1. A six people committee is chosen at random from 5 girls and 15 boys. Conditioned on there being at least one girl on the committee, what is the expected number and the variance of boys?

   Solution: Let G be the (unconditional) number of girls on the committee. By the conditional expectation formula, we have

   $$\mathrm{E}[G] = \mathrm{E}[G|G=0]\,\mathrm{P}(G=0) + \mathrm{E}[G|G>0](1-\mathrm{P}(G=0)).$$

   By linearity of expectation, $\mathrm{E}[G] = \mathrm{E}[G_1] + \mathrm{E}[G_2] + \mathrm{E}[G_3] + \mathrm{E}[G_4] + \mathrm{E}[G_5] + \mathrm{E}[G_6]$, where $G_i, i = \{1,2,3,4,5,6\}$ are random events of whether the i^{th} commitee member is a girl. $\mathrm{E}[G] = 6 \cdot (5/20) = 3/2$. On the other hand, $G = 0$ occurs only when all six members are boys, so by the product formula:

   $$\mathrm{P}(G=0) = \frac{15}{20} \cdot \frac{14}{19} \cdot \frac{13}{18} \cdot \frac{12}{17} \cdot \frac{11}{16} \cdot \frac{10}{15} \approx 0.129.$$

   Therefore :

   $$\mathrm{E}[G|G>0] = \frac{\mathrm{E}[G]}{1-\mathrm{P}(G=0)} \approx \frac{3/2}{1-0.129} \approx 1.722.$$

   If B is the number of boys then $B + G = 6$ so by linearity of expectation again

   $$\mathrm{E}[B|G>0] = 6 - \mathrm{E}[G|G>0] \approx 4.278.$$

x	1	2	3	4	5
$\mathrm{P}(B \mid G>0)$	$\frac{3}{6751}$	$\frac{105}{6751}$	$\frac{910}{6751}$	$\frac{2730}{6751}$	$\frac{3003}{6751}$

   $\mathrm{Var}(B \mid G>0) = \mathrm{E}(B^2 \mid G>0) - \mathrm{E}(B \mid G>0)^2 \approx 0.569.$

2. Let X and Y be independent random variables with PMFs $\mathrm{P}(X=1) = \mathrm{P}(X=2) = \mathrm{P}(X=3) = \mathrm{P}(X=4) = 1/4$ and $\mathrm{P}(Y=6) = \mathrm{P}(Y=7) = 1/2$. Let $M = X + Y$ and $N = Y - X$.

   (a) What is the PMF of M given N?

   Solution: We first calculate the joint PMF $\mathrm{P}_{MN}(m,n)$: of M and N:

$m \backslash n$	2	3	4	5	6
7	0	0	0	$\frac{1}{8}$	0
8	0	0	$\frac{1}{8}$	0	$\frac{1}{8}$
9	0	$\frac{1}{8}$	0	$\frac{1}{8}$	0
10	$\frac{1}{8}$	0	$\frac{1}{8}$	0	0
11	0	$\frac{1}{8}$	0	0	0

   Then we calculate the marginal PMF $\mathrm{P}_N(n)$ of N:

n:	2	3	4	5	6
$\mathrm{P}_N(n)$:	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{8}$

   The conditional PMF $\mathrm{P}_{M|N}(M=m|N=n)$ is then $\frac{\mathrm{P}_{MN}(m,n)}{\mathrm{P}_N(n)}$:

$m \backslash n$	2	3	4	5	6
7	0	0	0	$\frac{1}{2}$	0
8	0	0	$\frac{1}{2}$	0	1
9	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0
10	1	0	$\frac{1}{2}$	0	0
11	0	$\frac{1}{2}$	0	0	0

(b) Are M and N independent? Justify your answer.

Solution: No. For example, $\mathrm{P}_{MN}(7,2) = 0$, $\mathrm{P}_M(7) = 1/8$, and $\mathrm{P}_N(2) = 1/8$, so $\mathrm{P}_{MN}(7,2) \neq \mathrm{P}_M(7)\,\mathrm{P}_N(2)$.

(c) What is the expectation of M given $N < 2$?

Solution: Since the probability space of $N < 2$ is $\emptyset$, $\mathrm{P}(\emptyset) = 0$. $E(\emptyset) = 0$.

(d) What is the expectation of M given $N < 4$?

Solution: We first calculate the joint PMF $\mathrm{P}_{MN|N<4}(m,n)$ of M and N given $N < 4$. This is obtained from the joint PMF of M and N by discarding the columns $n = 4$, $n = 5$ and $n = 6$. And rescaling so that the probabilities add up to one:

$m \backslash n$	2	3
7	0	0
8	0	0
9	0	$\frac{1}{3}$
10	$\frac{1}{3}$	0
11	0	$\frac{1}{3}$

Then, we obtain the conditional PMF $\mathrm{P}_{M|N<4}(m)$ by adding up the rows of this table:

m:	9	10	11
$\mathrm{P}_{M|N<4}(m)$:	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

The conditional expectation is:

$$\mathrm{E}[M|N<4] = 9 \cdot \frac{1}{3} + 10 \cdot \frac{1}{3} + 11 \cdot \frac{1}{3} = 10.$$

3. You roll a fair 6-sided die until you get an ordered sequence of {3,5,5,1} in 4 recent rounds.

(a) What is the probability of you stopping at round n? Justify you answer.

Solution: We have $\mathrm{P}(n=4) = \frac{1}{6^4}$. For $n = 5$, since $n = 4$ will never be achieved if the last four rolls are {3,5,5,1}, then $\mathrm{P}(n=5) = \frac{1}{6^4}$. In this way, we can get $\mathrm{P}(n=6) = \frac{1}{6^4}$, $\mathrm{P}(n=7) = \frac{1}{6^4}$. Similarly, we can get the conclusion of start from $n = 4$, the following four successive events have the same probability.

Let $n \in \mathbf{N}^+, n \mod 4 = 0$ be the target event, we must ensure {3,5,5,1} didn't emerge in the all subsequences $n-4$, $n-4-4$, ..., which means $\mathrm{P}(n) = (1 - \mathrm{P}(n-4) - \mathrm{P}(n-4-4) - ...)\,\mathrm{P}_{seq}, \mathrm{P}_{seq} = \frac{1}{6^4}$. And we have $\mathrm{P}(n) = \mathrm{P}(n+1) = \mathrm{P}(n+2) = \mathrm{P}(n+3)$. E.g., $\mathrm{P}(n=8) = \mathrm{P}(n=9) = \mathrm{P}(n=10) = \mathrm{P}(n=11) = (1 - \frac{1}{6^4})\frac{1}{6^4}$.

Then we have:

$$\mathrm{P}(n)\,(n) = \begin{cases} 0, & 1 \leqslant n \leqslant 3 \\ (1 - \frac{1}{64^4})^{\lfloor \frac{n}{4} \rfloor - 1} \frac{1}{64^4}, & n \geqslant 4 \end{cases}$$

(Reminder: Since the sequence {3,5,5,1} has no identical subsequences, the event of n is not independent of the event of $n+1$.)

(b) What is the expected value of the number of total rounds?

Solution: Let E be the expected number of rolls until {3,5,5,1} and let E_{355} be the expected number of rolls until {3,5,5,1} when we start with a rolled {3,5,5}, let E_{35} be the expected number of rolls until {3,5,5,1} when we start with a rolled {3,5,5,1}, let E_3 be the expected number of rolls until {3,5,5,1} when we start with a rolled 3.

1) Start from E_3: If the next dice value is 3, the current subsequence is still {3}, the event continues. If it is 5, the new subsequence is still {3, 5}, the event transforms into

the same with E_{35}. Otherwise, there are no valid subsequences, the event transforms into the same with E.

2) Start from E_{35}: If the next dice value is 3, the new subsequence is $\{3\}$, the event transforms into the same with E_3. If it is 5, the new subsequence is still $\{3, 5, 5\}$, the event transforms into the same with E_{355}. Otherwise, the event transforms into the same with E.

3) Start from E_{355}: If the next dice value is 3, the new subsequence is $\{3\}$, the event transforms into the same with E_3. If it is 1, the new subsequence is $\{3, 5, 5, 1\}$, terminated. Otherwise, the event transforms into the same with E.

4) Start from E: If the next dice value is 3, the new subsequence is $\{3\}$, the event transforms into the same with E_3. Otherwise, the event is still E.

Then we have:

$$\begin{aligned}
E_3 &= \frac{1}{6}(E_3+1) + \frac{4}{6}(E+1) + \frac{1}{6}(E_{35}+1) \\
E_{35} &= \frac{1}{6}(E_3+1) + \frac{4}{6}(E+1) + \frac{1}{6}(E_{355}+1) \\
E_{355} &= \frac{1}{6}(E_3+1) + \frac{4}{6}(E+1) + \frac{1}{6} \\
E &= \frac{1}{6}(E_3+1) + \frac{5}{6}(E+1)
\end{aligned}$$

Solve the equation, we have $E = 1296, E_{355} = 1080, E_{35} = 1260, E_3 = 1290$.

(c) What is the expected value if the number of total rounds is less than or equal to 10?

Solution:

$$\begin{aligned}
\mathrm{P}(n \leqslant 10) &= \frac{1}{6^4} \cdot 4 + (1 - \frac{1}{6^4})\frac{1}{6^4} \cdot 3 \\
&= \frac{9069}{1679616} \approx 0.00539945.
\end{aligned}$$

$$\begin{aligned}
\mathrm{E}(n \mid n \leqslant 10) &= [\frac{1}{6^4} \cdot (4+5+6+7) + (1-\frac{1}{6^4})\frac{1}{6^4} \cdot (8+9+10)]/\,\mathrm{P}(n \leqslant 10) \\
&= \frac{63477}{9069} \approx 7.
\end{aligned}$$

4. You play 10 rounds of roulette with 1 green, 18 reds, 18 blacks. If you choose green, you may win 3 times of what you bet or lose 3 times of what you bet. For red and black, you may win 1 time of what you bet or lose 1 time of what you bet. You invest \$100 and bet 10% of your balance on each color randomly (with the same probability) in every round. What is your average balance after 10 rounds?

Solution: Let X_i be the average balance after round i. X_0 =100. W_i be the event of win in round i. By the conditional expectation formula, we have:

$$\begin{aligned}
\mathrm{E}\left(X_1\right) =& \mathrm{E}(X_1 \mid A)\,\mathrm{P}(A) + \mathrm{E}(X_1 \mid A^c)\,\mathrm{P}(A^c) \\
=& \frac{1}{3}(1.1 \times X_0 \times \frac{18}{37}) + \frac{1}{3}(1.1 \times X_0 \times \frac{18}{37}) + \frac{1}{3}(1.3 \times X_0 \times \frac{1}{37}) \\
&+ \frac{1}{3}(0.9 \times X_0 \times \frac{19}{37}) + \frac{1}{3}(0.9 \times X_0 \times \frac{19}{37}) + \frac{1}{3}(0.7 \times X_0 \times \frac{36}{37}) \\
=& \frac{1003}{1110} X_0 \approx 0.9036036 X_0
\end{aligned}$$

Then, $\mathrm{E}\left(X_{10}\right) = \frac{1003}{1110}^{10} X_0 = (\frac{1003}{1110}^{10})100 \approx 36.28937949.$

5. Let T be the number of times a 20-sided die is rolled until a 6 appears.

(a) What is your average value after 20 rounds?

Solution: Since dice value in the 20th round is known, if $T \leqslant 20$ happened, rolling has been terminated. Otherwise, $T > 20$, continue to roll the dice. By the conditional expectation formula, we have:

$$\mathrm{E}(T) = \mathrm{E}(T \mid T \leqslant 20)\,\mathrm{P}(T \leqslant 20) + \mathrm{E}(T \mid T > 20)\,\mathrm{P}(T > 20)$$

$\mathrm{P}(T \leqslant 20) = \sum_{i=0}^{19} \left(\frac{19}{20}\right)^{i} \left(\frac{1}{20}\right) = 1 - (\frac{19}{20})^{20} \approx 0.64151408$. $\mathrm{P}(T > 20) = 1 - \mathrm{P}(T \leqslant 20) = (\frac{19}{20})^{20} \approx 0.35848592$. $\mathrm{E}(T) = 20$. Then:

$\mathrm{E}(T \mid T \leqslant 20) = \sum_{i=1}^{20} \left(\frac{19}{20}\right)^{i-1} \left(\frac{1}{20}\right) i / \mathrm{P}(T \leqslant 20) = 20 - 2(\frac{19}{20})^{20} \approx 19.28302816$.

$\mathrm{E}(T \mid T > 20) = \frac{\mathrm{E}(T) - \mathrm{E}(T|T\leqslant 20)\mathrm{P}(T\leqslant 20)}{\mathrm{P}(T>20)} = \frac{20-(20-2(\frac{19}{20})^{20})(1-(\frac{19}{20})^{20})}{(\frac{19}{20})^{20}} = 22 - 2(\frac{19}{20})^{20} \approx 21.28302816$.

(b) What is the expected value of T conditioned on all rolls producing even numbers?

Solution: Let A be the event of all rolls producing even numbers. $\mathrm{E}(T \mid A) = \sum_{i=1}^{+\infty} i \cdot \mathrm{P}(T = i \mid A)$. Based on Bayes rule, we have:

$$\begin{aligned}
\mathrm{P}(A) &= \sum_{i=1}^{+\infty} \mathrm{P}(A \cap (T = i)) \\
&= \sum_{i=1}^{+\infty} (\frac{9}{20})^{i-1} \cdot \frac{1}{20} \\
&= \frac{1}{20} \cdot \frac{1}{1 - \frac{9}{20}} \\
&= \frac{1}{11}
\end{aligned}$$

Then,

$$\begin{aligned}
\mathrm{E}(T \mid A) &= \sum_{i=1}^{+\infty} \mathrm{P}(A \cap (T = i)) / \mathrm{P}(A) \\
&= \sum_{i=1}^{+\infty} i \cdot 11 \cdot (\frac{9}{20})^{i-1} \cdot \frac{1}{20} \\
&= \frac{20}{11} \approx 1.81818182.
\end{aligned}$$