21- Usamos la fórmula del producto para escribir la distribución de (X,Y):

$X \backslash Y$	0	1	2	P_X
0	$6/45$	$16/45$	$6/45$	$28/45$
1	$8/45$	$8/45$	0	$16/45$
2	$1/45$	0	0	$1/45$
P_Y	$15/45$	$24/45$	$6/45$	

Ej: $P_{(X,Y)}(0,0) = \frac{4}{10}\cdot\frac{3}{9}$

$= \frac{12}{90} = \frac{6}{45}$.

$$E(X) = 0\cdot\frac{28}{45} + 1\cdot\frac{16}{45} + 2\cdot\frac{1}{45} = \frac{2}{5}$$

$$E(Y) = 0\cdot\frac{15}{45} + 1\cdot\frac{24}{45} + 2\cdot\frac{6}{45} = \frac{4}{5}.$$

$$E(Y|X=0) = 0\cdot\frac{P(Y=0,X=0)}{P(X=0)} + 1\cdot\frac{P(Y=1,X=0)}{P(X=0)} + 2\,\frac{P(Y=2,X=0)}{P(X=0)}$$

$$= 0 + \frac{16/45}{28/45} + \frac{2\cdot 6/45}{28/45} = 1.$$

$E(Y|X=1) = \frac{1}{2}$; $E(Y|X=2) = 0$;

$E(X|Y=0) = \frac{2}{3}$; $E(X|Y=1) = \frac{1}{3}$; $E(X|Y=2) = 0$;

$$E(E(X|Y)) = \frac{2}{3}\cdot P(Y=0) + \frac{1}{3}P(Y=1) + 0\,P(Y=2)$$

$= \frac{2}{5} = E(X)$, como debe ser para la regla de la doble esperanza.

$$E(E(Y|X)) = 1\cdot P(X=0) + \frac{1}{2}P(X=1) + 0\cdot P(X=2)$$

$$= \frac{1}{28/45} + \frac{1}{2}\cdot\frac{16}{45} = \frac{36}{45} = \frac{4}{5} = E(Y).$$

(25-) Calculamos $E(Y)$ condicionando por los valores que toma N:

$$E(Y) = \sum_{j=0}^{\infty} P(N=j) E(Y \mid N=j)$$

$$= \sum_{j=1}^{\infty} P(N=j)\, E\left(\sum_{l=1}^{j} X_l\right)$$

$$= \sum_{j=1}^{\infty} P(N=j) \left[\sum_{l=1}^{j} E(X_l)\right]$$

$$= \sum_{j=1}^{\infty} P(N=j) .\, j\, E(X_1)$$

$$= E(X_1) \sum_{j=1}^{\infty} j\, P(N=j)$$

$$= \boxed{E(X_1)\, E(N)}$$

27.- Empezamos construyendo $P(U,V)$

$U \backslash V$	1	2	3	4	P_U
0	$1/16$	0	0	0	$1/16$
1	$2/16$	$1/16$	0	0	$3/16$
2	$2/16$	$2/16$	$1/16$	0	$5/16$
3	$2/16$	$2/16$	$2/16$	$1/16$	$7/16$
P_V	$7/16$	$5/16$	$3/16$	$1/16$	

a.- $E(U) = 0\cdot\frac{1}{16} + 1\cdot\frac{3}{16} + 2\cdot\frac{5}{16} + 3\cdot\frac{7}{16} = \frac{34}{16} = \frac{17}{8}$

$E(V) = 1\cdot\frac{7}{16} + 2\cdot\frac{5}{16} + 3\cdot\frac{3}{16} + 4\cdot\frac{1}{16} = \frac{30}{16} = \frac{15}{8}$

b.- $Var(U) = E(U^2) - [E(U)]^2$

$= 0\cdot\frac{1}{16} + 1\cdot\frac{3}{16} + 4\cdot\frac{5}{16} + 9\cdot\frac{7}{16} - \left(\frac{17}{8}\right)^2 \simeq 0{,}86$

$Var(V) = E(V^2) - [E(V)]^2$

$= 1\cdot\frac{7}{16} + 4\cdot\frac{5}{16} + 9\cdot\frac{3}{16} + 16\cdot\frac{1}{16} - \left(\frac{15}{8}\right)^2 \simeq 0{,}86$

c.- $Cov(U,V) = E(UV) - E(U)E(V)$

$= \sum_{j=0}^{3} \sum_{i=1}^{4} j\cdot i \; P_{(U,V)}(j,i) - \frac{17}{8}\cdot\frac{15}{8}$

$\simeq 0{,}39.$

$Corr(U,V) = \rho_{U,V} = \frac{Cov(U,V)}{\sigma_U \cdot \sigma_V} = \frac{0{,}39}{0{,}86}.$

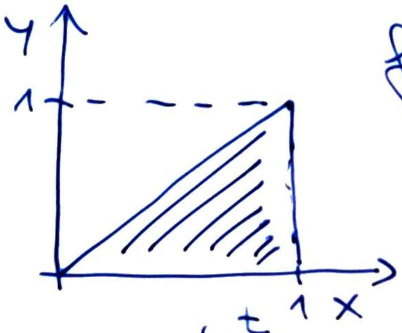
d- $E(U-V) = E(U) - E(V) = \frac{17}{8} - \frac{15}{8} = \frac{1}{4}$

$$\begin{aligned} Var(U-V) &= Var(U) + Var(-V) + 2\,Cov(U,-V) \\ &= Var(U) + Var(V) - 2\,Cov(U,V) \\ &= 0{,}86 + 0{,}86 - 2 \cdot 0{,}39 \approx 0{,}98. \end{aligned}$$

(28-) $\mathrm{Cov}(X+Y, X-Y)$

$$= E((X+Y)(X-Y)) - E(X+Y)E(X-Y)$$
$$= E(X^2-Y^2) - (E(X)+E(Y))(E(X)-E(Y))$$
$$= E(X^2) - E(Y^2) - ((E(X))^2 - (E(Y))^2)$$
$$= \mathrm{Var}(X) - \mathrm{Var}(Y),$$

para cualesquiera X e Y.



$$f_{(X,Y)}(t,s) = \begin{cases} 2, & (t,s) \in \triangle \\ 0, & \text{en otro caso.} \end{cases}$$

$$f_X(t) = \begin{cases} \int_0^t 2\,ds = 2t, & 0<t<1 \\ 0, & t \notin (0,1) \end{cases}$$

$$f_Y(s) = \begin{cases} \int_s^1 2\,dt = 2(1-s), & 0<s<1 \\ 0, & s \notin (0,1). \end{cases}$$

Como $f_Y(t) = f_X(1-t)$, $0<t<1$,

$\mathrm{Var}(Y) = \mathrm{Var}(X)$ (ejercicio).

$\Longrightarrow$

$\mathrm{Cov}(X+Y, X-Y) = 0;$

29.- Sabemos que $E(X_iY_j) = E(X_i)E(Y_j)$ siempre que $i \neq j$. Por tanto,

$$Cov(X,Y) = E(XY) - E(X)E(Y)$$

$$= E\left[\left(\sum_{i=1}^{n} X_i\right)\left(\sum_{j=1}^{n} Y_j\right)\right] - E\left[\sum_{i=1}^{n} X_i\right] E\left[\sum_{j=1}^{n} Y_j\right]$$

$$= E\left[\sum_{i=1}^{n}\sum_{j=1}^{n} X_iY_j\right] - \left(\sum_{i=1}^{n} E(X_i)\right)\left(\sum_{j=1}^{n} E(Y_j)\right)$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} E(X_iY_j) - \sum_{i=1}^{n}\sum_{j=1}^{n} E(X_i)E(X_j)$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} \left[E(X_iY_j) - E(X_i)E(Y_j)\right]$$

$$= \sum_{i=1}^{n} \left[E(X_iY_i) - E(X_i)E(Y_i)\right] =$$

$$= \sum_{i=1}^{n} Cov(X_i, Y_i)$$

30- Llamamos X_i a la puntuación del primer jugador en la partida i, e Y_i a la del segundo. Tenemos que

$$X = \sum_{i=1}^{n} X_i \ , \quad Y = \sum_{i=1}^{n} Y_i .$$

Además, puntuaciones de partidas distintas son independientes, y por tanto incorreladas. Por el ejercicio 29,

$$Cov(X,Y) = \sum_{i=1}^{n} Cov(X_i, Y_i)$$
$$= n\, Cov(X_1, Y_1).$$

Para una sola partida tenemos:

$X_1 \backslash Y_1$	0	1	2
0	0	0	q^2
1	0	$2pq$	0
2	p^2	0	0

$$Cov(X_1,Y_1) = E(X_1 Y_1) - E(X_1)E(Y_1)$$
$$= 2pq - (2pq + 2p^2)(2pq + 2q^2)$$
$$= 2pq - 4pq = -2pq .$$

Por tanto, $\boxed{Cov(X,Y) = -2npq}$