Reference for Assignment 6

2023/7/4

problem 6.1

By induction.

Basis:

$\frac{p_0}{q_0} = \frac{a_0}{1} = [a_0]$

$\frac{p_1}{q_1} = \frac{a_0 a_1 + 1}{a_1} = a_0 + \frac{1}{a_1} = [a_0, a_1]$

Hypothesis:

Assuming that $\forall i \leq k.\ \frac{p_i}{q_i} = [a_0, a_1, a_2, \ldots, a_i], i.e.,\ [a_0, \ldots, a_i] = \frac{p_i}{q_i} = \frac{a_i p_{i-1} + p_{i-2}}{a_i q_{i-1} + q_{i-2}}$

Inductive:

$$\begin{aligned}
[a_0, \ldots, a_k, a_k + 1] = [a_0, \ldots, a_k + \frac{1}{a_{k+1}}] \\
&= \frac{(a_k + \frac{1}{a_{k+1}}) p_{k-1} + p_{k-2}}{(a_k + \frac{1}{a_{k+1}}) q_{k-1} + q_{k-2}} \\
&= \frac{a_{k+1}(a_k p_{k-1} + p_{k-2}) - p_{k-1}}{a_{k+1}(a_k q_{k-1} + q_{k-2}) - q_{k-1}} \\
&= \frac{p_{k+1}}{q_{k+1}}
\end{aligned}$$

problem 6.2

By induction.

- $i = 0$, if $a_0 < a_0' \to [a_0] < [a_0']$.
- $i = 1$, if $a_1 < a_1' \to a_0 + \frac{1}{a_1} > a_0 + \frac{1}{a_1'} \to [a_0, a_1] > [a_0, a_1']$
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  - Supposing that $\forall i < 2k-1, [a_0, \dots, a_i] > [a_0, \dots, a_i']$ if $a_i < a_i'$. Then let $i = 2k+1$, $[a_0, \dots, a_{2k-1}, a_{2k}, a_{2k+1}] = [a_0, \dots, a_{2k-1} + \frac{1}{a_{2k}+\frac{1}{a_{2k+1}}}]$.
    If $a_n < a_n' \to a_{2k-1} + \frac{1}{a_{2k}+\frac{1}{a_{2k+1}}} < a_{2k-1} + \frac{1}{a_{2k}+\frac{1}{a_{2k+1}'}} \to [a_0, \dots, a_{2k-1} + \frac{1}{a_{2k}+\frac{1}{a_{2k+1}}}] > [a_0, \dots, a_{2k-1} + \frac{1}{a_{2k}+\frac{1}{a_{2k+1}'}}]$
  - When i is even, the method is similar.

problem 6.3

True. A construction method is using the method in Thm6.2 to get $\frac{p}{q} = [a_0, a_1, \dots, a_N]$, Then we can see $\frac{p}{q} = [a_0, a_1, \dots, a_N + 1, -1]$ which satisfies the requirement.

problem 6.4

Sketch: Consider $[a_k, \dots, a_{k+l-1}, \dots, a_{k+(n-1)l}, \dots, a_{k+nl-1}]$ as a whole and attempt to express the relationship between $[a_k, \dots, a_{k+nl-1}]$ and $[a_k, \dots, a_{k+(n+1)l-1}]$. Utilize the conclusion from problem 6.1, and then substitute $[a_k, \dots, a_{k+nl-1}]$ into $[a_0, \dots, a_k, \dots, a_{k+nl+1}]$.

problem 6.5

$|\mathbb{Q}| = \aleph_0$ implies $|\mathbb{A}| = \aleph_0$, while $|\mathbb{C}| = \aleph_1$, thus $|\mathbb{T}| = \aleph_1$, then $|\mathbb{T}| > |\mathbb{A}|$.

problem 6.6

Assume the contrary, so that for some integer k we have

$$k < 2^{n-1}\sqrt{3} < k + \frac{1}{2^{n-1}}.$$

Squaring gives

$$\begin{aligned} k^2 < 3 \cdot 2^{2n-2} &< k^2 + \frac{k}{2^n} + \frac{1}{2^{2n+2}} \\ &\leq k^2 + \frac{2^{n-1}\sqrt{3}}{2^n} + \frac{1}{2^{2n+2}} \\ &= k^2 + \frac{\sqrt{3}}{2} + \frac{1}{2^{2n+2}} \\ &\leq k^2 + \frac{\sqrt{3}}{2} + \frac{1}{16} \\ &< k^2 + 1 \end{aligned}$$

and this is a contradiction.

problem 6.7

For a fixed b

$-\frac{1}{b^a} < \xi - \frac{a}{b} < \frac{1}{b^a}$

$\Rightarrow \xi - \frac{1}{b^a} < \frac{a}{b} < \xi + \frac{1}{b^a}$

$\Rightarrow b(\xi - \frac{1}{b^a}) < a < b(\xi + \frac{1}{b^a})$

For sufficiently large b.

$b(\xi + \frac{1}{b^a}) - b(\xi - \frac{1}{b^a}) = \frac{1}{b^a} < 1$

So, there is at most one 'a' .

If ξ is a rational number and $\xi = \frac{p}{q}$

$\frac{bp}{q} - \frac{1}{b^{a-1}} < a < \frac{bq}{q} + \frac{1}{b^{a-1}}$

Also for sufficiently large b, $\frac{1}{b^{a-1}} < \frac{1}{q}$

So, $\frac{bp-1}{q} < a < \frac{bp+1}{q}$

There is no integer lies in $(\frac{bp-1}{q}, \frac{bp+1}{q})$ if $q \nmid b$

And if b=nq,then np-1<a<np+1, a=np, $\frac{a}{b} = \frac{p}{q} = \xi$

So there are finitely many 'b' which enable us to find some 'a'

problem 6.8

Solution: Let us start by writing out the first few terms in the sequence. They are $1, 2, \frac{1}{2}, \frac{3}{2}, \frac{2}{3}, \frac{5}{3}, \frac{3}{5}$. We notice that starting from the second term, the even-numbered terms are greater than 1, while the odd-numbered ones are less than 1. Proving this rigorously is straightforward with induction.

Now, every positive rational number can be written as $\frac{a}{b}$ where a, b are positive integers with $\gcd(a, b) = 1$. We will use double induction on a, b. The base case is $a = b = 1$ and is clear since $a_1 = 1$. Assume now for positive integers a, b, satisfying $\gcd(a, b)$, we want to show $\frac{a}{b} \in \{a_n\}_{n=1}^{\infty}$ and we know $\frac{c}{d} \in \{a_n\}_{n=1}^{\infty}$ for all positive integers c, d satisfying $c \leq a-1, d \leq b$ or $c \leq a, d \leq b-1$. We can assume $a \neq b$ (or else $a = b = 1$ and this case is already done.)

If $a > b$, then $\frac{a}{b} > 1$, so by our first observation, we want to prove that $\frac{a}{b} = a_{2n}$ for some n. But $a_{2n} = a_n + 1$, so we want to prove $a_n = \frac{a-b}{b}$ for some n - which we know is true by the induction assumption.

If $b > a$, then $\frac{a}{b} < 1$, so by our first observation, we want to prove $\frac{a}{b} = a_{2n+1}$ for some n. But $a_{2n+1} = \frac{1}{a_{2n}}$ so we want to prove $\frac{b}{a} = a_{2n}$ for some n - which we already know how to do - since $a_n = \frac{b-a}{a}$ for some n by the induction assumption. The induction step is complete, so every positive rational number occurs in the sequence at least once.

Finally, we need to prove that no two terms in the sequence are the same. Assume we have $a_m = a_n$ for some $m \neq n$ and $m + n$ minimal. It is clear that $m \neq 1$ and $n \neq 1$ (as $a_k \neq 1$ for $k > 1$). Then by our first observation m and n are both even, in which case $a_{\frac{m}{2}} = a_{\frac{n}{2}}$, or m and n are both odd, in which case $a_{m-1} = a_{n-1}$. Thus in both cases we get a contradiction to the fact that $m + n$ is minimal.

Comment: We see the initial observation that for $n \geq 1$, $a_n > 1$ if n is even, and $a_n < 1$ if n is odd, pretty much solved the problem. Playing around with the first few terms in the sequence is always a good idea, especially if they are reasonably nice small numbers. Induction is another thing to keep in mind when faced with these types of problems (although its application is often more tricky than in this problem.)