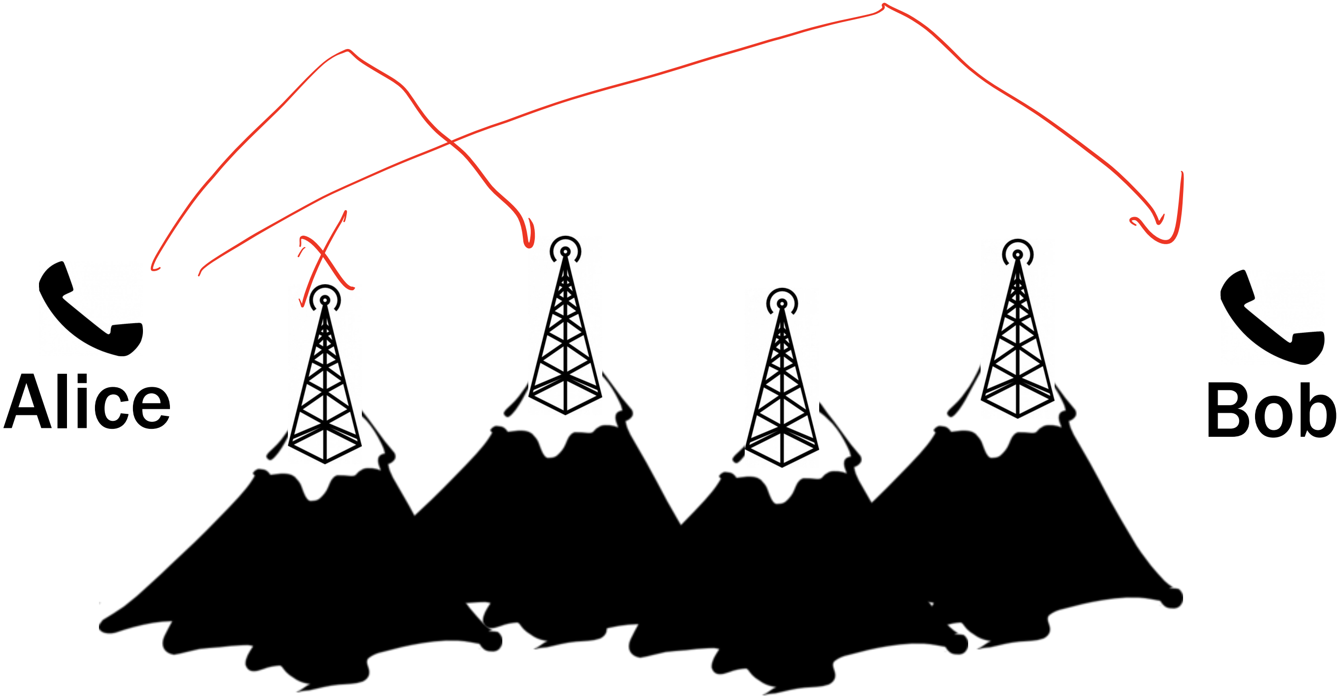
ENGG 2760A / ESTR 2018: Probability for Engineers

1. Probability and Counting

Prof. Hong Xu

Credit to Prof. Andrej Bogdanov



If 2 consecutive antennas broken,
NO CONNECTION

Can Alice and Bob make a connection?

In uncertain situations we want a number saying how likely something is

probability

The cheat sheet

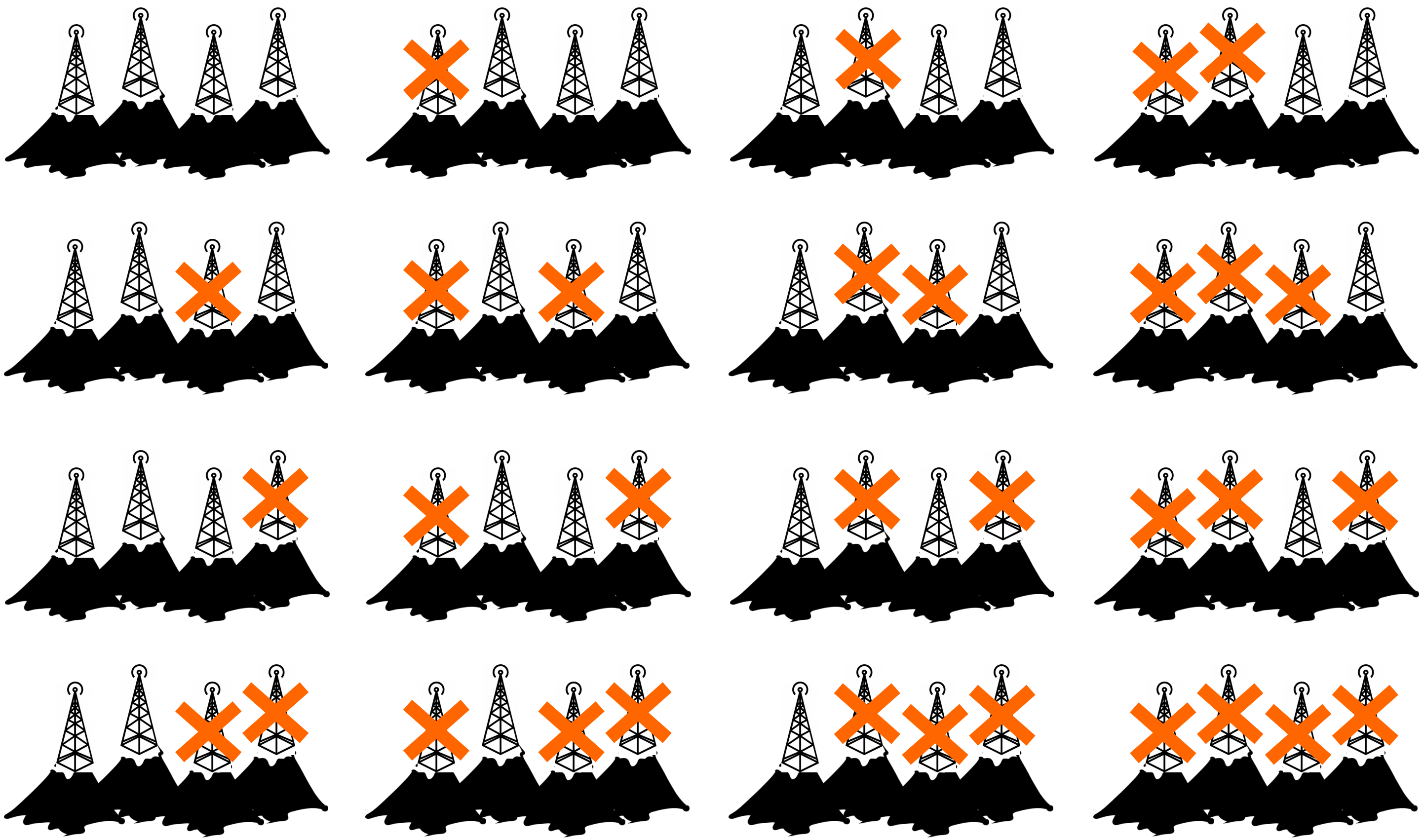
1. Specify all possible outcomes

2. Identify event(s) of interest

3. Assign probabilities

4. Shut up and calculate!

$2^4 = 16$



Sample spaces

The sample space is the set of all possible outcomes.

Examples

coin toss

$\Omega = \{H, T\}$

dice

$\Omega = \{1, 2, 3, 4, 5, 6\}$

Sample spaces

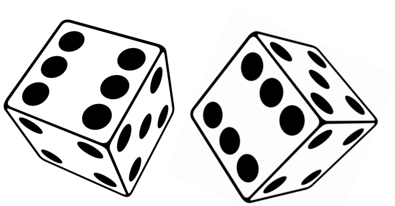
three coin tosses

product set

2^3

$$\Omega = \{ \texttt{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT} \}$$

$= \{H, T\} \times \{H, T\} \times \{H, T\}$



a pair of dice

$6^2 = 36$

$$\Omega = \{ \texttt{11, 12, 13, 14, 15, 16, 21, 22, 23, 24, 25, 26, 31, 32, 33, 34, 35, 36, 41, 42, 43, 44, 45, 46, 51, 52, 53, 54, 55, 56, 61, 62, 63, 64, 65, 66} \}$$

Events

An event is a subset of the sample space.

$$\Omega = \{ \text{HHH, HHT, HTH, HTT} \\ \text{THH, THT, TTH, TTT} \}$$

Exactly two heads: A = {HHT, HTH, THH}

No consecutive tails: B = {HHH, HHT, HTH, THH, THT}

Discrete probability

A probability model is an assignment of probabilities to elements of the sample space.

Probabilities are nonnegative and add up to one.

Example: three fair coins

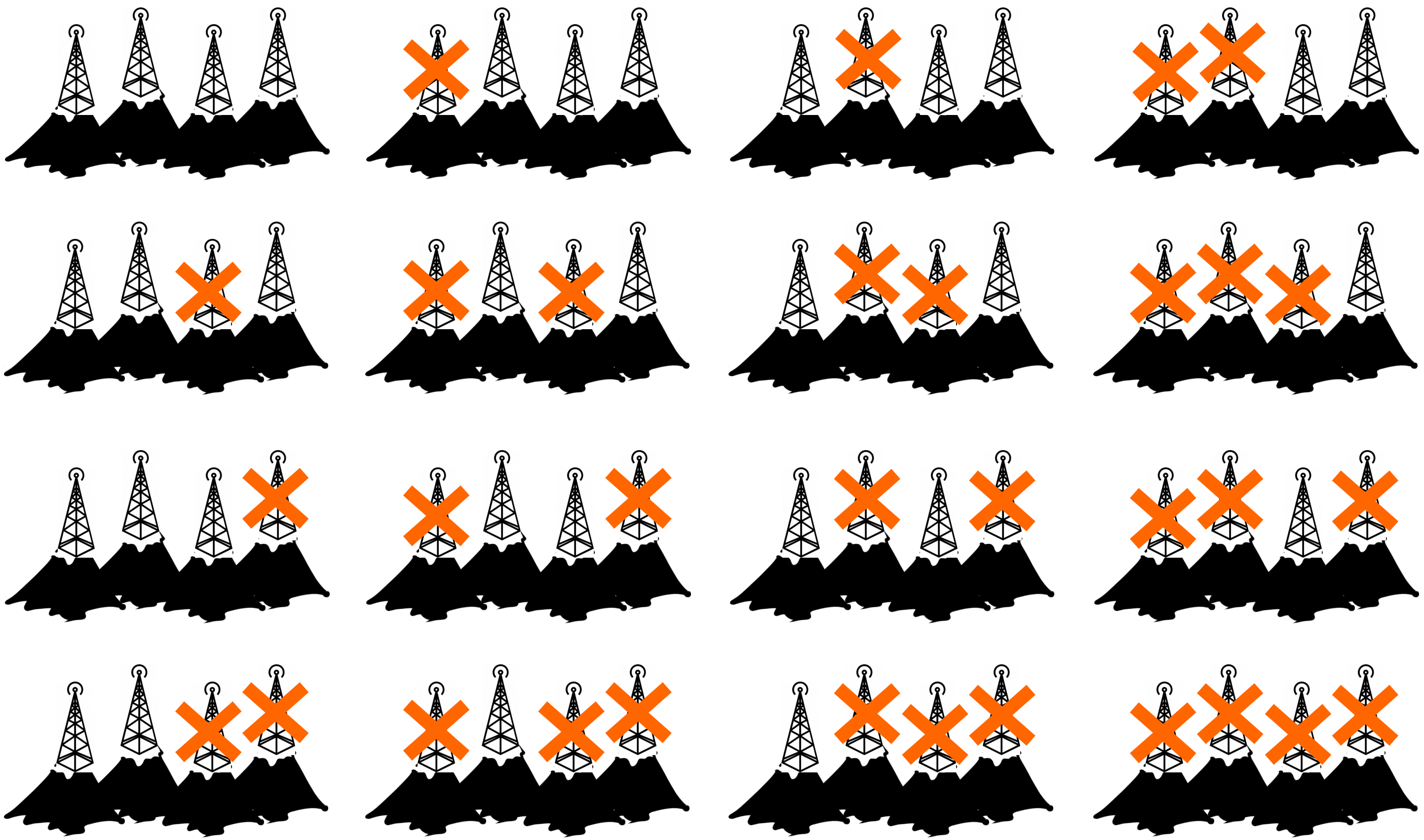
uniform prob. model

$$\Omega = \{ \mathrm{HHH}, \mathrm{HHT}, \mathrm{HTH}, \mathrm{HTT}, \mathrm{THH}, \mathrm{THT}, \mathrm{TTH}, \mathrm{TTT} \}$$

1/8 1/8 1/8 1/8

Example:

1/16 1/16 1/16 ...



Calculating probabilities

Exactly two heads:

$A = \{ \texttt{HHT}, \texttt{HTH}, \texttt{THH} \}$

$\mathbf{P}(A) = \frac{3}{8}$

No consecutive tails:

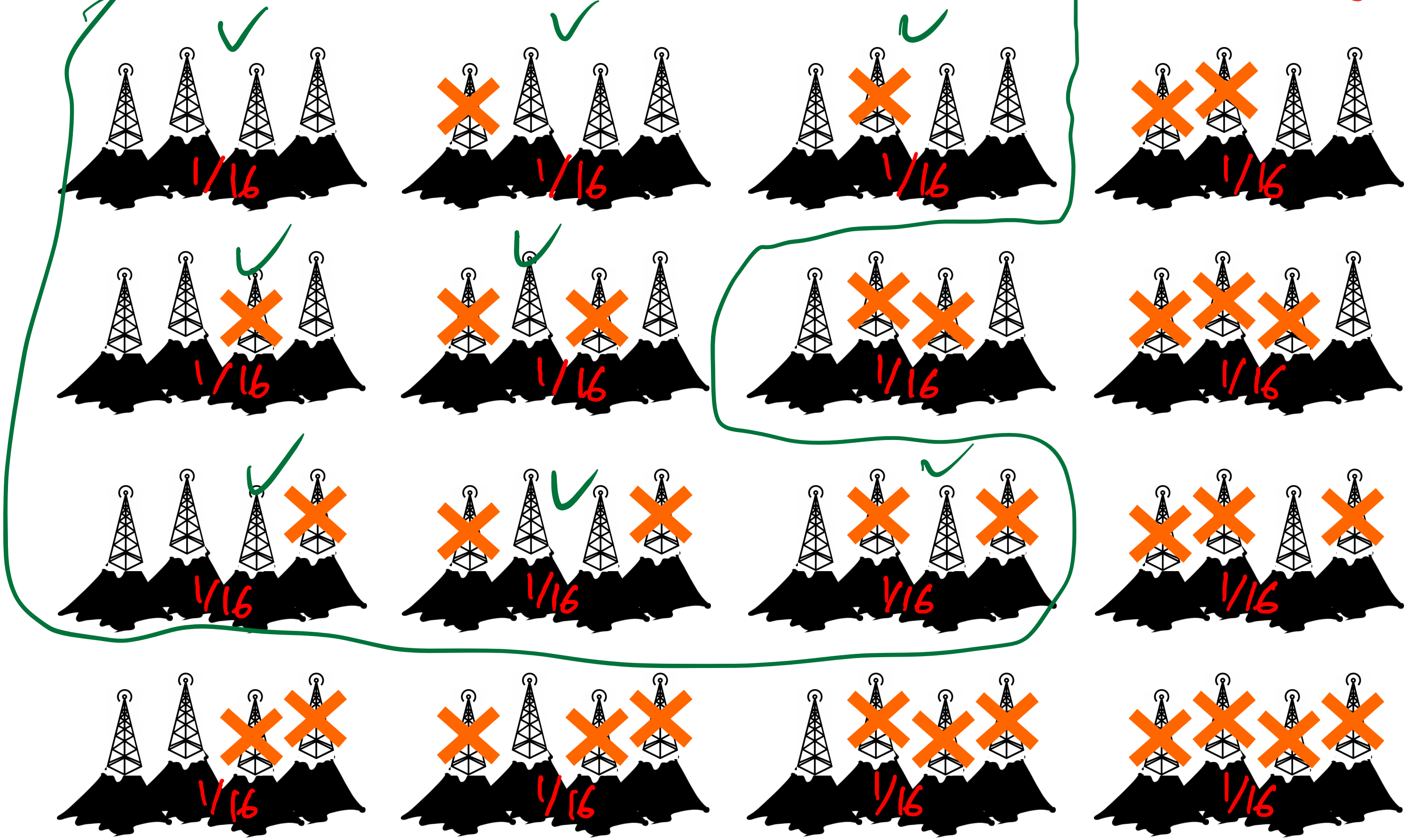
$B = \{ \texttt{HHT}, \texttt{HTH}, \texttt{THH}, \texttt{THH}, \texttt{THT} \}$

$\mathbf{P}(B) = \frac{5}{8}$

Example:

E = no 2 consecutive broken antennas

$P(E) = \frac{8}{16}$



Uniform probability law

If all outcomes are equally likely, then...

$$\mathbf{P}(A) = \frac{\text{number of outcomes in } A}{\text{number of outcomes in } \Omega}$$

...and probability amounts to counting.

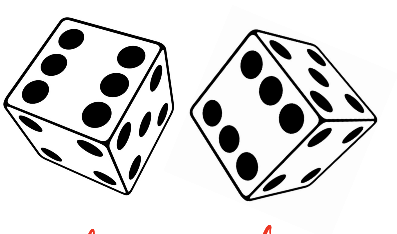
Product rule for counting

Experiment 1 has n possible outcomes. independent

Experiment 2 has m possible outcomes.

Together there are nm possible outcomes.

Examples



6×6

$2 \times 2 \times 2$

6×2

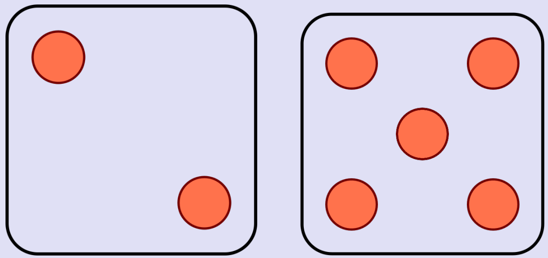
Generalized product rule

Experiment 1 has n possible outcomes.

For each such outcome,
experiment 2 has m possible outcomes.

Together there are nm possible outcomes.

You toss two dice. How many ways are there for the two dice to come out different?



A
15 ways

B
25 ways

C
30 ways

Solution 1:

$36 - 6 = 30$

$\cancel{11}$, 12, 13, 14, 15, 16,
21, $\cancel{22}$, 23, 24, 25, 26,
31, 32, $\cancel{33}$, 34, 35, 36,
41, 42, 43, $\cancel{44}$, 45, 46,
51, 52, 53, 54, $\cancel{55}$, 56,
61, 62, 63, 64, 65, $\cancel{66}$,

First dice : 6

Solution 2:

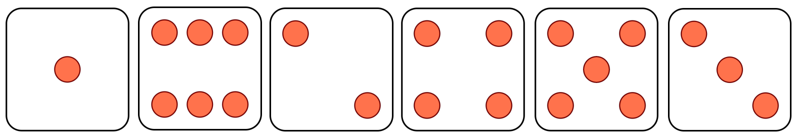
AFTER this,

second dice: 5

G.P. Rule : $6 \times 5 = 30$

Permutations

You toss six dice. How many ways are there for all six to come out different?



$6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$

The number of permutations of n different objects is

$n! = n \times (n-1) \times \cdots \times 1$

$0! = 1$

Equally likely outcomes

For two dice, the chance both come out different is

$\Omega = \{1, 2, \cdots, 6\}^2$ $\quad |\Omega| = 36$

$A = \{(a, b) : a \neq b\}$ $\quad |A| = 30$

$$P(A) = \frac{|A|}{|\Omega|} = \frac{30}{36} \approx 0.8366$$

For six dice, the chance they all come out different is

$\Omega = \{1, 2, \cdots, 6\}^6$

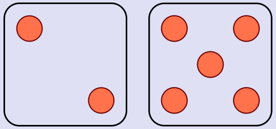
$B = \{(f_1, f_2, \cdots) : \text{All different}\}$

$$P(B) = \frac{|B|}{|\Omega|} = \frac{6!}{6^6} \doteq 0.015$$

$\Omega = \{1, \cdots, 6\}^2$

Toss two fair dice. What are the chances that...

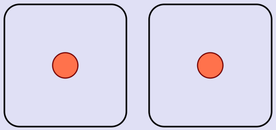
(a) The sum is equal to 7?



$A = \{(a,b) : a+b=7\}$

$|A| = 6 \times 1 = 6 \Rightarrow P(A) = \frac{6}{36}$

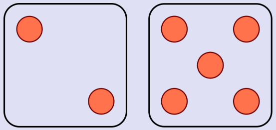
(b) The sum is even?



$B = \{(a,b) : a+b \text{ is even}\}$

$|B| = 6 \times 3 = 18, \quad P(B) = \frac{18}{36}$

(c) The second one is bigger?



$|C| = 15.$

11,	12,	13,	14,	15,	16,
21,	22,	23,	24,	25,	26,
31,	32,	33,	34,	35,	36,
41,	42,	43,	44,	45,	46,
51,	52,	53,	54,	55,	56,
61,	62,	63,	64,	65,	66

15

30/2

There are 3 brothers. What is the probability that

(a) All have birthdays on the same day of the week?

M T W T F S S

$\Omega = \{Mo, Tu, \dots, Su\}^3$

$A = \{(Mo, Mo, Mo), (Tu, \dots), \dots\}$

Equally likely outcomes:

$$P(A) = \frac{|A|}{|\Omega|} = \frac{7}{7^3} = \frac{1}{49}$$

(b) All have birthdays on different days of the week?

 M T  W T F  S S

$B = \{(a,b,c) : \text{all different}\}$

$|B| = 7 \cdot 6 \cdot 5$

$P(B) = \dfrac{7 \cdot 6 \cdot 5}{7^3} = \dfrac{30}{49}$

(c) **Exactly one** birthday is on the weekend?

T

T

F

S

$$C = C_A \cup C_B \cup C_C$$

$$C_A = \{(a, b, c) : a \in \{Sat, Sun\},\ b, c \in \{weekdays\}\}$$

$$C_B,\ C_C$$

$$|C| = |C_A| + |C_B| + |C_C|,$$

$$|C_A| = 2 \times 5 \times 5,\ C_B,\ C_C,$$

$$P(C) = \frac{3 \times 2 \times 5 \times 5}{7^3} \doteq 0.437$$

a classical, b jazz, and c pop CDs are arranged at random. What is the probability that all CDs of the same type are contiguous?

Eg, $C_1 C_2$ $J_1 J_2 J_3$ $P_1 P_2$, $C_1 J_2 P_1 C_2 P_2 J_1 J_3$ ✗

$J_3 J_2 J_1 C_1 C_2 P_2 P_1$ ✓

Ω = All permutations

E = CDs of same type are contiguous

$|\Omega| = (a+b+c)!$

$|E| = 3! \cdot a! \cdot b! \cdot c!$.

$$P(E) = \frac{6 \cdot a! \cdot b! \cdot c!}{(a+b+c)!}$$

Partitions

n choose k

$$\underline{\binom{n}{k}} = \frac{n!}{k!\,(n-k)!}$$

is the number of $\underline{\text{size-}k}$ subsets of a size-n set

$\{1,2,3,4,5\}$, $k=2$: 2-subsets: $\{1,2\}$, $\{3,4\}$,

$n=5$, $\{3,5\}$, ..., ,

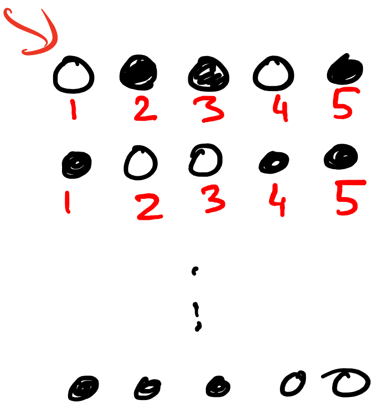
$$\binom{5}{2} = \frac{5!}{2!\,(5-2)!} = \frac{5\cdot 4}{2} = 10.$$

Partitions and arrangements

$$\binom{n}{k}$$

size-k subsets of a size-n set

arrangements of k white and $n-k$ black balls



$\to (1,4)$

$\to (2,3)$

$\vdots$

$\to (4,5)$

$\binom{5}{2} = 10$

Partitions and arrangements

$$\binom{n}{n_1, \ldots, n_t}$$

partitions of a size-n set into t subsets of sizes $n_1, \ldots, n_t$

E.g., 100 students, divide into 4 sessions, each with 25 students.

$$\binom{100}{25} \cdot \binom{75}{25} \cdot \binom{50}{25} \cdot \binom{25}{25} \Rightarrow \binom{n}{n_1, \ldots, n_t} = \frac{n!}{n_1!\, n_2! \cdots n_t!}$$

$T_1 \quad T_2 \quad T_3 \quad T_4$

$$= \frac{100!}{25!\, \cancel{75!}} \cdot \frac{\cancel{75!}}{25!\, \cancel{50!}} \cdot \cancel{\cdots} = \frac{100!}{25!\, 25!\, 25!\, 25!}$$

Partitions and arrangements

$$\binom{n}{n_1, \ldots, n_t} = \frac{(n_1+n_2+\cdots+n_t)!}{n_1!\, n_2! \cdots n_t!}$$

arrangements of n_1 red, n_2 blue, ..., n_t green balls

An urn has 10 white balls and 20 black balls. You draw two at random (without replacement). What is the probability that their colors are different?

Imagine drawing 30 balls, just look at the first two:

$$\Omega = \binom{30}{10}$$

A = subset so that the seq. start with

●○, or ○●

$$|A| = 2 \cdot \binom{28}{9} \; ; \; \binom{28}{19}$$

$$\underline{\frac{10}{30} \cdot \frac{20}{29} + \frac{20}{30} \cdot \frac{10}{29}}$$