Homework : Running Time and Algorithm Analysis

1. Write the following functions in increasing order by growth rate. Indicate which, if any, functions grow at the same rate.

$\cancel{4000}$, $\cancel{\frac{1}{n}}$, $\cancel{12n}$, $\cancel{3 \log 2n}$, $\cancel{n^3 + n - 100}$, $\cancel{\log (n^4)}$, $\cancel{4n^2}$, $\cancel{\sqrt{n}} \rightarrow n^{\frac{1}{2}}$

$\frac{1}{n}$, 4000 , $3\log 2n$, $\log n^4$, $12n$, $4n^2$, $n^3 + n - 100$

2. Give running time analysis for each of the following pseudocode fragments, assuming that the input size is *n*. Also determine a tight upper bound. Show your work.

2.1.

```
sum ← 0            } T1
i ← 0
WHILE i < n DO
    j ← 0
    WHILE j < n DO
        sum ← sum + 1
        sum ← sum + 1
        j ← j + 2
    END WHILE
    i ← i + 1
END WHILE
```

T_2

$k = 0, 1, 2, 3, \ldots$

$j = 0, 2, 4, 6, \ldots, n-1$

$j = 2k$, $k = 0, \ldots, \frac{(n-1)}{2}$

$$T(n) = T_1 + T_2$$
$$= C_1 + \sum_{i=0}^{n-1}\left(\sum_{k=0}^{\frac{n-1}{2}} C_2 + C_3\right)$$
$$= C_1 + \sum_{i=0}^{n-1}\left[\left(\frac{n-1}{2}+1\right)C_2 + C_3\right]$$
$$= C_1 + \sum_{i=0}^{n-1}\left[\left(\frac{n+1}{2}\right)C_2 + C_3\right]$$
$$= C_1 + \sum_{i=0}^{n-1}\left[n\frac{C_2}{2} + \frac{C_2}{2} + C_3\right]$$
$$= C_1 + n(n-1+1)C_4 + (n-1+1)C_5$$
$$= C_1 + n^2C_4 + nC_5 \Rightarrow O(n^2)$$

เจอตัวเอง
แล้วคูณ 2 กระเด้งออก

2.2.

```
sum ← 0            } T1
i ← n
WHILE i > 1 DO
    j ← n - 1
    WHILE j >= 0 DO
        sum ← sum + 1
        j ← j - 1
    END WHILE
    i ← i / 2
END WHILE
```

T_2

$i = \frac{n}{2}, \frac{n}{4}, \frac{n}{8}, \frac{n}{16}, \ldots, \frac{n}{\frac{n}{2}}$

2^0

$2^x \rightarrow 2^x = \frac{n}{2}$

$\log_2 \frac{n}{2} = x$

$\log_2 n + \log_2(\frac{1}{2}) = x$

$\log_2 n + (-1) = x$

$\boxed{x = \log_2 n - 1}$

$= \frac{n}{2^k}$; $k = 0, 1, \ldots, \log_2 n - 1$

$j = n-1, n-2, n-3, \ldots, n-n$

$= n - m$; $m = 1, \ldots, n$

$$T(n) = T_1 + T_2$$
$$= C_1 + \sum_{k=0}^{\log_2 n - 1}\left(\sum_{m=1}^{n} C_2 + C_3\right)$$
$$= C_1 + \sum_{k=0}^{\log_2 n - 1}\left(nC_2 + C_3\right)$$
$$= C_1 + [(\log_2 n - 1) + 1](n)C_2 + (\log_2 n - 1 + 1)C_3$$
$$= C_1 + n\log_2 n C_2 + \log_2 n C_3$$
$$\Rightarrow O(n\log_2 n)$$

➢ สูตร Log ที่สำคัญ จำได้ยัง !!

1. $\log_a 1 = 0$
2. $\log_a a = 1$
3. $\log_a xy = \log_a x + \log_a y$
4. $\log_a \frac{x}{y} = \log_a x - \log_a y$
5. $\log_a x^n = n \log_a x$
6. $\log_{a^n} x = \frac{1}{n} \log_a x$
7. $\log_a x = \frac{1}{\log_x a}$
8. $a^{\log_a x} = x$
9. $\log_a x = \frac{\log x}{\log a}$

เพิ่มเติม

10. $a^{\log_b c} = c^{\log_b a}$

$2^x = \frac{n}{2}$

$\log_2 2^x = \log_2 \frac{n}{2}$

$x \cancel{\log_2 2}^{1} = \log_2 \frac{n}{2}$

$x(1) = \log_2 \frac{n}{2}$

$x = \log_2 n(\frac{1}{2})$

$x = \log_2 n + \log_2(\frac{1}{2})$

$x = \log_2 n + \log_2 2^{-1}$

$x = \log_2 n + (-1)$

$= \log_2 n - 1$

$C_1 + \sum_{k=0}^{\log_2 n - 1} (\sum_{m=0}^{n} C_2 + C_3)$

$C_1 + \sum_{k=0}^{\log_2 n - 1} (nC_2 + C_3)$

$C_1 + (\log_2 n \cancel{-1} \cancel{+1})(nC_2) + (\log_2 n \cancel{-1} \cancel{+1})(C_3)$

$C_1 + n\log_2 n\, C_2 + \log_2 n\, C_3$

$O(n\log_2 n)$

$i=0$, $j=1$ — 2^0
$i=1$, $j=1,2$ — 2^1
$i=2$, $j=1,2,4$ — 2^2
$i=3$, $j=1,2,4,8$ — 2^3

$j=2^k$ $\Rightarrow$ $i^2=2^k$

$j=i^2$ $\quad k=\log_2 i^2$

2.3.

T_1:
```
sum ← 0
i ← 0
```
T_2:
```
WHILE i < n DO
    sum ← sum + 1
    i ← i + 1
END WHILE
```
T_3:
```
i ← 0
```
T_4:
```
WHILE i < n DO
    j ← 1
    WHILE j < i * i DO
        sum ← sum + 1
        j ← j * 2
    END WHILE
    i ← i + 1
END WHILE
```

i^2

$$T(n) = T_1 + T_2 + T_3 + T_4$$
$$= C_1 + \sum_{i=0}^{n-1} C_2 + C_3 + \sum_{i=0}^{n-1}\left(C_4 + \sum_{k=0}^{\log_2 i^2} C_5\right)$$
$$= C_1 + (n-1+1)C_2 + C_3 + \sum_{i=0}^{n-1}\left(C_4 + (\log_2 i^2 + 1)C_5\right)$$
$$= C_1 + nC_2 + \sum_{i=0}^{n-1}\left(\underbrace{C_4 + \log_2 i^2 C_5 + C_5}\right)$$
$$= C_1 + nC_2 + (n-1+1)C_6 + \sum_{i=0}^{n-1}(2C_5\log_2 i)$$
$$C_1 + nC_2 + (n-\cancel{1+1})C_6 + (n-\cancel{1+1})\,2C_5\log_2 i$$

Therefore $T(n) = O(n\log n)$

2.4.

T_1:
```
sum ← 0
i ← 0
```
T_2:
```
WHILE i < n DO
    if i % 4 = 0 THEN            C3
        j ← 0
        WHILE j < i * i DO       C4
            sum ← sum + 1
            j ← j + 1
        END WHILE
    END IF
    i ← i + 1
END WHILE
```

$i=0$, $j=0$ $\quad (4\times 0)^2$
$i=4$, $j=0,1,2,\dots,4^2$ $\quad (4\times 1)^2$
$i=8$, $j=0,\dots,8^2$ $\quad (4\times 2)^2$
$i=12$, $j=0,\dots,16^2$ $\quad (4\times 3)^2$

$$\frac{2n}{4}+1 \qquad \frac{n}{2}+1)$$

$$T(n) = T_1 + T_2$$
$$= C_1 + \sum_{i=0}^{n-1} C_2 + \sum_{i=0}^{n-1}\left(\tfrac{1}{4}C_3\right) + \sum_{k=0}^{n/4}(4k)^2 C_4$$
$$= C_1 + (n-1+1)C_2 + (n-1+1)C_3 + \cancel{16}^{\,8} C_4 \frac{\left(\frac{n}{4}\right)\left(\frac{n}{4}+1\right)\left(\frac{n}{2}+1\right)}{\cancel{6}_{\,3}}$$
$$= C_1 + nC_2 + nC_3 + \tfrac{8}{3}C_4\left(\tfrac{n^2}{4}+\tfrac{n}{4}\right)\left(\tfrac{n}{2}+1\right)$$
$$= C_1 + nC_2 + nC_3 + \tfrac{8}{3}C_4\left(\tfrac{n^3}{8}+\tfrac{n^2}{4}+\tfrac{n^2}{8}+\tfrac{n}{4}\right)$$
$$= C_1 + nC_5 + \frac{C_4}{3}n^3 + \frac{2C_4}{3}n^2 + \frac{C_4}{3}n^2 + \frac{2C_4}{3}n$$
$$= C_1 + nC_5 + C_6 n^3 + C_7 n^2 + C_8 n$$
$$\Rightarrow O(n^3)$$

$$\sum_{i=1}^{n} k = n \times k$$
$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$
$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$
$$\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$$

$$C_1 + \sum_{i=0}^{n-1}\left(\frac{1}{4}\sum_{i=0}^{n-1}\left(\sum_{j=0}^{i/4} C_2\right)\right)$$
$$C_1 + \sum_{i=0}^{n-1}\left(\frac{1}{4}\sum_{i=0}^{n-1}\left(\frac{i}{4}+1\,C_2\right)\right)$$
$$C_1 + \sum_{i=0}^{n-1}\left(\frac{1}{4}\sum_{i=0}^{n-1}\left(\frac{i}{4}C_2 + C_2\right)\right)$$
$$C_1 + \sum_{i=0}^{n-1}\left(\frac{1}{4}\sum_{i=0}^{n-1}\left(iC_3 + C_2\right)\right)$$
$$C_1 + \sum_{i=0}^{n-1}\left(\frac{1}{4}\frac{((n-1)(n-1+1)}{2}C_3 + \frac{1}{4}C_2(n-\cancel{1+1})\right)$$
$$C_1 + \sum_{i=0}^{n-1} \frac{(n-1)(n)}{2}C_4 + nC_5$$
$$C_1 + \sum_{i=0}^{n-1}\left(n^2 - n\,C_4 + nC_5\right)$$
$$C_1 + (n-1+1)(n^2-n)C_4 + (n-1+1)nC_5$$
$$C_1 + n(n^2-n)C_4 + n(n)C_5$$
$$C_1 + n^3 - n^2 C_4 + n^2 C_5$$
$$O(n^3)$$

$i \leq n \longrightarrow n$ รอบ

```
n = input
i = 0
while i^2 < n
    j = i+1
    while j < i+8
        k = j+1
        while k < j+8

        k++
        end
    j++
    end
i++
end
```

$T_1 = C$

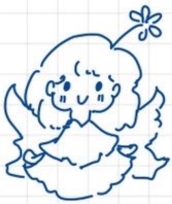
T_2

$\sqrt{n} - 1$

$i+8-1 = i+7$

$j+7$

$$T(n) = T_1 + T_2$$
$$= C_1 + \sum_{i=0}^{\sqrt{n}-1}\left(\sum_{j=i+1}^{i+7}\left(\sum_{k=j+1}^{j+7} C_2\right)\right)$$
$$= C_1 + \sum_{i=0}^{\sqrt{n}-1}\left(\sum_{j=i+1}^{i+7}\left(\overbrace{(\cancel{j}+7-(\cancel{j}+1))}^{6} C_2\right)\right)$$
$$= C_1 + \sum_{i=0}^{\sqrt{n}-1}\left(\sum_{j=i+1}^{i+7} C_3\right)$$
$$= C_1 + \sum_{i=0}^{\sqrt{n}-1}\left(\overbrace{(\cancel{i}+7-(\cancel{i}+1))}^{6} C_3\right)$$
$$= C_1 + \sum_{i=0}^{\sqrt{n}-1} C_4$$
$$= C_1 + (\sqrt{n} \cancel{-1} \cancel{+1}) C_4$$
$$= C_1 + \sqrt{n} C_4$$
$$\Longrightarrow O(\sqrt{n})$$



$i = 0, \ldots, \sqrt{n} - 1$

$i = 0, \quad j = 1, \ldots, 8 \rightarrow$ 6 ครั้ง

$i = 1, \quad j = 2, \ldots, 9 \rightarrow$ 6 ครั้ง

$100 \rightarrow 10$

$? \leftarrow 40$

$n = 100$ เวลา 10

$\sqrt{100} = 10$

$n = ?$ เวลา 40

$\sqrt{x} = 40$

$x = 1600$