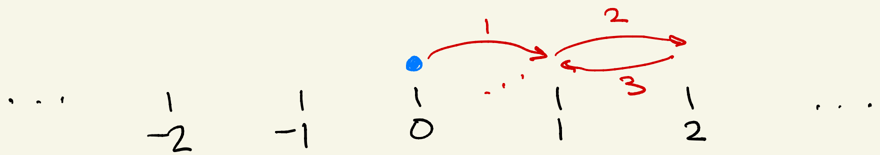
RANDOM WALK ON A LINE



<u>RULE</u>: AT EACH STEP • EQUALLY LIKELY TO MOVE 1 STEP LEFT & 1 STEP RIGHT.

<u>EVENT B_n</u>: "BACK AT ORIGIN AFTER n STEPS".

WHAT IS $P(B_n)$?

BY THE EQUALLY LIKELY OUTCOMES FORMULA,

$$P(B_n) = \frac{|B_n|}{|\Omega_n|} = \frac{\text{NUMBER OF } n\text{-STEP LOOPS}}{\text{NUMBER OF } n\text{-STEP PATHS}}.$$

THE DENOMINATOR IS 2^n. THE NUMERATOR IS 0 WHEN n IS ODD AND $\binom{2n}{n}$ WHEN n IS EVEN SO $P(B_n) = \binom{2n}{n}/2^n$ (FOR n EVEN). IN PARTICULAR $P(B_{100}) \approx 7.95\%$.

IF YOU PLOT $\log P(B_n)$ VERSUS $\log n$ YOU GET ALMOST A LINE WITH SLOPE $-1/2$. THIS SAYS $P(B_n)$ "SHOULD GROW LIKE" $1/\sqrt{n}$, SO WE'D EXPECT $P(B_{10000})$ TO BE ABOUT 1%. THIS IS NOT SUCH A SMALL PROBABILITY GIVEN THE LENGTH OF THE WALK.

NOW CONSIDER ANOTHER EVENT

F_n: "BACK AT ORIGIN FOR THE FIRST TIME AT STEP n".

THEN $P(F_n) = |F_n|/2^n$ SO WE NEED TO CALCULATE $|F_n|$. LET'S DO SOME SMALL EXAMPLES.

n	2	4	6	8
F_n	2	2	4	10

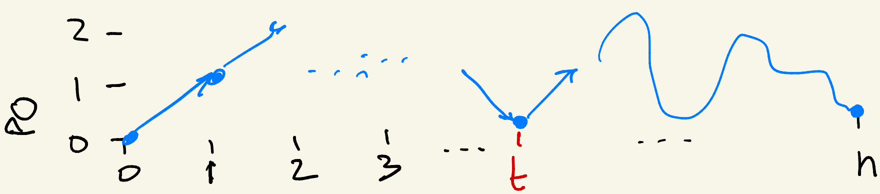
NOT CLEAR WHAT THE PATTERN IS. IT IS A BIT EASIER TO THINK ABOUT

P_n = "BACK AT THE ORIGIN AT STEP n & STAY NONNEGATIVE IN THE MEANTIME"

THEN $|F_n| = 2|P_{n-2}|$, SO

n	0	2	4	6
P_n	1	1	2	5

INSTEAD OF AIMING FOR A FORMULA LET'S SEE HOW WE WOULD GO ABOUT COMPUTING THESE NUMBERS. LET'S CALL THE OBJECTS OF INTEREST "POSITIVE LOOPS". A POSITIVE LOOP LOOKS LIKE THIS:



AT SOME TIME t THE POSITIVE LOOP WILL COME BACK TO 0 FOR THE FIRST TIME AND THEN IT WILL BOUNCE BACK.

SINCE STEP 1 HAS TO BE ↗ AND STEP $t-1$ HAS TO BE ↘ AND THE POSITION MUST STAY <u>STRICTLY POSITIVE</u> IN THE MEANTIME, THE NUMBER OF POSSIBLE SUB-PATHS BETWEEN 0 AND t IS P_{t-2}. THE NUMBER OF POSSIBLE SUBPATHS FROM t TO n IS P_{n-t}, SO BY THE PRODUCT RULE THE NUMBER OF PATHS FOR A GIVEN VALUE OF t IS $P_{t-2} \cdot P_{n-t}$. AS t CAN BE ANY (EVEN) NUMBER BETWEEN 0 AND n WE GET

$$|P_n| = |P_0| \cdot |P_{n-2}| + |P_2| \cdot |P_{n-4}| + \dots + |P_{n-2}| \cdot |P_0|.$$

THIS TYPE OF EQUATION IS CALLED A <u>RECURRENCE</u>: IT TELLS US HOW TO CALCULATE $|P_n|$ FOR ARBITRARY n STARTING FROM $|P_0| = 1$.

SINCE $P(F_n) = \frac{|F_n|}{2^n} = \frac{2|P_{n-2}|}{2^n}$, WE CAN CALCULATE ON THE COMPUTER THAT $P(F_{100}) \approx 0.08\%$.

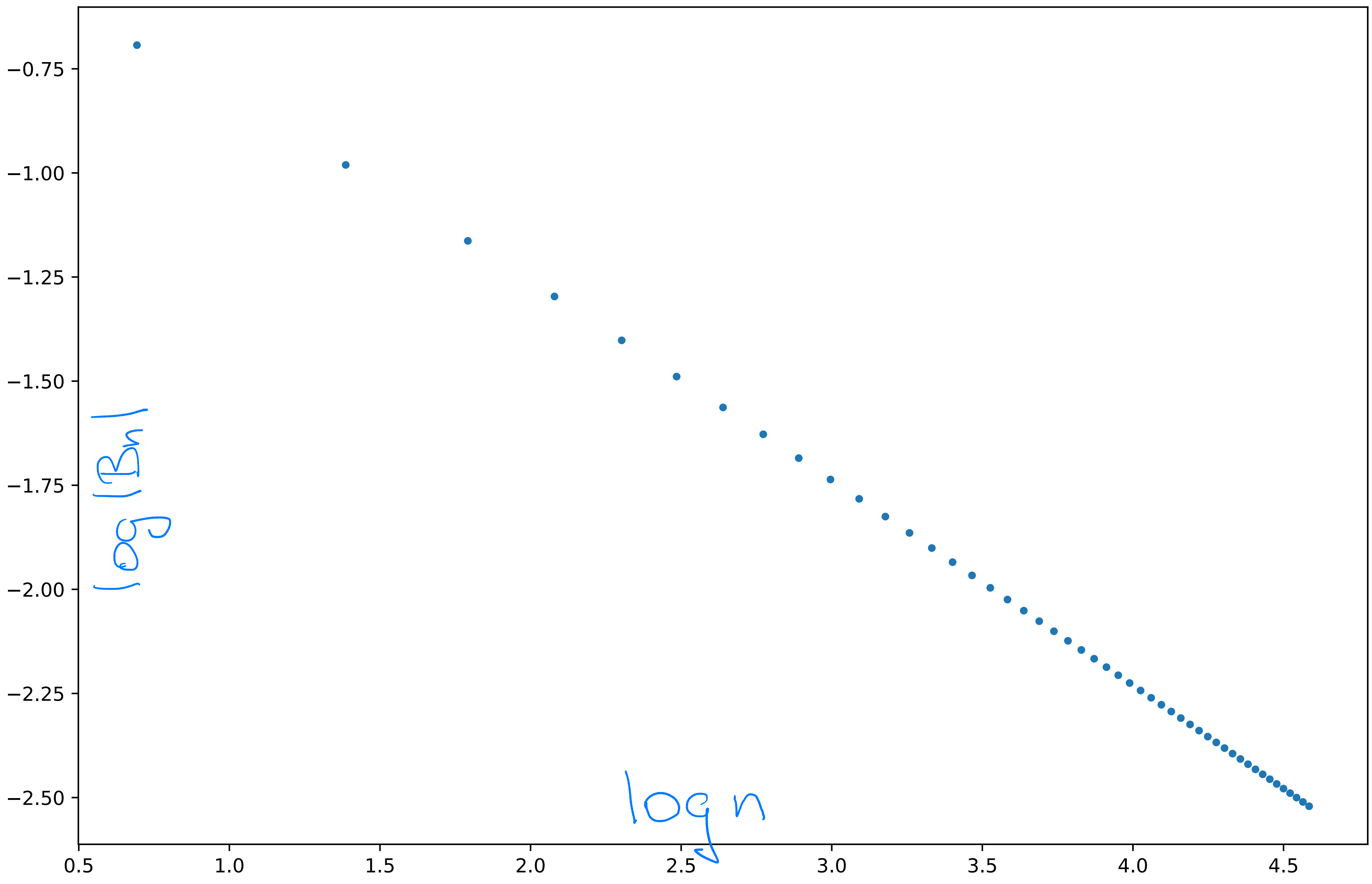
FINALLY, LET'S CONSIDER THE EVENT

R_n: "BACK AT ORIGIN <u>AT SOME POINT</u> WITHIN FIRST n STEPS."

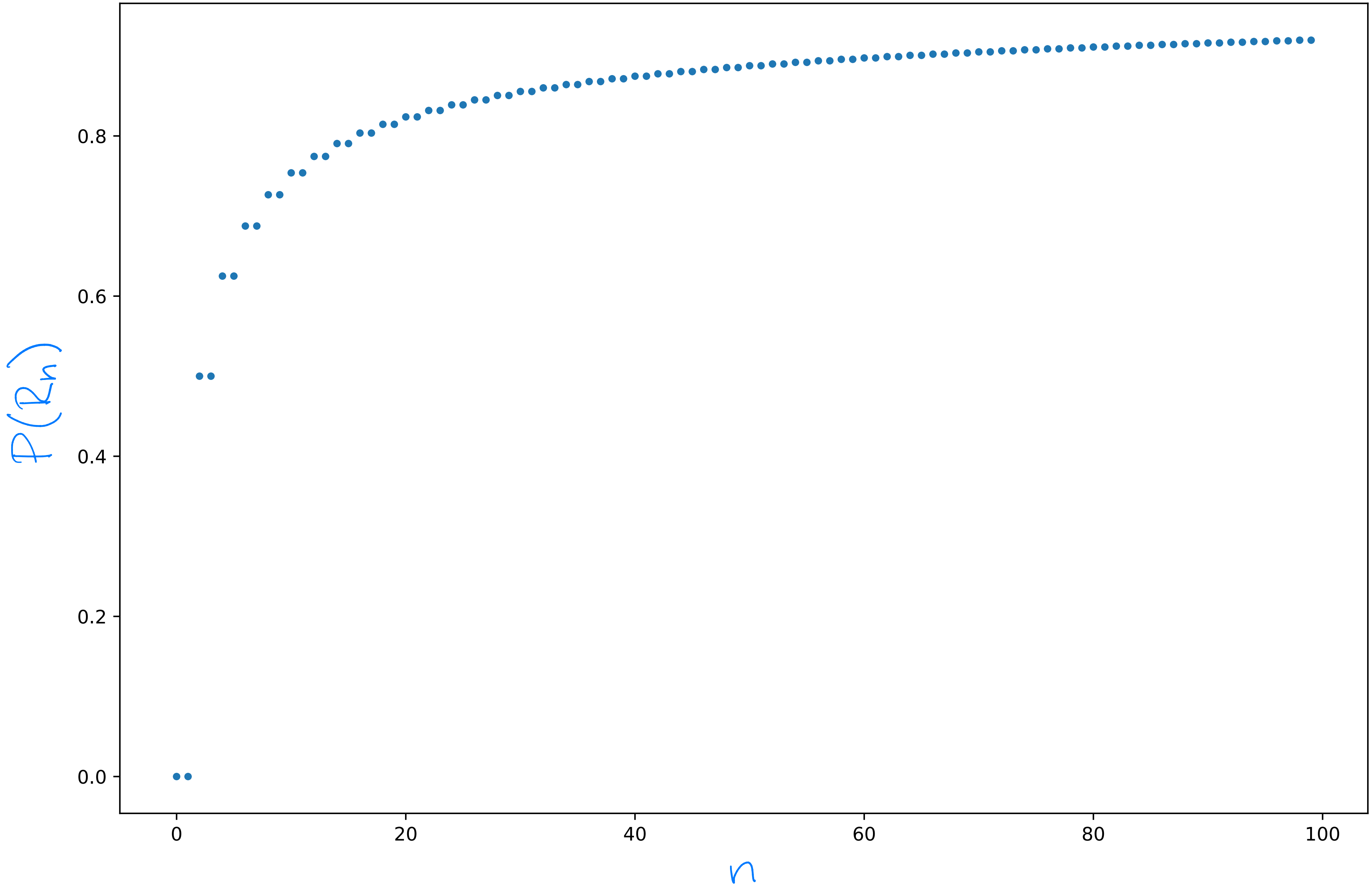
THEN R_n IS A <u>DISJOINT UNION</u> OF $F_1, F_2, \dots, F_n$ AND BY THE AXIOMS OF PROBABILITY (NEXT LECTURE) WE CAN CALCULATE

$$P(R_n) = P(F_1) + \dots + P(F_n).$$

IF YOU PLOT $P(R_n)$ VERSUS n YOU NOTICE IT APPROACHES 1: GIVEN ENOUGH STEPS, THE PARTICLE IS ALMOST SURE TO HAVE COME BACK HOME.



−0.75
−1.00
−1.25
−1.50
−1.75
−2.00
−2.25
−2.50
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
4.5
log n
log |B_n|



0.0
0.2
0.4
0.6
0.8
0
20
40
60
80
100
P(Rn)
n