PROBLEMAS HOJA 5

1- d. $X \sim BN(t,p)$.

$$P(X=k) = \binom{t+k-1}{k} p^t (1-p)^k, \quad k=0,1,2,\dots$$

$$\varphi_X(s) = E(e^{isX}) = \sum_{k=0}^{\infty} P(X=k) \cdot e^{isk}$$

$$= p^t \sum_{k=0}^{\infty} \binom{t+k-1}{k} \left[(1-p) e^{is}\right]^k = \circledast$$

Idea 1: generalizar el teorema del binomio.

Ejemplos: $(1-x)^{-1} = \sum_{k=0}^{\infty} x^k = \sum_{k=0}^{\infty} \binom{k}{0} x^k$

$$(1-x)^{-2} = \left[(1-x)^{-1}\right]' = \sum_{k=1}^{\infty} k x^{k-1}$$

$$= \sum_{k=0}^{\infty} (k+1) x^k$$

$$= \sum_{k=0}^{\infty} \binom{k+1}{k} x^k$$

$$= \sum_{k=0}^{\infty} \binom{k+2-1}{k} x^k$$

En general (ejercicio de derivadas)

$$(1-x)^{-m} = \sum_{k=0}^{\infty} \binom{k+m-1}{k} x^k, \text{ si } |x|<1.$$

$$\circledast = p^t \left[1-((1-p)e^{is})\right]^{-t} = \left(\frac{p}{1-(1-p)e^{is}}\right)^t$$

Idea 2: propiedades de f.c. Si $X \sim BN(t,p)$, entonces

$$X = \sum_{j=1}^{t} X_j, \text{ donde}$$

los $X_j \sim Geom(p)$ y son independientes.

$\Longrightarrow$

$$\varphi_X(s) = \varphi_{\sum_{j=1}^{t} X_j}(s) \underset{\text{Indep}}{=} \prod_{j=1}^{t} \varphi_{X_j}(s)$$

$$= \left[\varphi_{X_1}(s)\right]^t \underset{1c-}{=} \left[\frac{p}{1-(1-p)e^{is}}\right]^t$$

(g-) $Y \sim exp(\lambda)$. $f_Y(t) = \lambda e^{-\lambda t}$, $t \geq 0$.

$$\varphi_Y(s) = E(e^{isX}) = \int_0^\infty e^{its} \cdot \lambda e^{-\lambda t}\, dt$$

$$= \lambda \int_0^\infty \cos(ts) e^{-\lambda t}\, dt + i\lambda \int_0^\infty \operatorname{sen}(ts) e^{-\lambda t}\, dt.$$

$$= \lambda \left[I_1 \Big|_0^\infty + i\, I_2 \Big|_0^\infty \right].$$

$$I_1 = e^{-\lambda t} \cdot \operatorname{sen}(ts) \cdot s^{-1} + \int \frac{\lambda}{s} e^{-\lambda t} \operatorname{sen}(ts)\, dt = (*)$$

$u = e^{-\lambda t}$ $du = -\lambda e^{-\lambda t}$ || $u = e^{-\lambda t}$ $du = -\lambda e^{-\lambda t}$

$dv = \cos(ts)$ $v = \frac{\operatorname{sen}(ts)}{s}$ || $dv = \operatorname{sen}(ts)$ $v = -\frac{\cos(ts)}{s}$

$$(*) = \frac{e^{-\lambda t} \operatorname{sen}(ts)}{s} - \frac{\lambda}{s^2} \cos(ts) e^{-\lambda t} - \frac{\lambda^2}{s^2} I_1.$$

$\Longrightarrow$

$$I_1 = \frac{1}{1 + \frac{\lambda^2}{s^2}} \left[\frac{e^{-\lambda t} \operatorname{sen}(ts)}{s} - \frac{\lambda}{s^2} \cos(ts) \cdot e^{-\lambda t} \right]$$

$$I_2 = -e^{-\lambda t}\cos(ts)s^{-1} - \int \frac{\lambda}{s} e^{-\lambda t}\cos(ts)\,dt$$

$u = e^{-\lambda t}$ $\quad du = -\lambda e^{-\lambda t}$ $\quad\Big\|\quad$ $u = e^{-\lambda t}$ $\quad du = -\lambda e^{-\lambda t}$

$dv = \operatorname{sen}(ts)$ $\quad v = \dfrac{-\cos(ts)}{s}$ $\quad\Big\|\quad$ $dv = \cos(ts)$ $\quad v = \dfrac{\operatorname{sen}(ts)}{s}$

$$= \frac{-e^{-\lambda t}\cos(ts)}{s} - \frac{\lambda}{s^2}\operatorname{sen}(ts)e^{-\lambda t} - \frac{\lambda^2}{s^2} I_2$$

$$\Longrightarrow$$

$$I_2 = \left(\frac{1}{1+\frac{\lambda^2}{s^2}}\right)\left[-\frac{e^{-\lambda t}\cos(ts)}{s} - \frac{\lambda}{s^2}\operatorname{sen}(ts)e^{-\lambda t}\right]$$

$$\varphi_Y(s) = \lambda\left[I_1 + iI_2\right]_0^{\infty}$$

$$= \frac{\lambda}{1+\frac{\lambda^2}{s^2}}\left[\frac{\lambda}{s^2} + i\frac{1}{s}\right]$$

$$= \frac{\lambda^2}{s^2+\lambda^2} + i\frac{\lambda s}{s^2+\lambda^2} = \frac{\lambda}{\lambda - is}$$

(h-) Z con $f_Z(t) = \text{máx}\{1-|t|, 0\}$.

$$f_Z(t) = \begin{cases} 1-t, & 0<t\leq 1 \\ 1+t, & -1\leq t\leq 0 \\ 0, & |t|>1. \end{cases}$$

$$\varphi_Z(s) = E(e^{isZ}) = \int_{-1}^{0} e^{ist}(1+t)\,dt + \int_0^1 e^{ist}(1-t)\,dt$$

$$= \frac{1-e^{-is}+e^{is}-1}{is} + \int_{-1}^{0} t e^{ist}\,dt - \int_0^1 t e^{ist}\,dt$$

$$= \frac{2\operatorname{sen}s}{s} + \frac{1}{s^2}e^{ist}(1-ist)\Big|_{-1}^{0} - \frac{1}{s^2}e^{ist}(1-ist)\Big|_0^1$$

$$= \cancel{\frac{2\operatorname{sen}s}{s}} + \frac{2}{s^2} - \frac{e^{-is}}{s^2} + \cancel{\frac{i}{s}e^{-is}} - \frac{e^{is}}{s^2} + \cancel{\frac{i}{s}e^{is}}$$

$$= \frac{2}{s^2} - \frac{2\cos s}{s^2}$$

③- $\varphi(t) = \dfrac{\operatorname{sen}(4t)}{4t}$, X, Y indep, $\varphi_X = \varphi_Y = \varphi$

$$E(X+Y) = E(X) + E(Y) = 2E(X) = \frac{2\varphi'(0)}{i}$$

$$\operatorname{Var}(X+Y) \overset{\text{Indep}}{=} \operatorname{Var}(X) + \operatorname{Var}(Y) = 2\operatorname{Var}(X) = 2\left[\frac{\varphi''(0)}{i^2} + \frac{(\varphi'(0))^2}{i^2}\right]$$

$$\varphi'(0) = \lim_{h\to 0} \frac{\frac{\operatorname{sen}(4h)}{4h} - 1}{h} = \lim_{h\to 0} \frac{\frac{4h + C_1 h^3}{4h} - 1}{4h} = 0;$$

$$\varphi''(0) = \lim_{h\to 0} \frac{\varphi'(h) - 0}{h} =$$

$$= \lim_{h\to 0} \frac{16h\cos(4h) - 4\operatorname{sen}(4h)}{16h^3}$$

$$= \lim_{h\to 0} \frac{h \cdot 16\left[\cancel{1} - \frac{16h^2}{2} + C_1 h^4\right] - 4\left[\cancel{4h} - \frac{4^3h^3}{3!} + C_2 h^5\right]}{16h^3}$$

$$= \lim_{h\to 0} \frac{-16.8h^3 - 4^4h^3 + C_3 h^5}{16h^3}$$

$$= -24$$

$$E(X+Y) = 0$$

$$\operatorname{Var}(X+Y) = 48$$

4.- $\varphi_1(t) = (2e^{-it} - 1)^{-1}$

$\varphi_2(t) = (2 - e^{it})^{-1}$.

a.- • $\varphi_1(0) = \varphi_2(0) = 1$; ✓

• φ_1 y φ_2 son no integrables

$\Longrightarrow$

No son f.c. de v.a. continuas.

• Para φ_2: se parece a la geométrica

$$\varphi_2(t) = \frac{1}{2-e^{it}} = \frac{\frac{1}{2}}{1-\frac{1}{2}e^{it}} = \varphi_Y(t),$$

con $Y \sim \text{Geom}(\frac{1}{2})$

• Para φ_1: relacionamos con φ_2.

$$\varphi_1(t) = \frac{1}{2e^{-it}-1} = \frac{e^{it}}{2-e^{it}} = e^{it}\varphi_2(t)$$

$$= \varphi_{Y+1}(t).$$

b.- Sea $X = Y+1$; $2X+1 = 2Y+3$.

$$E(X) = 2E(Y)+3 = 2\frac{\varphi_2'(0)}{i} + 3$$

$$= \frac{2}{i}\left(\frac{ie^{it}}{(2-e^{it})^2}\right)\Bigg|_{t=0} + 3 = 5.$$

$$Var(X) = 4Var(Y) = 4(E(Y^2) - (E(Y))^2)$$

$$= 4\left(\frac{\varphi_2''(0)}{i^2} - \frac{\varphi'(0)}{i}\right)$$

$$= 4\left(\frac{1}{i^2}\frac{i^2e^{it}(2-e^{it})^2 + 2i^2e^{2it}(2-e^{it})}{(2-e^{it})^4}\Bigg|_{t=0} - 1\right) = 11$$

5- $\varphi(t) = \dfrac{e^{3it-2t^2}}{1+it}$.

a- $\varphi(t) = e^{3it} \cdot e^{-\frac{(2t)^2}{2}} \cdot \dfrac{1}{1-(-it)}$

Sea $Z \sim N(0;1)$.

$Y \sim Exp(1)$.

$\Longrightarrow$

$$\varphi(t) = e^{3it}\, \varphi_Z(2t) \cdot \varphi_Y(-t).$$

$$= e^{3it}\, \varphi_{2Z}(t) \cdot \varphi_{-Y}(t)$$

$$= \varphi_X(t),$$

donde $X = 2Z - Y + 3$, y Z e Y son indep.

$E(X) = 2E(Z) - E(Y) + 3 = 0 - 1 + 3 = 2$.

$Var(X) \overset{indep}{=} 4\,Var(Z) + Var(Y) = 4 + 1 = 5$.

8- $X, Y \sim \mathrm{Exp}(a)$, X, Y indep.

$U = X + Y$
$V = X - Y$.

Quiero demostrar que U y V no son indep.
Si lo fueran,

$$\varphi_{U+V}(t) = \varphi_U(t)\varphi_V(t)$$
$$= \varphi_{X+Y}(t)\varphi_{X-Y}(t)$$
$$= \varphi_X(t)\varphi_Y(t)\varphi_X(t)\varphi_Y(-t)$$
$$= \left(1 - \frac{it}{a}\right)^{-3}\left(1 + \frac{it}{a}\right)$$

Por otro lado,

$$\varphi_{U+V}(t) = \varphi_{X+Y+X-Y}(t) = \varphi_{2X}(t)$$
$$= \varphi_X(2t) = \left(1 - \frac{2it}{a}\right)^{-1}$$

Ejercicio: Los dos resultados no son iguales.

10- Se hacen casi todos igual:

a- $X \sim B(n,p)$, $Y \sim B(m,p)$.

$$\varphi_{X+Y}(t) = \varphi_X(t)\,\varphi_Y(t) =$$
$$= ((1-p)+pe^{it})^n((1-p)+pe^{it})^m$$
$$= ((1-p)+pe^{it})^{n+m}$$
$$= \varphi_Z(t),$$

con $Z \sim B(n+m,p)$.

e- $X \sim N(a,\sigma)$; $Y \sim N(b,\tau)$; $Z \sim N(0;1)$

$$\varphi_X(t) = \varphi_{\sigma Z+a}(t) = e^{ait}\,e^{-\frac{(\sigma t)^2}{2}}.$$
$$\varphi_Y(t) = \varphi_{\tau Z+b}(t) = e^{bit}\,e^{-\frac{(\tau t)^2}{2}}.$$

$$\varphi_{X+Y}(t) = \varphi_X(t)\,\varphi_Y(t)$$
$$= e^{ait}\,e^{-\frac{(\sigma t)^2}{2}}\,e^{bit}\,e^{-\frac{(\tau t)^2}{2}}$$
$$= e^{(a+b)it}\,e^{-\frac{(\sigma^2+\tau^2)t^2}{2}}$$
$$= \varphi_{\sqrt{\sigma^2+\tau^2}\,Z+(a+b)}(t),$$

así que $X+Y \sim N\left(a+b, \sqrt{\sigma^2+\tau^2}\right)$.

13- $X, Y, \frac{X+Y}{\sqrt{2}}$ tienen la misma distribución

$E(X)=0$, $Var(X)=1$; X, Y independientes

Sea $\varphi(t)=\varphi_X(t)=\varphi_Y(t)=\varphi_{\frac{X+Y}{\sqrt{2}}}(t)$.

- $$\varphi(t)=1+iE(X)t+i^2\frac{E(X^2)}{2}t^2+o(t^2)$$
$$=1-\frac{t^2}{2}+o(t^2).$$

- Por otro lado,
$$\varphi(t)=\varphi_{\frac{X+Y}{\sqrt{2}}}(t)=\varphi_X\left(\frac{t}{\sqrt{2}}\right)\varphi_Y\left(\frac{t}{\sqrt{2}}\right)$$
$$=\varphi\left(\frac{t}{\sqrt{2}}\right)^2.$$

Iterando, $$\varphi(t)=\varphi\left(\frac{t}{(\sqrt{2})^n}\right)^{2^n}$$
$$=\left(1-\frac{t^2}{2\cdot 2^n}+o\left(\frac{t^2}{2^n}\right)\right)^{2^n}$$

Tomamos límites:

$$\varphi(t)=\lim_{n\to\infty}\left(1-\frac{t^2}{2\cdot2^n}+o\left(\frac{t^2}{2^n}\right)\right)^{2^n}$$
$$=\lim_{n\to\infty}\left(1+\frac{-t^2+o\left(\frac{t^2}{2^n}\right)2^{n+1}}{2^{n+1}}\right)^{\frac{2^{n+1}}{-t^2+2^{n+1}o\left(\frac{t^2}{2^n}\right)}\cdot\frac{-t^2+2^{n+1}o\left(\frac{t^2}{2^n}\right)}{2}}$$
$$=e^{-\frac{t^2}{2}+\lim_{n\to\infty}2^{n+1}o\left(\frac{t^2}{2^n}\right)}=e^{-\frac{t^2}{2}}\Rightarrow X\sim N(0;1).$$