Causal inference of mediation mechanism

Part I: 反事實結果與可互換性

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A toy example

An intuitive definition of cause

- Ian smoked.
  - ➢ Ten years later, he has lung cancer.

God's POV:

- Had Ian not smoked,
  - ➢ Ten years later, he would not have developed lung cancer.

- Did smoking cause Ian's lung cancer?

An intuitive definition of cause

- Jim did not smoke.
  - Ten years later, he does not have lung cancer.

God's POV:

- Had Jim smoked,
  - ➢ Ten years later, he would not have developed lung cancer.

- Did smoking cause Jim's risk of lung cancer?

Notation for actual data

- $Y = 1$ if the patient died, 0 otherwise
  - $Y_i = 1, Y_j = 0$
- $S = 1$ if the patient treated, 0 otherwise
  - $S_i = 1, S_j = 0$

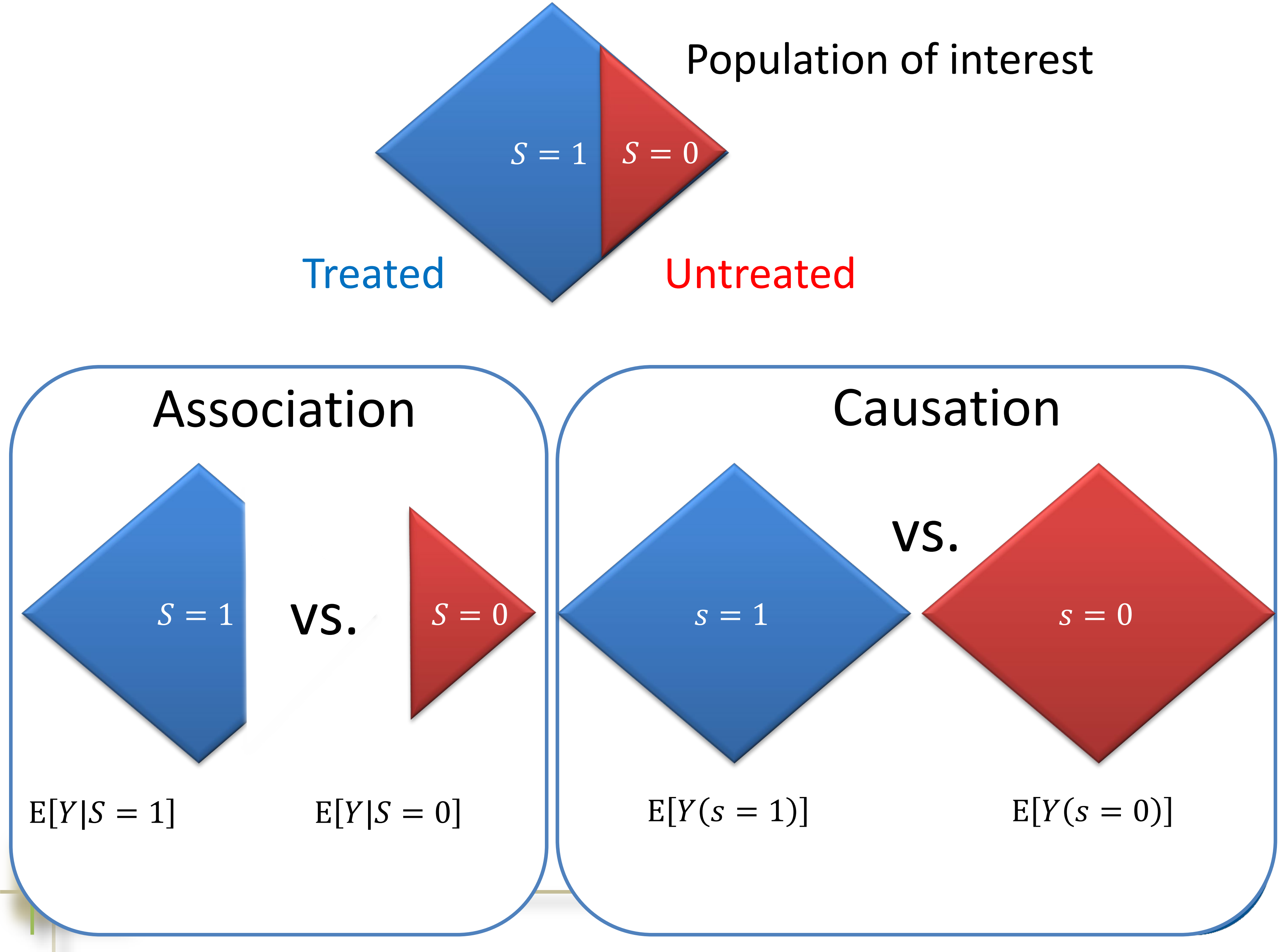
Subject	S	Y
Ian	1	1
Jim	0	0

Notation for ideal data

- $Y(s=0)=1$: if the patient had not smoked, he would have developed cancer
  - $Y_i(s=0)=0, Y_j(s=0)=0$
- $Y(s=1)=1$: if the patient had smoked, he would have developed cancer
  - $Y_i(s=1)=1, Y_j(s=1)=0$

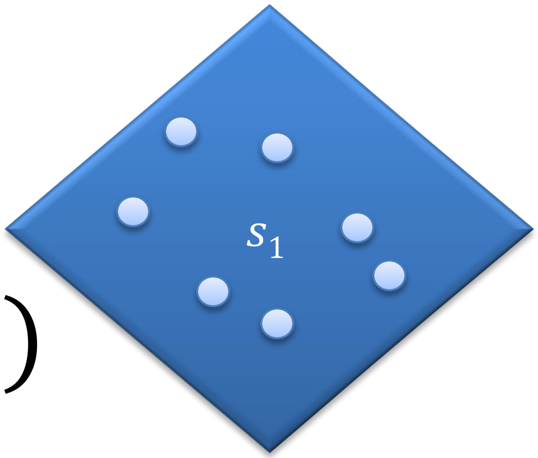
Subject	S	$Y(s=0)$	$Y(s=1)$
Ian	1	0	1
Jim	0	0	0

Causation vs. association


Population of interest
$S = 1$
$S = 0$
Treated
Untreated
Association
$S = 1$
vs.
$S = 0$
$\mathrm{E}[Y|S = 1]$
$\mathrm{E}[Y|S = 0]$
Causation
$s = 1$
vs.
$s = 0$
$\mathrm{E}[Y(s = 1)]$
$\mathrm{E}[Y(s = 0)]$

Potential outcome

$$s_1 \longrightarrow Y(s_1)$$

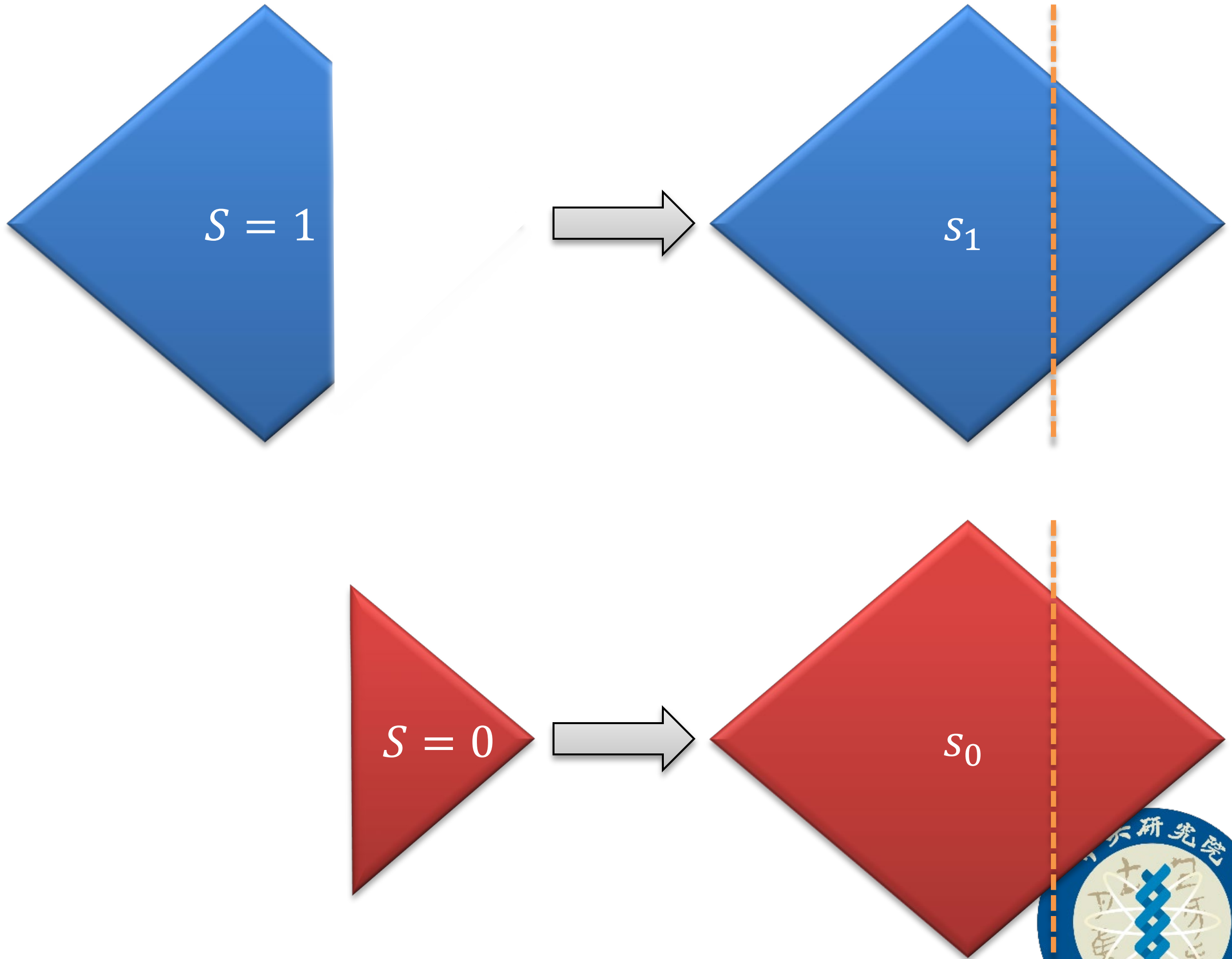


$$s_0 \longrightarrow Y(s_0)$$

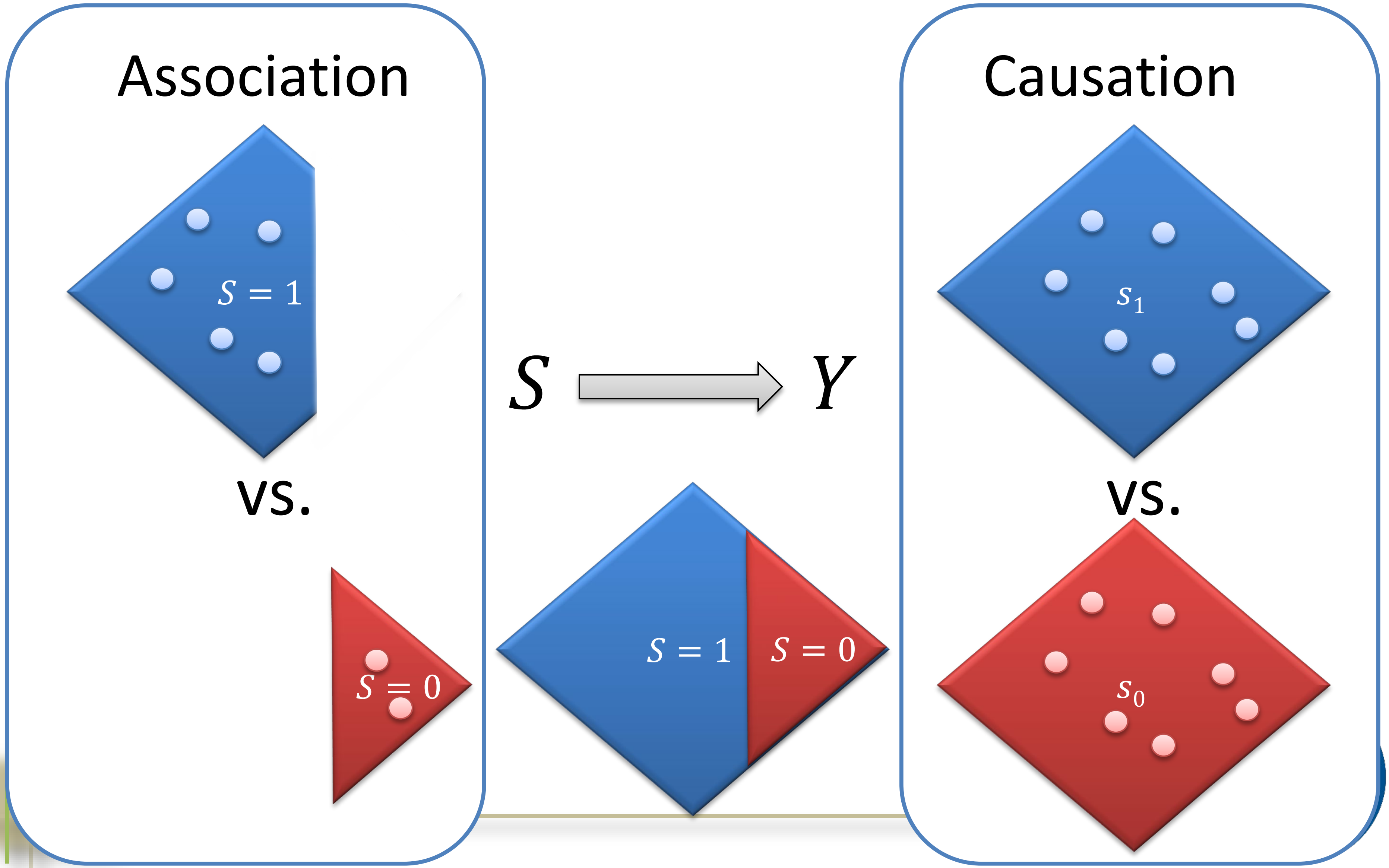
Exchangeability

- $Y(s_1) \perp S$

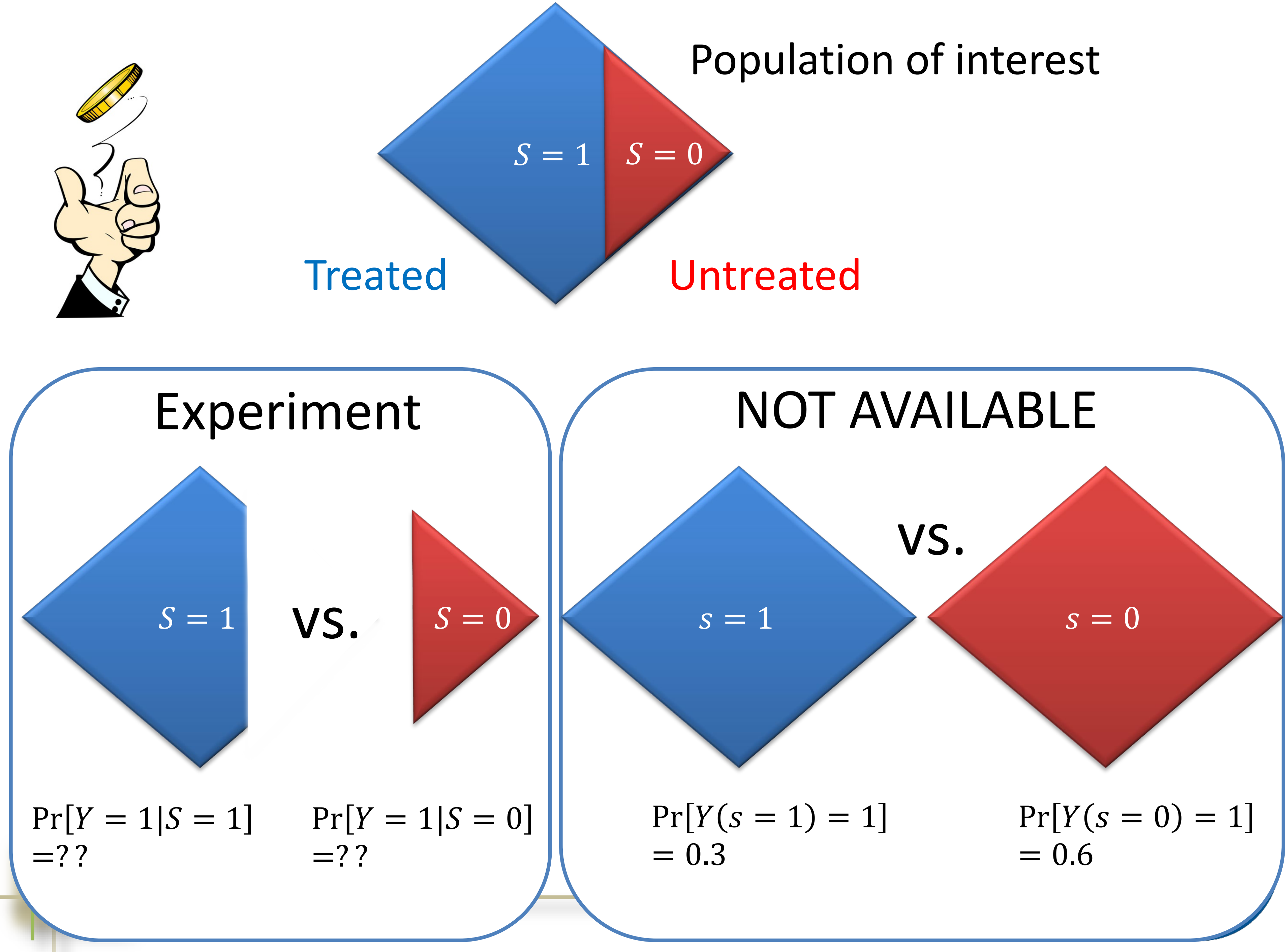
- $Y(s_0) \perp S$

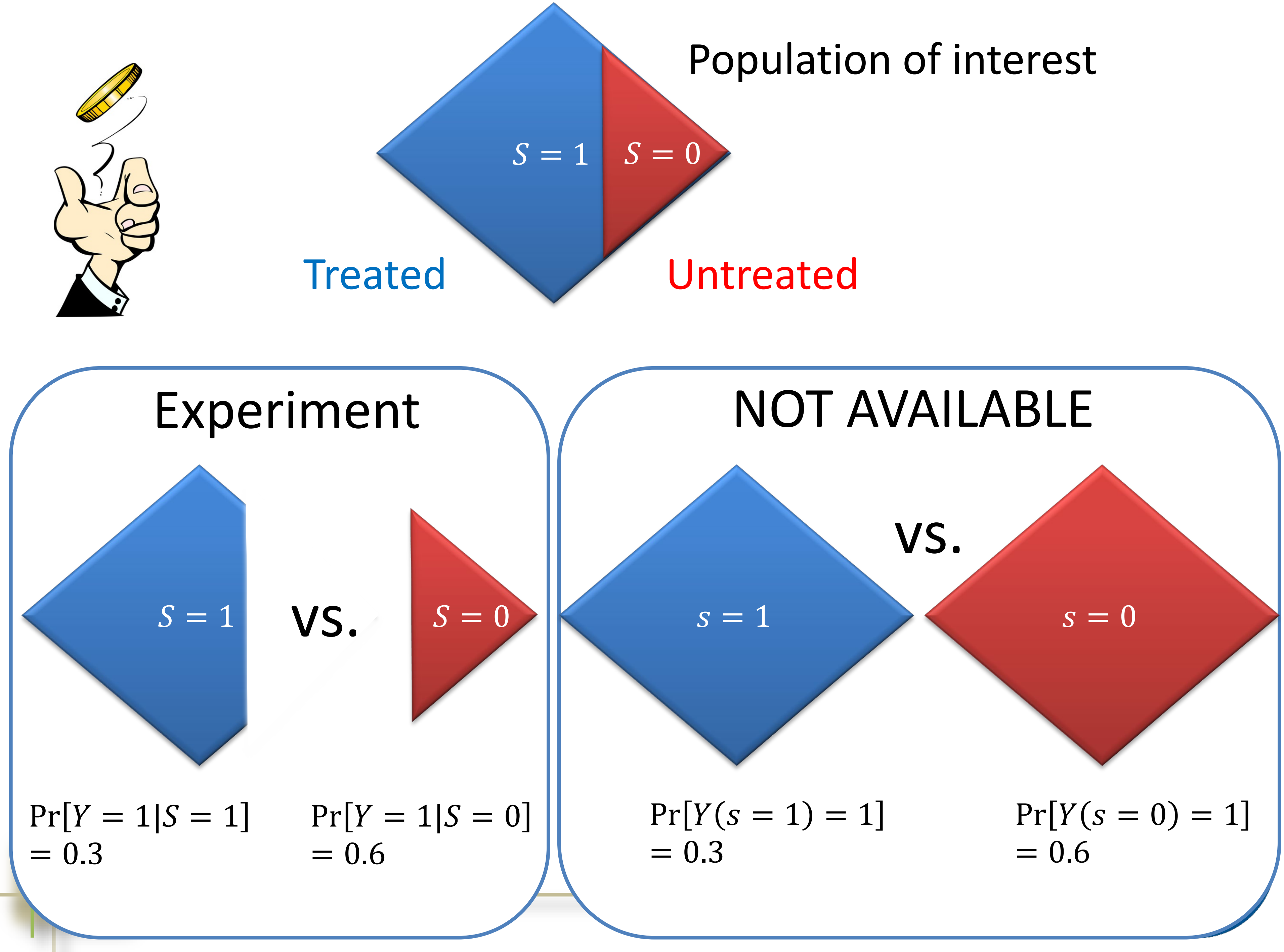


If exchangeable, association=causation



Exchangeability


Population of interest
$S = 1$
$S = 0$
Treated
Untreated
Experiment
$S = 1$
vs.
$S = 0$
$\Pr[Y = 1 | S = 1]$ $= ??$
$\Pr[Y = 1 | S = 0]$ $= ??$
NOT AVAILABLE
$s = 1$
vs.
$s = 0$
$\Pr[Y(s = 1) = 1]$ $= 0.3$
$\Pr[Y(s = 0) = 1]$ $= 0.6$


Population of interest
$S = 1$
$S = 0$
Treated
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NOT AVAILABLE
$s = 1$
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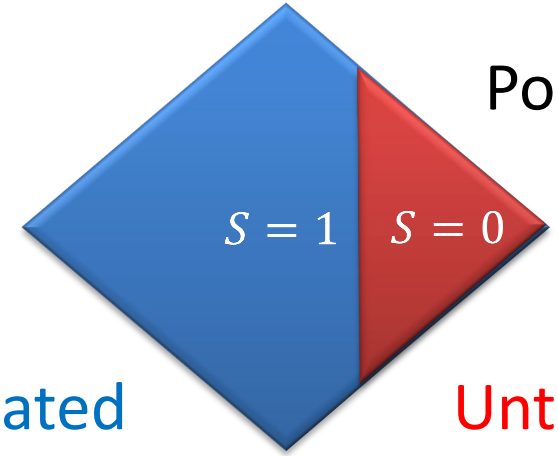
$\Pr[Y(s = 1) = 1] = 0.3$
$\Pr[Y(s = 0) = 1] = 0.6$

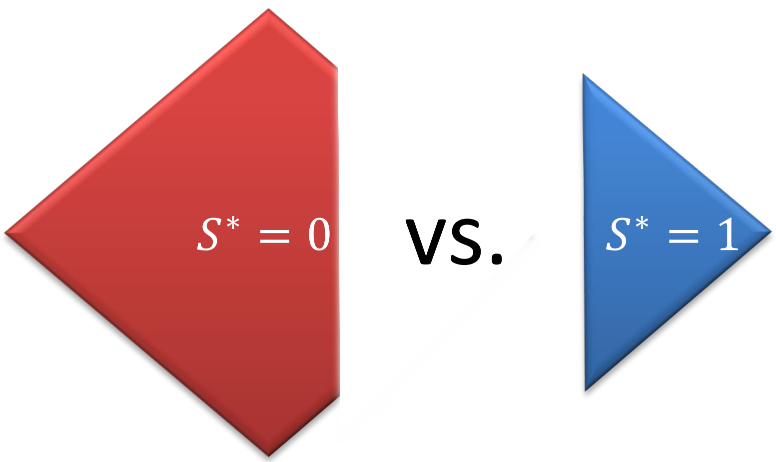
Population of interest



Treated

Untreated

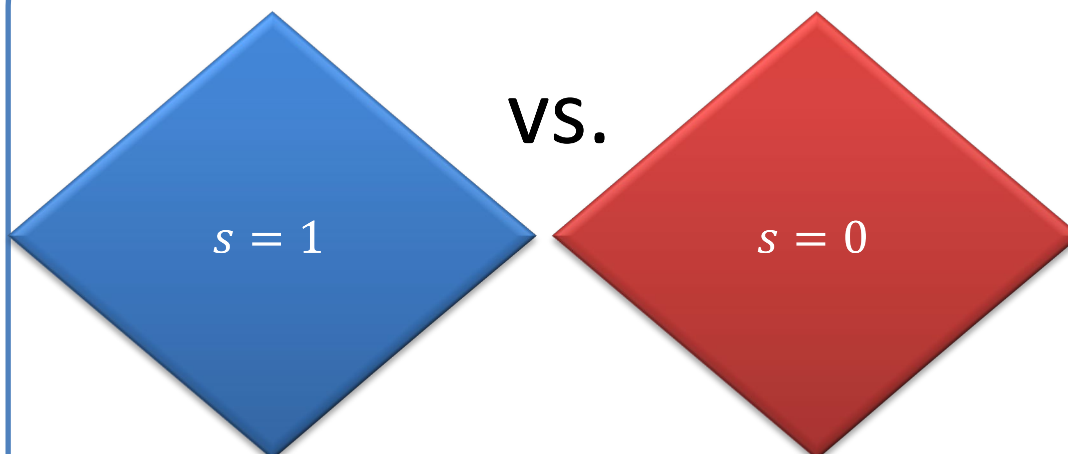
Experiment



$\Pr[Y = 1 | S^* = 0]$
$= ??$

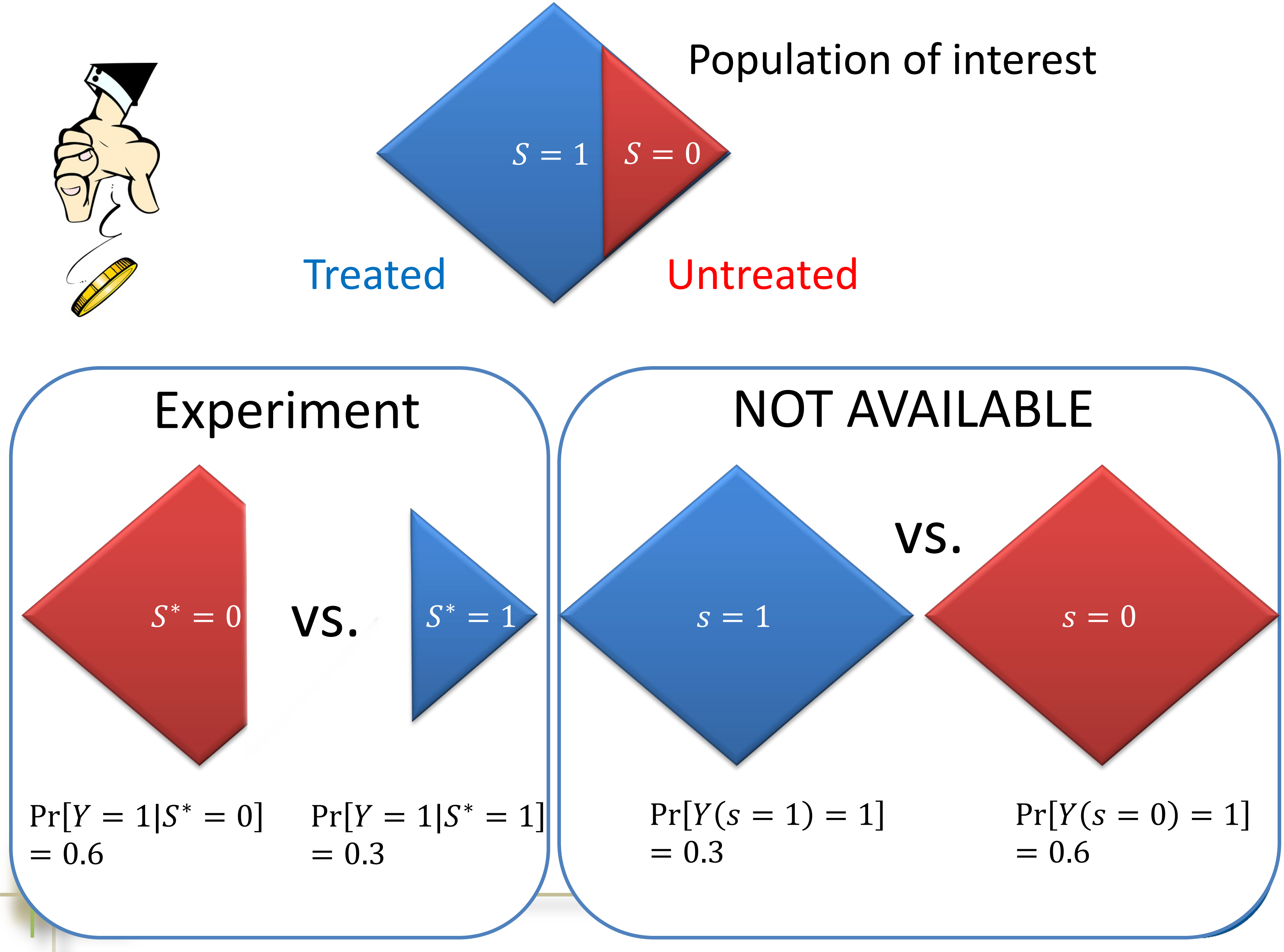
$\Pr[Y = 1 | S^* = 1]$
$= ??$

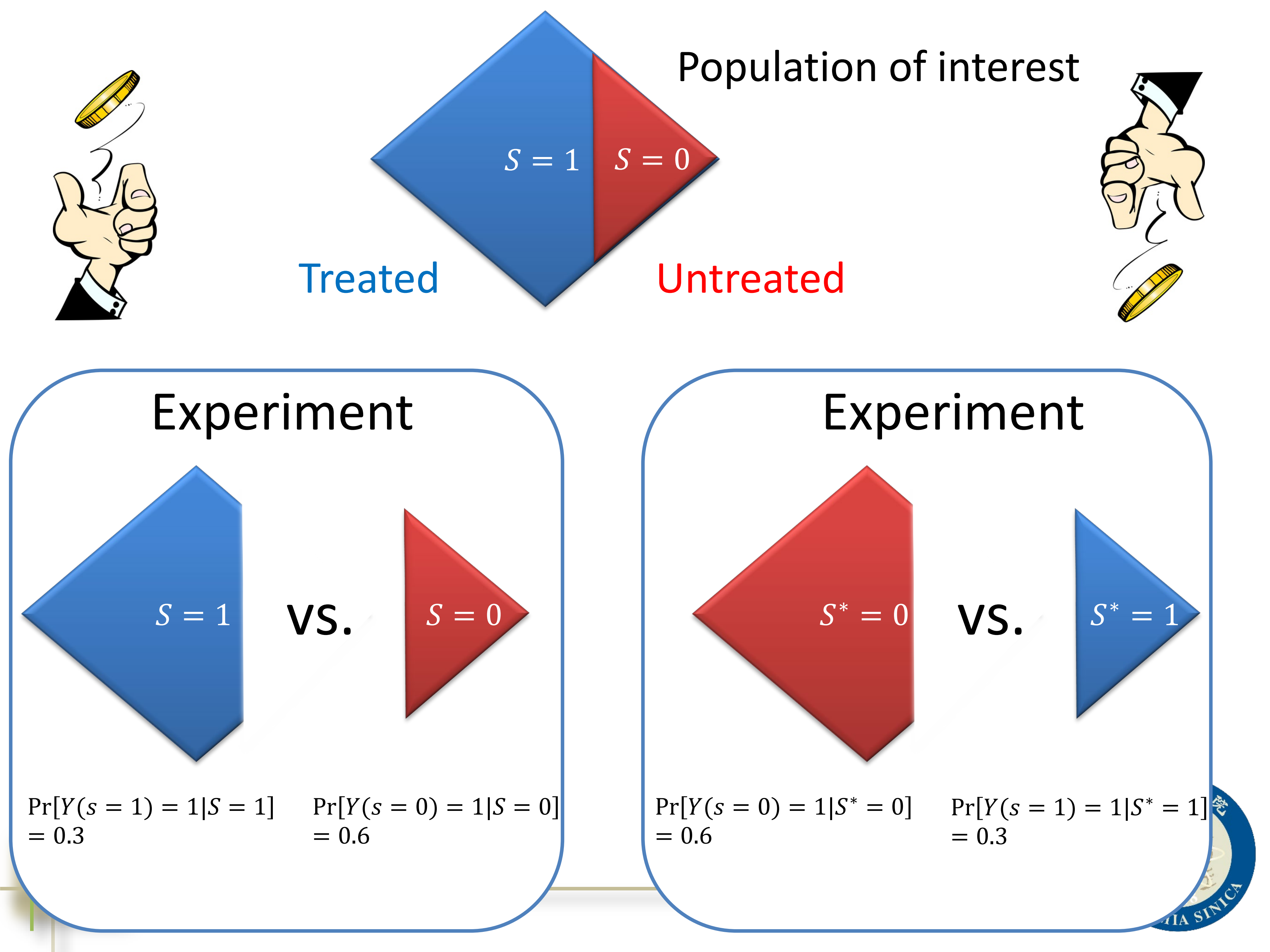
NOT AVAILABLE



$\Pr[Y(s = 1) = 1]$
$= 0.3$

$\Pr[Y(s = 0) = 1]$
$= 0.6$


Population of interest
$S = 1$
$S = 0$
Treated
Untreated
Experiment
$S^* = 0$
vs.
$S^* = 1$
$\Pr[Y = 1|S^* = 0] = 0.6$
$\Pr[Y = 1|S^* = 1] = 0.3$
NOT AVAILABLE
$s = 1$
vs.
$s = 0$
$\Pr[Y(s = 1) = 1] = 0.3$
$\Pr[Y(s = 0) = 1] = 0.6$



Population of interest
$S = 1$
$S = 0$
Treated
Untreated
Experiment
$S = 1$
vs.
$S = 0$
$\Pr[Y(s=1)=1|S=1]$
$= 0.3$
$\Pr[Y(s=0)=1|S=0]$
$= 0.6$
Experiment
$S^* = 0$
vs.
$S^* = 1$
$\Pr[Y(s=0)=1|S^*=0]$
$= 0.6$
$\Pr[Y(s=1)=1|S^*=1]$
$= 0.3$

Exchangeability

- The two groups are exchangeable.
- Why? Randomization.

Exchangeable

$S = ?$ vs. $S = ?$

Exchangeability

- The risk of death in the diamond group would have been the same as the risk of death in the triangle group had subjects in the diamond group received the treatment given to those in the triangle group.
- Formally, the counterfactual outcome $Y(s)$ is independent of what is really assigned to the subject S (or S^*):

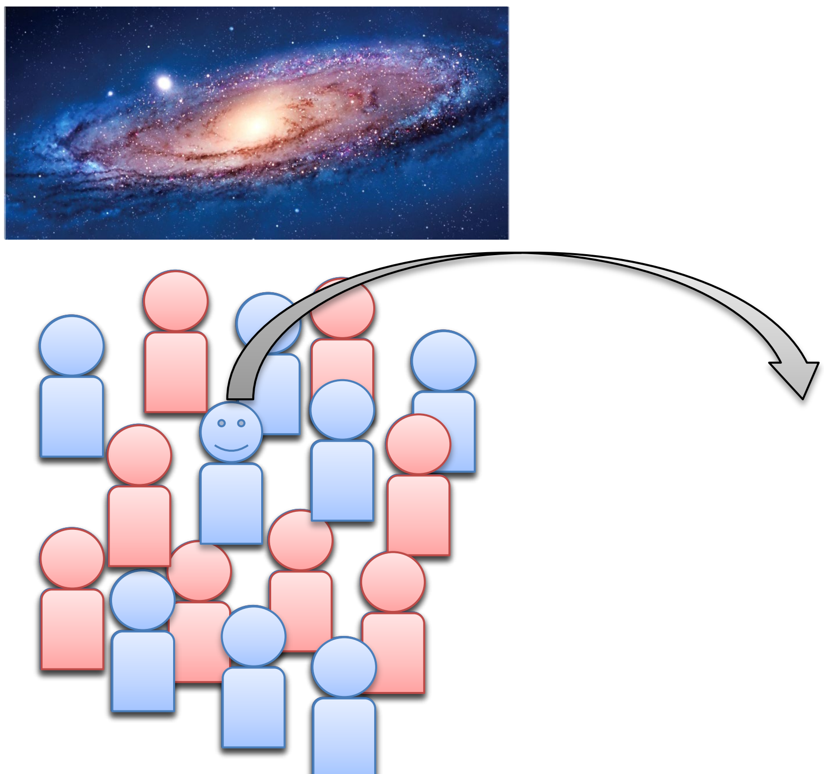
$$Y(s) \perp S$$

Exchangeability is different from $Y \perp S$

- Exchangeability: $Y(s) \perp S$
- Independence of two random variables: $Y \perp S$
- They are different.

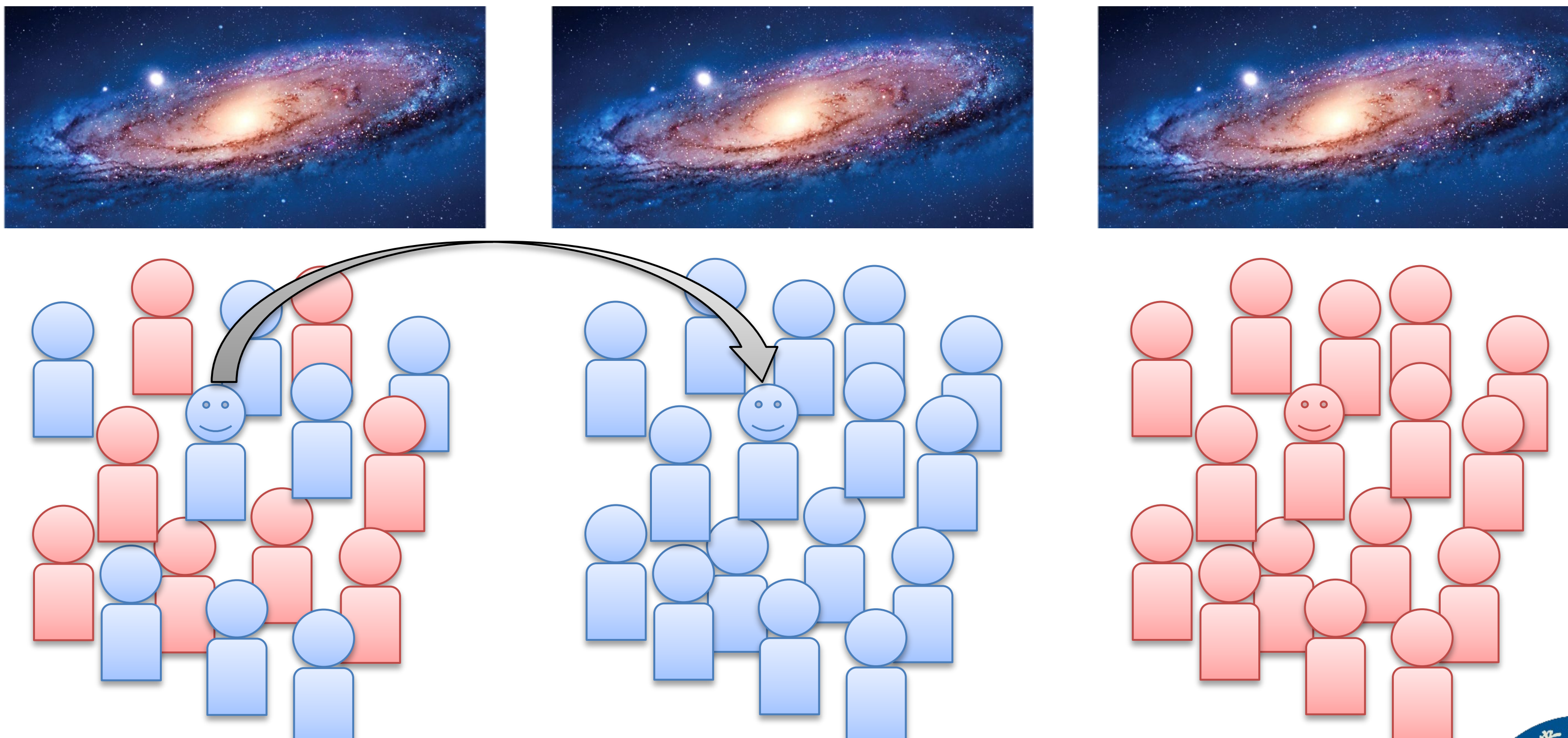
Causal consistency

(Causal) Consistency



(Causal) Consistency

$$\Pr[Y = 1|S = s] = \Pr[Y(s) = 1|S = s]$$



Association = causation

- Risk difference

$$\Pr[Y(s=1)=1] - \Pr[Y(s=0)=1]$$
$$= \Pr[Y=1|S=1] - \Pr[Y=1|S=0]$$

- Proof:

$$\Pr[Y(s)=1] = \Pr[Y(s)=1|S=s] \text{ by exchangeability}$$
$$= \Pr[Y=1|S=s] \quad \text{by consistency}$$

Individual vs. average causal effect

Available data set from a study

Subject	S	Y	$Y(s=0)$	$Y(s=1)$
Ian	1	1	?	1
Jim	0	0	0	?
Ken	1	0	?	0
Leo	0	1	1	?
Mike	1	1	?	1
Nick	0	0	0	?
…				

Fundamental problem of causal inference

- Individual causal effects cannot be determined
  - except under extremely strong (and generally unreasonable) assumptions
  - because only one counterfactual outcome is observed
  - Causal inference as a missing data problem
- Need another definition of causal effect that requires weaker assumptions

Average causal effect in the population

- Exposure S has an average causal effect on the outcome Y if

$$\mathrm{E}[Y(s=1)] \neq \mathrm{E}[Y(s=0)]$$

- **Average causal null hypothesis** holds if

$$\mathrm{E}[Y(s=1)] = \mathrm{E}[Y(s=0)]$$

Available data set from a study

Subject	S	Y	$Y(s=0)$	$Y(s=1)$
Ian	1	1	?	1
Jim	0	0	0	?
Ken	1	0	?	0
Leo	0	1	1	?
Mike	1	1	?	1
Nick	0	0	0	?

Complete data for causal inference

Subject	S	Y	$Y(s=0)$	$Y(s=1)$
Ian	1	1	0	1
Jim	0	0	0	0
Ken	1	0	1	0
Leo	0	1	1	0
Mike	1	1	1	1
Nick	0	0	0	1

Average causal effect

- $\Pr[Y(s=0)=1]=\frac{3}{6}$
- $\Pr[Y(s=1)=1]=\frac{3}{6}$
- $\Pr[Y(s=0)=1]=\Pr[Y(s=1)=1]$
  - There is no average causal effect.
- Is there any individual causal effect?
  - Hazardous: Ian, Nick
  - Protective: Ken, Leo

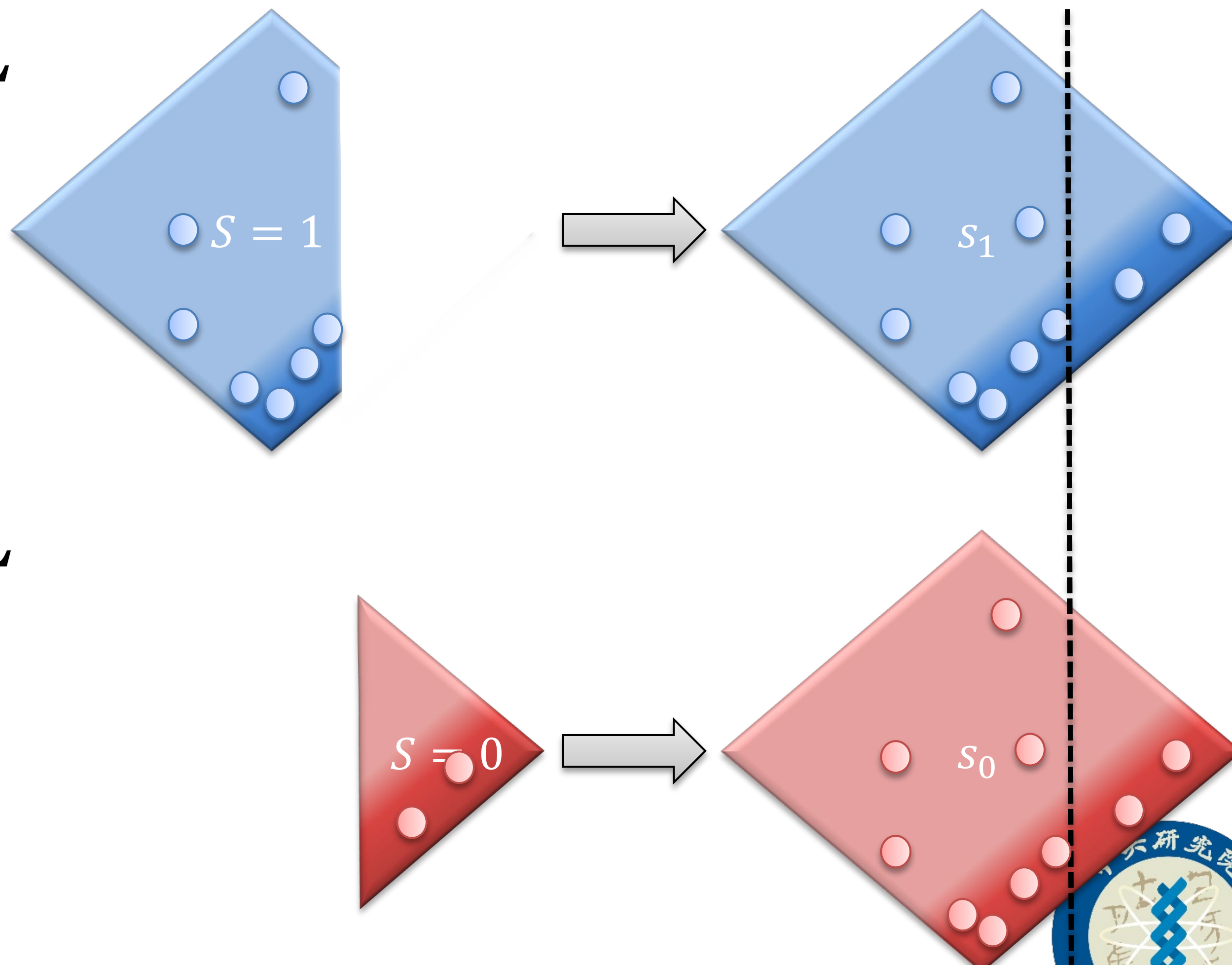
Individual vs. average effects

- Individual causal effects cannot be determined
  - except under quite restrictive assumptions
- Average causal effects can be determined under
  - no assumptions (*ideal* randomized studies)
  - strong assumptions (observational studies)

Conditional exchangeability

Conditional exchangeability

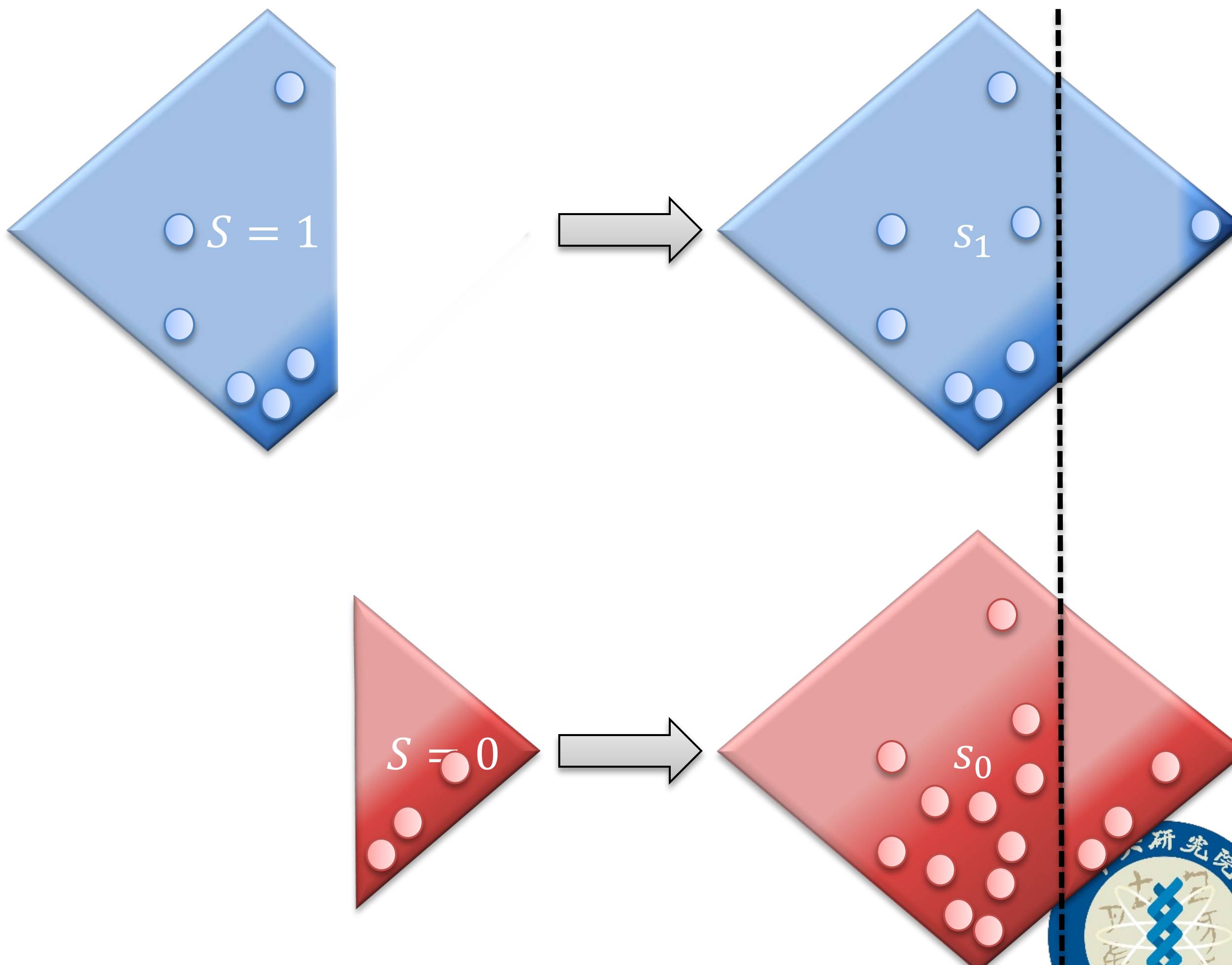
- $Y(s_1) \perp S|L$
- $Y(s_0) \perp S|L$



If (mistakenly) assuming marginal exchangeability

- $Y(s_1) \perp S$

- $Y(s_0) \perp S$



Conditional association = causation

- $\Pr[Y(s=1)=1|L] - \Pr[Y(s=0)=1|L]$
  $= \Pr[Y=1|S=1,L] - \Pr[Y=1|S=0,L]$

- Proof:

$$\begin{aligned} \Pr[Y(s)=1|L] &= \Pr[Y(s)=1|S=s,L] && \text{by } Y(s) \perp S|L \\ &= \Pr[Y=1|S=s,L] && \text{by consistency} \end{aligned}$$

Summary

- Association vs. causation
- Exchangeability
- Conditional exchangeability
- Consistency
- Under (conditional) exchangeability and consistency, association = causation.