ENGG 2760A / ESTR 2018: Probability for Engineers

2. Axioms of Probability. Conditional Probabilities

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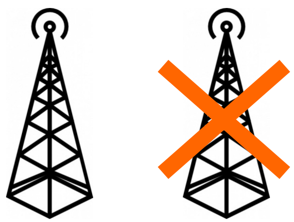
Credit to Prof. Andrej Bogdanov

How to come up with a model?



a pair of antennas

each can be working or defective

$\Omega = \{ \mathtt{WW}, \mathtt{WD}, \mathtt{DW}, \mathtt{DD} \}$

Model 1: Each antenna defective 10% of the time

Defects are "independent"

WW	WD	DW	DD	
0.81	0.09	0.09	0.01	=> 1

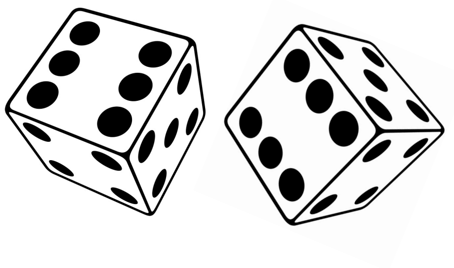
Model 2: Dependent defects

e.g. both antennas use same power supply

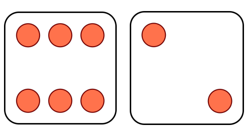
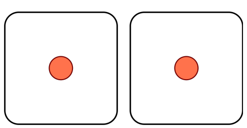
WW	WD	DW	DD	
0.9	0	0	0.1	=> 1

How to come up with a model?

Option 1: Use common sense

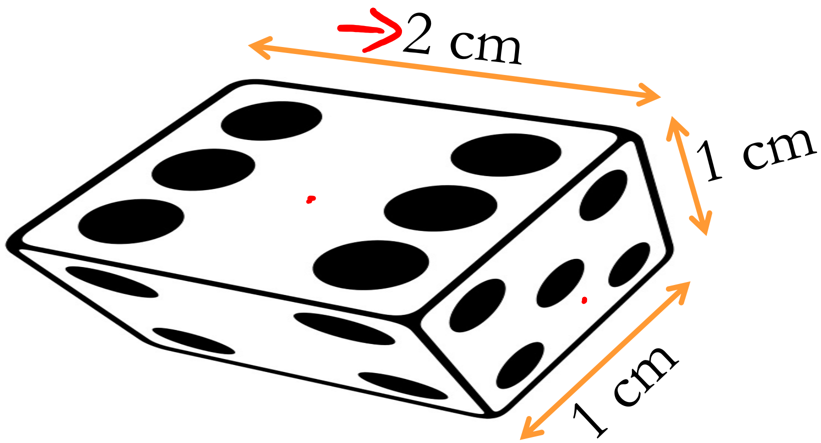


If there is no reason to favor one outcome over another, assign same probability to both

E.g.  and  should get same probability

So every outcome must be given probability $1/36$

The unfair dice



$$\Omega = \{\, 1, 2, 3, 4, 5, 6 \,\}$$

Common sense model: Probability $\propto$ surface area

	1	2	3	4	5	6	
outcome	1	2	3	4	5	6	
surface area (in cm^2)	2 +	1 +	2 +	2 +	1 +	2	= 10
probability	.2	.1	.2	0.2	0.1	0.2	$\Rightarrow$ 1

How to come up with a model?

Option 2: Frequency of occurrence

The probability of an outcome should equal the fraction of times that it occurs when the experiment is performed many times under the same conditions.

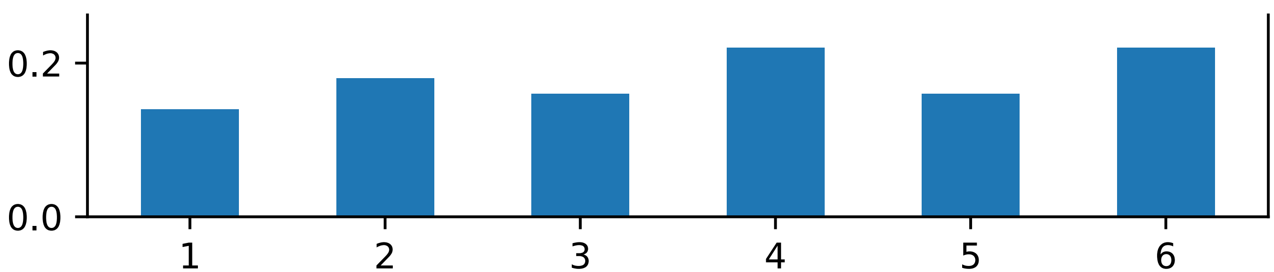
Frequency of occurrence

$$S = \{\ 1, 2, 3, 4, 5, 6\ \}$$

toss 50 times

44446163164351534251412664636216266362223241324453

outcome	1	2	3	4	5	6
occurrences	7	9	8	11	8	11
probability	.14	.18	.16	.22	.16	.22



Frequency of occurrence

The more times we repeat the experiment, the more accurate our model will be

toss 500 times

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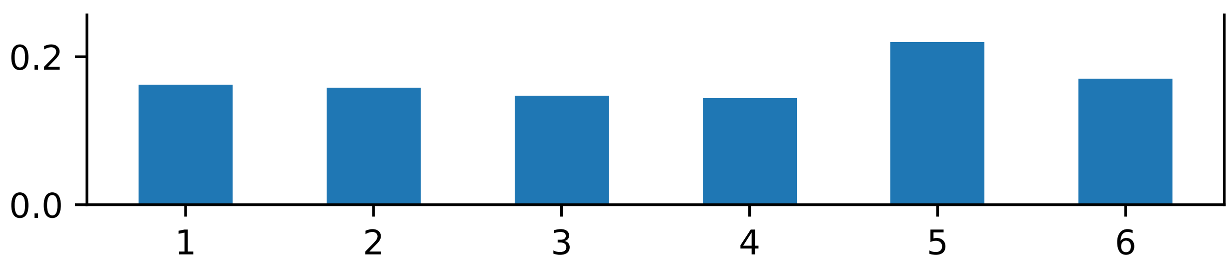
1356532511132365226434634623345663453543633514546423623551161445613441262462134541255656616436145465
5564544432666511115423226153655664335622316516625253424311263112466133443122113456244222324152625654
2435142565512653245554554435244153234535112232451656555551431435342225311453366652416621555663645155
1466565423451154611556156623152142224326265654263522234145214313453155221561523135262255633144613411
1115146113656156264255326331563211622355663545116144655216122656515362263456355232115565533521245536
```

outcome	1	2	3	4	5	6
occurrences	81	79	73	72	110	85
probability	.162	.158	.147	.144	.220	.170

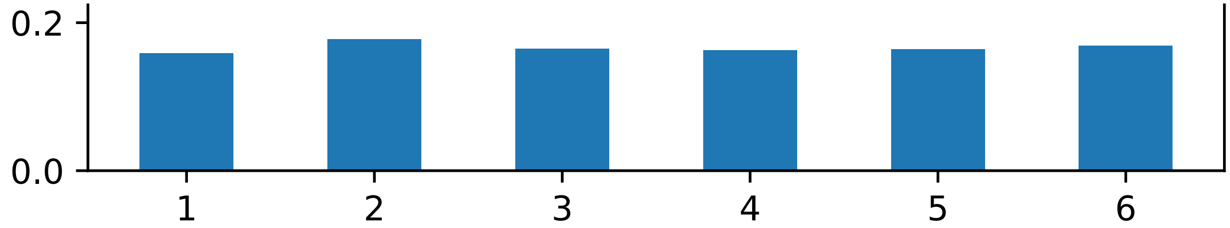


Frequency of occurrence

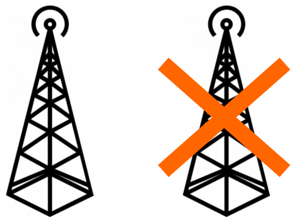
The more times we repeat the experiment, the more accurate our model will be

toss 5000 times

outcome	1	2	3	4	5	6
occurrences	797	892	826	817	821	847
probability	.159	.178	.165	.163	.164	.169



Frequency of occurrence



$$S = \{\, \mathtt{WW}, \mathtt{WD}, \mathtt{DW}, \mathtt{DD} \,\}$$

	M	T	W	T	F	S	S	M	T	W	T	F	S	S
WW	x	x	x	x		x		x			x	x		
WD														
DW														x
DD					x		x		x	x			x	

outcome	WW	WD	DW	DD
occurrences	8	0	1	5
probability	8/14	0	1/14	5/14

Frequency of occurrence

Give a probability model for the gender of Hong Kong young children.

sample space = { boy, girl }

Model 1:	common sense	1/2	1/2
Model 2:		.51966	.48034

1.2 按年齡組別及性別劃分的年中人口
Mid-year population by age group and sex

人數
Number of persons

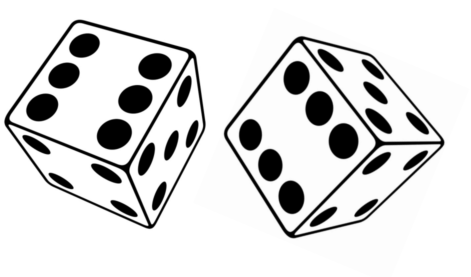
年齡組別（歲） Age group (years)	性別 Sex		2001	2006	2007	2008	2009	2010	2011
0 - 4	男性	M	142 000	110 400	111 300	114 000	117 700	124 200	129 500
	女性	F	130 800	102 600	103 200	105 200	108 300	113 800	119 700

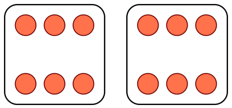
from *Hong Kong annual digest of statistics, 2012*

How to come up with a model

Option 3: Ask the market

The probability of an outcome should be proportional to the amount of money you are willing to bet on it.



Will you bet on 

... if the casino's odds are 35:1?

... how about 37:1?

Do you think that after their current term ...

Jack	Angus

...Xi will still be president of China?

100%	99.9%

...Biden will still be president of the USA?

10%	73%

...Xi and Biden will both still be presidents?

10%	73%

...Neither of them will be president?

0%	0.1%

Events

An event is a subset of the sample space.

Examples

$$\Omega = \{ \mathrm{HH}, \mathrm{HT}, \mathrm{TH}, \mathrm{TT} \}$$

both coins come out heads $E_1 = \{HH\}$

first coin comes out heads $E_2 = \{HH, HT\}$

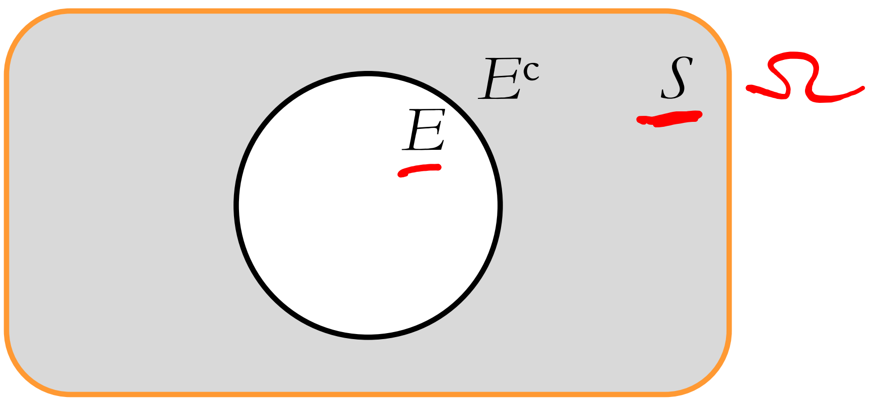
both coins come out same $E_3 = \{HH, TT\}$

Events

The complement of an event is the opposite event.

both coins come out heads $E_1 = \{ \texttt{HH} \}$

both coins do not come out heads $E_1{}^c = \{HT, TH, TT\}$



Events

The intersection of events happens when all events happen.

(a) first coin comes out heads $E_2 = \{\ \mathtt{HH}, \mathtt{HT}\ \}$

(b) both coins come out same $E_3 = \{\ \mathtt{HH}, \mathtt{TT}\ \}$

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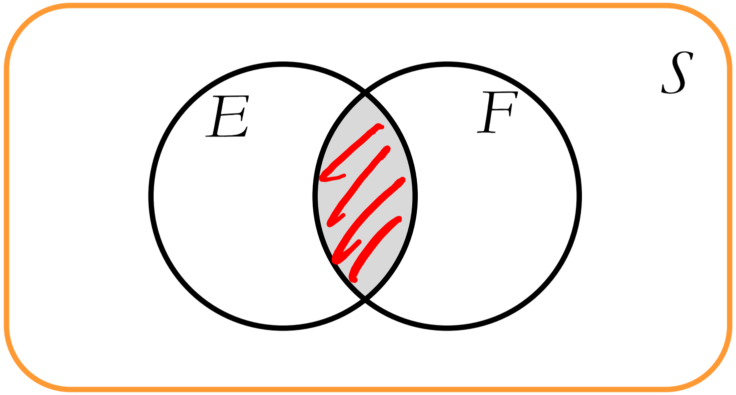
both (a) and (b) happen $E_2 \cap E_3 = \{HH\}$



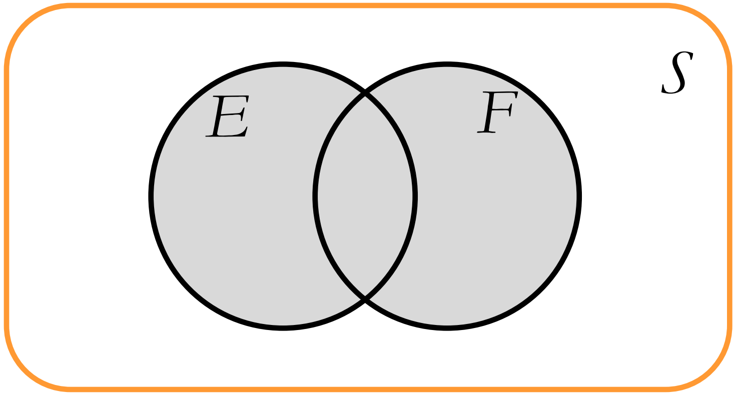
Events

The union of events happens when at least one of the events happens.

(a) first coin comes out heads $E_2 = \{\ \mathtt{HH}, \mathtt{HT}\ \}$

(b) both coins come out same $E_3 = \{\ \mathtt{HH}, \mathtt{TT}\ \}$

at least one happens $E_2 \cup E_3 =$ {HH, HT, TT}



Probability for finite spaces

The probability of an event is the sum of the probabilities of its elements

Example

$$\Omega = \{ \texttt{HH}, \texttt{HT}, \texttt{TH}, \texttt{TT} \}$$
$$\quad \tfrac{1}{4} \quad \tfrac{1}{4} \quad \tfrac{1}{4} \quad \tfrac{1}{4}$$

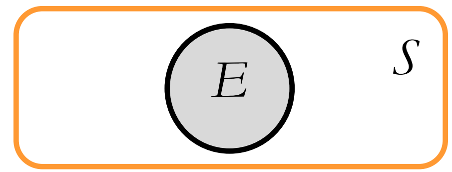
both coins come out heads	$E_1 = \{ \texttt{HH} \}$	$P(E_1) = 1/4$
first coin comes out heads	$E_2 = \{ \texttt{HH}, \texttt{HT} \}$	$P(E_2) = 1/2$
both coins come out same	$E_3 = \{ \texttt{HH}, \texttt{TT} \}$	$P(E_3) = 1/2$

Axioms of probability

A sample space Ω.

For every event E, a probability $\mathbf{P}(E)$ such that

1. for every E: $0 \le \mathbf{P}(E) \le 1$

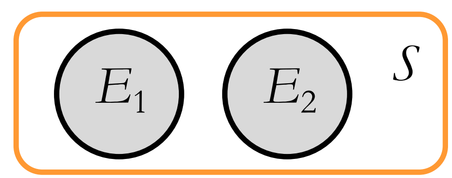


2. $\mathbf{P}(\Omega) = 1$

$E_1 \cap E_2 = \phi$

3. If $E_1, E_2, \ldots$ disjoint then

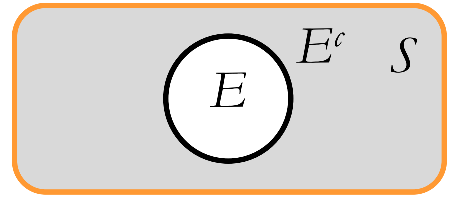
$$\mathbf{P}(\underline{E_1 \cup E_2 \cup \ldots}) = \mathbf{P}(E_1) + \mathbf{P}(E_2) + \ldots$$



Rules for calculating probability

Complement rule:

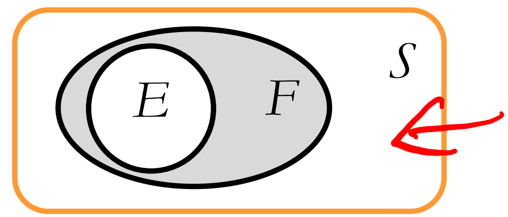
$\mathbf{P}(E^c) = 1 - \mathbf{P}(E)$



Difference rule: **If** $E \subseteq F$

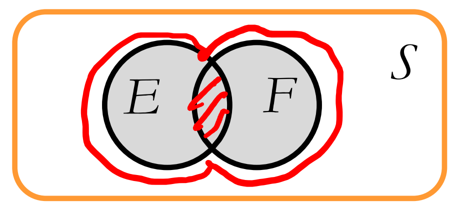
$\mathbf{P}(F \cap E^c) = \mathbf{P}(F) - \mathbf{P}(E)$

in particular, $\mathbf{P}(E) \leq \mathbf{P}(F)$



Inclusion-exclusion:

$\mathbf{P}(E \cup F) = \mathbf{P}(E) + \mathbf{P}(F) - \mathbf{P}(E \cap F)$



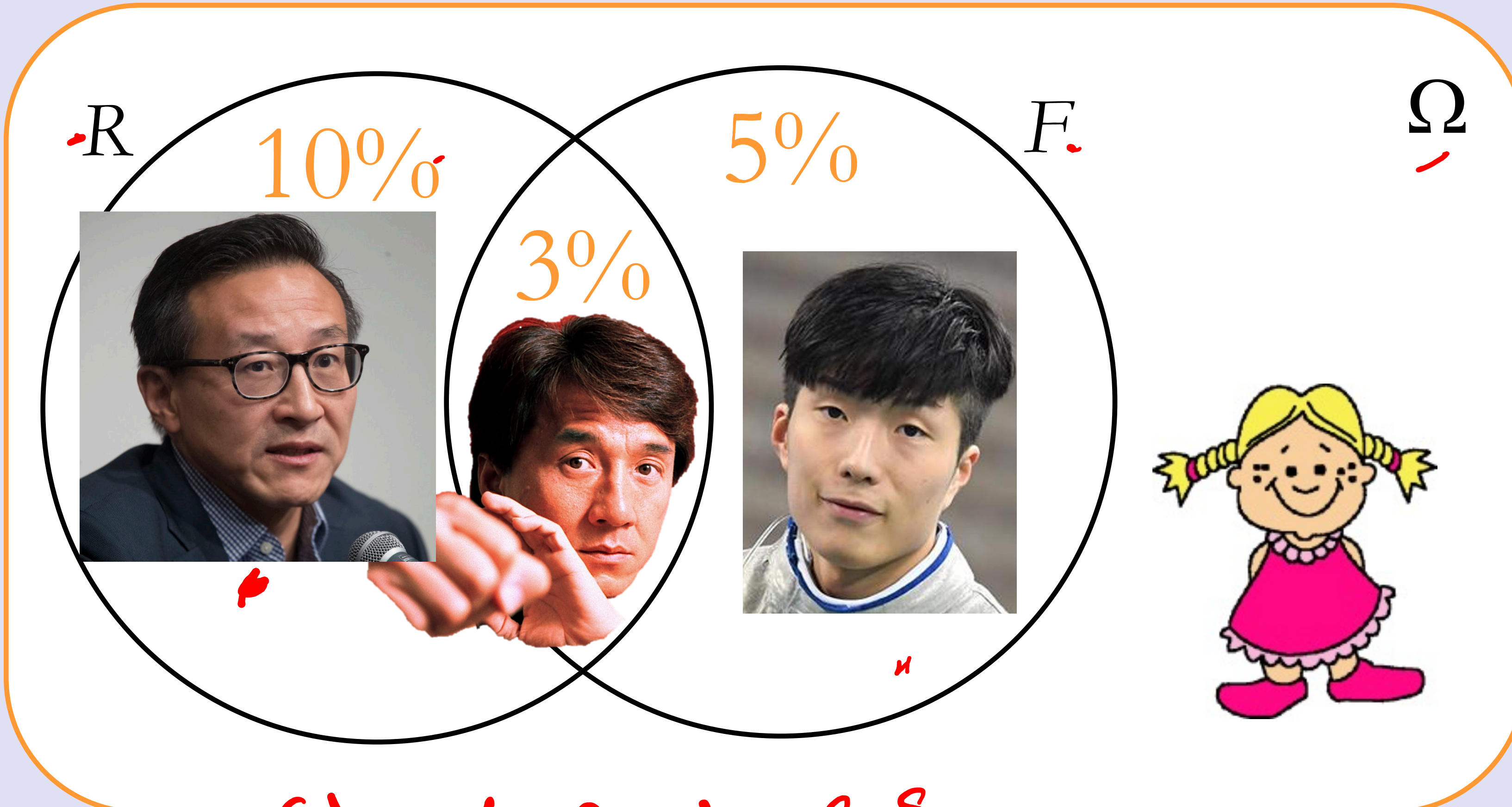
You can prove them using the axioms.

In some town 10% of the people are rich, 5% are famous, and 3% are rich and famous. For a random resident of the town what are the chances that:

(a) The person is not rich?

(b) The person is rich but not famous?

(c) The person is neither rich nor famous?

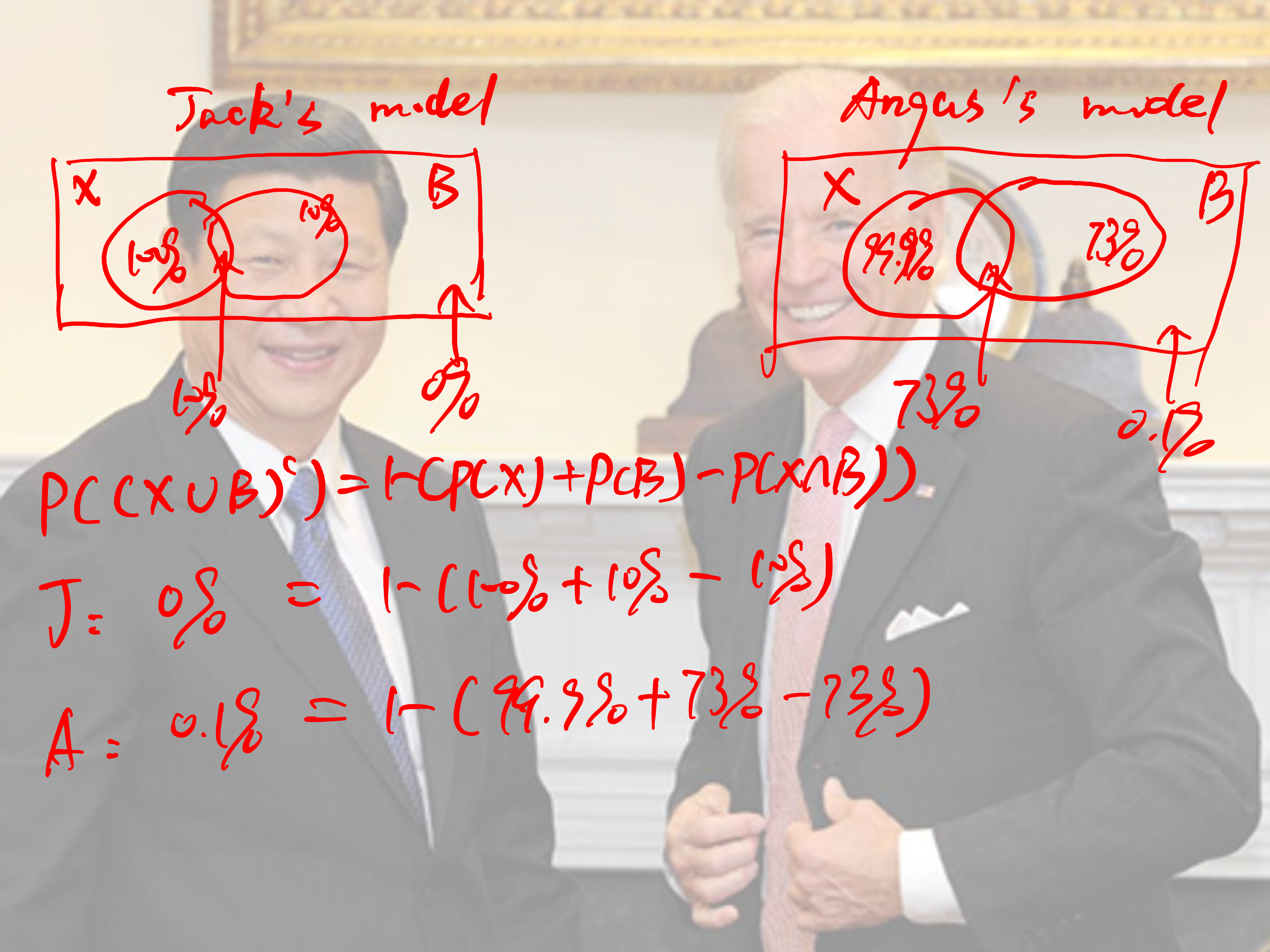


a. $P(R^c) = 1 - P(R) = 90\%$

b. $P(R \cap F^c) = P(R) - P(R \cap F) = 7\%$

c. $P((R \cup F)^c) = 1 - P(R \cup F) = 1 - (10\% + 5\% - 3\%)$

$= 88\%$



Jack's model

X

B

100%

10%

10%

0%

Angus's model

X

B

99.9%

73%

73%

0.1%

$$P((X \cup B)^c) = 1 - (P(X) + P(B) - P(X \cap B))$$

$$J: \; 0\% = 1 - (100\% + 10\% - 10\%)$$

$$A: \; 0.1\% = 1 - (99.9\% + 73\% - 73\%)$$

What are the chances that two people in this room have the same birthday?

A	B	C
< 10%	≈ 50%	> 90%

n students

$\Omega = \{1, 2, \cdots, 365\}^n$

E = " At least Two people have the same birthday".

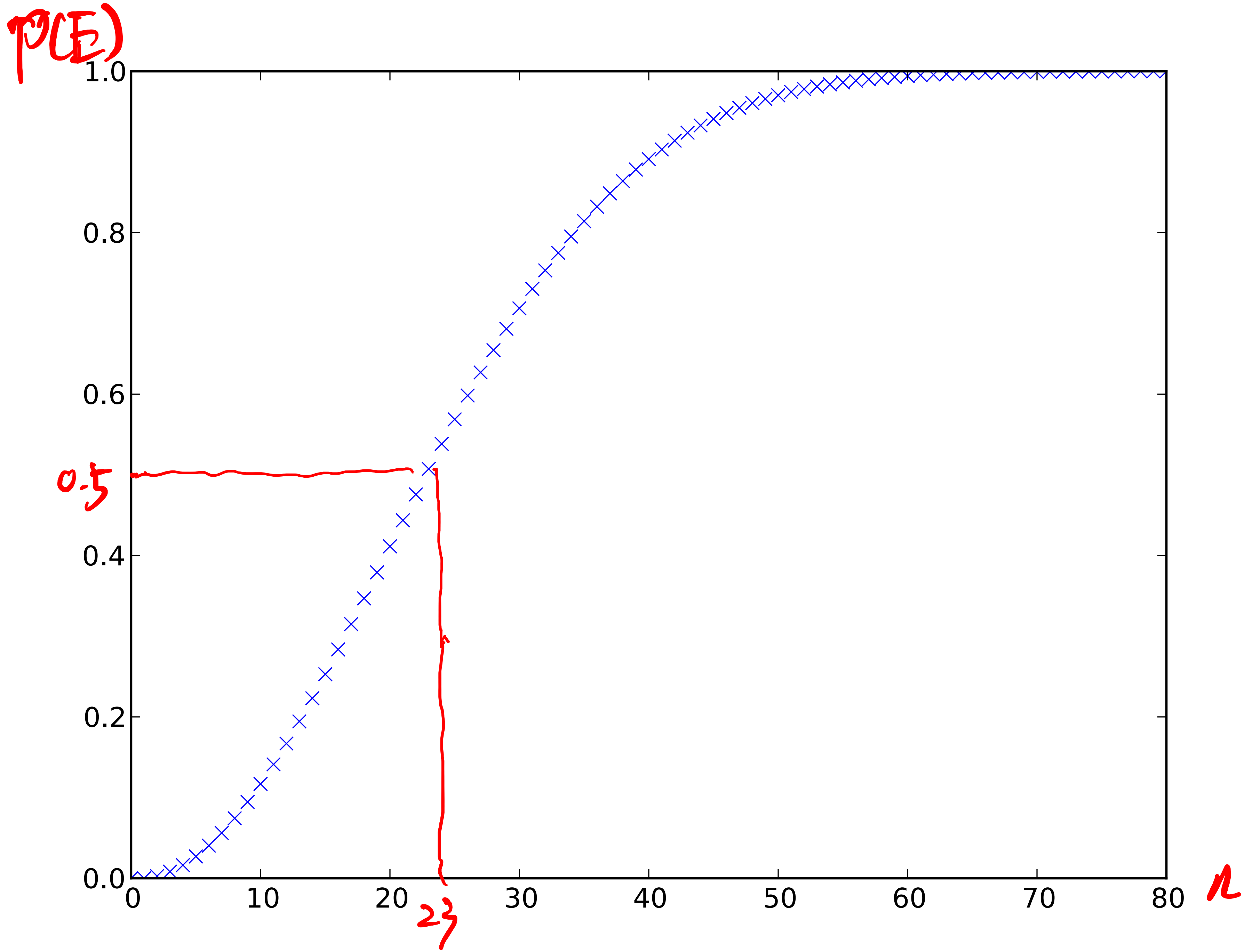
Eg.: $\omega = (31, 23, 174, \cdots, 55)$

$\omega' = (\cdots, 65, 65, , \cdots)$

$P(E) = 1 - P(E^c)$

$P(E^c) = \frac{|E^c|}{|\Omega|} = \frac{365 \cdot 364 \cdot 363 \cdots (365-n+1)}{365^n}$

equally likely outcomes



P(E)
1.0
0.8
0.6
0.4
0.2
0.0
0.5
0
10
20
30
40
50
60
70
80
23
n

Coins game

Toss 3 coins. You win if at least two come out heads.

$$S = \{\ \mathrm{HHH},\ \mathrm{HHT},\ \mathrm{HTH},\ \mathrm{HTT},\ \mathrm{THH},\ \mathrm{THT},\ \mathrm{TTH},\ \mathrm{TTT}\ \}$$

$$W = \{\ \mathrm{HHH},\ \mathrm{HHT},\ \mathrm{HTH},\ \mathrm{THH}\ \}$$

$$P(W) = \frac{|W|}{|S|} = \frac{1}{2}$$

Coins game

The first coin was just tossed and it came out heads. How does this affect your chances?

$$S = \{\ \mathrm{HHH},\ \mathrm{HHT},\ \mathrm{HTH},\ \mathrm{HTT},\ \cancel{\mathrm{THH}},\ \cancel{\mathrm{THT}},\ \cancel{\mathrm{TTH}},\ \cancel{\mathrm{TTT}}\ \}$$

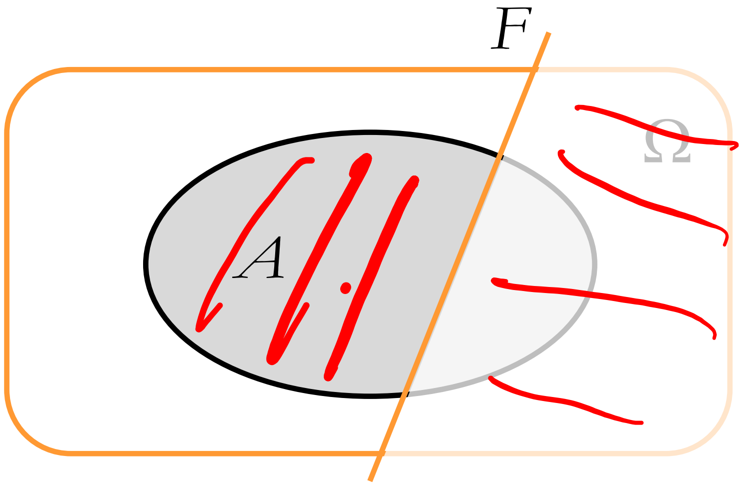
$$W = \{\ \boxed{\mathrm{HHH},\ \mathrm{HHT},\ \mathrm{HTH}},\ \cancel{\mathrm{THH}}\ \}$$

$$P(W|F) = \frac{3}{4}$$

$$P(W|F) = \frac{P(W \cap F)}{P(F)} = \frac{3/8}{1/2} = \frac{3}{4}$$

Conditional probability

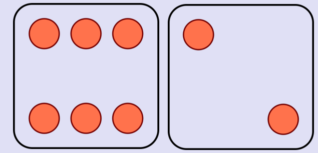
The conditional probability $P(A \mid F)$ represents the probability of event A assuming event F happened.



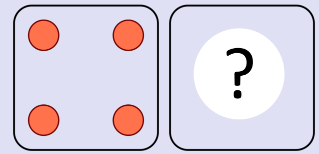
Conditional probabilities with respect to the reduced sample space F are given by the formula

$$\boxed{\mathbf{P}(A \mid F) = \frac{\mathbf{P}(A \cap F)}{\mathbf{P}(F)}}$$

Toss 2 dice. You win if the sum of the outcomes is 8. $= A$



The first dice toss is a 4. $= F$
Should you be happy?



11	12	13	14	15	16
21	22	23	24	25	26
31	32	33	34	35	36
41	42	43	44	45	46
51	52	53	54	55	56
61	62	63	64	65	66

A

P

$$P(A) = \frac{5}{36}$$

$$P(A|F) = \frac{P(A \cap F)}{P(F)}$$

$$= \frac{1/36}{6/36} = \frac{1}{6}$$

Now suppose you win if the sum is 7. $= B$
Your first toss is a 4. Should you be happy?
$= F$

11	12	13	14	15	16
21	22	23	24	25	26
31	32	33	34	35	36
41	42	43	44	45	46
51	52	53	54	55	56
61	62	63	64	65	66

$$P(B) = \frac{1}{6}$$

$$P(B/F) = \frac{P(B \cap F)}{P(F)}$$

$$= \frac{1}{6}$$

Properties of conditional probabilities

1. Conditional probabilities are probabilities:

$$\mathbf{P}(F \mid F) = 1$$

$$\mathbf{P}(A \cup B \mid F) = \mathbf{P}(A \mid F) + \mathbf{P}(B \mid F) \text{ if disjoint}$$

$P(A \cup B) = P(A) + P(B)$

2. Under equally likely outcomes,

$$\mathbf{P}(A \mid F) = \frac{\text{number of outcomes in } A \cap F / \Omega}{\text{number of outcomes in } F / \Omega}$$

Toss two dice. The $\boxed{\text{smaller value is a 2.}}^{F}$ What is the probability that the larger value is 1, 2, ..., 6?

L_i = "larger value is i", $i \in \{1, 2, \cdots 6\}$

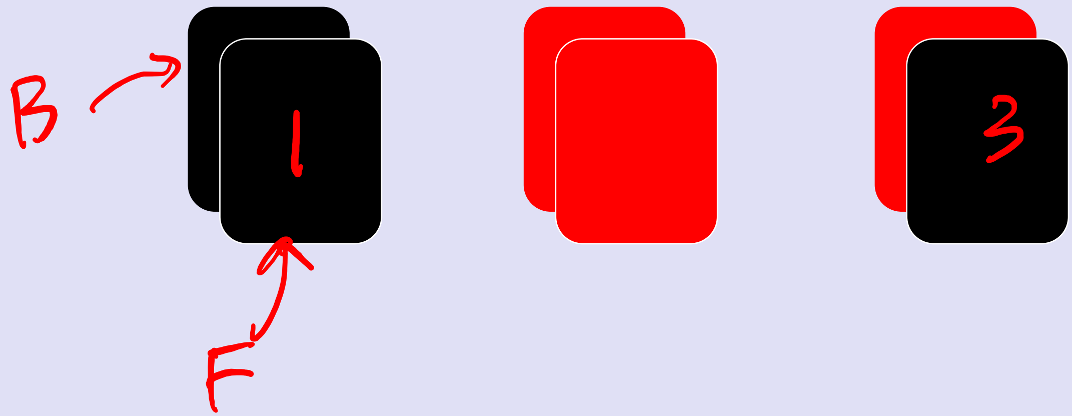
Ω

11	12	13	14	15	16
21	22	23	24	25	26
31	32	33	34	35	36
41	42	43	44	45	46
51	52	53	54	55	56
61	62	63	64	65	66

$$P(L_i \mid F) = \frac{|L_i \cap F|}{|F|}$$

i	1	2	3	4	5	6
$P(L_i \mid F)$	0	1/9	2/9	2/9	2/9	2/9

$\Sigma = 1$

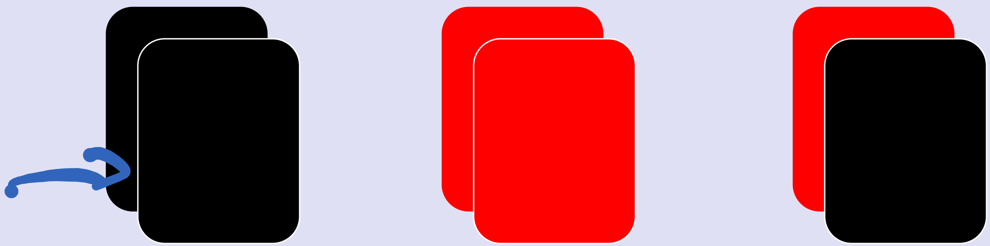


F

You draw a random card and see a black side. What are the chances the other side is red?

A: $1/4$ **B:** $1/3$ **C:** $1/2$

0 80%

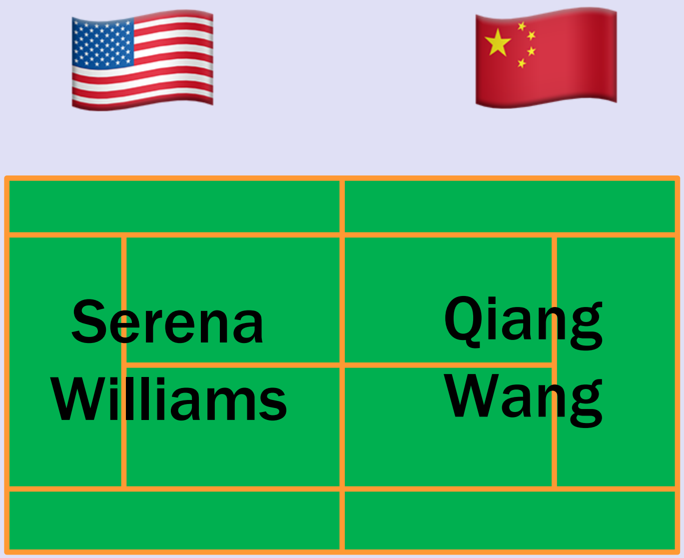


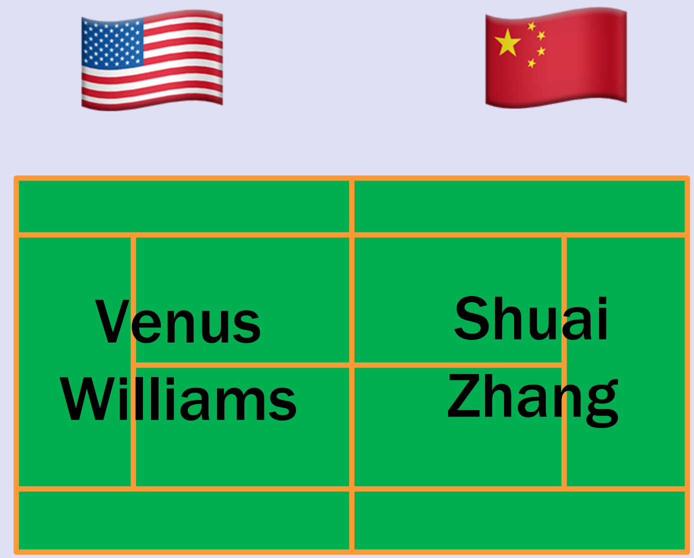
$\Omega = \{1F, 1B, 2F, 2B, 3B, 3F\}$

$F = \{1F, 1B, 3F\}$

$A = \{2F, 2B, 3F\}$ A: other side is red.

$$P(A|F) = \frac{|A \cap F|}{|F|} = \frac{1}{3}$$





$\mathbf{P}(\text{Serena wins}) = 2/3$

$\mathbf{P}(\text{Venus wins}) = 1/2$

$\mathbf{P}($🇨🇳 2: 🇺🇸 $0) = 1/4$

FINAL SCORE

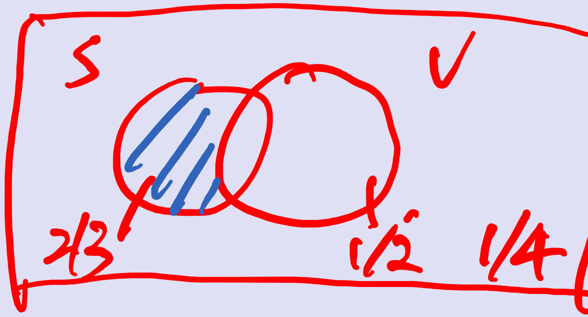
🇨🇳	1
🇺🇸	1

What is the probability Serena won her game?

$$\Omega = \{\overset{S}{W}\overset{V}{W},\ WL,\ LW,\ LL\}$$

$$S = \{WW,\ WL\}$$

$$F = \{WL,\ LW\}$$



$$\Rightarrow P(S|F) = \frac{P(S\cap F)}{P(F)} = \frac{P(\{W,L\})}{P(\{WL,\ LW\})}$$

$$P(\{WL\}) = P(S\cap V^c) = P(S\cup V) - P(V)$$

$$= (1-1/4) - \frac{1}{2} = \frac{1}{4}$$

$$P(\{LW\}) = P(V\cap S^c) = P(V\cup S) - P(S) = \frac{3}{4} - \frac{2}{3} = \frac{1}{12}$$

$$\Rightarrow P(S|F) = \frac{1/4}{1/4 - 1/12} = \frac{3}{4}$$

The multiplication rule

Using the formula $\mathbf{P}(E_2 | E_1) = \dfrac{\mathbf{P}(E_1 \cap E_2)}{\mathbf{P}(E_1)}$

We can calculate the probability of intersection

$$\mathbf{P}(E_1 \cap E_2) = \mathbf{P}(E_1) \cdot \mathbf{P}(E_2 | E_1)$$

In general

$$\mathbf{P}(E_1 \cap \ldots \cap E_n) = \mathbf{P}(E_1)\, \mathbf{P}(E_2 | E_1) \ldots \mathbf{P}(E_n | E_1 \cap \ldots \cap E_{n-1})$$

An urn has 10 white balls and 20 black balls. You draw two at random. What is the probability that their colors are different?

6 local and 3 non-local students are randomly split into three groups of 3. What is the probability that each group has a non-local?

VANILLA PROB. :

Ω = Partition 9 students into 3 sets

Equal likelihood :

E = Each set has 1 NL student.

$$P(E) = \frac{|E|}{|\Omega|} = \frac{3! \cdot \frac{6!}{2!\,2!\,2!}}{\frac{9!}{3!\,3!\,3!}} = \frac{9}{28}$$

6 local and 3 non-local students are randomly split into three groups of 3. What is the probability that each group has a non-local?

COND. PROB. :

E_1 = " NL_1 is the only nl in his group."

E_2, E_3.

$$P(E_1 \cap E_2 \cap E_3) = P(E_1) \cdot P(E_2 | E_1) \cdot P(E_3 | E_2, E_1)$$

$$= \frac{6}{8} \cdot \frac{5}{7} \cdot \left(\frac{4}{5} \cdot \frac{3}{4}\right) \cdot 1 = \frac{9}{28}$$