MEM 634 HW#1

Problem #1

Consider the following plant,

$$Y(s) = \frac{s+2}{s^2(s-1)}U(s)$$

(a) Find a state-space representation for the plant.
(b) Design an observer-based controller so that the closed-loop system is stable.
(c) Find the closed-loop poles and verify that the closed-loop poles are the regulator poles together with the observer poles.
(d) Assume the initial conditions are: $y(0) = 2,\ \dot{y}(0) = 0.5,\ \ddot{y}(0) = 0$, plot the state response and the output response for the closed-loop system.

Problem #2

Repeat Problem #1 with the following plant:

$$Y(s) = \frac{s-2}{s^2(s-1)}U(s)$$

MEM634 - HW1

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Prob 2.

$$\frac{Y(s)}{U(s)} = \frac{s-2}{s^2(s-1)} = \frac{s-2}{s^3-s^2}$$

a)

$$U \longrightarrow [\text{DEN}] \xrightarrow{X} [\text{NUM}] \longrightarrow Y$$

$$\frac{X}{U} = \frac{1}{s^3-s^2} \rightarrow s^3X - s^2X = U$$

$$\rightarrow u = \dddot{x} - \ddot{x} \qquad \mathcal{L}\{\}^{-1}$$

and

$$\frac{Y}{X} = s-2 \rightarrow Y = sX - 2X$$

$$y = \dot{x} - 2x \qquad \mathcal{L}\{\}^{-1}$$

set

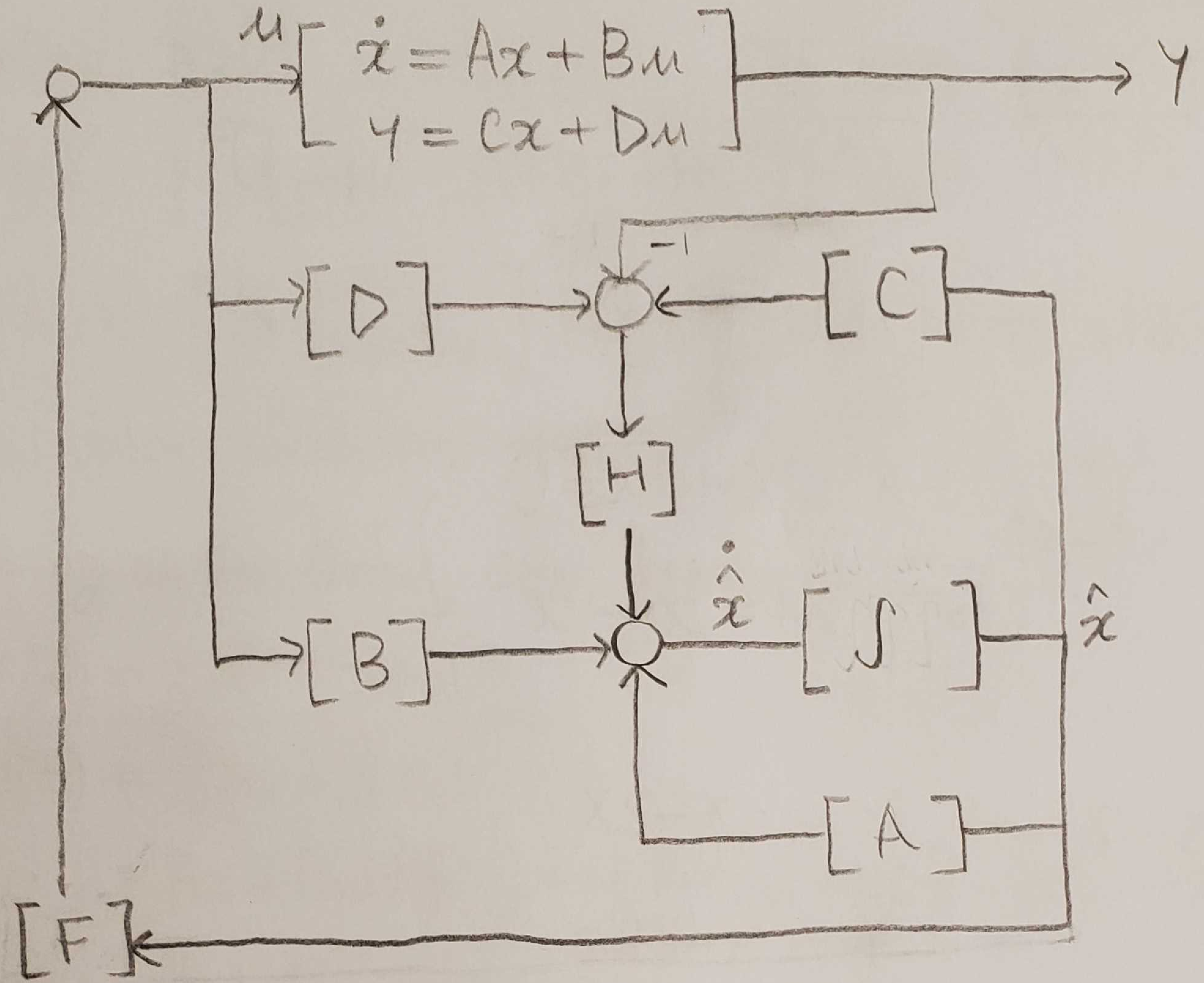
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \ddot{x} \\ \dot{x} \\ x \end{bmatrix}$$

Then

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 & -2 \end{bmatrix} x$$

b)



Given system's dynamics in state-space representation (A, B, C, D), we can now construct observer-based closed-loop controller as shown above

c) Where observer controller dynamics is...

$$\dot{\hat{x}} = A\hat{x} + Bu + H[C\hat{x} + Du - Y]$$

with state feedback $u = F\hat{x}$ and $Y = Cx + Du = Cx + DF\hat{x}$

Thus,

$$\dot{\hat{x}} = A\hat{x} + BF\hat{x} + HC\hat{x} - HCx$$

Also, since our original system is...

$$x = Ax + Bu = Ax + BF\hat{x}$$

Together it forms closed loop system as shown below

$$\begin{bmatrix} \dot{x}(t) \\ \dot{\hat{x}}(t) \end{bmatrix} = \begin{bmatrix} A & BF \\ -HC & A+BF+HC \end{bmatrix} \begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix}$$

Thus, if $\begin{bmatrix} A & BF \\ -HC & A+BF+HC \end{bmatrix}$ has the eigenvalues with the negative real value system's stability can be achieved

looking at the error, $\epsilon(t)$, dynamics...

$$\epsilon(t) = x(t) - \hat{x}(t)$$

$$\dot{\epsilon}(t) = \dot{x}(t) - \dot{\hat{x}}(t)$$

$$= Ax + Bu - [A\hat{x} + (B+HD)u + HC\hat{x} - HCx - HDu]$$

$$= (A+HC)(x-\hat{x})$$

$$= (A+HC)\,\epsilon(t)$$

$\mathcal{L}^{-1}()$

$$\epsilon(t) = e^{(A+HC)(t-t_0)}\,\epsilon(t_0)$$

Thus,

$\epsilon(t) \to 0$ as $t \to \infty$ for all $\epsilon(t_0)$

Therefore, we are interested in finding the closed-loop poles of the system by investigating $\begin{bmatrix} A & BF \\ -HC & A+BF+HC \end{bmatrix}$ matrix and make sure it has the eigenvalues with negative real value.

To do that...

$$\det\left(\begin{bmatrix} sI-A & BF \\ HC & sI-(A+BF+HC) \end{bmatrix}\right) = 0 \quad \text{must be satisfied}$$

Conveniently, above $\det([\sim]) = 0$ equation can be separated as shown below

$$\det([\sim]) = (sIsI - sIA - sIBF - sIHC$$
$$- sIA + AA + ABF + AHC)$$
$$- (BFHC)$$
$$= (sI - A - BF)(sI - A - HC) = 0$$

Thus,

$$\det(sI-(A+BF)) \cdot \det(sI-(A+HC)) = 0$$

above equation can be used to determine the regulator poles (λ of $A+BF$) and observer poles ($A+HC$)

Thus,..

$$A+BF = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} [f_1 \; f_2 \; f_3] = \begin{bmatrix} f_1+1 & f_2 & f_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\det(sI-(A+BF)) = \begin{vmatrix} s-f_1-1 & -f_2 & -f_3 \\ -1 & s & 0 \\ 0 & -1 & s \end{vmatrix} = s^3 - (f_1+1)s^2 - f_2 s - f_3 = 0$$

choose $F = [-5 \ \ -110 \ \ -200]$

such that $s^3 + 4s^2 + s + 2 = 0$

where $s = -1.8866, -1.0567 \pm 10.2419 i$

Next...

$$A + HC = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} h_1 \\ h_2 \\ h_3 \end{bmatrix} [0 \ \ 1 \ \ -2] = \begin{bmatrix} 1 & h_1 & -2h_1 \\ 1 & h_2 & -2h_2 \\ 0 & h_3+1 & -2h_3 \end{bmatrix}$$

$$\det(sI - (A+HC)) = s^3 + (-1 - h_2 + 2h_3)s^2 + (-h_1 + 3h_2 - 2h_3)s + (2h_1 - 2h_2) = 0$$

choose $H = \begin{bmatrix} 168 \\ 123 \\ 69 \end{bmatrix}$

such that $s = -3, -5, -6$

once F and H gain is found, let's construct full observer based closed loop system. $\dot{x} = \begin{bmatrix} A & BF \\ -HC & A+BF+HC \end{bmatrix} x$

$$\begin{bmatrix} \dot{x}(t) \\ \dot{\hat{x}}(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -5 & -110 & -200 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -168 & 336 & -4 & 58 & -536 \\ 0 & -123 & 246 & 1 & 123 & -246 \\ 0 & -69 & 138 & 0 & 70 & -138 \end{bmatrix} \begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix}$$

whose eigenvalues are... $\lambda = -1.0567 \pm 10.2419 i$, $-1.8866, -3, -5, -6$. Thus, system will be stable.

Also, given $\begin{bmatrix} y(0) \\ \dot{y}(0) \\ \ddot{y}(0) \end{bmatrix} = \begin{bmatrix} 2 \\ 0.5 \\ 0 \end{bmatrix}$, we find $\begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix}$

$$y(t) = [0 \quad 1 \quad -2]\, x(t)$$

$$\dot{y}(t) = [0 \quad 1 \quad -2]\, A\, x(t) = [1 \quad -2 \quad 0]\, x(t)$$

$$\ddot{y}(t) = [0 \quad 1 \quad -2]\, A^2\, x(t) = [-1 \quad 0 \quad 0]\, x(t)$$

Thus,

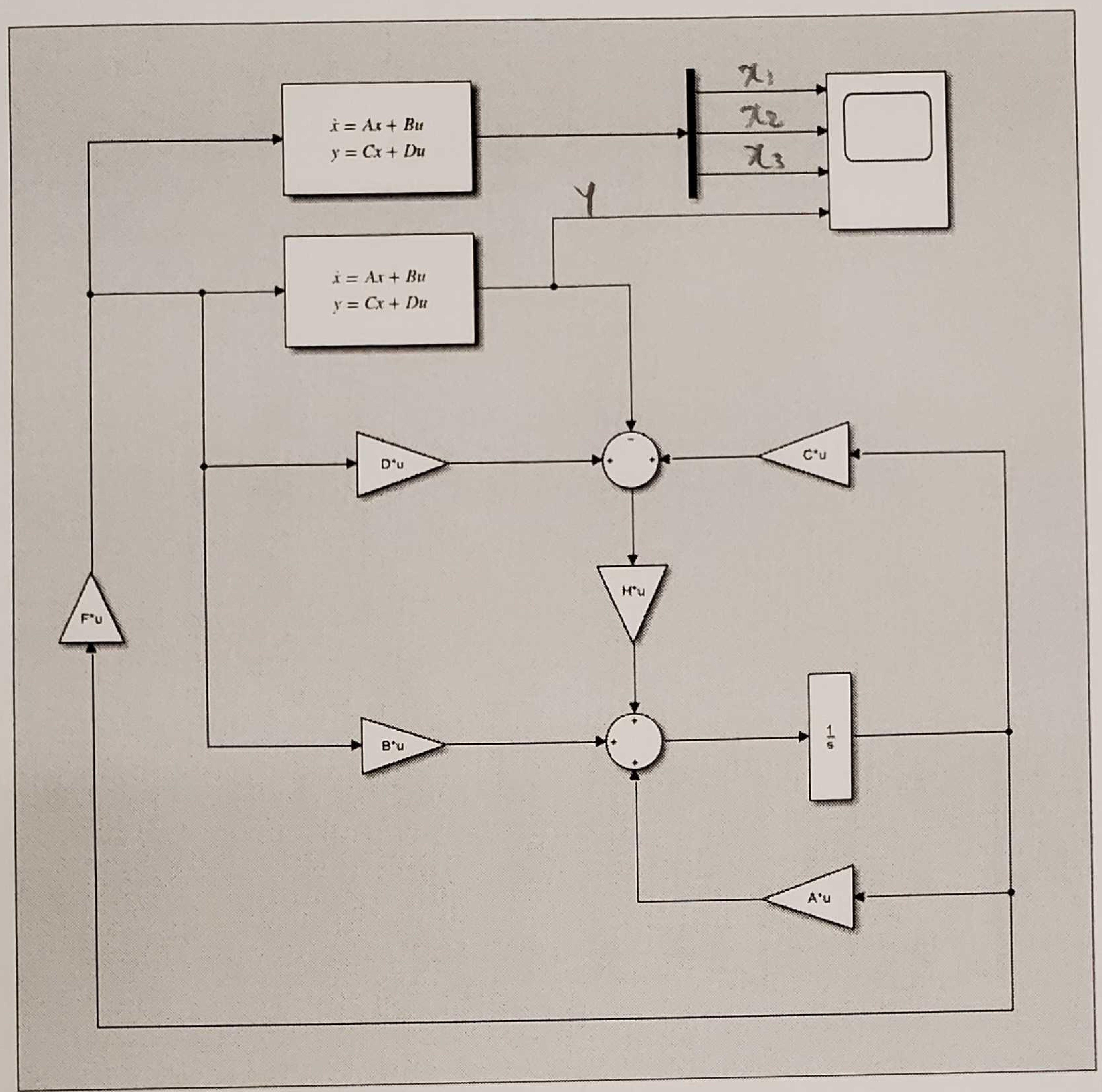
$$\begin{bmatrix} y(0) \\ \dot{y}(0) \\ \ddot{y}(0) \end{bmatrix} = \begin{bmatrix} 0 & 1 & -2 \\ 1 & -2 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix}$$

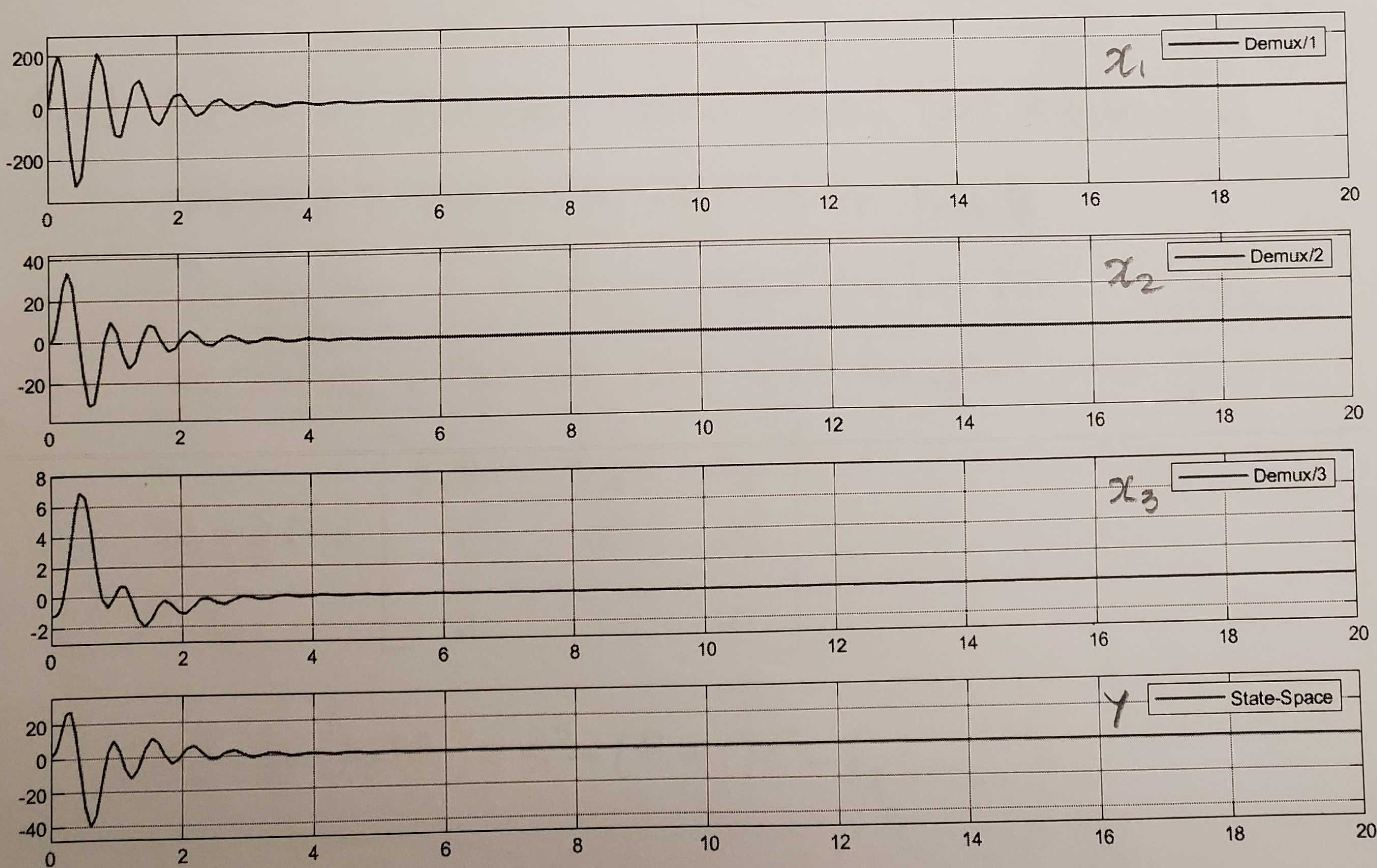
Therefore,

$$\begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} = \begin{bmatrix} 0 & 1 & -2 \\ 1 & -2 & 0 \\ -1 & 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 0.5 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -0.25 \\ -1.125 \end{bmatrix}$$

↑ was used for simulation!

Prob 2





All Stable!!!

Prob 1 was solved in a similar way.

a) $\frac{Y(s)}{U(s)} = \frac{s+2}{s^2(s-1)} = \frac{s+2}{s^3-s^2}$

set $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \ddot{x} \\ \dot{x} \\ x \end{bmatrix}$

Then

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 & 2 \end{bmatrix} x$$

b & c) Design of observer based closed loop system is covered in the same way as in Prob 2.

Thus, now, F and H need to be newly choosen because different transfer function was given.

$$A+BF = \begin{bmatrix} f_1+1 & f_2 & f_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\det(sI-(A+BF)) = s^3-(f_1+1)s^2-f_2 s-f_3 = 0$$

choose $F = [-5 \ \ -110 \ \ -200]$

Then, $s = -1.8866, \ -1.0567 \pm 10.2419\,i$

Next..

$$A+HC = \begin{bmatrix} 1 & h_1 & 2h_1 \\ 1 & h_2 & 2h_2 \\ 0 & h_3+1 & 2h_3 \end{bmatrix}$$

$$\det(sI-(A+HC)) = s^3 + (-1-h_2-2h_3)s^2 + (-h_1-h_2+2h_3)s + (-2h_1+2h_2) = 0$$

choose $H = \begin{bmatrix} -167 \\ -20 \\ -18 \end{bmatrix}$

Then $s = -52.216, \ -1.392 \pm 1.9217\,i$

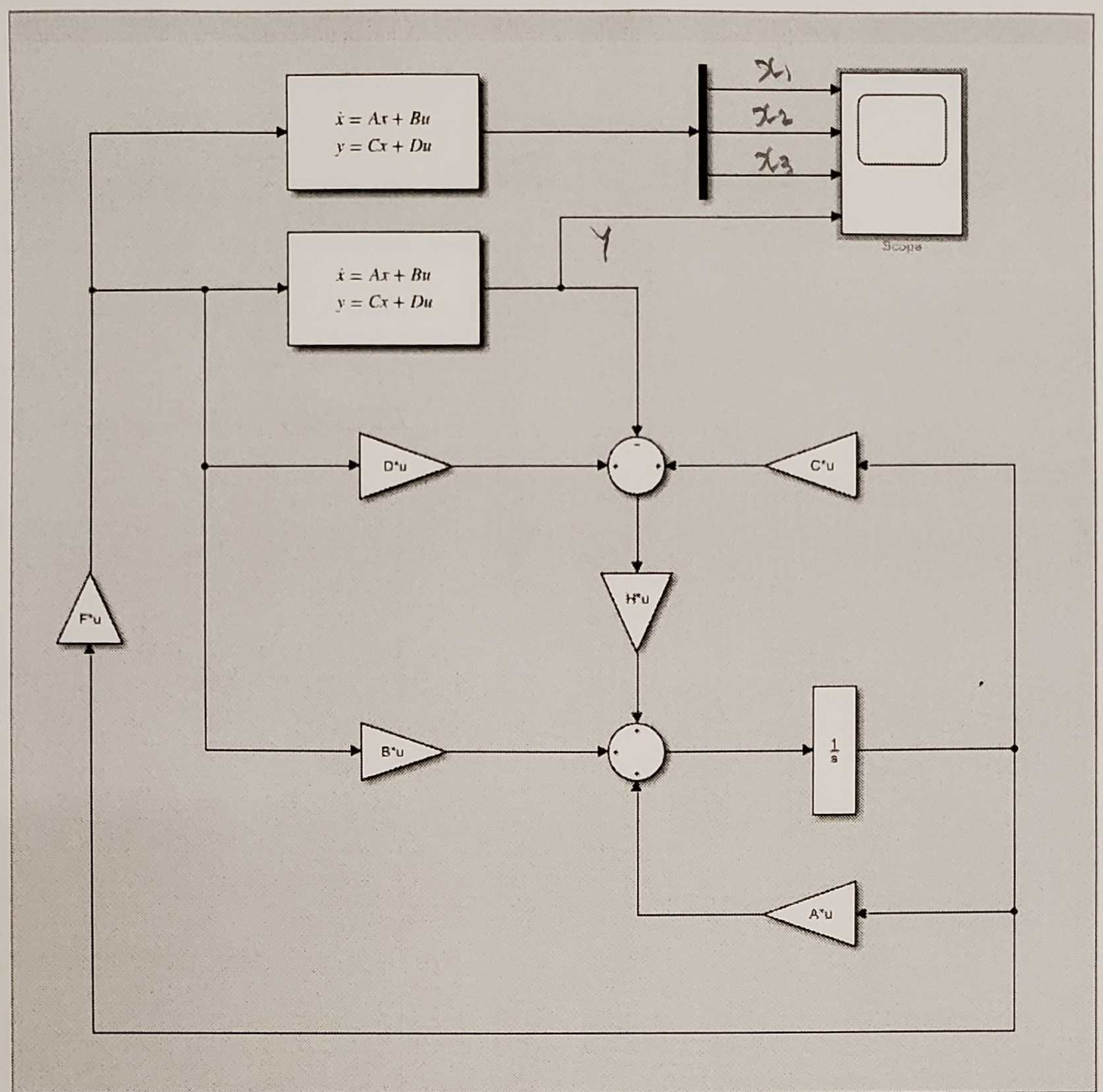
Thus..

$$\begin{bmatrix} \dot{x}(t) \\ \dot{\hat{x}}(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -5 & -110 & -200 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 167 & 334 & -4 & -277 & -534 \\ 0 & 20 & 40 & 1 & -20 & -40 \\ 0 & 18 & 36 & 0 & -17 & -36 \end{bmatrix} \begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix}$$

whose eigenvalues are $-1.8866, -1.0567 \pm 10.2419\,i$, $-52.216, -1.392 \pm 1.9217\,i$

Thus, system will be stable.

Prob 1



All Stable!!!