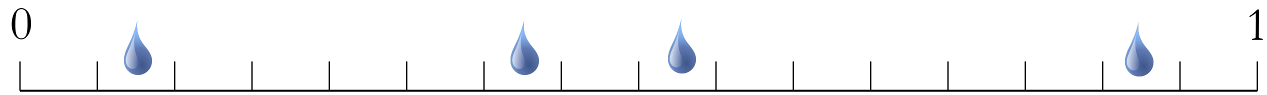
The Poisson random variable

$\mathrm{Poisson}(\lambda)$ **approximates** $\mathrm{Binomial}(n, p)$
when $\lambda = np$ is fixed and n is large (p is small)

$$\mathrm{P}(\mathrm{Poisson}(\lambda) = k) = \lim_{n \to \infty} \mathrm{P}(\mathrm{Binomial}(n, \lambda/n) = k)$$

The Poisson random variable

Rain is falling on your head at a rate of 3 drops/sec.



Functions of random variables

PMF of X:

x	0	1	2
$p(x)$	$1/3$	$1/3$	$1/3$

PMF of $X - 1$?

PMF of $(X-1)^2$?

If X is a random variable with p.m.f. p_X, then $Y = f(X)$ is a random variable with p.m.f.

$$p_Y(y) = \sum_{x:\, f(x) = y} p_X(x).$$

D is the difference of two 3-sided dice rolls. Calculate the PMF of $|D|$.

Expected value

The expected value (expectation) of a random variable X with p.m.f. p is

$$E[X] = \sum_x x\, p(x)$$

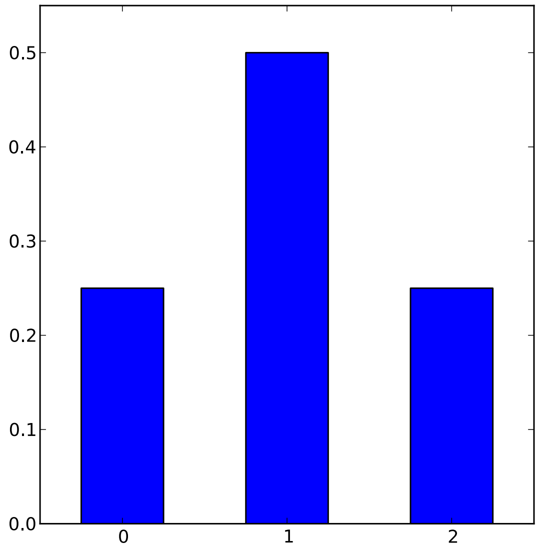
Example

$N =$ **number of** $\texttt{H}$**s**

Expected value

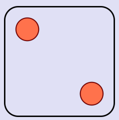
Example

$N =$ **number of** H**s**



The expectation is the average value the random variable takes when experiment is done many times

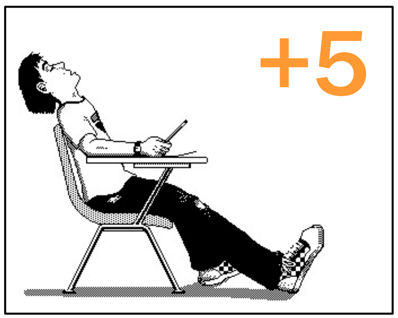
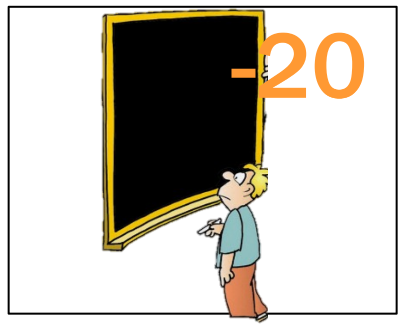
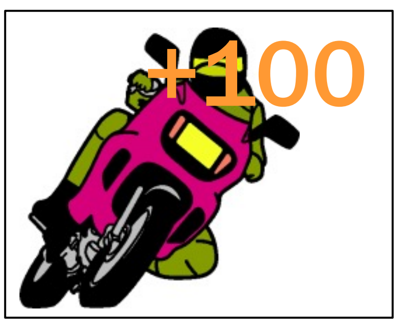
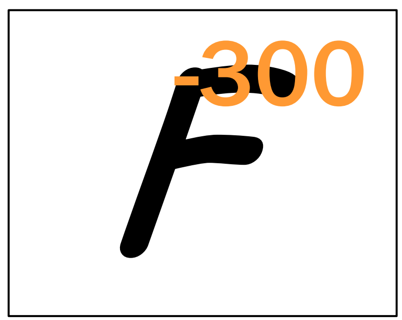
$F =$ **face value of fair 6-sided die**

If  appears k times, you win \$$k$.

If it doesn't appear, you lose \$1.

Utility

Should I go to tutorial?

	not called	called
*C*ome	 +5	 -20
*S*kip	 +100	 -300
	35/40	5/40


VIDEO GAMES
P(correct)
$200
80%
$400
50%
$600
$800
$1000