Solution 3

Problem3.1

$$
\begin{aligned}
T(n) &= n + \frac{2}{n}\sum_{k=0}^{n-1} T(k). \\
\Delta T(n) &= T(n) - T(n-1) \\
&= 1 + \frac{2}{n(n-1)}\sum_{k=0}^{n-2} T(k) + \frac{2}{n}T(n-1) \\
\Delta^2 T(n) &= \Delta T(n) - \Delta T(n-1) > 0.
\end{aligned}
$$

So T is convex.

Problem3.2

If two vertices u, v is separated by the min-cut, then the max-flow from u to v equals to the min-cut. So we can fix a vertex u and for all $v \in V$, then invoke the FF algorithm for u, v. We need to invoke FF algorithm n times and one invocation of FF algorithm needs $O(n^3)$ time, so we totally need $O(n^4)$ time.

Problem3.3

In the most trivial way, we can repeat the Karger's algorithm k times, then the failure probability is $p = (1 - \frac{2}{n(n-1)})^k$. So

$$
\begin{aligned}
p = (1 - \frac{2}{n(n-1)})^k &\leq \varepsilon \\
k &\geq \frac{\ln(1 - \frac{2}{n(n-1)})}{\ln(\varepsilon)}.
\end{aligned}
$$

Problem3.4

Monkey sort. Error probability is $1 - \frac{1}{n!}$.

Problem3.5

In the analysis of Karger's algorithm, we know that for any specific minimum cut, the algorithm outputs it with probability $\frac{1}{\binom{n}{x}}$. If there is more than $\binom{n}{x}$ distinct minimum cuts in the graph, the algorithm can only output one of them. So there must be one minimum the probability that the algorithm outputs it is less than $\frac{1}{\binom{n}{x}}$.