DREXEL UNIVERSITY
Department of Mechanical Engineering & Mechanics

MEM 633

Robust Control Systems I

Final Exam (Rev.1)

1. Hand in the Exam Report **hard copy** by 12:00Noon, Friday, 12/13/2019.
2. Upload **executable** computer programs to Bb Learn by 12:00Noon, 12/13/2019.

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Attention:

- The exam report should be **self-contained, complete,** clearly written and well organized according to the order of the problems. Do not write on the back of the paper. If you need more space, you can add only 8.5x11 letter size white paper.
- **All of you** are required to upload all **executable** computer files: m-files, slx-files, etc. to Drexel Bb Learn by 12:00Noon, Friday, 12/13/2019. These files should be grouped into one zip file with filename: MEM633Final_YourName.
- **Show detailed procedure!** A solution without detailed procedure will receive no credit even the answer is correct.
- Open books and notes.
- **Absolutely no discussions about the exam before, during, or even after the test until all the students turn in their exams.**
- **Use discreet judgment to determine if it is appropriate to use MATLAB (or other software) commands to obtain the solutions. Do NOT use a black-box command to answer the questions that may defeat the purpose of the test.**
- **Do NOT use MATLAB or any software to automatically generate a report for the exam.**

If there is any question about the exam, please contact Dr. Chang (changbc@drexel.edu).

Problem #1: (30%)

Consider the following transfer function matrix

$$H(s) = \frac{\begin{bmatrix} -s-2 & -2 \\ s+2 & s+3 \end{bmatrix}}{(s-4)(s+1)^2}$$

(a) Find a state-space realization of $H(s)$ in block controller form. (10%)

(b) Use PBH Test to check the controllability and observability of the realization. (5%)

(c) Is it a minimal realization? If not, find a similarity transform to transform the realization into either a controllability decomposition or an observability decomposition form. Then find a minimal realization by eliminating the noncontrollable and/or nonobservable parts of the canonical form. (10%)

(d) Determine the poles and zeros by using the (A, B, C) of the minimal realization in 1(c). (5%)
Note that the zeros of (A, B, C) are the values of s at which the rank of

$$\begin{bmatrix} sI - A & B \\ -C & 0 \end{bmatrix}$$

drops below its normal rank.

1-a) given

$$H(s) = \frac{\begin{bmatrix} -s-2 & -2 \\ s+2 & s+3 \end{bmatrix}}{(s-4)(s+1)^2} = \frac{N(s)}{d(s)}$$

$u(s) \rightarrow \boxed{H(s)} \rightarrow y(s)$

$Y = Hu$

we can rewrite $H(s)$ as

$$H(s) = N(s) D_R^{-1}(s)$$

where

$$D_R(s) = d(s) \cdot I_2$$

Then

$$Y(s) = N(s) D_R^{-1}(s) u(s)$$

Define

$$\xi(s) \triangleq D_R^{-1}(s) u(s)$$

Then

$$Y(s) = N(s) \xi(s)$$

ok, next, let's find $D_R(s)$

$$D_R(s) = d(s) \cdot I_2 = (s^3 - 2s^2 - 7s - 4) I_2$$

and

$$\xi(s) = D_R^{-1}(s) u(s) \longrightarrow D_R(s) \xi(s) = u(s)$$

Plug $D_R(s)$ to above eq, $D_R(s)\xi(s) = u(s)$

$$(s^3 - 2s^2 - 7s - 4) I_s \xi(s) = u(s)$$

$$(s^3\xi(s) - 2s^2\xi(s) - 7s\xi(s) - 4\xi(s)) I_s = u(s)$$

Inverse Laplace...

$$(\dddot{\xi}(t) - 2\ddot{\xi}(t) - 7\dot{\xi}(t) - 4\xi(t)) I_s = u(s)$$

good...

Let's define $N(s)$....

$$N(s) = N_1 s^2 + N_2 s + N_3 \quad (N_1 \text{ is identically zero here})$$

Plug $N(s)$ in eq, $Y(s) = N(s)\xi(s)$

$$Y(s) = N_1 s^2 \xi + N_2 s \xi + N_3 \xi$$

Taking inverse laplace ...

$$Y(t) = N_1 \ddot{\varsigma}(t) + N_2 \dot{\varsigma}(t) + N_3 \varsigma(t)$$

Thus, we have system ...

$$\dddot{\varsigma} I_2 = 2\ddot{\varsigma} I_2 + 7\dot{\varsigma} I_2 + 4\varsigma I_2 + u$$

$$Y = N_1 \ddot{\varsigma} + N_2 \dot{\varsigma} + N_3 \varsigma$$

In state space form... (Block Controller form)

Define

$$\begin{aligned} \varsigma_1 &= \ddot{\varsigma} \\ \varsigma_2 &= \dot{\varsigma} \\ \varsigma_3 &= \varsigma \end{aligned} \longrightarrow \begin{bmatrix} \dot{\varsigma}_1 \\ \dot{\varsigma}_2 \\ \dot{\varsigma}_3 \end{bmatrix} = \begin{bmatrix} 2I_2 & 7I_2 & 4I_2 \\ I_2 & 0 & 0 \\ 0 & I_2 & 0 \end{bmatrix} \begin{bmatrix} \varsigma_1 \\ \varsigma_2 \\ \varsigma_3 \end{bmatrix} + \begin{bmatrix} I_2 \\ 0 \\ 0 \end{bmatrix} u$$

$$Y = \begin{bmatrix} N_1 & N_2 & N_3 \end{bmatrix} \begin{bmatrix} \varsigma_1 \\ \varsigma_2 \\ \varsigma_3 \end{bmatrix}$$

$$\left(N = \underset{\substack{\uparrow \\ N_1}}{0} s^2 \varsigma + s \underbrace{\begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}}_{N_2} \varsigma + \underbrace{\begin{bmatrix} -2 & -2 \\ 2 & 3 \end{bmatrix}}_{N_3} \varsigma \right)$$

Thus...

$$A = \begin{bmatrix} 2 & 0 & 7 & 0 & 4 & 0 \\ 0 & 2 & 0 & 7 & 0 & 4 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 & -1 & 0 & -2 & -2 \\ 0 & 0 & 1 & 1 & 2 & 3 \end{bmatrix}$$

1-b) PBH Test

rank([sI-A B]) → convert it into hermiteform to check the rank manually

$$\rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & -s^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -s^2 & 0 & 0 \\ 0 & 0 & 1 & 0 & -s & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -s & 0 & 0 \\ 0 & 0 & 0 & 0 & (s^3-2s^2-7s-4) & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & (s^3-2s^2-7s-4) & 0 & 1 \end{bmatrix}$$

These two columns (5th & 6th) can be made of a linear combination of (1st, 2nd, 3rd, 4th, 7th, 8th) columns.

Thus, rank([sI-A B]) = 6 = n = 6

Thus, it is $\boxed{\text{controllable}}$

$\text{rank}\left(\begin{bmatrix} sI-A \\ C \end{bmatrix}\right)$ → convert it into hermite form to check the rank

$$\rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & s+1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

6th column can be made of a linear combination of (1st, 2nd, 3rd, 4th, 5th) columns.

However, when s = -1, that 6th column is linearly dependent. However, theorem requires "$\text{rank}\left(\begin{bmatrix} sI-A \\ C \end{bmatrix}\right) = n$ for all s"*

Because for s = -1, the 6th column is linearly dependent, the $\text{rank}\left(\begin{bmatrix} sI-A \\ C \end{bmatrix}\right) = 5 \neq n = 6$

Thus, $\boxed{\text{not observable}}$

1-c)

It is not a minimal realization because it is controllable, but not observable.

We can find the similarity matrix by...

$T^{-1} = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}$ where U_1 = any 5 linearly independent rows of observability matrix (obsv(A,C))

U_2 = 1 row that is linearly independent of U_1

$$T^{-1} = \begin{bmatrix} \text{first five rows of obsv(A,C)} \\ 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & -1 & 0 & -2 & -2 \\ 0 & 0 & 1 & 1 & 2 & 3 \\ -1 & 0 & -2 & -2 & 0 & 0 \\ 1 & 1 & 2 & 3 & 0 & 0 \\ -4 & -2 & -7 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Thus T is..

$$T = (T^{-1})^{-1} = \begin{bmatrix} 6 & 4.66 & 0.33 & -1.33 & -0.66 & -2 \\ -1 & -1 & 1 & 1 & 0 & 1 \\ -4 & -3.33 & -0.66 & 0.66 & 0.33 & 2 \\ 1 & 1 & 0 & 0 & 0 & -1 \\ 1.5 & 1.66 & 0.33 & -0.33 & -0.166 & -2 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Thus, with T_1 I can form my observability decomposition form.

$$\bar{A} = \left[\begin{array}{ccccc|c} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 4 & 4 & 3 & 3 & -1 & 0 \\ 4 & 0 & 0 & 0 & 2 & 0 \\ \hline 1 & 1 & 0 & 0 & 0 & -1 \end{array}\right], \quad \bar{B} = \left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \\ -1 & 0 \\ 1 & 1 \\ -4 & -2 \\ \hline 0 & 0 \end{array}\right]$$

(upper-left 5×5 block $\to \bar{A}_{11}$; upper 5×2 block $\to \bar{B}_1$)

$$\bar{C} = \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{array}\right]$$

(left 2×5 block $\to \bar{C}_1$)

By eliminating non-observable part, we have $\{\bar{A}_{11}, \bar{B}_1, \bar{C}_1\}$ that is a minimal realization!

$\left(\text{rank}(\bar{A}_{11}) = 5,\ \underset{= 5 \,\to\, \text{controllable}}{\text{rank}(\text{ctrb}(\bar{A}_{11}, \bar{B}_1))},\ \underset{= 5 \,\to\, \text{observable}}{\text{rank}(\text{obsv}(\bar{A}_{11}, \bar{C}_1))} \right)$

1-d)

$$\text{Pole} = \text{eig}(\bar{A}_{11}) = \underline{4, 4, -1, -1, -1}$$

$$\text{zero} \Rightarrow \left| \lambda \underbrace{\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}}_{E} - \underbrace{\begin{bmatrix} \bar{A}_{11} & -\bar{B}_1 \\ \bar{C}_1 & -D \end{bmatrix}}_{Z} \right| = 0 \Rightarrow \text{eig}\left(\begin{bmatrix} \bar{A}_{11} & -\bar{B}_1 \\ \bar{C}_1 & -D \end{bmatrix}, \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \right)$$

$$\left(D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right) \qquad \Downarrow$$

$$\underline{\text{zeros} = -2}$$

Problem #2: (37%)

For the transfer function matrix $H(s)$ shown in Problem #1, which can be represented as

$$H(s) = N(s)D(s)^{-1}$$

where

$$N(s) = \begin{bmatrix} -s-2 & -2 \\ s+2 & s+3 \end{bmatrix}, \quad D(s) = (s-4)(s+1)^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(a) Find a greatest common right divisor (gcrd) of $N(s)$ and $D(s)$. (10%)

(b) Find an irreducible right MFD for $H(s)$ by extracting the gcrd of $N(s)$ and $D(s)$. (4%)

(c) Determine the poles and zeros of the system based on the irreducible MFD in (b). (3%)

(d) Find a state-space realization for the right MFD in (c). (10%)

(e) Is the state-space realization controllable? observable? Is it a minimal realization? (5%)

(f) Can you find a similarity transformation which relates the realizations in Problem #1(c) and Problem #2(d)? If yes, show the results and procedure. If not, explain. (5%)

2-a) given

$$H(s) = N(s)D(s)^{-1} \quad \text{(right MFD)}$$

$$N(s) = \begin{bmatrix} -s-2 & -2 \\ s+2 & s+3 \end{bmatrix}, \quad D(s) = \begin{bmatrix} (s-4)(s+1)^2 & 0 \\ 0 & (s-4)(s+1)^2 \end{bmatrix}$$

using unimodular matrix U, the gcrd, R, can be found

$$\begin{bmatrix} U_{11}(s) & U_{12}(s) \\ U_{21}(s) & U_{22}(s) \end{bmatrix} \begin{bmatrix} D(s) \\ N(s) \end{bmatrix} = \begin{bmatrix} R(s) \\ 0 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} -\frac{1}{6} & 0 & (-\frac{s^2}{6}+\frac{s}{3}+\frac{3}{2}) & (\frac{5}{3}-\frac{s}{3}) \\ 0 & 0 & 1 & 1 \\ (s+2) & 0 & (s^3-13s-12) & (2s^2-6s-8) \\ 0 & 1 & (-s^2+3s+4) & (-s^2+3s+4) \end{bmatrix} \begin{bmatrix} (s-4)(s+1)^2 & 0 \\ 0 & (s-4)(s+1)^2 \\ -s-2 & -2 \\ s+2 & s+3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & (s+1) \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Thus, gcrd of $N(s)$ and $D(s)$ is...

$$R(s) = \begin{bmatrix} 1 & 2 \\ 0 & (s+1) \end{bmatrix}$$

Full Derivation of unimodular matrix and gcrd can be found in next pages...

2-b) By definining,

$$N_1 = N(s)R(s)^{-1} = \begin{bmatrix} -s-2 & 2 \\ s+2 & -1 \end{bmatrix}$$

$$D_1 = D(s)R(s)^{-1} = \begin{bmatrix} (s+1)^2(s-4) & (-2s^2+6s+8) \\ 0 & s^2-3s-4 \end{bmatrix}$$

We can find irreducible right MFD.

$$H = N_1(s)D_1(s)^{-1} = \downarrow$$

$$= \begin{bmatrix} -\dfrac{(s+2)}{(s+1)^2(s-4)} & \dfrac{2}{(-s^2+3s+4)(s+1)} \\[2ex] \dfrac{(s+2)}{(s+1)^2(s-4)} & -\dfrac{(s+3)}{(-s^2+3s+4)(s+1)} \end{bmatrix}_{\text{"}}$$

2-c) Numerator = 0 → zeros → $s = -2, -3, 0$

Denominator = 0 → poles → $s = -1, 4$

However! By definition, zeros are the values of s at which the rank(H) drops below its normal rank ($n=2$)

When $s_{zeros} = -3, 0$, rank(H) stays as 2. so no $s = -3, 0$

$$\boxed{\begin{aligned} &\text{Thus,} \\ &\quad s_{zeros} = -2 \\ &\quad s_{poles} = -1, 4 \end{aligned}}_{\text{"}}$$

(Also
· zeros can be found from N_1
Poles can be found from D_1)

2-d) State-space

$$N_1(s) = \begin{bmatrix} -s-2 & 2 \\ s+2 & -1 \end{bmatrix}$$

$$D_1(s) = \begin{bmatrix} s^3-2s^2-7s-4 & -2s^2+6s+8 \\ 0 & s^2-3s-4 \end{bmatrix}$$

$$= \begin{bmatrix} s^3 & -2s^2 \\ 0 & s^2 \end{bmatrix} + \begin{bmatrix} -2s^2-7s-4 & 6s+8 \\ 0 & -3s-4 \end{bmatrix}$$

$k_1 = 3$, $k_2 = 2$, $k_1 + k_2 = 5 = \deg(\det D_1(s))$ "

$$D_1(s) = D_{hc} S(s) + D_{Ic} \psi(s)$$

$$= \underbrace{\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}}_{D_{hc}} \underbrace{\begin{bmatrix} s^3 & 0 \\ 0 & s^2 \end{bmatrix}}_{S(s)} + \underbrace{\begin{bmatrix} -2 & -7 & -4 & 6 & 8 \\ 0 & 0 & 0 & -3 & -4 \end{bmatrix}}_{D_{Ic}} \underbrace{\begin{bmatrix} s^2 & 0 \\ s & 0 \\ 1 & 0 \\ 0 & s \\ 0 & 1 \end{bmatrix}}_{\psi(s)}$$

$$N_1(s) = N_{Ic} \psi(s)$$

$$= \underbrace{\begin{bmatrix} 0 & -1 & -2 & 0 & 2 \\ 0 & 1 & 2 & 0 & -1 \end{bmatrix}}_{N_{Ic}} \underbrace{\begin{bmatrix} s^2 & 0 \\ s & 0 \\ 1 & 0 \\ 0 & s \\ 0 & 1 \end{bmatrix}}_{\psi(s)}$$

Then we can form the state space form.

$$A_c^o = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad B_c^o = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$A_c = A_c^o - B_c^o D_{hc}^{-1} D_{Ic} = \begin{bmatrix} 2 & 7 & 4 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} //$$

$$B_c = B_c^o D_{hc}^{-1} = \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} //, \quad C_c = N_{Ic} = \begin{bmatrix} 0 & -1 & -2 & 0 & 2 \\ 0 & 1 & 2 & 0 & -1 \end{bmatrix} //$$

Thus,

$$\begin{cases} \dot{x} = A_c x + B_c u \\ y = C_c x \end{cases} //$$

2-e)

$$\mathrm{ctrb}(A_c, B_c) = \begin{bmatrix} B & AB & A^2B & A^3B & A^4B \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 2 & 4 & 11 & 22 & 40 & 80 & 165 & 330 \\ 0 & 0 & 1 & 2 & 2 & 4 & 11 & 22 & 40 & 80 \\ 0 & 0 & 0 & 0 & 1 & 2 & 2 & 4 & 11 & 22 \\ 0 & 1 & 0 & 3 & 0 & 13 & 0 & 51 & 0 & 205 \\ 0 & 0 & 0 & 1 & 0 & 3 & 0 & 13 & 0 & 51 \end{bmatrix}$$

$\mathrm{rank}(\mathrm{ctrb}(A_c, B_c)) = 5 = n \rightarrow$ controllable!

$$\mathrm{obsv}(A_c, C_c) = \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \\ CA^4 \end{bmatrix} = \begin{bmatrix} 0 & -1 & -2 & 0 & 2 \\ 0 & 1 & 2 & 0 & -1 \\ -1 & -2 & 0 & 2 & 0 \\ 1 & 2 & 0 & -1 & 0 \\ -4 & -9 & -4 & 6 & 8 \\ 4 & 7 & 4 & -3 & -4 \\ -15 & -32 & -16 & 26 & 24 \\ 15 & 32 & 16 & -13 & -12 \\ -62 & -121 & -60 & 102 & 104 \\ 62 & 121 & 60 & -51 & -52 \end{bmatrix} \rightarrow \mathrm{rank}(\mathrm{obsv}(A_c, C_c)) = 5 \downarrow \text{observable!}$$

Both controllable and observable (and minimal size) $\rightarrow$ Thus, the state-space realization is a minimal realization!

Also, degree of this minimal realization is 5, thus if $\deg(\det D_1) = 5$, then this realization is minimal.

Let's check...

$$\det D_1 = s^5 - 5s^4 - 5s^3 + 25s^2 + 40s + 16 \rightarrow \deg(\det D_1) = 5 \checkmark$$

2-f) Realization from 1-c)...

$$\bar{A}_{11} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 4 & 4 & 3 & 3 & -1 \\ 4 & 0 & 7 & 0 & 2 \end{bmatrix}, \quad \bar{B}_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -1 & 0 \\ 1 & 1 \\ -4 & -2 \end{bmatrix}, \quad \bar{C}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

Realization from 2-d)...

$$A_c = \begin{bmatrix} 2 & 7 & 4 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad B_c = \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad C_c = \begin{bmatrix} 0 & -1 & -2 & 0 & 2 \\ 0 & 1 & 2 & 0 & -1 \end{bmatrix}$$

So my similarity matrix from 1-c) to 2-d)...

$$T = C_{1-c} C_{2-d}^T \left(C_{2-d} C_{2-d}^T \right)^{-1}$$

$$= \text{ctrb}(\bar{A}_{11}, \bar{B}_1)\, \text{ctrb}(A_c, B_c)' \left(\text{ctrb}(A_c, B_c)\, \text{ctrb}(A_c, B_c)' \right)^{-1}$$

$$= \begin{bmatrix} 0 & -1 & -2 & 0 & 2 \\ 0 & 1 & 2 & 0 & -1 \\ -1 & -2 & 0 & 2 & 0 \\ 1 & 2 & 0 & -1 & 0 \\ -4 & -7 & -4 & 6 & 8 \end{bmatrix}$$

✓

Let's check...

$$T^{-1} \bar{A}_{11} T = \begin{bmatrix} 2 & 7 & 4 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} = A_c$$ ✓ Yes, it matches

$$T^{-1} \bar{B}_1 = \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = B_c$$ ✓ Yes, it matches

$$\tilde{C}_1 T = \begin{bmatrix} 0 & -1 & -2 & 0 & 2 \\ 0 & 1 & 2 & 0 & -1 \end{bmatrix} = C_c \checkmark$$ Yes, it matches.

"Similarity matrix between two minimal realization is unique"

2-a) Continued

$$\begin{bmatrix} D \\ N \end{bmatrix} = \begin{bmatrix} (s-4)(s+1)^2 & 0 \\ 0 & (s-4)(s+1)^2 \\ -s-2 & -2 \\ s+2 & s+3 \end{bmatrix} + \longrightarrow \begin{bmatrix} (s-4)(s+1)^2 & 0 \\ 0 & (s-4)(s+1)^2 \\ -s-2 & -2 \\ 0 & s+1 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} s^3-2s^2-7s-4 & 0 \\ 0 & 0 \\ -s-2 & -2 \\ 0 & s+1 \end{bmatrix} \times(s-2)^2 \longrightarrow \begin{bmatrix} -3s-12 & 0 \\ 0 & 0 \\ -s-2 & -2 \\ 0 & s+1 \end{bmatrix} \times -3 \longrightarrow \begin{bmatrix} -6 & 6 \\ 0 & 0 \\ -s-2 & -2 \\ 0 & s+1 \end{bmatrix} \begin{matrix} \times -(s+1)(s-4) \\ \leftarrow \div 6 \end{matrix} \longrightarrow$$

$$\begin{bmatrix} -1 & 1 \\ 0 & 0 \\ -s-2 & -2 \\ 0 & s+1 \end{bmatrix} -s \longrightarrow \begin{bmatrix} -1 & 1 \\ 0 & 0 \\ -2 & -s-2 \\ 0 & s+1 \end{bmatrix} \longrightarrow \begin{bmatrix} -1 & 1 \\ 0 & 0 \\ -2 & -1 \\ 0 & s+1 \end{bmatrix} \times -1 \longrightarrow$$

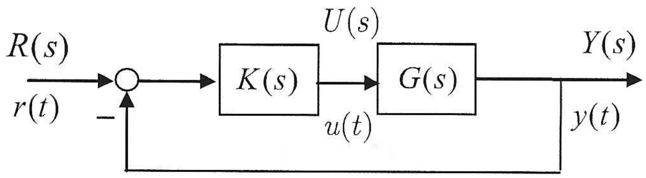
$$\begin{bmatrix} -1 & 1 \\ 0 & 0 \\ -1 & -2 \\ 0 & s+1 \end{bmatrix} \times -1 \longrightarrow \begin{bmatrix} 0 & 3 \\ 0 & 0 \\ -1 & -2 \\ 0 & s+1 \end{bmatrix} \leftarrow \times(s+1) \longrightarrow \begin{bmatrix} 0 & 3s+3 \\ 0 & 0 \\ -1 & -2 \\ 0 & s+1 \end{bmatrix} \times -3$$

$$\longrightarrow \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -1 & -2 \\ 0 & s+1 \end{bmatrix} \longrightarrow \begin{bmatrix} -1 & -2 \\ 0 & s+1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \leftarrow \times -1 \longrightarrow \begin{bmatrix} 1 & 2 \\ 0 & s+1 \\ \hline 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} R \\ \hline \phi \end{bmatrix}$$

✓

Problem #3: (13%)

Consider the following system,



where

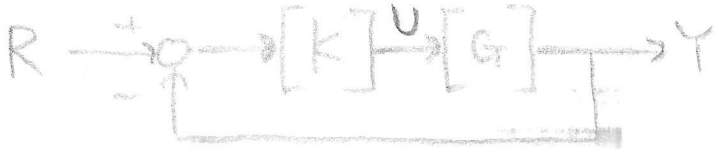
$$G(s) = \frac{s+1}{s^2 - 2s}.$$

(a) Design a proportional controller $K(s) = K_0$ so that the closed-loop system is stable with damping ratio $\varsigma = 0.9$, and then compute the corresponding natural frequency ω_n. (5%)

(b) Let $r(t) = u_s(t)$, the unit step function. Plot $y(t)$ and $u(t)$, and comment on the closed-loop system performance. (3%)

(c) Comment on how the change of K_0 will affect the damping ratio, the natural frequency, and the performance of the closed-loop system. (5%)

3-a) given



where $G = \dfrac{s+1}{s^2-2s}$

$$\Downarrow$$

$$R \longrightarrow \left[\frac{KG}{1+KG}\right] \longrightarrow Y$$

$\hookrightarrow H(s)$

$$H(s) = \frac{KG}{1+KG} = \frac{\frac{K(s+1)}{s^2-2s}}{\frac{s^2-2s}{s^2-2s} + \frac{K(s+1)}{s^2-2s}} = \frac{Ks+K}{s^2+(K-2)s+K}$$

and we compare with typical 2nd order system ....

$$\frac{W_n^2}{s^2+2\zeta W_n s + W_n^2} \quad \text{vs} \quad \frac{Ks+K}{s^2+(K-2)s+K}$$

given $\zeta = 0.9$ ...

$$\left.\begin{aligned} 1.8\,W_n &= K-2 \\ W_n^2 &= K \end{aligned}\right) \begin{matrix}\text{system} \\ \text{of} \\ \text{equation}\end{matrix} \longrightarrow 1.8\,W_n = W_n^2 - 2$$

$$\downarrow$$

$$W_n^2 - 1.8\,W_n - 2 = 0$$

$$\downarrow$$

$$W_n = 2.576,\ \cancel{-0.776}$$

$$K = 6.636,\ \cancel{0.602}$$

But as we can see, our closed loop system does not "exactly" the same as typical 2nd order system, thus, we had approximate $W_n = 2.576$ and $K = 6.636$. Let's use "matlab" to "exactly" investigate our closed loop system! $\rightarrow$ rltool$\left(\dfrac{s+1}{s^2-2s}\right)$

3-b) unit step response

$$R \rightarrow [H] \rightarrow Y$$

$$Y = HR$$

$$Y = \frac{ks+k}{s^2+(k-2)s+k} \cdot \overbrace{\frac{1}{s}}^{\text{unit step}} = \frac{ks+k}{s^3+(k-2)s^2+ks}$$

Inverse Laplace ...

$$\mathcal{L}^{-1}(Y(s)) = y(t) = 1 - e^{-t\left(\frac{k}{2}-1\right)}\left(\cosh\left(t\sqrt{\frac{k^2}{4}-2k+1}\right) - \frac{\sinh\left(t\sqrt{k^2/4-2k+1}\right)\left(k/2+1\right)}{\sqrt{k^2/4-2k+1}}\right)$$

With $k = 6.6391$, we can plot step response, $Y(t)$, of the system...

For $u(t)$...

$$U = kE, \quad E = R - Y, \quad Y = GU$$

$$\hookrightarrow U = k(R-Y) = kR - kY = kR - kGU$$

$$\hookrightarrow U + kGU = kR$$

$$\hookrightarrow U(1+kG) = kR \rightarrow U = \frac{kR}{1+kG}$$

$$U = \frac{\frac{k}{s}}{\frac{s^2-2s}{s^2-2s} + \frac{k(s+1)}{s^2-2s}} = \frac{k(s^2-2s)}{s^3+(k-2)s^2+ks}$$

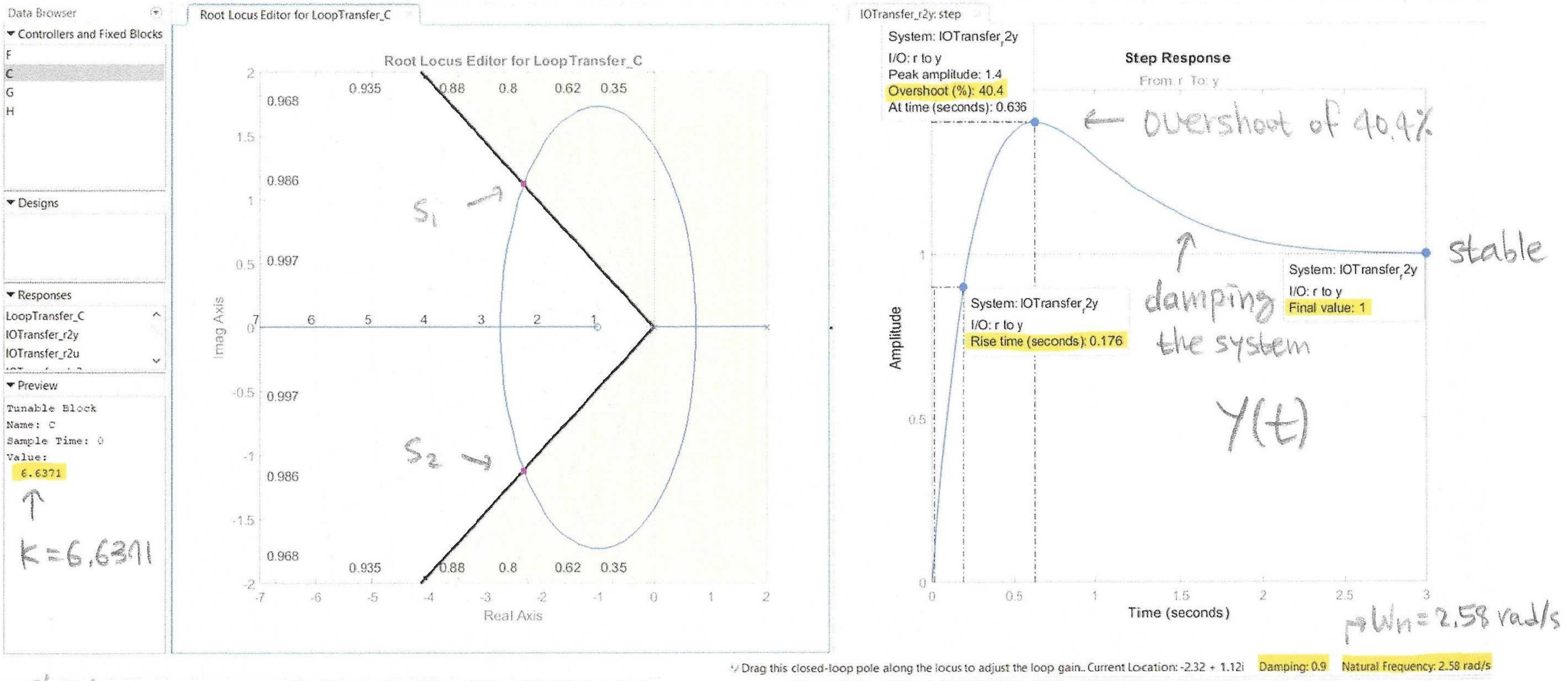
Taking inverse laplace transform ...

$$L^{-1}(U(s)) = u(t) = k e^{-t\left(\frac{k}{2}-1\right)}\left(\cosh\left(t\sqrt{\tfrac{k^2}{4}-2k+1}\right) - \frac{\sinh\left(t\sqrt{\tfrac{k^2}{4}-2k+1}\right)\left(\frac{k}{2}+1\right)}{\sqrt{\tfrac{k^2}{4}-2k+1}}\right)$$

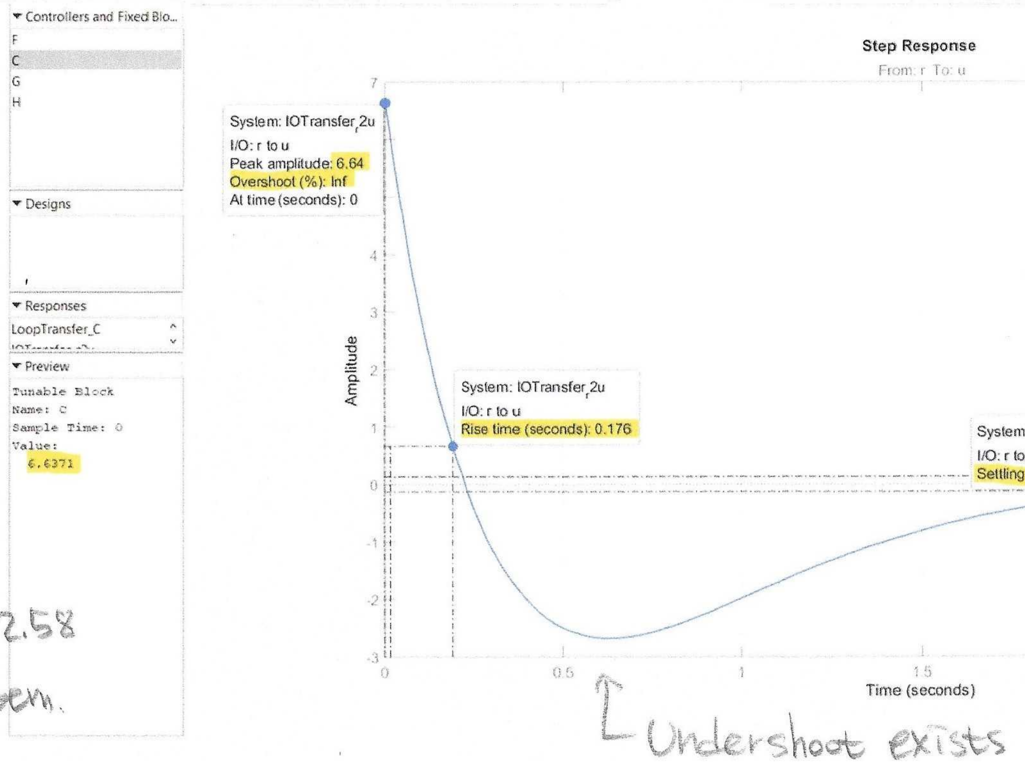
With $k = 6.6371$ found from 3-a), we can plot $u(t)$ ...

3-a,b)

With $k=6.6391$ system become stable, and we can see that the poles, s_1, s_2 stay on left side of complex plane



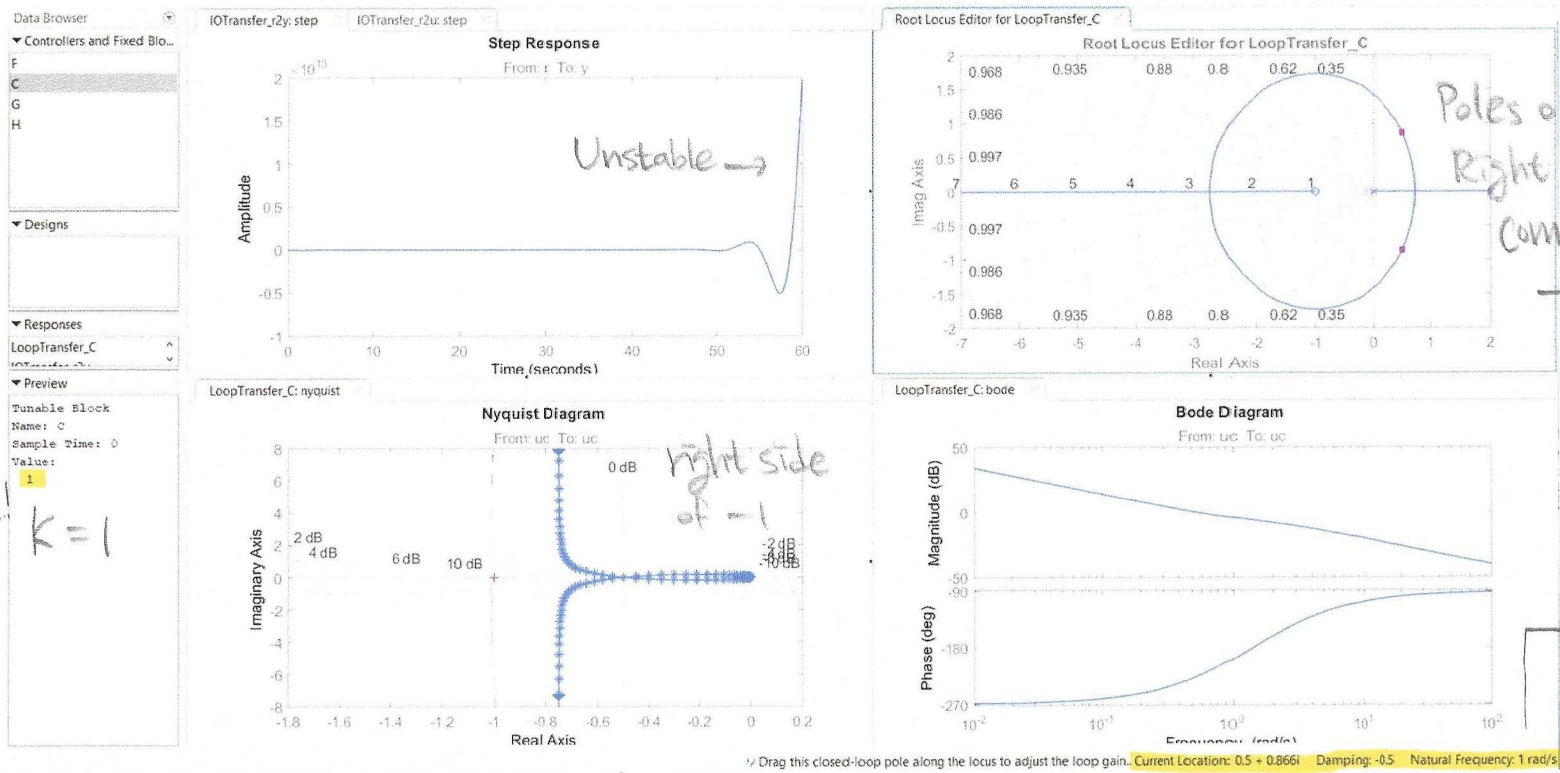
Y(t) gives overshoot of 40.4% during transient, but settle down in approximately 2.5~3 seconds. With $\zeta=0.9$ we will have $W_n=2.58$ rad/s for the system.



3 - c)

Let's increase k and see what happens!

K = 1



Unstable →

right side of −1

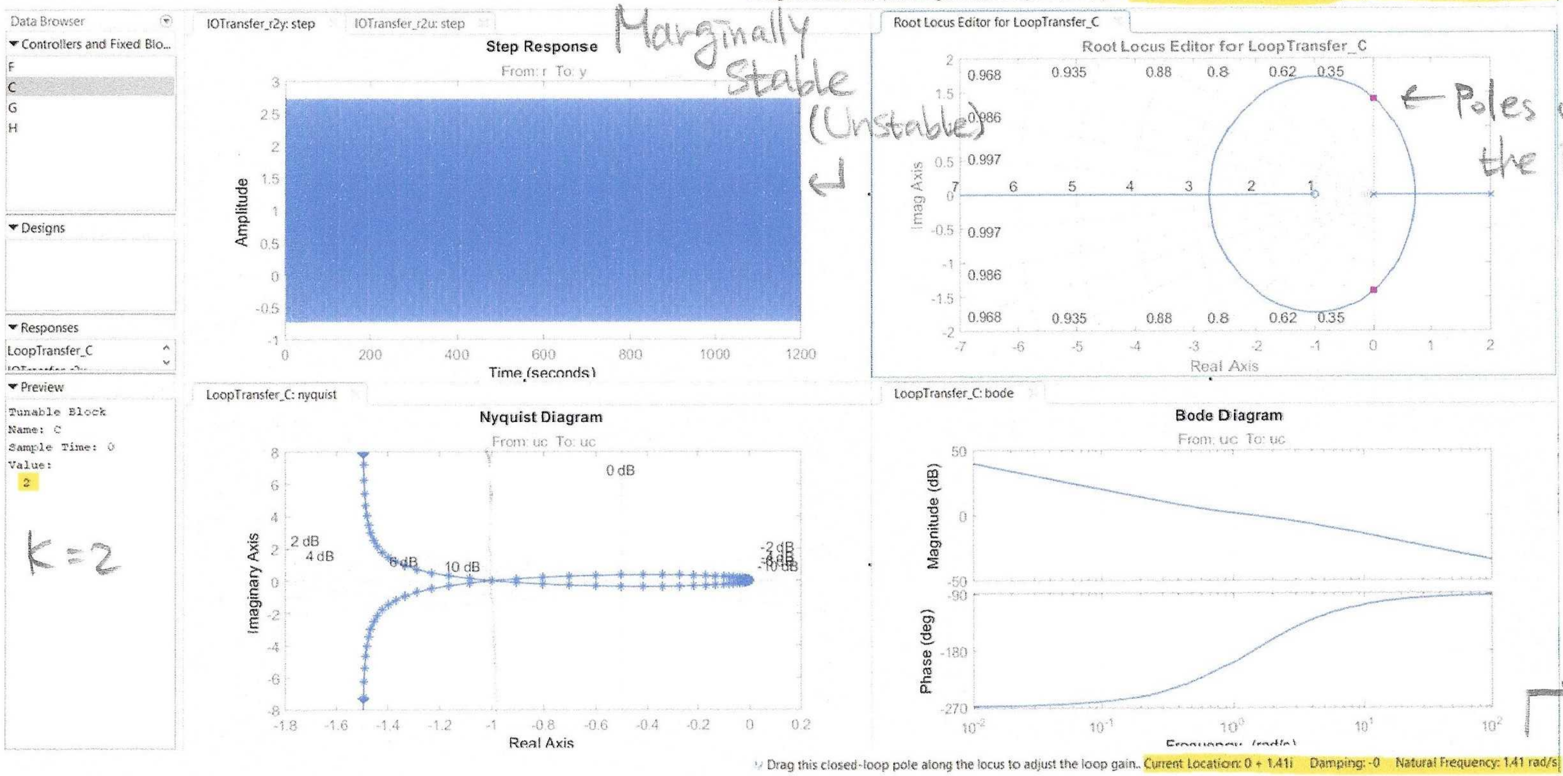
Poles on Right side of complex plane → unstable

$\zeta_1 = -0.5$

$W_{n_1} = 1$ rad/s

K = 2

Marginally stable (Unstable)



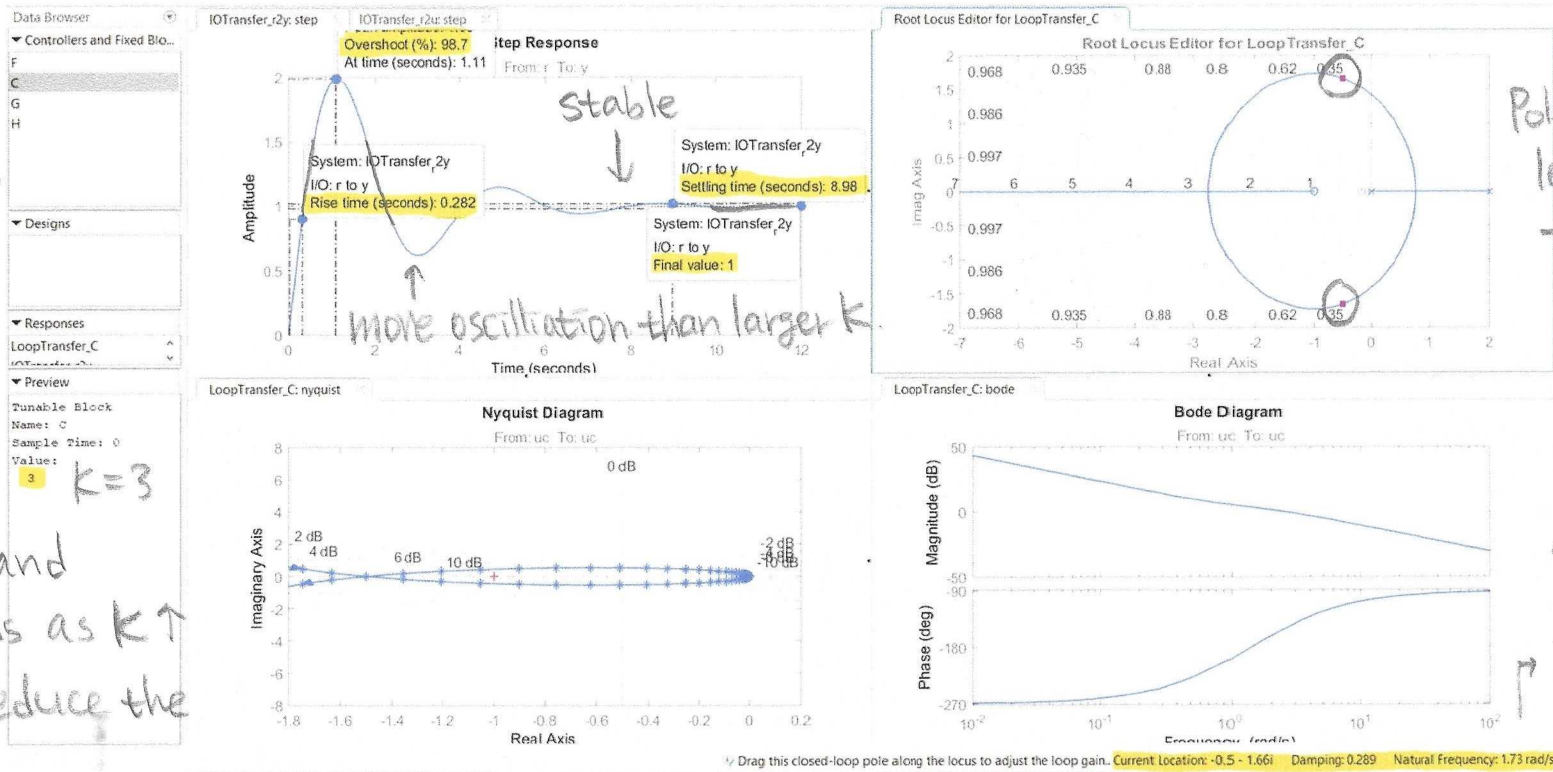
← Poles on the imaginary axis

$\zeta_2 = 0$

$W_{n_2} = 1.41$ rad/s

As can be seen from k=1,2,3,6 cases as k increases ζ damping ratio increases, thus system gets damped more and oscillates less as k↑ This will help reduce the settling time of the system.

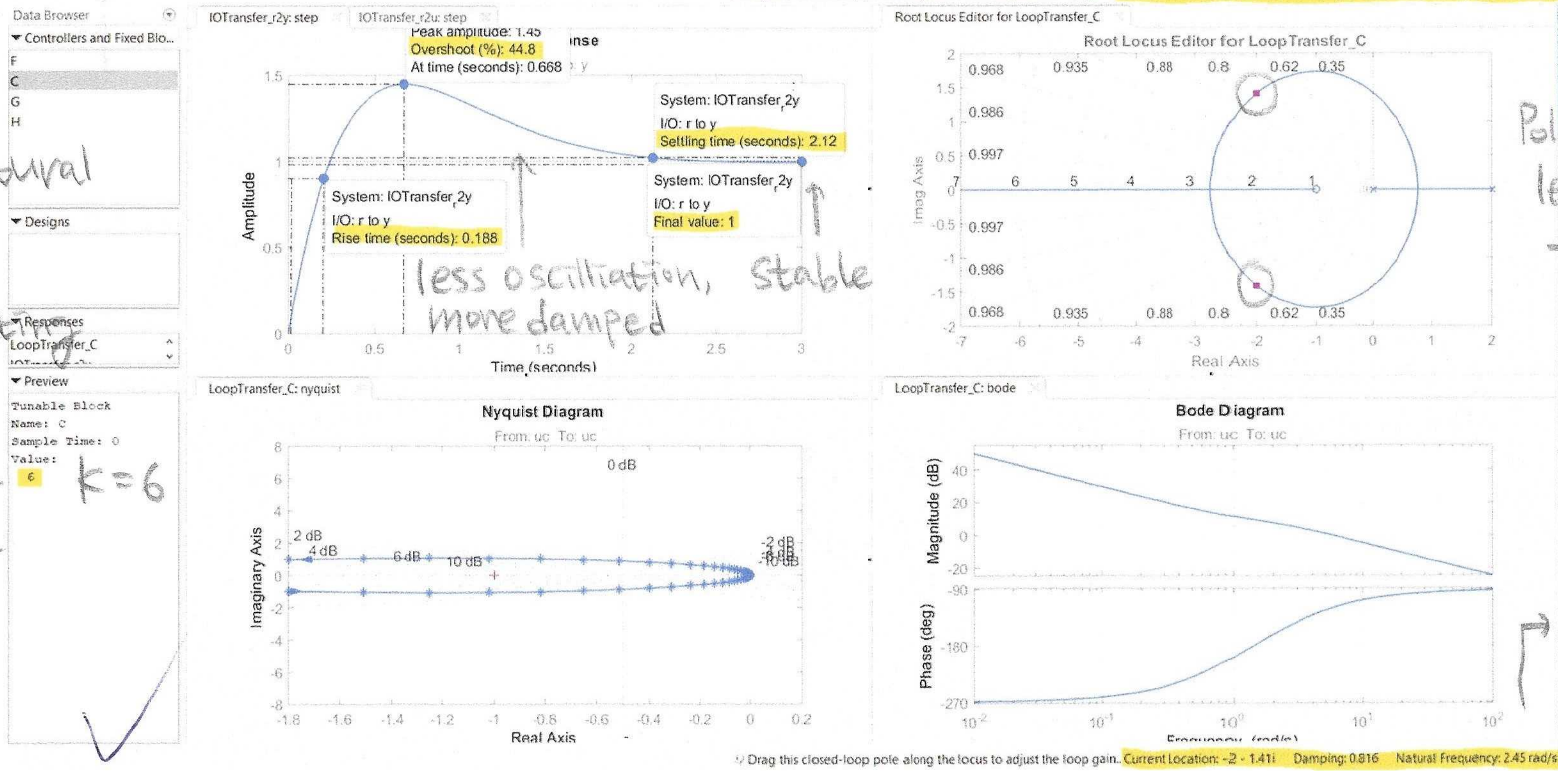
As k↑, Wn natural frequency also increases, affecting the system (overshoot, oscillation, settling time, etc.)



Poles on left side → stable

$\zeta_3 = 0.289$

$W_{n3} = 1.73$ rad/s



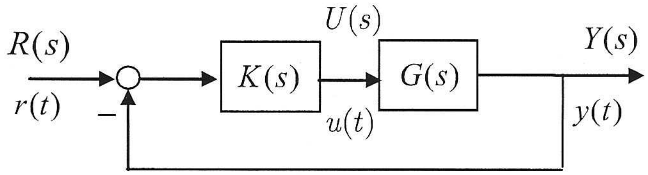
Poles on left side → stable

$\zeta_6 = 0.816$

$W_{n6} = 2.45$ rad/s

Problem #4: (20%)

Consider the following system,



where

$$G(s) = \frac{s-1}{s^2 - 2s}.$$

(a) Show that there exists no proportional controller $K(s) = K_0$ to stabilize the system. (3%)

(b) Show that there exists no PI controller $K(s) = K_p + \frac{K_i}{s}$ to stabilize the system. (3%)

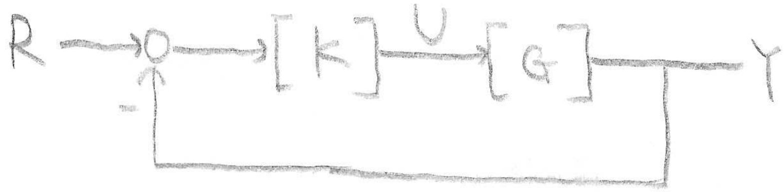
(c) Show that the first-order controller $K(s) = \frac{b_1 s + b_0}{s + a_0}$ can be employed to place the closed-loop system poles at any desired locations on the complex plane by choosing a_0, b_1, and b_0. (3%)

(d) Choose the values of a_0, b_1, and b_0 in the first-order controller so that the closed-loop system poles are at $s_1, s_2 = -\varsigma\omega_n \pm j\omega_n\sqrt{1-\omega_n^2}$ with $\varsigma = 0.9$ and $\omega_n = 1\,\text{rad/s}$, and $s_3 = -10$. Then plot the responses $y(t)$ and $u(t)$ due to $r(t) = u_s(t)$, the unit step function. (4%)

(e) Choose the values of a_0, b_1, and b_0 in the first-order controller so that the closed-loop system poles are at $s_1, s_2 = -\varsigma\omega_n \pm j\omega_n\sqrt{1-\omega_n^2}$ with $\varsigma = 0.9$ and $\omega_n = 2\,\text{rad/s}$, and $s_3 = -10$. Then plot the responses $y(t)$ and $u(t)$ due to $r(t) = u_s(t)$, the unit step function. (4%)

(f) Comment on how the choice of the natural frequency ω_n affects the closed-loop system performance based on the results of 4(d) and 4(e). (3%)

4-a) given



where $G = \frac{s-1}{s^2-2s}$

if I have a proportional gain...

$$H = \frac{Y}{R} = \frac{KG}{1+KG} = \frac{\frac{K(s-1)}{s^2-2s}}{\frac{s^2-2s}{s^2-2s} + \frac{K(s-1)}{s^2-2s}} = \frac{K(s-1)}{s^2+(K-2)s-K}$$

when you investigate the characteristic equation of the transfer function

$$s^2+(K-2)s-K=0 \rightarrow s = 1 - \frac{\sqrt{K^2+4}}{2} - \frac{K}{2} \rightarrow \text{always negative}$$

$$\rightarrow s = \frac{\sqrt{K^2+4}}{2} - \frac{K}{2} + 1 \rightarrow \text{always positive}$$

One of the poles will always stay on right side of the imaginary axis no matter what proportion gain value is.
Thus, proportional gain does not exist to stabilize the system.
(shown in root locus plot in next page)

4-b) given

$$K = K_p + \frac{K_i}{s}$$

$$H = \frac{Y}{R} = \frac{KG}{1+KG} = \frac{K(s-1)}{s^2+(K-2)s-K} = \frac{(K_p+\frac{K_i}{s})(s-1)}{s^2+(K_p+\frac{K_i}{s}-2)s-(K_p+\frac{K_i}{s})}$$

$$= \frac{k_p s - \frac{k_i}{s} - k_p + k_i}{s^2 + (k_p - 2)s - (k_p + \frac{k_i}{s} - k_i)} = \frac{k_p s^2 + (k_i - k_p)s - k_i}{s^3 + (k_p - 2)s^2 + (k_i - k_p)s - k_i}$$

Let's do routh table...

s^3	1	$k_i - k_p$
s^2	$k_p - 2$	$-k_i$
s^1	A	0
s^0	B	

where...

$$A = -\frac{\begin{vmatrix} 1 & k_i - k_p \\ k_p - 2 & -k_i \end{vmatrix}}{k_p - 2} = -\frac{k_p^2 - 2k_p - k_i k_p + k_i}{k_p - 2}$$

$$B = -\frac{\begin{vmatrix} k_p - 2 & -k_i \\ A & 0 \end{vmatrix}}{A} = -k_i$$

ok let's check → # of sign change in 1st column
= # of poles in right hand plane.

k_p	k_i	# of sign change
+	+	1
+	−	1
−	+	2
−	−	1

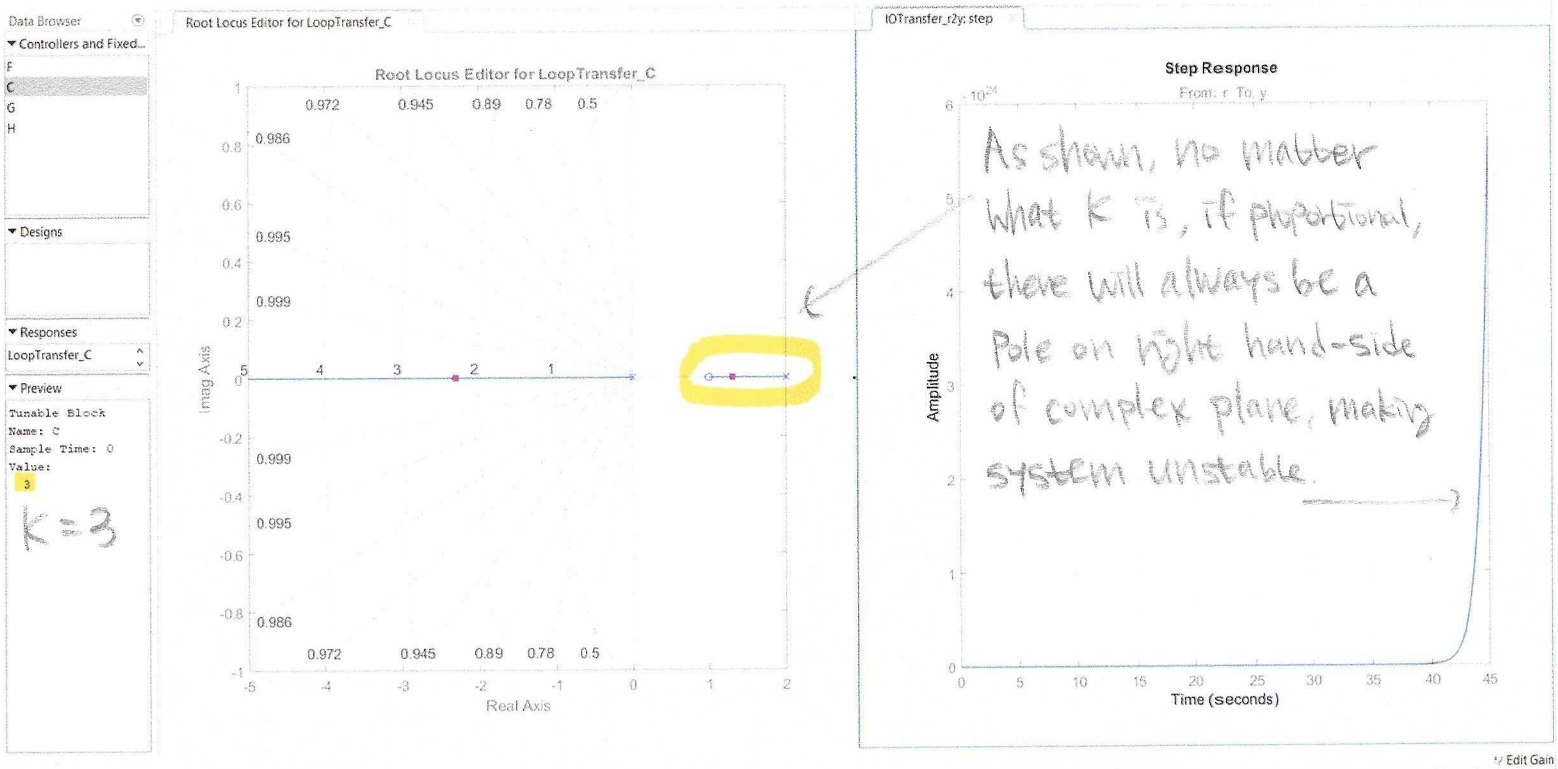
$(k_p > 2)$

k_p	k_i	# of sign change
+	+	2
+	−	2
−	+	3
−	−	2

$(k_p < 2)$

In all cases, there was at least "one" pole on the right hand side of the complex plane, making the system unstable. Thus, no matter what the value of k_p and k_i is, there exists <u>no</u> PI controller to stabilize the system. (please also see the root locus plot next page)

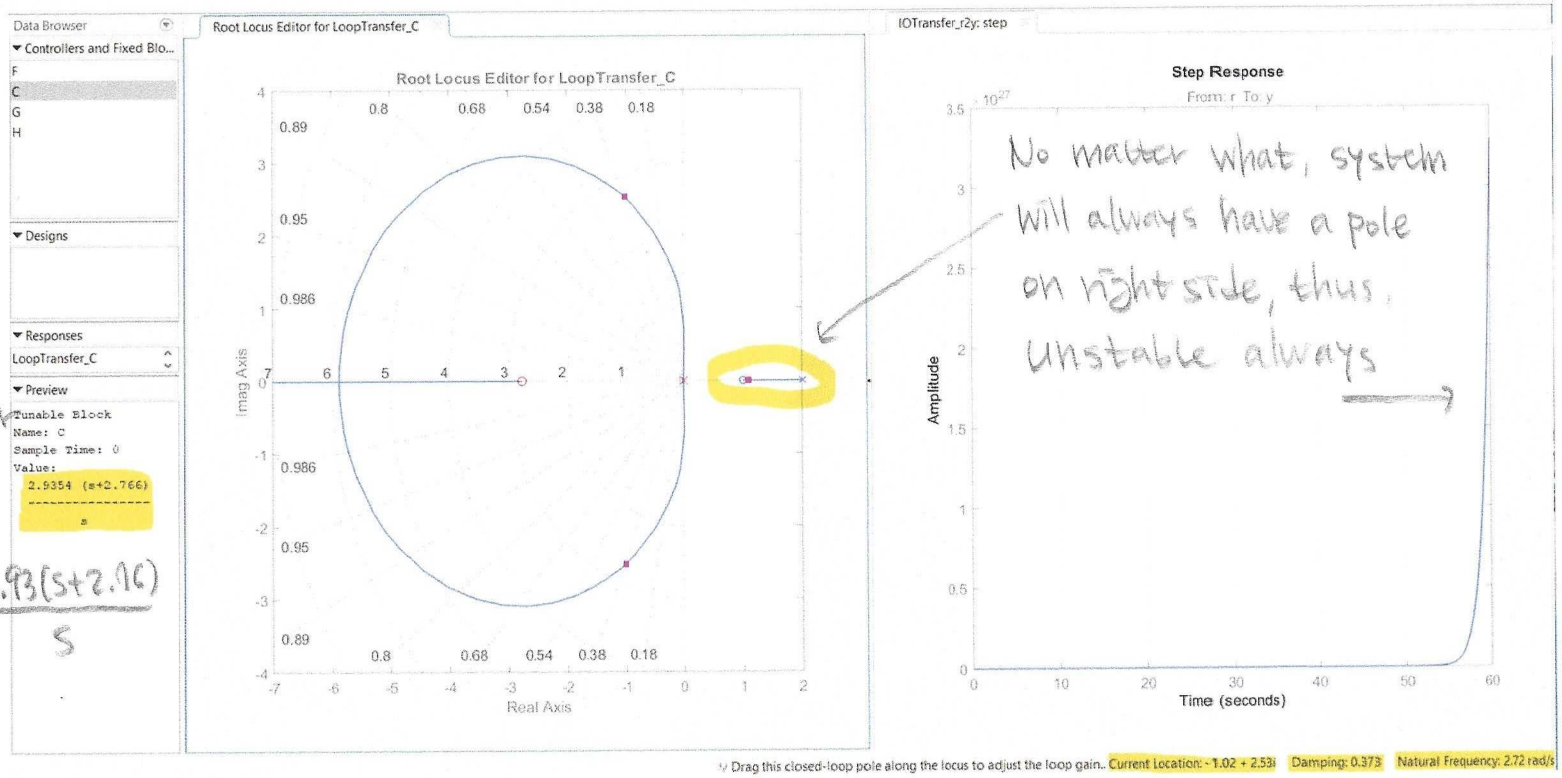
4-a)



K=3

As shown, no matter what K is, if proportional, there will always be a pole on right hand-side of complex plane, making system unstable.

4-b)



PI controller

$$K = \frac{2.93(s+2.76)}{s}$$

No matter what, system will always have a pole on right side, thus unstable always

4-c) given $K = \dfrac{b_1 s + b_0}{s + a_0}$

$$H = \frac{Y}{R} = \frac{KG}{1+KG} = \frac{K(s-1)}{s^2+(K-2)s-K}$$

$$= \frac{\left(\dfrac{b_1 s + b_0}{s+a_0}\right)(s-1)}{s^2+\left(\dfrac{b_1 s + b_0}{s+a_0}-2\right)s-\left(\dfrac{b_1 s+b_0}{s+a_0}\right)}$$

$$= \frac{\dfrac{b_1 s^2+(b_0-b_1)s-b_0}{s+a_0}}{\dfrac{s^3+a_0 s^2}{s+a_0}+\dfrac{b_1 s^2+b_0 s}{s+a_0}-\dfrac{2s^2+2a_0 s}{s+a_0}-\dfrac{b_1 s+b_0}{s+a_0}}$$

As shown, we can place poles at any desired location by choosing proper values for a_0, b_0, b_1

$$= \frac{b_1 s^2+(b_0-b_1)s-b_0}{s^3+(a_0+b_1-2)s^2+(b_0-2a_0-b_1)s-b_0} \quad //$$

Routh....

s^3	1	$(b_0-2a_0-b_1)$
s^2	(a_0+b_1-2)	b_0
s^1	A	0
s^0	b_0	

where

$$A = \frac{-\begin{vmatrix} 1 & (b_0-2a_0-b_1) \\ (a_0+b_1-2) & b_0 \end{vmatrix}}{(a_0+b_1-2)}$$

$$= \frac{4a_0-3b_0+2b_1+a_0b_0-3a_0b_1+b_0b_1-2a_0^2-b_1^2}{a_0+b_1-2}$$

4-d) $W_n = 1$ rad/s $\zeta = 0.9$

$$S_1 = -\zeta W_n + j W_n \sqrt{1-W_n^2} = -0.9$$

$$S_2 = -\zeta W_n - j W_n \sqrt{1-W_n^2} = -0.9 \qquad -1 \quad +1$$

$$S_3 = -10$$

Let's construct a desired characteristic equation...

$$(s+0.9)(s+0.9)(s+10) = s^3 + 11.8s^2 + 18.81s + 8.1 = 0$$

$\updownarrow$ compare with 4-c)...

$$s^3 + (a_0 + b_1 - 2)s^2 + (b_0 - 2a_0 - b_1)s - b_0 = 0$$

we have system of eq...

$$\begin{bmatrix} 1 & 0 & 1 \\ -2 & 1 & -1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} a_0 \\ b_0 \\ b_1 \end{bmatrix} = \begin{bmatrix} 13.8 \\ 18.81 \\ 8.1 \end{bmatrix} \sim> \begin{bmatrix} a_0 \\ b_0 \\ b_1 \end{bmatrix} = \begin{bmatrix} -40.71 \\ -8.1 \\ 54.51 \end{bmatrix}$$

✓

Thus, my first-order controller is...

$$K = \frac{54.51s - 8.1}{(S - 40.71)} = \frac{54.51(s - 0.1486)}{(S - 40.71)}$$

my closed-loop transfer function, $H(s)$, is...

$$R \longrightarrow \left[\frac{54.51s^2 - 62.61s + 8.1}{s^3 + 11.8s^2 + 18.81s + 8.1} \right] \longrightarrow Y$$

my $y(t)$ is ... to unit step...

$$y(t) = \mathcal{L}^{-1}(Y(s)) = \mathcal{L}^{-1}\left(\frac{H(s)}{s}\right) = \longrightarrow \text{please see } y(t) \text{ plot..}$$

my $u(t)$ is ... to unit step

$$u(t) = \mathcal{L}^{-1}(U(s)) = \mathcal{L}^{-1}\left(\frac{KR}{1+KG}\right) = \mathcal{L}^{-1}\left(\frac{K}{s(1+KG)}\right) = \longrightarrow \text{Please see } u(t) \text{ Plot...}$$

driven similarly from 3-b)

4-e) $W_n = 2$ rad/s $\zeta = 0.9$

$$S_1 = -\zeta W_n + \hat{j} W_n \sqrt{1-W_n^2} = -1.8 + 3.4641\hat{j}$$

$$S_2 = -\zeta W_n - \hat{j} W_n \sqrt{1-W_n^2} = -1.8 - 3.4641\hat{j}$$

$$S_3 = -10$$

Desired equation...

$$(s-(-1.8+3.4641\hat{j}))(s-(-1.8-3.4641\hat{j}))(s+10)$$

$$= s^3 + 13.6s^2 + 51.24s + 152.4$$

$\updownarrow$ compare with 4-c)

$$s^3 + (a_0+b_1-2)s^2 + (b_0-2a_0-b_1)s - b_0 = 0$$

System of eq...

$$\begin{bmatrix} 1 & 0 & 1 \\ -2 & 1 & -1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} a_0 \\ b_0 \\ b_1 \end{bmatrix} = \begin{bmatrix} 15.6 \\ 51.24 \\ 152.4 \end{bmatrix} \rightsquigarrow \begin{bmatrix} a_0 \\ b_0 \\ b_1 \end{bmatrix} = \begin{bmatrix} -219.24 \\ -152.4 \\ 234.84 \end{bmatrix}$$

Thus, my first-order controller is...

$$K = \frac{234.84s - 152.4}{s - 219.24} = \frac{234.84(s-0.6491)}{(s-219.24)}$$

my closed-loop transfer function, $H(s)$, is...

$$R \longrightarrow \left[\frac{234.84s^2 - 387.24s + 152.4}{s^3 + 13.6s^2 + 51.24s + 152.4} \right] \longrightarrow Y$$

my $Y(t)$ is ... to unit step

$$Y(t) = L(Y(s))^{-1} = L\left(\frac{H(s)}{s}\right)^{-1} = \rightarrow \text{Please see } Y(t) \text{ plot}$$

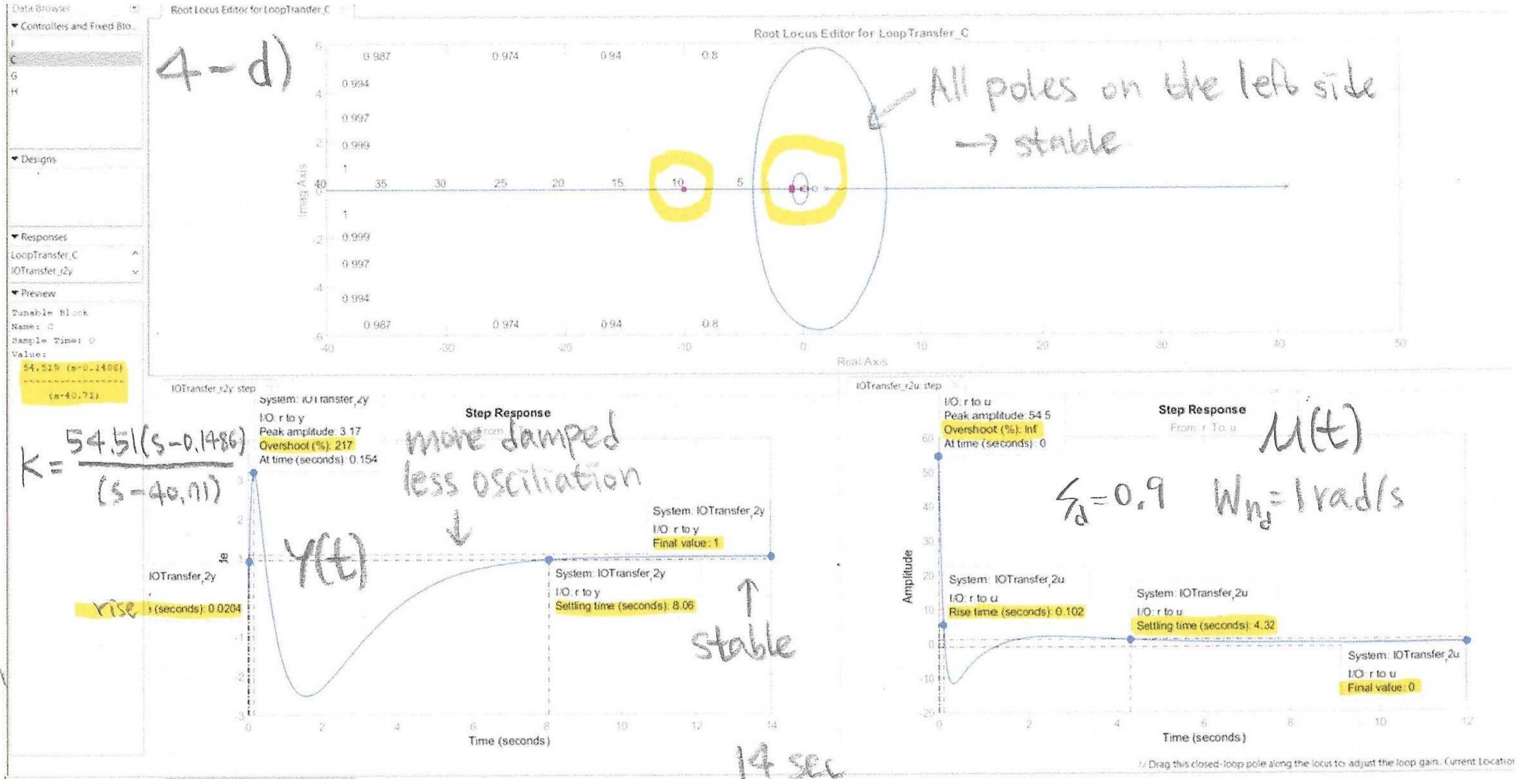
my $U(t)$ is ... to unit step

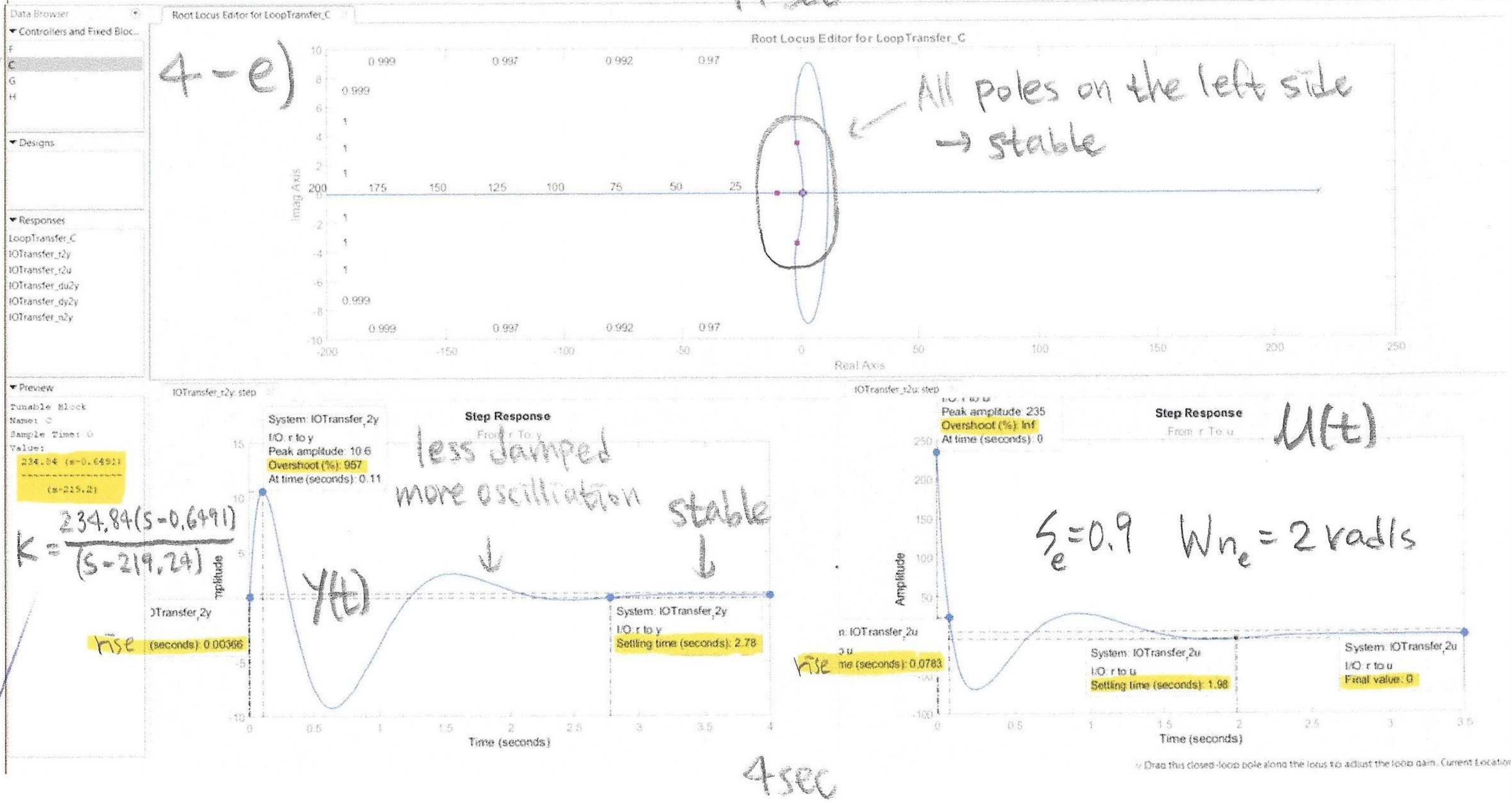
$$u(t) = L(U(s))^{-1} = L\left(\frac{KR}{1+KG}\right)^{-1} = L\left(\frac{K}{s(1+KG)}\right)^{-1} = \rightarrow$$ Please see $u(t)$ plot

driven similarly as 3-b)...

4-f)

As W_n, natural frequency increases, system's step response seems to have more oscillation and less damping. However, when considering the scale is different... Actually, as $W_n \uparrow$, although OS increased a lot but higher W_n increased the damping strength, which corelates to reduction in settling time from 8.06 s to 2.78. Thus, higher W_n damps the system faster but with high oscillation during transient.





DREXEL UNIVERSITY
Department of Mechanical Engineering & Mechanics

MEM 633

Robust Control Systems I

99 + 1

Final Exam (Rev.1)

1. Hand in the Exam Report **hard copy** by 12:00Noon, Friday, 12/13/2019.
2. Upload **executable** computer programs to Bb Learn by 12:00Noon, 12/13/2019.

NAME: June Kwon

Excellent.

Attention:

- The exam report should be **self-contained, complete,** clearly written and well organized according to the order of the problems. Do not write on the back of the paper. If you need more space, you can add only 8.5x11 letter size white paper.
- **All of you** are required to upload all **executable** computer files: m-files, slx-files, etc. to Drexel Bb Learn by 12:00Noon, Friday, 12/13/2019. These files should be grouped into one zip file with filename: MEM633Final_YourName.
- **Show detailed procedure!** A solution without detailed procedure will receive no credit even the answer is correct.
- Open books and notes.
- **Absolutely no discussions about the exam before, during, or even after the test until all the students turn in their exams.**
- **Use discreet judgment to determine if it is appropriate to use MATLAB (or other software) commands to obtain the solutions. Do NOT use a black-box command to answer the questions that may defeat the purpose of the test.**
- **Do NOT use MATLAB or any software to automatically generate a report for the exam.**

If there is any question about the exam, please contact Dr. Chang (changbc@drexel.edu).