DREXEL UNIVERSITY
Department of Mechanical Engineering & Mechanics

MEM 633

Robust Control Systems I

Mid-Term Exam

1. Hand in the Exam **hard copy** at 6:00PM, Monday, 11/11/2019, before the class.
2. Upload **executable** computer programs to Bb Learn before 11:59PM, 11/12/2019.

NAME: June Kwon

Attention:

- The exam report should be **self-contained, complete,** clearly written and well organized according to the order of the problems. Do not write on the back of the paper. If you need more space, you can add only 8.5x11 letter size white paper.
- **All of you** are required to upload all **executable** computer files: m-files, slx-files, etc. to Drexel Bb Learn by 11:59PM, Tuesday, 11/12/2019. These files should be grouped into one zip file with filename: MEM633midterm_YourName.
- **Show detailed procedure!** A solution without detailed procedure will receive no credit even the answer is correct.
- Open books and notes.
- **Absolutely no discussions about the exam before, during, or even after the test until all the students turn in their exams.**
- **Use discreet judgment to determine if it is appropriate to use MATLAB (or other software) commands to obtain the solutions. Do NOT use a black-box command to answer the questions that may defeat the purpose of the test.**
- **Do NOT use MATLAB or any software to automatically generate a report for the exam.**
- If there is any question about the exam, please contact Dr. Chang (changbc@drexel.edu).

4x3 3x1 4x1

$Ax = y$

Problem #1: (20%)

Consider the linear operator represented by the matrix

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & -1 \\ 3 & 0 & -3 \\ 4 & 2 & -2 \end{bmatrix}.$$

$A = [a_1\ a_2\ a_3]$ subspace. basis

(a) Find the range spaces $R(A)$. → rank

(b) Find the null spaces $N(A)$. → $Ax = 0$

→ 1 sol, 0 sol, ∞ sol? what is x?

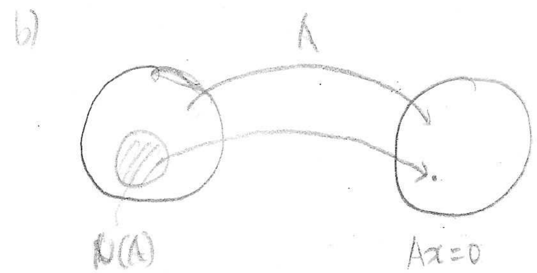
(c) Consider the equation $Ax = b$ with A matrix shown above. Find the condition on the vector b such that the equation has solutions.

7 4 3 8

(d) Consider the equation $Ax = b$ with $b^T = \cancel{[3\ \ 4\ \ 3\ \ 8]}$ and A matrix shown above. Find the full set of solutions for the equation.

7

b)



c) $b \in R(A)$

use Null Space

(Pick one for b combine with Null) → could be d)

1-a) $A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & -1 \\ 3 & 0 & -3 \\ 4 & 2 & -2 \end{bmatrix} \rightarrow A = [a_1\ a_2\ a_3]$ where $a_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$, $a_2 = \begin{bmatrix} 3 \\ 1 \\ 0 \\ 2 \end{bmatrix}$,

$a_3 = \begin{bmatrix} 2 \\ -1 \\ -3 \\ -2 \end{bmatrix} \rightarrow$ How many linearly independent vectors there are in A?

are there constants c_1, c_2, c_3 that satisfies

$$c_1 a_1 + c_2 a_2 + c_3 a_3 = 0 \quad ?$$

Yes, given $c_1 = -1$, $c_2 = 1$, $c_3 = -1$

we can have

$$-\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \\ 0 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ -1 \\ -3 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$ Thus, A has basis

$a_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$, $a_2 = \begin{bmatrix} 3 \\ 1 \\ 0 \\ 2 \end{bmatrix}$ that form 2-dimensional range space ✓

Also taking row reduced echelon form..

$\text{rref}(A) = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow$ clearly there 2 linearly independent vectors.

$$\left[\begin{array}{l} \text{Thus,} \\ \quad \text{Range Space } R(A) = 2 = \text{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 0 \\ 2 \end{bmatrix} \right\} \end{array}\right]$$

can be $\text{span}\{a_1, a_2\}$, $\text{span}\{a_1, a_3\}$, $\text{span}\{a_2, a_3\}$

1-b) Null Space

A being 4x3 matrix follows that $\underset{m\times n}{A} \rightarrow \dim(R(A)) + \dim(N(A)) = n$

Thus, $2 + \dim(N(A)) = 3 \rightarrow \underline{\dim(N(A)) = 1}$

Taking $Ax = 0$ to find the null space

also, when doing rref(A).. we have system like below

(taking $x = [x_1\ x_2\ x_3]'$)

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow \begin{matrix} x_1 = x_3 \\ x_2 = -x_3 \end{matrix} \rightarrow x = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} x_3$$

Thus, A has 1-dimensional null space that has

basis $[1\ -1\ 1]'$

1-c) $Ax = b$, taking $b = [b_1\ b_2\ b_3\ b_4]'$

We have

$$\left[\begin{array}{ccc|c} 1 & 3 & 2 & b_1 \\ 2 & 1 & -1 & b_2 \\ 3 & 0 & -3 & b_3 \\ 4 & 2 & -2 & b_4 \end{array}\right]$$

But, for the system $Ax = b$ to have solution (unique solution or ∞-many solution), not zero-solution

The reduced form of $[A|b]$ must be.

$b^* = [b_1^*\ b_2^*\ b_3^*\ b_4^*]'$ is b after reduction operation

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & b_1^* \\ 0 & 1 & 1 & b_2^* \\ 0 & 0 & 0 & b_3^* \\ 0 & 0 & 0 & b_4^* \end{array}\right]$$

here b^* must be...

b^* must be $b^* = [\alpha \quad \beta \quad 0 \quad 0]$ where α & β are any number, and b_3^* & b_4^* must be "zero" to ensure the existence of the solution (not zero-solution)

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & \alpha \\ 0 & 1 & 1 & \beta \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightarrow$$ This way, system will have ∞-many solution.

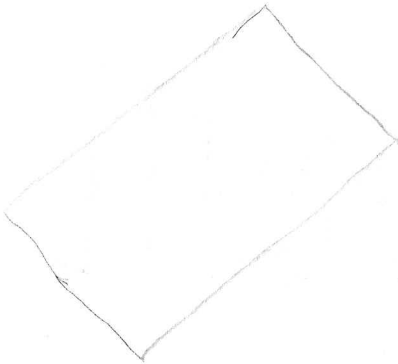
Also! from $[A|b]$, to ensure that solution exists

$$\left[\begin{array}{ccc|c} 1 & 3 & 2 & b_1 \\ 2 & 1 & -1 & b_2 \\ 3 & 0 & -3 & b_3 \\ 4 & 2 & -2 & b_4 \end{array}\right]$$

the $b = [b_1 \ b_2 \ b_3 \ b_4]^*$ must be

$$b = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 0 \\ 2 \end{bmatrix} \right\}$$

a linear combination of two basis of A



In plane of basis $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 0 \\ 2 \end{bmatrix}$, ✓

the b must lie in this plane!

so that x exists.

1-d) Given $Ax = b$ where $b = [7 \ 4 \ 3 \ 8]$ we have augmented

$$\left[\begin{array}{ccc|c} 1 & 3 & 2 & 7 \\ 2 & 1 & -1 & 4 \\ 3 & 0 & -3 & 3 \\ 4 & 2 & -2 & 8 \end{array}\right]$$

matrix $[A|b]$ as shown left if we take reduction form of $[A|b]$ ...

We will have

$$\text{rref}([A\ B]) = \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

3rd & 4th rows are $0=0$
Thus, there will be ∞-many solutions.

Thus $x_1 = 1 + x_3$
$x_2 = 2 - x_3$ $\longrightarrow x = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} x_3$

Particular Solution

Basis from Null space

Thus, the set of solutions for $Ax=b$ where $b = [7\ 4\ 3\ 8]'$ is

$$x = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} x_3$$

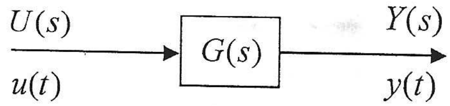
For ex

$$x_{s1}\Big|_{x_3=-1} = \begin{bmatrix} 0 \\ 3 \\ -1 \end{bmatrix}, \quad x_{s2}\Big|_{x_3=0} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad x_{s3}\Big|_{x_3=1} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

✓

Problem #2: (25%)

Consider the following block diagram



where $G(s)$ is given as

$$G(s) = \frac{s-3}{s^2+1}$$

(a) Find a bounded input that would cause the output to be unbounded, and use this example to verify that the system is not BIBO stable **based on** the **definition** of BIBO stability. Plot the output response due to this input. (10%)

(b) Find a state-space representation in the following form for the system $G(s)$. (5%)

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

(c) Explain why the system is not internally stable **based on** the **definition** of internal stability. Plot the state response, $x(t)$, of the system with initial state $x(0) = [2 \quad 2]^T$ and zero input. (10%)

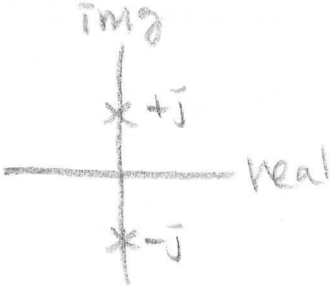
a) $s^2+1=0 \rightarrow s=\pm j \rightarrow$ Unstable ✓

b) Internal > BIBO
strength

2-a) Given transfer function $G = \frac{s-3}{s^2+1}$, let's investigate its poles.

$$s^2+1=0 \rightarrow s^2=-1 \rightarrow s=+j,-j$$

So the poles stay on the imaginary axis



Thus, the system is marginally stable (but, really it is unstable)

The definition of BIBO stability is [A system is said to BIBO stable (externally stable if for each $M_1<\infty$ there exists $M_2<\infty$ such that

$$|u(t)| \le M_1 \text{ implies } |y(t)| \le M_2 \,]$$

Thus, if there is an input that is bounded, but the output is unbounded, the system is BIBO unstable

let's find this input!

I picked $\sin(x)$ → bounded (in s-domain, $\frac{1}{s^2+1} = L^{-1}(\sin(x))$)

so the system output will be...

$$\frac{1}{s^2+1} \longrightarrow \boxed{\frac{s-3}{s^2+1}} \longrightarrow \frac{s-3}{s^4+2s^2+1}$$

let's take a inverse laplace of this input...

$$L^{-1}\left(\frac{s-3}{s^4+2s^2+1}\right) = -\frac{3\sin(t)}{2} + \frac{t\sin(t)}{2} + \frac{3t\cos(t)}{2}$$

Problem!!

We can see that in our output response, it contains x explicitly, thus! As $x \to \infty$, the output will go ∞. as shown in the plot next page!

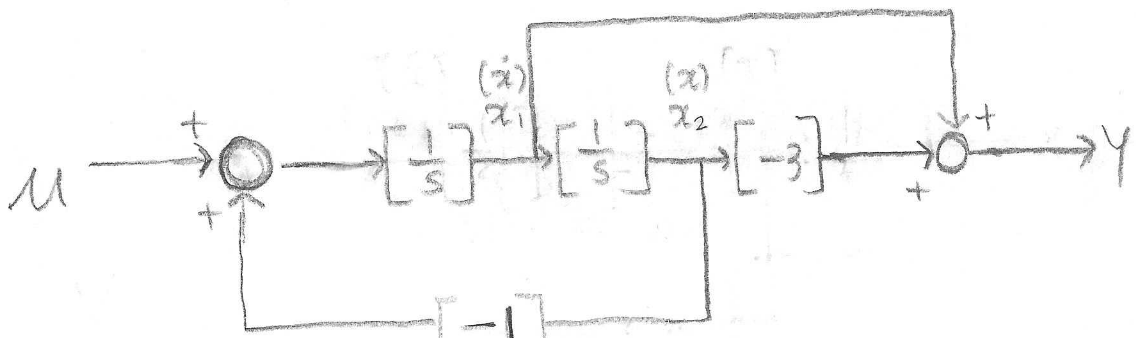
Thus, by definiton of BIBO stability, $M_1 < \infty$ gave $M_2 > \infty$, in other words, bounded input gave unbounded output. The system to this $\sin(x)$ input is <u>not BIBO stable</u>

2-b) Taking Controller Canonical Form...

$$\frac{\alpha s + \beta}{s^2 + As + B} \rightarrow \dot{x} = \begin{bmatrix} -A & -B \\ I & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u, \quad Y = \begin{bmatrix} \alpha & \beta \end{bmatrix} x$$

$$\frac{s-3}{s^2+1} \rightarrow \dot{x} = \underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{B} u, \quad Y = \underbrace{\begin{bmatrix} 1 & -3 \end{bmatrix}}_{C} x \qquad D = [\,]$$

In signal flow diagram...



using Mason... $TF = \Sigma(T_i \Delta_i / \Delta)$

$T = 2$ (Forward) $\qquad \Delta = 1 + \frac{1}{s^2}$

$T_1 = -\frac{3}{s^2} \qquad L_1 = -\frac{1}{s^2}$

$T_2 = \frac{1}{s} \qquad \Delta_{1,2} = 1$

Confirmed!

$$\rightarrow \frac{Y}{U} = \frac{-\frac{3}{s^2}}{1+\frac{1}{s^2}} + \frac{\frac{1}{s}}{1+\frac{1}{s^2}} = \frac{\frac{s-3}{s^2}}{1+\frac{1}{s^2}} = \left[\frac{s-3}{s^2+1}\right]$$

or the other way...

$$G(s) = \frac{s-3}{s^2+1} \quad \left(= \frac{Y(s)}{U(s)} \right)$$

Let $G(s) = b(s)a(s)^{-1}$ where $b(s) = s-3$, $a(s) = s^2+1$

Then $Y(s) = b(s)a(s)^{-1}U(s)$

Define new variable

$$X(s) = a(s)^{-1}U(s)$$

Then and

$$a(s)X(s) = U(s) \qquad Y(s) = b(s)X(s)$$

Inverse Laplace..

$$\ddot{x} + x = u(t) \quad \text{and} \quad y(t) = \dot{x} - 3x$$

so

Define

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \dot{x} \\ x \end{bmatrix} \longrightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -x_2 + u(t) \\ x_1 \end{bmatrix}$$

$$Y = x_1 - 3x_2$$

So the state-space representation of this system is..

$$\dot{x} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

$$Y = \begin{bmatrix} 1 & -3 \end{bmatrix} x$$

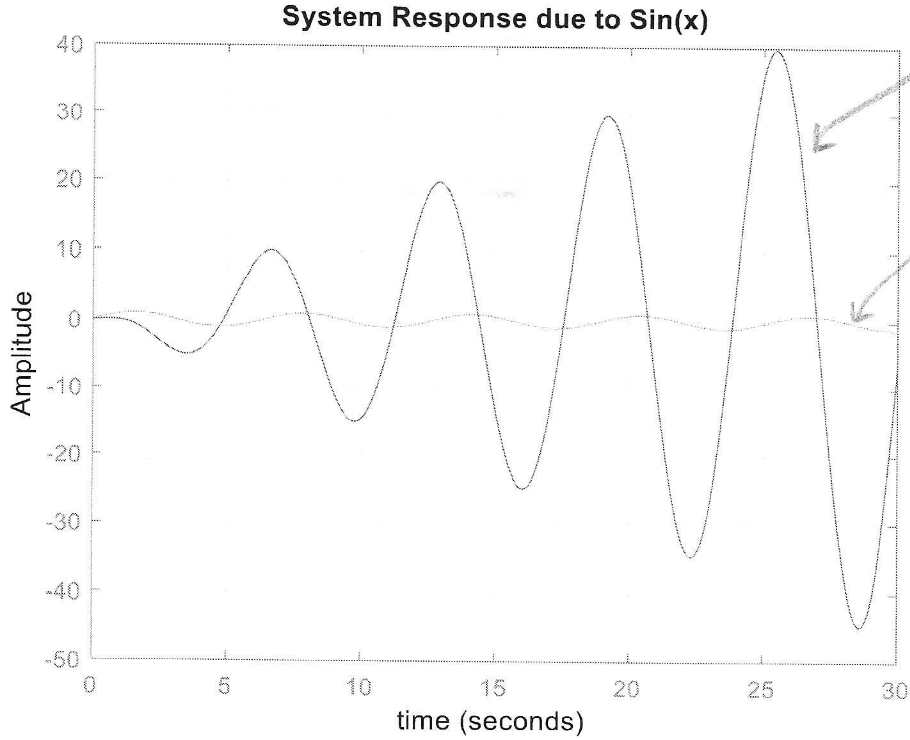
2-c) Our given system, $G(s)=\frac{s-3}{s^2+1}$, is not internally stable because, by definition of internal stability, the linear time-invariant system is not internally stable because the solution $x(t)$ of $\dot{x}(t)=Ax(t)$ with initial state $x(0)$ does not tend toward zero as $t\to\infty$ for arbitrary $x(0)$ as shown in the plot next page, the system only oscillate from ± 9 and never goes to zero as $t\to\infty$ (plot used $x(0)=[2 \ \ 2]'$)

✓

Also, by the theorem, the eigenvalue of this system is $\pm j$. Because there is no negative real parts in the eigenvalue, by theorem, the system is not internally stable

($\pm j$ means the pole of the system lies on the imaginary axis, thus, system will only produce oscillation)

2-a)

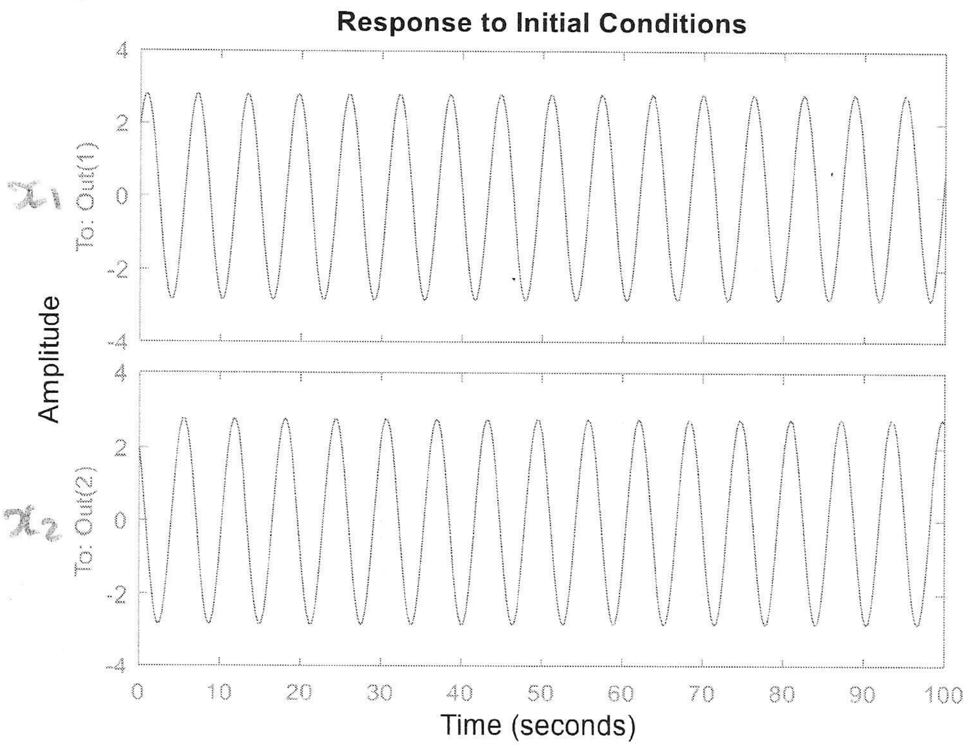


OUT: system.IN
(Unbounded)

IN: Sin(x)
(Bounded)

As can be seen system goes unbounded given bounded input

2-c)



As $t \to \infty$, system never reaches 0.

Problem #3: (30%)

For the nonlinear cart-inverted system shown in pages 6-11 of Chapter 1. If both of the rotational and translational are negligible, a linearized model at the upright stick equilibrium can be obtained as follows:

$$\dot{\bar{x}}(t) = \bar{A}\bar{x}(t) + \bar{B}\bar{u}(t)$$

$$\bar{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 25.15 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -0.42094 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -\gamma & 0 & 0 & 0 \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} 0 \\ -1.3547 \\ 0 \\ 0.5504 \end{bmatrix} = \begin{bmatrix} 0 \\ -\beta \\ 0 \\ \delta \end{bmatrix}$$

Assume the initial condition is

$$\bar{x}(0) = \begin{bmatrix} 0.2 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

$u(t) = 0 \rightarrow \bar{u}(t) = 0$ $\qquad$ $\dot{\bar{x}}(t) = \bar{A}\bar{x}(t)$

(a) Is the system stable? Controllable? Explain. (5%) λ, do it by hand!

(b) Let $u(t) = 0$. Solve for $x(t)$ and then plot it. (5%) $x(t) = e^{At} x_0$

(c) Find a state feedback $\bar{u}(t) = F\bar{x}(t)$ such that the closed-loop pole locations are at -3, -2, and $-1 \pm j$. (10%) Design Controller

(d) Assume the same initial condition, conduct a simulation to verify your controller works for the nonlinear model given in Chapter 1 notes. (5) works

(e) Repeat (d) with another initial condition $\bar{x}(0) = \begin{bmatrix} 0.5 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, and comment if the controller still works. (5%) Works

Equilibrium

Given: $x^* = [0\ 0\ 0\ 0]'$

$u^* = [0\ 0\ 0\ 0]'$

Thus,

$x = \bar{x}$

$u = \bar{u}$

3-a) Knowing that internal stability guarantees BIBO stability
(Internal Stability $\supset$ BIBO stability 
),

I will investigate the internal stability of this system.
We can compute the eigenvalues to do this.

$$\det\left(\begin{bmatrix} \lambda & -1 & 0 & 0 \\ -\alpha & \lambda & 0 & 0 \\ 0 & 0 & \lambda & -1 \\ -\beta & 0 & 0 & \lambda \end{bmatrix}\right) = \lambda \cdot \det\left(\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & -1 \\ 0 & 0 & \lambda \end{bmatrix}\right) - 1 \cdot -1 \cdot \begin{vmatrix} -\alpha & 0 & 0 \\ 0 & \lambda & -1 \\ -\beta & 0 & \lambda \end{vmatrix} + 0 + 0 =$$

$$\lambda(\lambda(\lambda^2)) + (-\alpha(\lambda^2)) = \underline{\lambda^4 - \alpha\lambda^2 = 0} \rightarrow \underline{\lambda^2(\lambda^2 - \alpha) = 0}$$

Thus $\lambda = 0, 0$ and $\lambda^2 = \alpha \rightarrow \lambda = \pm\sqrt{\alpha} = \pm\sqrt{25.15}$

Thus $\lambda = 0, 0, \underline{+5.015}, -5.015 \rightarrow$ $\boxed{\text{Internally Unstable}}$

Controllability of the system can be determined if, by definition, the state of the system can be transferred from the initial zero state $x(0^-) = 0$ to any final state $x(t_f)$ in a finite time $t_f \geq 0$, by some control $u(t)$

By theorem, system is controllable iff the column vectors of the controllability matrix

$$C = [B \; AB \cdots A^{n-1}B]$$

span the n-dimensional space

$$\text{Rank}(C) = n$$

✓

For out 4x4 system, controllability matrix is...

$$C = [B \ \ AB \ \ A^2B \ \ A^3B]$$

$$= \begin{bmatrix} 0 & -1.3547 & 0 & -34.0707 \\ -1.3547 & 0 & -34.0707 & 0 \\ 0 & 0.5504 & 0 & 0.5702 \\ 0.5504 & 0 & 0.5702 & 0 \end{bmatrix}$$

checking the rank or rref of this controllability matrix...

$\text{rank}(C) = 4$, $\text{rref}(C) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ thus, all column vectors in C are linearly independent, and thus, can span the 4-dimensional space

[Thus, the system is <u>controllable</u>.]

3-b) letting $\bar{u}(t) = 0$, we have $\dot{\bar{x}} = A\bar{x}(t) + B \cdot 0 = A\bar{x}(t)$

Taking Laplace Transform... $L\{\dot{\bar{x}}\} = L\{A\bar{x}(t)\}$

$$\rightarrow s\bar{X}(s) - \bar{x}_0 = A\bar{X}(s) \rightarrow s\bar{X}(s) - A\bar{X}(s) = \bar{x}_0$$

$$\rightarrow \bar{X}(s)(s-A) = \bar{x}_0 \rightarrow \bar{X}(s) = \bar{x}_0/(s-A)$$

Now taking Inverse Laplace Transform...

$$L^{-1}\left\{\bar{X}(s) = \frac{\bar{x}_0}{s-A}\right\} \rightarrow \left[\bar{x}(t) = e^{At}\bar{x}_0\right]$$

state-transition matrix

$$\begin{bmatrix} 0.2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

So we expect to have...

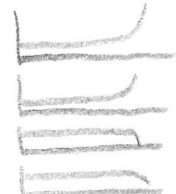
$$\left[(4\times4)\ e^{At}\right]\left[(4\times1) \begin{matrix} 0.2 \\ 0 \\ 0 \\ 0 \end{matrix}\right] = \left[(4\times1) \begin{matrix} eq\_1(t) \\ eq\_2(t) \\ eq\_3(t) \\ eq\_4(t) \end{matrix}\right] \rightarrow$$ there will be 4 plots

eq1

eq2

eq3

eq4



3-c) To find state feedback $\bar{u}(t) = -k\bar{x}(t)$ (where $F = -k$) such that poles are at $[-3, -2, -1+j, -1-j]$, we first have to find the transition matrix to convert original A system to companion form so that we can more easily manipulate & investigate the system.

The transition matrix (similarity transformation) can be defined as... for 4x4 A system.

$$T = \left[(A^3B + A^2a_4B + Aa_3B + a_2B) \; (A^2B + Aa_4B + a_3B) \; (AB + a_4B) \; B \right]$$

where a_1, a_2, a_3, a_4 are the coefficient of the characteristic polynomial A

$$\det(sI - A) = s^4 - \frac{503s^2}{20} = s^4 + a_4s^3 + a_3s^2 + a_2s + a_1$$

Thus $a_1 = 0,\ a_2 = 0,\ a_3 = -\frac{503}{20} = -25.15,\ a_4 = 0$

Thus, we can form transistion matrix.

$$T = \begin{bmatrix} 0 & 0 & -1.3547 & 0 \\ 0 & 0 & 0 & -1.3547 \\ -13.2123 & 0 & 0.5504 & 0 \\ 0 & -13.2123 & 0 & 0.5504 \end{bmatrix}$$

Now with T, let's find the companion form of the system

$$\hat{A} = T^{-1}AT = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 25.15 & 0 \end{bmatrix}$$

$$\hat{B} = T^{-1}B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Next, we know state feedback law $\bar{u} = -k\bar{x}$, plug it in original system $\rightarrow \dot{\bar{x}} = A\bar{x} + Bu = A\bar{x} - BK\bar{x} = (A-BK)\bar{x}$

so we want $(A-BK)$ to have a pole at $-3, -2, -1 \pm j$

To achieve that, first, we want to compute $\hat{A}_{CL}$ where $\hat{A}_{CL} = \hat{A} - \hat{B}\hat{K}$, and $\hat{K} = [k_1 \; k_2 \; k_3 \; k_4]$

$$\hat{A}_{CL} = \hat{A} - \hat{B}\hat{K} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -k_1 & -k_2 & \left(\frac{503}{20} - k_3\right) & -k_4 \end{bmatrix}$$

Taking determinant to find its characteristic equation

$$\det(sI - \hat{A}_{CL}) = s^4 + k_4 s^3 + \left(k_3 - \frac{503}{20}\right)s^2 + k_2 s + k_1$$

Now we want to compare this equation with "desired" characteristic equation to find k_1, k_2, k_3, k_4

$$(s+3)(s+2)(s+1+j)(s+1-j)$$

$$= s^4 + 7s^3 + 18s^2 + 22s + 12$$

compar

By inspection, we can find $k_1 = 12, k_2 = 22, k_3 = 43.15, k_4 = 7$

so, $\hat{K} = [12 \quad 22 \quad 43.15 \quad 7]$ (in companion form)

Now we can use similarity matrix, T, to represent k in original form.

$$k = \hat{K}T^{-1} = [-32.2194 \;\; -5.8407 \;\; -0.9041 \;\; -1.6576]$$

Now we know k, we can find A_{CL} in original form

$$A_{CL} = A - BK = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -18.4976 & -7.9123 & -1.2248 & -2.2455 \\ 0 & 0 & 0 & 1 \\ 17.3126 & 3.2147 & 0.4976 & 0.9123 \end{bmatrix}$$

To verify the pole placement was successful

$$\mathrm{eig}(A_{CL}) = -3, -2, -1+j, -1-j$$

Pole placement was successful.

Thus, a state feedback is defined as below

$$\bar{u}(t) = -K\bar{x}(t) \quad \text{where } -k = F$$
$$= F\bar{x}(t)$$

$$F = \begin{bmatrix} 32.2194 & 5.8407 & 0.9041 & 1.6576 \end{bmatrix}.$$

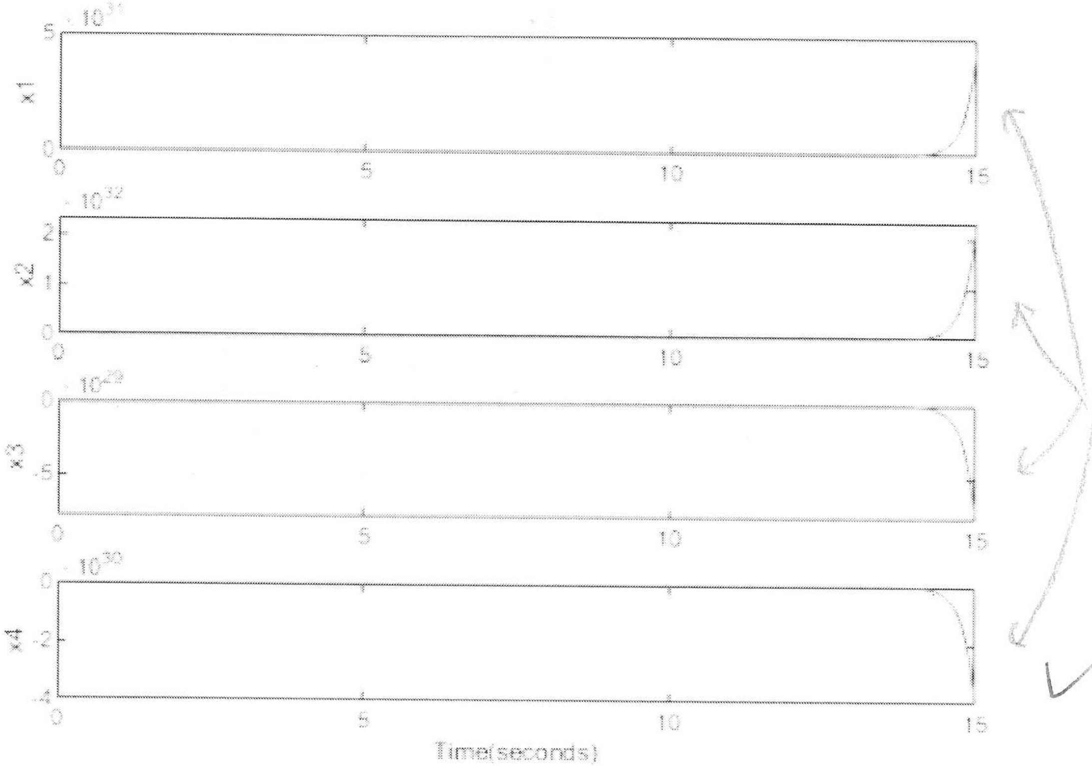
✓

3-d) The linear controller from 3-c) also works with the "non-linear" model with initial condition $x_0 = [0.2 \ 0 \ 0 \ 0]'$ as shown in the plot next page.

3-e) The linear controller found from 3-c), again, also works with the nonlinear model with initial condition $x_0 = [0.5 \ 0 \ 0 \ 0]'$ as shown in the plot next page

3-b)

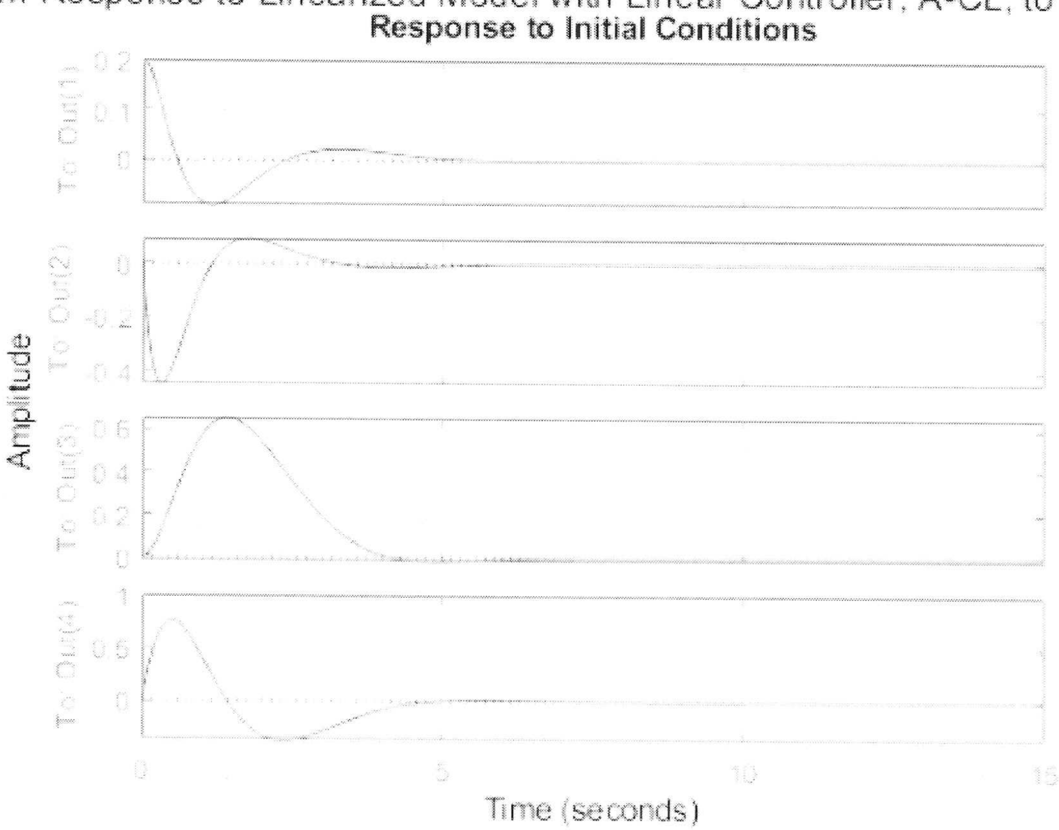
System Response of Original System, A, to Initial Condition



Goes Unstable

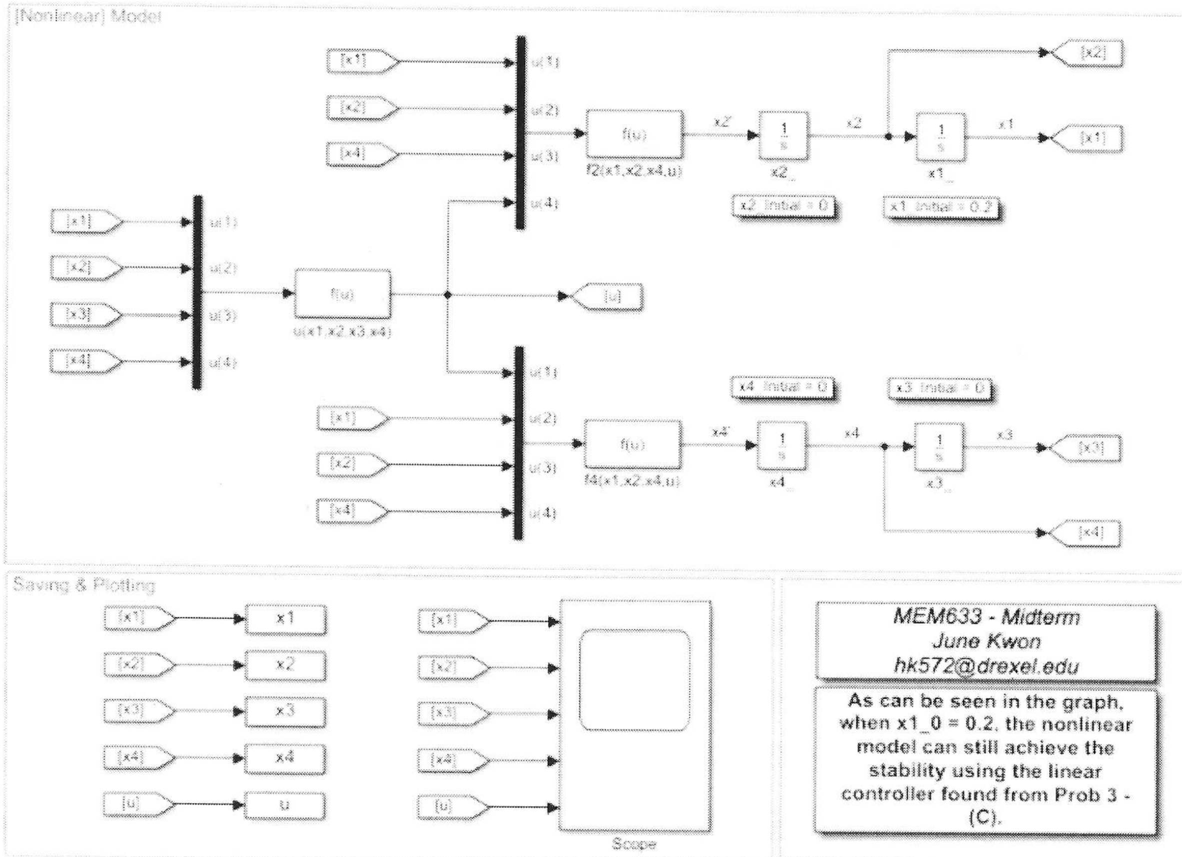
3-c)

System Response to Linearized Model with Linear Controller, A-CL, to $x1_o = 0.2$

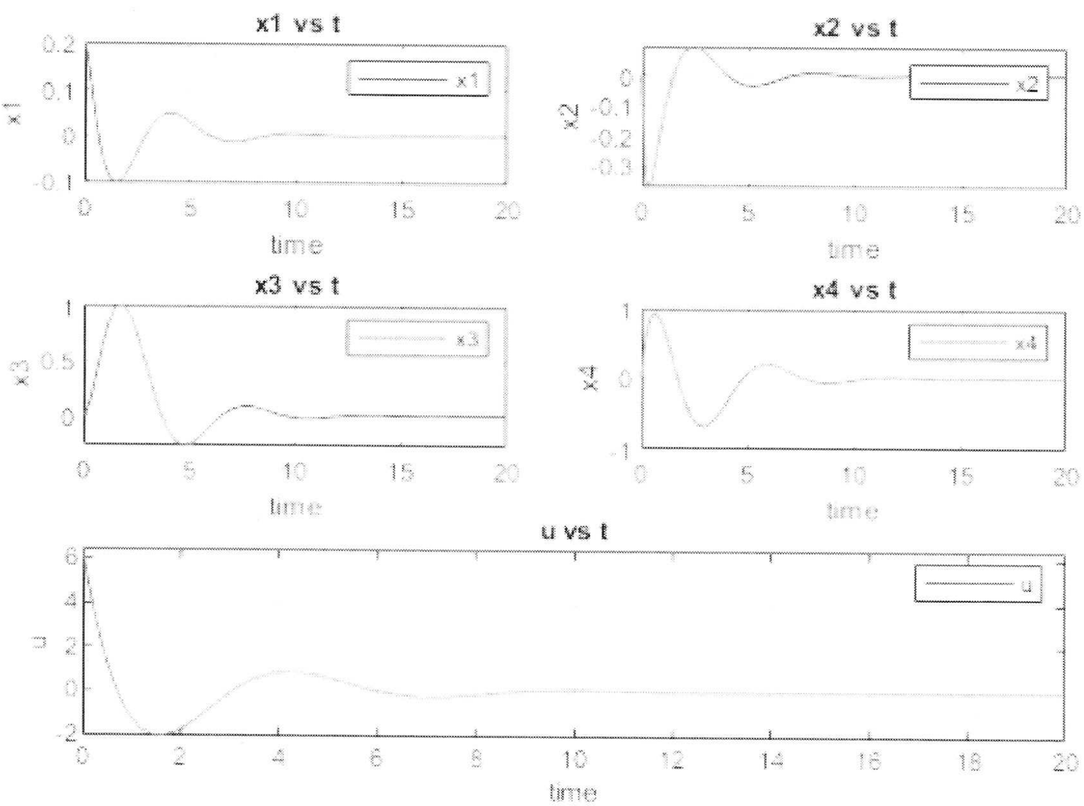


Stability achieved

3-d)

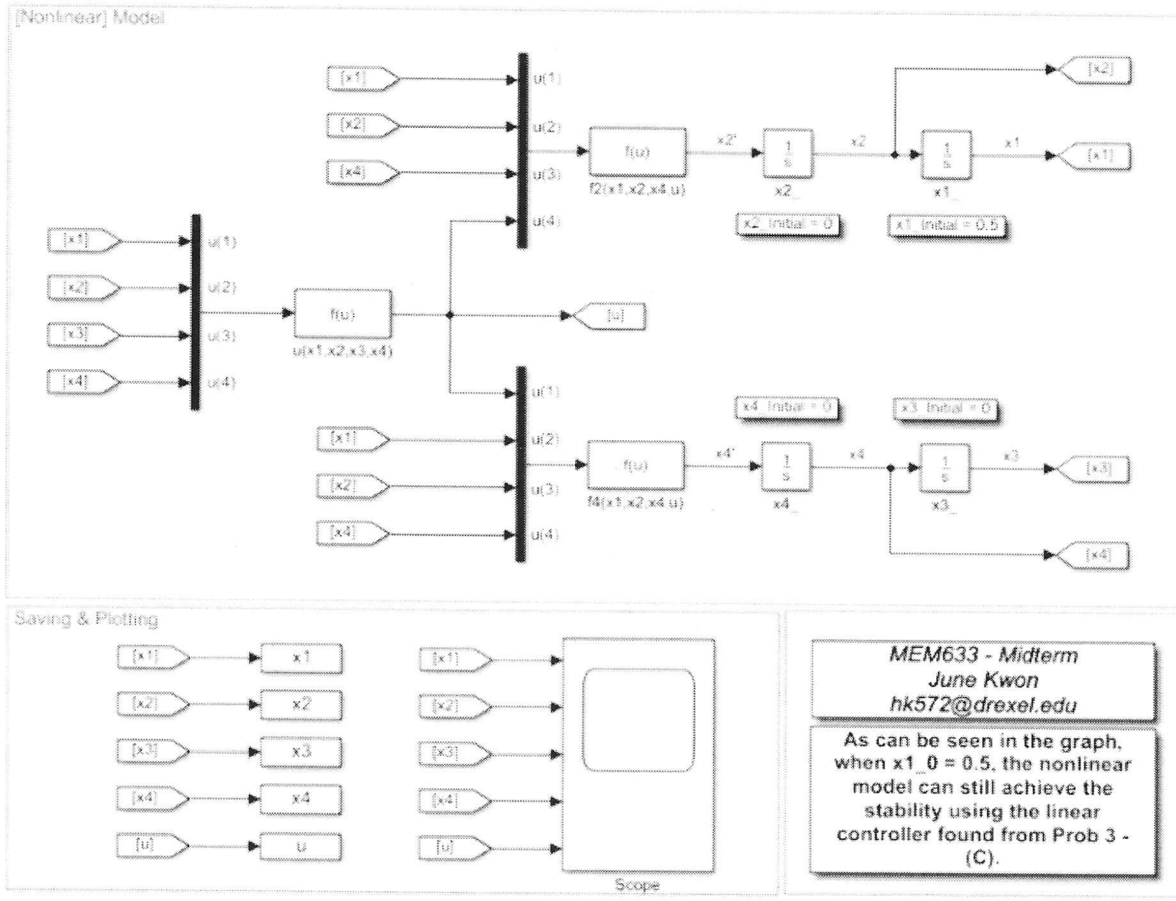


System Response of Nonlinear Model with Linear Controller to $x1_0 = 0.2$

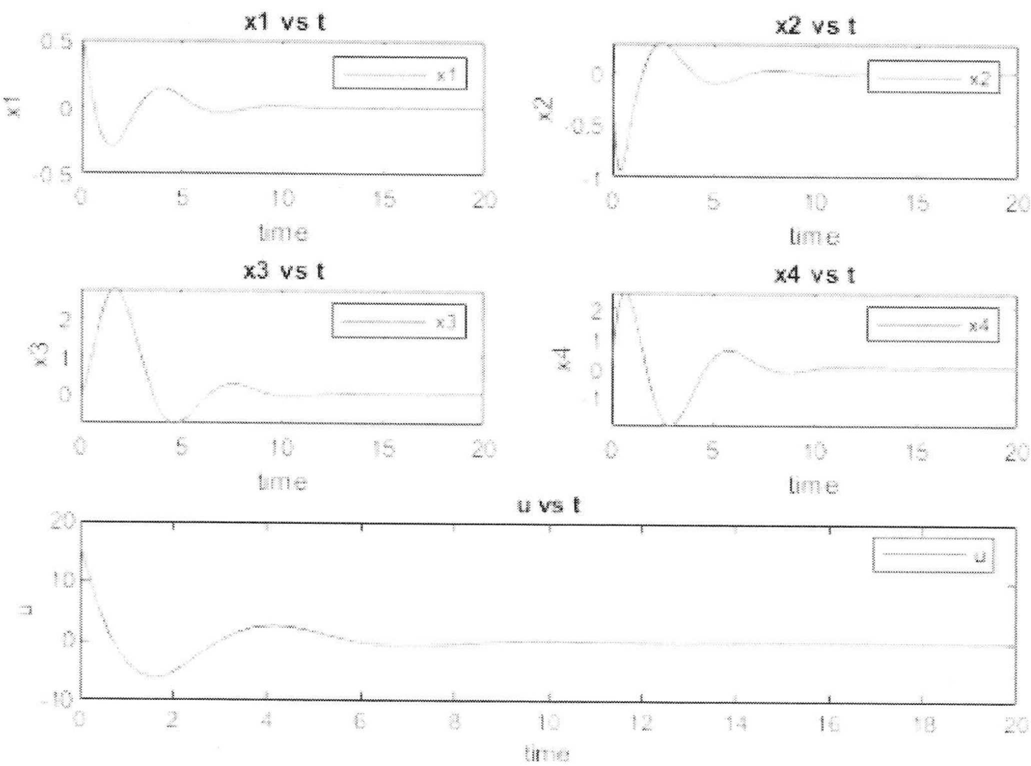


Stability achieved.

3-e)



System Response of Nonlinear Model with Linear Controller to $x1_o = 0.5$



Stability achieved

unc
uno → sta

unc → uns →

Detec → sta

Pg.31.

Problem #4: (25%)

Consider the system,

$$G(s) = \frac{s-1}{s^2-1}$$

(Don't cancel the common factor $s-1$, $G(s)$ is a second-order system.)

A_c (a) Realize G(s) in controller form. Is the realization controllable? Observable? Stabilizable? Detectable? (5%) similarity observer → stable

↳ obser → stable
↳ contb → un

A_{co} (b) Realize G(s) in controllability form. Is the realization controllable? Observable? Stabilizable? Detectable? (5%)

A_o (c) Realize G(s) in observer form. Is the realization controllable? Observable? Stabilizable? Detectable? (5%) no similarity

(d) Is there a similarity transformation between realizations (a) and (b)? If yes, find it. Otherwise, explain why. (5%) Yes

(e) Is there a similarity transformation between realizations (a) and (c)? If yes, find it. Otherwise, explain why. (5%) No

When there is a pole-zero cancellation in the original plant, it means system is either non-controllable or non-observable.

4-a)

$$G(s) = \frac{s-1}{s^2-1} \quad \left(= \frac{Y(s)}{U(s)}\right)$$

controller form,,

Let $G(s) = b(s) \cdot a(s)^{-1}$ where $b(s) = s-1$, $a(s) = s^2-1$

Then $Y(s) = b(s) \cdot a(s)^{-1} \cdot U(s)$

Now define new variable

$$X(s) = a(s)^{-1} U(s)$$

Then $a(s) X(s) = U(s)$ and $Y(s) = b(s) X(s)$

Inverse Laplace...

$$\ddot{x} - x = u(t) \text{ and } y(t) = \dot{x} - x$$

Define so...

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \dot{x} \\ x \end{bmatrix} \longrightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 + u(t) \\ x_1 \end{bmatrix}$$

$$Y = x_1 - x_2$$

So $G(s)$ in controller form is...

$$\dot{x} = \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{(A_c)} x + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{(B_c)} u(t)$$

$$y = \underbrace{\begin{bmatrix} 1 & -1 \end{bmatrix}}_{(C_c)} x$$

Now let's analyze the system..

$$\text{ctrb}(A_c, B_c) = \begin{bmatrix} B_c & A_c B_c \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \leftarrow \text{Linearly Independent}$$

$$\text{rank}(\text{ctrb}(A_c, B_c)) = 2 = n \longrightarrow [\text{controllable}]!!$$

$$\text{obsv}(A_c, C_c) = \begin{bmatrix} C_c \\ C_c A_c \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \leftarrow \text{Linearly Dependent}$$

$$\text{rank}(\text{obsv}(A_c, C_c)) = 1 \neq n$$

$\rightarrow$ Thus, [not observable]!

Next, the definition of stabilizable states!

"A system is stabilizable if all unstable states are controllerble"

So investigating the internal stability of the system by eigenvalue ...

$$\det(\lambda I - A_c) = \begin{vmatrix} \lambda & -1 \\ -1 & \lambda \end{vmatrix} = \lambda^2 - 1 = 0$$

$$\rightarrow \lambda = +1, -1$$

!!

The system is internally <u>unstable</u>. However! because the system is <u>controllable</u> ($\text{rank}(\text{ctrb}(A_c, B_c)) = 2 = n$)

It satisfies the definition, and system is

[stabilizable]

meaning that even the unstable states can be controlled

Next, the definition of detectable states!

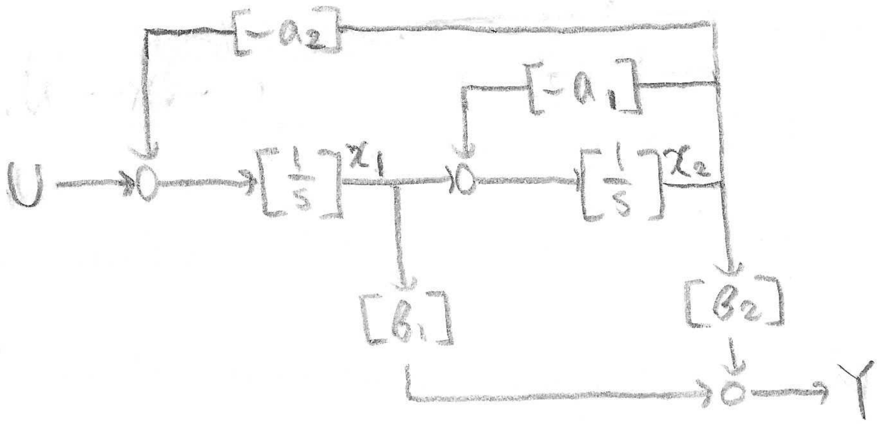
"A system is ~~detectable~~ detectable if all unstable states are observable"

We know the system is internally unstable for that it has one positive eigenvalue.

However, that is a state that is not observable.
(rank(obsv(A_c, C_c)) = 1 ≠ n). Thus, because there is unstable state that is not observable, system is [not detectable.]

✓

4-b) Controllability Form (Duality of Observability Form)



From the diagram, we can see that

$$\dot{x}_1 = -a_2 x_2 + u$$

$$\dot{x}_2 = x_1 - a_1 x_2$$

$$Y = \beta_1 x_1 + \beta_2 x_2$$

We know from 4-a)

$$\begin{pmatrix} a(s) = s^2 - 1 \\ b(s) = s - 1 \end{pmatrix} \text{ vs } \begin{pmatrix} a(s) = s^2 + a_1 s + a_2 \\ b(s) = b_1 s + b_2 \end{pmatrix} \longrightarrow \begin{matrix} a_1 = 0, \ a_2 = -1 \\ b_1 = 1, \ b_2 = -1 \end{matrix}$$

But from duality of Observability Form... β are...

$$\begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ a_1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

So...

$\dot{x}_1 = x_2 + u$
$\dot{x}_2 = x_1$
$y = x_1 - x_2$

$$\rightarrow \dot{x} = \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{(A_{co})} x + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{(B_{co})}$$

$$y = \underbrace{\begin{bmatrix} 1 & -1 \end{bmatrix}}_{(C_{co})} x$$

As you can see, it has the same form as 4-a) controller form
Thus, the answers for controllable? observable? Stabilizable? Detectable? will be the same

$\text{rank}(\text{ctrb}(A_{co}, B_{co})) = \text{rank}\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 2 = n \rightarrow$ [controllable]

$\text{eig}(A_{co}) = +1, -1 \rightarrow$ Internally Unstable, however since unstable state is controllable $\rightarrow$ [stabilizable]

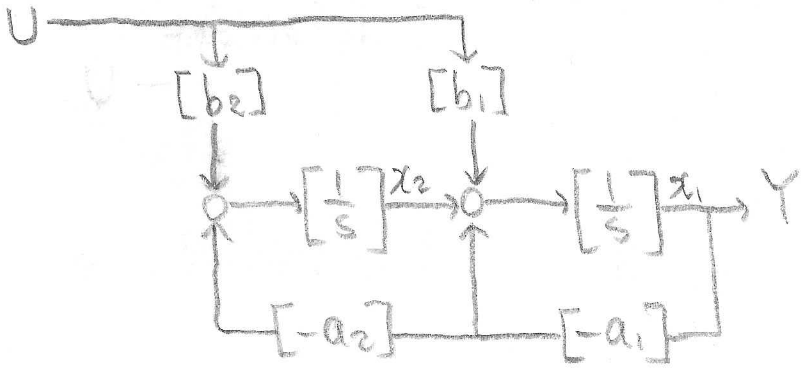
$\text{rank}(\text{obsv}(A_{co}, C_{co})) = \text{rank}\left(\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\right) = 1 \neq n \rightarrow$ [not observable]
$\text{eig}(A_{co}) = +1, -1 \rightarrow$ Internally Unstable, also, unstable state is not observable $\rightarrow$ it is [not detectable]

4-c) $G(s) = \frac{s-1}{s^2-1} \quad \left(= \frac{Y(s)}{U(s)}\right)$

Let $G(s) = b(s)a(s)^{-1}$ where $b(s) = s-1$, $a(s) = s^2-1$
Then $Y(s) = b(s)a(s)^{-1}U(s)$
with 2nd order system

$a(s) = s^2 + a_1 s + a_2$
$b(s) = b_1 s + b_2$
$\xrightarrow{\text{Thus,}}$ $a_1 = 0$, $a_2 = -1$
$b_1 = 1$, $b_2 = -1$

Also, from Observer Form block diagram, we know



$\dot{x}_1 = -a_1 x_1 + x_2 + b_1 u$

$\dot{x}_2 = -a_2 x_1 + b_2 u$

$Y = x_1$

Thus, plugging in we have observer form.

$$\dot{x} = \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{(A_o)} x + \underbrace{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}_{(B_o)} u \leftarrow \begin{aligned} \dot{x}_1 &= x_2 + u \\ \dot{x}_2 &= x_1 - u \\ Y &= x_1 \end{aligned}$$

$$Y = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{(C_o)} x$$

Now, let's analyze

$$\text{ctrb}(A_o, B_o) = \begin{bmatrix} B_o & A_oB_o \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \Big] \text{ Linearly Dependent}$$

$$\text{rank}(\text{ctrb}(A_o, B_o)) = 1 \neq n$$

Thus, this system, G, is $\boxed{\text{not controllable}}$ in observer form.

Taking eigenvalue of A_o

$$\text{eig}(A_o) = \text{eig}\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right) \rightarrow \det(\lambda I - A_o) = \begin{vmatrix} \lambda & -1 \\ -1 & \lambda \end{vmatrix} = \lambda^2 + 1 = 0$$

$\lambda = +1, -1$ $\rightarrow$ The system is internally unstable

!!

Thus, by definition of stabilizable,

Since the unstable state is <u>not</u> controllable,

the system is $\boxed{\text{not stabilizable}}$

$$\text{obsv}(A_o, C_o) = \begin{bmatrix} C_o \\ C_o A_o \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Big] \text{ Linearly Independent}$$

$$\text{rank}(\text{obsv}(A_o, C_o)) = 2 = n$$

Thus, the system is [observable] in this observer form.

We know the system is internally unstable ($\lambda = -1, +1$)

Thus, since the unstable state is observable, by definition of detectable, the system is [detectable]

4-d) C = controller form, CO = controllability form, O = Observer form

similarity transformation takes the form

$$T_d = \text{ctrb}(A_c, B_c)\, \text{ctrb}(A_{co}, B_{co})^{-1}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \frac{1}{1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \leftarrow \text{similarity matrix}$$

So once we find T_d, let's check if similarity matrix works with (a) and (b), knowing below realization property

$$\hat{A} = T^{-1}AT, \quad \hat{B} = T^{-1}B, \quad \hat{C} = CT$$

Now defining

$$\hat{A} \triangleq A_{co}, \quad \hat{B} \triangleq B_{co}, \quad \hat{C} = C_{co},$$

$$A \triangleq A_c, \quad B \triangleq B_c, \quad C \triangleq C_c$$

Thus.

$$A_{co} = T^{-1}A_cT = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}_{(A_{co})} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}_{(T^{-1}A_cT)} \quad \checkmark$$

$$B_{co} = T^{-1}B_c = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}_{(B_{co})} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}_{(T^{-1}B_c)} \quad \checkmark$$

$$C_{co} = C_cT = \begin{bmatrix} 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & -1 \end{bmatrix}_{(C_{co})} = \begin{bmatrix} 1 & -1 \end{bmatrix}_{(C_cT)} \quad \checkmark$$

similarity transformation is successful. Thus, there exists similarity transformation between (a) and (b).

4-e) computing similarity matrix..

$$T_e = ctrb(A_c, B_c)\, ctrb(A_o, B_o)^{-1}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \frac{1}{0} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} \infty & \infty \\ \infty & \infty \end{bmatrix}$$

!!!

because $ctrb(A_o, B_o)$ is not invertible, similarity transformation between (a) and (c) does not exist

Even if you compute

$$A_o = T^{-1} A_c T = \begin{bmatrix} \infty & \infty \\ \infty & \infty \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \infty & \infty \\ \infty & \infty \end{bmatrix}$$

→ undefined

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}_{(A_o)} \neq \begin{bmatrix} UDF & UDF \\ UDF & UDF \end{bmatrix} \quad \times$$

$$B_o = T^{-1} B_c = \begin{bmatrix} \infty & \infty \\ \infty & \infty \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix}_{(B_o)} \neq \begin{bmatrix} UDF \\ UDF \end{bmatrix} \quad \times$$

$$C_o = C_c T = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} \infty & \infty \\ \infty & \infty \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \end{bmatrix}_{(C_o)} = \begin{bmatrix} UDF & UDF \end{bmatrix} \quad \times$$

Thus, no similarity transformation exists beteen (a) and (c)

DREXEL UNIVERSITY
Department of Mechanical Engineering & Mechanics

MEM 633

Robust Control Systems I

100

Excellent!

Mid-Term Exam

1. Hand in the Exam **hard copy** at 6:00PM, Monday, 11/11/2019, before the class.
2. Upload **executable** computer programs to Bb Learn before 11:59PM, 11/12/2019.

NAME: June Kwon

Attention:

- The exam report should be **self-contained, complete,** clearly written and well organized according to the order of the problems. Do not write on the back of the paper. If you need more space, you can add only 8.5x11 letter size white paper.
- **All of you** are required to upload all **executable** computer files: m-files, slx-files, etc. to Drexel Bb Learn by 11:59PM, Tuesday, 11/12/2019. These files should be grouped into one zip file with filename: MEM633midterm_YourName.
- **Show detailed procedure!** A solution without detailed procedure will receive no credit even the answer is correct.
- Open books and notes.
- **Absolutely no discussions about the exam before, during, or even after the test until all the students turn in their exams.**
- **Use discreet judgment to determine if it is appropriate to use MATLAB (or other software) commands to obtain the solutions. Do NOT use a black-box command to answer the questions that may defeat the purpose of the test.**
- **Do NOT use MATLAB or any software to automatically generate a report for the exam.**
- If there is any question about the exam, please contact Dr. Chang (changbc@drexel.edu).