ENGG 2760A / ESTR 2018: Probability for Engineers

6. Conditional PMFs and Independent Random Variables

Prof. Hong Xu

Credit to Prof. Andrej Bogdanov

Conditional PMF

Let X be a random variable and A be an event.

The conditional PMF of X given A is

$$P(X = x \mid A) = \frac{P(X = x \text{ and } A)}{P(A)}$$

What is the PMF of a 6-sided die roll given that the outcome is even?

You flip 3 coins. What is the PMF of number of heads given that there is at least one?

Conditioning on a random variable

The conditional PMF of X given Y is

$$\mathbf{P}(X = x \mid Y = y) = \frac{\mathbf{P}(X = x \text{ and } Y = y)}{\mathbf{P}(Y = y)}$$

$$p_{X|Y}(x \mid y) = \frac{p_{XY}(x, y)}{p_Y(y)}$$

For fixed y, $p_{X|Y}$ is a PMF as a function of x.

Roll two 3-sided dice. What is the PMF of the sum given the first roll?

Roll two 3-sided dice. What is the PMF of the first roll given the sum?

Conditional Expectation

The conditional expectation of X given event A is

$$\mathbf{E}[X \mid A] = \sum_x x\,\mathbf{P}(X = x \mid A)$$

The conditional expectation of X given $Y = y$ is

$$\mathbf{E}[X \mid Y = y] = \sum_x x\,\mathbf{P}(X = x \mid Y = y)$$

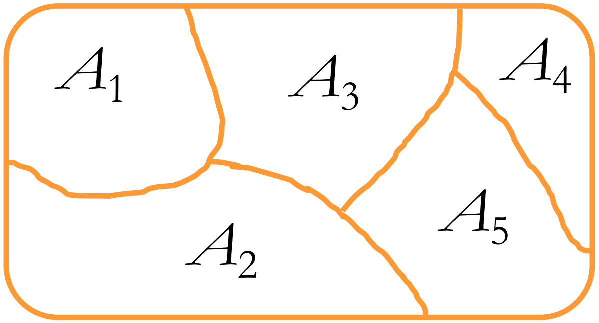
You flip 3 coins. What is the expected number of heads given that there is at least one?

Total Expectation Theorem

$$\boxed{\mathbf{E}[X] = \mathbf{E}[X \mid A]\ \mathbf{P}(A) + \mathbf{E}[X \mid A^c]\ \mathbf{P}(A^c)}$$

Proof

Total Expectation Theorem (general form)



If $A_1, \ldots, A_n$ **partition** Ω
then

$$\mathbf{E}[X] = \mathbf{E}[X \mid A_1]\mathbf{P}(A_1) + \ldots + \mathbf{E}[X \mid A_n]\mathbf{P}(A_n)$$

In particular

$$\mathbf{E}[X] = \sum_y \mathbf{E}[X \mid Y = y]\, \mathbf{P}(Y = y)$$

type	👩‍🏫	👵	🕵️‍♂️
average time on facebook	30 min	60 min	10 min
% of visitors	60%	30%	10%

average visitor time =

You play 10 rounds of roulette. You invest $100 and bet 10% of your balance on red in every round.

What is your average balance after 10 rounds?

You flip 3 coins. What is the expected number of heads given that there is at least one?

Mean of the Geometric

$X = \text{Geometric}(p)$ **random variable**

$\mathbf{E}[X] =$

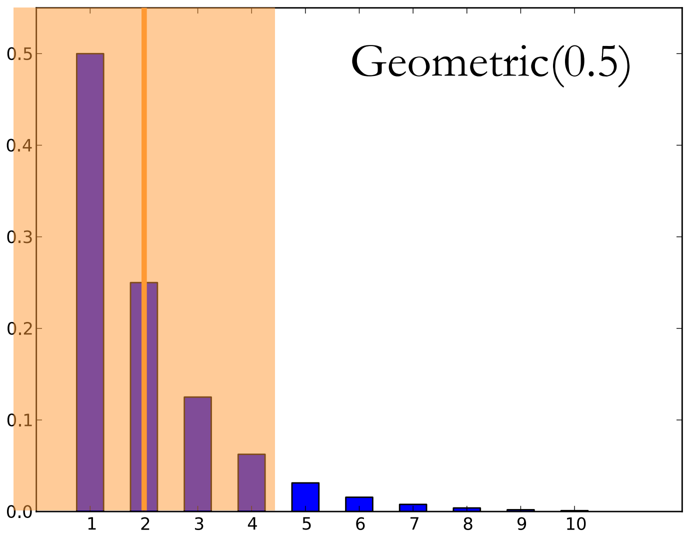
Variance of the Geometric

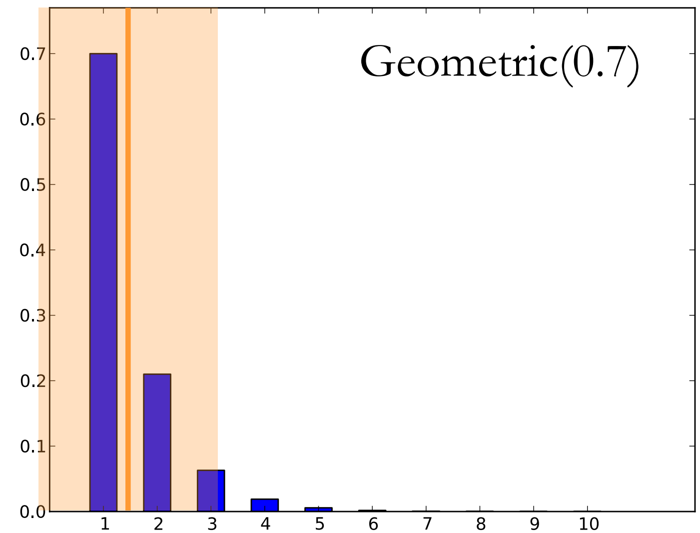
$X = \mathrm{Geometric}(p)$ **random variable**

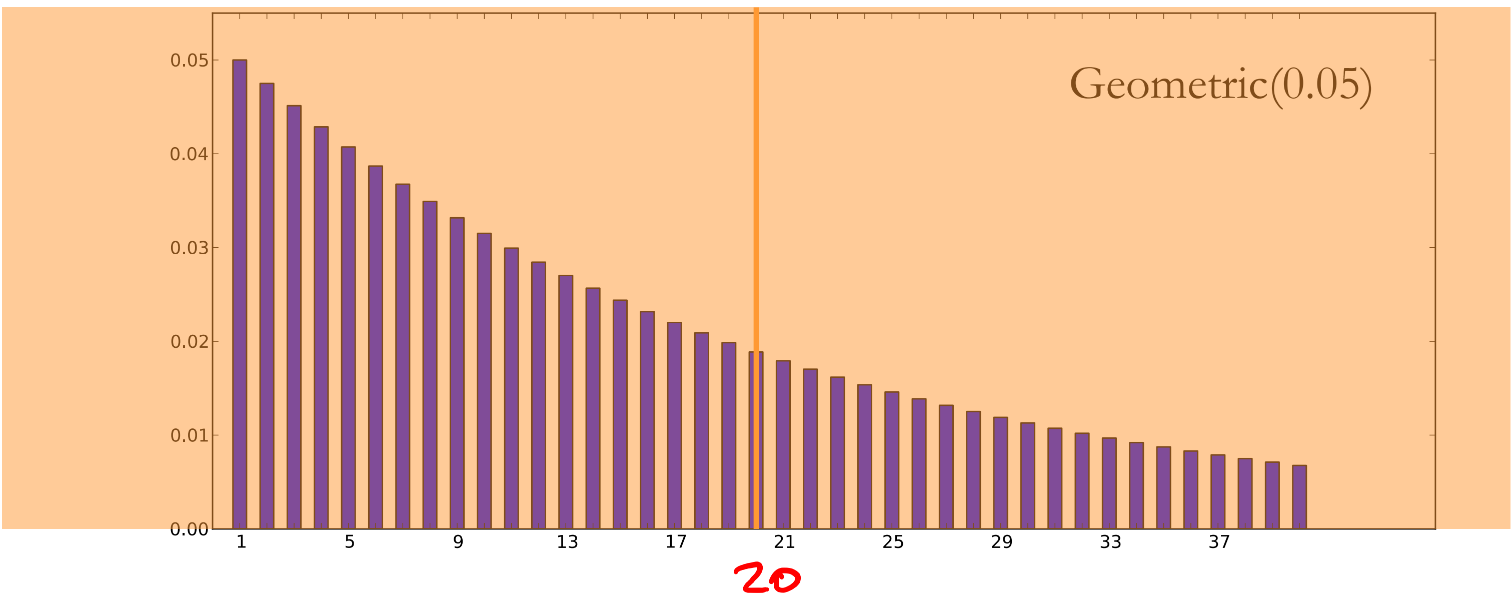
$\mathbf{Var}[X] = \dfrac{1-p}{p^2}$ $\qquad\qquad \sigma = \dfrac{\sqrt{1-p}}{p} < \dfrac{1}{p}.$

Proof (optional):

$$\begin{aligned}
\mathrm{Var}[X] &= E[(X-1/p)^2] \\
&= E[(X-1/p)^2 \mid W]\cdot p + E[(X-1/p)^2 \mid W^c]\cdot(1-p) \\
&= (1-1/p)^2\cdot p + E[(1+Y-1/p)^2 \mid W^c]\cdot(1-p) \\
&= (1-1/p)^2\cdot p + E[1+2(Y-1/p)+(Y-1/p)^2 \mid W^c](1-p) \\
&= (1-1/p)^2\cdot p + (1+\underbrace{\cancel{2E[Y-1/p \mid W]}}_{0}+\underbrace{E[(Y-1/p)^2 \mid W^c]}_{\mathrm{Var}[X]})(1-p) \\
&= (1-1/p)^2\cdot p + (1+\mathrm{Var}[X])\cdot(1-p)
\end{aligned}$$


Geometric(0.5)
0.5
0.4
0.3
0.2
0.1
0.0
1
2
3
4
5
6
7
8
9
10


Geometric(0.7)
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
1
2
3
4
5
6
7
8
9
10


Geometric(0.05)
0.05
0.04
0.03
0.02
0.01
0.00
1
5
9
13
17
21
25
29
33
37
20

Independent random variables

Let X and Y be discrete random variables.

X and Y are independent if

$$\mathbf{P}(X = x, Y = y) = \mathbf{P}(X = x)\,\mathbf{P}(Y = y)$$

for all possible values of x and y.

In PMF notation, $\mathrm{p}_{XY}(x, y) = p_X(x)\,p_Y(y)$ for all x, y.

Independent random variables

X and Y are independent if

$$\mathbf{P}(X = x \mid Y = y) = \mathbf{P}(X = x)$$

for all x and y such that $\mathbf{P}(Y = y) > 0$.

In PMF notation, $p_{X|Y}(x \mid y) = p_X(x)$ if $p_Y(y) > 0$.

Alice tosses three coins and so does Bob. Alice gets \$1 per head and Bob gets \$1 per tail.

Are their earnings independent?

Now they toss the same coin three times. Are their earnings independent?

Expectation and independence

X and Y are independent if and only if

$$\mathbf{E}[f(X)g(Y)] = \mathbf{E}[f(X)]\,\mathbf{E}[g(Y)]$$

for all real valued functions f and g.

Expectation and independence

In particular, if X and Y are independent then

$$E[XY] = E[X]\,E[Y]$$

Not true in general!

Variance of a sum

Recall $\mathbf{Var}[X] = \mathbf{E}[(X - \mathbf{E}[X])^2] = \mathbf{E}[X^2] - \mathbf{E}[X]^2$

$$\mathbf{Var}[X + Y] =$$

Variance of a sum

$$\mathbf{Var}[X_1 + \ldots + X_n] = \mathbf{Var}[X_1] + \ldots + \mathbf{Var}[X_n]$$

if every pair X_i, X_j is independent.

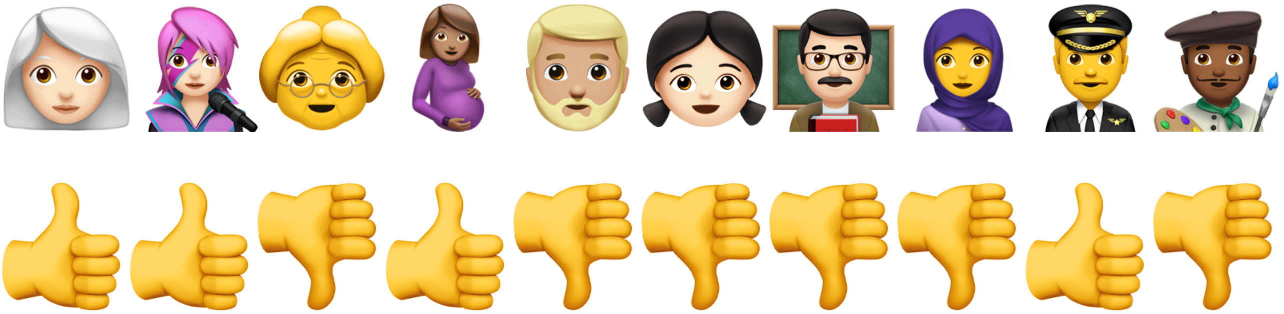
Not true in general!

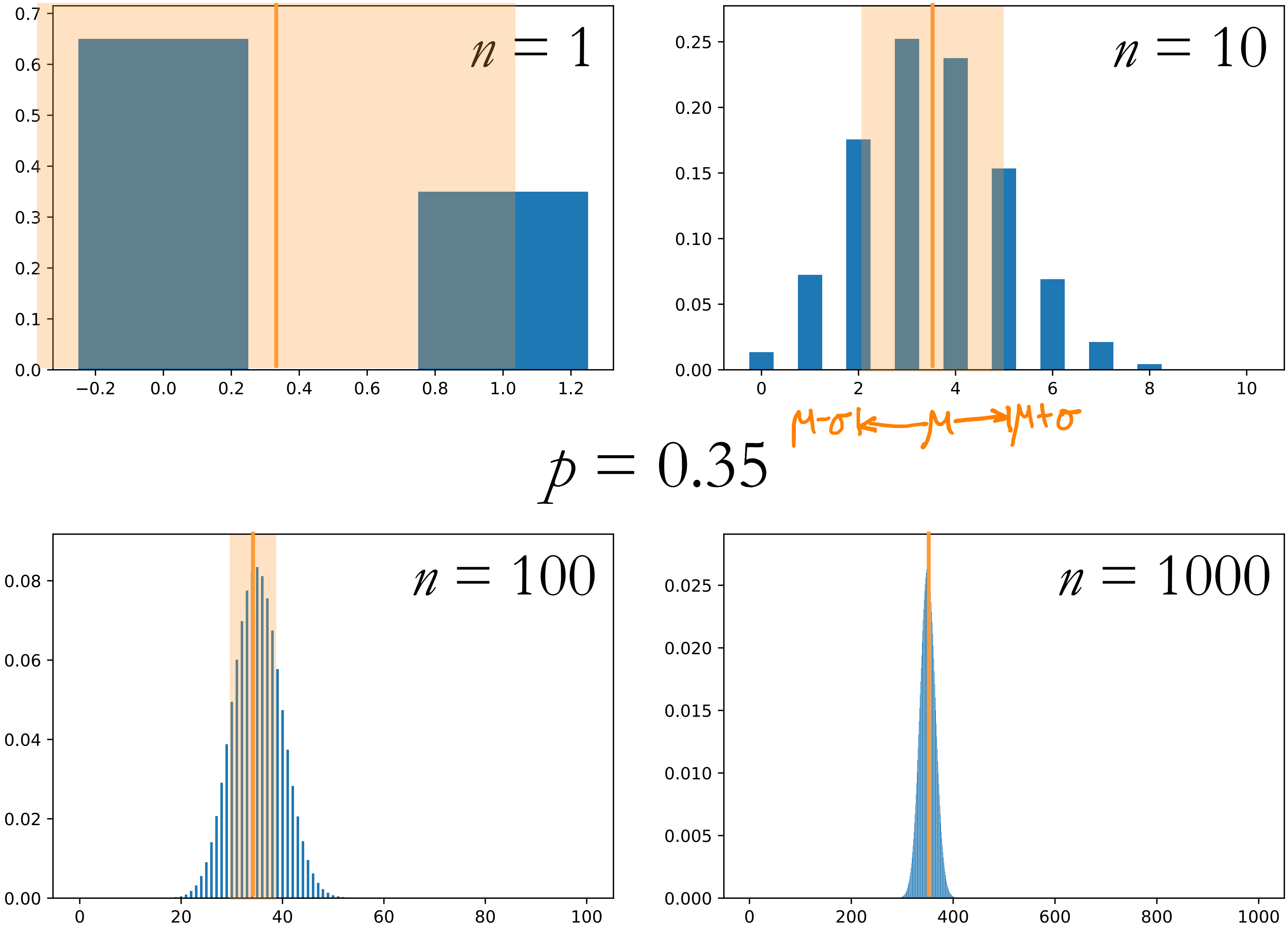
Keep tossing a fair coin until you get ten heads.
On average, how many times T will you toss?

What is the standard deviation of T?

Variance of the Binomial

Sample mean





n = 1
n = 10
n = 100
n = 1000
μ−σ
μ
μ+σ
p = 0.35

Variance of the Poisson

$\mathrm{Poisson}(\lambda)$ **approximates** $\mathrm{Binomial}(n, \lambda/n)$ **for large** n

$$\boxed{p(k) = e^{-\lambda}\, \lambda^k / k!} \qquad k = 0, 1, 2, 3, \ldots$$

Independence of multiple random variables

X, Y, Z independent if

$$\mathbf{P}(X = x, Y = y, Z = z) = \mathbf{P}(X = x)\ \mathbf{P}(Y = y)\ \mathbf{P}(Z = z)$$

for all possible values of x, y, z.

X, Y, Z independent if and only if

$$\mathbf{E}[f(X)g(Y)h(Z)] = \mathbf{E}[f(X)]\ \mathbf{E}[g(Y)]\ \mathbf{E}[h(Z)]$$

for all f, g, h.

Usual warnings apply.