DREXEL UNIVERSITY
Department of Mechanical Engineering & Mechanics

MEM 634

Robust Control Systems II

Final Exam

Upload to Drexel Bb Learn
before March 18, 2020 (Wednesday) 11:59PM

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Attention:

1. Open book and notes. If you use information other than the textbook and notes, please quote the source.
2. **Absolutely no discussions about the exam with anyone except Dr. Chang before, during, or even after the test until all the students turn in their exams.**
3. If there is any question about the exam, please contact Dr. Chang (changbc@drexel.edu).
4. **No hard copy will be accepted this time!!** The exam report should be **self-contained** and saved as **ONE** LEGIBLE unlocked pdf file with size less than 5Mb. The report should be clearly written and well organized according to the order of the problems. Be sure to include your name on the front page of the report. **In addition to the exam report, you also need to submit the executable computer programs (like m-files, slx-files or mdl-files) used to obtain the solutions.**
6. The exam report and all computer program files are required to be **zipped** and **uploaded to Drexel Bb Learn before 11:59PM, 3/18/2020, Wednesday.**
7. The name of the zipped file should include MEM634_MidTerm_YourName.
8. **Read the problems carefully** before answering the questions.
9. There will be **no credit if no detailed procedure is included** even the final solution is correct!
10. **Use discreet judgment to determine if it is appropriate to use MATLAB (or other software) commands to obtain the solutions. Do NOT use a black-box command to answer the questions that may defeat the purpose of the test.**

Problem #1: (20%)

Consider the following system with transfer function,

$$G(s)=\frac{4}{s^2+0.2s+4}$$

(a) Compute the H_∞ norm of the system using the frequency-domain approach, i.e.,

$$\|G\|_\infty := \sup_\omega |G(j\omega)|$$

(b) Compute the H_∞ norm of the system using the state-space approach, i.e., find the smallest γ such that the following Hamiltonian matrix has no eigenvalues on the $j\omega$-axis.

$$H_G=\begin{bmatrix} A+BR^{-1}D^TC & BR^{-1}B^T \\ -C^T(I+DR^{-1}D^T)C & -(A+BR^{-1}D^TC)^T \end{bmatrix}$$

where $R=\gamma^2 I-D^TD$.

1-a)

$$G(s) = \frac{4}{s^2+0.2s+4}$$

$$\|G\|_\infty = \sup_\omega |G(j\omega)| = \left|\frac{4}{-\omega^2+0.2j\omega+4}\right|$$

$$= \frac{4}{\sqrt{(-\omega^2+4)^2+(0.2\omega)^2}} = \frac{4}{\sqrt{\omega^4-7.96\omega^2+16}}$$

$$\text{set} \left(\frac{d|G(j\omega)|}{d\omega}\right) = 0$$

$$\rightarrow \frac{d}{d\omega}\left(\frac{4}{\sqrt{\omega^4-7.96\omega^2+16}}\right) = 0$$

$$\rightarrow -\frac{2(4\omega^3-15.92\omega)}{(\omega^4-7.96\omega^2+16)^{3/2}} = 0$$

$\rightarrow$

$$\omega = 0 \longrightarrow \sup_\omega |G(j\omega)| = 1$$

$$\omega = -1.995 \longrightarrow \sup_\omega |G(j\omega)| = 10.0125$$

$$\omega = 1.995 \longrightarrow \sup_\omega |G(j\omega)| = 10.0125$$

$\rightarrow$ Thus,

$$\|G\|_\infty = 10.0125$$

1-b)

. H_∞ norm of the system using state space approach (Hamiltonian matrix) was computed

$$\|G(s)\|_{\infty\text{-time domain}} = 10.0125 \checkmark$$

MEM634 - Final P1

June Kwon 3/17/2020

```
clc
clear
close all
```

Problem - B: H-infinity Norm of System

```
SYS = tf(4,[1 0.2 4]);
[A,B,C,D] = tf2ss(4,[1 0.2 4]);
% [A,B,C,D] = ssdata(ss(SYS));
```

Problem - B: Computation of H-infinity Norm in Time Domain

```
TOL = 1e-9;
BB = 50;
AA = 0;
i = 0;
while BB-AA > TOL
    G = (BB+AA)/2; % Gamma
    R = G^2 - D'*D;
    H = [      A+B/R*D'*C           B/R*B'
          -C'*(1+D/R*D')*C    -(A+B/R*D'*C)' ];
    E = eig(H);
    i = i+1;
    if any(abs(real(E))<TOL)
        AA = G;
    else
        BB = G;
    end
end
fprintf('H-infinity Norm = \n\n')
disp(G)
```

H-infinity Norm =

10.0125

Published with MATLAB® R2019b

Problem #2: (20%) Solution in the notes Ch9-2 Model Matching

Let

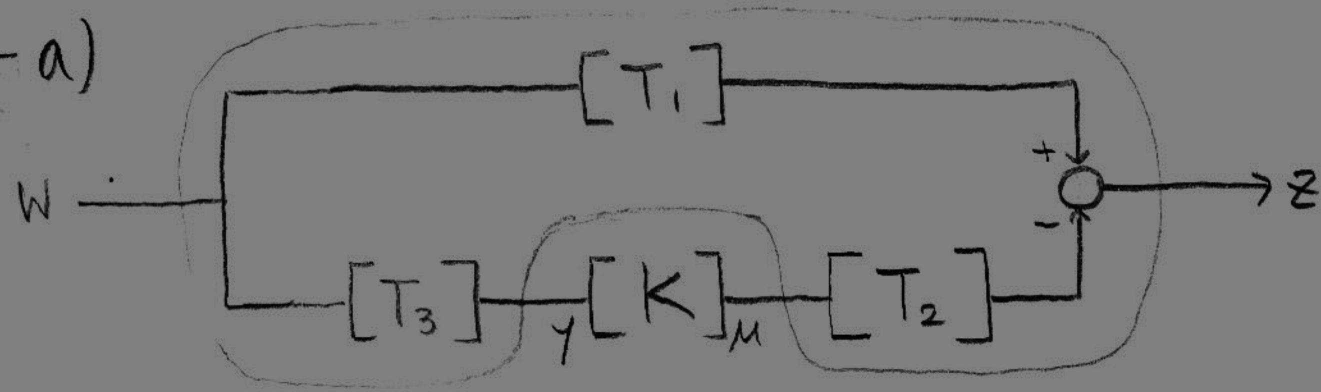
$$\Phi(s) = \frac{2}{s+4} - \frac{s-4}{s+4}K(s)$$

(a) Using the state-space approach in Glover/Doyle paper to find the minimal H_∞ norm of $\Phi(s)$.

(b) Using the state-space suboptimal H_∞ controller equations in Glover/Doyle paper to construct a proper optimal controller $K(s)$ so that $\Phi(s)$ is stable and the H_∞ norm of $\Phi(s)$ is minimized.

(c) Use the state-space approach in Problem #1(b) to compute the H_∞ norm of $\Phi(s)$ with the optimal $K(s)$ given in Problem #2(b).

2-a)



$$\begin{bmatrix} z \\ y \end{bmatrix} = \begin{bmatrix} T_1 & -T_2 \\ T_3 & 0 \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix}$$

$$\left\| \Phi(s) = \frac{2}{s+4} - \frac{s-4}{s+4} K(s) \right\|_\infty ,$$

$$T_1 = \frac{2}{s+4} = \left[\begin{array}{c|c} -4 & 1 \\ \hline 2 & 0 \end{array}\right]$$

$$-T_2 = \frac{-s+4}{s+4} = -1 + \frac{8}{s+4} = \left[\begin{array}{c|c} -4 & 4 \\ \hline 2 & -1 \end{array}\right]$$

$$T_3 = 1$$

Thus,

$$\begin{bmatrix} T_1 & -T_2 \\ T_3 & 0 \end{bmatrix} = \left[\begin{array}{c|cc} -4 & 1 & 4 \\ \hline 2 & 0 & -1 \\ 0 & 1 & 0 \end{array}\right]$$

$A = -4$, $B_1 = 1$, $B_2 = 4$

$C_1 = 2$, $D_{11} = 0$, $D_{12} = -1$

$C_2 = 0$, $D_{21} = 1$, $D_{22} = 0$

Next,

$$R = \begin{bmatrix} D_{11} \\ D_{12} \end{bmatrix} \begin{bmatrix} D_{11} & D_{12} \end{bmatrix} - \begin{bmatrix} \gamma^2 I & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -\gamma^2 & 0 \\ 0 & 1 \end{bmatrix} ///$$

Next,

$$H = \left\{ \begin{bmatrix} A & 0 \\ -C_1^* C_1 & -A^* \end{bmatrix} - \begin{bmatrix} B \\ -C_1^* D_1 \end{bmatrix} R^{-1} \begin{bmatrix} D_1^* C_1 & B^* \end{bmatrix} \right\}$$

$$= \left\{ \begin{bmatrix} -4 & 0 \\ -4 & 4 \end{bmatrix} - \begin{bmatrix} 1 & 4 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} -\gamma^{-2} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -2 & 4 \end{bmatrix} \right\}$$

$$= \begin{bmatrix} 4 & \gamma^{-2} - 16 \\ 0 & -4 \end{bmatrix}$$

(* This is for computing J...

$$\bar{R} = D_{\cdot 1} \cdot D_{\cdot 1}^* - \begin{bmatrix} \gamma^2 I & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} - \begin{bmatrix} \gamma^2 I & 0 \\ 0 & 0 \end{bmatrix}$$

$$D_{\cdot 1} = \begin{bmatrix} D_{11} \\ D_{21} \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} -\gamma^2 & 0 \\ 0 & 1 \end{bmatrix}}$$)

And

$$[(-4)I - H]e_1 = \begin{bmatrix} -8 & 16 - \gamma^{-2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{16\gamma^2 - 1}{8\gamma^2} \\ 1 \end{bmatrix} = 0$$

stable eigenvalue of H...

Thus,

$$X_\infty = \frac{8}{16 - \gamma^{-2}} \geq 0$$

$$J = \left\{ \begin{bmatrix} A^* & 0 \\ -B_1B_1^* & -A \end{bmatrix} - \begin{bmatrix} C^* \\ -B_1D_{\cdot 1}^* \end{bmatrix} \bar{R}^{-1} \begin{bmatrix} D_{\cdot 1}B_1^* & C \end{bmatrix} \right\}$$

$$= \left\{ \begin{bmatrix} -4 & 0 \\ -1 & 4 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -\gamma^{-2} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} \right\}$$

$$= \begin{bmatrix} -4 & 4\gamma^{-2} \\ 0 & 4 \end{bmatrix}$$

$$\left[(-4)I - J\right]e^2 = \begin{bmatrix} 0 & -4\gamma^{-2} \\ 0 & -8 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0$$

Thus,

$$Y_\infty = 0$$

Thus,

$$X_\infty Y_\infty = 0 \Rightarrow \rho\left[X_\infty Y_\infty\right] = 0 < \gamma^2$$

Denominator of X_∞

$$16 - \gamma^{-2} \geq 0$$

$$-\gamma^{-2} \geq -16$$

$$\gamma^{-2} \leq 16$$

$$1 \leq 16\gamma^2$$

$$\frac{1}{16} \leq \gamma^2$$

$$\left[\frac{1}{4} \leq \gamma\right]$$

Thus,

$$\left[\text{optimal norm} = \frac{1}{4}\right]_{\prime\prime}$$

2-b)

$$H = \begin{bmatrix} H_{11} & H_{12} & H_2 \end{bmatrix} = -\left(\overset{0}{Y_\infty C^T} + B_1 \overset{\begin{bmatrix} D_{11} \\ D_{21} \end{bmatrix}}{D_{\cdot 1}^T}\right) \bar{R}^{-1}$$

$$= -\begin{bmatrix} D_{11} \\ D_{21} \end{bmatrix}^T \bar{R}^{-1} = -\begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} -\gamma^{-2} & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 \end{bmatrix} \Rightarrow H_{12} = 0 \;,\; H_2 = -1$$

$$\begin{bmatrix} F_{11}^T & F_{12}^T & F_2^T \end{bmatrix} = \left(X_\infty B + C_1^T \overset{\begin{bmatrix} D_{11} & D_{12} \end{bmatrix}}{D_{1\cdot}}\right) R^{-1}$$

$$= \left(\frac{8}{16-\gamma^{-2}} \begin{bmatrix} 1 & 4 \end{bmatrix} + 2 \begin{bmatrix} 0 & -1 \end{bmatrix}\right) \begin{bmatrix} -\gamma^{-2} & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{8}{16-\gamma^{-2}} & \dfrac{32}{16-\gamma^{-2}} - 2 \end{bmatrix} \begin{bmatrix} -\gamma^{-2} & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{-8\gamma^{-2}}{16-\gamma^{-2}} & \dfrac{2\gamma^{-2}}{16-\gamma^{-2}} \end{bmatrix} \Rightarrow F_{12} = \frac{-8\gamma^{-2}}{16-\gamma^{-2}}$$

$$F_2 = \frac{2\gamma^{-2}}{16-\gamma^{-2}}$$

Now let's form optimal K controller...

$$Z = I$$

$$\hat{D} = 0$$

$$\hat{C} = F_2 Z = F_2 = \frac{2\gamma^{-2}}{16-\gamma^{-2}}$$

$$\hat{B} = -H = 1$$

$$\hat{A} = A + HC + (B_2 + H_{12})\hat{C}$$

$$= -4 + [0 \ \ -1]\begin{bmatrix} 2 \\ 0 \end{bmatrix} + (4+0)\frac{2\gamma^{-2}}{16-\gamma^{-2}}$$

$$= -4 + \frac{8\gamma^{-2}}{16-\gamma^{-2}} = \frac{-64+4\gamma^{-2}+8\gamma^{-2}}{16-\gamma^{-2}}$$

$$= \frac{-64+12\gamma^{-2}}{16-\gamma^{-2}}$$

Thus, my $K(s)$ controller is...

$$K(s) = \left[\begin{array}{c|c} \hat{A} & \hat{B} \\ \hline \hat{C} & \hat{D} \end{array}\right], \qquad \begin{aligned} \dot{x}_K &= \hat{A} x_K + \hat{B} y \\ u &= \hat{C} x_K \end{aligned}$$

Thus,

$$K(s) = \hat{C}(sI-\hat{A})^{-1}\hat{B} = \frac{\dfrac{2\gamma^{-2}}{16-\gamma^{-2}}}{s - \dfrac{-64+12\gamma^{-2}}{16-\gamma^{-2}}}$$

$$= \frac{2\gamma^{-2}}{(16-\gamma^{-2})s - (-64+12\gamma^{-2})}$$

Applying optimal $\gamma = \frac{1}{4}$ from 2-(a)... ($^{*}\gamma^{-2}=16$)

$$K_{opt}(s) = \frac{32}{(0)s-(-64+12(16))} = \frac{32}{64-192}$$

$$= \frac{32}{-128} = \boxed{-\frac{1}{4}}$$

2-C) Now apply $K_{opt}(s)$ to the given system..

$$\Phi_{opt}(s) = \frac{2}{s+4} - \frac{s-4}{s+4}\left(-\frac{1}{4}\right) = \frac{2}{s+4} + \frac{0.25(s-4)}{s+4}$$

$$= \boxed{\frac{0.25s+1}{s+4}}$$

Thus,

$$\boxed{\|\Phi_{opt}(s)\|_\infty = 0.25}$$

same as $\frac{1}{4}$ found in 2-(a)... ✓

MEM634 - Final P2

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Problem - C: Computation of H-infinity Norm in Time Domain

```
SYS = tf([0.25 1],[1 4])
[A,B,C,D] = tf2ss([0.25 1],[1 4]);

TOL = 1e-10;
BB = 50;
AA = 0;
i = 0;
while BB-AA > TOL
    G = (BB+AA)/2; % Gamma
    R = G^2 - D'*D;
    H = [     A+B/R*D'*C           B/R*B'
          -C'*(1+D/R*D')*C    -(A+B/R*D'*C)' ];
    E = eig(H);
    i = i+1;
    if any(abs(real(E))<TOL)
        AA = G;
    else
        BB = G;
    end
end
fprintf('H-infinity Norm = \n\n')
disp(G)
```

SYS =

$$\frac{0.25\,s + 1}{s + 4} = \frac{\frac{1}{4}(s+4)}{s+4} = \frac{1}{4}$$ Thus, H_∞ norm of 0.25 is 0.25...

Continuous-time transfer function.

H-infinity Norm =

0.2500

H_∞ LQG

~~generalized~~ LQR

~~Robust stability~~

$\bar{\sigma}[\ ]$: maximum singular value

no need to be square matrix like eigenvalue

it shows margins

$$\frac{1}{1-L} + \frac{L}{1+L} = 1$$

$\uparrow \quad \downarrow \quad = -$

Disturbance has nothing to do with stability

Problem #3: (60%)

Consider the mixed sensitivity optimization problem described in Section III of the AIAA paper by Chang, Li, Banda, and Yeh. P.125 EX 1

Let

$$P(s) = \frac{2}{s-2}, \quad W_1(s) = \frac{s+10}{100(s+0.1)}, \quad W_2(s) = 0.1s+1$$

The objective is to find a controller $K(s)$ so that the closed-loop system is internally stable and the H_∞ norm of $\Phi(s)$ is minimized where

$$\Phi(s) = \begin{bmatrix} W_1\underbrace{(I-PK)^{-1}}_{\text{sensitivity function}} \\ W_2PK(I-PK)^{-1} \end{bmatrix} \begin{matrix} \xrightarrow{\text{for}} \text{disturbance reduction} \\ \xrightarrow{\text{for}} \text{robust stability} \end{matrix}$$

(a) Find a generalized plant $G(s)$ so that the lower linear fractional transformation is $F_\ell(G,K) = \Phi(s)$. Also find a state-space representation for the generalized plant. (10%)

generalized plant

(b) Determine the minimal H_∞ norm of $\Phi(s)$, denoted by γ_o. (15%)

minimize

(c) Choose a γ that is one percent higher than γ_o and construct a suboptimal H_∞ controller $K_{subopt}(s)$ so that the closed-loop system is internally stable and $\| F_\ell(G,K) \|_\infty < \gamma$. (15%)

(d) Let complementary sensitivity function $\Phi_2(s) = P(s)K_{subopt}(s)\left[I - P(s)K_{subopt}(s)\right]^{-1}$. Plot $|\Phi_2(j\omega)|$ as a function of ω, and explain how this plot relates to the robust stability of the closed-loop system. (10%)

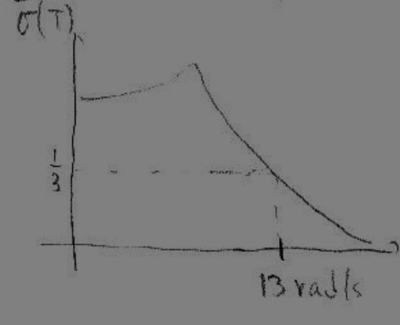
(e) Compute the gain and phase margins based on the Bode or Nyquist plots of the loop transfer function of the closed-loop system, and identify the two points on the complementary sensitivity plot in (d) that would give you the same gain and phase margins.(10%)

$\underbrace{\bar{\sigma}[T(j\omega)]}\,\bar{\sigma}[\Delta_m(j\omega)] < 1$

→ minimize → more robust stability

$20\log_{10} a = 40$ dB

$a = 100$



$\bar{\sigma}(T)\,\bar{\sigma}(\Delta) < 1$

↓

$\frac{1}{3}\cdot$(so this can be 3) < 1

↓ if

$\frac{1}{100}\cdot$(can be 100) < 1

↓

and still stable!

3-a)

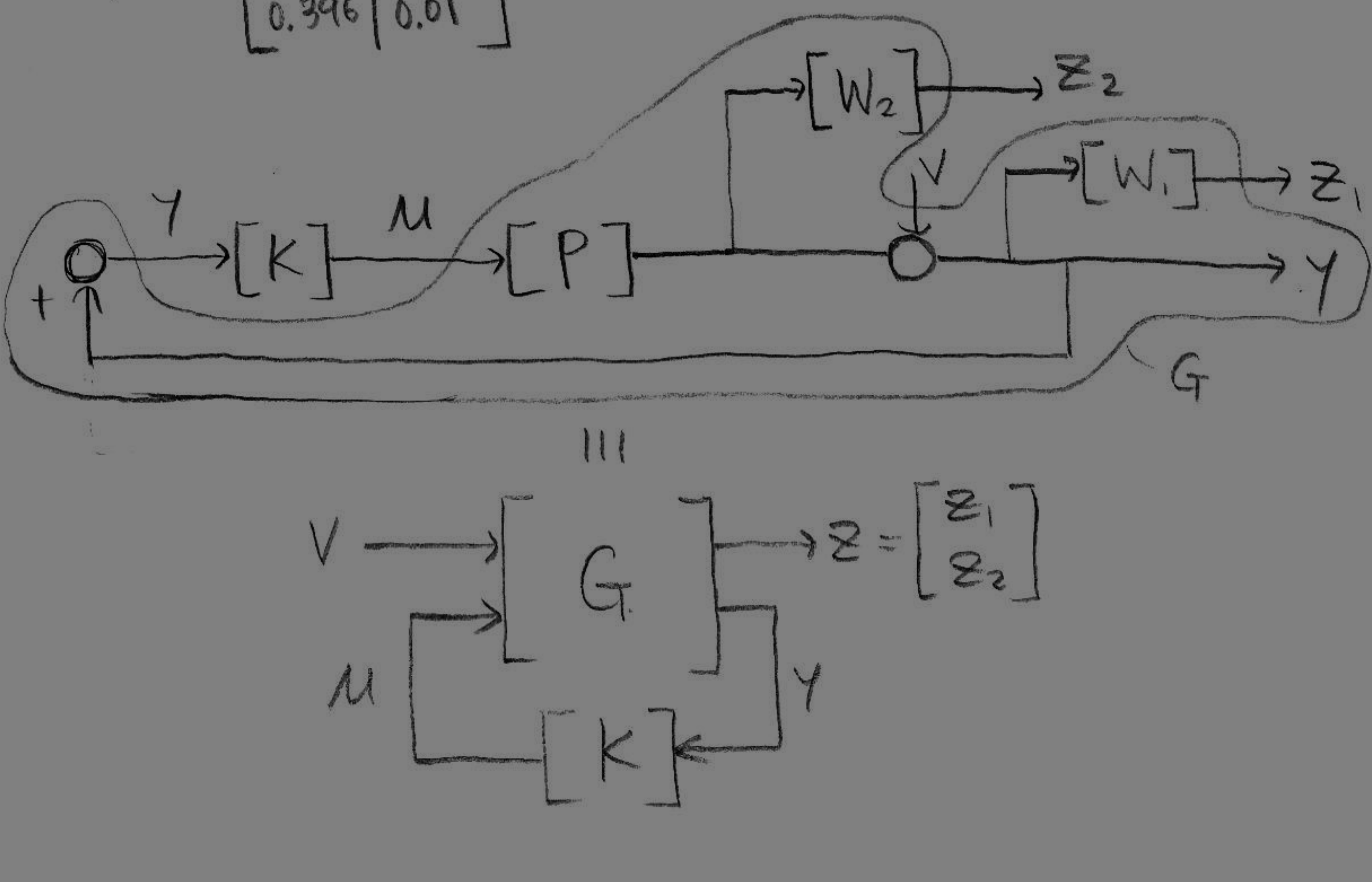
$$P(s) = \frac{2}{s-2}, \quad W_1(s) = \frac{s+10}{100(s+0.1)}, \quad W_2(s) = 0.1s + 1$$

$$\Phi(s) = \begin{bmatrix} W_1(I-PK)^{-1} \\ W_2PK(I-PK)^{-1} \end{bmatrix}.$$

State space...

$$P(s) = \left[\begin{array}{c|c} 2 & 2 \\ \hline 1 & 0 \end{array}\right], \quad W_2(s)P(s) = \left[\begin{array}{c|c} 2 & 2 \\ \hline 1.2 & 0.2 \end{array}\right]$$

$$W_1(s) = \left[\begin{array}{c|c} -0.1 & 0.25 \\ \hline 0.396 & 0.01 \end{array}\right]$$



Above structure is...

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} W_1(I-PK)^{-1} \\ W_2PK(I-PK)^{-1} \end{bmatrix} V = \Phi(s)V$$

Adding states of K controller...

$$\left[\begin{array}{c} z_1 \\ z_2 \\ \hline Y \end{array}\right] = \begin{bmatrix} W_1(V+Pu) \\ W_2Pu \\ V+Pu \end{bmatrix} = \left[\begin{array}{c|c} W_1 & W_1P \\ 0 & W_2P \\ \hline I & P \end{array}\right] \begin{bmatrix} V \\ u \end{bmatrix}$$

$$= \left[\begin{array}{c|c} G_{11} & G_{12} \\ \hline G_{21} & G_{22} \end{array}\right] \begin{bmatrix} V \\ u \end{bmatrix}$$

We are interested in knowing $\left[\begin{array}{c|c} G_{11} & G_{12} \\ \hline G_{21} & G_{22} \end{array}\right]$ ...

So let's form state-space realization of $G(s)$...

$$G(s) = \left[\begin{array}{c|c} W_1 & W_1P \\ 0 & W_2P \\ \hline I & P \end{array}\right] = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right]$$

Great. We know P, W_2P, W_1 from problem statement and..

$$P = \left[\begin{array}{c|c} A_p & B_p \\ \hline C_p & D_p \end{array}\right] = \left[\begin{array}{c|c} 2 & 2 \\ \hline 1 & 0 \end{array}\right]$$

$$W_2P = \left[\begin{array}{c|c} A_p & B_p \\ \hline C_{W_2} & D_{W_2} \end{array}\right] = \left[\begin{array}{c|c} 2 & 2 \\ \hline 1.2 & 0.2 \end{array}\right]$$

$$W_1 = \left[\begin{array}{c|c} A_{W_1} & B_{W_1} \\ \hline C_{W_1} & D_{W_1} \end{array}\right] = \left[\begin{array}{c|c} -0.1 & 0.25 \\ \hline 0.396 & 0.01 \end{array}\right]$$

and according to (3-5)

$$A = \begin{bmatrix} A_p & 0 \\ B_{W_1}C_p & A_{W_1} \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ B_{W_1} \end{bmatrix}, \quad B_2 = \begin{bmatrix} B_p \\ B_{W_1}D_p \end{bmatrix},$$

$$C_1 = \begin{bmatrix} D_{W_1}C_p & C_{W_1} \\ C_{W_2} & 0 \end{bmatrix}, \quad D_{11} = \begin{bmatrix} D_{W_1} \\ 0 \end{bmatrix}, \quad D_{12} = \begin{bmatrix} D_{W_1}D_p \\ D_{W_2} \end{bmatrix}$$

$$C_2 = \begin{bmatrix} C_p & 0 \end{bmatrix} \quad D_{21} = I \quad D_{22} = D_p$$

Thus, plugging in, my generalized plant $G(s)$ is..

$$
\begin{array}{cc}
 & \begin{array}{cccc} \quad n & & m_1 & m_2 \end{array} \\
G(s) = \begin{array}{c} n \\ \\ p_1 \\ \\ p_2 \end{array} &
\left[\begin{array}{cc|c:c}
2 & 0 & 0 & 2 \\
0.25 & -0.1 & 0.25 & 0 \\
\hline
0.01 & 0.3960 & 0.01 & 0 \\
1.2 & 0 & 0 & 0.2 \\
\hdashline
1 & 0 & 1 & 0
\end{array}\right]
\end{array}
$$

! It is not 1.

Notice we have 0.2 in D_{12}.

Therefore we need to scale the plant.

Since $D_{21} = I$ is already satisfying the assumption, we apply singular value decomposition only to D_{12}

$$D_{12} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0.2 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \end{bmatrix} \quad \text{Thus,} \quad L_{12} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$R_{12} = 5$$

$$\text{since } D_{21} = I \longrightarrow \left[\begin{array}{l} L_{21} = 1 \\ R_{21} = 1 \end{array} \right.$$

Thus, knowing $L_{12}, R_{12}, L_{21}, R_{21}$, we construct scaled general plant, $\hat{G}(s)$.

$$\hat{G}(s) = \left[\begin{array}{c|cc}
\hat{A} & \hat{B}_1 & \hat{B}_2 \\
\hline
\hat{C}_1 & \hat{D}_{11} & \hat{D}_{12} \\
\hat{C}_2 & \hat{D}_{21} & \hat{D}_{22}
\end{array}\right]$$

According to (4-4)...

$$\hat{A} = A, \quad \hat{B}_1 = B_1 R_{21}, \quad \hat{B}_2 = B_2 R_{12}$$

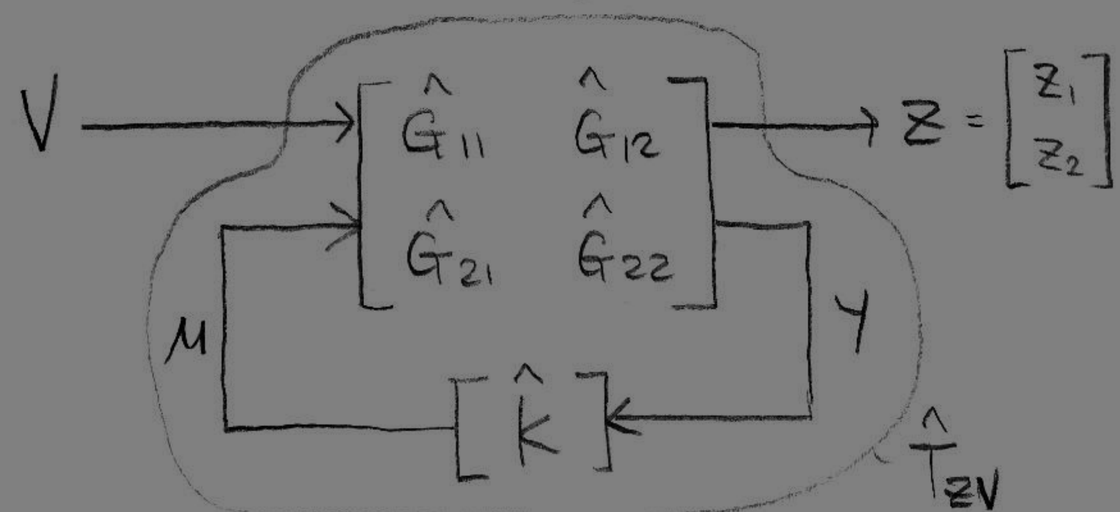
$$\hat{C}_1 = L_{12} C_1, \quad \hat{D}_{11} = L_{12} D_{11} R_{21}, \quad \hat{D}_{12} = \begin{bmatrix} 0 \\ I \end{bmatrix}$$

$$\hat{C}_2 = L_{21} C_2, \quad \hat{D}_{21} = \begin{bmatrix} 0 & I \end{bmatrix}, \quad \hat{D}_{22} = L_{21} D_{22} R_{12}$$

Thus, my scaled generalized plant $\hat{G}(s)$ is...

$$\hat{G}(s) = \left[\begin{array}{cc|c:c} 2 & 0 & 0 & 10 \\ 0.25 & -0.1 & 0.25 & 0 \\ \hline -0.01 & -0.3960 & -0.01 & 0 \\ 1.2 & 0 & 0 & 1 \\ \hdashline 1 & 0 & 1 & 0 \end{array}\right]_{//}$$

Great... Now we have following structure below..



Where $\hat{T}_{zv}$ is our lower linear fractional transformation...

$$\hat{T}_{zv} = \hat{F}_{\ell}(G,K) = \hat{G}_{11} + \hat{G}_{12}\hat{K}(I - \hat{G}_{22}\hat{K})^{-1}\hat{G}_{21} = \hat{\Phi}(s)$$

where

$$\hat{G}_{11} = \left[\begin{array}{c|c} \hat{A} & \hat{B}_1 \\ \hline \hat{C}_1 & \hat{D}_{11} \end{array}\right] = \left[\begin{array}{cc|c} 2 & 0 & 0 \\ 0.25 & -0.1 & 0.25 \\ \hline -0.01 & -0.3960 & -0.01 \\ 1.2 & 0 & 0 \end{array}\right]$$

$$\hat{G}_{12} = \left[\begin{array}{c|c} \hat{A} & \hat{B}_2 \\ \hline \hat{C}_1 & \hat{D}_{12} \end{array}\right] = \left[\begin{array}{cc|c} 2 & 0 & 10 \\ 0.25 & -0.1 & 0 \\ \hline -0.01 & -0.396 & 0 \\ 1.2 & 0 & 1 \end{array}\right]$$

$$\hat{G}_{21} = \left[\begin{array}{c|c} \hat{A} & \hat{B}_1 \\ \hline \hat{C}_2 & \hat{D}_{21} \end{array}\right] = \left[\begin{array}{cc|c} 2 & 0 & 0 \\ 0.25 & -0.1 & 0.25 \\ \hline 1 & 0 & 1 \end{array}\right]$$

$$\hat{G}_{22} = \left[\begin{array}{c|c} \hat{A} & \hat{B}_2 \\ \hline \hat{C}_2 & \hat{D}_{22} \end{array}\right] = \left[\begin{array}{cc|c} 2 & 0 & 10 \\ 0.25 & -0.1 & 0 \\ \hline 1 & 0 & 0 \end{array}\right]$$

and we want to find a stabilizing controller $\hat{K}$ such that $\|\hat{T}_{zv}\|_\infty < \gamma$.

once we find $\hat{K}$ then we will scale back

$$\hat{K}(s) = \left[\begin{array}{c|c} \hat{A}_K & \hat{B}_K \\ \hline \hat{C}_K & \hat{D}_K \end{array}\right]$$

$$\downarrow$$

$$K(s) = \left[\begin{array}{c|c} \hat{A}_K & \hat{B}_K L_{21} \\ \hline R_{12}\hat{C}_K & R_{12}\hat{D}_K L_{21} \end{array}\right]$$

2-b Knowing $\|\hat{T}_{zv}\|_\infty = \|T_{zv}\|_\infty$

- Thus, we will proceed with "scaled" generalized plant.

$$\hat{T}_{zv} = \hat{\Phi}(s) = \hat{G}_{11} + \hat{G}_{12}\hat{K}(I - \hat{G}_{22}\hat{K})^{-1}\hat{G}_{21}$$

$\uparrow$ loop transfer function of the closed-loop system.

To find optimal $\hat{K}_{opt}$ controller that yields optimal norm, we need to first find the optimal norm.

To find optimal norm, we must analyze the (scaled) generalized plant, $\hat{G}(s)$!

First,

$$X_\infty = Ric(H_\infty(\gamma))$$

(* where

$$H_\infty(\gamma) = \begin{bmatrix} A & 0 \\ -C_1^T C_1 & -A^T \end{bmatrix} - \begin{bmatrix} B \\ -C_1^T D_1 \end{bmatrix} R^{-1} \begin{bmatrix} D_1^T C_1 & B^T \end{bmatrix}$$
)

Thus

$$X_\infty = \begin{bmatrix} \frac{\gamma^2}{10\nabla} & \frac{99\gamma^2}{25\nabla} \\ \frac{99\gamma^2}{25\nabla} & \frac{19602\gamma^2}{125\nabla} \end{bmatrix}$$

where $\nabla = \sqrt{(100\gamma-1)(100\gamma+1)(2\gamma^2-1)} + 100\gamma^2 - 1$

so I am interested in "not making denominator into zero"... so

$$\text{solve}(\nabla, \gamma) \quad \left\{ \begin{array}{l} \nabla > 0 \\ [\gamma > 0] \end{array} \right.$$

$$\left(\text{* also...} \quad \left\{ \begin{array}{l} \nabla/\gamma^2 > 0 \\ \gamma > 0 \end{array} \right. \right)$$

The same...

Next,

$$Y_\infty = \mathrm{Ric}(J_\infty(\gamma))$$

$$\left(\text{* where} \quad J_\infty(Y) = \begin{bmatrix} A^T & 0 \\ -B_1B_1^T & -A \end{bmatrix} - \begin{bmatrix} C^T \\ -B_1D_{\cdot 1}^T \end{bmatrix} \bar{R}^{-1} \begin{bmatrix} D_{\cdot 1}B_1^T & C \end{bmatrix} \right)$$

Then,

$$Y_\infty = \begin{bmatrix} \frac{100\gamma^2}{25\gamma^2-36} & 0 \\ 0 & 0 \end{bmatrix}$$

we are interested in the first term..

$$\frac{100\gamma^2}{25\gamma^2-36} \longrightarrow \frac{100}{25-36\gamma^{-2}}$$

we want denominator not to go 0...

$$25 - 36\gamma^{-2} > 0$$

$$-36\gamma^{-2} > -25$$

$$36\gamma^{-2} < 25$$

$$\gamma^{-2} < \frac{25}{36}$$

$$1 < \frac{25}{36}\gamma^{2}$$

$$\frac{36}{25} < \gamma^{2}$$

$$1.2 = \sqrt{\frac{36}{25}} < \gamma$$

$$\boxed{\gamma > 1.2}$$

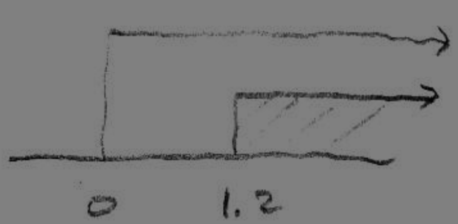
okay we have two conditions..

$\gamma > 0$ from X_∞

$\gamma > 1.2$ from Y_∞

$\rightarrow$ Thus, $\langle \gamma > 1.2 \rangle$

$\downarrow$



Thus,

$$\boxed{\gamma_{optimal} = 1.2}$$

↳ but should be slightly bigger than 1.2

if $\gamma = 1.2$, then denominator goes to 0, which we don't like...

So something like 1.21 is sufficient...!

3-c)

From 3-b) we have $\gamma_0 = 1.2$

Now $\gamma = \gamma_0 \times 1.01 = 1.2120$ //

Now we construct a suboptimal H_∞ controller $K_{supopt}(s)$ using "scaled" generalized plant from 3-a)...

$\bar{K}_{sub}(s)$ is given as follows...

$$\bar{K}_{sub}(s) = \left[\begin{array}{c|c} \bar{A} & \bar{B} \\ \hline \bar{C} & \bar{D} \end{array}\right]$$

where

$$X = Ric(H_\infty(\gamma = 1.212)) = \begin{bmatrix} 0.0005 & 0.0185 \\ 0.0185 & 0.7322 \end{bmatrix}$$

$$Y = Ric(J_\infty(\gamma = 1.212)) = \begin{bmatrix} 203.005 & 0 \\ 0 & 0 \end{bmatrix}$$

$$Z = (\underset{2\times 2}{I} - \gamma^{-2} Y X)^{-1} = \begin{bmatrix} 1.0690 & 2.7315 \\ 0 & 1 \end{bmatrix}$$

$$F^T = [\underset{0\times 2}{F_{11}^T},\ \underset{1\times 2}{F_{12}^T},\ \underset{1\times 2}{F_2^T}] = -(XB + C_1^T D_{1\cdot})R^{-1}$$

$$= \begin{bmatrix} 0.0032 & -1.2047 \\ 0.1273 & -0.1849 \end{bmatrix}$$

$$F_{12} = \begin{bmatrix} 0.0032 & 0.1273 \end{bmatrix}$$

$$F_2 = \begin{bmatrix} -1.2047 & -0.1849 \end{bmatrix}$$

$$H = \begin{bmatrix} \underset{2\times 1}{H_{11}} & \underset{2\times 1}{H_{12}} & \underset{2\times 1}{H_2} \end{bmatrix} = -(YC^T + B_1 D_{\cdot 1}^T)\bar{R}^{-1}$$

$$= \begin{bmatrix} 0 & 165.8375 & -203.005 \\ -0 & 0 & -0.25 \end{bmatrix}$$

$$H_{11} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$H_{12} = \begin{bmatrix} 163.8375 \\ 0 \end{bmatrix}$$

$$H_2 = \begin{bmatrix} -203.005 \\ -0.25 \end{bmatrix}$$

Thus,

$$\bar{D} = -D_{1121} D_{1111}^T (\gamma^2 I - D_{1111} D_{1111}^T)^{-1} D_{1112} - D_{1122}$$

where

$$\left(D_{11} = \begin{bmatrix} -0.01 \\ 0 \end{bmatrix} \text{ Thus, } \begin{matrix} D_{1111} = [\,], & D_{1112} = -0.01 \\ D_{1121} = [\,], & D_{1122} = 0 \end{matrix} \right)$$

Thus,

$$= (\gamma^2)^{-1} D_{1112} - D_{1122} = -0.0068 \,_{\prime\prime}$$

$$\bar{C} = \left(F_2 - \bar{D}(C_2 + F_{12})\right)Z$$

$$= \begin{bmatrix} -1.2805 & -3.4559 \end{bmatrix}$$ //

$$\bar{B} = -H_2 + (B_2 + H_{12})\bar{D}$$

$$= \begin{bmatrix} 201.8079 \\ 0.25 \end{bmatrix}$$ //

$$\bar{A} = A + HC + (B_2 + H_{12})\bar{C}$$

$$= \begin{bmatrix} -227.1534 & -607.6794 \\ 0 & -0.1 \end{bmatrix}$$ //

Thus, my scaled $K_{supoptimal}(s)$ controller is ...

$$\hat{K}_{subopt}(s) = \left[\begin{array}{cc|c} -227.2 & -607.7 & 201.8 \\ 0 & 0.1 & 0.25 \\ \hline -1.28 & -3.456 & -0.006808 \end{array}\right]$$

$$= \frac{-0.0068076(s + 3.83\times10^{4})(s + 0.1063)}{(s + 227.2)(s + 0.1)}$$ //

And I would like to scale back!

Thus, knowing $A_K = \hat{A}_K$, $B_K = \hat{B}_K L_{21}$

$$C_K = R_{12}\hat{C}_K, \quad D_K = R_{12}\hat{D}_K L_{21}$$

we can build our non-scaled $K_{supoptimal}$ Controller...

$$K_{subopt} = \left[\begin{array}{cc|c} -227.2 & -607.7 & 201.8 \\ 0 & -0.1 & 0.25 \\ \hline -6.402 & -17.28 & -0.03404 \end{array}\right]$$

$$= \frac{-0.034038(s+3.83\times10^4)(s+0.1063)}{(s+227.2)(s+0.1)}$$

Next,
we check our stability of the closed loop transfer function... $\Phi(s)$ ← (non-scaled)

$$\Phi(s) = \begin{bmatrix} W_1(I-PK)^{-1} \\ \\ W_2PK(I-PK)^{-1} \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{0.01(s+227.2)(s+10)(s-2)(s+0.1)}{(s+215.2)(s+10)(s+0.1077)(s+0.1)} \\ \\ \dfrac{-0.0068076(s+3.83\times10^4)(s+227.2)(s+10)\cancel{(s-2)}(s+0.1063)(s+0.}{(s+227.2)(s+215.2)(s+10)\cancel{(s-2)}(s+0.1077)(s+0.1)} \end{bmatrix}$$

$$\text{Pole}(\Phi(s)) = \text{eig}(\Phi(s))$$

$$= \left.\begin{matrix} -215.2137 \\ -10 \\ -0.1077 \\ -0.1 \\ -227.1534 \\ -215.2137 \\ -10 \\ -0.1077 \\ -0.1000 \end{matrix}\right\} \rightarrow \text{Internally stable}$$

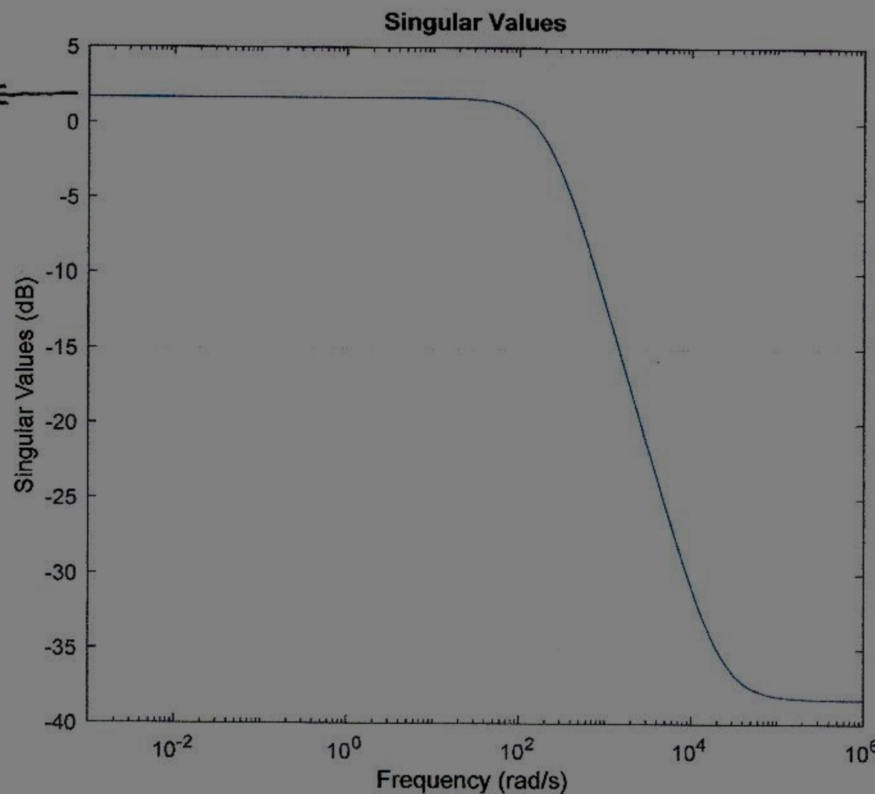
Now, we check H_∞ norm of $\Phi(s)$ ...

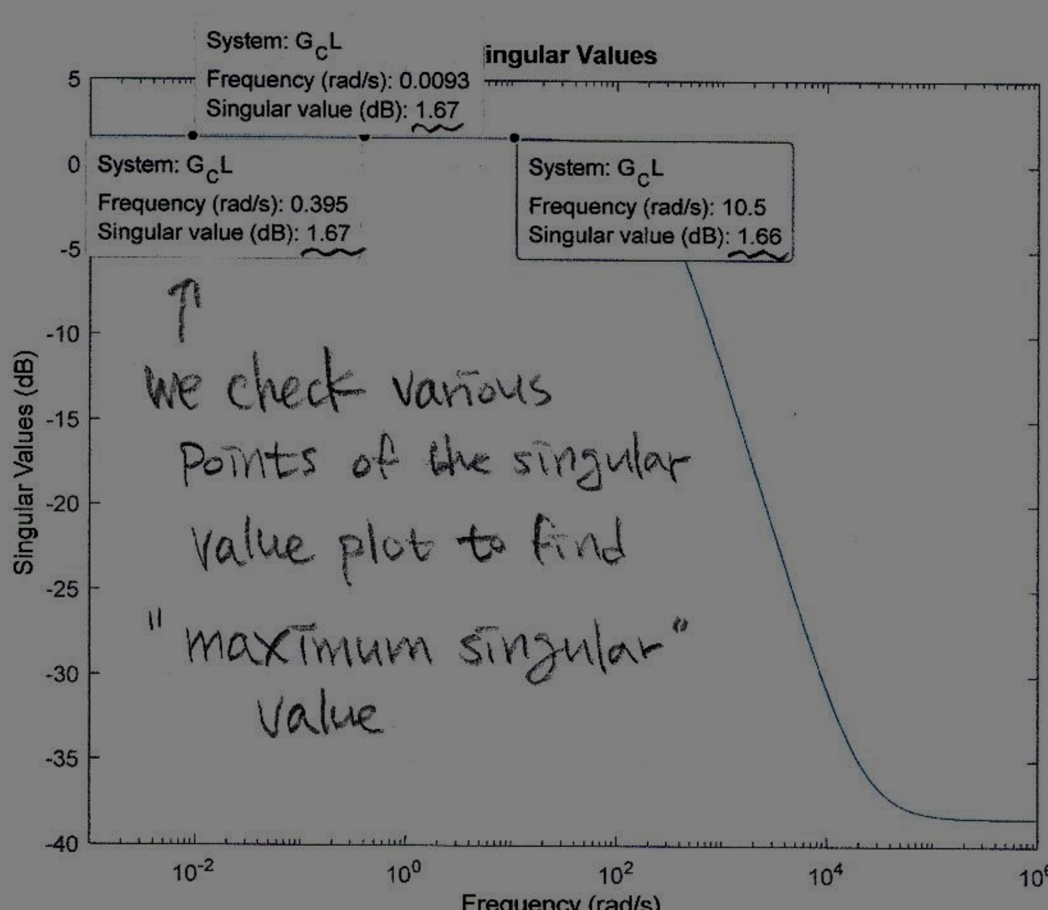
We plot our closed loop transfere function's, $\Phi(s)$

```
figure;
sigma(Φ(s));
grid on;
```

To do this, we investigate the singular value plot of $\Phi(s)$

1.67dB ←





↑
we check various points of the singular value plot to find "maximum singular" value

As can be seen in the figure,

maximum singular value was 1.67dB

and any singular value did not exceed

1.67dB. so we first convert dB into linear number

db2mag(1.67dB) $\Longrightarrow$ "1.2120"

Thus, our $\Phi(s)$ is achieving ($\Phi(s) = F_\ell(G,K)$)

$$\| F_\ell(G,K) \|_\infty < 1.2120 \quad (=\gamma)$$

Yes!

3-d)

Complementary Sensitivity Function

$$\Phi_2(s) = P(s)K_{subopt}(s)\left[I - P(s)K_{subopt}(s)\right]^{-1}$$

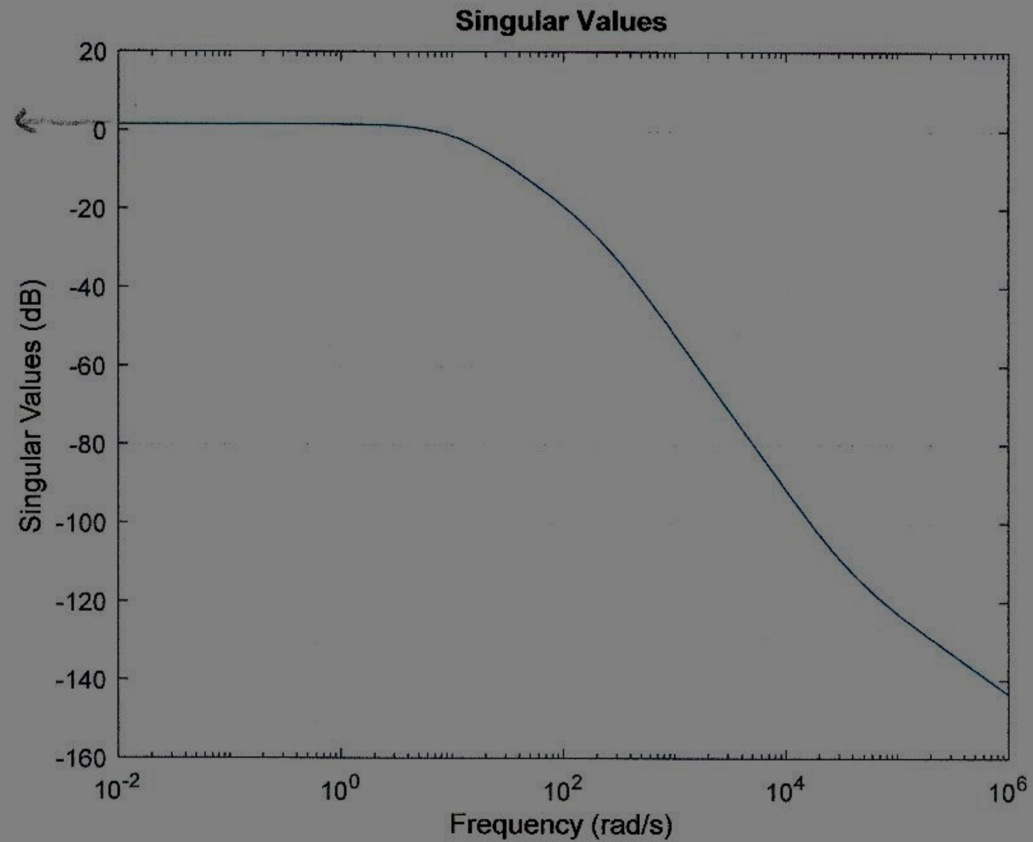
Plugging our $P(s)$ and $K_{subopt}(s)$ found from 2-c)... Then

$$\Phi_2(s) = \frac{-0.068076(s+3.83\times10^{4})(s+227.2)\cancel{(s-2)}(s+0.1063)(s+0.1)}{(s+227.2)(s+215.2)(s+10)\cancel{(s-2)}(s+0.1077)(s+0.1)}$$

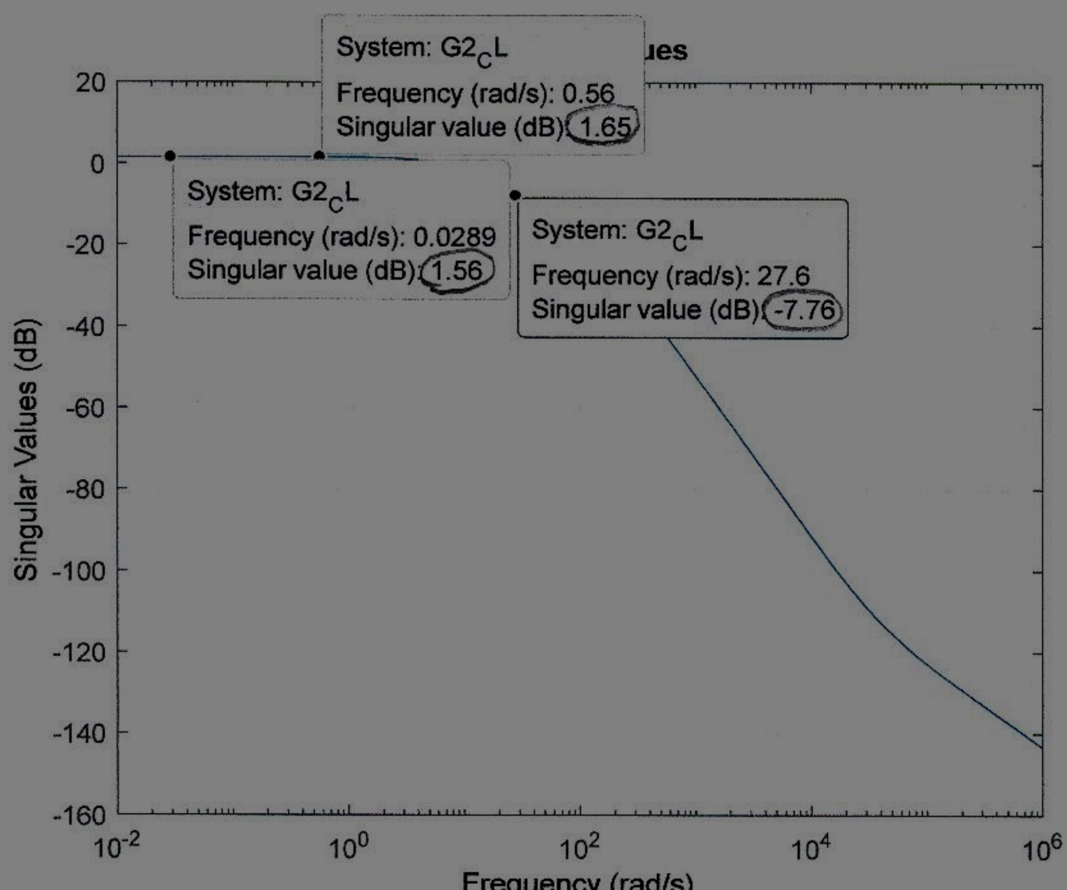
and we plot $|\Phi_2(jw)|$ as function of w (frequency)

```
figure;
sigma(Φ2(s));
grid on;
```

1.65 ≈
(dB)



Singular Values
Singular Values (dB)
Frequency (rad/s)



System: G2$_C$L
Frequency (rad/s): 0.56
Singular value (dB): 1.65
System: G2$_C$L
Frequency (rad/s): 0.0289
Singular value (dB): 1.56
System: G2$_C$L
Frequency (rad/s): 27.6
Singular value (dB): -7.76
Singular Values (dB)
Frequency (rad/s)

As can be seen from the plot max upper bound is approximately 1.65dB ($=1.2092$)

when we see our $\Phi_2(s)$...,

$$\Phi_2(s) = PK(1-PK)^{-1}$$

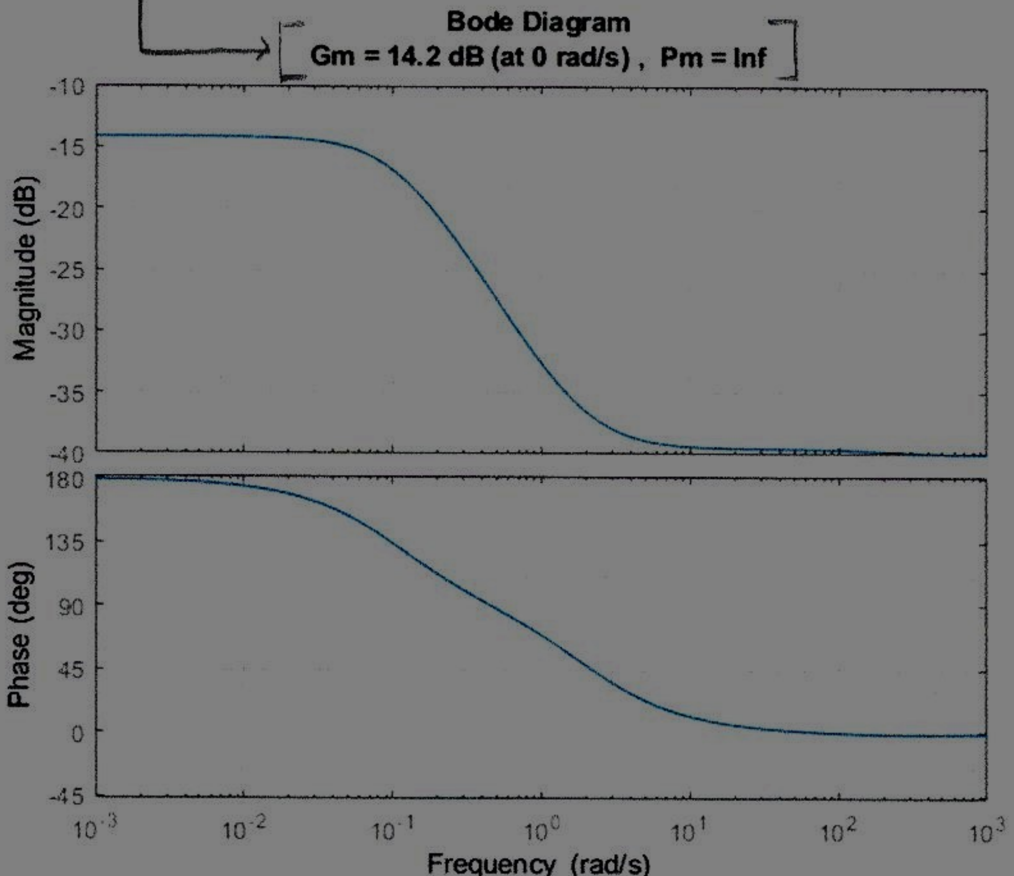
$\Phi_2(s)$ is called "output complementary sensitivity function" Its H_∞ norms indicate the "stability robustness of the closed loop system for the multiplicative plant uncertainty introduced at the output.

Therefore, what the plot is showing is that $K_{subopt}(s)$ controller found in 2-c) is working well in making $\|\Phi_2(s)\|_\infty$ always less than $\gamma = 1.2120$!

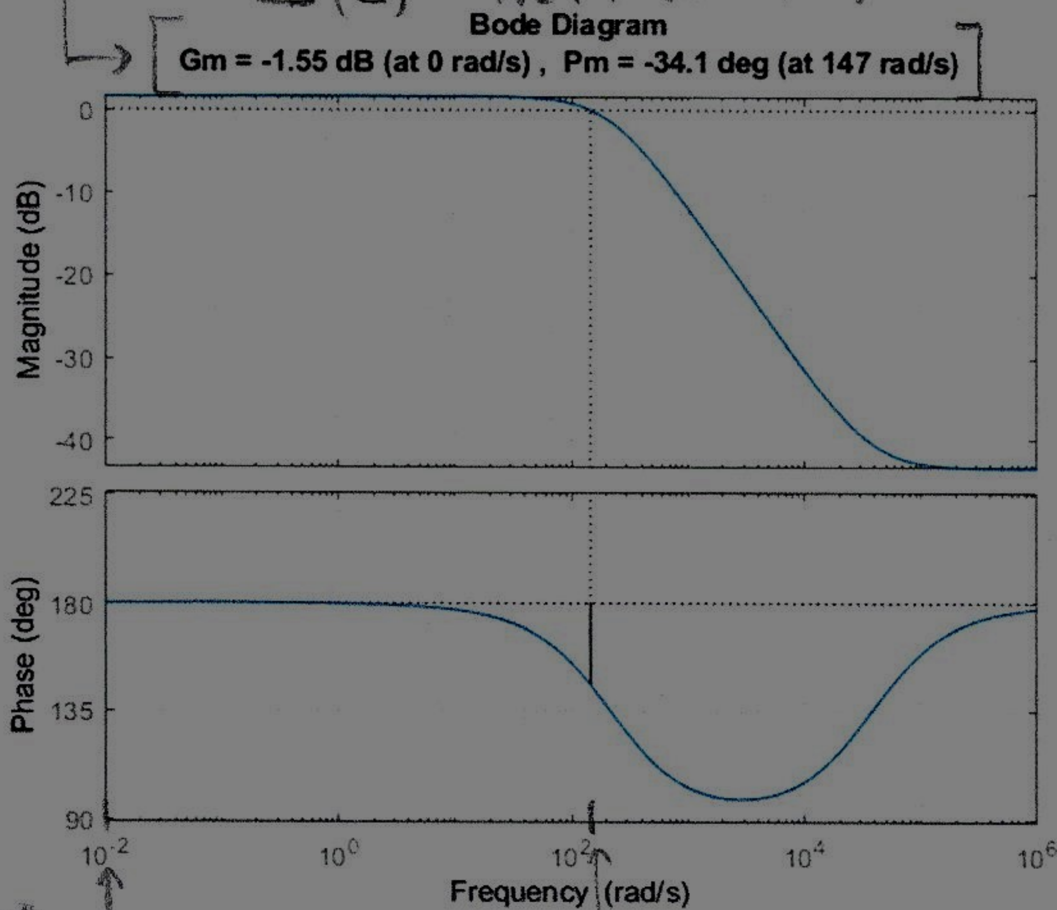
$$\|\Phi_2(s)\|_\infty < 1.2120 \; (=\gamma)$$

3-e)

$$\Phi(1) = W_1(I-PK)^{-1}$$



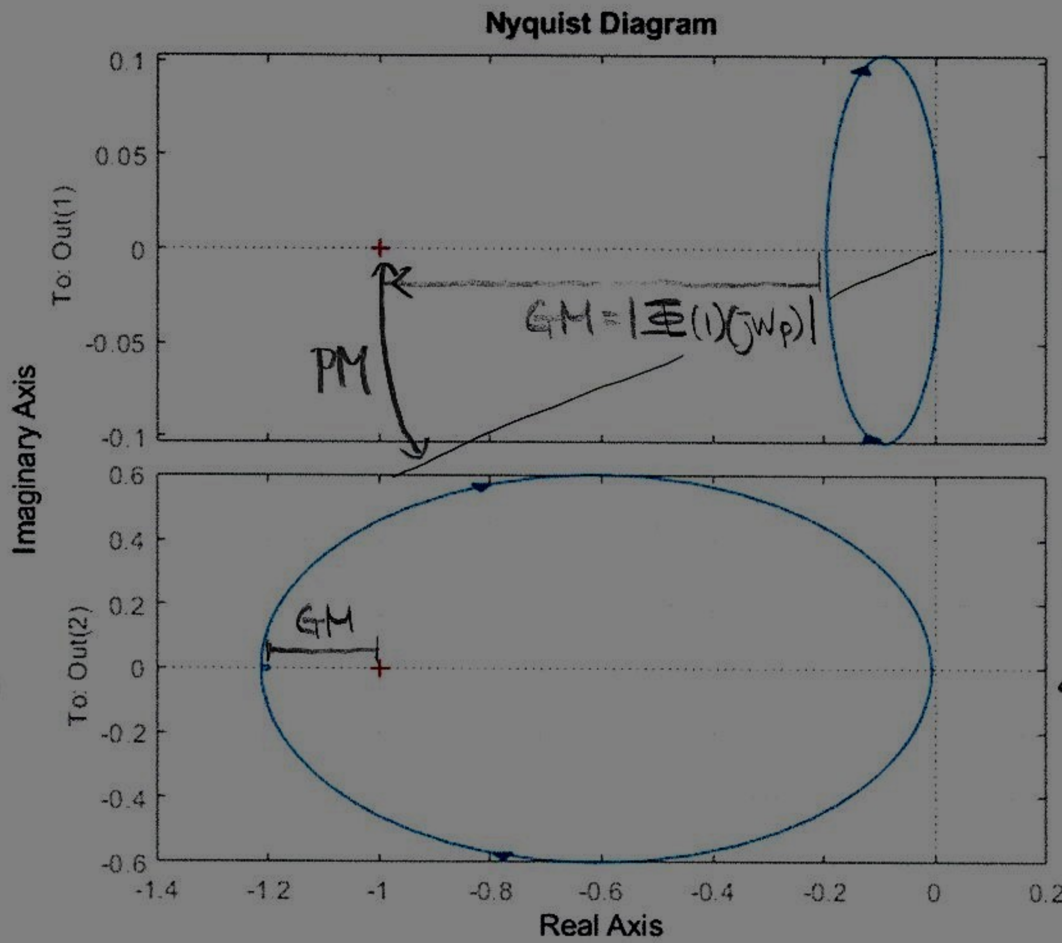
$$\Phi(2) = W_2PK(I-PK)^{-1}$$



$W_g = 0$

$W_p = 147$

$\Phi(1)$



$\Phi(2)$

$$GM = 20\log_{10}\frac{1}{|\Phi(1)(jW_p)|}$$

$$PM = \angle\Phi(2)(jW_g) - 180^\circ$$

$GM_{on}\ \Phi(1) = 14.2\ dB$ ($WP = 0$ rad/s)

$PM_{on}\ \Phi(1) = \infty$ ($WG = \infty$ rad/s)

$GM_{on}\ \Phi(2) = -1.55\ dB$ ($WP = 14.7$ rad/s)

$PM_{on}\ \Phi(2) = -34.1^\circ$ ($WG = 0$ rad/s)

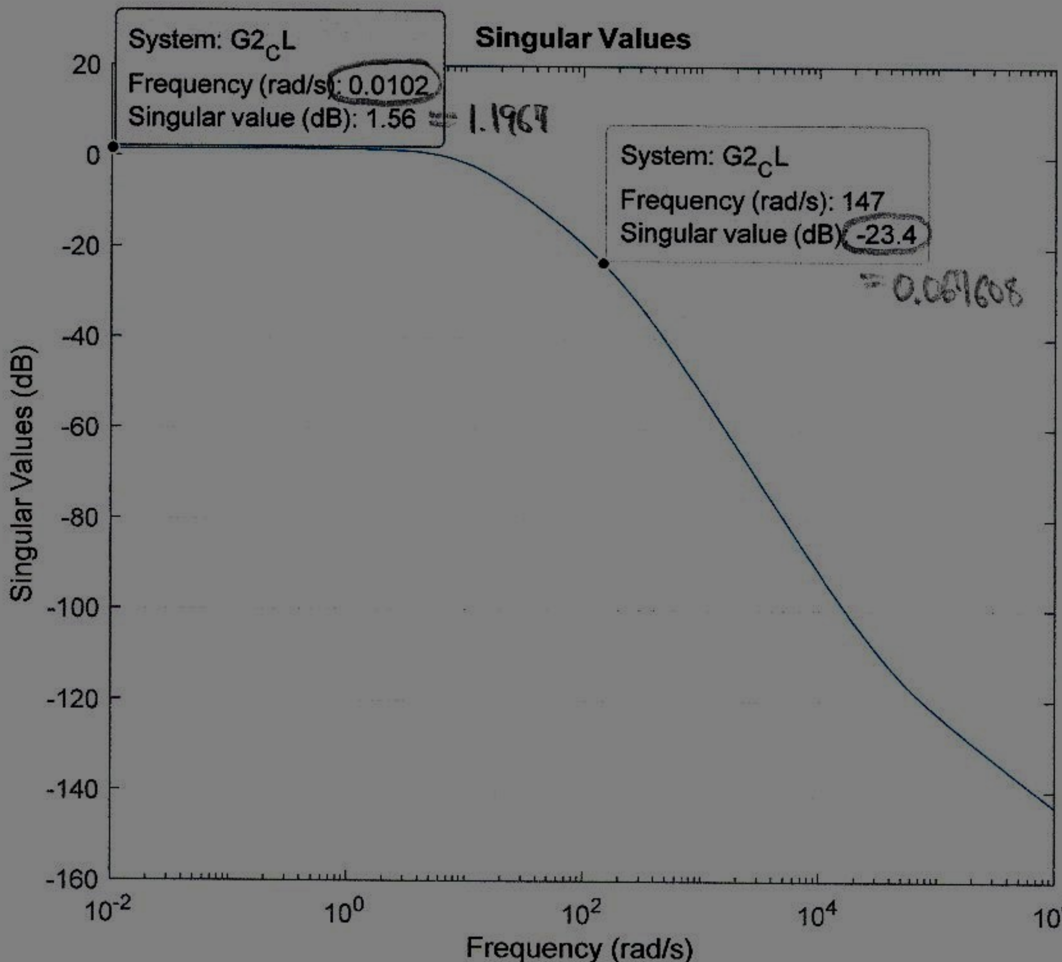
Thus, we choose

two points at $W_g = 0$, $W_p = 14.7$ (rad/s)

gain crossover freq

Phase crossover freq

Thus, two points are...



$\omega_g = 0$

$\text{sigma}(\Phi_2(s), \omega_g) = 1.196$

$\theta_m = -2\sin^{-1}\left(\frac{1}{2(1.196)}\right) = -49.425$ (← close enough)

$W_p = 147$

$\text{sigma}(\Phi_2(s), W_p) = 0.067921$

$G_m = 20\log_{10}\left(1 + \frac{1}{0.067921}\right) = 23.931\ dB$

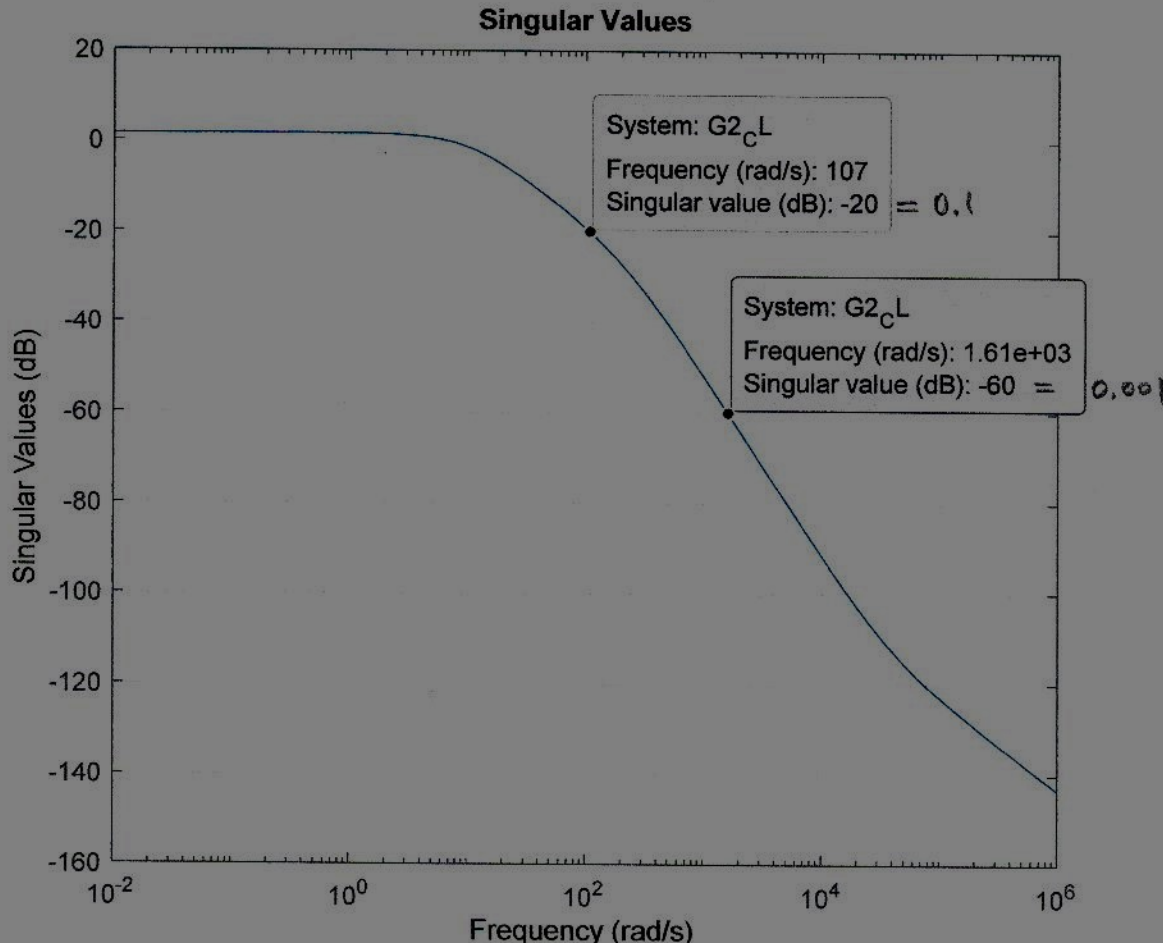
Considering Small Gain Theorem...

$$\bar{\sigma}[T(jw)] < \frac{1}{\bar{\sigma}[\Delta_M(jw)]} \quad \text{for all } w$$

(*smaller $\bar{\sigma}[T(jw)] \rightarrow$ better robust stability)

This means...

$\Phi_2(s)$ plot



At $\omega = 107$ rad/s, $\bar{\sigma}[T(j\omega)] < \dfrac{1}{\gamma(\omega)} = 0.1$

such that $\bar{\sigma}[\Delta_m(j\omega)] < \gamma(\omega) = 0.1$

Also,

At $\omega = 1613$ rad/s, $\bar{\sigma}[T(j\omega)] < \dfrac{1}{\gamma(\omega)} = 0.001$

such that $\bar{\sigma}[\Delta_m(j\omega)] < \gamma(\omega) = 0.001$