DREXEL UNIVERSITY
Department of Mechanical Engineering & Mechanics
Applied Engineering Analytical & Numerical Methods II

MEM 592 - Winter 2019

HOMEWORK #4: Due Friday, February 15, 2019 at 2:00 PM

1. [50 points]

The governing equation associated with the system of mass, spring, and damper of Fig. 1 can be written as follows:

$$mu" + \gamma u' + ku = 0, \quad u(0) = 1\ cm, u'(0) = 1\ cm/s.$$

Assume that m=2 kg, $\gamma = 2\ kg/$sec, and $k = 1$ N/m.

(a) For $0 \leq t \leq 20\ s$, and for three different time-steps $h =$ 0.5,1 and 2, solve the IVP with the following numerical methods. For each one of these methods you need to write the equations associated with that scheme, discuss the stability of that method, and write a MATLAB code to solve the problem. **[25 points]**

(i) Forward Euler Method
(ii) Backward Euler Method
(iii) Trapezoidal Method

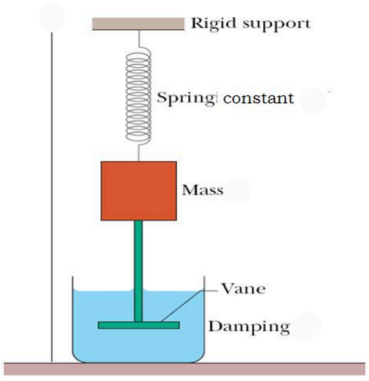


Figure 1

(b) Find the exact solution of the problem. **[10 points]**
(c) For each time step ($h = 0.5,1$ and 2), plot the displacement (u) versus time ($0 \leq t \leq 20$). **[15 points]**

(You need to plot 3 figures associated with three time-steps, each figure needs to contain 4 curves associated with Forward Euler, Backward Euler, Trapezoidal, and Exact methods)

Hint: To solve the given IVP numerically, first convert the second-order ODE to a system of two first-order ODE. Then apply suggested schemes to derive associate update formulas. Show the details of your derivation.

2. [25 points]

(a) Find the exact solution of the following IVP. **[5 points]**

$$y' = y - t^2 + 2,$$
$$y(0) = 0.5.$$

(b) Write a MATLAB code to solve the IVP for $0 \leq t \leq 2$ by using the second-order Adams-Bashforth method (Slide 93 of chapter 4 of the lecture notes). Assume h=0.2. **[15 points]**

Hint: You can find y(0.2) to use in the Adams-Bashforth method by the result of part a.

(c) Plot and compare the results of part a and b. **[5 points]**

3. [25 points]

(a) Write a MATLAB code to solve the following IVP for $0 \leq x \leq 3$ by using the Runge-Kutta method of order 4. Use $h = 0.1$. **[15 points]**

$$\frac{dy}{dx} = \frac{7x^6 - y}{e^{x^3+y}}$$

$$y(0) = 1.$$

(b) Use the built-in function "ode23" in MATLAB to solve the IVP. **[5 points]**

(c) In one figure, compare the results of part a and b. **[5 points]**

HW4 Hyukjun Kwon

1) $m u'' + \delta u' + k u = 0, \quad u(0) = 1, \quad u'(0) = 1$

$m = 2 \qquad \delta = 2 \qquad k = 1$

$\rightarrow \quad 2u'' + 2u' + u = 0$

using

$u = x_1 \rightarrow x_1' = x_2$

$u' = x_2 \rightarrow 2x_2' + 2x_2 + x_1 = 0 \rightarrow x_2' = -\frac{1}{2}x_1 - x_2$

$\rightarrow \quad x_1' = x_2$

$x_2' = -\frac{1}{2}x_1 - x_2$

In Matrix....

$$x' = \begin{bmatrix} 0 & 1 \\ -\frac{1}{2} & -1 \end{bmatrix} x \quad \rightarrow \quad \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{2} & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

a-i) Forward Euler

$(Y_{n+1} = Y_n + hf(t_n, Y_n))$

$0 < h < \frac{2}{\lambda_R}$

$x_{n+1} = x_n + h x_n'$

$$x_{n+1} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_n + h \begin{bmatrix} 0 & 1 \\ -\frac{1}{2} & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_n$$

$$= \begin{bmatrix} 1 & h \\ -\frac{1}{2}h & 1-h \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_n$$

With initial condition

$u = x_1 \rightarrow \langle x_1(0) = 1 \rangle$

$u' = x_2 \rightarrow \langle x_2(0) = 1 \rangle$

$$\det\left(\begin{bmatrix} 1 & h \\ -\frac{1}{2}h & 1-h \end{bmatrix}\right) = 1-h+\frac{1}{2}h^2 < 1$$

$$h^2-2h+2<2$$

$$h^2-2h<0$$

$$\downarrow$$

$0<h<2$ ensures stability.

a-ii) Back ward Euler

$$(Y_{n+1} = Y_n + h f(t_{n+1}, Y_{n+1}))$$

So...

$$x_{n+1} = x_n + h x'_{n+1}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{n+1} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_n + h\begin{bmatrix} 0 & 1 \\ -\frac{1}{2} & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{n+1}$$

(rearranging

$$\begin{bmatrix} 1 & -h \\ \frac{1}{2}h & 1+h \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{n+1} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_n$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{n+1} = \begin{bmatrix} 1 & -h \\ \frac{1}{2}h & 1+h \end{bmatrix}^{-1}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_n$$

$$= \frac{\begin{bmatrix} 2h+2 & 2h \\ -h & 2 \end{bmatrix}}{h^2+2h+2}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_n$$

$$\det\left(\frac{\begin{bmatrix} 2h+2 & 2h \\ -h & 2 \end{bmatrix}}{h^2+2h+2}\right) = \frac{2}{(h^2+2h+2)} < 1$$ (h, time step, will always be greater than zero)

↳ This will always produce value greater 2 → Thus fraction will always be $\left[\frac{2}{\#>2} < 1\right]$ → BE always stable

a-iii) Trapezoidal

$$(y_{n+1} = y_n + \frac{h}{2}(f(y_{n+1}, t_{n+1}) + f(y_n, t_n))$$

So..

$$x_{n+1} = x_n + \frac{h}{2}(x'_{n+1} + x'_n)$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{n+1} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_n + \frac{h}{2}\left(\begin{bmatrix} 0 & 1 \\ -\frac{1}{2} & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{n+1} + \begin{bmatrix} 0 & 1 \\ -\frac{1}{2} & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_n\right)$$

$$= \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_n + \begin{bmatrix} 0 & h/2 \\ -h/4 & -h/2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{n+1} + \begin{bmatrix} 0 & h/2 \\ -h/4 & -h/2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_n$$

↳ rearranging

$$\begin{bmatrix} 1 & -h/2 \\ h/4 & 1+h/2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{n+1} = \begin{bmatrix} 1 & h/2 \\ -h/4 & 1-h/2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_n$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{n+1} = \begin{bmatrix} 1 & -h/2 \\ h/4 & 1+h/2 \end{bmatrix}^{-1}\begin{bmatrix} 1 & h/2 \\ -h/4 & 1-h/2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_n$$

$$\det([\,]^{-1}[\,]) = \frac{h^2-4h+8}{h^2+4h+8}$$ → denominator will always be greater than numerator

$$\frac{h^2-4h+8}{h^2+4h+8} < 1$$

Thus, trapezoidal is always Stable ←

1-b) Exact Solution

$$2u'' + 2u' + u = 0$$

$$2\lambda^2 + 2\lambda + \lambda = 0 \rightarrow \lambda = -\frac{1}{2} \pm \frac{1}{2}\hat{i}$$

So...

$$u = e^{-\frac{1}{2}t}\left(A\cos\left(\tfrac{1}{2}t\right) + B\sin\left(\tfrac{1}{2}t\right)\right)$$

$$u' = A\left(-\tfrac{1}{2}e^{-\frac{1}{2}t}\cos\left(\tfrac{1}{2}t\right) - \tfrac{1}{2}e^{-\frac{1}{2}t}\sin\left(\tfrac{1}{2}t\right)\right) +$$

$$B\left(-\tfrac{1}{2}e^{-\frac{1}{2}t}\sin\left(\tfrac{1}{2}t\right) + \tfrac{1}{2}e^{-\frac{1}{2}t}\cos\left(\tfrac{1}{2}t\right)\right)$$

$$u(0) = A = 1$$

$$u'(0) = A\left(-\tfrac{1}{2}\right) + B\left(\tfrac{1}{2}\right) = \frac{B}{2} - \frac{1}{2} = 1 \Rightarrow B = 3$$

So...

$$\left[u = e^{-0.5t}\left(\cos(0.5t) + 3\sin(0.5t)\right)\right].$$

1-C) Matlab

2) $y' = y - t^2 + 2$, $y(0) = 0.5$

2-a) Exact

$$y' - y = -t^2 + 2$$

y_h) $\lambda - 1 = 0 \rightarrow \lambda = 1$

$$y_h = C_1 e^t$$

y_p) $r(x) = -t^2 + 2$

$$y_p = k_2 t^2 + k_1 t + k_0$$

$$y_p' = 2k_2 t + k_1$$

Substitute

$$2k_2 t + k_1 - k_2 t^2 - k_1 t - k_0 = -t^2 + 2$$

$k_2 = 1$

$k_1 = 2$

$k_0 = 0$

so...

$$y_p = t^2 + 2t$$

$y = y_p + y_h$)

$$y = C_1 e^t + t^2 + 2t \rightarrow y(0) = C_1 = 0.5$$

$$\boxed{y = 0.5 e^t + t^2 + 2t}$$

MEM 592 - Homework 1

Author: Hyukjun Kwon

Date: February 13, 2019

Problem #1 - Mass & Spring & Damper System 1

Problem #2 - 2nd order Adams Bashforth 4

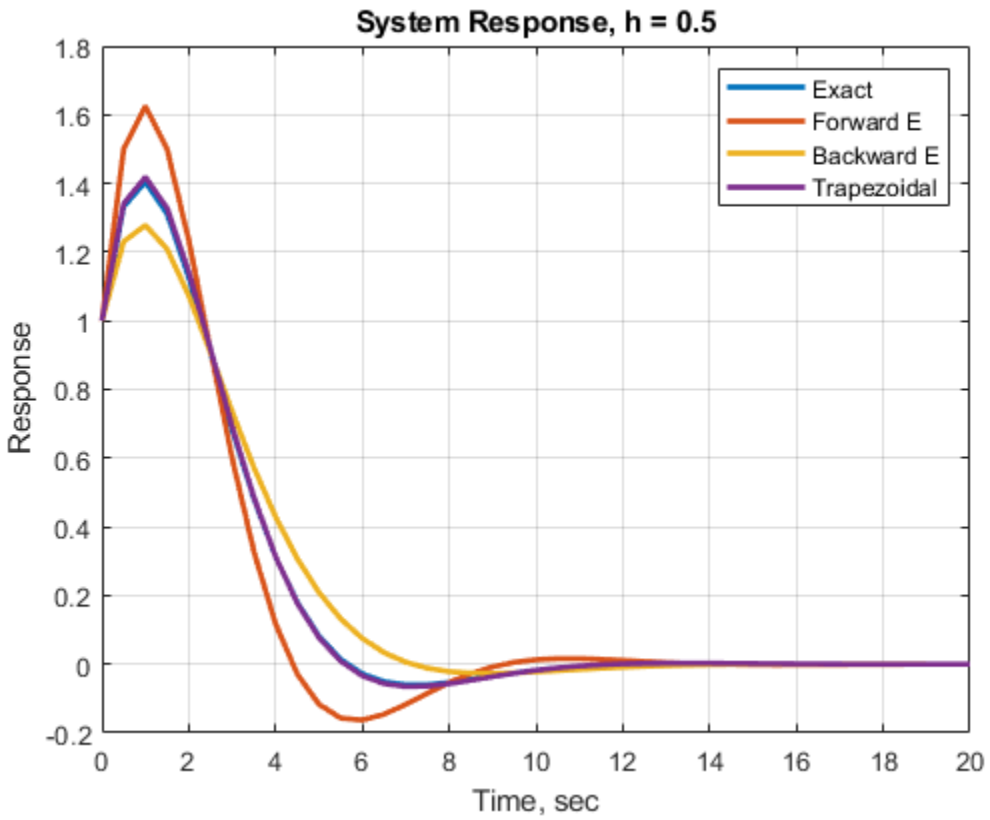
Problem #3 - 4th order Runge-Kutta 5

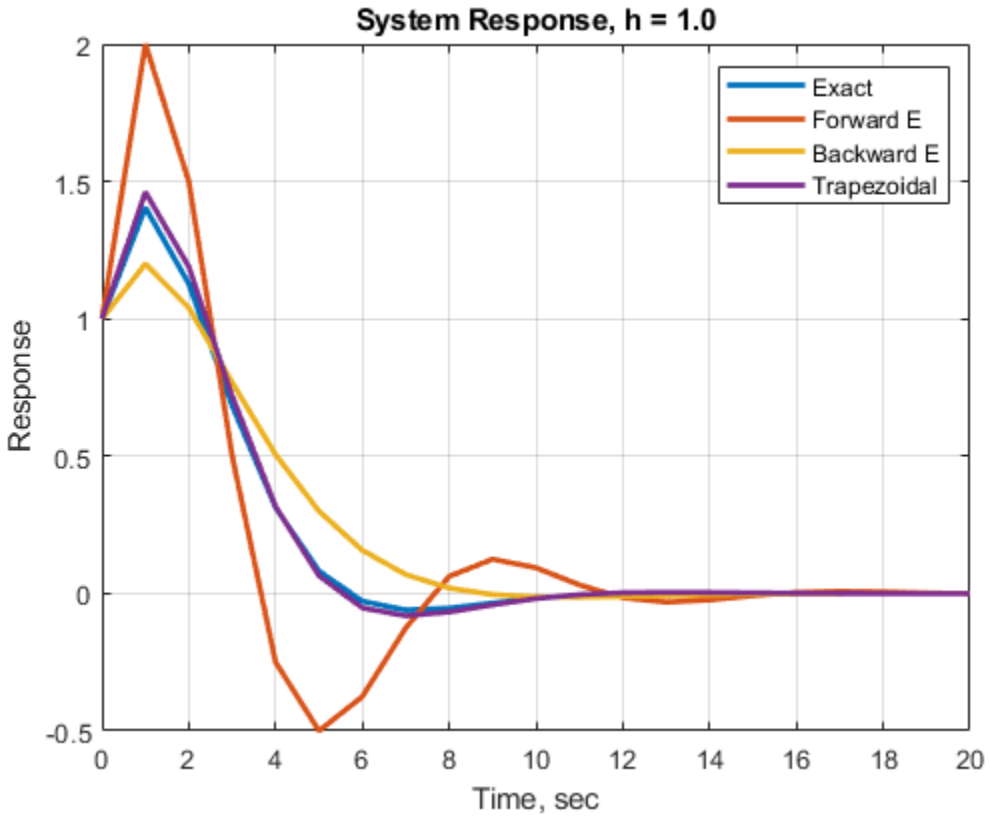
```
clear,clc,close all
```

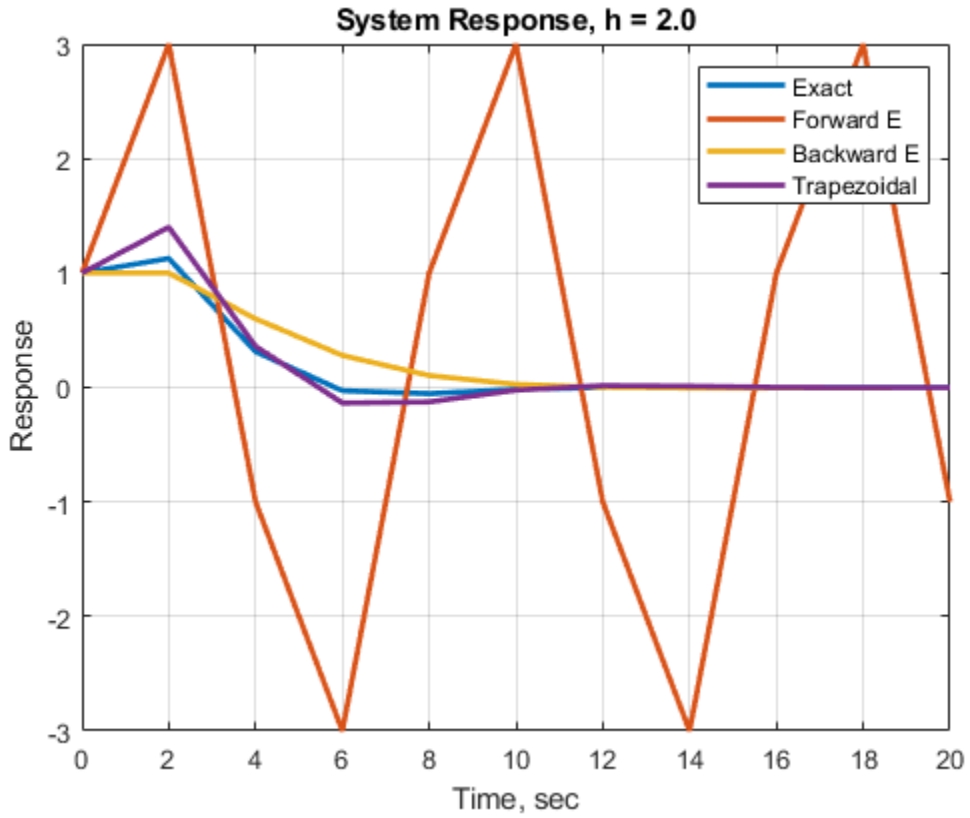
Problem #1 - Mass & Spring & Damper System

```
for j = 1:3
    figure(j)
    time_step = [0.5 1 2];
    h = time_step(j);
    t = 0:h:20;

    % Exact Solution
    Exact_Sol = exp(-0.5.*t).*(cos(0.5.*t)+3.*sin(0.5.*t));

    % Forward Euler
    A_FE = [1 h;-1/2*h 1-h];
    FE_Sol = zeros(2,length(t));
    FE_Sol(:,1) = 1;
    for i = 1:length(t)-1
        FE_Sol(:,i+1) = A_FE * FE_Sol(:,i);
    end

    % Backward Euler
    A_BE = [1 -h; 1/2*h 1+h];
    BE_Sol = zeros(2,length(t));
    BE_Sol(:,1) = 1;
    for i = 1:length(t)-1
        BE_Sol(:,i+1) = A_BE\BE_Sol(:,i);
    end

    % Trapezoidal
    A_TZ = [1 -1/2*h; h/4 1+(h/2)]\[1 h/2; -h/4 1-(h/2)];
    TZ_Sol = zeros(2,length(t));
    TZ_Sol(:,1) = 1;
    for i = 1:length(t)-1
        TZ_Sol(:,i+1) = A_TZ * TZ_Sol(:,i);
    end

    plot(t,Exact_Sol,t,FE_Sol(1,:),t,BE_Sol(1,:),t,TZ_Sol(1,:),'linewidth',2)
    grid on
```

```
    xlabel('Time, sec')
    ylabel('Response')
    title(sprintf('System Response, h = %.1f',h))
    legend('Exact','Forward E','Backward E','Trapezoidal')
end
```

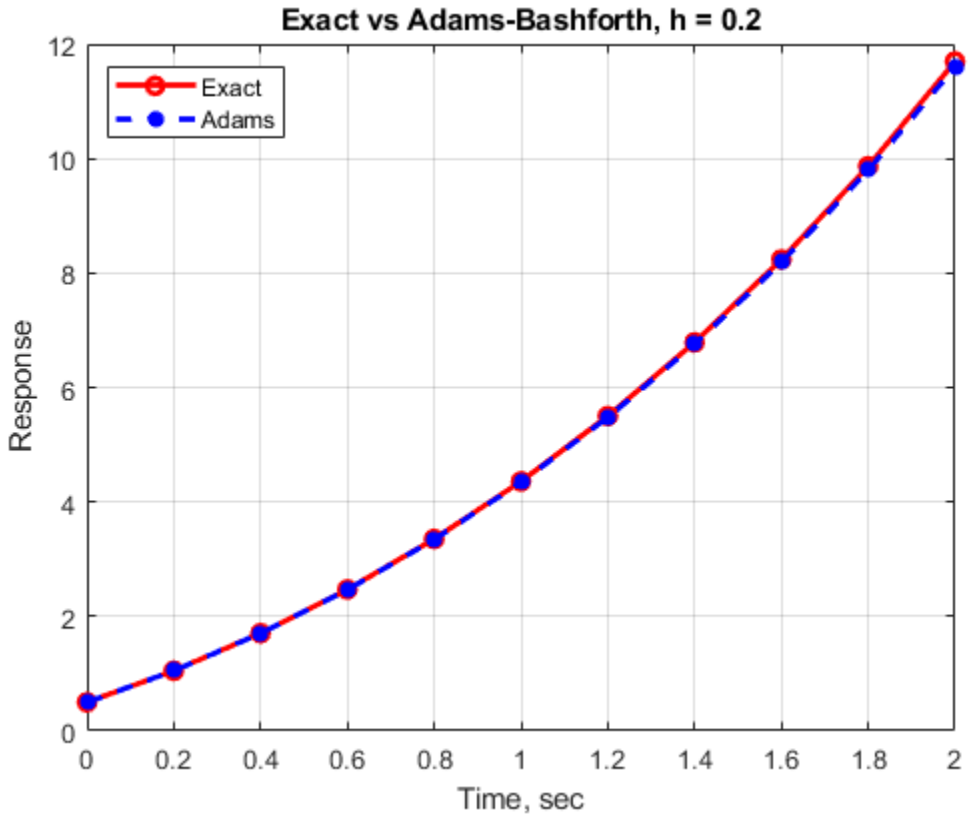



System Response, h = 1.0
Exact
Forward E
Backward E
Trapezoidal
Response
Time, sec
2
1.5
1
0.5
0
-0.5
0
2
4
6
8
10
12
14
16
18
20



Problem #2 - 2nd order Adams Bashforth

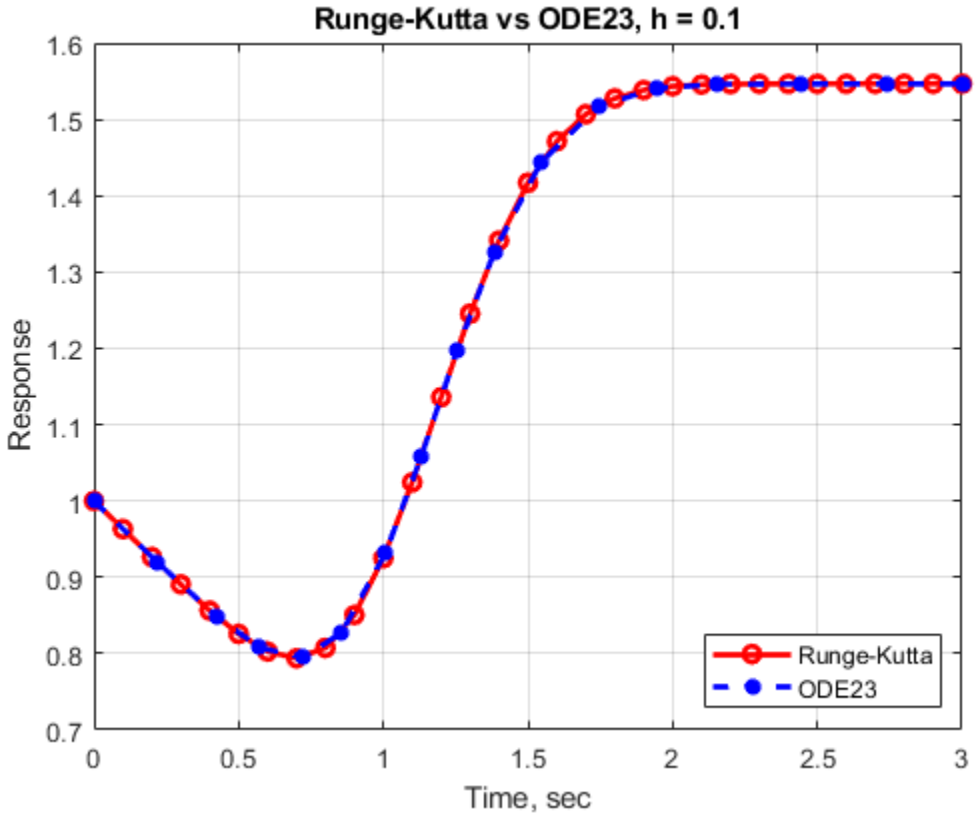
```
figure(4)
h_2 = 0.2;
t_2 = 0:h_2:2;
AB_FT = @(y,t) y-(t^2)+2;

% Exact Solution
Exact_Sol_2 = 0.5.*exp(t_2) + (t_2.^2) + (2.*t_2);

% Adams Bashforth
AB_Sol = zeros(1,length(t_2));
AB_Sol([1 2]) = [0.5 Exact_Sol_2(2)];
for i = 1:length(t_2)-2
    AB_Sol(i+2) = AB_Sol(i+1)...
                  + (3/2)*h_2*AB_FT(AB_Sol(i+1),t_2(i+1))...
                  - (h_2/2)*AB_FT(AB_Sol(i),t_2(i));
end

plot(t_2,Exact_Sol_2,'ro-',t_2,AB_Sol,'b*--','linewidth',2)
grid on
xlabel('Time, sec')
ylabel('Response')
title(sprintf('Exact vs Adams-Bashforth, h = %.1f',h_2))
legend('Exact','Adams','location','northwest')
```



Problem #3 - 4th order Runge-Kutta

```
figure(5)
h_3 = 0.1;
t_3 = 0:h_3:3;
RG_FT = @(y,t) (7*(t^6)-y)/(exp((t^3)+y));

% Runge_Kutta
RG_Sol = zeros(1,length(t_3));
RG_Sol(1) = 1;
for i = 1:length(t_3)-1
    RG_K1 = h_3*RG_FT(RG_Sol(i),t_3(i));
    RG_K2 = h_3*RG_FT(RG_Sol(i)+((1/2)*RG_K1),t_3(i)+(h_3/2));
    RG_K3 = h_3*RG_FT(RG_Sol(i)+((1/2)*RG_K2),t_3(i)+(h_3/2));
    RG_K4 = h_3*RG_FT(RG_Sol(i)+RG_K3,t_3(i)+h_3);
    RG_Sol(i+1) = RG_Sol(i) + (1/6)*RG_K1 + (1/3)*(RG_K2+RG_K3)...
        + (1/6)*RG_K4;
end

% ode23
[t_3_ode,ODE_Sol] = ode23(@(t,y) (7*(t^6)-y)/(exp((t^3)+y)),[0 3],1);

plot(t_3,RG_Sol,'ro-',t_3_ode,ODE_Sol,'b*--','linewidth',2)
grid on
xlabel('Time, sec')
```

```
ylabel('Response')
title(sprintf('Runge-Kutta vs ODE23, h = %.1f',h_3))
legend('Runge-Kutta','ODE23','location','southeast')
```



Published with MATLAB® R2018b