DREXEL UNIVERSITY
Department of Mechanical Engineering & Mechanics
Applied Engineering Analytical & Numerical Methods II

MEM 592 - Winter 2019

HOMEWORK #7: Due Friday, March 15, 2019 at 2:00 PM

1. [40 points]

Consider the problem of determining the steady-state heat distribution in a thin square metal plate 0.5 meters on a side. Two adjacent boundaries are held at 0°C, and the heat on the other boundaries increases linearly from 0°C at one corner to 100°C where the sides meet. If we place the sides with the zero boundary conditions along the x- and y-axes, the problem is expressed as:

$$\frac{\partial^2 u}{\partial x^2}(x,y) + \frac{\partial^2 u}{\partial y^2}(x,y) = 0,$$

for (x, y) in the set R = {(x, y) | 0 < x < 0.5; 0 < y < 0.5}, with the boundary conditions

$$u(0,y) = 0, \quad u(x,0) = 0, \quad u(x,0.5) = 200x, \qquad u(0.5,y) = 200y.$$

a) Write the Finite Difference equation using the centered-difference formula.
b) Use the formula in part (a) to find $u(x_i, y_j)$ values, where $x_i = a + ih$ and $y_j = c + jk$ for each $i = 1, \dots, n-1$ and $j = 1, \dots, m-1$. Use $m = 4$ and $n = 4$. $(a < x < b, c < y < d,\ h = \frac{b-a}{n}, k = \frac{d-c}{m})$
   - Fill a table like the following for your results.
   - Create a three-dimensional surface plot for your results to show the temperature distribution.

$\boldsymbol{u(x_i, y_j)}$	
$\boldsymbol{y_j}$	
$\boldsymbol{x_i}$	

c) Use $m = 40$ and $n = 40$ for part (b) and then create a three-dimensional surface plot for your results to show the temperature distribution.

2. [40 points]

Sagar and Payne [SP] analyze the stress-strain relationships and material properties of a cylinder subjected alternately to heating and cooling and consider the equation

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} = \frac{1}{4K}\frac{\partial T}{\partial t}, \qquad 0.5 < r < 1, \qquad T > 0$$

where T = T(r, t) is the temperature, r is the radial distance from the center of the cylinder, t is time, and K is a diffusivity coefficient.

a) Find the Backward-Difference scheme for the equation. (Use second-order central difference for first and second derivatives)
b) Find approximations to $T(r, 10)$ using the scheme in part (a) for a cylinder with outside radius 1, given the initial and boundary conditions:
$T(1,t) = 100 + 40t, \qquad T(0.5,t) = t, \qquad 0 \le t \le 10$
$T(r,0) = 200(r - 0.5), \qquad 0.5 \le r \le 1$
Use $\Delta t = 0.5, \Delta r = 0.1$ and write your results for $T(r, 10)$ in a table.
c) Do part (b) with $\Delta t = 0.5, \Delta r = 0.01$ and then plot your results for $T(r, 10)$.

3. [20 points]

Consider the following parabolic partial-differential equation

$$\frac{\partial^2 u}{\partial x^2}(x,t) - \frac{\partial u}{\partial t}(x,t) = 0, \qquad 0 < x < 1, \qquad t \ge 0$$

with boundary condition:
$u(1,t) = u(0,t) = 0, \qquad 0 \le t$
$u(x,0) = \sin(\pi x), \qquad 0 \le x \le 1$
has the exact solution $u(x,t) = \exp(-\pi^2 t)\sin(\pi x)$.

a) Use Backward FD to approximate the solutions at $u(x, 0.5)$. (Use $\Delta t = 0.01, \Delta x = 0.1$)
b) Use Crank-Nicolson method to approximate the solutions at $u(x, 0.5)$. (Use $\Delta t = 0.01, \Delta x = 0.1$)
c) Plot your results in (a) and (b) and compare them with the exact solution in one figure.
d) Do part (c) with $\Delta t = 0.01, \Delta x = 0.01$.

HW7 Hyukjun Kwon

1) $$\frac{\partial^2 u}{\partial x^2}(x,y) + \frac{\partial^2 u}{\partial y^2}(x,y) = 0 \qquad 0 < x < 0.5, \quad 0 < y < 0.5$$

BC

$u(0,y) = 0 \qquad u(x,0) = 0$

$u(0.5,y) = 200y \qquad u(x,0.5) = 200x$

a) Centeral-difference finite difference

$$\frac{u_{i+1}^{(j)} - 2u_i^{(j)} + u_{i-1}^{(j)}}{\Delta x^2} + \frac{u_i^{(j+1)} - 2u_i^{(j)} + u_i^{(j-1)}}{\Delta y^2} = 0 \qquad x \to i, \quad y \to j$$

$$(\Delta y^2) u_{i+1}^{(j)} + (\Delta y^2) u_{i-1}^{(j)} + (-2\Delta x^2 - 2\Delta y^2) u_i^{(j)}$$

$$+ (\Delta x^2) u_i^{(j+1)} + (\Delta x^2) u_i^{(j-1)} = 0$$

2) $$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} = \frac{1}{4k}\frac{\partial T}{\partial t}$$ $0.5 < r < 1$, $T > 0$

a) $$\frac{T_{j+1}^{(n)} - 2T_j^{(n)} + T_{j-1}^{(n)}}{\Delta r^2} + \frac{1}{r_j}\frac{T_{j+1}^{(n)} - T_{j-1}^{(n)}}{2\Delta r} = \frac{1}{4k}\frac{T_j^{(n)} - T_j^{(n-1)}}{\Delta t}$$

$$\left[\left(\frac{1}{\Delta r^2} + \frac{1}{r_j 2\Delta r}\right)T_{j+1}^{(n)} + \left(-\frac{2}{\Delta r^2} - \frac{1}{4k\Delta t}\right)T_j^{(n)} + \left(\frac{1}{\Delta r^2} - \frac{1}{r_j 2\Delta r}\right)T_{j-1}^{(n)} = \left(-\frac{1}{4k\Delta t}\right)T_j^{(n-1)}\right]$$

$j \to r$
$n \to t$

Tridiagonal System

$$\begin{bmatrix} \left(-\frac{2}{\Delta r^2} - \frac{1}{4k\Delta t}\right) & \left(\frac{1}{\Delta r^2} + \frac{1}{r_j 2\Delta r}\right) & & \\ \left(\frac{1}{\Delta r^2} - \frac{1}{r_j 2\Delta r}\right) & \left(-\frac{2}{\Delta r^2} - \frac{1}{4k\Delta t}\right) & \left(\frac{1}{\Delta r^2} + \frac{1}{r_j 2\Delta r}\right) & \\ & \ddots & \ddots & \ddots \\ & & \left(\frac{1}{\Delta r^2} - \frac{1}{r_j 2\Delta r}\right) & \left(-\frac{2}{\Delta r^2} - \frac{1}{4k\Delta t}\right) \end{bmatrix} \begin{bmatrix} T_{j-1} \\ T_j \\ T_{j+1} \end{bmatrix}_n = \left(\frac{-1}{4k\Delta t}\right)\begin{bmatrix} T_j \end{bmatrix}_{n-1}$$

BC

$T(0.5, t) = t$

$T(1, t) = 100 + 40t$

$(0 \le t \le 10)$

IC

$T(r, 0) = 200(r - 0.5)$

$(0.5 \le r \le 1)$

$\Delta t = 0.5$

$\Delta r = 0.1, 0.01$

$k = 0.1$

3) $\frac{\partial^2 u}{\partial x^2}(x,y) - \frac{\partial u}{\partial t}(x,t) = 0$ $\quad 0 < x < 1$, $t \geq 0$

BC $\quad u(0,t) = 0$, $u(1,t) = 0$ $\quad (0 \leq t)$

IC $\quad u(x,0) = \sin(\pi x)$ $\quad (0 \leq x \leq 1)$

Exact Solution: $u(x,t) = \exp(-\pi^2 t)\sin(\pi x)$

a) $$\frac{u_{j+1}^{(n)} - 2u_j^{(n)} + u_{j-1}^{(n)}}{\Delta x^2} - \frac{u_j^{(n)} - u_j^{(n-1)}}{\Delta t} = 0$$

$j \to x$

$n \to t$

$$\left(\frac{1}{\Delta x^2}\right)u_{j+1}^{(n)} + \left(-\frac{2}{\Delta x^2} - \frac{1}{\Delta t}\right)u_j^{(n)} + \left(\frac{1}{\Delta x^2}\right)u_{j-1}^{(n)} = \left(-\frac{1}{\Delta t}\right)u_j^{(n-1)}$$

Tridiagonal..

$$\begin{bmatrix} \left(-\frac{2}{\Delta x^2} - \frac{1}{\Delta t}\right) & \left(\frac{1}{\Delta x^2}\right) & & \\ \left(\frac{1}{\Delta x^2}\right) & \left(-\frac{2}{\Delta x^2} - \frac{1}{\Delta t}\right) & \left(\frac{1}{\Delta x^2}\right) & \\ & \ddots & \ddots & \ddots \\ & & \left(\frac{1}{\Delta x^2}\right) & \left(-\frac{2}{\Delta x^2} - \frac{1}{\Delta t}\right) \end{bmatrix} \begin{bmatrix} u_{j-1} \\ u \\ u_{j+1} \end{bmatrix}_n = \left(-\frac{1}{\Delta t}\right)\begin{bmatrix} u_j \end{bmatrix}_{n-1}$$

b) Applying trapezoidal method,

$$\frac{1}{2}\left[\frac{\partial^2 u^{(n+1)}}{\partial x^2} + \frac{\partial^2 u^{(n)}}{\partial x^2}\right]_j - \frac{u_j^{(n+1)} - u_j^{(n)}}{\Delta t} = 0$$

using 2nd-order FD..

$$u_j^{(n+1)} - u_j^{(n)} = \frac{\Delta t}{2}\left[\frac{u_{j+1}^{(n+1)} - 2u_j^{(n+1)} + u_{j-1}^{(n+1)}}{\Delta x^2} + \frac{u_{j+1}^{(n)} - 2u_j^{(n)} + u_{j-1}^{(n)}}{\Delta x^2}\right]$$

let $\beta = \frac{\Delta t}{2\Delta x^2}$ and rearrange

$$\left[\begin{aligned} -\beta u_{j+1}^{(n+1)} + (1+2\beta) u_j^{(n+1)} - \beta u_{j-1}^{(n+1)} \\ = \beta u_{j+1}^{(n)} + (1-2\beta) u_j^{(n)} + \beta u_{j-1}^{(n)} \end{aligned}\right]$$

$j \to x$
$n \to t$

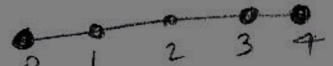
→ Tridiagonal System of Equation

→ So

$u(x, 0.5) = ?$

$\Delta t = 0.01$, $\Delta x = 0.1$, $u_j^{(n)}(x,t) = e^{-\pi^2 t}\sin(\pi x)$

$$\underbrace{\begin{bmatrix} 1+2\beta & -\beta & & \\ -\beta & 1+2\beta & -\beta & \\ & \ddots & \ddots & \ddots \\ & & -\beta & 1+2\beta \end{bmatrix}}_{3\times3}\begin{bmatrix} u_{j,} \\ u_{j,} \\ \vdots \end{bmatrix}^{n+1} = \begin{bmatrix} 1-2\beta & \beta & & \\ \beta & 1-2\beta & \beta & \\ & \ddots & \ddots & \ddots \\ & & \beta & 1-2\beta \end{bmatrix}\begin{bmatrix} u_{j,} \\ u_{j,} \\ \vdots \end{bmatrix}$$



$N+1 = 5$ $\qquad x_i = x_{i-1} + \Delta x$

2
3 = ?

$x_1 = ?$
$x_2 = ?$
$x_3 = ?$

MEM 592 - Homework 5

Author: Hyukjun Kwon

Date: March 15, 2019

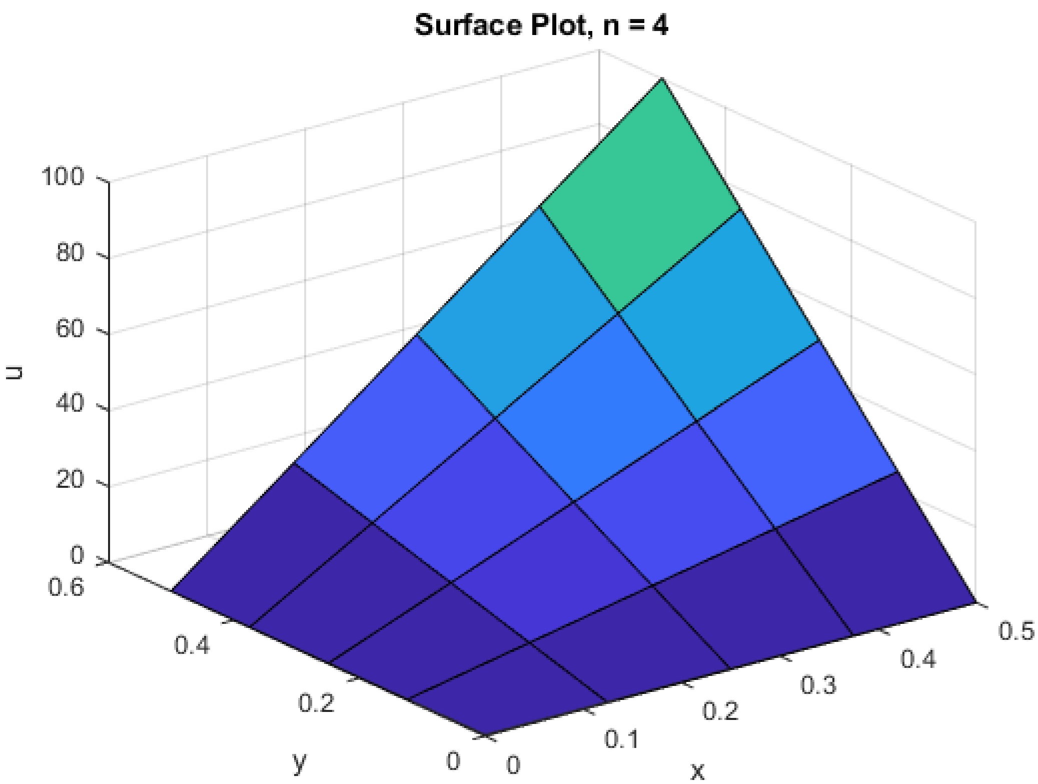

```
clear,clc,close all
```

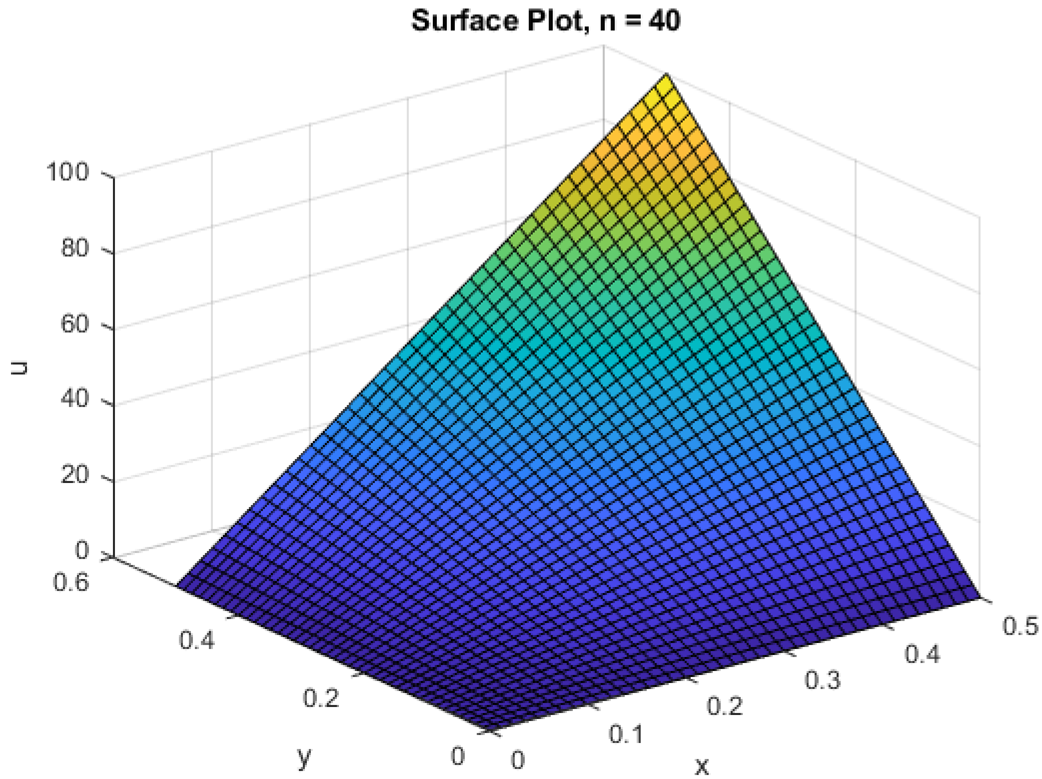
Problem #1 - Heat Distribution

```
LC = 0;
RC = @(y)(200*y);
TC = @(x)(200*x');
BC = @(x)(x'*0);

for n = [4 40]
    m=n;

    x_step=(0.5-0)/n;
    y_step=(0.5-0)/m;
    x=(0:x_step:0.5)';
    y=(0:y_step:0.5)';

    U = zeros(m+1,n+1);
    U(1,:)=BC(x);
    U(end,:)=TC(x);
    U(:,1)=LC;
    U(:,end)=RC(y);

    ROW=@(i_interval,j_interval)((i_interval-1)*(m-1)+j_interval);
    A = diag(-ones((n-1)*(m-1),1)*2*(x_step^2+y_step^2),0);
    RIGHT = zeros((n-1)*(m-1),1);

    for i=1:n-1
        for j=1:m-1
            A_row=ROW(i,j);
            switch i
                case 1
                    RIGHT(A_row)=-U(j+1,i)*y_step^2;
                    A(A_row,ROW(i+1,j))=y_step^2;
                case (n-1)
                    RIGHT(A_row)=-U(j+1,i+2)*y_step^2;
                    A(A_row,ROW(i-1,j))=y_step^2;
                otherwise
                    A(A_row,ROW(i+1,j))=y_step^2;
```

```
                A(A_row,ROW(i-1,j))=y_step^2;
            end
            switch j
                case 1
                    RIGHT(A_row)=RIGHT(A_row)-U(j,i+1)*x_step^2;
                    A(A_row,ROW(i,j+1))=x_step^2;
                case (m-1)
                    RIGHT(A_row)=RIGHT(A_row)-U(j+2,i+1)*x_step^2;
                    A(A_row,ROW(i,j-1))=x_step^2;
                otherwise
                    A(A_row,ROW(i,j+1))=x_step^2;
                    A(A_row,ROW(i,j-1))=x_step^2;
            end
        end
    end


    FLAG = 0;
    LEFT = A\RIGHT;
    UVECTOR = zeros((m+1)*(n+1),1);
    XVECTOR = UVECTOR;
    YVECTOR = UVECTOR;


    for i = 0:n
        XVECTOR(((m+1)*i+1):((m+1)*(i+1)))=ones(m+1,1)*x(i+1);
        YVECTOR(((m+1)*i+1):((m+1)*(i+1)))=y;
        for j = 0:m
            FLAG=FLAG+1;
            if i==0 || i==n || j==0 || j==m
                UVECTOR(FLAG)=U(j+1,i+1);
            else
                UVECTOR(FLAG)=LEFT(ROW(i,j));
                U(j+1,i+1)=UVECTOR(FLAG);
            end
        end
    end
    figure
    surf(x,y,U)
    grid on
    xlabel('x')
    ylabel('y')
    zlabel('u')
    title('Surface Plot, n = ' + string(n))
    if n==4
        TABLE = table(XVECTOR,YVECTOR,UVECTOR);
        TABLE.Properties.VariableNames={'x_values','y_values','u_values'};
        disp(TABLE)
    end
end
```

x_values	y_values	u_values
0	0	0
0	0.125	0
0	0.25	0
0	0.375	0
0	0.5	0
0.125	0	0
0.125	0.125	6.25
0.125	0.25	12.5
0.125	0.375	18.75
0.125	0.5	25
0.25	0	0
0.25	0.125	12.5
0.25	0.25	25
0.25	0.375	37.5
0.25	0.5	50
0.375	0	0
0.375	0.125	18.75
0.375	0.25	37.5
0.375	0.375	56.25
0.375	0.5	75
0.5	0	0
0.5	0.125	25
0.5	0.25	50
0.5	0.375	75
0.5	0.5	100

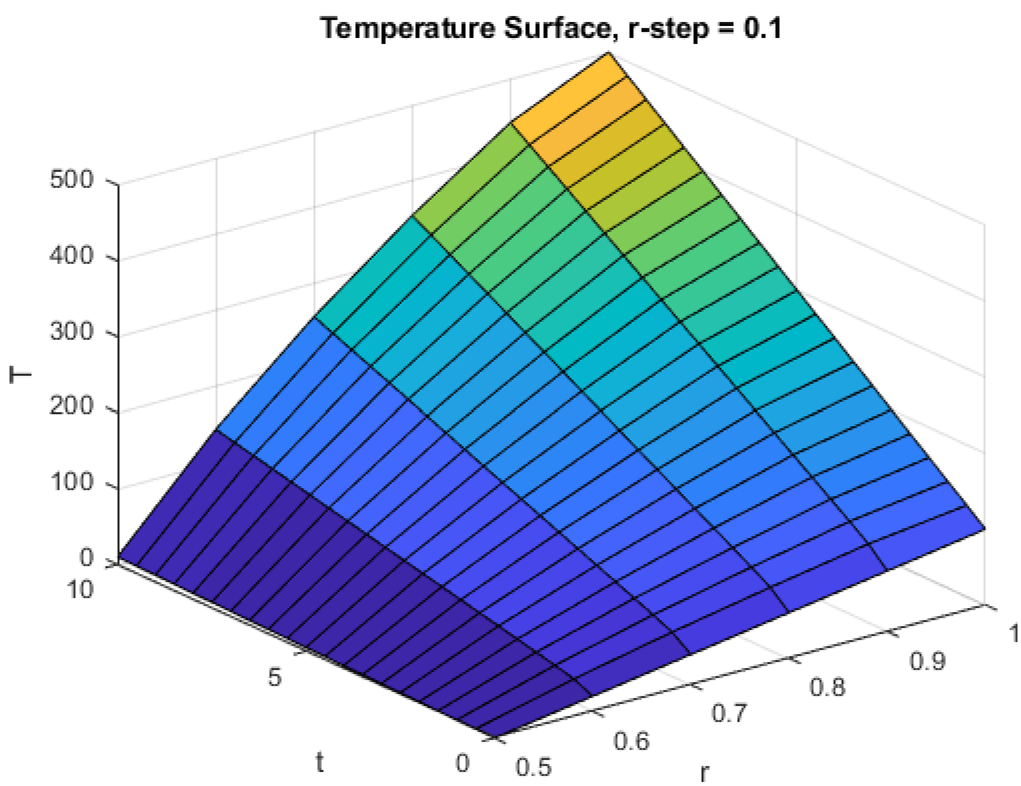

Surface Plot, n = 4
u
100
80
60
40
20
0
0.6
0.4
0.2
0
y
0
0.1
0.2
0.3
0.4
0.5
x

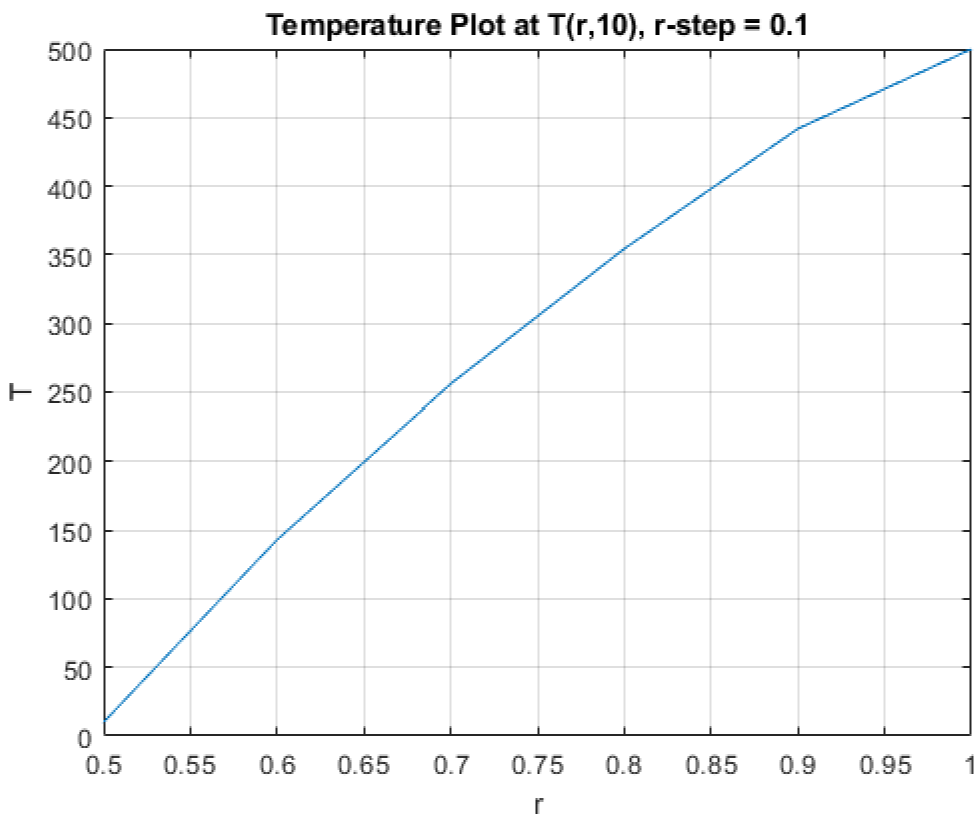


Problem #2 - Sagar and Payne

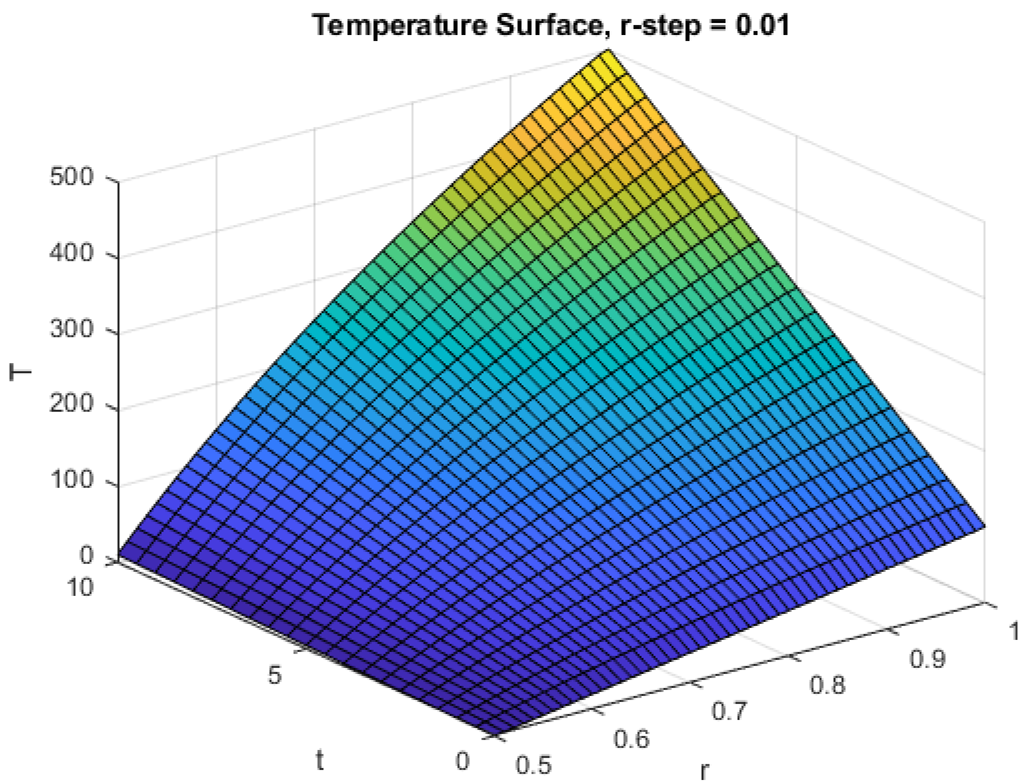
```
K = 0.1;
T_STEP=0.5;
LC = @(t)(t);
RC = @(t)(100+40*t);
BC = @(r)(200*(r'-0.5));


for R_STEP = [0.1 0.01]

    n=(1-0.5)/R_STEP;
    m=(10-0)/T_STEP;
    R_SPAN = (0.5:R_STEP:1)';
    T_SPAN = (0:T_STEP:10)';

    T = zeros(m+1,n+1);
    T(1,:)=BC(R_SPAN);
    T(:,1)=LC(T_SPAN);
    T(:,end)=RC(T_SPAN);

    A = diag(ones(n-1,1)*(-2/R_STEP-R_STEP/(4*K*T_STEP)),0)...
        + diag(1/R_STEP+0.5./R_SPAN(2:n-1),1)...
        + diag(1/R_STEP-0.5./R_SPAN(3:n),-1);
```

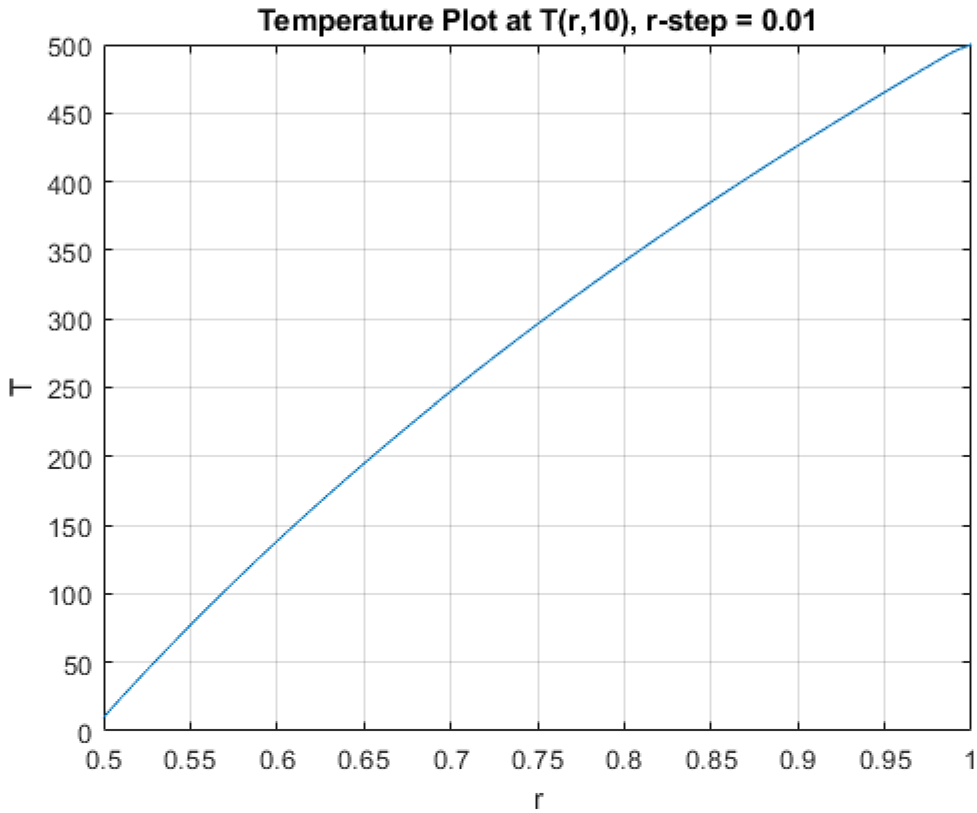
```
    for j = 1:m
        RIGHT = -T(j,2:n)'*R_STEP/(4*K*T_STEP);
        RIGHT(1)=RIGHT(1)-(1/R_STEP-0.5/R_SPAN(1))*T(j+1,1);
        RIGHT(end)=RIGHT(end)-(1/R_STEP+0.5/R_SPAN(1))*T(j+1,end);
        T(j+1,2:n)=(A\RIGHT)';
    end
    figure
    surf(R_SPAN,T_SPAN,T)
    xlabel('r')
    ylabel('t')
    zlabel('T')
    grid on
    title('Temperature Surface, r-step = ' + string(R_STEP))
    figure
    plot(R_SPAN,T(end,:))
    xlabel('r')
    ylabel('T')
    grid on
    title('Temperature Plot at T(r,10), r-step = ' + string(R_STEP))
end
```





Temperature Plot at T(r,10), r-step = 0.1
T
r
0
50
100
150
200
250
300
350
400
450
500
0.5
0.55
0.6
0.65
0.7
0.75
0.8
0.85
0.9
0.95
1


Temperature Surface, r-step = 0.01
T
500
400
300
200
100
0
10
5
0
t
0.5
0.6
0.7
0.8
0.9
1
r

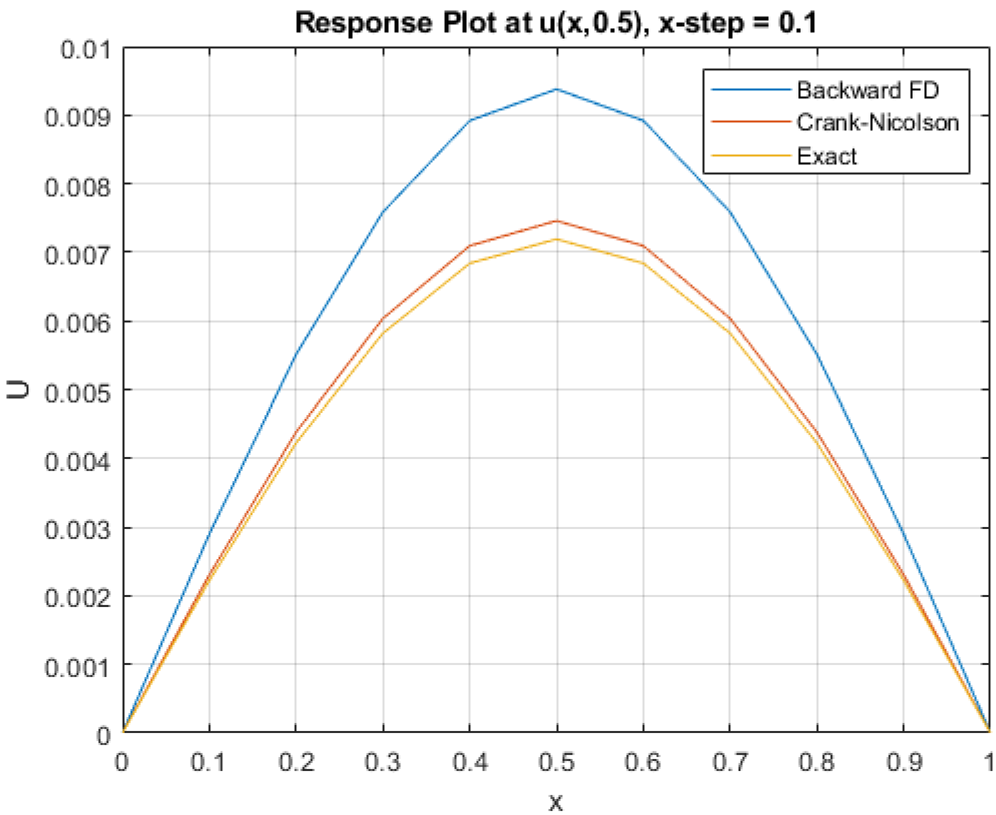


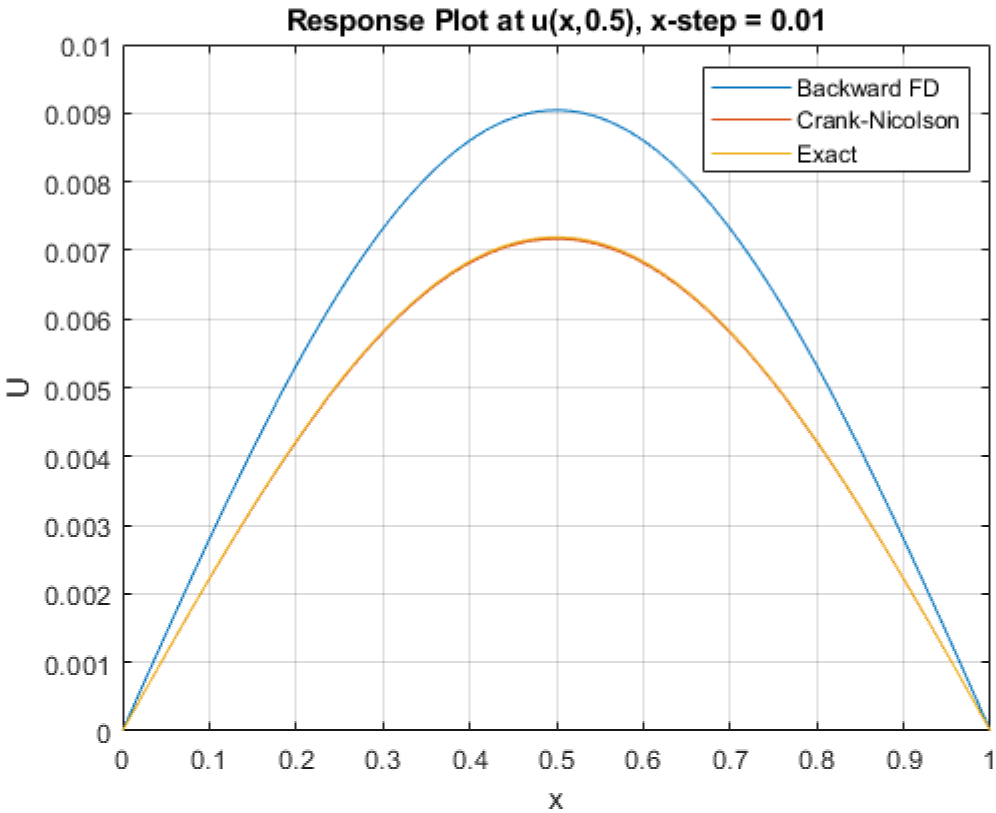
Problem #3 - Crank Nicolson

```
for x_step = [0.1 0.01]

t_step = 0.01;
t_desired = 0.5;
t = 0:t_step:t_desired;
x = 0:x_step:1;
beta = t_step/(2*(x_step^2));
beta_BFD = -2/(x_step^2)-(1/t_step);
EX = @(x,t) exp(-(pi^2)*t)*sin(pi*x);

% Backward FD
A_BFD = zeros(length(x)-2,length(x)-2);
A_BFD(1,1) = beta_BFD; A_BFD(1,2) = 1/(x_step^2);
A_BFD(end,end) = beta_BFD; A_BFD(end,end-1) = 1/(x_step^2);
for j = 2:length(x)-3
    A_BFD(j,j-1) = 1/(x_step^2);
    A_BFD(j,j) = beta_BFD;
    A_BFD(j,j+1) = 1/(x_step^2);
end

U_1_BFD = EX(x',0); %initial condition
U_1_BFD = U_1_BFD(2:end-1);
```

```matlab
clear U_BFD
U_BFD(:,1) = U_1_BFD;
for i = 1:length(t)-1
    U_BFD(:,i+1) = A_BFD\((-1/t_step)*U_BFD(:,i));
end

% Now add boundary condition
U_Temp_BFD = zeros(length(x),length(t));
for i = 2:length(x)-1
    U_Temp_BFD(i,:) = U_BFD(i-1,:);
end

% Crank-Nicolson
% Defining A
A = zeros(length(x)-2,length(x)-2);
A(1,1) = 1+2*beta; A(1,2) = -beta;
A(end,end) = 1+2*beta; A(end,end-1) = -beta;
for j = 2:length(x)-3
    A(j,j-1) = -beta;
    A(j,j) = 1+2*beta;
    A(j,j+1) = -beta;
end

% Defining B
B = zeros(length(x)-2,length(x)-2);
B(1,1) = 1-2*beta; B(1,2) = beta;
B(end,end) = 1-2*beta; B(end,end-1) = beta;
for j = 2:length(x)-3
    B(j,j-1) = beta;
    B(j,j) = 1-2*beta;
    B(j,j+1) = beta;
end

% Defining U
U_1 = EX(x',0); %initial condition
U_1 = U_1(2:end-1);

clear U
U(:,1) = U_1;
for i = 1:length(t)-1
    U(:,i+1) = A\B*U(:,i);
end

% Now add boundary condition
U_Temp = zeros(length(x),length(t));
for i = 2:length(x)-1
    U_Temp(i,:) = U(i-1,:);
end

figure
plot(x,U_Temp_BFD(:,end),x,U_Temp(:,end),x,EX(x',t_desired))
grid on
```

```
xlabel('x')
ylabel('U')
title('Response Plot at u(x,0.5), x-step = ' + string(x_step))
legend('Backward FD','Crank-Nicolson','Exact')
end
```





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